

Mathematical modelling of elastoplasticity at high stress



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This thesis is dedicated to
Gillian Thomson.

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Abstract

This thesis is concerned with the mathematical modelling of elastic-plastic deformation in regimes of stress far exceeding the yield stress. Such scenarios are typically encountered in violent impact testing, where millimetre-thick samples of metal are subjected to pressures on the order of the bulk modulus of the material.

We begin with an overview of violent impact testing, with particular attention paid to a specific class of experiments known as isentropic compression experiments (ICEs), which will provide motivation for the mathematical modelling and analysis in subsequent chapters. In chapter 2, by appealing to sound notions from rational mechanics and thermodynamics, we construct a mathematical model which aims to encapsulate the essential phenomena involved in violent elastic-plastic deformation. This is followed in chapter 3 with a numerical analysis of the mathematical model in uniaxial strain, which is the geometry relevant ICEs. In chapters 4 and 5, we corroborate the observations made in chapter 3 via a systematic mathematical analysis. In particular, our focus will be on the elastic and plastic waves that can propagate through finite metal samples during isentropic compression. Finally, in chapter 6, we explore the applicability of our model to other geometries, specifically the radially axisymmetric expansion of a circular cavity embedded in an infinite elastic-plastic medium. We conclude with a summary of our findings and suggest some avenues for future investigation.

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Chapter 1

Introduction

In this thesis we explore the mathematical modelling and analysis of violent elastic-plastic deformation that occurs in solids when the stress applied to the material is far in excess of the yield stress. Modelling this phenomena has various applications in defence and engineering, such as high-velocity impact testing of metals and the penetration of a solid object by a shaped-charge device.

Under extreme conditions, the ability of a solid material to resist shear is limited by the yield strength of the material, which is typically much smaller than the applied stress. The most rudimentary models of violent elastic-plastic deformation therefore assume that the material is so highly plasticised that it can be modelled as an inviscid compressible fluid [20, 50]. However, the resulting equations of motion, namely the Euler equations, require the specification of an *a priori* unknown equation of state (EoS) relating the pressure, p , to the density, ρ , and entropy, S . The challenge then is to determine this EoS using experimental evidence. Traditionally, shock waves, generated using a gas-gun, have been used to determine this information theoretically and experimentally [21, 45, 44, 20, 11, 25]. More recently, however, so-called “isentropic compression experiments” (ICEs) have become a standard method by which to extract EoS information in the absence of shock waves [18, 64, 65]. We note in passing that the term “isentropic compression” is something of a misnomer. A more accurate name for such experiments is “quasi-isentropic”, owing to the small amount of entropy production through dissipation as a result of plastic flow (we will return to this phenomenon in §2.3.2).

ICEs have several advantages over shock wave experiments, one of which is that the entire isentrope (see Figure 1.1(a)) for a particular material can be obtained in a single experiment. In comparison, a single shock wave experiment gives only one point in (p, ρ, S) -space corresponding to a shock of a particular strength. Multiple experiments, using shocks of varying strength, are then required to generate many such points, the locus of which is known as a Hugoniot (see Figure 1.1(b)). Another advantage is that the temperature in shock wave experiments can be sufficiently high as to melt the

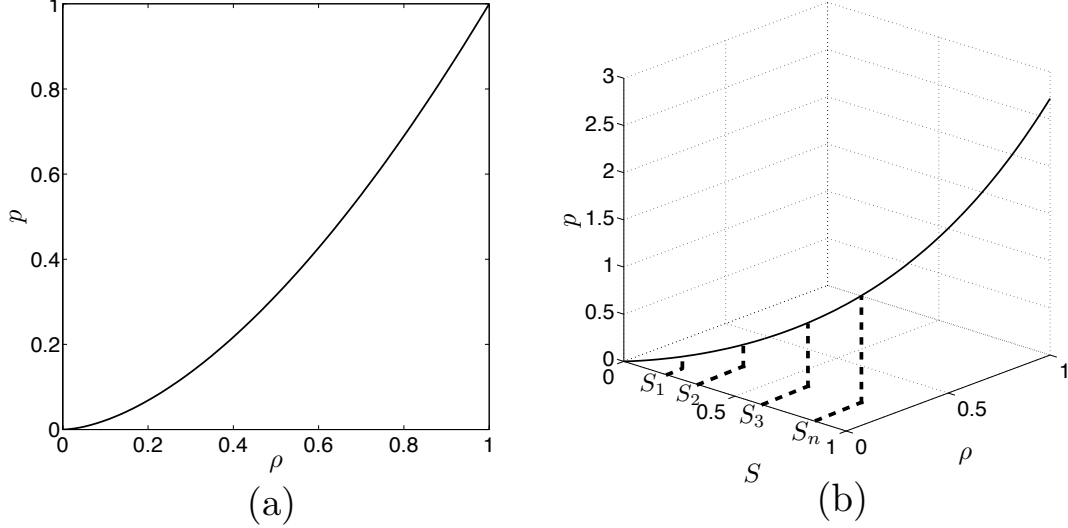


Figure 1.1: Schematic of (a) an isentrope and (b) a Hugoniot. The entire EoS can be mapped out in a single experiment using ICEs, however one shock wave experiment gives only one point on the Hugoniot corresponding to a single value of the entropy, labelled by the values S_n . Multiple experiments are then required to interpolate between these values to obtain the entire EoS.

material, while in ICEs the material remains in the solid phase. This allows EoS data for the solid phase to be obtained at much higher pressures.

Facilities such as the “Z”-machine at Sandia National Laboratories, New Mexico test and analyse the mechanical response of materials under conditions of extreme temperature and pressure, typically on the order of 10^2 K and $10\text{--}10^2$ GPa. A schematic of a typical experiment is shown in Figure 1.2(a) to aid in the understanding of the basic operation of ICEs (for a more detailed account the reader is referred to [29]). A large current is passed between the anode and the cathode, which induces a strong magnetic field directed into the page. The resulting Lorentz force then provides a multi-megabar pressure drive against the front face of the target sample. The target is machined in such a way as to be thin compared to its lateral extent, so that waves generated at the outer edge of the sample do not have time to reach the centre during the time-scale of the experiment. Thus, the material at the centre of the sample initially undergoes purely uniaxial deformation, with displacement varying only in the direction of impact. After a short time, the velocity at the rear face of the target material is recorded using velocity interferometry (VISAR). The results from a recent experiment conducted by Rothman et al. [64] on the “Z”-machine are shown in Figure 1.2(b), where the different colours correspond to varying thicknesses of lead used in the experiment. A striking feature of these profiles is the sheer magnitude of attainable particle velocities in violent impact testing (tens of

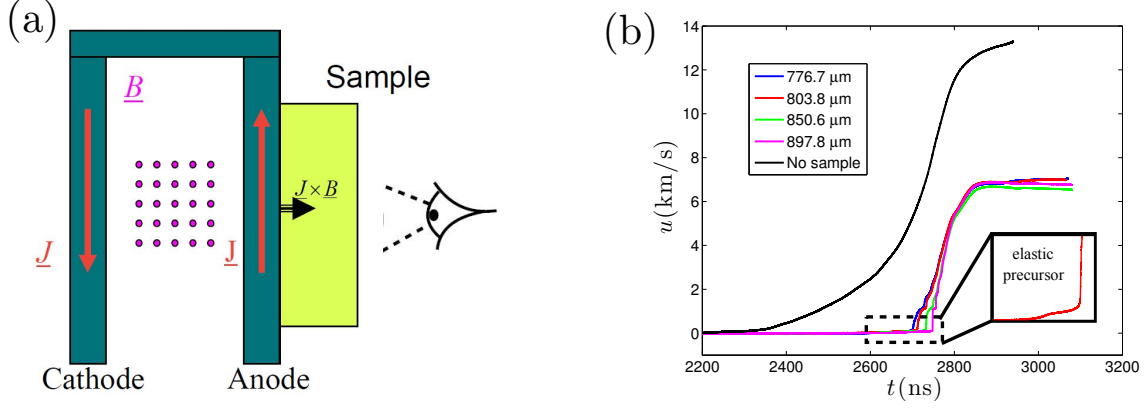


Figure 1.2: (a) Schematic of a ramped isentropic compression experiment. A current passes between the cathode and anode, perpendicular to a magnetic field directed into the page. The resulting Lorentz force provides a pressure drive against the front face of the target sample and (b) boundary velocimetry data obtained from the 3 MBar compression of lead [64]. The coloured curves show the free-surface velocity u measured for different thicknesses of the sample. A reference curve with no lead sample is plotted in black. The inset shows the arrival of a small amplitude elastic disturbance before the arrival of the larger amplitude plastic wave.

kilometres per second), with a rise time of only a few hundred nanoseconds.

Given a set of experimental data like that shown in Figure 1.2(b), the goal is then to reconstruct these velocity profiles by tuning certain parameters in the constitutive assumptions of the model; in this way, the problem is specified as an inverse problem for the *a priori* unknown EoS of the material. Assuming that plastically flowing metal can be modelled as an inviscid compressible fluid, several methods have been developed by which to obtain the EoS of the material from free surface velocity measurements, including the widely used Inverse Lagrangian Analysis method (ILA) [19], and more recently the methods of Platt et al. [51] and Hinch [31]. To provide an outline of these methods, we first recall the Euler equations for an inviscid compressible fluid, written in Lagrangian coordinates X and t :

$$\rho_0 \frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial X} = 0, \quad (1.0.1)$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial X} = 0 \quad (1.0.2)$$

where ρ is the density, ρ_0 is the initial density (assumed to be constant), u is the velocity, and $p(\rho)$ is the pressure. These equations are endowed with the Riemann invariants [49]

$$u \pm \int_{\rho_0}^{\rho} \frac{c \, d\rho}{\rho} = u \pm \phi = \text{constant} \quad \text{along} \quad \frac{dX}{dt} = \pm \frac{\rho c}{\rho_0}, \quad (1.0.3)$$

where the speed of sound

$$c^2 = \frac{\partial p}{\partial \rho}. \quad (1.0.4)$$

A schematic of the characteristic diagram for Equations (1.0.1)–(1.0.2) is shown in Figure 1.3(a). In region A, the material is unstressed and stationary with $u = \phi = 0$, while in region B there is a simple wave with $u = \phi$ and the positive characteristics are straight lines. The first response at the free surface is recorded at $t = t_1$ upon the arrival of the leading characteristic $X = t$ (solid red line). In region C, waves are reflected from the free surface at $X = L_1$, which begin to interact with the boundary at $X = 0$ at time $t = t_2$. In order to determine $p = p(\rho)$, the basic idea behind all of the aforementioned methods of analysis is to apply the same load to two different sample lengths L_1 and L_2 , and then use the free surface velocity profiles obtained to infer the unknown function $p(\rho)$. This relies on the fact that the positive characteristics in regions B and B' will give rise to the same simple wave in both regions, up to the first reflect characteristic (dashed red line). By demanding that the solutions be identical along this line, $p(\rho)$ can be determined.

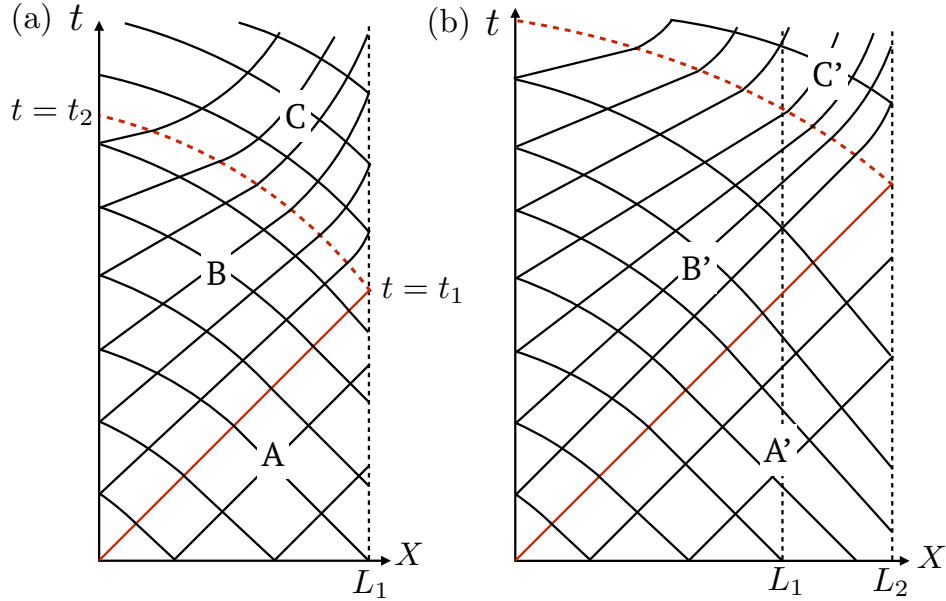


Figure 1.3: Characteristic diagram for the longitudinal motion of a highly plasticised metal, modelled as an inviscid compressible fluid, when subject to a compressive force.

By assuming an EoS relating the pressure to the density, Ockendon et al. [51] transform the governing equations into the hodograph plane and are able to find the “polytropic equation of state that is in best agreement with the (boundary velocimetry) measurements” by solving an overdetermined Goursat problem. A numerical approach is

adopted in the ILA and Hinch methods where one begins at $X = 1$ and integrates backward along positive characteristics, utilising the fact that the solution on the first reflected characteristic is the same for both samples.

Methods such as those outlined above have been widely applied in the analysis of ICEs, yielding somewhat satisfactory results in the multi-megabar regimes of pressure that we are interested in. However, assuming that the metal flows in much the same way as a gas under extreme conditions belies the basic nature of an elastic solid. As the inset of Figure 1.2(b) shows, careful scrutiny of the velocity profiles reveals the arrival of a small amplitude disturbance, now known to be an elastic wave, before the arrival of the larger amplitude plastic wave. The former propagate through the material as the stress increases toward the yield stress, and the latter as the material is compressed further beyond the yield surface. The existence of these elastic waves is well known, however their influence on the accuracy of EoS measurements can be profound. Since the elastic wave speed is greater than the plastic wave speed, the elastic wave has time to reach the free surface and reflect off it back toward the incoming plastic wave. Although small in amplitude compared to the bulk plastic wave, in [19] Davis et al. remark, during their analysis of an experiment of the multi-megabar compression of tantalum, that:

“The most significant source of systematic error...was found to arise from the...free surface reflected wave interactions and the deviatoric stress response of tantalum. This could cause local, possible temporary, unloading of material from a ramp compressed state, and thus multi-value response in wave speed that invalidates the free surface to in-material velocity mapping step of ILA.”

Therefore, a proper account of violent elastic-plastic deformation requires one to account for both the compressibility of the material and the small but measurable effects of elasticity. Despite this acknowledgement, it appears that little attention has been paid to a “forward” study of ICEs to investigate the nature of the combined elastic-plastic deformation, and the phenomena to emerge as a result. Thus we arrive at the central theme of this thesis: to construct and analyse a mathematical model of extreme elastic-plastic deformation which incorporates and predicts the salient phenomenon observed during violent impact testing. A central precept of our modelling philosophy will be that, in the limit of zero yield strength, any model we propose must reduce to the equations of barotropic flow, which are currently used in the analysis of ICEs.

1.1 Outline of thesis

This thesis will be arranged as follows. Chapter 2 will begin with a review of the relevant literature in the modelling of elastic-plastic deformation. We will then derive a

one-dimensional mathematical model of extreme plasticity, based on sound notions from continuum mechanics and thermodynamics. Using this model as a template, we will generalise our formulation to three-dimensions, thus enhancing the potential utility and applicability of our model. In chapter 3, numerical solutions of some problems relevant to violent impact testing will be computed, in order to investigate the predictions of our model on the elastic and plastic waves that propagate through a metal sample during violent impact. Our results will also be compared to those computed using computational hydrocodes ubiquitously employed in industry. Motivated by the results of chapter 3, in chapter 4 we present a systematic mathematical analysis of the elastic and plastic waves that can propagate through a sample subject to loading. This analysis will be limited to small displacements of the material which, although resulting in a simplified system of equations, gives rise to a series of fascinating problems, including a previously unexplored phenomenon whereby fast-moving elastic waves reflect off slower moving plastic waves. We then present a weakly nonlinear analysis in chapter 5 in order to shed light on the influence of nonlinearity and rate-dependence. Surprisingly, the introduction of these two phenomena can give rise to significant differences over the linear model. Our mathematical analysis will conclude in chapter 6 with an exploration into the application of our model to geometries which are not uniaxial, using the expansion of a cylindrical cavity in an infinite elastic-plastic medium as a prototype. chapter 7 will summarise the findings of this thesis and propose some topics for future work.

Chapter 2

Mathematical model

2.1 Synopsis

In this chapter we construct from first principles a mathematical model for elastoplasticity applicable to regimes in which the applied stress greatly exceeds the yield stress. Motivated by experimental evidence, we first focus our attention on one-dimensional motion with uniaxial strain. Our approach is to exploit the smallness of the ratio δ between the yield stress and the applied stress to derive a model which treats the effects of elasticity as a small perturbation to a leading order barotropic flow model. Using the one-dimensional formulation as a guide, we then extend our ideas to develop a fully three-dimensional model for extreme plasticity.

We note that finite-element based hydrocodes currently employed in industry were originally formulated to solve problems in shock hydrodynamics [3, Chp. 3]. When it became necessary to include elastic deformations, these effects were superimposed onto the existing hydrodynamic models which, for simplicity, was achieved by assuming that the strain tensor could be linearised and the stress tensor could be additively decomposed into an isotropic thermodynamic part plus a deviatoric part. However, there is no reason to believe *a priori* that such an approximation for the strain is valid when considering the large deformations in which we are interested. Therefore, in §2.5, we attempt to establish an equivalence between the “hydrodynamic-plus-strength” models employed in industry and the one-dimensional model we derive in §2.3.

Some of the contents of this chapter have been published in Thomson and Howell [78].

2.2 Literature review

There exists a vast body of literature on the physical nature of metal plasticity, in part due to its applications in the defence and engineering communities, but principally due to its complex multi-scale nature. Macroscopic theories of plasticity build on the sound

notions established in the study of elastic deformation, but differ in two crucial ways: first, there exists a critical yield stress σ_Y above which the material no longer deforms elastically, and second, once the applied load has been released, the material is endowed with a permanent non-zero strain. Hence, any theory that we propose must be able to account for both of these phenomenon.

Under careful experimental measurement, it is observed that metals begin to deform plastically at stresses far below those predicted from macroscopic properties of the metal derived from considerations of the lattice structure and inter-atomic forces [30, 32, 84]. This vital observation gave rise to the notion of *dislocations* which, loosely speaking, are defects present in the lattice structure. It is now undisputed that it is the nucleation and motion of dislocations that is the key microscopic physical phenomenon underpinning plastic deformation. However, even on the length-scale of a few millimetres there are an extraordinary number of dislocations, occupying many possible configurations. Incorporating such phenomena is a formidable task, prompting the development of so-called “multi-scale models” which attempt to reconcile the physical behaviour of the plastically flowing metal at the atomistic level with the response of the material at the macroscopic level [5, 67]. While the more sophisticated of such models are able to capture a broad range of phenomena, and have achieved some success in reproducing the basic observations in modern high strain-rate experiments [57, 68, 86], they incur a cost, generally being too complicated to be susceptible to a systematic mathematical analysis. Moreover, the microscopic phenomena which some models aim to incorporate remain poorly understood and appear as adjustable parameters in the constitutive assumptions of the model [35, 71, 72]. Nevertheless, they often form the basis of computational hydrocodes used to predict macroscopic properties of the material such as the equation of state. With all of this in mind, our approach is therefore to construct continuum models based on sound notions derived from mechanics and thermodynamics, which are also simple enough to allow for a systematic mathematical analysis using a mixture of asymptotic and numerical techniques.

The key physical phenomenon underpinning much of our modelling is the large, irreversible plastic deformation that occurs when metals are subject to stresses far in excess of the yield stress. The majority of experimental evidence in such regimes comes from high velocity planar impact, with the material initially undergoing uniaxial deformation and with displacement varying only in the direction of impact. In this case, it is relatively straightforward to write down macroscopic conservation equations of mass, momentum and energy relating the density, velocity and stress. The stress is then assumed to be derivable from a strain energy density function, which is itself a function of the elastic deformation and the entropy. To model the plastic deformation, it is assumed by some

authors [3] that the strain tensor may be decomposed additively into an elastic and plastic part, where the plastic contribution is assumed to be incompressible. However, it is well-known that such a decomposition is only properly valid when the deformation of the material is limited to small displacements [32]. A more realistic model is constructed by employing a multiplicative decomposition of the deformation gradient, first proposed by Lee and Liu [40]. This approach is followed by Plohr et al. [59, 60] and Howell et al. [33], whose work will provide the basis of our modelling in §2.3 and §2.4. General theories of elastic-plastic continua have also been approached from a rational mechanics viewpoint, with important contributions from Willis [88], Green and Naghdi [28], Wang et al. [83], and Germain & Lee [25].

2.3 Mathematical model: uniaxial strain

2.3.1 Governing equations

In this section we derive the simplest possible one-dimensional, macroscopic model of violent elastic-plastic deformation. We focus our attention on isotropic materials, and do not attempt to incorporate phenomena such as work hardening [59, 60], elastic precursor decay [12], or any other more elaborate microscopic descriptions of the plastically flowing metal.

We consider the uniaxial deformation of an elastic-plastic medium in which a material element at Lagrangian position (X, Y, Z) is displaced to Eulerian position $(x(X, t), Y, Z)$ after time t . The Lagrangian deformation gradient for the motion under consideration is therefore

$$\mathbf{F} = \text{diag} \left(\frac{\partial x}{\partial X}, 1, 1 \right), \quad (2.3.1)$$

and hence the Jacobian, which measures the local change in volume of a material element, is

$$J = \frac{\partial x}{\partial X} = \rho_0 / \rho = \nu, \quad (2.3.2)$$

where ρ is the density and ρ_0 is the initial density, assumed to be constant. The dimensionless quantity ν is referred to as the specific volume.¹ In one-dimensional strain, the kinematic equation relates the specific volume ν to the axial velocity $u = \partial x / \partial t$. Differentiating (2.3.2) with respect to time we have

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0. \quad (2.3.3)$$

¹In literature [20], the specific volume is sometimes defined as $\eta = 1/\rho$, and hence $\nu = \rho_0/\rho = \eta/\eta_0$ where $\eta_0 = 1/\rho_0$. However, this distinction is academic and we will proceed with the definition given in (2.3.2).

We now make the common assumption [33, 40, 60, 83, 88] that the total deformation gradient may be decomposed multiplicatively² into an elastic deformation tensor $\mathbf{F}^e$ and a plastic deformation tensor $\mathbf{F}^p$. This decomposition is subject to some degree of controversy, as there appears to be an ambiguity in the ordering of the successive deformations. In the uniaxial configuration considered here, the distinction is academic but, in what follows, we follow the convention depicted in Figure 2.1. First, the body, in its reference configuration Ω_r , is mapped to an intermediate configuration Ω_i . This deformation maps neighbourhoods of each material point to a locally stress-free configuration, where the stress-free elements are not required to fit together to make a continuous body. Continuity is achieved by the action of $\mathbf{F}^e$ which maps the intermediate configuration to the final configuration Ω_c . We therefore define

$$\mathbf{F} = \text{diag}(\nu, 1, 1) = \mathbf{F}^e \mathbf{F}^p, \quad (2.3.4)$$

where the elastic and plastic deformations are each assumed to be uniaxial, so that

$$\mathbf{F}^e = \text{diag}(F_{11}^e, F_{22}^e, F_{22}^e), \quad \mathbf{F}^p = \text{diag}(F_{11}^p, F_{22}^p, F_{22}^p). \quad (2.3.5)$$

We thus obtain the relations

$$F_{11}^e F_{11}^p = \nu, \quad F_{22}^e F_{22}^p = 1. \quad (2.3.6)$$

Since we are considering uniaxial deformation, the Cauchy stress tensor is of the form

$$\boldsymbol{\sigma} = \text{diag}(\sigma_{11}(X, t), \sigma_{22}(X, t), \sigma_{22}(X, t)). \quad (2.3.7)$$

In Lagrangian coordinates, conservation of momentum implies

$$\rho_0 \frac{\partial u}{\partial t} = \frac{\partial \mathcal{T}_{11}}{\partial X}, \quad (2.3.8)$$

where $\mathcal{T}_{11}$ is the leading diagonal component of the (first) Piola-Kirchhoff stress tensor

$$\boldsymbol{\mathcal{T}} = J \boldsymbol{\sigma} (\mathbf{F}^T)^{-1}. \quad (2.3.9)$$

In the case of uniaxial deformation

$$\boldsymbol{\mathcal{T}} = \text{diag}(\sigma_{11}, \nu \sigma_{22}, \nu \sigma_{22}), \quad (2.3.10)$$

and hence $\mathcal{T}_{11} = \sigma_{11}$.

²An analogous decomposition appears in the study of morphoelasticity in which the reference configuration is mapped to the intermediate state by a *growth tensor* $\mathbf{G}$, and then to the current configuration by an elastic deformation tensor $\mathbf{A}$ [2, 27, 63].

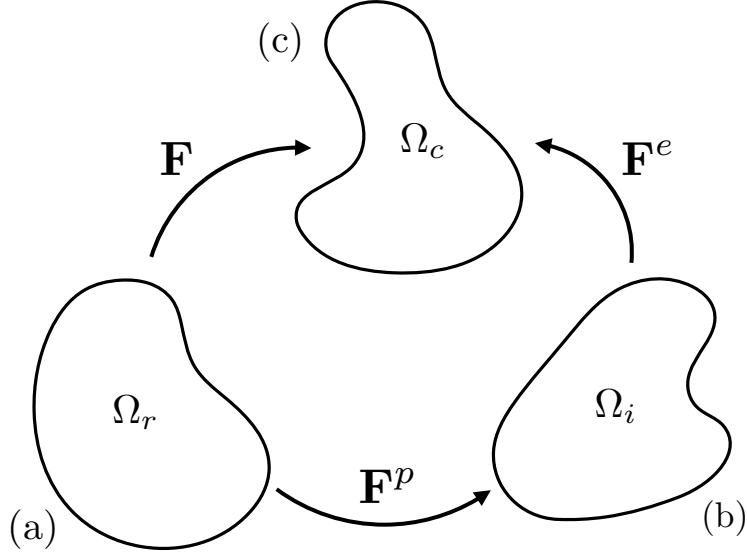


Figure 2.1: The body in its reference configuration (a) is mapped to the intermediate, stress-free configuration (b) by the action of $\mathbf{F}^p$. The elastic deformation $\mathbf{F}^e$ manipulates the stress-free elements of (b) in such a way as to achieve a continuous body in the current configuration (c).

Provided there are no heat sources and thermal conduction is negligible, the equation for conservation of energy reads

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 u^2 + \mathcal{W} \right) - \frac{\partial}{\partial X} (u \sigma_{11}) = 0, \quad (2.3.11)$$

where $\mathcal{W}$ is the internal energy per unit reference volume. In the language of solid mechanics, the function $\mathcal{W}$ is a scalar potential called the strain energy which is a measure of the energy stored in the material as it deforms. Expanding out (2.3.11) and using (2.3.3) and (2.3.8), we can alternatively write

$$\frac{\partial \mathcal{W}}{\partial t} = \sigma_{11} \frac{\partial \nu}{\partial t}. \quad (2.3.12)$$

We now assume that $\mathcal{W}$ is a function of the elastic deformation components F_{11}^e and F_{22}^e , and the entropy S . On physical grounds, this is motivated by the observation that only the elastic deformation contributes to the storage of energy, while the plastic deformation is a purely dissipative mechanism. This description is a generalization of a hyperelastic material where $\mathcal{W}$ is a function of the elastic deformation only.

The stress components and the temperature T are then given by

$$\rho_0 T = \frac{\partial \mathcal{W}}{\partial S}, \quad \sigma_{11} = \frac{1}{\nu} F_{11}^e \frac{\partial \mathcal{W}}{\partial F_{11}^e}, \quad \sigma_{22} = \frac{1}{\nu} \frac{F_{22}^e}{2} \frac{\partial \mathcal{W}}{\partial F_{22}^e}, \quad (2.3.13)$$

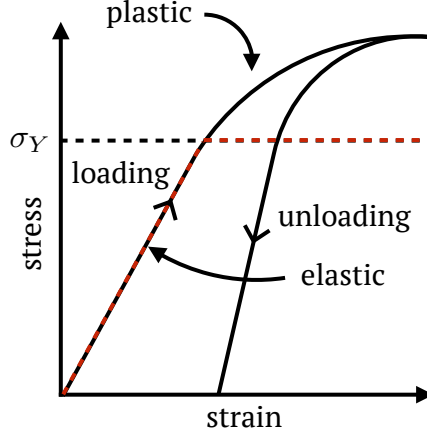


Figure 2.2: A typical stress-strain curve for an elastic-plastic material during loading and unloading. As a result of plastic deformation, the material does not return along the same loading curve as the stress is relaxed to zero. The red dashed line demarcates the response of an elastic-perfectly plastic material.

where the factor 2 in the expression for σ_{22} arises from the assumption of uniaxiality. Using the relations (2.3.13) we can write the energy equation (2.3.12) in the form

$$\rho_0 T \frac{\partial S}{\partial t} = \frac{\nu \sigma_{11}}{F_{11}^p} \frac{\partial F_{11}^p}{\partial t} + \frac{2\nu \sigma_{22}}{F_{22}^p} \frac{\partial F_{22}^p}{\partial t} = \Phi, \quad (2.3.14)$$

where we define Φ to be the dissipation function. Note that when $\Phi = 0$, S is conserved when the plastic strain is zero. When $\Phi \neq 0$ we must ensure $\Phi \geq 0$ to preclude negative dissipation.³

A material begins to flow plastically if the stress placed on it reaches, and possibly exceeds, the yield stress σ_Y .⁴ Below this point, the material deforms elastically, returning to its reference configuration once the stress placed on it has been relaxed to zero. Plastic deformation, however, is irreversible and the material is left with a permanent non-zero strain once the loading placed on it has been removed. This behaviour is described schematically in Figure 2.2.

Mathematically we express the aforementioned ideas in the form of a yield function, which relates some measure of the stress tensor to the yield stress. The simplest scenario is that of rate-independent (perfect) plasticity, in which the stress may never exceed the yield stress (sketched by the red dashed line in Figure 2.2). This leads to a yield condition of the form

$$Y(\sigma_{11}, \sigma_{22}) \leq 0, \quad (2.3.15)$$

³Note that the energy equation (2.3.14) demonstrates the erroneous nature of the term “isentropic compression” since, when the plastic strain is non-zero, we have $\partial S / \partial t \geq 0$.

⁴A hypothesis first attributed to Tresca c. 1864.

where $Y(\sigma_{11}, \sigma_{22})$ is called the yield function. In what follows we use the Tresca⁵ yield function

$$Y(\sigma_{11}, \sigma_{22}) = |\sigma_{11} - \sigma_{22}| - \sigma_Y, \quad (2.3.16)$$

where σ_Y is the yield stress. A variety of models for σ_Y have been proposed in the literature which attempt to incorporate temperature and a plethora of material parameters; these include the widely used empirical models due to Steinberg et al. [71, 72] and Johnson and Cook [35]. While we do not venture into this level of complication here, we note that our model is sufficiently flexible to allow for the incorporation of more sophisticated models for σ_Y .

We are now in a position to specify a flow rule for the irreversible plastic flow which occurs when the stress on the material reaches the yield stress. Following [33] and maximising the dissipation function Φ subject to the yield constraint (2.3.16) we define

$$\frac{1}{F_{11}^p} \frac{\partial F_{11}^p}{\partial t} = -\frac{2}{F_{22}^p} \frac{\partial F_{22}^p}{\partial t} = \Lambda \operatorname{sgn}(\sigma_{11} - \sigma_{22}), \quad (2.3.17)$$

where Λ is a Lagrange multiplier. This allows us to express the dissipation function as

$$\Phi = \Lambda \nu |\sigma_{11} - \sigma_{22}|. \quad (2.3.18)$$

We note that a first integral of equations (2.3.17) renders the plastic flow incompressible; that is

$$\det \mathbf{F}^p = F_{11}^p F_{22}^{p2} = 1. \quad (2.3.19)$$

We can therefore write (2.3.14) as

$$\rho_0 T \frac{\partial S}{\partial t} = \frac{2\nu(\sigma_{11} - \sigma_{22})}{F_{22}^e} \frac{\partial F_{22}^e}{\partial t}. \quad (2.3.20)$$

For elastic behaviour $\Lambda = 0$ and the inequality (2.3.15) is strict. Furthermore, this means that $\partial S / \partial t \equiv 0$, so S is constant if it is uniform initially. When the material is flowing plastically, we require Λ to be non-negative to deny the possibility of negative dissipation. In other words, to ensure that $\partial S / \partial t \geq 0$, we require $\Phi \geq 0$, and hence that $\Lambda \geq 0$ by (2.3.18). To encompass all of these possibilities, we extend (2.3.16) to the complementarity conditions

$$\Lambda (|\sigma_{11} - \sigma_{22}| - \sigma_Y) = 0, \quad \Lambda \geq 0, \quad |\sigma_{11} - \sigma_{22}| \leq \sigma_Y. \quad (2.3.21)$$

When the deviatoric stress is below the yield stress σ_Y , the material is assumed to be elastic with $\Lambda = 0$. In rate-independent perfect plasticity, the deviatoric stress is

⁵We note that, due to our uniaxiality assumption, the von Mises yield function also reduces to the form (2.3.16).

