Improper colourings of graphs

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Abstract

We consider a generalisation of proper vertex colouring of graphs, referred to as *improper colouring*, in which each vertex can only be adjacent to a bounded number t of vertices with the same colour, and we study this type of graph colouring problem in several different settings.

The thesis is divided into six chapters. In Chapter 1, we outline previous work in the area of improper colouring. In Chapters 2 and 3, we consider improper colouring of unit disk graphs — a topic motivated by applications in telecommunications — and take two approaches, first an algorithmic one and then an average-case analysis. In Chapter 4, we study the asymptotic behaviour of the improper chromatic number for the classical Erdős-Rényi model of random graphs. In Chapter 5, we discuss acyclic improper colourings, a specialisation of improper colouring, for graphs of bounded maximum degree. Finally, in Chapter 6, we consider another type of colouring, *frugal colouring*, in which no colour appears more than a bounded number of times in any neighbourhood.

Throughout the thesis, we will observe a gradient of behaviours: for random unit disk graphs and "large" unit disk graphs, we can greatly reduce the required number of colours relative to proper colouring; in Erdős-Rényi random graphs, we do gain some improvement but only when t is relatively large; for acyclic improper chromatic numbers of bounded degree graphs, we discern an asymptotic difference in only a very narrow range of choices for t.

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Chapter 1

Introduction

Graph colouring, the origins of which can be traced back for over a century and a half, is a wide-ranging field that touches many subjects, not only within mathematics, but also in operations research, engineering, as well as the physical and biological sciences. For more background into the field, consult the monograph by Jensen and Toft [50]

Here is the simple premise. First off, a *graph* is an abstract set V of points — called *vertices* — together with a set E of unordered pairs of points — called *edges*. If two vertices share an edge, then we say they are *adjacent*. Given a graph $G = (V, E)$, the object of graph colouring, in the traditional sense, is to deliver an assignment of integers — or *colours* — to all of V so that no two adjacent vertices receive the same colour. Such an assignment is called a *proper (vertex) colouring* and it is desirable to use as few colours as possible.

The most famous graph colouring problem is the Four Colour Problem (cf. [31, 83]), proposed by Francis Guthrie in 1852. It asks, is it possible to colour any given map using only four colours so that no two neighbouring countries receive the same colour? This question is innocuous in appearance, but has turned out to be extremely difficult, testing the ingenuity of some of the most brilliant mathematical minds for well over a century. The first valid solution was given thirty years ago by Appel and Haken [9, 10], who gave a proof involving a computer-aided case analysis that tested hundreds of thousands of possibilities. Even after significant contributions from Robertson, Sanders, Seymour and Thomas [76, 77], it remains open whether there exists an argument that could be verified by hand within a human lifespan. The deceptive simplicity of the Four Colour Problem is a common feature

of many graph colouring problems.

Although the Four Colour Theorem is not a particularly good example, many graph colouring problems, especially the more recent ones, originate in real-world applications. This is because graphs are treated as a basic model of networks and graph colouring as a way of measuring resource conflicts. Frustratingly, but also expectedly, even the most basic of these problems often turn out to be extremely complex on any practical (large) scale. In the language of computational complexity theory, these problems are usually NP-hard; that is, they are likely to require an inordinate amount of computational resources for large instances. For this reason, researchers have turned to a number of different approaches for coping with NP-hard problems, some of which we employ in this work — for instance, we will consider some approximation algorithms, perform average-case analysis, and study asymptotic behaviour. Consult the classic text by Garey and Johnson [34] for further background into computational complexity.

For most of this thesis, we will focus on a natural modification of proper colouring — one we will refer to as *improper colouring* — that can be considered to be a generalisation of the original notion. Before we delve into more precise definitions, let us describe it as follows. We shall be given a class of graphs together with a parameter t. The parameter t can be treated as a measure of how far away from proper we allow our vertex colourings to be. Our goal is to determine what improvements if any (in terms of t and the class of graphs under consideration) we can achieve to reduce the number of colours required, in comparison to proper colourings.

In the remainder of the chapter, we formally introduce improper colouring and review some basic results that will be used several times throughout the thesis. We will assume some elementary graph theory and computational complexity. For any unfamiliar concepts or notation, please refer to the standard texts of West [82] and Garey and Johnson [34], respectively.

1.1 Improper colouring

Given a graph $G = (V, E)$ and some positive integer k, we shall say that a *(vertex) k-colouring* of G is a map $c : V \rightarrow \{1, \ldots, k\}$. For $v \in V$, the *colour* of v (under c) is precisely $c(v)$. A colouring c may be considered as a partition of the vertex set V into *colour classes* $V_1, \ldots, V_k$ where $V_i = c^{-1}(i)$ for any $i \in \{1, \ldots, k\}$, i.e. a colour class is a set of all vertices with the same colour.

We say that a colouring c of G is *proper* if no vertex has the same colour as any of its neighbours. Alternatively, a colouring of G is proper if each colour class is an *independent set* — that is, a subset of V that induces a subgraph of G with no edges. A graph G is said to be *(properly) k-colourable* if there exists a proper k-colouring of G. We define the *chromatic number* of G to be

$$\chi(G) := \min\{k \mid G \text{ is } k\text{-colourable}\}.$$

The related *independence number* of G is defined as

$$\alpha(G) := \max\{|X| \mid X \subseteq V \text{ is an independent set}\}.$$

Given a graph $G = (V, E)$, a colouring c of G, and a subset S of V, the *impropriety* of a vertex v restricted to S under c, denoted $\text{im}_S^c(v)$, is the number of neighbours of v in S with the same colour as v. The *impropriety* of c in S is $\text{im}_S(c) := \max\{\text{im}_S^c(v) \mid v \in S\}$. We will often drop the superscript or subscript if the context of the impropriety is clear. We denote $\text{im}_V(c)$ by $\text{im}(c)$.

A colouring c is *t-improper* if $\text{im}(c) \leq t$. Alternatively, a colouring of G is t-improper if each colour class is a *t-dependent set* — that is, a subset of V that induces a subgraph with maximum degree at most t. A graph G is *t-improperly k-colourable* if there exists a t-improper k-colouring of G. The *t-improper chromatic number* of G is defined to be

$$\chi^t(G) := \min\{k \mid G \text{ is } t\text{-improperly } k\text{-colourable}\}.$$

We shall also consider the related *t-dependence number* of G, defined as

$$\alpha^t(G) := \max\{|X| \mid X \subseteq V \text{ is a } t\text{-dependent set}\}.$$

The notion of t-improper colouring is identical to that of proper colouring in the particular case $t = 0$; thus, $\chi(G) = \chi^0(G)$ and $\alpha(G) = \alpha^0(G)$ for any graph G.

The t-improper chromatic number was introduced two decades ago independently by Andrews and Jacobson [8], Harary and Fraughnaugh (née Jones) [42,43], and Cowen, Cowen and Woodall [22]. In the first paper, the authors considered various general lower bounds for the t-improper chromatic number; in the second, the authors studied χ^t as part of the larger setting of generalised chromatic numbers; in the third, the authors established best upper bounds on χ^t for planar graphs to generalise the Four Colour Theorem.

Improper colouring of planar graphs

Let us briefly discuss improper colouring of planar graphs as first studied by Cowen, Cowen and Woodall [22]. They asked the following: for fixed t, what is the smallest p_t such that every planar graph is t-improperly p_t-colourable? It was pleasantly surprising that this question could be fully answered.

Theorem 1.1 (Cowen, Cowen and Woodall [22])**.** $p_0 = p_1 = 4$ *and* $p_t = 3$ *for all* $t \geq 2$.

Here, we highlight a related problem that is of continuing interest. The list colouring analogue of the above problem is informally stated as follows: what is the smallest p_t^* such that for any planar graph G, if every vertex is assigned an arbitrary list of at least p_t^* distinct colours, then G admits a t-improper colouring whose colours are chosen from the lists? Due to Voigt [81] and Thomassen [80], we know that $p_0^* = 5$; independently, Eaton and Hull [26] and Škrekovski [79] showed that $p_2^* = 3$; however, it is is still open whether p_1^* is 4 or 5.

It was not until a decade after the first papers on improper colouring appeared when Cowen, Goddard and Jesurum [23] considered the computational complexity of improper colourability of (planar) graphs. The proof of the Four Colour Theorem naturally extends to an algorithm for 4-colouring a planar graph [11]; indeed, a quadratic-time algorithm

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has been developed [76]. The proof of planar 2-improper 3-colourability [22] naturally extends to a linear-time algorithm. It is well-known that it is NP-complete to determine if a planar graph is 3-colourable and that 2-colouring is polynomial-time for general graphs. The remaining complexity questions are answered as follows.

Theorem 1.2 (Cowen, Goddard and Jesurum [23])**.**

(i) *It is NP-complete to determine if a planar graph is 1-improperly 3-colourable.*

(ii) *For any fixed positive integer t, it is NP-complete to determine if a planar graph is t-improperly 2-colourable.*

Furthermore, for general graphs and fixed t, it is known that (as for proper colouring) there exists $\varepsilon > 0$ such that χ^t cannot be approximated to within a factor of n^ε, unless P = NP [23].

Basic bounds

Next, let us review some general bounds that relate the t-improper chromatic number to other fundamental graph properties. First of all, observe that if a graph is t-improperly k-colourable, then it is t_1-improperly k_1-colourable for any $t_1 \geq t$ and $k_1 \geq k$. As a consequence, the following trivial upper bound holds.

Proposition 1.3. *For any graph G and any integer $t \geq 0$, $\chi(G) \geq \chi^t(G) \geq \chi^{t+1}(G)$.*

Recall the following basic upper bound in terms of the maximum degree $\Delta(G)$ of G.

Proposition 1.4. *For any graph G, $\chi(G) \leq \Delta(G) + 1$.*

In 1966, Lovász [56] demonstrated a graph decomposition result for graphs of bounded maximum degree which, rephrased in terms of improper colouring, is essentially the following.

Proposition 1.5 (Lovász [56], cf. [23])**.** *Let $G = (V, E)$ be a graph with maximum degree $\Delta \leq k(t+1)-1$ for fixed non-negative integers t and k. There is an algorithm to t-improperly k-colour G in $O(\Delta \cdot |E)|)$ time.*

A t-improper analogue to Proposition 1.4 is a corollary.

Corollary 1.6. *For any graph G and any integer $t \geq 0$, $\chi^t(G) \leq \left\lceil \frac{\Delta(G)+1}{t+1} \right\rceil$.*

We remark here that not all bounds for the chromatic number are easily generalised to the t-improper chromatic numbers. Recall that Brooks' Theorem states that every graph G that is not a clique or an odd cycle satisfies $\chi(G) \leq \Delta(G)$; however, it is shown in [23] that, if $t \geq 1$ and $k = 2$, then it is NP-complete to determine if a given graph of maximum degree $k(t+1)$ is t-improperly k-colourable. In other words, there is no t-improper analogue of Brooks' Theorem unless P = NP. As another example, recall that the *degeneracy* of G is $\delta^*(G) = \max\{\delta(H) \mid H \subseteq G\}$ (where $\delta(H)$ denotes the minimum degree of H) and that, for any graph G, $\chi(G) \leq \delta^*(G)+1$; however, no nontrivial t-improper version of this bound is known.

As for lower bounds, recall that a *clique* is a set of pairwise adjacent vertices and the *clique number* $\omega(G)$ of G is the size of the largest clique in G. Here are two elementary bounds for $\chi(G)$ in terms of the clique and independence numbers.

Proposition 1.7. *For any graph $G = (V, E)$, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V|}{\alpha(G)}$.*

The following are generalisations of these bounds to t-improper colouring.

Proposition 1.8. *For any graph G and any integer $t \geq 0$, $\chi^t(G) \geq \frac{\chi(G)}{t+1} \geq \frac{\omega(G)}{t+1}$.*

Proposition 1.9. *For any graph $G = (V, E)$ and any integer $t \geq 0$, $\chi^t(G) \geq \frac{|G|}{\alpha^t(G)}$.*

To see that the first bound holds, notice that each colour class of a t-improper colouring induces a subgraph with maximum degree at most t and hence can be partitioned into at most $t+1$ independent sets (by Proposition 1.4). The second bound holds since the size of any colour class of a t-improper colouring of G is at most $\alpha^t(G)$.

Propositions 1.8 and 1.3 demonstrate that, as a function of $\chi(G)$, the t-improper chromatic number of G lies between $\chi(G)/(t+1)$ and $\chi(G)$ and we will often find it interesting to determine where it belongs along this range.

1.2 Probabilistic tools

In a large proportion of this thesis, we employ probabilistic techniques. We here state two of the basic tools. The reader may consult the texts of Molloy and Reed [66] and Janson,

Łuzcak and Rucinski [49] for further reference.

We use two versions of the Lovász Local Lemma [27].

Symmetric Lovász Local Lemma ([27], cf. [66], page 40)**.** *Let $\mathcal{E}$ be a set of (typically bad) events such that for each $A \in \mathcal{E}$*

(i) $\Pr(A) \leq p < 1$, *and*

(ii) *A is mutually independent of a set of all but at most δ of the other events.*

If $ep(\delta + 1) < 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

General Lovász Local Lemma ([27], cf. [66], page 222)**.** *Let $\mathcal{E} = \{A_1, \ldots, A_n\}$ be a set of (typically bad) events such that each A_i is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_i \cup \{A_i\})$ for some $\mathcal{D}_i \subseteq \mathcal{E}$. If there are real weights $0 \leq x_i < 1$ such that for all i*

$$\Pr(A_i) \leq x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j),$$

then the probability that none of the events in $\mathcal{E}$ occur is at least $\prod_{i=1}^{n}(1 - x_i) > 0$.

We also use the following bounds on the tails of the binomial distribution $\mathrm{Bin}(n, p)$.

Some Chernoff Bounds (cf. inequalities (2.4–6) and (2.9) of [49])**.** *Suppose $X \in \mathrm{Bin}(n, p)$, i.e. X has a binomial distribution with parameters n and p. If $0 \leq t \leq n - np$, then*

$$\Pr(X \geq np + t) \leq \left(\frac{np}{np + t}\right)^{np+t} \left(\frac{n - np}{n - np - t}\right)^{n-np-t}. \tag{1.1}$$

The following simpler bounds are corollaries to (1.1), but usually suffice for most purposes. If $t \geq 0$, then

$$\Pr(X \geq np + t) \leq \exp\left(-\frac{t^2}{2(np + t/3)}\right) \quad and \tag{1.2}$$

$$\Pr(X \leq np - t) \leq \exp\left(-\frac{t^2}{2np}\right). \tag{1.3}$$

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If $0 < \varepsilon \leq 3/2$, then

$$\Pr(|X - np| \geq \varepsilon np) \leq 2 \exp\left(-\frac{\varepsilon^2 np}{3}\right). \tag{1.4}$$

1.3 Asymptotic notation

We will frequently make use of standard notation to compare the relative asymptotic be-haviour of two real sequences $f, g : \mathbb{Z}^+ \to \mathbb{R}$ that depend on a parameter $n \to \infty$. We write that (as $n \to \infty$)

- $f(n) = O(g(n))$ if there exist constants $C > 0$ and n_0 such that $|f(n)| < C|g(n)|$ for all $n \geq n_0$;

- $f(n) = \Omega(g(n))$ if there exist constants $C > 0$ and n_0 such that $C|g(n)| < |f(n)|$ for all $n \geq n_0$;

- $f(n) = \Theta(g(n))$ if there exist constants $C, C' > 0$ and n_0 such that $C|g(n)| < |f(n)| < C'|g(n)|$ for all $n \geq n_0$, i.e. if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$;

- $f(n) \sim g(n)$ if $\lim_{n \to \infty} f(n)/g(n) = 1$;

- $f(n) = o(g(n))$ if $\lim_{n \to \infty} f(n)/g(n) = 0$, i.e. if for any constant $C > 0$ there exists n_0 such that $|f(n)| < C|g(n)|$ for all $n \geq n_0$; and

- $f(n) = \omega(g(n))$ if $\lim_{n \to \infty} f(n)/g(n) = \infty$, i.e. if for any constant $C > 0$ there exists n_0 such that $C|g(n)| < |f(n)|$ for all $n \geq n_0$.

Chapter 2

Improper colouring of unit disk graphs

Given a set V of points in the plane and a distance threshold $r > 0$, we let $G(V, r)$ denote the following graph. The vertex set is V and distinct vertices are joined by an edge whenever the Euclidean distance between them is less than r. Any graph isomorphic to such a graph is called a *unit disk graph*. Alternatively, a unit disk graph is the intersection graph of equal-sized open disks in the the plane. Note that by rescaling we may assume that $r = 1$: this motivates the name *unit* disk graph.

Any set V together with the distance parameter r (or alternatively, the set of radius $r/2$ disks centred on the points of V) which gives rise to a unit disk graph G is called a *(unit disk) representation* of G. It is interesting to note that it is NP-hard to determine whether a given graph has a representation, i.e. whether it is a unit disk graph [19]. We shall assume, though, that each unit disk graph is accompanied by one of its representations. We remark here that for each representation of a unit disk graph using open disks, there exists an equivalent representation using an intersection model of closed disks, and *vice versa*; therefore, unless otherwise stated, we will assume when describing representations that all disks are open disks.

The study of the class of unit disk graphs stems partly from applications in communication networks. In particular, the problem of finding a proper vertex colouring of a given

unit disk graph is closely associated with the so-called *frequency allocation problem* [41]. For example, one might perceive the disks to be the transmission areas of a fixed set of radio towers, and a proper colouring of the associated unit disk graph would be an interference-free allocation of radio frequencies to these towers. Consult Leese and Hurley [55] for a more general treatment of this important problem. The problem of colouring unit disk graphs is one of the simplest models for radio channel assignment and also one of the most well-studied (for example, see [21, 37, 41, 58, 59]).

In this chapter and the next, we consider the problem of t-improperly colouring unit disk graphs. This is a problem which arises in practice for instance when modelling certain satellite communications problems. More precisely, Alcatel Industries has proposed the following problem. A satellite sends information to receivers on Earth and, because it is technically difficult to precisely focus the satellite's signal upon a receiver, part of the signal spills over into the surrounding area creating noise for nearby receivers listening on the same frequency. A receiver is able to distinguish its particular signal from the noise if the sum of total noise does not exceed a certain threshold. The problem is to simultaneously assign frequencies to the receivers in such a way that each receiver can obtain its intended signal properly and use as few frequencies as possible.

In the simplest model of this problem, we assume that each receiver's signal contributes noise to other receivers within a disk-shaped area surrounding it, and that the radii of the noise disks and the intensity of the noise created by the signals are independent of the frequency and the receiver. Hence, to distinguish its signal from the noise, a receiver must be in the noise disks of at most t receivers (where t is a fixed integer depending on the noise threshold and the intensity of the signals) listening on the same frequency. It is clear that, under this model, we are precisely asking to find χ^t for a given unit disk graph.

In this chapter, we study the *unit disk graph t-improper colourability problem* defined for any fixed non-negative integer t as follows.

UD t-IMPROPER CHROMATIC NUMBER

INSTANCE: a unit disk graph G.

QUESTION: what is $\chi^t(G)$?

Our aim is to show that this computational problem is NP-hard. We also consider restricted classes and discuss (in)approximability.

2.1 An overview of unit disk graphs

Classes of graphs related to unit disk graphs

Besides the class of unit disk graphs, we consider some related sub- and superclasses. Denote the class of unit disk graphs by $\mathcal{UD}$. The class of intersection graphs on disks of arbitrary radii in the plane is the class $\mathcal{D}$ of *disk graphs*. This superclass of $\mathcal{UD}$ is relevant to radio channel assignment in that it models radio towers with varying transmission power. Another generalisation of $\mathcal{UD}$ would be to consider higher dimensions; for instance, consider the intersection graph of open (unit) balls in space. We have not yet considered such classes. On the other hand, it is important to consider the restrictions of $\mathcal{UD}$ and $\mathcal{D}$ to one dimension: the class $\mathcal{UI}$ of *unit interval graphs* (a.k.a. *indifference graphs*) and the class $\mathcal{I}$ of *interval graphs* have been extensively studied. These two classes are chordal (hence perfect) classes and many computational problems are feasible when restricted to these classes; consult Golumbic [36] for more background into chordal graphs. An important restricted class of unit disk graphs (specific to radio channel assignment) is that of weighted induced subgraphs of the triangular lattice, or *hexagonal graphs* [6, 63, 69]. This class is related to a common placement pattern of radio transmission towers in a cellular communications network: for efficient coverage, the transmitters are only placed on points of a (unit edge length) triangular lattice, possibly with more than one transmitter corresponding to one point. We denote the class of hexagonal graphs by $\mathcal{TL}$. We denote the class of planar graphs by $\mathcal{P}$.

Complexity on restriction to unit disk graphs

For channel assignment, three computational problems are of particular interest. Besides CHROMATIC NUMBER — given a graph G, what is the chromatic number $\chi(G)$ of G — the most relevant problems are MAX CLIQUE — given a graph G, what is the clique number $\omega(G)$ of G — and MAX INDEPENDENT SET — given a graph G, what is the

independence number $\alpha(G)$ of G. See Table 2.1 for a summary of what is known about these problems, comparing the restrictions to interval graphs, to unit disk graphs, to planar graphs, to disk graphs and to weighted induced subgraphs of the triangular lattice. Later, we shall be able to add two rows to this table that correspond to t-IMPROPER CHROMATIC NUMBER and MAX t-DEPENDENT SET.

Table 2.1: Relative complexity for certain problems restricted to the graph classes $\mathcal{I}$, $\mathcal{UD}$, $\mathcal{P}$, and $\mathcal{TL}$.

Problem	$\mathcal{I}$	$\mathcal{UD}$	$\mathcal{P}$	$\mathcal{D}$	$\mathcal{TL}$
CHROMATIC NUMBER	P [70]	NPc [21]	NPc	NPc	NPc [63]
MAX CLIQUE	P [70]	P [21]	P	Open	P
MAX INDEPENDENT SET	P [35]	NPc [21]	NPc	NPc	NPc

Of particular note is the polynomial-time algorithm for UD MAX CLIQUE. Clark, Colbourn and Johnson [21] exhibited a clever algorithm which utilises the geometry to find subgraphs that can be solved using the polynomial-time algorithm for cobipartite graphs. There is also a polynomial-time algorithm for UD MAX CLIQUE that does not require a representation [75].

Colouring of unit disk graphs

Let us delve into more detail for the problem of computing the chromatic number of unit disk graphs. Clark *et al.* [21] demonstrated NP-hardness of this problem by showing the slightly stronger result that, for the class of unit disk graphs, the computational problem of 3-COL — given a graph G, is G 3-colourable — is NP-complete. They used a reduction from 3-colourability of planar graphs with maximum degree 4. Using a different reduction, from k-colourability for fixed $k \geq 3$, Gräf, Stumpf and Weißenfels [37] showed the following generalisation of this result.

Theorem 2.1 (Gräf, Stumpf and Weißenfels [37])**.** *For any fixed integer $k \geq 3$, the problem UD k-COL is NP-complete.*

It is natural to ask how closely we can approximate $\chi(G)$. The NP-completeness of UD 3-COL implies that $\chi(G)$ is inapproximable in polynomial time to within a factor of less

than 4/3, unless P = NP. The following proposition implies that $\chi(G)$ is approximable to within a factor of 3.

Proposition 2.2 (Peeters [72]). *There is a polynomial-time algorithm that, for any unit disk graph G, finds a proper colouring of G using at most $3\omega(G)-2$ colours. This is a polynomial-time approximation algorithm for UD CHROMATIC NUMBER with performance guarantee of 3.*

The relatively simple proof of this assertion uses geometric ideas to show that the degeneracy $\delta^*(G)$ is at most $3\omega(G) - 3$. Since $\delta^*(G) \leq 3\omega(G) - 3$, a proper colouring using at most $3\omega(G) - 2$ colours can be found inductively. (We note that, for any integer $k \geq 2$, there exists a $(3k - 3)$-regular unit disk graph G with $\omega(G) = k$ [58].) Similar arguments show that (a) for any unit disk graph G, the maximum degree $\Delta(G)$ is at most $6\omega(G) - 7$, and (b) there is a polynomial-time algorithm that, for any disk graph G, finds a proper colouring of G using at most $6\omega(G) - 6$ colours. Proposition 2.2 was given in 1991, but there has been no tangible improvement of this approximation result since then. Gräf *et al.* [37] provide a more sophisticated heuristic called the STRIPE algorithm, which also has performance guarantee of 3. It is still unknown whether the best approximation ratio for computing the chromatic number of unit disk graphs is closer to 4/3 or 3.

A related problem is to consider the best upper bound on the ratio $\chi(G)/\omega(G)$ for unit disk graphs. Malesińska, Piskorz and Weißenfels [58] showed that there are classes of unit disk graphs with $\chi(G) \geq \frac{3}{2}\omega(G)$; however, the question of whether this parameter is closer to 3/2 or 3 is an enticing open problem. Results on colouring of random unit disk graphs show that this parameter is significantly lower than 3 for "most" unit disk graphs [64].

Colouring of hexagonal graphs

In light of the particular application of frequency assignment in cellular telephone networks, McDiarmid and Reed [63] studied the weighted colouring problem on induced subgraphs of the triangular lattice T.

Given a graph $G = (V, E)$, a *weight assignment* w is an association of each vertex $v \in V$ with a non-negative weight w_v. A *weighted graph* is an ordered pair (G, w) where G is a

graph and w is a weight assignment for G. A *weighted colouring c* of a weighted graph (G, w) is an assignment to each vertex v of a multiset $c_w(v)$ of w_v colours. A *weighted k-colouring* is a weighted colouring where the colours are chosen from $\{1, \ldots, k\}$. (We deviate slightly from the definition given by McDiarmid and Reed [63] to eventually allow impropriety.) A weighted k-colouring of (G, w) is *proper* if each multiset $c_w(v)$ is a set of w_v distinct colours, and adjacent vertices receive disjoint multisets c_w.

Note that, given a weighted unit disk graph (G, w), there is a natural corresponding graph G_w obtained by replacing each vertex v of G by a clique of size w_v. If G is a unit disk graph, then G_w is also a unit disk graph. Also, the above definitions associated with weighted (proper) colouring of (G, w) correspond directly with those associated with (proper) colouring of G_w.

McDiarmid and Reed [63] proved that the weighted (proper) 3-colourability problem restricted to hexagonal graphs is NP-complete. They also provided a 4/3-approximation algorithm for the weighted chromatic number of such graphs.

2.2 Improper colouring of unit disk graphs

In this subsection, we present our analysis for the unit disk graph improper colourability problem. Some of the more unwieldy NP-completeness proofs are postponed to the appendix.

Since the k-colourability problem for unit disk graphs is NP-complete for any fixed integer $k \geq 3$ (cf. Theorem 2.1), it is natural to enquire whether the corresponding t-improper k-colourability problem is also NP-complete for any fixed positive integer t. To see that this is correct, we use a reduction similar to that of Gräf *et al.* [37], i.e. from k-colourability.

Theorem 2.3. *Unit disk graph t-improper k-colourability is NP-complete for any fixed integers t and k such that $t \geq 0$ and $k \geq 3$.*

Our approach, outlined in the appendix, generalises that of Gräf *et al.* [37] and our key contribution is to produce auxiliary graphs — in particular, the crossing auxiliary graph — that are more general than those of the proof for unit disk graph k-colourability.

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It is not clear if we should expect t-improper 2-colourability for unit disk graphs to be NP-complete, as 2-colourability is polynomial-time in general, while the planar t-improper 2-colourability problem, for any fixed positive integer t, is NP-complete (cf. Theorem 1.2). As we shall see later, we can in fact reduce from the latter problem to show NP-completeness.

Theorem 2.4. *Unit disk graph t-improper 2-colourability is NP-complete for any fixed positive integer t.*

The reduction from planar t-improper 2-colourability does not require crossing auxiliary graphs. However, the auxiliaries must transmit information about impropriety; also, we need to take care of high-degree vertices. The task of constructing such auxiliary graphs is the crux of the reduction.

Detailed proofs of Theorems 2.3 and 2.4 can be found in Appendix A. These two results show that, as for unit disk graph (proper) colourability, the unit disk graph improper colourability problem is NP-hard in a relatively strong sense. In light of these negative results, our next question is to consider approximability. By Theorem 2.4, the t-improper chromatic number (for $t \geq 1$) is inapproximable in polynomial time to within a factor less than $3/2$, unless P = NP. Since there is no improper analogue for colouring graphs with bounded degeneracy, the only known positive approximation result is the following.

Proposition 2.5. *For any fixed non-negative integer t, there is a polynomial-time approximation algorithm that, given a unit disk graph G, finds a t-improper colouring of G using at most $\left\lceil \frac{6\omega(G)-6}{t+1} \right\rceil$ colours. This is a polynomial-time approximation algorithm for UD t-IMPROPER CHROMATIC NUMBER with performance guarantee of 6.*

This is a direct consequence of the bound $\Delta(G) \leq 6\omega(G) - 7$ for unit disk graphs, and Propositions 1.8 and 1.5. It is unknown whether the best approximation ratio for computing χ^t of unit disk graphs is closer to $3/2$ or 6, if t is a fixed positive integer.

A related problem is to consider the best upper bound for unit disk graphs on the ratio between the t-improper chromatic number and the trivial lower bound, i.e. $\left\lceil \frac{\omega}{t+1} \right\rceil$. We mentioned at the end of Subsection 2.1 that for $t = 0$ this bound is between $3/2$ and 3. By the last proposition, we know that for positive integers t this bound is at most 6.

Proposition 2.6. *There exist unit disk graphs G_n such that*

$$\frac{\chi^t(G_n)}{\omega(G_n)/(t+1)} \geq \begin{cases} 2 & \text{if } t \text{ is odd, and} \\ 2(t+1)/(t+2) & \text{if } t \geq 2 \text{ is even.} \end{cases}$$

Proof. Fix an arbitrary integer $n > t/2 + 1$. Consider the graph G_n whose vertices are the $2n$ points equally spaced on a circle. Join each point to all other points on the circle except for the one directly opposite it. It can be verified that G_n is a unit disk graph and that $\omega(G_n) = n$. Since each vertex is adjacent to all but one vertex in G_n, there can be no t-dependent set with more than $t + 2$ vertices, giving that $\alpha^t(G_n) \leq t + 2$. When t is odd, we can reduce this estimate by one: suppose t is odd and there is a t-dependent set S of size $t + 2$. Since t is odd, there must be two opposite (non-adjacent) vertices u and v such that only one of them, say v, is in S. Then v must be adjacent to all other vertices in S and hence have degree $t + 1$, a contradiction. We now apply Proposition 1.9 and the result follows. $\square$

These examples are inspired by the unit disk graphs that show the ratio $3/2$ can be attained in the case $t = 0$ (cf. Malesińska *et al.* [58]) — these are also formed by equally spaced points around a circle. That we can obtain higher ratios for all other cases (except $t = 2$) gives us further evidence to believe that, for unit disk graphs, the improper chromatic numbers (i.e. when $t \geq 1$) are harder to approximate than the chromatic number.

In the next chapter, we will perform some asymptotic analysis for the problem of t-improperly colouring unit disk graphs. Among other things, we show that in the standard model for random unit disk graphs, for nearly all asymptotic choices for the distance parameter $r(n)$, that the t-improper chromatic number tends to a value at most $2\sqrt{3}/\pi \approx 1.103$ times the trivial lower bound as $n \to \infty$. One interpretation of this is that, given large instances G of randomly generated unit disk graphs, returning the polynomial-time computable value of $2\sqrt{3}/\pi \cdot \left\lceil \frac{\omega(G)}{t+1} \right\rceil$ is a reasonable estimate for $\chi^t(G)$.

2.3 Improper colouring of hexagonal graphs

In this subsection, we investigate weighted improper colouring on induced subgraphs of the triangular lattice T. A weighted colouring c of (F, w) is *t-improper* if, for every vertex $v \in V(F)$ and each colour $x \in c_w(v)$, the number of times (counted with multiplicities) the colour x is assigned to v and to any neighbour of v is at most $t + 1$. We define the *weighted t-improper chromatic number* $\chi(F, w)$ to be the least k needed in a weighted t-improper k-colouring. Note that $\chi^t(F, w)$ is precisely $\chi^t(F_w)$, the improper chromatic number of the corresponding unweighted graph. We consider the weighted t-improper colouring problem on the triangular lattice and, for any fixed non-negative integers t and k, we define the following computational problem.

TL t-IMPROPER k-COL

INSTANCE: an induced subgraph F of the triangular lattice T together with a corresponding weight assignment w.
QUESTION: does (F, w) have a weighted t-improper k-colouring?

We show NP-completeness of this problem for $k = 3$.

Theorem 2.7. *For any fixed non-negative integer t, TL t-IMPROPER 3-COL is NP-complete*

The proof, given in Appendix A, is a generalisation of the NP-completeness proof of McDiarmid and Reed [63], and is a reduction from 3-colourability of planar graphs with maximum degree 4. This is the shortest proof of the fact that computing the t-improper chromatic number of unit disk graphs is NP-hard.

Let us now consider approximability for computation of χ^t for hexagonal graphs. The previous theorem shows that χ^t is inapproximable in polynomial time to within a factor of less than $4/3$ for such graphs, unless P $=$ NP. We point out that a 3-approximation algorithm follows from Propositions 1.8 and 1.5 and the fact that $\Delta(F_w) \leq 3\omega(F_w) - 3$ for any $\mathcal{TL}$ graph (F, w); however, we can obtain a better approximation algorithm by pursuing the method in the second half of the paper by McDiarmid and Reed [63].

Theorem 2.8. *For any fixed non-negative integer t, there is a polynomial-time approximation algorithm that, given a weighted induced subgraph (F, w) of the triangular lattice, finds a weighted t-improper colouring of F using at most $\left(\frac{4\omega(F_w)+1}{3} + 4(2t+1) \right) \Big/ (t+1)$ colours.*

When $\omega(F)$ is large relative to t, this approximation algorithm is essentially optimal for χ^t of $\mathcal{TL}$ graphs in the sense that its approximation ratio approaches $4/3$ as $\omega(F) \to \infty$.

Proof Outline. We will closely follow the approach of McDiarmid and Reed [63] and the reader is advised to consult that paper for the details that have been omitted. We begin with an analogue of Lemma 1 in McDiarmid and Reed [63].

Lemma 2.9. *For any fixed non-negative integer t, there is a polynomial-time algorithm that, given a weighted bipartite graph (G, w), finds a weighted t-improper colouring of G using at most $\hat{k} = \left\lceil \frac{\omega(G_w)}{t+1} \right\rceil + 1$ colours.*

Proof. We may assume that G is connected. Let $G = (A, B)$ be the bipartition and determine $\omega(G_w)$. To specify a *near* optimal weighted colouring, let $T_v = \left\{ 1, \ldots, \left\lceil \frac{w_v}{t+1} \right\rceil \right\}$ if $v \in A$ and $T_v = \left\{ \hat{k} - \left\lceil \frac{w_v}{t+1} \right\rceil + 1, \ldots, \hat{k} \right\}$ if $v \in B$. If $u \in A$ and $v \in B$ are adjacent, then $\left\lceil \frac{w_u}{t+1} \right\rceil + \left\lceil \frac{w_v}{t+1} \right\rceil \leq \hat{k}$ so that $T_u \cap T_v = \emptyset$. Thus, for each vertex v, it is adequate to assign colours from T_v with multiplicity up to $t+1$ to obtain a weighted t-improper colouring. $\quad\square$

For the rest of the proof, we follow the proof of Theorem 2 in McDiarmid and Reed [63], except for a few modifications. Their proof is a two-stage colouring procedure, using 3κ colours in the first stage and, in the second stage, applying their Lemma 1 to optimally colour the remaining weighted bipartite graph U, using at most $\omega(U) \leq \omega(F_w) - 2\kappa$ additional colours, where $\kappa = \left\lfloor \frac{\omega(F_w)+1}{3} \right\rfloor$. The modifications we will make are to use $3 \cdot \left(\left\lceil \frac{\kappa}{t+1} \right\rceil + 1 \right)$ colours (using a similar method to the above lemma) in the first stage and, in the second stage, apply our lemma above to U, using at most $\left\lceil \frac{\omega(F_w)-2\kappa}{t+1} \right\rceil + 1$ additional colours. Some routine integer-part calculations show that the number of colours used overall is at most the announced quantity. $\quad\square$

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2.4   Improper colouring of interval graphs

In this subsection, we investigate improper colouring for interval and unit interval graphs.

Proposition 2.10. *For any fixed non-negative integers t and k, there exists a unit interval graph $I_{t,k}$ with maximum degree and clique number equal to $k(t+1)$ which is not t-improperly k-colourable.*

Proof. To construct $I_{t,k}$, just start with a $(k(t+1))$-clique $K = K_{k(t+1)}$ and add a vertex u linked to exactly $(k-1)(t+1)+1$ vertices of K. Suppose $I_{t,k}$ has a t-improper k-colouring: K must have exactly $(t+1)$ vertices of each colour. Thus any vertex of K has impropriety t in K. As u has $(k-1)(t+1)+1$ neighbours in K it must have at least one neighbour of each colour and hence cannot be coloured, a contradiction. $I_{t,k}$ is clearly a unit interval graph. $\qquad\square$

This proposition raises the question of the complexity of t-improperly k-colouring unit interval graphs for fixed non-negative integers t and k. We prove now that this problem is polynomial time for general interval graphs, and we provide a dynamic programming algorithm.

Theorem 2.11. *The t-improper k-colourability problem restricted to interval graphs is in P for any fixed non-negative integers t and k.*

Proof. Let G be an interval graph. We preprocess the graph by computing $\omega(G)$ (and this can be done in polynomial time). We may assume that $\omega(G) \leq k(t+1)$; otherwise, G is not t-improperly k-colourable by Proposition 1.8. Now assume we have an interval representation for G. Let $v_1, \ldots, v_n$ be the vertices of G ordered by the left endpoints of the respective intervals. We consider the vertices one-by-one according to this order and assign v_1 colour 1.

For this algorithm, we maintain all valid partial t-improper k-colourings of the induced subgraph processed so far; however, we discard vertices that are not required. More precisely, suppose v records the next vertex to be processed and we wish to extend all of the partial colourings (and discard ones that are impossible to extend). We need only maintain

a list of all valid partial t-improper k-colourings (together with accumulated improprieties) of a set S, where S contains all previously coloured neighbours of v.

