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ESSAYS ON INFORMATION DISCLOSURE IN AUCTIONS AND MONOPOLY PRICING

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I declare that this thesis represents my own work, except where otherwise specified, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any institution for a degree, diploma or other qualifications.

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ABSTRACT

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The existing literature on information disclosure commonly assumes full commitment to truthful disclosure and therefore revelations are always credible, which can be quite unrealistic in many circumstances. This thesis mainly contributes to the literature by studying information disclosure in the form of cheap-talk in auctions and monopoly pricing, which allows for mis-reporting and false disclosure.

The thesis is composed mainly of three research papers. The first paper (Chapter 2), also the major chapter of this thesis, investigates cheap-talk information disclosure in auctions, where bidders' preferences are horizontally differentiated. The seller may reveal information of product attributes before the auction, and the disclosure policy is characterized by a partition of the attribute space. In a symmetric setting, I first show that informative equilibria reveal the feature that more precise information is provided for less popular product attributes. Second, I prove an equilibrium existence theorem that an informative equilibrium can be supported by an information partition of any degree, as long as the number of bidders is sufficiently large. And finally, the optimal disclosure policy reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. In this chapter, optimal information structure is endogenously determined.

In the second paper (Chapter 3), I turn to study how a monopoly seller should reveal a product's horizontal attributes, when consumer preferences conform to a mixture distribution. I show that the optimal disclosure policy largely depends on the characteristics of the mixture distribution. Specifically, when consumer preferences are highly heterogenous, it is better for the seller to reveal information and serve different groups of consumers separately. And when the preference distribution becomes more asymmetric, cheap-talk disclosure is more likely to be dominated by no disclosure at all. In this chapter, information structure is taken as given.

The third paper (Chapter 4) studies optimal regulation of risk-averse producers, in a setting of complementary production with independent cost realization. The production can be organized in the form of either *component*, or *integrated* production. I show in this paper that the relative virtues of these two forms of production depend on the degree of risk aversion of the producers.

Dedicated to my parents

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CHAPTER 1

Introduction

In everyday of our lives, we are exposed to huge amounts of public information, such as product advertising by retailers, policy announcements by government, and financial disclosure of listed companies, and there has been a vast literature exploring these issues of information disclosure and advertising in economics and business. A typical assumption in the existing literature, however, is that agents always commit to truthful disclosure and false announcement is assumed impossible, and therefore revelations are always credible (Milgrom, 1982; Lewis and Sappington, 1994; Myatt and Johnson, 2006; Ganuza and Penalva, 2010).

Although this assumption may seem reasonable sometimes, it can be quite unrealistic in many circumstances, especially when the information is not verifiable, or costless, or there is no effective legal recourses for disclosure fraud. Actually, false announcement and disclosure fraud are rather pervasive in the real world. To investigate information disclosure in these cases where the credibility of information is a serious concern, we should certainly forgo this full commitment assumption.

This thesis contributes to the literature by studying information disclosure in the form of cheap-talk in auctions and monopoly pricing, which allows for the possibilities of mis-reporting and false announcement. Cheap-talk means that information is costless, non-verifiable and non-binding (Crawford and Sobel, 1982; Farrell and Rabin, 1996). Besides the clear meaning of costless and non-verifiable, *non-binding* implies that a seller is not committed to a disclosure rule, and can send any possible signals she likes. Therefore, under cheap-talk, mis-reporting is possible.

The thesis is composed mainly of two inter-related papers on cheap-talk disclosure, and a third paper on optimal regulation of risk-averse producers. In the first paper (Chapter 2), which is also the major chapter of this thesis, I investigate cheap-talk information disclosure in auctions where bidders' preferences are horizontally differentiated. In the auction, bidder's valuations depend on the matching between their preferences and product attributes. The seller first announces a disclosure policy, which is characterized by a partition of the attribute space, and then send a public message after observing the realization of product attributes. Bidders update their beliefs after the announcement, and thereafter an auction is conducted.

Under cheap-talk, an informative equilibrium requires that interim expected revenues for each possible message be equal to each other, otherwise the seller will deviate to report the message generating the highest expected revenue.

In our model, the interim expected revenue for a message is jointly determined by the message precision and the expected product attribute conditional on that message, which are respectively termed as the *precision effect* and *location effect* of a message in this chapter. The interim expected revenue increases with the precision of the message, and decreases as the conditional expected attribute moves towards extremes, when the product attributes become less popular. Furthermore, when the number of bidders increases, the location effects across different messages become more and more equalized.

I first provide an characterization of informative equilibria in a symmetric setting, which shows that equilibrium partitions reveal the feature that the seller provides more precise messages for less popular product attributes. The intuition is that, for less popular attributes, the interim expected revenue is lower *ceteris paribus*, and to compensate that revenue deficit, the seller needs to provide more precise messages so as to get the equilibrium conditions satisfied.

The second also the main result of this paper is an existence theorem of informative equilibria. I prove that, under cheap-talk information disclosure, an informative equilibrium can be supported by a partition of *any degree*,[1] as long as the number of bidders is sufficiently large. Moreover, when the number of bidders converges to infinity, equilibrium partitions all converge to equal partitions.

The intuition is that, as the number of bidders increases, the gaps between the location effects across different messages converge to zero, and it is easier for the seller to adjust the precision of messages and achieve equilibria in higher degree partitions. In the limit, when the number of bidders converges to infinity and all the product attributes are of the same popularity, any equal partition can constitute an informative equilibrium, and an equilibrium can be supported by an equal partition of any finite degree.

As the third result in this chapter, I show that the optimal disclosure policy reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. That is, when the number of bidders increases, it is better for the seller to construct an informative equilibrium using higher degree partitions, for the purpose of revenue maximization. The intuition behind this result is similar to that behind the existence result.

The major technical challenge in this chapter lies in the proof of the existence theorem. It requires us to show that, under certain conditions, there exists a partition such that *all* the interim expected revenues are equal. In proving this theorem, I make use of the Intermediate Value Theorem (IVT) defined on partial order sets (Guillerme, 1995), which imposes restrictions on the monotonicity properties of vector-valued functions. Compared with the common approach of Fixed Point Theorems, though the conditions for applying IVT on a partial order set are

[1]For interval partition, the degree of a partition is defined as the number of interior cutting points.

more restrictive, the results are also more informative, since it provides additional information about the locations of the informative equilibria.

In Chapter 3, I examine the question of how a seller should advertise a product's attributes, when consumer preferences conform to mixture distributions. Unlike Chapter 2, where information structures, characterized as partitions of product attribute space, are endogenously determined, in Chapter 3, I focus on disclosure policies in the form of binary messages.

Chapter 3, entitled "Information Disclosure in a Hotelling Model with Mixture Distribution", explores this question in a Hotelling model in a context of monopoly pricing, and, to my knowledge, it is the first paper studying a Hotelling model with mixedly distributed preferences. Compared with the typical, yet not very realistic, assumption that consumer preferences conform to either uniform or binary distribution, mixture distribution obviously better characterizes consumer preferences in reality.

I identify two basic functions of informative advertising in this model: first, inducing consumers to adjust their expectations, and second, reducing their risk in purchase decisions. In this paper, I first show that advertising increases trade possibilities, in the sense that it makes possible some trades which are initially impossible if there is no information disclosure. This is largely due to the risk-reduction effect mentioned above. Second, information disclosure under full commitment generates higher expected profit than corresponding cheap-talk disclosure. This is because, under cheap-talk, to make the revealed information to be credible, some additional conditions need to be satisfied such that the seller won't deviate to mis-reporting, which therefore reduces the expected profits.

Furthermore, I show that optimal disclosure policies largely depend on the characteristics of the mixture distribution of consumer preferences. First, when different groups of consumers' preferences become sufficiently heterogenous, it is better for the seller to reveal product attributes and serve different groups separately. Second, when the composition of different groups of consumers becomes more asymmetric, cheap-talk disclosure is more likely to be dominated by no disclosure at all, in terms of profit maximization.

Chapter 4 is an extra paper on optimal regulation of risk-averse producers. In this chapter, I study a setting of complementary production with independent cost realizations. A regulator can choose between two production arrangements: one is integrated production, where a single producer supplies both products, and the other is component production, where each product is separately supplied by a producer.

The existing literature shows that when producers are risk neutral, integrated production generates strictly higher revenue than component production, due to informational economy of scope under integrated production. However, this is not necessarily true in the case of risk-averse producers.

I show that, under component production, the regulator strictly prefers production by risk-averse producers rather than by risk-neutral ones. Furthermore, in the case of CRRA preferences, I prove that when the risk coefficient is higher than some threshold, component production dominates integrated production in terms of revenue maximization. This is because, when producers are risk-averse, they are willing to render some information rents for the reduction in production uncertainties associated with component production. As a result, the relative virtues of integrated and component production largely depend on the degree of risk-aversion of producers. This paper, entitled "Regulating Risk-Averse Producers: the Case of Complementary Products", is already published at *Economics Letters* in 2010.

CHAPTER 2

Cheap-Talk Information Disclosure in Auctions

2.1. Introduction

In the vast literature on information disclosure, surprisingly little attention has been paid to cheap-talk, where mis-reporting or lying is possible. A typical assumption in the literature is that agents fully commit to truthful disclosure, and therefore false announcement is impossible and revelations are always credible. Though this assumption might be reasonable in some cases, yet it is obviously unrealistic still in many other circumstances, especially when the information is non-verifiable, or costless, or there is no effective legal recourses to deter false announcement. To explore real disclosure issues in these circumstances where the credibility of information becomes a serious concern, we certainly need to forgo this full commitment assumption.[1]

This paper contributes to the literature by studying cheap-talk disclosure in an auction context where bidders' preferences are horizontally differentiated. Cheap-talk implies that information is costless, non-binding and non-verifiable (Farrell and Rabin, 1996). Besides the clear meaning of costless and non-verifiable information, *non-binding* implies that there is no commitment to truthful information disclosure, and a seller can send any possible signals she likes. Therefore, under cheap-talk, mis-reporting is possible. In our model, a seller reveals the information about product attributes to bidders, whose valuations depend on the matching between

[1]Without causing confusion, *full commitment* in this paper always means full commitment to truthful information disclosure. This is distinct from the full commitment assumption in mechanism design literature, where a principal fully commits to implement the announced mechanism after knowing agents' types.

the product attributes and their preferences.[2] In this paper, we will investigate the conditions for credible cheap-talk disclosure, prove a general result about the existence of informative equilibria, and characterize the optimal disclosure policies. We will also compare the cheap-talk results with those under full commitment to truthful disclosure.

Information disclosure in auctions itself is not a new topic in the literature, and the early classic work dates back to Milgrom and Weber (1982). They show that in interdependent value auctions with affiliated signals, revealing information to the bidders *on average* increases the expected auction revenues, and the best disclosure policy is full information disclosure. In their symmetric auction model, the public signal affects bidders' valuations in the same manner, and the seller pre-commits to a disclosure policy. Our model departs from theirs in three major aspects: first, in our model, the public signal on product attributes has differentiated impacts on bidders' valuations; second, under cheap-talk, the seller does not commit to truthfully implement the disclosure policy; and finally, in stead of interdependent value auctions, we study standard independent private value auctions in this paper.[3]

More recently, the literature focuses more on optimal information structure in disclosure games (Lewis and Sappington, 1994; Johnson and Myatt, 2006; Board, 2009; Ganuza and Penalva, 2010). These papers emphasize the result that more precise signals result in more dispersed posterior evaluations of consumers, which implicitly assumes the differentiation in consumers' tastes. However, since there are no

[2]A product's attributes can be divided into vertical and horizontal ones. Vertical attributes are the attributes that all the consumers share the same preference over, such as quality. Horizontal attributes, however, can not be ordered in the same way, and different consumers may have different preferences over them, such as colour, design, etc. So only over a product's horizontal attributes could consumers have heterogenous preferences. Without specification, we always refer to "attributes" as a product's horizontal attributes in this paper.

[3]Actually, independent private value auction can be seen as an extreme case of interdependent value auction with affiliated signals, as shown in Milgrom and Weber (1982). However, in this extreme case, public information disclosure won't affect the *ex ante* expected revenue of the auction in their model, since the signals are statistically independent.

explicit signals in their models, the differentiation in preferences is not endogenously modelled in their models. Instead, they simply assume the relationship between the informativeness of information structures and the distribution of posterior evaluations. In this paper, we explicitly model the matching between product attributes and consumers' preferences, and demonstrate the mechanism how a common signal affects bidders' posterior valuations in differentiated ways.

In our model, a seller sells a product to a number of *ex ante* homogenous and risk-neutral bidders using an auction mechanism. Bidders' preferences are distributed on a horizontal line, and their valuations depend on the matching between their preferences and product attributes. Before knowing the product attributes herself, the seller announces the disclosure policy which is characterized by a partition of the attribute space. She then sends a public signal after observing the realization of the attributes, upon which the bidders update their beliefs and form posterior valuations.[4] Thereafter the auction is conducted, and final payoffs are implemented.

In our setting, after receiving the public signal, the bidders are in a standard independent private value auction, where the one with the highest bid wins the auction, and the expected revenue is equal to the expected value of the second highest valuation (Milgrom, 2004, Chapter 3). Therefore, the effects of disclosure on the expected auction revenues depend on how it affects the second highest valuation. Intuitively, a credible signal may affect a bidder's valuation through two channels: one is to adjust his expectation on the location of the attributes, and the other is to reduce the variance of his posterior estimate. The expected value of the highest valuation is obviously increasing in the precision of the signals, but this is not necessarily true for the second highest valuation, which depends on the number of bidders in the auction.

[4] In this paper, we use "she" to denote the auctioneer, and "he" to denote a bidder.

Moreover, the credibility of information is another and more important concern in our model, as mis-reporting is possible under cheap-talk. And the major problem in this paper is to explore the conditions for credible information disclosure, and thus for the existence of informative equilibria.

In our model of cheap-talk disclosure, the equilibrium condition requires that the interim expected revenues be equal to each other, otherwise the seller will deviate and always send the signal that generates the highest level of expected revenues, as also shown in Crawford and Sobel (1982). And a direct implication of this is that full information disclosure is normally impossible in a cheap-talk equilibrium, as the seller is constrained by the equal revenue constraints in equilibria, and normally she is not able to send arbitrarily precise signals. This result is in sharp contrast to the extreme ones of either no or full information disclosure in the cases of information disclosure under full commitment (Milgrom and Weber, 1982; Lewis and Sappington, 1994; Johnson and Myatt, 2006; Board, 2009).

In our model, the interim expected revenue for a signal is determined by two things. One is the precision of the signal, measured by the length of corresponding subinterval in the partition. Intuitively, the interim expected revenue increases in the precision of the signal, and we term this as *precision effect* of a signal. The other is the expected location of the product attributes conditional on the signal. When it moves toward the extremes, the product attributes become less popular in our symmetric setting, and the interim expected revenue decrease *ceteris paribus.* We term this as *location effect* of a signal. An interesting property is that, when the number of bidders increases, the difference in popularity across different product attributes gradually converges to zero.

Our first result in this paper is regarding an interesting characterization of the informative equilibria in a symmetric setting. We show that under cheap-talk, the equilibrium partitions are asymmetric with the feature that for less popular products, the seller provides more precise signals. The intuition is that, for less popular product attributes, fewer bidders are attracted and willing to pay high prices for it, and therefore the expected value of the second highest valuation is lower *ceteris paribus*, if compared with the more popular ones. To compensate for the reduced valuations, the seller needs to provide more precise information for the less popular products, so as to increase the interested bidders' valuations and get the non-deviation condition satisfied.[5]

The second also the major result of this paper is an existence theorem of informative equilibria. We prove that, under cheap-talk information disclosure, an informative equilibrium can be supported by a partition of *any degree*, as long as the number of bidders is sufficiently large.[6] On the other hand, for given number of bidders, if a $J-$degree partition supports an informative equilibrium, then there also exists informative equilibria supportable by a lower degree partition. Furthermore, we also show a convergence property in our symmetric setting, that is, when the number of bidders converges to infinity, equilibrium partitions all converge to equal partitions.

The intuition is that, with increasing number of bidders, the difference in popularity across different product attributes converges to zero, and the differences in interim expected revenues are increasingly determined by the precision effects of the

[5]An interesting observation relating to this result is the product catalog on online stores. For example, the online bookstore of Amazon provides a catalog for 31 categories of books, varying from the popular ones such as *Travel&Holiday* to the more specialized ones such as *Science&Nature.* Under *Travel&Holiday*, there are 7 sub-categories, and under *Science&Nature,* there are 31 sub-categories. And for the more specified categories, Amazon provides more detailed classifications. For instance, under *Chemistry*, there are 18 sub-categories varying from *Analytical Chemistry* to *Surface Chemistry*, and under *Physics*, there are even more, 29 sub-categories varying from *Applied Physics* to *Thermodynamics.* Interestingly, for the more popular categories, it seems that less detailed information is provided at Amazon. For instance, under *Travel Writing*, there is no further sub-categories provided, though there are nearly 37,000 results under it!

[6]In our case of interval partition, the degree of a partition is defined as the number of interior cutting points.

signals. As the number of bidders increases, the location effects of different signals become more and more equalized, and it becomes easier for the seller to adjust the precision of signals and construct equilibria using higher degree partitions. In the limit, when the number of bidders converge to infinity and all the product attributes are of the same level of popularity, any given equal partition can constitute an informative equilibrium, and an equilibrium can be supported by an equal partition of any finite degree.

As the third result, we show that the optimal disclosure policy reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. That is, when the number of bidders increases, it is better for the seller to construct an equilibrium using higher degree partitions, for the purpose of revenue maximization. The intuition behind this result is similar to that behind the existence theorem. When the number of bidders increase, the difference in popularity across different product attributes vanishes, and it is easier to construct informative equilibria using more precise signals, which in turn increase the valuations of the bidders and thus the expected revenue of the auction. This result of complementarity between the number of bidders and the precision of signals is also reported in other researches (Ganuza, 2004; Board, 2009; Ganuza and Penalva, 2010), which study information disclosure in the cases of full commitment.[7]

We then proceed to study the case of information disclosure under full commitment. Our model differs from other models under full commitment assumptions in at least two aspects: first, we explicitly model the horizontal differentiation in bidders' preferences, and second, the disclosure policy in our model is characterized by a partition of the state space of the product attributes. Under full commitment,

[7] Actually, Ganuza (2004), Ganuza and Penalva (2010) studies this in an auction context with costly information disclosure, and thus there is the trade-off between the cost and benefit of information disclosure. If we assume costless information disclosure, we will reach the extreme result of either no or full information disclosure. And Board (2009) considers just the two extreme cases of either full or no information disclosure.

the equilibrium partitions show some distinct features from those under cheap-talk. In our symmetric setting, the equilibrium partitions bear the feature that a signal for the more popular attributes generates greater interim expected revenues than those of less popular attributes. In the limit when the number of bidders converges to infinity, the equilibrium partitions also converge to equal partitions, but from an opposite direction of the case of cheap-talk.

The remaining parts of this paper are organized as below: Section 2.2 is a short review of related literature; Section 2.3 is model set-up; Section 2.4 studies the properties of interim expected revenues and then provides a characterization of informative equilibria; Section 2.5 proves the existence theorem of informative equilibria, the main result of this paper. Section 2.6 shows the complementarity property of the optimal disclosure policies. In Section 2.7, we explore the case of information disclosure under full commitment, and compare its results with those of cheap-talk disclosure. Finally, Section 2.8 is a short conclusion.

2.2. Related Literature

There has been a strand of literature on information disclosure in auctions and monopoly pricing, and recently more attentions are paid to the problem of optimal information structure in information disclosure games. Normally, full commitment to truthful disclosure is a common assumption in most of the existing literature.

About information disclosure in auctions, in an early and classic paper, Milgrom and Weber (1982) show that, in an interdependent value auction with affiliated signals, revealing information to bidders can *on average* increase the expected revenue of the auction, and no reporting policy is better than the one of full information revelation. There are two major assumptions behind this full disclosure result: one is the positive affiliation between the seller and bidders' private signals, and the other is the full commitment assumption, by assuming that the seller commits to truthfully reveal the information.

In a recent paper, Board (2009) studies the effects of disclosing *independent* information to bidders in an private value auction, and finds that, under the assumption that there exists horizontal differentiation among the bidders, revealing information may increase auction revenues, if the number of bidders n is sufficiently large. Specifically, when $n = 3$, the seller is indifferent between revealing information or not, and when $n \geq 4$, it is better for the seller reveal all the information. Intuitively, this is because in a standard auction, the expected revenue is equal to the value of the second highest valuation, and when $n = 3$ it is just equal to the *ex ante* expected valuation, and therefore, information disclosure has no effects on the expected revenue of the auction. However, in this paper the author just compares the two extreme cases of no and full information disclosure, and furthermore, there is no explicit signal in his model.

Information disclosure is also explored in many other contexts, and a central question is to explore how much information to reveal to consumers for the purpose of profit maximization. Lewis and Sappington (1994) is an early paper for the case of monopoly pricing. In their model, a product's value can be either high or low. If there is no information disclosure, all the buyers have a common expectation of the product value and are willing to pay a medium price for it. Under information disclosure, however, some buyers will receive a high signal, while others receive a low one, and their ex post valuations of the product thus become more dispersed. Since the buyers receiving a high signal are willing to pay a high price for it, the seller can charge a high price and serve only parts of the buyers. Therefore, the seller faces the trade-off between a mass market with a medium price and a niche market with a high price, where only a portion of the buyers are served. They show that the expected profit is convex in the informativeness of advertising, and the optimal advertising policy is either no or full information disclosure.

Johnson and Myatt (2006) explore the same question as Lewis and Sappington (1994), yet in a more general framework. They explain how information disclosure can result in the rotation of demand curves in a market, and define the rotation order of the informativeness of information structures, which is based on the intuition that more informative signals result in more dispersed conditional distributions of consumers' posterior valuations. Also in a model of monopoly pricing, they show that the seller faces the same trade-off between niche and mass markets, and the expected profit is quasi-convex in the informativeness of the advertising information structure. So they also get the similar result that the seller prefers extreme advertising policies.

Recently, Ganuza and Penalva (2010) study a similar issue in the context of auction. They introduce two nested information order criteria (supermodular and integral precision), which is also based on the intuition that more informative signals lead to greater dispersion of conditional expectations. They then apply these new criteria in a model of private information provision, where the precision of the signals is known by all while each bidder receives an idiosyncratic signal from the seller. Using integral precision, the authors show two intuitive results that a more precise signal yields a more efficient allocation and the seller provides less than efficient level of information. Furthermore, when information is *costly*, under supermodular precision, they find there is a complementarity relationship between information and competition, and both social efficient and auction optimal levels of information precision increase with the number of bidders. However, the setting-up of their model of private information seems weird, since in reality it is rare to see a seller provides idiosyncratic signals to bidders, and furthermore, all the bidders know that the signals are of the same level of precision. Second, the complementarity result depends on the assumption of costly information, otherwise we return to the former result of extreme disclosure policies.

Moreover, in the papers studying information structure (Johnson and Myatt,2006; Ganuza and Penalva, 2010; Board, 2009), though the horizontal differentiation in consumers' preferences is commonly emphasized, yet it is not modelled in an explicit way. Instead, they simply assume that more informative signals result in more dispersed distributions of posterior evaluations. One exception is Ganuza (2004), which explicitly models the horizontal differentiation among bidders in a Hotelling model, and studies the impacts of information disclosure in an auction context. In his model, it is more costly to reveal more informative information, and he shows that the optimal amount of information increases with the number of bidders in the auction. In modeling horizontal differentiation, Ganuza (2004) adopts a Hotelling model on a circle, and makes the assumptions that both bidders' preferences and product attribute are uniformly distributed on the circle. This setting greatly simplifies the technical complexity, since the location of the product attribute is irrelevant, and what matters is only the informativeness of the signals. In this paper, we adopt a Hotelling model on an interval, which is more realistic since it allows the possibility of difference in popularity of the product attributes.[8]

[8] Sun (2008) and Chapter 3 of this thesis also study information disclosure in Hotelling models, but they pay more attention to the tradeoff between mass and niche markets, while not much to the information structure issues.

All the papers above study information disclosure or revelation under the assumption of full commitment[9].[10] With this assumption, if information is costless, we normally reach some extreme results whether in monopoly pricing or auctions. In this paper, in contrast, we study cheap-talk information disclosure in auctions, and explicitly model the horizontal differentiation of consumer preferences. We show in cheap-talk information disclosure, even with costless information, there exists a complementarity relationship between the number of bidders and the informativeness of signals, that is, it is better for the seller to provide more precise information as the number of bidders increases.

2.3. The Model

A seller sells a product to n *ex ante* homogenous bidders, indexed by $i = 1, 2, \cdots, n$, using a standard auction mechanism. Both the seller and the bidders are risk-neutral. The product for sale is characterized by its quality, V, and attribute, S. The value of V is commonly known by all the players, but the product attribute S is a random variable, whose realization, s, is privately observed only by the seller.

Bidders' valuations of the product depend on the matching between their preferences and the product attributes. Let θ_i be bidder i's ideal product attribute,

[9]Other researches include Bar-Issac et al (2009) and Hoffmann and Inderst (2009). Bar-issac et al (2009) study a specific setting where consumers are ex ante heterogenous and advertising is acquired at a cost, and show that *intermediate* advertising policy can be optimal, which is different from the extreme result of Lewis and Sappington (1994) and Johnson and Myatt (2006). Hoffmann and Inderst (2009), using the rotation order of information structure, study the problem of selling information to consumers before a selling mechanism, and show that the consumers with valuation equal to the price have the maximum incentives to buy information. Their result is similar to that of information acquisition in auctions literature, such as Shi (2009).

[10]On exception is Chakraborty and Harbaugh (2010), where they also study advertising and information disclosure in cheap-talk. But different from the model in this paper, where horizontal differentiation in preferences is emphasized, they study in the framework of multi-dimensional cheap talk, and show that informative disclosure is always achievable in their model.

which also represents his type, and the valuation function for bidder i is thus

$$v_i(s;\theta_i) = V - \tau\,(s-\theta_i)^2 \tag{2.1}$$

where τ is a coefficient measuring the degree of disutility of mis-matching. The seller's valuation of the product is zero, and she is a revenue maximizer.

2.3.1. Information Structure

The product attributes S conform to the cumulative distribution of $G(s)$, and the types of the bidders, θ_i's, are independent draws from an identical cumulative distribution of $F(\theta)$. S and θ_i's are also independent from each other. The distribution functions of $G(s)$ and $F(\theta)$ are defined on a common support of $I = [\underline{s}, \bar{s}]$, and both are common knowledge among the players. When there is no information disclosure by the seller, bidder i's *ex ante* expected valuation is therefore

$$v^i = \mathbb{E}_s v_i(s;\theta_i) = \int_I v_i(s;\theta_i)dG\,(s)$$

and he will bid according to the expected valuation of v^i.

Prior to the auction, the seller has the option of revealing public information about s to the bidders, according to some disclosure policy. In our model, a disclosure policy is characterized by a partition of the state space of the product attribute, which is an interval, I. We then define interval partition as below:

Definition 1. *A J-**partition** of the interval $I = [\underline{s}, \bar{s}]$, denoted as P_J, is a finite sequence of $\{p_1, p_2, \cdots, p_J\}$ in the following form*

$$P_J: \qquad \underline{s} = p_0 < p_1 < \cdots < p_{J-1} < p_J < p_{J+1} = \bar{s} \tag{2.2}$$

*and J is the **degree** of the partition.*

Therefore, an interval partition is represented by a sequence of cutting points of the interval. For an interval partition P_J, there are J interior cutting points which divide the interval into $J+1$ subintervals. We denote the space of all possible interval partitions of degree J as $\mathcal{P}_J$, and $P_J \in \mathcal{P}_J$. Let Δ_j denote the length of the subinterval $[p_j, p_{j+1}]$, and

$$\Delta_j = |p_{j+1} - p_j| \tag{2.3}$$

and then we define equal partition, which is a partition where all the subintervals are of the same length.

Definition 2. *An **equal** J**-partition**, denoted as $\underline{P}_J$, is a J-partition such that*

$$\Delta_j = \Delta_{j'} \textit{ for any } j,j' \in \{0, 1, 2, \cdots, J\}$$

It is standard to define information structures on state space partitions (Hirshleifer and Riley, 1994, Chapter 5; Mas-colell, Whinston and Green, 1995, Chapter 15). In our model, as is common in the cheap-talk literature, we also define the disclosure policies on information partitions. Specifically, in an information partition of P_J, for a realization of the product attribute $s \in [p_j, p_{j+1}]$, the seller will send a public message of m_j to the bidders, which is costless in our model. Let $D_{P_J}(s)$ denote the disclosure policy defined on the information partition of P_J, and M_J the corresponding message space such that $M_J = \{m_0, m_1, \cdots, m_J\}$, and for any $j \neq j'$, $m_j \neq m_{j'}$.

Definition 3. *A **disclosure policy** under an information partition of P_J is a mapping from the state space I to the message space M_J, that is, $D_{P_J}(s) : I \longrightarrow M_J$. Specifically,*

$$D_{P_J}(s) = m_j \quad \textit{iff } s \in [p_j, p_{j+1}) \tag{2.4}$$

Trivially, for $s \in [p_J, p_{J+1})$, a message m_J is reported. It is worth attention that all the messages are different from each other, and thus it is not possible that for realizations on two dis-connected subintervals, a common message is reported. In other words, the partition is connected[11]. We also define the precision of a message under the intuition that a shorter subinterval corresponds to a more precise message.