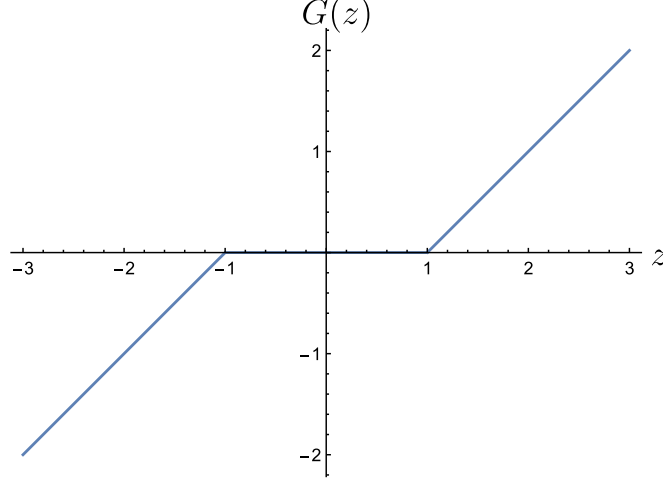


Figure 2.3: Plot of the yield function $G(z)$ versus z .

precisely equal to the yield stress during plastic flow. However, in reality we are interested in scenarios where the stress greatly exceeds the yield stress and this strict yield criterion may be violated. We therefore look to relax the yield condition (2.3.15) and generalize (2.3.21) to adopt a model in which Λ is an increasing function of the amount by which the deviatoric stress exceeds the yield stress. We therefore posit the constitutive relation

$$\Lambda = \frac{r}{\sigma_Y} [Y(\sigma_{11}, \sigma_{22})]^+, \quad (2.3.22)$$

where

$$[x]^+ = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0, \end{cases} \quad (2.3.23)$$

and r is a rate constant with units of inverse time. We note that while the choice of the constitutive relation (2.3.22) is not unique, models of this type are well-substantiated and may be found in the literature as far back as Perzyna [58]. Combining equations (2.3.17) and (2.3.22) we obtain

$$\frac{1}{F_{11}^p} \frac{\partial F_{11}^p}{\partial t} = rG\left(\frac{\sigma_{11} - \sigma_{22}}{\sigma_Y}\right), \quad (2.3.24)$$

where

$$G(z) = z + \frac{1}{2}|z - 1| - \frac{1}{2}|z + 1| \quad (2.3.25)$$

as shown in Figure 2.3. Equation (2.3.24) reduces to rate-independent perfect plasticity in the limit $r \rightarrow \infty$.

2.3.2 Equation of state

To close our model, we must specify the form of the constitutive law for the strain energy density $\mathcal{W}$. For the material to undergo large plastic deformations with $O(1)$ variations

in density, we see that to balance the terms in (2.3.3) and (2.3.8) the axial stress must be $O(\rho_0 c_0^2)$ where

$$c_0 = \sqrt{\frac{K_b}{\rho_0}} \quad (2.3.26)$$

is the bulk sound speed of the material and K_b is the bulk modulus [56]. However, the flow rule (2.3.24) implies that the deviatoric stress $\sigma_{11} - \sigma_{22}$ must be of order σ_Y which is typically much smaller. This in turn implies that, although the net volumetric deformation ν may undergo $O(1)$ variations, the deviatoric elastic strain must be much smaller. It is therefore useful to decompose the elastic strain into isotropic and deviatoric parts by setting

$$F_{11}^e = \nu^{\frac{1}{3}} \sqrt{1 + 2\mathcal{E}}, \quad F_{22}^e = \frac{\nu^{\frac{1}{3}}}{(1 + 2\mathcal{E})^{\frac{1}{4}}} \quad (2.3.27)$$

where $\mathcal{E}$ is a measure of the deviatoric elastic strain. We note that

$$\mathbf{F}^e = \nu^{1/3} \text{diag} \left(\sqrt{1 + 2\mathcal{E}}, (1 + 2\mathcal{E})^{-1/4}, (1 + 2\mathcal{E})^{-1/4} \right) = \nu^{1/3} \tilde{\mathbf{F}}^e \quad (2.3.28)$$

with $\det \tilde{\mathbf{F}}^e = 1$. Thus the elastic deformation may be decomposed multiplicatively into a dilatation, followed by an incompressible shear deformation [59].

If we now think of $\mathcal{W}$ as a function of ν , $\mathcal{E}$ and S , we can express the stress components as

$$\sigma_{11} = \frac{\partial \mathcal{W}}{\partial \nu} + \frac{2(1 + 2\mathcal{E})}{3\nu} \frac{\partial \mathcal{W}}{\partial \mathcal{E}}, \quad \sigma_{11} - \sigma_{22} = \frac{1 + 2\mathcal{E}}{\nu} \frac{\partial \mathcal{W}}{\partial \mathcal{E}}. \quad (2.3.29)$$

The energy equation (2.3.14) then becomes

$$\frac{\partial \mathcal{W}}{\partial S} \frac{\partial S}{\partial t} = \frac{\partial \mathcal{W}}{\partial \mathcal{E}} \left(\frac{2(1 + 2\mathcal{E})}{3\nu} \frac{\partial \nu}{\partial t} - \frac{\partial \mathcal{E}}{\partial t} \right) \quad (2.3.30)$$

and the rate-dependent plastic flow rule (2.3.24) becomes

$$\frac{2}{3\nu} \frac{\partial \nu}{\partial t} - \frac{1}{1 + 2\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} = rG \left(\frac{1 + 2\mathcal{E}}{\sigma_Y \nu} \frac{\partial \mathcal{W}}{\partial \mathcal{E}} \right). \quad (2.3.31)$$

The foregoing discussion suggests that the deviatoric elastic strain will be small, specifically of order

$$\delta = \frac{\sigma_{Y0}}{K_b} \ll 1, \quad (2.3.32)$$

where σ_{Y0} denotes a representative value of σ_Y to allow for situations where σ_Y is not constant.⁶ This motivates us to define

$$\varepsilon = \frac{\mathcal{E}}{\delta} = O(1). \quad (2.3.33)$$

⁶For example, models such as [71, 72] have σ_Y dependent on the plastic strain rate and the entropy or temperature.

We can therefore expand $\mathcal{W} = \mathcal{W}(\delta\varepsilon, \nu, S)$ in a Taylor series about $\delta = 0$:

$$\frac{\mathcal{W}}{K_b} \sim \mathcal{W}_0(\nu, S) + \delta^2 \mathcal{W}_2(\nu, S) \varepsilon^2 + O(\delta^3) \quad (2.3.34)$$

noting that $\varepsilon = 0$ is a local minimum of the elastic energy. For clarity, we define

$$\mathcal{W}_0(\nu, S) = \frac{\mathcal{W}(0, \nu, S)}{K_b}, \quad \mathcal{W}_2(\nu, S) = \frac{1}{2K_b} \frac{\partial^2 \mathcal{W}}{\partial \varepsilon^2}(0, \nu, S). \quad (2.3.35)$$

The leading order term in the expression (2.3.34) can be thought of as the thermodynamic part of the equation of state.

We can now infer from Equation (2.3.30) that the entropy production through dissipation is small:

$$S = S_0(1 + \delta s) \quad (2.3.36)$$

where $s = O(1)$ and the initial entropy S_0 is assumed to be constant. We can therefore restrict Equation (2.3.34) further to

$$\frac{\mathcal{W}}{K_b} \sim h(\nu) + \delta s \theta(\nu) + \delta^2 \left(s^2 S_0^2 \frac{\partial^2 \mathcal{W}_0}{\partial S^2}(S_0, \nu) + \frac{1}{2} k(\nu) \varepsilon^2 \right) + O(\delta^3), \quad (2.3.37)$$

where

$$h(\nu) = \mathcal{W}_0(S_0, \nu), \quad \theta(\nu) = S_0 \frac{\partial \mathcal{W}_0}{\partial S}(S_0, \nu), \quad \text{and} \quad k(\nu) = 2\mathcal{W}_2(S_0, \nu) \quad (2.3.38)$$

are in principle known functions of ν alone.

Finally, we find from (2.3.29) the following approximations for the stress components

$$\frac{\sigma_{11}}{K_b} \sim h'(\nu) + \delta \left(s \theta'(\nu) + \frac{2k(\nu)}{3\nu} \varepsilon \right) + O(\delta^2), \quad \frac{\sigma_{11} - \sigma_{22}}{\sigma_{Y0}} \sim \frac{k(\nu)}{\nu} \varepsilon + O(\delta). \quad (2.3.39)$$

In the presence of shocks, we may expect $O(1)$ variations in entropy and thus the approximation (2.3.36) breaks down. The preceding approximations also break down in the very unlikely event that the yield stress is comparable to the elastic modulus of the material.

2.3.3 Non-dimensionalisation

We non-dimensionalise as follows:

$$X = L\tilde{X}, \quad t = \frac{L}{c_0}\tilde{t}, \quad u = c_0\tilde{u}, \quad (2.3.40)$$

where L is a typical sample thickness. In dimensionless form, to leading order in δ , the governing equations (2.3.3), (2.3.8), (2.3.30) and (2.3.31) become

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0, \quad (2.3.41)$$

$$\frac{\partial u}{\partial t} - h''(\nu) \frac{\partial \nu}{\partial X} = 0, \quad (2.3.42)$$

$$\theta(\nu) \frac{\partial s}{\partial t} = \frac{2k(\nu)\varepsilon}{3\nu} \frac{\partial \nu}{\partial t}, \quad (2.3.43)$$

$$\frac{2a}{3\nu} \frac{\partial \nu}{\partial t} = G \left(\frac{\varepsilon}{\varepsilon_c(\nu)} \right), \quad (2.3.44)$$

where

$$\varepsilon_c(\nu) = \frac{\sigma_Y(\nu)\nu}{\sigma_{Y0}k(\nu)} \quad (2.3.45)$$

represents the normalized yield strain, and

$$a = \frac{c_0}{rL} \quad (2.3.46)$$

is a dimensionless rate constant. We note that ν , s , and ε are already non-dimensional.

The leading order energy equation (2.3.43) and the flow rule (2.3.44) decouple since ε and s can in principle be found once ν and u are determined from Equations (2.3.41)–(2.3.42).

2.3.4 Small amplitude oscillations

The equations (2.3.41) and (2.3.42) satisfied by the specific volume ν and the velocity u are equivalent to the equations of one-dimensional barotropic flow where the pressure $p = -h'(\nu)$. The entropy production and the deviatoric strain are then determined in principle by the decoupled energy equation (2.3.43) and the flow rule (2.3.44). However, the form of the function G means that equation (2.3.44) is degenerate when the left-hand side is zero, and suggests the existence of a distinguished asymptotic limit when $\partial \nu / \partial t$ is small.

To examine the behaviour of the governing equations when the displacement is small, we perform the re-scalings

$$\nu = 1 + \delta\eta, \quad u = \delta U, \quad s = \delta\zeta, \quad (2.3.47)$$

before taking the limit $\delta \rightarrow 0$. The leading order governing equations in this case are

$$\frac{\partial \eta}{\partial t} - \frac{\partial U}{\partial X} = 0, \quad (2.3.48)$$

$$\frac{\partial U}{\partial t} - \frac{\partial \eta}{\partial X} - \frac{2k_0}{3} \frac{\partial \varepsilon}{\partial X} = 0, \quad (2.3.49)$$

$$\theta(1) \frac{\partial \zeta}{\partial t} = k_0 \varepsilon \left(\frac{2}{3} \frac{\partial \eta}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right), \quad (2.3.50)$$

$$a\delta \left(\frac{2}{3} \frac{\partial \eta}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right) = G(\varepsilon), \quad (2.3.51)$$

where $k_0 = k(1)$, and we assume $h''(1) = \varepsilon_c(1) = 1$ by the choice of scalings c_0 and σ_{Y0} . If $a\delta = O(1)$, equations (2.3.48), (2.3.49) and (2.3.51) provide a coupled semi-linear hyperbolic system for η , U and ε , while the entropy change $\delta^2 \zeta$ is again negligible to lowest order and may be determined *a posteriori* from Equation (2.3.50).

For future reference, we note that the characteristics of (2.3.48)–(2.3.51) are

$$\frac{dX}{dt} = \pm c_e \quad \text{and} \quad \frac{dX}{dt} = 0, \quad (2.3.52)$$

where

$$c_e = \sqrt{1 + \frac{4k_0}{9}} \quad (2.3.53)$$

is the *elastic wave speed*.

2.3.5 Uniformly valid model

All of the preceding analysis suggests that entropy variations are universally negligible in the absence of shocks, and a uniformly valid model which captures all of the above behaviour is

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0, \quad (2.3.54)$$

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial X} \left(h'(\nu) + \frac{2\delta k(\nu)}{3\nu} \varepsilon \right) = 0, \quad (2.3.55)$$

$$a \left(\frac{2}{3\nu} \frac{\partial \nu}{\partial t} - \delta \frac{\partial \varepsilon}{\partial t} \right) = G \left(\frac{\varepsilon}{\varepsilon_c(\nu)} \right). \quad (2.3.56)$$

Equations (2.3.54)–(2.3.56) constitute a 3×3 system of hyperbolic conservation laws with characteristics

$$\frac{dX}{dt} = \pm \sqrt{\frac{\partial f}{\partial \nu} + \frac{2}{3\delta\nu} \frac{\partial f}{\partial \varepsilon}}, \quad f = h'(\nu) + \frac{2\delta k(\nu)}{3\nu} \varepsilon, \quad \text{and} \quad \frac{dX}{dt} = 0. \quad (2.3.57)$$

In regions where the strain is $O(1)$, terms of $O(\delta)$ are negligible, and we retrieve the equations of barotropic flow. However in regions where the strain is $O(\delta)$, the deviatoric

and isotropic contributions to the strain balance, and hence we expect elastic effects to be noticeable near the tails of propagating waves.

We note that while the models of Plohr et al. [59, 60, 83] are also expressed as systems of conservation laws, a systematic mathematical analysis of elastic-plastic deformation using such models appears to be limited. Furthermore, the appearance of the non-dimensional parameters a and δ in our model motivate the use of techniques from asymptotic analysis which, as we shall see in forthcoming chapters, can provide crucial insight into the physical nature of elastic-plastic deformation. The use of such techniques is not afforded by existing models in the literature. In Chapters 3 to 5 we will present numerical and asymptotic solutions of the system (2.3.54)–(2.3.56), but before that we will demonstrate how our one-dimensional formulation may be generalised to arbitrary three-dimensional deformations.

2.4 Three-dimensional elastoplasticity

2.4.1 Governing equations

The deformation considered in §2.3 was motivated by the one-dimensional compression of a metal sample during ICEs. It is then natural to ask whether our modelling philosophy can be generalised to three dimensions. As well as being of interest mathematically, a fully three-dimensional model could, in principle, find applications in describing other high strain-rate phenomena in more complex geometries, such as the penetration of a metal target by a shaped-charged device (see chapter 6).

To describe fully three-dimensional motion, we consider a particle initially at Lagrangian position $\mathbf{X} = (X, Y, Z)$ in the reference configuration, which is displaced to Eulerian position $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ after time t in the current configuration. In what follows, we will make regular use of Einstein's summation convention, where repetition of indices indicates summation.

The Lagrangian deformation gradient tensor of the motion is

$$[\mathbf{F}]_{ij} = F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad (2.4.1)$$

where the specific volume ν and density ρ are again given by

$$J = \det \mathbf{F} = \nu = \frac{\rho_0}{\rho}. \quad (2.4.2)$$

The components of $\mathbf{F}$ are related to the velocity $\mathbf{u}$ via the kinematic equations

$$\dot{F}_{ij} = \frac{\partial u_i}{\partial X_j}, \quad (2.4.3)$$

where the dot notation denotes differentiation with respect to time and

$$\mathbf{u} = \dot{\mathbf{x}}. \quad (2.4.4)$$

We once more adopt the assumption that the deformation gradient may be decomposed multiplicatively into an elastic and plastic part. Thus

$$F_{ij} = F_{ik}^e F_{kj}^p. \quad (2.4.5)$$

In the absence of external body forces, conservation of momentum in the Lagrangian frame states

$$\rho_0 \frac{\partial u_i}{\partial t} = \frac{\partial \mathcal{T}_{ij}}{\partial X_j} \quad (2.4.6)$$

where we recall that

$$\mathcal{T}_{ij} = J \sigma_{ik} F_{jk}^{-1} \quad (2.4.7)$$

is the first Piola-Kirchoff stress tensor and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. We note that $\boldsymbol{\mathcal{T}}$ is not necessarily symmetric, even though $\boldsymbol{\sigma}$ is. However, the components of $\boldsymbol{\mathcal{T}}$ are related to the second Piola-Kirchoff tensor $\boldsymbol{\mathcal{S}}$ by

$$\boldsymbol{\mathcal{T}} = \mathbf{F} \boldsymbol{\mathcal{S}}, \quad (2.4.8)$$

where $\boldsymbol{\mathcal{S}}$ is symmetric and therefore endowed with 6 independent components instead of the 9 possessed by $\boldsymbol{\mathcal{T}}$.

Neglecting heat sources and thermal conduction, conservation of energy states

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 u_i u_i + \mathcal{W} \right) = \frac{\partial}{\partial X_j} (u_i \mathcal{T}_{ij}), \quad (2.4.9)$$

where $\mathcal{W}$ is again the internal energy per unit reference volume. Using (2.4.6), we can alternatively write (2.4.9) as

$$\frac{\partial \mathcal{W}}{\partial t} = \mathcal{T}_{ij} \dot{F}_{ij}. \quad (2.4.10)$$

Making the assumption that the internal energy is a function of the elastic deformation $\mathbf{F}^e$ and the entropy S , we find that the temperature T and the components of the Piola-Kirchoff stress tensor are given by

$$\rho_0 T = \frac{\partial \mathcal{W}}{\partial S}, \quad \frac{\partial \mathcal{W}}{\partial F_{ij}^e} = \mathcal{T}_{ik} F_{jk}^p = [\boldsymbol{\mathcal{T}} \mathbf{F}^{pT}]_{ij}. \quad (2.4.11)$$

Using the chain rule, we can then write (2.4.10) as

$$\rho_0 T \frac{\partial S}{\partial t} = \mathcal{T}_{ij} F_{ik}^e \frac{\partial F_{kj}^p}{\partial t} = \text{tr} \left[\boldsymbol{\mathcal{T}}^T \mathbf{F}^e \frac{\partial \mathbf{F}^p}{\partial t} \right] = \Phi \quad (2.4.12)$$

where Φ is again the dissipation function.

To model the plastic flow which occurs when the stress on the material exceeds the yield stress, we again invoke the yield criterion

$$Y(\boldsymbol{\sigma}) \leq 0. \quad (2.4.13)$$

In three-dimensional problems, the von Mises yield function is often preferred [32]:

$$Y = ||\text{dev } \boldsymbol{\sigma}|| - \sqrt{\frac{2}{3}}\sigma_Y \quad (2.4.14)$$

where

$$\text{dev } \boldsymbol{\sigma} = \boldsymbol{\sigma} - \frac{1}{3}(\text{tr } \boldsymbol{\sigma})\mathbf{I} \quad (2.4.15)$$

is the (trace-free) deviatoric part of $\boldsymbol{\sigma}$. Note the factor multiplying σ_Y ensures that (2.4.14) is consistent with the Tresca yield criterion (2.3.16) in uniaxial strain.

We specify a flow rule which dictates the evolution of the plastic strain when the material is undergoing irreversible plastic deformation. We note that our approach follows the work of Plohr and Sharp [59, 60]. We first recall the dissipation function Φ in (2.4.12). Using (2.4.5) and (2.4.8) we can re-write Φ as

$$\Phi = \mathcal{S}_{jp} F_{ip} F_{in} (F_{nk}^p)^{-1} \frac{\partial F_{kj}^p}{\partial t} = \text{tr} \left[\boldsymbol{\mathcal{S}} \mathbf{F}^T \mathbf{F} \mathbf{F}^{p-1} \frac{\partial \mathbf{F}^p}{\partial t} \right]. \quad (2.4.16)$$

It is now convenient to introduce the Cauchy-Green strain tensors [60]

$$\boldsymbol{\mathcal{C}} = \mathbf{F}^T \mathbf{F}, \quad \boldsymbol{\mathcal{C}}^e = \mathbf{F}^{eT} \mathbf{F}^e, \quad \boldsymbol{\mathcal{C}}^p = \mathbf{F}^{pT} \mathbf{F}^p. \quad (2.4.17)$$

By expressing the equations in terms of these symmetric tensors, we reduce the number of unknowns in the problem. This allows us to write (2.4.16) in the form

$$\Phi = \mathcal{S}_{jp} \mathcal{C}_{pn} (C_{nm}^p)^{-1} F_{km}^p \frac{\partial F_{kj}^p}{\partial t} = \text{tr} \left[\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1} \mathbf{F}^{pT} \frac{\partial \mathbf{F}^p}{\partial t} \right]. \quad (2.4.18)$$

Furthermore, using (2.4.17) and the symmetry of $\boldsymbol{\mathcal{S}}$, we can deduce that

$$\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1} = \boldsymbol{\mathcal{C}}^{p-1} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{S}} = [\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1}]^T; \quad (2.4.19)$$

that is, $\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1}$ is symmetric. Hence we can write (2.4.18) in terms of $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}^p$:

$$\Phi = \frac{1}{2} \text{tr} \left[\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1} \left(\mathbf{F}^{pT} \dot{\mathbf{F}}^p + \dot{\mathbf{F}}^{pT} \mathbf{F}^p \right) \right] = \frac{1}{2} \text{tr} \left[\boldsymbol{\mathcal{S}} \boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{C}}^{p-1} \dot{\boldsymbol{\mathcal{C}}}^p \right] = \frac{1}{2} \text{tr} \left[\boldsymbol{\mathcal{C}} \boldsymbol{\mathcal{S}} \dot{\boldsymbol{\mathcal{C}}}^p \boldsymbol{\mathcal{C}}^{p-1} \right] \quad (2.4.20)$$

where in the last equality we have used the property that $\text{tr} [\mathbf{A} \mathbf{B}^T] = \text{tr} [\mathbf{A}^T \mathbf{B}]$. Equivalently, using (2.4.7) and (2.4.8) we can write (2.4.20) in terms of the Cauchy stress tensor:

$$\Phi = \frac{1}{2} J \text{tr} \left[\mathbf{F}^T \boldsymbol{\sigma} \mathbf{F}^{-T} \dot{\boldsymbol{\mathcal{C}}}^p \boldsymbol{\mathcal{C}}^{p-1} \right] = \frac{1}{2} J \text{tr} \left[\boldsymbol{\sigma} \mathbf{F}^{-T} \dot{\boldsymbol{\mathcal{C}}}^p \boldsymbol{\mathcal{C}}^{p-1} \mathbf{F}^T \right]. \quad (2.4.21)$$

If we now seek to maximise (2.4.21) subject to the constraint (2.4.13), we obtain the flow rule

$$\frac{1}{2} \left(\mathbf{F}^{-T} \dot{\mathbf{C}}^p \mathbf{C}^{p-1} \mathbf{F}^T \right) = \sqrt{\frac{3}{2}} \Lambda \frac{\partial Y}{\partial \boldsymbol{\sigma}}, \quad (2.4.22)$$

where Λ is a Lagrange multiplier. Using (2.4.14) we find that

$$\frac{\partial Y}{\partial \boldsymbol{\sigma}} = \frac{\text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} \quad (2.4.23)$$

and thus we obtain

$$\frac{1}{2} \left(\mathbf{F}^{-T} \dot{\mathbf{C}}^p \mathbf{C}^{p-1} \mathbf{F}^T \right) = \sqrt{\frac{3}{2}} \Lambda \frac{\text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|}. \quad (2.4.24)$$

Then by using (2.4.24), the equation for conservation of energy (2.4.12) may alternatively be written

$$\rho_0 T \frac{\partial S}{\partial t} = \sqrt{\frac{3}{2}} J \Lambda \|\text{dev } \boldsymbol{\sigma}\|. \quad (2.4.25)$$

We note that (2.4.24) is trace-free, rendering the plastic flow *incompressible*. For elastic behaviour $\Lambda = 0$ and the inequality in (2.4.13) is strict. When the material has yielded $\Lambda \geq 0$ to ensure that $\partial S / \partial t \geq 0$, as discussed in §2.3.1.

Finally, in an analogous way to (2.3.22), to extend our model to a rate-dependent formulation we posit the rate-dependent flow rule

$$\Lambda = \sqrt{\frac{3}{2}} \frac{r}{\sigma_Y} [Y(\boldsymbol{\sigma})]^+. \quad (2.4.26)$$

In summary, from considerations of kinematics, conservation of momentum and energy, and a plastic flow rule, we obtain a system of 19 equations

$$\mathbf{F} = \text{Grad } \mathbf{x}, \quad (2.4.27)$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = \text{Div } \mathbf{F} \boldsymbol{\mathcal{S}}, \quad (2.4.28)$$

$$\rho_0 T \frac{\partial S}{\partial t} = \frac{1}{2} J \text{tr} \left[\boldsymbol{\sigma} \mathbf{F}^{-T} \dot{\mathbf{C}}^p \mathbf{C}^{p-1} \mathbf{F}^T \right], \quad (2.4.29)$$

$$\frac{1}{2} \left(\mathbf{F}^{-T} \dot{\mathbf{C}}^p \mathbf{C}^{p-1} \mathbf{F}^T \right) = \sqrt{\frac{3}{2}} \Lambda \frac{\text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} \quad (2.4.30)$$

for the 25 scalar unknowns in $\mathbf{F}$, $\mathbf{u}$, $\boldsymbol{\mathcal{S}}$, S , and $\mathbf{C}^p$, where Λ is given by (2.4.26). To close the problem we must consider the derivation of the stress from the equation of state $\mathcal{W}$.

2.4.2 Equation of state

We restrict our attention to isotropic, non-hardening materials for which the strain energy $\mathcal{W}$ is a function of the elastic deformation $\mathbf{F}^e$ and the entropy S . The isotropy of the

material implies that the constitutive relation must be invariant under rigid body motions, that is

$$\mathcal{W}(\mathbf{F}^e, S) = \mathcal{W}(\mathbf{Q}\mathbf{F}^e, S) \quad (2.4.31)$$

for all orthogonal matrices $\mathbf{Q}$ [32, 55]. Since the elastic Cauchy-Green tensor $\mathcal{C}^e$ is clearly invariant under rigid body motions, we can instead write $\mathcal{W}$ in terms of the six components $\mathcal{C}^e$ and S , while satisfying (2.4.31). Therefore, referring to equation (2.4.11), we can write

$$\frac{\partial \mathcal{W}}{\partial F_{ij}^e} = 2F_{ik}^e \frac{\partial \mathcal{W}}{\partial \mathcal{C}_{kj}^e}. \quad (2.4.32)$$

The components of the second Piola-Kirchhoff stress tensor are then derivable from

$$\mathbf{S} = 2\mathbf{F}^{p-1} \frac{\partial \mathcal{W}}{\partial \mathcal{C}^e} \mathbf{F}^{p-T}, \quad (2.4.33)$$

and hence the Cauchy stress tensor is given by

$$\boldsymbol{\sigma} = \frac{2}{J} \mathbf{F}^e \frac{\partial \mathcal{W}}{\partial \mathcal{C}^e} \mathbf{F}^{eT}. \quad (2.4.34)$$

For an isotropic material, the strain energy density can be written as a function of the three isotropic strain invariants of $\mathcal{C}^e$ and the entropy [32, 55]. Hence

$$\mathcal{W} = \hat{\mathcal{W}}(I_1, I_2, I_3, S), \quad (2.4.35)$$

where

$$I_1 = \text{Tr}(\mathcal{C}^e), \quad (2.4.36)$$

$$I_2 = \frac{1}{2} [\text{Tr}(\mathcal{C}^e)^2 - \text{Tr}(\mathcal{C}^{e2})], \quad (2.4.37)$$

$$I_3 = \det(\mathcal{C}^e), \quad (2.4.38)$$

and

$$\frac{\partial I_1}{\partial \mathcal{C}^e} = \mathcal{I}, \quad (2.4.39)$$

$$\frac{\partial I_2}{\partial \mathcal{C}^e} = I_1 \mathcal{I} - \mathcal{C}^e, \quad (2.4.40)$$

$$\frac{\partial I_3}{\partial \mathcal{C}^e} = I_3 \mathcal{C}^{e-1}, \quad (2.4.41)$$

where $\mathcal{I}$ is the identity.

Using (2.4.39)–(2.4.41), we find that

$$\frac{\partial \mathcal{W}}{\partial \mathcal{C}_{ij}^e} = f_0 \delta_{ij} + f_1 C_{ji}^e + f_2 C_{ji}^{e-1} \quad (2.4.42)$$

where

$$f_0 = \frac{\partial \hat{\mathcal{W}}}{\partial I_1} + I_1 \frac{\partial \hat{\mathcal{W}}}{\partial I_2}, \quad f_1 = -\frac{\partial \hat{\mathcal{W}}}{\partial I_2}, \quad f_2 = I_3 \frac{\partial \hat{\mathcal{W}}}{\partial I_3}. \quad (2.4.43)$$

Hence we can write the components of the second Piola-Kirchoff tensor as

$$\mathcal{S} = 2\mathbf{F}^{p-1} (f_0 \mathcal{I} + f_1 \mathcal{C}^e + f_2 \mathcal{C}^{e-1}) \mathbf{F}^{p-T}, \quad (2.4.44)$$

$$= 2 (f_0 \mathcal{C}^{p-1} + f_1 \mathcal{C}^{p-1} \mathcal{C} \mathcal{C}^{p-1} + f_2 \mathcal{C}^{-1}), \quad (2.4.45)$$

which now depends only on $\mathcal{C}$ and $\mathcal{C}^p$, but not independently on $\mathcal{C}^e$. Thus (2.4.45) provides the final 6 constitutive equations to close the problem (2.4.27)–(2.4.30).

For future reference, we note that the Cauchy stress may be written as

$$\boldsymbol{\sigma} = \frac{2}{J} (f_0 \mathcal{B}^e + f_1 \mathcal{B}^{e2} + f_2 \mathcal{I}) \quad (2.4.46)$$

where

$$\mathcal{B}^e = \mathbf{F}^e \mathbf{F}^{eT} \quad (2.4.47)$$

is the left elastic Green deformation tensor, or the elastic Finger tensor [32].

2.4.3 Strain decomposition

As in §2.3.2, we decompose the elastic deformation multiplicatively into an isotropic and deviatoric part:

$$\mathbf{F}^e = J^{1/3} \tilde{\mathbf{F}}^e, \quad (2.4.48)$$

where

$$\det \tilde{\mathbf{F}}^e = 1 \quad (2.4.49)$$

so the shear deformation is volume preserving. The elastic Green deformation tensor and the elastic Finger tensor are then given by

$$\mathcal{C}^e = J^{2/3} \tilde{\mathbf{F}}^{eT} \tilde{\mathbf{F}}^e = J^{2/3} \tilde{\mathcal{C}}^e, \quad \mathcal{B}^e = J^{2/3} \tilde{\mathbf{F}}^e \tilde{\mathbf{F}}^{eT} = J^{2/3} \tilde{\mathcal{B}}^e. \quad (2.4.50)$$

Using the polar decomposition theorem [32, 55], $\tilde{\mathbf{F}}^e$ may be decomposed as

$$\tilde{\mathbf{F}}^e = \mathbf{P}^T \tilde{\Lambda} \mathbf{Q} = \mathbf{P}^T \tilde{\Lambda} \mathbf{P} \mathbf{R}, \quad \tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) \quad (2.4.51)$$

for orthogonal matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$, where $\tilde{\lambda}_i^2$ are the eigenvalues of $\tilde{\mathcal{C}}^e$. Analogously with (2.3.27), we define the principal deviatoric stretches to be

$$\tilde{\lambda}_i = \sqrt{1 + 2\mathcal{E}_i} \quad (2.4.52)$$

where $\mathcal{E}_i$ are the principal deviatoric strains, which are expected to be small. For future reference, the elastic Finger tensor becomes

$$\mathcal{B}^e = J^{2/3} (\mathcal{I} + 2\tilde{\mathcal{E}}), \quad \tilde{\mathcal{E}} = \mathbf{P}^T \text{diag}(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) \mathbf{P}. \quad (2.4.53)$$

Note that the deviatoric strain tensor $\tilde{\mathcal{E}}$ is symmetric.