If the vertex v_j is not adjacent to $v = v_s$, where $j < s$, then v_j is not adjacent to v_i with $i \geq s$ (and hence we can safely remove v_j from S). Furthermore, the maximum number of vertices in S at any given point in time is $\omega - 1$, since S together with v induces a clique. Thus, a list of size $(kt)^{\omega(G)} \leq (kt)^{k(t+1)}$ is sufficient. Note that the step of colouring a vertex and updating the list is polynomial in time. $\square$

This result does not fully answer the complexity question for improper colouring of unit interval graphs: it is unknown whether, for fixed $t > 0$, there is a polynomial-time algorithm to find $\chi^t(G)$ given a unit interval graph G. The following result, however, shows that only two values are possible: the lower bound given by Proposition 1.8, or this number plus one.

Theorem 2.12. *For any fixed non-negative integer t, there is a linear-time algorithm that, given a unit interval graph G, finds a t-improper colouring of G using at most $\hat{k} = \left\lceil \frac{\omega(G)}{t+1} \right\rceil + 1$ colours.*

Proof. Let $v_1, \ldots, v_n$ be a unit interval representation for G. Under this ordering, our colouring procedure proceeds by assigning colour 1 to the first $t+1$ vertices, colour 2 to the next $t+1$, and so on until colour $\hat{k}$ has been assigned whereupon it begins assigning colour 1 again. If we now have an invalid colouring, we can suppose without loss of generality that v_{t+1} and $v_{(t+1)\hat{k}+1}$ (both coloured 1) are adjacent. But, because G is a unit interval graph, this implies that $\left\{ v_{t+1}, \ldots, v_{(t+1)\hat{k}+1} \right\}$ induces a clique in G and this contradicts the choice of $\hat{k}$. $\square$

When only t is fixed, one can think of applying the algorithm of Theorem 2.11 with $k = \left\lceil \frac{\omega}{t+1} \right\rceil$; however, this may be polynomial neither in space nor in time, since space and time complexity both are $O\left((kt)^{k(t+1)} \right)$ and ω, hence k, can be linear in the number of vertices. In light of this, we pose the following problem.

Problem 2.13. *Let G be a unit interval graph and t a positive integer. The preceding result states that $\chi^t(G) \in \left\{ \left\lceil \frac{\omega(G)}{t+1} \right\rceil, \left\lceil \frac{\omega(G)}{t+1} \right\rceil + 1 \right\}$. Is there a polynomial-time algorithm to decide which value is correct?*

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The following problems also remain open.

Problem 2.14. *For any fixed positive integer t, is there a polynomial-time algorithm that, given an interval graph G, computes $\chi^t(G)$?*

Problem 2.15. *For any fixed positive integer t, what is the largest ratio between χ^t and $\omega/(t+1)$ for interval graphs?*

2.5   Maximum t-dependent set for unit disk graphs

Since $\chi^t(G) \geq |V|/\alpha^t(G)$ for any $G = (V, E)$, computing the t-dependence number could be helpful in finding a lower bound for the t-improper chromatic number. Dessmark, Janson and Lingas [25] were the first to study the problem of computing the size of a maximum t-dependent set. Among other results, they showed that MAX t-DEPENDENT set for bipartite planar graphs is NP-complete. It is natural to expect the same for unit disk graphs.

Theorem 2.16. *MAX t-DEPENDENT SET for unit disk graphs is NP-complete, for any fixed non-negative integer t.*

Note. The following reduction uses a special embedding which is used in the proof of Theorem 2.4.

Proof. Our reduction is from PLANAR INDEPENDENT SET. Let $G = (V, E)$ be a planar graph. Our first step is to find a special embedding of a subdivision of G that is realisable as a unit disk graph.

As a side note, the reduction of Clark *et al.* [21] for showing NP-hardness of UD CHROMATIC NUMBER use an orthogonal embedding of G, i.e. a planar embedding of G such that each edge corresponds to an arc made up of horizontal and vertical line segments, but this requires that G has maximum degree at most 4. In the reduction of Gräf *et al.* [37], each edge corresponds to an arc made up of horizontal and vertical line segments in the embedding of G; however, G need not be planar and, to take account of high-degree vertices, each vertex is represented by a (possibly degenerate) line segment. In this proof here (and

later in the proof of Theorem 2.4), we use what is called a box-orthogonal embedding for our reduction.

A *box-orthogonal embedding* of G is a planar embedding of G such that each edge is represented by alternate horizontal and vertical line segments and each vertex is represented by a (possibly degenerate) rectangle, called a box (See Figure 2.1). We assume that all line segments, including those at the perimeter of a box, lie on lines of the integer grid. There is a box-orthogonal embedding for each planar graph and one with area at most $|E(G)|^2$ can be computed in polynomial time [30, 71].

<div align="center">(a)</div> <div align="center">(b)</div>
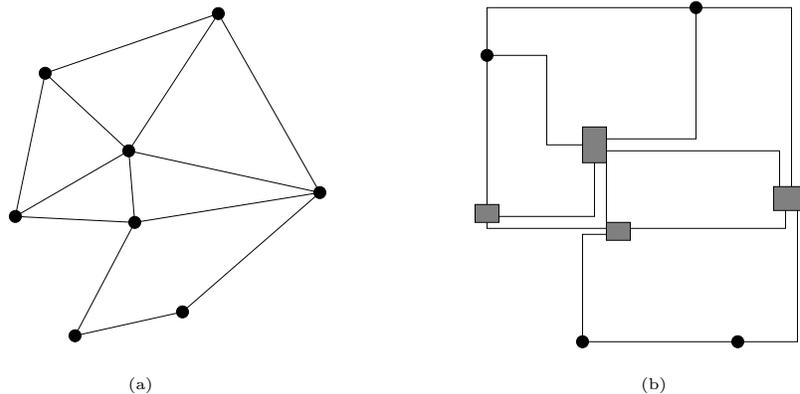
Figure 2.1: (a) An arbitrary planar graph G and (b) a box-orthogonal embedding of G.

Now, let us generate such an embedding of G (whose area is at most $|E|^2$). Let us remove degenerate boxes as follows: expand each by distance $1/2$ in each of the four directions and then double the scale of the grid. We double the scale of the grid once more to ensure that all line segments are of even length and all incidences between the edge segments and the boxes occur at even coordinates. After this rescaling, the box-orthogonal embedding has area at most $16(|E| + 1)^2$.

For each vertex $v \in V$, we have replaced v by a box $Box(v)$ of perimeter $2b(v)$. The perimeter of $Box(v)$ is incident with $\deg(v)$ line segments. Of the grid points that lie on the perimeter of $Box(v)$, we choose a grid point that has an odd coordinate (and in particular is not one of the $\deg(v)$ incidence points), and at each of the $2b(v) - 1$ other grid points we centre exactly $t + 1$ *closed disks of diameter* 1. (It is more convenient here to use a closed disk intersection representation.)

For each edge $e \in E$, we have replaced e by a piecewise linear curve $Line(e)$ composed of alternating horizontal and vertical line segments whose lengths sum to $2l(e)$. Of the $2l(e) - 1$ grid points contained in the interior of $Line(e)$ (i.e. all grid points of $Line(v)$ other than the endpoints), we arbitrarily choose $2l(e) - 2$ of them and centre exactly $t + 1$ closed disks of diameter 1 at each such point. For the one grid point x_e of $Line(e)$ omitted, denote its nearest grid point neighbours in $Line(e)$ by x_1 and x_2 — one of these might be an endpoint of $Line(e)$. We now choose two points x_e^1 and x_e^2 such that x_e^1 is within distance 1 of x_1, x_e^2 is within distance 1 of x_2, x_e^1 and x_e^2 are within distance 1 of each other, and x_e^1 and x_e^2 are not within distance 1 of any other grid point of $Line(e)$ or $Box(v)$ for some v. It is clear that a choice for x_e^1 and x_e^2 always exists. Now we centre $t + 1$ disks at point x_e^1 and $t + 1$ disks at point x_e^2.

It is helpful to visualise the resulting unit disk graph $\widehat{G}$ as composed of "chains" of $(t+1)$-cliques. The chains along the perimeter of a box each have an odd number of cliques; the chains "along" $Line(e)$ for an edge $e \in E$ (quotations due to the minor bobble at points x_e^1 and x_e^2) each have an even number of cliques. Since the area of the box-orthogonal embedding of G is at most $16(|E| + 1)^2$, the construction of $\widehat{G}$ takes polynomial time.

Now, suppose that X is an independent set of G of size m. We can construct a t-dependent set of $\widehat{G}$ of size $(t + 1)\left(m + \sum_{v \in V}(b(v) - 1) + \sum_{e \in E} l(e)\right)$ as follows. For each $v \in X$, we include $b(v)$ of the $(t + 1)$-cliques, by alternately including the cliques along the perimeter of $Box(v)$ and, in particular, including all of the cliques at the incidence points. For each $v \notin X$, we include only $b(v) - 1$ of the $(t + 1)$-cliques, again by alternately including the cliques along the perimeter, but this time not including any of the cliques at the incidence points. For each $e \in E$, since X is an independent set, we can always include half of the $(t + 1)$-cliques of the chain along $Line(e)$.

Conversely, suppose we are given a maximum t-dependent set $\widehat{X}$ of $\widehat{G}$ of size $\hat{m}$. Suppose $v \in \widehat{X}$. First, we remark that v has at least t neighbours in $\widehat{X}$, for if not we could just add to $\widehat{X}$ another member of the $(t+1)$-clique to which v belongs, contradicting the maximality of $\widehat{X}$. Second, we may assume that the t neighbours of v are all in the same $(t+1)$-clique as v, for if any are not we may simply move them into the same $(t+1)$-clique as v. We remark that it is equivalent now to consider $\widehat{X}$ as a maximum independent set of size $\hat{m}/(t + 1)$

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in the underlying unit disk graph (where each $(t + 1)$-clique is substituted with a single point). Now suppose $\hat{m} = (t + 1)\left(m + \sum_{v \in V}(b(v) - 1) + \sum_{e \in E} l(e)\right)$. By this choice of $\hat{m}$, it is clear that half of the cliqes along $Line(e)$ for any $e \in E$ are included in $\widehat{X}$ and that, for precisely m vertices $v \in V$, more than half of the cliques along the perimeter of $Box(v)$ are included in $\widehat{X}$. This set of m vertices forms an independent set in V.

Thus, the problem of finding a maximum independent set in G is polynomially equivalent to the problem of finding a maximum t-dependent set in $\widehat{G}$. $\square$

Note that the reduction just used is actually to the MAX t-DEPENDENT SET for weighted induced subgraphs of the grid graph. It is technically possible (but laborious) to show a similar reduction to MAX t-DEPENDENT SET for the class $\mathcal{TL}$.

Although the problem is NP-hard, there is a polynomial-time approximation scheme (PTAS) for UD MAX t-DEPENDENT SET.

Proposition 2.17. *Given $\varepsilon > 0$, there is a polynomial-time algorithm which can approximate UD MAX t-DEPENDENT set to within a factor of $1 - \varepsilon$.*

Proof. This is done using a so-called *shifting strategy* that splits the plane into large disconnected square pieces and solving exactly on those pieces (cf. Hunt *et al.* [47]). The approach follows Hunt *et al.* closely and in fact is similar to an argument used in the next chapter (Section 3.2), so we omit the proof. It is crucial to notice that, in a square of side length k, any t-dependent set has size at most $O((t + 1)k^2)$, by a simple packing argument. $\square$

2.6   Conclusion

We have established NP-completeness in nearly all cases of the decision version of the improper colourability problem of unit disk graphs. For weighted colouring of hexagonal graphs, we showed NP-completeness for all possible values of t but we have not analysed the decision problem for $k > 3$.

On the positive side, there is a 6-approximation for the t-improper chromatic number of unit disk graphs and an approximation within 1 for unit interval graphs. There is a

Problem	$\mathcal{I}$	$\mathcal{UD}$	$\mathcal{P}$	$\mathcal{D}$	$\mathcal{TL}$
CHROMATIC NUMBER	P	NPc	NPc	NPc	NPc
MAX CLIQUE	P	P	P	Open	P
MAX INDEPENDENT SET	P	NPc	NPc	NPc	NPc
MAX t-DEPENDENT SET	Open	NPc	NPc	NPc	NPc
t-IM CHROMATIC NUMBER	Open	NPc	NPc	NPc	NPc

straightforward 3-approximation and a more-complicated, "near" 4/3-approximation for computing the weighted t-improper chromatic number of hexagonal graphs.

These positive results are partial, however. We believe that the approximation ratio for unit disk graphs should be closer to 3, if not lower. Also, the complexity of determining the t-improper chromatic number for unit interval graphs is still unknown.

We showed that MAX t-DEPENDENT SET is NP-complete for unit disk graphs, and even for weighted induced subgraphs of the square or triangular lattices. Also, this problem admits a PTAS.

We end the chapter with two tempting problems.

Problem 2.18. *Is the following problem NP-complete?*
INSTANCE: *an induced subgraph G of the triangular lattice.*
QUESTION: *is the graph G 1-improperly 2-colourable?*

Problem 2.19. *Is there a constant $r > 0$ such that, for any fixed positive integer t, there is a polynomial-time approximation algorithm for computing the t-improper chromatic number of disk graphs with approximation ratio at most r?*

Related to this last problem, a previous comment on page 13 implies a polynomial-time approximation algorithm for computing the t-improper chromatic number of disk graphs with approximation ratio at most $6(t+1)$. We also mention that, due to an adaptation of Proposition 2.6, there exist disk graphs H_t such that

$$\frac{\chi^t(H_t)}{\omega(H_t)/(t+1)} \geq \begin{cases} 2 + \frac{t-1}{t(t+1)+1} & \text{if } t \text{ is odd, and} \\ 2 + \frac{t-3}{(t+1)^2} & \text{if } t \geq 2 \text{ is even.} \end{cases}$$

Chapter 3

Improper colouring of random unit disk graphs

The authors McDiarmid, Reed, and Müller [61,62,64] investigated the chromatic number χ for unit disk graphs in two related cases. The first case is the asymptotic limit of χ relative to the clique number ω where V is countably infinite and the distance threshold r approaches infinity: for countable sets V with finite upper density (to be defined below), the ratio of chromatic number over clique number approaches $\frac{2\sqrt{3}}{\pi}$ as $r \to \infty$ [62]. The second case is the asymptotic behaviour of χ for unit disk graphs based on randomly chosen points in the plane (where the distance threshold r approaches 0 as the number of points n approaches infinity). The papers [61,64] establish almost sure (and in probability) convergence results for these random instances of unit disk graphs. Recall that a sequence of random variables $(X_n)_n$ converges towards X *in probability* if $\lim_{n\to\infty} \Pr(|X_n - X| \geq \epsilon) = 0$ for every fixed $\epsilon > 0$ and $(X_n)_n$ converges towards X *almost surely* if $\Pr(\lim_{n\to\infty} X_n = X) = 1$.

Our aim, in this chapter, is to determine χ^t for unit disk graphs in the two cases mentioned above. From the last chapter, we know that for any fixed t the problem of computing the t-improper chromatic number of a given unit disk graph is NP-hard. It even seems difficult to give a reasonable approximation for this problem, since the best known approximation for χ^t of unit disk graphs has approximation ratio 6 (for $t \geq 1$). This is, of course, bad news for any practical applications which rely on computing χ^t; however,

results for proper colouring just mentioned above offer a small glimmer of hope. The results of [61, 62, 64] suggest that for large instances of unit disk graphs, it suffices to compute the clique number (which, as we mentioned in the last chapter, is computationally feasible) and return $\frac{2\sqrt{3}}{\pi}$ times that value, and this is likely to be a good estimate for the number of colours required.

Under the asymptotic models of unit disk graphs considered in this chapter the lower bound of Proposition 1.8 more or less gives the right answer. We will see that in both models $(t+1)\chi^t$ approaches χ (in an appropriate sense, with some small exceptions). We mention here that under the standard asymptotic model for general graphs, i.e. the Erdős-Rényi $\mathbb{G}(n, p)$ random graph model, there is qualitatively different behaviour. The perhaps rather counterintuitive result that, if $t = o(\ln(np))$ and $np \to \infty$ then $\chi^t(\mathbb{G}(n, p))/\chi(\mathbb{G}(n, p)) \to 1$ in probability, will be shown in Chapter 4.

The chapter is divided as follows. In Section 3.1, we consider the extensions of [62], stating definitions and results, then later giving proofs. Similarly, in Section 3.2, we analyse improper colouring for random geometric graphs to extend results of [61, 64].

3.1   Asymptotically, improperly colouring unit disk graphs

This section discusses our extensions of [62]. Let V be any countable set of points in the plane. For $x > 0$, let $f(x)$ be the supremum of the ratio $\frac{|V \cap S|}{x^2}$ over all open $(x \times x)$ squares S with sides aligned with the axes. The *upper density* of V is $\sigma^+(V) = \inf_{x>0} f(x)$.

Theorem 3.1 (McDiarmid and Reed [62])**.** *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$.*

(i) $\frac{\omega(G(V,r))}{r^2} \geq \sigma\frac{\pi}{4}$ *and* $\frac{\chi(G(V,r))}{r^2} \geq \sigma\frac{\sqrt{3}}{2}$ *for any $r > 0$;*

(ii) $\frac{\Delta(G(V,r))}{r^2} \to \sigma\pi,$ $\frac{\omega(G(V,r))}{r^2} \to \sigma\frac{\pi}{4}$ *and* $\frac{\chi(G(V,r))}{r^2} \to \sigma\frac{\sqrt{3}}{2}$ *as $r \to \infty$.*

We extend this theorem as follows.

Theorem 3.2. *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$, and let $\gamma = \sigma\frac{\sqrt{3}}{2}$. Then*

(i) $\frac{\chi^t(G(V,r))}{r^2} \geq \frac{\gamma}{t+1}$ for any $r > 0$; and

(ii) as $r \to \infty$, $(t+1)\frac{\chi^t(G(V,r))}{r^2} \to \gamma$ if $t = o(r)$.

In particular, the following holds.

Corollary 3.3. *Let $V \subseteq \mathbb{R}^2$ be a set with upper density $\sigma \in (0, \infty)$ and suppose that t satisfies $t = o(r)$. Then*

$$\frac{(t+1)\chi^t(G(V,r))}{\chi(G(V,r))} \to 1,$$

as $r \to \infty$.

It also holds that, for any countable set V of points in the plane with finite positive upper density, the ratio of $\chi^t(G(V,r))$ to $\frac{\omega(G(V,r))}{(t+1)}$ tends to $\frac{2\sqrt{3}}{\pi}$ as r approaches infinity. When t is zero, this result was proved in [62] and conjectured for the triangular lattice in [32]. We remark here that we shall allow t to vary as a function of r.

McDiarmid and Reed also tighten the upper bounds in Theorem 3.1 for the case where the points are approximately uniformly spread over the plane. Given a set V of points in the plane, a *cell structure* of V with density σ and radius ρ is a family $(C_v \mid v \in V)$ of sets that partition the plane and such that each C_v has area $\frac{1}{\sigma}$ and is contained in a ball of radius ρ about v.

Theorem 3.4 (McDiarmid and Reed [62]). *Let the set V of points in the plane have a cell structure with density σ and radius ρ, and let $\gamma = \sigma\frac{\sqrt{3}}{2}$. Then, for any $r > 0$,*

$$\omega(G(V,r)) \leq \frac{\sigma\pi}{4}(r + 2\rho)^2 \text{ and}$$

$$\chi(G(V,r)) < \left(\gamma^{1/2}(r + 2\rho) + \frac{2}{\sqrt{3}} + 1\right)^2.$$

Thus, combined with Theorem 3.1,

$$\omega(G(V,r)) = \sigma\frac{\pi}{4}r^2 + O(r) \text{ and}$$

$$\chi(G(V,r)) = \gamma r^2 + O(r) \text{ as } r \to \infty.$$

We extend this theorem as follows.

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Theorem 3.5. *Let the set V of points in the plane have a cell structure with density σ and radius ρ, and let $\gamma = \sigma \frac{\sqrt{3}}{2}$. Then, if $r \geq \frac{3t}{2}$,*

$$\chi^t(G(V,r)) < \frac{\left(\gamma^{1/2}(r + 2\rho) + \frac{2}{\sqrt{3}} + 2t + 1\right)\left(\gamma^{1/2}(r + 2\rho) + \frac{2}{\sqrt{3}} + \frac{t}{2} + 1\right)}{(t+1)}.$$

Thus, if $r \geq \frac{3t}{2}$, then $(t+1)\chi^t(G(V,r)) = \gamma r^2 + O(tr)$ as $r \to \infty$.

The key to all of the above theorems is the special case when V is the triangular lattice T, which, in this chapter, is defined as the integer linear combinations of the vectors $a = (1,0)$ and $b = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. In what follows, we will frequently make use of the observation that for $x, y \in \mathbb{R}$

$$\| ax + by \| = \sqrt{x^2 + xy + y^2}. \tag{3.1}$$

Note that the Dirichlet-Voronoï cells of the set T constitute a cell structure with density $\frac{2}{\sqrt{3}}$ and radius $\frac{1}{\sqrt{3}}$, and hence Theorem 3.5 above gives good bounds on $\chi^t(G(V,r))$; however, better results hold and indeed there is an exact result for $t = 0$.

For any $r > 0$, let $\hat{r}$ be the minimum distance between two points in T subject to that distance being at least r. From (3.1) it can be seen that $\hat{r}$ is the least value of $\sqrt{x^2 + xy + y^2}$ greater than or equal to r so that x and y are integers. Note that $r \leq \hat{r} \leq \lceil r \rceil$, and the value of $\hat{r}^2$ can be computed in $O(r)$ arithmetic operations.

Theorem 3.6 (McDiarmid and Reed [62])**.** *For any $r > 0$, $\chi(G(T,r)) = \hat{r}^2$.*

Consult [62] for the origin of this result. Unfortunately, when we consider t-improper colouring, we do not obtain an exact result such as Theorem 3.6, but we give a good bound in the following theorem.

Theorem 3.7. *Suppose $t \geq 0$. If $r \geq \frac{3t}{2}$, then*

$$\chi^t(G(T,r)) \leq \left\lceil \frac{r-1}{t+1} + 1 \right\rceil \left\lceil r + \frac{t}{2} \right\rceil < \frac{(r + 2t + 1)\left(r + \frac{t}{2} + 1\right)}{t+1};$$

furthermore, if $r < \lceil \frac{t+1}{2} \rceil$, then $\chi^t(G(T,r)) \leq \left\lceil \frac{2r}{\sqrt{3}} \right\rceil$.

Proofs

We now proceed with the detailed proofs. As mentioned above, the main results rest on the special case when V is the set of points on the triangular lattice T so we will first focus our attention here. Theorem 3.7 follows from the following slightly more general result.

Theorem 3.8. *Suppose $t \geq 0$ and $1 \leq \kappa \leq \sqrt{t+1}$. Let $x_0 := \left\lfloor \frac{t+1}{\kappa} \right\rfloor$. If $r \geq \frac{3}{2}(x_0 - 1)$, then*

$$\chi^t(G(T,r)) \leq \left\lceil \frac{1}{x_0} \left(r + \frac{\kappa - 1}{2} + x_0 - 1 \right) \right\rceil \left\lceil \frac{1}{\kappa} \left(r + \frac{x_0 - 1}{2} + \kappa - 1 \right) \right\rceil ;$$

furthermore, if $r < \left\lceil \frac{t+1}{2} \right\rceil$, then $\chi^t(G(T,r)) \leq \left\lceil \frac{2r}{\sqrt{3}} \right\rceil$.

Proof. We just need to exhibit a t-improper colouring of T that satisfies the bound. It turns out that a *strict tiling* of T — a colouring such that each colour class is a translate $v + T'$ of some sublattice T' of T — suffices. We can describe such a colouring succinctly by using one of its "tiles", i.e. a finite subset $V' \subseteq T$ such that $V' + T'$ both covers and packs T.

Let us first define the tile V' and the sublattice T'. Set

$$x_1 := \alpha x_0 \text{ and } y_1 := \beta \kappa,$$

where $\alpha := \left\lceil \frac{1}{x_0} \left(r + \frac{\kappa - 1}{2} + x_0 - 1 \right) \right\rceil$ and $\beta := \left\lceil \frac{1}{\kappa} \left(r + \frac{x_0 - 1}{2} + \kappa - 1 \right) \right\rceil$. We define V' to be all points $xa + yb$ such that $0 \leq x < x_1$ and $0 \leq y < y_1$. We let T' be all integer linear combinations of $x_1 a$ and $y_1 b$. Clearly, $V' + T'$ both covers and packs T.

Define

$$V'_{i,j} := \{ ai' + bj' \mid ix_0 \leq i' \leq (i+1)x_0 - 1 \text{ and } j\kappa \leq j' \leq (j+1)\kappa - 1 \}$$

for $0 \leq i < \alpha$ and $0 \leq j < \beta$, and assign each set $V'_{i,j} + T'$ a distinct colour. Observe that this colouring uses $\alpha\beta$ colours, as required. To prove that it is a t-improper colouring, it suffices to show that the distance between any point in $V'_{0,0}$ and any point in $V'_{0,0} + ax_1, V'_{0,0} + by_1$ or $V'_{0,0} + ax_1 + by_1$ is at least r and that the distance between any point in $V'_{0,0} + ax_1$ and any point in $V'_{0,0} + by_1$ is at least r (see Figure 3.1).

As can be easily seen from Figure 3.1, the distance between any point in $V'_{0,0} + ax_1$ and

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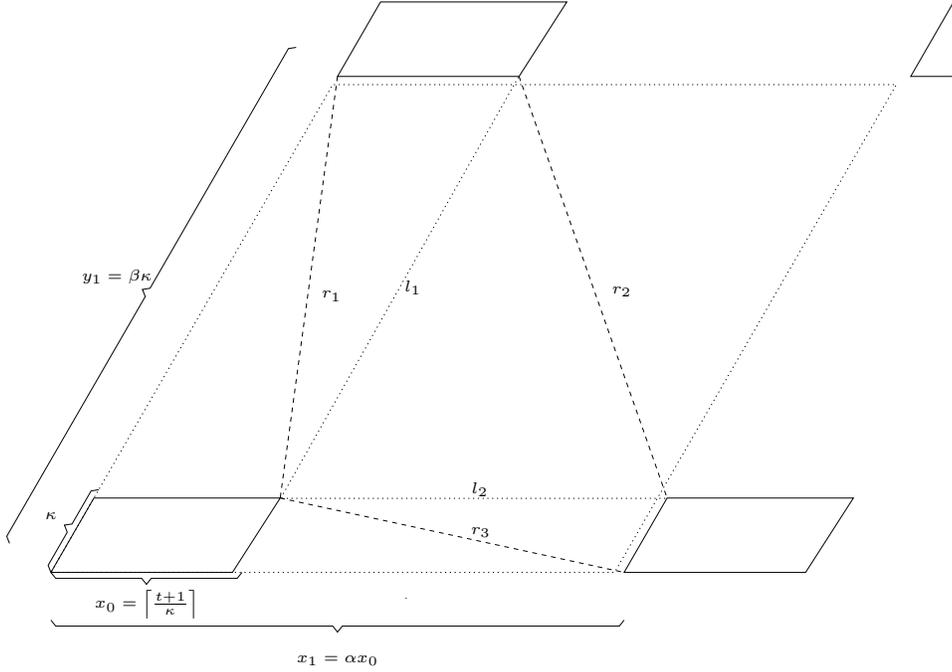
Figure 3.1: Illustration of the tiling of Theorem 3.8.

any point in $V'_{0,0} + by_1$ is at least r_2 shown in Figure 3.1.

The rightmost point of $V'_{0,0}$ has Cartesian x-coordinate $x_0 - 1 + \frac{1}{2}(\kappa - 1)$ and the leftmost point of $V'_{0,0} + y_1 b$ has Cartesian x-coordinate $\frac{1}{2}y_1 = \frac{1}{2}\beta\kappa \geq \frac{1}{2}(r + \frac{x_0-1}{2} + \kappa - 1) \geq x_0 - 1 + \frac{1}{2}(\kappa - 1)$, using that $r \geq \frac{3}{2}(x_0 - 1)$ by assumption. Thus, $V'_{0,0} + by_1$ lies completely to the right of $V'_{0,0}$, and the distance between a point in $V'_{0,0}$ and any point in $V'_{0,0} + by_1$ is therefore at least r_1 shown in Figure 3.1.

If we consider the projections onto the linear subspace generated by b (or alternatively, if we rotate the picture clockwise by 60 degrees and consider the Cartesian x-coordinates), then an analogous argument applies to $V'_{0,0}$ and $V'_{0,0} + ax_1$, showing that the distance between these sets is at least r_3 if $r \geq \frac{3}{2}(\kappa - 1)$ (and this condition is satisfied as $x_0 \geq \kappa$ and $r \geq \frac{3}{2}(x_0 - 1)$ by assumption).

Clearly $V'_{0,0} + ax_1 + by_1$ lies completely to the right of $V'_{0,0}$ so that the distance between these two sets is also at least r_1.

Thus, it suffices to show that $r_1, r_2, r_3 \geq r$. Using the formula (3.1) we see that

$$r_1 = \sqrt{l_1^2 - (x_0 - 1)l_1 + (x_0 - 1)^2},$$

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where l_1 is as shown in Figure 3.1. Thus $r_1 \geq r$ if and only if $l_1^2 - (x_0 - 1)l_1 + (x_0 - 1)^2 - r^2 \geq 0$ and by the quadratic formula this holds if

$$l_1 \geq \frac{1}{2}\left((x_0 - 1) + \sqrt{(x_0 - 1)^2 + 4(r^2 - (x_0 - 1)^2)}\right)$$

$$= \frac{x_0 - 1}{2} + \sqrt{r^2 - \frac{3}{4}(x_0 - 1)^2}.$$

Now notice that $l_1 = \beta\kappa - (\kappa - 1) \geq r + \frac{x_0 - 1}{2} \geq \frac{x_0 - 1}{2} + \sqrt{r^2 - \frac{3}{4}(x_0 - 1)^2}$ by the choice of β, so that indeed $r_1 \geq r$.

Analogous computations yield $r_3 \geq r$. Finally notice that $l_1 \geq r_1 \geq r$ (this can be seen by noting that l_1 is the distance between some point in $V'_{0,0}$ and some point in $V'_{0,0} + by_1$) and similarly $l_2 \geq r_3 \geq r$. Thus, $r_2 = \sqrt{l_1^2 - l_1 l_2 + l_2^2} \geq r$ and we see that the colouring defined by the tiling is indeed a t-improper colouring.

The "furthermore" condition implies that we may use one colour per row of T, and hence we need no more that $\lceil 2r/\sqrt{3} \rceil$ colours in total. $\qquad \square$

Theorem 3.7 is just Theorem 3.8 for $\kappa = 1$. Note that other choices of κ will give better bounds when $t + 1$ is composite. Indeed, the best bound is when $t + 1$ is a square.

Corollary 3.9. *Suppose $t \geq 0$ and $t + 1 = \kappa^2$ is a square. If $r \geq \frac{3}{2}(\kappa - 1)$, then*

$$\chi^t(G(T, r)) \leq \left\lceil \frac{1}{\kappa}\left(r + \frac{3\kappa - 3}{2}\right)\right\rceil^2 < \frac{\left(r + (5\sqrt{t + 1} - 3)/2\right)^2}{t + 1}.$$

Although Theorem 3.7 suffices for the remaining proofs of this section, it would also be interesting to know what is the value of $\chi^t(G(T, r))$ for all choices of t and r.

Let us continue with the proofs. One way to prove the lower bound of Theorem 3.2 (and hence of Theorem 3.5), would be to mimic the approach given in [62], by establishing a lower bound on a t-improper analogue of the *stability quotient* (i.e. the maximum over all induced subgraphs $H \subseteq G$ of $\frac{|V(H)|}{\alpha(H)}$). It is, however, sufficient to apply Proposition 1.8 to Theorem 3.1; therefore, we just need to prove the upper bounds, for which we shall generalise the arguments of [62].

Let us recall a definition from [62]. Given two sets A and B of points in the plane, and

$w > 0$, we say that a function $\phi : A \to B$ is *w-wobbling* if the Euclidean distance $\| a - \phi(a) \|$ is at most w for each $a \in A$. Observe that, if there is a w-wobbling injection from A into B, then $\chi^t(G(A, r)) \le \chi^t(G(B, r + 2w))$ for any $r > 0$.

Proof of Theorem 3.2. Part (i) follows immediately from part (i) of Theorem 3.1 together with Proposition 1.8.

For the proof of part (ii), we shall adapt the proof of Lemma 11 in [62]. Let $\varepsilon > 0$. We wish to show that $\frac{\chi^t(G)}{r^2} \le (\sigma + \varepsilon)\frac{\sqrt{3}}{2(t+1)}$ if r is sufficiently large. First, we set T' to be T scaled so that its density is $(\sigma + \varepsilon/2)$, i.e. let $T' := \xi^{-1}T$ where ξ is $\left(\frac{(\sigma + \varepsilon/2)\sqrt{3}}{2} \right)^{1/2}$.

Let S denote the half-open unit square $S = [0, 1)^2$. For any x sufficiently large, every translate of the square xS contains at least $(\sigma + \varepsilon/4)x^2$ points of T' and at most this number of points of V. If we partition the plane into a square grid with side length x, then for each grid square X there is a w-wobbling injection from $V \cap X$ into $T' \cap X$ where $w = \sqrt{2}x$. We may patch these injections together to obtain a w-wobbling injection $\phi : V \to T'$.

Now, using Theorem 3.7, we obtain

$$
\begin{aligned}
\chi^t(G(V, r)) &\le \chi^t(G(T', r + 2w)) \\
&= \chi^t(G(T, \xi(r + 2w))) \\
&< \frac{(\xi(r + 2w) + 2t + 1)\left(\xi(r + 2w) + \frac{t}{2} + 1\right)}{(t + 1)} \\
&< r^2(\sigma + \varepsilon)\frac{\sqrt{3}}{2(t + 1)}
\end{aligned}
$$

if r is sufficiently large. $\square$

The following lemma, proved by McDiarmid and Reed [62], will be used in the proof of Theorem 3.5. For all $w > 0$, two sets A and B are *w-close* if there exists a w-wobbling bijection between A and B.

Lemma 3.10 (McDiarmid and Reed [62])**.** *Let A (respectively, B) be a set with a cell structure of density σ and radius r_A (respectively, r_B). The sets A and B are $(r_A + r_B)$-close.*

Proof of Theorem 3.5. We apply the proof of (3) in Theorem 2 in [62]. We first recall that the cells of the triangular lattice T constitute a cell structure with density $\frac{2}{\sqrt{3}}$ and radius $\frac{1}{\sqrt{3}}$. Let $\xi = \sqrt{\gamma} = \left(\sigma\frac{\sqrt{3}}{2}\right)^{1/2}$. Observe that ξV has the same density $\frac{2}{\sqrt{3}}$, but has radius $\xi\rho$. By Lemma 3.10, ξV and T are w-close where $w = \frac{1}{\sqrt{3}} + \xi\rho$ and hence, for any $r > 0$,

$$\chi^t(G(V,r)) = \chi^t\left(G\left(\xi V, \xi r\right)\right) \leq \chi^t(G(T,D))$$

where $D = \xi r + 2w = \xi(r + 2\rho) + \frac{2}{\sqrt{3}}$, whence,

$$\chi^t(G(V,r)) \leq \chi^t(G(T,D)) < \frac{(D + 2t + 1)\left(D + \frac{t}{2} + 1\right)}{t + 1}$$

for r sufficiently large by Theorem 3.7. $\qquad\square$

3.2 Improper colouring of random unit disk graphs

This section discusses our generalisations of [64] and [61]. We consider a sequence of graphs $(G_n)_n$ obtained as follows. We pick points $X_1, X_2, \ldots$ of $\mathbb{R}^2$ at random (i.i.d. according to some probability distribution ν on $\mathbb{R}^2$) and we set $G_n = G(\{X_1, \ldots, X_n\}, r(n))$, where we assume we are given a sequence of distances $r(n)$ that satisfies $r(n) \to 0$ as $n \to \infty$ and we will allow any choice of ν that has a bounded probability density function. We are interested in the behaviour of the clique number, the chromatic number, and the t-improper chromatic number of G_n as n grows large.

In this model, the distance $r(n)$ plays a role similar to that of $p(n)$ in the Erdős-Rényi $\mathbb{G}(n, p)$ model. Depending on the choice of $r(n)$, qualitatively different types of behaviour can be observed. We prefer to describe the various cases in terms of the quantity nr^2, because nr^2 can be considered a measure of the average degree of the graph similar to np in the Erdős-Rényi $\mathbb{G}(n, p)$ model. Intuitively, this should be obvious (consider for instance the case ν is uniform on $[0, 1]^2$, so that the probability of an edge between X_1 and X_2 is $\approx \pi r^2$ when r is small and the expected degree of X_1 is therefore $\approx \pi(n - 1)r^2$). For a somewhat more rigorous treatment of the relationship between nr^2 and the average degree, see [68].

In this section we will only consider the case when the parameter t is fixed. It is, however, possible to generalise Theorem 3.11 below to growing t as long as t does not grow too quickly. The results fully extend to arbitrary norm and dimension, i.e. the case when points are drawn from some distribution on $\mathbb{R}^d$ (replacing 2 by d appropriately in what follows) and an arbitrary norm is used to measure the distance between points; however, the scope of this chapter is unit disk graphs in the plane.

We alluded to the following result in the introduction.

Theorem 3.11. *For $t \geq 0$ fixed and G_n as before, the following holds.*

(i) *If $nr^2 = \omega\left(n^{-\delta}\right)$ for all $\delta > 0$ then*

$$\frac{(t+1)\chi^t(G_n)}{\chi(G_n)} \to 1 \text{ almost surely;}$$

(ii) *If $nr^2 = o\left(n^{-\delta}\right)$ for some $\delta > 0$ then*

$$\Pr\left(\chi^t(G_n) \in \left\{ \left\lceil \frac{\chi(G_n)}{t+1} \right\rceil, \left\lceil \frac{\chi(G_n)}{t+1} \right\rceil + 1 \right\} \text{ for all but finitely many } n \right) = 1.$$

The following proposition shows that the two-point range for $\chi^t(G_n)$ in item (ii) cannot be reduced in general.