Definition 4. *For two messages m_j and $m_{j'}$, m_j reveals* ***more precise*** *information than $m_{j'}$ iff*

$$\Delta_j \leq \Delta_{j'}$$

The seller will not necessarily commit to truthfully follow the announced disclosure policy, under cheap-talk. However, if the message is credible, upon receiving a message m_j and with a knowledge of the disclosure policy $D_{P_J}(s)$, a bidder will update his belief and form a posterior estimate of his valuation, which is

$$v^i(m_j) = \mathbb{E}_{s|m_j} v_i(s; \theta_i) = \int_I v_i(s; \theta_i) dG(s|m_j) \tag{2.5}$$

The value of $v^i(m_j)$ depends both on the realization of the product attributes s, and on the information partition of P_J.

2.3.2. Timing

In our model, the seller first announces the disclosure policy of $D_{P_J}(s)$ before she knows the realization of the product attribute S. Then nature selects the realization of S and bidders' types, θ_i's, which are respectively observed by the seller and corresponding bidders. Observing the realization $s \in [p_J, p_{J+1})$, the seller sends a public message of m_j to all the bidders, according to the disclosure policy of $D_{P_J}(s)$. Then, receiving the public message, if it is credible, the bidders update

[11]Indeed, it is easy to show that non-connected partitions can not support optimal informative equilibria in our model.

their beliefs and form posterior estimates of their valuations of the product, based on the knowledge of m_j and P_J. Finally, a standard auction is conducted, and final payoffs are implemented. The time line for this game is shown in Figure 2.1 below.

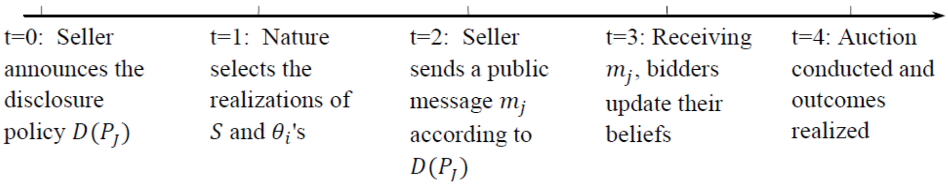


Figure 2.1: Timeline of the Game

2.3.3. Seller's Problem

It is noteworthy that a disclosure policy of $D_{P_J}(s)$ implies $J+1$ messages m_j, $j = 0, 1, \cdots, J$, and each message induces a conditional distribution of a bidder's posterior expectations of his valuation. In other words, an information partition of P_J generates a set of $J+1$ posteriors: one for each possible message.

For a message m_j under the partition of P_J, we let $\mathbb{E}R(m_j)$ denote the corresponding interim expected revenue of the auction. And the seller's problem is to determine the optimal partition that maximizes the *ex ante* expected revenue of the auction. The optimal partition depends both on the degree of the partition, J, and the positions of the interior cutting points, $(p_1, p_2, \cdots, p_J)$.

We will study two distinct cases: one is cheap-talk, and the other is information disclosure under full commitment. In the case of full commitment, the seller will truthfully implement the announced disclosure policy, $D_{P_J}(s)$, and the her problem is thus to

$$\max_{P_J \in \mathcal{P}_J} \quad \mathbb{E}R = \textstyle\sum_{j=0}^{J} \Pr(m_j)\, \mathbb{E}R(m_j) \tag{2.6}$$

For cheap-talk information disclosure, the seller does not commit to truthfully implement the disclosure policy. In this case, if an informative equilibrium is supported by a disclosure policy, $D_{P_J}(s)$, it is necessary that at any realization of the product attribute s, truthful disclosure generates higher revenue than misreporting, that is ,

$$\mathbb{E}R(m_j|\, s) \geq \mathbb{E}R(m_{j'}|\, s) \quad \text{for } \forall s \in [p_j, p_{j+1}) \text{ and } j' \neq j$$

We will later show that the interim expected revenue $\mathbb{E}R(m_j|\, s)$ does not depend on s, and therefore in an informative equilibrium, it is necessary that

$$\mathbb{E}R(m_j) = \mathbb{E}R(m_{j'}) \qquad \text{for any } j,\, j' \tag{2.7}$$

otherwise, the seller has incentive to misreport the message generating the highest level of expected revenue. The non-deviation condition of (3.7) implies that, in an informative equilibrium, the *ex ante* expected revenue is equal to any of the interim expected revenues, that is, $\mathbb{E}R = \mathbb{E}R(m_j)$ for any j. The seller's problem is thus to

$$\begin{aligned} \max\nolimits_{P_J \in \mathcal{P}_J} \quad & \mathbb{E}R = \mathbb{E}R(m_j) \\ \text{s.t.} \qquad & \mathbb{E}R(m_j) = \mathbb{E}R(m_{j'}) \qquad \text{for any } j,\, j' \end{aligned} \tag{2.8}$$

To solve problem (4.8), it is helpful for us to proceed in two steps. The first is to investigate the outcomes at an arbitrary message of m_j. For example, how does the interim expected revenue $\mathbb{E}R(m_j)$ depend on the cutting points of (p_j, p_{j+1}) and n, the number of the bidders? Second, with these results, we then explore the properties of equilibrium partitions under cheap-talk. The solution of optimal equilibrium partition is determined by the partition degree, J, and the positions of the interior cutting points, $(p_1, p_2, \cdots, p_J)$.

2.4. Informative Equilibrium: Characterizations

To focus our attention on the properties of the informative equilibria themselves, we will make some simplification and assume that both $G(s)$ and $F(\theta)$ are uniform distributions on a common support of $[-1, 1]$. And the *ex ante* expected location of the product attributes is therefore 0.

In this section, we first provide some characterizations of the informative equilibria under this symmetric setting, and in the following sections, we will investigate the conditions for equilibrium existence and the seller's optimal disclosure policy.

As for the main results in this sections, we first show that equilibrium partitions are symmetric to 0, with the feature that more precise messages are provided for less popular product attributes. Second, in the limit when the number of bidders converges to infinity, equilibrium partitions all converge to equal partitions. The characterizations of equilibrium partitions are directly related to the non-deviation condition of (3.7) as well as the properties of the interim expected revenues, $\mathbb{E}R(m_j)$'s.

2.4.1. Interim Expected Revenue

For an arbitrary partition of P_J, a message m_j is reported if the realization of the product attributes, $s \in [p_j, p_{j+1})$, according to (3.4).[12] We define $\mu_j = \mathbb{E}\left(S|m_j\right) = \frac{1}{2}\left(p_j + p_{j+1}\right)$, which is the expected location of product attributes conditional on m_j. In our symmetric setting, we assume $\mu_j \geq 0$ without loss of generality. And therefore p_{j+1} is positive, though p_j can be negative in some cases. Obviously, a subinterval with a message of m_j can be equivalently represented either by (p_j, p_{j+1}),

[12]It is noteworthy that the benchmark case of no information disclosure just corresponds to the case of P_0, where there is no interior cutting point and the only subinterval is just the entire interval as a whole.

the two cutting points, or by $\left(\mu_j, \Delta_j\right)$, the conditional expected location and the length of the subinterval.

When receiving a credible message of m_j under the disclosure policy of $D_{P_J}(s)$, bidder i will update his belief. It is easy to show that the bidder's posterior evaluation of the product when receiving the message of m_j is

$$v^i(m_j) = \int_I v_i(s;\theta_i)dG(s\,|m_j) = V - \tau\left[\frac{\Delta_j^2}{12} + \left(\theta_i - \mu_j\right)^2\right] \tag{2.9}$$

And $v^i(m_j)$, the posterior evaluation of bidder i, depends both on the precision of the message, Δ_j^2, which is common to all the bidders, and on the matching between his preference and the expected attribute, measured by $\left(\theta_i - \mu_j\right)^2$, which is however type specific.

For convenience in exposition, we introduce a new random variable here,

$$\beta_i\left(\mu_j\right) = \left(\theta_i - \mu_j\right)^2 \tag{2.10}$$

which measures the distance between bidder i's ideal product attribute, θ_i, and the conditional expectation of the product attributes, μ_j. For all the possible realization of θ_i, we have $\beta_i \in \left[0, \bar{\beta}\right]$, where the upper bound $\bar{\beta} = \left(1 + \mu_j\right)^2$. The cumulative distribution of β_i, denoted as $H(x)$, is

$$H(x) = \Pr(\beta_i \leq x) = \begin{cases} \sqrt{x} & x \in [0, \beta^o] \\ \frac{1}{2}\left(\sqrt{\beta^o} + \sqrt{x}\right) & x \in \left(\beta^o, \bar{\beta}\right] \end{cases} \tag{2.11}$$

where $\beta^o = \left(1 - \mu_j\right)^2$ and $H(x)$ is a continuous function[13].

[13]Without causing confusion, we abbreviate the expression of $\beta_i\left(\mu_j\right)$ to β_i, for the purpose of economy in expressions.

For a given message of m_j, β_i's are n independent draws from the same distribution of $H(x)$. Let $\beta_n^{(1)}$, $\beta_n^{(2)}$, $\cdots$, $\beta_n^{(n)}$ be a rearrangement of these such that

$$\beta_n^{(1)} \leq \beta_n^{(2)} \leq \cdots \leq \beta_n^{(n)}$$

and $\beta_n^{(k)}$, $k = 1, 2, \cdots, n$, are the **order statistics** of β_i's when the number of bidders is n. Specifically, $\beta_n^{(k)}$ is the kth smallest of the β_i's.

In a standard auction, the bidder with the highest valuation wins the auction, which corresponds to $\beta_n^{(1)}$. Naturally, under a message of m_j, the bidder whose type θ_i is nearest to the conditional expected location, μ_j, wins the auction, who has the highest posterior valuation.

Furthermore, the expected revenue of the auction is equal to the expected valuation of the bidder with the second-smallest β_i, that is, $\beta_n^{(2)}$. The cumulative distribution function of $\beta_n^{(2)}$ is denoted as $H_n^{(2)}(x)$, and it can be derived from the following property of the order statistics of $\beta_n^{(2)}$

$$\Pr\left(\beta_n^{(2)} > x\right) = \Pr\left(\beta_i > x\right)^n + n \Pr\left(\beta_i > x\right)^{n-1} \Pr\left(\beta_i \leq x\right) \tag{2.12}$$

$H_n^{(2)}(x)$ is also a continuous function on the subintervals of $[0, \beta^o]$ and $\left(\beta^o, \bar{\beta}\right]$. Given the accumulative distribution of $\beta_n^{(2)}$, the expected value of $\beta_n^{(2)}$ for a message of m_j is

$$\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] = \int_0^{\beta^o} x dH_n^{(2)}(x) + \int_{\beta^o}^{\bar{\beta}} x dH_n^{(2)}(x)$$

which is a function of just μ_j and n. From (3.8), the interim expected revenue of the auction under the message m_j is

$$\mathbb{E}R\left(m_j\right) = V - \tau\left[\frac{\Delta_j^2}{12} + \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]\right] \tag{2.13}$$

An interesting observation is that, in (3.11), the effects of a message m_j on the interim expected revenue of the auction can be divided into two parts. One is the precision of the information revealed by the message, which is measured by Δ_j^2. The more precise the information is, the greater the value of the interim expected revenue. And we term this effect as **precision effect** of a message. The other is the expected location of the product attributes, as measured by $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$. It is a function of μ_j, the expected location of the product attributes conditional on m_j, and n, the number of bidders in the auction, and we term this as **location effect** of a message. The following proposition demonstrates some properties of the location effect.

Lemma 5. *Under a message m_j for $s \in [p_j, p_{j+1})$, in the setting of uniform distribution, $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is strictly decreasing in n, strictly increasing in μ_j, and its cross derivative is negative, that is,*

$$\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial n} < 0 \qquad \frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j} > 0 \qquad \frac{\partial^2\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j \partial n} < 0 \tag{2.14}$$

Furthermore, when n converges to infinity,

$$\lim_{n \longrightarrow \infty} \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] = 0 \quad \lim_{n \longrightarrow \infty} \frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j} = 0 \tag{2.15}$$

Proof. All the omitted proofs in this Section are relegated to Appendix 2.A. □

$\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ decreases strictly in n is obvious, since with increasing number of bidders in the auction, the type space of the bidders becomes denser, and therefore, the expected value of $\beta_n^{(2)}$, the second shortest distance between the types and the expected attribute becomes smaller. In the limit, that value converges to zero, and bidder's types can be infinitely close to the expected attribute.

Second, when the value of μ_j becomes closer to the extreme points, say $\mu_j \longrightarrow 1$, the value of $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ increases. This is because, the product attributes at the center are more popular than those at the extremes, and for the less popular attributes, less bidders are attracted and the value of $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ becomes therefore larger. For example, when the expected attribute is at the center, the demand for that attribute comes symmetrically from both sides, while when it is at the extreme points, the demand comes just from one side and thus the demand is smaller. Third, the convergence property shows that, when the number of bidders converges to infinity, the location effects across different μ_j's becomes negligible.

Furthermore, the cross derivative of $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ with respect to n and μ_j is negative, which shows that as the number of bidders increases, the slope of $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ with respect to μ_j decreases. The properties of $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ are shown in the following diagrams.

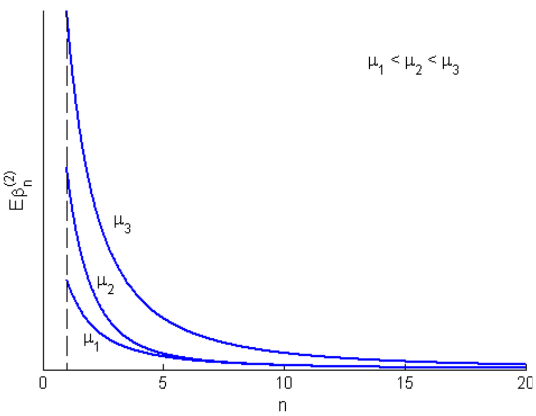


Figure 2.2.1. $\mathbb{E}\beta_n^{(2)}$ vs. n

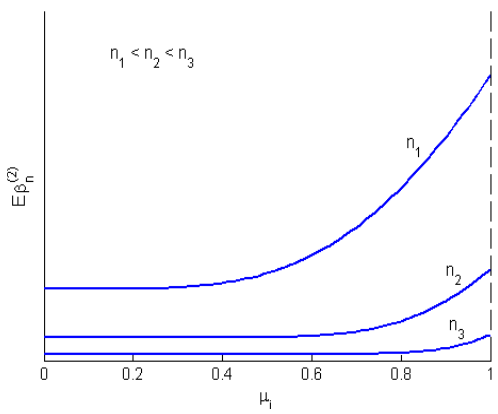


Figure 2.2.2. $\mathbb{E}\beta_n^{(2)}$ vs. μ_j

The result below shows that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is convex in μ_j, and uniformly converges to 0 when the number of bidders converges to infinity. The convexity property implies that its value increases at an accelerating rate when μ_j moves towards the extremes. And from the property of uniform convergence, we know for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that for all $n \geq N(\epsilon)$

$$\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] < \epsilon \text{ for any } \mu_j$$

which is equivalent to $\mathbb{E}\left[\beta_n^{(2)}(1)\right] < \epsilon$ given that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is strictly increasing. Uniform convergence is a desirable property of the location effect, and it is important for our proof of the existence theorem for informative equilibria in the next section. These results are formally summarized in the corollary below:

Lemma 6. $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$*: (1) is strictly convex in* μ_j*, and (2) uniformly converges to* 0*.*

Lemma 5 and Lemma 6 provide some characterizations of the location effects of $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$, and Figure 2.2 above illustrates how $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ changes with μ_j across different n's. We see that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is convex in μ_j, and when the number of bidders increases, its value converges to zero and the location effects across different μ_j's are also become more and more equalized.

As mentioned, the interim expected revenue, $\mathbb{E}R(m_j)$, can be seen as function of either $\left(\Delta_j, \mu_j\right)$ or (p_j, p_{j+1}). In the first case, it is obvious that $\mathbb{E}R(m_j)$ is increasing in the precision of the message, Δ_j, and decreasing in μ_j, the expected location of the product attributes, from (3.12). And we have the following result of the interim expected revenue.

Proposition 7. *The interim expected revenue, $\mathbb{E}R(m_j)$, is decreasing in both Δ_j and μ_j*

$$\frac{\partial \mathbb{E}R(m_j)}{\partial \Delta_j} < 0 \qquad \frac{\partial \mathbb{E}R(m_j)}{\partial \mu_j} < 0 \tag{2.16}$$

Therefore, for a given message m_j, if the conditional expected value of μ_j remains unchanged, we can increases the value of the interim expected revenue by increasing the precision of the message m_j. On the other hand, if the precision of the message is given, when the expected product attribute moves towards the center, the value of the interim expected revenue also increases.

If $\mathbb{E}R(m_j)$ is represented as function of (p_j, p_{j+1}), then for given p_j, it is obvious that $\mathbb{E}R(m_j)$ is decreasing in p_{j+1}, since both Δ_j and μ_j increases in this case. However, when p_{j+1} is given, increasing p_j can result in two opposite effects: on the one hand, the information revealed by the message becomes more precise, since Δ_j decreases; on the other hand, the expected product attribute becomes more closer to the extremes. These opposite effects make the effects of increasing p_j on the value of $\mathbb{E}R(m_j)$ ambiguous. However, we can still derive the following result

Corollary 8. *When $m = m_j$ for $s \in [p_j, p_{j+1})$*

$$\frac{\partial \mathbb{E}R(m_j)}{\partial p_{j+1}} < 0 \tag{2.17}$$

For $p_{j+1} > 0$, (1) for any n, there exist $\bar{p}_j \in (-p_{j+1}, p_{j+1})$ such that for any $p_j > \bar{p}_j$, $\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} < 0$; (2) for any n, there exist $\underline{p}_j \in (-p_{j+1}, p_{j+1})$ such that for any $p_j < \underline{p}_j$, $\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} > 0$.

Compared with the result of part (1), we see that the condition for the existence of $\underline{p}_j$ of part (2) is much more restrictive, since for an arbitrary $J-$partition, for all the subintervals but one we have p_j and p_{j+1} are of the same sign. As a result, for an arbitrary $J-$partition, we can always find a $\bar{p}_j$ for each interval such that

$\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} < 0$ for $p_j > \bar{p}_j$, but it is not necessary that we can also always find a $\underline{p}_j$ for each interval such that $\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} > 0$ for $p_j < \underline{p}_j$.

2.4.2. Properties of Informative Equilibria

In this subsection, we provide a simple yet interesting characterization of informative equilibria, which is based on the non-deviation condition in (3.7) for informative equilibria and the properties of the interim expected revenue, $\mathbb{E}R\left(m_j\right)$, in Proposition 7. In the next section, we will prove an existence theorem for informative equilibria.

We first introduce some notations. Under the symmetric setting of our model, we will focus on the partitions that are symmetric to the center of 0. For a J-partition of the state space, J can be either an even or odd numbers. If $J = 2K+1$, $K \in \mathbb{Z}^+$, then the symmetric partition is

$$-1 = -p_{K+1} < -p_K < \cdots < -p_1 < 0 < p_1 < \cdots < p_K < p_{K+1} = 1 \tag{2.18}$$

and $\Delta_0 = |p_1|$. If $J = 2K$, then the symmetric partition is

$$-1 = -p_{K+1} < -p_K < \cdots < -p_1 < p_1 < \cdots < p_K < p_{K+1} = 1 \tag{2.19}$$

and we define $\Delta_0 = |2p_1|$. In both cases, when $j = 1, 2, \cdots, K$, $\Delta_j = |p_{j+1} - p_j|$, as defined before. We then define the popularity of a product attribute as below:

Definition 9. *For two product attributes, s and s', s is **more popular** than s', if at any positive price, the demand for s is greater than that for s'.*

Specifically, in our symmetric setting, the product attribute of 0 is most popular, since its demand is derived symmetrically from both sides. When s is approaching to the extremes of the interval, the attribute becomes less and less popular. Therefore, under our symmetric setting, the popularity of a product attribute, s, increases when it is closer to the central point of 0.

With these notations and the definition of popularity, a characterization of the informative equilibria can be summarized as below.

Proposition 10. *For given number of bidders n, in our setting of uniform distribution, if a $J-$partition P_J supports an informative equilibrium under cheap-talk, then it is necessary that*

$$\Delta_0 > \Delta_1 > \cdots > \Delta_K \tag{2.20}$$

$J = 2K$ or $2K+1$, $K \in \mathbb{Z}^+$. That is, in an informative equilibrium, more precise information is provided for the less popular product attributes.

Under the symmetric setting of our model, Proposition 10 says that if a partition P_J supports an informative equilibrium, it is necessary that the lengths of subintervals become shorter and shorter when they are moving from the center to the extremes. This result is implied by the non-deviation condition for informative equilibria and the properties of the location effect. The intuition is that, when the subintervals move to the extremes, μ_j increases and so does the value of $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$, and therefore the interim expected revenue decreases *ceteris paribus.* However, the non-deviation condition requires that all the interim expected revenues be equal to each other. So when μ_j is approaching to the extremes, to compensate the revenue deficit of $\mathbb{E}\left[R\left(m_j\right)\right]$, the seller needs to provide increasingly more precise information for the subintervals that are closer to the extremes.

Another direct implication is that full information is not possible under cheap-talk, if the number of bidders is finite. Full information disclosure implies that the seller will truthfully report each realization of the product attributes, implying that $\Delta_j = 0$ for any j. However, when n is given, $\mathbb{E}\left(\beta_n^{(2)}\right)$ is strictly increasing in μ_j.

Therefore, under full information disclosure, for any two realization $0 \leq s_1 < s_2$, $\mathbb{E}R(s_1) > \mathbb{E}R(s_2)$ necessarily and there is no way to get the non-deviation condition of (3.7) satisfied. This result is formally stated as below:

Corollary 11. *For given finite number of bidders, full information disclosure is not possible in an informative equilibrium under cheap-talk disclosure.*

The result below demonstrates a convergence property of the informative equilibria, which shows that when the number of bidders converges to infinity, all equilibrium partitions converge to equal partition. This is because, in the limit, the location effects of μ_j become fully equalized across different messages, and what matters for the interim expected revenue is only the precision effect of the messages. The non-deviation condition then implies that equilibrium partition will converge to equal partition in the limit, where all the messages are of equal precision of information.

Proposition 12. *When $n \longrightarrow \infty$, any informative equilibrium partition, P_J, converges to equal partition where*

$$\Delta_j = \Delta_{j'} \text{ for any } j,j' \in \{0,1,2,\cdots,K\}$$

Example 13. *We provide an numerical example when the partition degree $J = 2$. Under our symmetric setting, this implies that the partition is $-1 < -p_1 < p_1 < 1$, and when $p_1 = \frac{1}{3}$ the partition is an equal partition. First Proposition 10 implies that in any equilibrium partitions of degree 3, $p_1^E > \frac{1}{3}$ since it is necessary that $\Delta_0 > \Delta_1$. Second, from Proposition 12, when $n \longrightarrow \infty$, the equilibrium partition will converge to equal partition, that is*

$$\lim_{n \longrightarrow \infty} p_1^E = \frac{1}{3}$$

We set $V = 10$ and $\tau = 1$, the following diagram shows how the expected revenue in equilibrium and the cutting point p_1 changes with n, the number of bidders.

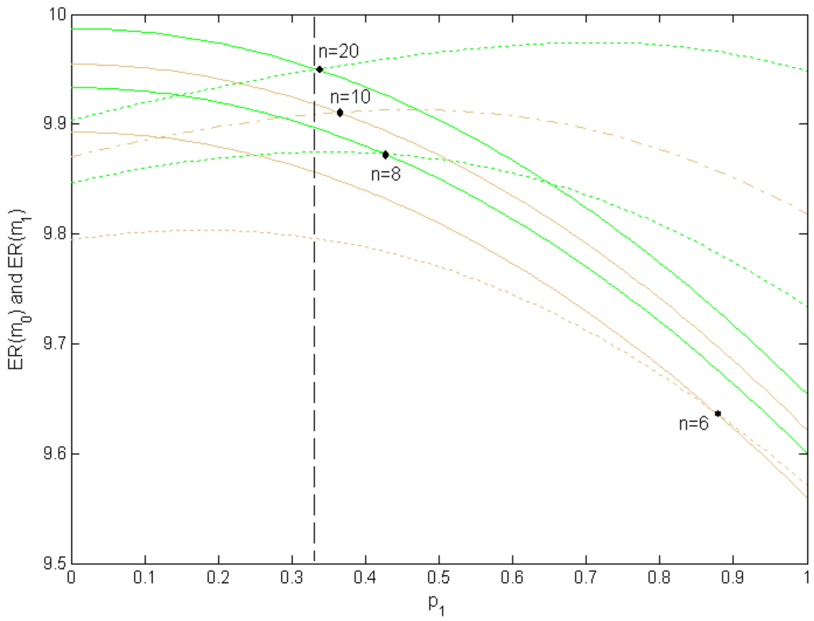


Figure 2.3. Equilibrium Partition and Convergence in P_2

In Figure 2.3, we provides an example for P_2. The solid lines shows how the interim expected revenue $\mathbb{E}\left[R\left(m_0\right)\right]$ changes with the cutting point p_1 for different n's. Similarly, the dashed lines shows the trends of the interim expected partition of $\mathbb{E}\left[R\left(m_1\right)\right]$. For a given n, when these two lines cross, the crossing point represents an informative equilibrium of this game, since $\mathbb{E}\left[R\left(m_0\right)\right] = \mathbb{E}\left[R\left(m_1\right)\right]$ at this point.

Figure 2.3 shows four crossing points of the interim expected revenue curves, corresponding to $n = 6, 8, 10$ and 20 respectively. An obvious observation is that in all the equilibria, $p_1^E > \frac{1}{3}$, which implies that $\Delta_0 > \Delta_1$ in equilibrium, and therefore more precise information is provided to the attributes that are closer to the extremes, as shown in Proposition 10. Second, when the number of bidders increases, the crossing point moves up-left, and becomes more and more close to the equal partition point of $p_1 = \frac{1}{3}$. This has two implications: first, when the number of bidders increases, $p_1 \longrightarrow \frac{1}{3}$ from the right hand side, and second, the equilibrium revenue increases with the number of bidders, for a given degree of equilibrium partitions.

2.5. Informative Equilibrium: An Existence Theorem

2.5.1. The Existence Theorem

The above section provides some interesting characterizations of the informative equilibria in our game of cheap-talk disclosure. However, the conditions for the existence of an informative equilibrium haven't been explored. In this section, we will prove an existence theorem, which demonstrates that an informative equilibrium can be supported by a partition of any degree, as long as the number of bidders is sufficiently large.

To prove the existence of an informative equilibrium, we need to show that, under certain conditions, there exists a partition such that all the interim expected revenues are equal to each other. The main difficulty lies in the fact that, for a general partition of P_J, there are $J+1$ subintervals and correspondingly the same number of interim expected revenues. Then how to construct a partition such that all the interim expected revenues are equal, and what are the necessary elements to make that partition possible?

To prove the existence result, we make use of the Intermediate Value Theorem defined on partial order sets, such as vector spaces (Guilerme, 1995). The main objective is to construct a vector-valued function in $\mathbb{R}^J$, whose element values correspond to the revenue difference between two consecutive subintervals, and then show that, under certain conditions, there exists a partition such that the value of the vector-valued function is $\mathbf{0}$.

The outline of the proof constitutes the following five steps: (1) to define a metric space of partitions, $(\mathcal{P}_J, d)$, and the partial order on it; (2) to show that it is convex and thus connected; (3) then to define a continuous vector valued function $\mathbf{f} : \mathcal{P}_J \longrightarrow \mathbb{R}^K$, and its jth element is $f_j(P_J) = \frac{1}{\tau}[\mathbb{E}R(m_j) - \mathbb{E}R(m_{j-1})]$, $j = 1, \cdots, K$; (4) to construct two special partitions, $\underline{P}_J$ and $\bar{P}_J$, and show that for n large enough, $\mathbf{f}(\underline{P}_J) < \mathbf{0} < \mathbf{f}(\bar{P}_J)$; (5) By applying the Intermediate Value Theorem on a poset, we then prove that for n large enough, there exists a partition $\hat{P}_J$, such that $\underline{P}_J < \hat{P}_J < \bar{P}_J$ and $\mathbf{f}\left(\hat{P}_J\right) = \mathbf{0}$.

In the general case, the set of all possible $J-$partitions is denoted as $\mathcal{P}_J$, and $P_J \in \mathcal{P}_J$ is a particular partition of degree J. Under our symmetric setting, we focus on the partitions symmetric to the center, and a partition P_J, $J = 2K$ or $2K + 1$, $K \in \mathbb{Z}^+$, can be represented by a sequence of K positive numbers, that is

$$P_J := \{p_1, p_2, \cdots, p_K\} \tag{2.21}$$

where $0 < p_1 < p_2 < \cdots < p_K < 1$, as shown in (3.18) and (2.22).

We define a metric on the partition space, $d : \mathcal{P}_J \times \mathcal{P}_J \longrightarrow \mathbb{R}_+$, in the usual way as in K-dimension Euclidean space, that is, for any $P_J, P_J' \in \mathcal{P}_J$, $d(P_J, P_J') = \left[\sum_{j=1}^K (p_j - p_j')^2\right]^{\frac{1}{2}}$, and the metric space of $(\mathcal{P}_J, d)$ is then defined. We second introduce a partial order, $\geq$, into the space $\mathcal{P}_J$ in the usual way as in $\mathbb{R}_+^K$, that is, for any $P_J, P_J' \in \mathcal{P}_J$, $P_J \geq P_J'$ iff $p_j \geq p_j'$ for all $j = 1, \cdots, K$, and thus we define the *poset* (partially ordered set) of $(\mathcal{P}_J; \geq)$.

