

# A categorified excision principle for elliptic symbol families

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## Abstract

We develop a categorical index calculus for elliptic symbol families. The categorified index problems we consider are a secondary version of the traditional problem of expressing the index class in  $K$ -theory in terms of differential-topological data. They include orientation problems for moduli spaces as well as similar problems for skew-adjoint and self-adjoint operators. The main result of this paper is an excision principle which allows the comparison of categorified index problems on different manifolds. Excision is a powerful technique for actually solving the orientation problem; applications appear in the companion papers Joyce–Tanaka–Upmeyer [16], Joyce–Upmeyer [17], and Cao–Gross–Joyce [8].

## 1 Introduction

Index theory assigns the numerical invariant  $\text{ind } P = \dim \text{Ker } P - \dim \text{Coker } P$  to an elliptic operator  $P$  on a compact manifold  $X$ . The Atiyah–Singer index theorem [4] solves the index problem of identifying this invariant in terms of topological data on  $X$ . A key technique in its proof is the following excision principle of Seeley [21, A.1]. Let  $U$  be contained as an open subset  $U_{\pm}$  in two compact manifolds  $X_{\pm}$ . Let  $P_{\pm}, Q_{\pm}$  be a pair of elliptic operators on each of the manifolds  $X_{\pm}$ , each pair differing at most on  $U$ . Then, deforming the problem through pseudo-differential operators, the index difference  $\text{ind } Q_{\pm} - \text{ind } P_{\pm}$  can be concentrated on  $U_{\pm}$ , see for example [11, §7.1]. In particular, if we assume  $P_+ = P_-$  and  $Q_+ = Q_-$  over  $U$  it is not hard to believe the excision formula

$$\text{ind } Q_- - \text{ind } P_- = \text{ind } Q_+ - \text{ind } P_+. \quad (1.1)$$

The excision principle is one of the most powerful techniques of index theory. Once it is established, one can compare the index problem on any compact manifold to that on a sphere, where it is solved using Bott periodicity.

In proving (1.1) and extending to families the key is to establish a formal calculus for the index map in terms of topological  $K$ -theory. In this calculus  $\text{ind } P$  depends only on the principal symbol, is functorial, and can be extended to open manifolds. These properties lead to (1.1) and, combined with the multiplicative property, to the families index theorem.

In his study of orientations in Yang–Mills theory, Donaldson introduced in [10, (3.10)] a formally similar excision isomorphism for the determinant

$$\text{Det } D_- \longrightarrow \text{Det } D_+,$$

where  $D_{\pm}$  are first order differential operators on  $X_{\pm}$ , skew-adjoint outside  $U_{\pm}$ . It is based on an adiabatic construction of solutions to the differential equation  $D_+ f_+ = 0$ , given a solution  $D_- f_- = 0$ . An alternative proof closer to our discussion here in terms of pseudo-differential operators is explained in Donaldson–Kronheimer [11, §7.1].

In this paper, we shall expand these ideas and develop a new ‘categorical index calculus’ in Theorem 2.8. In a categorified index problem one assigns, rather than numbers,  $G$ -torsors to Fredholm operators  $P$  ( $G = \mathbb{Z}_2, \mathbb{Z}$ ). We shall consider the following three basic cases  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$ :

- (or) The *orientation problem* for real Fredholm operators to which we assign the  $\mathbb{Z}_2$ -torsor of orientations  $\text{or}(P)$  on the vector space  $\text{Ker } P \oplus (\text{Coker } P)^*$ .
- (pf) The *skew orientation problem* for real skew-adjoint Fredholm operators to which we assign the  $\mathbb{Z}_2$ -torsor of orientations  $\text{pf}(P)$  on  $\text{Ker } P$ .
- (sp) The *spectral orientation problem* for self-adjoint Fredholm operators to which we assign their spectral  $\mathbb{Z}$ -torsor  $\text{sp}(P)$ , see Definition 3.14.

Thus  $\lambda(P)$  plays the role of  $\text{ind}(P)$  and the category of  $G$ -torsors plays the role of the topological  $K$ -theory group. A key point is that all three orientation problems are deformation invariant. In particular, restricted to pseudo-differential operators they only depend on principal symbols, see Section 3.6.1, which is the starting point for the categorical index calculus in Section 3.6.

Rather than to connected components  $\pi_0$  these problems now correspond to  $\pi_1 \mathfrak{Fred}_{\mathbb{R}} = \mathbb{Z}_2$ ,  $\pi_1 \mathfrak{Fred}_{\mathbb{R}}^{\text{skew}} = \mathbb{Z}_2$ , and  $\pi_1 \mathfrak{Fred}_{\mathbb{C}}^{\text{sa}} = \mathbb{Z}$ . In the first two cases, the universal covers are known as the real determinant line bundle and the real Pfaffian line bundle, see for example Freed [14], turned into double covers. The universal cover of  $\mathfrak{Fred}_{\mathbb{C}}^{\text{sa}}$  is less familiar and in Section 3.4 we construct the spectral cover for self-adjoint families, a principal  $\mathbb{Z}$ -bundle. This construction is new and in Theorem 3.21 we establish also its connection to the transgression of the complex determinant line bundle. The last case (sp) has potential applications to orientations graded over  $\mathbb{Z}_n$  for  $n \in \mathbb{N}$ .

The categorical index calculus is then applied to prove Theorem 2.10. In all basic cases  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$  we establish a *canonical excision isomorphism*

$$\text{Ex: } \lambda(P_+)^* \otimes \lambda(Q_+) \longrightarrow \lambda(P_-)^* \otimes \lambda(Q_-)$$

together with its functoriality properties and its compatibilities with other constructions. These properties will be crucial in Joyce–Upmeyer [17] to solve the orientation problem for twisted Dirac operators  $P = \not{D}_{\text{Ad } E}$  on a 7-dimensional compact spin manifold in terms of a flag structure on  $X$ , with applications to the study of moduli of  $G_2$ -instantons [12].

Categorified index problems are secondary index problems, meaning they only make sense when the corresponding traditional index vanishes. For (or) this is the orientability of the real determinant line bundle, which amounts to the vanishing of the first Stiefel–Whitney class of the index class  $\text{ind } P \in KO^0(Y)$  from [6]. For (pf) and (sp) we have by [1] and by [3, §3] an index in  $KO^{-1}(Y)$  and  $K^{-1}(Y)$  whose images in  $H^1(Y; \mathbb{Z}_2)$  and  $H^1(Y; \mathbb{Z})$  must vanish. Orientability questions can thus be treated by classical index theory, but for questions of picking actual orientations this is no longer sufficient. To do this, traditional equalities must be replaced by *canonical isomorphisms*.

## Outline of the paper

In Section 2.1 we set up the necessary terminology for elliptic symbols to formulate the categorical index calculus. This is done in Section 2.2 as Theorem 2.8, where we state also our other main result, Theorem 2.10, on excision. A simplified version of the excision principle for gauge theory is stated as Theorem 2.13. Assuming the categorical index calculus, we prove Theorem 2.10 in Section 2.3.

The rest of the paper establishes the categorical index calculus in the three basic cases. Our orientation conventions are fixed in Section 3.1. Sections 3.2 and 3.3 review known results for the determinant and Pfaffian line bundles. The spectral cover for self-adjoint Fredholm operators is constructed in Section 3.4. As an aside, the relationship with the transgression of the complex determinant line is clarified in Theorem 3.21. After these preparations, we then prove Theorem 2.8, the categorical symbol calculus, in Section 3.6.

For convenience, we have collected in Appendix A some elementary background on compactly supported pseudo-differential operators.

This is the first of a series of papers on orientations in gauge theory. The second paper Joyce–Tanaka–Upmeyer [16] establishes the general theory and gives examples of solutions to orientation problems in dimensions up to 6, some of which are new. The third paper Joyce–Upmeyer [17] solves the orientation problem for Dirac operators in dimension 7. In the fourth paper, Cao–Gross–Joyce [8] will prove orientability of the moduli space of  $\text{Spin}(7)$ -instantons in dimension 8 using our excision principle.

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## 2 Categorical index calculus

### 2.1 Elliptic symbol families

#### 2.1.1 The category of elliptic symbol families

For the precise meaning of (continuous)  $Y$ -families of smooth objects parameterized by a topological space  $Y$  we refer to Appendix A.1.

**Definition 2.1** (Atiyah–Singer [4, p. 491]). Let  $Y$  be a space,  $X$  a manifold, not necessarily compact, and  $\pi: (T^*X \setminus 0_X) \times Y \rightarrow X \times Y$  the projection. A  $Y$ -family  $\{p_y\}_{y \in Y}$  of elliptic symbols over  $X$  of order  $m \in \mathbb{R}$  consists of Hermitian vector bundles  $E^0, E^1 \rightarrow X \times Y$  and a  $Y$ -family of isomorphisms

$$p: \pi^* E^0 \longrightarrow \pi^* E^1$$

satisfying  $p_{\lambda \cdot \xi, y} = \lambda^m \cdot p_{\xi, y}$  for all  $0 \neq \xi \in T^*X, y \in Y, \lambda > 0$  on the fibers. The  $Y$ -families of  $m$ -th order elliptic symbols form the set  $\mathcal{E}\ell_Y^m(X; E^0, E^1)$ .

Direct sums and adjoints of elliptic symbols are formed pointwise. We consider the following *types* of elliptic symbols:

- (or)  $p$  is *real* if  $E^0, E^1$  have orthogonal real structures and  $\overline{p_{\xi,y}(e)} = p_{-\xi,y}(\bar{e})$ .  
(pf)  $p$  is *real skew-adjoint* if  $E^0 = E^1$ ,  $p$  is real, and  $(p_{\xi,y})^\dagger = -p_{\xi,y}$ ,  
(sp)  $p$  is *self-adjoint* if  $E^0 = E^1$  and  $(p_{\xi,y})^\dagger = p_{\xi,y}$ .

**Definition 2.2.** Let  $p_\pm \in \mathcal{E}\ell_Y^m(X_\pm; E_\pm^0, E_\pm^1)$  be families of elliptic symbols of the same order  $m$  on manifolds  $X_\pm$ , and  $\phi: X_- \rightarrow X_+$  an open embedding. An *identification*  $\Phi: p_- \rightarrow p_+$  of symbols over  $\phi$  is given by two  $Y$ -families of unitary isomorphisms  $\Phi^\bullet: E_-^\bullet \rightarrow \phi^* E_+^\bullet$ ,  $\bullet = 0, 1$ , satisfying

$$\forall 0 \neq \xi \in T^*X_-, y \in Y: (p_+)_{d\phi(\xi),y} \circ \Phi^0 = \Phi^1 \circ (p_-)_{\xi,y}.$$

When  $p_\pm$  have type  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$  we consider the following *types* of  $\Phi$ :

- (or)  $\Phi$  is *real* if  $\Phi^0, \Phi^1$  commute with the given real structures on  $E_\pm^\bullet$ ,  
(pf)  $\Phi$  is *real skew-adjoint* if  $\Phi$  is real,  $E_\pm^0 = E_\pm^1$ , and  $\Phi^0 = -\Phi^1$ ,  
(sp)  $\Phi$  is *self-adjoint* if  $E_\pm^0 = E_\pm^1$  and  $\Phi^0 = \Phi^1$ .

There is then a category  $\mathcal{E}\ell_Y^m(X)$  whose objects are  $m$ -th order  $Y$ -families of elliptic symbols with morphisms the identifications over  $\phi = \text{id}_X$ . An embedding  $\phi: X_- \rightarrow X_+$  induces a functor  $\phi^*: \mathcal{E}\ell_Y^m(X_+) \rightarrow \mathcal{E}\ell_Y^m(X_-)$ , so  $\mathcal{E}\ell_Y^m(\cdot)$  organizes into a presheaf of groupoids on the site of manifolds. There is a de-categorification map  $\pi_0 \mathcal{E}\ell_Y^m(X) \rightarrow K_{\text{cpt}}^0(Y \times T^*X)$  given by the families version of the symbol class construction in [19, Ch. III, (1.7)].

**Proposition 2.3.** *We have a reduction of order functor:*

- (i) If  $p \in \mathcal{E}\ell_Y^m(X; E^0, E^1)$  then  $p(p^\dagger p)^{-1/2} \in \mathcal{E}\ell_Y^0(X; E^0, E^1)$ .  
(ii) If  $\Phi: p_- \rightarrow p_+$  in  $\mathcal{E}\ell_Y^m(X)$  then  $\Phi: p_-(p_-^\dagger p_-)^{-1/2} \rightarrow p_+(p_+^\dagger p_+)^{-1/2}$  in  $\mathcal{E}\ell_Y^0(X)$ .

Finally, we introduce the following terminology for dealing with zeroth-order symbols on open manifolds.

**Definition 2.4.** Let  $L \subset X$  be a compact set and  $p \in \mathcal{E}\ell_Y^0(X; E^0, E^1)$ . Then  $p$  is *compactly supported* in  $L$  if there exists a (unique)  $Y$ -family of bundle isomorphism  $\tilde{p}: E^0|_{(X \setminus L) \times Y} \rightarrow E^1|_{(X \setminus L) \times Y}$  satisfying the following conditions:

$$\begin{aligned} \forall (x, y) \in (X \setminus L) \times Y, \xi \in T_x^*X: p_{\xi,y} &= \tilde{p}_{x,y}, \\ \forall (x, y) \in (X \setminus L) \times Y, \exists c, C > 0: c \leq \|\tilde{p}_{x,y}\| &\leq C, \\ \forall \varepsilon > 0, y_0 \in Y \text{ there exists a neighborhood } V \text{ of } y_0 \text{ such that:} \\ (x, y) \in (X \setminus L) \times V \implies \|\tilde{p}_{x,y} - \tilde{p}_{x,y_0}\| &\leq \varepsilon. \end{aligned} \tag{2.1}$$

More generally, for  $p \in \mathcal{E}\ell_{Y \times Z}^0(X; E^0, E^1)$  with  $Z$  compact Hausdorff we allow the support to be any compact subset  $K \subset X \times Z$ . Then  $(X \setminus L) \times Y$  above is replaced by  $(X \times Y \times Z) \setminus \pi_{X \times Z}^{-1}(K)$  for the projection  $\pi_{X \times Z}: X \times Y \times Z \rightarrow X \times Z$ , and  $(X \setminus L) \times V$  by  $(X \times U) \setminus \pi_{X \times Z}^{-1}(K)$  for  $U \subset Y \times Z$  a neighborhood of  $(y_0, z_0)$ .

### 2.1.2 The category of relative pairs

The terminology of this section is not needed for the categorical index calculus, but is used in the formulation of the excision theorem.

An identification corresponds to the data that identifies pseudo-differential operators  $P_\pm$  on  $X_\pm$ . As such, they induce isomorphisms  $\Phi^*: \lambda(p_+) \rightarrow \lambda(p_-)$ , see Theorem 2.8(i). Excision extends this *global* functoriality to the case where we have a pair  $(P_\pm, Q_\pm)$  of operators on each of the spaces  $X_\pm$  and where the diffeomorphism  $\phi$  only needs to be defined *locally* where the operators differ, see Figure 2.1. The idea is that the index-theoretic information contained in  $P_-$  and  $Q_-$  can be compressed close to where the operators differ. Similarly for  $P_+$  and  $Q_+$ . The compressed solutions to the (pseudo-)differential equations can then be mapped back and forth using a only locally defined diffeomorphism.

The data needed to perform the compression in a canonical way promotes the pair  $(p, q)$  of principal symbols to a relative pair  $(p, \Xi, q)$ :

**Definition 2.5.** Let  $p \in \mathcal{E}\ell_Y^m(X; E^0, E^1)$ ,  $q \in \mathcal{E}\ell_Y^m(X; F^0, F^1)$ . An identification  $\Xi: p|_{X \setminus L} \rightarrow q|_{X \setminus L}$  over the identity defined outside a compact subset  $L \subset X$  is called a *relative  $Y$ -pair*  $(p, \Xi, q)$  with *support*  $L$ . A relative pair has *type*  $(\lambda)$  if all of  $p, q, \Xi$  have type  $(\lambda)$ .

The motivation for this terminology is that  $\Xi$  promotes  $[p] - [q] \in K_{\text{cpt}}^0(Y \times T^*X)$  to a relative class in  $K_{\text{cpt}}^0(Y \times T^*X, Y \times T^*(X \setminus L))$ . We shall see that  $\Xi$  allows us to ‘compress’  $\lambda(p)^* \otimes \lambda(q)$  into a neighborhood of the support of  $\Xi$ . To make this idea precise, we introduce the following notion.

**Definition 2.6.** Let  $(p_\pm, \Xi_\pm, q_\pm)$  be relative  $Y$ -pairs on manifolds  $X_\pm$  with compact supports  $L_\pm$  and  $\phi: U_- \rightarrow U_+$  a diffeomorphism of open sets  $U_\pm \subset X_\pm$  containing  $L_\pm$  with  $\phi(L_-) = L_+$ . An *isomorphism of relative  $Y$ -pairs* over  $\phi$ ,

$$(p_-, \Xi_-, q_-)|_{U_-} \xrightarrow{(\phi, \Pi, K)} (p_+, \Xi_+, q_+)|_{U_+}, \quad (2.2)$$

consists of two identifications  $\Pi: p_-|_{U_-} \rightarrow p_+|_{U_+}$ ,  $K: q_-|_{U_-} \rightarrow q_+|_{U_+}$  over  $\phi$  satisfying  $\Xi_+ \circ \Pi|_{U_- \setminus C} = K \circ \Xi_-|_{U_- \setminus C}$  outside a compact subset  $C \subset U_-$  with  $L_- \subset C$  and  $L_+ \subset \phi(C)$ . We then say the *support* of (2.2) is contained in  $C$ . An isomorphism (2.2) is of *type*  $(\lambda)$  if  $(p_\pm, \Xi_\pm, q_\pm)$  and  $\Pi, K$  are all of type  $(\lambda)$ .

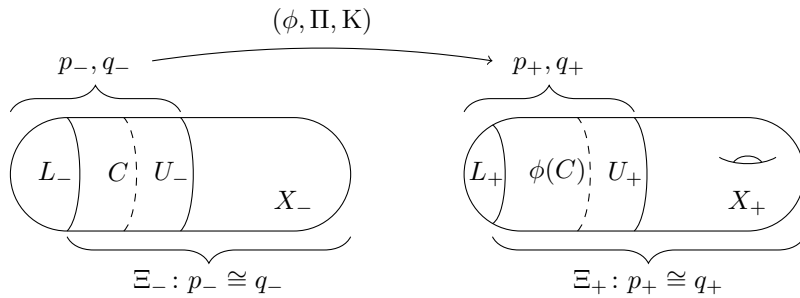


Figure 2.1: An isomorphism of relative pairs.