Proposition 3.12. *If $t \geq 1$ is fixed and r is chosen so that $nr^2 = \gamma n^{-\frac{1}{m(t+1)}}$ for some $\gamma > 0, m \in \mathbb{Z}^+$, then there exists $c = c(\gamma, m) \in (0,1)$ such that*

$$\Pr\left(\chi^t(G_n) = \left\lceil \frac{\chi(G_n)}{t+1} \right\rceil + 1\right) \to c.$$

When $nr^2 = o\left(n^{-\delta}\right)$ for some $\delta > 0$ then it can be shown that $\chi^t(G_n)$ will remain bounded in the sense that $\Pr(\chi^t(G_n) \leq m$ for all but finitely many $n) = 1$ for some $m = m(\delta)$. Thus, Proposition 3.12 shows that when $nr^2 = o\left(n^{-\delta}\right)$, almost sure convergence of the ratio $(t+1)\chi^t(G_n)/\chi(G_n)$ to 1 does not hold in general.

In contrast it was shown in [61] that for proper colouring it holds that $\Pr(\chi(G_n) = \omega(G_n)$ for all but finitely many $n) = 1$ whenever $nr^2 = o\left(n^{-\delta}\right)$ for some $\delta > 0$.

It follows from the proof of Theorem 3.11 that if $nr^2 = o\left(n^{-\delta}\right)$ for some $\delta > 0$, then there exists a sequence $m(n)$ such that

$$\Pr(\chi^t(G_n) \in \{m(n), m(n) + 1\} \text{ for all but finitely many } n) = 1.$$

Thus, the probability distribution of χ^t becomes concentrated on two consecutive integers as n grows large in the sense that $\Pr(\chi^t(G_n) \in \{m(n), m(n) + 1\}) \to 1$.

This phenomenon (of the probability measure becoming concentrated on two consecutive integers) is called focusing in [73, 74] and is well known to occur for various graph parameters in Erdős-Rényi random graphs. Recently, Müller [68] proved a conjecture of Penrose stating that when $nr^2 = o(\ln n)$ then the clique number of G_n becomes focused and the same was shown to hold for the chromatic number. It is a straightforward exercise to adapt the proof in [68] to yield the analogous result for improper colouring as well: if $nr^2 = o(\ln n)$ then there exists a sequence $m(n)$ such that $\Pr(\chi^t(G_n) \in \{m(n), m(n) + 1\}) \to 1$.

Proofs

We shall use the following result from [61].

Theorem 3.13 (McDiarmid and Müller [61]). *If* $nr^2 = o\left(n^{-\delta}\right)$ *for some* $\delta > 0$ *then the following holds.*

(i) *There exists a sequence* $m(n)$ *such that* $\Pr(\omega(G_n) \in \{m(n), m(n) + 1\}$ *and* $\Delta(G_n) \in \{m(n), m(n) - 1\}$ *for all but finitely many* $n) = 1$;

(ii) $\Pr(\chi(G_n) = \omega(G_n)$ *for all but finitely many* $n) = 1$.

Part (ii) of Theorem 3.11 is an easy consequence of this last theorem.

Proof of part (ii) *of Theorem 3.11.* It follows from part (i) of Theorem 3.13 that, when $nr^2 = o\left(n^{-\delta}\right)$,

$$\Pr(\Delta(G_n) \in \{\omega(G_n), \omega(G_n) - 1\} \text{ for all but finitely many } n) = 1.$$

Combining this with part (ii) of Theorem 3.13, we see that also

$$\Pr(\chi(G_n) = \omega(G_n) \text{ and } \Delta(G_n) \leq \omega(G_n) \text{ for all but finitely many } n) = 1.$$

Now note that if $\chi(G_n) = \omega(G_n)$ and $\Delta(G_n) \leq \omega(G_n)$ then Corollary 1.6 and Proposition 1.8 give

$$\left\lceil \frac{\chi(G_n)}{t+1} \right\rceil \leq \chi^t(G_n) \leq \left\lceil \frac{\chi(G_n)+1}{t+1} \right\rceil \leq \left\lceil \frac{\chi(G_n)}{t+1} \right\rceil + 1,$$

which concludes the proof. $\qquad\square$

For the proof of Proposition 3.12 we will rely on some results from Chapter 3 of Penrose [73]. Recall that if Z, Z' are two integer-valued random variables, then their *total variational distance* is defined as

$$d_{TV}(Z, Z') := \sup_{A \subseteq \mathbb{Z}} |\Pr(Z \in A) - \Pr(Z' \in A)|.$$

Recall also that a sequence of k-dimensional random vectors $(x_n)_n$ converges towards x *in distribution*, and we write $x_n \xrightarrow{\mathcal{D}} x$ if $\mathbb{E}h(x_n) \to \mathbb{E}h(x)$ as $n \to \infty$ for any bounded continuous $h : \mathbb{R}^k \to \mathbb{R}$.

Proposition 3.14 (Penrose [73], Chapter 3). *Let H be a connected unit disk graph with $l \geq 2$ vertices, let N denote the number of induced subgraphs of G_n isomorphic to H and let Z be a Poisson variable with mean $\mathbb{E}N$. The following hold.*

(i) *There exists a constant $\mu = \mu(H) > 0$ such that $\mathbb{E}N \sim \mu n^l r^{2(l-1)}$;*

(ii) *There exists a constant $c = c(H)$ such that $d_{TV}(N, Z) \leq cnr^2$.*

Proposition 3.15 (Penrose [73], Chapter 3). *Let $H_1, \ldots, H_s$ be non-isomorphic connected unit disk graphs with $l \geq 2$ vertices. Let N_i denote the number of induced subgraphs of G_n isomorphic to H_i. Let $\mu_1 = \mu(H_1), \ldots, \mu_s = \mu(H_s)$ be as given by part (i) of Proposition 3.14. Suppose that $nr^2 \sim \gamma n^{-\frac{1}{l-1}}$ with $\gamma > 0$. Let $Z_1, \ldots, Z_s$ be independent Poisson variables with $\mathbb{E}Z_i = \gamma^{l-1}\mu_i$. Then*

$$(N_1, \ldots, N_s) \xrightarrow{\mathcal{D}} (Z_1, \ldots, Z_s).$$

Proof of Proposition 3.12. According to part (ii) of Theorem 3.13 it suffices to consider $\Pr\left(\chi^t(G_n) = \left\lceil \frac{\omega(G_n)}{t+1} \right\rceil + 1\right)$, as $\Pr(\chi(G_n) \neq \omega(G_n)) \to 0$. Set $l := m(t+1) + 1$. By the choice of $r(n)$, we have $n^l r^{2(l-1)} = \gamma^{l-1}$, and $n^{l+1} r^{2l} \to 0, n^{l-1} r^{2(l-2)} \to \infty$. If we denote the order of the largest component of G_n by $L(G_n)$ then Proposition 3.14 implies that

$$\Pr(\omega(G_n) \geq l - 1, L(G_n) \leq l) \to 1.$$

To see this let N be the number of induced subgraphs of G_n isomorphic to K_{l-1} and let N' be the number of connected subgraphs of order $l + 1$. Part (i) of Proposition 3.14 gives $\mathbb{E}N' = O(n^{l+1} r^{2l}) = o(1)$. Clearly, the largest component has size greater than l if and only if there exists a connected subgraph of order $l + 1$; therefore,

$$\Pr(L(G_n) > l) = \Pr(N' > 0) \leq \mathbb{E}N' = o(1).$$

On the other hand using part (ii) of Proposition 3.14 we get

$$\Pr(\omega(G_n) < l - 1) = \Pr(N = 0) \leq \Pr(Z = 0) + cnr^2$$

for some Poisson variable Z with mean $\mathbb{E}N$ and some constant $c = c(K_{l-1})$. Furthermore, $\mathbb{E}N = \Omega(n^{l-1} r^{2(l-2)}) \to \infty$ and $cnr^2 \to 0$ as $n \to \infty$; therefore,

$$\Pr(\omega(G_n) < l - 1) \leq \exp(-\mathbb{E}N) + cnr^2 = o(1).$$

Let $H_1, \ldots, H_s$ be all non-isomorphic connected unit disk graphs of order l that satisfy $\chi^t(H_i) = m + 1$ yet H_i is not (isomorphic to) K_l. There exists at least one such graph, the unit disk graph $H := G(\{(\frac{i}{l-1}, 0) \mid i = 0, \ldots, l-1\}, 1)$, as depicted in Figure 3.2. This is simply the complete graph on l vertices with one edge removed. To see that there is no t-improper colouring of H with m colours, note that its vertices can be partitioned into a clique of size $l - 1 = m(t+1)$ and a vertex v_0 which is adjacent to all but one of the other nodes. If there were a t-improper colouring with m colours then every colour would have to occur $t + 1 \geq 2$ times amongst the vertices of the clique. Hence whichever of the m colours
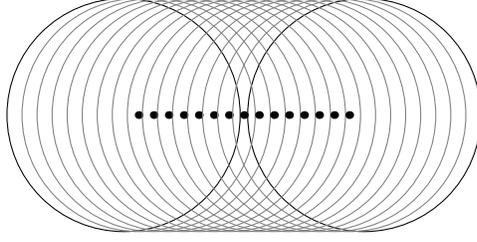
Figure 3.2: For Proposition 3.12, $H = K_{m(t+1)+1} - e$ satisfies $\omega(H) = m(t+1)$ and $\chi^t(H) = m + 1$.

we assign to v_0 there will be a node in the clique adjacent to $t+1$ nodes of the same colour.

Let N_0 be the number of induced subgraphs of G_n isomorphic to K_l and let N_i be the number of induced subgraphs isomorphic to H_i. Observe that if both $\omega(G_n) \geq l - 1$ and $L(G_n) \leq l$ hold then $\chi^t(G_n) = \left\lceil \frac{\omega(G_n)}{t+1} \right\rceil + 1$ holds if and only if $N_0 = 0$ and $N_i > 0$ for some $1 \leq i \leq s$. Thus, we infer that the probability that $\chi^t(G_n)$ equals $\left\lceil \frac{\omega(G_n)}{t+1} \right\rceil + 1$ is

$$\Pr\left(N_0 = 0 \text{ and } N_i > 0 \text{ for some } 1 \leq i \leq s\right) + o(1).$$

Using Proposition 3.15, we may therefore conclude that

$$\Pr\left(\chi^t(G_n) = \left\lceil \frac{\omega(G_n)}{t+1} \right\rceil + 1\right) \to e^{-\mu_0 \gamma^{l-1}}(1 - e^{-(\mu_1 + \cdots + \mu_s)\gamma^{l-1}}),$$

for some $\mu_0, \ldots, \mu_s > 0$. $\square$

It should be noted that although the use of part (ii) of Proposition 3.14 is not crucial to show that $\Pr(\omega(G_n) < l - 1) = o(1)$, since Proposition 3.15 will suffice instead, its use shortens the proof of Proposition 3.12.

The proof of part (i) of Theorem 3.11 relies on some results from [61] that were developed to study the behaviour of $\chi(G_n)$.

One important ingredient of the proof is the connection between graph colouring and integer linear programming. Recall that the chromatic number of a graph G is the objective

value of the following integer linear program (ILP for short).

$$\min \quad 1^T x$$

$$\text{subject to} \quad Ax \geq 1,$$

$$x \geq 0, x \text{ integral,}$$

where A is the *vertex-independent set incidence matrix* of G. This is a $(0,1)$-matrix whose rows are indexed by the vertices of G and whose columns correspond to all possible independent sets in G. It has $a_{ij} = 1$ if vertex v_i is in the independent set corresponding to the j-th column and $a_{ij} = 0$ otherwise. Now, given a non-negative integer vector $b = (b_1, \ldots, b_n)$ where n is the order of G, let the graph G' be obtained from G by replacing vertex $v_i \in G$ by a clique of size b_i and the vertices in the cliques corresponding to v_i and v_j are joined in G' if and only if v_i and v_j are joined in G. Then $\chi(G')$ is the objective value of the following ILP.

$$\min \quad 1^T x$$

$$\text{subject to} \quad Ax \geq b,$$

$$x \geq 0, x \text{ integral.}$$

Furthermore, $\chi^t(G')$ does not exceed the objective value of the following ILP.

$$\min \quad 1^T x$$

$$\text{subject to} \quad (t+1)Ax \geq b,$$

$$x \geq 0, x \text{ integral.}$$

This is because taking $t+1$ copies of each node in a stable set in G gives a t-dependent set in G' (but not every t-dependent set can be constructed in this way of course).

As mentioned earlier in this section, we assume that the probability measure ν on the plane used to generate the X_i has a bounded density function f. Let us denote the *essential*

supremum of f by $f_{\max}$, i.e.

$$f_{\max} := \sup\{\tau \mid \mathrm{vol}(\{x \mid f(x) > \tau\}) > 0\},$$

where vol denotes the Lebesgue measure. We say that a measurable set $A \subseteq \mathbb{R}^2$ has a *small neighbourhood* if $\lim_{\epsilon \to 0} \mathrm{vol}(A + B(0, \epsilon)) = \mathrm{vol}(A)$ where $B(0, \epsilon)$ denotes the closed ball of radius ϵ centred at the origin.

We shall denote by $\mathcal{F}$ the collection of all bounded, non-negative functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with bounded support that satisfy the regularity conditon that $\{x \mid \varphi(x) > \tau\}$ has a small neighbourhood for all τ, and that are not almost everywhere zero — i.e. their support has nonzero area. For $\varphi \in \mathcal{F}$, let us define the random variable M_φ as

$$M_\varphi := \sup_{x \in \mathbb{R}^2} \sum_{i=1}^{n} \varphi\left(\frac{X_i - x}{r}\right).$$

It turns out that the random variables M_φ play an important role when studying the (t-improper) chromatic number of G_n, see [61].

Proposition 3.16 (McDiarmid and Müller [61])**.** *Let $\varphi = 1_W$ for some bounded set $W \subset \mathbb{R}^2$ with a small neighbourhood and nonempty interior. For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $n^{-\delta} \le nr^2 \le \delta \ln n$ then*

$$\Pr\left((1 - \epsilon)m \le M_\varphi \le (1 + \epsilon)m \text{ for all but finitely many } n\right) = 1,$$

where $m = m(n) := (\ln n)/\ln\left(\frac{\ln n}{nr^2}\right)$.

Proposition 3.17 (McDiarmid and Müller [61])**.** *Pick $\varphi \in \mathcal{F}$. For every $\epsilon > 0$ there exists $T = T(\epsilon)$ such that if $nr^2 \ge T \ln n$ then*

$$\Pr\left((1 - \epsilon)m \le M_\varphi \le (1 + \epsilon)m \text{ for all but finitely many } n\right) = 1,$$

where $m = m(n) = f_{\max} nr^2 \int_{\mathbb{R}^2} \varphi(x) dx$.

Proposition 3.18 (McDiarmid and Müller [61])**.** *Pick $\varphi \in \mathcal{F}$. If $nr^2 \sim \tau \ln n$ for some*

$\tau \in (0, \infty)$ *then*

$$\frac{M_\varphi}{nr^2} \to f_{\max} \int_{\mathbb{R}^2} \varphi(x) e^{\varphi(x)s} dx \ \ almost \ surely,$$

where $s = s(\varphi, \tau) \geq 0$ *solves*

$$\int_{\mathbb{R}^2} (s\varphi(x) e^{\varphi(x)s} - e^{\varphi(x)s} + 1) dx = \frac{1}{\tau f_{\max}}. \tag{3.2}$$

For $\varphi \in \mathcal{F}, 0 < \tau < \infty$, we shall set

$$\xi(\varphi, \tau) := \int_{\mathbb{R}^2} \varphi(x) e^{\varphi(x)s(\varphi,\tau)} dx,$$

where $s(\varphi, \tau)$ is the unique non-negative solution of (3.2) above. To see that (3.2) indeed has a unique non-negative solution (unless φ is almost everywhere 0, in which case there is no solution to (3.2)), notice that the left-hand side of (3.2) is finite for all $s > 0$ (as $\varphi \in \mathcal{F}$ is bounded and has bounded support), and is increasing with s for $s \geq 0$ (as the integrand $s\varphi(x)e^{\varphi(x)s} - e^{\varphi(x)s} + 1 = H(e^{\varphi(x)s})$ with $H(x) := x \ln x - x + 1$ is strictly increasing in s for any fixed x with $\varphi(x) > 0$). We will need the following observation from [61].

Lemma 3.19 (McDiarmid and Müller [61]). *For $\varphi \in \mathcal{F}$ and $\lambda \in (0, 1)$, let φ_λ be given by $\varphi_\lambda(x) := \varphi(\lambda x)$. Then*

$$\xi(\varphi_\lambda, \tau) \leq \lambda^{-2} \xi(\varphi, \tau).$$

Proof of part (i) *of Theorem 3.11.* We adapt a proof from [61]. We will first derive an upper bound on $\chi^t(G_n)$. To this end, let us fix $\epsilon > 0$ and consider the graphs G'_n constructed as follows. For $x \in \mathbb{R}^2$ let S_x denote the square $x + [0, \epsilon r)^2$ of side ϵr and lower left hand corner x. So the squares $S_p, p \in \epsilon r \mathbb{Z}^2$ form a dissection of $\mathbb{R}^2$. Let Γ be the graph with vertex set $\epsilon r \mathbb{Z}^2$ and an edge $pq \in E(\Gamma)$ if $\|p - q\| < r(1 + \epsilon\sqrt{2})$. For $W \subseteq \mathbb{R}^2$, we denote by $N(W)$ the number of points in W, i.e.

$$N(W) := |\{X_1, \ldots, X_n\} \cap W|.$$

We will now consider the graph G'_n, constructed by replacing each point p of Γ by a clique of size $N(S_p)$. Note that G_n is a subgraph of G'_n, so in particular $\chi^t(G_n) \leq \chi^t(G'_n)$. Let us

fix $K > 0$ (large) such that ϵ divides K. For $p \in \epsilon r \mathbb{Z}^2$, denote by H_p the subgraph of G_n induced by the points of Γ inside $p + [0, Kr)^2$, and by H_p' the corresponding subgraph of G_n'. By remarks made before the start of the proof we know that $\chi(H_p')$ is no more than the objective value of the following ILP.

$$\min \quad 1^T x$$

$$\text{subject to} \quad Ax \geq \tfrac{1}{t+1} b(p),$$
$$x \geq 0, x \text{ integral},$$

where A is the vertex-independent set incidence matrix of the subgraph Γ_K of Γ induced by the points of Γ inside $p + [0, Kr)^2$, and $b(p) = (N(S_{p+p_1}), \ldots, N(S_{p+p_l}))$ is the (random) vector whose entries are the number of points in each of the squares S_{p+p_i} for $p_i \in [0, Kr)^2 \cap \epsilon r \mathbb{Z}^2$. We now consider the LP-relaxation of this program (we drop the condition that the variables need to be integral), and denote by $M(p)$ the objective value of this LP-relaxation. As A depends only on ϵ and K, there is a constant $c = c(K, \epsilon)$ such that

$$\chi^t(H_p') \leq M(p) + c(K, \epsilon), \tag{3.3}$$

because rounding up all the variables of a feasible point of the LP-relaxation gives a feasible point of the ILP. So, in particular, we may take $c(K, \epsilon)$ equal to the number of columns of A, which equals the number of stable sets in Γ_K. This upper bound on the difference $\chi(H_p') - M(p)$ can be further improved, but it does not concern us here as for the proof it is only relevant that $c(K, \epsilon)$ is a constant that does not depend on n. By LP-duality, $M(p)$ is also equal to the value of the program

$$\max \quad \tfrac{1}{t+1} b^T y$$

$$\text{subject to} \quad A^T y \leq 1,$$
$$y \geq 0.$$

This formulation has the advantage that the polytope defined by $A^T y \leq 1$ depends only on

ϵ, K. Given ϵ, K we can therefore list the vertices of this polytope, which we will denote by $y_1, \ldots, y_m$. Because the objective value of the LP will be attained in one of the vertices, we can write

$$M(p) = \frac{1}{t+1} \max_i y_i^T b(p). \tag{3.4}$$

We now remark that the y_i correspond to functions in a natural way. Let $\psi_i : \mathbb{R}^2 \to [0,1]$ be the function which is given by

$$\psi_i(x) := \begin{cases} (y_i)_j & \text{if } x \in S_{p_j} \text{ for } 1 \leq j \leq l, \\ 0 & \text{if } x \notin [0, Kr)^2. \end{cases}$$

Here we used the same enumeration $p_1, \ldots, p_l$ of $[0, Kr)^2 \cap \epsilon \mathbb{Z}^2$ used earlier in the construction of the ILP. It is not hard to see that

$$b^T(p)y_i = \sum_{j=1}^{l}(y_i)_j N(S_{p+p_j}) = \sum_{t=1}^{n} \psi_i(X_t - p).$$

Let us now set $\varphi_i(x) := \psi_i(rx)$. The functions $\varphi_1, \ldots, \varphi_m$ in fact do not depend on r (or n) anymore, but merely on ϵ, K. Furthermore $\varphi_i \in \mathcal{F}$ for all i and

$$b^T(p)y_i = \sum_{j=1}^{n} \varphi_i\left(\frac{X_j - p}{r}\right).$$

Together with (3.3) and (3.4) this shows that for all p we have

$$\chi^t(H'_p) \leq \frac{1}{t+1} \max_{i=1,\ldots,m} M_{\varphi_i} + c(K, \epsilon).$$

We now remark that not only H'_p can be coloured with this many colours for any $p \in \epsilon r \mathbb{Z}^2$, but also the subgraph of G_n induced by the points in the set $W_p := p + [0, Kr)^2 + (K+1)r\mathbb{Z}^2$, as depicted in Figure 3.3. To see this, note that if $x \in p + [0, Kr)^2 + (K+1)rz$ and $y \in p + [0, Kr)^2 + (K+1)rz'$ for $z \neq z' \in \mathbb{Z}^2$, then $\|x - y\| \geq r$.

We may assume without loss of generality that $K \in \mathbb{N}$. Now consider the collection of all $\mathcal{W} := \{W_p \mid p \in r\{0, \ldots, K\}^2\}$ and note that each small square S_q, $q \in \epsilon r \mathbb{Z}^2$, is covered by exactly K^2 of the $(K+1)^2$ sets $W_p \in \mathcal{W}$ considered.
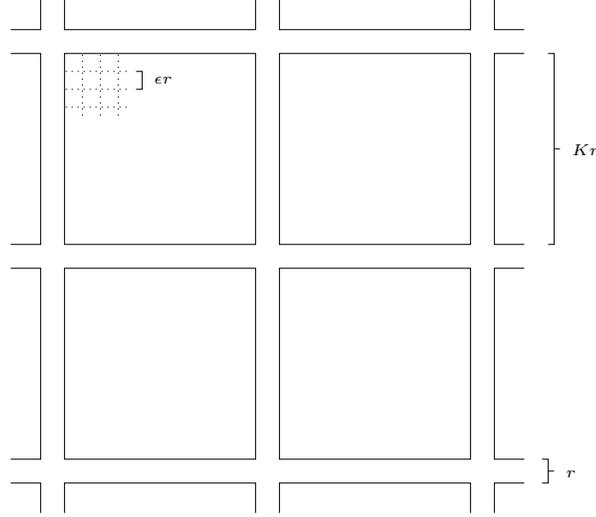
Figure 3.3: Artist's impression of the sets W_p.

Let us now consider the graphs $G'_{n,p}$ which are obtained by replacing each point q of $\Gamma \cap W_p$ by a clique of size $\left\lceil \frac{N(S_q)}{K^2} \right\rceil$ rather than $N(S_q)$. The subgraph $G'_{n,p}$ can be t-improperly coloured with no more than

$$\max_p \frac{1}{t+1} \max_i \left(\left\lceil \frac{N(S_{p+p_1})}{K^2} \right\rceil, \ldots, \left\lceil \frac{N(S_{p+p_l})}{K^2} \right\rceil \right) \cdot y_i + c(K, \epsilon)$$

$$\leq \frac{1}{(t+1)K^2} \max_i M_{\varphi_i} + c(K, \epsilon) + l$$

colours (as $(y_i)_j \leq 1$ for all i, j, so that the difference due to rounding is at most l). The colourings of the $G'_{n,p}$ can be combined to give a colouring of G_n with a total of at most

$$\frac{1}{t+1} \left(\frac{K+1}{K} \right)^2 \max_{i=1,\ldots,m} M_{\varphi_i} + (K+1)^2 (c(K, \epsilon) + l)$$

colours.

Next we wish to lower bound $\chi(G_n)$. For convenience let us set $r' := r\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}}$ and $S'_y := y + [0, \epsilon r')^2$ for $y \in \mathbb{R}^2$. For $x \in \mathbb{R}^2$ consider the subgraph H_x of G_n induced by the points in the square $x + [0, r'K)^2$.

Let Γ'_K be the graph with vertices $x + p'_i$, with p'_i running through $[0, Kr')^2 \cap \epsilon r'\mathbb{Z}^2$ and an edge $yz \in E(\Gamma'_K)$ if $\|y - z\| < r'(1 + \epsilon\sqrt{2}) = r(1 - \epsilon\sqrt{2})$. Then Γ'_K is in fact isomorphic to Γ_K and in particular has the same vertex-independent set incidence matrix A. Let H'_x

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be the graph we get by replacing a vertex $x + p'_i$ of Γ'_K by a clique of size $N(S'_{x+p'_i})$. Note that H'_x is a subgraph of G_n and $\chi(H'_x)$ is at least the objective value of the linear program

$$\max \quad b'(x)^T y$$

$$\text{subject to} \quad A^T y \leq 1,$$
$$y \geq 0,$$

where $b'(x) := (N(S'_{x+p'_1}), \dots, N(S'_{x+p'_l}))$. The vertices $y_1, \dots, y_m$ of the polytope are still the same; however, they now correspond to the sums

$$b'(x)^T y_i = \sum_{j=1}^{n} \varphi'_i \left(\frac{X_j - x}{r} \right),$$

where we set $\varphi'_i(x) := \varphi_i \left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}} x \right)$. Maximising over all choices of $x \in \mathbb{R}^2$ we get

$$\chi(G_n) \geq \max_{i=1,\dots,m} M_{\varphi'_i}. \tag{3.5}$$

It follows that

$$1 \leq \frac{(t+1)\chi^t(G_n)}{\chi(G_n)} \leq \left(\frac{K+1}{K} \right)^2 \frac{\max_i M_{\varphi_i}}{\max_i M_{\varphi'_i}} + \frac{\alpha}{\max_i M_{\varphi'_i}}, \tag{3.6}$$

where $\alpha = \alpha(K, \varepsilon) := (K+1)^2(c(K, \epsilon) + l)$.

Let us pick $t_0 < t_1 < \cdots < t_a$ with $t_{i+1} \leq (1 + \epsilon)t_i$, t_0 small and t_a large (to be made precise later) and let us "split" the sequence r into subsequences $r_0, \dots, r_{a+1}$:

$$r_0(n) := \begin{cases} r(n) & \text{if } nr^2 \leq t_0 \ln n, \\ \sqrt{\frac{t_0 \ln n}{n}} & \text{otherwise.} \end{cases} \quad , \quad r_{a+1}(n) := \begin{cases} r(n) & \text{if } nr^2 \geq t_a \ln n, \\ \sqrt{\frac{t_a \ln n}{n}} & \text{otherwise.} \end{cases}$$

$$r_i(n) := \begin{cases} r(n) & \text{if } t_{i-1} \ln n \leq nr^2 \leq t_i \ln n, \\ \sqrt{\frac{t_i \ln n}{n}} & \text{otherwise.} \end{cases} \quad , \quad \text{for } 1 \leq i \leq a,$$

and let us set $G_n^i := G(\{X_1, \ldots, X_n\}, r_i(n))$. We claim that for all i

$$\Pr\left(1 \leq \frac{(t+1)\chi^t(G_n^i)}{\chi(G_n^i)} \leq \gamma(K, \epsilon) \text{ for all but finitely many } n\right) = 1, \qquad (3.7)$$

where $\gamma(K, \epsilon) := \frac{(1+\epsilon)^2}{1-\epsilon} \left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right)^2 \left(\frac{K+1}{K}\right)^2 + \epsilon$. From this it will immediately follow that also

$$\Pr\left(1 \leq \frac{(t+1)\chi^t(G_n)}{\chi(G_n)} \leq \gamma(K, \epsilon) \text{ for all but finitely many } n\right) = 1,$$

as G_n always coincides with one of the G_n^i and the intersection of finitely many events of probability one has probability one. Taking $K \to \infty, \epsilon \to 0$ will then show that

$$\frac{(t+1)\chi^t(G_n)}{\chi(G_n)} \to 1 \text{ almost surely,}$$

as required. Thus it remains to establish (3.7) for all i (for a suitable choice of $t_0, \ldots, t_a$).

We first consider G_n^0. Let us set $\psi_1 := 1_{B(0,\frac{1}{2})}, \psi_2 := 1_{B(0,1)}$ and notice that M_{ψ_1}, M_{ψ_2} are respectively the maximum number of points of G_n in a disk of radius $\frac{r}{2}$ and r. Notice that we must have

$$M_{\psi_1} \leq \omega(G_n) \leq \chi(G_n) \leq \Delta(G_n) + 1 \leq M_{\psi_2}.$$

Applying Corollary 1.6 and Proposition 3.16, we get

$$\Pr\left(\chi(G_n^0) \geq (1-\epsilon)b \text{ for all but finitely many } n\right) = 1,$$
$$\Pr\left(\chi^t(G_n^0) \leq \frac{1}{t+1}(1+\epsilon)b + 1 \text{ for all but finitely many } n\right) = 1,$$

where $b = b(n) := (\ln n)/\ln\left(\frac{\ln n}{nr_0^2}\right)$. Now notice that $b \to \infty$, since $nr_0^2 \geq n^{-\delta}$ implies that $b \geq \frac{1}{\delta} + o(1)$. We see that

$$\Pr\left(\frac{(t+1)\chi^t(G_n^0)}{\chi(G_n^0)} \leq \frac{1+\epsilon}{1-\epsilon} + \epsilon \text{ for all but finitely many } n\right) = 1.$$

Let us now consider G_n^{a+1}. Provided t_a was chosen large enough, we have by Proposition 3.17

$$\Pr(M_{\varphi_i} \leq (1+\epsilon)b_i \text{ for all but finitely many } n) = 1 \text{ and}$$

$$\Pr(M_{\varphi_i'} \geq (1-\epsilon)b_i' \text{ for all but finitely many } n) = 1,$$

for $i \in \{1, \ldots, m\}$, where

$$b_i := f_{\max} n r_i^2 \int_{\mathbb{R}^2} \varphi_i(x) dx \text{ and}$$

$$b_i' := f_{\max} n r_i^2 \int_{\mathbb{R}^2} \varphi_i'(x) dx = \left(\frac{1 - \epsilon\sqrt{2}}{1 + \epsilon\sqrt{2}}\right)^2 b_i.$$

We have used the substitution $y = \left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right) x$ for the last identity. By (3.6) and the fact that $b_i' \to \infty$, we deduce that, with probability one,

$$\frac{(t+1)\chi^t(G_n^{a+1})}{\chi(G_n^{a+1})} \leq \frac{1+\epsilon}{1-\epsilon} \left(\frac{1 + \epsilon\sqrt{2}}{1 - \epsilon\sqrt{2}}\right)^2 \left(\frac{K+1}{K}\right)^2 + \epsilon \text{ for all but finitely many } n.$$

Now let us consider G_n^i for $1 \leq i \leq a$. We may assume that the t_i have been chosen in such a way that $t_i \leq (1+\epsilon)t_{i-1}$. Let us set $r_i^- := \sqrt{\frac{t_{i-1} \ln n}{n}}, r_i^+ := \sqrt{\frac{t_i \ln n}{n}}$. Because $\chi^t(G(V, \rho)), \chi(G(V, \rho))$ are both increasing in ρ,

$$\chi(G_n^i) \geq \chi(G(\{X_1, \ldots, X_n\}, r_i^-)) \text{ and } \chi^t(G_n^i) \leq \chi^t(G(\{X_1, \ldots, X_n\}, r_i^+)).$$

Applying Proposition 3.18 and (3.5) we see that (with probability one, for all but finitely many n)

$$\chi(G_n^i) \geq (1-\epsilon) f_{\max} n (r_i^-)^2 \max_j \xi(\varphi_j', t_{i-1}),$$

$$\chi^t(G_n^i) \leq \frac{1}{t+1} \left(\frac{K+1}{K}\right)^2 (1+\epsilon) f_{\max} n (r_i^+)^2 \max_j \xi(\varphi_j, t_{i-1}) + \alpha.$$

In the language of Lemma 3.19 we have $\varphi_i = (\varphi_i')_\lambda$ with $\lambda := \left(\frac{1-\epsilon\sqrt{2}}{1+\epsilon\sqrt{2}}\right)$. So we see that

(with probability one, for all but finitely many n)

$$\frac{(t+1)\chi^t(G_n^i)}{\chi(G_n^i)} \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{t_i}{t_{i-1}}\left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right)^2\left(\frac{K+1}{K}\right)^2 + \epsilon$$

$$\leq \frac{(1+\epsilon)^2}{1-\epsilon}\left(\frac{1+\epsilon\sqrt{2}}{1-\epsilon\sqrt{2}}\right)^2\left(\frac{K+1}{K}\right)^2 + \epsilon.$$

$\square$

3.3   Conclusion

In Section 3.1, we studied the asymptotic behaviour of χ^t when $r \to \infty$ and V is countably infinite to extend results of [62]. For these results, the bound for the t-improper chromatic number of the generalised triangular lattice given in Theorem 3.7 suffices; however, we would be interested to know an exact expression for $\chi^t(G(T,r))$ for any t and r. In Section 3.2, we studied the t-improper chromatic number of random unit disk graphs to extend results in [61,64]. An essential element of one of the proofs was an LP-formulation of the problem and an appropriate partition of the space. An issue that we have not studied for random unit disk graphs, but that might be of practical importance, is the rates of convergence of our results.

In both cases, with minor exceptions, we have seen that χ^t is well-approximated by Proposition 1.8, in other words $(t+1)\chi^t$ approaches χ in some appropriately defined manner. This behaviour differs notably from that of Erdős-Rényi random graphs, where $\chi^t/\chi \to 1$ in probability if $t(n) = o(\ln(np))$ and $np \to \infty$, as we will see in Chapter 4.

Due to the motivating application in satellite communications, we have focused upon the case with Euclidean norm and dimension two (i.e. unit disk graphs); however, we note here that our results naturally generalise to arbitrary norm and higher dimensions (cf. [67]).

A major purpose of this study was to gain insight into the problem of finding the t-improper chromatic number of unit disk graphs. The results of this chapter suggest that, for fixed t, given randomly generated unit disk graphs G_n, the polynomial-time computable value $\omega(G_n)/(t+1)$ multiplied by the factor $2\sqrt{3}/\pi \approx 1.103$ (which is much smaller than 6) is a reasonable estimate for the t-improper chromatic number when n is large enough.

Chapter 4

Improper colouring of random graphs

In this chapter, we consider the t-improper chromatic number of the Erdős-Rényi random graph $\mathbb{G}\left(n,p\right)$. As usual, $\mathbb{G}\left(n,p\right)$ denotes a random graph with vertex set $[n] = \{1,\ldots,n\}$ in which the edges are included independently at random with probability p.

Clearly, when $t = 0$, we are simply considering the ordinary notion of the chromatic number of random graphs, and this topic is well studied. Fix $0 < p < 1$ and let $\gamma = 2/\ln\frac{1}{1-p}$. In 1975, Grimmett and McDiarmid [38] showed that, for any $\varepsilon > 0$, the expected number $C_{n,j}$ of j-colourings of $\mathbb{G}\left(n,p\right)$ satisfies

$$C_{n,j} \to \begin{cases} 0 & \text{if } j \leq (1-\varepsilon)\frac{n}{\gamma \ln n} \\ \infty & \text{if } j \geq (1+\varepsilon)\frac{n}{\gamma \ln n} \end{cases} \tag{4.1}$$

as $n \to \infty$ thus showing that $\chi(\mathbb{G}\left(n,p\right)) \geq (1-\varepsilon)\frac{n}{\gamma \ln n}$ asymptotically almost surely (a.a.s.) They conjectured that $\chi(\mathbb{G}\left(n,p\right)) \sim \frac{n}{\gamma \ln n}$ a.a.s. This conjecture remained a major open problem in random graph theory for over a decade, until independently Bollobás [16] and Matula and Kučera [60] used martingale techniques to establish the conjecture. Łuczak [57] extended the result to sparse random graphs. For further background into the colouring of random graphs, consult [17,49].

By Proposition 1.3, Corollary 1.6 and Proposition 1.8, it follows that for any integer t,

$$\frac{\chi(\mathbb{G}\,(n,p))}{t+1} \le \chi^t(\mathbb{G}\,(n,p)) \le \min\left\{\left\lceil\frac{\Delta(\mathbb{G}\,(n,p))+1}{t+1}\right\rceil, \chi(\mathbb{G}\,(n,p))\right\}. \qquad (4.2)$$

In the dense case — i.e. when the edge probability p is a positive constant — we show that $\chi^t(\mathbb{G}\,(n,p))$ is likely to be close to the upper end of this range, as long as $t(n) = o(\ln n)$ or $t(n) = \omega(\ln n)$. Recall that $\Delta(\mathbb{G}\,(n,p)) \sim np$ a.a.s. in this case. More fully, we have

Theorem 4.1. *For fixed edge probability $0 < p < 1$, the following hold:*

(i) *if $t(n) = o(\ln n)$, then $\chi^t(\mathbb{G}\,(n,p)) \sim \left(\frac{1}{2}\ln\frac{1}{1-p}\right)\frac{n}{\ln n}$ a.a.s.;*

(ii) *if $t(n) = \Theta(\ln n)$, then $\chi^t(\mathbb{G}\,(n,p)) = \Theta\left(\frac{n}{\ln n}\right) = \Theta\left(\frac{np}{t}\right)$ a.a.s.;*

(iii) *if $t(n) = \omega(\ln n)$ and $t(n) = o(n)$, then $\chi^t(\mathbb{G}\,(n,p)) \sim \frac{np}{t}$ a.a.s.;*

(iv) *if $t(n) \sim \frac{np}{x}$, where $x > 0$ is fixed and not integral, then $\chi^t(\mathbb{G}\,(n,p)) = \lceil x \rceil$ a.a.s.*

In Sections 4.1 and 4.2, we prove Theorem 4.1 part by part: part (i) follows from Theorem 4.3, parts (iii) and (iv) follow from Theorem 4.4, and part (ii) follows from Corollary 4.8.