The decomposition (2.4.48) leads to

$$I_1 = J^{2/3} [3 + 2(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)], \quad (2.4.54)$$

$$I_2 = J^{4/3} [3 + 4(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_3)], \quad (2.4.55)$$

$$I_3 = J^2(1 + 2\mathcal{E}_1)(1 + 2\mathcal{E}_2)(1 + 2\mathcal{E}_3) = J^2, \quad (2.4.56)$$

where, from (2.4.49), we get the consistency condition

$$(1 + 2\mathcal{E}_1)(1 + 2\mathcal{E}_2)(1 + 2\mathcal{E}_3) = 1. \quad (2.4.57)$$

Thus, we can instead write

$$\hat{\mathcal{W}}(I_1, I_2, I_3, S) = \tilde{\mathcal{W}}(J, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, S), \quad (2.4.58)$$

where, for the two descriptions to be consistent, we require

$$(\mathcal{E}_2 - \mathcal{E}_3) \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_1} + (\mathcal{E}_3 - \mathcal{E}_1) \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_2} + (\mathcal{E}_1 - \mathcal{E}_2) \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_3} \equiv 0. \quad (2.4.59)$$

For the constitutive relation (2.4.45), we require the partial derivatives $\partial \hat{\mathcal{W}} / \partial I_j$. By using the chain rule, we find

$$J \frac{\partial \tilde{\mathcal{W}}}{\partial J} = \frac{2}{3} \sum_{j=1}^3 j I_j \frac{\partial \hat{\mathcal{W}}}{\partial I_j}, \quad (2.4.60)$$

$$\frac{\partial \hat{\mathcal{W}}}{\partial I_1} = \frac{(1 + \mathcal{E}_i + \mathcal{E}_k) \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_i} - (1 + \mathcal{E}_j + \mathcal{E}_k) \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_j}}{2J^{2/3}(\mathcal{E}_i - \mathcal{E}_j)}, \quad (2.4.61)$$

for even permutations of $(i, j, k) = (1, 2, 3)$, and

$$\frac{\partial \hat{\mathcal{W}}}{\partial I_2} = \frac{1}{4(\mathcal{E}_i - \mathcal{E}_j)J^{4/3}} \left(\frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_j} - \frac{\partial \tilde{\mathcal{W}}}{\partial \mathcal{E}_i} \right), \quad (2.4.62)$$

for $i \neq j$.

We now Taylor expand (2.4.58) and (2.4.57) in powers of $\mathcal{E}_i$, requiring that $\mathcal{E}_i = 0$ be a global minimum of the elastic energy. We then find

$$\tilde{\mathcal{W}} \sim \tilde{W}_0(J, S) + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \mathcal{E}_i \mathcal{E}_j \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_i \partial \mathcal{E}_j} \Big|_{\mathcal{E}_k \equiv 0} + O(\|\mathcal{E}\|^3) \quad (2.4.63)$$

and

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + 2(\mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_3) = O(\|\mathcal{E}\|^3). \quad (2.4.64)$$

Equivalently, we can write (2.4.63) as a quadratic form, namely

$$\tilde{\mathcal{W}} \sim \tilde{\mathcal{W}}_0(J, S) + \frac{1}{2} \boldsymbol{\varepsilon}'^T \mathbf{D} \boldsymbol{\varepsilon}' \quad (2.4.65)$$

where

$$\mathbf{D} = \left[\begin{array}{ccc} \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_1^2} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_1 \mathcal{E}_2} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_1 \mathcal{E}_3} \\ \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_1 \mathcal{E}_2} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_2^2} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_2 \mathcal{E}_3} \\ \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_1 \mathcal{E}_3} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_2 \mathcal{E}_3} & \frac{\partial^2 \tilde{\mathcal{W}}}{\partial \mathcal{E}_3^2} \end{array} \right]_{\boldsymbol{\varepsilon}'=0}, \quad \boldsymbol{\varepsilon}' = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)^T. \quad (2.4.66)$$

Since we require $\mathcal{W}$ to be isotropic, and hence invariant under the permutation of the indices $i, j = 1, 2, 3$, we can write the matrix $\mathbf{D}$ in terms of two scalars α and β , that is

$$\mathbf{D} = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{bmatrix}. \quad (2.4.67)$$

We can restrict (2.4.65) further by using (2.4.64) to eliminate the diagonal entries of $\mathbf{D}$. That is, we can write

$$\tilde{\mathcal{W}} = \mathcal{W}_0(J, S) + \frac{1}{2} \boldsymbol{\varepsilon}'^T \hat{\mathbf{D}} \boldsymbol{\varepsilon}' \quad (2.4.68)$$

where

$$\hat{\mathbf{D}} = (\beta - \alpha) \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (2.4.69)$$

Thus, without loss of generality, we can set the trace of $\mathbf{D}$ to zero.

Finally, writing $\alpha - \beta = \Omega(J, S)$, we have

$$\tilde{\mathcal{W}} = \mathcal{W}_0(J, S) - \frac{1}{2} \Omega(J, S) \sum_{\substack{i,j=1 \\ i \neq j}}^3 \mathcal{E}_i \mathcal{E}_j. \quad (2.4.70)$$

Therefore

$$\frac{\partial \hat{\mathcal{W}}}{\partial I_1} = \frac{\Omega(J, S)}{2J^{2/3}}, \quad \frac{\partial \hat{\mathcal{W}}}{\partial I_2} = -\frac{\Omega(J, S)}{4J^{4/3}}, \quad \frac{\partial \hat{\mathcal{W}}}{\partial I_3} = \frac{1}{2J} \frac{\partial \mathcal{W}_0}{\partial J}(J, S) \quad (2.4.71)$$

and

$$f_0 = -\frac{\Omega(J, S)}{4J^{2/3}}, \quad f_1 = \frac{\Omega(J, S)}{4J^{4/3}}, \quad f_2 = \frac{J}{2} \frac{\partial \mathcal{W}_0}{\partial J} \quad (2.4.72)$$

up to $O(\|\boldsymbol{\varepsilon}\|^2)$, which we can now use to find the second Piola-Kirchhoff stress tensor (2.4.44)

$$\boldsymbol{\mathcal{S}} = J \left[\frac{\partial \mathcal{W}_0}{\partial J} \boldsymbol{\mathcal{I}} + \frac{\Omega(J, S)}{J} \tilde{\boldsymbol{\varepsilon}} \right] \boldsymbol{\mathcal{C}}^{-1} + O(\|\boldsymbol{\varepsilon}\|^2) \quad (2.4.73)$$

and the Cauchy stress tensor (2.4.46)

$$\boldsymbol{\sigma} = \frac{\partial \mathcal{W}_0}{\partial J} \mathbf{I} + \frac{\Omega(J, S)}{J} \tilde{\boldsymbol{\varepsilon}} + O(\|\boldsymbol{\varepsilon}\|^2) \quad (2.4.74)$$

where $\tilde{\boldsymbol{\varepsilon}}$ is as defined in (2.4.53).

Finally, we recall the plastic flow rule (2.4.24), which may alternatively be written as

$$\mathcal{D}^p = \sqrt{\frac{3}{2}} \Lambda \mathbf{F}^{eT} \frac{\text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} \mathbf{F}^{e-T} \equiv \sqrt{\frac{3}{2}} \Lambda \mathbf{F}^{e-1} \frac{\text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} \mathbf{F}^e \quad (2.4.75)$$

where the (symmetric) plastic rate-of-strain tensor is

$$\mathcal{D}^p = \frac{1}{2} \left(\dot{\mathbf{F}}^p \mathbf{F}^{p-1} + \mathbf{F}^{p-T} \dot{\mathbf{F}}^{pT} \right). \quad (2.4.76)$$

If we now rearrange (2.4.75), it can be shown that

$$\begin{aligned} \sqrt{\frac{3}{2}} \frac{\Lambda \text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} &= \frac{1}{2} \left(\mathbf{F}^{-T} \dot{\mathcal{C}}^p \mathcal{C}^{p-1} \mathbf{F}^T \right) = \frac{1}{2} (\boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}^T) - \frac{\dot{J}}{3J} \mathbf{I} - \left(\dot{\tilde{\boldsymbol{\varepsilon}}} + \tilde{\boldsymbol{\varepsilon}} \boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}} \tilde{\boldsymbol{\varepsilon}} \right), \quad (2.4.77) \\ &\equiv \frac{1}{2} (\boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}^T) - \frac{\dot{J}}{3J} \mathbf{I} - \left(\dot{\tilde{\boldsymbol{\varepsilon}}} - \tilde{\boldsymbol{\varepsilon}} \boldsymbol{\mathcal{L}}^T + \boldsymbol{\mathcal{L}}^T \tilde{\boldsymbol{\varepsilon}} \right), \quad (2.4.78) \end{aligned}$$

where

$$\boldsymbol{\mathcal{L}} = \dot{\mathbf{F}} \mathbf{F}^{-1} = \nabla \mathbf{u}^T \quad (2.4.79)$$

is the velocity gradient. Taking the average of (2.4.77) and (2.4.78) leads to

$$\sqrt{\frac{3}{2}} \frac{\Lambda \text{dev } \boldsymbol{\sigma}}{\|\text{dev } \boldsymbol{\sigma}\|} = \mathcal{D} - \frac{\dot{J}}{3J} \mathbf{I} - \dot{\tilde{\boldsymbol{\varepsilon}}} \quad (2.4.80)$$

where the (symmetric) rate-of-strain tensor and the (anti-symmetric) rotation tensor are

$$\mathcal{D} = \frac{1}{2} (\boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{L}}^T) \quad \text{and} \quad \boldsymbol{\Omega} = \frac{1}{2} (\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}^T) \quad (2.4.81)$$

respectively. The operator

$$\dot{\tilde{\boldsymbol{\varepsilon}}} = \frac{\partial \tilde{\boldsymbol{\varepsilon}}}{\partial t} \Big|_{\mathbf{x}} - \boldsymbol{\Omega} \tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}} \boldsymbol{\Omega} \quad (2.4.82)$$

is the so-called *Jaumann derivative* which, unlike the conventional convective derivative, is objective, and hence is the same for all observers.⁷ We can therefore write (2.4.80) as

$$\text{dev } \mathcal{D} - \dot{\tilde{\boldsymbol{\varepsilon}}} = \sqrt{\frac{3}{2}} \frac{\Lambda \tilde{\boldsymbol{\varepsilon}}}{\|\tilde{\boldsymbol{\varepsilon}}\|} \quad (2.4.83)$$

where we have used the identity

$$\dot{J} = J \text{tr} \left(\dot{\mathbf{F}} \mathbf{F}^{-1} \right) = J \text{tr}(\boldsymbol{\mathcal{L}}). \quad (2.4.84)$$

⁷The Jaumann derivative (2.4.82) is a special case of the more general convected derivative $\hat{\mathcal{G}} = \text{D}\mathcal{G}/\text{D}t - \boldsymbol{\Omega}\mathcal{G} + \mathcal{G}\boldsymbol{\Omega} - \mu(\mathcal{D}\mathcal{G} + \mathcal{G}\mathcal{D})$ which appears frequently in models pertaining to viscoelasticity. The Jaumann derivative corresponds to the choice of parameter $\mu = 0$. For a general viscoelastic material, the value of μ has to be determined experimentally.

2.4.4 Summary

In an analogous way to §2.3.4, we find that entropy changes are universally negligible in the absence of shocks, and thus our three-dimensional model consists of the following 24 equations:

$$\dot{\mathbf{F}} = \text{Grad } \mathbf{u}, \quad (2.4.85)$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = \text{Div } J \boldsymbol{\sigma} \mathbf{F}^{-T}, \quad (2.4.86)$$

$$\text{dev } \mathcal{D} - \overset{\circ}{\mathcal{E}} = \sqrt{\frac{3}{2}} \frac{\Lambda \tilde{\mathcal{E}}}{\|\tilde{\mathcal{E}}\|}, \quad \Lambda = \sqrt{\frac{3}{2}} r \left[\frac{\|\tilde{\mathcal{E}}\|}{\mathcal{E}_Y} - \sqrt{\frac{2}{3}} \right]^+, \quad (2.4.87)$$

$$\boldsymbol{\sigma} = \frac{\partial \mathcal{W}_0}{\partial J}(J, S_0) \mathcal{I} + \frac{\Omega(J, S_0)}{J} \tilde{\mathcal{E}}, \quad (2.4.88)$$

for the 24 unknowns $\mathbf{F}$, $\mathbf{u}$, $\tilde{\mathcal{E}}$, and $\boldsymbol{\sigma}$, where S_0 is the initial entropy and we define

$$\mathcal{E}_Y = \frac{J \sigma_Y}{\Omega} \quad (2.4.89)$$

to be the yield strain.

The model (2.4.85)–(2.4.87) may alternatively be written in Eulerian coordinates:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.4.90)$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = \nabla \cdot \boldsymbol{\sigma}, \quad (2.4.91)$$

$$\text{dev } \mathcal{D} - \overset{\circ}{\mathcal{E}} = \sqrt{\frac{3}{2}} \frac{\Lambda \tilde{\mathcal{E}}}{\|\tilde{\mathcal{E}}\|}, \quad \Lambda = \sqrt{\frac{3}{2}} r \left[\frac{\|\tilde{\mathcal{E}}\|}{\mathcal{E}_Y} - \sqrt{\frac{2}{3}} \right]^+, \quad (2.4.92)$$

where $\rho = \rho_0/J$. Note that we now have the operators

$$\text{dev } \mathcal{D} = \mathcal{D} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \mathcal{I} \quad \text{and} \quad \overset{\circ}{\mathcal{E}} = \frac{\partial \tilde{\mathcal{E}}}{\partial t} + (\mathbf{u} \cdot \nabla) \tilde{\mathcal{E}} + \boldsymbol{\Omega} \tilde{\mathcal{E}} + \tilde{\mathcal{E}} \boldsymbol{\Omega}. \quad (2.4.93)$$

2.4.5 Uniaxial strain

As confirmation of the consistency of our model, we briefly demonstrate that the model (2.4.85)–(2.4.88) reduces to the one-dimensional model of §2.3.

In uniaxial strain, the deformation gradient, velocity, and deviatoric strain tensor are

$$\mathbf{F} = \text{diag}(\nu, 1, 1), \quad \mathbf{u} = \text{diag}(u, 0, 0), \quad \tilde{\mathcal{E}} = \text{diag} \left(\mathcal{E}, -\frac{\mathcal{E}}{2}, -\frac{\mathcal{E}}{2} \right), \quad (2.4.94)$$

where the specific volume ν , the axial velocity u , and the deviatoric strain $\mathcal{E}$ are all functions of the Lagrangian variable X and time t . Using (2.4.94) and (2.4.88), we find

$$\text{dev } \mathcal{D} = \text{diag} \left(\frac{2}{3\nu} \frac{\partial \nu}{\partial t}, -\frac{1}{3\nu} \frac{\partial \nu}{\partial t}, -\frac{1}{3\nu} \frac{\partial \nu}{\partial t} \right), \quad \overset{\circ}{\mathcal{E}} = \text{diag} \left(\frac{\partial \mathcal{E}}{\partial t}, -\frac{1}{2} \frac{\partial \mathcal{E}}{\partial t}, -\frac{1}{2} \frac{\partial \mathcal{E}}{\partial t} \right), \quad (2.4.95)$$

and

$$\sigma_{11} = \frac{\partial \mathcal{W}_0}{\partial \nu}(\nu, S_0) + \frac{\Omega(\nu, S_0)}{\nu} \mathcal{E}, \quad \sigma_{11} - \sigma_{22} = \frac{3\Omega(\nu, S_0)}{2\nu} \mathcal{E}. \quad (2.4.96)$$

Comparing (2.4.96) with (2.3.39) we see that

$$\frac{W_0(\nu, S_0)}{K_b} = h(\nu) \quad \text{and} \quad \frac{\Omega(\nu, S_0)}{K_b} = \frac{2k(\nu)}{3}. \quad (2.4.97)$$

Using the above relations, and after non-dimensionalising in the same fashion as §2.3.3, it is straightforward to see that Equations (2.4.85)–(2.4.87) become

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0, \quad (2.4.98)$$

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial X} \left(h'(\nu) + \delta \frac{2k(\nu)}{3\nu} \varepsilon \right) = 0, \quad (2.4.99)$$

$$a \left(\frac{2}{3\nu} \frac{\partial \nu}{\partial t} - \delta \frac{\partial \varepsilon}{\partial t} \right) = G \left(\frac{\varepsilon}{\varepsilon_c(\nu)} \right), \quad (2.4.100)$$

where we have scaled $\mathcal{E} = \delta \varepsilon$, and

$$\varepsilon_c(\nu) = \frac{\mathcal{E}_Y}{\sigma_{Y0}}. \quad (2.4.101)$$

Hence our three-dimensional formulation reduces to (2.3.54)–(2.3.56) in uniaxial strain.

2.5 Comparison with hydrocode models

To enhance the utility of our model in industry, it is useful to make a comparison between it and the models currently employed in industry standard computational hydrocodes [3]. The main purpose of this investigation is to reconcile any differences in modelling philosophy, so that an equivalence can be established between the two models. For simplicity, we focus our attention on comparing the two models in the one-dimensional, uniaxial framework.

2.5.1 The strain tensor

We begin our discussion by considering the definition of the strain tensor $\mathbf{E}$. In §2.3 and §2.4, we adopt a multiplicative decomposition of the deformation gradient as in (2.3.4), so the strain tensor is written as

$$\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathcal{I}) = \frac{1}{2} (\mathbf{F}^{pT} \mathbf{F}^{eT} \mathbf{F}^e \mathbf{F}^p - \mathcal{I}). \quad (2.5.1)$$

In contrast, models currently employed in industrial hydrocodes assume a linearisation of the strain tensor, followed by a further additive decomposition into its elastic and plastic parts; that is

$$\mathbf{E} = \frac{1}{2} (\nabla \mathbf{q} + \nabla \mathbf{q}^T) = \mathbf{E}^e + \mathbf{E}^p \quad (2.5.2)$$

where $\mathbf{q} = \mathbf{x} - \mathbf{X}$ is the displacement. However, it is well known from the theory of elasticity (see, for example, [32, Chp. 2]) that a linearisation is only truly justified when the deformation is limited to small displacements, and hence higher order terms in (2.5.1) may be neglected.

In the hydrocode model, the elastic part of the strain tensor is further decomposed into a volumetric and deviatoric part

$$\mathbf{E}^e = e^v \mathcal{I} + \mathbf{e} \quad (2.5.3)$$

where the deviator is trace-free

$$\text{tr } \mathbf{e} = 0. \quad (2.5.4)$$

The plastic deformation is assumed to be incompressible and hence is also trace-free

$$\text{tr } \mathbf{E}^p = 0. \quad (2.5.5)$$

Equations (2.5.4) and (2.5.5) may be used to eliminate e_{22} and E_{22}^p and then, after some rearranging, we obtain

$$e_{11} = 2e^v - E_{11}^p, \quad (2.5.6)$$

$$e_{22} = -e^v + \frac{1}{2}E_{11}^p, \quad (2.5.7)$$

$$E_{22}^p = -\frac{1}{2}E_{11}^p, \quad (2.5.8)$$

$$e_{22} - e_{11} = -3e^v + \frac{3}{2}E_{11}^p. \quad (2.5.9)$$

We now recall that the Lagrangian deformation gradient is given by (2.3.4) and hence from (2.5.1) we have

$$E_{11} = \frac{\nu - 1}{\nu}, \quad E_{22} = E_{33} = 0, \quad (2.5.10)$$

which leads to

$$e^v = \frac{1}{3} \frac{\nu - 1}{\nu}, \quad (2.5.11)$$

$$e_{11} = \frac{2}{3} \frac{\nu - 1}{\nu} - E_{11}^p, \quad (2.5.12)$$

$$e_{22} = -\frac{1}{3} \frac{\nu - 1}{\nu} + \frac{1}{2}E_{11}^p, \quad (2.5.13)$$

$$E_{22}^p = -\frac{1}{2}E_{11}^p, \quad (2.5.14)$$

$$e_{22} - e_{11} = -\frac{1}{3} \frac{\nu - 1}{\nu} + \frac{3}{2}E_{11}^p. \quad (2.5.15)$$

Hence, through (2.5.11)–(2.5.15), an equivalence between the components of the strain tensors (2.5.1) and (2.5.2) may be established.

2.5.2 Constitutive modelling

The definition of the Cauchy stress tensor in the hydrocode model is given by an additive decomposition into an isotropic and deviatoric part

$$\boldsymbol{\sigma} = -p(\nu, U)\mathbf{I} + \mathbf{s} \quad (2.5.16)$$

where U is the internal energy and the deviatoric stress $\mathbf{s}$ is trace-free:

$$\text{tr } \mathbf{s} = 0. \quad (2.5.17)$$

The deviatoric stress is given by the shear strength model

$$\mathbf{s} = 2\mu\mathbf{e}, \quad (2.5.18)$$

where μ is the shear modulus. The equation of state is given by a Mie-Grüneisen equation of state [20] of the form

$$p(\nu, U) = p_{\text{ref}}(\nu) + \rho_0 \frac{\Gamma_{\text{ref}}(\nu)}{\nu} (U - U_{\text{ref}}(\nu)) \quad (2.5.19)$$

where $p_{\text{ref}}(\nu)$ is a reference curve. We note that (2.5.19) is nothing more than a Taylor expansion of p about $U = U_{\text{ref}}$, if the *Grüneisen coefficient* [20] is

$$\Gamma_{\text{ref}} = \rho_0 \nu \left. \frac{\partial p}{\partial U} \right|_{\nu}. \quad (2.5.20)$$

By making the definition

$$p_{\text{ref}}(\nu) = \left. \frac{1}{\rho_0} \frac{\partial U_{\text{ref}}}{\partial \nu} \right|_S \quad (2.5.21)$$

the reference curve is identified as the principal isentrope, which is defined as the curve in state space on which S is constant and equal to its initial value S_0 (see Figure 1.1(a)). We note that other choices of reference curve are possible, such as the principal Hugoniot.⁸ However the choice (2.5.21) will allow a direct comparison between the hydrocode model and our elastic-plastic model to be made.

By Taylor expanding $p(\nu, S)$ about the principal isentrope and using the thermodynamic relations

$$\left. \frac{\partial p}{\partial S} \right|_{\nu} = \left. \frac{\partial p}{\partial U} \right|_{\nu} \left. \frac{\partial U}{\partial S} \right|_{\nu} = T \left. \frac{\partial p}{\partial U} \right|_{\nu} = \frac{T}{\nu} \Gamma, \quad (2.5.22)$$

we can alternatively write (2.5.19) in terms of ν and S ; hence

$$p(\nu, S) = p_{\text{ref}}(\nu) + (S - S_0) \left. \frac{\partial p}{\partial S} \right|_{\nu}(\nu, 0), \quad (2.5.23)$$

$$= p_{\text{ref}}(\nu) + \frac{T_{\text{ref}}(\nu) \Gamma_{\text{ref}}(\nu)}{\nu} S, \quad (2.5.24)$$

⁸The principal Hugoniot is obtained if the initial state of the material prior to shock loading is the unstressed, undeformed configuration.

where $S = 0$ corresponds to the principal isentrope and T is the temperature. The relationship between the entropy and the internal energy is therefore

$$T_{\text{ref}}(\nu)S = U - U_{\text{ref}}(\nu). \quad (2.5.25)$$

We now recall the equations (2.3.39) for the stress components, which in dimensional form read

$$\sigma_{11} = K_b \left(h'(\nu) + \delta\theta'(\nu)s + \frac{2\delta k(\nu)}{3\nu}\varepsilon \right), \quad (2.5.26)$$

$$\sigma_{11} - \sigma_{22} = K_b \frac{\delta k(\nu)}{\nu} \varepsilon. \quad (2.5.27)$$

Comparing (2.5.26) with (2.5.24) leads to the identifications

$$h'(\nu) = \frac{1}{K_b} p_{\text{ref}}(\nu), \quad \theta'(\nu) = \frac{S_0}{c_0^2} T_{\text{ref}}(\nu) \frac{\Gamma_{\text{ref}}(\nu)}{\nu}. \quad (2.5.28)$$

Hence, the function $h'(\nu)$ corresponds to the pressure on the principal isentrope, while $\theta'(\nu)$ defines the off-isentrope part of the equation of state.

2.5.3 Deviatoric stresses and strains

In the hydrocode model, the transverse stress is given by

$$\sigma_{11} - \sigma_{22} = -2\mu(e_{11} - e_{22}) = 2\mu \left(\frac{\nu - 1}{\nu} - \frac{3}{2} E_{11}^p \right) \quad (2.5.29)$$

where we recall that e_{11} and e_{22} are the deviatoric contributions to the elastic strain. Comparing (2.5.29) to (2.5.27) gives

$$K_b \frac{\delta k(\nu)}{\nu} \varepsilon = 2\mu \left(\frac{\nu - 1}{\nu} - \frac{3}{2} E_{11}^p \right). \quad (2.5.30)$$

In order to obtain a relationship between the function k and μ , we consider the case where there is no plastic deformation, and hence

$$E_{11}^p = 0, \quad F_{11}^p = 1. \quad (2.5.31)$$

We therefore have that

$$F_{11}^p = \frac{\nu}{F_{11}^e} = \frac{\nu^{2/3}}{\sqrt{1 + 2\delta\varepsilon}} = 1 \quad (2.5.32)$$

and therefore

$$\varepsilon = \frac{\nu^{4/3} - 1}{2\delta}. \quad (2.5.33)$$

Substituting into (2.5.30), we obtain

$$k(\nu) = \frac{4\mu}{K_b} \left(\frac{\nu - 1}{\nu^{4/3} - 1} \right). \quad (2.5.34)$$

Hence for small compressions when $\nu \sim 1$

$$k(\nu) \sim 3 \frac{\mu}{K_b} = k_0 \quad (2.5.35)$$

and for large compressions as $\nu \rightarrow 0$

$$k(\nu) \sim 4 \frac{\mu}{K_b}. \quad (2.5.36)$$

We note that the equivalence (2.5.35) means that we can write the elastic wave speed (2.3.53) as

$$c_e = \frac{1}{c_0} \sqrt{\frac{K_b + 4\mu/3}{\rho_0}} = \frac{1}{c_0} \sqrt{\frac{\lambda + 2\mu}{\rho_0}}, \quad (2.5.37)$$

where, in the final equality, we have used the relation $K_b = \lambda + 2\mu/3$ between the bulk modulus and the Lame moduli. Hence (2.3.53) is consistent with the classical p-wave speed of an elastic solid [32].

2.5.4 Yield stress and plastic flow rule

The yield condition used in hydrocode calculations is the von-Mises yield criterion

$$Y(\boldsymbol{\sigma}) = s_{ij}s_{ij} - \sqrt{\frac{2}{3}}\sigma_Y \leq 0, \quad (2.5.38)$$

where we recall from (2.5.16) that s_{ij} are the components of the deviatoric stress tensor. The yield stress σ_Y is generally an increasing function of the plastic strain (to allow for the inclusion of phenomena such as work hardening) and the entropy. For example, the simplest and most rudimentary yield strength model of this type is the reduced Johnson-Cook model [35]

$$\sigma_Y = \sigma_{Y0} \left(1 + c \log \left(\frac{|\dot{\epsilon}^p|}{\dot{\epsilon}_0^p} \right) \right), \quad (2.5.39)$$

where $c > 0$ is a constant, and the quantity

$$\dot{\epsilon}^p = \sqrt{\frac{2}{3} \dot{E}_{ij}^p \dot{E}_{ij}^p} \quad (2.5.40)$$

is the so-called equivalent plastic strain rate. The constant $\dot{\epsilon}_0^p > 0$ is a threshold value of the strain rate above which (2.5.39) is valid. Note that, in uniaxial strain, (2.5.40) simply reduces to

$$\dot{\epsilon}^p = \dot{E}_{11}^p. \quad (2.5.41)$$

The plastic flow rule used in hydrocode calculations is of Reuss type

$$\dot{E}_{ij}^p = \Lambda \frac{\partial Y}{\partial \sigma_{ij}}, \quad (2.5.42)$$

where the function $\Lambda = 0$ during elastic deformation, and $\Lambda \geq 0$ while the material is deforming plastically. The total strain rate is then given by

$$\dot{E}_{ij} = \dot{\epsilon}^v \delta_{ij} + \dot{\epsilon}_{ij} + \dot{E}_{ij}^p. \quad (2.5.43)$$

Equations (2.5.43) are solved numerically by calculating incremental stresses and strains through a series of time-steps Δt , where the discretised version of (2.5.43) is commonly referred to as the Prandtl-Reuss model [32]. At each time-step, the components of the deviatoric stress tensor are constrained to the yield surface using Wilkin's radial return algorithm [3, 87]. The basic idea behind this method is to compute a trial value for the stress, assuming that the material is elastic. If this value is found to exceed the yield stress, then the stress is projected back onto the yield surface via a path running along a line perpendicular to the von Mises yield circle, which lies in the π -plane $s_{11} + s_{22} + s_{33} = 0$ (see, for example, [87, Fig. 1(b)] or [32, Chp. 8]).

In an attempt to reconcile this approach toward plastic flow and the one set out in §2.3, we demonstrate a procedure of how models of the type (2.5.39) may be incorporated into the model (2.3.54)–(2.3.56). In uniaxial strain, the Johnson-Cook model (2.5.39) gives

$$\frac{|\sigma_{11} - \sigma_{22}|}{\sigma_{Y0}} \leq 1 + c \log \left(\frac{|\dot{\epsilon}^p|}{\dot{\epsilon}_0^p} \right). \quad (2.5.44)$$

Note, however, that this model fails when $|\dot{\epsilon}^p|$ is small, since (2.5.44) implies that $|\sigma_{11} - \sigma_{22}|$ can become negative. Thus, we posit that the model (2.5.44) can only hold when the deviatoric stress exceeds the yield stress σ_{Y0} . This then suggests a flow rule of the form

$$\dot{\epsilon}^p = \begin{cases} 0 & \text{for } |\sigma_{11} - \sigma_{22}| \leq \sigma_{Y0}, \\ r \operatorname{sgn} \left(\frac{\sigma_{11} - \sigma_{22}}{\sigma_{Y0}} \right) \exp \left[\beta \left(\left| \frac{\sigma_{11} - \sigma_{22}}{\sigma_{Y0}} \right| - 1 \right) \right] & \text{for } |\sigma_{11} - \sigma_{22}| > \sigma_{Y0}, \end{cases} \quad (2.5.45)$$

where $\beta = 1/c$ and $r = \dot{\epsilon}_0^p$. If we now identify $\dot{\epsilon}^p$ with $\dot{F}_{11}^p / F_{11}^p$ in (2.3.24) we arrive at

$$\frac{1}{F_{11}^p} \frac{\partial F_{11}^p}{\partial t} = r H \left(\frac{\sigma_{11} - \sigma_{22}}{\sigma_Y} \right) \quad (2.5.46)$$

where the function H is shown in Figure 2.4. We note that in this case, by analogy with (2.3.22), the function

$$\Lambda = r \mathcal{H}(x - 1) \exp [\beta ([x]^+ - 1)] \quad (2.5.47)$$

where $\mathcal{H}$ is the Heaviside function and $[x]^+$ is defined by (2.3.23).

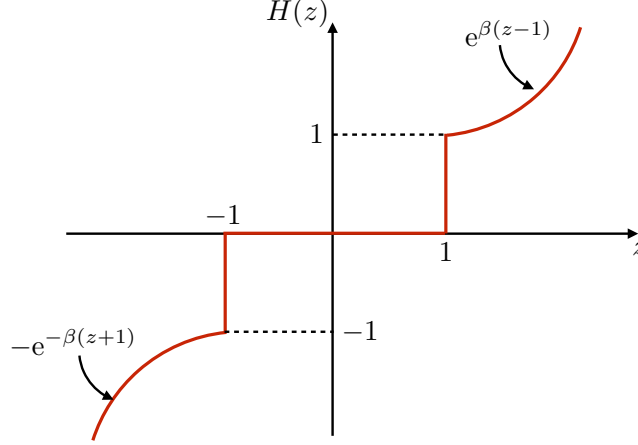


Figure 2.4: Schematic of the yield function $H(z)$ versus z .

2.6 Conclusions

In §2.3 we derived a one-dimensional mathematical model for elastoplasticity applicable when the applied stress is much greater than the yield stress. This ratio is manifested through the non-dimensional parameter δ which is assumed to be small throughout. Our model is valid in the absence of shocks, which is important since isentropic compression experiments are designed in such a way that shock formation occurs outwith the lifetime of the experiment. However the methodology we apply here may be easily generalised to incorporate shocks by reinstating the energy equation (2.3.12) to account for jumps in the entropy S . In the limit $\delta \rightarrow 0$, our model reduces to the equations of barotropic flow with an equation of state relating the pressure to the density. In regions where the strain is $O(\delta)$, our model predicts that the deviatoric and isotropic contributions to the strain are comparable, hence we expect elastic effects to be noticeable near the tails of propagating elastic waves.

Using the one-dimensional model as a guide, we then generalised our formulation to a fully three-dimensional model in §2.4. This model is complemented with a generalisation of the equation of state $\mathcal{W}$, which reduces to a leading order function $\mathcal{W}_0$ of the specific volume and the entropy in the limit of vanishingly small deviatoric strain. The three-dimensional formulation greatly expands the applicability of our model, and could in principle find use in the modelling of other high strain-rate phenomena such as the penetration of a metal target by shaped charge device (see chapter 6).