To this setup, depicted in Figure 2.1, we shall attach an excision isomorphism. We will also prove its independence under the following deformations.

**Definition 2.7.** Let  $Y$  be a topological space and  $Z$  compact Hausdorff. There is a version of Definition 2.5 in which the support  $K \subset X \times Z$  may change along

the compact parameter  $z \in Z$  and the symbol families  $p, q$  are supported in  $K$  as in the last paragraph of Definition 2.4. Similarly, in Definition 2.6 we can more generally take  $U_{\pm} \subset X_{\pm} \times Z$  open and  $\phi: U_- \rightarrow U_+$  to be a  $Z$ -family of diffeomorphisms. In this case we speak of a  $Z$ -deformation of relative  $Y$ -pairs

$$(p_{-,z}, \Xi_{-,z}, q_{-,z}) \xrightarrow{(\phi_z, \Pi_z, K_z)} (p_{+,z}, \Xi_{+,z}, q_{+,z}), \quad z \in Z. \quad (2.3)$$

## 2.2 Statement of results

### 2.2.1 Index calculus

We now present a categorical version of the  $K$ -theory calculus for the index map. Proofs will be given in Section 3.6. We first need some notation.

When  $G$  is abelian we have the *tensor product*  $P_1 \otimes P_2 = (P_1 \times P_2)/G$  of principal  $G$ -bundles, quotienting by the  $G$ -action  $(p_1 g, p_2 g^{-1})$  by all  $g \in G$  on  $(p_1, p_2) \in P_1 \times P_2$ . The *dual* principal  $G$ -bundle has the same underlying space, but  $G$ -action twisted by the inversion  $G \rightarrow G$ . To keep track of signs we use graded principal  $G$ -bundles, which in addition have a *degree*  $\deg P$  in the (multiplicative) group  $\mathbb{Z}_2 = \{\pm 1\}$ . For the tensor product and dual we set

$$\deg(P_1 \otimes P_2) = \deg P_1 \cdot \deg P_2, \quad \deg(P^*) = \deg P.$$

**Theorem 2.8.** *Let  $X$  be a manifold,  $Y$  a topological space, and  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$ . In Propositions 3.5, 3.11, 3.16, and Definition 3.24 we construct for every  $Y$ -family of elliptic symbols  $p \in \mathcal{E}\ell_Y^m(X)$  of type  $(\lambda)$ , zeroth-order compactly supported when  $X$  is not compact, a graded principal  $G$ -bundle  $\lambda(p) \rightarrow Y$  with*

$$\deg \lambda(p) = \begin{cases} (-1)^{\text{ind}_{\mathbb{R}}(p)}, \\ (-1)^{\dim_{\mathbb{R}} \text{Ker}(p)}, \\ +1, \end{cases} \quad G = \begin{cases} \mathbb{Z}_2 & (\text{or}), \\ \mathbb{Z}_2 & (\text{pf}), \\ \mathbb{Z} & (\text{sp}), \end{cases} \quad (2.4)$$

and the following formal properties:

- (i) An identification  $\Phi: p_- \rightarrow p_+$  of type  $(\lambda)$  covering a diffeomorphism  $\phi: X_- \rightarrow X_+$  induces a functoriality isomorphism

$$\Phi^*: \lambda(p_+) \longrightarrow \lambda(p_-) \quad (2.5)$$

satisfying  $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$  and  $\text{id}^* = \text{id}$ .

- (ii) For  $p, q \in \mathcal{E}\ell_Y^m(X)$  of type  $(\lambda)$  we have a direct sum isomorphism

$$\lambda(p \oplus q) \longrightarrow \lambda(p) \otimes \lambda(q), \quad (2.6)$$

which is associative, graded commutative, and natural for (2.5).

- (iii) For  $p \in \mathcal{E}\ell_Y^m(X)$  of type  $(\lambda)$  we have an adjointness isomorphism

$$\lambda(-p^\dagger) \longrightarrow \lambda(p)^*, \quad (2.7)$$

natural for (2.5) and compatible with (2.6).

- (iv) For  $p \in \mathcal{E}\ell_Y^0(X)$  of type  $(\lambda)$  and compactly supported in the image of an open embedding  $i: U \hookrightarrow X$  we have a pushforward isomorphism

$$i_! : \lambda(i^*p) \longrightarrow \lambda(p), \quad (2.8)$$

natural for (2.5), compatible with (2.6) and (2.7), and satisfying

$$i_! j_! = (ij)_!, \quad \text{id}_! = \text{id}.$$

- (v) For  $p \in \mathcal{E}\ell_Y^m(X)$  of type  $(\lambda)$  we have  $p(p^\dagger p)^{-1/2} \in \mathcal{E}\ell_Y^0(X)$  of type  $(\lambda)$ . There is a reduction of order isomorphism

$$\lambda(p) \longrightarrow \lambda(p(p^\dagger p)^{-1/2}), \quad (2.9)$$

compatible with (2.6)–(2.8). An identification  $\Phi: p_- \rightarrow p_+$  of type  $(\lambda)$  is also an identification  $p_-(p_-^\dagger p_-)^{-1/2} \rightarrow p_+(p_+^\dagger p_+)^{-1/2}$  of the same type, and then (2.9) is natural for (2.5).

In Definition 2.4 the condition that the support  $K \subset X \times Z$  of a symbol be a product  $L \times Z$  was relaxed in case  $Z$  is compact Hausdorff. But  $p$  is then also supported in the compact product  $L \times Z$  with  $L = \pi_X(K)$ , so this case is already covered by Theorem 2.8. We record this for later reference.

**Corollary 2.9.** *Let  $Z$  be compact Hausdorff. For  $p \in \mathcal{E}\ell_{Y \times Z}^m(X)$  of type  $(\lambda)$ , compactly supported if  $X$  is not compact, Theorem 2.8 still applies to  $p$ .*

*In particular, when  $Z = [0, 1]$  fiber transport in the principal  $G$ -bundle  $\lambda(p) \rightarrow Y \times [0, 1]$  defines an isomorphism, compatible with (2.5)–(2.9),*

$$\lambda\{p_z\}_{z \in [0, 1]} : \lambda(p_0) \longrightarrow \lambda(p_1). \quad (2.10)$$

### 2.2.2 Excision

**Theorem 2.10.** *Let  $X_\pm$  be compact manifolds and  $Y$  a topological space. Every isomorphism  $(\phi, \Pi, K)$  of relative pairs of type  $(\lambda)$  as in (2.2) over a base diffeomorphism  $\phi: U_- \rightarrow U_+$  of open subsets  $U_\pm \subset X_\pm$  induces an excision isomorphism of graded principal  $G$ -bundles over  $Y$ ,*

$$\text{Ex}(\phi, \Pi, K) : \lambda(p_-)^* \otimes \lambda(q_-) \longrightarrow \lambda(p_+)^* \otimes \lambda(q_+), \quad (2.11)$$

uniquely determined by the following properties:

- (i) (Empty support.) *If  $(\phi, \Pi, K)$  has empty support, then  $\lambda(p_\pm)^* \otimes \lambda(q_\pm)$  are both identified with the trivial cover. In this identification the excision isomorphism  $\text{Ex}(\phi, \Pi, K)$  becomes the identity map.*
- (ii) (Restriction.) *Let  $U_\pm \subset \tilde{U}_\pm$  be open supersets. Assume that  $(\phi, \Pi, K)$  has an extension  $(\tilde{\phi}, \tilde{\Pi}, \tilde{K})$  of same type  $(\lambda)$  over a base diffeomorphism  $\tilde{\phi}: \tilde{U}_- \rightarrow \tilde{U}_+$ . Then  $\text{Ex}(\phi, \Pi, K) = \text{Ex}(\tilde{\phi}, \tilde{\Pi}, \tilde{K})$ .*
- (iii) (Sums.) *In addition to  $(\phi, \Pi, K)$  let  $(\tilde{p}_-, \tilde{\Xi}_-, \tilde{q}_-) \xrightarrow{(\phi, \tilde{\Pi}, \tilde{K})} (\tilde{p}_+, \tilde{\Xi}_+, \tilde{q}_+)$  be another isomorphism of relative pairs over the same  $\phi$  and type  $(\lambda)$ . For the sum of symbols and identifications we have a commutative diagram*

$$\begin{array}{ccc} \lambda(p_-)^* \otimes \lambda(q_-) \otimes \lambda(\tilde{p}_-)^* \otimes \lambda(\tilde{q}_-) & \xrightarrow{(2.12)} & \lambda(p_- \oplus \tilde{p}_-)^* \otimes \lambda(q_- \oplus \tilde{q}_-) \\ \downarrow \text{Ex}(\phi, \Pi, K) \otimes \text{Ex}(\phi, \tilde{\Pi}, \tilde{K}) & & \downarrow \text{Ex}(\phi, \Pi \oplus \tilde{\Pi}, K \oplus \tilde{K}) \\ \lambda(p_+)^* \otimes \lambda(q_+) \otimes \lambda(\tilde{p}_+)^* \otimes \lambda(\tilde{q}_+) & \xrightarrow{(2.12)} & \lambda(p_+ \oplus \tilde{p}_+)^* \otimes \lambda(q_+ \oplus \tilde{q}_+). \end{array}$$

- (iv) (Functoriality.) Let  $X_-, X_0, X_+$  be compact manifolds. Given composable isomorphisms of relative pairs of the same type  $(\lambda)$

$$(p_-, \Xi_-, q_-) \xrightarrow{(\phi_{-0}, \Pi_{-0}, K_{-0})} (p_0, \Xi_0, q_0) \xrightarrow{(\phi_{0+}, \Pi_{0+}, K_{0+})} (p_+, \Xi_+, q_+)$$

we have for the horizontal composition  $(\phi_{-+} := \phi_{0+} \circ \phi_{-0})$  and so forth

$$\text{Ex}(\phi_{0+}, \Pi_{0+}, K_{0+}) \circ \text{Ex}(\phi_{-0}, \Pi_{-0}, K_{-0}) = \text{Ex}(\phi_{-+}, \Pi_{-+}, K_{-+}).$$

- (v) (Global excision.) If the isomorphism  $(\phi, \Pi, K)$  of relative pairs is defined over a global diffeomorphism  $\phi: X_- \rightarrow X_+$ ,  $U_\pm = X_\pm$ , then excision coincides with the global functoriality  $\text{Ex}(\phi, \Pi, K) = (\Pi^*)^* \otimes (K^*)^{-1}$  of (2.5).
- (vi) (Deformations.) Consider a  $Z$ -deformation (2.3). Then (2.11) is continuous in the topology of Corollary 2.9. In particular, when  $Z = [0, 1]$  and both relative pairs  $(p_{\pm, z}, \Xi_{\pm, z}, q_{\pm, z})$  are constant in  $z$  we get a homotopy

$$\text{Ex}(\phi_z, \Pi_z, K_z): \lambda(p_-)^* \otimes \lambda(q_-) \longrightarrow \lambda(p_+)^* \otimes \lambda(q_+), \quad z \in [0, 1].$$

- (vii) (Reduction of order.) Regarding  $(\phi, \Pi, K)$  also as an isomorphism of the zeroth-order relative symbols, we have a commutative diagram

$$\begin{array}{ccc} \lambda(p_-)^* \otimes \lambda(q_-) & \xrightarrow{(2.9)} & \lambda\left(p_-(p_-^\dagger p_-)^{-1/2}\right)^* \otimes \lambda\left(q_-(q_-^\dagger q_-)^{-1/2}\right) \\ \text{Ex}(\phi, \Pi, K) \downarrow & & \downarrow \text{Ex}(\phi, \Pi, K) \\ \lambda(p_+)^* \otimes \lambda(q_+) & \xrightarrow{(2.9)} & \lambda\left(p_+(p_+^\dagger p_+)^{-1/2}\right)^* \otimes \lambda\left(q_+(q_+^\dagger q_+)^{-1/2}\right). \end{array}$$

If the involved symbols are zeroth-order compactly supported everything holds for non-compact manifolds. In (ii) we allow  $\tilde{U}_\pm$  to be contained in larger manifolds  $\tilde{X}_\pm$ . Assuming extensions  $\tilde{p}_\pm, \tilde{q}_\pm$  to  $\tilde{X}_\pm$  and  $(\tilde{\phi}, \tilde{\Pi}, \tilde{K})$  to  $\tilde{U}_\pm$ , the isomorphism  $\text{Ex}(\phi, \Pi, K)$  then identifies with  $\text{Ex}(\tilde{\phi}, \tilde{\Pi}, \tilde{K})$  under (2.8) for  $i_\pm: X_\pm \rightarrow \tilde{X}_\pm$ .

**Remark 2.11.** Our sign convention in Section 3.1 fixes the map in (iii) as

$$\begin{aligned} & \lambda(p)^* \otimes \lambda(q) \otimes \lambda(\tilde{p})^* \otimes \lambda(\tilde{q}) \xrightarrow{1 \otimes (3.2) \otimes 1} \lambda(p)^* \otimes \lambda(\tilde{p})^* \otimes \lambda(q) \otimes \lambda(\tilde{q}) \\ & \xrightarrow{(3.3) \otimes 1} (\lambda(\tilde{p}) \otimes \lambda(p))^* \otimes \lambda(q) \otimes \lambda(\tilde{q}) \xrightarrow{(3.2)^* \otimes 1 \otimes 1} (\lambda(p) \otimes \lambda(\tilde{p}))^* \otimes \lambda(q) \otimes \lambda(\tilde{q}) \\ & \xrightarrow{(2.6)^* \otimes 1 \otimes (2.6)} \lambda(p \oplus \tilde{p})^* \otimes \lambda(q \oplus \tilde{q}). \quad (2.12) \end{aligned}$$

**Remark 2.12.** Applying (2.11) for  $\lambda = \text{sp}$  and  $Y = S^1$  we deduce on the level of isomorphism classes an excision formula for the spectral flow around a loop of self-adjoint elliptic pseudo-differential operators on  $X$ :

$$\text{SF}\{Q_t^+\}_{t \in S^1} - \text{SF}\{P_t^+\}_{t \in S^1} = \text{SF}\{Q_t^-\}_{t \in S^1} - \text{SF}\{P_t^-\}_{t \in S^1} \quad (2.13)$$

For first order elliptic differential operators this is already known also a consequence of the Atiyah–Patodi–Singer index theorem [2, Th. 3.10] which allows one to express the spectral flow of a periodic family as an index on  $X \times S^1$ ,



see [3, p. 95]. This depends on a detailed analysis of elliptic boundary value problems using heat kernels. Once the spectral flow has been expressed as an index, one can apply the classical excision formula (1.1) to get (2.13). Although this direct proof for (2.13) is compelling, it does not strictly require the categorical point of view. A key point is that (2.11) includes a generalization of (2.13) for non-periodic families that is not expressible in terms of spectral flow.

### 2.2.3 Specialization for gauge theory

For convenience, we formulate Theorem 2.10 in the case relevant to gauge theory. Let  $G$  be a Lie group,  $X$  a compact manifold,  $P \rightarrow X$  a principal  $G$ -bundle, and  $\text{Ad } P \rightarrow X$  the associated bundle of Lie algebras. Let

$$D: C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$$

be an elliptic differential operator on  $X$ , which is also denoted  $E^\bullet = (E^0, E^1, D)$ . Given fixed connections on  $E^0, E^1, T^*X$ , we can use connections  $\nabla^P$  on  $P$  to define as in [16, Def. 1.2] the  $\nabla^P$ -twisted differential operator

$$D^{\nabla_{\text{Ad } P}}: C^\infty(X, E^0 \otimes \text{Ad } P) \longrightarrow C^\infty(X, E^1 \otimes \text{Ad } P).$$

Let  $O_P^{E^\bullet} = \text{or}(D^{\nabla_{\text{Ad } P}})$  be its set of orientations. More generally, a  $Y$ -family of connections determines a double cover of  $Y$ . Using the trivial bundle  $\underline{G}$  we define the *normalized orientation torsor*  $\check{O}_P^{E^\bullet} := (O_P^{E^\bullet})^* \otimes_{\mathbb{Z}_2} O_{\underline{G}}^{E^\bullet}$ .

**Theorem 2.13.** *Let  $X_\pm$  be compact manifolds. The data consisting of*

- (a) *open covers  $X_\pm = U_\pm \cup V_\pm$ ,*
- (b) *principal  $G$ -bundles  $P_\pm \rightarrow X_\pm$  and  $G$ -frames  $\tau_\pm$  of  $P_\pm|_{V_\pm}$  over  $V_\pm$ ,*
- (c) *elliptic operators  $E_\pm^\bullet = (E_\pm^0, E_\pm^1, D_\pm)$  on  $X_\pm$ ,*
- (d) *bundle isomorphisms  $\Phi^\bullet: E_-^\bullet|_{U_-} \rightarrow E_+^\bullet|_{U_+}$  covering a diffeomorphism  $\phi: U_- \rightarrow U_+$  identifying  $D_\pm$  in the sense that*

$$\forall s \in C_{\text{cpt}}^\infty(U_-, E_-^0): \quad \Phi^1 \circ D_-(s) = D_+(\Phi^0 \circ s \circ \phi^{-1}) \circ \phi,$$

- (e) *an isomorphism  $\Psi: P_-|_{U_-} \rightarrow P_+|_{U_+}$  of principal  $G$ -bundles covering  $\phi$ , satisfying  $\Psi \circ \tau_- = \tau_+ \circ \phi$  outside a compact subset of  $U_- \cap V_- \cap \phi^{-1}(V_+)$ ,*

*induces a canonical excision isomorphism of normalized orientation torsors*

$$\text{Ex}_{-+}: \check{O}_{P_-}^{E_-^\bullet} \longrightarrow \check{O}_{P_+}^{E_+^\bullet}. \quad (2.14)$$

*These have the following properties:*

- (i) (Empty set.) *If  $U_\pm = \emptyset$  then we have canonical identifications  $\check{O}_{P_\pm}^{E_\pm^\bullet} = \mathbb{Z}_2$  under which  $\text{Ex}_{-+}$  becomes  $\text{id}_{\mathbb{Z}_2}$ .*
- (ii) (Restriction.) *Assume  $(\phi, \Phi^\bullet)$  can be extended over open supersets  $U_- \subset \tilde{U}_-$  and  $U_+ \subset \tilde{U}_+$  to  $(\tilde{\phi}, \tilde{\Phi}^\bullet)$ . Then  $\text{Ex}_{-+} = \tilde{\text{Ex}}_{-+}$ .*

- (iii) (Sums.) In addition to (a)–(e) let  $H$  be a Lie group,  $Q_\pm \rightarrow X_\pm$  principal  $H$ -bundles,  $\rho_\pm$  frames of  $Q_\pm|_{V_\pm}$ , and  $\Xi: Q_-|_{U_-} \rightarrow Q_+|_{U_+}$  an isomorphism of  $H$ -bundles over  $\phi$  with  $\Xi \circ \rho_- = \rho_+ \circ \phi$  outside a compact subset of  $U_- \cap V_- \cap \phi^{-1}(V_+)$ . Then we have a commutative diagram

$$\begin{array}{ccc} \check{O}_{P_-}^{E_\bullet} \otimes \check{O}_{Q_-}^{E_\bullet} & \xrightarrow{(2.6)} & \check{O}_{P_- \times_{X_-} Q_-}^{E_\bullet} \\ \text{Ex}_{-+}^P \otimes \text{Ex}_{-+}^Q \downarrow & & \downarrow \text{Ex}_{-+}^{P \times Q} \\ \check{O}_{P_+}^{E_\bullet} \otimes \check{O}_{Q_+}^{E_\bullet} & \xrightarrow{(2.6)} & \check{O}_{P_+ \times_{X_+} Q_+}^{E_\bullet} \end{array}$$

- (iv) (Functoriality.) Given three sets of data  $X_{\pm 0} = U_{\pm 0} \cup V_{\pm 0}$ ,  $P_{\pm 0} \rightarrow X_{\pm 0}$ ,  $E_{\pm 0}^\bullet$ ,  $\tau_{\pm 0}$  as above, diffeomorphisms  $\phi_{-0}: U_- \rightarrow U_0$ ,  $\phi_{0+}: U_0 \rightarrow U_+$  that identify  $D_-$ ,  $D_0$ ,  $D_+$ , and  $G$ -bundle isomorphisms  $\Psi_{-0}$ ,  $\Psi_{0+}$ , we have

$$\text{Ex}_{0+} \circ \text{Ex}_{-0} = \text{Ex}_{-+}.$$

Both (ii) and (iii) are natural for this functoriality.