The first result is to show that the partition space $\mathcal{P}_J$ is connected. Actually, we can get a stronger result that $\mathcal{P}_J$ is convex, since for any two partitions, $P_J, P_J \in \mathcal{P}_J$, and $\alpha \in [0, 1]$, $\alpha P_J + (1 - \alpha) P_J$ is also a $J-$partition, since

$(\alpha p_1 + (1-\alpha) p'_1, \cdots, \alpha p_K + (1-\alpha) p'_K)$ is also a sequence of K ordered positive numbers. We then get the following Lemma.

Lemma 14. *$\mathcal{P}_J \subset \mathbb{R}_+^K$ is convex, and thus connected.*

Knowing that $\mathcal{P}_J$ is connected, we then next construct two ordered special partitions with desirable properties, denoted as $\underline{P}_J$ and $\bar{P}_J$. We will show that $\bar{P}_J > \underline{P}_J$ according to the defined partial order on $\mathcal{P}_J$.

Actually, the partition of $\underline{P}_J$ is just the equal partition, as shown in Definition 2. In an equal partition, the length of all subintervals are equal, which is denoted as $\underline{\Delta}(J)$. It is easy to show that this common length of subintervals

$$\underline{\Delta}(J) = \frac{2}{J+1} = \begin{cases} \frac{2}{2K+1} & \text{if } J = 2K \\ \frac{1}{K+1} & \text{if } J = 2K+1 \end{cases} \tag{2.22}$$

and in the equal partition $\underline{P}_J$, the locations of the interior cutting points, denoted as $\underline{p}_j$'s, are

$$\underline{p}_j = \begin{cases} \left(j - \frac{1}{2}\right) \frac{2}{2K+1} & \text{if } J = 2K \\ \frac{j}{K+1} & \text{if } J = 2K+1 \end{cases} \qquad j = 1, \cdots, K \tag{2.23}$$

The other partition, denoted as $\bar{P}_J$, is the partition that satisfies

$$\Delta_j = \Delta_{j+1} + \delta(J) \text{ for } j = 0, 1, \cdots, K-1 \text{ and } \Delta_K = B \tag{2.24}$$

$\delta(J)$ is an positive incremental length when the subintervals are moving from the extremes to the center, and B is the length of the terminal subinterval. For this partition to be compatible and $\delta(J) > 0$, it is necessary that $B < \underline{\Delta}(J)$. It is easy to derive that the length of a subinterval in $\bar{P}_J$

$$\Delta_j = B + (K-j)\,\delta(J) \tag{2.25}$$

where the incremental term

$$\delta(J)=\begin{cases}\frac{2}{K^2}\left[1-\left(K+\frac{1}{2}\right)B\right]=\frac{4}{(J-1)^2}(2-BJ) & \text{if } J=2K\\ \frac{2}{K(K+1)}\left[1-(K+1)B\right]=\frac{4}{(J-1)^2-1}(2-BJ) & \text{if } J=2K+1\end{cases} \tag{2.26}$$

For partition $\bar{P}_J$, the locations of the interior partition points, denoted as $\bar{p}_j$'s, satisfy

$$\bar{p}_j=\begin{cases}\left(j-\frac{1}{2}\right)\left[B+\delta(J)\left(K-\frac{1}{2}j+\frac{1}{4}\right)\right]+\frac{1}{8}\delta(J) & \text{if } J=2K\\ j\left[B+\frac{1}{2}\delta(J)(2K-j+1)\right] & \text{if } J=2K+1\end{cases} \tag{2.27}$$

The following Lemma provides a result about the ordering between the two partitions, $\underline{P}_J$ and $\bar{P}_J$. And Figure 2.3 provides an illustrative example for the case of $K=2$ and $J=5$.

Lemma 15. $\bar{P}_J > \underline{P}_J$.

Proof. All the omitted proofs in this Section are relegated to Appendix 2.B. $\square$

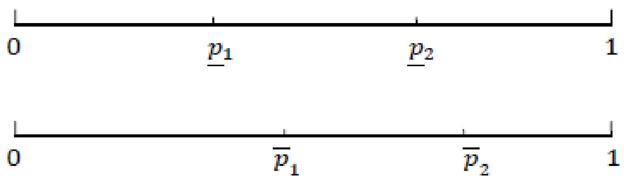


Figure 2.4. An Example of $\underline{P}_J$ (upper) and $\bar{P}_J$ (lower)

We then define a vector-valued function $\mathbf{f}:\mathcal{P}_J\longrightarrow\mathbb{R}^K$, which is a mapping from the partition space $\mathcal{P}_J$ to the $\mathbb{R}^K$ space. The function $\mathbf{f}(\cdot)=(f_1(\cdot),f_2(\cdot),\cdots,f_K(\cdot))$, and its jth element function is

$$f_j(P_J)=\frac{1}{12}\left[\Delta_{j-1}^2-\Delta_j^2\right]-\left[\mathbb{E}\left(\beta_n^{(2)}(\mu_j)\right)-\mathbb{E}\left(\beta_n^{(2)}(\mu_{j-1})\right)\right]$$

for $j = 1, 2, \cdots, K$. Obviously, for a partition P_J, $f_j(P_J) = \frac{1}{\tau}[\mathbb{E}R(m_j) - \mathbb{E}R(m_{j-1})]$ and $f_j(P_J) = 0$ implies that the two interim expected revenues are equal to each other. And a desirable property of the two ordered partitions, $\bar{P}_J$ and $\underline{P}_J$, is that

Lemma 16. $\mathbf{f}(\underline{P}_J) < \mathbf{0}$ *for any* n. *And there exists an* $N(\bar{P}_J)$ *such that for any* $n \geq N(\bar{P}_J)$,

$$\mathbf{f}(\bar{P}_J) > \mathbf{0}$$

Proof. see Appendix 2.B. □

So far we have constructed two partitions, $\underline{P}_J$ and $\bar{P}_J$, such that $\underline{P}_J < \bar{P}_J$, and define a vector valued function $\mathbf{f}(P_J)$ which corresponds to the differences in interim expected revenues across all the subintervals. We show that $\mathbf{f}(\underline{P}_J) < \mathbf{0}$, and for n large enough, $\mathbf{f}(\bar{P}_J) > \mathbf{0}$. To prove the existence of an informative equilibrium, we just need to show that under certain conditions, there exist a equilibrium partition $\hat{P}_J$, such that $\underline{P}_J < \hat{P}_J < \bar{P}_J$ and $\mathbf{f}(\hat{P}_J) = \mathbf{0}$.

Before turning to the proof of the existence theorem, we provide an addition result about a partition P_J between $\underline{P}_J$ and $\bar{P}_J$. We know that when J is given, $\bar{P}_J$ is fully determined by B, the length of the terminal subinterval. We can then show there exists a value of $\hat{B}$, the length of the terminal subinterval, such that in any $J-$partition P_J, such that $\underline{P}_J < P_J < \bar{P}_J$, the length of any subinterval, Δ_j, is greater than $\frac{1}{J}$.

Lemma 17. *There exists* $\hat{B} \in (0, \underline{\Delta}(J))$, *such that for a partition* P_J *satisfying* $\underline{P}_J < P_J < \bar{P}_J$

$$\min_{j \in \{0,1,\cdots,K-1\}} \Delta_j = \frac{1}{J}$$

With the above preparation, we now turn to the proof of the existence theorem. To prove this result, we make use of the Intermediate Value Theorem (IVT) for semi-continuous functions defined in product spaces (Guillerme,1995). The theorem is as below

Theorem 18. *(Intermediate Value Theorem, Guillerme (1995)) Let* $\mathbf{h} := (h_i)_{i \in I}$ *be a function from an interval of* $[\mathbf{u}, \mathbf{v}]$ *in* $\mathbb{R}^I$ *with* $\mathbf{u} \leq \mathbf{v}$*. Suppose that, for each* i *in* I *and each* $\mathbf{x}$ *in* $[\mathbf{u}, \mathbf{v}]$*, the following properties are fulfilled:*

(1) the function $h_i(\cdot, x_{-i})$ *is upper semi-continuous on the right on* $[u_i, v_i]$*;*

(2) the function $h_i(\cdot, x_{-i})$ *is lower semi-continuous on the left on* $[u_i, v_i]$*;*

(3) the function $h_i(x_i, \cdot)$ *is nonincreasing on* $[\mathbf{u}_{-i}, \mathbf{v}_{-i}]$*.*

Then the interval $[\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{v})]$ *is contained in the set* $\mathbf{h}([\mathbf{u}, \mathbf{v}])$*.*

Different from the IVT theorem for real value functions defined on an interval, the IVT theorem for vector valued functions defined on vector spaces not only require the continuity of the function, but impose monotonicity restrictions on its element real-value functions. In our model, these conditions are all satisfied, and we provide the following existence theorem result.

Theorem 19. ***(Existence Theorem)*** *For any* $J \in \mathbb{Z}^+$*, there exists a minimum number of bidders,* $N(J)$*, such that an informative equilibrium can be supported by a partition of* J *for any* $n \geq N(J)$*.*

Proof. see Appendix 2.B. □

The intuition is that, with increasing number of bidders, the difference in popularity across different product attributes gradually converge to zero, since $\mathbb{E}\left(\beta_n^{(2)}(\mu_j)\right)$ uniformly converges to zero as n increases. The interim expected revenues, $\mathbb{E}R(m_j)$'s, are increasingly determined by the precision of messages, Δ_j. Therefore, when the number of bidders increases, the *location effect* of μ_j gradually reduces, the difference in popularities across different attributes gradually vanish, and it becomes easier to adjust the precision of messages, Δ_j's, so as to get the non-deviation conditions satisfied. In the limit, when the number of bidders converge to infinity and all the product attributes are of the same popularity, any equal partition can constitute an informative equilibrium, and we can construct an equilibrium using equal partitions of any finite degree.

Corollary 20. *In a n bidder auction, if there exists an informative equilibrium at a partition of degree J, P_J, then for any $n' > n$, there is also an equilibrium supportable by a same degree partition, denoted as $\tilde{P}_J$. Moreover, $\tilde{P}_J < P_J$.*

Corollary 20 seems straightfoward. If there exists an informative equilibrium supported by a partition of P_J, then all the interim expected revenues are equal to each other. If we increase the number of bidders in this case, as the value of $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ decreases more for greater μ_j, then the interim expected revenues are no longer equal to each other. Corollary 20 just shows that, when the number of bidders increases, there also exists an equilibrium supportable by a same degree partition, and specifically the new equilibrium partition is closer to equal partition.

Remarks: (1) For any given number of bidders, there exists a maximum degree of partition, $\bar{J}(n)$, that can support an informative equilibrium; (2) Under cheap-talk, a seller's ability to reveal credible information is largely determined by the size of the market, and full information disclosure is not possible under cheap-talk, for any given n; (3) $\underline{P}_J < \hat{P}_J < \bar{P}_J$, also implies the characterization of the informative equilibria.

2.5.2. An Additional Result

In the above subsection, we have shown that for given J, there exists an informative equilibrium supportable by a $J-$partition as long as the number of bidders are sufficiently large.

Here we will show the result that, for given n, if an informative equilibrium is supportable by a partition of degree J, then informative equilibria can also be supported by a partition of any lower degree J', with $J' < J$. To prove this, we will first show it is true for $J' = J - 1$ and $J - 2$, and then as a result of mathematical induction, it is true for any $J' < J$.

For the cases of J being an odd ($J = 2K + 1$) or even number ($J = 2K$), the corresponding partitions are shown in (3.15) and (3.16). In both cases, as in the Existence Theorem, we use $\hat{P}_J = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_K)$ to represent the corresponding $J-$partition, that supports an informative equilibrium.

Corresponding to the given partition $\hat{P}_J$, we introduce the following special partition of degree $J - 2$, by removing the two cutting points of $\pm p_1$, which is denoted as below

$$\bar{P}_{J-2} = \left(p'_1, p'_2, \cdots, p'_{K-1}\right) = (\hat{p}_2, \hat{p}_3, \cdots, \hat{p}_K)$$

and as before, we use $\underline{P}_{J-2}$ to denote the corresponding equal partition of degree $J - 2$. It is easy to show that

$$\underline{P}_{J-2} < \bar{P}_{J-2}$$

Example 21. *For example, when* $J = 2K + 1$, $\underline{P}_{J-2} = \left(\frac{1}{K}, \frac{2}{K}, \cdots, \frac{K-1}{K}\right)$, *and because* $\bar{P}_J$ *supports an informative equilibrium, we have, for* $\bar{P}_{J-2}$

$$p'_k = \hat{p}_{k+1} > \frac{k+1}{K+1} \quad k \leq K - 1$$

as implied by the equilibrium property of (3.10). Therefore, $p'_k > \frac{k}{K}$ *and* $\underline{P}_{J-2} < \bar{P}_{J-2}$*. And we can easily show that this is also true for the case of* $J = 2K$*.*

For the case of partitions of degree $J-1$, we similarly introduce the follow specific partitions. For $J = 2K+1$, by removing the cutting point of $p_0 = 0$, we get the new partition of degree $2K$, denoted as $\bar{P}_{J-1} = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_K)$. For $J = 2K$, by removing the two cutting point of $\pm p_1$ and introducing the new cutting point of $p_0 = 0$, we get the new partition of degree $2K-1$, denoted as $\bar{P}_{J-1} = (\hat{p}_2, \hat{p}_3, \cdots, \hat{p}_K)$. It is easy to show that in both cases,

$$\underline{P}_{J-1} < \bar{P}_{J-1}$$

where $\underline{P}_{J-1}$ is again the corresponding equal partition of degree $J-1$.

Proposition 22. *For given n, if there exists a partition of degree J,* P_J*, that supports an informative equilibrium, then there also exists an informative equilibrium supportable by a partition of degree* $J' = J-1$ *and* $J-2$*, denoted as* $\hat{P}_{J'}$*, with the property that*

$$\underline{P}_{J'} \le \hat{P}_{J'} \le \bar{P}_{J'}$$

The proof of this result is similar to that of the Existence Theorem. We first construct two specific partitions of degree J', $\underline{P}_{J'}$ and $\bar{P}_{J'}$ such that $\underline{P}_{J'} < \bar{P}_{J'}$. Second, we show the corresponding defined vector-valued function $\mathbf{f}(P_{J'})$ has the property that $\mathbf{f}(\underline{P}_{J'}) < 0$ and $\mathbf{f}\left(\bar{P}_{J'}\right) > 0$. Then, by applying the Intermediate Value Theorem defined on vector spaces, we have there exists a partition $\tilde{P}_{J'}$ on the interval of $\left[\underline{P}_{J'}, \bar{P}_{J'}\right]$, such that $\mathbf{f}\left(\hat{P}_{J'}\right) = \mathbf{0}$.

As an implication of Proposition 22, we have the following corollary, the proof of which is just a direct application of the method of mathematical induction.

Corollary 23. *For given n, if there exists an informative equilibrium supported by a partition of degree J, then for any $J' < J$, there also exist an informative equilibrium supportable by a partition of degree J'.*

2.6. Optimal Disclosure Policy

In this section, the central question to explore is: what is the *optimal* cheap-talk equilibrium? In the above section we have provided an existence theorem which shows that an informative equilibrium can be supported by a partition of any degree, if the number of bidders is large enough. Therefore, for given number of bidders in the auction, there may exists more than one informative equilibrium, corresponding to $J = 0, 1, 2, \cdots$ and so on.

Specifically, when $J = 0$, it corresponds to the case of no information disclosure, since the only subinterval is just the entire interval. Naturally, in an optimal informative equilibrium, the expected revenue should be greater than that of no information disclosure. Our first result demonstrates that when the number of bidders is greater than 3, it is optimal for the seller to reveal information to the bidders.

Proposition 24. *When $n \geq 3$, revealing information generates higher expected revenues than in the case of no information disclosure.*

Proof. All the omitted proofs for this section are in Appendix 2.C. □

Interestingly, When $n = 3$, the seller strictly prefers revealing the information to the bidders, and $n = 3$ is no longer a threshold where the seller is indifferent between disclosing or not the information, as in the case of Ganuza (2004) and Board (2009). This is because in the case of Hotelling model, the expected revenue depends on the expected value of $\beta_N^{(2)}$, which is in quadratic form of θ_i. When $n = 3$, we always have $\mathbb{E}v_3^{(2)} = \mathbb{E}v$, but this is not true in the case of Hotelling model.

From the equilibrium existence results, we already know that, on the one hand, for any given partition degree of J, there exists an informative equilibrium supportable by a partition P_J, as long as the number of bidders is sufficiently large. On the other hand, for given number of bidders, if there exists an equilibrium supported by a J-partition, then informative equilibria are also supportable at lower degree partitions.

Roughly speaking, informative equilibria supported by higher degree partitions correspond to finer partition of the attribute space, and thus more precise messages. And more precise messages increase the expected revenue of the auction. About the optimal disclosure policy, then a natural question is, will the optimal degree of equilibrium partitions increases with the number of bidders in the auction? The answer is yes, and we will provide a proof of this result here.

Let $J^*(n)$ denote the optimal degree of equilibrium partition when the number of bidders is n, and we will show it is an non-decreasing function in n. The proof is composed of several steps. First, we will investigate the property of $\mathbb{E}R(m_K)$, which is the interim expected revenue under a message of m_K, corresponding to an attribute realization on the subinterval of $[p_K, 1]$. $\mathbb{E}R(m_K)$ is thus a function of p_K, and an appealing feature of it is that, different from $\mathbb{E}R(m_0)$, it does not depend on whether J is even or not.

The first result is that $\mathbb{E}R(m_K)$ is a strictly concave in p_K, and when p_K approaches to 1, it is decreasing. Moreover, let p_K^* denote the cutting point that maximizes the value of $\mathbb{E}R(m_K)$, and we show that, p_K^* increases with the number of bidders. The results are summarized as below

Lemma 25. *(1)* $\mathbb{E}R(m_K)$ *is strictly concave in* p_K*, and*

$$\lim_{p_K \longrightarrow 1} \frac{\partial \mathbb{E}R(m_K)}{\partial p_K} < 0$$

and $\left.\frac{\partial \mathbb{E}R(m_K)}{\partial p_K}\right|_0 > 0$ *iff* $n \geq 5$. *(2) Let* $p_K^* = \arg\max \mathbb{E}R\,(m_K)$, *then for* $n \geq 5$

$$\frac{\partial p_K^*}{\partial n} > 0$$

From the Lemma above, we know $\mathbb{E}R\,(m_K)$ is a decreasing function in p_K when $n \leq 4$. And when $n \geq 5$, it is first increasing and then decreasing in p_K. And p_K^* at which $\mathbb{E}R\,(m_K)$ achieves its maximum is increasing in the number of bidders. Figure 2.5 below shows how $\mathbb{E}R\,(m_K)$, as a function of p_K, changes with the number of bidders.

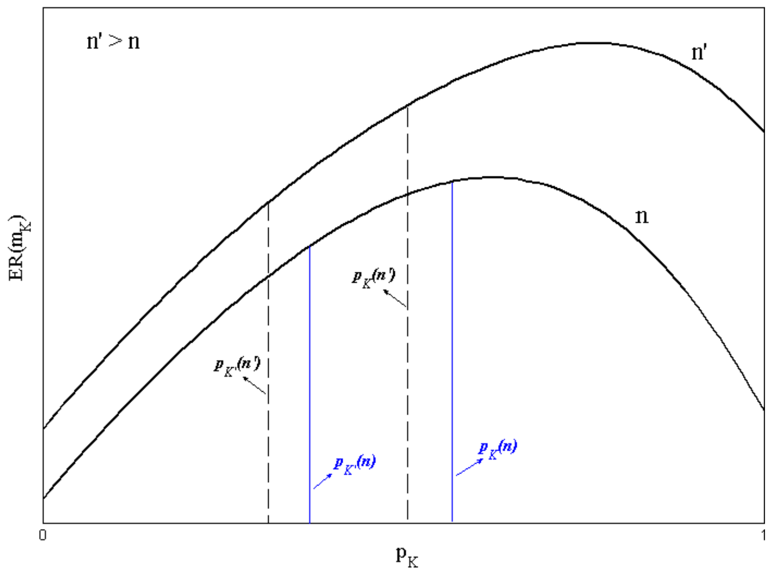


Figure 2.5. How $\mathbb{E}R\,(m_K)$ changes with n

With the result of Lemma 25, we next show a property of optimal partitions. Let $J^*(n)$ be the optimal degree of equilibrium partition when the number of bidders is n. We already know that, for any $J' < J^*(n)$, there also exists an informative equilibrium supportable by a partition of degree J', denoted as $\hat{P}_{J'} = (\hat{p}_{1'}, \hat{p}_{2'}, \cdots, \hat{p}_{k'})$. Then we have the following result:

Lemma 26. *For $J' < J^*(n)$ and a partition $\hat{P}_{J'}$ that supports an informative equilibrium, we have*

$$\left.\frac{\partial \mathbb{E}R\left(m_K\right)}{\partial p_K}\right|_{\hat{p}_{K'}} > 0$$

or equivalently, $\hat{p}_{K'} < p_K^*$.

As shown also in Figure 2.5, Lemma 26 states that, at the cutting point of $p_{K'}$ in the lower degree equilibrium partition of $\hat{P}_{J'}$, $\mathbb{E}R\left(m_K\right)$ must be increasing. Think otherwise, if it is decreasing, we must have $\mathbb{E}R\left(m_K\right)|_{\hat{p}_{K'}} > \mathbb{E}R\left(m_K\right)|_{p_K}$, as $\hat{p}_{K'} < p_K$ and $\mathbb{E}R\left(m_K\right)$ is strictly concave, which contradicts the presumption that $J^*(n)$ is the optimal degree of equilibrium partition.

With these two lemmas, we then now approach to the main result about the optimal disclosure policy. The result is that $J^*(n)$ is a non-decreasing function of n, which reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. Roughly speaking, the result states that when the number of bidders increase, it is better for the seller to provide more precise information about product attributes to the bidders.

Theorem 27. *The optimal degree of equilibrium partition, $J^*\left(n\right)$, is non-decreasing in n. That is, for $n' > n$*

$$J^*\left(n'\right) \geq J^*\left(n\right)$$

The result states that, if $J^*\left(n\right)$ is the optimal degree of equilibrium partition at n, then when n increases, the new optimal degree of equilibrium partition is no smaller than $J^*(n)$. The result is rather intuitive. As we already know that when the number of bidders increases, the seller is able to construct equilibrium using higher degree partitions. And roughly speaking, higher degree partitions correspond to more precise messages, which will increase the interim expected revenues in equilibrium. As a result, when the number of bidders increases, the optimal degree of equilibrium partitions should also increase.

This result reveals a kind of complementarity relationship between the number of bidders, n, and the optimal degree of equilibrium partitions. It is similar to the complementarity results in Ganuza (2004) and Ganuza and Panelva (2010), both of which show that the precision of revealed information increases with the number of bidders, under the assumption that revealing information is costly.

Though the result of our model is similar to theirs, yet our model setup differ from theirs in at least two ways. First, we study cheap-talk, while they study information disclosure under full commitment. Second, their result depends on the assumption of costly information. If information is costless, their models will predict extreme disclosure policies, that is, in optimality, the seller will either reveal no or full information to the bidders, as in Lewis and Sappington (1994) and Johnson and Myatt(2006).

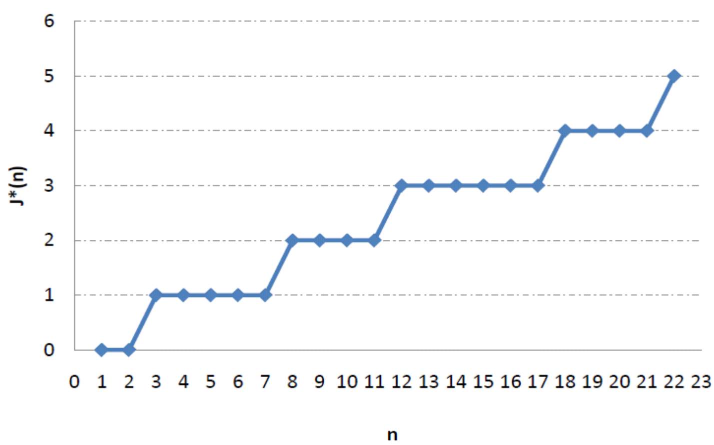


Figure 2.6. Optimal Degree of Equilibrium Partition: an Example

Example 28. *Actually, we can directly derive numerical results about the optimal degrees of equilibrium partition in our model, where we have assumed that both product attributes and bidders' types conform to uniform distribution on* $[-1,1]$*. The method is similar to that in Example 13, we derive the curves of the interim expected revenues as functions of the cutting points, and then identify the crossing points of them, which constitute part of an informative equilibrium. For* $n \leq 22$*, we have the optimal degree of equilibrium partitions,* $J^*(n)$*, in our model as below*

$$J^*(n) = \begin{cases} 0 & \textit{if } n \in \{1,2\} \\ 1 & \textit{if } n \in \{3,\cdots,7\} \\ 2 & \textit{if } n \in \{8,\cdots,11\} \\ 3 & \textit{if } n \in \{12,\cdots,17\} \\ 4 & \textit{if } n \in \{18,21\} \\ 5 & \textit{if } n = 22 \end{cases}$$

We did not provide the result for $n > 22$*. The numerical result shows that* $J^*(n)$*, the optimal degree of equilibrium partition, is non-decreasing in* n*, as shown in Proposition 21. Figure 2.6. above provides an illustration of the numerical results.*

2.7. Information Disclosure under full Commitment

In this section, we proceed to study the case of information disclosure under full commitment, and then compare its results with those under cheap-talk. Our model differs from other models on information disclosure under full commitment in at least two aspects. First, adopting a Hotelling model on an interval, we explicitly model the matching between product attributes and consumers' preferences. Second, the information structure in our model is characterized by a partition of the state space of product attributes.

Again we consider an information structure of J-partition, and try to characterize the equilibrium in this case of full commitment. As before, if $J = 2K + 1$ number, then the partition is $-1 < -p_K < \cdots < -p_1 < 0 < p_1 < \cdots < p_K < 1$, and the expected revenue of the auction is

$$\mathbb{E}R = 2\sum_{j=0}^{K} (p_{j+1} - p_j)\,\mathbb{E}R\,(m_j)$$

where $p_0 = 0$. If $J = 2K$, then the partition is $-1 < -p_K < \cdots < -p_1 < p_1 < \cdots < p_K < 1$, and the expected revenue of the auction is

$$\mathbb{E}R = 2\sum_{j=0}^{K} (p_{j+1} - p_j)\,\mathbb{E}R\,(m_j)$$

where $p_0 = 0$ and m_0 corresponds to $s \in [-p_1, p_1]$. And the interim expected revenue is shown in (3.4).

The existence of an optimal partition that maximizes the *ex ante* expected revenue is guaranteed by the Maximum Value Theorem, as the function of *ex ante* expected revenue is continuous in the partition P_J, and the set of $J-$partition itself is a bounded close set. Therefore, we have the following result.

Proposition 29. *(Existence Theorem) For information disclosure under full commitment, for any $J-$partition there are always exists unique interior partition points, p_j^*'s, that maximizes the ex ante expected revenue of the auction.*

An interesting characterization of the equilibria under full commitment reveals that, in equilibrium, a message for more popular product attributes generates strictly higher interim expected revenue than those of less popular attributes. Therefore, the equilibrium partitions under full commitment distinct from those under cheap-talk, where all the interim expected revenues in a partition are equal to each other. The result is as below.

Proposition 30. *Given the number of bidders n, if a J-partition of the state space constitutes an equilibrium, then it is necessary that*

$$\mathbb{E}R(m_0) > \mathbb{E}R(m_1) > \cdots > \mathbb{E}R(m_K) \tag{2.28}$$

Proof. The omitted proofs of this section is in Appendix 2.D. □

The other result is about a convergence property of the equilibrium partitions under full commitment, which also shows that in the limit when the number of bidders converges to infinity, the equilibrium partitions all converge to equal partitions, just as in the case of cheap-talk information disclosure.

Proposition 31. *For disclosure under full commitment, when the number of bidders converges to infinity, the optimal J-partition converges to equal partition, that is,*

$$\lim_{n \longrightarrow \infty} P_J^* = \underline{P}_J$$

2.8. Conclusion Remarks

The existing literature on information disclosure commonly makes the assumption of full commitment to truthful disclosure, and thus lying and false announcement are simply assumed impossible. This assumption is obviously unrealistic in many circumstances, especially for our understanding of real disclosure issues. In this paper, we avoid this full commitment assumption, and study cheap-talk information disclosure in an auction context, when bidders' preferences are horizontally differentiated. Cheap-talk implies that information is costless, non-binding and non-verifiable, and thus lying and disclosure fraud are possible.

We first provide an characterization of informative equilibria in a symmetric setting, which reveals the feature that the seller provides more precise information for less popular product attributes. The intuition is simple: for less popular attributes, the interim expected revenue is lower *ceteris paribus*, and to compensate that revenue deficit, the seller needs to provide more precise messages so as to get the non-deviation conditions satisfied.

The second also the main result of this paper is an existence theorem of informative equilibria. We prove that, under cheap-talk information disclosure, an informative equilibrium can be supported by a partition of any degree, as long as the number of bidders is sufficiently large. Moreover, when the number of bidders converges to infinity, equilibrium partitions all converge to equal partitions.

The intuition is that, as the number of bidders increases, the gaps between the location effects across different messages converge to zero, and it is easier for the seller to adjust the precision of messages and achieve equilibria in higher degree partitions. In the limit, when the number of bidders converge to infinity and all the product attributes are of the same popularity, any equal partition can constitute an informative equilibrium, and an equilibrium can be supported by an equal partition of any finite degree.

As the third result, we show that the optimal disclosure policy reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. That is, when the number of bidders increases, it is better for the seller to construct an informative equilibrium using higher degree partitions, for the purpose of revenue maximization. The intuition behind this result is similar to that behind the existence result.