In order to enhance the utility of our model for AWE, in §2.5 we compared our uniaxial model with the “hydrodynamic-plus-strength” models currently employed in industry standard hydrocodes [3]. A principal difference between the models is the decomposition of the strain tensor, which is done additively in hydrocode models. However, this

decomposition is only valid in the limit of small displacements of the material. In comparison, we adopt a multiplicative decomposition of the deformation gradient and hence implicitly retain higher order terms in the strain tensor which are neglected when making the additive decomposition. Nevertheless, an equivalence between the components of the stress and strain tensors can be made in the appropriate limits which is intended to aid the future application of our model. Another principal difference between the two models is the approach taken in accounting for plastic flow. The hydrocode model employs a Prandtl-Reuss flow model combined with Wilkin's radial return algorithm to constrain the components of the deviatoric stress to the yield surface. To reconcile our approach with the hydrocode model, we proposed a modification to the flow rule (2.3.24), namely Equation (2.5.46), in which the function H is an exponentially increasing function of the amount by which the deviatoric stress exceeds the yield stress. If opportunity permits, a validation of this model against numerical results from hydrocode calculations could be made in the future.

Chapter 3

Numerical methods

3.1 Synopsis

In this chapter we exploit contemporary numerical methods to explore the predictions of the one-dimensional model derived in chapter 2 for the elastic-plastic deformation of a metal sample subject to a compressive force at $X = 0$. This analysis will allow us to investigate the nature of the elastic and plastic waves that can propagate through the sample, and how these waves give rise to the free surface velocity profile obtained from a typical ICE. Furthermore, to enhance the utility of our model, we will compare it to solutions obtained from finite-element based hydrocodes ubiquitously employed in industry. A hydrocode package based on the work of Barlow [3] was used by the author to compute the numerical solutions presented in §3.4 via an AWE finite-element implementation.

The author would like to thank the authors of [19] for providing the experimental data used in Figure 3.13.

3.2 Unidirectional wave propagation

The governing p.d.e (2.3.54)–(2.3.56) are of hyperbolic type, so we use the method of Kurganov and Tadmor [39] to compute numerical solutions. Unless otherwise stated, a grid spacing of $\Delta X = 0.001$ is used in the numerical calculations throughout this chapter. We refer the reader to Appendix A for validation that decreasing grid size any further offers no significant refinement of the solution.

3.2.1 Compressive wave propagation

Our goal is to simulate the conditions under which a material is compressed during violent impact testing. The simplest initial and boundary conditions we can impose to replicate

this scenario are

$$\nu = 1, \quad u = 0, \quad \varepsilon = 0 \quad \text{at} \quad t = 0, \quad X > 0, \quad (3.2.1)$$

$$u = U_0 (1 - e^{-t}) \quad \text{at} \quad X = 0, \quad t > 0. \quad (3.2.2)$$

These correspond to an initially unstressed semi-infinite sample in $X \in (0, \infty)$, subject to an imposed velocity at the boundary $X = 0$ which ramps up to a constant value U_0 . To proceed with this forward problem, we also need to make specific choices for the constitutive functions h , k and ε_c . We note that a wide variety of empirical constitutive relations may be found in the literature [20, 24, 71, 72], the more sophisticated of which attempt to incorporate microscopic behaviour at the dislocation level, such as work hardening and elastic precursor decay. However, for simplicity, and to compare with a leading order barotropic flow model that resembles homentropic gas dynamics, we make the following constitutive assumptions:

$$h(\nu) = \frac{1}{\gamma(\gamma - 1)\nu^{\gamma-1}}, \quad \varepsilon_c(\nu) = 1, \quad k(\nu) = k_0 \quad (3.2.3)$$

with $k_0 > 0$ and $\gamma > 1$ constant. We stress that the constitutive functions and parameter values chosen here are for convenience, and hence should not be interpreted in the context of any particular material. We also note from (2.3.45) that the choice of ε_c corresponds to a non-constant yield stress σ_Y , which is a function of the specific volume ν .

Figure 3.1 shows plots of numerical solutions for (a) the velocity $u(X, t)$, (b) the specific volume $\nu(X, t)$, and (c) the deviatoric strain $\varepsilon(X, t)$. The solid red curves show solutions to the full elastic-plastic model (2.3.54)–(2.3.56), while the solid blue curves in (a) and (b) show the corresponding numerical solutions of the barotropic flow model, where δ is set to zero (see the caption for a summary of the parameter values used in these computations). These blue curves show the expected propagation of a large-amplitude plastic wave into the undisturbed material as the velocity at $X = 0$ is ramped up. Initially, the elastic-plastic solution remains close to the barotropic flow prediction, apart from a small elastic precursor wave which propagates ahead of the much larger plastic wave. This confirms our expectation that elastic effects should be noticeable in regions where the strain is small, while plastic effects dominate wherever the strain is $O(1)$. Figure 3.1(c) shows that the deviatoric strain varies rapidly from zero to -1 across the elastic wave front. Between the elastic and plastic wave fronts, there is a “plateau” region where $\varepsilon \approx -1$ and the material remains close to the yield surface, before fully yielding when the plastic wave arrives. We note that the numerical experiment is terminated before a shock has had time to form, at which point the model (2.3.54)–(2.3.56) becomes invalid.

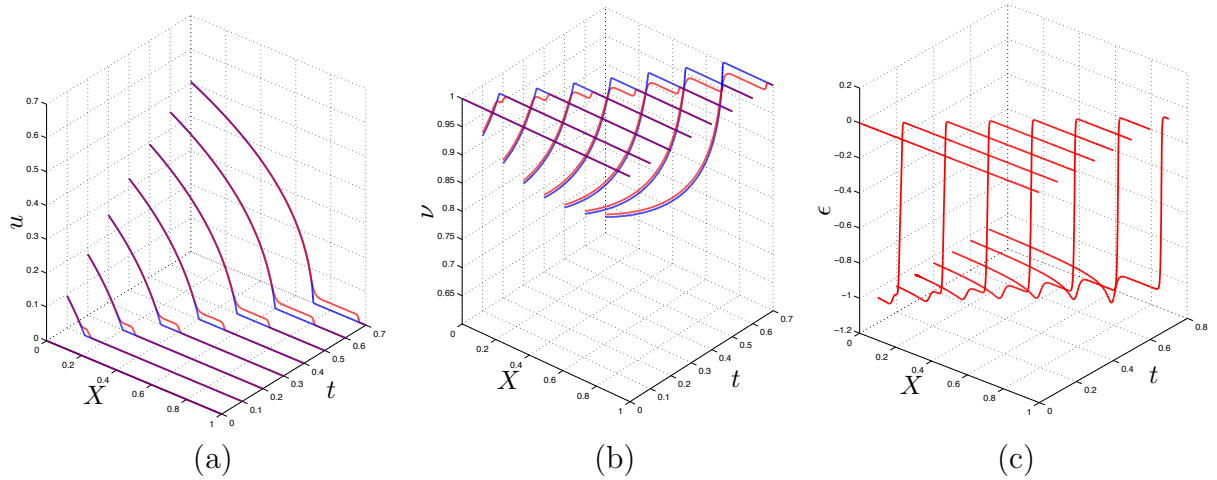


Figure 3.1: Plots of the numerical solutions (solid red) of the problem (2.3.54)–(2.3.56) subject to (3.2.1)–(3.2.2) for (a) velocity u , (b) specific volume ν , (c) deviatoric strain ε , plotted versus Lagrangian position X at times $t = 0, 0.1, \dots, 0.7$. The parameter values are $U_0 = 1$, $\gamma = 5/3$, $k_0 = 2$, $a = 0.1$, $\delta = 0.01$. In (a) and (b), the solid blue curves show the corresponding results for the barotropic flow model (with $\delta = 0$).

In Figure 3.2, we plot the axial velocity u at $t = 0.5$ to explore the consequences of varying the parameters (a) δ (at fixed $k_0 = 2$), and (b) k_0 (at fixed $\delta = 0.01$). All other parameter values are unchanged from Figure 3.1. Figure 3.2(a) shows that the amplitude of the elastic precursor decreases as δ decreases, with the solution approaching the barotropic flow solution as $\delta \rightarrow 0$. The solutions are otherwise identical, except from in those regions where $u = O(\delta)$. In Figure 3.2(b), we observe that varying k_0 leads to a larger amplitude, faster moving elastic precursor (since the elastic wave speed c_e varies with k_0).

To elucidate the influence of the rate parameter a , in Figure 3.3 we plot (a) the deviatoric strain ε and (b) the normalised deviatoric stress $(\sigma_{11} - \sigma_{22})/\sigma_{Y0}$ versus X for different values of the rate parameter a at fixed time ($t = 0.5$). In both plots the dashed grey line corresponds to the rate-independent solution $a = 0$. The coloured lines all correspond to different values of a (see legend). As a is increased we see that the departure of ε and the deviatoric stress from the yield surface increases, and hence the time scale over which the material relaxes back to the yield surface upon plasticising also increases. The question of what constitutes a suitable value of a will be addressed in §3.4.2.

3.2.2 Loading and unloading

We now investigate the response of the material when subject to an applied velocity which is at first increased and then decreased after reaching a maximum value $u = T$.

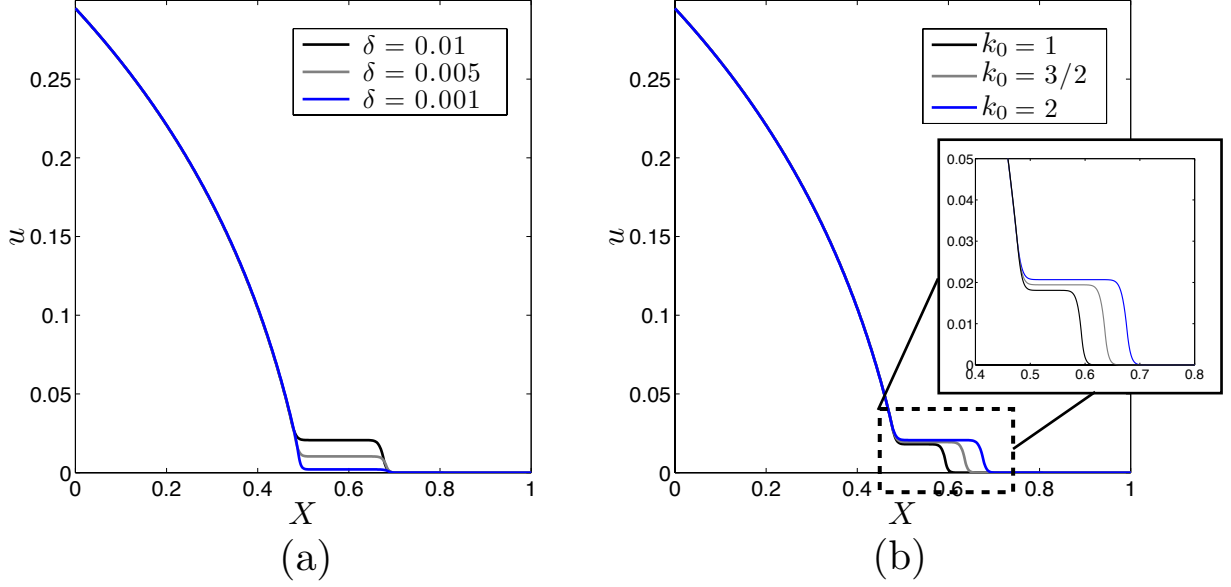


Figure 3.2: Plots of the axial velocity u versus the Lagrangian variable X at $t = 0.5$ for varying (a) δ and (b) k_0 . All other parameters values are equal to those stated in Figure 3.1.

More specifically, we impose the simple boundary condition

$$u = \begin{cases} t & \text{for } 0 \leq t \leq T, \\ T - \lambda(t - T) & \text{for } T < t \leq T(\lambda + 1)/\lambda, \\ 0 & \text{for } t > T(\lambda + 1)/\lambda. \end{cases}$$

The limit $\lambda \rightarrow 0$ corresponds to a velocity which is increased to its maximum value and then held fixed at this value, while $\lambda \rightarrow \infty$ corresponds to a rapid deceleration of the face $X = 0$.

In Figure 3.4 we plot a heat map of the deviatoric strain $\varepsilon(X, t)$ for $X \in (0, 2)$ and $t \in (0, 1)$ for $T = 0.5$, $a = 0.01$ and (a) $\lambda = 0.01$ (b) $\lambda = 1$, and (c) $\lambda = 100$. All other parameter values remain unchanged. The material is undisturbed (U) wherever $\varepsilon = 0$, elastic (E) where $|\varepsilon| < 1$, and plastic (P) where $|\varepsilon| \geq 1$. We observe that an elastic wave in region EI propagates ahead of a slower moving plastic wave in region PI, and this remains the case until $u(0, t) = T$. It is clear from the figures that case (a) differs significantly from (b) and (c). In the latter two cases, as the velocity is decreased, the value of ε transits from $\varepsilon = -1$ to $\varepsilon = +1$ over a thin elastic region EII propagating away from the boundary $X = 0$, the width of which depends on the size of λ . The foregoing observations suggest that the material initially undergoes *elastic* deformation upon unloading, and can then *re-yield* to give rise to a second plastic region PII. In contrast, the region EII never appears over the considered time-scale in (a), suggesting that the material remains plastic

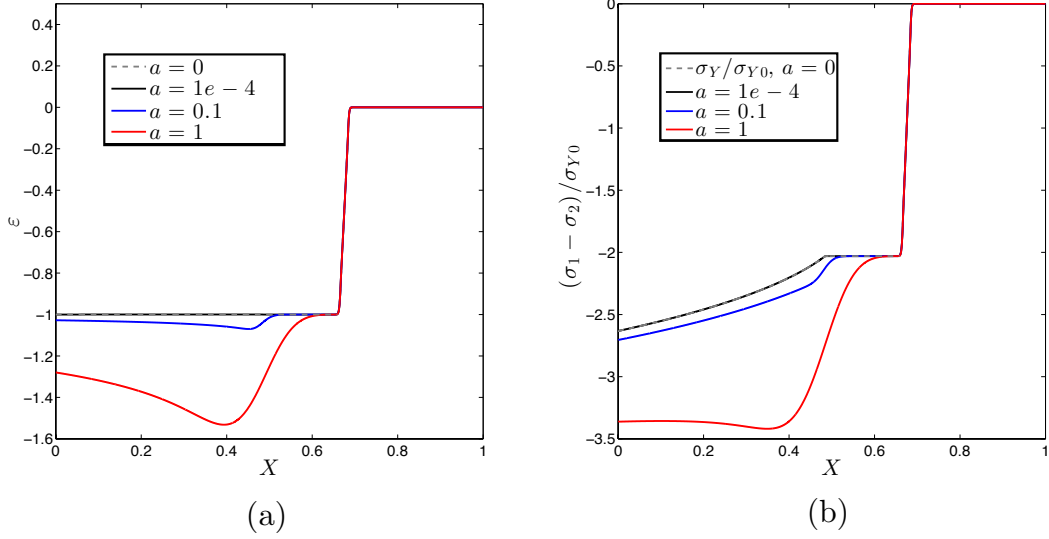


Figure 3.3: Plots of (a) the deviatoric strain ε and (b) the normalised deviatoric stress versus X for different values of the parameter a at $t = 0.5$. All other parameter values are the same as in Figure 3.1.

if the velocity is held fixed and does not undergo significant unloading. We will explore this phenomenon in more detail in chapter 4.

3.3 Free surface velocity

In ICEs, the quantity which can be measured is the velocity of the stress-free end of the target material upon the arrival of the incident elastic and plastic waves. To model such experiments, we solve the model (2.3.54)–(2.3.56) on a finite domain $X \in [0, 1]$ subject to (3.2.1) and (3.2.2) plus the additional boundary condition

$$\sigma_1(1, t) = \left[h'(\nu) + \frac{2\delta k_0}{3\nu} \varepsilon \right]_{X=1} = 0. \quad (3.3.1)$$

The dynamics of the elastic waves are again best visualised through the deviatoric strain ε . Figure 3.5(a) shows a heat map of $\varepsilon(X, t)$ for $X \in (0, 1)$ and $t \in (0, 2)$, computed using the elastic-plastic model (2.3.54)–(2.3.56) with the same parameter values as in Figure 3.1. Once more, the material is undisturbed where $\varepsilon = 0$, elastic where $|\varepsilon| < 1$ and plastic where $|\varepsilon| \geq 1$. Initially an elastic wave, labelled E, propagates ahead of a plastic wave of much larger amplitude, labelled P. When the elastic wave reaches the free surface at $X = 1$, it reflects back toward the incoming plastic wave. During the interaction of the two waves (see the inset), part of the elastic wave is transmitted through the plastic wave, while part is reflected back toward the free surface. The elastic wave thus reverberates

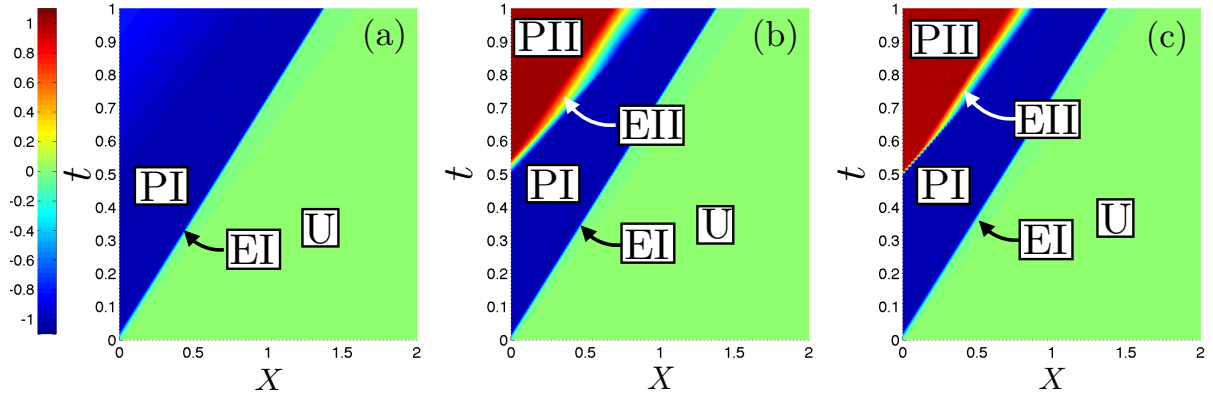


Figure 3.4: Plots of the deviatoric strain ε for $X \in (0, 2)$, $t \in (0, 1)$. In (a) $\lambda = 0.01$, (b) $\lambda = 1$, and (c) $\lambda = 100$, while $T = 0.5$ and $a = 0.01$ in all three plots. All other parameter values remain unchanged from Figure 3.1.

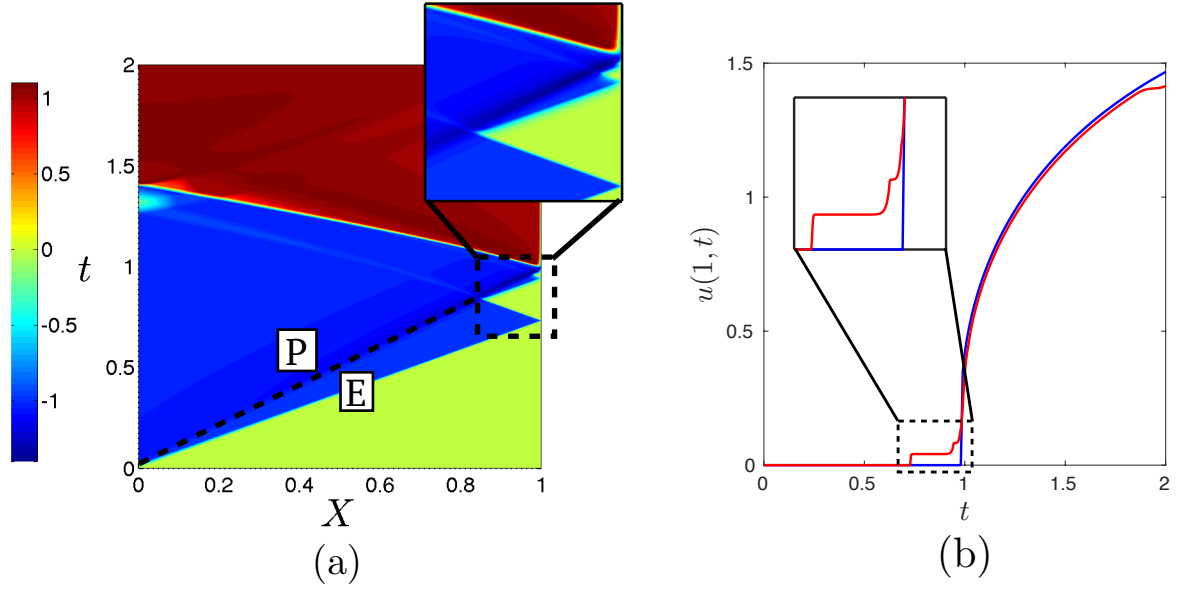


Figure 3.5: (a) Numerical solution of the model (2.3.54)–(2.3.56) for the deviatoric strain $\varepsilon(X, t)$ using the same parameter values as Figure 3.1 and (b) free surface velocity $u(1, t)$ computed using the elastic-plastic model (2.3.54)–(2.3.56) (solid red), both for $t \in (0, 2)$; corresponding barotropic flow solution in blue.

back and forth until the eventual arrival of the large amplitude plastic wave at the free surface. Interestingly, we note that there is always an unyielded region close to the free surface $X = 1$ and that, while they are not interacting, the elastic and plastic waves are separated by a plateau region in which $|\varepsilon| \approx 1$ and the material sits approximately on the yield surface.

The presence of the unyielded region close to $X = 1$ is a phenomenon not found in gas dynamics, owing to the fact that on the leading characteristic $X = t$ the pressure is zero, as it is on the free surface, so the bulk wave can reach and reflect from the free surface $X = 1$. However, in our elastic-plastic model, the plastic wave is no longer propagating into quiescent material, as the material has already undergone elastic deformation. Thus it appears the plastic wave can never reach the free surface, and is prevented from doing so by the presence of the elastic edge layer near $X = 1$. We will explore this phenomenon further in chapter 4. We also note that, as in Figure 3.4, in Figure 3.5(a) for $t \gtrsim 1$, the value of ε switches sign from -1 to $+1$ over a thin elastic region which propagates away from the free surface. This implies that the influence of the stress-free boundary is to force the material to unload, and therefore cross the yield surface to create a new plastic region in which $\varepsilon \approx +1$ (shaded red).

The velocity at the free surface $u(1, t)$ is plotted by the red line in Figure 3.5(b), and compared with the corresponding results for barotropic flow in blue. A small elastic precursor arrives ahead of the large plastic wave, and the reverberation leads to a fascinating staircase-like feature, with the velocity increasing through a series of small steps.¹ None of the aforementioned significant features can be explained or resolved by a barotropic flow model alone.

Finally, in Figures 3.6 (a)–(c), we plot the the free surface velocity $u(1, t)$ for varying δ , a , and k_0 . In Figure 3.6(a), we observe (as in Figure 3.2), that smaller amplitude elastic precursors correspond to smaller values of δ , but the arrival time of the elastic wave at the free surface is unchanged. Figure 3.6(b) shows that, for small values of a , the features of the “staircase” are well defined, whereas for larger values of a these features are smoothed out and it becomes harder to distinguish the reverberation of the elastic wave. Furthermore, we see that for $a = 1$, the amplitude of the first disturbance is larger than that for smaller values of a , which is a direct consequence of rate-dependence and the fact that the deviatoric stress in the material may exceed the yield stress. Lastly, in Figure 3.6(c), we see that increasing k_0 results in an earlier arrival of the elastic precursor at the free surface, with the amplitude of each step of the staircase also increasing.

¹We note that the small deviation in the elastic-plastic model from the barotropic model near $t = 2$ is a consequence of a reflected elastic wave from $X = 0$ reaching $X = 1$ over the time-scale of the calculation.

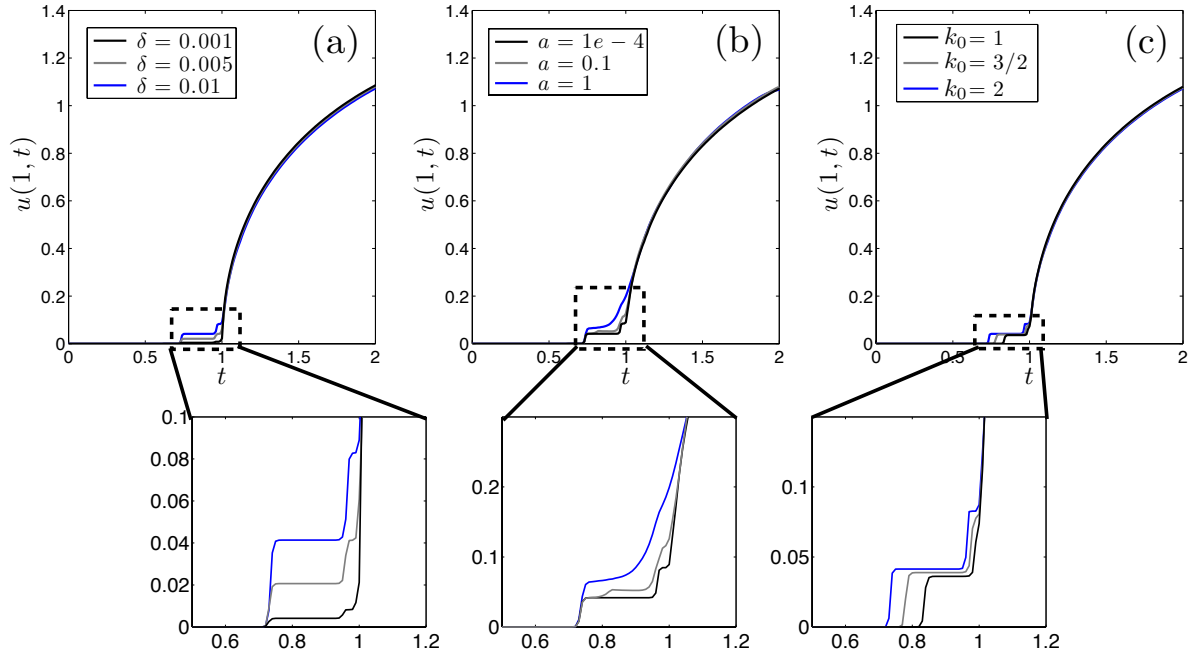


Figure 3.6: Plots of the free surface velocity $u(1, t)$, $t \in (0, 2)$, for varying (a) δ (fixing $a = 0.1$, $k_0 = 2$), (b) a (fixing $\delta = 0.01$ and $k_0 = 2$), and (c) k_0 (fixing $\delta = 0.01$, $a = 0.1$).

3.4 Comparison with hydrocode models

To enhance the utility of our model for AWE, we now compare numerical solutions of (2.3.54)–(2.3.56) (in dimensional form) with solutions obtained from finite-element based hydrocodes. To fix ideas, we will simulate a synthetic isentropic compression experiment involving Tantalum (Ta). For simplicity, and to ensure that the models agree in the absence of more sophisticated modelling assumptions, we will restrict our attention to rate-independent plasticity using a known equation of state, a constant shear modulus, and a constant yield strength. Unless otherwise stated, we approximate rate-independence numerically in §3.4.1 by setting the parameter a to be as small as feasibly possible, in this case $a = 0.001$.²

3.4.1 Model comparison

In what follows, we will employ the well-known Birch-Murnaghan equation of state

$$h'(\nu) = -3f(2f + 1)^{\frac{5}{2}}(1 + \alpha f), \quad f = \frac{1}{2}(\nu^{-2/3} - 1) \quad (3.4.1)$$

²This value was chosen by decreasing a until no further qualitative refinement of the solution could be observed. Decreasing a any further merely results in unwanted numerical stiffness, without an improvement in the resolution of the results.

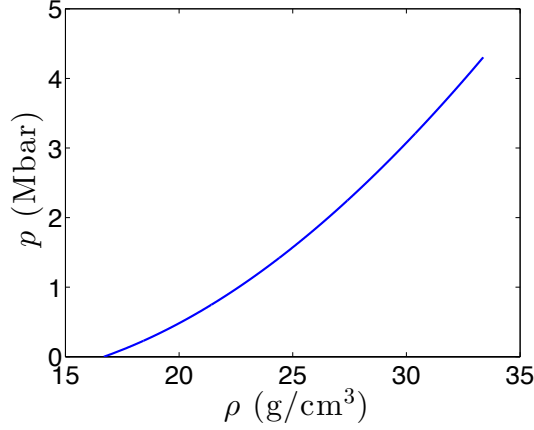


Figure 3.7: Birch-Murnaghan equation of state with $\alpha = -0.7054$.

where α is a constant. The function (3.4.1) with $\alpha = -0.7054$ is shown in Figure 3.7. This value for α was obtained by fitting (3.4.1) to the principal isentrope calculated from the Steinberg equation of state given in [70]. We also take the shear modulus and yield strength to be constant and equal to their values at ambient conditions as given in Table 3.1.³ The function $k(\nu)$ in (2.3.54)–(2.3.56) is then given by the relation (2.5.34).

Property	Value
Ambient density, ρ_0	16.69 g/cm ³ (16.69 × 10 ³ kg/m ³)
Bulk modulus, K_b	1.94 Mbar (1.94 × 10 ¹¹ Pa)
Shear modulus, μ	0.690 Mbar (6.90 × 10 ¹⁰ Pa)
Yield strength, σ_{Y0}	7.7 × 10 ⁻³ Mbar (7.7 × 10 ⁸ Pa)
Bulk sound speed, $c_0 = \sqrt{K_b/\rho_0}$	0.341 cm/μs (3.41 × 10 ³ m/s)
Elastic wave speed, $c_{\text{elastic}} = \sqrt{(\lambda + 2\mu)/\rho_0}$	0.414 cm/μs (4.14 × 10 ³ m/s)
Ratio of yield stress to bulk modulus, δ	~ 0.004

Table 3.1: Some properties of Tantalum.

We obtain a realistic velocity drive by fitting an exponential function to the drive calculated by Davis et al. in [19], where the authors subject samples of Ta to pressures on the order of 3 Mbar. Assuming that, in the absence of elastic strength, the material may be modelled as an inviscid compressible fluid with an equation of state (3.4.1), it is straightforward to estimate the pressure drive from the velocity; both the velocity drive and pressure drive are shown in Figure 3.8.

³The values given in Table 3.1 are in units adopted by AWE and those which commonly appear in literature pertaining to isentropic compression. The corresponding values in S.I. units are noted in brackets.

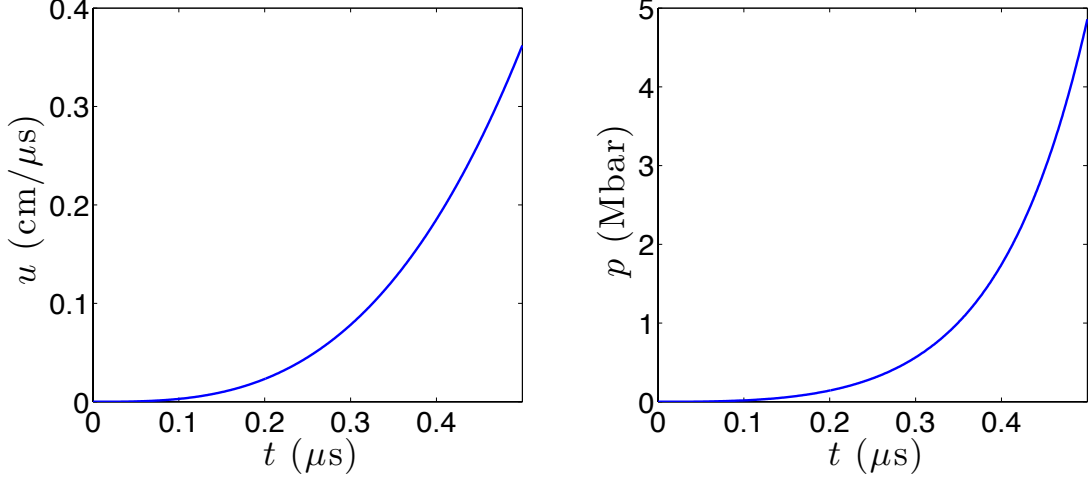


Figure 3.8: Plots of the velocity and pressure drive employed in our analysis.

Our comparison begins by first ignoring the presence of the free surface and calculating in-situ profiles for the density, velocity, pressure, and axial stress at $X = 0.5$ (that is, at the midpoint of the sample). In the first instance, we compute solutions in the absence of elastic strength to verify that the two models agree in this simplest case. In Figure 3.9 we plot solutions for the (dimensional) density ρ , axial velocity u , and the pressure p , versus time t . Shown in black are the solutions of the model (2.3.54)–(2.3.56) with $\delta = 0$, and in dashed grey are the solutions obtained from the hydrocode calculation. Unsurprisingly, the solutions from the two numerical calculations are virtually superimposed and hence agree extremely well.