- (v) (Global excision.) If  $\phi: X_- \rightarrow X_+$  is a global diffeomorphism,  $U_\pm = X_\pm$ , then excision coincides with the global functoriality defined by mapping the kernels of the differential operators  $D^{\nabla_{\text{Ad } P_\pm}}$  and  $D^{\nabla_{\text{Ad } G}}$  using  $\phi$  and  $\Phi^\bullet$ .
- (vi) (Families.) Let  $Y$  be compact Hausdorff. Given a  $Y$ -family of data as above, where  $U_\pm, V_\pm \subset X_\pm \times Y$  and all the other data are allowed to change in  $Y$ , (2.14) becomes a continuous map of coverings over  $Y$ .

*Proof.* Define  $p_\pm = \sigma(D_\pm) \otimes \text{id}_{\text{Ad } P_\pm}: \pi^*(E_\pm^0 \otimes \text{Ad } P_\pm) \rightarrow \pi^*(E_\pm^1 \otimes \text{Ad } P_\pm)$  and  $q_{\pm, \xi} = \sigma(D_\pm) \otimes \text{id}_{\text{Ad } G}: \pi^*(E_\pm^0 \otimes \text{Ad } G) \rightarrow \pi^*(E_\pm^1 \otimes \text{Ad } G)$  by twisting the principal symbols  $\sigma(D_\pm)$  of the elliptic operators with  $\text{Ad } P_\pm \rightarrow X_\pm$  and the trivial bundle  $\text{Ad } G \rightarrow X_\pm$ . Here  $\pi: T^*X \setminus \{0\} \rightarrow X$  denotes the projection.

The  $G$ -frames  $\tau_\pm$  induce isomorphisms  $\text{Ad } \tau_\pm: \text{Ad } P_\pm|_{V_\pm} \rightarrow \text{Ad } G|_{V_\pm}$  and thus identifications  $\Xi_\pm: p_\pm|_{V_\pm} \rightarrow q_\pm|_{V_\pm}$ ,  $\Xi_\pm = \text{id}_{E_\pm^\bullet} \otimes \text{Ad } \tau_\pm$ . In other words,  $(p_\pm, \Xi_\pm, q_\pm)$  are relative pairs supported in the compact subset  $X \setminus V_\pm \subset U_\pm$ . Also,  $\Phi^\bullet \otimes \text{Ad } \Psi$  defines an identification  $\Pi: p_-|_{U_-} \rightarrow p_+|_{U_+}$  and  $\Phi^\bullet \otimes \text{id}_{\text{Ad } G}$  an identification  $K: q_-|_{U_-} \rightarrow q_+|_{U_+}$ . Hence  $(\phi, \Pi, K)$  is an isomorphism of relative pairs  $(p_-, \Xi_-, q_-) \rightarrow (p_+, \Xi_+, q_+)$ . Applying Theorem 2.10 with  $Y = \{\text{pt}\}$  gives (2.14). The compatibilities stated in Theorem 2.10 lead to (i)–(vi).  $\square$

## 2.3 Proof of Theorem 2.10

Assuming the categorical index calculus of Theorem 2.8, we will perform a series of reductions until Theorem 2.10 reduces completely to Theorem 2.8, verifying in each step that the properties (i)–(vii) claimed in Theorem 2.10 are preserved. We prove existence in §2.3.1–2.3.3. Uniqueness is similar and proven in §2.3.4.

### 2.3.1 Reduction to zeroth order

Assume Theorem 2.10 in the special case of zeroth-order families. Consider an isomorphism  $(p_-, \Xi_-, q_-)|_{U_-} \xrightarrow{(\phi, \Pi, K)} (p_+, \Xi_+, q_+)|_{U_+}$  of relative pairs of type  $(\lambda)$  over a diffeomorphism  $\phi: U_- \rightarrow U_+$ , where  $(p_\pm, \Xi_\pm, q_\pm)$  are relative

$Y$ -pairs, compactly supported in  $L_{\pm} \subset U_{\pm}$ . Recall from Proposition 2.3(ii) that an identification  $\Xi: p \rightarrow q$  is also an identification of the zeroth-order symbols. In particular, we may regard  $(\phi, \Pi, K)$  also as an isomorphism between the relative pairs  $(p_{\pm}(p_{\pm}^{\dagger}p_{\pm})^{-1/2}, \Xi_{\pm}, q_{\pm}(q_{\pm}^{\dagger}q_{\pm})^{-1/2})$ . Using the corresponding excision isomorphism for zeroth order families, define  $\text{Ex}(\phi, \Pi, K)$  in general by

$$\begin{array}{ccc} \lambda(p_{-})^{*} \otimes \lambda(q_{-}) & \xrightarrow{\text{Ex}(\phi, \Pi, K)} & \lambda(p_{+})^{*} \otimes \lambda(q_{+}) \\ \downarrow (2.9)^{*,-1} \otimes (2.9) & & \downarrow (2.9)^{*,-1} \otimes (2.9) \\ \lambda(p_{-}(p_{-}^{\dagger}p_{-})^{-1/2})^{*} & \xrightarrow{\text{Ex}(\phi, \Pi, K)} & \lambda(p_{+}(p_{+}^{\dagger}p_{+})^{-1/2})^{*} \\ \otimes \lambda(q_{-}(q_{-}^{\dagger}q_{-})^{-1/2}) & & \otimes \lambda(q_{+}(q_{+}^{\dagger}q_{+})^{-1/2}). \end{array}$$

Then Theorem 2.10(vii) holds by definition and (i)–(vi) follow from the compatibilities in Theorem 2.8(v) and the assumed properties for zeroth-order families.

### 2.3.2 Deformation to compactly supported symbols

The reduction to Theorem 2.8 is based on the following deformation:

**Proposition 2.14.** *In the notation of Definition 2.5, let  $(p, \Xi, q)$  be a relative  $Y$ -pair of order zero with compact support  $L$  over a manifold  $X$ . Let  $U \subset X$  be an open set with  $L \subset U$  and pick  $\chi \in C_{\text{cpt}}^{\infty}(U, [0, 1])$  with  $\chi|_L \equiv 1$ . Then*

$$(p, \Xi, q)_{\chi}^t = \begin{pmatrix} -(1-t+t\chi)p^{\dagger} & t(1-\chi)\pi^{*}\Xi^{0\dagger} \\ t(1-\chi)\pi^{*}\Xi^1 & (1-t+t\chi)q \end{pmatrix}: \pi^{*}(E^1 \oplus F^0) \longrightarrow \pi^{*}(E^0 \oplus F^1)$$

for  $t \in [0, 1]$  has the following properties:

- (i) For each  $t \in [0, 1]$ ,  $(p, \Xi, q)_{\chi}^t$  is a zeroth-order family of elliptic symbols.
- (ii)  $(p, \Xi, q)_{\chi}^0 = -p^{\dagger} \oplus q$ .
- (iii)  $(p, \Xi, q)_{\chi}^1$  has compact support  $\chi^{-1}(1) \subset U$ .
- (iv) When  $p, q$  are skew-adjoint symbol families and  $\Xi^0 = -\Xi^1$ , all  $(p, \Xi, q)_{\chi}^t$  are skew-adjoint. Similarly when  $p, q$  are self-adjoint and  $\Xi^0 = \Xi^1$ .
- (v) Let  $(p_{-}, \Xi_{-}, q_{-}) \xrightarrow{(\phi, \Pi, K)} (p_{+}, \Xi_{+}, q_{+})$  be an isomorphism of relative pairs with support  $L_{\pm}$  over the open embedding  $\phi: U_{-} \rightarrow U_{+}$ . Fix cut-offs  $\chi_{\pm} \in C_{\text{cpt}}^{\infty}(U_{\pm}, [0, 1])$  with  $\chi_{\pm}|_{L_{\pm}} \equiv 1$  and  $\chi_{+} \circ \phi = \chi_{-}$ . Then  $(\Gamma^0, \Gamma^1) = (\Pi^1 \oplus K^0, \Pi^0 \oplus K^1)$  defines for each  $t \in [0, 1]$  an identification over  $\phi$

$$\Gamma: (p_{-}, \Xi_{-}, q_{-})_{\chi_{-}}^t \longrightarrow (p_{+}, \Xi_{+}, q_{+})_{\chi_{+}}^t, \quad (2.15)$$

functorial for the composition of isomorphisms of relative pairs.

*Proof.* Note that  $(p, \Xi, q)_{\chi}(t)$  is well-defined since  $\chi|_L \equiv 1$  and  $\Xi$  is defined outside  $L$ . On  $\chi^{-1}(1)$  we have  $(p, \Xi, q)_{\chi}(t) = -p^{\dagger} \oplus q$  and (i) is clear. To prove (i) for  $x \notin \chi^{-1}(1)$ , let  $0 \neq \xi \in T_x^*X$  and write

$$(p, \Xi, q)_{\chi}^t(\xi) = \begin{pmatrix} 0 & (\Xi^0)^{\dagger} \\ \Xi^1 & 0 \end{pmatrix} \circ \left[ t(1-\chi)\text{id} + (1-t+t\chi) \cdot \begin{pmatrix} 0 & (\Xi^1)^{\dagger}q_{\xi} \\ -\Xi^0p_{\xi}^{\dagger} & 0 \end{pmatrix} \right].$$

Since by assumption  $(-\Xi^0 p_\xi^\dagger)^\dagger = -(\Xi^1)^\dagger q_\xi$ , the latter summand is skew-adjoint and invertible. Hence all of its spectral values are non-zero and purely imaginary. It follows that the endomorphism given by the inner square brackets does not have zero in its spectrum. The rest are trivial verifications.  $\square$

This deformation  $(p, \Xi, q)_\chi^t$  determines the *compression isomorphism*

$$\lambda(p)^* \otimes \lambda(q) \xrightarrow{(2.6), (2.7)} \lambda(-p^\dagger \oplus q) \xrightarrow{(2.10)} \lambda((p, \Xi, q)_\chi^1). \quad (2.16)$$

If  $\chi_0, \chi_1, \dots, \chi_r$  are cut-offs as in Proposition 2.14, so are all convex combinations  $\sum_{i=0}^r s_i \chi_i$  for  $s \in \Delta^r$ . For  $r = 1$  we thus have a commutative diagram

$$\begin{array}{ccc} & & \lambda(i^*(p, \Xi, q)_{\chi_0}^1) \\ & \nearrow & \downarrow \lambda\{(p, \Xi, q)_{(1-s)\chi_0 + s\chi_1}\}_{s \in [0,1]} \\ \lambda(p)^* \otimes \lambda(q) & & \lambda((p, \Xi, q)_{\chi_1}^1). \end{array} \quad (2.17)$$

### 2.3.3 Proof of Theorem 2.10 for zeroth-order families

We must define  $\text{Ex}(\phi, \Pi, K)$  and verify Theorem 2.10 in the  $m = 0$  case.

Let  $(\phi, \Pi, K): (p_-, \Xi_-, q_-) \rightarrow (p_+, \Xi_+, q_+)$  be an isomorphism of relative  $Y$ -pairs of zeroth order. Pick  $\chi_+ \in C_{\text{cpt}}^\infty(U_+, [0, 1])$  with  $\chi_+|_{L_+} \equiv 1$  and let  $\chi_- := \chi_+ \circ \phi$ . Depending on this choice, we have by Proposition 2.14 a deformation  $(p_\pm, \Xi_\pm, q_\pm)_\chi^t$  through symbols of type  $(\lambda)$ , beginning with  $-p_\pm^\dagger \oplus q_\pm$  and ending with a family  $(p_\pm, \Xi_\pm, q_\pm)_{\chi_\pm}^1$  of elliptic symbols compactly supported in  $U_\pm$  and two compression isomorphisms (2.16). Moreover, the isomorphism  $(\phi, \Pi, K)$  induces identifications  $(\phi, \Pi^1 \oplus K^0, \Pi^0 \oplus K^1)$  by (2.15). Define

$$\begin{array}{ccc} \lambda(p_-)^* \otimes \lambda(q_-) & \xrightarrow{\text{Ex}(\phi, \Pi, K)} & \lambda(p_+)^* \otimes \lambda(q_+) \\ \downarrow (2.16) & & \downarrow (2.16) \\ \lambda((p_-, \Xi_-, q_-)_{\chi_-}^1) & & \lambda((p_+, \Xi_+, q_+)_{\chi_+}^1) \\ \uparrow (2.8) i_-^- & & \uparrow (2.8) i_+^+ \\ \lambda((p_-, \Xi_-, q_-)_{\chi_-}^1|_{U_-}) & \xrightarrow[(2.5)]{(\phi, \Pi^1 \oplus K^0, \Pi^0 \oplus K^1)^*} & \lambda((p_+, \Xi_+, q_+)_{\chi_+}^1|_{U_+}). \end{array} \quad (2.18)$$

The composition  $\text{Ex}(\phi, \Pi, K)$  is independent of the choice of  $\chi_+$  by (2.17), since according to Corollary 2.9 both (2.5), (2.8) are compatible with deformations.

Property (i) follows by using  $\chi_+ = 0$ . Similarly (ii) follows by reusing the same cut-offs for the supersets. Since all of the maps used in the construction of (2.18) are functorial and compatible with direct sums and deformations, we see that (iii)–(v) follow from the corresponding properties in Theorem 2.8.

In Theorem 2.10(vi) the support  $K_\pm \subset X_\pm \times Z$  and  $U_\pm \subset X_\pm \times Z$  are allowed to vary in  $Z$ . To check continuity at each point  $z_0 \in Z$ , we reduce to the (known) product case. By elementary point-set topology,  $z_0$  has a compact neighborhood  $Z_0$  for which there exist subsets  $L_\pm \subset V_\pm \subset X_\pm$  with  $L_\pm$  compact and  $V_\pm$  open satisfying  $K_\pm \cap (X_\pm \times Z_0) \subset L_\pm \times Z_0 \subset V_\pm \times Z_0 \subset U_\pm$ . If we restrict  $(\phi, \Pi, K)$  to a  $Z_0$ -family then it has compact support in  $V_\pm$  in the previous sense, so Theorem 2.10 yields the continuity of  $\text{Ex}(\phi, \Pi, K)|_{Z_0}$ .

### 2.3.4 Uniqueness

The proof is largely parallel to the existence proof. As in §2.3.1, an isomorphism  $(p_-, \Xi_-, q_-)|_{U_-} \xrightarrow{(\phi, \Pi, K)} (p_+, \Xi_+, q_+)|_{U_+}$  of relative pairs induces a corresponding isomorphism of zeroth order symbols, and by Theorem 2.10(vii) the diagram in §2.3.1 commutes with vertical isomorphisms. Hence  $\text{Ex}(\phi, \Pi, K)$  is uniquely determined by the zeroth order case. In this case we have identifications  $\Gamma^t: (p_-, \Xi_-, q_-)_{\chi_-}^t \rightarrow (p_+, \Xi_+, q_+)_{\chi_+}^t$  from Proposition 2.14. Beginning with the isomorphism of relative pairs  $(p_-, \text{id}, p_-)|_{U_-} \xrightarrow{(\phi, \Pi, \Pi)} (p_+, \text{id}, p_+)|_{U_+}$  we similarly obtain identifications  $\tilde{\Gamma}^t: (p_-, \text{id}, p_-)_{\chi_-}^t \rightarrow (p_+, \text{id}, p_+)_{\chi_+}^t$ . Together these define a  $[0, 1]$ -family of isomorphisms  $(\phi, \tilde{\Gamma}^t, \Gamma^t)$  between the relative pairs  $((p_{\pm}, \text{id}, p_{\pm})_{\chi_{\pm}}^t, \text{id} \oplus \Xi_{\pm}, (p_{\pm}, \Xi_{\pm}, q_{\pm})_{\chi_{\pm}}^t)|_{U_{\pm}}$ . According to Theorem 2.10(vi),  $\text{Ex}(\phi, \tilde{\Gamma}^0, \Gamma^0), \text{Ex}(\phi, \tilde{\Gamma}^1, \Gamma^1)$  are identified under fiber transport isomorphisms. Let  $\Pi^\dagger = (\Pi^1, \Pi^0)$ . Since  $(\phi, \tilde{\Gamma}^0, \Gamma^0)$  is the direct sum of the isomorphisms  $(\phi, \Pi^\dagger, \Pi^\dagger): (-p_-^\dagger, \text{id}, -p_-^\dagger)_{U_-} \rightarrow (-p_+^\dagger, \text{id}, -p_+^\dagger)_{U_+}$  and  $(\phi, \Pi, K)$ , we find by Theorem 2.10(iii) that  $\text{Ex}(\phi, \Pi, K)$  is uniquely determined by  $\text{Ex}(\phi, \Pi^\dagger, \Pi^\dagger)$  and  $\text{Ex}(\phi, \tilde{\Gamma}^0, \Gamma^0)$ . By Theorem 2.10(i),  $\text{Ex}(\phi, \Pi^\dagger, \Pi^\dagger)$  becomes the identity under the obvious trivializations. In conclusion,  $\text{Ex}(\phi, \Pi, K)$  is uniquely determined by  $\text{Ex}(\phi, \tilde{\Gamma}^1, \Gamma^1)$ , where all of the symbols are compactly supported in  $U_{\pm}$ .