We also compare the cheap-talk results with those under full commitment. In our symmetric setting, the equilibrium partitions under full commitment reveal the feature that messages for more popular attributes generates greater revenues than those for less popular attributes. This is a distinct feature from the partition equilibria under cheap-talk. Second, in the limit when the number of bidders converges to infinity, the equilibrium partitions also converge to equal partitions, but from an opposite direction if compared with the case of cheap-talk.

This paper contributes to the literature on information disclosure in several aspects. First, we study cheap-talk information disclosure, thus avoiding the typical yet unrealistic assumption of full commitment to truthful disclosure, and make it possible to investigates the issue of disclosure frauds. Second, we explicitly model the horizontal differentiation in bidders' preferences, and thus make transparent the mechanism that how bidders respond differently to a common message. Finally, the analytical framework developed in this paper can be readily extended to other contexts, such as monopoly pricing and advertising.

Cheap-talk information disclosure is a relatively new topic in the literature, and there are two possible directions to extend our current framework. One is to explore the case of multiple product information disclosure, which is of great practical importance. In the real world, from eBay to Amazon, from manufacturers to retailers, a seller commonly has many products for sale. Then how should a seller advertise the product attributes to the consumers in this multiple product environment? And what are the determinants for the optimal disclosure policies? These are interesting and important questions to investigate. This research is related to the fast-growing literature on multiple-product auctions and internet auctions, and it is imaginable that it will be quite technically challenging.

The other direction is to extend the current static model to a model of dynamic information disclosure.. We commonly observe that a seller may be willing to reveal some true yet bad information about the products, which is detrimental to contemporary profit, even through there is no commitment for the seller to do so. How to explain this? An intuition is that a seller is willing to do so because of the concern of long-run reputation, and his current profit loss can be compensated in the long run if a good reputation is maintained. Obviously, for dynamic information disclosure, the trade-off in a dynamic context can be distinct from that in a static game. This research is related to the recent literature on dynamic cheap-talk, such as Jullien and Park (2011).

2.A. Omitted Proofs for Section 2.4

Proof. [**Lemma 5**] We first derive the cumulative distribution function of $\beta_n^{(2)}$ from the following property of (4.10) that

$$\begin{aligned}\Pr\left(\beta_n^{(2)} > x\right) &= \Pr\left(\beta_i > x\right)^n + n\Pr\left(\beta_i > x\right)^{n-1}\Pr\left(\beta_i \leq x\right) \\ &= \begin{cases}\left(1-\sqrt{x}\right)^n + n\sqrt{x}\left(1-\sqrt{x}\right)^{n-1} & x \in [0,\beta^o] \\ \left[1-\phi(x)\right]^n + n\phi(x)\left[1-\phi(x)\right]^{n-1} & x \in \left(\beta^o,\bar{\beta}\right]\end{cases} \qquad (2.29)\end{aligned}$$

where $\phi(x) = \frac{\left(\sqrt{\beta^o}+\sqrt{x}\right)}{2}$. Therefore, the cumulative distribution function of $\beta_n^{(2)}$ is $H_n^{(2)}(x) = 1-\Pr\left(\beta_n^{(2)} > x\right)$, which is continuous function on $[0,\beta^o]$ and $\left(\beta^o,\bar{\beta}\right]$. The expected value of the order statistics $\beta_n^{(2)}$ is thus

$$\begin{aligned}\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] &= \int_0^{\beta^o} xdH_n^{(2)}(x) + \int_{\beta^o}^{\bar{\beta}} xdH_n^{(2)}(x) \\ &= \frac{6}{(n+1)(n+2)} + 2\mu_j^n - 2\mu_j^{n+1}\left(2-\frac{5}{n+1}\right) \qquad (2.30) \\ &\quad +2\mu_j^{n+2}\left[1-\frac{5}{n+1}+\frac{9}{(n+1)(n+2)}\right]\end{aligned}$$

To prove that the location effect, $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$, is increasing in μ_j, we simply take the derivative and get that

$$\frac{1}{2}\frac{\partial\mathbb{E}\beta_n^{(2)}}{\partial\mu_j} = \left[\left(n-3+\frac{4}{n+1}\right)\mu_j^2 + (3-2n)\mu_j + n\right]\mu_j^{n-1} = g_1\left(\mu_j,n\right)\mu_j^{n-1} \qquad (2.31)$$

and $\frac{\partial \mathbb{E}\beta_n^{(2)}}{\partial \mu_j} > 0$ for $\mu_j \in (0,1]$ is equivalent to $g_1\left(\mu_j, n\right) > 0$.

$$\frac{\partial g_1\left(\cdot\right)}{\partial \mu_j} = 2\mu_j\left(n - 3 + \frac{4}{n+1}\right) + (3 - 2n)$$

The maximum value of this derivative is achieved at $\mu_j = 1$ since $\left(n - 3 + \frac{4}{n+1}\right) > 0$ for $n \geq 2$, and the maximum value is negative. We then get $\frac{\partial g_1(\cdot)}{\partial \mu_j} < 0$ for all $\mu_j \in (0,1]$, and

$$\min g_1\left(\mu_j\right) = g_1\left(1\right) = \frac{4}{n+1} > 0 \tag{2.32}$$

Therefore, $g_1\left(\mu_j\right) > 0$ for $\mu_j \in (0,1]$. We the prove that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is strictly increasing in μ_j. It is easy to show that whether $\mu_j = 1$ or not, $\lim_{n\longrightarrow\infty} \frac{\partial \mathbb{E}\left(\beta_N^{(2)}\middle| m_j\right)}{\partial \mu_j} = 0$.

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 \mathbb{E}\beta_n^{(2)}}{\partial \mu_j \partial n} &= \frac{\partial\left[g_1\left(\mu_j, n\right)\mu_j^{n-1}\right]}{\partial n} \\
&= \mu^{n-1}\left\{g_1\left(\mu_j, n\right)\ln \mu_j + \left[\left(1 - \frac{4}{(n+1)^2}\right)\mu_j^2 - 2\mu_j + 1\right]\right\} \\
&= \mu^{n-1} h\left(\mu_j, n\right)
\end{aligned}$$

where $h\left(\mu_j, n\right) = g_1\left(\mu_j, n\right)\ln \mu_j + \left[\left(1 - \frac{4}{(n+1)^2}\right)\mu^2 - 2\mu + 1\right]$, achieving its maximum at $\mu_j = 1$. The maximum of $h\left(\mu_j, n\right)$ is $h\left(1, n\right) = -\frac{4}{(n+1)^2} < 0$, and therefore $\frac{\partial^2 \mathbb{E}\beta_n^{(2)}}{\partial \mu_j \partial n} < 0$.

Second, for the result that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is decreasing in n, if we could show the cumulative distribution function of $\beta_n^{(2)}$, which is $H_N^{(2)}\left(x\right)$, is increasing in n, then we get the result proved since

$$\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] = \bar{\beta} - \int_0^{\beta^o} H_n^{(2)}(x)dx - \int_{\beta^o}^{\bar{\beta}} H_n^{(2)}(x)dx \tag{2.33}$$

When $x \in [0, \beta^o]$, that is $\sqrt{x} \in [0, 1 - \mu_j]$,

$$\frac{\partial H_n^{(2)}(x)}{\partial n} = -\left(1-\sqrt{x}\right)^{n-1} \underbrace{\left\{\sqrt{x} + \left[(n-1)\sqrt{x}+1\right] \ln\left(1-\sqrt{x}\right)\right\}}_{\leq 0 \text{ with equality only at } x=0} \geq 0$$

with equality only at $x = 0$. When $x \in \left(\beta^o, \bar{\beta}\right]$, that is $\sqrt{x} \in \left[1 - \mu_j, 1 + \mu_j\right]$,

$$\begin{aligned}
\frac{\partial H_n^{(2)}(x)}{\partial n} &= -\left[1-\phi(x)\right]^n \ln\left[1-\phi(x)\right] - \phi(x)\left[1-\phi(x)\right]^{n-1} \\
&\quad -n\phi(x)\left[1-\phi(x)\right]^{n-1} \ln\left[1-\phi(x)\right] \\
&= -\left[1-\phi(x)\right]^{n-1} \underbrace{\left\{\phi(x) + \left[(n-1)\phi(x)+1\right]\ln\left[1-\phi(x)\right]\right\}}_{<0 \text{ for } x\in\left(\beta^o,\bar{\beta}\right]} > 0
\end{aligned}$$

where $\phi(x) = \frac{\left(\sqrt{\beta^o}+\sqrt{x}\right)}{2}$, as shown above. Therefore, $\mathbb{E}\left[\beta_N^{(2)}\left(\mu_j\right)\right]$ is strictly decreasing in n for any given p. Whether $\mu_j < 1$ or $\mu_j = 1$, the limit of $\mathbb{E}\left[\beta_N^{(2)}\left(m_R\right)\right]$ is obviously zero. □

Proof. [**Lemma 6**] **Part (1)**: To prove convexity, we just take the second order derivative, from (2.31)

$$\frac{1}{2}\frac{\partial^2 \mathbb{E}\left(\beta_n^{(2)}\right)}{\partial \mu_j^2} = \left[(n-1)^2 \mu_j^2 + n(3-2n)\mu_j + n(n-1)\right]\mu_j^{n-2} = g_2\left(\mu_j\right)\mu_j^{n-2}$$

where $\mu_R \in [0,1]$, and $\frac{\partial^2 \mathbb{E}\left(\beta_n^{(2)}\right)}{\partial \mu_j^2} > 0$ is equivalent to $g_2\left(\mu_j\right) > 0$.

$$\frac{\partial g_2\left(\mu_j\right)}{\partial \mu_j} = 2(n-1)^2 \mu_j + n(3-2n)$$

and the maximum value of it is achieved at $\mu_j = 1$ since $2(n-1)^2 > 0$ for $n \geq 2$, and the value is equal to $2 - n \leq 0$. We the get $\frac{\partial g_1\left(\mu_j\right)}{\partial \mu_j} < 0$ for all μ_j, and

$$\min g_2\left(\mu_j\right) = g_2(1) = 1 > 0$$

Therefore $\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\right]}{\partial \mu_j}$ is strictly increasing in μ_j, and $\mathbb{E}\left[\beta_n^{(2)}\right]$ is strictly convex in μ_j.

Part (2): We have $\lim_{n \longrightarrow \infty} \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] = 0$, and therefore $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ converges pointwise to the continuous function $\beta^{(2)}\left(\mu_j\right) = 0$ for $\mu_j \in [0,1]$. For any n, $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ is continuous on the support of $\mu_j \in [0,1]$. Since $\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial n} < 0$, $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] \geq \mathbb{E}\left[\beta_{n+1}^{(2)}\left(\mu_j\right)\right]$ for $\forall \mu_j \in [0,1]$ and $n = 1, 2, 3, \cdots$. The limit function is continuous on the support of $\mu_j \in [0,1]$, and then from the Dini Theorem we know that $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ uniformly converges to 0 for $\mu_j \in [0,1]$. $\square$

Proof. [**Corollary 8**] (1) How $\mathbb{E}R\left(m_j\right)$ changes with p_{j+1}: According to our assumptions, $\mu_j \in [0, p_{j+1}]$ for given p_{j+1}, and

$$\begin{aligned}
\frac{1}{\tau}\frac{\partial \mathbb{E}R\left(m_j\right)}{\partial p_{j+1}} &= \underbrace{\frac{1}{6}\left(p_{j+1} - p_j\right)}_{\text{positive}} - \underbrace{\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial p_{j+1}}}_{\text{positive}} \\
&= \frac{1}{6}\left(p_{j+1} - p_j\right) - \underbrace{\left[\left(n - 3 + \frac{4}{n+1}\right)\mu_j^2 + \mu_j\left(3 - 2n\right) + n\right]\mu_j^{n-1}}_{=g_1\left(\mu_j\right) \text{ and } \min g_1\left(\mu_j\right) = g_1\left(p_{j+1}\right) > 0 \text{ from } (2.32)} \\
&\leq \frac{1}{6}\left(p_{j+1} - p_j\right) - g_1\left(p_{j+1}\right)\mu_j^{n-1} = \bar{\phi}\left(p_j\right)
\end{aligned}$$

Obviously $\bar{\phi}\left(p_j\right)$ is strictly decreasing in p_j with

$$\begin{aligned}
\min \bar{\phi}\left(p_j\right) &= \bar{\phi}\left(p_{j+1}\right) = -g_1\left(p_{j+1}\right)p_{j+1}^{n-1} < 0 \\
\max \bar{\phi}\left(p_j\right) &= \bar{\phi}\left(-p_{j+1}\right) = \frac{1}{3}p_{j+1} > 0
\end{aligned}$$

Therefore, for given $p_{j+1} > 0$, there exists $\bar{p}_j \in \left(-p_{j+1}, p_{j+1}\right)$ such that $\bar{\phi}\left(\bar{p}_j\right) = 0$, and for any $p_j > \bar{p}_j$, $\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} < 0$.

(2) How $\mathbb{E}R(m_j)$ changes with p_j: Again, we take the derivative

$$\begin{aligned}
\frac{1}{\tau}\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} &= \frac{1}{6}(p_{j+1}-p_j) - \underbrace{\left[\left(n-3+\frac{4}{n+1}\right)\mu_j^2+\mu_j(3-2n)+n\right]}_{=g_1(\mu_j),\ g_1'(\mu_j)<0 \text{ and } \max g_1(\mu_j)=g_1(0)=n \text{ from (2.32)}} \mu_j^{n-1} \\
&\geq \frac{1}{6}(p_{j+1}-p_j) - n\mu_j^{n-1} = \underline{\phi}(p_j)
\end{aligned}$$

Obviously $\underline{\phi}(p_j)$ is strictly decreasing in p_j with

$$\begin{aligned}
\min \underline{\phi}(p_j) &= \underline{\phi}(p_{j+1}) = -np_{j+1}^{n-1} < 0 \\
\max \underline{\phi}(p_j) &= \underline{\phi}(-p_{j+1}) = \frac{1}{3}p_{j+1} > 0
\end{aligned}$$

Therefore, for given $p_{j+1} > 0$, there exists $\underline{p}_j \in (-p_{j+1}, p_{j+1})$ such that $\underline{\phi}\left(\underline{p}_j\right) = 0$, and for any $p_j < \bar{p}_j$, $\frac{\partial \mathbb{E}R(m_j)}{\partial p_j} > 0$. $\square$

Proof. [**Proposition 10**] For the existence of an informative equilibrium, it is necessary that for any j and j',

$$\mathbb{E}R(m_j) = \mathbb{E}R(m_{j'})$$

which is just the non-deviation condition of (3.7). We know that $\mathbb{E}R(m_j) = V - \tau\left[\frac{\Delta_j^2}{12} + \mathbb{E}\left(\beta_n^{(2)}\right)\right]$. Without loss of generality, $j' > j$, and therefore in an informative equilibrium

$$\frac{\Delta_j^2}{12} + \mathbb{E}\left[\beta_n^{(2)}(\mu_j)\right] = \frac{\Delta_{j'}^2}{12} + \mathbb{E}\left[\beta_n^{(2)}(\mu_{j'})\right] \tag{2.34}$$

Since $\mathbb{E}\left[\beta_n^{(2)}(\mu_{j'})\right] > \mathbb{E}\left[\beta_n^{(2)}(\mu_j)\right]$, it is necessary that

$$\Delta_j > \Delta_{j'}$$

for the equality of (2.34) to hold. $\square$

Proof. [**Corollary 11**] Suppose not, and then there exists an informative equilibrium where the auctioneer truthfully reveals the real state, that is, $m = s$ for all $s \in [-1, 1]$. Without loss of generality, let $s' > s$, and

$$\mathbb{E}R(s) = V - \tau\mathbb{E}\left(\beta_n^{(2)}(s)\right) \quad \text{and} \quad \mathbb{E}R(s') = V - \tau\mathbb{E}\left(\beta_n^{(2)}(s')\right)$$

Since $\mathbb{E}\left(\beta_n^{(2)}(s)\right) < \mathbb{E}\left(\beta_n^{(2)}(s')\right)$, $\mathbb{E}R(s) > \mathbb{E}R(s')$ and therefore it's always profitable for the seller to deviate to report s when the true state is s', which contradicts the assumption that full information disclosure constitutes an informative equilibrium. □

Proof. [**Proposition 12**] When $n \longrightarrow \infty$, $\lim_{n\longrightarrow\infty} \mathbb{E}\left[\beta_n^{(2)}(\mu_j)\right] = 0$ for any m_j, and the equilibrium condition, $\mathbb{E}R(m_j) = \mathbb{E}R(m_{j'})$, reduces to

$$\frac{1}{12}\Delta_j^2 = \frac{1}{12}\Delta_{j'}^2$$

And therefore, when $n \longrightarrow \infty$, in an informative equilibrium, it is necessary that

$$\Delta_j = \Delta_{j'}$$

which constitutes an equal J-partition. □

2.B. Omitted Proofs for Section 2.5 (Existence Theorem)

Proof. [**Lemma 15**] For any $j = 1, 2, \cdots, K$ and $J = 2K + 1$

$$\begin{aligned}
\frac{\bar{p}_j}{\underline{p}_j} &= (K+1)\left[B + \frac{1}{2}\delta(J)(2K - j + 1)\right] \\
&= (K+1)\left[\frac{(2K - j + 1)}{K(K+1)} + \frac{(j - K - 1)}{K}B\right] \\
&> (K+1)\left[\frac{(2K - j + 1)}{K(K+1)} + \frac{(j - K - 1)}{K(K+1)}\right] = 1
\end{aligned}$$

the inequality is based on the fact that $B < \frac{1}{K+1}$ and $j - K - 1 < 0$. $\square$

Proof. [**Lemma 16**] For equal partition $\underline{P}_J$, $\Delta_j = \Delta_{j+1}$ for any $j = 0, 1, \cdots, K$. Therefore,

$$f_j\left(P_J\right) = \mathbb{E}\left(\beta_n^{(2)}\left(\mu_j\right)\right) - \mathbb{E}\left(\beta_n^{(2)}\left(\mu_{j+1}\right)\right) < 0$$

since $\mu_j < \mu_{j+1}$ and $\mathbb{E}\left(\beta_n^{(2)}\left(\mu_j\right)\right)$ is strictly increasing in μ_j. Second, the partition $\bar{P}_J$ is fully determined by the value of B and $\delta\left(J\right)$. It is easy to show that in partition $\bar{P}_J$,

$$\min_j\left(\Delta_j^2 - \Delta_{j+1}^2\right) = \Delta_{K-1}^2 - \Delta_K^2 = 2\delta\left(J\right)\left[B + \bar{\Delta}\left(J\right)\right] = \epsilon\left(\bar{P}_J\right)$$

Since $\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]$ uniformly converges to 0, there exists a $N(\bar{P}_J)$ such that for all $n \geq N\left(\bar{P}_J\right)$, $0 < \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right] < \frac{1}{12}\epsilon\left(\bar{P}_J\right)$ for $\mu_j \in [0,1]$. Therefore, for $n \geq N\left(\bar{P}_J\right)$,

$$\begin{aligned} f_j\left(P_J\right) &= f_j\left(P_J\right) = \frac{1}{12}\left[\left(p_j - p_{j-1}\right)^2 - \left(p_{j+1} - p_j\right)^2\right] - \left[\mathbb{E}\left(\beta_n^{(2)}\left(\mu_j\right)\right) - \mathbb{E}\left(\beta_n^{(2)}\left(\mu_{j-1}\right)\right)\right] \\ &> \frac{1}{12}\epsilon\left(\bar{P}_J\right) - \frac{1}{12}\epsilon\left(\bar{P}_J\right) = 0 \end{aligned}$$

$\square$

Proof. [**Lemma 17**] For any $\bar{P}_J \geq P_J \geq \underline{P}_J$, we have when $J = 2\left(K+1\right)$

$$\begin{aligned} \frac{j}{K+1} &\leq p_j \leq j\left[B + \frac{1}{2}\delta\left(J\right)\left(2K - j + 1\right)\right] \\ \frac{j+1}{K+1} &\leq p_{j+1} \leq \left(j+1\right)\left[B + \frac{1}{2}\delta\left(J\right)\left(2K - j\right)\right] \end{aligned}$$

Let $B = \frac{\alpha}{K+1}$ with $\alpha \in [0,1]$, then we have

$$\begin{aligned} p_{j+1} - p_j &\geq \frac{j+1}{K+1} - j\left[B + \frac{1}{2}\delta\left(J\right)\left(2K - j + 1\right)\right] \\ &= \frac{j+1}{K+1} - j\left[B + \frac{1}{K(K+1)}\left[1 - \left(K+1\right)B\right]\left(2K - j + 1\right)\right] \\ &= \frac{j+1}{K+1} - j\frac{(2K - j + 1)}{K(K+1)} + j\frac{(K - j + 1)}{K(K+1)}\alpha \\ &= \frac{1}{K(K+1)}\left\{K\left(j+1\right) + j\left[\left(K - j + 1\right)\alpha - \left(2K - j + 1\right)\right]\right\} \end{aligned}$$

Since

$$\begin{aligned} K(K+1)\frac{\partial \Delta_j}{\partial j} &= K + \left[\left(K - j + 1\right)\alpha - \left(2K - j + 1\right)\right] + j\left(-\alpha + 1\right) \\ &= \left(K - 2j + 1\right)\left(\alpha - 1\right) = \begin{cases} < 0 & j < \frac{1}{2}\left(K+1\right) \\ > 0 & j > \frac{1}{2}\left(K+1\right) \end{cases} \end{aligned}$$

$\Delta_j = p_j - p_{j-1}$ reaches its minimum at $j = \frac{1}{2}\left(K+1\right)$ which is

$$\min \Delta_j = \frac{1}{2K(K+1)}\left\{K\left(K+3\right) + \left(K+1\right)\left[\frac{1}{2}\left(K+1\right)\alpha - \frac{1}{2}\left(3K+1\right)\right]\right\}$$

and it is monotonically increasing in α. When $\alpha = 1$, $\min \Delta p_j = \frac{1}{K+1} = \frac{2}{J}$, when $\alpha = 0$, $\min \Delta p_j = -\frac{(K-1)^2}{4K(K+1)} \leq 0$. We let $\epsilon\left(J\right) = \frac{1}{J}$, by intermediate value theorem, there always exists an $\hat{\alpha} \in (0,1)$ such that $\min \Delta_j|_{\alpha=\hat{\alpha}} = \frac{1}{J}$. □

Proof. [**Theorem 19**] (**Existence Theorem**) We already know that the set of all possible partitions of J, $\mathcal{P}_J$ is a convex hull in $\mathbb{R}_+^K$. Define the function $\mathbf{f} : \mathcal{P}_J \longrightarrow \mathbb{R}^K$, and $\mathbf{f}\left(\cdot\right) = \left(f_1\left(\cdot\right), f_2\left(\cdot\right), \cdots, f_K\left(\cdot\right)\right)$ where

$$f_j\left(P_J\right) = \frac{1}{12}\left[\left(p_j - p_{j-1}\right)^2 - \left(p_{j+1} - p_j\right)^2\right] - \left[\mathbb{E}\left(\beta_n^{(2)}\left(\mu_j\right)\right) - \mathbb{E}\left(\beta_n^{(2)}\left(\mu_{j-1}\right)\right)\right]$$

for $j = 1, 2, \cdots, K$. Obviously, $f_j\left(P_J\right)$ is continuous in $\mathcal{P}_J$ and thus conditions (1) and (2) of Theorem 18 are satisfied.

According to the partial order we defined, $\underline{P}_J < \bar{P}_J$, and for any $n \geq N\left(\bar{P}_J\right)$,

$$\mathbf{f}\left(\underline{P}_J\right) < \mathbf{0} < \mathbf{f}\left(\bar{P}_J\right)$$

we just need to show, for enough large n, there exists a partition $\hat{P}_J \in \left[\underline{P}_J, \bar{P}_J\right]$ such that $\mathbf{f}\left(\hat{P}_J\right) = \mathbf{0}$. We then check condition (3) of Theorem 18.

$$\frac{\partial f_j\left(P_J\right)}{\partial p_{j+1}} = -\frac{1}{6}\Delta_{j+1} - \frac{1}{2}\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j} < 0$$

which is negative every where, and

$$\frac{\partial f_j\left(P_J\right)}{\partial p_{j-1}} = -\frac{1}{6}\left(p_j - p_{j-1}\right) + \frac{1}{2}\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j} < 0$$

Denote $\epsilon' = \frac{1}{3}\left(p_j - p_{j-1}\right) \geq \frac{1}{3}\delta(J) = \frac{1}{3J}$, and there exists an $N\left(\epsilon'\right)$ such that for any $n \geq N\left(\epsilon'\right)$, $\frac{\partial \mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]}{\partial \mu_j} < \epsilon'$ and therefore $\frac{\partial f_j\left(P_J\right)}{\partial p_{j-1}} < 0$. Let $N\left(J\right) = \max\left\{N\left(\epsilon'\right), N\left(\epsilon\right)\right\}$. Then for any $n \geq N\left(J\right)$, conditions (1), (2) and (3) of Theorem 18 are all satisfied, and therefore, there exists a partition of J, denoted as $\hat{P}_J$, such that

$$\mathbf{f}\left(\hat{P}_J\right) = \mathbf{0}$$

So an informative equilibrium can be supported by a partition of degree J, as long as the number of bidders $n \geq N\left(J\right)$. □

Proof. **[Corollary 20]** Suppose in a n bidder auction, the partition of degree J that supports an informative equilibrium is $\hat{P}_J$ defined by the sequence of cutting points, $\{\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_J\}$. Then we know

$$f_j\left(\hat{P}_J\right) = \frac{1}{12}\left[\left(\hat{p}_j - \hat{p}_{j-1}\right)^2 - \left(\hat{p}_{j+1} - \hat{p}_j\right)^2\right] - \left[\mathbb{E}\left(\beta_n^{(2)}\left(\hat{\mu}_j\right)\right) - \mathbb{E}\left(\beta_n^{(2)}\left(\hat{\mu}_{j-1}\right)\right)\right] = 0$$

for any $j = 1, 2, \cdots J+1$. For any $n' > n$, at the partition $\hat{P}_J$, we have

$$\begin{aligned} f_j\left(\hat{P}_J\right) &= \frac{1}{12}\left[(\hat{p}_j - \hat{p}_{j-1})^2 - (\hat{p}_{j+1} - \hat{p}_j)^2\right] - \left[\mathbb{E}\left(\beta_{n'}^{(2)}\left(\hat{\mu}_j\right)\right) - \mathbb{E}\left(\beta_{n'}^{(2)}\left(\hat{\mu}_{j-1}\right)\right)\right] \\ &> \frac{1}{12}\left[(\hat{p}_j - \hat{p}_{j-1})^2 - (\hat{p}_{j+1} - \hat{p}_j)^2\right] - \left[\mathbb{E}\left(\beta_{n}^{(2)}\left(\hat{\mu}_j\right)\right) - \mathbb{E}\left(\beta_{n}^{(2)}\left(\hat{\mu}_{j-1}\right)\right)\right] > 0 \end{aligned}$$

because $\left[\mathbb{E}\left(\beta_{n}^{(2)}\left(\hat{\mu}_j\right)\right) - \mathbb{E}\left(\beta_{n}^{(2)}\left(\hat{\mu}_{j-1}\right)\right)\right]$ is strictly decreasing in n, from (3.12). Therefore, $\mathbf{f}\left(\hat{P}_J\right) > \mathbf{0}$ and we also have $\mathbf{f}\left(\underline{P}_J\right) < \mathbf{0}$. By applying the Intermediate Value Theorem again, we know there exists a partition $\tilde{P}_J$ between $\underline{P}_J$ and $\hat{P}_J$ such that $\mathbf{f}\left(\tilde{P}_J\right) = \mathbf{0}$ when the number of bidders is greater than n. $\square$

Proof. [**Proposition 22**] As before, we define the corresponding vector-valued function, $\mathbf{f} : \mathcal{P}_{J'} \longrightarrow \mathbb{R}^{K'}$, with its jth element function being

$$f_j\left(P_{J'}\right) = \frac{1}{12}\left[\Delta_{j-1}'^2 - \Delta_j'^2\right] - \left[\mathbb{E}\left(\beta_n^{(2)}\left(\mu_j'\right)\right) - \mathbb{E}\left(\beta_n^{(2)}\left(\mu_{j-1}'\right)\right)\right]$$

We already know that $\underline{P}_{J'} < \bar{P}_{J'}$, and $\mathbf{f}\left(\underline{P}_{J'}\right) < \mathbf{0}$ as $\underline{P}_{J'}$ is equal partition and therefore $f_j\left(\underline{P}_{J'}\right) = -\left[\mathbb{E}\beta_n^{(2)}\left(\mu_j'\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_{j-1}'\right)\right] < 0$. If we show $\mathbf{f}\left(\bar{P}_{J'}\right) > \mathbf{0}$, then by applying the Intermediate Value Theorem, we know there exists a partition $\tilde{P}_{J'} \in \left[\underline{P}_{J'}, \bar{P}_{J'}\right]$, such that $\mathbf{f}\left(\tilde{P}_{J'}\right) = \mathbf{0}$.