In Section 4.2, we more closely examine the intermediate case (ii) of Theorem 4.1, where we can say more. Suppose that $t(n) \sim \tau \ln n$ for some fixed $\tau > 0$. In Theorem 4.6 below, we identify a fixed $\kappa > 0$ (depending on p and τ) such that the expected number of t-improper j-colourings of $\mathbb{G}\,(n,p)$ tends to 0 if $j \le (1-\varepsilon)\frac{n}{\kappa \ln n}$ and to ∞ if $j \ge (1+\varepsilon)\frac{n}{\kappa \ln n}$. Note that this result is analogous to (and extends) (4.1). It is natural now to conjecture that $\chi^t(\mathbb{G}\,(n,p)) \sim \frac{n}{\kappa \ln n}$ a.a.s. This is more fully detailed in Conjecture 4.9.

We study the t-improper chromatic number of sparse random graphs as well, i.e. when $p(n) = o(1)$:

Theorem 4.2. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$.*

(i) *Fix $\varepsilon > 0$. There exist constants d_0 and $\tau > 0$ such that, if $d(n) \ge d_0$ and $t(n) \le \tau \ln d$, then $(1-\varepsilon)\frac{d}{2\ln d} \le \chi^t(\mathbb{G}\,(n,p)) \le (1+\varepsilon)\frac{d}{2\ln d}$ a.a.s.*

(ii) *If $d(n) = \omega(1)$ and $t(n) = o(\ln d)$, then $\chi^t(\mathbb{G}\,(n,p)) \sim \frac{d}{2\ln d}$ a.a.s.*

Furthermore, if $d(n) = \omega(\sqrt{\ln n})$, then the following hold:

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(iii) *if $t(n) = \Theta(\ln d)$, then $\chi^t(\mathbb{G}(n, p)) = \Theta\left(\frac{d}{\ln d}\right) = \Theta\left(\frac{d}{t}\right)$ a.a.s.;*

(iv) *if $t(n) = \omega(\ln d)$ and $t(n) = o(d)$, then $\chi^t(\mathbb{G}(n, p)) \sim \frac{d}{t}$ a.a.s.;*

(v) *if $t(n) \sim \frac{d}{x}$, where $x > 0$ is fixed and not integral, then $\chi^t(\mathbb{G}(n, p)) \in \{\lceil x \rceil, \lceil x \rceil + 1\}$ a.a.s.*

In Section 4.3, we prove Theorem 4.2 part by part: parts (i) and (ii) are shown in Theorem 4.11, parts (iv) and (v) follow from Theorems 4.12 and 4.13, and part (iii) follows from Theorems 4.13 and 4.15.

We are unable to extend our results to a fuller range of probabilities when $t(n) = \Omega(\ln d)$. The major difficulty arises when the expected maximum degree is much larger than the expected average degree (in which case our upper bounds are perhaps insufficient). Although we give a conjecture for the asymptotic constant suggested by part (iii), our support for this is weaker than in the fixed p case.

We should mention here that we obtain our lower bounds by estimates on the *weak t-dependence number $\hat{\alpha}^t(G)$*, i.e. the maximum size of a *weakly t-dependent set* — a vertex subset which induces a subgraph of average degree at most t. The *weakly t-improper chromatic number $\hat{\chi}^t(G)$* is analogously defined. It is clear that $\alpha^t(G) \leq \hat{\alpha}^t(G)$ and $\hat{\chi}^t(G) \leq \chi^t(G)$. Analysis of the weakly improper chromatic number of $\mathbb{G}(n, p)$ is more tractable and we can pin down the behaviour more precisely than for the ordinary improper chromatic number. These results are summarised in Section 4.4.

When it is not explicitly indicated, we adopt the convention of denoting $q = 1 - p$ and $d = np$. To avoid confusion, we use the notation $\deg(v)$ or $\deg_G(v)$ to denote the degree of a vertex v in G.

4.1 Dense random graphs: $t(n) = o(\ln n)$ or $t(n) = \omega(\ln n)$

Our first result — for the case when $t(n) = o(\ln n)$ and $0 < p < 1$ fixed — is perhaps surprising, but even more so is the fact that it can be proven using the first moment method and straightforward estimates.

Theorem 4.3. *Fix $0 < p < 1$. If $t(n) = o(\ln n)$, then $\chi^t(\mathbb{G}(n,p)) \sim \frac{n}{\gamma \ln n}$ a.a.s., where $\gamma = 2/\ln \frac{1}{1-p}$.*

Proof. Since the result holds for $t = 0$ and $\chi(G) \geq \chi^t(G)$, we assume $t \geq 1$ and we need only establish the lower bound on $\chi^t(\mathbb{G}(n,p))$. We will estimate α^t and use Proposition 1.9. Fix $0 < \varepsilon < 1$ and set $k = k(n) = \left\lceil \frac{1}{1-\varepsilon} \gamma \ln n \right\rceil$. We will show that the expected number of t-dependent k-sets approaches zero.

Clearly, the number of graphs on $[k]$ with *maximum* degree at most t is at most the number $g(k,t)$ of graphs on $[k] = \{1, \ldots, k\}$ with *average* degree at most t. Passing from maximum degree to average degree is an important simplification not only here but also in later results. Now, since a graph on k vertices with average degree d' has $d'k/2$ edges, it is clear that

$$g(k,t) = \sum_{s=0}^{s_0} \binom{\binom{k}{2}}{s},$$

where $s_0 = \lfloor \frac{tk}{2} \rfloor$. If $t \leq \frac{k-1}{3}$, then $\frac{tk}{2} \leq \frac{1}{3}\binom{k}{2}$; furthermore, if $s \leq \frac{1}{3}(x+1)$, then $\binom{x}{s-1} = \frac{s}{x-s+1}\binom{x}{s} \leq \frac{1}{2}\binom{x}{s}$; therefore, since $t(n) = o(k)$, it follows that for large enough n

$$g(k,t) \leq \sum_{s=0}^{s_0} 2^{-s}\binom{\binom{k}{2}}{s_0} \leq 2\binom{\binom{k}{2}}{s_0} \leq 2\left(\frac{ek(k-1)}{2s_0}\right)^{s_0} \leq 2\left(\frac{e(k-1)}{t}\right)^{\frac{tk}{2}}.$$

The last inequalities use the facts that $\binom{x}{s} \leq \left(\frac{ex}{s}\right)^s$ and the function $(a/x)^x$ is increasing for $0 < x < a/e$.

Since $p < 1$ and at least $\binom{k}{2} - \frac{tk}{2}$ of the possible edges are missing from a graph on $[k]$ with average degree at most t, the expected number of t-dependent k-sets in $\mathbb{G}(n,p)$ is at most

$$\binom{n}{k} q^{\binom{k}{2} - \frac{tk}{2}} g(k,t) \leq \left(\frac{en}{k} q^{\frac{k-t-1}{2}} 2^{\frac{1}{k}} \left(\frac{ek}{t}\right)^{\frac{t}{2}}\right)^k.$$

Let us examine the expression $B \equiv \frac{en}{k} q^{\frac{k-t-1}{2}} 2^{\frac{1}{k}} \left(\frac{ek}{t}\right)^{\frac{t}{2}}$. Taking its logarithm, we obtain

$$\ln B = \ln n - \ln k - \frac{k-t}{\gamma} + \frac{t}{2}\ln\frac{k}{t} + \frac{t}{2} + O(1)$$

$$\leq -\frac{\varepsilon}{1-\varepsilon}\ln n + O\left(t \ln \frac{\ln n}{t}\right).$$

Now $t \ln \frac{\ln n}{t} = \ln n \cdot \frac{\ln \frac{\ln n}{t}}{\frac{\ln n}{t}} = o(\ln n)$ since $\frac{\ln n}{t} \to \infty$. Thus, $\ln B \to -\infty$ and the expected number of t-dependent k-sets in $\mathbb{G}\,(n,p)$ approaches zero as $n \to \infty$. So, with probability going to one, $\alpha^t(\mathbb{G}\,(n,p)) \leq \frac{1}{1-\varepsilon}\gamma \ln n$ and $\chi^t(\mathbb{G}\,(n,p)) \geq (1-\varepsilon)\frac{n}{\gamma \ln n}$. $\qquad\square$

Next, we consider the case where $t(n) = \omega(\ln n)$ and $0 < p < 1$ fixed. The following theorem immediately implies parts (iii) and (iv) for Theorem 4.1.

Theorem 4.4. *Fix $0 < p < 1$ and $\varepsilon > 0$. If $t/\ln n \to \infty$, then $(1-\varepsilon)\frac{np}{t} \leq \chi^t(\mathbb{G}\,(n,p)) \leq \left\lceil (1+\varepsilon)\frac{np}{t} \right\rceil$ a.a.s.*

Proof. s By Corollary 1.6 and the fact that $\Delta(\mathbb{G}\,(n,p)) \sim np$ a.a.s., we need only establish a lower bound. Again, we want to estimate the maximum size α^t of a t-dependent set in $\mathbb{G}\,(n,p)$ and we will pass to average degree (or, equivalently, the number of edges).

Now let $k = k(n) = \left\lceil \frac{t}{(1-\varepsilon')p} \right\rceil + 1$ for some $0 < \varepsilon' < \varepsilon$ so that $k < \frac{t}{(1-\varepsilon)p}$ for large enough n. Clearly, $p(k-1) \geq \frac{t}{1-\varepsilon'}$. Let $E = E([k])$ be the set of edges induced on $[k]$. Then $|E| \in \text{Bin}\left(\binom{k}{2},p\right)$ and, using the Chernoff bound of (1.3) (and the fact that $t \leq p(k-1)$),

$$
\begin{aligned}
\Pr(\Delta(\mathbb{G}\,(k,p)) \leq t) &\leq \Pr(\deg_{\text{avg}}(\mathbb{G}\,(k,p)) \leq t) = \Pr(|E| \leq kt/2) \\
&= \Pr\left(|E| - p\binom{k}{2} \leq -\frac{k}{2}(p(k-1) - t) \right) \\
&\leq \exp\left(-\left(\frac{k}{2}(p(k-1)-t)\right)^2 \Big/ \left(2p\binom{k}{2}\right) \right) \\
&\leq \exp\left(-(p(k-1) - t)^2/(4p) \right).
\end{aligned}
$$

We have that $p(k-1) - t \geq \frac{t}{1-\varepsilon'} - t = \frac{\varepsilon'}{1-\varepsilon'}t = \Omega(k)$ and hence the expected number of t-dependent k-sets in $\mathbb{G}\,(n,p)$ is at most

$$
\binom{n}{k} e^{-\Omega(k^2)} \leq n^k e^{-\Omega(k^2)} = \exp(k \ln n - \Omega(k^2)) \to 0
$$

as $n \to \infty$ since $k/\ln n \to \infty$. So, with probability going to one, $\alpha^t(\mathbb{G}\,(n,p)) \leq \frac{t}{(1-\varepsilon)p}$ and $\chi^t(\mathbb{G}\,(n,p)) \geq (1-\varepsilon)\frac{np}{t}$. $\qquad\square$

We note here that the above argument can be adapted to give also the remaining part (ii) for Theorem 4.1. We will analyse this case in more detail, however, in the following section.

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4.2 Dense random graphs: $t(n) = \Theta(\ln n)$

Here, we consider the case $t(n) = \Theta(\ln n)$ and $0 < p < 1$ fixed. Due to the monotonicity of the t-improper chromatic number with respect to t — if $t_1 \le t_2$, then $\chi^{t_1}(G) \ge \chi^{t_2}(G)$ for any graph G — it suffices to assume that $t(n) \sim \tau \ln n$ for some fixed $\tau > 0$. By the trivial bound of Corollary 1.6 and the fact that $\Delta(\mathbb{G}(n,p)) \sim np$ a.a.s, we have in this case that, for any fixed $\varepsilon > 0$, $\chi^t(\mathbb{G}(n,p)) \le (1 + \varepsilon)\frac{np}{t} = O\left(\frac{n}{\ln n}\right)$ a.a.s. We believe, however, that this upper bound can be improved. We propose a conjectured value for the asymptotic constant suggested by Theorem 4.1(ii) by using large deviation methods to give a more precise estimate for $\alpha^t(\mathbb{G}(n,p))$.

For further background into large deviations, consult [24]; we borrow some notation from this reference. In the following, we denote

$$\Lambda^*(x) = \begin{cases} x \ln \dfrac{x}{p} + (1 - x) \ln \dfrac{1 - x}{q} & \text{for } x \in [0, 1] \\ \infty & \text{otherwise} \end{cases}$$

(where $\Lambda^*(0) = \ln \frac{1}{q}$ and $\Lambda^*(1) = \ln had\frac{1}{p}$). This is the Fenchel-Legendre transform of the logarithmic moment generating function associated with the Bernoulli distribution with probability p (cf. Exercise 2.2.23(b) of [24]). Some easy calculus checks that $\Lambda^*(x)$ has a global minimum of 0 at $x = p$, is strictly decreasing on $[0, p)$ and strictly increasing on $(p, 1]$. The next lemma will be used extensively in the rest of this chapter as it gives us a precise estimate for the number of (weakly) t-dependent k-sets.

Lemma 4.5. *Suppose $0 < p = p(n) < 1$ and suppose the positive integers $t = t(n)$ and $k = k(n)$ satisfy that $t \le p(k-1)$. Then*

(i) $\Pr(\deg_{\text{avg}}(\mathbb{G}(k,p)) \le t) \le \exp\left(-\binom{k}{2}\Lambda^*\left(\dfrac{t}{k-1}\right)\right)$; *and*

(ii) $\Pr(\deg_{\text{avg}}(\mathbb{G}(k,p)) \le t) \ge \exp\left(-\binom{k}{2}\left(\Lambda^*\left(\dfrac{kt-1}{k(k-1)}\right) + O\left(\dfrac{\ln kt}{k^2}\right)\right)\right).$

In particular, the expected number of t-dependent k-sets in $\mathbb{G}(n,p)$ is at most

$$A_{n,t,k} = \binom{n}{k}\exp\left(-\binom{k}{2}\Lambda^*\left(\dfrac{t}{k-1}\right)\right).$$

Proof. Let E be the set of edges induced on $[k]$ and let $\bar{E}$ be the set of non-edges. Clearly, $\Pr(\deg_{\mathrm{avg}}(\mathbb{G}(k,p)) \leq t) = \Pr(|E| \leq \lfloor kt/2 \rfloor) = \Pr\left(|\bar{E}| \geq \binom{k}{2} - \lfloor kt/2 \rfloor\right)$. We have that $|\bar{E}| \in \mathrm{Bin}\left(\binom{k}{2}, q\right)$. It follows from the Chernoff bound of (1.1) that

$$
\Pr(\deg_{\mathrm{avg}}(\mathbb{G}(k,p)) \leq t) = \Pr\left(|\bar{E}| \geq \binom{k}{2} - \left\lfloor \frac{kt}{2} \right\rfloor\right) \leq \left(\frac{\binom{k}{2}q}{\binom{k}{2} - \frac{kt}{2}}\right)^{\binom{k}{2} - \frac{kt}{2}} \left(\frac{\binom{k}{2} - \binom{k}{2}q}{\frac{kt}{2}}\right)^{\frac{kt}{2}}
$$

$$
= \left(\frac{(k-1)q}{k-t-1}\right)^{\binom{k}{2} - \frac{kt}{2}} \left(\frac{(k-1)p}{t}\right)^{\frac{kt}{2}} = \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right)\right).
$$

This confirms part (i).

For part (ii), the following Stirling's formula calculations are sufficient. If $m \leq n/2$, then by inequality (1.5) of [17],

$$
\binom{n}{m}p^m q^{n-m} \geq \left(\frac{np}{m}\right)^m \left(\frac{nq}{n-m}\right)^{n-m} e^{-\frac{1}{6m}} \sqrt{\frac{n}{2\pi m(n-m)}}
$$

$$
= \exp\left(-n\left(\Lambda^*\left(\frac{m}{n}\right) + O\left(\frac{\ln m}{n}\right)\right)\right).
$$

Since $|E| \in \mathrm{Bin}\left(\binom{k}{2}, p\right)$,

$$
\Pr(\deg_{\mathrm{avg}}(\mathbb{G}(k,p)) \leq t) \geq \Pr\left(|E| = \left\lfloor \frac{kt}{2} \right\rfloor\right) = \binom{\binom{k}{2}}{\lfloor \frac{kt}{2} \rfloor} p^{\lfloor \frac{kt}{2} \rfloor} q^{\binom{k}{2} - \lfloor \frac{kt}{2} \rfloor}
$$

and part (ii) follows. $\qquad\square$

The main result of this section is to show that, if $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) > 0$, then the expected number of t-dependent $(\kappa \ln n)$-sets goes to infinity and, if $1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) < 0$, then it goes to zero. There is similar behaviour for the expected number of t-improper j-colourings, where $j \sim \frac{n}{\kappa \ln n}$. More precisely, we have the following.

Theorem 4.6. *Fix $0 < p < 1$. Fix $\tau, \kappa > 0$ with $\kappa > \tau/p$ and suppose $t(n) \sim \tau \ln n$, $k(n) \sim \kappa \ln n$ and $j(n) = \lceil n/k \rceil$. Let $E_{n,t,k}$ be the expected number of t-dependent k-sets in $\mathbb{G}(n,p)$ and $C_{n,t,j}$ be the expected number of t-improper j-colourings of $\mathbb{G}(n,p)$. Then*

(i) $E_{n,t,k} = \exp\left(k \ln n \left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right);$ *and*

(ii) $C_{n,t,j} = \exp\left(n \ln n \left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right).$

As alluded to earlier, this theorem is intended to be analogous to Theorem 4 of [38], cf. (4.1) on page 50. We note that, by t-improper j-colourings, we mean ordered partitions of the vertex set into j t-dependent sets (though it would not make any difference if we counted unordered partitions instead).

Proof of part (i) *of Theorem 4.6.* Clearly, since $\kappa > \tau/p$, it follows that $t(n) \leq p(k(n) - 1)$ for large enough n. Thus, since $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, it follows from Lemma 4.5(i) that

$$E_{n,t,k} \leq A_{n,t,k} = \exp\left(\kappa(\ln n)^2 \left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

and so we just need to show the reverse inequality.

Our approach will be to bound the probability that a k-set is t-dependent with an appropriately chosen conditional probability. First, we will give an estimate for the conditional probability

$$P_{n,\varepsilon} = \Pr(\Delta(\mathbb{G}\,(k,p)) > t \mid \deg_{\mathrm{avg}}(\mathbb{G}\,(k,p)) \leq (1 - \varepsilon)t)$$

for $0 < \varepsilon < 1$. Note that, if we condition on a fixed number $m \in \{0, \ldots, \binom{k}{2}\}$ of edges in $\mathbb{G}\,(k,p)$, this is essentially the uniform random graph model $\mathbb{G}\,(k,m)$ (where we choose among all $\binom{\binom{k}{2}}{m}$ possible subgraphs with m edges). Thus,

$$\Pr(\Delta(\mathbb{G}\,(k,p)) > t \mid |E(\mathbb{G}\,(k,p))| = m) = \Pr(\Delta(\mathbb{G}\,(k,m)) > t).$$

Also, it is clear by a coupling argument that, if there are more edges, then it is more likely that the maximum degree will be higher, i.e.

$$\Pr(\Delta(\mathbb{G}\,(k,m-1)) > t) \leq \Pr(\Delta(\mathbb{G}\,(k,m)) > t).$$

Now let $\hat{m} = \lfloor (1-\varepsilon)kt/2 \rfloor$. It follows that

$$P_{n,\varepsilon} = \Pr\left(\Delta(\mathbb{G}\left(k,p\right)) > t \mid |E(\mathbb{G}\left(k,p\right))| \le \hat{m}\right)$$

$$\le \Pr\left(\Delta(\mathbb{G}\left(k,\hat{m}\right)) > t\right)$$

$$\le k\Pr\left(\deg(v) > t \text{ in } \mathbb{G}\left(k,\hat{m}\right)\right).$$

The degree of a vertex in $\mathbb{G}\left(k,\hat{m}\right)$ has a hypergeometric distribution with parameters $\binom{k}{2}$, $k-1$ and $\hat{m}$ (with expected value $\lambda = (k-1)\hat{m}/\binom{k}{2}$, so that $(1-\varepsilon)t - 2/k \le \lambda \le (1-\varepsilon)t$) and thus, by the Chernoff-Hoeffding analogue of (1.2) (cf. Theorem 2.10 of [49]),

$$P_{n,\varepsilon} \le k\exp\left(-\frac{\varepsilon^2 t^2}{2((1-\varepsilon)t + \varepsilon t/3 + 2/(3k))}\right) \le k\exp\left(-\frac{\varepsilon^2}{2}t\right).$$

If we choose $\varepsilon = \varepsilon_n$ approaching zero slowly enough, say, $\varepsilon_n = (\ln n)^{-1/3}$, then this conditional probability is $o(1)$. Then, furthermore, using Lemma 4.5(ii),

$$\Pr(\Delta(\mathbb{G}\left(k,p\right)) \le t) \ge \Pr(\Delta(\mathbb{G}\left(k,p\right)) \le t \mid \deg_{\text{avg}}(\mathbb{G}\left(k,p\right)) \le (1-\varepsilon_n)t)$$

$$\times \Pr(\deg_{\text{avg}}(\mathbb{G}\left(k,p\right)) \le (1-\varepsilon_n)t)$$

$$= (1 - P_{n,\varepsilon_n})\Pr(\deg_{\text{avg}}(\mathbb{G}\left(k,p\right)) \le (1-\varepsilon_n)t)$$

$$\ge (1 - o(1))\exp\left(-\binom{k}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

$$[\text{since } \varepsilon = \varepsilon_n \to 0 \text{ and slowly enough}]$$

$$= \exp\left(-\binom{k}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

and the expected number of t-dependent k-sets satisfies

$$E_{n,t,k} \ge \binom{n}{k}\exp\left(-\binom{k}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

$$= \exp\left(k\ln n\left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proof of part (ii) *of Theorem 4.6.* We shall begin by proving the lower bound. We will estimate $C_{n,t,j}$ by counting the number C_j of partitions of $[n]$ into j sets of size $k-1$ or k. If k divides n, then $C_j = \frac{n!}{(k!)^j j!}$ and in any case clearly $C_j \geq \frac{n!}{(k!)^j j!}$. Since $\frac{n!}{(k!)^j j!} = n^{(1+o(1))n}$, it follows that

$$C_{n,t,j} \geq C_j \Pr(\Delta(\mathbb{G}\,(k,p)) \leq t)^j \geq n^{(1+o(1))n} \left(\exp\left(-\binom{k}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)\right)^j$$

using the estimate for $\Pr(\Delta(\mathbb{G}\,(k,p)) \leq t)$ implied by the proof of part (i). Therefore,

$$C_{n,t,j} \geq \exp\left((1+o(1))n \ln n - j\frac{k^2}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$
$$= \exp\left(n \ln n \left(1 - \frac{\kappa}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)\right),$$

as required.

Next, to prove the upper bound, we bound the probability that an arbitrary j-colouring is t-improper. Given a partition P of $[n]$ into j parts $P_1, \ldots, P_j$, let $\mathbb{G}\,(n,p)\,[P]$ denote the disjoint union of the subgraphs induced by $P_1, \ldots, P_j$. Thus, the probability that P is t-improper is

$$\Pr(\Delta(\mathbb{G}\,(n,p)\,[P]) \leq t) \leq \Pr(\deg_{\mathrm{avg}}(\mathbb{G}\,(n,p)\,[P]) \leq t).$$

Since the number of edges in $\mathbb{G}\,(n,p)\,[P]$ has a binomial distribution, the latter probability is minimised when the number of "candidate" edges, i.e. the number of edges in $K_n[P]$, is minimized. This occurs when P is a balanced partition Q of the vertices; in particular, the least number of candidate edges is at least $j\binom{k-1}{2}$ which, by the choice of j, is at least $\frac{n(k-3)}{2}$. Thus,

$$\Pr(\Delta(\mathbb{G}\,(n,p)\,[P]) \leq t) \leq \Pr(\deg_{\mathrm{avg}}(\mathbb{G}\,(n,p)\,[Q]) \leq t)$$
$$\leq \Pr\left(\mathrm{Bin}\left(\frac{n(k-3)}{2}, p\right) \leq \frac{nt}{2}\right)$$
$$= \exp\left(-\frac{n(k-3)}{2}\left(\Lambda^*\left(\frac{t}{k-3}\right) + o(1)\right)\right)$$

by the Chernoff bound of (1.1). The total number of j-partitions P is at most j^n (since

each $x \in [n]$ can choose j parts); thus, the expected number of t-improper j-colourings is

$$C_{n,t,j} \leq j^n \exp\left(-\frac{n(k-3)}{2}\left(\Lambda^*\left(\frac{t}{k-3}\right) + o(1)\right)\right)$$

$$\leq \exp\left(n \ln n - n \ln k - n \ln n \cdot \frac{\kappa}{2}\left(\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right)$$

$$= \exp\left(n \ln n \left(1 - \frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) + o(1)\right)\right).$$

$\square$

Lemma 4.7. *For any $0 < p < 1$ and $\tau > 0$, there is a unique $\kappa_0 = \kappa_0(p,\tau) > \tau/p$ such that*

$$\frac{\kappa}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) \begin{cases} < 1 & \text{if } \tau/p < \kappa < \kappa_0 \\ = 1 & \text{if } \kappa = \kappa_0 \\ > 1 & \text{if } \kappa > \kappa_0 \end{cases}.$$

Proof. Consider the function $f : (0, \infty) \to \mathbb{R}$ defined as follows:

$$f(\kappa) = \frac{1}{2}\Lambda^*\left(\frac{\tau}{\kappa}\right) - \frac{1}{\kappa}.$$

Then f is continuous and strictly increasing on $\kappa \in (\tau/p, \infty)$, $f(\tau/p) = -p/\tau < 0$ and $f(\kappa) \to \frac{1}{2}\ln\frac{1}{q} > 0$ as $\kappa \to \infty$. $\square$

Due to the monotonicity of χ^t with respect to t, we obtain the required lower bound in part (ii) of Theorem 4.1 as a corollary of either part of Theorem 4.6.

Corollary 4.8. *Fix $0 < p < 1$. Suppose $t(n) \sim \tau \ln n$ for some fixed $\tau > 0$. If $\kappa > \kappa_0(p,\tau)$ (where κ_0 is as in Lemma 4.7), then $\chi^t(\mathbb{G}(n,p)) \geq \frac{n}{\kappa \ln n}$ a.a.s.*

Theorem 4.6 suggests the following.

Conjecture 4.9. *Fix $0 < p < 1$. Suppose $t(n) \sim \tau \ln n$ for some fixed $\tau > 0$. If $\kappa_0 = \kappa_0(p,\tau)$ is as in Lemma 4.7, then $\chi^t(\mathbb{G}(n,p)) \sim \frac{n}{\kappa_0 \ln n}$ a.a.s.*

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4.3   Sparse random graphs

In the previous sections, we considered dense graphs, in which the expected average degree of $\mathbb{G}(n,p)$ is $np = \Theta(n)$. In this section, we will consider random graphs with smaller expected average degree $d(n) = np(n) = o(n)$. In particular, we would like to show an analogue of the following result.

Theorem 4.10 (Łuczak [57]). *Suppose $0 < p(n) < 1$, $p(n) = o(1)$ and $\varepsilon > 0$. Set $d(n) = np(n)$. There exists constant d_0 such that, if $d(n) \geq d_0$, then $(1-\varepsilon)\frac{d}{2\ln d} \leq \chi(\mathbb{G}(n,p)) \leq (1+\varepsilon)\frac{d}{2\ln d}$ a.a.s.*

For t small enough in terms of d, we can extend the above theorem using first moment method arguments; however, the simple argument used for Theorem 4.3 is not strong enough here. Instead, we adapt the large deviations estimate of Lemma 4.5(i) to an appropriate choice of $k(n)$.

Theorem 4.11. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$.*

(i) *Fix $\varepsilon > 0$. There exist constants d_0 and $\tau > 0$ such that, if $d(n) \geq d_0$ and $t(n) \leq \tau \ln d$, then $(1-\varepsilon)\frac{d}{2\ln d} \leq \chi^t(\mathbb{G}(n,p)) \leq (1+\varepsilon)\frac{d}{2\ln d}$ a.a.s.*

(ii) *If $d(n) = \omega(1)$ and $t(n) = o(\ln d)$, then $\chi^t(\mathbb{G}(n,p)) \sim \frac{d}{2\ln d}$ a.a.s.*

Proof. First of all, note that part (ii) is a corollary to part (i); therefore, it will suffice to prove (i). Since the conclusion holds for the case $t = 0$ due to Theorem 4.10, we may assume $t \geq 1$. Fix $\varepsilon, \varepsilon'$ such that $0 < \varepsilon' < \varepsilon$. As alluded to above, we will follow the first moment arguments of Section 4.2 but where we set $k = k(n) = \left\lceil \frac{1}{1-\varepsilon'} \frac{2\ln d}{p} \right\rceil + 1$.

Note that $k < \frac{1}{1-\varepsilon} \frac{2\ln d}{p}$ and $t \leq p(k-1)$ for large enough d and small enough τ so that we may apply Lemma 4.5(i). Thus, in this case, we have $(\ln A_{n,t,k})/k$ is at most

$$\ln n - \ln k + 1 - \frac{t}{2}\ln\left(\frac{t}{k-1} \cdot \frac{1}{p}\right) - \frac{k-t-1}{2}\ln\left(\left(1 - \frac{t}{k-1}\right) \cdot \frac{1}{q}\right).$$

Now,f $\ln \frac{1}{q} = (1 + o(1))p$ and also

$$\frac{k-t-1}{2}\ln\left(1 - \frac{t}{k-1}\right) = \frac{k-t-1}{2} \cdot \frac{-t}{k-1}(1 + o(1)) = -\frac{t}{2}(1 + o(1))$$

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so, therefore, for large enough n and d,

$$(\ln A_{n,t,k})/k \leq \ln n - \ln k + 1 + \frac{t}{2}\ln\frac{kp}{t} - \frac{kp}{2} + \frac{t}{2} + o(\ln d)$$

$$\leq \ln d - \ln\ln d + \frac{t}{2}\ln\frac{2\ln d}{t} - \frac{1}{1-\varepsilon'}\ln d + \frac{t}{2}\left(\ln\frac{1}{1-\varepsilon} + 1\right) + o(\ln d)$$

$$\leq -\frac{\varepsilon'}{1-\varepsilon'}\ln d + \frac{\ln\frac{2\ln d}{t}}{\frac{2\ln d}{t}}\ln d + \frac{t}{2}\left(\ln\frac{1}{1-\varepsilon} + 1\right) + o(\ln d).$$

By choosing $\tau > 0$ small enough, we can force

$$\frac{\ln\frac{2\ln d}{t}}{\frac{2\ln d}{t}} + \frac{\tau}{2}\left(\ln\frac{1}{1-\varepsilon} + 1\right) < \frac{\varepsilon'}{1-\varepsilon'},$$

giving $(\ln A_{n,t,k})/k < 0$ for large enough d. Therefore, there exists d_0 and $\tau > 0$ such that if $d \geq d_0$ and $t \leq \tau \ln d$ then, with probability going to one, $A_{n,t,k} \to 0$ and $\chi^t(\mathbb{G}(n,p)) \geq (1-\varepsilon)\frac{d}{2\ln d}$. Combining this with Theorem 4.10 gives the desired result. $\qquad\square$

The above theorem for sparse random graphs is analogous to Theorem 4.3 (i.e. part (i) of Theorem 4.1) for dense random graphs. For sparse graphs in the case when t grows quickly, we can again establish a lower bound for the t-improper chromatic by a first moment argument, but we require the more sophisticated estimate introduced in Section 4.2 to prove Theorem 4.12 below. As for the upper bound, we can no longer exclusively rely on Corollary 1.6 (particularly for $d(n) = O(\ln n)$ when the maximum degree of $\mathbb{G}(n,p)$ is asymptotically larger than np); however, we can still extend the upper bound to the range of probabilities satisfying $d(n) = \omega(\sqrt{\ln n})$ by colouring two parts of the graph separately, as shown in Theorem 4.13 below. Together, Theorems 4.12 and 4.13 imply parts (iv) and (v) of Theorem 4.2, but for part (v) we can only give a two-point range of values for χ^t due to an extra round-up that is introduced when we partition the graph in Theorem 4.13.

Theorem 4.12. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Fix $\varepsilon > 0$. If $t(n) = \omega(\max\{\ln d(n), 1\})$, then $(1-\varepsilon)\frac{d}{t} \leq \chi^t(\mathbb{G}(n,p))$ a.a.s.*

Proof. We will follow the first moment arguments of Section 4.2 but where we instead set $k = k(n) = \left\lceil\frac{1}{1-\varepsilon'}\frac{t}{p}\right\rceil + 1$ for some $0 < \varepsilon' < \varepsilon$ so that $k < \frac{1}{1-\varepsilon}\frac{t}{p}$ for large enough n. Clearly, $t \leq p(k-1)$ and so Lemma 4.5(i) applies. For our choice of k and since $p(n) = o(1)$, it can

be shown using Taylor expansion that

$$\Lambda^* \left(\frac{t}{k-1} \right) \geq \Lambda^*((1-\varepsilon')p)$$

$$= p(1-\varepsilon')\ln(1-\varepsilon') + (q+\varepsilon'p)\ln\left(1 + \frac{\varepsilon'p}{q}\right)$$

$$= (C + o(1))p$$

where $C = \left(\varepsilon' - (1-\varepsilon')\ln \frac{1}{1-\varepsilon'} \right) > 0$ fixed. Hence, by Lemma 4.5(i), there is some $C' > 0$ fixed such that

$$(\ln A_{n,t,k})/k \leq \ln n - \ln k - C't + o(t)$$

$$= \ln n - \ln t - \ln(1/p) - C't + o(t)$$

$$= \ln d - C't + o(t).$$

Since $t = \omega(\max\{\ln d(n), 1\})$, this expression approaches $-\infty$ as $n \to \infty$. Therefore, with probability going to one, the maximum size of a t-dependent set is less than $\frac{1}{1-\varepsilon} \frac{t}{p}$, as required. $\qquad\square$

Theorem 4.13. *Suppose* $0 < p(n) < 1$ *and* $p(n) = o(1)$. *Set* $d(n) = np(n)$. *Fix* $\varepsilon > 0$. *If* $d(n) = \omega(\sqrt{\ln n})$, *then* $\chi^t(\mathbb{G}(n,p)) \leq \left\lceil (1+\varepsilon)\frac{d}{t} \right\rceil + 1$ *a.a.s.*

Proof. By the use of Corollary 1.6 and the fact that $\Delta(\mathbb{G}(n,p)) \sim np$ for $p(n) = \omega(\ln n/n)$, we may assume now that $d(n) = O(\ln n)$.

Let $W = \left\{ v \mid \deg(v) \geq \left(1 + \frac{\varepsilon}{2}\right)d \right\}$ and $H = \mathbb{G}(n,p)[W]$ (the subgraph of $\mathbb{G}(n,p)$ induced by the set W). We want to estimate the expected number of vertices in W with degree at least k in H, where $k = k(n) = \left\lceil \frac{\varepsilon}{4}d(n) \right\rceil$.

Let us calculate $\Pr(\deg_H(v) \geq k \land v \in W)$. First observe that this probability is at most the probability that v has a set U of k neighbours such that each $u \in U$ has at least

$\left(1 + \frac{\varepsilon}{4}\right) d$ neighbours in $V \setminus \{\{u\} \cup U\}$.

$$\Pr(\deg_H(v) \geq k \wedge v \in W) \leq \binom{n-1}{k} p^k \cdot \Pr\left(\deg(u) \geq \left(1 + \frac{\varepsilon}{4}\right) d\right)^k$$

$$\leq \binom{n-1}{k} p^k \left(\exp\left(-\frac{\varepsilon^2}{48} d\right)\right)^k \quad \text{[by the Chernoff bound (1.2)]}$$

$$\leq \left(\frac{ed}{k} \exp\left(-\frac{\varepsilon^2}{48} d\right)\right)^k.$$

Therefore, the expected number of vertices in W with degree at least k in H is

$$E_k = n \cdot \Pr\left(\deg_H(v) \geq k \wedge v \in W\right) \leq n \left(\frac{ed}{k} \exp\left(-\frac{\varepsilon^2}{48} d\right)\right)^k$$

$$\leq \exp\left(\ln n - \frac{\varepsilon^2}{48} kd + O(k)\right).$$

Now $E_k \to 0$ as $n \to \infty$, since $d(n) = \omega(\sqrt{\ln n})$. This shows that, with probability going to one, the maximum degree of H is at most $\frac{\varepsilon}{4} d$.

Now, if we t-improperly colour H and $\mathbb{G}(n,p) \setminus W$ with disjoint sets of colours using Corollary 1.6, then the desired result follows. $\qquad \square$

For $d(n) = O(\sqrt{\ln n})$, we have the upper bound implied by Corollary 1.6, but as long as $d(n) > c$ for some fixed $c > 0$, we know that $\Delta(\mathbb{G}(n,p)) \geq \ln n / \ln \ln n$ a.a.s. (cf. Exercise 3.5 of [17]). The previous theorem implies a bound that is asymptotically lower than $\ln n/(t \ln \ln n)$; furthermore, with more care, we can obtain an upper bound of $O(\sqrt{\ln n}/t)$.

Theorem 4.14. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Fix $\delta, \kappa > 0$ such that $\delta < \kappa^3/2 - \kappa/3$. If $d(n) = \omega(1)$ and $d(n) \leq \delta\sqrt{\ln n}$ for large enough n, then $\chi^t(\mathbb{G}(n,p)) \leq \left\lceil \frac{(\delta+4\kappa)\sqrt{\ln n}}{t} \right\rceil + 1$ a.a.s.*

Proof. Let $k = k(n) = \left\lceil \kappa\sqrt{\ln n} \right\rceil$, $W = \{v \mid \deg(v) \geq d + 2k\}$ and $H = \mathbb{G}(n,p)[W]$. We want to estimate the expected number of vertices in W with degree at least k in H.

As before, we have

$$\Pr(\deg_H(v) \geq k \wedge v \in W) \leq \binom{n-1}{k} p^k \cdot \Pr\left(\deg(u) \geq d+k\right)^k$$

$$< \left(\frac{ed}{k} \exp\left(-\frac{\kappa^2}{2(\delta+\kappa/3)}\sqrt{\ln n}\right)\right)^k.$$

Therefore, the expected number of vertices in W with degree at least k in H is

$$E_k \leq n\left(\frac{ed}{k}\exp\left(-\frac{\kappa^2}{2(\delta+\kappa/3)}\sqrt{\ln n}\right)\right)^k$$

$$= \left(\exp\left(\left(\frac{1}{\kappa} - \frac{\kappa^2}{2(\delta+\kappa/3)}\right)\sqrt{\ln n} + o\left(\sqrt{\ln n}\right)\right)\right)^k.$$

Since $\delta < \kappa^3/2 - \kappa/3$, it follows that $E_k \to 0$ as $n \to \infty$ and this shows that, with probability going to one, the maximum degree of H is at most $k-1$.