Thus suppose  $(\phi, \Pi, K)$  is an isomorphism in which  $p_{\pm}, q_{\pm}$  are compactly supported in  $U_{\pm} \subset X_{\pm}$ . In this case Theorem 2.10 applies also to non-compact manifolds and properties (ii)&(iv) imply that  $\text{Ex}(\phi, \Pi, K)$  can be identified with  $\text{Ex}(\phi|_{U_-}, \Pi|_{U_-}, K|_{U_-})$ , which by Theorem 2.10(v) is given by the global functoriality  $(\Pi^*)^* \otimes (K^*)^{-1}$  over the diffeomorphism  $\phi: U_- \rightarrow U_+$  of non-compact manifolds. Hence  $\text{Ex}(\phi, \Pi, K)$  is uniquely determined by the axioms.

## 3 Determinant, Pfaffian, and spectral covers

### 3.1 Sign convention and spectral preliminaries

#### 3.1.1 Supersymmetric sign convention

The top exterior power ‘ $\bigwedge$ ’ of a finite-dimensional vector space has the property that a short exact sequence  $\Sigma: 0 \rightarrow U \xrightarrow{i} V \xrightarrow{p} W \rightarrow 0$  induces an isomorphism

$$\det_{\Sigma}: \bigwedge U \otimes \bigwedge W \longrightarrow \bigwedge V, \quad u \otimes p_*(w) \mapsto i_*(u) \wedge w, \quad (3.1)$$

where  $u \in \Lambda^{\dim U} U$ ,  $w \in \Lambda^{\dim W} W$ , and  $i_*, p_*$  are the maps induced by  $i, p$  on exterior powers. This expresses our orientation convention that in such a sequence  $U$ -coordinates are regarded to come before  $W$ -coordinates in  $V$ . The data (3.1) defines a determinant functor, in the sense of Deligne [9, §4.13], which has a unique extension to bounded complexes [18], subject to a sign convention.

We use the sign convention of supersymmetry, where vectors and co-vectors are viewed as odd and scalars as even when commuting them. This rule determines various isomorphisms involving tensor products and dualization, which for convenience we make explicit. Thus we regard the determinant as a graded line in degree  $(-1)^{\dim V} \in \mathbb{Z}_2 = \{\pm 1\}$  use the graded tensor product, and braiding

$$\sigma: L_1 \otimes L_2 \longrightarrow L_2 \otimes L_1, \quad \sigma(x_1 \otimes x_2) = (-1)^{\deg L_1 \cdot \deg L_2} x_2 \otimes x_1. \quad (3.2)$$

We agree to evaluate functionals on the left  $\text{ev}: L \otimes L^* \rightarrow \mathbb{R}$ ,  $(x, \alpha) \mapsto \langle x, \alpha \rangle = \alpha(x)$ , so that evaluation on the right introduces a sign  $(-1)^{\deg L}$ . This convention matches (3.4) in that the dual appears there also on the right. Instead of the naïve one, we insist on the identification

$$\tau: L_1^* \otimes L_2^* \longrightarrow (L_2 \otimes L_1)^*, \quad (x_2 \otimes x_1)(\alpha_1 \otimes \alpha_2) := \langle x_1, \alpha_1 \rangle \langle x_2, \alpha_2 \rangle. \quad (3.3)$$

In the same way, to identify  $\bigwedge(V^*)$  with  $(\bigwedge V)^*$  we must use

$$\mathcal{P}: \alpha_1 \wedge \dots \wedge \alpha_n \longmapsto (v_n \wedge \dots \wedge v_1 \longmapsto \det[v_j(\alpha_i)]_{i,j=1}^n),$$

which differs by  $(-1)^{n(n-1)/2}$  from the naïve convention. For the dual of an exact sequence  $\Sigma^*: 0 \rightarrow W^* \rightarrow V^* \rightarrow U^* \rightarrow 0$  we then get commutative diagrams

$$\begin{array}{ccc} \bigwedge(W^*) \otimes \bigwedge(U^*) & \xrightarrow{\text{Det}_{\Sigma^*}} & \bigwedge(V^*) \\ (\mathcal{P} \otimes \mathcal{P}) \downarrow & & \downarrow \mathcal{P} \\ (\bigwedge W)^* \otimes (\bigwedge U)^* & \xrightarrow{(\text{Det}_{\Sigma}^{-1})^* \circ \tau} & (\bigwedge V)^*. \end{array}$$

### 3.1.2 Spectral theory of Fredholm operators

We shall use a definition of the Quillen determinant for Hilbert spaces due to Bismut–Freed [7, Sect. f)] in terms of spectral theory.

Recall that the *essential spectrum* of a bounded operator is defined as its spectrum in the Calkin algebra  $\mathcal{B}/\mathcal{K}$  modulo compact operators. By definition,  $0 \notin \text{spec}_{\text{ess}}(A_0)$  for a Fredholm operator  $A_0$ . As the spectrum in  $\mathcal{B}/\mathcal{K}$  is a closed set, we can then find an essential spectral gap  $(-\delta, \delta)$  with  $\delta > 0$ .

**Lemma 3.1** (see [20]). *For a self-adjoint Fredholm operator  $A_0: H \rightarrow H$  let  $\delta > 0$  be such that  $\text{spec}_{\text{ess}}(A_0) \cap (-\delta, \delta) = \emptyset$ . For all  $-\delta < \nu < 0 < \mu < \delta$  with  $\mu, \nu \notin \text{spec } A_0$  there exists a neighborhood  $U$  of  $A_0$  in the space of self-adjoint Fredholm operators with the following properties:*

- (i)  $\forall A \in U: \nu, \mu \notin \text{spec } A$  and  $\text{spec}_{\text{ess}}(A) \cap (-\delta, \delta) = \emptyset$ .
- (ii) The direct sum  $V_{(\nu, \mu)}(A) \subset H$  of all eigenspaces of  $A$  with eigenvalue  $\nu < \lambda < \mu$  defines a vector bundle over  $U$  of finite locally constant rank.

*Proof.* Pick  $\varepsilon > 0$  such that both  $(\nu - \varepsilon, \nu + \varepsilon)$  and  $(\mu - \varepsilon, \mu + \varepsilon)$  are disjoint from  $\text{spec } A_0$ . This remains true in a neighborhood  $U$  of  $A_0$  where we require also  $\text{spec}_{\text{ess}}(A) \cap (-\delta, \delta) = \emptyset$ . Then all  $A - \lambda$  with  $\nu < \lambda < \mu$  are Fredholm and  $A$  has discrete eigenvalues near zero. Hence  $V_{(\nu, \mu)}(A)$  is finite-dimensional. The projection  $\chi_{(\nu, \mu)}(A)$  onto  $V_{(\nu, \mu)}(A)$  can be formed using functional calculus. Let  $f$  be the continuous function with  $f|_{[\nu, \mu]} \equiv 1$ ,  $f_{(-\infty, \nu - \varepsilon]} \equiv 0$ ,  $f_{[\mu + \varepsilon, \infty)} \equiv 0$ , and that is otherwise affine-linear. Then  $\chi_{(\nu, \mu)}(A) = f(A)$  for all  $A \in U$  and the map  $A \mapsto f(A)$  from  $U$  into the bounded projections is continuous. In particular it has locally constant finite rank, so its image is a vector subbundle.  $\square$

**Definition 3.2.** Let  $\mathcal{H}^0, \mathcal{H}^1$  be Hilbert bundles over a space  $Y$ . The bounded operators in each fiber define a Banach bundle  $\mathcal{B}(\mathcal{H}^0, \mathcal{H}^1)$  in the operator norm. A  $Y$ -family of Fredholm operators is a continuous section  $P: Y \rightarrow \mathcal{B}(\mathcal{H}^0, \mathcal{H}^1)$  that is fiberwise Fredholm. Let  $P_{\pm}: Y \rightarrow \mathcal{B}(\mathcal{H}_{\pm}^0, \mathcal{H}_{\pm}^1)$  be families of Fredholm operators. An *isomorphism*  $F^{\bullet}: P_- \rightarrow P_+$  is a pair of continuous sections  $F^{\bullet}: Y \rightarrow \mathcal{B}(\mathcal{H}_{-}^{\bullet}, \mathcal{H}_{+}^{\bullet})$  of metric-preserving operators with  $F^1 \circ P_- = P_+ \circ F^0$ .

## 3.2 Determinant cover of real Fredholm operators

### 3.2.1 Determinant line bundle

**Definition 3.3.** Let  $H^0, H^1$  be Hilbert spaces. The *determinant line* of a Fredholm operator  $P: H^0 \rightarrow H^1$  (regarded as a two term complex) is

$$\text{Det } P := \bigwedge \text{Ker } P \otimes \left( \bigwedge \text{Ker } P^\dagger \right)^*, \quad (3.4)$$

a  $\mathbb{Z}_2$ -graded line in degree  $(-1)^{\text{ind } P}$ . More generally, for a Fredholm operator between Banach spaces, replace  $\text{Ker } P^\dagger$  by  $\text{Coker } P$ .

For a family of Fredholm operators the disjoint union over all (3.4) will be topologized as a line bundle using Lemma 3.1. It generalizes that of Freed [14] for Dirac operators. One may alternatively use ‘stabilization’ to topologize the determinant line bundle for general Banach spaces, see for example Zinger [23], paying attention to a tedious sign convention, as in (3.13).

We say  $\mu > 0$  is *sufficiently small* for  $P$  if  $[0, \mu] \cap \text{spec}_{\text{ess}}(P^\dagger P) = \emptyset$ . For  $P$  Fredholm and  $0 < \mu < \nu$  sufficiently small, set  $Q = P(P^\dagger P)^{-1/2}$  and define

$$\begin{aligned} \bigwedge V_{[0, \nu]}(P^\dagger P) \otimes \left( \bigwedge V_{[0, \nu]}(PP^\dagger) \right)^* &\xrightarrow{\text{stab}_{\nu, \mu}} \bigwedge V_{[0, \mu]}(P^\dagger P) \otimes \left( \bigwedge V_{[0, \mu]}(PP^\dagger) \right)^* \\ \text{stab}_{\nu, \mu}(v \wedge w \otimes \beta \wedge \alpha) &= \langle \Lambda(Q)w, \beta \rangle v \otimes \alpha, \end{aligned}$$

for  $v \in \bigwedge V_{[0, \mu]}(P^\dagger P), \alpha \in \left( \bigwedge V_{[0, \mu]}(PP^\dagger) \right)^*, w \in \bigwedge V_{(\mu, \nu]}(P^\dagger P)$ , and  $\beta \in \left( \bigwedge V_{(\mu, \nu]}(PP^\dagger) \right)^*$ . Here  $(P^\dagger P)^{-1/2}$  denotes the unique positive square root on each of the eigenspaces  $V_\lambda(P^\dagger P)$  with eigenvalue  $\lambda > 0$  and  $Q$  restricts to an isomorphism  $V_\lambda(P^\dagger P) \rightarrow V_\lambda(PP^\dagger)$ , so the top exterior power  $\Lambda(Q)$  makes sense on  $(\mu, \nu]$ . One checks

$$\text{stab}_{\mu, 0} \circ \text{stab}_{\nu, \mu} = \text{stab}_{\nu, 0}. \quad (3.5)$$

**Definition 3.4.** Let  $P$  be a  $Y$ -family of Fredholm operators. The *determinant line bundle* of  $P$  is the  $Y$ -family of one-dimensional vector spaces  $\{\text{Det } P_y\}_{y \in Y}$  with the following topology. Let  $y_0 \in Y$  and pick  $\mu > 0$  sufficiently small for  $P_{y_0}$  with  $\mu \notin \text{spec } P_{y_0}^\dagger P_{y_0}$ . Pick a neighborhood  $U$  of  $y_0$  over which the Hilbert bundles are trivial. By shrinking  $U$  we may also assume the conclusions of Lemma 3.1. Then we can transport the bundle topology provided by Lemma 3.1(ii) on exterior powers along the isomorphisms

$$\text{stab}_{\mu, 0}: \bigwedge V_{[0, \mu]}(P^\dagger P) \otimes \left( \bigwedge V_{[0, \mu]}(PP^\dagger) \right)^* \longrightarrow \text{Det } P|_U. \quad (3.6)$$

This topology is independent of  $\mu$  by (3.5) and since  $\text{stab}_{\nu, \mu}$  are homeomorphisms. It is also easily seen to be independent of the trivializations of the Hilbert bundles on overlaps. Hence we get a line bundle  $\text{Det } P \rightarrow Y$ .

**Proposition 3.5.** *The determinant line bundle has the following properties:*

- (i) (Functoriality.) Let  $P_\pm$  be  $Y$ -families of Fredholm operators. An isomorphism  $F^\bullet: P_- \rightarrow P_+$  induces an isomorphism

$$\bigwedge F^0 \otimes \left( \bigwedge F^1 \right)^{*, -1}: \text{Det } P_- \longrightarrow \text{Det } P_+. \quad (3.7)$$

- (ii) (Direct sums.) For  $Y$ -families of Fredholm operators  $P, Q$  there is a canonical isomorphism, natural for (3.7),

$$\det_{P,Q}: \text{Det } P \otimes \text{Det } Q \longrightarrow \text{Det}(P \oplus Q). \quad (3.8)$$

These are associative. They are graded commutative in the sense of a commutative diagram

$$\begin{array}{ccc} \text{Det } P \otimes \text{Det } Q & \xrightarrow{\det_{P,Q}} & \text{Det}(P \oplus Q) \\ \sigma \downarrow & & \downarrow \text{Det}(\text{swap}) \\ \text{Det } Q \otimes \text{Det } P & \xrightarrow{\det_{Q,P}} & \text{Det}(Q \oplus P). \end{array} \quad (3.9)$$

Here  $\sigma$  includes  $(-1)^{\text{ind } P \cdot \text{ind } Q}$  and the isomorphism  $\text{swap}$  exchanges the Hilbert spaces without a sign. More generally, a short exact sequence

$$\Sigma: 0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 0$$

of Fredholm operators, meaning a diagram of bounded operators

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_-^0 & \xrightarrow{i^-} & H_-^1 & \xrightarrow{p^-} & H_-^2 \longrightarrow 0 \\ & & P^0 \downarrow & & P^1 \downarrow & & P^2 \downarrow \\ 0 & \longrightarrow & H_+^0 & \xrightarrow{i^+} & H_+^1 & \xrightarrow{p^+} & H_+^2 \longrightarrow 0 \end{array} \quad (3.10)$$

with exact rows and  $P^0, P^1, P^2$  Fredholm, induces an isomorphism

$$\det_\Sigma: \text{Det } P^0 \otimes \text{Det } P^2 \longrightarrow \text{Det } P^1.$$

- (iii) (Adjoint.) There is a canonical isomorphism  $\tau: \text{Det } P^\dagger \rightarrow (\text{Det } P)^*$ . It is natural for (3.7):

$$\begin{array}{ccc} \text{Det } P^\dagger & \xrightarrow{\tau} & (\text{Det } P)^* \\ \text{Det } f^\dagger \uparrow & & \uparrow (\text{Det } f)^* \\ \text{Det } Q^\dagger & \xrightarrow{\tau} & (\text{Det } Q)^*. \end{array} \quad (3.11)$$

For the exact sequence  $\Sigma^\dagger: 0 \rightarrow P_2^\dagger \rightarrow P_1^\dagger \rightarrow P_0^\dagger \rightarrow 0$  adjoint to (3.10) and the isomorphisms we have a commutative diagram

$$\begin{array}{ccc} \text{Det } P_2^\dagger \otimes \text{Det } P_0^\dagger & \xrightarrow{\det_{\Sigma^\dagger}} & \text{Det } P_1^\dagger \\ \tau \otimes \tau \downarrow & & \downarrow \tau \\ (\text{Det } P_2)^* \otimes (\text{Det } P_0)^* & & (\text{Det } P_1)^* \\ (3.3) \downarrow & & \downarrow (\det_\Sigma)^* \\ (\text{Det } P_0 \otimes \text{Det } P_2)^* & \xleftarrow{(\det_\Sigma)^*} & (\text{Det } P_1)^*. \end{array} \quad (3.12)$$

- (iv) (Composition.) When  $P = P^+ \circ P^-$  factors into two Fredholm operators  $P^-: H^- \rightarrow H^0$  and  $P^+: H^0 \rightarrow H^+$ , we get an isomorphism

$$\det_{P^-, P^+}: \text{Det } P^- \otimes \text{Det } P^+ \longrightarrow \text{Det } P.$$



- (v) (Invertible.) For a  $Y$ -family of invertible operators  $P$  the determinant line bundle has a canonical continuous trivialization  $\det(P) = 1 \otimes 1^*$ .

*Proof.* (i) To prove the continuity of (3.7) we claim the commutativity of

$$\begin{array}{ccc} \bigwedge V_{[0,\mu]}(P_-^\dagger P_-) \otimes \left( \bigwedge V_{[0,\mu]}(P_- P_-^\dagger) \right)^* & \xrightarrow{\text{stab}_{\mu,0}(P_-)} & \bigwedge \text{Ker } P_- \otimes \left( \bigwedge \text{Ker } P_-^\dagger \right)^* \\ \downarrow \bigwedge F^0 \otimes (\bigwedge F^1)^{*, -1} & & \downarrow \bigwedge F^0 \otimes (\bigwedge F^1)^{*, -1} \\ \bigwedge V_{[0,\mu]}(P_+^\dagger P_+) \otimes \left( \bigwedge V_{[0,\mu]}(P_+ P_+^\dagger) \right)^* & \xrightarrow{\text{stab}_{\mu,0}(P_+)} & \bigwedge \text{Ker } P_+ \otimes \left( \bigwedge \text{Ker } P_+^\dagger \right)^* \end{array}$$

The horizontal isomorphisms are continuous by (3.6) and the left vertical map is continuous in the vector bundle topologies, hence so is the right map. Let  $v \in \bigwedge \text{Ker } P_-$ ,  $\alpha \in (\bigwedge \text{Ker } P_-^\dagger)^*$ ,  $w \in \bigwedge V_{[0,\mu]}(P_-^\dagger P_-)$ ,  $\beta \in (\bigwedge V_{[0,\mu]}(P_- P_-^\dagger))^*$ . Mapping  $v \wedge w \otimes \beta \wedge \alpha$  through the upper right gives  $\langle \Lambda(Q_-)w, \beta \rangle \Lambda(F^0)(v) \otimes (\Lambda F^1)^{*, -1}(\alpha)$ , while the lower left gives  $\langle \Lambda(F^1)^{-1} \Lambda(Q_+) \Lambda(F^0)w, \beta \rangle \Lambda(F^0)(v) \otimes (\Lambda F^1)^{*, -1}(\alpha)$ . We have  $(P_+^\dagger P_+)^{1/2} = F^0(P_-^\dagger P_-)^{1/2}(F^0)^\dagger$  by uniqueness of positive roots. This implies  $Q_+ F^0 = F^1 Q_-$ , so  $\Lambda(F^1)^{-1} \Lambda(Q_+) \Lambda(F^0) = \Lambda(Q_-)$ .