If $J' = J-2$, then $\bar{P}_{J-2} = \left(p_1', p_2', \cdots, p_{K-1}'\right) = \left(p_2, \cdots, p_K\right)$, and therefore $f_1\left(\bar{P}_{J-2}\right) = \frac{1}{12}\left[\Delta_0'^2 - \Delta_1'^2\right] - \left[\mathbb{E}\beta_n^{(2)}\left(\mu_1'\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_0'\right)\right] = \frac{1}{12}\left[\Delta_0'^2 - \Delta_1^2\right] - \left[\mathbb{E}\beta_n^{(2)}\left(\mu_1\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_0'\right)\right] > \frac{1}{12}\left[\Delta_0^2 - \Delta_1^2\right] - \left[\mathbb{E}\beta_n^{(2)}\left(\mu_1\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_0\right)\right] = 0$. This is because $\Delta_0'^2 > \Delta_0^2$ and $\mu_0' > \mu_0$. And for other $j \neq 1$, $f_1\left(\bar{P}_{J-2}\right) = 0$, because $\bar{P}_J$ constitutes an informative equilibrium. Therefore,

$$\mathbf{f}\left(\bar{P}_{J'}\right) = \begin{pmatrix} \frac{1}{\tau}\left[\mathbb{E}R\left(m_1'\right) - \mathbb{E}R\left(m_0'\right)\right] \\ 0 \\ \vdots \\ 0 \end{pmatrix} > \mathbf{0}$$

If $J' = J - 1$, we consider the case of $J = 2K + 1$ and the partition $\bar{P}_{J'}$ is thus

$$\bar{P}_{J'} : -1 < -p_K < \cdots < -p_1 < p_1 < \cdots < p_K < 1$$

In this new partition,

$$\begin{aligned}
& \mathbb{E}\beta_n^{(2)}\left(\mu_K'\right) - \mathbb{E}\beta_n^{(2)}(0) \\
= \; & \mathbb{E}\beta_n^{(2)}\left(\mu_K\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_0\right) + \mathbb{E}\beta_n^{(2)}\left(\mu_0\right) - \mathbb{E}\beta_n^{(2)}(0) \\
< \; & 2\left[\mathbb{E}\beta_n^{(2)}\left(\mu_K\right) - \mathbb{E}\beta_n^{(2)}\left(\mu_0\right)\right] \\
= \; & 2 \cdot \frac{1}{12}\left[\Delta_0^2 - \Delta_K^2\right] \qquad (\bar{P}_J \text{ supports equilibrium}) \\
< \; & \frac{1}{12}\left[\left(\Delta_0'\right)^2 - \Delta_K^2\right] \qquad (\Delta_0' = 2\Delta_0)
\end{aligned}$$

The inequality on line 3 is because $\mathbb{E}\beta_n^{(2)}\left(\mu_j\right)$ is convex and $\mu_K - \mu_0 > \mu_0 - 0$. And therefore, $f_1\left(\bar{P}_{J-1}\right) > 0$ and $\mathbf{f}\left(\bar{P}_{J'}\right) > \mathbf{0}$. Following the similar approach, we also show that it is also true for the case of $J = 2K$.

Therefore, we have $\underline{P}_{J'} < \bar{P}_{J'}$, $\mathbf{f}\left(\underline{P}_{J'}\right) < \mathbf{0}$ and $\mathbf{f}\left(\bar{P}_{J'}\right) > \mathbf{0}$ for $J' = J - 1$ and $J - 2$. By applying the Intermediate Value Theorem above, we have that there exists a partition of degree J', $\tilde{P}_{J'}$, such that $\mathbf{f}\left(\tilde{P}_{J'}\right) = \mathbf{0}$. $\square$

2.C. Omitted Proofs for Section 2.6

Proof. [**Proposition 24**] When no information disclosure, $J = 0$, and the expected revenue is $\mathbb{E}R_0 = V - \tau\left[\frac{1}{3} + \frac{6}{(n+1)(n+2)}\right]$. We just need to show that an equal partition of $J = 1$ generates higher expected revenue than $\mathbb{E}R_0$ for each $n \geq 3$. Given our symmetric settings, an equal partition of $J = 1$ naturally supports an informative equilibrium, and the expected revenue is

$$\mathbb{E}R_1 = V - \tau\left\{\frac{1}{12} + \frac{6}{(n+1)(n+2)} + \frac{\left[1 - \frac{5}{n+1} + \frac{9}{(n+1)(n+2)}\right]}{2^{n+1}} - \frac{\left(2 - \frac{5}{n+1}\right)}{2^n} + \frac{1}{2^{n-1}}\right\}$$

To prove this, we just need to show for $n \geq 3$

$$\begin{aligned}
\frac{1}{3} &\geq \frac{1}{12} + 2^{-(n+1)}\left[1 - \frac{5}{n+1} + \frac{9}{(n+1)(n+2)}\right] - 2^{-n}\left(2 - \frac{5}{n+1}\right) + 2^{-n+1} \\
&= \frac{1}{12} + 2^{-(n+1)}\left[1 + \frac{5}{n+1} + \frac{9}{(n+1)(n+2)}\right]
\end{aligned}$$

the left hand side of the inequality is decreasing in n, and thus its maximum is achieved at $n = 3$, which is $\frac{121}{480}$. And it is strictly smaller than $\frac{1}{3}$. □

Proof. [Lemma 25] (1) $\mathbb{E}R(m_K) = V - \tau\left[\frac{(1-p_K)^2}{12} + \mathbb{E}\left(\beta_n^{(2)}(m_K)\right)\right]$

$$\begin{aligned}
\frac{1}{\tau}\frac{\partial \mathbb{E}R(m_K)}{\partial p_K} &= \frac{(1-p_K)}{6} - \frac{\partial \mathbb{E}\left(\beta_n^{(2)}(m_K)\right)}{\partial p_K} \\
\frac{1}{\tau}\frac{\partial^2 \mathbb{E}R(m_K)}{\partial p_K^2} &= -\frac{1}{6} - \frac{\partial^2 \mathbb{E}\left(\beta_n^{(2)}(m_K)\right)}{\partial p_K^2} < 0
\end{aligned}$$

since $\frac{\partial^2 \mathbb{E}\left(\beta_N^{(2)}(m_K)\right)}{\partial p_K^2} > 0$ from Corollary 6. We know

$$\frac{\partial \mathbb{E}\left(\beta_n^{(2)}(m_K)\right)}{\partial p_K} = \left[\left(n - 3 + \frac{4}{n+1}\right)\mu_K^2 + (3-2n)\mu_K + n\right]\mu_K^{n-1}$$

where $\mu_K = \frac{1}{2}(1+p_K)$. When $p_K = 1$, $\frac{1}{\tau}\frac{\partial \mathbb{E}R(m_K)}{\partial p_K} = 0 - \frac{4}{n+1} < 0$. When $p_K = 0$,

$$\frac{1}{\tau}\frac{\partial \mathbb{E}R(m_K)}{\partial p_K} = \frac{1}{6} - \left[\frac{n}{4} + \frac{1}{n+1} + \frac{3}{4}\right]\left(\frac{1}{2}\right)^{n-1} > 0 \quad \text{if } n \geq 5$$

and negative if $n < 5$. (2) For $n = 1, \cdots, 4$, we already knows that $p_K^* = 0$. When $n \geq 5$, we know at p_K^*, $\frac{\partial \mathbb{E}R(m_K)}{\partial p_K} = 0$ and it is unique.

$$\left.\frac{1}{\tau}\frac{\partial \mathbb{E}R(m_K)}{\partial p_K}\right|_{p_K^*} = \frac{(1-\hat{p}_K)}{6} - \frac{\partial \mathbb{E}\left(\beta_n^{(2)}(m_K; p_K^*)\right)}{\partial p_K} = 0$$

By taking total differentiation with respect to $\hat{p}_K$ and n,

$$\underbrace{\left[\frac{1}{6}+\frac{\partial^2\mathbb{E}\left(\beta_n^{(2)}\left(m_K;p_K^*\right)\right)}{\partial p_K^2}\right]}_{\text{positive}}d\hat{p}_K=-\underbrace{\frac{\partial^2\mathbb{E}\left(\beta_n^{(2)}\left(m_K;p_K^*\right)\right)}{\partial p_K\partial n}}_{\text{negative}}dn$$

and therefore $\frac{\partial p_K^*}{\partial n}>0$. □

Proof. [**Lemma 26**] From Proposition 21, we know that for two partitions of degree J' and $J^*(n)$, with $J'<J^*(n)$, we have $p_{K'}<p_K$, as $P_{j'}<\bar{P}_{j'}$. If $p_{K'}\geq p_K^*$, then $\mathbb{E}R\left(m_K\right)\big|_{p_{K'}}>\mathbb{E}R\left(m_K\right)\big|_{p_K}$ because $\mathbb{E}R\left(m_K\right)$ is strictly concave in p_K, which results in the contradiction that $J^*(n)$ is the optimal degree of partition. □

Proof. [**Theorem 27**] We prove by contradiction. When the number of bidders is n, the optimal degree of equilibrium partition is $J^*\left(n\right)$ and the corresponding expected revenue is $\mathbb{E}R^*\left(n\right)_{J=J^*(n)}$. When the number of bidders increases to $n'>n$, suppose the optimal degree is $J^*\left(n'\right)$ and

$$J^*\left(n'\right)<J^*\left(n\right)$$

Let $\mathbb{E}R^*\left(n'\right)_{J=J^*(n')}$ denote the optimal expected revenue in this case.

First, as $J^*\left(n'\right)<J^*\left(n\right)$, from Corollary 23, we know at n there also exists an informative equilibrium that is supportable by a partition of degree $J^*\left(n'\right)$. Furthermore, from Proposition 22, we have $p_{K'}\left(n\right)<p_K\left(n\right)$. In addition, as $J^*\left(n\right)$ is the optimal degree of equilibrium partition, from Lemma 26, we know that at n,

$$\frac{\partial\mathbb{E}R\left(m_K;n\right)}{\partial p_{K'}\left(n\right)}>0$$

Second, when the number of bidders increases from n to n', from Corollary 20, there are also exist informative equilibria supported by partitions of degree $J^*\left(n'\right)$

and $J^*(n)$. And as $J^*(n') < J^*(n)$, we have

$$p_{K'}(n') < p_K(n')$$

again from Proposition 22. Furthermore, for given degree of partition, the equilibrium partition converges to equal partition as the number of bidders increases, and therefore

$$p_{K'}(n') < p_{K'}(n) \text{ and } p_K(n') < p_K(n)$$

From Lemma 25, we know p_K^* increases in n. As a result, at n' we also have

$$\frac{\partial \mathbb{E}R(m_K; n')}{\partial p_{K'}(n')} > 0$$

Finally, we know at n, $\mathbb{E}R(m_K; n)|_{p_K(n)} > \mathbb{E}R(m_K; n)|_{p_{K'}(n)}$ as $J^*(n)$ is the optimal degree of equilibrium partition. In addition, as the number of bidders increases to $n' > n$, we already have that

$$p_{K'}(n') < p_K(n') < p_K(n)$$

and $p_{K'}(n') < p_K^*(n')$. Therefore, we get the following result that

$$\mathbb{E}R(m_K; n')|_{p_K(n')} > \mathbb{E}R(m_K; n')|_{p_{K'}(n')}$$

as $\frac{\partial \mathbb{E}R(m_K)}{\partial p_K \partial n} > 0$ from (3.14). And this implies that

$$\mathbb{E}R^*(n')_{J=J^*(n)} > \mathbb{E}R^*(n')_{J=J^*(n')}$$

which results in a contradiction! $\square$

2.D. Omitted Proofs in Section 2.7

Proof. [**Proposition 30**] For any two adjacent subintervals corresponding to m_{j-1} and m_j, we will show only when $\mathbb{E}R(m_{j-1}) > \mathbb{E}R(m_j)$ is it possible for the optimality condition to hold. First, if $\mathbb{E}R(m_{j-1}) = \mathbb{E}R(m_j)$, then $\frac{(p_{j+1}-p_j)^2}{12} - \frac{(p_j-p_{j-1})^2}{12} = \mathbb{E}\left[\beta_n^{(2)}(\mu_j)\right] - \mathbb{E}\left[\beta_n^{(2)}(\mu_j)\right] < 0$ which also implies $p_j - p_{j-1} > p_{j+1} - p_j > 0$

$$\begin{aligned}
\frac{1}{2\tau}\frac{\partial \mathbb{E}R}{\partial p_j} &= \frac{1}{\tau}\Delta_j \frac{\partial \mathbb{E}R(m_j)}{\partial p_j} + \frac{1}{\tau}\Delta_{j-1}\frac{\partial \mathbb{E}R(m_{j-1})}{\partial p_j} \\
&= \frac{\Delta_j^2}{6} - \Delta_j \frac{\partial \mathbb{E}\beta_n^{(2)}(m_j)}{\partial p_j} - \frac{\Delta_{j-1}^2}{6} - \Delta_{j-1}\frac{\partial \mathbb{E}\beta_n^{(2)}(m_{j-1})}{\partial p_j} \\
&= 2\underbrace{\left[\mathbb{E}\left[\beta_n^{(2)}(m_j)\right] - \mathbb{E}\left[\beta_n^{(2)}(m_j)\right]\right]}_{<0} - \Delta_{j-1}\underbrace{\frac{\partial \mathbb{E}\beta_N^{(2)}(m_{j-1})}{\partial p_j}}_{>0} - \Delta_j \underbrace{\frac{\partial \mathbb{E}\beta_N^{(2)}(m_j)}{\partial p_j}}_{>0} \\
&< 0
\end{aligned}$$

Second, if $\mathbb{E}R(m_{j-1}) < \mathbb{E}R(m_j)$, then it is necessary that $\Delta_{j-1} > \Delta_j > 0$

$$\begin{aligned}
&\frac{1}{2\tau}\frac{\partial \mathbb{E}R}{\partial p_j} \\
&= \frac{1}{\tau}\left[\mathbb{E}R(m_{j-1}) - \mathbb{E}R(m_j)\right] + \frac{1}{\tau}\Delta_j \frac{\partial \mathbb{E}R(m_j)}{\partial p_j} + \frac{1}{\tau}\Delta_{j-1}\frac{\partial \mathbb{E}R(m_{j-1})}{\partial p_j} \\
&= \frac{1}{\tau}\left[\mathbb{E}R(m_{j-1}) - \mathbb{E}R(m_j)\right] + \Delta_j\left[\frac{\Delta_j}{6} - \frac{\partial \mathbb{E}\beta_N^{(2)}(m_j)}{\partial p_j}\right] + \Delta_{j-1}\left[-\frac{\Delta_{j-1}}{6} - \frac{\partial \mathbb{E}\beta_N^{(2)}(m_{j-1})}{\partial p_j}\right] \\
&= \frac{1}{\tau}\underbrace{\left[\mathbb{E}R(m_{j-1}) - \mathbb{E}R(m_j)\right]}_{<0} - \Delta_j \underbrace{\frac{\partial \mathbb{E}\beta_N^{(2)}(m_j)}{\partial p_j}}_{>0} - \Delta_{j-1}\underbrace{\frac{\partial \mathbb{E}\beta_N^{(2)}(m_{j-1})}{\partial p_j}}_{>0} + \underbrace{\left[\frac{\Delta_j^2}{6} - \frac{\Delta_{j-1}^2}{6}\right]}_{<0} \\
&< 0
\end{aligned}$$

Therefore, in optimality it is necessary that

$$\mathbb{E}R(m_{j-1}) > \mathbb{E}R(m_j)$$

□

Proof. [**Proposition 31**] We already knows that $\lim_{n\longrightarrow\infty}\mathbb{E}\beta_n^{(2)}(m_j)=0$ and $\lim_{n\longrightarrow\infty}\frac{\partial\mathbb{E}\beta_N^{(2)}(m_j)}{\partial p_j}=0$. Taking limit of the first order condition, we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\frac{1}{2\tau}\frac{\partial\mathbb{E}R}{\partial p_j}\\
=\;&\lim_{n\longrightarrow\infty}\left[\frac{\Delta_j^2}{4}-\frac{\Delta_{j-1}^2}{4}\right]-\lim_{n\longrightarrow\infty}\left[\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]+\Delta_{j-1}\frac{\partial\mathbb{E}\beta_N^{(2)}\left(m_{j-1}\right)}{\partial p_j}\right]\\
&+\lim_{n\longrightarrow\infty}\left[\mathbb{E}\left[\beta_n^{(2)}\left(\mu_j\right)\right]-\Delta_j\frac{\partial\mathbb{E}\beta_N^{(2)}\left(m_j\right)}{\partial p_j}\right]=0
\end{aligned}$$

Therefore, when $n\longrightarrow\infty$, in optimality we have

$$\Delta_j^*=\Delta_{j-1}^*$$

which constitutes an equal partition of the interval. $\square$

CHAPTER 3

Information Disclosure in a Hotelling Model with Mixture Population

3.1. Introduction

It is common that there are different groups of consumers present in a market, who can be classified by their genres, occupations and so on, and their preferences over a certain product attribute can be highly heterogeneous. For example, for the product of shampoo, some consumers may prefer oily-hair shampoos, while others not.

By saying a product attribute, we actually mean here a product's horizontal attribute. A product's attributes can be divided into vertical and horizontal ones. For vertical attributes, such as quality, all the consumers have the same preference over. A product's horizontal attributes, however, can not be ordered in the same way, and different consumers normally have different preferences over them, such as color, design, etc.

Naturally, when there are different groups of consumers present in a market, revealing a product's horizontal attribute appeals just parts of the consumers, but not all. An interesting question is then how to optimally reveal a product's attributes. More interestingly, how is the optimal disclosure policy related to the distribution of consumer preferences? In this paper, we will explore these questions in a Hotelling model.

Hotelling model is a natural candidate for studying product horizontal differentiation. In the huge literature on Hotelling models, however, consumer preferences are commonly assumed to conform to either binary or uniform distributions, which is obviously unrealistic in many contexts. The main departure in this paper, if compared with the standard Hotelling literature, is that we assume consumer preferences conform to mixture distributions, which is more realistic and can better characterize real consumer preferences in a market.

Mixture distribution means that population distribution is a mixture of several component, usually simpler, distributions. In the literature of statistics, there is a growing interest in studying mixture distribution and finite mixture models. Initial researches focus on how to describe and characterize a mixture distribution, and later more attentions are casted into how to identify and estimate different mixture distributions (Titterington, et al, 1985; Everitt and Hand, 1981; McLachlan and Peel, 2000; Došlá, 2009).

About information disclosure, there has been a strand of literature on how to advertise product attributes to consumers. Lewis and Sappington (1994) is one of the earliest paper showing the trade-off between mass and niche market strategies under information disclosure. In their model, a product's value can be either high or low. When there is no information, consumers are willing to pay just a medium price for it. However, under information disclosure, some consumers receive a *high* message and are willing to pay higher prices for it, while others receive a *low* message. Therefore, when revealing information, the seller faces the trade-off between a medium price in a mass market and higher prices in a niche market, where only a portion of the consumers are served. They show that the seller will either reveal all or no information of the product. Johnson and Myatt (2006) study a similar question yet in a general framework. They also show the same trade-off between niche and mass markets, and the seller also adopts extreme information disclosure policies. Bar-Issac et al (2009) study the problem in a different setting, where consumers are *ex ante* heterogenous and information disclosure is acquired at a cost, and in contrast to the above results, they show in their model that intermediate disclosure strategy can be optimal. In these papers, it is the vertical values of a product that are advertised.

The paper of Sun (2009) is nearest to ours in model setting. She explores the issue of how to disclose a product's multiple attributes in a Hotelling model, and the main result is that when a product's vertical value is known to be high, it is less likely for the seller to reveal the horizontal attributes. This result, in fact, also reflects the trade-off between mass and niche markets. This is because revealing the horizontal attributes appeals only parts of the consumers, and when the vertical value is high, the market is already fully covered and it is better to choose a mass market strategy. In her model, consumers' preferences are assumed to be uniformly distributed.

A common assumption in the papers reviewed above is that the seller commits to truthful information disclosure, and therefore lying or mis-reporting is not possible. However, this assumption can be problematic in many situations, especially when the information is not verifiable, such as in the case of cheap-talk. Cheap-talk means that information is costless, non-verifiable and non-binding (Farell and Rabin, 1996), and non-binding implies that the seller is not committed to an announced disclosure rule. Under cheap-talk, the central issue is always the condition for credible or informative information.

In this paper, we will investigate how a monopoly seller should reveal a product's attributes to consumers, when their preferences are mixedly distributed. The vertical value of the product is assumed commonly known, and a consumer's valuation therefore depends on the matching between his preference and the product attribute. Specifically, we study the simple case of dichotomal disclosure, and the seller can send just two possible messages regarding the product attributes[1]. Informative information revealed by the seller assumes two basic functions: first, reducing the risks of mismatching for the consumers, and second, indicating the expected location of the product attribute. We will compare the results of disclosure with full commitment and those under cheap talk, where mis-reporting is possible. Furthermore, we will explore how optimal disclosure policy is related to the characteristics of the mixture distribution.

There are many real examples of advertising and information disclosure that fit the situations in our model, where a seller commits to a dichotomal reporting policy, but not to truthful disclosure. One example is shampoo, as a shampoo producer normally indicates it is for "men" or "women" users. The other example is COD liver oil, whose consumers are normally classified into different age ranges, such as adult, child, etc. We can still enumerate many other examples. A common feature of these examples is that labelling a product is easy and costless, yet it is highly costly or nearly impossible to verify whether the revealed information is true or not. In the case of shampoo and COD liver oil, it depends on product ingredients and nutrients, which is hardly verifiable by common consumers.

With this model setting, we first show that information disclosure increases trade possibility, in the sense that it makes possible some trades which are initially impossible if there is no disclosure. This is largely due to the risk-reduction effect mentioned above. Second, information disclosure under full commitment generates higher expected profit than corresponding cheap-talk disclosure. This is because, under cheap-talk, to make the revealed information to be credible, some additional non-deviation conditions need to be satisfied such that the seller won't deviate to mis-reporting, which therefore reduces the expected profits for the seller.

[1]Chapter 2 in this thesis studies cheap-talk information disclosure in auctions, where it is possible for the seller to send abitrary number of signals.

Furthermore, we show that optimal disclosure policies largely depend on the characteristics of the mixture distribution of consumer preferences, in an example of mixture normals. First, when different groups of consumers' preferences become sufficiently heterogenous, it is better for the seller to reveal product attributes and serve different groups separately. For example, in the case of mixture normal distribution, information disclosure becomes more profit-enhancing when the modes of the two component distributions are farther away from each other. Second, when the composition of different groups of consumers becomes more asymmetric, cheap-talk disclosure is more likely to be dominated by no disclosure at all, in terms of profit maximization. For example, in the case of mixture normal distribution, when the mixture proportion converges to 0 or 1, compared with cheap-talk disclosure, it's better for the seller to withhold information and reveal no information at all.[2]

The contributions of this paper to the literature are mainly twofold. First, to my best knowledge, it is the first paper studying information disclosure in a Hotelling model with *mixture* population. Compared with the standard assumption of either binary or uniform distributions, mixture distribution obviously better characterizes real consumer preferences in a market. Second, in addition to studying information disclosure with full commitment, this paper also investigates cheap-talk disclosure, where lying or mis-reporting is possible. By comparing the two scenarios, we are able to evaluate the impacts of commitment on the outcomes of information disclosure.

The remaining parts of this paper are organized as below. Section 3.2 describes the model setup. In Section 3.3, we provide some general results about information disclosure and trade possibility, and the impacts of commitment on disclosure outcomes. In Section 3.4, in an example of mixture normal distribution, we show

[2] Furthermore, we also show in a numerical example that when the vertical value is sufficiently high, it is better for the seller to withhold the information and send no messages. This is because when the vertical value is sufficiently high, the market is nearly fully covered even no information is revealed, and it is better for the seller adopt a mass market strategy. This result is also consistent with the main result of Sun (2009).

how optimal information disclosure policy is related to the features of the mixture distribution. And Section 3.5 is a short conclusion.

3.2. The Model

Consider a market where a profit-maximizing monopoly seller sells a product to a continuum of consumers. The product is produced at constant marginal cost, c, and is characterized by its vertical value, V, and horizontal attribute, S. The vertical value V is commonly known, but the horizontal attribute S is a random variable, whose realization s is known only by the seller.

We normalize the population of the continuum of consumers to one, and each consumer has a unit demand, say he buys either one or zero unit of the product. A consumer's valuation of the product depends on the matching between the product's horizontal attribute s and his own preference θ. Let θ be the type of a consumer, and the vNM utility of a type θ consumer buying one unit of product is

$$u(s,p;\theta) = V - \tau\,(s-\theta)^2 - p \tag{3.1}$$

where p is the price and τ a coefficient measuring the degree of disutility of mismatching.

It is worth attention that consumers are risk averse in s, the realization of the horizontal attribute, since $u(\cdot)$ is strictly concave in s. Consumer's reservation utility is normalized to 0, so a type θ consumer will buy the product if and only if the expected utility $U(p;\theta) = \mathbb{E}_s u(s,p;\theta) \geq 0$. For simplicity, we assume the horizontal attribute S is uniformly distributed on $[-N, N]$, $N > 0$.

The population of the consumers consists of two subgroups, denoted as 1 and 2, and their probability density functions are $f_1(\theta)$ and $f_2(\theta)$ respectively. The mixture proportion of group 1 is $\alpha \in (0,1)$. Therefore, the density function of the mixture population, denoted as $g(\theta)$, is

$$g(\theta) = \alpha f_1(\theta) + (1-\alpha) f_2(\theta) \tag{3.2}$$

We assume $f_1(\theta)$ and $f_2(\theta)$ are both uni-modal, differentiable and strictly positive. For simplicity, we assume $f_1(\theta)$ has a unique mode at $-M$, and $f_2(\theta)$ at M with $0 < M \leq N$.

Before setting the price, the seller may send a public message regarding the horizontal attribute of the product, in the following dichotomal form[3]

$$m = \begin{cases} L & \text{if } s \in [-N, s^*] \\ R & \text{if } s \in (s^*, N] \end{cases} \tag{3.3}$$

and the cutoff value s^* is a policy variable of the seller. Obviously, when $s^* = \pm N$, , it is strategically equivalent to no information disclosure.

After announcing a message m, the seller set a price for the product, denoted as $p^m = \left\{p^L, p^R\right\}$. Therefore, p^m implies both a message of m announced and a price of p^m charged. At p^m, the expected demand for the product is

$$D(p^m) = \int_{\theta \in B(p^m)} g(\theta) d\theta$$

where $B(p, m)$ is the set of consumer types with positive demand, so

$$B(p^m) = \left\{\theta : V - p^m - \tau \mathbb{E}_s \left[(s-\theta)^2 |m\right] \geq 0\right\}$$

[3]In this paper, we focus on the disclosure rule in the form of (3.3), where there are two *connected* subintervals corresponding to the two distinct messages. There are other dichotomal disclosure rules where the partition are not connected, which is disregarded here. For example, there are two cutting points s_1 and s_2 such that $-N < s_1 < s_2 < N$: for attributes $s \in [-N, s_1]$ or $[s_2, N]$, a message L is reported, and for $s \in (s_1, s_2)$ a message R reported.

At the interim stage when the seller observes value of s, the interim expected profit for a message m is

$$\mathbb{E}\pi(p^m) = (p^m - c)\, D(p; m) \tag{3.4}$$

Except for s, the realization of the horizontal attribute, all other information is assumed to be common knowledge. And the timing of the game is as below:

(1) The seller announces the disclosure policy in the dichotomal form of (3.3);

(2) Nature selects the realization of S, which is then privately observed by the seller;

(3) The seller decides whether or not to reveal information. If yes, she will send a public message, m, according to (3.3), and then she sets the price p^m;

(4) Receiving the message m, the consumers decide whether or not purchase the product at p^m.

In this chapter, we will investigate only the pure strategy equilibria in our model, and focus on the specific equilibrium outcomes under the binary disclosure policy and pricing strategies specified above.

3.3. Disclosure, Commitment and Trade Possibility

We study in this section, first, how information disclosure affects trade possibility, and then the impacts of committing to truthful disclosure on equilibrium outcomes, in a general setting. In next section, we will explore how optimal disclosure policy is related to the characteristics of the mixture preference distribution. The main results of this section are, first, information disclosure increases trade possibility, and second, disclosure under full commitment dominates cheap-talk disclosure in terms of profit maximization.

3.3.1. Benchmark: No Information Disclosure

If there is no information revealed by the seller, the consumers will keep their prior of S unchanged. Let p be the price charged, and a type θ consumer's expected utility in this case is

$$U(p;\theta) = V - p - \tau \mathbb{E}_s (s-\theta)^2 = V - p - \tau \left(\theta^2 + \frac{1}{3} N^2 \right)$$

Given this, it is easy to derive that the consumers with type

$$|\theta| \leq \sqrt{\frac{1}{\tau}(V-p) - \frac{1}{3}N^2} = \upsilon(p) \tag{3.5}$$

have strictly positive demand for the product, and the threshold $\upsilon(p)$ is decreasing in both p and N.

The seller chooses p to maximize the expected profit

$$\mathbb{E}\pi(p) = (p-c)\, D(p) = (p-c) \int_{-\upsilon(p)}^{\upsilon(p)} g(\theta) d\theta \tag{3.6}$$

By charging a higher p, the seller faces the trade-off between increased unit profit and reduced total demand for the product, as increasing price reduces the value of $\upsilon(p)$ and thus the demand $D(p)$ in this case. We use π^o denote the maximum profit under no information disclosure, and its existence is guaranteed by the maximum value theorem.

The minimum possible price for the seller is the marginal cost c, at which $\upsilon(p)$ achieves its maximum, $\upsilon(c)$. If $\frac{1}{\tau}(V-c) - \frac{1}{3}N^2 < 0$, $\upsilon(c)$ is not well-defined and therefore trade is not possible. We then get the following result on the possibility of trade in this case.