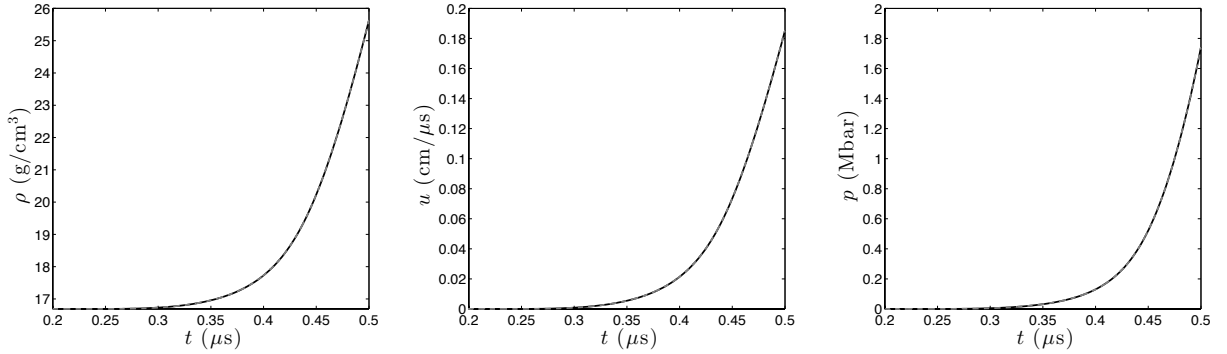


Figure 3.9: Comparison of the (dimensional) in-situ density ρ , axial velocity u , and pressure p for a model based on barotropic flow, calculated at the midpoint ($X = 0.5$) of the sample in the Lagrangian frame. The dimensional time $t \in (0, 0.5) \mu\text{s}$. The solid black line is the prediction of (2.3.54)–(2.3.55) when $\delta = 0$, while the dashed grey line is the prediction of the hydrocode model in the absence of elastic strength. Both solutions use the constitutive law (3.4.1) with $\alpha = -0.7054$.

Next we compare the predictions of our model and hydrocode results when elastic strength is included. In Figure 3.10 we plot the density, velocity, and deviatoric stress $|\sigma_1 - \sigma_2|$ versus time for the model (2.3.54)–(2.3.56) (solid black) and the hydrocode model (dashed grey). When the material is deforming plastically, the two models once more agree very well. There is small discrepancy in the elastic precursor (see inset), however the salient features of the elastic deformation are clearly captured by both models; that is, an increase in stress towards the yield stress and the existence of a plateau where the material remains precisely on the yield surface. This is followed by a large deformation due to the arrival of the plastic wave.

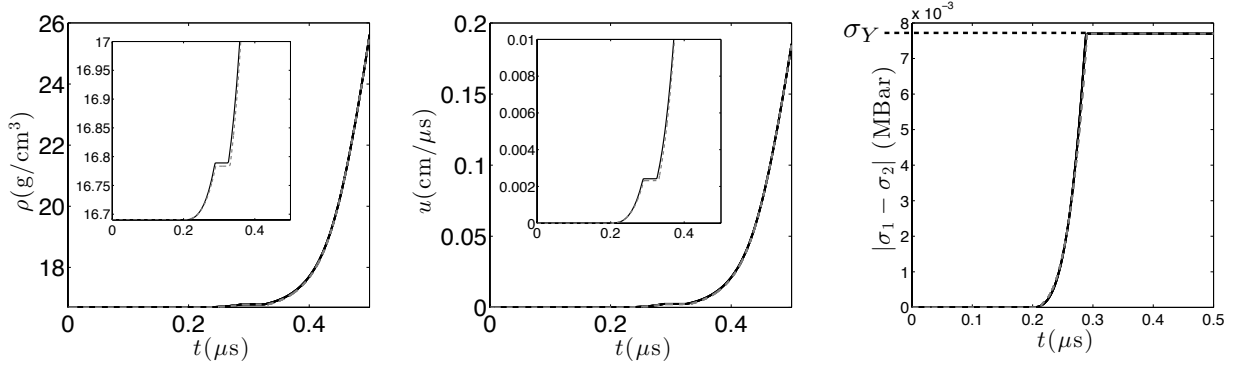


Figure 3.10: Comparison of the (dimensional) in-situ density ρ , axial velocity u , and deviatoric stress $|\sigma_1 - \sigma_2|$, calculated at the midpoint ($X = 0.5$) of the sample in the Lagrangian frame. The dimensional time $t \in (0, 0.5) \mu s$. The solid black line is the prediction of (2.3.54)–(2.3.56), while the dashed grey line is the prediction of the hydrocode model. Both solutions use the constitute law (3.4.1), while the function $k(\nu)$ in our model is related to the shear modulus *via* (2.5.34). The yield stress in both models is constant and equal to its value given in Table 3.1.

Finally, we explore the predictions of the hydrocode calculations on the free surface velocity profiles obtained from this experiment. To fix ideas, we select one thickness of Ta, in this case $L = 0.08$ cm, and apply the same velocity drive as in Figures 3.9 and 3.10. In Figure 3.11, we plot the free surface velocity for the elastic-plastic model (2.3.54)–(2.3.56) in black, the hydrocode result in dashed grey, and the corresponding result in the absence of elastic strength ($\delta = 0$). As is now familiar, the first arrival at the free surface is that of the elastic precursor. Like Figure 3.5(b), the reverberation of the elastic wave leads to a staircase like feature with the velocity increasing in a series of small steps, a feature which is well captured by both models. As the velocity increases, we see that the results from all three models begin to converge as the isotropic component of the strain begins to dominate. Thus, the effects of elastic strength are only a small perturbation to the leading order model based on barotropic flow.

All of the foregoing observations suggest that the two models, despite the disagreement in the decomposition of the strain outlined in §2.5, are essentially equivalent in the regimes of stress which we are interested in. This gives us further confidence that the modelling approach detailed in §2.3 faithfully describes the type of high strain rate phenomena of interest in the defence community, without having to resort to *ad hoc* assumptions regarding the additive decomposition of the strain tensor.

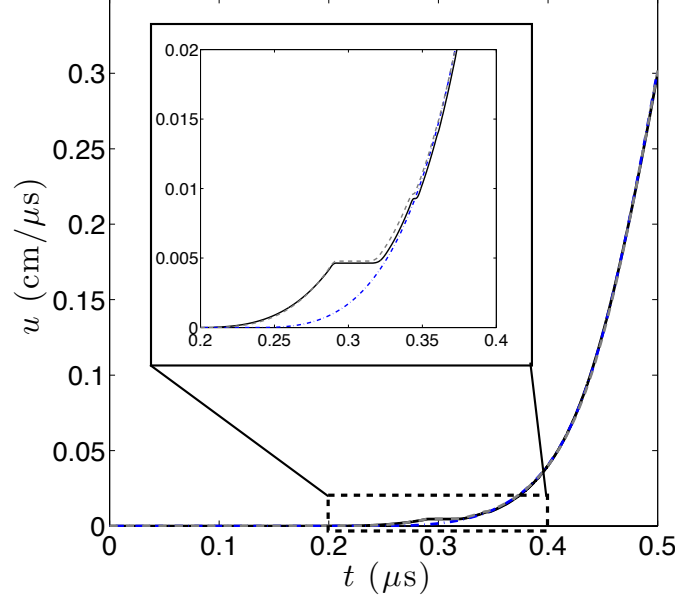


Figure 3.11: Prediction of the free surface velocity for a Ta sample of thickness $L = 0.08$ cm. In black is the prediction of the model (2.3.54)–(2.3.56), while the dashed grey line is the prediction of the hydrocode model. For comparison, we plot the solution of (2.3.54)–(2.3.55) with $\delta = 0$ in dashed blue. The parameter values are the same as those used in Figure 3.10.

3.4.2 Estimation of a

In the hydrocode model, rate dependence is incorporated by assuming that the yield strength is dependent on the plastic strain rate. This modelling assumption is adopted, for example, in the Johnson-Cook yield strength model (2.5.39). In Figure 3.12(a) we plot the in-situ deviatoric stress $|\sigma_1 - \sigma_2|$ versus t at $X = 0.5$ cm for $t \in (0, 0.5)$ μs . The grey-dashed line plots the result from the hydrocode model using the constitutive assumption (2.5.39), while the solid lines plot the result using the elastic-plastic model (2.3.54)–(2.3.56) with a constant yield strength σ_Y (given by its value in Table 3.1) for varying a . The results using the elastic-plastic model clearly do not provide satisfactory results simply by varying the value of a . However, we note that a value of $a = 1$

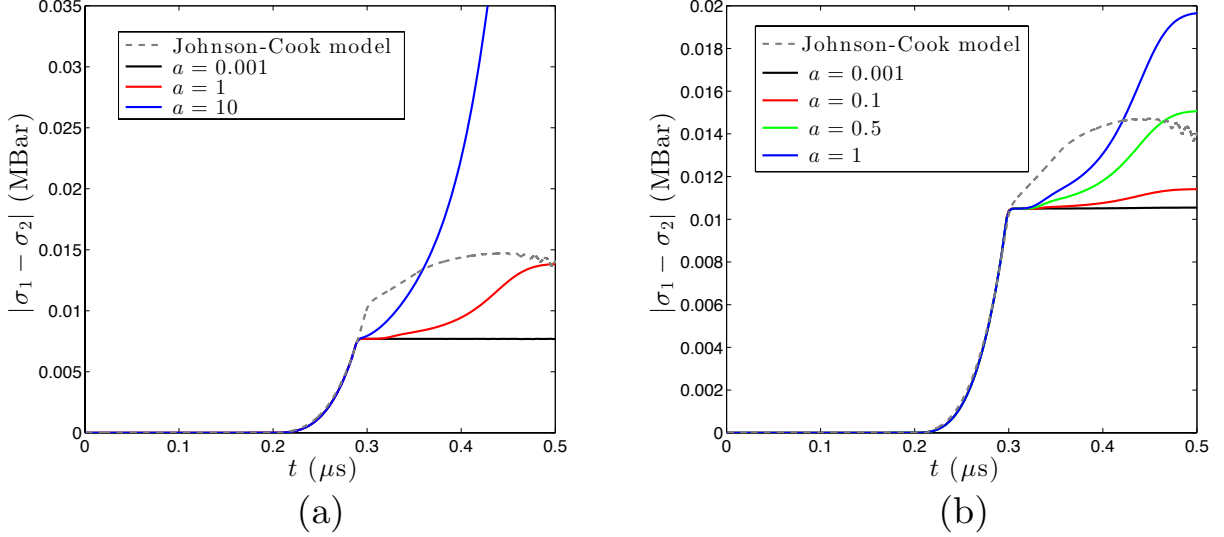


Figure 3.12: Plots comparing the in-situ deviatoric stress $|\sigma_1 - \sigma_2|$ at the midpoint of the sample ($X = 0.5$ cm) by varying (a) the value of a at fixed $\sigma_{Y0} = 0.0077$ MBar and (b) changing the value of σ_{Y0} and a .

provides a reasonable estimate to predict the peak overshoot of the deviatoric stress from the yield surface. In Figure 3.12(b), we again plot the deviatoric stress, but now with $\sigma_Y = 0.0105$ Mbar. Again, simply assuming a constant yield strength in the elastic-plastic model provides an unsatisfactory prediction of the result from the hydrocode, however a refined estimate of $a = 1/2$ for matching the peak overshoot is obtained in this example.

To reconcile the two models, the yield function used in the elastic-plastic model (2.3.54)–(2.3.56) must accommodate dependence on the plastic strain rate. In §2.5.4, we proposed a method of how to incorporate the Johnson-Cook yield strength model into (2.3.54)–(2.3.56). Unfortunately, implementing this model numerically proved to be more difficult than anticipated, but ought to be attempted in the future.

3.4.3 Experimental data

Many of the features described in the foregoing analysis are present in raw experimental data obtained from experiments. In Figure 3.13 we plot the free surface velocity for two thicknesses L of Ta obtained from the experiment conducted by Davis et al. in reference [19]. The black line represents a sample $L_1 = 0.0911$ cm thick and the red line a sample $L_2 = 0.1208$ cm thick. The first response measured at the free surface is the arrival of a small amplitude elastic precursor (see inset), which results in a slight increase in the velocity. Closer scrutiny of this region reveals further small increases in

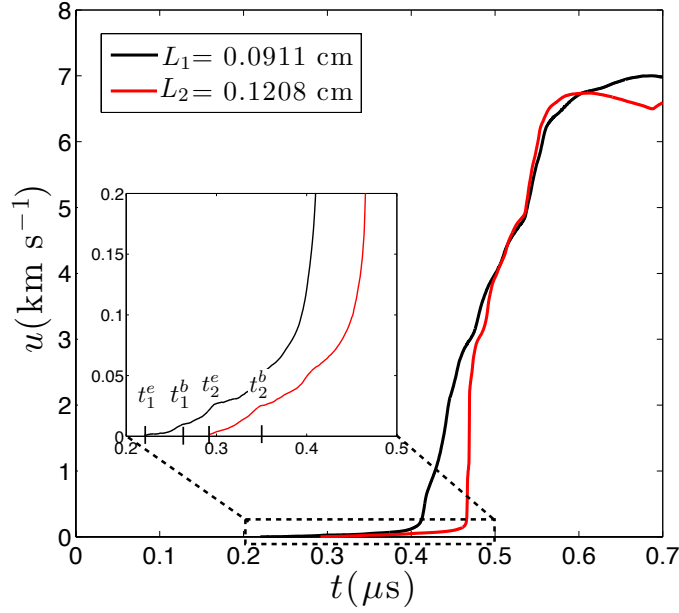


Figure 3.13: Experimental data for the multi-megabar compression of tantalum for two different thicknesses (see legend) [19].

the velocity, which are thought to be the result of the reverberation of the elastic wave between the free surface and the plastic wave. Note that in reality these are hard to distinguish from other non-uniformities in the velocity profile, which are thought to be due to non-smooth drive velocity.

Using the material parameters in Table 3.1, we find that $c_{\text{elastic}} = 0.4140 \text{ cm}/\mu\text{s}$. Thus for sample lengths L_1 and L_2 , the precursor will arrive at time $t_1 = L_1/c_{\text{elastic}} \approx 0.22 \mu\text{s}$ for the thinner sample, and $t_2 = L_2/c_{\text{elastic}} \approx 0.2918 \mu\text{s}$ for the thicker sample. The bulk wave is predicted to arrive later at the free surface at $t_1^b = 0.2672 \mu\text{s}$ for $L = L_1$, and $t_2^b = 0.3542 \mu\text{s}$ for $L = L_2$. Notice that there are features of the velocity profile before the expected arrival of the bulk wave due to the arrival and reverberation of the elastic wave. This is significant in the analysis of isentropic compression experiments since, as discussed in chapter 1, methods such as Inverse Lagrangian Analysis begin at $t = t^b$ and hence do not account for features due to the elasticity of the material.

3.5 Conclusions

In this chapter we have presented numerical solutions of the governing p.d.e (2.3.54)–(2.3.56) when applied to scenarios relevant to isentropic compression experiments. We found that large scale plastic compression waves, which are approximately described by a barotropic flow model, are accompanied by small amplitude but fast moving elastic

waves. Although the elastic waves are small, repeated interactions between the elastic and plastic waves may result in significant cumulative effects which cannot be explained by a purely barotropic theory. For example, Figure 3.5(b) indicates a non-zero velocity at the free surface *before* the arrival of the plastic wave, which would be impossible in purely barotropic flow. This is of great significance to the analysis of isentropic compression experiments as current methods such as Inverse Lagrangian analysis begin when $t = 1$ (i.e. when the leading characteristic of the barotropic flow model $X = t$ intersects the boundary), and therefore miss the features of the velocity profile attributed to the elastic waves.

We then compared our model with the predictions of an AWE finite-element based hydrocode model. To fix ideas, we simulated an isentropic compression experiment involving the common metal tantalum. We found excellent agreement between our model and the hydrocode model in both in-situ and free surface velocity profiles, laying further credence to the equivalence of the two models in the regimes of stress in which we are interested. Finally, we sought to address the outstanding question of the magnitude of the rate parameter a . Our simple analysis suggests that $a \sim O(1)$ is a reasonable estimate for the experiments in which we are interested, however a more accurate model requires the yield function to depend of the plastic strain rate. Such models could in principle be tested using a modified yield function in (2.3.54)–(2.3.56) in the future.

Chapter 4

Elastic-plastic wave propagation: linear analysis

4.1 Synopsis

In this chapter, we attempt to corroborate some of the observations made in chapter 3 of the elastic and plastic waves that can propagate through a metal sample when subject to a compressive force at $X = 0$. We focus here on the simplest possible case of rate-independent, linear elastoplasticity which, despite yielding a simplified system of governing equations, allows us to investigate in detail a fascinating series of problems encompassing many of the key phenomena underpinning elastic-plastic wave propagation. Our analysis begins with the study of a prototypical problem relating to the loading and unloading of a semi-infinite elastic-plastic material. The phenomena encountered in this problem will then help to inform a more realistic scenario in which a stress free boundary is introduced at $X = 1$, from which elastic waves may reflect. In accounting for the multiple reverberations of the elastic wave that can occur between the free surface and the plastic wave, we will find a fascinating analogy to the classic pantograph problem [23, 54, 75].

4.2 Literature review

Open an introductory textbook on the mathematical foundations of elasticity [32, 42, 79] and one is likely to find an exposition on elastodynamics, which concerns the time dependent deformation of elastic media and the ensuing propagation of elastic waves. An area about which perhaps less is known is the onset and subsequent propagation of plastic waves, which propagate through a material once the yield stress has been reached and possibly exceeded. Research output in this area spiked around the time of Second World War, partly due to applications in explosives and ballistics, with much of

the early theoretical work focusing on infinitesimal strains and shock waves. Important contributions around this time are due to Craggs, Pack, Taylor, and von Kármán [17, 56, 76, 81].

Present day research into dynamic plasticity has remained intense, particularly as experimental capabilities have become more sophisticated, allowing researchers to test the mechanical and thermodynamic response of materials at increasingly high strain rates and stresses [11, 18, 19, 64, 65, 66]. As we alluded to in chapter 1, shock wave experiments provide much of the earlier experimental data for the violent compression of metals, and as such much is known about the physical nature of shock processes in these materials [20, 24, 25, 36, 45, 82]. However, comparatively little attention appears to have been paid to the nature of elastic and plastic waves that can propagate through metal samples during shockless compression. Pioneering work in this area was conducted by Morland et al. [46, 47] and by von Kármán et al. [81], whose modelling paved the way for much of present day analyses of elastic-plastic wave propagation. We note, however, these early models do not include the presence of an explicit flow rule, dictating the time evolution of the plastic strain, which form the basis of modern day understanding of plastic flow. Instead the stress is assumed to follow a path based on a Preisach-type hysteresis model [62], from which field equations for the elastic and plastic deformations are derived. By analogy with the classical piston problem from gas dynamics, Howell et al. [33, 34] recently studied the time-dependent deformation of a semi-infinite, mechanically linear slab of material, based on a model derived from sound notions from rational mechanics. This work gave rise to some fascinating predictions regarding the phenomenon of overdriven shocks, and also to some fundamental mathematical questions regarding singular asymptotic perturbations. A similar problem was recently studied by Cohen et al. [14], who consider the propagation of longitudinal shock waves in hardening elastoplastic solids, and derive conditions for the onset of elastic-plastic shocks.

4.3 Rate-independent linear model

To examine the behaviour of (2.3.54)–(2.3.56) when the displacement is small, we expand

$$u = \delta U + O(\delta^2), \quad \nu = 1 + \delta\eta + O(\delta^2) \quad (4.3.1)$$

before taking the limit $\delta \rightarrow 0$. The leading order governing equations are then the following semi-linear system

$$\frac{\partial \eta}{\partial t} - \frac{\partial U}{\partial X} = 0, \quad (4.3.2)$$

$$\frac{\partial U}{\partial t} - \frac{\partial \eta}{\partial X} - \frac{2k_0}{3} \frac{\partial \varepsilon}{\partial X} = 0, \quad (4.3.3)$$

$$\text{and } \delta a \left(\frac{2}{3} \frac{\partial \eta}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right) = G(\varepsilon), \quad (4.3.4)$$

where we recall $k_0 = k(1)$. In the case of rate-independent plasticity ($a \equiv 0$), (4.3.4) further reduces to

$$\frac{\partial}{\partial t} \left(\frac{2}{3} \eta - \varepsilon \right) = 0, \quad |\varepsilon| < 1 \quad (\text{elastic}) \quad \text{or} \quad |\varepsilon| = 1 \quad (\text{plastic}). \quad (4.3.5)$$

When the material is deforming elastically, (4.3.5) states that rate of plastic strain is zero.

As we shall see in §§4.5 and 4.6, (4.3.5) is not on its own sufficient to discriminate between elastic and plastic behaviour. The selection criterion is provided by the hitherto neglected equation for the entropy (2.3.50), namely

$$\theta(1) \frac{\partial \zeta}{\partial t} = k_0 \varepsilon \left(\frac{2}{3} \frac{\partial \eta}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right) \quad (4.3.6)$$

where we recall the entropy $S = S_0(1 + \delta^2 \zeta)$. For elastic behaviour, (4.3.5) implies $\partial \zeta / \partial t \equiv 0$, while for plastic behaviour, $|\varepsilon| = 1$, and therefore

$$\varepsilon \frac{\partial \eta}{\partial t} \geq 0 \quad (4.3.7)$$

in order to preclude thermodynamic unfeasability.

4.4 Elastic-plastic free boundary conditions

As the stress on a material increases, it first deforms elastically, before the yield stress is reached, heralding the onset of plastic deformation. This switch in behaviour from elastic to plastic occurs across a free boundary which we denote $X = \xi(t)$. When considering the appropriate conditions to apply across the elastic-plastic free boundary, we note that conservation of mass (4.3.2) and momentum (4.3.3) require

$$\frac{d\xi}{dt} = \dot{\xi} = -\frac{[U]}{[\eta]} = -\frac{\left[\eta + \frac{2k_0}{3} \varepsilon \right]}{[U]} \quad (4.4.1)$$

while (4.3.5) requires

$$\dot{\xi} \left[\varepsilon - \frac{2}{3}\eta \right] = 0, \quad (4.4.2)$$

where the bracket notation denotes the jump in the enclosed quantity across the elastic-plastic free boundary. This implies that, unless $\dot{\xi} \equiv 0$, we must have continuity of plastic strain, that is

$$[\varepsilon] = \frac{2}{3}[\eta]. \quad (4.4.3)$$

From (4.4.1), we then have

$$[U] + \dot{\xi}[\eta] = \dot{\xi}[U] + c_e^2[\eta] = 0, \quad (4.4.4)$$

where we recall that c_e is the elastic wave speed given by (2.3.53).

We are then presented with two possible cases; either $\dot{\xi} = \pm c_e$, which implies that discontinuities in η and U can only occur across elastic characteristics, or η , U , and ε are all continuous across $X = \xi(t)$. In this chapter we restrict our attention to the latter case.

4.5 Unidirectional wave propagation

In this section, we analyse the deformation of an initially unstressed material occupying the half-space $X \in (0, \infty)$, which is subject to an imposed velocity at $X = 0$. An analysis of wave reflections from the free surface at $X = 1$ is deferred to §4.6.

4.5.1 Compressive wave propagation

We start by considering a compressive acceleration of the face $X = 0$. We therefore solve (4.3.2), (4.3.3), and (4.3.5) subject to the initial and boundary conditions

$$\eta = 0, \quad U = 0, \quad \varepsilon = 0 \quad \text{at} \quad t = 0, \quad X \geq 0, \quad (4.5.1)$$

$$U = U_0(t)/\delta \quad \text{at} \quad X = 0, \quad t > 0. \quad (4.5.2)$$

The factor of δ in (4.5.2) arises from the scaling (4.3.1). To balance the boundary condition (4.5.2), we restrict our attention to the small time deformation of the sample, and hence we make the scalings $t = \delta t'$, $X = \delta X'$ (note the governing equations (4.3.2), (4.3.3), and (4.3.5) are unchanged in this new coordinate system). The boundary condition (4.5.2) then becomes

$$U_0(t) \sim U'_0(0)t' = \alpha t' \quad (4.5.3)$$

which corresponds to a constant acceleration of the face $X = 0$. We note that α may be set to unity without loss of generality, however we retain it for now to facilitate the

discussion in §4.6.2 of cases where the acceleration is not constant. Henceforth, in this section, we drop the prime notation.

Initially, the material deforms elastically, and thus we are required to solve the hyperbolic system

$$\frac{\partial \eta}{\partial t} - \frac{\partial U}{\partial X} = 0, \quad (4.5.4)$$

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial X} \left(\eta + \frac{2k_0}{3} \varepsilon \right) = 0, \quad (4.5.5)$$

$$\frac{\partial}{\partial t} \left(\frac{2}{3} \eta - \varepsilon \right) = 0, \quad (4.5.6)$$

with $|\varepsilon| < 1$. The characteristics of the system (4.5.4)–(4.5.6) are

$$\frac{dX}{dt} = 0, \quad \frac{dX}{dt} = \pm c_e. \quad (4.5.7)$$

It is straightforward to show that the solution of the system (4.5.4)–(4.5.6) subject to (4.5.1) and (4.5.3) is

$$\eta = -\frac{\alpha}{c_e} \left(t - \frac{X}{c_e} \right), \quad U = \alpha \left(t - \frac{X}{c_e} \right), \quad \varepsilon = -\frac{2\alpha}{3c_e} \left(t - \frac{X}{c_e} \right) \quad (4.5.8)$$

for $t > \frac{X}{c_e}$. The material remains elastic until $|\varepsilon| = 1$. We can see that this will first happen on $X = 0$ and, since $\partial \eta / \partial t < 0$ at $X = 0$, Equation (4.3.7) requires $\varepsilon = -1$. We find from the third equation in (4.5.8) that $\varepsilon = -1$ on the line

$$X_* = c_e (t - t_*), \quad (4.5.9)$$

where

$$t_* = 3c_e/2\alpha. \quad (4.5.10)$$

Also, on this line

$$\eta = -\frac{3}{2} = \eta_*, \quad U = \frac{3c_e}{2} = U_*. \quad (4.5.11)$$

In the region $t > t^* + X/c_e$ we are left to solve

$$\frac{\partial \eta}{\partial t} - \frac{\partial U}{\partial X} = 0, \quad (4.5.12)$$

$$\frac{\partial U}{\partial t} - \frac{\partial \eta}{\partial X} = 0, \quad (4.5.13)$$

with $\varepsilon = -1$. The characteristics of (4.5.12)–(4.5.13) are $dX/dt = \pm 1$, and hence the material deforms plastically in the region $t > t_* + X$. It is once more trivial to show that the solution of (4.5.12)–(4.5.13) in the region $0 < X < t - t_*$ is

$$\eta = -\alpha(t - X) + \frac{3}{2}(c_e - 1), \quad U = \alpha(t - X), \quad \varepsilon = -1. \quad (4.5.14)$$

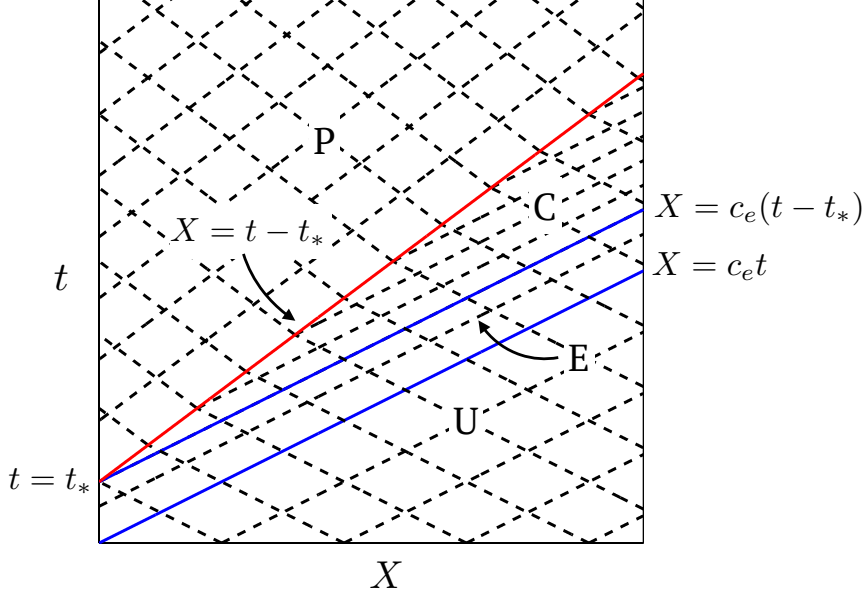


Figure 4.1: Characteristic diagram depicting the solution structure for the early time deformation in the case $a = 0$. A fast moving elastic wave E propagates into quiescent material, ahead of a slower moving plastic wave P. The two wave fronts are separated by a constant region C in which the material remains on the yield surface.

Finally, in the region $t - t_* < X < c_e(t - t_*)$ between the elastic and plastic waves, there is a constant region where

$$\eta = \eta_*, \quad U = U_*, \quad \text{and} \quad \varepsilon = -1. \quad (4.5.15)$$

The basic solution structure is therefore that depicted in Figure 4.1. A fast moving elastic wave E propagates into quiescent material labelled U, ahead of a slower moving plastic wave P, with the two wave fronts separated by a plateau C where the material sits precisely on the yield surface ($\varepsilon = -1$). An analogous picture was described by Howell et al. in [33]. The characteristics in regions U, E and C are $X \pm c_e t = \text{constant}$ and in region P they are $X \pm t = \text{constant}$.

4.5.2 Loading and unloading

Having resolved the initial deformation of the material as the velocity at $X = 0$ ramps up, it is then natural to investigate the deformation of the sample if the velocity is decreased after reaching a maximum value. Motivated by §3.2.2, we impose the boundary condition

$$U(0, t) = \begin{cases} \alpha t & \text{for } 0 \leq t \leq T, \\ \alpha T - \lambda(t - T) & \text{for } t > T, \\ 0 & \text{for } t > T(\lambda + \alpha)/\lambda = t_E \end{cases} \quad (4.5.16)$$

where $T > t_*$, $\lambda > 0$ are constants, and t_E denotes the time at which the velocity at $X = 0$ returns to zero. As $\lambda \rightarrow 0$, the velocity of the material boundary at $X = 0$ increases to a constant $U = \alpha T$, while in the limit $\lambda \rightarrow \infty$, the velocity is decreased instantaneously to zero.

The initial deformation is once again described by (4.5.8), (4.5.14), and (4.5.15) in regions EI, PI, and C of Figure 4.2. When the velocity reaches its maximum and begins to decrease, it is at first sight unclear whether the material remains plastic, or reverts back to being elastic. If we first assume that the material initially remains plastic during unloading, by imposing continuity of η , U , and ε across the plastic characteristic $X = t - T$, we find that in the region $0 < X < t - T$ we have

$$\eta = -(\lambda + \alpha)T + \lambda(t - X) + \frac{3}{2}(c_e - 1), \quad (4.5.17)$$

$$U = (\lambda + \alpha)T - \lambda(t - X), \quad (4.5.18)$$

$$\varepsilon = -1. \quad (4.5.19)$$

Since $\varepsilon\eta_t = -\lambda < 0$ in the plastic region, this solution violates the condition (4.3.7), and so can be discounted. Thus the material must instantaneously revert to being elastic upon unloading to prevent negative dissipation.

We therefore have a new region of elastic material, labelled EIIa in Figure 4.2, separated from plastic material PI by an elastic-plastic free boundary $X = V_I(t - T)$ (red solid line), where the speed V_I is to be determined. By appealing to causality, we can infer an upper and lower bound for V_I . The characteristics in region PI are $X \pm t = \text{constant}$, while in EIIa they are $X \pm c_e t = \text{constant}$. We are required to determine 6 *a priori* unknowns, namely η and U either side of the free boundary, ε in the new elastic region, and V_I . Hence, in order for the free boundary $X = V_I(t - T)$ to be causal, we must have three families of characteristics propagating into the free boundary, in addition to continuity of η , U , and ε . This implies the inequality

$$1 < V_I < c_e. \quad (4.5.20)$$

If V_I satisfies (4.5.20), there are two families of characteristics intersecting the free boundary from the plastic side, and one from the elastic side. Furthermore, this implies that there is no reflected plastic wave as the material unloads, but there will be reflected elastic waves in region EIIa. The solution in PI is therefore always described by (4.5.14).