(ii) A short exact sequence (3.10) determines a snake lemma exact sequence

$$\begin{array}{ccccccccc} \text{Ker } P^0 & \xrightarrow{i_*^-} & \text{Ker } P^1 & \xrightarrow{p_*^-} & \text{Ker } P^2 & \xrightarrow{\delta} & \text{Coker } P^0 & \xrightarrow{i_*^+} & \text{Coker } P^1 & \xrightarrow{p_*^+} & \text{Coker } P^2 \\ a & & & & & & & & & & \zeta \\ & & \searrow & \nearrow p_*^- & \searrow & \nearrow \delta & \searrow & \nearrow i_*^+ & & & \\ & & \text{Coker } i_*^- & & \text{Coker } p_*^- & & \text{Coker } \delta & & & & \\ & & d & & e, \gamma & & \beta & & & & \end{array}$$

that we splice as indicated into short exact sequences  $\Sigma_{\text{Ker } P^1}$ ,  $\Sigma_{\text{Ker } P^2}$ ,  $\Sigma_{\text{Coker } P^0}$ ,  $\Sigma_{\text{Coker } P^1}$  named after their middle term. Using (3.1), we define, denoting the dual homomorphisms by  $(\cdot)^*$ ,

$$\begin{aligned} \det_\Sigma: \text{Det } P^0 \otimes \text{Det } P^2 &\longrightarrow \text{Det } P^1, \\ a \otimes (\det_{\Sigma_{\text{Coker } P^0}}^{-1})^* (\beta \otimes \gamma) &\otimes \det_{\Sigma_{\text{Ker } P^2}}(d \otimes e) \otimes \zeta \\ \longmapsto \epsilon \cdot \gamma(e) \cdot \det_{\Sigma_{\text{Ker } P^1}}(a \otimes d) &\otimes (\det_{\Sigma_{\text{Coker } P^1}}^{-1})^* (\zeta \otimes \beta), \end{aligned} \quad (3.13)$$

using the sign  $\epsilon = (-1)^{\gamma + \gamma d + \beta(d + \zeta)}$  dictated by our supersymmetry conventions. This sign convention agrees with that of Zinger [23, (4.10)]. We refer to [23, (2.27) and Cor. 4.13] for the tedious verification of associativity. For a direct sum the coboundary  $\delta$  vanishes and (3.13) reduces to

$$\begin{aligned} \text{Det } P^0 \otimes \text{Det } P^2 &= \bigwedge \text{Ker } P^0 \otimes \left( \bigwedge \text{Coker } P^0 \right)^* \otimes \bigwedge \text{Ker } P^2 \otimes \left( \bigwedge \text{Coker } P^2 \right)^* \\ &\stackrel{(1 \otimes 1 \otimes \tau)(1 \otimes \sigma \otimes 1)}{\cong} \bigwedge \text{Ker } P^0 \otimes \bigwedge \text{Ker } P^2 \otimes \left( \bigwedge \text{Coker } P^2 \otimes \bigwedge \text{Coker } P^0 \right)^* \\ &\stackrel{1 \otimes 1 \otimes \sigma^*}{\cong} \bigwedge \text{Ker } P^0 \otimes \bigwedge \text{Ker } P^2 \otimes \left( \bigwedge \text{Coker } P^0 \otimes \bigwedge \text{Coker } P^2 \right)^*, \end{aligned} \quad (3.14)$$

so to naïve rearrangement with sign  $(-1)^{\dim \text{Coker } P^0 \text{ ind } P^2}$ . In this case, which is all we need, associativity and graded commutativity are easy. For example, (3.9) is straight-forward when rearranging naïvely and we discuss the sign. The horizontal maps in (3.9) contain  $(-1)^{\dim \text{Coker } P^0 \text{ ind } P^2}$  and  $(-1)^{\dim \text{Coker } P^2 \text{ ind } P^0}$ ,

and  $\text{Det}(\text{swap})$  exchanges the factors in  $\bigwedge \text{Ker } P^0 \otimes \bigwedge \text{Ker } P^2$  and  $\bigwedge \text{Coker } P^0 \otimes \bigwedge \text{Coker } P^2$  in the last line of (3.14), introducing  $(-1)^{\dim \text{Ker } P^0 \dim \text{Ker } P^2}$  and  $(-1)^{\dim \text{Coker } P^0 \dim \text{Coker } P^2}$ . Multiplying these four signs gives  $(-1)^{\text{ind } P^0 \text{ind } P^2}$ .

(iii) We define  $\tau$  to be (3.3). The verifications of (3.11) and (3.12) are straightforward, inserting signs whenever commuting symbols and using that the snake lemma exact sequence of  $\Sigma^\dagger$  is the adjoint of the snake lemma sequence of  $\Sigma$ . The continuity of  $\tau$  follows from the commutativity of (which uses  $Q^\dagger Q = 1$ )

$$\begin{array}{ccc} \bigwedge V_{[0,\mu]}(PP^\dagger) \otimes \left( \bigwedge V_{[0,\mu]}(P^\dagger P) \right)^* & \xrightarrow{\text{stab}_{\mu,0}(P^\dagger)} & \bigwedge \text{Ker } P^\dagger \otimes \left( \bigwedge \text{Ker } P \right)^* \\ \downarrow \tau & & \downarrow \tau \\ \left( \bigwedge V_{[0,\mu]}(P^\dagger P) \otimes \left( \bigwedge V_{[0,\mu]}(PP^\dagger) \right)^* \right)^* & \xleftarrow{(\text{stab}_{\mu,0}(P))^*} & \left( \bigwedge \text{Ker } P \otimes \left( \bigwedge \text{Ker } P^\dagger \right)^* \right)^* \end{array}$$

The horizontal isomorphisms are continuous by (3.6) and  $\tau$  on the left is continuous in the vector bundle topologies.

(iv) This follows by applying (3.1) to the splittings of the exact sequence

$$0 \rightarrow \text{Ker } P^- \rightarrow \text{Ker } P \rightarrow \text{Ker } P^+ \rightarrow \text{Coker } P^- \rightarrow \text{Coker } P \rightarrow \text{Coker } P^+ \rightarrow 0,$$

inserting again signs when rearranging. One may also regard (iv) as a special case of (ii) using the short exact sequence  $\Sigma : 0 \rightarrow P^- \rightarrow P \oplus \text{id}_{H^0} \rightarrow P^+ \rightarrow 0$  and then define  $\det_{P^-, P^+}$  as  $\det_\Sigma$  followed by  $\text{Det}(P \oplus \text{id}_{H^0}) \cong \text{Det}(P) \otimes \text{Det}(\text{id}_{H^0}) \cong \text{Det}(P)$ , again using (ii).

(v) Obviously (3.4) is canonically trivial and  $\det(P) := 1 \otimes 1^* \in \text{Det}(P)$  is a continuous section, by (3.6) for  $\mu$  smaller than the least eigenvalue of  $P^\dagger P$ .  $\square$

### 3.2.2 Determinant cover of real Fredholm operators

**Definition 3.6.** The *determinant cover* of a  $Y$ -family  $P$  of real Fredholm operators is the principal  $\mathbb{Z}_2$ -bundle  $\text{or } P := (\text{Det } P \setminus \{\text{zero section}\}) / \mathbb{R}_{>0}$  of  $Y$ , regarded as being  $\mathbb{Z}_2$ -graded in degree  $(-1)^{\text{ind } P}$ , where  $\mathbb{Z}_2 = \{\pm 1\}$ .

**Remark 3.7.** For  $P$  real or complex self-adjoint Fredholm, evaluation defines a canonical trivialization of (3.4). For real operators we thus get a canonical basepoint  $o_{\text{taut}, P} \in \text{or } P$ . However, in families we still get non-trivial determinant covers, since they vary discontinuously. To understand the discontinuity, suppose for simplicity that  $P_t$  is a family with a single eigenvalue  $\lambda_t$  in  $[-\mu, \mu]$  which crosses zero once and upwards at  $t_0$ . There is a corresponding continuous eigenvector  $v_t$ . By definition of the topology,  $\text{or } P_t$  is the set of orientations of the vector space of endomorphisms of  $\bigwedge V_{[0,\mu^2]} P_t^2 \cong \mathbb{R}$  (identified using  $v_t$ ). For  $t \neq t_0$  the tautological element is represented by the identity of  $\text{Ker } P_t = \{0\}$ , viewed as an endomorphism of  $V_{[0,\mu^2]} P_t^2$  via  $P_t$ . In terms of this identification,  $o_{\text{taut}, P_t}$  becomes the orientation of the multiplication map  $\lambda_t : \mathbb{R} \rightarrow \mathbb{R}$ . At  $t = t_0$  we take the orientation of the identity map of  $\text{Ker } P_{t_0} = \mathbb{R}$ . It follows that

$$o_{\text{taut}, P_t} = \begin{cases} -1 & (t < t_0), \\ +1 & (t \geq t_0). \end{cases}$$

This agrees with the parity of the spectral flow along  $\{P_t\}$ , see [20]. We therefore get a continuous section  $(-1)^{\text{SF}\{P_s\}_{s \in [0,t]}} \cdot o_{\text{taut}, P_t}$  of  $\text{or}(P)$ .

In the self-adjoint case we can thus represent  $\text{or}(P)$  by pairs  $[\mu, \epsilon]$  of  $\mu > 0$  with  $\pm\mu \notin \text{spec } P$  and  $\epsilon \in \{\pm 1\}$ . Here  $[\mu, \epsilon] = [\nu, \delta]$  where  $\delta = \epsilon \cdot (-1)^{N_{] \mu, \nu[}(P)}$  and  $N_{] \mu, \nu[}(P)$  is the number of eigenvalues of  $P$  in  $] \mu, \nu[$ , counted with multiplicity.

**Proposition 3.8.** *The properties (i)–(v) of Proposition 3.5 hold analogously for the determinant cover. The monodromy of the determinant cover around a loop of Fredholm operators is the parity of the spectral flow.*

### 3.3 Pfaffian cover of real skew-adjoint Fredholm operators

#### 3.3.1 Pfaffian line bundle

Initially we can work over the real or the complex numbers.

**Definition 3.9.** A bounded operator  $P: H \rightarrow H$  is *skew-adjoint* if  $P^\dagger = -\overline{P}$  for the complex conjugate operator  $\overline{P}: \overline{H} \rightarrow \overline{H}$ . The *Pfaffian line* of a skew-adjoint Fredholm operator is the graded line in degree  $(-1)^{\dim \text{Ker } P}$  defined by

$$\text{Pf } P := \bigwedge \text{Ker } P.$$

The inner product defines a two-form  $\langle P \cdot, \cdot \rangle$  that is non-degenerate on each finite-dimensional eigenspace  $V_\lambda(P^\dagger P)$ . This induces a preferred volume element  $\omega_\lambda \in \bigwedge V_\lambda(P^\dagger P)$ . For  $0 < \mu < \nu$  sufficiently small for  $P$  define a map  $\text{stab}_{\nu, \mu}^{\text{skew}}: \bigwedge V_{[0, \nu]}(P^\dagger P) \rightarrow \bigwedge V_{[0, \mu]}(P^\dagger P)$  by  $\text{stab}_{\nu, \mu}^{\text{skew}}(v \wedge \omega_{\lambda_1} \wedge \dots \wedge \omega_{\lambda_k}) = v$ , where  $v \in \bigwedge V_{[0, \mu]}(P^\dagger P)$  and where  $\mu < \lambda_1 < \dots < \lambda_k \leq \nu$  are all non-zero eigenvalues of  $P^\dagger P$  in this range. Then

$$\text{stab}_{\mu, 0}^{\text{skew}} \circ \text{stab}_{\nu, \mu}^{\text{skew}} = \text{stab}_{\nu, 0}^{\text{skew}}.$$

**Definition 3.10.** Let  $P$  be a family of skew-adjoint Fredholm operators. For  $\nu, \mu \notin \text{spec}(P^\dagger P)$  sufficiently small for  $P$  the maps  $\text{stab}_{\nu, \mu}^{\text{skew}}$  are homeomorphisms, using the topology provided by Lemma 3.1. As before, we get a well-defined topology on  $\text{Pf } P$  by transporting the topology along

$$\text{stab}_{\mu, 0}^{\text{skew}}: \bigwedge V_{[0, \mu]}(P^\dagger P) \xrightarrow{\sim} \bigwedge \text{Ker } P. \quad (3.15)$$

An isomorphism  $F: P_- \rightarrow P_+$  of skew-adjoint operators is an isomorphism  $F^\bullet$  as in Definition 3.2 with the added condition  $F^0 = F^1 = F$ .

**Proposition 3.11.** *The Pfaffian has the following properties:*

- (i) (Functoriality.) *Let  $P_\pm$  be  $Y$ -families of skew-adjoint Fredholm operators. Every isomorphism  $F: P_- \rightarrow P_+$  induces an isomorphism*

$$\bigwedge F: \text{Pf } P_- \longrightarrow \text{Pf } P_+. \quad (3.16)$$

- (ii) (Direct sums.) *For  $Y$ -families of skew-adjoint Fredholm operators  $P, Q$  there is a canonical isomorphism, natural for (3.16),*

$$\text{pf}_{P, Q}: \text{Pf } P \otimes \text{Pf } Q \longrightarrow \text{Pf}(P \oplus Q). \quad (3.17)$$

*These are graded commutative as in (3.9).*

- (iii) (Root.) For a  $Y$ -family of skew-adjoint operators there is a canonical isomorphism, functorial for isomorphisms as in (i),

$$\mathrm{Pf} P \otimes \mathrm{Pf} P \longrightarrow \mathrm{Det} P.$$

Since the square of a metric line bundle is trivial over the real numbers, this shows in particular that  $\mathrm{Det} P$  is trivial for real skew-adjoint families.

- (iv) (Invertible.) For a  $Y$ -family of invertible skew-adjoint Fredholm operators the Pfaffian line bundle has a canonical trivialization.

*Proof.* (i) Since  $F^\dagger = F^{-1}$  and  $FP_- = P_+F$ , we have  $F(P_-^\dagger P_-)F^{-1} = P_+^\dagger P_+$ . Therefore  $F$  maps  $V_\lambda(P_-^\dagger P_-)$  to  $V_\lambda(P_+^\dagger P_+)$ . This shows that  $\bigwedge F$  can be extended to the left-hand side of (3.15), proving continuity of (3.16). The remaining assertions follow similarly.  $\square$

### 3.3.2 Pfaffian cover of real skew-adjoint Fredholm operators

**Definition 3.12.** The *Pfaffian cover* of a family  $P$  of real skew-adjoint Fredholm operators is the principal  $\mathbb{Z}_2$ -bundle  $\mathrm{pf} P := (\mathrm{Pf} P \setminus \{\text{zero section}\})/\mathbb{R}_{>0}$ , regarded as being  $\mathbb{Z}_2$ -graded in degree  $(-1)^{\dim_{\mathbb{R}} \mathrm{Ker} P}$ .

Properties (i)–(iv) of Proposition 3.11 hold also for the Pfaffian cover. In particular, for real skew-adjoint elliptic symbol families the orientation cover is canonically trivial, since  $\mathrm{or} P = \mathrm{pf} P \otimes_{\mathbb{Z}_2} \mathrm{pf} P$  by (iii) and since the square of any double cover is canonically trivial.

**Remark 3.13.** Since the eigenspaces for small non-zero eigenvalues  $\lambda$  have the symplectic form  $\omega_\lambda$ , they all have even multiplicity. It follows that the spectral flow around a loop of skew-adjoint Fredholm operators is always even.

## 3.4 Spectral cover of self-adjoint Fredholm operators

The constructions in the section are taken over the complex numbers, but they apply equally to real operators by complexifying, since  $\pi_1 \mathfrak{Fred}_{\mathbb{R}}^{\mathrm{sa}} = \pi_1 \mathfrak{Fred}_{\mathbb{C}}^{\mathrm{sa}}$ .

### 3.4.1 Construction

For a self-adjoint operator  $P$  and bounded  $I \subset \mathbb{R} \setminus \mathrm{spec}_{\mathrm{ess}}(P)$  let  $N_I(P)$  be the number of eigenvalues  $\lambda \in \mathrm{spec} P$  in  $I$ , counted with multiplicity. We also adopt the notation  $N_{] \nu, \mu ]}(P) = -N_{] \mu, \nu ]}(P)$  for  $\mu < \nu$ . Below, all sums and products over eigenvalues  $\lambda$  are taken with multiplicity.

**Definition 3.14.** Let  $P$  be a self-adjoint Fredholm operator. An element of the *spectral torsor*  $\mathrm{sp}(P)$  is represented by a pair  $(\mu, n)$  of  $\mu > 0$  and  $n \in \mathbb{Z}$ , where  $\pm\mu \notin \mathrm{spec}(P)$  and  $[-\mu, \mu] \cap \mathrm{spec}_{\mathrm{ess}}(P) = \emptyset$ . Here  $(\mu, n)$  is equivalent to  $(\nu, m)$  if  $m - n = N_{] \mu, \nu ]}(P)$ . Write  $[\mu, n]$  for an equivalence class.

**Definition 3.15.** Let  $P$  be a  $Y$ -family of self-adjoint Fredholm operators. The *spectral cover* is the disjoint union  $\mathrm{sp}(P) = \bigcup_{y \in Y} (\{y\} \times \mathrm{sp}(P_y))$  with the following topology. Let  $U \subset Y$  be an open subset and  $\mu > 0$  with  $\pm\mu \notin \mathrm{spec}(P_y)$  for all  $y \in U$ . Suppose also there is  $\delta > \mu$  with  $\mathrm{spec}_{\mathrm{ess}}(P_y) \cap (-\delta, \delta) = \emptyset$  for all  $y \in U$ . We then call  $(U, \mu)$  an *admissible neighborhood* for  $P$ . By Lemma 3.1, each

point has an admissible neighborhood. Every admissible neighborhood determines a bijection  $h_{U,\mu}^P: U \times \mathbb{Z} \rightarrow \text{sp}(P)|_U, (y, n) \mapsto (y, [\mu, n])$ . Given admissible  $(U, \mu), (V, \nu)$  and  $y \in U \cap V$ , we have  $(h_{V,\nu}^P)^{-1} h_{U,\mu}^P(y, n) = (y, n + N_{[\mu,\nu]}(P_y))$ . By our assumptions on the spectrum,  $N_{[\mu,\nu]}(P_y)$  is constant on  $U \cap V$ , so the transition maps are homeomorphisms. Hence we get a principal  $\mathbb{Z}$ -bundle  $\text{sp}(P) \rightarrow Y$ .