Now, we t-improperly colour H and $\mathbb{G}(n,p) \setminus W$ with disjoint sets of colours using Corollary 1.6, and it follows that

$$\chi^t(\mathbb{G}(n,p)) \leq \left\lceil \frac{d+2k}{t+1} \right\rceil + \left\lceil \frac{k}{t+1} \right\rceil \leq \left\lceil \frac{(\delta+4\kappa)\sqrt{\ln n}}{t} \right\rceil + 1 \text{ a.a.s.}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Theorem 4.13 shows that $\chi^t(\mathbb{G}(n,p)) = O(\frac{d}{t})$ when $t(n) = \Theta(\ln d)$ and $d(n) = \omega(\sqrt{\ln n})$, and the following theorem shows that $\chi^t(\mathbb{G}(n,p)) = \Omega(\frac{d}{t})$ in the same setting (and more); therefore, together, Theorem 4.13 and Theorem 4.15 imply part (iii) of Theorem 4.2.

Given fixed $\tau > 0$, let f be the real-valued function defined by $f(\kappa) = \frac{1}{2}\left(\kappa - \tau - \tau\ln\frac{\kappa}{\tau}\right)$; note that $f(\tau) = 0$, $f(\kappa) \to \infty$ as $\kappa \to \infty$, and $f(\kappa)$ is an increasing function for $\kappa > \tau$. Thus, there is a unique $\kappa_0 = \kappa_0(\tau) > \tau$ satisfying that

$$\frac{1}{2}\left(\kappa - \tau - \tau\ln\frac{\kappa}{\tau}\right) \begin{cases} < 1 & \text{if } \tau < \kappa < \kappa_0 \\ = 1 & \text{if } \kappa = \kappa_0 \\ > 1 & \text{if } \kappa > \kappa_0 \end{cases}.$$

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Theorem 4.15. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Suppose that $d(n) = \omega(1)$ and $t(n) \sim \tau \ln d(n)$ for some constant $\tau > 0$. If $\kappa > \kappa_0(\tau)$ (where κ_0 is as defined above) and $\varepsilon > 0$ fixed, then $(1-\varepsilon)\frac{d}{\kappa \ln d} \leq \chi^t(\mathbb{G}(n,p))$ a.a.s.*

Proof. Let $0 < \varepsilon' < \varepsilon$ and set $k = k(n) = \left\lceil \frac{1}{1-\varepsilon'}\frac{\kappa}{p(n)} \ln d(n) \right\rceil + 1$. We have that $k < \frac{1}{1-\varepsilon}\frac{\kappa}{p(n)} \ln d(n)$ for large enough n. Also, since $\kappa > \tau$, it follows that $t(n) \leq p(n)(k(n)-1)$ for large enough n. As usual, we apply Lemma 4.5(i):

$$\frac{\ln A_{n,t,k}}{k} \leq \ln n - \ln \ln d + \ln p - \frac{k}{2}\Lambda^*\left(\frac{t}{k-1}\right) + O(1)$$

$$\leq \left(1 - \frac{\kappa}{2p}\Lambda^*\left(\frac{\tau}{\kappa}p\right)\right)\ln d - \ln \ln d + O(1).$$

It suffices now to show that $1 - \frac{\kappa}{2p}\Lambda^*(\frac{\tau}{\kappa}p) \leq 0$ and then this last expression will approach $-\infty$ as $n \to \infty$.

It is easy to verify, using elementary calculus, that $(1+x)\ln(1+x) - x \geq 0$ for any $x \geq 0$. Thus, we have that

$$\Lambda^*\left(\frac{\tau}{\kappa}p\right)\Big/ p = \frac{\tau}{\kappa}\ln\frac{\tau}{\kappa} + \frac{1 - \frac{\tau}{\kappa}p}{p}\ln\left(\frac{1 - \frac{\tau}{\kappa}p}{q}\right)$$

$$= \frac{\tau}{\kappa}\ln\frac{\tau}{\kappa} + 1 - \frac{\tau}{\kappa} + \frac{q}{p}\left((1+x)\ln(1+x) - x\right) \qquad (4.3)$$

$$\left[\text{where } x = \left(1 - \frac{\tau}{\kappa}\right)\frac{p}{q}\right]$$

$$\geq \frac{\tau}{\kappa}\ln\frac{\tau}{\kappa} + 1 - \frac{\tau}{\kappa}$$

giving

$$1 - \frac{\kappa}{2p}\Lambda^*\left(\frac{\tau}{\kappa}p\right) \leq 1 - \frac{\kappa}{2}\left(\frac{\tau}{\kappa}\ln\frac{\tau}{\kappa} + 1 - \frac{\tau}{\kappa}\right)$$

$$= 1 - \frac{1}{2}\left(\kappa - \tau - \tau\ln\frac{\kappa}{\tau}\right) \leq 0,$$

This completes the proof. $\qquad\qquad\square$

Note that our calculations in (4.3) can be used to show, using Taylor expansion, that $\Lambda^*\left(\frac{\tau}{\kappa}p\right)/p = \frac{\tau}{\kappa}\ln\frac{\tau}{\kappa} + 1 - \frac{\tau}{\kappa} + o(1)$. This will be used in the proof of part (i) of Theorem 4.17.

We have a one-sided conjecture for the case when $t(n) = \Theta(\ln d)$.

Conjecture 4.16. *Suppose $0 < p(n) < 1$ and set $d(n) = np(n) = o(n)$ and suppose $t(n) \sim \tau \ln d(n)$ for some fixed $\tau > 0$. If $\kappa_0 = \kappa_0(\tau)$ is as defined before Theorem 4.15, then $\chi^t(\mathbb{G}(n,p)) \sim \frac{d}{\kappa_0 \ln d}$ a.a.s.*

We now provide an analogue of Theorem 4.6 to lend support to the above formulation. Unfortunately, the result only applies to a limited range of probabilities: $p(n) = n^{-o(1)}$.

Theorem 4.17. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Assume $p(n) = n^{-o(1)}$. Fix $\kappa > \tau > 0$ and suppose $t(n) \sim \tau \ln d(n)$, $k(n) \sim \frac{\kappa}{p(n)} \ln d(n)$ and $j(n) = \lceil n/k \rceil$ (in particular, $j(n) \sim \frac{d(n)}{\kappa \ln d(n)}$). Let $E_{n,t,k}$ be the expected number of t-dependent k-sets in $\mathbb{G}(n,p)$ and $C_{n,t,j}$ be the expected number of t-improper j-colourings of $\mathbb{G}(n,p)$. Then*

(i) $E_{n,t,k} = \exp\left(k \ln d \left(1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1) \right) \right)$; *and*

(ii) $C_{n,t,j} = \exp\left(n \ln d \left(1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1) \right) \right)$.

Proof of part (i) *of Theorem 4.17.* Theorem 4.15 implies that

$$E_{n,t,k} \leq A_{n,t,k} \leq \exp\left(k \left(\left(1 - \frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) \right) \ln d - \ln \ln d + O(1) \right) \right)$$
$$= \exp\left(k \ln d \left(1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1) \right) \right)$$

so we just need to establish the reverse inequality.

As for part (i) of Theorem 4.6, we will give an estimate for the conditional probability

$$P_{n,\varepsilon} = \Pr(\Delta(\mathbb{G}(k,p)) > t \mid \deg_{\mathrm{avg}}(\mathbb{G}(k,p)) \leq (1 - \varepsilon)t)$$

for $0 < \varepsilon < 1$. Previous arguments give that

$$P_{n,\varepsilon} \leq k \exp\left(-\frac{\varepsilon^2}{2} t \right)$$

and we want to choose $\varepsilon = \varepsilon_n$ approaching zero slowly enough. Note that the additional condition, $p(n) = n^{-o(1)}$, is necessary only for the following computations and not part (ii).

In the following, we will suppose that $p(n) = n^{-f(n)}$ where $f(n)$ is a non-negative, real function and $f(n) = o(1)$. Note that $\ln d = (1 - f(n)) \ln n$.

Put $\varepsilon_n = \sqrt{\frac{2}{t(n)} \ln \frac{\kappa \ln n}{p(n)}}$. First, let us show that $\varepsilon_n = o(1)$:

$$\varepsilon_n = \sqrt{\frac{2}{t} \ln(\kappa n^{f(n)} \ln n)} = \sqrt{\frac{2}{\tau(1 - f(n))} \left(f(n) + \frac{\ln(\kappa \ln n)}{\ln n} \right)} = o(1).$$

Next, observe the following:

$$P_{n,\varepsilon_n} \le k \exp\left(-\frac{\varepsilon_n^2}{2} t \right) = k \exp\left(-f(n) \ln n - \ln(\kappa \ln n) \right)$$

$$= k \exp\left(-\ln \frac{\kappa \ln n}{p} \right) = 1 - f(n).$$

Finally, note that, since $p(n) = o(1)$ and $f(n) = o(1)$, it follows that $\frac{1}{f(n)} = \frac{\ln n}{\ln \frac{1}{p}} = O(\ln n)$ and hence that $\ln \frac{1}{f(n)} = o(\ln n) = o(k^2)$. Then, tying the strands together, we have, using Lemma 4.5(ii),

$$\Pr(\Delta(\mathbb{G}(k,p)) \le t) \ge (1 - P_{n,\varepsilon_n}) \Pr(\deg_{\text{avg}}(\mathbb{G}(k,p)) \le (1 - \varepsilon_n)t)$$

$$\ge f(n) \exp\left(-\binom{k}{2} \left(\Lambda^*\left(\frac{\tau}{\kappa}p\right) + o(1) \right) \right)$$

[since $\varepsilon_n \to 0$ and slowly enough]

$$= \exp\left(-k \ln d \left(\frac{\kappa}{2p} \Lambda^*\left(\frac{\tau}{\kappa}p\right) + o(1) \right) \right)$$

and the expected number of t-dependent k-sets satisfies

$$E_{n,t,k} \ge \binom{n}{k} \exp\left(-k \ln d \left(\frac{\kappa}{2p} \Lambda^*\left(\frac{\tau}{\kappa}p\right) + o(1) \right) \right)$$

$$= \exp\left(k \ln d \left(1 - \frac{\kappa}{2p} \Lambda^*\left(\frac{\tau}{\kappa}p\right) + o(1) \right) \right)$$

$$= \exp\left(k \ln d \left(1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1) \right) \right)$$

using the calculations at the end of the proof of Theorem 4.15 (see the comment just afterwards). $\square$

Proof of part (ii) *of Theorem 4.17.* This proof will mirror that of Theorem 4.6(ii). We first

prove the lower bound. Since the number of partitions of $[n]$ into j sets of size $k-1$ or k is at least $\frac{n!}{(k!)^j j!}$ and $\frac{n!}{(k!)^j j!} = d^{(1+o(1))n}$, we have

$$C_{n,t,j} \geq d^{(1+o(1))n} \Pr(\Delta(\mathbb{G}(k,p)) \leq t)^j$$

$$\geq d^{(1+o(1))n} \left(\exp \left(-k \ln d \left(\frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right) \right)^j$$

using the estimate for $\Pr(\Delta(\mathbb{G}(k,p)) \leq t)$ implied by the proof of part (i). Therefore,

$$C_{n,t,j} \geq \exp \left((1+o(1))n \ln d - jk \ln d \left(\frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right)$$

$$= \exp \left(n \ln d \left(1 - \frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right).$$

The calculation at the end of the proof of Theorem 4.15 shows that $1 - \frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) = 1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1)$ and this establishes the lower bound.

Next, to prove the upper bound, we bound the probability that an arbitrary j-colouring is t-improper. With the same arguments as for Theorem 4.6(ii), we can show that, if Q is a balanced partition of the vertices,

$$\Pr(\Delta(\mathbb{G}(n,p)[P]) \leq t) \leq \Pr(\deg_{\mathrm{avg}}(\mathbb{G}(n,p)[Q]) \leq t)$$

$$\leq \exp \left(-\frac{n(k-3)}{2} \left(\Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right).$$

The total number of j-partitions P is at most j^n and hence the expected number of t-improper j-colourings is

$$C_{n,t,j} \leq j^n \exp \left(-\frac{n(k-3)}{2} \left(\Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right)$$

$$\leq \exp \left(n \ln d - n \ln \ln d - n \ln d \cdot \left(\frac{\kappa}{2p} \Lambda^* \left(\frac{\tau}{\kappa} p \right) + o(1) \right) \right)$$

$$= \exp \left(n \ln d \left(1 - \frac{1}{2} \left(\kappa - \tau - \tau \ln \frac{\kappa}{\tau} \right) + o(1) \right) \right).$$

$\square$

4.4 Weakly improper chromatic number

In this section, we give the analysis for the weakly improper chromatic number (as defined at the end of the introduction). We show that the following holds.

Theorem 4.18. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$.*

(i) *Fix $\varepsilon > 0$. There exist constants d_0 and $\tau > 0$ such that, if $d(n) \geq d_0$ and $t(n) \leq \tau \ln d$, then $(1 - \varepsilon)\frac{d}{2 \ln d} \leq \hat{\chi}^t(\mathbb{G}(n,p)) \leq (1 + \varepsilon)\frac{d}{2 \ln d}$ a.a.s.*

   If $d(n) = \omega(1)$, then the following holds:

(ii) *if $t(n) = o(\ln d)$, then $\hat{\chi}^t(\mathbb{G}(n,p)) \sim \frac{d}{2 \ln d}$ a.a.s.;*

(iii) *if $t(n) = \Theta(\ln d)$, then $\hat{\chi}^t(\mathbb{G}(n,p)) = \Theta\left(\frac{d}{\ln d}\right) = \Theta\left(\frac{d}{t}\right)$ a.a.s.;*

(iv) *if $t(n) = \omega(\ln d)$ and $t(n) = o(d)$, then $\hat{\chi}^t(\mathbb{G}(n,p)) \sim \frac{d}{t}$ a.a.s.;*

(v) *if $t(n) \sim \frac{d}{x}$, where $x > 0$ is fixed and not integral, then $\hat{\chi}^t(\mathbb{G}(n,p)) = \lceil x \rceil$ a.a.s.*

Observe that the lower bounds of Theorems 4.11, 4.12 and 4.15 were indeed bounds for the weakly improper chromatic number. Thus, by those results, it suffices to establish the following upper bound so as to complete the missing analysis (i.e. essentially for the case $d(n) = O(\sqrt{\ln n})$) in parts (iii), (iv) and (v).

Theorem 4.19. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Fix $\varepsilon > 0$ and suppose $d(n) = \omega(1)$. If $t(n) = \Omega(\ln d(n))$, then $\hat{\chi}^t(\mathbb{G}(n,p)) \leq \lceil (1 + \varepsilon)\frac{d}{t} \rceil$ a.a.s.*

Proof. The idea is to analyse a simple colouring procedure. Fix $0 < \varepsilon' < \varepsilon$ and let $k = k(n) = \left\lfloor \frac{1}{1+\varepsilon'} \frac{t(n)}{p(n)} \right\rfloor + 1$ so that $\lceil n/k \rceil < (1 + \varepsilon)\frac{d}{t}$ for large enough n. Our procedure will be to arbitrarily partition $[n]$ into $\lceil n/k \rceil$ colour classes of size k. We begin by calculating the probability that $\mathbb{G}(k,p)$ is not weakly t-dependent. Checking $t \geq p(k-1)$, it follows that

$$\Pr(\deg_{\text{avg}}(\mathbb{G}(k,p)) > t) = \Pr(\deg_{\text{avg}}(\mathbb{G}(k,q)) < k - t - 1)$$
$$\leq \exp\left(-\binom{k}{2}\bar{\Lambda}^*\left(1 - \frac{t}{k-1}\right)\right)$$

by Lemma 4.5(i), where $\bar{\Lambda}^*$ is Λ^* with p and q interchanged. By Taylor expansion, it can be verified that

$$\bar{\Lambda}^* \left(1 - \frac{t}{k-1} \right) \geq \bar{\Lambda}^*(q - \varepsilon'p) = (C + o(1))p$$

where $C = (1+\varepsilon')\ln(1+\varepsilon') - \varepsilon' > 0$ fixed. Hence, the probability that our partition is not a valid weakly t-improper colouring is at most

$$\left\lceil \frac{n}{k} \right\rceil \cdot \Pr(\deg_{\text{avg}}(\mathbb{G}(k,p)) > t) \leq (1+\varepsilon)\frac{d}{t} \exp\left(-\binom{k}{2}(C+o(1))p \right)$$

$$= \exp\left(\ln d - \frac{C}{2(1+\varepsilon')^2}\frac{t^2}{p} + o\left(\frac{t^2}{p}\right) \right)$$

for large enough n and this probability goes to zero as $n \to \infty$ since $t^2/p = \omega(\ln d)$. $\qquad\square$

We are unable to show the weak analogues to Conjectures 4.9 and 4.16. The following is in support of a weak analogue to Conjecture 4.16. Notice that unlike Theorem 4.17, there is no awkward condition on $p(n)$. Since this result is nearly a direct corollary of Lemma 4.5 and the arguments employed for Theorems 4.6 and 4.17, we omit the proof.

Theorem 4.20. *Suppose $0 < p(n) < 1$ and $p(n) = o(1)$. Set $d(n) = np(n)$. Fix $\kappa > \tau > 0$ and suppose $t(n) \sim \tau \ln d(n)$, $k(n) \sim \frac{\kappa}{p(n)} \ln d(n)$ and $j(n) = \lceil n/k \rceil$ (in particular, $j(n) \sim \frac{d(n)}{\kappa \ln d(n)}$). Let $\hat{E}_{n,t,k}$ be the expected number of weakly t-dependent k-sets in $\mathbb{G}(n,p)$ and $\hat{C}_{n,t,j}$ be the expected number of weakly t-improper j-colourings of $\mathbb{G}(n,p)$. Then*

(i) $\hat{E}_{n,t,k} = \exp\left(k \ln d \left(1 - \frac{1}{2}\left(\kappa - \tau - \tau \ln\frac{\kappa}{\tau} \right) + o(1) \right) \right)$; *and*

(ii) $\hat{C}_{n,t,fj} = \exp\left(n \ln d \left(1 - \frac{1}{2}\left(\kappa - \tau - \tau \ln\frac{\kappa}{\tau} \right) + o(1) \right) \right)$.

4.5 Conclusion

The division into dense and sparse cases was purely aesthetic. Theorems 4.1 and 4.2 could be combined into Theorem 4.21, while Conjectures 4.9 and 4.16 could be combined into Conjecture 4.22 below.

Theorem 4.21. *Suppose $0 < p(n) < p'$ for some fixed $p' < 1$ and set $d(n) = np(n)$.*

(i) *Fix $\varepsilon > 0$. There exist constants d_0 and $\tau > 0$ such that, if $d(n) \geq d_0$ and $t(n) \leq \tau \ln d$,*

then $(1 - \varepsilon) \left(\frac{1}{2} \ln \frac{1}{1-p} \right) \frac{n}{2 \ln d} \leq \chi^t(\mathbb{G}(n, p)) \leq (1 + \varepsilon) \left(\frac{1}{2} \ln \frac{1}{1-p} \right) \frac{n}{2 \ln d}$ *a.a.s.*

(ii) *If $d(n) = \omega(1)$ and $t(n) = o(\ln d)$, then $\chi^t(\mathbb{G}(n, p)) \sim \left(\frac{1}{2} \ln \frac{1}{1-p} \right) \frac{n}{\ln d}$ a.a.s.*

Furthermore, if $d(n) = \omega(\sqrt{\ln n})$, then the following hold:

(iii) *if $t(n) = \Theta(\ln d)$, then $\chi^t(\mathbb{G}(n, p)) = \Theta \left(\ln \frac{1}{1-p} \cdot \frac{n}{\ln d} \right) = \Theta \left(\frac{d}{t} \right)$ a.a.s.;*

(iv) *if $t(n) = \omega(\ln d)$ and $t(n) = o(d)$, then $\chi^t(\mathbb{G}(n, p)) \sim \frac{d}{t}$ a.a.s.;*

(v) *if $t(n) \sim \frac{d}{x}$, where $x > 0$ is fixed and not integral, then $\chi^t(\mathbb{G}(n, p)) \in \{\lceil x \rceil, \lceil x \rceil + 1\}$*

a.a.s.

Conjecture 4.22. *Suppose $0 < p(n) < 1$ and set $d(n) = np(n) = o(n)$ and suppose $t(n) \sim \tau \ln d(n)$ for some fixed $\tau > 0$. Let κ be the unique value satisfying $\kappa > \tau$ and $1 - \frac{\kappa}{2p} \Lambda^* \left(\frac{\kappa}{\tau} p \right) = o(1)$. Then $\chi^t(\mathbb{G}(n, p)) \sim \frac{d}{\kappa \ln d}$ a.a.s.*

Crucial to our approach were the large deviation inequalities for the binomial distribution introduced in Section 4.2. Their use in the sparse case was essential. We used these tools to provide results that support our conjectures by estimating the numbers of t-dependent sets and t-improper colourings. These inequalities were also used to complete a more complete analysis of the behaviour of weakly improper chromatic numbers.

We remark that, for fixed t, the property of a set being t-dependent is an hereditary property. In this case, the results of Scheinerman [78] and Bollobás and Thomason [15] apply; however, their results do not cover the cases in which $t = t(n)$ is allowed to vary.

To conclude this chapter, we reiterate the gaps in our analysis of the t-improper chromatic number of $\mathbb{G}(n, p)$:

(i) Conjectures 4.9 and 4.16; and

(ii) the case $d(n) = O(\sqrt{\ln n})$ and $t(n) = \Omega(\ln d)$, for which we believe our asymptotic upper bounds are inadequate.

In this last case, we have the lower bound $\chi^t(\mathbb{G}(n, p)) = \Omega(d/t)$ implied by Theorem 4.12 and the upper bound $\chi^t(\mathbb{G}(n, p)) = O(\sqrt{\ln n}/t)$ implied by Theorem 4.14.

Chapter 5

Acyclic improper colouring of graphs

Given a graph $G = (V, E)$, a proper colouring $(V_1, \ldots, V_k)$ of G is *acyclic* if for all $1 \leq i < j \leq k$, the subgraph of G induced by $V_i \cup V_j$, which we denote $G[V_i \cup V_j]$, contains no cycles (i.e. is a forest). A graph G is *acyclically k-colourable* if there exists an acyclic proper k-colouring of G. The *acyclic chromatic number* of G is defined to be

$$\chi_a(G) := \min\{k \ \mid \ G \text{ is acyclically } k\text{-colourable}\}.$$

For an integer $d \geq 0$, we define

$$\chi_a(d) := \max\{\chi_a(G) \ \mid \ \Delta(G) \leq d\}.$$

This notion was introduced by Grünbaum [39] and he showed that $\chi_a(3) = 4$. Burnstein [20] showed that $\chi_a(4) = 5$.

The more general question of determining $\chi_a(d)$, even asymptotically, is still open and is apparently difficult. On the other hand, it is easy to see that $\chi_a(d) \leq d^2 + 1$, as any proper colouring of the square G^2 of a graph G is *de facto* a proper acyclic colouring of G, and G^2 has maximum degree at most $\Delta(G)^2$. In 1976, Erdős (cf. [3]) conjectured that $\chi_a(d) = o(d^2)$; this conjecture was proved by Alon *et al.* [5], who showed, with the use

of the General Lovász Local Lemma, the existence of a fixed constant $c < 50$ such that $\chi_a(d) \leq cd^{4/3}$. Alon *et al.* [5] also showed that their bound was close to optimal by proving via probabilistic arguments that $\chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$.

When studying the asymptotics of $\chi_a(d)$, the restriction that the colouring be *proper* is in a sense not of great importance. Indeed, suppose we define the *laid-back acyclic chromatic number* $\chi_\ell(G)$ to be the smallest value k for which there exists a colouring $(V_1, \ldots, V_k)$ of G such that, for all $1 \leq i < j \leq k$, $G[V_i \cup V_j]$ is a forest (placing no further restriction on edges within a given block $G[V_i]$). Clearly, $\chi_\ell(G) \leq \chi_a(G)$. On the other hand, given such a colouring, it follows in particular that for all $1 \leq i \leq k$, $G[V_i]$ is a forest, so $\chi(G[V_i]) \leq 2$. By splitting V_i into stable sets $V_i^{(1)}$ and $V_i^{(2)}$ (for each $1 \leq i \leq k$), we may then obtain an acyclic *proper* colouring of G with at most $2k$ colours. It follows that $\chi_a(G)$ and $\chi_\ell(G)$ are within a factor of two of each other, so that $\chi_a(d)$ and the corresponding parameter $\chi_\ell(d)$ are within a factor of two of each other.

In this chapter, we investigate a different kind of relaxation of the acyclic chromatic number, one that exhibits more interesting asymptotic behaviour. In order to define it we first note that we may reformulate the definition of $\chi_a(G)$ by observing that if V_i and V_j are distinct stable sets in G, then $G[V_i \cup V_j]$ is exactly the bipartite graph $G[V_i, V_j]$ containing all edges if G with one endpoint in V_i and one endpoint in V_j. We may then equivalently define $\chi_a(G)$ as the smallest value k for which there exists a proper colouring $(V_1, \ldots, V_k)$ of V such that for all $1 \leq i < j \leq k$, $G[V_i, V_j]$ is a forest (i.e. such that with this colouring, G contains no *alternating cycle*).

Starting from this definition instead, we may now relax the requirement that $(V_1, \ldots, V_k)$ be a proper colouring while continuing to impose that G contain no alternating cycle. To wit: we say a colouring is *acyclic* if it does not admit an alternating cycle in G and a graph G is *acyclically t-improperly k-colourable* if there exists a k-colouring of G that is simultaneously acyclic and t-improper. The *acyclic t-improper chromatic number* of G is defined to be

$$\chi_a^t(G) := \min\{k \ \mid \ G \text{ is acyclically } t\text{-improperly } k\text{-colourable}\}.$$

This notion of acyclic improper colouring was first considered by Boiron, Sopena and Vignal [13, 14].

Here is an observation analogous to the trivial lower and upper bounds for the t-improper chromatic number.

Proposition 5.1. *For any graph G and any integer $t \geq 0$,*

$$\frac{\chi_a(G)}{50t^{4/3}} \leq \chi_a^t(G) \leq \chi_a(G).$$

Proof. The upper bound is immediate. Given an acyclic t-improper colouring, by applying the first of the results from [5] mentioned above, we can acyclically colour each colour class with at most $50t^{4/3}$ new colours to obtain an acyclic proper colouring of the entire graph. This shows the lower bound. □

We would like to gain a qualitative understanding of how χ_a^t behaves relative to the range implied by Proposition 5.1.

Although we are mainly interested in the case of graphs of bounded maximum degree, we briefly depart from this tack to mention some of the work on planar graphs. The proper acyclic chromatic number in this setting has an interesting history: due to work of many authors [4, 18, 39, 54, 65], it was shown that $\chi_a(\mathcal{P}) := \max\{\chi_a(G) \mid G \text{ is planar}\}$ is five. The work of Borodin [18] to ultimately bring the upper bound down to five is a 25-page *tour de force* that executes a discharging procedure by inspecting over 400 reducible configurations. When considering the corresponding question of determining the value of $\chi_a^t(\mathcal{P}) := \max\{\chi_a^t(G) \mid G \text{ is planar}\}$ for any $t \geq 0$, Boiron *et al.* [14] found, for every fixed $t \geq 0$, a planar graph G_t such that $\chi_a^t(G_t) \geq 5$. Thus, $\chi_a^t(\mathcal{P}) = \chi_a(\mathcal{P}) = 5$ for any $t \geq 0$; informally, the t-impropriety has no effect on reducing the number of colours needed to acyclically colour planar graphs.

Now, for an integer $d \geq 0$, we define

$$\chi_a^t(d) := \max\{\chi_a^t(G) \mid \Delta(G) \leq d\}.$$

The object of this chapter is to study how $\chi_a^t(d)$ varies as a function of t and of d. Clearly,

for any d, $\chi_a^0(d) \geq \chi_a^1(d) \geq \cdots \geq \chi_a^d(d) = 1$.

It is easily seen that $\chi_a^t(d) = \Omega\left((d/t)^{4/3}/(\ln d)^{1/3}\right)$ due to Proposition 5.1 and the lower bound $\chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$. Our first result is to show that this trivial lower bound on $\chi_a^t(d)$ can be much improved upon asymptotically, as long as $t \leq d - 10\sqrt{d \ln d}$. More fully, we show the following in Section 5.1.

Theorem 5.2. *If $t \leq d - 10\sqrt{d \ln d}$, then $\chi_a^t(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right)$.*

In particular, if $t = (1-\varepsilon)d$ for any fixed constant ε, $0 < \varepsilon \leq 1$, then we obtain the same asymptotic lower bound as Alon *et al.* [5]. Comparing this lower bound with the upper bound $\chi_a^t(d) = O(d^{4/3})$, we see the surprising fact that even allowing $t = \Omega(d)$ does not greatly reduce the number of colours needed for improper acyclic colourings of graphs with large maximum degree.

At some point, $\chi_a^t(d)$ must drop significantly as t increases, because $\chi_a^d(d) = 1$. Although we are unable to pin down the asymptotic behaviour of $\chi_a^t(d)$ viewed as a function of t, our second main result is to prove the following theorem, which improves upon the upper bound of Alon *et al.* [5] when t is very close to d (more precisely, when $d - t = o(d^{1/3})$).

Theorem 5.3. $\chi_a^t(d) = O(d \ln d + (d-t)d)$.

This result will follow from Theorem 5.6 below.

As for lower bounds on $\chi_a^t(d)$ when $d - t = o(d)$, we first note that Boiron *et al.* [14] showed $\chi_a^{d-2}(d) \geq 3$; we can straightforwardly generalise this result by showing that $\chi_a^t(d) \geq d - t + 1$. This is done as follows: if K_{d+1} is the complete graph on $d+1$ vertices, then $\chi_a^t(K_{d+1}) \geq d - t + 1$, since, in any acyclic t-improper colouring of K_{d+1}, at most one colour class has more than one vertex and no colour class has more than $t + 1$ vertices. We can, however, improve upon this further and, in the final section, we exhibit a set of examples showing the following lower bound.

Theorem 5.4. $\chi_a^{d-1}(d) = \Omega(d)$.

We would like to reduce the gaps between the lower and upper bounds on $\chi_a^t(d)$. For $t = d - 1$, the problem is particularly tantalising and, in this case, the lower bound of

Theorem 5.4 and the upper bound of Theorem 5.3 differ asymptotically by a multiple of $\ln d$.

Parallel with the study of the acyclic t-improper chromatic number, we also consider a different but related parameter. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. The *star chromatic number* $\chi_s(G)$ is the least number of colours needed in a proper star colouring of G. We analogously define the *t-improper star chromatic number* $\chi_s^t(G)$ and the parameter $\chi_s^t(d)$ in the natural way. Clearly, any star colouring is acyclic; thus, $\chi_a^t(d) \leq \chi_s^t(d)$. The star chromatic number was one of the main motivations for the original study of acyclic colourings [39]. We note that Fertin, Raspaud and Reed [29] showed that $\chi_s(d) = O(d^{3/2})$ and that $\chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$. We shall demonstrate the following results for the t-improper star chromatic number. (Observe that the second implies Theorem 5.3.)

Theorem 5.5. *If* $t \leq d - 16\sqrt{d \ln d}$, *then* $\chi_s^t(d) = \Omega\left((d-t)^{3/2}/(\ln d)^{1/2}\right)$.

Theorem 5.6. $\chi_s^t(d) = O(d \ln d + (d-t)d)$.

5.1 Probabilistic lower bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$

In this section, we prove Proposition 5.9 below, a more explicit version of Theorem 5.2. Our argument mirrors that of Alon *et al.* [5] but uses upper bounds on the t-dependence number $\alpha^t(\mathbb{G}(n,p))$ for the random graph $\mathbb{G}(n,p)$. We also outline the analogous lower bound for the t-improper star chromatic number.

Lemma 5.7. *Fix an integer* $n \geq 1$ *and* $p \in \mathbb{R}$ *with* $4(\ln n/n)^{1/4} \leq p \leq 1$. *Let* $m = \lfloor n - 128 \ln n/p^4 \rfloor$. *Then a.a.s. and uniformly over* p *in the above range, any colouring of* $\mathbb{G}(n,p)$ *with* $k \leq (n-m)/4$ *colours and in which each colour class contains at most* m *vertices contains an alternating 4-cycle.*

Proof. As there are at most $k^n \leq n^n$ possible k-colourings of $\mathbb{G}(n,p)$, to prove the lemma it suffices to show that for any fixed k-colouring of the vertices of $\mathbb{G}(n,p)$ (which we denote

$\{v_1, \ldots, v_n\}$) with colour classes $C_1, \ldots, C_k$ in which $|C_i| \leq m$ for all $1 \leq i \leq k$, the probability that $\mathbb{G}(n, p)$ does not contain an alternating 4-cycle is $o(n^{-n})$.

Fix a colouring as above, and let q be minimal such that $|C_1 \cup \cdots \cup C_q| \geq (n - m)/2$. Let $A = C_1 \cup \cdots \cup C_q$ and let $B = C_{q+1} \cup \cdots \cup C_k$. As no colour class has size greater than m, $|A| \leq (n + m)/2$ and so $|B| \geq (n - m)/2$. By interchanging A and B, we may also assume that $|A| \geq n/2$.

Next, let $P = \{\{x_1, x_1'\}, \ldots, \{x_r, x_r'\}\}$ be a maximal collection of pairs of elements of A such that for $1 \leq i \leq r$, x_i and x_i' have the same colour, and for $1 \leq i < j \leq r$, $\{x_i, x_i'\}$ and $\{x_j, x_j'\}$ are disjoint. As we may place all but perhaps one vertex from each colour class C_i in some such pair (with one vertex left over precisely if $|C_i|$ is odd), it follows that

$$r \geq \frac{1}{2}(|A| - q) \geq \frac{1}{2}\left(\frac{n}{2} - k\right) \geq \frac{n}{8}.$$

Similarly, let $Q = \{\{y_1, y_1'\}, \ldots, \{y_s, y_s'\}\}$ be a maximal collection of pairs of elements of B satisfying identical conditions; by an identical argument to that above, it follows that $s \geq (n - m)/8$.

Let E be the event that for all $1 \leq i \leq r$, $1 \leq j \leq s$, $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle, and let E' be the event that $\mathbb{G}(n, p)$ contains no alternating 4-cycle; clearly $E' \subseteq E$. For *fixed* $1 \leq i \leq r$ and $1 \leq j \leq s$, the probability that $\{x_i, y_j, x_i', y_j'\}$ is not an alternating 4-cycle is $1 - p^4$ and this event is independent from all other such events. As $n - m \geq 128 \ln n / p^4$ it follows that

$$\Pr(E') \leq \Pr(E) \leq (1 - p^4)^{rs} \leq e^{-p^4 rs}$$
$$\leq \exp\left(-\frac{p^4 n(n - m)}{64}\right) \leq e^{-2n \ln n} = o(n^{-n}),$$

as required. $\qquad \qquad \square$

Using this lemma, we next bound below the acyclic t-improper chromatic number of $\mathbb{G}(n, p)$ for p in the range allowed in Lemma 5.7.

Lemma 5.8. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n / n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n / p^4 \rfloor$ and let $t(n, p) = p(m - 1) - 2\sqrt{np}$. Then a.a.s., for all integers $t \leq t(n, p)$,*

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$\chi_a^t(\mathbb{G}(n,p)) \geq 32\ln n/p^4$, *uniformly over p and t in the above ranges.*

Proof. Fix n and p as above, and choose $t \leq t(n,p)$. We will show that a.a.s. $\mathbb{G}(n,p)$ contains no t-dependent set of size greater than m, from which the claim follows immediately by applying Lemma 5.7 as $(n-m)/4 \geq 32\ln n/p^4$. Let $V(\mathbb{G}(n,p)) = \{v_1, \ldots, v_n \text{ and } G[m]$ represent the subgraph of $\mathbb{G}(n,p)$ induced by $\{v_1, \ldots, v_m\}$. By symmetry and subadditivity of probabilities, we have

$$\Pr(\alpha^t(\mathbb{G}(n,p)) \geq m) \leq \binom{n}{m} \Pr(\Delta(G[m]) \leq t) \leq 2^n \Pr(\Delta(G[m]) \leq t).$$

Since, if $\Delta(G[m]) \leq t$ then $G[m]$ has at most $tm/2$ edges, it follows that

$$\Pr(\alpha^t(\mathbb{G}(n,p)) \geq m) \leq 2^n \Pr\left(|E(G[m])| \leq \frac{tm}{2}\right)$$
$$\leq 2^n \Pr\left(|E(G[m])| - p\binom{m}{2} \leq \frac{tm}{2} - p\binom{m}{2}\right)$$

Finally, by the Chernoff bound of (1.3) and by the definition of $t(n,p)$, we conclude that

$$\Pr(\alpha^t(\mathbb{G}(n,p)) \geq m) \leq 2^n \exp\left(-\left(\frac{tm}{2} - p\binom{m}{2}\right)^2 \bigg/ \left(2p\binom{m}{2}\right)\right)$$
$$\leq 2^n \exp\left(-\frac{(t-p(m-1))^2}{4p}\right) \leq (2/e)^n = o(1),$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using Lemma 5.8, it is a straightforward calculation to bound $\chi_a^t(d)$ below for d sufficiently large and t sufficiently far from d.