Proposition 32. *When there is no information disclosure, trade is not possible iff*

$$V - c < \frac{\tau}{3} N^2 \tag{3.7}$$

The left hand side of the inequality, $V - c$, can be interpreted as net value of the product, and N is a parameter that measures the risk facing the consumers. For example, when N increases, the consumers face greater uncertainty in the realization of the horizontal attribute. So the intuition behind the above statement is that, in the case of no information disclosure, when the product net value can not compensate the risk facing the consumers, no trade will happen even at the minimum possible price c.

3.3.2. Information Disclosure with Full Commitment

We then study information disclosure with full commitment, which implies that the seller will truthfully follow the disclosure policy of (3.3), if she decides to reveal information. Therefore, the seller's message is credible in this case.

When receiving a message of L, the consumers will know that the conditional distribution of S is now a uniform distribution on $U[-N, s^*]$, according to (3.3). The expected utility for a type θ consumer is thus

$$U(p; \theta | \, m = L) = V - p - \tau \left[\left(\theta - \frac{s^* - N}{2} \right)^2 + \frac{1}{12} \left(N + s^* \right)^2 \right]$$

which is symmetric to the conditional expectation of the horizontal attribute, $\left(s^* - N \right)/2$. And it is easy to show that for the consumers with type θ located in the following interval

$$\underline{v}^L(p) \leq \theta \leq \bar{v}^L(p) \tag{3.8}$$

have positive demand, where

$$
\begin{aligned}
\underline{v}^L(p) &= \frac{s^*-N}{2} - \sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N+s^*)^2} \\
\bar{v}^L(p) &= \frac{s^*-N}{2} + \sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N+s^*)^2}
\end{aligned}
$$

Following the same calculation, we can get the corresponding result for a R message: the consumers with θ such that

$$
\underline{v}^R(p) \leq \theta \leq \bar{v}^R(p) \tag{3.9}
$$

have positive demand for the product, where

$$
\begin{aligned}
\underline{v}^R(p) &= \frac{N+s^*}{2} - \sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N-s^*)^2} \\
\bar{v}^R(p) &= \frac{N+s^*}{2} + \sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N-s^*)^2}
\end{aligned}
$$

So far we have derived the intervals of consumer type, θ's, for each message, on which the consumers have positive demand for the product, as shown in (3.5), (3.8) and (3.9). We then study how information disclosure can affect trade possibilities. Let range$^{\varnothing}$, range^L and range^R represent the lengths of the corresponding type intervals for $m = \varnothing$ (no information disclosure), $m = L$ and R respectively, and

$$
\begin{aligned}
\text{range}^{\varnothing} &= 2v(p) = 2\sqrt{\frac{1}{\tau}(V-p) - \frac{1}{3}N^2} \\
\text{range}^{L} &= \bar{v}^L(p) - \underline{v}^L(p) = 2\sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N+s^*)^2} \\
\text{range}^{R} &= \bar{v}^R(p) - \underline{v}^R(p) = 2\sqrt{\frac{1}{\tau}(V-p) - \frac{1}{12}(N-s^*)^2}
\end{aligned} \tag{3.10}
$$

and the following result is obvious.

Proposition 33. *For* $\forall s^* \in [-N, N]$*, at a given price*

$$\min\left\{range^L, range^R\right\} \geq range^{\varnothing} \tag{3.11}$$

$range^L$ *decreases in* s^**, and* $range^R$ *increases in* s^**.*

This result illustrates the role of risk reduction in information disclosure. In the above expressions (3.10), the terms $\frac{1}{3}N^2$, $\frac{1}{12}(N+s^*)^2$ and $\frac{1}{12}(N-s^*)^2$ measure the risks facing the consumers for $m = \varnothing$, L and R respectively. By introducing a cutoff value $s^* \in [-N, N]$, information disclosure reduces consumers' risks in purchase, since the values of both $\frac{1}{12}(N+s^*)^2$ and $\frac{1}{12}(N-s^*)^2$ are no greater than that of $\frac{1}{3}N^2$ for any $s^* \in [-N, N]$. Roughly speaking, as a result of risk reduction, more types of consumers are willing to buy the product under information disclosure, for given price. It is worth attention that this range expansion doesn't necessarily imply greater demand for the production, which still depends on the mixture distribution itself.

We already know that, when there is no disclosure, trade is not possible if and only if $V - c < \frac{\tau}{3}N^2$. However, a direct implication of Proposition 2 is that information disclosure increases trade possibilities, as a result of the range expansion effect, which is formalized as below:

Corollary 34. *For information disclosure under full commitment, trade is possible iff*

$$V - c > 0 \tag{3.12}$$

Intuitively, this is because as the cutoff point, s^*, approaching the terminal points of either N or $-N$, the risks facing the consumers gradually reduces to zero at one message,and trade is possible for a product with positive net values.

We now turn to the optimal expected profit. Naturally, after a message m is reported in this case, the seller will charge an optimal price, denoted as $\hat{p}^m$, to maximize the interim expected profit

$$\mathbb{E}\pi(p^m) = (p^m - c) \int_{\underline{v}^m}^{\bar{v}^m} g(\theta)\, d\theta$$

We denote the optimal interim expected profit as $\hat{\pi}^m$. Both $\hat{\pi}^m$ and $\hat{p}^m$ are functions of the cutoff value s^*. Then the seller's problem is to select the optimal s^*, to maximize the *ex ante* expected profit, as below

$$\mathbb{E}\pi(s^*) = \frac{N + s^*}{2N} \cdot \hat{\pi}^L + \frac{N - s^*}{2N} \hat{\pi}^R \tag{3.13}$$

It is worth attention that, when fully committing to the disclosure policy, the seller maximizes the expected profit at the *ex ante* stage, before she knows the realization of the horizontal attribute S.

We denote this optimal cutoff value as $\hat{s}^*$, and the corresponding maximum *ex ante* expected profit as $\hat{\pi}^C$, where the superscript C means full commitment. Because $s^* \in [-N, N]$ and $\mathbb{E}\pi(s^*)$ is continuous in s^*, the existence of $\hat{\pi}^C$ is guaranteed by the maximum value theorem. As mentioned above, the benchmark case is strategically equivalent to setting $s^* = \pm N$ in the case of full commitment, and we then get the following result:

Lemma 35. *For information disclosure with full commitment, there exists a triple $\left(\hat{s}^*, \hat{p}^L, \hat{p}^R\right)$ that maximizes the expected profit, and the maximum expected profit, $\hat{\pi}^C$, is greater than that under no information disclosure, that is*

$$\hat{\pi}^C \geq \pi^o$$

This result is natural and reflects an important property of costless disclosure, that is, profit-enhancing. A seller decides to reveal product information just because it increases the expected profit.

Figure 3.1 below shows an example of a mixture distribution of consumer preferences, and the demands for the product at different messages in the case of full commitment. The shaded areas correspond to the market demands when the messages are L and R respectively.

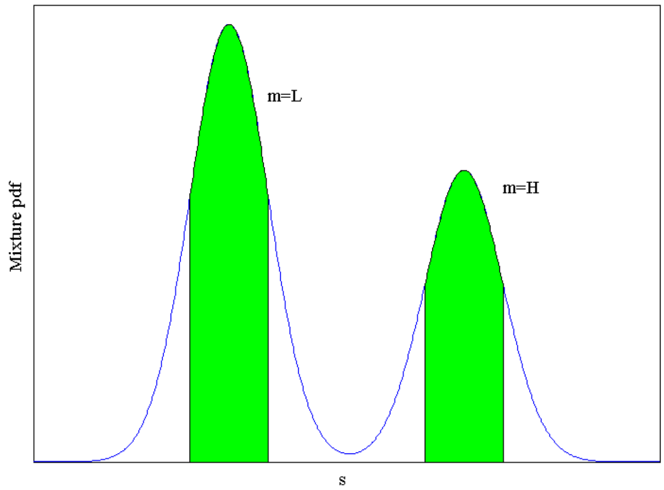


Figure 3.1. Market Demand under Disclosure with Commitment

3.3.3. Cheap-talk Information Disclosure

We now investigate the case of cheap-talk information disclosure. Under cheap-talk, although the seller has announced the dichotomal reporting policy of (3.3), she can still freely switch between R and L message, *after* observing the realization of S. Therefore, mis-reporting is possible in this case, and the central issue here is the condition for credible information.

In cheap-talk games, equilibria can generally be divided into two groups. One is pooling equilibrium, also known as *babbling equilibrium*, in which the messages sent by the seller are not credible and therefore no informative information is conveyed. In this case, the consumers will disregard the messages sent by the seller and keep their priors unchanged, which is equivalent to the benchmark case of no information disclosure. An a babbling equilibrium always exists in this kind of cheap-talk games (Gibbons, 1992, Chapter 4). The other is *informative equilibrium*, where, under certain conditions, the seller truthfully follows the disclosure rule and thus convey credible information. In this case, when receiving a message, the consumers will update their beliefs and adjust their strategies accordingly. Due to the triviality of babbling equilibria, we will focus on informative equilibria.

Specifically, we are interested in *non-degenerated* informative equilibrium, where the cutoff point is strictly interior, that is, $s^* \in (-N, N)$. In the *degenerated* case of $s^* = \pm N$, the seller returns to the benchmark case of no information disclosure and achieves the profit of π^o. For example, by setting $s^* = N$, with probability zero the seller reports a message of R, and a message of L reveals no more information than in the benchmark case. As setting $s^* = \pm N$ is always an option available for the seller, if a non-degenerated informative equilibrium exists, it is necessarily (weakly) profit-enhancing. We then first explore the conditions for the existence of a non-degenerated informative equilibrium, and then study its profit properties.

Without any commitment requirement, for truthful information disclosure to be possible, it is necessary that the disclosure policy itself is self-enforcing for the seller, that is, the seller has no incentive to deviate from truthful reporting, which implies the following non-deviation conditions

$$\begin{aligned} \mathbb{E}\pi\left(p^L \left| s \leq s^* \right.\right) \geq \mathbb{E}\pi\left(p^R \left| s \leq s^* \right.\right) \\ \mathbb{E}\pi\left(p^R \left| s > s^* \right.\right) \geq \mathbb{E}\pi\left(p^L \left| s > s^* \right.\right) \end{aligned} \tag{3.14}$$

which simply puts that truthful reporting is preferable to mis-reporting. We observe that the interim expected profits depends just on the message m, not s, and therefore condition (3.14) must be held with equalities.

We denote the maximum profit in an informative equilibrium as $\hat{\pi}^{NC}$, where the superscript "NC" represents no commitment, and the corresponding optimal cutoff value and prices are $\tilde{s}^*$, $\tilde{p}^L$ and $\tilde{p}^R$. Then we have the following result:

Proposition 36. *Under cheap-talk information disclosure, if there exists a triple* $\left(\tilde{s}^*, \tilde{p}^L, \tilde{p}^R\right)$ *that supports a (non-degenerated) informative equilibrium, it is necessary that*

1) $\mathbb{E}\pi\left(\tilde{p}^L \left| s \leq s^*\right.\right) = \mathbb{E}\pi\left(\tilde{p}^R \left| s > s^*\right.\right)$ *(non-deviation condition);*

2) $\hat{\pi}^{NC} \geq \pi^o$ *(profit-enhancing property).*

An interesting question is regarding the comparison of the maximum profits achieved under disclosure with and without commitments. Intuitively, information disclosure with commitment always generates greater profit than one under cheap-talk. This is because, under cheap-talk, the interim non-deviation conditions need to be satisfied in an informative equilibrium, which is absent in the case of disclosure with full commitment.

And the non-deviation condition implies that, in a non-denegerated informative equilibrium supported by $\left\{\tilde{s}^*, \tilde{p}^L, \tilde{p}^R\right\}$, the maximum *ex ante* expected profit, $\hat{\pi}^{NC}$, is just equal to the interim expected profit $\mathbb{E}\pi(\tilde{p}^L)$ or $\mathbb{E}\pi(\tilde{p}^R)$, because they are equal in equilibrium.

Moreover, we already know that, in the case of disclosure with *full commitment*, at the same cutoff point of $\tilde{s}^*$, the maximum interim expected profits achievable are $\hat{\pi}^L(\tilde{s}^*)$ and $\hat{\pi}^R(\tilde{s}^*)$ respectively, and the corresponding prices charged are $\hat{p}^L(\tilde{s}^*)$ and $\hat{p}^R(\tilde{s}^*)$. In an informative equilibrium under cheap-talk, the interim expected profits must be equal, and therefore the maximum *ex ante* expected profit, $\hat{\pi}^{NC}$, must be equal to the minimum of $\hat{\pi}^L(\tilde{s}^*)$ and $\hat{\pi}^R(\tilde{s}^*)$, otherwise the seller can increase the profit without violating the non-deviation condition. A further implication is that, for the prices charged in an informative equilibrium under cheap-talk, $\tilde{p}^L$ and $\tilde{p}^R$, we must have either $\tilde{p}^L = \hat{p}^L$ or $\tilde{p}^R = \hat{p}^R$. We formalize the results in the following proposition

Proposition 37. *For dichotomal information disclosure, the one under full commitment always generates greater expected profit than that under cheap-talk, that is,*

$$\hat{\pi}^C \geq \hat{\pi}^{NC}$$

And in a non-degenerated informative equilibrium supported by the triple of $\left\{\tilde{s}^*, \tilde{p}^L, \tilde{p}^R\right\}$,

$$\hat{\pi}^{NC} = \min\left\{\hat{\pi}^L(\tilde{s}^*), \hat{\pi}^R(\tilde{s}^*)\right\}$$

and either $\tilde{p}^L = \hat{p}^L$ *or* $\tilde{p}^R = \hat{p}^R$ *or both.*

The existence of a (non-degenerated) informative equilibrium, however, is not necessarily guaranteed, which depends on the shape of the mixture distribution $g(\theta)$, as well as the distribution of the horizontal attributes S. In the next section, we will show some examples for the non-existence, as well as the existence, of informative equilibria with mixture normal distribution.

3.4. Optimal Disclosure: An Example of Mixture Normals

So far we haven't explored how optimal disclosure policy is related to the mixture distribution of consumer preferences. In this section, we try to answer this question in an example of mixture normal distribution. Specifically, the population distribution is a mixture of two normal distributions with equal variance, normalized to be 1, with mean $-\mu$ and μ respectively. The density functions of the component distributions are thus

$$\varphi_1(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta+\mu)^2} \qquad \varphi_2(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\theta-\mu)^2} \tag{3.15}$$

with corresponding cumulative distribution functions $\Phi_1(\theta)$ and $\Phi_2(\theta)$. The density function of the mixture population is

$$g(\theta) = \alpha\varphi_1(\theta) + (1-\alpha)\varphi_2(\theta) \tag{3.16}$$

with cumulative distribution function $G(\theta) = \alpha\Phi_1(\theta) + (1-\alpha)\Phi_2(\theta)$.

For simplicity, we assume the horizontal attribute S conforms to a binary distribution, and takes the values of $-\mu$ and μ with equal probability. Under the symmetric setting, without loss of generality, we assume $\alpha \geq \frac{1}{2}$. Throughout this section, we assume that $c = 0$ and $\tau = 1$ for simplicity. The timing and the structure of the game remain the same as before. We first have the following result of component distributions:

Lemma 38. *For any $\theta \in \mathbb{R}$, we have $\varphi_1(\theta) = \varphi_2(-\theta)$ and*

$$\Phi_1(\theta) - \Phi_1(-\theta) = \Phi_2(\theta) - \Phi_2(-\theta) \tag{3.17}$$

3.4.1. Benchmark: No disclosure

We already know that, for the case of no disclosure, the consumers with $|\theta| \leq \upsilon(p)$ have positive demand for the product. It is easy to show that, in the example of mixture normals

$$\upsilon(p) = \sqrt{V - p - \mu^2} \tag{3.18}$$

If $V < \mu^2$, then $\upsilon(p)$ is not well defined for any positive price p.

Therefore, when there is no disclosure, trade is not possible if and only if $V < \mu^2$. This result is quite intuitive, since μ^2 is a measure of the risk facing the consumers. If $V < \mu^2$, then the vertical value is not high enough to compensate the risk facing the consumers, and therefore no trade happens in this case.

For $V \geq \mu^2$, the expected profit is

$$\mathbb{E}\pi(p) = p\left[G\left(\upsilon(p)\right) - G\left(-\upsilon(p)\right)\right] = p\left[\Phi_1\left(\upsilon(p)\right) - \Phi_1\left(-\upsilon(p)\right)\right]$$

where the last equality is a direct result of Lemma 38. $\Phi_1\left(\upsilon(p)\right) - \Phi_1\left(-\upsilon(p)\right)$ is the total demand for the product, which is strictly decreasing in p. An increase in price will increase unit profit, yet reduce the total demand. When the first effect dominates the second one, the profit will increase with price. Otherwise, increasing price will reduce the profit. The first order condition is

$$\frac{\partial\pi}{\partial p} = \left[\Phi_1(\upsilon) - \Phi_1(-\upsilon)\right] - \frac{p}{2\upsilon}\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right] = 0$$

and the second-order condition is satisfied[4]. We then have the following result:

Lemma 39. *When there is no disclosure and $V \geq \mu^2$, the profit maximizing price is*

$$p^o = 2\upsilon \cdot \frac{\Phi_1(\upsilon) - \Phi_1(-\upsilon)}{\varphi_1(\upsilon) + \varphi_1(-\upsilon)} \tag{3.19}$$

and

$$\frac{\partial\pi^o}{\partial V} > 0 \quad \frac{\partial\pi^o}{\partial\mu} < 0 \quad \frac{dp^o}{d\mu} < 0 \tag{3.20}$$

[4]The second order derivative is

$$\begin{aligned}\frac{\partial^2\pi^o}{\partial p^2} &= \left[\varphi_1(\upsilon) + \varphi_2(\upsilon)\right](2\upsilon' + p\upsilon'') + p\left[\varphi_1'(\upsilon) + \varphi_2'(\upsilon)\right]\upsilon'^2 \\ &= \left[\varphi_1(\upsilon) + \varphi_2(\upsilon)\right](2\upsilon' + p\upsilon'') + p\upsilon'^2\left[-(\upsilon+\mu)\varphi_1(\upsilon) + (\mu-\upsilon)\varphi_2(\upsilon)\right] \lesseqgtr 0\end{aligned}$$

where $\varphi_1'(\upsilon) = -(\upsilon+\mu)\varphi_1(\upsilon)$ and $\varphi_2'(\upsilon) = (\mu-\upsilon)\varphi_2(\upsilon)$. Except for the last term $(\mu-\upsilon)\varphi_2(\upsilon)$, which is positive when $\mu > \upsilon$, all the other terms are negative. Using Matlab to plot the profit function, it is easy to see that this is a concave function in p in numerical examples.

Moreover, both π^o and p^o are independent of α.

We observe that, in the case of no disclosure, first, neither the expected profit nor the optimal price depends on the mixture ratio, α, because it is a mixture distribution of two symmetric normal distributions, and the binary distribution of S is also symmetric to zero. The second and obvious result is that, the expected profit increases with V, since for a higher value of V, the demand for the product is greater *ceteris paribus*. Finally, increasing μ reduces the maximum expected profit π^o. This is because increasing μ increases the risk of mismatching for the consumers, and thus reduces the demand. Interestingly, increasing μ also decreases the optimal price p^o, because when μ increases the tastes of the two groups of consumers become more heterogeneous, and therefore the seller should charge lower prices so as to increase the market coverage.

The two figures below show numerically how the expected profit changes with V and μ respectively. For given μ, the optimal profit increases with V, and when the price is low enough, the profit increases in the proportion with the price, since the market is nearly fully covered in this case. The figure on the right shows that for given price, the profit decreases with μ, as a result of increasing risk.

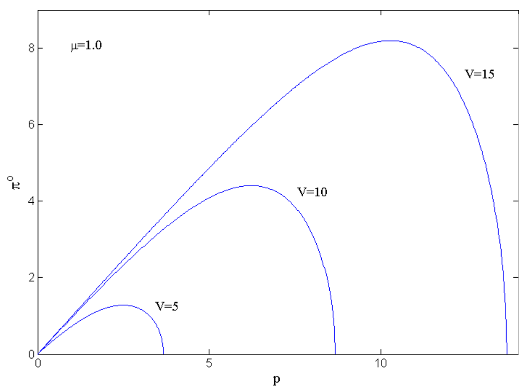
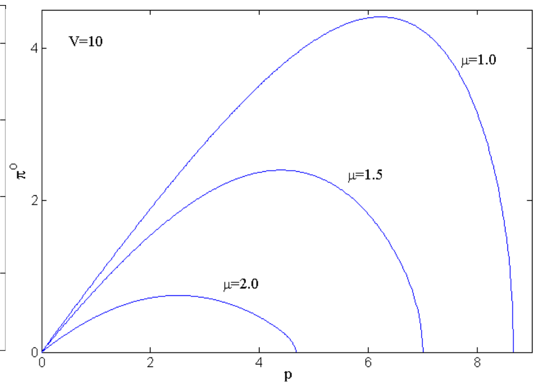


Figure 3.2. Profit vs. V



Profit vs. μ

3.4.2. Information Disclosure with Full Commitment

From this subsection on, we investigate how optimal profits under information disclosure is related to the characteristics of the mixture normal distribution. As before, the seller may advertise the horizontal attributes in the following dichotomal form

$$m = \begin{cases} L & \text{if } s = -\mu \\ R & \text{if } s = \mu \end{cases} \tag{3.21}$$

Under full commitment, it is easy to show that, for a message L and price p, the consumers with type $\theta \in \left[\underline{\upsilon}^L(p), \bar{\upsilon}^L(p)\right]$ will purchase the product, where $\underline{\upsilon}^L(p) = -\mu - \sqrt{V-p}$ and $\bar{\upsilon}^L(p) = -\mu + \sqrt{V-p}$. Similarly, for $m = R$, correspondingly we have $\underline{\upsilon}^R(p) = \mu - \sqrt{V-p}$ and $\bar{\upsilon}^R(p) = \mu + \sqrt{V-p}$.

It is obvious that range^L =range^R $= 2\sqrt{V-p}$, which is positive if $V \geq p$. Therefore, for dichotomal information disclosure under full commitment, trade is possible if and only if $V \geq 0$. As before, this result is due to the risk reduction effects of information disclosure, but what is different here is that, in our example, a truthful message of R or L implies removing all the risk facing the consumers.

For a message $m = L$ or R, the interim expected profit is

$$\mathbb{E}\pi^m = p^m \int_{\underline{\upsilon}^m}^{\bar{\upsilon}^m} g(\theta)d\theta = p^m \int_{\underline{\upsilon}^m}^{\bar{\upsilon}^m} \left[\alpha\varphi_1(\theta) + (1-\alpha)\varphi_2(\theta)\right] d\theta$$

and the first order condition for profit maximization is

$$\frac{\partial \mathbb{E}\pi^m}{\partial p} = \left[G(\bar{\upsilon}^m) - G(\underline{\upsilon}^m)\right] + p^m \left[g(\bar{\upsilon}^m) + g(\underline{\upsilon}^m)\right] \frac{\partial \bar{\upsilon}^m}{\partial p} = 0$$

from which we get the optimal price, $\hat{p}^m$, and expected profit, $\hat{\pi}^m$, as follows

$$\begin{cases} \hat{p}^m = \left(\bar{\upsilon}^m - \underline{\upsilon}^m\right) \frac{G(\bar{\upsilon}^m) - G(\underline{\upsilon}^m)}{g(\bar{\upsilon}^m) + g(\underline{\upsilon}^m)} \\ \hat{\pi}^m = \hat{p}^m \left[G(\bar{\upsilon}^m(\hat{p}^m)) - G(\underline{\upsilon}^m(\hat{p}^m))\right] \end{cases} \qquad m = L, R$$

Obviously, the values of the optimal interim profit, $\hat{\pi}^m$, is determined by the values of the parameters $\{V, \mu, \alpha\}$. V is the vertical value of the product which is common to all the consumers, and μ and α are parameters that fully determine the shape of the mixture distribution. μ measures the distance between these two component normal distributions, and α is the mixture proportion.

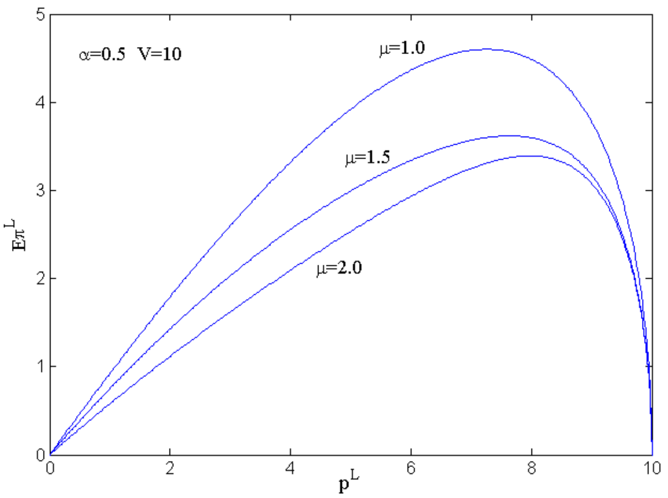


Figure 3.3. $\mathbb{E}\pi^L$ vs. μ

Figure 3.3 above provides numerical results on how the profit $\mathbb{E}\pi^L$ varies with the price p for different values of μ. We first get the following two results related to the interim expected profits

Lemma 40. *(1)* $\hat{\pi}^L(\alpha) = \hat{\pi}^R(1-\alpha)$, *and for* $\mu > 0$,

$$\frac{\partial \hat{\pi}^L}{\partial \alpha} > 0 \quad and \quad \frac{\partial \hat{\pi}^R}{\partial \alpha} < 0 \tag{3.22}$$

(2) For $m = L$ *and* R, $\frac{\partial \hat{\pi}^m}{\partial \mu} < 0$ *and* $\frac{\partial \hat{\pi}^m}{\partial V} > 0$.

The interim profits $\hat{\pi}^L$ and $\hat{\pi}^R$ are symmetric to $\alpha = 0.5$, which is a direct result of our symmetric setting. $\hat{\pi}^L$ increases in α because α is the mixture proportion of group 1 consumers and a message of L corresponds to $s = -\mu$, which is exactly the mode of the component distribution of group 1. Furthermore, $\hat{\pi}^m$ decreases in μ is due to the demand reduction effect of increasing μ. For example, for $m = L$ and given price, the demand from group 1 consumers remains unchanged, but that from group 2 consumers will decrease with increasing μ, since they are farther away from their ideal horizontal attribute.

Let $\pi^C(V, \mu, \alpha)$ denote the profit in this case of information disclosure under full commitment, and

$$\pi^C(V, \mu, \alpha) = \frac{1}{2}\hat{\pi}^L(V, \mu, \alpha) + \frac{1}{2}\hat{\pi}^R(V, \mu, \alpha) \tag{3.23}$$

The proposition below provides some comparative statics results of the relationship between π^C and the values of the parameters $\{V, \mu, \alpha\}$.

Proposition 41. *For information disclosure under full commitment, (1) π^C increases with V, decreases in μ*

$$\frac{\partial \pi^C}{\partial V} > 0 \quad \frac{\partial \pi^C}{\partial \mu} < 0$$

and $\lim_{\mu \to \infty} \frac{\partial \pi^C}{\partial \mu} = 0$*; and (2)* $\pi^C(\alpha) = \pi^C(1-\alpha)$ *and*

$$\frac{\partial \pi^C(\alpha)}{\partial \alpha} = -\frac{\partial \pi^C(1-\alpha)}{\partial \alpha}$$

Specifically, $\left.\frac{\partial \pi^C(\alpha)}{\partial \alpha}\right|_{\alpha=0.5} = 0.$

The result of V is quite natural, since for given prices, increasing V results in greater demand for the product, and thus higher values of $\hat{\pi}^L$ and $\hat{\pi}^R$. This result of (μ, α) shows that increasing the distance between the two component distributions will reduce the value of π^C. In our setting, S takes the values of $-\mu$ and μ respectively, so under information disclosure, increasing μ won't reduce the demand from the *targeted* group, but that from the *un-targeted* groups. For example, increasing μ, the decrease in $\hat{\pi}^L$ is mainly due to the demand reduction of group 2 consumers, and the decrease in $\hat{\pi}^R$ is mainly due to the demand reduction of group 1 consumers.

The symmetric result of (2) stems also from the symmetric setting in our model. It is noteworthy that the value of π^C is not necessarily greater than π^o, which depends on the values of the parameters $\{V, \alpha, \mu\}$. Let $\hat{\pi}^C$ denote the *optimal* profit under information disclosure with full commitment, and we get

$$\hat{\pi}^C = \max\left\{\pi^C, \pi^o\right\} \tag{3.24}$$

Only when $\pi^C \geq \pi^o$ should the seller choose to reveal the product's horizontal attributes in the case of full commitment. And in Subsection 3.4.4 below, we will discuss the conditions under which the inequality is held strictly.

3.4.3. Cheap-talk Information Disclosure

From Proposition 5 we know that, for the existence of a non-degenerated informative equilibrium, it is necessary that the non-deviation condition is satisfied. In this example of mixture normals, it is $\mathbb{E}\pi^L(\tilde{p}^L) = \mathbb{E}\pi^R(\tilde{p}^R)$, that is, the interim expected profits are equal for $m = L$ and R. Since s^* is no longer a choice variable, the only control variables for the seller are the prices, $\tilde{p}^L$ and $\tilde{p}^R$. The result below states that the non-deviation condition can always be satisfied at some prices.

Lemma 42. *For $V > 0$, there always exists a price pair $\left\{\tilde{p}^L, \tilde{p}^R\right\}$ such that $\pi^L(\tilde{p}^L) = \pi^R(\tilde{p}^R) = \pi^{NC}$.*

It is possible that $\pi^{NC} < \pi^o$, depending on the values of the parameters $\{V, \alpha, \mu\}$. If $\pi^{NC} \geq \pi^o$, the informative equilibrium supported by $\{\tilde{p}^L, \tilde{p}^R\}$ is therefore profit-enhancing.