The governing equations in EIIa are

$$\frac{\partial \eta}{\partial t} = \frac{\partial U}{\partial X}, \quad (4.5.21)$$

$$\frac{\partial U}{\partial t} = \frac{\partial \eta}{\partial X} + \frac{2k_0}{3} \frac{\partial \varepsilon}{\partial X}, \quad (4.5.22)$$

$$\frac{2}{3}\eta - \varepsilon = f(X). \quad (4.5.23)$$

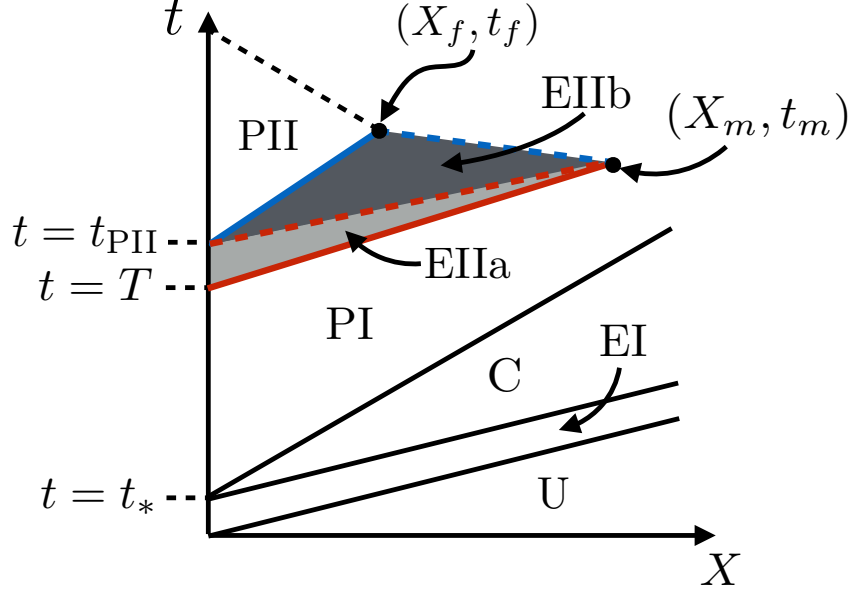


Figure 4.2: A schematic of the characteristic diagram for the loading/unloading of a linearly elastic-plastic material when $a = 0$. The solid red and blue lines demarcate the positions of the elastic-plastic free boundaries $X = V_I(t - T)$ and $X = V_{II}(t - t_{PII})$ respectively.

We note that, unlike the initial deformation of the sample, the function $f(X)$ is not zero since the material is now endowed with residual plastic strain having already undergone plastic deformation. Knowledge of η and ε in region PI allows us to determine

$$f(X) = \frac{2\alpha}{3} \left(\frac{V_I - 1}{V_I} X - T \right) + c_e. \quad (4.5.24)$$

Across the free boundary $X = V_I(t - T)$, we apply continuity of η , U , and ε , from which it may be easily shown that the velocity of the free boundary satisfies the following quadratic equation (shown in Figure 4.3):

$$Q_I(V_I) = (\lambda + \alpha)V_I^2 + \alpha(c_e^2 - 1)V_I - (\lambda + \alpha)c_e^2 = 0, \quad (4.5.25)$$

where we choose the positive root for V_I , namely

$$V_I = \frac{\alpha(c_e^2 - 1)}{2(\lambda + \alpha)} \left[\sqrt{1 + \frac{4c_e^2(\lambda + \alpha)^2}{\alpha^2(c_e^2 - 1)^2}} - 1 \right]. \quad (4.5.26)$$

We note that in the limit $\lambda \rightarrow \infty$, which corresponds to a rapid deceleration, $V_I \rightarrow c_e$. On the other hand, in the limit $\lambda \rightarrow 0$, where the velocity is held constant after reaching a maximum, $V_I \rightarrow 1$. For all other values of $\lambda > 0$, the speed of the free boundary is intermediate between these two values, and therefore adheres to the inequality (4.5.20).

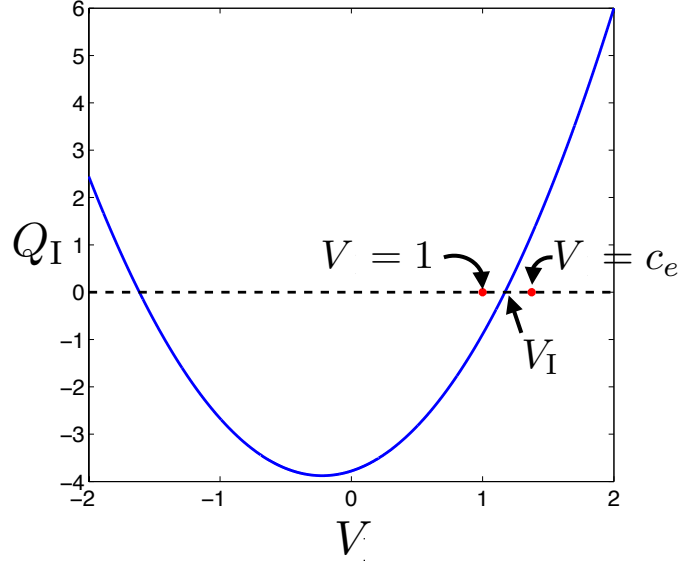


Figure 4.3: Plot of the quadratic function (4.5.25) for $\lambda = \alpha = 1$, $k_0 = 2$ ($c_e = 1.4$). Since $Q_I(1) < 0 < Q_I(c_e)$, the positive root V_I lies in the range $1 < V_I < c_e$, which is consistent with the causality criterion (4.5.20).

To complete the picture, the solution in the elastic region EIIa is

$$\eta = \frac{\alpha}{V_I^2}(V_I^2 - 1)X - \frac{\alpha(V_I - 1)}{V_I}t - \frac{\alpha}{V_I}T + \frac{\lambda}{V_I^2}(V_I(t - T) - X) + \frac{3}{2}(c_e - 1), \quad (4.5.27)$$

$$U = \alpha T - \frac{\alpha}{V_I}(V_I - 1)X - \frac{\lambda}{V_I}(V_I(t - T) - X), \quad (4.5.28)$$

$$\varepsilon = -1 + \frac{2}{3V_I^2}(\lambda - \alpha(V_I - 1))(V_I(t - T) - X), \quad (4.5.29)$$

where from (4.5.25)

$$\lambda = \frac{\alpha(c_e^2 + V_I)(V_I - 1)}{c_e^2 - V_I^2}. \quad (4.5.30)$$

Equation (4.5.29) tells us that on $X = 0$, $\varepsilon_t > 0$ for $t > T$, and hence ε increases from -1 as the material unloads. Therefore ε can in principle transit from $\varepsilon = -1$ to $\varepsilon = +1$, which implies that the material can re-yield during unloading, and hence return to being plastic behind another elastic-plastic free boundary

$$X = V_{II}(t - t_{PII}), \quad (4.5.31)$$

marked by the solid blue line in Figure 4.2, where

$$t_{PII} = T + \frac{3(c_e^2 - V_I^2)}{2\alpha V_I(V_I - 1)} \quad (4.5.32)$$

and V_{II} is to be determined. In the limit $\lambda \rightarrow 0$, $t_{PII} \rightarrow \infty$, and as $\lambda \rightarrow \infty$, $t_{PII} \rightarrow T$. This is consistent with the observations made in §3.2.2 that the width of the elastic region

created by re-yielding of the material is dependent on the rate at which the surface $X = 0$ is decelerated.

It is tempting to think that the solution EIIa persists throughout the entire elastic region, and that the free boundary between EIIa and PII propagates at speed V_I . However, we find that the solution (4.5.27)–(4.5.29) is only valid in the region bounded by $X = V_I(t - T)$ and the elastic characteristic $X = c_e(t - t_{PII})$ (red dashed line in Figure 4.2) up until $(X, t) = (X_m, t_m)$. After this time, the information travelling along positive elastic characteristics changes, since they emanate not from $X = 0$ but from the new elastic-plastic free boundary $X = V_{II}(t - t_{PII})$. Therefore, there exists another elastic region, labelled EIIb in Figure 4.2. The solution in this region, along with the solution in the new plastic region PII and the speed V_{II} may once again be determined by appealing to continuity of η , U , and ε across $X = c_e(t - t_{PII})$ and $X = V_{II}(t - t_{PII})$. The exact form of the solutions is omitted here, however we note that the region EIIb is analogous to the region C, but is endowed with a non-zero residual plastic strain. After some algebra, we find that the speed V_{II} is a root of the quadratic function

$$Q_{II}(V_{II}) = c_e^2 V_{II}^2 + (c_e - V_I)V_{II} - c_e V_I = 0, \quad (4.5.33)$$

shown in Figure 4.4. Again, choosing the positive root for V_{II} we have

$$V_{II} = \frac{c_e - V_I}{2c_e^2} \left[\sqrt{1 + \frac{4c_e^3 V_I}{(c_e - V_I)^2}} - 1 \right]. \quad (4.5.34)$$

We note that, since $1 < V_I < c_e$ we have

$$\frac{V_I}{c_e} < V_{II} < 1. \quad (4.5.35)$$

Since there is a reflected elastic wave in region EIIa/b, the solution EIIb itself is only valid until the negative blue-dashed characteristic in Figure 4.2 intersects $X = V_{II}(t - t_{PII})$ at $(X, t) = (X_f, t_f)$. There can therefore be multiple elastic regions enclosed by elastic-plastic free boundaries, all separated by elastic characteristics. To compute solutions analytically in successive regions quickly becomes tedious, and so for times $t_m < t < t_E$ we continue the solutions numerically.

In Figure 4.5, we plot η , U , and ε for (a)–(c) $t = t_{PII}$, (d)–(f) $t = (t_{PII} + t_m)/2$, (g)–(i) $t = t_m$, (j)–(l) $t = t_f$, and (m)–(o) $t = t_E$. In black is a numerical solution of the problem (4.3.2)–(4.3.4) subject to (4.5.1) and (4.5.16) with $a = 0.001$ (solved using the method described in chapter 3), while the grey-dashed lines in (a)–(i) represent the analytical solutions previously discussed. Figures (a)–(c) clearly show the existence of five distinct regions; the undisturbed region U, into which the elastic wave EI is propagating,

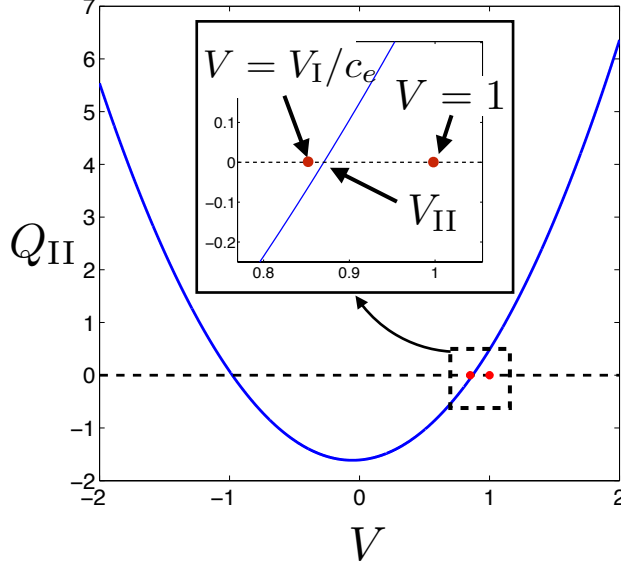


Figure 4.4: Plot of the quadratic equation (4.5.33) for $\lambda = 1$, $k_0 = 2$ ($c_e = 1.4$). The positive root V_{II} lies in the range $V_I/c_e < V_{\text{II}} < 1$ (see inset).

the plastic region PI separated from the elastic wave by a plateau C, and the new elastic region EIIa as the velocity at $X = 0$ begins to decrease (see inset). In (d)–(f), the solution structure is the same apart from a noticeable “kink” in the solution due to the elastic region EIIb. This is followed by the plastic region PII. Then, in (g)–(i) at $t = t_m$ we notice that when the characteristic $X = c_e(t - t_{\text{PII}})$ catches up with $X = V(t - T)$, the elastic region EIIa has disappeared and we are left with EIIb in the entire elastic region. The numerical solutions in (j)–(o) continue for $t_f < t < t_E$, where we see alternating elastic and plastic regions, all separated by elastic-plastic free boundaries.

Finally, we validate our predictions for V_I and V_{II} by plotting a numerical solution of (4.3.2)–(4.3.4) for ε subject to (4.5.1) and (4.5.16) in Figure 4.6 with $a = 0.001$. This figure shows the expected initial propagation of an elastic wave labelled EI, followed by a plastic wave labelled PI. These solutions persist until $t = 50$ where the unloading begins, and we observe the propagation of the elastic plastic boundary $X = V_I(t - T)$ demarcated by the first solid black line. As the unloading progresses, the deviatoric strain ε reaches +1 at $X = 0$, $t = t_{\text{PII}}$ at which point the second elastic-plastic free boundary $X = V_{\text{II}}(t - t_{\text{PII}})$ forms. In between the two free boundaries, $|\varepsilon| < 1$, and hence the material is deforming elastically, with the regions EIIa and EIIb separated by an elastic characteristic $X = c_e(t - t_{\text{PII}})$ (dashed black line). For $t > t_m$ (lower white marker in Figure 4.6), we see that the numerical solution begins to diverge from the prediction for $X = V_I(t - T)$, and similarly for $t > t_f$ (upper white marker) for the prediction of $X = V_{\text{II}}(t - t_{\text{PII}})$, due to the internal reflections of the elastic waves between the two free

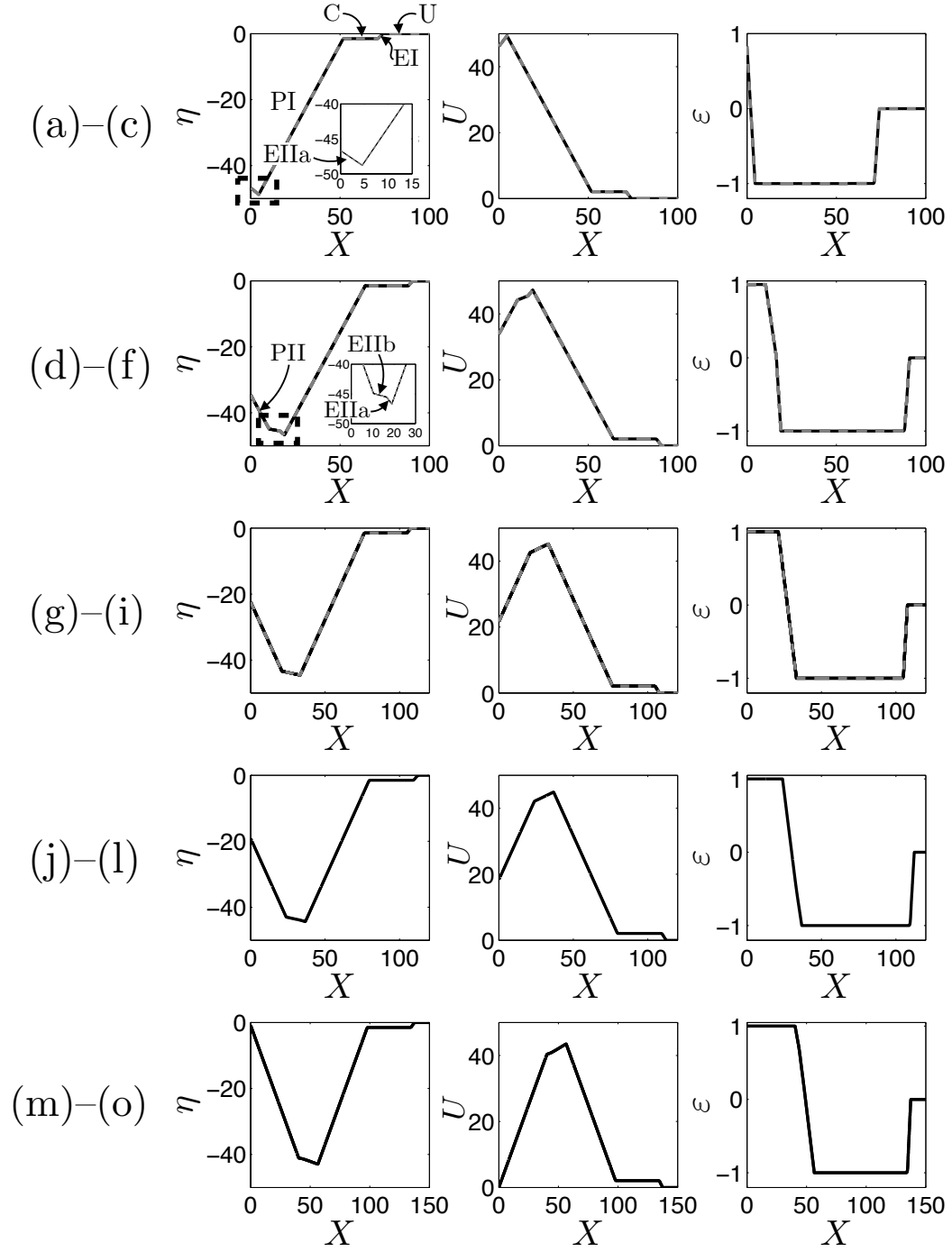


Figure 4.5: Plots of η , U , and ε for different stages of unloading for (a)–(c) $t = t_{\text{PII}}$, (d)–(f) $t = (t_{\text{PII}} + t_m)/2$, (g)–(i) $t = t_m$, (j)–(l) $t = t_f$, and (m)–(o) $t = t_E$. The parameter values used are $T = 50$, $\lambda = \alpha = 1$, $k_0 = 2$ ($c_e = 1.4$), and in the numerical solution $a = 0.001$.

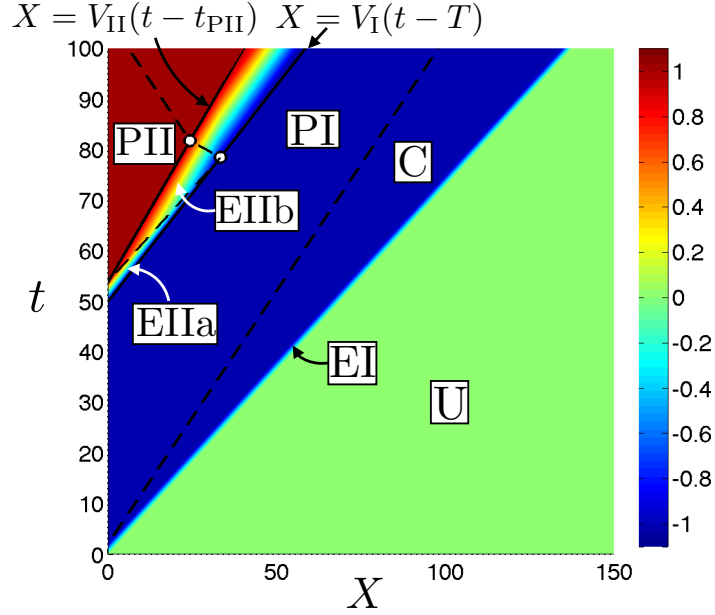


Figure 4.6: Heat map of the deviatoric strain $\varepsilon(X, t)$ using the same parameter values as Figure 4.5. An elastic wave EI propagates into undisturbed material U, followed by a plastic wave PI. As the material unloads, two elastic-plastic free boundaries propagate outward from $X = 0$, bounding a region of elastic material.

boundaries.

Even for this ostensibly simplest problem (the loading and unloading of a linear, rate-independent, elastic-plastic material) we uncover a rich ensemble of phenomena, all of which we will encounter again in the course of this chapter. These include the emergence of elastic and plastic regions separated by free boundaries which can propagate at non-characteristic velocities; internal reflections of elastic and plastic waves; and the possibility of re-yielding during unloading. This problem therefore provides a prototype for the more complex reflection problems analysed below. Finally, we note that the agreement between the analytic and numerical solutions evident in Figures 4.5 and 4.6 provides reassurance of both the accuracy of our numerical scheme, and the efficacy of the rate-dependent model with small non-zero a in selecting thermodynamically consistent rate-independent solutions.

4.6 Elastic-plastic wave interactions

The problem analysed in §4.5 was an idealised scenario in which the elastic-plastic medium was considered to be semi-infinite, thus removing the possibility of reflections from the free surface of the material at $X = 1$. In this section we consider the propagation of elastic and plastic waves through a sample of finite length when a stress free boundary

is introduced to the problem. We therefore solve (4.3.2), (4.3.3), and (4.3.5) subject to the initial and boundary conditions

$$\eta = U = \varepsilon = 0 \quad \text{at} \quad t = 0, \quad 0 \leq X \leq 1, \quad (4.6.1)$$

$$U = U_0(t)/\delta \quad \text{at} \quad X = 0, \quad t \geq 0, \quad (4.6.2)$$

$$\text{and} \quad \sigma_{11} = \eta + \frac{2k_0}{3}\varepsilon = 0 \quad \text{at} \quad X = 1, \quad t \geq 0. \quad (4.6.3)$$

We emphasise that, owing to the scalings (4.3.1), we have incurred a factor of $1/\delta$ in the boundary condition (4.6.2).

4.6.1 Constant acceleration

We begin by considering the simplest scenario in which we impose a constant compressive acceleration

$$U_0(t) = \alpha t \quad (4.6.4)$$

of the face $X = 0$. As is familiar from §4.5.1, the solution in region E_1 , bounded by the red characteristics, of Figure 4.7 is

$$\eta = -\frac{\alpha}{\delta c_e} \left(t - \frac{X}{c_e} \right), \quad U = \frac{\alpha}{\delta} \left(t - \frac{X}{c_e} \right), \quad \varepsilon = \frac{2}{3}\eta. \quad (4.6.5)$$

The material remains elastic until $\varepsilon = -1$, and we find that this happens on the line

$$X = c_e(t - \delta t_*), \quad t_* = \frac{3c_e}{2\alpha}. \quad (4.6.6)$$

Thus the elastic wave is very narrow when $\delta \ll 1$. In the plastic region P_1 , the solution is

$$\eta = -\frac{\alpha}{\delta}(t - X) + \frac{3}{2}(c_e - 1), \quad U = \frac{\alpha}{\delta}(t - X), \quad \varepsilon = -1 \quad (4.6.7)$$

with the elastic and plastic regions separated by a plateau in which η , U , and ε are given by (4.5.15).

Since $c_e > 1$, the elastic wave has time to reach the free surface and reflect from it before the arrival of the plastic wave. To determine the solution in the region E_2 where the incoming and outgoing elastic waves interact, we note that the elastic equations (4.5.4)–(4.5.6) are endowed with the Riemann invariants

$$U \mp c_e \eta = \text{constant} \quad \text{on} \quad \dot{X} = \pm c_e. \quad (4.6.8)$$

On $X = c_e(t - T)$, where $0 < T < \delta t_*$, we have

$$U - c_e \eta = 2\alpha T/\delta \quad (4.6.9)$$

in region E_3

$$\eta = \frac{\alpha}{\delta c_e} \left(t + \frac{X-2}{c_e} \right) - \frac{3}{2}, \quad (4.6.13)$$

$$U = \frac{\alpha}{\delta} \left(t + \frac{X-2}{c_e} \right) + \frac{3c_e}{2}, \quad (4.6.14)$$

$$\varepsilon = \frac{2}{3}\eta. \quad (4.6.15)$$

Finally, there is a constant region C_2 behind the advancing characteristic $X = -c_e(t - \delta t_*) + 2$ in which

$$\eta = 0, \quad U = 3c_e, \quad \varepsilon = 0. \quad (4.6.16)$$

The solutions (4.6.7) and (4.6.13)–(4.6.15) are valid until the leading plastic characteristic $X = t - \delta t_*$ intersects with the first reflected elastic characteristic $X = -c_e t + 2$, at which point the waves start to interact. This happens at

$$t = t_i^{(1)} = \frac{2 + \delta t_*}{c_e + 1}, \quad X = X_i^{(1)} = \frac{2 - \delta c_e t_*}{c_e + 1}. \quad (4.6.17)$$

In order to investigate the collision and subsequent interaction of the incoming elastic and plastic waves, we move into a frame centred at $(X, t) = (X_i^{(1)} + \delta X', t_i^{(1)} + \delta t')$ and consider the interaction depicted in Figure 4.8; namely, the elastic wave, where the solution is given by (4.6.13)–(4.6.15), from region E_3 propagates from right-to-left toward the incoming plastic wave P_1 , with solution (4.6.7), with the wave fronts first meeting at $(X', t') = (0, 0)$.

Before the interaction begins, the incoming plastic wave solution (4.6.7) in local coordinates (X', t') is of the form

$$\eta = -\alpha(t' - X') - \frac{3}{2}, \quad U = \alpha(t' - X') + \frac{3c_e}{2}, \quad \varepsilon = -1, \quad (4.6.18)$$

while the incoming elastic wave solution (4.6.13)–(4.6.15) is

$$\eta = \frac{\alpha}{c_e} \left(t' + \frac{X'}{c_e} \right) - \frac{3}{2}, \quad U = \alpha \left(t' + \frac{X'}{c_e} \right) + \frac{3c_e}{2}, \quad \varepsilon = \frac{2}{3}\eta. \quad (4.6.19)$$

During the interaction of the two wave types, there will in general be elastic and plastic waves travelling in both directions, separated by an elastic/plastic free boundary $X' = \xi(t')$. On the plastic side, $X' < \xi(t')$, the general solution is

$$\eta = -\alpha(t' - X') + f(t' + X') - \frac{3}{2}, \quad (4.6.20)$$

$$U = \alpha(t' - X') + f(t' + X') + \frac{3c_e}{2}, \quad (4.6.21)$$

$$|\varepsilon| = 1, \quad (4.6.22)$$

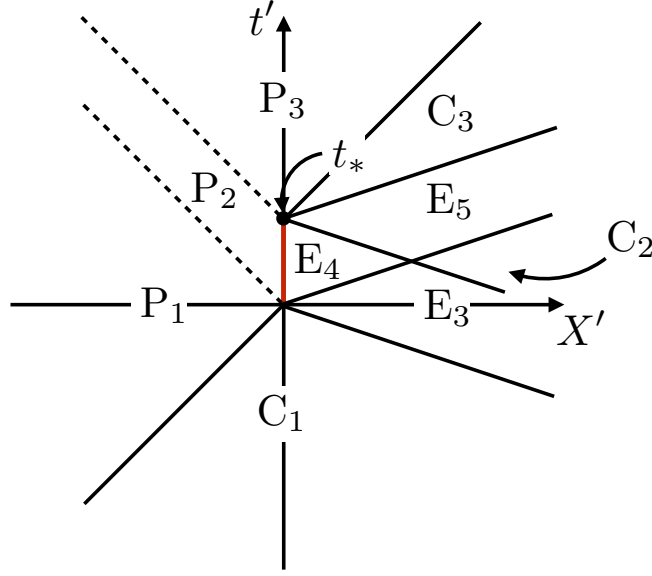


Figure 4.8: Close-up schematic of the interaction when $\dot{\xi} = 0$. The labels correspond to the respective regions in Figure 4.7.

while for $X' > \xi(t')$ on the elastic side

$$\eta = \frac{\alpha}{c_e} \left(t' + \frac{X'}{c_e} \right) - \frac{1}{c_e} g \left(t' - \frac{X'}{c_e} \right) - \frac{3}{2}, \quad (4.6.23)$$

$$U = \alpha \left(t' + \frac{X'}{c_e} \right) + g \left(t' - \frac{X'}{c_e} \right) + \frac{3c_e}{2} \quad (4.6.24)$$

and we assume for the moment that η and ε are related by

$$\varepsilon = \frac{2}{3}\eta \quad (4.6.25)$$

in the entire elastic region (in §4.6.2 we will see that this is not always true).

Following §4.4, we impose continuity of η , U , and ε across $X' = \xi(t')$, which leads to

$$\xi = 0 \quad \text{for} \quad 0 < t' < t_* \quad (4.6.26)$$

as shown by the solid red line in Figure 4.8. Therefore, the elastic-plastic free boundary is momentarily stationary as a result of the interaction between the two wave types. We also find that in region E_4 where the incoming and outgoing elastic waves interact

$$\eta = \frac{2\alpha}{c_e^2} X' - \frac{3}{2}, \quad U = 2\alpha t' + \frac{3c_e}{2}, \quad \varepsilon = \frac{2}{3}\eta \quad (4.6.27)$$

while in region P_2

$$\eta = 2\alpha X' - \frac{3}{2}, \quad U = 2\alpha t' + \frac{3c_e}{2}, \quad \varepsilon = -1. \quad (4.6.28)$$

The incoming and outgoing waves cease to interact at time $t' = t_*$, and we are left with an outgoing elastic wave E_5

$$\eta = -\frac{\alpha}{c_e} \left(t' - \frac{X'}{c_e} \right), \quad U = \alpha \left(t' - \frac{X'}{c_e} \right) + 3c_e, \quad \varepsilon = \frac{2}{3}\eta, \quad (4.6.29)$$

and an outgoing plastic wave P_3

$$\eta = -\alpha(t' - X') + \frac{3}{2}(c_e - 1), \quad U = \alpha(t' - X') + 3c_e, \quad \varepsilon = -1 \quad (4.6.30)$$

with the wave fronts separated by another constant region C_3 in which

$$\eta = \eta_*, \quad U = 3U_*, \quad \text{and} \quad \varepsilon = -1. \quad (4.6.31)$$

In summary, the interaction of the elastic and plastic wave results in a perturbation to the position of the elastic-plastic free boundary from its initial path along $X' = t'$. After the incoming waves have stopped interacting, a reflected elastic wave propagates back toward the free surface. The profile of the outgoing plastic wave is unchanged from its initial profile at $t' = 0$, although the plastic wave front now propagates along the line $X' = t' - t_*$. The elastic wave produced from this interaction will eventually reflect again from the free surface, leading to further interactions with the plastic wave; we will consider the cumulative effect of this reverberation in §4.7.

The solution presented in this section appears to be a very special case in which $\dot{\xi} = 0$, and hence the free boundary propagates along a characteristic of the elastic equations (4.5.4)–(4.5.6). Heuristically, the elastic and plastic waves have equivalent “strength” and cancel each other out, resulting in η being independent of t , and U being independent of X , in regions E_4 and P_2 . In fact, the region E_4 is analogous to the constant region C_1 , but is endowed with non-zero residual strain. This raises the question of how robust the structure depicted in Figure 4.8 is if we vary the strengths of the incident elastic and plastic waves; this question will be addressed in the next section.

4.6.2 Non-constant acceleration

We now consider the nature of the interaction if the acceleration at $X = 0$ is not constant, but is made concave or convex. The simplest boundary condition we can impose to model this scenario is the following:

$$U(t) = \begin{cases} \alpha_1 t & \text{for } 0 < t < \delta t_* \\ \alpha_2(t - \delta t_*) + \frac{3c_e}{2}\delta & \text{for } t > \delta t_*, \end{cases} \quad (4.6.32)$$

with now $t_* = 3c_e/2\alpha_1$. This corresponds to an acceleration α_1 when the material is deforming elastically, and an instantaneous switch to an acceleration α_2 when the material

has plasticised. This boundary condition is clearly artificial, and we would not have been able to posit its form had we not resolved the problem detailed in §4.5.1 to determine the point at which the material yields. However, as we shall see, this simple boundary condition gives insight into the different types of interaction that can occur between an elastic and plastic wave of different profiles.