**Proposition 3.16.** *The spectral cover has the following properties:*

- (i) (Functoriality.) *Let  $P^\pm$  be  $Y$ -families of self-adjoint Fredholm operators. For every isomorphism  $F^\bullet: P^- \rightarrow P^+$  of Fredholm operators with  $F^0 = F^1$  the identity map may be viewed as an isomorphism*

$$\text{sp } P^- \longrightarrow \text{sp } P^+. \quad (3.18)$$

- (ii) (Direct sums.) *For  $Y$ -families of self-adjoint Fredholm operators  $P, Q$  there is a canonical isomorphism*

$$\text{sp}(P) \otimes \text{sp}(Q) \longrightarrow \text{sp}(P \oplus Q). \quad (3.19)$$

*These are natural for (3.18) and commutative as in (3.9) without a sign.*

- (iii) (Negative.) *There is a canonical isomorphism  $\text{sp}(-P) \rightarrow \text{sp}(P)^*$ , natural for (3.18) and compatible with (3.19).*
- (iv) (Invertible.) *Let  $P$  be a family of invertible self-adjoint operators. Then  $\text{sp}(P)$  is canonically trivial. In particular, for positive definite families.*
- (v) *The monodromy of the covering  $\text{sp}(P)$  is the spectral flow around loops.*

*Proof.* (i) By assumption,  $F_y^0 P_y^- (F_y^0)^\dagger = P_y^+$ , so  $\text{spec}(P_y^+) = \text{spec}(P_y^-)$  for all  $y \in Y$  and similarly for the essential spectrum. We thus get a well-defined bijection (3.18) by mapping  $(y, [\mu, n])$  identically. Pick an admissible neighborhood  $(U, \mu)$  for  $P^-$ , which is then also admissible for  $P^+$ . Then  $h_{U,\mu}^{P^+} \circ h_{U,\mu}^{P^-} = \text{id}_{U \times \mathbb{Z}}$ , which shows that our map is indeed continuous.

(ii) The tensor product in (3.19) is the quotient of  $\text{sp}(P) \times \text{sp}(Q)$  by the  $\mathbb{Z}$ -action  $((y, [\mu, n]), (y, [\nu, m])) \mapsto ((y, [\mu, n+k]), (y, [\nu, m-k]))$  of  $k \in \mathbb{Z}$ , where  $\pm\mu \notin \text{spec}(P_y), \pm\nu \notin \text{spec}(Q_y)$ . Assuming also  $\pm\mu \notin \text{spec}(Q_y)$ , we may suppose  $\mu = \nu$  and define (3.19) by  $((y, [\mu, n]), (y, [\mu, m])) \mapsto (y, [\mu, n+m])$ , observing  $\mu \notin \text{spec}(P_y \oplus Q_y)$ . This is invariant under the  $\mathbb{Z}$ -action. When passing from  $\mu$  to  $\nu$  the image of  $([\mu, n], [\mu, m]) = ([\nu, n + N_{[\mu,\nu]}(P_y)], [\nu, m + N_{[\mu,\nu]}(Q_y)])$  is  $[\nu, n + m + N_{[\mu,\nu]}(P_y) + N_{[\mu,\nu]}(Q_y)] = [\nu, n + m + N_{[\mu,\nu]}(P_y \oplus Q_y)] = [\mu, n + m]$ . Hence (3.19) is well-defined. To check continuity near  $y$  use Lemma 3.1 to pick  $\delta > \mu > 0$  and a neighborhood  $y \in U$  with  $(-\delta, \delta) \cap (\text{spec}_{\text{ess}}(P_y) \cup \text{spec}_{\text{ess}}(Q_y)) = \emptyset$  and  $\pm\mu \notin \text{spec}(P_y) \cup \text{spec}(Q_y)$ . Then  $(U, \mu)$  is admissible for  $P, Q, P \oplus Q$  and (3.19) becomes  $((y, n), (y, m)) \mapsto (y, n+m)$  in the corresponding trivializations.

(iii) This follows since the spectrum of  $-P$  is the negative of the spectrum of  $P$ . The isomorphism maps  $(y, [\mu, n]) \in \text{sp}(-P_y)$  to  $(y, [\mu, -n + N_{[-\mu,\mu]}(P_y)]) \in \text{sp}(P_y)^*$ , which is invariant under the replacement  $[\mu, n] = [\nu, n + N_{[\mu,\nu]}(-P_y)]$ . Here we use  $N_{[-\nu,-\mu]}(-P) = N_{[\mu,\nu]}(P)$ .

(iv) Define a canonical section  $s: Y \rightarrow \text{sp}(P)$  by picking  $\mu > 0$  with  $[-\mu, \mu] \cap \text{spec}(P_y) = \emptyset$  for each  $y \in Y$  and setting  $s(y) = (y, [\mu, 0])$ , which is independent of the choice of  $\mu$ . In a suitable trivialization,  $s$  is constant, hence continuous.

(v) Let  $Y = S^1 = \mathbb{R}/\mathbb{Z}$ . Use Lemma 3.1 and compactness of  $Y$  to pick a partition  $0 = y_0 < y_1 < \dots < y_{n-1} < y_n = 1$  and admissible neighborhoods  $(U_k, \mu_k)$  with  $U_k \supset [y_{k-1}, y_k] + \mathbb{Z}$ . To compute the monodromy, start at  $h_{U_1, \mu_1}(y_0, 0)$  and take the fiber transport  $h_{U_1, \mu_1}(y_1, 0) = h_{U_2, \mu_2}(y_1, N_{\mu_1, \mu_2}(P_{y_1}))$  to  $y_1$ . Continuing to  $y_2, \dots, y_n$ , we reach  $h_{U_n, \mu_n}(y_n, N_{\mu_1, \mu_2}(P_{y_1}) + \dots + N_{\mu_{n-1}, \mu_n}(P_{y_{n-1}})) = h_{U_1, \mu_1}(y_n, N_{\mu_1, \mu_2}(P_{y_1}) + \dots + N_{\mu_{n-1}, \mu_n}(P_{y_{n-1}}) + N_{\mu_n, \mu_1}(P_{y_n}))$ . This last integer can be rewritten  $-\sum_{k=1}^n [N_{[0, \mu_k]}(P_{y_k}) - N_{[0, \mu_k]}(P_{y_{k-1}})]$ , which is (negative of) the spectral flow along  $\{P_y; y \in S^1\}$  as defined in Phillips [20, p. 462].  $\square$

**Remark 3.17.** Let  $P$  be a first order elliptic differential operator of Dirac type. Then the spectral torsor can also be obtained by a reduction of  $\xi(P) := \frac{1}{2}(\eta(P, 0) + \dim \text{Ker } P)$  modulo integers. Recall here that

$$\eta(P, s) := \sum_{0 \neq \lambda \in \text{spec}(P)} \frac{\text{sgn}(\lambda)}{|\lambda|^s}, \quad \Re(s) > \dim X,$$

is the  $\eta$ -invariant of  $P$ , continued analytically to  $s = 0$  as in Atiyah–Patodi–Singer [2]. A canonical isomorphism of  $\mathbb{Z}$ -torsors is given by

$$\text{sp}(P) \longrightarrow \xi(P) + \mathbb{Z} \subset \mathbb{R}, \quad [\mu, n] \longmapsto \xi(P) + n - N_{[0, \mu]}(P).$$

It is continuous, since both  $\xi(P)$ ,  $N_{[0, \mu]}(P)$  jump according to the spectral flow.

### 3.4.2 Orientations graded over $\mathbb{Z}_n$

Let  $n \in \mathbb{N}$ ,  $P$  be a  $Y$ -family of self-adjoint Fredholm operators, and  $\sigma$  a  $\mathbb{Z}_n$ -orientation of this family, meaning a section  $\sigma: Y \rightarrow \text{sp}(P) \otimes_{\mathbb{Z}} \mathbb{Z}_n$ . Suppose each  $P_y$  is invertible, which is the expected generic situation under perturbation. For example,  $Y$  could be a moduli space of connections that satisfy a non-linear elliptic equation whose linearization is self-adjoint.

We have two elements of  $\text{sp}(P_y) \otimes_{\mathbb{Z}} \mathbb{Z}_n$ , namely  $\sigma(y)$  and the trivialization of Proposition 3.16(iv) and these differ by  $\mu(y) \in \mathbb{Z}_n$ , which may be used as weights for counting. When  $Y$  is finite such numbers  $p = \mu(y)$  can, given also a finite set of ‘preferred’ trajectories, be used to define chain groups  $C^p$  graded over  $p \in \mathbb{Z}_n$  in the style of Floer [13].

**Example 3.18.** Let  $P \rightarrow S^7$  be an  $\text{SU}(4)$ -bundle. Let  $\not{D}$  be the real Diracian on  $S^7$ , a self-adjoint operator. Let  $\not{D}^{\nabla_{\text{Ad } P}}$  be the Dirac operator twisted by a connection  $\nabla_P \in \mathcal{A}_P$  on  $P$ . The spectral cover of this  $\mathcal{A}_P$ -family is then a  $\mathbb{Z}$ -cover of  $\mathcal{A}_P$ , which is trivial since the base space is contractible. Any trivialization descends to a trivialization of  $\text{sp}(\not{D}^{\nabla_{\text{Ad } P}}) \otimes \mathbb{Z}_8$  over the configuration space of connections modulo gauge. This is because every gauge transformation acts by a multiple of 8. To see this, let  $g \in \text{Aut}(P)$  and pick a path of connections  $\nabla_P^t$  from  $\nabla_P$  to  $g^* \nabla_P$ . The mapping torus bundle  $Q \rightarrow S^7 \times S^1$ , obtained by identifying endpoints of  $P \times [0, 1]$  using  $g$ , can be used to calculate the spectral flow  $\text{SF}\{\not{D}^{\nabla_{\text{Ad } P}^t}\}_{t \in [0, 1]}$  as in Walpuski [22], based on Atiyah–Patodi–Singer [3]:

$$\text{ind } \not{D}^{\text{Ad } Q} = \int_{S^7 \times S^1} \frac{5}{3} c_2(Q)^2 - \frac{4}{3} c_4(Q) + \frac{1}{3} c_2(Q) p_1(S^7) = -\frac{4}{3} \int_{S^7 \times S^1} c_4(Q).$$

The Euler number  $\int_{S^7 \times S^1} c_4(Q)$  of any  $\text{SU}(4)$ -bundle  $Q \rightarrow S^7 \times S^1$  is divisible by six. This can be seen as follows. Represent  $Q$  by gluing two trivial  $\text{SU}(4)$ -bundles

along the equator  $S^7 \times \{1/2\}$  using a map  $f: S^7 \rightarrow \text{SU}(4)$ . By counting zeros of a section constructed from  $f$  the Euler number of  $Q$  is seen to be equal to the degree of  $f$  composed with the projection  $\pi: \text{SU}(4) \rightarrow \text{SU}(4)/\text{SU}(3) = S^7$ . But from the long exact sequence of homotopy groups of a fibration

$$\pi_7(\text{SU}(4)) \xrightarrow{\pi_*} \pi_7(S^7) = \mathbb{Z} \xrightarrow{\partial} \pi_6(\text{SU}(3)) = \mathbb{Z}_6 \longrightarrow \pi_6(\text{SU}(4)) = 0$$

the image of  $\pi_*$  is  $6\mathbb{Z}$ , so the image of  $(\pi \circ f)_* = \pi_* \circ f_*$  is also divisible by 6.

### 3.5 Transgression and the spectral cover

We now establish a relation between spectral torsors and the orientation cover.

**Definition 3.19.** The *transgression* of a complex line bundle  $L \rightarrow X$  is the principal  $\mathbb{Z}$ -bundle  $S \rightarrow \text{Map}(S^1, X)$  whose fibers

$$S_\gamma := \{\tau: S^1 \rightarrow \gamma^*L \setminus \{0\}\} / \simeq, \quad \forall \gamma \in \text{Map}(S^1, X),$$

are the homotopy classes of trivializations along  $\gamma$ . The  $\mathbb{Z}$ -action  $e^{2\pi i n t} \tau(t)$  is freely transitive, since any two  $\tau$  differ by the mapping degree of some  $S^1 \rightarrow \mathbb{C}^*$ . To define the topology on  $S$ , let  $\mathcal{U} \subset \text{Map}(S^1, X)$  be an open set. Then a continuous section  $\tau: S^1 \times \mathcal{U} \rightarrow L \setminus \{0\}$  with  $\tau(t, \gamma) \in L_{\gamma(t)}$  determines, by definition, a continuous section  $\mathcal{U} \rightarrow S, \gamma \mapsto [\tau(\cdot, \gamma)]$  of the covering  $S$ .

**Lemma 3.20.** For a self-adjoint Fredholm operator  $A_0: H \rightarrow H$ , let  $\delta > \mu > 0$ ,  $\pm\mu \notin \text{spec } A_0$ ,  $\nu = -\mu$ , and  $U$  be a neighborhood of  $A_0$  in the self-adjoint Fredholm operators as in Lemma 3.1. Then the restriction of the determinant line bundle to  $U$  is trivial.

*Proof.* Consider the  $Y = U \times [0, 1]$ -family  $A_t := A_0 - t\mu$ . By Proposition 3.5(v) the determinant line bundle of this family is canonically trivial at  $t = 1$  and the restrictions of the bundle to either endpoints of  $[0, 1]$  are isomorphic.  $\square$

**Theorem 3.21.** Under the equivalence of Atiyah–Patodi–Singer [1], [3],

$$\mathfrak{Fred}_{\mathbb{C}}^{\text{sa}} \xrightarrow{\simeq} \Omega \mathfrak{Fred}_{\mathbb{C}}, \quad P \mapsto P_t := \begin{cases} \cos(t) + i \sin(t)P & (0 \leq t \leq \pi), \\ \cos(t) + i \sin(t) & (\pi \leq t \leq 2\pi), \end{cases}$$

we have that the spectral cover  $\text{sp}(P)$  is canonically isomorphic to the transgression  $S \rightarrow \text{Map}(S^1, \mathfrak{Fred}_{\mathbb{C}})$  of the complex determinant line bundle  $\text{Det } P_t$  restricted to the based loop space  $\Omega \mathfrak{Fred}_{\mathbb{C}} \subset \text{Map}(S^1, \mathfrak{Fred}_{\mathbb{C}})$ .

Any two connected  $\mathbb{Z}$ -coverings of  $\mathfrak{Fred}_{\mathbb{C}}^{\text{sa}}$  are clearly isomorphic, but a canonical isomorphism fixes an integer choice here.

*Proof.* Recall as in Lemma 3.1 that near zero all eigenvalues of a self-adjoint Fredholm operator  $P$  are discrete with finite-dimensional eigenspaces. By the spectral theorem, the eigenvalues of the normal operator  $P_t^\dagger P_t$  for  $t \in [0, \pi]$  are  $\lambda(t) := \cos^2(t) + \sin^2(t)\lambda^2$ , where  $\lambda$  ranges over the spectrum of  $P$ . In particular,  $P_t$  is invertible unless  $t = \pi/2$ . Moreover, for  $|\lambda|$  small we have

$$V_{\cos^2(t) + \sin^2(t)\lambda^2} \left( P_t^\dagger P_t \right) = V_{\lambda^2}(P^2) = V_\lambda(P) \oplus V_{-\lambda}(P). \quad (3.20)$$

(one of the spaces on the right can be trivial.) This determines a map  $\phi_t(\mu)$

$$\begin{aligned} \text{Det } P_t &\xrightarrow{(3.6)} \bigwedge V_{[0,\mu(t)]}(P_t^\dagger P_t) \otimes \left( \bigwedge V_{[0,\mu(t)]}(P_t P_t^\dagger) \right)^* \\ &\stackrel{(3.20)}{=} \bigwedge V_{[0,\mu]}(P^\dagger P) \otimes \left( \bigwedge V_{[0,\mu]}(P P^\dagger) \right)^* \xrightarrow{(3.6)} \text{Det } P \end{aligned}$$

for  $t \in [0, \pi]$  and  $\mu > 0$  sufficiently small,  $\pm\mu \notin \text{spec } P$ . Since  $P_t$  is invertible,  $\det P_t := 1 \otimes 1^* \in \text{Det } P_t$  is a canonical trivialization. Unravelling the definition of (3.6), it is easy to check the formulas

$$\begin{aligned} \phi_t(\nu) &= \prod_{\substack{\lambda \in \text{spec } P \\ \mu < |\lambda| < \nu}} (\lambda^{-1} \cos t + i \sin t) \cdot \phi_t(\mu), \\ \phi_t(\mu) (\det P_t) &= \prod_{\substack{\lambda \in \text{spec } P \\ 0 < |\lambda| < \mu}} \frac{\lambda^{-1} \cos(t) + i \sin(t)}{\lambda^{-1} \cos(s) + i \sin(s)} \cdot \phi_s(\mu) (\det P_s). \end{aligned} \quad (3.21)$$

By deforming, any element of the transgression  $S_{t \rightarrow P_t}$  can be represented by a map  $t \mapsto \tau(t) \in \text{Det } P_t$  with  $\tau(t) = \det P_t$  for  $t \in [0, 2\pi] \setminus ]0, \pi[$ . We can thus represent these by maps  $\gamma_\mu(t) = \phi_t(\mu) \circ \tau(t)$ ,  $\gamma_\mu: [0, \pi] \rightarrow \text{Det } P \setminus \{0\}$  satisfying by (3.21) the periodicity condition  $\gamma_\mu(0) = (-1)^{N_{]-\mu, \mu[}} \cdot \gamma_\mu(\pi)$ , modulo homotopy through such maps. By Lemma 3.20 the choice of  $\mu$  determines also a trivialization  $\Phi: \mathbb{C} \rightarrow \text{Det } P$ . In this trivialization we can describe  $\gamma_\mu$  up to homotopy by its mapping degree. In other words, we can put any  $\gamma_\mu$  into standard form  $e^{i(-N_{]-\mu, \mu[}(P) + 2n)t}$ . The sought-for isomorphism is then

$$\text{sp } P \longrightarrow S_{t \rightarrow P_t}, \quad [n, \mu] \longmapsto \left[ \phi_t(\mu)^{-1} \circ \Phi \left( e^{i(-N_{]-\mu, \mu[}(P) + 2n)t} \right) \right].$$

Another choice of trivialization differs by multiplication by some  $\kappa \in \mathbb{C} \setminus \{0\}$  which does not affect the homotopy class on the right. It is independent of  $\mu$  by (3.21), where we can use also the homotopic  $\exp([N_{]\mu, \nu[}(P) - N_{]-\nu, -\mu[}(P)] \cdot it)$  as prefactor. According to Lemma 3.20 we can perform all this over open sets  $U$ , which proves continuity of the isomorphism there.  $\square$

**Remark 3.22.** Using this result one may alternatively reduce Theorem 2.10 for  $\lambda = \text{sp}$  to the case  $\lambda = \text{or}$ .