The results below show some properties of the expected profit π^{NC}. Just as π^C, it increases with V and decreases with μ. Moreover, π^{NC} first increases and then decreases in α, and achieves its maximum value at $\alpha = 0.5$. This is because $\pi^{NC} = \min\{\hat{\pi}^L, \hat{\pi}^R\}$, which is equal to $\hat{\pi}^L$ for $\alpha \leq 0.5$ and $\hat{\pi}^R$ for $\alpha > 0.5$, and $\hat{\pi}^L$ increases in α while $\hat{\pi}^R$ decreases in α. The economic intuition is that, if in a market, a consumer group becomes more populous than the other, then the seller needs to sacrifice more profit in the market for the more populous group to make cheap-talk disclosure credible.

Corollary 43. *(1) Other things equal, $\pi^{NC} \leq \pi^C$, and $\pi^{NC} = \pi^C$ only for equal mixture distribution ($\alpha = 0.5$); (2) π^{NC} achieves its highest value at $\alpha = 0.5$, and*

$$\begin{cases} \frac{\partial \pi^{NC}}{\partial \alpha} > 0 & \textit{for } \alpha < \frac{1}{2} \\ \frac{\partial \pi^{NC}}{\partial \alpha} < 0 & \textit{for } \alpha > \frac{1}{2} \end{cases} \tag{3.25}$$

and $\pi^{NC}(\alpha) = \pi^{NC}(1-\alpha)$; (3) $\frac{\partial \pi^{NC}}{\partial V} > 0$. $\frac{\partial \pi^{NC}}{\partial \mu} < 0$ and $\lim_{\mu \to \infty} \frac{\partial \pi^{NC}}{\partial \mu} = 0$.

As before, no disclosure is always an option available to the seller, and the optimal profit in the case of information disclosure under no commitment is thus

$$\hat{\pi}^{NC} = \max\{\pi^{NC}, \pi^o\} \tag{3.26}$$

3.4.4. Comparison of π^C, π^{NC} and π^o

An important question is when the seller should reveal the product's horizontal attributes to the buyers. Obviously, the answer to this question depends on the relative values of π^C, π^{NC} and π^o, all of which, in our example, are functions of

(V, α, μ). V is the vertical value of the product, and (α, μ) totally characterizes the mixture normal distribution in our example. Specifically, we are particularly interested in how the optimal disclosure decisions depend on the shape of the mixture distribution, that is, (α, μ).

In our example of mixture normal distribution with equal variance, α is the mixture proportion. If $\alpha = 0.5$, then it is a equal mixture of two subgroups of consumers. When α moves farther from 0.5, then the mixture distribution becomes more *asymmetric*. Specifically, when $\alpha = 0$ or 1, there is only one group of consumers present in the market. μ and $-\mu$ are respectively the modes of the two component distributions. Increasing μ means the two component distributions are farther away from each other, and thus the preferences of different groups of consumers in the market become more *heterogenous*.

The first result in this section is that, when the consumer preferences are sufficiently heterogenous, i.e. μ is great enough, it is better for the monopolistic seller to advertise the product horizontal attributes, and serve the two subgroups separately. This result is formally summarized as below:

Proposition 44. *For any $V > 0$ and $\alpha \in [0, 1]$, there exists a μ^o such that for any $\mu > \mu^o$*

$$\pi^C \geq \pi^{NC} > \pi^o \tag{3.27}$$

$\mu^o = \sqrt{V}$ guarantees the results, but it can be smaller than $\sqrt{V}$, as shown in the numerical results in Figure 3.4 below. For $\alpha = 0.5$, $\pi^C = \pi^{NC}$ and therefore the two profit curves coincide; for $\alpha = 0.75$, $\pi^C \geq \pi^{NC}$, as the figure below on the right shows. In both figures, with the increase in μ, all the profits decreases, but π^o decreases more quickly than π^C and π^{NC}. This is because there is double reduction of π^o when μ increases: one is due to increasing risk, which is absent under disclosure in our example; the other is reducing demand, because when μ

increases, the component distributions become farther apart, and the demand now cover thinner tails of both component distributions. And when $\mu^2 > V$, trade is not possible under no information disclosure. This result is quite intuitive. When the tastes of different groups of consumers are quite different from each other, it is better for the seller to reveal the horizontal attributes of the product, which contributes to the matching between consumer's tastes and product attributes.

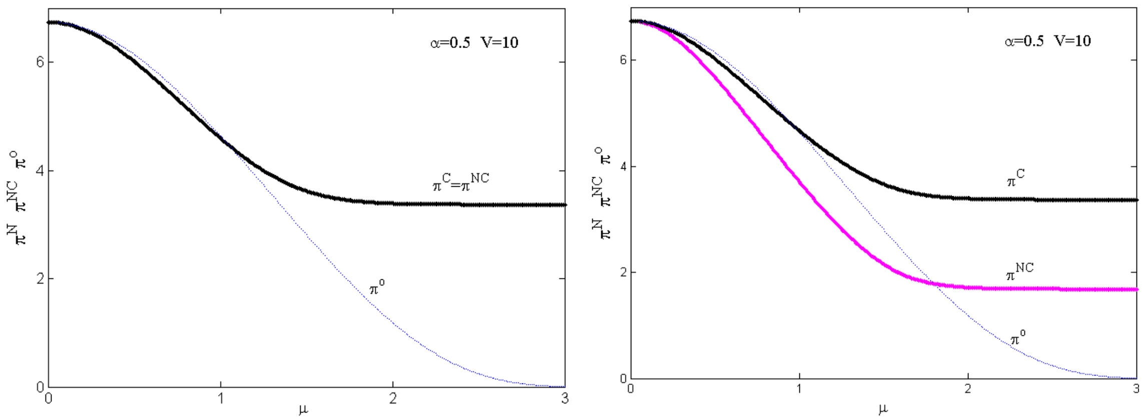


Figure 3.4. π vs. μ ($\alpha = 0.5$) $\qquad$ π vs. μ ($\alpha = 0.75$)

The second result is about how optimal information disclosure under no commitment is related to the mixture proportion α. We know π^{NC} first increases and then decreases in α, and achieves its maximum at $\alpha = 0.5$, and $\pi^{NC} \leq \pi^{C}$. Moreover, π^{NC} is symmetric to $\alpha = 0.5$ since $\pi^{NC}(\alpha) = \pi^{NC}(1-\alpha)$, and reaches its minimum values at either $\alpha = 0$ or 1. On the other hand π^{o} is independent from α. So π^{NC} is more likely to be dominated by π^{o} when α is close to 0 and 1. In other words, when the mixture distribution is sufficiently asymmetric, it is more likely for the seller to withhold the information of product attributes, rather than advertise it under no commitment. The result is formally summarized as below:

Proposition 45. *For given* (μ, V)*, if* $\hat{\pi}^L(\alpha = 0, \mu, V) < \pi^o(\mu, V)$*, then there exists* $\alpha^o \in \left[0, \frac{1}{2}\right]$ *such that for any* $\alpha \leq \alpha^o$ *or* $\alpha \geq 1 - \alpha^o$

$$\pi^o(\mu, V) \geq \pi^{NC}(\alpha, \mu, V) \tag{3.28}$$

Figure 3.5 below provides numerical examples for this result. For information disclosure under no commitment, to make advertised information credible, the non-deviation condition must be satisfied, which implies that $\pi^{NC} = \min\left\{\hat{\pi}^L, \hat{\pi}^R\right\}$. If in a market, one group of consumers is much more populous than the other, the seller needs to sacrifice much profit in the market for the more populous group to make disclosure credible. In this case, she would prefer no disclosure rather than information disclosure under no commitment.

It is interesting in Figure 3.5 below that no disclosure can also dominate information disclosure when μ becomes smaller. This is because, under information disclosure, the seller appeals one consumer group and attracts just a thin tail of the other. On the other hand, without information disclosure, moderate parts of both groups will be attracted into the market. And the results show that, when μ is small enough, the latter dominates the former case of disclosure, in terms of demand for the product.

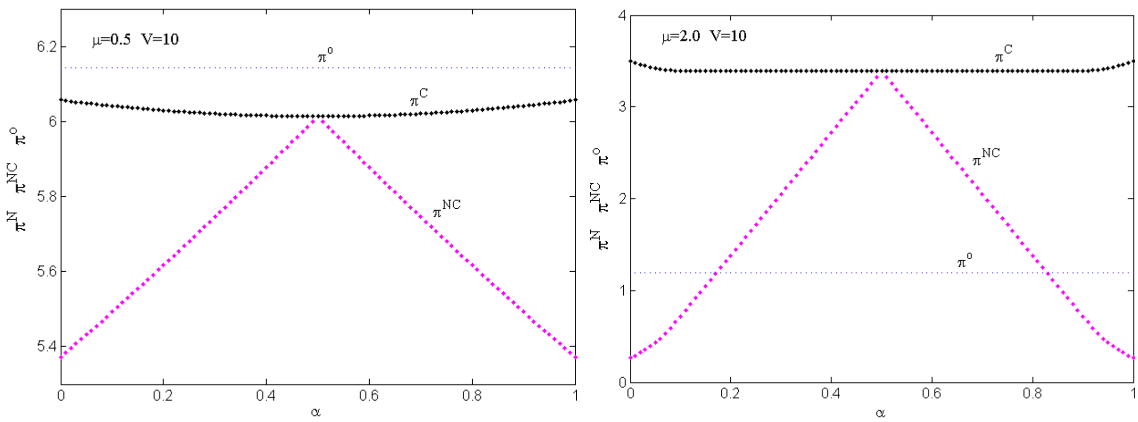


Figure 3.5. π vs. α ($\mu = 0.5$) π vs. α ($\mu = 2.0$)

In addition, the numerical examples below further show that for given mixture distribution, if the vertical value V becomes large enough, no information disclosure can dominate information disclosure with and under no commitment. We do not give a formal proof here, but the intuition is very simple and straightforward. When V is large enough, the market is nearly fully covered even if there is no disclosure, and a large part of both groups of consumers are willing to buy the product at some price. However, in this case, revealing the value of horizontal attributes though will appeal one group, yet it will also discourage the other group of consumers. As a result, compared with the case of no disclosure, the seller will lose a "fat tail" of either group of consumers for a given information disclosure message, L or R.

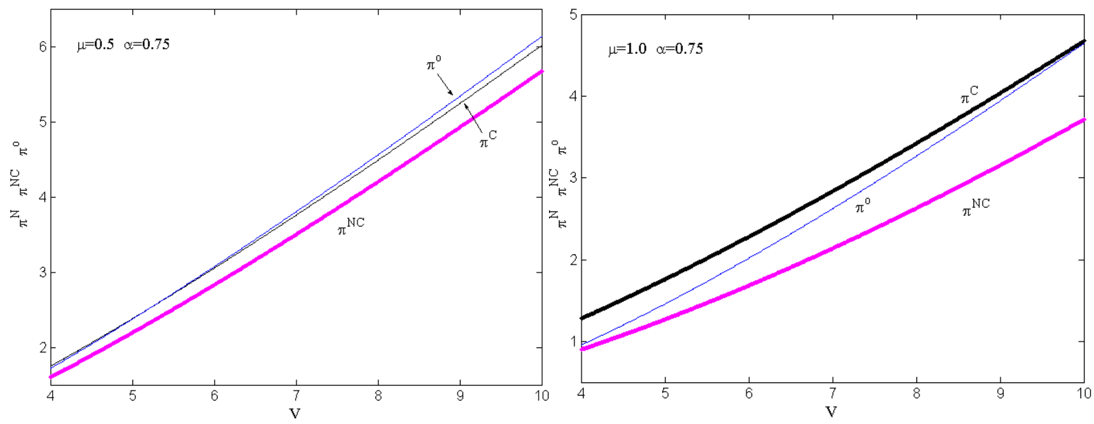


Figure 3.6. π vs. V $(\mu = 0.5)$ $\quad\quad$ π vs. V $(\mu = 1.0)$

3.5. Conclusion

This paper explores the question of how a seller should advertise a product's horizontal attributes to the consumers, when the consumer preferences are mixedly distributed, in a Hotelling model. It also studies the impacts of commitment on the outcomes of information disclosure. In our model, a consumer's valuation of the product depends on the matching between his preference and the product attribute, at a given price. And an informative message sent by the seller, first, can reduce the risks of mismatching for the consumers, and second, can indicate the expected location of the product attribute.

We shows that information disclosure increases trade possibility in the sense that it may make possible some trades which are impossible when there is no information disclosure. This is mainly due to the risk-reduction function of information disclosure. Second, information disclosure under full commitment generates higher expected profit than one under no commitment. This is because in the case of no commitment, to make information disclosure credible to the consumers, the seller needs to satisfy interim non-deviation conditions, which is absent in the case of information disclosure under full commitment.

Moreover, we show that optimal information disclosure policies largely depend on the shape of the mixture distribution. First, when the preferences of different groups of consumers become sufficiently heterogenous, it is better for the seller to advertise product attributes and serve different groups separately. Second, when the composition of different groups of consumers becomes more asymmetric, information disclosure under no commitment is more likely to be dominated by no disclosure. And finally, when the vertical value is great enough, it is more likely for the seller to choose not to advertise the horizontal attributes.

To our knowledge, this is the first research that introduces mixture distribution into the study of the classical Hotelling model. And the results of our model provide important implications for the real practice of disclosure.

3.A. Appendix: Mathematical Proofs

Proof. [**Corollary 34**] Take $m = L$ as an example. The minimum possible price is c, which corresponds to the maximum value of range^L, which is $2\sqrt{\frac{1}{\tau}\left(V-c\right)-\frac{1}{12}\left(N+s^*\right)^2}$, for given s^*. Since range^L decreases in s^*, the maximum value of range^L =

$2\sqrt{\frac{1}{\tau}(V-c)}$ when $s^* = -N$. We have assumed that the density function of the mixture distribution, $g(\theta)$, is strictly positive, and therefore, whenever $V - c \geq 0$, there is positive demand for the product for at least some cutoff value s^*. □

Proof. [**Proposition 36**] (1) If $\tilde{s}^* \in (-N, N)$, then $\hat{\pi}^{NC} = \min\left\{\pi(\hat{p}^L), \pi(\hat{p}^R)\right\} \leq \frac{N+\tilde{s}^*}{2N} \cdot \pi(\hat{p}^L) + \frac{N-\tilde{s}^*}{2N}\pi(\hat{p}^R) \leq \hat{\pi}^C$. With a little abuse of notation, for $\tilde{s}^* = \pm N$, we also use $\hat{\pi}^{NC}$ to denote the expected profit in a degenerated informative equilibrium. When the seller sets $\tilde{s}^* = \pm N$, it is equivalent to the case of no information revelation, and therefore $\hat{\pi}^{NC} = \pi^o \leq \hat{\pi}^C$.

(2) We know that in a non-denegerated informative equilibrium supported by $\left\{\tilde{s}^*, \tilde{p}^L, \tilde{p}^R\right\}$,

$$\mathbb{E}\pi(\tilde{s}^*) = \frac{N+\tilde{s}^*}{2N} \cdot \mathbb{E}\pi(\tilde{p}^L) + \frac{N-\tilde{s}^*}{2N}\mathbb{E}\pi(\tilde{p}^R)$$

and $\mathbb{E}\pi(\tilde{p}^L) \leq \hat{\pi}^L(\tilde{s}^*)$ and $\mathbb{E}\pi(\tilde{p}^R) \leq \hat{\pi}^R(\tilde{s}^*)$, as $\hat{\pi}^L(\tilde{s}^*)$ and $\hat{\pi}^R(\tilde{s}^*)$ are respectively maximum interim expected profits achievable at the cutting point $\tilde{s}^*$, under full commitment. The non-deviation condition implies that in equilibrium, $\mathbb{E}\pi(\tilde{p}^L) = \mathbb{E}\pi(\tilde{p}^R)$, and therefore $\mathbb{E}\pi(\tilde{s}^*) \leq \min\left\{\hat{\pi}^L(\tilde{s}^*), \hat{\pi}^R(\tilde{s}^*)\right\}$.

In optimality, we must have $\hat{\pi}^{NC} = \min\left\{\hat{\pi}^L(\tilde{s}^*), \hat{\pi}^R(\tilde{s}^*)\right\}$. If not, suppose without loss of generality that $\hat{\pi}^{NC} < \hat{\pi}^L(\tilde{s}^*) \leq \hat{\pi}^R(\tilde{s}^*)$, then by setting $\tilde{p}^L = \hat{p}^L$ we have $\mathbb{E}\pi(\tilde{p}^L) = \hat{\pi}^L(\tilde{s}^*)$ and there exists a $\tilde{p}^R$ such that $\mathbb{E}\pi(\tilde{p}^R) = \hat{\pi}^L(\tilde{s}^*) \leq \hat{\pi}^R(\tilde{s}^*)$, which is guaranteed by the continuity of $\mathbb{E}\pi(\tilde{p}^R)$ and the intermediate value theorem. □

Proof. [**Lemma 38**] It is obvious that $\varphi_1(\theta) = \varphi_2(-\theta)$. We also have $\Phi_2(\theta) - \Phi_2(-\theta) = \int_{-\theta}^{\theta} \varphi_2(x)dx = -\int_{-\theta}^{\theta} \varphi_1(-x)d(-x) = -\int_{\theta}^{-\theta} \varphi_1(y)dy = \int_{-\theta}^{\theta} \varphi_1(y)dy = \Phi_1(\theta) - \Phi_1(-\theta)$. □

Proof. [**Lemma 39**] (1) The expected profit function is $\pi^o(p;V,\mu) = p\int_{-\upsilon(p,V,\mu)}^{\upsilon(p,V,\mu)} \varphi_1(\theta;\mu)d\theta$, which is independent of α. Using envelope theorem, we have for V

$$\frac{\partial \pi^o}{\partial V} = p\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right]\frac{\partial \upsilon}{\partial V} > 0$$

where $\frac{\partial \upsilon}{\partial V} = \frac{1}{2\upsilon}$. And for μ,

$$\begin{aligned}
\frac{\partial \pi^o}{\partial \mu} &= p\left\{\varphi_1(\upsilon)\frac{\partial \upsilon}{\partial \mu} + \varphi_1(-\upsilon)\frac{\partial \upsilon}{\partial \mu} + \int_{-\upsilon(p;V,\mu)}^{\upsilon(p;V,\mu)} \frac{\partial \varphi_1(\theta;\mu)}{\partial \mu}d\theta\right\} \\
&= p\left\{-\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right]\frac{\mu}{\upsilon} - \int_{-\upsilon(p;V,\mu)}^{\upsilon(p;V,\mu)} \varphi_1(\theta;\mu)(\theta+\mu)d\theta\right\} \\
&= p\left\{-\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right]\frac{\mu}{\upsilon} - \left[\varphi_1(-\upsilon) - \varphi_1(\upsilon)\right]\right\} < 0
\end{aligned}$$

where $\frac{\partial \upsilon}{\partial \mu} = -\frac{\mu}{\upsilon} < 0$. (2) We know $p\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right] - 2\upsilon\left[\Phi_1(\upsilon) - \Phi_1(-\upsilon)\right] = 0$ and by taking total differentiation, we have

$$dp\left\{\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right] + A(\cdot)\frac{d\upsilon}{dp}\right\} + d\mu \cdot A(\cdot)\frac{d\upsilon}{d\mu} + d\alpha \cdot 0 = 0$$

where $A(\cdot) = p\left[\varphi_1'(\upsilon) - \varphi_1'(-\upsilon)\right] - 2\left[\Phi_1(\upsilon) - \Phi_1(-\upsilon)\right] - 2\upsilon\left[\varphi_1(\upsilon) + \varphi_1(-\upsilon)\right] < 0$ and $\frac{d\upsilon}{dp} < 0$ and $\frac{d\upsilon}{d\mu} < 0$. Therefore, $\frac{dp^o}{d\mu} < 0$.

Proof. [**Lemma 40:** Part 1] $\hat{\pi}^L(\alpha) = \hat{\pi}^R(1-\alpha)$ is a direct result of the symmetric setting of our model. $\hat{\pi}^L(V,\mu,\alpha) = \hat{p}^L\int_{\underline{\upsilon}^L(p,V,\mu)}^{\bar{\upsilon}^L(p,V,\mu)} g(\theta;\mu,\alpha)d\theta$. By envelope theorem and Newton-Leibniz Formula

$$\begin{aligned}
\frac{1}{\hat{p}^L}\frac{\partial \hat{\pi}^L}{\partial \alpha} &= \int_{\underline{\upsilon}^L}^{\bar{\upsilon}^L} \left[\varphi_1(\theta) - \varphi_2(\theta)\right]d\theta \\
&= \left[\Phi_1(\bar{\upsilon}^L) - \Phi_1(\underline{\upsilon}^L)\right] - \left[\Phi_2(\bar{\upsilon}^L) - \Phi_2(\underline{\upsilon}^L)\right] \\
&= \left[\Phi(A) - \Phi(-A)\right] - \left[\Phi(-2\mu + A) - \Phi(-2\mu - A)\right] \\
&= \left[\Phi(A) - \Phi(A - 2\mu)\right] - \left[\Phi(-A) - \Phi(-2\mu - A)\right]
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution and $A = \sqrt{V - p^L} > 0$. We know $\Phi(x)$ is strictly concave for $x \geq 0$ and strictly convex for $x < 0$. (1) If $A - 2\mu \geq 0$, then $\Phi(A) - \Phi(A - 2\mu) > \varphi(A) \cdot 2\mu$ and $\Phi(-A) - \Phi(-2\mu - A) < \varphi(-A) \cdot 2\mu$, where $\varphi(\cdot)$ is the density function of standard normal distribution and $\varphi(-A) = \varphi(A)$. So

$$\frac{1}{\hat{p}^L}\frac{\partial \hat{\pi}^L}{\partial \alpha} > \varphi(A) \cdot 2\mu - \varphi(-A) \cdot 2\mu = 0$$

(2) If $A - 2\mu < 0$, then $\Phi(0) - \Phi(-A) > \Phi(-2\mu + A) - \Phi(-2\mu) > \Phi(-2\mu) - \Phi(-2\mu - A)$ by concavity. Therefore

$$\begin{aligned}
\Phi(A) - \Phi(-A) &= 2\left[\Phi(0) - \Phi(-A)\right] \\
&> \left[\Phi(-2\mu + A) - \Phi(-2\mu)\right] + \left[\Phi(-2\mu) - \Phi(-2\mu - A)\right] \\
&= \Phi(-2\mu + A) - \Phi(-2\mu - A)
\end{aligned}$$

Therefore, $\frac{\partial \hat{\pi}^L}{\partial \alpha} > 0$. Since $\hat{\pi}^L(\alpha) = \hat{\pi}^R(1 - \alpha)$, the result for $\hat{\pi}^R$ is obvious. $\square$

[**Lemma 40**: Part 2] (1) μ: For $m = L$, $\hat{\pi}^L(V, \mu, \alpha) = \hat{p}^L \int_{\underline{\upsilon}^L(p,V,\mu)}^{\bar{\upsilon}^L(p,V,\mu)} g(\theta; \mu, \alpha) d\theta$. By Envelope theorem and Newton-Leibniz formula

$$\begin{aligned}
\frac{1}{\hat{p}^L}\frac{\partial \hat{\pi}^L}{\partial \mu} &= \frac{\partial \int_{\underline{\upsilon}^L(p,V,\mu)}^{\bar{\upsilon}^L(p,V,\mu)} g(\theta; \mu, \alpha) d\theta}{\partial \mu} \\
&= g(\bar{\upsilon}^L)\frac{\partial \bar{\upsilon}^L}{\partial \mu} - g(\underline{\upsilon}^L)\frac{\partial \underline{\upsilon}^L}{\partial \mu} + \int_{\underline{\upsilon}^L}^{\bar{\upsilon}^L} \frac{\partial g}{\partial \mu} d\theta \\
&= g(\underline{\upsilon}^L) - g(\bar{\upsilon}^L) + \int_{\underline{\upsilon}^L}^{\bar{\upsilon}^L} \left[-\alpha(\theta + \mu)\varphi_1(\theta) + (1 - \alpha)(\theta - \mu)\varphi_2(\theta)\right] d\theta \\
&= (1 - \alpha)\left[\varphi_2(\underline{\upsilon}^L) - \varphi_2(\bar{\upsilon}^L)\right] + \alpha\left[\varphi_1(\bar{\upsilon}^L) - \varphi_1(\underline{\upsilon}^L)\right] + (1 - \alpha)\left[\varphi_2(\underline{\upsilon}^L) - \varphi_2(\bar{\upsilon}^L)\right] \\
&= 2(1 - \alpha)\left[\varphi_2(\underline{\upsilon}^L) - \varphi_2(\bar{\upsilon}^L)\right] < 0
\end{aligned}$$

where $\varphi_1(\bar{v}^L) - \varphi_1(\underline{v}^L) = 0$ and

$$
\begin{aligned}
-\int_{\underline{v}^L}^{\bar{v}^L} (\theta+\mu)\varphi_1(\theta)d\theta &= \frac{-1}{\sqrt{2\pi}} \int_{\underline{v}^L}^{\bar{v}^L} (\theta+\mu)e^{-\frac{1}{2}(\theta+\mu)^2} d\theta = \varphi_1(\bar{v}^L) - \varphi_1(\underline{v}^L) \\
\int_{\underline{v}^L}^{\bar{v}^L} (\theta-\mu)\varphi_2(\theta)d\theta &= \frac{1}{\sqrt{2\pi}} \int_{\underline{v}^L}^{\bar{v}^L} (\theta-\mu)e^{-\frac{1}{2}(\theta-\mu)^2} d\theta = \varphi_2(\bar{v}^L) - \varphi_2(\underline{v}^L)
\end{aligned}
$$

Similarly, for $\hat{\pi}^R$, by symmetry we have $\frac{1}{\hat{p}^R}\frac{\partial \hat{\pi}^R}{\partial \mu} = 2\alpha\left[\varphi_1(\underline{v}^R) - \varphi_1(\bar{v}^R)\right] < 0$. (2) V:

$$
\frac{\partial \pi^L}{\partial V} = g(\bar{v}^L)\frac{\partial \bar{v}^L}{\partial V} - g(\underline{v}^L)\frac{\partial \underline{v}^L}{\partial V} > 0
$$

since $\frac{\partial \bar{v}^L}{\partial V} > 0$ and $\frac{\partial \underline{v}^L}{\partial V} < 0$, and so is $\frac{\partial \pi^R}{\partial V}$. □

Proof. [**Proposition 41**] (1) For V, $\frac{\partial \pi^C}{\partial V} = \frac{1}{2}\frac{\partial \pi^L}{\partial V} + \frac{1}{2}\frac{\partial \pi^R}{\partial V} > 0$. For μ, $\frac{\partial \pi^C}{\partial \mu} = \frac{1}{2}\left(\frac{\partial \hat{\pi}^L}{\partial \mu} + \frac{\partial \hat{\pi}^R}{\partial \mu}\right) < 0$, and since $\hat{p}^m$ is bounded by V and $\lim_{\mu\to\infty}\left[\varphi_1(\underline{v}^m) - \varphi_1(\bar{v}^m)\right] = 0$ for $m = L, R$, from the Lemma above, we know $\lim_{\mu\to\infty}\frac{\partial \pi^C}{\partial \mu} = 0$. (2) For α, $2\pi^C(\alpha) = \pi^L(\alpha) + \pi^R(\alpha) = \pi^R(1-\alpha) + \pi^L(1-\alpha) = 2\pi^C(1-\alpha)$. Since $2\pi^C(\alpha) = \pi^L(\alpha) + \pi^L(1-\alpha)$, for $\alpha = \frac{1}{2}$,

$$
2\frac{\partial \hat{\pi}^C}{\partial \alpha} = \frac{\partial \hat{\pi}^L(\alpha)}{\partial \alpha} - \frac{\partial \hat{\pi}^L(1-\alpha)}{\partial \alpha} = 0
$$

□

Proof. [**Lemma 42**] Without loss of generality, we assume $\hat{\pi}^L \geq \hat{\pi}^R$. By setting $\tilde{p}^R = \hat{p}^R$, we have $\hat{\pi}^R = \pi^R(\tilde{p}^R) = \pi^{NC}$. Furthermore, since π^L is continuous in p^L and when $p^L = 0$ or V, $\pi^L = 0$. By intermediate value theorem, there always exists a price $\tilde{p}^L$ such that $\pi^L(\tilde{p}^L) = \hat{\pi}^R$. □

Proof. [**Corollary 43**] Since $\pi^C = \frac{1}{2}\left(\hat{\pi}^L + \hat{\pi}^R\right)$ and

$$
\pi^{NC} = \min\left\{\hat{\pi}^L, \hat{\pi}^R\right\} = \begin{cases} \hat{\pi}^L & \text{for } \alpha \leq \frac{1}{2} \\ \hat{\pi}^R & \text{for } \alpha > \frac{1}{2} \end{cases}
$$

it's obvious that $\pi^{NC} \leq \pi^{C}$, with equality only when $\alpha = 0.5$ such that $\hat{\pi}^{L} = \hat{\pi}^{R}$. Since $\hat{\pi}^{L}(\alpha) = \hat{\pi}^{R}(1-\alpha)$ and $\hat{\pi}^{L} \nearrow$ in α, $\hat{\pi}^{L}(\alpha) \leq \hat{\pi}^{R}(\alpha)$ if and only if $\alpha \leq \frac{1}{2}$. Therefore, $\frac{\partial \pi^{NC}}{\partial \alpha} > 0$ for $\alpha < \frac{1}{2}$, $\frac{\partial \pi^{NC}}{\partial \alpha} < 0$ for $\alpha > \frac{1}{2}$, and π^{NC} achieves its maximum value at $\alpha = 0.5$. $\pi^{NC}(\alpha) = \pi^{NC}(1-\alpha)$ stems from the simple fact that $\hat{\pi}^{L}(\alpha) = \hat{\pi}^{R}(1-\alpha)$. $\square$

Proof. [**Proposition 44**] It is self-evident that $\pi^{C} \geq \pi^{NC}$. For the existence of μ^{o}, we can simply select $\mu^{o} = \sqrt{V}$. Therefore, for any $\mu > \mu^{o}$, trade is not possible when there is no disclosure. However, for any $V > 0$ and $\alpha \in [0,1]$, trade is possible under information disclosure, either with or under no commitment, and the optimal profit is positive. $\square$

Proof. [**Proposition 45**] We know that $\hat{\pi}^{L}(\alpha) = \hat{\pi}^{R}(1-\alpha)$, and for $\mu > 0$, $\frac{\partial \hat{\pi}^{L}}{\partial \alpha} > 0$ and $\frac{\partial \hat{\pi}^{R}}{\partial \alpha} < 0$. Furthermore,

$$\pi^{NC} = \min\left\{\hat{\pi}^{L}, \hat{\pi}^{R}\right\} = \begin{cases} \hat{\pi}^{L} & \text{if } \alpha \leq \frac{1}{2} \\ \hat{\pi}^{R} & \text{if } \alpha > \frac{1}{2} \end{cases}$$

If $\hat{\pi}^{L}(\alpha = 0, \mu, V) < \pi^{o}(\mu, V)$, then the result is self-evident. $\square$

CHAPTER 4

Regulating Risk-Averse Producers: the Case of Complementary Products

4.1. Introduction

In multiproduct industry, a regulator often faces the problem of selecting a proper production form, for example, between *integrated production* and *component production*. Integrated production means a single producer supplies all products, while component production means different products are separately supplied by different producers. The relative virtues of these two production forms depend on specific industrial features and production technologies. If there is significant effect of technological economy of scope, then integrated production can dominate component production. On the other hand, if there is no such effect, it is shown in the case of independent products with correlated costs that, when the correlation is low, the regulator prefers integrated production, due to informational economy of scope under integrated production, but when the correlation is sufficiently high, component production is preferred instead, since the regulator can exploit the benefits of yardstick competition under component production (Dana, 1993; Armstrong, 1999).