4.6.2.1 Case (i): $\alpha_2 > \alpha_1$

We begin by considering the case when $\alpha_2 > \alpha_1$, and hence the material is accelerated further upon plasticising. The initial deformation of the sample, and the form of the reflected elastic wave from $X = 1$ may be determined in an identical way to §4.6.1. Moving again into the frame (X', t') , the solution of the incoming elastic wave in region E_3 of Figure 4.9 is

$$\eta = \frac{\alpha_1}{c_e} \left(t' + \frac{X'}{c_e} \right) - \frac{3}{2}, \quad U = \alpha_1 \left(t' + \frac{X'}{c_e} \right) + \frac{3c_e}{2}, \quad \varepsilon = \frac{2}{3}\eta, \quad (4.6.33)$$

and the incoming plastic wave solution in P_1 is

$$\eta = -\alpha_2(t' - X') - \frac{3}{2}, \quad U = \alpha_2(t' - X') + \frac{3c_e}{2}, \quad \varepsilon = -1. \quad (4.6.34)$$

By imposing continuity of η , U , and ε across the elastic-plastic free boundary $X' = \xi(t')$, as before, we find

$$\dot{\xi} = \frac{c_e(\alpha_2 - \alpha_1)}{c_e\alpha_2 + \alpha_1} = V_I. \quad (4.6.35)$$

Therefore, provided $\alpha_2 > \alpha_1$, the speed of the free boundary $V_I > 0$ as shown by the solid red line in Figure 4.9. In the special case $\alpha_1 = \alpha_2$, we recover the result for constant acceleration, namely $V_I = 0$. On the other hand, if $\alpha_2 \gg \alpha_1$ we see that $V_I \rightarrow 1$. This is consistent with the requirements for the free boundary to be causal. Either side of the free boundary, both η and U are unknown, as well as the position $X' = \xi(t')$. We therefore require 5 pieces of information on the free boundary, three of which are provided by continuity of η , U and ε . This implies that we must have two families of characteristics propagating into the free boundary; one from the elastic side and one from the plastic side (and hence there is one family of characteristics propagating away from the free boundary on each side). The speed of the free boundary therefore satisfies

$$0 \leq V_I \leq 1. \quad (4.6.36)$$

Again, we can show that in region E_4 where the incoming and outgoing elastic wave

interact we have

$$\eta = \frac{\alpha_1}{c_e} \left(t' + \frac{X'}{c_e} \right) - \gamma \frac{\alpha_1}{c_e} \left(t' - \frac{X'}{c_e} \right) - \frac{3}{2}, \quad (4.6.37)$$

$$U = \alpha_1 \left(t' + \frac{X'}{c_e} \right) + \gamma \alpha_1 \left(t' - \frac{X'}{c_e} \right) + \frac{3c_e}{2}, \quad (4.6.38)$$

$$\varepsilon = \frac{2}{3}\eta, \quad (4.6.39)$$

where $\gamma = (c_e + V_I)/(c_e - V_I)$, while in region P_2

$$\eta = -\alpha_2(t' - X') - \alpha_2 \frac{(V_I - 1)}{(V_I + 1)}(t' + X') - \frac{3}{2}, \quad (4.6.40)$$

$$U = \alpha_2(t' - X') - \alpha_2 \frac{(V_I - 1)}{(V_I + 1)}(t' + X') + \frac{3c_e}{2}, \quad (4.6.41)$$

$$\varepsilon = -1. \quad (4.6.42)$$

The two waves cease to interact at time

$$t' = \frac{t_*}{1 + V_I/c_e}. \quad (4.6.43)$$

It is then straightforward to show that in region E_5 we have an outgoing elastic wave of the form

$$\eta = -\alpha_1 \gamma \left(t' - \frac{X'}{c_e} \right), \quad (4.6.44)$$

$$U = \alpha_1 \gamma \left(t' - \frac{X'}{c_e} \right) + 3c_e, \quad (4.6.45)$$

$$\varepsilon = \frac{2}{3}\eta, \quad (4.6.46)$$

and in the plastic region P_3

$$\eta = -\alpha_2(t' - X') + \frac{3}{2}(c_e - 1), \quad (4.6.47)$$

$$U = \alpha_2(t' - X') + 3c_e, \quad (4.6.48)$$

$$\varepsilon = -1 \quad (4.6.49)$$

with the two wave fronts separated by a plateau C_3 in which

$$\eta = \eta_*, \quad U = 3U_*, \quad \text{and} \quad \varepsilon = -1. \quad (4.6.50)$$

Notice that in comparison to (4.6.29), the outgoing elastic wave has steepened by a factor of γ where, from the inequality (4.6.36), we find that $1 < \gamma < (c_e + 1)/(c_e - 1)$. When $\alpha_1 = \alpha_2$, we recover the analogous results in the case of constant acceleration detailed in §4.6.1, where $V_I = 0$ and hence $\gamma = 1$.

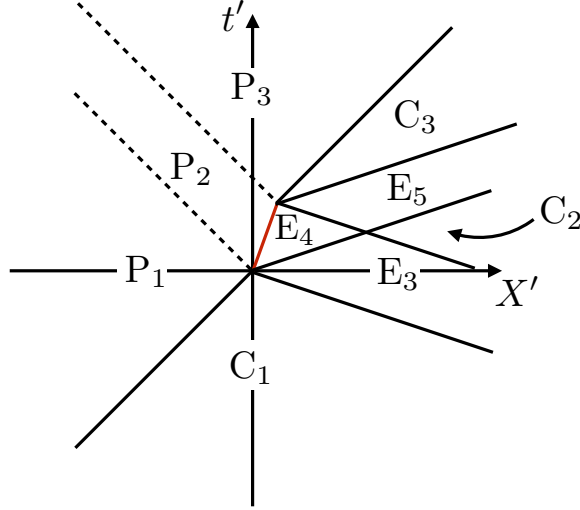


Figure 4.9: Schematic of the characteristic diagram when the free boundary propagates in the positive X direction.

We note that (4.6.35) appears to offer the possibility that $V_I < 0$ if $\alpha_2 < \alpha_1$, however we find that in the plastic region

$$\left. \frac{\partial \eta}{\partial t'} \right|_{X=V_I t} = -\frac{2\alpha V_I}{(V_I + 1)} \quad (4.6.51)$$

which is positive if $-1 < V_I < 0$; this solution therefore violates the dissipation condition (4.3.7) since $\varepsilon \eta_{t'} < 0$ and can be discounted. We also notice that if $V_I < -1$, then the dissipation condition is apparently satisfied since $\eta_{t'} < 0$. However, this leads to an incorrect prediction of the free boundary speed. As we shall see, both of these results are hardly surprising since, in arriving at (4.6.35), we assumed that $\varepsilon = 2\eta/3$ in the entire elastic region. We will see below that if the free boundary propagates in the negative- X' direction, this assumption becomes invalid as the elastic region inherits residual strain from material which has already been plasticised.

4.6.2.2 Case (ii): $\alpha_2 < \alpha_1$

The case where $\alpha_2 < \alpha_1$ is somewhat more complex than the case considered above, as the free boundary must now propagate in the negative- X' direction, with velocity $-V_{II}$ say. The structure of the interaction is shown in Figure 4.10. The elastic interaction region E_4 is decomposed into four sub-regions, E_{4a} – E_{4d} , in which the solution takes different forms. The characteristics $\dot{X}' = 0$ in E_{4a} carry information away from the free boundary $X' = -V_{II}t'$ (marked in red), while in E_{4b} , information is propagated away from the line

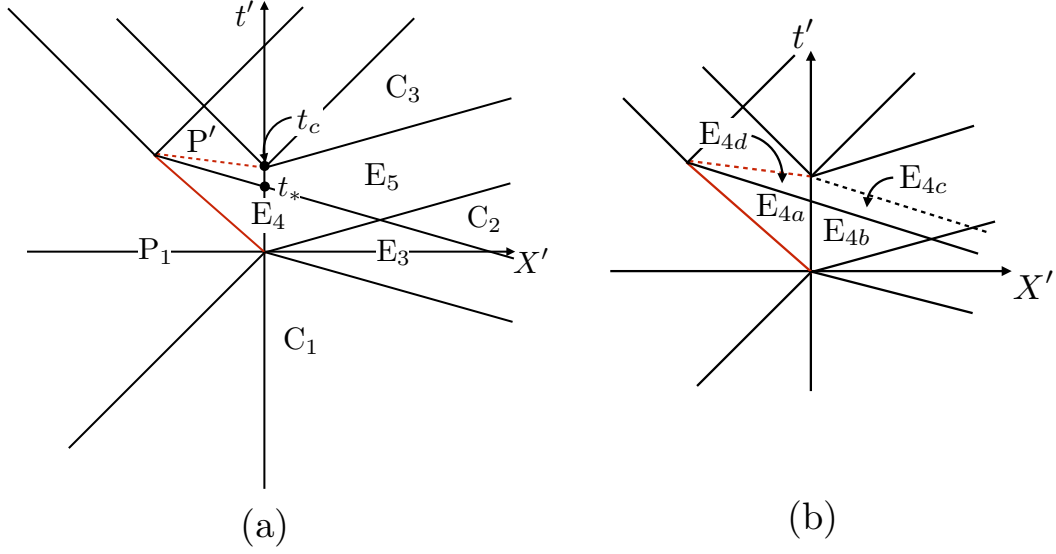


Figure 4.10: (a) A schematic of the characteristic diagram when the free boundary propagates in the negative X' direction and (b) a close-up of the region E_4 to highlight the sub-regions E_{4a-d} . The solid red line indicates the line $\dot{X} = -V_{II}$ and the dashed red line indicates $\dot{X} = -V_{III}$.

$X = c_e t'$. Therefore,

$$\frac{\partial}{\partial t'} \left(\frac{2}{3} \eta - \varepsilon \right) = 0 \implies \frac{2}{3} \eta - \varepsilon = r(X') \quad (4.6.52)$$

where $r(X')$ is no longer zero everywhere as in the cases considered in §4.6.1 and §4.6.2.1. As a result of the interaction between the elastic and plastic wave, the material can *locally unload* to become elastic, in contrast to the previous case where the material is always plastic for $X' < 0$.

We once more invoke causality to select a physically plausible solution for the scenario depicted in Figure 4.10. The unknowns are η and U in both the elastic and plastic regions, ε in the elastic region, and position of the free boundary $X = -V_{II} t'$. We therefore require six pieces of information on the free boundary, which can only be realised if three families of characteristics propagate into the free boundary to compliment the three continuity conditions for η , U and ε . This requires that two families of characteristics intersect the free boundary from the plastic side and one from the elastic side. Furthermore, we note that this implies there is no reflected plastic wave.

The incoming elastic characteristics that propagate into the free boundary $X' = -V_{II} t'$ (red solid line) are of the form

$$X' = -c_e(t' - T) \quad \text{for } T \in (0, t_*) \quad (4.6.53)$$

and intersect with the free boundary at $t_i = c_e T / (c_e - V_{II})$, and hence $t_i \in [0, c_e t_*/(c_e - V_{II})]$. The outgoing positive elastic characteristics are

$$X = c_e(t' - \gamma T) = c_e(t' - \tau) \quad \text{for } \tau \in (0, \gamma t_*) \quad (4.6.54)$$

for $t' \geq c_e \tau / (c_e + V_{II})$, where $\gamma = (c_e + V_{II}) / (c_e - V_{II})$. The Riemann invariants in the elastic region are now

$$c_e U \mp \sigma = \text{constant} \quad \text{on} \quad \dot{X}' = \pm c_e, \quad \sigma = \eta + \frac{3}{2}(c_e^2 - 1)\varepsilon. \quad (4.6.55)$$

We therefore have

$$c_e U + \sigma = 2c_e \alpha_1 T \quad (4.6.56)$$

on negative characteristics, while on the positive characteristics

$$c_e U - \sigma = c_e \alpha_2 \frac{(c_e + 1)(V_{II} + 1)}{c_e + V_{II}} \tau + 3c_e^2 \quad (4.6.57)$$

where we have used the fact that the solution in P_1 is entirely specified and given by (4.6.34). Hence, in the region where the incoming and outgoing elastic waves interact, labelled E_4 , we have

$$\sigma = c_e \alpha_1 \left(t' + \frac{X'}{c_e} \right) - c_e \alpha_2 \frac{(c_e + 1)(V_{II} + 1)}{2(c_e + V_{II})} \left(t' - \frac{X'}{c_e} \right) - \frac{3}{2} c_e^2. \quad (4.6.58)$$

In region E_{4a} we then have

$$\frac{2}{3} \eta - \varepsilon = r(X') = \frac{2\alpha_2}{3V_{II}} (V_{II} + 1) X' \quad (4.6.59)$$

from which we determine that

$$\varepsilon = \frac{1}{c_e^2} \left(\frac{2}{3} \sigma - r(X') \right). \quad (4.6.60)$$

Then, by imposing continuity of ε on $X' = -V_{II} t'$, we have

$$V_{II} = c_e - \frac{\alpha_2(c_e^2 - 1)}{\alpha_2(c_e - 1) + 2\alpha_1}. \quad (4.6.61)$$

We note that in the limit $0 < \alpha_2 \ll \alpha_1$, $V_{II} \rightarrow c_e$. We also find that when $\alpha_2 = \alpha_1$, $V_{II} = 1$ and the solution in E_{4a} reduces to (4.6.28) with residual strain

$$\frac{2}{3} \eta - \varepsilon = \frac{4\alpha}{3} X, \quad (4.6.62)$$

while the solution in E_{4b} reduces to the solution (4.6.27). Interestingly, we also see that the solution (4.6.28) satisfies both the plastic equations and the elastic equations, with residual strain (4.6.62).

In the entire elastic region E_4 , we find that $\varepsilon > -1$, and hence the material always remains elastic. Eventually, for $T > t_*$, the negative characteristics originate from the constant region C_2 in which $U = 3c_e$ and $\sigma = 0$. We therefore have $c_e U + \sigma = 3c_e^2$ on $\dot{X}' = -c_e$, while on the positive characteristics we still have (4.6.57). Therefore, in regions E_{4c} and E_{4d} , we have

$$\sigma = -c_e \alpha_2 \frac{(c_e + 1)(V_{II} + 1)}{2(c_e + V_{II})} \left(t' - \frac{X'}{c_e} \right). \quad (4.6.63)$$

From this, we find that ε returns to -1 on $X' = 0$ at time

$$t' = \frac{3c_e(c_e + V_{II})}{\alpha_2(c_e + 1)(V_{II} + 1)} = t_c. \quad (4.6.64)$$

The material therefore begins to deform plastically again, and we find that this occurs behind another elastic-plastic free boundary $X' = -V_{III}(t' - t_c)$ propagating outward from $X' = 0$ (indicated by the red dashed line in Figure 4.10). Note that when $\alpha_1 = \alpha_2$, $t_c = t_*$. By appealing once more to causality, we realise at once that the solution in the light grey region of Figure 4.10 is entirely specified, hence we must have two families of characteristics propagating into the free boundary from the elastic side. Furthermore, since $\varepsilon = -1$ in region P' , it only remains to find η and U in the plastic region, along with V_{III} . Continuity of η , U , and ε is therefore sufficient to determine the aforementioned three unknowns. We must then have two families of plastic characteristics leaving the free boundary. The immediate implication of this analysis is that the free boundary must propagate at a speed *faster* than the elastic wave speed, in order to preserve causality.

Imposing continuity of ε along $X' = -V_{III}(t' - t_c)$ allows us to find

$$V_{III} = \frac{V_{II}(c_e + 1)}{2c_e - (c_e - 1)V_{II}}. \quad (4.6.65)$$

Although the unwieldy details are omitted here, it can be shown that the solution in the plastic region P' satisfies the dissipation condition (4.3.7). Finally, we note that both free boundaries $X' = -V_{II}t'$ and $X = -V_{III}(t' - t_c)$ intersect the elastic characteristic $X = -c_e(t' - t_*)$ simultaneously at

$$t_f = \frac{c_e t_*}{c_e - V_{II}}. \quad (4.6.66)$$

4.6.3 Comparison with numerical solutions

To verify the foregoing predictions for V_I , V_{II} , and V_{III} , in Figure 4.11 we plot a direct numerical solution for ε versus X' and t' for the cases (a) $\alpha_1 = \alpha_2$, (b) $\alpha_2 > \alpha_1$, and (c) $\alpha_2 < \alpha_1$. In the numerical solution we approximate rate-independence by setting

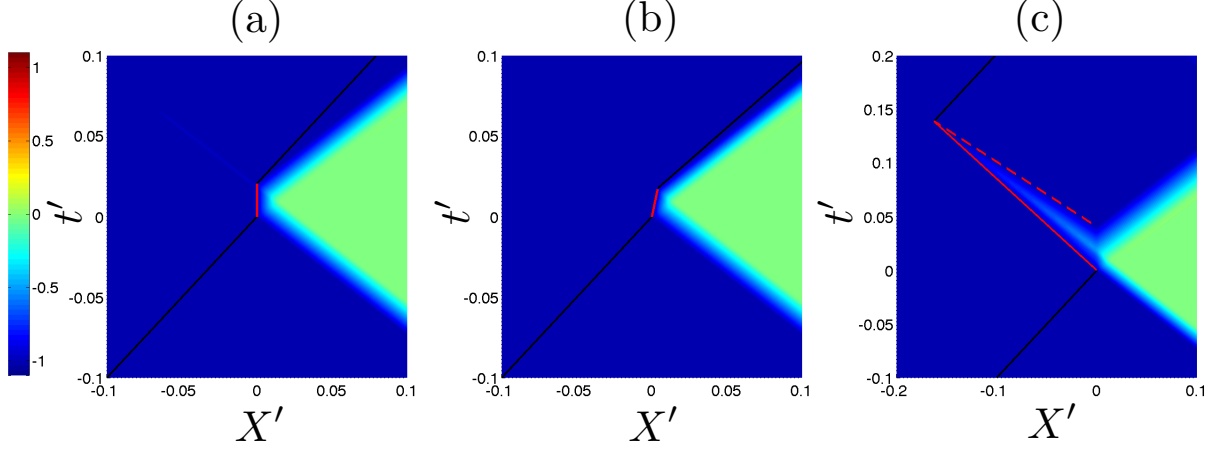


Figure 4.11: Plots of three heat maps of the deviatoric strain $\varepsilon(X', t')$ for (a) $\alpha_1 = \alpha_2 = 1$, (b), $\alpha_2 = 3/2 > \alpha_1 = 1$, and (c) $\alpha_2 = 1/2 < \alpha_1 = 1$. The black solid lines indicate the leading incoming and outgoing plastic characteristics for reference, while the solid and dashed red lines correspond to the positions of the three elastic-plastic free boundaries derived in the text.

the parameter $a = 0.001$. In case (a), the elastic-plastic free boundary (solid red line) clearly propagates along $X' = 0$ for $0 < t' < t_*$, while in case (b) the free boundary is slightly held back as a result of the interaction. Case (c) is more complex; initially the plastic wave front (marked by the solid black line) approaches $X' = 0$ and the interaction begins. As a result of the interaction there is a region of elastic material in $X' < 0$, which is bounded by the two free boundaries $X' = -V_{II}t'$ and $X' = -V_{III}(t' - t_c)$, indicated by the solid and dashed red lines respectively. Closer inspection of the dashed red line clearly shows that this free boundary does indeed propagate at a speed greater than c_e . Eventually, the elastic region disappears as the two free boundaries intersect the elastic characteristic $X' = -c_e(t' - t_*)$. In all three cases, we have extremely good agreement between the analytical predictions of this section and the numerical solutions.

4.6.4 Discussion

The free boundary problems discussed in this section are very delicate in nature. When the acceleration is not constant, as in §4.6.2, the cases $\alpha_2 > \alpha_1$ and $\alpha_2 < \alpha_1$ have entirely different structures, however both collapse onto the solution of constant acceleration when $\alpha_2 = \alpha_1$. In case (i), we found that $0 < V_I < 1$, while in case (ii) we identified one free boundary velocity satisfying $-c_e < -V_{II} < -1$ while the other, V_{III} , can travel faster than the elastic wave speed c_e . These observations are consistent with Morland and Cox who study the phenomena of interacting elastic and plastic waves, using a linear

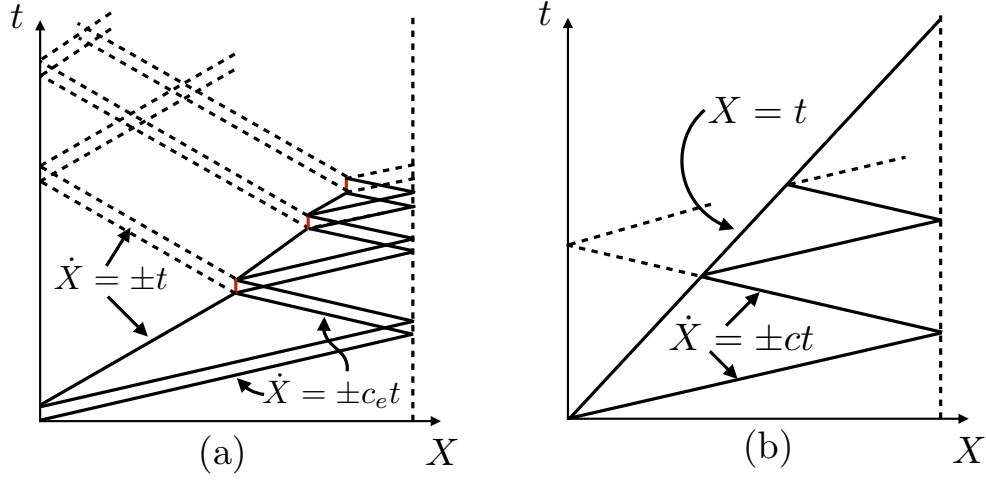


Figure 4.12: Schematic of the characteristic diagram after multiple reflections for (a) the elastic-plastic problem and (b) the pantograph problem.

elastic-plastic model derived from a heuristic engineering stress-strain relation. These fascinating, and somewhat unexpected, phenomena were corroborated with a comparison with direct numerical solutions, giving more reassurance that the structure of each of these interactions is thermodynamically feasible.

4.7 Multiple reflections: an analogue with the pantograph problem

So far we have only considered a single interaction between an elastic wave reflected from the free surface at $X = 1$ and the incoming plastic wave. We demonstrated that this results in a retardation of the elastic-plastic free boundary, and the production of an outgoing elastic wave which propagates back towards the free surface. One can then easily envisage the scenario after multiple reflections depicted in Figure 4.12(a), in which there are many such interactions leading to a reverberation of the elastic wave between the plastic wave front and $X = 1$. This scenario is reminiscent of the classic *pantograph problem* (Figure 4.12(b)) which concerns the modelling of the overhead electricity collection system for a train [23, 54]. In this problem, the role of the elastic-plastic free boundary is replaced with the given motion of the train path $X = t$, and the second wave speed c is the elastic wave speed in the wire.

4.7.1 Single interaction

We solve the system (4.3.2), (4.3.3), and (4.3.5), subject to the initial and boundary conditions (4.6.1)–(4.6.3). Informed by §4.6.2, we also assume that the boundary condition

at $X = 0$ satisfies

$$U_0(0) = 0, \quad U'_0(t) > 0, \quad \text{and} \quad U''_0(t) \geq 0 \quad (4.7.1)$$

so as to ensure that the elastic-plastic free boundary never travels backwards, and therefore

$$\frac{2}{3}\eta - \varepsilon = h(X) \equiv 0 \quad (4.7.2)$$

wherever the material is deforming elastically. The general solution of the p.d.e (4.3.2), (4.3.3), and (4.3.5) in the elastic region is therefore

$$\eta = -\frac{1}{c_e}f_e\left(t - \frac{X}{c_e}\right) + \frac{1}{c_e}g_e\left(t + \frac{X}{c_e}\right), \quad (4.7.3)$$

$$U = f_e\left(t - \frac{X}{c_e}\right) + g_e\left(t + \frac{X}{c_e}\right), \quad (4.7.4)$$

$$\varepsilon = \frac{2}{3}\eta, \quad (4.7.5)$$

and the general solution in the plastic region is

$$\eta = -f_p(t - X) + g_p(t + X), \quad (4.7.6)$$

$$U = f_p(t - X) + g_p(t + X), \quad (4.7.7)$$

$$\varepsilon = -1. \quad (4.7.8)$$

The functions f_e , g_e , f_p , and g_p are all assumed to be continuous, but with discontinuities in slope along the black solid lines in Figure 4.12.

By the initial conditions (4.6.1) we have

$$f_e(-s) = g_e(s) = 0 \quad \text{for} \quad 0 < s < \frac{1}{c_e} \quad (4.7.9)$$

while from the boundary condition (4.6.3) we have

$$f_e\left(t - \frac{1}{c_e}\right) = g_e\left(t + \frac{1}{c_e}\right) \quad \text{for} \quad t \geq 0 \quad (4.7.10)$$

and hence

$$g_e(s) = f_e\left(s - \frac{2}{c_e}\right) \quad \text{for} \quad s > \frac{1}{c_e}. \quad (4.7.11)$$

In particular, we note that $g_e \equiv 0$ at $X = 1$ for $0 < t \leq 1/c_e$, hence

$$g_e(s) \equiv 0 \quad \text{for} \quad 0 < s < \frac{2}{c_e}. \quad (4.7.12)$$

As is now familiar, as the velocity ramps up at $X = 0$, the material first deforms elastically. From (4.6.2) we have

$$f_e(s) = U(s)/\delta \quad \text{for} \quad 0 < s < t_c \quad (4.7.13)$$

where the constant t_c denotes the time at which the material turns plastic. Note that we expect $t_c \ll 2/c_e$, and hence in the interval $0 < t < t_c$, $g_e(t) \equiv 0$ from (4.7.12). The material remains elastic until $\varepsilon = -1$, from which we can determine

$$U(t_c) = \frac{3c_e}{2}\delta. \quad (4.7.14)$$

For $t > t_c$, the material begins to deform plastically, and hence

$$f_p(s) + g_p(s) = U(s)/\delta \quad \text{for } s > t_c. \quad (4.7.15)$$

In summary, the solution when the material is deforming elastically is

$$\eta = -\frac{1}{c_e}f_e\left(t - \frac{X}{c_e}\right) + \frac{1}{c_e}f_e\left(t + \frac{X-2}{c_e}\right), \quad (4.7.16)$$

$$U = f_e\left(t - \frac{X}{c_e}\right) + f_e\left(t + \frac{X-2}{c_e}\right), \quad (4.7.17)$$

$$\varepsilon = \frac{2}{3}\eta, \quad (4.7.18)$$

for $t > (1-X)/c_e$, while in the plastic region the solution is

$$\eta = -\frac{1}{\delta}U(t-X) + g_p(t-X) + g_p(t+X), \quad (4.7.19)$$

$$U = \frac{1}{\delta}U(t-X) - g_p(t-X) + g_p(t+X), \quad (4.7.20)$$

$$\varepsilon = -1 \quad (4.7.21)$$

for $t-X \geq t_c$. We note that regions in which η , U , and ε are constant are included in (4.7.16)–(4.7.18).

We now impose continuity of η , U and ε across the elastic/plastic free boundary $X = \xi(t)$, which leads to

$$-f_e\left(t - \frac{\xi}{c_e}\right) + f_e\left(t + \frac{\xi-2}{c_e}\right) = -\frac{3c_e}{2}, \quad (4.7.22)$$

$$-\frac{1}{\delta}U(t-\xi) + g_p(t-\xi) + g_p(t+\xi) = -\frac{3}{2}, \quad (4.7.23)$$

$$f_e\left(t - \frac{\xi}{c_e}\right) + f_e\left(t + \frac{\xi-2}{c_e}\right) = \frac{1}{\delta}U(t-\xi) - g_p(t-\xi) + g_p(t+\xi) \quad (4.7.24)$$

where

$$\xi = 0 \quad \text{for } 0 < t < t_c \quad (4.7.25)$$

and

$$f_e(s) = \begin{cases} 0 & \text{for } s < 0 \\ U(s)/\delta & \text{for } 0 < s < t_c. \end{cases}$$

Initially $\xi = t - t_c$, and the continuity conditions (4.7.22)–(4.7.24) are satisfied provided

$$g_p(s) = \frac{3}{4}(c_e - 1) \quad \text{for} \quad t_c < s < \frac{4}{c_e + 1} - \frac{c_e - 1}{c_e + 1}t_c = T_1. \quad (4.7.26)$$

Then from (4.7.24)

$$f_e\left(t - \frac{\xi}{c_e}\right) = \frac{3c_e}{2}, \quad f_e\left(t + \frac{\xi - 2}{c_e}\right) = 0, \quad (4.7.27)$$

and hence

$$f_e(s) \equiv \frac{3c_e}{2} \quad \text{for} \quad t_c < s < \frac{2t_c}{c_e + 1} + \frac{2(c_e - 1)}{c_e(c_e + 1)}. \quad (4.7.28)$$

In summary, in the interval $t_c \leq t - \xi \leq T_1$ we have

$$g_p(t - \xi) = \frac{3}{4}(c_e - 1), \quad g_p(t + \xi) = -\frac{3}{4}(c_e + 1) + \frac{1}{\delta}U(t - \xi) \quad (4.7.29)$$

and hence by adding and subtracting (4.7.22) and (4.7.24) we arrive at

$$f_e\left(t - \frac{\xi}{c_e}\right) = \frac{1}{\delta}U(t - \xi) = f_e\left(t + \frac{\xi - 2}{c_e}\right) + \frac{3c_e}{2}. \quad (4.7.30)$$

Equations (4.7.30) comprise a *functional equation* for f_e and the position of the free boundary ξ in terms of the applied velocity U , subject to

$$U(t_c) = \frac{3c_e}{2}\delta, \quad (4.7.31)$$

$$f_e(s) = \begin{cases} 0 & \text{for } s < 0 \\ U(s)/\delta & \text{for } 0 < s < t_c, \end{cases} \quad (4.7.32)$$

$$\xi(t_c) = 0. \quad (4.7.33)$$

For the first interaction between the elastic and plastic wave, $\xi = t - t_c$ until $t = t_i = (2 + t_c)/(c_e + 1)$ where, from (4.7.32), we have that

$$f_e\left(t + \frac{\xi - 2}{c_e}\right) = \frac{1}{\delta}U\left(t + \frac{\xi - 2}{c_e}\right). \quad (4.7.34)$$

Thus, we can determine ξ from

$$U(t - \xi) = U\left(t + \frac{\xi - 2}{c_e}\right) + \frac{3c_e}{2}\delta. \quad (4.7.35)$$

Henceforth we focus on the simplest case from §4.6.1, where $U_0(t) = \alpha t$, and we find that $t_c = \frac{3c_e}{2\alpha}\delta = \delta t_* \ll 1$, and then

$$\xi = \frac{2 - c_e t_c}{c_e + 1} = \xi_i = t_i - t_c \quad \text{for} \quad t_i < t < t_i + t_c, \quad (4.7.36)$$

which is precisely the result (4.6.17). This case is summarised in Figure 4.13.

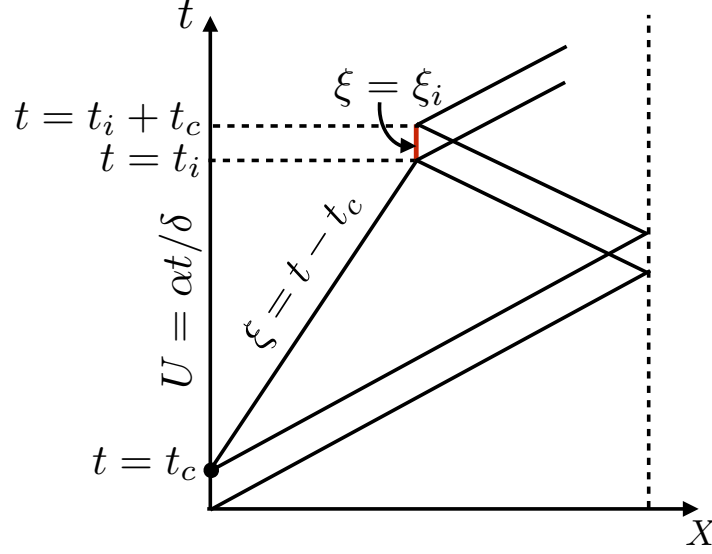


Figure 4.13: Schematic of the first interaction between the elastic wave and plastic wave described in §4.7.1.

4.7.2 Multiple interactions

4.7.2.1 Leading-order pantograph problem

The preceding analysis showed that $t_c \ll 1$, specifically $O(\delta)$. We therefore posit that the free boundary is of the form

$$X = \xi(t) = t - \delta\beta(t) \quad (4.7.37)$$

where $\beta(t)$ is the time-dependent perturbation to the free boundary position and $\beta(t_c) = 3c_e/2\alpha$.

In the limit $\delta \rightarrow 0$, the functional equation (4.7.30) reads

$$f_e\left(\frac{c_e - 1}{c_e}t\right) = \alpha\beta(t) = f_e\left(\frac{(c_e + 1)t - 2}{c_e}\right) + \frac{3c_e}{2} \quad (4.7.38)$$

while the boundary condition (4.7.32) becomes

$$f_e(s) = \begin{cases} 0 & \text{for } s < 0 \\ \frac{\alpha s}{\delta} & \text{for } 0 < s < \delta t_*. \end{cases} \quad (4.7.39)$$

This boundary condition is identical to the constant acceleration boundary condition we imposed in §4.6.1, and hence satisfies the condition (4.7.1). Therefore, while the free boundary remains close to $\xi = t$, we expect each interaction and perturbation to the free boundary to be described by the analysis of §4.6.1.

From (4.7.38) we have

$$\beta(t) = \beta \left(\frac{c_e + 1}{c_e - 1} \left(t - \frac{2}{c_e + 1} \right) \right) + \frac{3c_e}{2\alpha} \quad (4.7.40)$$

subject to

$$\beta(s) = \begin{cases} 0 & \text{for } s < 0 \\ \frac{s}{\delta} & \text{for } 0 < s < \delta t_*. \end{cases} \quad (4.7.41)$$

Alternatively, setting $t = 1 - \tau$, $\beta(1 - \tau) = \frac{3c_e}{2\alpha} y(\tau)$ we obtain

$$y(\tau) = y(q\tau) + 1 \quad (4.7.42)$$

where $q = (c_e + 1)/(c_e - 1)$ and

$$y(s) = \begin{cases} 0 & \text{for } s > 1 \\ \frac{1 - s}{\delta t_*} & \text{for } 1 - \delta t_* < s < 1. \end{cases} \quad (4.7.43)$$

Equation (4.7.42) is a special case of the more general functional differential equation for the pantograph problem considered by Ockendon, Fox et al. [23, 38, 75].