## 3.6 Proof of Theorem 2.8

### 3.6.1 The covering of an elliptic symbol family

Let  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$ . By Propositions 3.5, 3.11, and 3.16 we can assign to every  $Y$ -family of Fredholm operators  $P$  of type  $(\lambda)$  a graded principal  $G$ -bundle  $\lambda(P) \rightarrow Y$  with the degree  $\deg \lambda(p)$  and structure group  $G$  given by (2.4).

**Definition 3.23.** A deformation  $P = \{P_t\}_{t \in [0,1]}$  of type  $(\lambda)$  is a  $Y \times [0,1]$ -family of Fredholm operators of type  $(\lambda)$ . Then  $P$  determines a covering of  $Y \times [0,1]$  in which fiber transport gives an isomorphism

$$\lambda\{P_t\}_{t \in [0,1]}: \lambda(P_0) \longrightarrow \lambda(P_1), \quad (3.22)$$

functorial for isomorphisms of deformations, compatible with direct sums of deformations and with adjoints.



By general properties of covering maps, (3.22) depends on  $\{P_t\}_{t \in [0,1]}$  only up to homotopy through operators of type  $(\lambda)$  relative endpoints  $\{0, 1\}$  and is functorial for the juxtaposition of deformations. Moreover, a  $[0, 1]$ -deformations of isomorphisms  $F_t^\bullet: P_t \rightarrow Q_t$  as in Definition 3.2 induces a commutative diagram

$$\begin{array}{ccc} \lambda(P_0) & \xrightarrow{(3.22)} & \lambda(P_1) \\ \lambda(F_0^\bullet) \downarrow & & \downarrow \lambda(F_1^\bullet) \\ \lambda(Q_0) & \xrightarrow{(3.22)} & \lambda(Q_1) \end{array}$$

using the functoriality defined in part (i) of Propositions 3.5, 3.11, and 3.16. Moreover, the naturality of parts (ii) and (iii) shows that (3.22) is compatible in the obvious sense with direct sums and adjoints.

**Definition 3.24.** Let  $p \in \mathcal{E}\ell_Y^m(X; E^0, E^1)$  be a  $Y$ -family of elliptic symbols of type  $(\lambda)$  over a manifold  $X$ . Suppose  $X$  is compact or that  $m = 0$  and  $p$  is compactly supported. The set  $\sigma^{-1}(p)_{(\lambda)} \subset \Psi\text{DO}_Y^m(X; E^0, E^1)$  of compactly supported  $Y$ -families of  $\Psi\text{DO}$ s  $P$  of type  $(\lambda)$  with principal symbol  $p$  is non-empty and convex, by Theorem A.6(iii). By Lemma A.3 the straight line between  $P_0, P_1 \in \sigma^{-1}(p)_{(\lambda)}$  is a  $Y \times [0, 1]$ -family  $P$  in  $\sigma^{-1}(p)_{(\lambda)}$  of compactly supported operators. Then (3.22) defines an isomorphism  $t_{P_0, P_1}: \lambda(P_0) \rightarrow \lambda(P_1)$ . For  $P_0, P_1, P_2 \in \sigma^{-1}(p)_{(\lambda)}$  we have  $t_{P_1, P_2} \circ t_{P_0, P_1} = t_{P_0, P_2}$ . The limit over these is the set of compatible families:

$$\lambda(p) := \left\{ (x_P \in \lambda(P))_{P \in \sigma^{-1}(p)_{(\lambda)}} \mid \forall P_0, P_1 \in \sigma^{-1}(p)_{(\lambda)} : t_{P_0, P_1}(x_{P_0}) = x_{P_1} \right\}.$$

For any  $P \in \sigma^{-1}(p)_{(\lambda)}$  the projection defines an isomorphism  $\text{ev}_P: \lambda(p) \rightarrow \lambda(P)$ . In particular,  $\lambda(p)$  has a canonical topology of principal  $G$ -bundle.

### 3.6.2 General properties

We now summarize the general properties of  $\lambda(p)$ . These are straightforward consequences of Propositions 3.5, 3.11, and 3.16. On the level of symbols, functoriality takes the following form. An identification  $\Phi: p_- \rightarrow p_+$  of type  $(\lambda)$  over a diffeomorphism  $\phi: X_- \rightarrow X_+$  induces an isomorphism of coverings

$$\Phi^*: \lambda(p_-) \longrightarrow \lambda(p_+). \quad (3.23)$$

When  $X_\pm = \emptyset$  we take (3.23) to mean the identity map.

To see (3.23) choose  $P_+ \in \sigma^{-1}(p_+)_{(\lambda)}$ . Then  $P_- := (\phi, \Phi)^* P_+$  (see Definition A.4) represents  $p_-$  and is of type  $(\lambda)$ , using our type assumption on  $\Phi$ . Moreover  $\Phi: P_- \rightarrow P_+$  is an isomorphism of Fredholm operators, and (3.23) is defined as  $\lambda(\Phi)$ , meaning (3.7), (3.16), or (3.18), composed with the canonical projections. Similarly we have, from the corresponding properties of Fredholm operators, direct sum and adjointness isomorphisms, natural for (3.23),

$$\lambda(p) \otimes \lambda(q) \longrightarrow \lambda(p \oplus q), \quad (3.24)$$

$$\lambda(-p^\dagger) \longrightarrow \lambda(p)^*. \quad (3.25)$$

Recall then (2.10) as a consequence. Thus, for a deformation  $\{p_z\}_{z \in [0,1]}$  we have

$$\lambda\{p_z\}_{z \in [0,1]}: \lambda(p_0) \longrightarrow \lambda(p_1). \quad (3.26)$$

### 3.6.3 Proofs of Theorem 2.8(iv) and (v)

The functoriality isomorphisms in (3.7), (3.16), and (3.18) were defined for isomorphisms  $F^\bullet: H_-^\bullet \rightarrow H_+^\bullet$  of Hilbert spaces, but actually the same construction works for maps defined only on eigenspaces whose eigenvalue is close to zero. Unravelling the definitions, the precise statement that ensures the continuity in the topologies of Definitions 3.4, 3.10, and 3.15 is as follows.

**Proposition 3.25.** *Let  $P_\pm(y)$  be  $Y$ -families of Fredholm operators of type  $(\lambda)$ .*

(or) *Define  $\Delta_\pm^0 = (P_\pm)^\dagger P_\pm, \Delta_\pm^1 = P_\pm (P_\pm)^\dagger$ . Let  $0 \in I_\pm \subset [0, \infty)$  be half-open intervals disjoint from  $\text{spec}_{\text{ess}} \Delta_\pm^\bullet(y)$  and let  $f: I_- \rightarrow I_+$  be a homeomorphism with  $f(0) = 0$  and  $f(I_- \cap \text{spec} \Delta_-^\bullet(y)) = I_+ \cap \text{spec} \Delta_+^\bullet(y)$  for all  $y \in Y$  and  $\bullet = 0, 1$ . Let  $F_\lambda^\bullet(y): V_\lambda(\Delta_-^\bullet(y)) \rightarrow V_{f(\lambda)}(\Delta_+^\bullet(y))$  be metric-preserving isomorphisms with  $F_\lambda^1(y) \circ P_-(y)|_{V_\lambda(\Delta_-^0)} = P_+(y)|_{V_\lambda(\Delta_+^0)} \circ F_\lambda^0(y)$  for all  $y \in Y, \lambda \in I_\pm$ . Assume: there is a covering by open subsets  $U \subset Y$  and  $\mu_\pm \in I_\pm^\circ$  with  $\mu_+ = f(\mu_-), \pm\mu_\pm \notin \text{spec} \Delta_\pm^\bullet(y) (\forall y \in U)$  for which  $\bigoplus_{\lambda \in I_- \cap [0, \mu_-]} F_\lambda^\bullet: V_{I_- \cap [0, \mu_-]}(\Delta_-^\bullet) \rightarrow V_{I_+ \cap [0, \mu_+]}(\Delta_+^\bullet)$  are continuous in the eigenbundle topologies. Then  $\bigwedge F_0^0(y) \otimes (\bigwedge F_0^1(y))^{*, -1}: \text{Det } P_-(y) \rightarrow \text{Det } P_+(y)$  define a continuous map  $\text{Det } P_- \rightarrow \text{Det } P_+$ .*

(pf) *Define  $\Delta_\pm = (P_\pm)^\dagger P_\pm$ . Let  $0 \in I_\pm \subset [0, \infty)$  be half-open intervals disjoint from  $\text{spec}_{\text{ess}}(\Delta_\pm(y))$  and let  $f: I_- \rightarrow I_+$  be a homeomorphism with  $f(0) = 0, f(I_- \cap \text{spec} \Delta_-(y)) = I_+ \cap \text{spec} \Delta_+(y) (\forall y \in Y)$ . Let  $F_\lambda(y): V_\lambda(\Delta_-(y)) \rightarrow V_{f(\lambda)}(\Delta_+(y))$  be metric-preserving isomorphisms with  $F_\lambda(y) \circ P_-(y)|_{V_\lambda(\Delta_-)} = P_+(y) \circ F_\lambda(y)|_{V_\lambda(\Delta_+)}$  for all  $y \in Y, \lambda \in I_\pm$ . Assume: there is a covering by open  $U \subset Y$  and  $\mu_\pm \in I_\pm^\circ$  with  $\mu_+ = f(\mu_-), \pm\mu_\pm \notin \text{spec} \Delta_\pm(y) (\forall y \in U)$  for which the map  $\bigoplus_{\lambda \in I_- \cap [0, \mu_-]} F_\lambda$  is continuous in the vector bundle topologies on  $V_{I_\pm \cap [0, \mu_\pm]}(\Delta_\pm)$ . Then  $\bigwedge F_0(y): \text{Pf } P_-(y) \rightarrow \text{Pf } P_+(y)$  define a continuous map  $\text{Pf } P_- \rightarrow \text{Pf } P_+$ .*

(sp) *Let  $I_\pm \subset \mathbb{R} \setminus \text{spec}_{\text{ess}}(P_\pm(y))$  be open intervals, symmetric about 0. Let  $f: I_- \rightarrow I_+$  be an odd homeomorphism with  $f(I_- \cap \text{spec} P_-(y)) = I_+ \cap \text{spec} P_+(y) (\forall y \in Y)$ . Assume  $V_\lambda(P_-(y)) \cong V_{f(\lambda)}(P_+(y))$  for all  $y \in Y, \lambda \in I_\pm$ . Then  $\text{sp } P_-(y) \rightarrow \text{sp } P_+(y), (y, [\mu, n]) \mapsto (y, [f(\mu), n])$  for  $\pm\mu \notin \text{spec} P_-(y), \pm\mu \in I_-$  is a continuous map  $\text{sp } P_- \rightarrow \text{sp } P_+$ .*

*Proof.* (or) Note  $V_0(\Delta_\pm^0) = \text{Ker } P_\pm, V_0(\Delta_\pm^1) = \text{Ker } P_\pm^\dagger$  so that  $\bigwedge F_\lambda^0, \bigwedge F_\lambda^1$  for  $\lambda = 0$  indeed define a bundle homomorphism  $\bigwedge \text{Ker } P_- \otimes (\bigwedge \text{Ker } P_-^\dagger)^* \rightarrow \bigwedge \text{Ker } P_+ \otimes (\bigwedge \text{Ker } P_+^\dagger)^*$ . The diagram

$$\begin{array}{ccc}
 \bigwedge V_{I_- \cap [0, \mu_-]}(\Delta_-^0) \otimes (\bigwedge V_{I_- \cap [0, \mu_-]}(\Delta_-^1))^* & \xrightarrow[\text{stab}_{\mu, 0}(P_-)]{} & \text{Det } P_-|_U \\
 \downarrow \bigwedge (\bigoplus_{\lambda \in I_- \cap [0, \mu_-]} F_\lambda^0) & & \downarrow \bigwedge F_0^0 \otimes (\bigwedge F_0^1)^{*, -1} \\
 \otimes \bigwedge (\bigoplus_{\lambda \in I_- \cap [0, \mu_-]} F_\lambda^1)^{*, -1} & & \\
 \bigwedge V_{I_+ \cap [0, \mu_+]}(\Delta_+^0) \otimes (\bigwedge V_{I_+ \cap [0, \mu_+]}(\Delta_+^1))^* & \xrightarrow[\text{stab}_{\mu, 0}(P_+)]{} & \text{Det } P_+|_U.
 \end{array}$$

commutes by the same argument as for Proposition 3.5(i) and all arrows except the right vertical one are known to be continuous. The proof of (pf) is analogous.

(sp) By assumption,  $f: I_- \cap \text{spec } P_-(y) \rightarrow I_+ \cap \text{spec } P_+(y)$  correspond bijectively with multiplicity. Hence  $N_{[\mu, \nu]}(P_-(y)) = N_{[f(\mu), f(\nu)]}(P_+(y))$  for all  $0 < \mu < \nu$  in  $I_-$  so that  $\alpha(y): \text{sp } P_-(y) \rightarrow \text{sp } P_+(y)$ ,  $(y, [\mu, n]) \mapsto (y, [f(\mu), n])$  is well-defined. To prove continuity, let  $(U, \mu), \mu \in I_-$  be an admissible neighborhood for  $P_-$ . Then  $(U, f(\mu))$  is admissible for  $P_+$  and  $\alpha \circ h_{U, \mu}^{P_-} = h_{U, f(\mu)}^{P_+}$ . This implies continuity of  $\alpha$ , since the other two isomorphisms are continuous.  $\square$

**Corollary 3.26.** *Let  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$ . Let  $P \in \Psi\text{DO}^0(X; E^0, E^1)$  be a  $Y$ -family of zeroth-order  $\Psi\text{DO}$ s of type  $(\lambda)$ , compactly supported in the image of an open embedding  $i: U \hookrightarrow X$  and with  $\sigma(P)$  elliptic and compactly supported. Then extension by zero defines a continuous isomorphism  $i_!: \lambda(i^*P) \rightarrow \lambda(P)$ . We allow  $U = \emptyset$  in which case  $\lambda(i^*P)$  is trivial. These are functorial and hence compatible with deformations (3.22). For all  $p \in \mathcal{E}\ell_Y^0(X)$  of type  $(\lambda)$  compactly supported in  $i(U)$  we therefore obtain a continuous isomorphism*

$$i_!: \lambda(i^*p) \rightarrow \lambda(p), \quad (3.27)$$

*functorial for (3.23), compatible with direct sums (3.24), and compatible with adjoints (3.25). Moreover, we have  $i_! \circ j_! = (i \circ j)_!$  and  $\text{id}_! = \text{id}$ .*

*Proof.* This follows from Theorem A.6(ii) which allows us to apply Proposition 3.25 with  $I_\pm = \mathbb{R}$ ,  $f(x) = x$ , and the extension by zero isomorphisms  $F_\lambda^\bullet$  to construct  $i_!: \lambda(i^*P) \rightarrow \lambda(P)$ . This construction can be performed in families, so is compatible with deformations and descends to an isomorphism (3.27).  $\square$

**Proposition 3.27.** *Let  $\lambda \in \{\text{or}, \text{pf}, \text{sp}\}$ . Let  $p$  be a  $Y$ -family of elliptic symbols of type  $(\lambda)$ . Then we have canonical reduction of order isomorphisms*

$$\lambda(p) \rightarrow \lambda(p(p^\dagger p)^{-1/2}). \quad (3.28)$$

*These are compatible with the isomorphisms (3.23)–(3.26).*

*Proof.* Choose a pseudo-differential operator  $P$  representing  $p$ . Then  $1 + P^\dagger P$  is a positive definite operator and we may define  $(1 + P^\dagger P)^{-1/2}$  using unbounded functional calculus. The first step is to construct an isomorphism  $\lambda(P) \cong \lambda(R)$  for  $R = P(1 + P^\dagger P)^{-1/2}$  using Proposition 3.25. For (or) take  $I_- = [0, \infty)$ ,  $I_+ = [0, 1)$ ,  $f(x) = x/(1+x)$ . Since  $R^\dagger R = f(P^\dagger P)$  the identity map defines isomorphisms  $V_\lambda(P^\dagger P) \rightarrow V_{f(\lambda)}(R^\dagger R)$  for all  $\lambda \in \text{spec}(P^\dagger P)$  and similarly  $V_\lambda(P P^\dagger) \rightarrow V_{f(\lambda)}(R R^\dagger)$ . These have the continuity properties required in Proposition 3.25, being restrictions of a globally defined continuous map. Hence we get the required isomorphism. The case (pf) follows by the exact same argument. For (sp) take  $I_- = \mathbb{R}$ ,  $I_+ = (-1, 1)$ , and  $g(x) = x(1+x^2)^{-1/2}$ . Then  $R = g(P)$  and  $V_\lambda(P) \cong V_{g(\lambda)}(R)$  by the spectral theorem, and we may again apply Proposition 3.25 to get the isomorphism. Thus we obtain  $\lambda(P) \cong \lambda(P(1 + P^\dagger P)^{-1/2})$  in every case, compatible with the versions of (3.23)–(3.26) for operators.

The second step is to pick a pseudo-differential operator  $Q$  with the property that  $S := (1 + P^\dagger P)^{-1/2} - Q$  is a smoothing operator and  $\sigma(Q) = (p^\dagger p)^{-1/2}$ . The operator  $P(1 + P^\dagger P)^{-1/2}$  can be deformed along the straight line  $P \circ [Q + tS]$  to  $P \circ Q$ , a pseudo-differential operator representing  $p(p^\dagger p)^{-1/2}$ . We can assume  $Q$  to be self-adjoint and when  $P$  is self- or skew-adjoint we can perform

the deformation through operators of the same type. Then (3.22) gives an isomorphism  $\lambda(P(1 + P^\dagger P)^{-1/2}) \rightarrow \lambda(P \circ Q) \cong \lambda(p(p^\dagger p)^{-1/2})$ , continuous over  $Y$  and compatible with (3.23)–(3.26).  $\square$

## A Compactly supported $\Psi$ DOs

We review the theory of compactly supported  $\Psi$ DOs, whose theory reduces quickly to finitely many charts like in the compact case.