Gilbert and Riordan (1995, thereafter GR) study the problem of regulating complementary products with independent cost realizations. It is obvious that, in their setting, there are neither technological economy of scope nor the benefits of yardstick competition. In this case, they prove that the regulator prefers integrated production over component production, due to the benefit of informational economy of scope.

To our knowledge, the existing literature commonly assumes that the producers are risk-neutral. However, risk-averse producers seems a more realistic assumption in many cases. For example, a producer may worry about not only the expected value, but the volatility of his revenue streams. Does producers' risk-aversion have any impact on the relative virtues of integrated and component productions? This paper tries to answer this question in the case of complementary products.

Following the same setting as GR, we prove two important results. First, component production by risk-averse producers generates higher revenue than that by risk-neutral producers. This is because the aversion to outcome uncertainty relaxes the incentive constraints, and thus reduces the information-rent to the producers. Second, when the producers are risk-averse, the relative virtues of integrated and component production depend on the degree of risk aversion. Specifically, in the case of CRRA preference, we prove that component production generates higher expected revenue than integrated production iff the coefficient of risk-aversion is higher than some threshold.

The paper is organized as follows. Section 4.2 presents the model. Section 4.3 investigates the relative virtues of integrated and component productions under risk-aversion producers. And section 4.4 is conclusion.

4.2. The Model

A regulator contracts for a final good made up of two complementary intermediate products. The cost of a unit of final good is the sum of the costs of a unit of each intermediate product. The regulator's problem is two folds, selecting a proper production form and designing optimal production contacts. Both products are produced at constant marginal costs, which are drawn independently from the same distribution

$$c_i = \begin{cases} \underline{c} & \Pr = p \\ \bar{c} & \Pr = 1-p \end{cases} \qquad i = 1,2$$

and $\Delta c = \bar{c} - \underline{c} > 0$. The realization of c_i is observable only to the producer of product i. We also keep GR's assumption that when cost realizations $(c_1, c_2) = (\bar{c}, \bar{c})$, the production is not profitable.

For integrated production, the contract is a scheme of combination of transfers and outputs, $\{T(\hat{\mathbf{c}}), Q(\hat{\mathbf{c}})\}$, conditional on the producer's cost announcement $\hat{\mathbf{c}} = (\hat{c}_1, \hat{c}_2)$. $T(\hat{\mathbf{c}})$ is the transfer to the producer, and $Q(\hat{\mathbf{c}})$ is the output of the final good. The regulator is risk neutral, and his revenue is

$$W_j^I(\hat{\mathbf{c}}) = S(Q(\hat{\mathbf{c}})) - T(\hat{\mathbf{c}})$$

where $j = n, a$ denotes the cases of risk-neutral (n) and risk-averse (a) producers, and $S(\cdot)$ is increasing and strictly concave. When the producer is risk neutral, his vNM utility is just the amount of net monetary transfer, that is, $U(\hat{\mathbf{c}}|\mathbf{c}) = T(\hat{\mathbf{c}}) - (c_1 + c_2)Q(\hat{\mathbf{c}})$, where $\mathbf{c} = (c_1, c_2)$ is the true realization of the marginal costs. When the producer is risk averse, his vNM utility is

$$U(\hat{\mathbf{c}}|\mathbf{c}) = u\left[T(\hat{\mathbf{c}}) - (c_1 + c_2)Q(\hat{\mathbf{c}})\right]$$

and $u(\cdot)$ is a strictly concave and increasing function satisfying Inada conditions, with the normalization that $u(0) = 0$.

For component production, the contract is $\{t_1(\hat{\mathbf{c}}), t_2(\hat{\mathbf{c}}), q(\hat{\mathbf{c}})\}$, where t_i is the transfer to the producer of product i and $q(\cdot)$ is the output level. The regulator's revenue is thus

$$W_j^C(\hat{\mathbf{c}}) = S(q(\hat{\mathbf{c}})) - t_1(\hat{\mathbf{c}}) - t_2(\hat{\mathbf{c}})$$

For risk- neutral producers, the vNM utility is $U_i(\hat{\mathbf{c}}|c_i) = t_i(\hat{\mathbf{c}}) - c_i q(\hat{\mathbf{c}})$ for $i = 1, 2$. For risk-averse producers, they are

$$U_i(\hat{\mathbf{c}}|c_i) = u\left[t_i(\hat{\mathbf{c}}) - c_i q(\hat{\mathbf{c}})\right] \quad i = 1, 2$$

We consider only symmetric equilibrium in our model, which implies that $q(\underline{c}, \bar{c}) = q(\bar{c}, \underline{c})$, $t_1(\underline{c}, c) = t_2(c, \underline{c})$ and $t_1(\bar{c}, c) = t_2(c, \bar{c})$ for $c = \underline{c}, \bar{c}$.

The timing of the game is as follows: (1)The regulator selects the production form and proposes the contract to the producer(s); (2) c_i is observed by relevant producer, and $(\hat{c}_1, \hat{c}_2)$ is report; (3) If $\hat{\mathbf{c}} = (\bar{c}, \bar{c})$, the game is ended and each player receives a payoff of 0; otherwise, the project is implemented. (4) Production is finished and transfers are paid according to the contract.

4.3. Regulating Risk-averse Producers

For the case of risk-neutral producers, GR have shown under the same setting that the regulator prefers integrated production to component production. We will show that this is not necessarily true in the case of risk-averse producers.

4.3.1. Component Production

We first introduce some new notations for output and net monetary transfer: $\underline{q} = q(\underline{c}, \underline{c})$, $\hat{q} = q(\underline{c}, \bar{c}) = q(\bar{c}, \underline{c})$; $\underline{m} = t_1(\underline{c}, \underline{c}) - \underline{c} \cdot \underline{q}$, $\hat{m} = t_1(\underline{c}, \bar{c}) - \underline{c} \cdot \hat{q}$, $\bar{m} = t_1(\bar{c}, \underline{c}) - \bar{c} \cdot \hat{q}$.[1] In equilibrium, the following interim incentive compatible (IC) conditions for producers should be satisfied

$$\underline{IC} \; (c_i = \underline{c}) \quad pu(\underline{m}) + (1-p)u(\hat{m}) \geq pu\left(\bar{m} + \Delta c \cdot \hat{q}\right) \tag{4.1}$$

$$\overline{IC} \; (c_i = \bar{c}) \quad pu(\bar{m}) \geq pu\left(\underline{m} - \Delta c \underline{q}\right) + (1-p)u\left(\hat{m} - \Delta c \hat{q}\right) \tag{4.2}$$

[1] In symmetric equilibrium, for the producer of product 2, we have $\underline{m} = t_2(\underline{c}, \underline{c}) - \underline{c} \cdot \underline{q}$, $\hat{m} = t_2(\bar{c}, \underline{c}) - \underline{c} \cdot \hat{q}$, $\bar{m} = t_2(\underline{c}, \bar{c}) - \bar{c} \cdot \hat{q}$.

So should the following interim individual rationality (IR) conditions.

$$\underline{IR}\ (c_i = \underline{c}) \qquad p \cdot u(\underline{m}) + (1-p) \cdot u(\hat{m}) \geq 0 \tag{4.3}$$

$$\overline{IR}\ (c_i = \bar{c}) \qquad pu(\bar{m}) \geq 0 \tag{4.4}$$

And the regulator's ex ante expected revenue is

$$EW_a^C \ = p^2 \left[S(\underline{q}) - 2\underline{c}\underline{q} - 2\underline{m}\right] + 2p(1-p)\left[S(\hat{q}) - (\underline{c} + \bar{c})\hat{q} - \hat{m} - \bar{m}\right]$$

and his objective is to maximize EW_a^C subject to the IC and IR constraints, that is

$$(\mathcal{P}_1) \quad \begin{array}{ll} \max & EW_a^C \\ \text{s.t.} & (4.1) - (4.4) \end{array}$$

We need first to clarify which constraints are binding in $(\mathcal{P}_1)$, and the result is as below.

Lemma 46. *In the case of risk averse producers, only constraint* (4.1) *and* (4.4) *are binding in the optimality of* $(\mathcal{P}_1)$.

Proof. From (4.1), we have

$$\begin{aligned} pu(\underline{m}) + (1-p)u(\hat{m}) \ &\geq p \cdot \{u(\bar{m}) + u'(\bar{m} + \Delta c\hat{q}) \cdot \Delta c\hat{q}\} \\ &= p \cdot u(\bar{m}) + A \end{aligned}$$

where $u(\bar{m}) \geq 0$ and $A > 0$. Thus (4.4) is binding but (4.3) is not. From (4.2), we have

$$\begin{aligned} pu(\bar{m}) \ &\geq p\left\{u(\underline{m}) - u'\left(\underline{m} - \Delta c\underline{q}\right) \cdot \Delta c\underline{q}\right\} + (1-p)\left\{u(\hat{m}) - u'(\hat{m} - \Delta c\hat{q}) \cdot \Delta c\hat{q}\right\} \\ &= pu(\underline{m}) + (1-p)u(\hat{m}) - \Delta c\left\{p\underline{q} \cdot u'\left(\underline{m} - \Delta c\underline{q}\right) + (1-p)\hat{q} \cdot u'(\hat{m} - \Delta c\hat{q})\right\} \\ &= pu(\underline{m}) + (1-p)u(\hat{m}) - B \end{aligned}$$

From $u(\bar{m}) = 0$, we have $B \geq p \cdot u(\underline{m}) + (1-p) \cdot u(\hat{m}) \geq A$. The regulator wants to reduce the rent of the producers, and therefore (4.1) is binding but (4.2) is not. $\square$

Given this result, we can easily write down the Lagrangian of $(\mathcal{P}_1)$, and characterizes the optimal contract for component production as below.

Proposition 47. *The optimal allocation of* $(\mathcal{P}_1)$, $\{\underline{q}^{Ca}, \hat{q}^{Ca}; \underline{m}^{Ca}, \hat{m}^{Ca}, \bar{m}^{Ca}\}$, *satisfies the following conditions*[2]*:*

$$S'(\underline{q}^{Ca}) = 2\underline{c} \tag{4.5}$$

$$S'(\hat{q}^{Ca}) = (\underline{c} + \bar{c}) + \frac{p}{1-p} \cdot \frac{u'(\Delta c \hat{q}^{Ca})}{u'(\underline{m}^{Ca})} \Delta c \tag{4.6}$$

$$u(\underline{m}^{Ca}) = pu(\Delta c \hat{q}^{Ca}) \tag{4.7}$$

$$\bar{m}^{Ca} = 0, \ \underline{m}^{Ca} = \hat{m}^{Ca} \tag{4.8}$$

From Proposition 47, we can easily derive that, for component production by risk-neutral producers, an optimal allocation $\{\underline{q}^{Cn}, \hat{q}^{Cn}; \underline{m}^{Cn}, \hat{m}^{Cn}, \bar{m}^{Cn}\}$ satisfies:[3]

$$\begin{aligned} S'(\underline{q}^{Cn}) &= 2\underline{c} \\ S'(\hat{q}^{Cn}) &= (\underline{c} + \bar{c}) + \frac{p}{1-p} \cdot \Delta c && (4.9) \\ \underline{m}^{Cn} &= p\Delta c \cdot \hat{q}^{Cn} && (4.10) \\ \bar{m}^{Cn} &= 0, \ \underline{m}^{Cn} = \hat{m}^{Cn} && (4.11) \end{aligned}$$

It is noteworthy that, in both cases, the producers are fully insured if they truthfully report their types, as revealed in condition (4.8) and (4.11). The result below shows

[2]The superscripts Ca and Cn mean "component production" by "risk-averse" and "risk-neutral" producers respectively.

[3]It worths attention that in the case of risk-neutral producers, there are multiple solutions for $\{\underline{m}^{Cn}, \hat{m}^{Cn}\}$, since only the expected value of $p\underline{m}^{Cn} + (1-p)\hat{m}^{Cn}$ that matters.

that, under component production, risk-averse producers produce more than risk-neutral producers in equilibrium.

Corollary 48. *$\underline{m}^{Ca} < p\Delta c \cdot \hat{q}^{Ca}$ and $\hat{q}^{Ca} > \hat{q}^{Cn}$.*

Proof. From condition (4.7) we have $u(\underline{m}^{Ca}) = pu(\Delta c\hat{q}^{Ca}) < u(p \cdot \Delta c\hat{q}^{Ca})$, so $\underline{m}^{Ca} < p\Delta c\hat{q}^{Ca}$, and therefore $u\prime(\underline{m}^{Ca}) < u'(\Delta c\hat{q}^{Ca})$. Comparing condition (4.6) with (4.9), we have $\hat{q}^{Ca} > \hat{q}^{Cn}$. □

The first important result of this paper, regarding the relative virtues of component production by risk-averse and risk-neutral producers, is summarized as follows.

Proposition 49. *Under complementary production with independent cost realization, a component production by risk-averse producers generates strictly higher expected revenue than that by risk-neutral producers.*

Proof. For the problem of component production by risk-neutral producers, substituting the binding constraints (4.10) and (4.11) into the objective function, we get the value function as

$$EW_n^C(\underline{q}^{Cn}, \hat{q}^{Cn}) = \max_{\underline{q},\hat{q}} \; p^2\left[S(\underline{q}) - 2\underline{c}\underline{q}\right] + 2p(1-p)\left[S(\hat{q}) - (\underline{c}+\bar{c})\hat{q}\right] - 2p^2 \cdot \Delta c\hat{q}$$

Similarly, substituting (4.7) and (4.8), we get the value function for component production by risk-averse producers as

$$\begin{aligned} EW_a^C(\underline{q}^{Ca}, \hat{q}^{Ca}) &= \max\nolimits_{\underline{q},\hat{q}} \; p^2\left[S(\underline{q}) - 2\underline{c}\underline{q}\right] + 2p(1-p)\left[S(\hat{q}) - (\underline{c}+\bar{c})\hat{q}\right] - 2p \cdot \underline{m} \\ &> \max\nolimits_{\underline{q},\hat{q}} \; p^2\left[S(\underline{q}) - 2\underline{c}\underline{q}\right] + 2p(1-p)\left[S(\hat{q}) - (\underline{c}+\bar{c})\hat{q}\right] - 2p \cdot p\Delta c\hat{q} \\ &= EW_n^C(\underline{q}^{Cn}, \hat{q}^{Cn}) \end{aligned}$$

Since both (4.1) and (4.4) are binding, and $\underline{m} = \hat{m}$ at optimality, we have

$$\underline{m} = u^{-1}[pu(\Delta c\hat{q})] < u^{-1}[u(p\Delta c\hat{q})] = p\Delta c\hat{q}$$

□

The result is quite intuitive. Though the optimal contact exposes the truth-telling producers no risk, the deviation payoff is risky for the producers. Specifically, when a low cost producer deviates by reporting $\bar{c}$, he will receive $\Delta c\hat{q}^{Ca}$ with probability p, and 0 otherwise. And the aversion to outcome uncertainty reduces the information rent to the low cost producers, as $\underline{m}^{Ca} < p\Delta c \cdot \hat{q}^{Ca}$ shows. As a result, risk-averse producers generate higher expected revenue than risk-neutral producers.

4.3.2. Integrated Production

In the case of integrated production, the marginal cost of producing final good is $c^I = c_1 + c_2$, and its distribution is

$$c^I = \begin{cases} 2\underline{c} & \Pr = p^2 \\ \underline{c} + \bar{c} & \Pr = 2p(1-p) \\ 2\bar{c} & \Pr = (1-p)^2 \end{cases}$$

Again we introduce some notations: $\underline{Q} = Q(2\underline{c})$, $\hat{Q} = Q(\underline{c} + \bar{c})$; $\underline{M} = T(2\underline{c}) - 2\underline{c} \cdot \underline{Q}$, and $\hat{M} = T(\underline{c} + \bar{c}) - (\underline{c} + \bar{c}) \cdot \hat{Q}$. The IC and IR constraints are:

$$\underline{IC}: \quad u(\underline{M}) \geq u(\hat{M} + \Delta c \cdot \hat{Q}) \tag{4.12}$$

$$\widehat{IC}: \quad u(\hat{M}) \geq U(\underline{M} - \Delta c \cdot \underline{Q}) \tag{4.13}$$

$$\underline{IR}: \quad u(\underline{M}) \geq 0 \tag{4.14}$$

$$\widehat{IR}: \quad u(\hat{M}) \geq 0 \tag{4.15}$$

The regulator's expected revenue is

$$EW_a^I \quad = p^2(S(\underline{Q}) - 2\underline{c}\underline{Q} - \underline{M}) + 2p(1-p)(S(\hat{Q}) - (\underline{c} + \bar{c})\hat{Q} - \hat{M})$$

and the problem is thus

$$(\mathcal{P}_2) \qquad \begin{array}{rl} \max & EW_a^I \\ \text{s.t.} & (4.12)-(4.15) \end{array}$$

It's easy to prove that at optimality, only constraints (4.12) and (4.15) are binding, and the solution to $(\mathcal{P}_2)$ is summarized as below:

Proposition 50. *The optimal allocation of* $(\mathcal{P}_2)$, $\{\underline{Q}^{Ia}, \hat{Q}^{Ia}; \underline{M}^{Ia}, \hat{M}^{Ia}\}$,*satisfies*[4]*:*

$$S'(\underline{Q}^{Ia}) = 2\underline{c} \tag{4.16}$$

$$S'(\hat{Q}^{Ia}) = (\underline{c}+\bar{c}) + \frac{p}{2(1-p)} \cdot \Delta c \tag{4.17}$$

$$\hat{M}^{Ia} = 0 \tag{4.18}$$

$$\underline{M}^{Ia} = \Delta c \cdot \hat{Q}^{Ia} \tag{4.19}$$

It worth attention that, under integrated production, the optimal contract for a risk-averse producer is the same as that for a risk-neutral producer, because there is no risk facing the producer indeed. We next investigate the impacts of producers' risk-aversion on the relative virtues of integrated and component production.

We have shown that the value function of component production is

$$EW_a^C(\underline{q}^{Ca}, \hat{q}^{Ca}) = \max_{\underline{q},\hat{q}} \ p^2 \left[S(\underline{q}) - 2\underline{c}\underline{q}\right] + 2p(1-p)\left[S(\hat{q}) - (\underline{c}+\bar{c})\hat{q} - \frac{u^{-1}(pu(\Delta c\hat{q}))}{1-p}\right]$$

and that of integrated production is

$$EW_a^I(\underline{Q}^{Ia}, \hat{Q}^{Ia}) = \max_{\underline{Q},\hat{Q}} \ p^2 \left[S(\underline{Q}) - 2\underline{c}\underline{Q}\right] + 2p(1-p)\left[S(\hat{Q}) - (\underline{c}+\bar{c})\hat{Q} - \frac{p\Delta c\hat{Q}}{2(1-p)}\right]$$

For risk-neutral producers, it is obvious that $u^{-1}(pu(\Delta c\hat{q})) = p\Delta c\hat{q}$ and $EW_n^C < EW_n^I$, as shown in GR. For risk-averse producers, the difference between EW_a^C

[4]The superscript Ia means "integrated production" by "risk-averse" producers.

and EW_a^I depends on the functional form of $u(\cdot)$. The regulator faces the trade-off between the reduction of information rent under component production, due to producers' risk-aversion, and the benefit of information economy of scope under integrated production. Intuitively, when the degree of risk-aversion is low, we should still have $EW_a^C < EW_a^I$, but when the producers are sufficiently risk-averse, it is possible that $EW_a^C \geq EW_a^I$? We next provide a clear result for this in the case of CRRA preference.

The producer's vNM utility is

$$u(x) = \frac{x^{1-r}}{1-r} \tag{4.20}$$

where r is the coefficient of relative risk-aversion, and $u^{-1}(pu(\Delta c\hat{q})) = p^{\frac{1}{1-r}}\Delta c\hat{q}$. The second main result of this paper is summarized as below.

Proposition 51. *Under complementary production by CRRA producers with independent cost realization, component production generates higher expected revenue than integrated production iff*

$$r \geq \frac{\ln 2}{\ln 2 - \ln p} \tag{4.21}$$

Proof. Since $\underline{q}^{Ca} = \underline{Q}^{Ia}$, the difference between EW_a^C and EW_a^I stems from the difference of revenues generated from $\hat{q}^{Ca}$ and $\hat{Q}^{Ca}$. That of component production is $S(\hat{q}) - (\underline{c} + \bar{c})\hat{q} - \frac{1}{1-p} \cdot p^{\frac{1}{1-r}}\Delta c\hat{q}$, and that of integrated production is $S(\hat{Q}) - (\underline{c} + \bar{c})\hat{Q} - \frac{1}{1-p} \cdot \frac{1}{2}p\Delta c\hat{Q}$. Obviously, a necessary and sufficient condition for

$$EW_a^C(\underline{q}^{Ca}, \hat{q}^{Ca}) \geq EW_a^I(\underline{Q}^{Ia}, \hat{Q}^{Ia})$$

is $p^{\frac{1}{1-r}} \geq \frac{1}{2}p$, which is equivalent to (4.21). $\square$

Proposition 6 shows that the relative virtues of component and integrated production depends on the degree of producers' risk-aversion. When the producers are sufficiently risk-reverse, the reduction of information rent under component production may outweigh the benefit of informational economy of scope under integrated production, and component production generates higher expected revenue to the regulator.

4.4. Conclusion

We study the regulation of risk-averse producers in the context of complementary products with independent cost realization. The production can be organized either in the form of component production or integrated production. The regulator faces the trade-off between the benefit of informational economy of scope under integrated production, and the reduction in information rent under component production, due to the producers' risk-aversion. We first show that a regulator prefers component production by risk-averse producers than that by risk-neutral producers. And we then prove in the case of CRRA preference that, when the producers are sufficiently risk-averse, component production generates strictly higher expected revenue than integrated production. The results of this paper provide important implications for regulating complementary product industries in real practice.

CHAPTER 5

Conclusion and Extension

Information disclosure itself is not a new topic in economics. However, a typical assumption in the existing literature is that agents commit to truthful disclosure, and therefore there is no possibility of false announcement, and revelations are always credible. This assumption is obviously unrealistic in many circumstances, especially when the information is not easily verifiable, or there is no effective legal recourses to deter disclosure fraud. In fact, false announcement and disclosure fraud are never rare in the real world. To investigate real disclosure problems in these cases, we should certainly abandon this full commitment assumption.

Departure from the full commitment assumption, this thesis contributes to the literature by studying cheap-talk information disclosure in auctions and monopoly pricing, which allows for the possibilities of mis-reporting or false announcement.

5.1. Main Findings

As the first step of exploration, Chapter 3 investigates and compares the differences between cheap-talk disclosure and disclosure with full commitment, in a Hotelling model. And the main research question of this chapter is that how a monopoly seller should reveal a product's horizontal attributes to consumers, when their preferences conform to mixture distributions. In contrast to the standard assumption in Hotelling models, that consumer preferences conform to either uniform or binary distributions, mixture distributions could obviously better characterize consumer preferences in reality.

I first show in Chapter 3 that optimal disclosure policies largely depend on the characteristics of the mixture distribution. Specifically, when consumer preferences are highly heterogenous, it is better for the seller to reveal information and serve different groups of consumers separately. And when the preference distribution becomes more asymmetric, cheap-talk disclosure is more likely to be dominated by no disclosure at all. Second, I also show that cheap-talk disclosure is dominated by disclosure with full commitment, in terms of profit maximization. The intuition behind this result is that commitment makes effective communication easier, which in turn increases the expected profit under information disclosure.

Chapter 3 focuses on the disclosure policies in the form of binary messages, and information structure is therefore taken as given. However, in optimal disclosure problems, it should be determined endogenously.

In Chapter 2, the major chapter of this thesis, I make one step forward further by studying optimal cheap-talk disclosure in auctions, with endogenously determined information structures. In the auction, bidders' preferences are horizontally differentiated, and the seller's disclosure policy is characterized by partitions of the product's attribute space. In a symmetric setting, I show that informative equilibria reveal the feature that more precise information is provided for less popular product attributes. Second, I prove an equilibrium existence theorem that an informative equilibrium can be supported by an information partition of any degree, as long as the number of bidders is sufficiently large. And finally, the optimal disclosure policy reveals a complementarity relationship between the number of bidders and the optimal degree of equilibrium partitions. Moreover, I also compare the cheap-talk outcomes with those of information disclosure under full commitment.

Compared with the recent literature that focuses on optimal information structure in information disclosure, Chapter 2 makes important contributions to the literature. First, it study information disclosure under cheap-talk, avoiding the typical assumption of full commitment, and thus makes it possible to evaluate the impacts of commitment on disclosure outcomes. Second, although differentiated preferences is commonly emphasized in the recent literature, it is not explicitly modelled yet. Instead, in a Hotelling model, Chapter 2 explicitly models the horizontal differentiation in bidders' preferences, and thus endogenizes the distributions of bidder's posterior valuations under a public signal. Finally, without introducing restrictive assumptions, the analytical framework developed in Chapter 2 can be readily extended to other situations, such as monopoly pricing and advertising.

5.2. Possible Directions for Extensions

This thesis contributes to the existing literature on information disclosure and advertising by studying cheap-talk disclosure, and it has the potentials to initiate a new line of researches in the related literature. The thesis, especially Chapter 2, develops a clean and tractable framework to investigate the issues of revealing product attributes to consumers under cheap-talk, and this framework can be extended to many other situations.

At this stage, there are at least two possible directions for the theoretical extension of the current model. One is to explore the case of multiple product disclosure, as this thesis just investigates the one product case. And the other, also more important, is to extend the current static framework to a model of dynamic information disclosure.

The first extension to multiple products is of great practical importance. In the real economy, from eBay to Amazon, from manufacturers to retailers, a seller commonly has many products for sale. The products can belong to different categories, or in the same category yet with different attributes. And they can be substitutes or complements to each other. In this multiple product environment, how should a seller optimally advertise the product attributes? How does the optimal advertising policy under cheap-talk depends on the structure of the market? These are important questions to investigate, and it is related to the fast-growing literature on multiple-product or combinatorial auctions.

The second direction is to investigate cheap-talk disclosure in a dynamic context. We commonly observe that a seller may be willing to reveal some true yet negative information about the products, which is detrimental to contemporary profit, even through there is no commitment or obligation for the seller to do so. For example, the rating system at most online stores is a platform of intertemporal information disclosure, which keeps records of both positive and negative comments. Why should an online store has incentives to keep the negative comments on his products. An intuition is that the seller is willing to do so under the concern of long-run *reputation*, and his current profit loss can be compensated in the future if a good reputation is maintained. This research is related to the recent literature on dynamic cheap-talk games.

Besides the two directions of extension mentioned above, there is still a theoretical challenge in the studies of cheap-talk disclosure, that is, how to order information structures in terms of informativeness, especially in the case of information partitions. When we investigate optimal cheap-talk disclosure, we should be able to order different equilibrium partitions according to some economically meaningful terms, such as informativeness. Two recent papers, Ganuza and Penalva (2010) and Quah and Strulovici (2009), explore the question of how to order information structures. They introduce stochastic order in the literature of statistics into their studies and provide some preliminary results, but neither of them investigate the case of information partitions. I think this is a challenging theoretical question for the research on cheap-talk information disclosure.

Finally, to check the validity of a theory, empirical verification is indispensable. There are many interesting empirical questions to explore, and one of them would be how the advertising policies change with market structures. And I plan to collect data, and conduct some empirical studies jointly with empirical researchers in the future.

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