Proposition 5.9. *For all sufficiently large integers d and all non-negative integers $t \leq d - 10\sqrt{d\ln d}$,*

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

Proof. Choose n so that

$$2^{13}n^3\ln n \leq d^3(d-t) \leq 2^{14}n^3\ln n; \qquad\qquad\qquad\qquad\qquad (5.1)$$

such a choice of n clearly exists as long as d is large enough. Let $p = (d - 4\sqrt{d \ln d})/n$; we first check that p and t satisfy the requirements of Lemma 5.8. Presuming d is large enough that $np \geq d/2$, by the lower bound in (5.1) we have

$$p \geq \frac{d}{2n} \geq \frac{(d^3(d-t))^{1/4}}{2n} \geq \frac{8n^{3/4}(\ln n)^{1/4}}{2n} = 4\left(\frac{\ln n}{n}\right)^{1/4}. \tag{5.2}$$

Furthermore, letting $m = \lfloor n - 128 \ln n/p^4 \rfloor$, we have

$$p(m-1) - 2\sqrt{np} \geq np - \frac{128 \ln n}{p^3} - 2\sqrt{np} - 2 = d - 4\sqrt{d \ln d} - 2\sqrt{np} - 2 - \frac{128 \ln n}{p^3}$$
$$\geq d - 8\sqrt{d \ln d} - \frac{128 \ln n}{p^3}. \tag{5.3}$$

Since $p \geq d/2n$ and by the lower bound in (5.1),

$$\frac{128 \ln n}{p^3} \leq \frac{2^{10} n^3 \ln n}{d^3} \leq \frac{d-t}{8},$$

which combined with (5.3) yields

$$p(m-1) - 2\sqrt{np} > d - 8\sqrt{d \ln d} - \frac{(d-t)}{8}$$
$$= t + \frac{7(d-t)}{8} - 8\sqrt{d \ln d} \geq t, \tag{5.4}$$

the last inequality holding since $t \leq d - 10\sqrt{d \ln d}$. As (5.2) and (5.4) hold we may apply Lemma 5.8 to bound $\chi_a^t(\mathbb{G}(n, p))$ with this choice of t and p; as $n > d$, it follows that as long as d is sufficiently large,

$$\Pr\left(\chi_a^t(\mathbb{G}(n, p)) \geq \frac{32 \ln n}{p^4}\right) \geq \frac{3}{4}, \tag{5.5}$$

say. Furthermore, by subadditivity of probabilities and the Chernoff bound of (1.2),

$$\Pr(\Delta(\mathbb{G}(n, p)) > d) \leq n \Pr\left(\text{Bin}\left(n, \frac{d - 4\sqrt{d \ln d}}{n}\right) > d\right)$$
$$\leq ne^{-16 \ln d/3} \leq \frac{1}{n}, \tag{5.6}$$

the last inequality holding as $\ln d \geq \ln n / 2$ (which is an easy consequence of (5.1)). Combining (5.5) and (5.6), we obtain that

$$\Pr\left(\chi_a^t(\mathbb{G}(n,p)) \geq \frac{32 \ln n}{p^4}, \Delta(\mathbb{G}(n,p)) \leq d\right) \geq \frac{3}{4} - \frac{1}{n} \geq \frac{1}{2}$$

as long as $n \geq 4$, so there is some graph G with maximum degree at most d and with $\chi_a^t(G) \geq 32 \ln n / p^4$. Since χ_a^t is monotonically increasing in d, it follows that

$$\chi_a^t(d) \geq \frac{32 \ln n}{p^4} \geq \frac{32 n^4 \ln n}{d^4}. \tag{5.7}$$

An easy calculation using the upper bound in (5.1) and the fact that $\ln n < 2 \ln d$ gives the bound

$$d^4 \leq \frac{2^{19} n^4 (\ln d)^{1/3} \ln n}{(d-t)^{4/3}},$$

so $32 n^4 \ln n / d^4 > (d-t)^{4/3} / 2^{14} (\ln d)^{1/3}$. By (5.7), it follows that

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14} (\ln d)^{1/3}},$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

For the t-improper star chromatic number, the proof of Theorem 5.5 is similar to that of Theorem 5.2. We state the analogues for Lemmas 5.7, 5.8 and Proposition 5.9, but without proof as the calculations are very similar. Proposition 5.12 below implies Theorem 5.5.

Lemma 5.10. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n / n)^{1/3} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n / p^3 \rfloor$. Then a.a.s. and uniformly over p in the above range, any colouring of $\mathbb{G}(n,p)$ with $k \leq (n-m)/4$ colours and in which each colour class contains at most m vertices contains an alternating path of length three.*

Lemma 5.11. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n / n)^{1/3} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n / p^3 \rfloor$ and let $t(n,p) = p(m-1) - 2\sqrt{np}$. Then a.a.s., for all integers $t \leq t(n,p)$, $\chi_s^t(\mathbb{G}(n,p)) \geq 32 \ln n / p^3$, uniformly over p and t in the above ranges.*

Proposition 5.12. *For all sufficiently large integers d and all non-negative integers $t \leq d - 16\sqrt{d \ln d}$,*

$$\chi_s^t(d) \geq \frac{(d-t)^{3/2}}{2^{12}(\ln d)^{1/2}}.$$

5.2   A probabilistic upper bound for $\chi_s^t(d)$

In this section our aim is to prove Theorem 5.3. We shall in fact establish the stronger Theorem 5.6 as a by-product of our method. In the situation when $d - t = o(d^{1/2})$, this result gives an upper bound for $\chi_s^t(d)$ that improves upon the trivial bound implied by the bound $\chi_s(d) = O(d^{3/2})$ of Fertin *et al.* [29]. Similarly, in the situation when $d - t = o(d^{1/3})$, this gives an upper bound for $\chi_a^t(d)$ that improves upon the bound $\chi_a^t(d) = O(d^{4/3})$ due to Alon *et al.* [5].

Given a graph $G = (V, E)$ of maximum degree d, the idea behind our method for improved upper bounds is to find a dominating set $\mathcal{D}$ and a function $g = g(d) = o(d^{3/2})$ such that $\left|(N(v) \cup N^2(v)) \cap \mathcal{D}\right| \leq g$ for all $v \in V$. Given such a set $\mathcal{D}$ in G, we assign colours to the vertices in $\mathcal{D}$ by greedily colouring $\mathcal{D}$ in the square of G (i.e. vertices in $\mathcal{D}$ at distance at most two in G receive different colours) with at most $g + 1$ colours; then we give the vertices of $G \setminus \mathcal{D}$ the colour $g + 2$. It can be verified that this colouring prevents any alternating paths of length three (and so prevents alternating cycles) and ensures that every vertex has at least one neighbour of a different colour. Furthermore, we can generalise this idea by prescribing that our set $\mathcal{D}$ is *k-dominating* — each vertex outside of $\mathcal{D}$ has at least k neighbours in $\mathcal{D}$ — to give a bound on $\chi_s^{d-k}(d)$. Indeed, we can impose the stronger requirement that $\mathcal{D}$ is *total k-dominating* — *every* vertex has at least k neighbours in $\mathcal{D}$.

The following proposition is a more explicit version of Theorem 5.6.

Proposition 5.13. *For any $t = t(d) \geq 0$ and sufficiently large d,*

$$\chi_s^t(d) \leq d \cdot \max\{3(d-t), 37 \ln d\} + 2.$$

This proposition is an easy consequence of the following lemma. Given a d-regular graph $G = (V, E)$ and $1 \leq k \leq d$, let $\psi(G, k)$ be the least integer $k \leq k' \leq d$ such that there exists

a total k-dominating set $\mathcal{D}$ for which $|N(v) \cap \mathcal{D}| \leq k'$ for all $v \in V$. The quantity $\psi(G, k)$ is well-defined due to the fact that V is a total k-dominating set in G for $1 \leq k \leq d$. Let $\psi(d, k)$ be the maximum over all d-regular graphs G of $\psi(G, k)$.

Lemma 5.14. *For all d sufficiently large and $1 \leq k \leq d$, $\psi(d, k) \leq \max\{3k, 37 \ln d\}$.*

We postpone the proof of this lemma, first using it to prove Proposition 5.13:

Proof of Proposition 5.13. We first remark that the function χ_s^t is monotonic with respect to graph inclusion in the following sense: if $G = (V, E)$ and $G' = (V', E')$ are graphs with $V' \subseteq V'$, and $E \subset E'$, then $\chi_s^t(G) \leq \chi_s^t(G')$. As any graph G of maximum degree d is a subgraph of a d-regular graph (possibly with a greater number of vertices), to prove that $\chi_s^t(d) = O(d \ln d + (d - t)d)$ it therefore suffices to show that $\chi_s^t(G) = O(d \ln d + (d - t)d)$ for d-regular graphs G. We hereafter assume G is d-regular and d is large enough to apply Lemma 5.14. Let $k = d - t$. We will show that $\chi_s^t(G) \leq d\psi(d, k) + 2$, which proves the theorem.

By the definition of $\psi(d, k)$, there is a k-dominating set $\mathcal{D}$ such that $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V$. Fix such a dominating set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V$, H has maximum degree at most $d\psi(d, k)$.

To colour G, we first greedily properly colour H by choosing colours from the set $\{1, \ldots, d\psi(d, k) + 1\}$ and assign each vertex v of $\mathcal{D}$ the colour it received in H. We next assign colour $d\psi(d, k) + 2$ to all vertices of $V \setminus \mathcal{D}$. We remind the reader that $\mathrm{im}(v)$ denotes the number of neighbours of v of the same colour as v. If $v \in \mathcal{D}$ then $\mathrm{im}(v) = 0$, and if $v \in V \setminus \mathcal{D}$ then $\mathrm{im}(v) \leq d - |N(v) \cap \mathcal{D}| \leq d - k = t$, so the resulting colouring is t-improper.

Furthermore, given any path $P = v_1 v_2 v_3 v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case $c(v_i) = c(v_{i+1})$ and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case $c(v_i) \neq c(v_{i+2})$ and P is not alternating). Thus, the above colouring is a t-improper star colouring of G using at most $\psi(d, k) + 2$ colours; as G was an arbitrary d-regular graph, it follows that $\chi_s^t(d) \leq d\psi(d, k) + 2$, as claimed. $\square$

We next prove Lemma 5.14 with the aid of the Symmetric Lovász Local Lemma:

Proof of Lemma 5.14. We may clearly assume that k is at least $(37/3)\ln d$, since, if the claim of the lemma holds for such k, then it also holds for smaller k. Let $p = 2k/d$ and let $\mathcal{D}$ be a random set obtained by independently choosing each vertex v with probability p. We claim that, with positive probability, $\mathcal{D}$ is a total k-dominating set such that $|N(v) \cap \mathcal{D}| \leq 3k$ for all $v \in V$; we will prove our claim using the local lemma.

For $v \in V$, let A_v be the event that either $|N(v) \cap \mathcal{D}| < k$ or $|N(v) \cap \mathcal{D}| > 3k$. By the mutual independence principle, cf. [66], page 41, A_v is mutually independent of all but at most d^2 events A_w (with $w \neq v$). Furthermore, since $|N(v) \cap \mathcal{D}|$ has a binomial distribution with parameters d and p, we have by the Chernoff bound of (1.4) that

$$\Pr(A_v) = \Pr(||N(v) \cap \mathcal{D}| - \mathbb{E}(|N(v) \cap \mathcal{D}|)| > k) \leq 2e^{-k/6} = o(d^{-2})$$

so $e \Pr(A_v)(d^2 + 1) < 1$ for d large enough. By applying the Symmetric Lovász Local Lemma with $\mathcal{E} = \{A_v \mid v \in V\}$, it follows that with positive probability none of the events A_v occur, i.e. $\mathcal{D}$ has the desired properties. $\square$

5.3 A deterministic lower bound for $\chi_a^{d-1}(d)$

In this section, we concentrate on the case $t = d-1$ and exhibit examples for the lower bound of Theorem 5.4. Let n, m be integers and let us define a graph $G_{n,m} = (V, E)$ with $2nm$ vertices as follows. Set $V = \{v_{i,j}^1, v_{i,j}^2 \mid i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ and add $v_{i,j}^x v_{i',j'}^{x'}$ to E if and only if $i = i'$ or $j = j'$. The graph $G_{n,m}$ may be envisaged as a $(n \times m)$-matrix with two vertices in each entry, where distinct vertices are adjacent if and only if they share the same row or column. Let us also define $H_{n,m} = G_{n,m} \setminus \{v_{i,m}^2 \mid i \in \{1, \ldots, n\}\}$, i.e. $H_{n \times m}$ is the same as $G_{n,m}$ except that it has only one vertex in each entry of the last column. Thus, $G_{n,m}$ is a regular graph with degree $2(n + m) - 3$, and $H_{n,m}$ has maximum degree $2(n + m) - 4$.

Lemma 5.15. *If $n \leq m$, then $\chi_a^{2(n+m)-4}(G_{n,m}) \geq n/2$ and $\chi_a^{2(n+m)-3}(H_{n,m+1}) \geq n/2$.*

Let us first show how this lemma implies Theorem 5.4.

Proof of Theorem 5.4. Let d be an arbitrary positive integer. There is a positive integer n such that we can write either $d = 4n - 4$, $d = 4n - 3$, $d = 4n - 2$ or $d = 4n - 1$; thus, d is the maximum degree of one of $H_{n,n}$, $G_{n,n}$, $H_{n,n+1}$, or $G_{n,n+1}$, respectively. By Lemma 5.15, it follows that $\chi_a^{d-1}(d) \geq n/2 \geq (d+1)/8$, so that $\chi_a^{d-1}(d) = \Omega(d)$ as required.   □

Proof of Lemma 5.15. We shall focus on the case of $H_{n,m+1}$, since the case of $G_{n,m}$ is similar. Let d be the maximum degree of $H_{n,m+1}$ and suppose that there exists a $(d-1)$-improper colouring $c : V \to \{1, \ldots, k\}$ for some $k < n/2$.

In any row, there is at most one colour that occurs more than once, because if two distinct colours occur more than once in the same row, there is an alternating 4-cycle. As the number of colours used is less than $n/2$, in each row there is some colour that appears at least $2m + 1 - (k - 1) > 2m + 2 - n/2 > 3(m + 1)/2$ times and we call this the "dominant colour" of that row. In particular, for any $i \in \{1, \ldots, n\}$, there are more than $(m + 1)/2$ values $j \in \{1, \ldots, m\}$ for which both vertices $v_{i,j}^1, v_{i,j}^2$ are coloured by the dominant colour.

Now consider rows i, i' for $i \neq i'$. By the above, there must exist $j \in \{1, \ldots, m\}$ such that the pair $v_{i,j}^1, v_{i,j}^2$ both have the dominant colour of row i and the pair $v_{i',j}^1, v_{i',j}^2$ both have the dominant colour of row i'. We conclude that rows i and i' must have the same dominant colour, for otherwise the 4-cycle $v_{i,j}^1 v_{i',j}^2 v_{i,j}^2, v_{i',j}^1$ is alternating. As i and i' were arbitrary, it follows that all rows have the same dominant colour. By similar aguments, there is a single dominant colour for the columns 1 to m; furthermore, the dominant colour for the rows and the dominant colour for the columns must coincide and we may assume this colour is, say, 1.

Because the colouring is $(d-1)$-improper, it must either hold that none of the rows is monochromatic or that none of columns 1 to m is monochromatic, for if both row i and column j (with $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$) are monochromatic then the vertices $v_{i,j}^1, v_{i,j}^2$ and their d neighbours all have colour 1. Let us assume none of columns 1 through m is monochromatic. (The case when no row is monochromatic is similar.) For technical reasons, let us assume by permuting the rows that if column $m + 1$ is not monochromatic with colour 1, then a colour different from 1 occurs in the intersection of row 1 and column $m + 1$.

Now let $A_1 \subseteq \{2, \ldots, k\}$ be the set of non-dominant colours appearing in the first row, and let $C_1 \subseteq \{1, \ldots, m+1\}$ be the set of columns in which these colours appear. Note that either $m + 1 \in C_1$ or column $m + 1$ is monochromatic with colour 1, by assumption. If a colour from A_1 appears in column $j \in \{1, \ldots, m\} \setminus C_1$ then there is an alternating 4-cycle through the vertices $v_{1,j}^1, v_{1,j}^2$, both of colour 1; thus, colours from A_1 appear only in the columns from C_1. For $i \in \{2, \ldots, n\}$, let $A_i \subseteq \{2, \ldots, k\}$ be the set of colours that appear in the row i and columns $\{1, \ldots, m+1\} \setminus \bigcup_{j=1}^{i-1} C_j$; let C_i be the corresponding set of columns in which these colours appear. By the same logic, the colours from A_i do not appear outside the columns from C_i. Observe that $|A_i| \geq |C_i|$ and the sets $A_1, \ldots, A_n$ are mutually disjoint. Since none of the columns 1 to m is monochromatic, each is a member of exactly one C_i and hence

$$k - 1 \geq |A_1| + \cdots + |A_n| \geq |C_1| + \cdots + |C_m| \geq m \geq n.$$

But this contradicts the assumption that $k < n/2$. $\qquad\square$

5.4 Conclusion

In our view, the most surprising result of this chapter is the implication (of Theorem 5.2) that the same asymptotic lower bound for ordinary acyclic chromatic number by Alon *et al.* [5] also holds for the acyclic t-improper chromatic number for any $t = t(d)$ satisfying $d - t = \Theta(d)$.

In the case that t is very close to d, Theorem 5.6 improves upon upper bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$ implied by the results of Alon *et al.* [5] and Fertin *et al.* [29], respectively, giving for instance that $\chi_s^t(d) = O(d \ln d)$ for $d - t = O(\ln d)$. On the other hand, we showed that $\chi_a^{d-1}(d) = \Omega(d)$ by a deterministic construction.

There is much remaining work in the case $d - t = o(d)$. Table 5.1 is a rough summary of the current bounds on $\chi_a^t(d)$ and $\chi_s^t(d)$ when d is large. A case of particular interest is when $d - t = 1$; in this case, it is unknown if $\chi_a^{d-1}(d)$ is $\Theta(d)$, $\Theta(d \ln d)$ or lies somewhere strictly between these extremes.

Table 5.1: Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$.

$d-t$	$\chi_a^t(d)$		$\chi_s^t(d)$	
	lower	upper	lower	upper
$\Theta(d)$	$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$		$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$	
$\omega(d^{3/4}(\ln d)^{1/4})$	$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$	$O(d^{4/3})$	$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$	$O(d^{3/2})$
$\omega(d^{2/3}(\ln d)^{1/3})$				
$O(d^{1/2})$	$\Omega(d)$		$\Omega(d)$	$O((d-t)d)$
$O(d^{1/3})$		$O((d-t)d)$		
$O(\ln d)$		$O(d\ln d)$		$O(d\ln d)$
0	1	1	1	1

Chapter 6

Acyclic frugal colouring of graphs

In this chapter, we consider *frugal colourings*. This is a type of vertex partition that is not a generalisation of proper colouring, but in certain situations shares qualitative features with improper colouring, at least in the setting of graphs with bounded maximum degree.

Given a colouring c of G, the *frugality* of a vertex v under c is the size of the largest monochromatic set of neighbours of v in G, and the *frugality* of c is the value of the largest frugality among all vertices of G. For $t \geq 1$, a colouring c is *t-frugal* if its frugality is at most t. Alternatively, a colouring of G is *t-frugal* if no colour appears more than t times in any neighbourhood. Notice that t-frugality is a stronger requirement than t-impropriety, i.e. any colouring that is t-frugal is necessarily t-improper, but not conversely.

This notion was introduced a decade ago by Hind, Molloy and Reed [45]. They considered t-frugal proper colourings as a way to improve bounds for total colouring (cf. [46]).

Note that, as we have done throughout the thesis, we do not presuppose our colourings to be proper; therefore, we next define four new colouring parameters. For $t \geq 1$, a graph G is *t-frugally k-colourable* if there exists a t-frugal k-colouring of G. The *t-frugal chromatic number* $\varphi^t(G)$ is defined to be

$$\varphi^t(G) := \min\{k \mid G \text{ is } t\text{-frugally } k\text{-colourable}\}.$$

Analogously, for $t \geq 1$, we define the *proper t-frugal chromatic number* $\chi_\varphi^t(G)$, the *acyclic t-frugal chromatic number* $\varphi_a^t(G)$ and the *acyclic proper t-frugal chromatic number* $\chi_{\varphi,a}^t(G)$.

We are mainly interested in studying these parameters for graphs G of bounded maximum degree. Therefore, for integers $d \geq 0$, $t \geq 1$, we define

$$\varphi^t(d) := \max\{\varphi^t(G) \mid \Delta(G) \leq d\},$$

and analogously define $\chi^t_\varphi(d)$, $\varphi^t_a(d)$ and $\chi^t_{\varphi,a}(d)$. We frequently use the monoticity of these parameters with respect to d: $\varphi^t(d-1) \leq \varphi^t(d)$, and so on.

In this chapter, our aim is to study the asymptotic behaviour of these parameters as a function of t and d, as well as compare them to the analogous parameters $\chi^t(d)$, $\chi^t_a(d)$. We begin with the study of frugal colourings in Section 6.1, then give analysis for acyclic frugal colourings in Section 6.2, and conclude in Section 6.3 by considering some related deterministic questions. In cases when there is asymptotically more than a constant multiple difference between upper and lower bounds, we often make no attempt to optimise constants. Before we proceed, let us outline some straightforward observations.

Proposition 6.1. *For any graph G and any $t \geq 1$, the following hold:*

(i) $\chi^1_\varphi(G) = \chi^1_{\varphi,a}(G) = \chi(G^2)$, *where G^2 denotes the square of G;*

(ii) $\chi^t(G) \leq \varphi^t(G) \leq \chi^t_\varphi(G)$ *and* $\chi^t_a(G) \leq \varphi^t_a(G) \leq \chi^t_{\varphi,a}(G)$;

(iii) $\varphi^t(G) \leq \varphi^t_a(G)$ *and* $\chi^t_\varphi(G) \leq \chi^t_{\varphi,a}(G)$;

(iv) $\varphi^{t+1}(G) \leq \varphi^t(G)$, $\chi^{t+1}_\varphi(G) \leq \chi^t_\varphi(G)$, $\varphi^{t+1}_a(G) \leq \varphi^t_a(G)$, $\chi^{t+1}_{\varphi,a}(G) \leq \chi^t_{\varphi,a}(G)$; *and*

(v) $\varphi^t(G) \geq \Delta(G)/t$.

Proof. Part (i) holds because 1-frugality ensures that vertices at distance two have distinct colours; therefore, a 1-frugal proper colouring is precisely a colouring of the square of the graph (and such a colouring forbids alternating cycles). Part (ii) holds since any proper t-frugal colouring is t-frugal and any t-frugal colouring is t-improper. Part (iii) holds since forbidding alternating cycles can only make the corresponding chromatic number larger. Part (iv) holds as any t-frugal colouring is $(t+1)$-frugal. Part (v) is simply the observation that, in any t-frugal colouring, the $\deg(v)$ vertices in the neighbourhood of any vertex v require at least $\deg(v)/t$ colours. $\square$

6.1 Frugal colourings

As mentioned above, Hind, Molloy and Reed [45] studied proper frugal colouring as a way to attack problems in total colouring (cf. [46]) and, using sophisticated probabilistic techniques (including use of the Local Lemma), developed the following result.

Theorem 6.2 (Hind *et al.* [45])**.** *For sufficiently large d,*

$$\chi_\varphi^{(\ln d)^5}(d) \leq d+1.$$

For smaller frugalities, they also showed the following bound that is tight up to a constant multiple as long as $t = o(\ln d / \ln \ln d)$.

Theorem 6.3 (Hind *et al.* [45])**.** *For any $t \geq 1$ and sufficiently large d,*

$$\chi_\varphi^t(d) \leq \max\left\{(t+1)d, \left\lceil e^3 \frac{d^{1+1/t}}{t} \right\rceil\right\}.$$

Both of these results are asymptotically tight in a sense, but there is a logarithmic gap between them — specifically, the behaviour of $\chi_\varphi^t(d)$ between $t = \Theta(\ln d / \ln \ln d)$ and $t = \Theta((\ln d)^5)$ is undetermined. Complete closure of this gap would be likely to require significant effort and ingenuity.

By dropping the condition that the colouring be proper, we can obtain a smaller upper bound for $\varphi^t(d)$ than those implied by Theorems 6.2 and 6.3:

Theorem 6.4. *For any fixed $t \geq 1$, let γ be such that*

$$\gamma^t > \frac{e(t+1)}{\sqrt{2\pi t}e^{1/(12t+1)}};$$

if $t \to \infty$ as $d \to \infty$, then let $\gamma = 1 + \varepsilon$ for some fixed $\varepsilon > 0$. For sufficiently large d,

$$\varphi^t(d) \leq \left\lceil \gamma e \frac{d^{1+1/t}}{t} \right\rceil.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \left\lceil \gamma e d^{1+1/t}/t \right\rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$

is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$.

For vertices $v, v_1, \ldots, v_{t+1}$ with $\{v_1, \ldots, v_{t+1}\} \subseteq N(v)$, let $A_{\{v, v_1, \ldots, v_{t+1}\}}$ be the event that $f(v_1) = \cdots = f(v_{t+1})$. If none of these events hold, then f is t-frugal.

Clearly, $\Pr\left(A_{\{v, v_1, \ldots, v_{t+1}\}}\right) = 1/x^t$. Furthermore, each vertex participates in at most $d\binom{d-1}{t}$ of these events; thus, each event is independent of all but at most $(t+1)d\binom{d-1}{t}$ other events. We have that

$$e \Pr\left(A_{\{v, v_1, \ldots, v_{t+1}\}}\right) \left((t+1)d\binom{d-1}{t} + 1\right) < e\frac{t^t}{\gamma^t e^t d^{t+1}}(t+1)\frac{d^{t+1}}{t!} = \frac{e(t+1)}{\gamma^t}\frac{(t/e)^t}{t!}.$$

By a precise form of Stirling's formula (cf. (1.4) of [17]), $t! \geq (t/e)^t \sqrt{2\pi t}e^{1/(12t+1)}$; therefore,

$$\frac{e(t+1)}{\gamma^t}\frac{(t/e)^t}{t!} \leq \frac{e(t+1)}{\gamma^t \sqrt{2\pi t}e^{1/(12t+1)}}.$$

It follows that $e \Pr\left(A_{\{v, v_1, \ldots, v_{t+1}\}}\right) \left((t+1)d\binom{d-1}{t} + 1\right) < 1$ (for sufficiently large d, if $t \to \infty$ as $d \to \infty$); thus, by the Symmetric Lovász Local Lemma, f is t-frugal with positive probability. $\qquad\square$

To get a better feeling of this bound, notice that for example when $t = 2$, we need $\gamma > \left(\frac{3}{\sqrt{4\pi}}e^{24/25}\right)^{1/2} \approx 1.487$, but when $t = 1000$, γ may be as low as 1.004.

The following example due to Alon (cf. [45]) shows that this result is asymptotically best possible up to a constant multiple ($< 1.49 \cdot e$) when $t = o(\ln d)$.

Proposition 6.5. *For any $t \geq 1$ and any prime power n,*

$$\varphi^t(n^t + \cdots + 1) \geq \frac{n^{t+1} + \cdots + 1}{t}.$$

Proof. Set $d = n^t + \cdots + 1$ and $m = n^{t+1} + \cdots + 1$. Let P be a $(t+2)$-dimensional projective geometry with m points. We form a bipartite graph G with parts A and B, where A is the set of points in P and B is the set of $(t+1)$-flats (hyperplanes), and an edge between two vertices $a \in A, b \in B$ if the point a lies in the hyperplane b.

Every hyperplane contains exactly d points in P, so G has maximum degree d by pro-jective geometry duality. Since every set of $t+1$ points lies in a $(t+1)$-flat, no colour can

appear more than t times on A in any t-frugal colouring of G (*whether proper or not*); thus, at least m/t colours are required. $\qquad\square$

Corollary 6.6. *Suppose that $t = t(d) \geq 2$ and $t = o(\ln d / \ln \ln d)$. Then, for any $\varepsilon > 0$, it holds that*

$$\varphi^t(d) \geq (1 - \varepsilon)\frac{d^{1+1/t}}{t}$$

for sufficiently large d.

Proof. Let x solve $d = x^{t(d)} + \cdots + 1$ where d is chosen large enough to satisfy certain inequalities specified below. Set $m = x^{t(d)+1} + \cdots + 1$. Note that, since $t = o(\ln d)$, $x \to \infty$ as $d \to \infty$. It follows that $d = (1 + o(1))x^{t(d)}$ and $x = (1 + o(1))d^{1/t(d)}$.

Due to a classical result of Ingham [48] on the gaps between primes, there is a prime n between $x - Cx^{5/8}$ and x, for some absolute constant C. Let $d' = n^{t(d)} + \cdots + 1$ and $m' = n^{t(d)+1} + \cdots + 1$. We have, using Proposition 6.5,

$$\varphi^{t(d)}(d) \geq \varphi^{t(d)}(d') \geq \frac{m'}{t(d)} \geq \left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1}\frac{m}{t(d)} \tag{6.1}$$

Since $x = (1 + o(1))d^{1/t(d)}$, we have

$$\left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1} \geq 1 - \frac{C(t(d) + 1)}{x^{3/8}} \geq 1 - \frac{2C(t(d) + 1)}{d^{0.375/t(d)}} \geq 1 - \varepsilon/2$$

for d sufficiently large, with this last inequality due to $t(d) = o(\ln d / \ln \ln d)$. Also, using $d = (1 + o(1))x^{t(d)}$ and $x = (1 + o(1))d^{1/t(d)}$, we have that

$$m = (x^{t(d)+2} - 1)/(x - 1) = (1 + o(1))x^{t(d)+1} = (1 + o(1))d^{1+1/t(d)}.$$

Substituting these last two estimates into Inequality (6.1), we obtain that

$$\varphi^{t(d)}(d) \geq (1 - \varepsilon/2)^2\frac{d^{1+1/t(d)}}{t(d)} \geq (1 - \varepsilon)\frac{d^{1+1/t(d)}}{t(d)}$$

for large enough d, as claimed. $\qquad\square$

Recall from Proposition 6.1(v) that $\varphi^t(d) \geq d/t$. We now give an essentially optimal

upper bound for $\varphi^t(d)$ in the case $t = \omega(\ln d)$.

Theorem 6.7. *Suppose $t = \omega(\ln d)$. For any $\varepsilon > 0$, it holds that*

$$\varphi^t(d) \leq \left\lceil (1+\varepsilon)\frac{d}{t} \right\rceil$$

for sufficiently large d.

Proof. We first remark that if G is a subgraph of G', then $\varphi^t(G) \leq \varphi^t(G')$. As any graph of maximum degree d is contained in a d-regular graph, it therefore suffices to show the theorem holds for d-regular graphs. Let $G = (V, E)$ be any d-regular graph and let $x = \lceil (1+\varepsilon)d/t \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$.

For a vertex v and a colour $i \in \{1, \ldots, x\}$, let $A_{v,i}$ be the event that v has more than t neighbours with colour i. If none of these events hold, then f is t-frugal. Each event is independent of all but at most $d^2 x \ll d^3$ other events.

By the Chernoff bound of (1.2), we have that

$$\Pr(A_{v,i}) = \Pr(\mathrm{Bin}(d, 1/x) > t) \leq \Pr(\mathrm{Bin}(d, 1/x) > d/x + ct)$$
$$\leq \exp\left(-c^2 t^2/(2d/x + 2ct/3)\right)$$

where $c = \varepsilon/(1+\varepsilon)$. Thus, $e \Pr(A_{v,i})(d^3 + 1) = \exp(-\Omega(t))d^3 < 1$ for large enough d, and by the Symmetric Lovász Local Lemma, f is t-frugal with positive probability for large enough d. □

We see from this last theorem that when $t = \omega(\ln d)$, the behaviour of $\varphi^t(d)$ is closely tied to that of $\chi^t(d)$. Because cliques satisfy $\chi(K_{d+1}) = d+1$, it follows from Corollary 1.6 and Proposition 1.8 that $\chi^t(d) = \lceil (d+1)/(t+1) \rceil$ for any $t \geq 0$. Therefore, $\varphi^t(d) \sim \chi^t(d) \sim \lceil d/t \rceil$ if $t = \omega(\ln d)$ and $t = o(d)$. If $d/t \to x$ for some $0 < x < \infty$, then $\chi^t(d) = \lceil x \rceil$ while Theorem 6.7 implies that $\varphi^t(d) = \lceil x \rceil$ if x is not integral and $\varphi^t(d) \in \{x, x+1\}$ otherwise.

When $t = \Theta(\ln d)$, our bounds are weaker. For convenience, let $t \sim \tau \ln d$. If $\tau > 2$, then by following the proof of Theorem 6.7 and performing some straightforward calculations,

we obtain that $\varphi^t(d) \leq \left\lceil \left(1 + (4 + \sqrt{4 + 6\tau})/(\tau - 2)\right) d/t \right\rceil$, while if $\tau \leq 2$, then Theorem 6.4 implies that $\varphi^t(d) \leq \left\lceil (e + \varepsilon) e^{\frac{1}{\tau}} d/t \right\rceil$. In either case, we suspect the behaviour of $\varphi^t(d)$ to be closer to $d^{1+1/t}/t = e^{\frac{1}{\tau}} d/t$ (cf. Conjecture 6.19 below).

6.2 Acyclic frugal colourings

For acyclic frugal colourings, we start by considering the smallest cases $t = 1, 2, 3$ and establish upper bounds for proper acyclic frugal colourings. Later in the section, we consider larger values of t and concentrate our attention upon acyclic frugal colourings that are not necessarily proper.

For $t = 1, 2, 3$, Corollary 6.6 implies the bounds $\varphi_a^1(d) \geq (1 - \varepsilon) d^2$, $\varphi_a^2(d) \geq (1/2 - \varepsilon) d^{3/2}$ and $\varphi_a^3(d) \geq (1/3 - \varepsilon) d^{4/3}$, for fixed $\varepsilon > 0$ and large enough d. Since $\chi_{\varphi,a}^1(G) = \chi(G^2) \leq \Delta(G)^2 + 1$, it follows that $\varphi_a^1(d) \sim \chi_{\varphi,a}^1(d) \sim d^2$.

We mentioned in the last chapter that Alon et $al.$ [5] showed that $\chi_a(d) \leq cd^{4/3}$ for some absolute constant $c < 50$. In fact, any constant c satisfying

$$\left(1 - \frac{20}{c}\right)\left(1 - \frac{30}{c^3}\right)\left(1 - \frac{10}{c^2}\right) > \frac{1}{2},$$

e.g. $c = 40.27$, suffices. Yuster [84] considered acyclic proper 2-colourings of graphs and showed that $\chi_{\varphi,a}^2(d) = \Theta(d^{3/2})$. In particular, by an adaptation of the abovementioned bound (Theorem 1.1) of Alon et $al.$, he showed that $\chi_{\varphi,a}^2(d) \leq \lceil \max\{50d^{4/3}, 10d^{3/2}\} \rceil$. We note that a more precise analysis of Yuster's proof gives the following.

Theorem 6.8 (Yuster [84]). *For sufficiently large d, $\chi_{\varphi,a}^2(d) \leq \lceil 5.478 d^{3/2} \rceil$.*

The above statement still holds if 5.478 is replaced by any fixed constant greater than $\sqrt{30}$. We give two extensions to Theorem 6.8, one for the case $t = 2$ for star colouring, and the other for the case $t = 3$ for acyclic colouring. In both cases we employ the General Lovász Local Lemma.

Denote $\chi_{\varphi,s}^t(\cdot)$ to be the *proper t-frugal star chromatic number*, the least number of colours needed in a proper t-frugal star colouring. For the following result, we give an upper bound for the proper 2-frugal star chromatic number. With this bound, we provide

a slightly simpler proof of the fact that $\chi^2_{\varphi,a}(d) = O(d^{3/2})$. Our proof is an extension of the proof of Theorem 8.1 in Fertin *et al.* [29].

Theorem 6.9. *For sufficiently large d, $\chi^2_{\varphi,s}(d) \leq \lceil 6.325d^{3/2} \rceil$.*

The above statement still holds if 6.325 is replaced by any fixed constant greater than $2\sqrt{10}$.

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil 6.325d^{3/2} \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$. We define three types of events, the first two of which are from Fertin *et al.* [29]:

I For adjacent vertices u, v, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.

II For a path of length three $v_1 v_2 v_3 v_4$, let $B_{\{v_1, \ldots, v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

III For vertices v, v_1, v_2, v_3 with $\{v_1, v_2, v_3\} \subseteq N(v)$, let $A_{\{v_1, v_2, v_3\}}$ be the event that $f(v_1) = f(v_2) = f(v_3)$.

It is clear that if none of these events occur, then f is a proper 2-frugal star colouring. Furthermore, $\Pr(A) = 1/x$ and $\Pr(B) = \Pr(C) = 1/x^2$, where A, B, C are events of Types I, II, III, respectively. Also, since G has maximum degree d, each vertex participates in at most $d \cdot \binom{d-1}{2} < d^3/2$ events of Type III. It is routine to check that in Table 6.1, the (i, j) entry is an upper bound on the number of nodes corresponding to events of Type j which are adjacent in the dependency graph to a node corresponding to an event of Type i.

Table 6.1: Upper bounds in the dependency graph for Theorem 6.9.

	I	II	III
I	$2d$	$4d^3$	d^3
II	$4d$	$8d^3$	$2d^3$
III	$3d$	$6d^3$	$3d^3/2$

We define the weight x_i of each event i to be twice its probability and we want to show

that each of the following inequalities hold:

$$\frac{1}{x} \leq \frac{2}{x}\left(1 - \frac{2}{x}\right)^{2d}\left(1 - \frac{2}{x^2}\right)^{5d^3}, \tag{6.2}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1 - \frac{2}{x}\right)^{4d}\left(1 - \frac{2}{x^2}\right)^{10d^3}, \text{ and} \tag{6.3}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1 - \frac{2}{x}\right)^{3d}\left(1 - \frac{2}{x^2}\right)^{15d^3/2}. \tag{6.4}$$

Inequalities (6.2), (6.3) and (6.4) correspond to events of Type I, II and III, respectively. Inequality (6.3) implies the other two and it is valid for sufficiently large d since

$$\left(1 - \frac{2}{x}\right)^{4d}\left(1 - \frac{2}{x^2}\right)^{10d^3} \geq \left(1 - \frac{8}{6.325\sqrt{d}}\right)\left(1 - \frac{20}{6.325^2}\right) > \frac{1}{2}$$

if $d \geq 9 \cdot 10^7$. Therefore, by the General Lovász Local Lemma, f is an acyclic proper 3-frugal colouring with positive probability. $\qquad\square$

Corollary 6.10. *For any $t \geq 2$ and sufficiently large d, $\chi^t_{\varphi,s}(d) \leq \lceil 6.325d^{3/2}\rceil$.*

Since $\chi^t_{\varphi,s}(d) \geq \chi_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$ for any $t \geq 1$, this corollary is correct up to a logarithmic multiple. Note that $\chi^1_{\varphi,s}(G) = \chi(G^2)$ for any graph G, so that $\chi^1_{\varphi,s}(d) \sim d^2$.

Next, for the case $t = 3$, we show that $\chi^3_{\varphi,a}(d) = O(d^{4/3})$, giving a bound that is within a constant multiple of the asymptotic lower bound for $\varphi^3_a(d)$. Our proof is an extension of the proof of Theorem 1.1 in Alon *et al.* [5].