### A.1 Families of pseudo-differential operators

Let  $Y$  be a topological space. A  $Y$ -family of manifolds is a space  $E$  with a continuous map  $\pi: E \rightarrow Y$  and a collection of smooth structures on each fiber  $E_y = \pi^{-1}(y)$ . Let  $(E_\pm, \pi_\pm)$  be  $Y$ -families of manifolds. A  $Y$ -family of smooth maps is a continuous map  $\phi: E_- \rightarrow E_+$  with  $\pi_+ \circ \phi = \pi_-$  whose restriction  $\phi_y$  to every fiber is smooth. For a  $Y$ -family of open embeddings, diffeomorphisms, or bundle isomorphisms we additionally require  $\phi$  to be a homeomorphism onto its image and every  $\phi_y$  to be of the corresponding type.

For open subsets  $\Omega \subset \mathbb{R}^d$ , Hörmander [15] is a good reference on  $\Psi$ DOs. In this paper, we work with the following combination of definitions from Atiyah–Singer [4, p. 509], [5, p. 123], [6, p. 141] and Hörmander [15, p. 153].

**Definition A.1.** Let  $Y$  be a space,  $X$  a manifold,  $m \in \mathbb{R}$ , and  $E^0, E^1 \rightarrow X \times Y$  Hermitian vector bundles. The set  $\Psi\text{DO}_Y^m(X; E^0, E^1)$  of  $Y$ -families of  $m$ -th order pseudo-differential operators over  $X$  consists of families of  $\mathbb{C}$ -linear maps

$$P_y: C_{\text{cpt}}^\infty(X \times \{y\}, E^0) \longrightarrow C^\infty(X \times \{y\}, E^1), \quad y \in Y,$$

having a Schwartz kernel of prescribed local form as follows. Let  $x: U \rightarrow \Omega \subset \mathbb{R}^d$  be a coordinate on an open subset  $U \subset X$  (possibly disconnected),  $f \in C_{\text{cpt}}^\infty(U)$ ,  $V \subset Y$  open, and  $\tau^0$  and  $\tau^1$  unitary trivializations of  $E^0|_{U \times V}$  and  $E^1|_{U \times V}$ . Given this, there should exist a  $V$ -family of *total symbols*, homomorphisms

$$p_y^{(f)}: U \times \mathbb{R}^d \longrightarrow \text{Hom}_{\mathbb{C}}(E^0|_{U \times \{y\}}, E^1|_{U \times \{y\}}), \quad \forall y \in V,$$

such that for all  $s \in C_{\text{cpt}}^\infty(U \times \{y\}, E^0)$  we can write

$$P_y(fs)(x) = (2\pi)^{-n} \int_{\mathbb{R}^d} p_y^{(f)}(x, \xi) \hat{s}(\xi) e^{i\langle x, \xi \rangle} d\xi, \quad \forall x \in U. \quad (\text{A.1})$$

(The left hand side need not be supported in  $U$ .) Here the integral and Fourier transform are performed on  $\Omega$  using  $x, \tau^0, \tau^1$ . The following conditions are required in order for the oscillatory integral (A.1) to be well-behaved:

(i)  $\forall \alpha, \beta \in \mathbb{N}^d, y \in V$  we have finite bounds

$$\|p_y^{(f)}\|_{\alpha, \beta} := \sup_{x \in U, \xi \in \mathbb{R}^d} \frac{\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \xi^\beta} p_y^{(f)}(x, \xi) \right\|}{(1 + \|\xi\|)^{m-\beta}} < +\infty.$$

(ii)  $\forall \alpha, \beta \in \mathbb{N}^d, y_0 \in V, \varepsilon > 0$  there exists a neighborhood  $V_0 \subset V$  of  $y_0$  with

$$\|p_y^{(f)} - p_{y_0}^{(f)}\|_{\alpha, \beta} \leq \varepsilon, \quad \forall y \in V_0. \quad (\text{A.2})$$

(iii) The limit  $\sigma_{x,\xi}^{(f)}(P_y) := \lim_{\lambda \rightarrow +\infty} \lambda^{-m} \cdot p_y^{(f)}(x, \lambda \cdot \xi)$  exists.

Since  $p_y^{(f)}$  is uniquely determined by  $P$ ,  $f$  and the trivializations  $x, \tau$ , we can define the *principal symbol family*  $\sigma(P): \pi^*E^0 \rightarrow \pi^*E^1$  by  $\sigma_\xi(P_y) := \sigma_{x,\xi}^{(f)}(P_y)$  for any  $\xi \in T_x^*U$  and  $f \in C_{\text{cpt}}^\infty(U)$  with  $f \equiv 1$  near  $x$ .

We consider the following *types* of  $P \in \Psi\text{DO}_Y^m(X; E^0, E^1)$ : (or)  $P$  is *real* if  $E^0$  and  $E^1$  are equipped with orthogonal real structures and  $P_y(\bar{s}) = \overline{P_y(s)}$ , (pf)  $P$  is *real skew-adjoint* if  $P$  is real and  $P^\dagger = -P$  for the formally adjoint operator, and (sp)  $P$  is *self-adjoint* if  $P^\dagger = P$ .

## A.2 Compactly supported operators

**Definition A.2.** A zeroth-order family  $P \in \Psi\text{DO}_Y^0(X; E^0, E^1)$  is *compactly supported* in an open  $U \subset X$  if there exists  $\phi \in C_{\text{cpt}}^\infty(U, [0, 1])$  and a  $Y$ -family of bundle homomorphisms  $\tilde{p}: E^0|_{(X \setminus L) \times Y} \rightarrow E^1|_{(X \setminus L) \times Y}$ , where  $L := \phi^{-1}(1)$ , satisfying the following conditions:

$$\forall y \in Y, s \in C_{\text{cpt}}^\infty(X \times \{y\}, E^0) : P_y(s) = \phi P_y(\phi s) + (1 - \phi^2) \tilde{p}_y \circ s. \quad (\text{A.3})$$

$$\forall y \in Y \exists C > 0 \forall x \notin L : \|\tilde{p}_{x,y}\| \leq C. \quad (\text{A.4})$$

$$\begin{aligned} \forall \varepsilon > 0, y_0 \in Y \text{ there exists a neighborhood } V \text{ of } y_0 \text{ such that} \\ (x, y) \in (X \setminus L) \times V \implies \|\tilde{p}_{x,y} - \tilde{p}_{x,y_0}\| \leq \varepsilon. \end{aligned} \quad (\text{A.5})$$

**Lemma A.3.** Assume  $\psi \in C^\infty(X)$  satisfies  $\psi|_{\text{supp } \phi} \equiv 1$ . Then (A.3) holds with  $\psi$  in place of  $\phi$ . In particular, a finite convex combination of elements in  $\Psi\text{DO}_Y^0(X; E^0, E^1)$  compactly supported in  $U$  is again compactly supported in  $U$ .

*Proof.* By assumption,  $\phi\psi = \phi$ . Then

$$\begin{aligned} \psi P_y(\psi s) + (1 - \psi^2) \tilde{p}_y \circ s &\stackrel{(\text{A.3})}{=} \psi (\phi P_y(\phi \psi s) + (1 - \phi^2) \tilde{p}_y \circ \psi s) + (1 - \psi^2) \tilde{p}_y \circ s \\ &= \phi P_y(\phi s) + (1 - \phi^2) \tilde{p}_y \circ s = P_y(s). \end{aligned}$$

In a finite convex combination we may therefore use a common cut-off  $\phi$  for each of the operators to check condition (A.3).  $\square$

Pseudo-differential operators are not local in general, but assuming (A.3) we can at least restrict to  $U$ .

**Definition A.4.** Let  $i: U \rightarrow X$  be an open embedding of manifolds, not necessarily compact and  $P \in \Psi\text{DO}_Y^0(X; E^0, E^1)$  compactly supported in  $i(U)$ . For  $y \in Y$  and  $s \in C_{\text{cpt}}^\infty(U \times \{y\}, E^0)$  the section  $P_y(s \circ i^{-1})$  is supported in  $i(U)$ , by (A.3). The *pullback family* in  $\Psi\text{DO}_Y^0(U; i^*E^0, i^*E^1)$  is

$$(i^*P)_y(s) := P_y(s \circ i^{-1}) \circ i.$$

Given, in addition, a pair of unitary bundle isomorphisms  $\Phi^\bullet: F^\bullet \rightarrow i^*E^\bullet$ , we may define  $(i, \Phi)^*P_y(s) := (\Phi^1)^{-1} \circ P_y(\Phi^0 \circ s \circ i^{-1}) \circ i$  acting on sections of  $F^\bullet$ .

**Definition A.5.** Let  $P \in \Psi\text{DO}_Y^0(X; E^0, E^1)$ . Then  $P$  is *compactly supported elliptic* in  $U \subset X$  if  $P$  is compactly supported in  $U$  and  $\sigma(P)$  is an elliptic symbol family supported in a compact subset  $L \times Y \subset U$ .

### A.3 Fredholm results

We next summarize the main properties of compactly supported families. These are well-known for  $X$  and  $Y$  compact, see Atiyah–Singer [5].

**Theorem A.6.** *Let  $Y$  be space,  $X$  an oriented Riemannian manifold, and let  $E^0, E^1 \rightarrow X \times Y$  be Hermitian vector bundles. Let  $L^2(X, E^0), L^2(X, E^1)$  be the Hilbert space bundles over  $Y$  of  $L^2$ -sections in  $X$ -direction.*

- (i) *Let  $P \in \Psi\text{DO}_Y^0(X; E^0, E^1)$  be a compactly supported  $Y$ -family. Then  $P$  can be extended to a continuous bundle map*

$$P: L^2(X, E^0) \longrightarrow L^2(X, E^1) \quad (\text{A.6})$$

*restricting fiberwise to bounded operators.*

- (ii) *Let  $P \in \Psi\text{DO}_Y^0(X; E^0, E^1)$  be compactly supported elliptic in  $U$ . Then each  $\text{Ker } P_y$  is finite-dimensional and contained in  $C_{\text{cpt}}^\infty(U, E^0)$ . The operator  $P^\dagger$  is also compactly supported elliptic in  $U$ . Hence the restrictions of (A.6) to all of the fibers are Fredholm.*

*When  $E^0 = E^1$ , the operator  $P - \lambda$  is compactly supported elliptic in  $U$  for all  $|\lambda|$  sufficiently small, depending on the constant  $c$  in (2.1). In particular, all eigenspaces of  $P_y$  with eigenvalue  $\lambda$  near zero are finite-dimensional and all eigenfunctions belong to  $C_{\text{cpt}}^\infty(U, E^0)$ .*

- (iii) *Every family of elliptic symbols  $p \in \mathcal{E}\ell_Y^0(X; E^0, E^1)$  is the principal symbol of a family  $P$  in  $\Psi\text{DO}_Y^0(X; E^0, E^1)$ . Moreover, a real, skew-adjoint, or self-adjoint family  $p$  can be realized by a family  $P$  of corresponding type. When  $p$  is compactly supported in  $L \subset U \subset X$  and  $U$  is open, the operator  $P$  can be chosen to be compactly supported elliptic in  $U$ . In each case there is a convex space of choices for  $P$ .*

*Proof.* (i) The key point is the following local estimate. For  $y_0 \in Y$  and functions  $f, g \in C_{\text{cpt}}^\infty(V)$  supported in a chart neighborhood  $V$  we have

$$\|gP_{y_0}(fs)\|_{L^2} \leq D_{y_0} \cdot \|s\|_{L^2}, \quad \forall s \in C_{\text{cpt}}^\infty(V \times \{y_0\}, E^0). \quad (\text{A.7})$$

Here the constant  $D_{y_0}$  is a homogeneous linear polynomial in  $\|p_{y_0}^{(f)}\|_{\alpha,0}$  for  $|\alpha| \leq \dim X + 1$  whose coefficients depend on  $\dim X$ , lower and upper bounds for the density  $\text{vol}|_V$  with respect to the Lebesgue measure (e.g. if  $V$  is relatively compact), and uniform bounds for the first  $\dim X + 1$  derivatives of  $g$ . This estimate is an adaption of Hörmander’s proof for [15, Th. 3.1] on p. 154.

Assume now  $P$  is compactly supported in  $U$ . Using a Lebesgue number we find finitely many relatively compact chart neighborhoods  $\{V_a\}_{a \in A}$  covering  $L$  such that for all  $a, b \in A$  the union  $V_a \cup V_b$  is also a chart. Pick  $\chi_a \in C_{\text{cpt}}^\infty(V_a)$  with  $\sum_a \chi_a^2 \equiv 1$  on  $L$ . For  $s \in C_{\text{cpt}}^\infty(\{y_0\} \times X, E^0)$  we find

$$\|P_{y_0}s\|_{L^2} \leq \|\phi P_{y_0}(\phi s)\|_{L^2} + \|(1 - \phi^2)\tilde{p}_{y_0} \circ s\|_{L^2} \quad (\text{A.3})$$

$$\leq \sum_{a,b} \|\chi_a^2 \phi P_{y_0}(\chi_b^2 \phi s)\|_{L^2} + C_{y_0} \cdot \|s\|_{L^2} \quad (\text{A.4})$$

$$\leq \sum_{a,b} D_{y_0,a,b} \cdot \|\chi_b s\|_{L^2} + C_{y_0} \cdot \|s\|_{L^2} \quad (\text{A.7})$$

$$\leq \text{const} \cdot \|s\|_{L^2}.$$

Hence  $P_{y_0}$  extends to a bounded operator on  $L^2$ -sections. To prove continuity we can use the same estimates for  $\|P_y s - P_{y_0} s\|_{L^2}$  as above, now using (A.2) and (A.5) to see that the constants tend to zero for  $y$  in a neighborhood of  $y_0$ .

(ii) A convenient parametrix  $Q_{y_0}$  for  $P_{y_0}$  is obtained by patching local parametrices on the charts  $U_a$  with the *exact* inverse  $\tilde{p}_{y_0}^{-1}$  outside  $L$ , which is a bounded operator by the lower bound of (2.1).

Then we find  $Q_{y_0} P_{y_0} - 1 = R_1$ ,  $P_{y_0} Q_{y_0} - 1 = R_2$  with  $R_1, R_2$  having compactly supported  $C^\infty$ -kernels. These define compact operators on  $L^2$ . The remaining assertions are easily verified.

(iii) This is true locally, since we can define  $P$  by (A.1) for  $p_y^{(f)}(x, \xi) := f(x)p_{y,\xi}$ , and then patched to a global result using a partition of unity on  $X$ . When the elliptic symbol family  $p$  is supported in  $L \subset U \subset X$ , by definition we have  $\tilde{p}$  outside  $L$ . Pick  $\phi \in C_{\text{cpt}}^\infty(U, [0, 1])$  with  $\phi|_L \equiv 1$ . Then, beginning with an arbitrary  $Y$ -family  $P$  in  $\Psi\text{DO}_Y^0(X; E^0, E^1)$  with principal symbol family  $p$ , replace it by  $\phi_y P_y(\phi s) + (1 - \phi^2)\tilde{p}_y \circ s$  to get one that is compactly supported in  $U$ . The reality condition can be ensured by passing to  $\frac{1}{2}(\overline{P_y(s)} + P_y(\bar{s}))$  and similarly  $\frac{1}{2}(P_y(s) \pm P_y^\dagger(s))$  ensures skew-adjointness or self-adjointness. Finally, a straight-line interpolation between two families in  $\Psi\text{DO}_Y^0(X; E^0, E^1)$  with given principal symbols remains a family in  $\Psi\text{DO}_Y^0(X; E^0, E^1)$  with that symbol. In the compactly supported case we use Lemma A.3 here.  $\square$

## References

- [1] M.F. Atiyah and I.M. Singer, *Index theory for skew-adjoint Fredholm operators*, Inst. Hautes Études Sci. Publ. Math. 37 (1969), 5–26.
- [2] M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69.
- [3] M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 71–99.
- [4] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators: I*, Ann. of Math. 87 (1968), 484–530.
- [5] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators: IV*, Ann. of Math. 92 (1970), 119–138.
- [6] M.F. Atiyah and I.M. Singer, *The Index of Elliptic Operators: V*, Ann. of Math. 93 (1971), 139–149.
- [7] J.M. Bismut and D.S. Freed, *The analysis of elliptic families. I. Metrics and connections on determinant bundles*, Comm. Math. Phys. 106 (1986), 159–176.
- [8] Y. Cao, J. Gross and D. Joyce, *Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi–Yau 4-folds*, arXiv:1811.09658, 2018.

- [9] P. Deligne, *Le déterminant de la cohomologie*, pages 93–177 in *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, Contemp. Math. 67, AMS, Providence, R.I., 1987.
- [10] S.K. Donaldson, *The orientation of Yang–Mills moduli spaces and 4-manifold topology*, J. Diff. Geom. 26 (1987), 397–428.
- [11] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, OUP, 1990.
- [12] S.K. Donaldson and E. Segal, *Gauge Theory in Higher Dimensions, II*, Surveys in Diff. Geom. 16 (2011), 1–41. arXiv:0902.3239.
- [13] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. 28 (1988), 513–547.
- [14] D.S. Freed, *On determinant line bundles*, pages 189–238 in *Mathematical Aspects of String Theory*, Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore, 1987.
- [15] L. Hörmander, *Pseudo-differential operators and hypoelliptic equations*, pages 138–183 in *Singular integrals*, Proc. Sympos. Pure Math. Vol. 10, A.M.S., Providence, RI, 1967.
- [16] D. Joyce, Y. Tanaka and M. Upmeyer, *On orientations for gauge-theoretic moduli spaces*, arXiv:1811.01096, 2018.
- [17] D. Joyce and M. Upmeyer, *Canonical orientations for moduli spaces of  $G_2$ -instantons with gauge group  $SU(m)$  or  $U(m)$* , arXiv:1811.02405, 2018.
- [18] F.F. Knudsen, *Determinant functors on exact categories and their extensions to categories of bounded complexes*, Michigan Math. J. 2 (2002), 407–444.
- [19] H.B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Math. Series 38, Princeton Univ. Press, Princeton, NJ, 1989.
- [20] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. 39 (1996), 460–467.
- [21] R.T. Seeley, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. 117 (1965), 167–204.
- [22] T. Walpuski, *Gauge theory on  $G_2$ -manifolds*, PhD Thesis, Imperial College London, 2013.
- [23] A. Zinger, *The determinant line bundle for Fredholm operators: construction, properties, and classification*, Math. Scand. 118 (2016), 203–268. arXiv:1304.6368

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