Theorem 6.11. *For sufficiently large d, $\chi^3_{\varphi,a}(d) \leq \lceil 40.27d^{4/3}\rceil$.*

The above statement still holds if 40.27 is replaced by any constant C satisfying

$$\left(1 - \frac{20}{C}\right)\left(1 - \frac{95/3}{C^3}\right)\left(1 - \frac{10}{C^2}\right) > \frac{1}{2}.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil 40.27d^{4/3}\rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$. We define five types of events, the first four of which are from [5]:

I For adjacent vertices u, v, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.

II For an induced path of length four $v_1 v_2 v_3 v_4 v_5$, let $B_{\{v_1,\dots,v_5\}}$ be the event that $f(v_1) = f(v_3) = f(v_5)$ and $f(v_2) = f(v_4)$.

III For an induced 4-cycle $v_1 v_2 v_3 v_4$ such that v_1, v_3 share at most $d^{2/3}$ common neighbours and v_2, v_4 share at most $d^{2/3}$ common neighbours, let $C_{\{v_1,v_2,v_3,v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

IV For non-adjacent vertices u, v that share more than $d^{2/3}$ common neighbours, let $D_{\{u,v\}}$ be the event that $f(u) = f(v)$.

V For vertices v, v_1, v_2, v_3, v_4 with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1,\dots,v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$.

In [5], it was shown that if none of the events of Type I–IV occur, then f is an acyclic proper colouring. If no event of Type V occurs, then f is 3-frugal. Thus, if none of these events occur, then f is an acyclic proper 3-frugal colouring.

Clearly, $\Pr(A) = \Pr(D) = 1/x$, $\Pr(B) = \Pr(E) = 1/x^3$, $\Pr(C) = 1/x^2$, where A, B, C, D, E are events of Types I, II, III, IV, V, respectively. Also, since G has maximum degree d, each vertex participates in at most $d \cdot \binom{d-1}{3} < d^4/6$ events of Type V. It is easy to check that in Table 6.2, the (i, j) entry is an upper bound on the number of nodes corresponding to events of Type j which are adjacent in the dependency graph to a node corresponding to an event of Type i.

Table 6.2: Upper bounds in the dependency graph for Theorem 6.11.

	I	II	III	IV	V
I	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	$d^4/3$
II	$5d$	$15d^4$	$5d^{8/3}$	$5d^{4/3}$	$5d^4/6$
III	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$	$2d^4/3$
IV	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$	$d^4/3$
V	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$	$2d^4/3$

We define the weight x_i of each event i to be twice its probability and we want to show

that each of the following inequalities hold:

$$\frac{1}{x} \leq \frac{2}{x}\left(1 - \frac{2}{x}\right)^{2d+2d^{4/3}} \left(1 - \frac{2}{x^3}\right)^{19d^4/3} \left(1 - \frac{2}{x^2}\right)^{2d^{8/3}}, \tag{6.5}$$

$$\frac{1}{x^3} \leq \frac{2}{x^3}\left(1 - \frac{2}{x}\right)^{5d+5d^{4/3}} \left(1 - \frac{2}{x^3}\right)^{95d^4/6} \left(1 - \frac{2}{x^2}\right)^{5d^{8/3}}, \text{ and} \tag{6.6}$$

$$\frac{1}{x^2} \leq \frac{2}{x^2}\left(1 - \frac{2}{x}\right)^{4d+4d^{4/3}} \left(1 - \frac{2}{x^3}\right)^{26d^4/3} \left(1 - \frac{2}{x^2}\right)^{4d^{8/3}}. \tag{6.7}$$

Inequality (6.5) corresponds to events of Types I and IV, inequality (6.6) to events of Types II and V, and inequality (6.7) to events of Type III. Inequality (6.6) implies the other two and it is valid since

$$\left(1 - \frac{2}{x}\right)^{5d+5d^{\frac{4}{3}}} \left(1 - \frac{2}{x^3}\right)^{95d^4/6} \left(1 - \frac{2}{x^2}\right)^{5d^{\frac{8}{3}}}$$
$$\geq \left(1 - \frac{20}{40.27}\right)\left(1 - \frac{95/3}{40.27^3}\right)\left(1 - \frac{10}{40.27^2}\right) > \frac{1}{2}$$

and therefore, by the General Lovász Local Lemma, f is an acyclic proper 3-frugal colouring with positive probability. $\qquad\square$

Corollary 6.12. *For any $t \geq 3$ and sufficiently large d, $\chi_{\varphi,a}^t(d) \leq \lceil 40.27d^{4/3} \rceil$.*

Since $\chi_{\varphi,a}^t(d) \geq \chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$ for any $t \geq 1$, this corollary is correct up to a logarithmic multiple. This partially answers a question by Esperet, Montassier and Raspaud [28].

For acyclic proper frugal colourings (respectively, proper frugal star colourings), Corollary 6.12 (respectively, Corollary 6.10) gives fairly reasonable answers, so now we would like to consider what happens when we no longer prescribe that the colourings be proper. In this setting, we adapt the methods of Section 5.2 to obtain an analogous asymptotic improvement upon Corollary 6.12 when $t = t(d)$ is close to d. As in Section 5.2, instead of acyclic colourings, we deal with the stronger notion of star colourings and denote $\varphi_s^t(\cdot)$ to be the *t-frugal star chromatic number*, the least number of colours needed in a t-frugal star colouring. The following theorem implies that $\varphi_a^t(d)$ is asymptotically smaller than $\chi_{\varphi,a}^t(d)$ when $d - t(d) = o\left(d^{1/3}/(\ln d)^{1/3}\right)$. The proof is only a slight modification of the proof of

Theorem 5.6.

Theorem 6.13. *For any $t = t(d) \geq 1$ and sufficiently large d,*

$$\varphi_s^t(d) \leq d \cdot \max\{3(d-t), 37\ln d\} + 2.$$

Proof. We first remark that if G is a subgraph of G', then $\varphi_s^t(G) \leq \varphi_s^t(G')$. As any graph of maximum degree d is contained in a d-regular graph, it therefore suffices to show the theorem holds for d-regular graphs. We hereafter assume $G = (V, E)$ is d-regular and d is large enough to apply Lemma 5.14. Let $k = d - t$. We will show that $\varphi_s^t(G) \leq d\psi(d, k) + 2$ (where $\psi(d, k)$ is the quantity defined before Lemma 5.14), which proves the theorem.

By the definition of $\psi(d, k)$, if d is sufficiently large, there is a set $\mathcal{D}$ such that $k \leq |N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$. Fix such a set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$, H has maximum degree at most $d\psi(d, k)$.

To colour G, we first properly colour H by using the greedy algorithm to choose colours from the set $\{1, \ldots, d\psi(d, k) + 1\}$ and then assign each vertex v of $\mathcal{D}$ the colour it received in H. We next assign colour $d\psi(d, k) + 2$ to all vertices of $V \setminus \mathcal{D}$. Since $k \leq |N(v) \cap \mathcal{D}|$ for any $v \in V$, colour $d\psi(d, k) + 2$ appears at most $d - k = t$ times in any neighbourhood. Since the vertices of H at distance two have distinct colours, each colour other than $d\psi(d, k) + 2$ appears at most once in any neighbourhood. So the resulting colouring is t-frugal.

Furthermore, given any path $P = v_1 v_2 v_3 v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case v_i and v_{i+1} have the same colour and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case v_i and v_{i+2} have different colours and P is not alternating). Thus, the above colouring is a star colouring of G with frugality at most t and using at most $d\psi(d, k) + 2$ colours. $\square$

Corollary 6.14. *For any $t = t(d) \geq 1$, $\varphi_a^t(d) = O(d\ln d + (d-t)d)$.*

The best lower bounds known for $\varphi_a^t(d)$ (respectively, $\varphi_s^t(d)$) are also the best lower bounds known for $\chi_a^t(d)$ (respectively, $\chi_s^t(d)$); by Proposition 6.1(ii) (and its star colouring

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analogue), it follows that Theorems 5.2, 5.4 and 5.5 imply the following:

(i) if $t \le d - 10\sqrt{d \ln d}$, then $\varphi_a^t(d) = \Omega\left((d-t)^{4/3}/(\ln d)^{1/3}\right)$;

(ii) $\varphi_a^{d-1}(d) = \Omega(d)$; and

(iii) if $t \le d - 16\sqrt{d \ln d}$, then $\varphi_s^t(d) = \Omega\left((d-t)^{3/2}/(\ln d)^{1/2}\right)$.

Table 6.3: Asymptotic bounds for $\chi_{\varphi,a}^t(d)$ and $\chi_{\varphi,s}^t(d)$.

t	$\chi_{\varphi,a}^t(d)$		$\chi_{\varphi,s}^t(d)$	
	lower	upper	lower	upper
1	$\Omega\left(d^2\right)$	$O\left(d^2\right)$	$\Omega\left(d^2\right)$	$O\left(d^2\right)$
2	$\Omega\left(d^{3/2}\right)$	$O\left(d^{3/2}\right)$	$\Omega\left(d^{3/2}\right)$	$O(d^{3/2})$
3	$\Omega\left(d^{4/3}\right)$	$O(d^{4/3})$	$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$	
≥ 4	$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$			

What we have demonstrated in this section is, first, that the asymptotic behaviour of the acyclic proper t-frugal chromatic number and the proper t-frugal star chromatic number can be determined up to at most a logarithmic multiple. Second, we showed that the asymptotic behaviour of the acyclic t-frugal chromatic number (respectively, t-frugal star chromatic number) of graphs of bounded maximum degree seems closely tied to that of their acyclic t-improper chromatic number (respectively, t-improper star chromatic number) as long as $t \ge 3$ (respectively, $t \ge 2$). Tables 6.3 and 6.4 give a summary of the bounds we have obtained (cf. Table 5.1).

6.3 Deterministic questions

All of the upper bounds obtained in this chapter so far are probabilistic in nature and rely on the Lovász Local Lemma. Although there are methods to convert applications of the Local Lemma into deterministic algorithms (cf. [12]), it remains an interesting problem to find *elementary* deterministic algorithms that do better than the most obvious bounds.

Table 6.4: Asymptotic bounds for $\varphi_a^t(d)$ and $\varphi_s^t(d)$.

$d-t$	$\varphi_a^t(d)$		$\varphi_s^t(d)$	
	lower	upper	lower	upper
$d-1$	$\Omega\left(d^2\right)$	$O\left(d^2\right)$	$\Omega\left(d^2\right)$	$O\left(d^2\right)$
$d-2$	$\Omega\left(d^{3/2}\right)$	$O\left(d^{3/2}\right)$	$\Omega\left(d^{3/2}\right)$	
$d-3$	$\Omega\left(d^{4/3}\right)$		$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$	$O(d^{3/2})$
$\omega(d^{3/4}(\ln d)^{1/4})$	$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$	$O(d^{4/3})$		
$\omega(d^{2/3}(\ln d)^{1/3})$				
$O(d^{1/2})$	$\Omega(d)$		$\Omega(d)$	$O((d-t)d)$
$O(d^{1/3})$		$O((d-t)d)$		
$O(\ln d)$		$O(d\ln d)$		$O(d\ln d)$
0	1	1	1	1

For instance, a simple way to acyclically $(d-1)$-frugally colour a graph of maximum degree d is to colour the square of the graph, using at most $d^2 + 1$ colours; however, Corollary 6.14 implies that we should be able to use many fewer colours. It is vexing that there does not appear to be an elementary deterministic algorithm giving $\varphi_a^{d-1}(d) = o(d^2)$.

Similarly, we can ask if there is any elementary deterministic algorithm that t-frugally colours a graph of maximum degree d with fewer colours than the following simple greedy algorithm.

Proposition 6.15. *If $t \geq (d-1)/k$, then $\varphi^t(d) \leq kd + 1$.*

Proof. Let $G = (V, E)$ be a graph of maximum degree d. Colour the vertices according to an arbitrary ordering $(u_1, \ldots, u_{|V|})$ of V and, at each step, assign the smallest available colour (i.e. greedily). Suppose we are at step i, for $i \in \{1, \ldots, |V|\}$. For each $v \in N(u_i)$, there are at most k colour classes which have t members in the set $N(v) \setminus \{u_i\}$. As there are at most d such v, at most kd choices of colour are forbidden for u_i. Thus, at most $kd + 1$ colours are required by this procedure. $\square$

We remark that, for $t = 1$ (setting $k = d - 1$), this algorithm also produces an acyclic colouring; thus, $\varphi^1(d) \leq \varphi_a^1(d) \leq d^2 - d + 1$ and the projective planes show via Proposition 6.5 that $\varphi^1(d) = \varphi_a^1(d) = d^2 - d + 1$ for infinitely many d. It was shown by Hahn et $al.$ [40] that these are the only graphs G which attain $\varphi^1(G) = \Delta(G)^2 - \Delta(G) + 1$.

For $t(d) = d - 1$ (setting $k = 1$), Proposition 6.15 gives that $\varphi^{d-1}(d) \leq d + 1$. We can give the following minor improvement which is essentially a backtracking version of the greedy algorithm:

Proposition 6.16. *If $d \geq 3$, then $\varphi^{d-1}(d) \leq d$.*

Proof. Let $G = (V, E)$ be a graph of maximum degree d. Without loss of generality, we may assume that G is d-regular. As in Proposition 6.15, let $\sigma = (u_1, \ldots, u_{|V|})$ be an arbitrary ordering of V and start by colouring the vertices greedily according to σ. Let i be the smallest $i \in \{1, \ldots, |V|\}$ such that at stage i we are forced to use colour $d + 1$ under the greedy algorithm.

We will now perform a search to find a recolouring that will allow u_i to be coloured from $\{1, \ldots, d\}$. As we search, we will define a sequence of vertex subsets $V_0, V_1, \ldots$. For convenience, let v_{-1} denote u_i.

Stage 0. It must be that for each $v \in N(u_i)$, there exists a colour $c_v \in \{1, \ldots, d\}$ such that each vertex of $N(v) \setminus \{u_i\}$ is coloured c_v; furthermore, the set $\{c_v \mid v \in N(u_i)\}$ is precisely the set of integers $\{1, \ldots, d\}$. Let $V_0 = \bigcup_{v \in N(u_i)} N(v) \setminus \{u_i\}$. For each member $x \in V_0$, there is a unique vertex $p_0(x) \in N(x) \cap N(u_i)$, which we call the *parent* of x.

For any $x \in V_0$, if we can recolour x to another colour without violating the frugality of its neighbours other than $p_0(x)$, then we can safely assign the colour $c_{p_0(x)} \in \{1, \ldots, d\}$ to u_i. So assume that none of the members of V_0 may be recoloured and pick an arbitrary $v_0 \in V_0$.

Stage j, $j \geq 1$. Assuming that V_{j-1}, $p_{j-1}(\cdot)$, and v_{j-1} were defined in previous stages, we form V_j as follows. Since v_{j-1} could not be recoloured at Stage $j-1$, it must be that for each $v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}$, there exists a colour $c_v \in \{1, \ldots, d\} \setminus \{c_{p_{j-1}(v)}\}$ such that each vertex of $N(v) \setminus \{v_{j-1}\}$ is coloured c_v; furthermore, the set $\{c_v \mid v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}\}$ is precisely the set $\{1, \ldots, d\} \setminus \{c_{p_{j-1}(v_{j-1})}\}$. Let $V_j = \bigcup_{v \in N(v_{j-1}) \setminus \{p_{j-1}(v_{j-1})\}} N(v) \setminus \{v_{j-1}\}$.

For each member $x \in V_j$, there is a unique vertex $p_j(x) \in N(x) \cap N(v_{j-1})$, which we call the *parent* of x.

For any $x \in V_j$, if we can recolour x to another colour without violating the frugality of its neighbours other than $p_j(x)$, then we can safely recolour v_{j-1} to the colour $c_{p_j(x)} \neq c_{p_{j-1}(v_{j-1})}$, and then recolour v_{j-2}, and so on until we recolour u_i. So assume that none of the members of V_j may be recoloured, pick an arbitrary $v_j \in V_j$, and continue to Stage $j+1$.

We want to show that this sequence terminates at some stage so that u_i is recoloured eventually; for a contradiction, suppose not. It is important to observe that, for any $j \geq 0$, the colour assigned to v_{j-1} is distinct from the colours assigned to the vertices in V_j (otherwise, we could have recoloured v_{j-1} at Stage $j-1$).

If it exists, let j_* be the smallest index such that for some $0 \leq j < j_*$ there exists $v \in V_j \cap V_{j_*}$. It follows that $v_{j-1} \neq v_{j_*-1}$; otherwise, either there is a smaller choice of j_* or $v_{j-1} = v_{j_*-1} = v_{-1}$ which is impossible since V_{j_*-1} contains only vertices with colours in $\{1, \ldots, d\}$. We consider two cases.

(i) Suppose that $p_j(v) = p_{j_*}(v)$. Since v and v_{j_*-1} are in $N(p_j(v)) \setminus \{v_{j-1}\}$, they must both have colour $c_{p_j(v)}$; however, by an observation above, v_{j_*-1} and $v \in V_{j_*}$ cannot have the same colour, a contradiction.

(ii) Suppose that $p_j(v) \neq p_{j_*}(v)$. Note that $p_j(v) \in N(v) \setminus \{p_{j_*}(v)\}$. Since $d \geq 3$, it follows from the definition of V_j that $N(p_j(v)) \setminus \{v\}$ contains v_{j-1} and at least one vertex v' coloured $c_{p_j(v)}$. Now, v_{j-1} and $v' \in V_j$ have different colours (again by the above observation). Now consider Stage j_*. The vertex $v \in V_{j_*}$ has a neighbour w other than $p_{j_*}(v)$ (namely $p_j(v)$) such that $N(w) \setminus \{v\}$ is not monochromatic. This means we could have recoloured v at Stage j_* without violating the frugality of its neighbours other than $p_{j_*}(v)$, a contradiction.

Now no such j_* exists, but this means that the V_j's are pairwise disjoint, and we have a contradiction as G is finite. $\qquad\square$

Note that if G is an odd cycle, then $\varphi^1(G) = 3$. It can be checked that the point-line

incidence graph for the Fano plane is a 3-regular graph verifying $\varphi^2(3) = 3$; however, by Theorem 6.7 we know that $\varphi^{d-1}(d) = 2$ for all but finitely many values of d.

Problem 6.17. *Are $d = 2, 3$ the only obstructions to $\varphi^{d-1}(d) = 2$?*

Subcubic graphs

The next remarks treat the smallest nontrivial case, $d = 3$, for the problems we have been considering. It is well known that $\chi^1_{\varphi,a}(3) = 10$ with attainment due to the Petersen graph. Note that Esperet *et al.* [28] showed $\chi^2_{\varphi,a}(3) = 5$ with attainment due to the complete bipartite graph $K_{3,3}$. As mentioned earlier, $\varphi^1(3) = \varphi^1_a(3) = 7$ due to the greedy algorithm and the Fano plane. We also have $\varphi^2(3) = 3$ by Proposition 6.16 and attainment due to the Fano plane. Except for the parameters $\varphi^t_s(3)$ and $\chi^t_{\varphi,s}(3)$, which we have yet to consider, the following result completely settles the case of subcubic graphs.

Proposition 6.18. $\varphi^2_a(3) = 3.$

Proof. Let G be a graph of maximum degree 3. By Proposition 6.16, there is a 2-frugal colouring of G using colours from $\{1, 2, 3\}$. Take such a colouring with the least number of alternating cycles. Suppose for a contradiction that, under this colouring, there is an alternating cycle C in G. Take a segment of length four $u_1 u_2 u_3 u_4 u_5$ along C (with possibly $u_1 = u_5$ if C has length four). Without loss of generality, suppose u_1, u_3, u_5 have colour 1 and u_2, u_4 have colour 2. For each $i \in \{2, 3, 4\}$, denote v_i to be the single vertex (if it exists) in $N(u_i) \setminus \{u_{i-1}, u_{i+1}\}$. We aim to recolour u_3 so that C is no longer alternating.

We first try to give u_3 the colour 2, for this recolouring destroys the alternating cycle on C and creates no others. Under this recolouring, u_2 and u_4 have frugality at most two; thus, it must be that v_3 has frugality three and in particular has two neighbours w and x other than u_3 that have colour 2.

Next, we try a different recolouring: try to give u_3 the colour 3. Again, u_2 and u_4 have frugality at most two under this recolouring and v_3 has frugality exactly two by way of w and x. This recolouring is 2-frugal, but C is no longer an alternating cycle; by our original choice of colouring, there must be a new alternating cycle C' through u_3 that alternates between the colours 2 and 3. The new alternating cycle C' does not pass through v_3 as w

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and x do not have colour 3. We deduce, therefore, that C' contains the path $v_2u_2u_3u_4v_4$; in particular, this means v_2 has at least one neighbour z other than u_2 of colour 2.

We now try another recolouring: try to give u_2 the colour 1. This recolouring destroys the alternating cycle on C and creates no others. Under this recolouring, we notice just as before that u_1 and u_3 have frugality at most two, but also v_2 has frugality at most two due to z. Now we have a 2-frugal colouring that has one fewer alternating cycles than C. This is a contradiction, showing that $\varphi_a^2(3) \leq 3$. The clique on four vertices K_4 has $\varphi_a^2(K_4) \geq 3$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

6.4   Conclusion

We believe the following conjecture to be natural in light of the results we obtained in Section 6.1.

Conjecture 6.19. $\varphi^t(d) \sim \lceil d^{1+1/t}/t \rceil$ *for any* $t = t(d) \geq 1$.

This conjecture holds for $t = \omega(\ln d)$, but, when $t = O(\ln d)$, the upper and lower bounds that we outlined are separated by at least a constant multiple.

In Sections 6.1 and 6.2, by dropping the condition that the colourings be proper, we demonstrated what seems to be a close qualitative link between t-frugal and t-improper colourings for t large enough. In Section 6.1, we established a threshold for t, namely $t = \Theta(\ln d)$, above which, the t-frugal chromatic number is asymptotically equal to the t-improper chromatic number. For $\varphi_a^t(d)$ (respectively, $\varphi_s^t(d)$), the threshold of convergence with $\chi_a^t(d)$ (respectively, $\chi_s^t(d)$) may possibly be $t = 3$ (respectively, $t = 2$). Indeed, we conjecture the following.

Conjecture 6.20. $\varphi_a^t(d) = \Theta(\chi_a^t(d))$ *for any* $t = t(d) \geq 1$ *unless* $t \in \{1, 2\}$. *Analogously,* $\varphi_s^t(d) = \Theta(\chi_s^t(d))$ *for any* $t = t(d) \geq 1$ *unless* $t = 1$.

We point out here that, in the setting of planar graphs, such an asymptotic "convergence" does not occur. The acyclic t-improper chromatic number of planar graphs is bounded by a constant (namely, 5); whereas, the t-frugal chromatic number of a planar graph G is at least

$\Delta(G)/t$ and thus can be arbitrarily large. It remains interesting to determine, for fixed t,

$$\varphi^t(\mathcal{P}, d) := \max\{\varphi^t(G) \ \mid \ G \text{ is planar and } \Delta(G) \leq d\}$$

and, in particular, what is the smallest constant $K \geq 1$ such that $\varphi^t(\mathcal{P}, d) \leq Kd/t + o(d)$. That such a constant K exists is implied by work of Amini, Esperet and van den Heuvel [7].

Another interesting line of inquiry is to determine the asymptotically slowest-growing choice of $t = t(d)$ for which $\varphi^t(d)$ is asymptotically smaller than $\chi_\varphi^t(d)$ or for which $\varphi_a^t(d)$ (respectively, $\varphi_s^t(d)$) is asymptotically smaller than $\chi_{\varphi,a}^t(d)$ (respectively, $\chi_s^t(d)$). Theorem 6.4 implies that the answer in the former case is in the range $t = \Theta(\ln d/\ln\ln d)$. Theorem 5.2 and Corollary 6.14 (respectively, Theorems 5.5 and 6.13) suggest that t, for the latter question, is in the range such that $d - t = \Omega(d^{1/3}/(\ln d)^{1/3})$ and $d - t = o(d)$ (respectively, $d - t = \Omega(d^{1/2}/(\ln d)^{1/2})$ and $d - t = o(d)$).

There are also intriguing questions concerning the case $t = d - 1$, for example, Problem 6.17. Another natural challenge (which is a subproblem of Conjecture 6.20) is to determine the asymptotic values of $\varphi_a^{d-1}(d)$ and $\varphi_s^{d-1}(d)$, as these parameters lie in the range $\Omega(d)$ and $O(d \ln d)$.

As far as we are aware, t-frugal colourings (i.e. ones that are not necessarily proper) have not been studied much previously. The parameter $\varphi^1(G)$, referred to as the *injective chromatic number*, was considered by Hahn *et al.* [40]. There are a handful of works that have examined proper t-frugal colourings [7, 28, 84], but there are still many open questions in this line of research.

Appendix A

NP-completeness proofs

Before continuing with descriptions of the NP-completeness proofs, we give the following lemma.

Lemma A.1. *Suppose K_1 is a $(t+1)$-clique, K_2 is a $((k-1)(t+1))$-clique, and K_3 is a j-clique, $1 \leq j \leq t+1$. Let H be the graph formed by including all possible edges between K_1 and K_2 and between K_2 and K_3. Then H is t-improperly k-colourable, and in any t-improper k-colouring of H, any vertex of K_1 and any vertex of K_3 must receive the same colour.*

Proof. Suppose we have a t-improper k-colouring of H, let u be a vertex of K_3 and assume without loss of generality that u has colour 1. The subgraph induced by $K_1 \cup K_2$ is an $(k(t+1))$-clique, so every colour appears exactly $t+1$ times in this clique and each vertex v in $K_1 \cup K_2$ has impropriety t in $K_1 \cup K_2$. Hence, the colour 1 may not appear on the vertices of K_2. Thus, the $t+1$ vertices of the clique $K_1 \cup K_2$ that are coloured 1 are those of K_1. As the vertex u of K_3 is arbitrary, this concludes the proof. □

A.1   Improper colouring of hexagonal graphs

Proof of Theorem 2.7. We shall generalise the proof of McDiarmid and Reed [63] and reduce the problem from 3-colourability of planar graphs with maximum degree 4. Suppose we are given a planar graph G with maximum degree 4. We construct an induced subgraph F of

the triangular lattice T and a corresponding weight vector w so that G is 3-colourable if and only if (F, w) admits a weighted t-improper 3-colouring.

The construction of F is the same as in McDiarmid and Reed [63], and we recall it here for completeness: for any $v \in T$, let H denote the subgraph of T induced by all the vertices at distance at most 3 from v. The infinite face of H is bounded by a regular hexagon; the "contact points" of H are the six extreme points of this hexagon. For each vertex v of G, we make a copy H_v of the hexagon H, and place them suitably far apart in the lattice. For each edge $e = uv$ of G, we put an induced path P^e between one of the contact points of H_u and one of the contact points of H_v. It is possible to make these paths P^e completely disjoint. Furthermore, we can suppose that they all have odd length. First subdivide each edge e of G by adding a vertex ν_e. Then observe that between any two contact points of the hexagon H_{ν_e} there are induced paths of both parities within the hexagon. Thus, we are always able to make only odd-length paths.

Now we define the weight w to be $t + 1$ for each vertex of F, except for every second internal vertex of each path P^e for which we give weight $2t + 2$. We will show that F_w is t-improperly 3-colourable if and only if G is 3-colourable. In any t-improper 3-colouring of the hexagon H_w, each of the $(t + 1)$-cliques corresponding to a vertex $v \in H$ is coloured with a single colour; moreover, all the $(t + 1)$-cliques induced by the contact points have the *same* colour. It follows by applying Lemma A.1 that the only t-improper 3-colouring of H_w (up to permutations of colours) is the one induced by the only proper 3-colouring of H.

In any t-improper 3-colouring of a path P^e_w, Lemma A.1 shows that the $(2t + 2)$-cliques induced by every second internal vertex each use exactly two colours, and any remaining $(t + 1)$-clique uses exactly one colour, which is the same for all of them. Thus, the terminal vertices must be coloured differently (and the extremal cliques must have only one colour since they both belong to a hexagon H).

So any t-improper 3-colouring of (F, w) induces a proper 3-colouring of G by applying to any vertex v of G the colour assigned to any of the contact points of H_v in F. Conversely, given a proper 3-colouring c of G, if we assign to each contact point of H_v the multiset $\{c(v)^{t+1}\}$, it is possible to extend this partial colouring in a t-improper 3-colouring of (F, w). These observations end the proof. $\square$

A.2 Unit disk graph t-improper k-colourability, $k \geq 3$

Our approach will generalise that of Gräf *et al.* [37] and we want to show how, given any graph G, to construct a corresponding unit disk graph $\widehat{G} = (\widehat{V}, \widehat{E})$ which is t-improperly k-colourable if and only if G is k-colourable. The key to our approach is to generalise what Gräf *et al.* call the auxiliary graphs. We will describe t-improperly k-colourable analogues for each of the four auxiliary graphs that they employ. We use the same embedding for the given graph G, and the unit disk graph embedding needs only a slight technical modification to accommodate a larger auxiliary graph for crossings. Here is the scheme of the proof: first, we use an embedding of G in the plane that allows us to replace, in a systematic way, its edges by well-chosen unit disk graphs. Our choice of unit disk graphs ensures that the existence of a k-colouring of G is equivalent to the existence of a t-improper k-colouring of $\widehat{G}$. There are two major issues in such an approach, in comparison to a reduction from 3-colourability of planar graphs with maximum degree 4. First, as G is not necessarily planar, there may be crossing edges. Second, as the maximum degree of G is not bounded, we have to deal with vertices of arbitrarily large degree. These two issues are solved by using two types of auxilary unit disk graphs, the crossing gadgets and the vertex gadgets.

Construction of the auxiliary graphs

First, we introduce the graphs that replace the edges in an embedding of G. All of these graphs are unit disk graphs and, except for the last one, use the same embeddings as in Gräf *et al.* [37]. The remaining properties are usually given without justification since they generally follow immediately from the construction or a simple application of Lemma A.1. Like in the cited reference, our construction makes frequent use of cliques. In figures, these cliques will be represented by circles using the following convention:

- a small circle with a + represents a $(t+1)$-clique;

- a large circle with a $\star$ represents a $(k-2)(t+1)$-clique; and

- a large circle with a $\times$ represents a $(k-1)(t+1)$-clique.

If cliques of other size are needed, they will be represented by a large circle with the order of the clique written inside. An edge between two cliques means that all possible edges between the two cliques are present.

Definition A.2. *A* (t, k)*-wire of order* m*, denoted* $W_{t,k}^m$*, consists of* $m + 1$ $(t + 1)$*-cliques* $WV_0, \ldots, WV_m$ *and* m $((k - 1)(t + 1))$*-cliques* $WC_1, \ldots, WC_m$*. For each* $i \in \{1, \ldots, m\}$*, all members of the clique* WC_i *are connected to the members of both* WV_{i-1} *and* WV_i*. The cliques* WV_0 *and* WV_m *are called output cliques.*

A (t, k)-wire of order 3 is shown in Figure A.1.
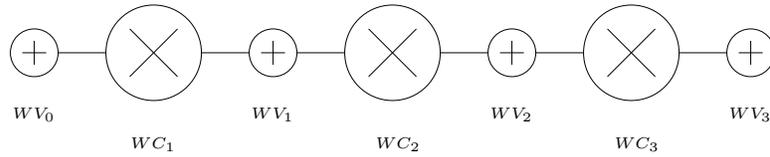
Figure A.1: The (t, k)-wire $W_{t,k}^3$.

Proposition A.3. *A* (t, k)*-wire of order* m *has the following properties:*

(i) $W_{t,k}^m$ *has* $m(k - 1)(t + 1) + (m + 1)(t + 1) = (mk + 1)(t + 1)$ *vertices;*

(ii) *a* (t, k)*-wire is* t*-improperly* k*-colourable, but not* t*-improperly* $(k - 1)$*-colourable;*

(iii) *each* t*-improper* k*-colouring assigns the same colour to all members of* $WV_0, \ldots, WV_m$*, and, in particular, the output cliques receive the same colour; and*

(iv) *a* (t, k)*-wire is a unit disk graph.*

Definition A.4. *A* (t, k)*-chain of order* m*, denoted* $K_{t,k}^m$*, consists of a* $W_{t,k}^m$ *together with an additional* j*-clique* WF *connected with* WV_m*, for some* $1 \leq j \leq (k - 1)(t + 1)$*. The clique* WV_0 *is called the fixed output clique while* WF *is called the forced output clique.*

A (t, k)-chain of order 3 is shown in Figure A.2.

Proposition A.5. *A* (t, k)*-chain of order* m *has the following properties:*

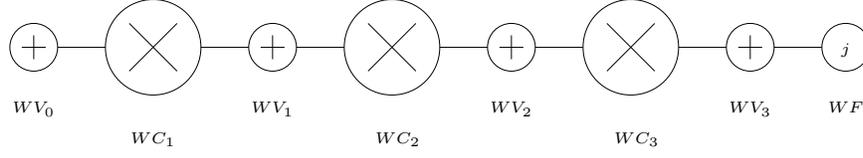(i) $K_{t,k}^m$ *has* $(mk + 1)(t + 1) + j$ *vertices, where* j *is the size of* WF*;*

Figure A.2: A (t, k)-chain of order 3 $K_{t,k}^3$.

(ii) *a (t, k)-chain is t-improperly k-colourable, but not t-improperly $(k-1)$-colourable;*

(iii) *each t-improper k-colouring assigns the same colour i to all members of any of the cliques $WV_x, 1 \leq x \leq m$, and each member of the clique WF must receive a colour that is different from i;*

(iv) *for each pair of different colours (i_1, i_2) from the set $\{1, 2, \ldots, k\}$ there exists a t-improper k-colouring in which the forced and fixed output cliques receive colours i_1 and i_2, respectively; and*

(v) *a (t, k)-chain is a unit disk graph.*

We now introduce the graphs that replace the vertices of G (including those of arbitrarily large degree).

Definition A.6. *A (t, k)-clone of size $m \geq 2$, denoted $C_{t,k}^m$, consists of the $7m - 7$ $(t+1)$-cliques $CV_1, \ldots, CV_{7m-7}$, the $7m - 6$ $((k-1)(t+1))$-cliques $CC_0, \ldots, CC_{7m-7}$, and the m $(t+1)$-cliques $O_0, \ldots, O_{m-1}$. For $1 \leq i \leq 7m - 7$, all members of the clique CV_i are connected to the members of both CC_{i-1} and CC_i. For each $i \in \{1, \ldots, m-1\}$, all members of O_i are connected to the members of CC_{7i}. The cliques $O_0, \ldots, O_{m-1}$ are called output cliques.*

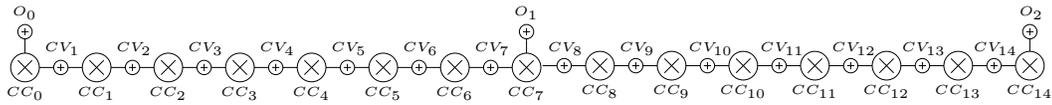A (t, k)-clone of size 3 is shown in Figure A.3.

Figure A.3: The (t, k)-clone $C_{t,k}^3$.

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Note that, in the corresponding auxiliary graph described in Gräf *et al.* [37], every third clique was connected to an output vertex for technical reasons. For similar reasons, every seventh $((k-1)(t+1))$-clique is connected to an output clique in our construction.

Proposition A.7. *A (t,k)-clone of size m has the following properties:*

(i) $C_{t,k}^m$ *has* $(7m-6)(k-1)(t+1)+(7m-7)(t+1)+m(t+1) = ((7m-6)k+m-1)(t+1)$ *vertices;*

(ii) *a (t,k)-clone is t-improperly k-colourable, but not t-improperly $(k-1)$-colourable;*

(iii) *each t-improper k-colouring assigns the same colour to all members of the output cliques; and*

(iv) *a (t,k)-clone is a unit disk graph.*

Finally, we introduce the graphs $H_{t,k}$ that replace the edge crossings in an embedding of G. This construction is based on the graph H_k used in Gräf *et al.* [37], which in turn is based on a construction by Fisher [33] that was used to prove NP-completenesss for the problem of 3-colourability of planar graphs. We replace all vertices of H_k by $(t+1)$-cliques and we replace all edges of H_k by (t,k)-chains of the appropriate order (either 1 or 2) so that the resulting graph has a unit disk representation.

When replacing edges in H_k, we have taken care to orient the (t,k)-chains so that we do not introduce cliques of size greater than $k(t+1)$; in particular, only the forced output cliques of the (t,k)-chains may be incident with the $((k-2)(t+1))$-cliques C_i of $H_{t,k}$. Note then that each $(t+1)$-clique representing a former vertex of H_k is incident to a $((k-1)(t+1))$-clique of some (t,k)-chain, and this ensures that, in a t-improper k-colouring of $H_{t,k}$, each $(t+1)$-clique is assigned a single colour.

See Figure A.4 for a description of how $H_{t,k}$ is derived.

Definition A.8. *Let a (t,k)-crossing, denoted $H_{t,k}$, $k \geq 3$ be the graph in Figure A.5. The cliques $V_0, \ldots, V_3$ are called output cliques.*

The graph H_k has output vertices $v_0, \ldots v_3$ (that correspond to the output cliques $V_0, \ldots, V_3$ of $H_{t,k}$) and Gräf *et al.* showed the following: (a) each k-colouring f of H_k satisfies $f(v_0) = f(v_2)$ and $f(v_1) = f(v_3)$ and (b) there exist two k-colourings f_1 and f_2 which
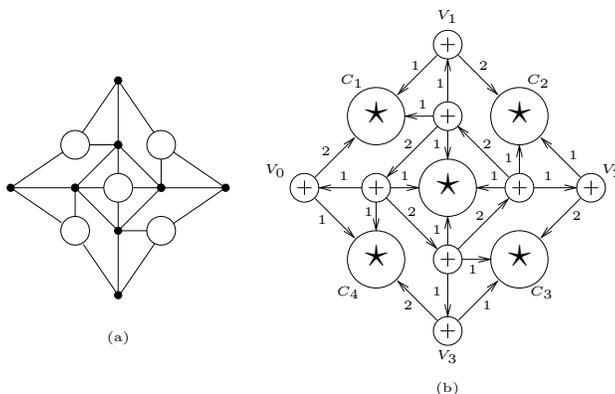
Figure A.4: The derivation of the (t,k)-crossing $H_{t,k}$ from the k-crossing H_k: (a) H_k, where the circles represent $(k-2)$-cliques and (b) a schematic figure of $H_{t,k}$, where each (t,k)-chain is represented by a directed edge (the edge is directed from the fixed output vertex to the forced output vertex of the chain) together with an integer (the order of the chain).

satisfy $f_1(v_0) = f_1(v_2) = f_1(v_1) = f_1(v_3)$ and $f_2(v_0) = f_2(v_2) \neq f_2(v_1) = f_2(v_3)$. Since $H_{t,k}$ is derived from H_k in a natural way, it is therefore routine, with the use of Proposition A.5, to verify the analogous properties (iii) and (iv) of Proposition A.9 below for the graph $H_{t,k}$.

Proposition A.9. *A (t,k)-crossing has the following properties:*

(i) $H_{k,t}$ *has $(37k-2)(t+1)$ vertices;*

(ii) *a (t,k)-crossing is t-improperly k-colourable, not t-improperly $(k-1)$-colourable;*

(iii) *each t-improper k-colouring c satisfies*
  $c(V_0) = c(V_2)$ *and* $c(V_1) = c(V_3)$;

(iv) *there exist two t-improper k-colourings c_1 and c_2 which satisfy*
  $c_1(V_0) = c_1(V_2) = c_1(V_1) = c_1(V_3)$ *and*
  $c_2(V_0) = c_2(V_2) \neq c_2(V_1) = c_2(V_3)$; *and*

(v) *a (t,k)-crossing is a unit disk graph.*

Embedding of the unit disk graph

As mentioned earlier, we shall use the same embedding of the given graph G, or rather, the embedding of a graph G' obtained from G. We aim at having the following properties:
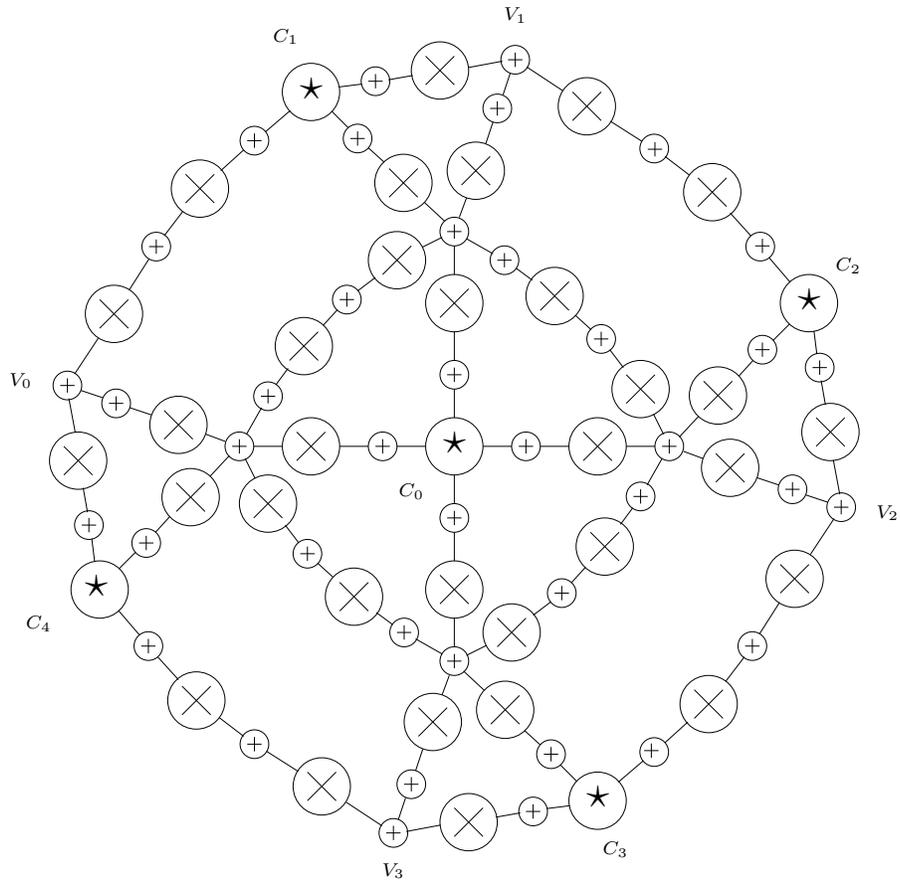
Figure A.5: The (t, k)-crossing $H_{t,k}$.

- all the edges are made of vertical and horizontal line segments;

- certain minimal distances are preserved between parallel line segments, vertices and crossings; and

- the embedding can be computed in a simple and systematic fashion.

However, for our auxiliary graphs, we must accommodate for the necessity of a larger unit disk representation for $H_{t,k}$.

First, each vertex v of G is replaced by an independent set $M(v)$ of order $d(v)$, the degree of v. Next, a vertex of $M(v)$ is linked to a vertex of $M(u)$ if and only if uv is an edge in G. The edges are added so that the maximum degree of the obtained graph $G' = (V', E')$ is one.

To describe the embedding of G', let $n(\cdot)$ be an order of the vertices of V' from 1 up to $|V'|$, in a such a way that the vertices of each set $M(v)$ are numbered by consecutive integers. The vertices of G' all lie on the x-axis: the coordinates of the vertex $v \in V'$ are $X(v) := (56n(v), 0)$.

An edge uv of G' is represented by the three following line segments:

$$\{(x, y) \mid x = X(u) \text{ and } y \in [0, X(u) + 8]\},$$

$$\{(x, y) \mid x \in [X(u), X(v)] \text{ and } y = X(u) + 8\}, \text{ and}$$

$$\{(x, y) \mid x = X(v) \text{ and } y \in [0, X(u) + 8]\}.$$

An example is given by Figure A.6. Such an embedding depends on the chosen numbering n, but is unique once n is chosen.

Let us state now the construction of $\widehat{G}$, and exhibit a representation thereby showing that it is a unit disk graph.

Notice that each clique of the auxiliary graphs can be represented by a single disk, since every vertex u belongs to a clique $C(u)$, and if u and v are adjacent then all the vertices of $C(u)$ and $C(v)$ are. So, using elementary properties of disks in the plane, it is sufficient to represent each clique by a vertex, called its *representative*. We shall now give a representation of G with disks of radius 3.
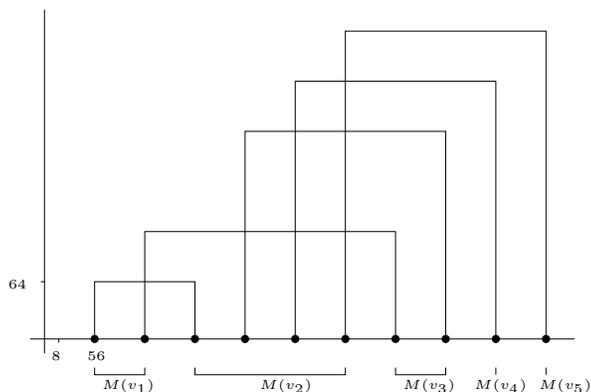
Figure A.6: Embedding of the graph G'.

The (t, k)-wires and the (t, k)-chains are embedded similarly to the k-wires and k-chains, respectively, of Gräf *et al.* [37]. These auxiliary graphs replace line segments in the embedding of G'. Observe that a (t, k)-wire of order m can be embedded so that the distance between the center of the disks of the representative of the output cliques is $8m$, and all the centers lie on a line. With a slight modification (since it contains one more clique), a (t, k)-chain of order m can also be embedded so that the distance between the representative of the output cliques is $8m$, and all the centers are on the same line. Moreover, the embedding of a (t, k)-wire can be modified so that the union of line segments joining consecutive centers makes a right angle (Figure A.7).
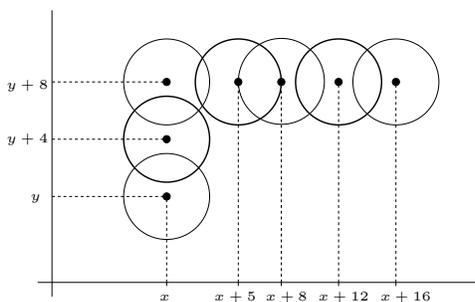

Figure A.7: Embedding of a (t, k)-wire making a right angle.

Now follows the embedding used for a (t, k)-clone. Let (x, y) be the coordinates of the center of the representative of the ouput clique O_0: the coordinates of the centers of the representative of the clique CC_i are $(x + 8i, y - 5)$, $i \in \{0, 1, \ldots, 7m - 6\}$, those of the representative of the clique CV_i are $(x + 4i, y - 5)$, $i \in \{1, 2, \ldots, 3n - 3\}$, and those of the representative of the ouput clique O_i are $(x + 56i, y)$, $i \in \{1, 2, \ldots, m - 1\}$ (see Figure A.8).
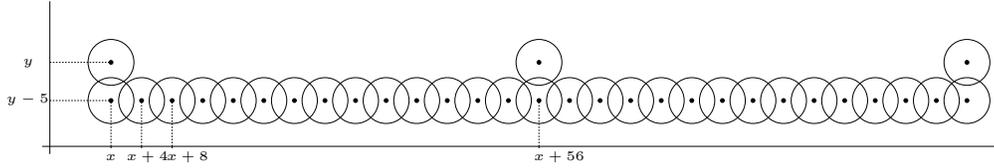
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Figure A.8: Embedding of a (t, k)-clone of order 3.

Figure A.9 shows an embedding of a (t, k)-crossing with disks of radius 3. The centres ot the representative of the output cliques lie at distance 24 of the centre.
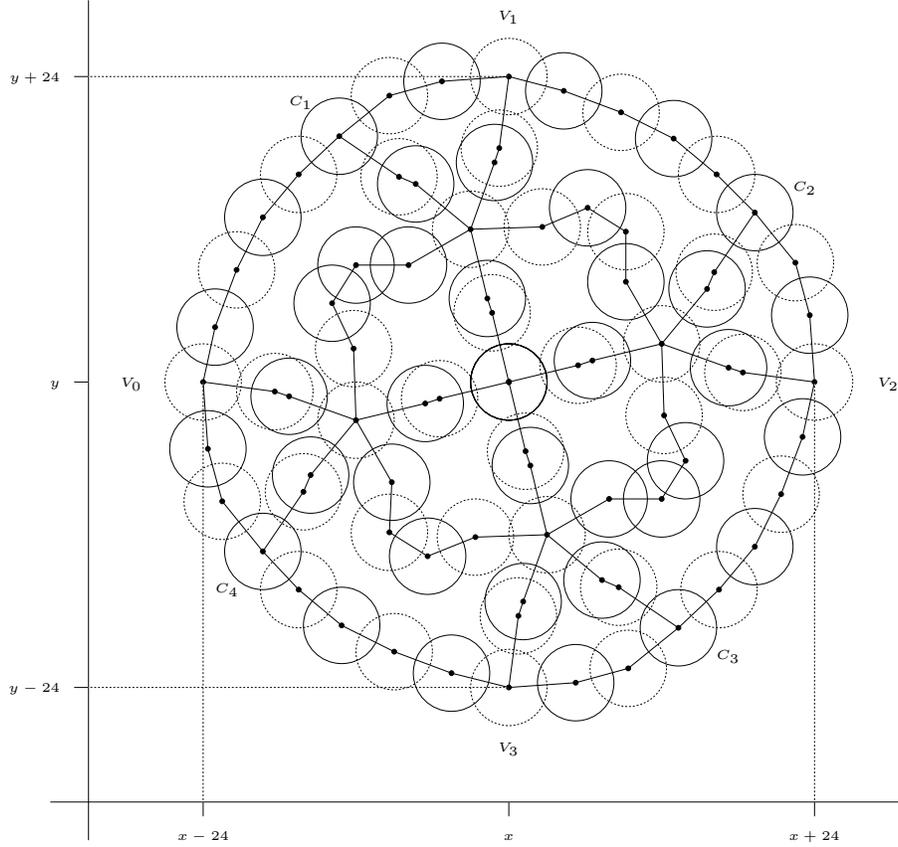

Figure A.9: An embedding of the (t, k)-crossing: a bold-lined disk represents $(k - 1)(t + 1)$ copies of the same disk, a dash-lined disk represents $t + 1$ copies of the same disk while each of the five remaining disks represents $(k - 2)(t + 1)$ copies of the same disk.

The final steps now are not difficult: for every vertex v of G, a (t, k)-clone of order $d(v)$ is embedded in such a way that the coordinates of the center of the representative of O_0 are $(x(v), 0)$, where $x(v)$ is the smallest x-coordinate of a vertex of $M(v)$. The coordinates of the output cliques therefore are exactly those of the vertices in $M(v)$. It only remains

to replace the edges: let uv an edge of G'. If it crosses no other edge, then a (t,k)-chain of appropriate order is embedded along the three line segments representing the edge uv so that the representative of the output cliques identify with the vertices u and v. More precisely, if the x-coordinate of u is $8s$, and the one of v is $8s'$ with $s < s'$, then the order of the (t,k)-chain is $s + 1 + (s' - s - 2) + 1 + s = s + s'$.

If the edge uv crosses at least one other edge, then let (x,y) be the coordinates of a crossing. The four points of coordinate $(x - 24, y), (x + 24, y), (x, y - 24)$ and $(x, y + 24)$ are replaced by the ouput cliques of a (t,k)-crossing. The remaining line segments are all replaced by (t,k)-wires of appropriate orders, except the one containing the vertex v, which is replaced by a (t,k)-chain. This is possible since the length of each line segment is a multiple of 8.

Proof of Theorem 2.3

It only remains to prove that the graph G is k-colourable if and only if the graph $\widehat{G}$ is t-improperly k-colourable. For each vertex v of G, let $I(v)$ denote the set of vertices of the output clique of the (t,k)-clone replacing the vertex v.

Let c be a k-colouring of G. Each vertex of $I(v)$ is given the colour $c(v)$, and this colouring is extended to a t-improper k-colouring of the corresponding (t,k)-clone by Proposition A.7(ii) and (iii). Consider now an edge uv of G: if the corresponding output cliques are linked by a single (t,k)-chain, then the colouring can be extended by Proposition A.5(iv), since $c(u) \neq c(v)$. Otherwise, the colouring is extended to each (t,k)-wire using Proposition A.3(iii), i.e. all the vertices of each output clique are assigned the same colour. Once this is done for every edge, the colouring is extended to each (t,k)-crossing by Proposition A.9(iv). Finally, each yet uncoloured (t,k)-chain is coloured by using Proposition A.5(iv), since the output cliques are coloured differently (one having the colour $c(u)$ and the other $c(v)$ for two adjacent vertices u, v of G).

Let $\hat{c}$ be a t-improper k-colouring of $\widehat{G}$. Each vertex v of G is given the colour of any vertex belonging to the output clique replacing v. By the construction, and Propositions A.3(iii), A.5(iii), A.7(iii) and A.9(iii), the obtained k-colouring c of G is proper. $\square$

A.3 Unit disk graph t-improper 2-colourability, $t \geq 1$

For Theorem 2.4, our reduction is from t-improper 2-colourability of planar graphs. Given any planar graph G, we show how to construct, in polynomial time, a unit disk graph $\widehat{G}$ which is t-improperly 2-colourable if and only if G is. Our construction is based on Gräf *et al.* [37], but, for the embedding, we have added the condition of planarity. Hence, we do not require a crossing auxiliary graph. On the other hand, since we are dealing entirely with t-improper 2-colouring, we must take care to handle impropriety appropriately.

Construction of the auxiliary graphs

These graphs are unit disk graphs. We give the corresponding unit disk representations later. First, we introduce the graphs that replace the edges in an embedding of G.

Definition A.10. *A $(t, 2)$-bond, denoted $B_{t,2}$, has vertex set $\{v_0, \ldots, v_{2t+2}\}$. For the edge set, the vertices $\{v_1, \ldots, v_{2t+1}\}$ induce a clique, v_0 is adjacent to any $v_i, i \leq t+1$, and v_{2t+2} is adjacent to any $v_i, i \geq t+1$. The vertices v_0 and v_{2t+2} are called output vertices.*
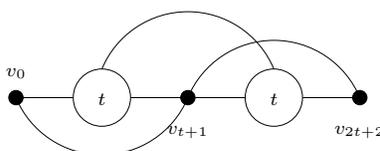
A $(t, 2)$-bond is shown in Figure A.10.

Figure A.10: The $(t, 2)$-bond $B_{t,2}$.

Proposition A.11. *A $(t, 2)$-bond has the following properties:*

(i) *$B_{t,2}$ has $2t + 3$ vertices;*

(ii) *a $(t, 2)$-bond is t-improperly 2-colourable, not t-improperly 1-colourable;*

(iii) *each t-improper 2-colouring of $B_{t,2}$ assigns the same colour to v_0 and v_{2t+2};*

(iv) *suppose v_0 is adjacent to $j \in \{0, \ldots, t\}$ additional vertices $u_1, \ldots, u_j$ and furthermore suppose that $v_0, u_1, \ldots, u_j$ are precoloured with the same colour: then there exists a*

t-improper 2-colouring of $B_{t,2}$ such that v_{2t+2} has impropriety j, but there is no such colouring such that v_{2t+2} has impropriety less than j; and

(v) a $(t,2)$-*bond is a unit disk graph.*

Note that, under the hypothesis of property (iv), we say that v_0 is coloured *with external impropriety j*.

Proof. The first two properties immediately follow from the definition, so we focus on proving properties (iii)–(iv). Assume that c is a t-improper 2-colouring of $B_{t,2}$.

To prove property (iii), suppose that $c(v_0) = 1$ and $c(v_{2t+2}) = 2$. Note that, as $\{v_1, \ldots, v_{2t+1}\}$ induces a $(2t+1)$-clique, then one colour, say 2, must appear exactly $t+1$ times. Hence, any such vertex coloured 2 has impropriety t in the clique, and so cannot be a neighbour of v_{2t+2}. However, among $v_1, \ldots, v_{2t+1}$, there are only t non-neighbours of v_{2t+2}. This is a contradiction.

To prove property (iv), suppose that $c(v_0) = c(u_1) = \cdots = c(u_j) = 1$. For the first part, set $c(v_1) = c(v_2) = \cdots = c(v_{t-j}) = 1$, $c(v_{t-j+1}) = c(v_{t-j+2}) = \cdots = c(v_{2t-j+1}) = 2$, and $c(v_{2t-j+2}) = c(v_{2t-j+3}) = \cdots = c(v_{2t+2}) = 1$. It is routine to check that this colouring satisfies our requirement. For the second part, since v_0 has impropriety j, colour 1 appears at most $t-j$ times among $v_1, \ldots, v_{t+1}$. As $v_1, \ldots, v_{2t+1}$ is a $(2t+1)$-clique, there are at least t vertices of colour 1. We deduce that there are at least j vertices among $\{v_{t+1}, \ldots, v_{2t+1}\}$ coloured 1. Since $c(v_{2t+2}) = 1$ by proposition (iii), v_{2t+2} has impropriety at least j.

For property (v), we describe the embedding of $B_{t,2}$ in the next section. □

Definition A.12. *A $(t,2)$-wire of order m, denoted $W_{t,2}^m$, is the left-to-right concatenation of m $(t,2)$-bonds $B_1, \ldots, B_m$. The extreme vertices, v_0 of B_1 and v_{2t+2} of B_m, are called output vertices.*

A $(t,2)$-wire of order 3 is shown in Figure A.11. The following properties follow from Proposition A.11.

Proposition A.13. *A $(t,2)$-wire of order m has the following properties:*
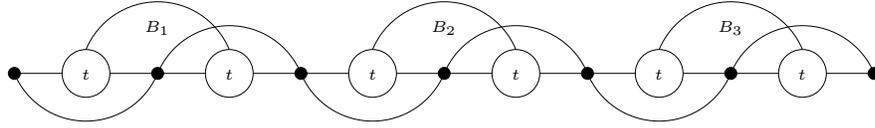
(i) $W_{t,2}^m$ *has $m(2t+2)+1$ vertices;*

Figure A.11: The $(t, 2)$-wire of order 3 $W_{t,2}^3$.

(ii) *a $(t, 2)$-wire is t-improperly 2-colourable, not t-improperly 1-colourable;*

(iii) *each t-improper 2-colouring of $W_{t,2}^m$ assigns the same colour to the output vertices;*

(iv) *if an output vertex v of B_i has external impropriety $j \in \{0, \ldots, t\}$, then there exists a t-improper 2-colouring of $W_{t,2}^m$ such that the other output vertex of B_i has impropriety j, but there is no such colouring such that the other output vertex of B_i has impropriety less than j; and*

(v) *a $(t, 2)$-wire is a unit disk graph.*

Definition A.14. *A $(t, 2)$-clone of size $m \geq 2$, denoted $C_{t,2}^m$, consists of m output vertices $o_1, \ldots, o_m$ such that there is a $(t, 2)$-wire W_i between o_i and o_{i+1} for each $i \in \{1, \ldots, m-1\}$.*

A $(2, 2)$-clone of size 3 is shown in Figure A.12. Note that we have defined the $(t, 2)$-clone to have arbitrary order, but we will apply $(t, 2)$-clones of bounded order to our embedding.
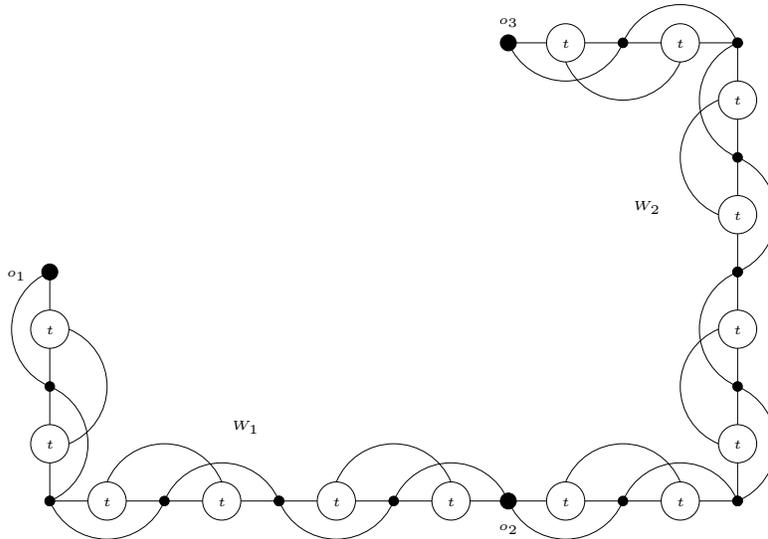

Figure A.12: A $(2, 2)$-clone of size 3 $C_{2,2}^3$.

Proposition A.15. *A $(t, 2)$-clone has the following properties:*

(i) *$C^m_{t,2}$ has $k(2t + 2) + 1$ vertices, for some $k \geq m$;*

(ii) *a $(t, 2)$-clone is t-improperly 2-colourable, not t-improperly 1-colourable;*

(iii) *each t-improper 2-colouring of $C^m_{t,2}$ assigns the same colour to all output vertices;*

(iv) *in any t-improper 2-colouring of $C^m_{t,2}$, the sum of external improprieties of the output vertices (cf. the remark following Proposition A.11) is at most t;*

(v) *given a sequence $s_1, \ldots, s_m$ of non-negative integers whose sum is at most t, there is a t-improper 2-colouring of $C^m_{t,2}$ such that the external impropriety of o_i is s_i, $1 \leq i \leq m$; and*

(vi) *a $(t, 2)$-clone is a unit disk graph.*

Proof. For property (v), we colour the vertices of $C^m_{t,2}$ starting at o_1. Suppose $c(o_1) = 1$. By Proposition A.13(iv), since o_1 and o_2 are output vertices of W_1, there exists a t-improper 2-colouring of W_1 such that o_2 has impropriety s_1. Now, $c(o_2) = 1$ and, if we set the external impropriety of o_2 in W_2 to $s_1 + s_2$, then o_2 has external impropriety s_2 in $C^m_{t,2}$. By Proposition A.13(iv), there exists a t-improper 2-colouring of W_2 such that o_3 has impropriety $s_1 + s_2$. We can carry on like this until we have coloured all of $C^m_{t,2}$, since $s_1 + s_2 + \cdots + s_m \leq t$.

The other properties use similar applications of Proposition A.11. $\qquad\square$

Definition A.16. *For any odd positive integer m, a $(t, 2)$-link of order m, denoted $K^m_{2,t}$, is defined as follows. The vertex set is $\{v_0, \ldots, v_{x(t,m)+1}\}$, where*

$$x(t, m) = \begin{cases} mt(t + 1) & \text{if } t \text{ is even} \\ mt(t + 1) + t + 1 & \text{if } t \text{ is odd} \end{cases}$$

For the edge set, we join v_i and v_j if and only if $|i - j| \leq t + 1$. The vertices v_0 and $v_{x(t,m)+1}$ are called output vertices.
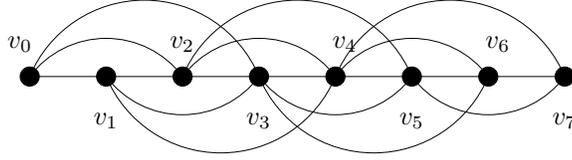
A $(2, 2)$-link of order 1 is shown in Figure A.13.

Figure A.13: The $(2,2)$-link of order 1 $K^1_{2,2}$.

Proposition A.17. *A $(t,2)$-link has the following properties:*

(i) $K_{t,2}$ *has $x(t,m)+2$ vertices;*

(ii) *a $(t,2)$-link is t-improperly 2-colourable, not t-improperly 1-colourable;*

(iii) *for any t-improper 2-colouring of $K_{t,2}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*

(iv) *there exists a t-improper 2-colouring of $K_{t,2}$ such that the output vertices receive different colours and both vertices have impropriety zero;*

(v) *there exists a t-improper 2-colouring of $K_{t,2}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*

(vi) *a $(t,2)$-link is a unit disk graph.*

Proof. We focus on proving properties (iii)–(vi).

For property (iii), suppose c is a t-improper 2-colouring of $K_{t,2}$ such that both output vertices are coloured 1. By symmetry, suppose that v_0 has impropriety 0. Then we must have $c(v_i) = 2$ for each $i \in \{1, \ldots, t+1\}$. In particular, note that $\text{im}_{\{v_1,\ldots,v_t\}}(v_{t+1}) = t$ so any vertex v_i with $i \in \{t+2, \ldots, 2t+2\}$ must be coloured 1. More generally, we see that the only possibility is that $c(v_i) = 1$ if and only if $(m-1)(t+1)+1 \le i \le m(t+1)$ for m an even integer. However, since $\frac{x(t,m)}{t+1}$ is even, the $t+1$ vertices with indices between $\left(\frac{x(t,m)}{t+1} - 1\right)(t+1)+1$ and $x(t,m)$ are coloured $1 = c(v_{x(t,m)+1})$. Since these $t+1$ vertices are adjacent to $v_{x(t,m)+1}$ we have a contradiction.

For property (iv), we use the above forced colouring. In other words, set $c(v_0) = 1$, $c(v_{x(t,m)+1}) = 2$ and for $1 \le i \le x(t,m)$, set $c(v_i) = 1$ if and only if $(m-1)(t+1)+1 \le$

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$i \leq m(t+1)$ for m an even integer. It is simple to check that the output vertices have impropriety zero.

For property (v), we use the following colouring. Set $c(v_0) = c(v_{x(t,m)+1}) = 1$. For each $i \in \{1, \ldots, x(t,m)\}$, set $c(v_i) = 1$ if and only if the index i is between $(m-1)t+1$ and mt for m an even integer. Under this colouring, v_0 is adjacent to exactly one vertex with colour 1, namely, v_{t+1}. For the impropriety of $v_{x(t,m)+1}$, we have to check the parity cases for t. If t is even, then $\frac{x(t,m)}{t}$ is odd and the only neighbour of $v_{x(t,m)+1}$ with colour 1 is $v_{x(t,m)-t}$; if t is odd, then $\frac{x(t,m)-1}{t}$ is odd and the only neighbour of $v_{x(t,m)+1}$ with colour 1 is $v_{x(t,m)}$. In either case, $v_{x(t,m)+1}$ has impropriety one.

For property (vi), we describe the embedding of $K_{t,2}$ in the next section. $\qquad\square$

Definition A.18. *A $(t,2)$-chain of order (m,n), denoted $K_{t,2}^{(m,n)}$, consists of the concatenation of a $(t,2)$-wire of order j $(B_1 B_2 \cdots B_j)$ with a single $(t,2)$-link of order n (K_1) then with another $(t,2)$-wire of order $m-j$ $(B_{j+1}B_{j+2}\cdots B_m)$ for some $j \in \{2, \ldots, m-1\}$. The extreme vertices, v_0 of B_1 and v_{2t+2} of B_m, are called output vertices.*

A $(2,2)$-chain of order $(2,1)$ is shown in Figure A.14. The following properties follow from Propositions A.11 and A.17.
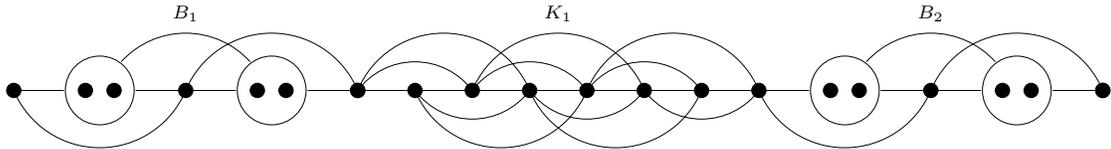
Figure A.14: The $(2,2)$-chain of order $(2,1)$ $K_{2,2}^{(2,1)}$.

Proposition A.19. *A $(t,2)$-chain of order (m,n) has the following properties:*

(i) *$K_{t,2}^{(m,n)}$ has $m(2t+2) + x(t,m) + 2$ vertices;*

(ii) *a $(t,2)$-chain is t-improperly 2-colourable, not t-improperly 1-colourable;*

(iii) *for any t-improper 2-colouring of $K_{t,2}^{(m,n)}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*

(iv) *there exists a t-improper 2-colouring of $K_{t,2}^{(m,n)}$ such that the output vertices receive different colours and both vertices have impropriety zero;*

(v) *there exists a t-improper 2-colouring of $K_{t,2}^{(m,n)}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*

(vi) *a $(t,2)$-chain is a unit disk graph.*

Embedding of the unit disk graph

Given any planar graph G, we now show how to construct and embed a unit disk graph $\widehat{G}$ which is t-improperly 2-colourable if and only if G is t-improperly 2-colourable. In the unit disk representation that we describe, each of the open disks have unit diameter. First, we embed G in the plane in a suitable way. Then we construct $\widehat{G}$ so that the vertices and edges of the original graph are replaced by the auxiliary graphs described above. Because of the definition of the $(t,2)$-chain, there are naturally two different classes of unit disk embeddings depending on the parity of t. We only fully describe the case of even t since the other case is similar.

Let $G = (V,E)$ be a planar graph. As for Theorem 2.16, we generate a box-orthogonal embedding of G. Let us assume that no two edges meet at a point, i.e. no box is degenerate and no two edges meet at the corner of a box. (We can do this by expanding each box by distance $1/2$ in each of the four directions then doubling the scale of the grid).

Each vertex $v \in V$ is replaced by a box $Box(v)$, and we denote the $\deg(v)$ points of contact with edges by $M(v)$. We aim to embed a $(t,2)$-clone in the perimeter of $Box(v)$ so that its output vertices replace the vertices in $M(v)$. We can do this by starting at an arbitrary point of $M(v)$ and proceed in clockwise direction about the perimeter. We extend the $(t,2)$-clone with an embedding of a $(t,2)$-wire to the next grid point in the perimeter and continue until all members of $M(v)$ have been included. It only remains to describe the unit disk embedding of some $(t,2)$-wire between two adjacent grid points.

Each edge $e \in E$ is replaced by a line $A(e)$ consisting of alternate horizontal and vertical line segments of the grid. It follows that $A(e)$ has integer grid length. We aim to embed a $(t,2)$-chain along $A(e)$. Since we use $(t,2)$-wires to extend a $(t,2)$-chain to arbitrary length,

it suffices to describe the unit disk embedding of some $(t, 2)$-chain between two adjacent grid points.

We first describe unit disk embeddings for the elementary auxiliary graphs: the $(t, 2)$-bonds and $(t, 2)$-links.

We denote the embedding of a $(t, 2)$-link of order m by E_K^m. Each centre of the disk replacing a vertex of $K_{t,2}^m$ lies on a line. The points are distributed equidistant from each other. Let the distance between adjacent vertices v_i and v_{i+1} be $d = \frac{mt}{mt(t+1)+1}$. Since $(t+2)^{-1} \leq d < (t+1)^{-1}$, v_i is adjacent to v_j if and only if $|i-j| \leq t+1$. Also, one can check that the distance in E_K^m between output vertices is precisely mt. See Figure A.15.
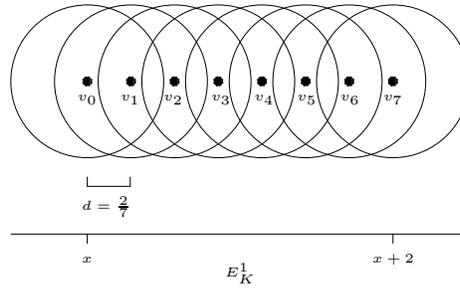

Figure A.15: An embedding of the $(2, 2)$-link of order 1.

We use two different embeddings for the $(t, 2)$-bonds. In the first embedding, denoted by E_B^a, the disks for the output vertices of $B_{t,2}$ are touching (but not intersecting) and, hence, the distance between the output vertices is 1. The first embedding is illustrated in Figure A.16(a). Note that the two bold disks represent cliques of size t. In the second embedding, denoted by E_B^b, all of the disks lie on a line. The output vertices of $B_{t,2}$ are at distance $2 - 2d'$, where $d' = \frac{1}{t+3}$ and the central vertex v_{t+1} of $B_{t,2}$ is midway between them. The centres of the two t-clique disks are at distance $1 - \frac{d+d'}{2}$ from the nearer respective output vertices. See Figure A.16(b).

E_B^a can be concatenated with itself, as can E_B^b. See Figure A.17. Also, E_B^b can be concatenated with E_K^m and with E_B^a. See Figure A.18.

We use these constructions to show that there are embeddings of some $(t, 2)$-wire and of some $(t, 2)$-chain between two adjacent grid points. We first scale the grid so that two adjacent grid points are distance $u = 5t + 8$ apart. We embed a $(t, 2)$-wire W^* of order

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(a) E_B^a

(b) E_B^b

Figure A.16: Two embeddings of a $(t, 2)$-bond: (a) E_B^a and (b) E_B^b.

(a) $E_B^a E_B^a$

(b) $E_B^b E_B^b$

Figure A.17: The concatenation of (a) two copies of E_B^a and of (b) two copies of E_B^b.

(a) $E_B^b E_K^1$

(b) $E_B^b E_B^a$

Figure A.18: The concatenation of (a) E_B^b with E_K^1 and of (b) E_B^b with E_B^a.

$3t + 6$ by concatenating $t + 3$ copies of E_B^b with t copies of E_B^a with $t + 3$ more copies of E_B^b. This embedding has length $(t+3)(2-2d') + t + (t+3)(2-2d') = 5t+8$, as required. We embed a $(t, 2)$-chain K^* of order $2(t+3)+1$ by concatenating $t + 3$ copies of E_B^b with E_K^1 with $t + 3$ more copies of E_B^b. This embedding has also has length $5t + 8$, as required.
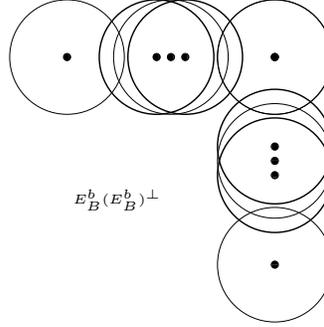

Figure A.19: The embedding of $W_{t,2}^2$ around a right-angle turn.

Since, in E_B^b, the distance between an output vertex and any other vertex is at least $\frac{1}{2}(\frac{2}{3} + \frac{3}{4}) > \frac{1}{\sqrt{2}}$, E_B^b can be concatenated with a perpendicular copy of itself (to bend around corners). See Figure A.19. Now, for each vertex v, we embed W^* between grid points along the perimeter of $Box(v)$ to obtain a $(t, 2)$-clone whose output vertices are precisely $M(v)$. Also, for each edge e we embed W^* between grid points along $A(e)$, except for one pair of grid points between which we embed a K^*, to obtain an embedding of a $(t, 2)$-chain along $A(e)$. The resulting graph is $\widehat{G}$.

We remark that for the case of odd t, we choose the values $d = \frac{mt+1}{(mt+1)(t+1)+1}$, $d' = \frac{1}{t+3}$ and $u = 5t + 9$.

Proof of Theorem 2.4

Let $G = (V, E)$ be a planar graph. One can verify that the construction of the corresponding unit disk graph $\widehat{G}$ and its embedding can be performed in polynomial time. It remains to show that G is t-improperly 2-colourable if and only if $\widehat{G}$ is. Each vertex $v \in V$ is replaced by a box, and then the points of contact with edges are denoted by $M(v)$. These points are then replaced by the output vertices of a $(t, 2)$-clone if $|M(v)| \geq 2$. The set of output vertices is denoted $I(v)$, where $I(v) = \{v\}$ if $|M(v)| = 1$.

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Let c be a t-improper 2-colouring of G. We want to construct a t-improper 2-colouring of $\widehat{G}$. First, for each vertex v of G, we colour the vertices of $I(v)$ by $c(v)$.

Second, for each edge $e = xy$ of G, let K_e be the $(t, 2)$-chain that connects $I(x)$ to $I(y)$ in $\widehat{G}$. If $c(x) \neq c(y)$, then we apply Proposition A.19(iv) to colour K_e. If $c(x) = c(y)$, then we apply Proposition A.19(v).

Last, for each vertex v of G, let C_v be the $(t, 2)$-clone whose output vertices are $I(v)$. Since c is a t-improper 2-colouring, we can apply Proposition A.15(v) to colour C_v. In this way, we obtain a t-improper 2-colouring of $\widehat{G}$.

Conversely, let $\hat{c}$ be a t-improper 2-colouring of $\widehat{G}$. We want to construct a t-improper 2-colouring of G. By Proposition A.15(iii), for any vertex v of G, we can assign the colour of the vertices of $I(v)$. The colouring c generated is a t-improper 2-colouring of G, for otherwise there is a vertex v with $t + 1$ neighbours $v_1, \ldots, v_{t+1}$ such that $c(v) = c(v_1) = \cdots = c(v_{t+1})$. Thus, $c(\{I(v)\}) = c(\{I(v_1)\}) = \cdots = c(\{I(v_{t+1})\})$ and, by Proposition A.19(iii), the sum of the external improprieties for the clone corresponding to v is at least $t + 1$. This violates Proposition A.15(iv). $\qquad\square$

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