In the limit $\delta \rightarrow 0$, we see from (4.7.43) that $y(\tau) \equiv 0$ until $\tau = 1$, at which point there is unit jump in y . The function y can therefore be described by

$$y(\tau) = \mathcal{H}(1 - \tau) \quad (4.7.44)$$

where $\mathcal{H}$ is the Heaviside function. This solution is valid in the interval $1/q < \tau < 1$, where there is a further unit jump in the value of y at $\tau = 1/q$, and so on. The solution of (4.7.42)–(4.7.43) can therefore be written as a sum of Heaviside functions

$$y(\tau) = \sum_{n=0}^{\infty} \mathcal{H} \left(\frac{1}{q^n} - \tau \right). \quad (4.7.45)$$

Equation (4.7.45) implies that $y(\tau) = n + 1$ in the interval $1/q^{n+1} < \tau < 1/q^n$. Hence, we can alternatively write

$$y(\tau) = \left\lfloor \frac{\log(1/\tau)}{\log q} \right\rfloor + 1 \sim \frac{1}{2} + \frac{\log(1/\tau)}{\log q} \quad \text{as } \tau \rightarrow 0. \quad (4.7.46)$$

The perturbation to the free boundary is then

$$\beta(t) = \frac{3c_e}{2\alpha} \left(1 + \left\lfloor \frac{\log(1/(1-t))}{\log q} \right\rfloor \right). \quad (4.7.47)$$

We plot the solution $y(\tau)$ in Figure 4.14, and overlay the asymptote (4.7.46), which demonstrates the correct asymptotic behaviour as $\tau \rightarrow 0$.

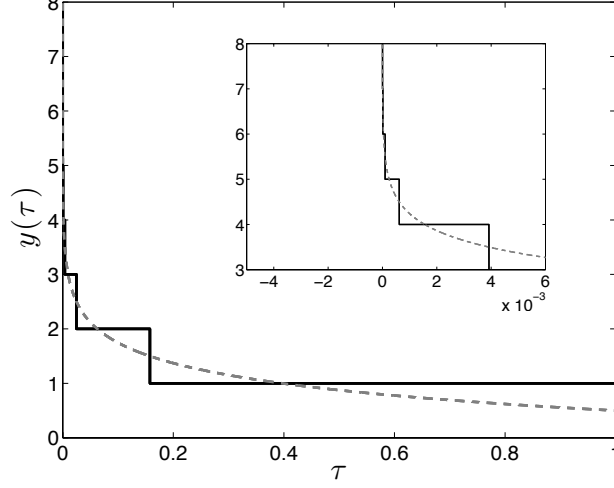


Figure 4.14: Plot of y versus τ where $q = (c_e + 1)/(c_e - 1)$ and $c_e = 1.4$. As $\tau \rightarrow 0$ the solution is well described by the asymptote (4.7.46) indicated by the dashed grey line.

4.7.2.2 First-order corrections

In taking the limit $\delta \rightarrow 0$ in the previous subsection, we saw that the jumps in $\beta(t)$ were described by Heaviside functions. We note, however, that the size of the interval between the jumps in $y(\tau)$ scales with $1/q^n$, and decreases exponentially with n . Eventually, when $n = O(-\log \delta)$, the size of the intervals become comparable with δ , and the jumps in $y(\tau)$ are no longer described by Heaviside functions. Instead, β varies continuously over a region of size δ around each point $t_n = 1 - 1/q^n$. To capture the behaviour of y when $t = 1 - \tau = O(\delta)$, we then need to include first-order corrections to (4.7.61).

To describe the details of the jumps in $\beta(t)$ we define

$$h(s) = \begin{cases} 0, & s < 0, \\ s, & 0 < s < 1, \\ 1, & s > 1. \end{cases}$$

Initially, we then have

$$\beta(t) = \frac{t_c}{\delta} h\left(\frac{t}{t_c}\right), \quad 0 < t < t_c \quad (4.7.48)$$

and from (4.7.30), now retaining terms of $O(\delta)$, we have

$$f_e\left(\frac{c_e - 1}{c_e}t + \frac{\delta\beta}{c_e}\right) \sim \alpha\beta(t) \sim f_e\left(\frac{c_e + 1}{c_e}t - \frac{2}{c_e} - \frac{\delta\beta}{c_e}\right) + \frac{3c_e}{2}h\left(\frac{t}{t_c}\right). \quad (4.7.49)$$

To reduce clutter we set $t = 1 - \tau$, $\epsilon = t_c \ll 1$, and

$$f_e(s) = \tilde{f}\left(1 - \frac{c_e s}{c_e - 1}\right), \quad \beta(\tau) = \frac{t_c}{\delta} y(1 - \tau); \quad (4.7.50)$$

hence we obtain

$$\tilde{f}\left(\tau - \frac{q-1}{2}\epsilon y(\tau)\right) = y(\tau) = \tilde{f}\left(q\tau + \frac{q-1}{2}\epsilon y(\tau)\right) + h\left(\frac{1-\tau}{\epsilon}\right) \quad (4.7.51)$$

where initially

$$y(\tau) = h\left(\frac{1-\tau}{\epsilon}\right). \quad (4.7.52)$$

Note that the function h generalises the Heaviside function in (4.7.44)–(4.7.45).

Inspired by (4.7.45), we seek a solution of (4.7.51)–(4.7.52) in the form

$$y(\tau) = \sum_{n=0}^{\infty} h\left(\frac{\tau_n - \tau}{\epsilon}\right), \quad (4.7.53)$$

where the scalars τ_n are to be determined. In original variables, (4.7.53) takes the form

$$\xi = t - t_* \sum_{n=0}^{\infty} h\left(\frac{t + \tau_n - 1}{\epsilon}\right). \quad (4.7.54)$$

For $\tau_n < \tau < \tau_{n-1} - \epsilon$, we have $y(\tau) = n$, and hence

$$\tilde{f}\left(\tau - \frac{q-1}{2}\epsilon n\right) = \tilde{f}\left(q\tau + \frac{q-1}{2}\epsilon n\right) + 1 = n \quad (4.7.55)$$

which implies that

$$\tilde{f}(s) = \tilde{f}\left(qs + \frac{\epsilon n}{2}(q^2 - 1)\right) + 1 = n \quad (4.7.56)$$

for $\tau_n - \epsilon n(q-1)/2 < s < \tau_{n-1} - \epsilon - (q-1)\epsilon n/2$. Equally we have

$$\tilde{f}(\eta) = n - 1 \quad \text{for} \quad q\tau_n + \frac{q-1}{2}\epsilon n < \eta < q\tau_{n-1} - \epsilon q + \frac{\epsilon n}{2}(q-1). \quad (4.7.57)$$

Requiring (4.7.56) and (4.7.57) to be consistent then leads to the following difference equation:

$$q\tau_n - \tau_{n-1} + \frac{(2n-1)\epsilon(q-1)}{2} = 0 \quad (4.7.58)$$

where $\tau_0 = 1$. We therefore find

$$\tau_n = -\epsilon n + \frac{\epsilon(q+1)}{2(q-1)}(1 - q^{-n}) + q^{-n}. \quad (4.7.59)$$

We note that as $\epsilon \rightarrow 0$, $\tau \sim q^{-n}$, which agrees with (4.7.45). However, when $n \gg 1$, the leading term $-\epsilon n$ begins to dominate.

All of the preceding analysis then suggests that for $n \gg 1$ and $\epsilon \ll 1$,

$$y \sim n \quad \text{when} \quad 1 - t \sim \frac{1}{q^n} - \epsilon n + \frac{\epsilon(q+1)}{2(q-1)} \quad (4.7.60)$$

and thus

$$\xi(t) \sim t - \epsilon n \sim 1 - \frac{1}{q^n} - \frac{\epsilon(q+1)}{2(q-1)} \sim 1 - \frac{\epsilon(q+1)}{2(q-1)} - \exp\left[-\frac{(t-1)\log q}{\epsilon}\right] \quad \text{for} \quad t \gtrsim 1. \quad (4.7.61)$$

We therefore find that the elastic-plastic free boundary never reaches $X = 1$, and that close to the free surface there always exists a region of material which remains unyielded.

4.7.2.3 Comparison with numerics

We compare our prediction for the position of the free boundary (4.7.54) and the large t approximation (4.7.61) in three cases:

$$(i) U_0(t) = t, \quad (ii) U_0(t) = e^t - 1, \quad \text{and} \quad (iii) U_0(t) = 1 - e^{-t}. \quad (4.7.62)$$

Case (i) is precisely the case analysed above with $\alpha = 1$, while case (ii) satisfies the inequalities (4.7.1), and hence we expect the elastic-plastic free boundary to always propagate forwards, with plasticised material never returning to being elastic. In case (iii), the applied acceleration decreases so the plastic wave grows weaker with time, and we expect the situation shown in Figure 4.10, with the elastic-plastic free boundary moving backwards during interactions. Note that in all three cases $U'(0) = 1$.

In Figure 4.15, we plot a heat map of the numerical solution of (4.3.2)–(4.3.4) for the deviatoric strain ε (again $a = 0.001$ to approximate rate-independence). The solid black line shows the predicted position of the free boundary (4.7.54) and the dashed grey line plots the asymptote (4.7.61). Initially the free boundary propagates along the line $X = t - \delta t_*$ until the elastic wave reflects from the free surface and meets the plastic wave. The interaction is shown in inset (a), where we see that the interaction results in the free boundary momentarily propagating along the line $\xi = \xi_i$ as in §4.6.1. There is then a reflected elastic wave, which propagates back toward the free surface, and an outgoing plastic wave. Inset (b) shows that, after multiple reverberations of the elastic wave, there is a condensation of incoming and outgoing elastic waves, leading to an unyielded region of material close to the free surface of width $O(\delta)$. The width of this region is well described by the asymptote (4.7.61). In this case, we have excellent agreement between the analytical predictions detailed in this section and the numerical solution. Note however that this situation is markedly different to that shown in Figure 3.5(a), since there is no reflected plastic release wave and $\varepsilon = -1$ in the entire plastic region.

In Figure 4.16, we again plot a heat map of the deviatoric strain ε , using the boundary condition (4.7.62)(ii) (left) and (4.7.62)(iii) (right). Insets (a) and (c) again show that, as the two waves interact, the elastic-plastic free boundary is perturbed onto the line $\xi = \xi_i$. Recall that, even though the boundary conditions in both cases do not correspond to a constant acceleration, the solution in the vicinity of the plastic wave front $X \sim t$, depends on the behaviour of $U_0(t)$ when $t = O(\delta)$, and therefore $U_0(t)$ is approximately linear in t . However, as the wave front approaches $X = 1$, it begins to sample U at larger values of its argument and begins to notice the concavity/convexity of the velocity drive $U_0(t)$, and hence the solutions begins to depart from the situation shown in Figure 4.15.

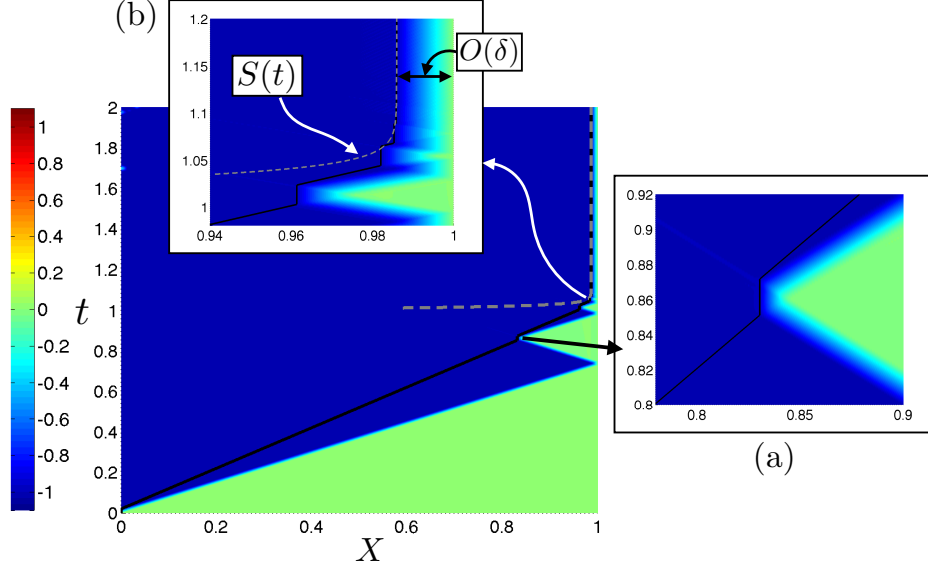


Figure 4.15: Heat map of the deviatoric strain $\varepsilon(X, t)$ for $U_0(t) = t$, $c_e = 1.4$, and $\delta = 0.01$. The solid black line shows the position of the elastic-plastic free boundary, which is perturbed upon each interaction with an elastic wave. As $t \rightarrow 1$, these interactions begin to condense to produce a region of unyielded material close to $X = 1$ of width $O(\delta)$. The asymptotic prediction (4.7.61) in dashed grey is shown to agree extremely well with the numerical prediction of the width of the elastic region close to $X = 1$.

In case (ii), where $U_0(t)$ is a convex function, the picture is similar to the case of constant acceleration, namely that the reverberation of the elastic wave leads to an unyielded region of material close to $X = 1$, and otherwise the material remains plastic. The picture in case (iii) is dramatically different. In the vicinity of $X = t = 1$, we see that initially the position of the elastic plastic free boundary is well described by the asymptote (4.7.61) as the elastic waves begin to coalesce as shown in inset (d). However, eventually, we observe that after a finite time the material re-yields over a thin elastic region propagating away from $X = 1$ in an analogous way to Figure 3.5(a). This occurs because after multiple interactions, and hence repeated perturbations to the position of the free boundary, the cumulative effect of the perturbations is such that $\xi - t = O(1)$. This in turn implies that the plastic wave begins to notice the deceleration of the velocity at $X = 0$. There is then a transition into a region where the interaction between the two wave types is such that the material locally unloads as in §4.6.2, the ultimate effect of which is the rapid transition of ε from -1 to $+1$ over the elastic region propagating away from the free surface.

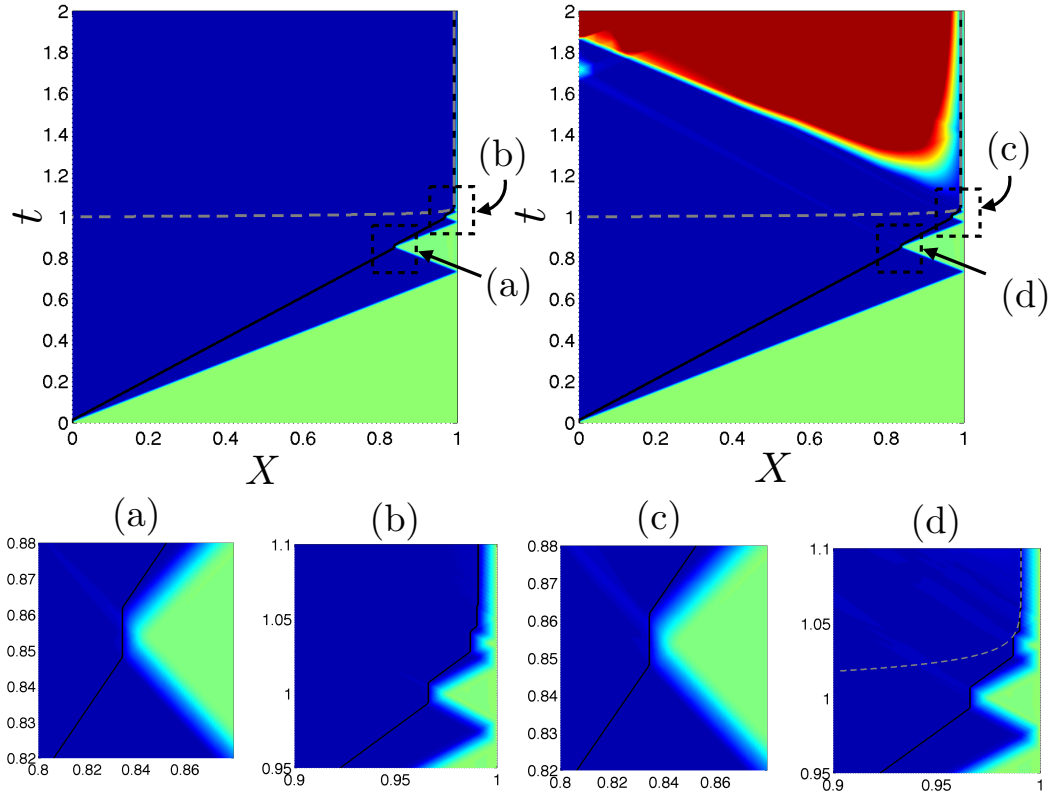


Figure 4.16: Heat map of the deviatoric strain ε for $U_0(t) = e^t - 1$ (left) and $U_0(t) = 1 - e^{-t}$ (right). The parameter values are otherwise the same as in Figure 4.15. Initially, in both cases, the position of the free boundary (solid black line) is well described by (4.7.54). As $t \rightarrow 1$, the elastic waves begin to coalesce in the same way as Figure 4.15. Eventually, however, the solutions diverge as the material re-yields in case (iii) as the boundary at $X = 0$ is decelerated.

4.8 Conclusions

By studying the behaviour of the model (2.3.54)–(2.3.56) in the simplest possible case of linear, rate-independent elastoplasticity, we have succeeded in shedding light on many of the key mechanisms underpinning elastic-plastic wave propagation. Our analysis began by considering the sequential loading and unloading of a semi-infinite elastic-plastic material. As expected, we found that as the stress on the material increases toward the yield stress, an elastic wave is produced which propagates with wave speed $c_e > 1$. This is followed by a larger amplitude but slower moving plastic wave as the stress is increased further beyond the yield surface. During unloading, the material first has to deform elastically, but then may create a new plastic region due to the phenomenon of re-yielding. This gives rise to two elastic-plastic interfaces, whose speed is determined by the solution of a free boundary problem.

Our next step was to introduce a stress-free boundary at $X = 1$, and then to consider the ensuing interaction of plastic waves and reflected elastic waves from the free surface. This led to an intricate free boundary problem, where elastic and plastic regions are separated by interfaces propagating at non-characteristic speeds. The exception to this was the very special case of constant acceleration, where the elastic-plastic interface propagates along a line $X = \text{constant}$, and the solution in the region where the two waves interact satisfies both the elastic and plastic equations with residual strain. These solutions were shown to be consistent with a direct numerical solution of the linear model (4.3.2)–(4.3.4), and hence gives us confidence that the free boundaries identified in §4.6 are thermodynamically feasible.

Once the two wave types cease to interact, we are left with an outgoing elastic wave, which propagates back toward the free surface, and can thus reverberate multiple times before the plastic wave arrives at $X = 1$. We found that this process of reverberation is a generalisation of the classic pantograph problem. In the cases where the boundary is subject to a constant acceleration and an acceleration which is convex, we found that coalescence of elastic waves leads to an unyielded region of material close to $X = 1$. However, if the boundary is decelerated, the picture is dramatically different, with the material re-yielding in an analogous way to Figure 3.5(a), and a plastic release wave propagating back into the domain.

In summary, the nature of the interaction between the elastic and plastic waves is extremely sensitive to a number of phenomena. In §4.6, our local analysis revealed that whether or not the plastic region relaxes and becomes elastic again depends on the relative “strengths” of the interacting waves. The consequences of this extend further into the ultimate release of the plastic wave from the free boundary, which otherwise seems to be universally observed in numerical solutions, e.g. Figure 3.5, and in experiments. In this rate-independent, linear model, whether or not this release behaviour manifests relies on the deceleration of the boundary at $X = 0$. We also neglected the influence of important non-linear effects, including the steepening of the plastic wave due to the nonlinearity of the function $h(\nu)$. We will see in chapter 5 that this means the compressive plastic wave always overcomes the elastic wave, and there can be no local unloading of the plastic material as in §4.6.2. Finally, the inclusion of rate-dependence would mean that $|\varepsilon| > 1$ in the plastic region, and hence this would possibly inhibit the plastic material reverting to being elastic. Such is the delicacy of the elastic-plastic interactions, that we anticipate such modifications to the model, even when apparently small, may affect the nature of the solutions profoundly. In the next chapter, we will analyse the interaction problem when non-linearity and rate-dependence are reinstated into the model.

Chapter 5

Elastic-plastic wave propagation: weakly nonlinear analysis

5.1 Synopsis

Our analysis of elastic-plastic wave propagation in chapter 4 neglected the effects of nonlinearity and rate-dependence. In this chapter, we provide a systematic mathematical analysis of how the inclusion of these phenomena gives rise to behaviour unforeseen by the linear rate-independent model. We exploit the smallness of the parameter δ to focus on regions where elastic effects are most prominent, and quantify how, in these regions, the model differs from one based on barotropic flow.

5.2 Early time behaviour

To investigate the early time deformation of the material we make the scalings $X = \delta X'$, $t = \delta t'$ and expand

$$\nu = 1 + \delta \nu' + O(\delta^2), \quad u = \delta u' + O(\delta^2). \quad (5.2.1)$$

After dropping the prime notation, the governing equations (2.3.54)–(2.3.56) reduce to the following semi-linear system

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0, \quad (5.2.2)$$

$$\frac{\partial u}{\partial t} - \frac{\partial \nu}{\partial X} - \frac{2k_0}{3} \frac{\partial \varepsilon}{\partial X} = 0, \quad (5.2.3)$$

$$\text{and} \quad a \left(\frac{2}{3} \frac{\partial \nu}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right) = G(\varepsilon), \quad (5.2.4)$$

where $h''(1) = \varepsilon_c(1) = 1$ and $k_0 = k(1)$. Following the above scalings, the initial and boundary conditions (3.2.1) and (3.2.2) reduce to the following as $\delta \rightarrow 0$:

$$\nu = u = \varepsilon = 0 \quad \text{at} \quad t = 0, \quad X > 0, \quad (5.2.5)$$

$$u \sim U'(0)t \quad \text{at} \quad X = 0, \quad t > 0, \quad (5.2.6)$$

where we assume $U'(0) > 0$. In what follows, we set $U'(0) = 1$ without loss of generality.

As is now familiar, initially the material deforms elastically upon loading, and hence the solution in the elastic region is

$$\nu = -\frac{1}{c_e} \left(t - \frac{X}{c_e} \right), \quad u = t - \frac{X}{c_e}, \quad \varepsilon = \frac{2}{3}\nu. \quad (5.2.7)$$

The material remains elastic until $\varepsilon = -1$, which happens on the elastic characteristic $X = c_e(t - t_*)$ with $t_* = 3c_e/2$ as in §4.5.1. In the region $t > t_* + X/c_e$ we are then left to solve

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial X} = 0, \quad (5.2.8)$$

$$\frac{\partial u}{\partial t} - \frac{\partial \nu}{\partial X} - \frac{2k_0}{3} \frac{\partial \varepsilon}{\partial X} = 0, \quad (5.2.9)$$

$$a \left(\frac{2}{3} \frac{\partial \nu}{\partial t} - \frac{\partial \varepsilon}{\partial t} \right) = \varepsilon + 1, \quad (5.2.10)$$

where, in (5.2.10), we have used the fact that the yield function $G = \varepsilon + 1$ when the material is deforming plastically.

In the case of rate-independent plasticity ($a \equiv 0$), the solution is nothing more than (4.5.14) with $\alpha = 1$, with the elastic and plastic wave fronts separated by the constant region (4.5.11). Before we proceed to the solution of (5.2.8)–(5.2.10) for finite a , it is interesting to note that these equations may be combined into a single equation for the velocity u , namely

$$a \frac{\partial}{\partial t} (u_{tt} - c_e^2 u_{XX}) + u_{tt} - u_{XX} = 0. \quad (5.2.11)$$

The properties of equations of the form (5.2.11) have been studied by numerous authors, most notably Whitham [85] in connection with magnetohydrodynamics and subsequently by Ockendon and Spence [52] in the context of relaxing gas flow. Among the observations made in [85], the Whitham theory predicts that the fastest wave speed is governed by the highest derivatives, in this context the elastic wave speed c_e , and the elastic characteristics $dX/dt = \pm c_e$ are called *supercharacteristics* of the p.d.e (5.2.2)–(5.2.4), while the plastic characteristics $dX/dt = \pm 1$ are *subcharacteristics*.

The system (5.2.8)–(5.2.10) is susceptible to a Laplace transform if we first switch to the variables $x = X$ and $T = t - t_* - X/c_e$ so that the problem now lies in the domain

$(x, T) \in [0, \infty] \times [0, \infty]$. We also rescale the variables $\nu = -3/2 + \tilde{\nu}$, $u = 3c_e/2 + \tilde{u}$ and $\varepsilon = -1 + \tilde{\varepsilon}$ so that $\tilde{\nu} = \tilde{u} = \tilde{\varepsilon} = 0$ along $T = 0$. The governing equations become

$$\tilde{\nu}_T - \tilde{u}_x + \frac{\tilde{u}_T}{c_e} = 0, \quad (5.2.12)$$

$$\tilde{u}_T - \tilde{\nu}_x + \frac{\tilde{\nu}_T}{c_e} - \frac{3}{2}(c_e^2 - 1) \left(\tilde{\varepsilon}_x - \frac{\tilde{\varepsilon}_T}{c_e} \right) = 0, \quad (5.2.13)$$

$$a \left(\frac{2}{3} \tilde{\nu}_T - \tilde{\varepsilon}_T \right) = \tilde{\varepsilon}, \quad (5.2.14)$$

where the subscripts x and T are short-hand for partial derivatives, with the boundary conditions

$$\tilde{u} = T \quad \text{at} \quad x = 0, \quad (5.2.15)$$

$$\tilde{\nu} = \tilde{u} = \tilde{\varepsilon} = 0 \quad \text{at} \quad T = 0. \quad (5.2.16)$$

The Laplace transform of a function f and its associated inverse are defined

$$\bar{f}(x, p) = \int_0^\infty f(x, T) e^{-pT} dT, \quad f(x, T) = \frac{1}{2\pi i} \int_C \bar{f} e^{pT} dp \quad (5.2.17)$$

where C is the Bromwich contour [10]. Taking the Laplace transform of Equations (5.2.12)–(5.2.15) and using (5.2.16) yields the following system of ordinary differential equations

$$\bar{u}_x = \frac{p}{c_e} \bar{u} + p\bar{\nu}, \quad (5.2.18)$$

$$\bar{\nu}_x - \frac{p}{c_e} \bar{\nu} + \frac{3}{2}(c_e^2 - 1) \left(\bar{\varepsilon}_x - \frac{p}{c_e} \bar{\varepsilon} \right) = p\bar{u}, \quad (5.2.19)$$

$$ap \left(\frac{2}{3} \bar{\nu} - \bar{\varepsilon} \right) = \bar{\varepsilon} \quad (5.2.20)$$

with

$$\bar{u} = \frac{1}{p^2} \quad \text{at} \quad x = 0. \quad (5.2.21)$$

We note that we have been somewhat fortuitous in that the linearity of the yield function G , when the material is deforming plastically, permitted us to take a Laplace transform of (5.2.14) and hence make analytical progress. Had this not been the case (for example, if $G(\varepsilon) \sim \varepsilon^n$ in the plastic region) then this procedure becomes invalid, and we would potentially have to resort to numerical solutions of (5.2.12)–(5.2.14) instead.

The third equation (5.2.20) is an algebraic relation for $\bar{\varepsilon}$, hence we have

$$\begin{pmatrix} \bar{\nu}_x \\ \bar{u}_x \end{pmatrix} = \begin{pmatrix} \frac{p}{c_e} & \frac{p(1+ap)}{1+c_e^2 ap} \\ p & \frac{p}{c_e} \end{pmatrix} \begin{pmatrix} \bar{\nu} \\ \bar{u} \end{pmatrix}. \quad (5.2.22)$$

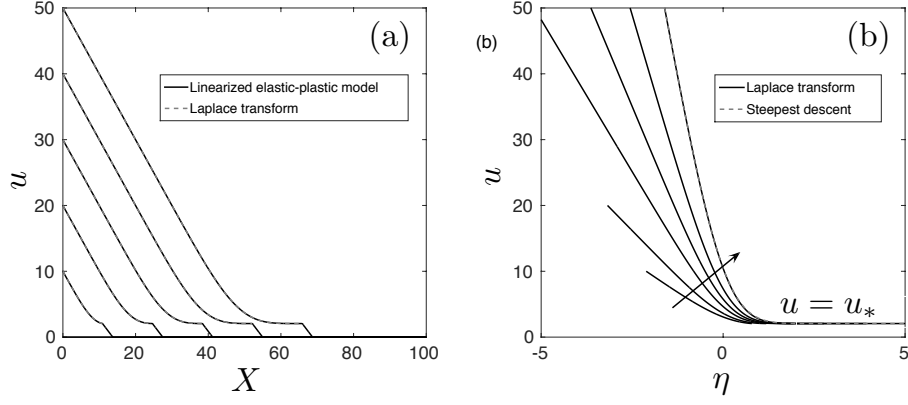


Figure 5.1: (a) Numerical solutions of the problem (5.2.2)–(5.2.4) subject to (5.2.5)–(5.2.6) (solid black) are plotted alongside the solution of the inversion integral (5.2.17) plus the elastic solutions (5.2.7) (dashed grey) for $t = 0, 10, \dots, 50$, $a = 1$, and $k_0 = 2$. (b) The solutions obtained from the inversion integral (5.2.17) and the solution (5.2.42) versus $\eta = z/\sqrt{2a(c_e^2 - 1)\tau}$. The solutions obtained by Laplace transform collapse onto (5.2.42) as $\tau \rightarrow \infty$. Here $t = 10, 20, 50, 100, 200, 500$ with the other parameter values the same as in (a).

This system has solutions

$$\begin{pmatrix} \bar{v} \\ \bar{u} \end{pmatrix} = \mathbf{c} e^{mx} \quad (5.2.23)$$

where

$$m_{\pm} = \frac{p}{c_e} \pm p \sqrt{\frac{1 + ap}{1 + c_e^2 ap}}, \quad \mathbf{c} = \begin{pmatrix} \sqrt{1 + ap} \\ \pm \sqrt{1 + ac_e^2 p} \end{pmatrix}. \quad (5.2.24)$$

We note that

$$m_+ = \frac{2p}{c_e} + \frac{c_e^2 - 1}{2ac_e^3} + O(1/p), \quad m_- = -\frac{c_e^2 - 1}{2ac_e^3} + O(1/p) \quad \text{as } p \rightarrow \infty \quad (5.2.25)$$

so (taking $\text{Re}\{p\} > 0$), we select the negative square root and omit m_+ . Thus we have that

$$\begin{pmatrix} \bar{v} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{1 + ap}}{p^2 \sqrt{1 + c_e^2 ap}} \\ \frac{1}{p^2} \end{pmatrix} \exp \left[px \left(\frac{1}{c_e} - \sqrt{\frac{1 + ap}{1 + ce^2 ap}} \right) \right]. \quad (5.2.26)$$

We invert the integrand (5.2.26) numerically using the method described in [22], to solve for the velocity u . In Figure 5.1(a) we compare the solutions for the velocity u computed by inverting (5.2.26) and via a direct numerical calculation of (5.2.2)–(5.2.4) subject to (5.2.5)–(5.2.6) using the method described in [39]. Comparing these curves suggests that the numerical scheme [39] provides an extremely accurate solution of this problem. As expected, we observe the propagation of an elastic precursor travelling at

speed c_e ahead of a slower moving, but larger amplitude, plastic wave as the velocity at $X = 0$ ramps up. In comparison with the solutions in the case $a \equiv 0$, we see that rate-dependence manifests itself by locally smoothing the solution in regions where $u = O(1)$. This is a consequence of the fact that the stress on the material is permitted to exceed the yield stress, while in rate-independent plasticity, the material remains precisely on the yield surface upon plasticising resulting instead in the “plateau” region in which $\varepsilon \equiv -1$.

As $t \rightarrow \infty$ the linear approximation (5.2.2)–(5.2.4) will break down because of the cumulative effect of nonlinear terms in the governing equations. For matching purposes, we study the large t behaviour of the solution by applying the method of steepest descents to the Laplace inversion integral [1, 6].

We first re-write the integrand (5.2.26) in terms of the original variables X and $T = \tau - X/c_e$ where $\tau = t - t^*$, and then in terms of $z = X - \tau$. The Laplace inversion integral is then

$$\tilde{u} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(p) e^{\phi(p)\tau} dp \quad (5.2.27)$$

where

$$f(p) = \frac{1}{p^2} \exp \left[-p \sqrt{\frac{ap+1}{ac_e^2 p+1}} z \right], \quad \phi(p) = p \left(1 - \sqrt{\frac{ap+1}{ac_e^2 p+1}} \right) \quad (5.2.28)$$

and c is a constant chosen to ensure that all singularities of the integrand lie to the left of the inversion contour. In the present case, c is any positive constant.

The theory of the method of steepest descents tells us that the dominant contribution to the integral (5.2.27) arises in the vicinity of saddle points of the function $\phi(p)$, which are located at the points where $\phi'(p) = 0$. Hence we find that one saddle point lies at $p = 0$, while another lies at $p = -\hat{p}$, where $\hat{p} \in (1, 2)$ is the positive root of the following cubic polynomial:

$$4a^3 c_e^4 \hat{p}^3 - 4a^2 c_e^2 (c_e^2 + 1) \hat{p}^2 + a(11c_e^2 + 9) \hat{p} - 8 = 0. \quad (5.2.29)$$

Furthermore, we find that $\phi''(0) = a(c_e^2 - 1) > 0$ and $\phi''(-\hat{p}) < 0$, and hence the directions of steepest descent are $\arg p = \pi/2, 3\pi/2$ and $\arg p = 0, \pi$ at $p = 0$ and $p = -\hat{p}$ respectively (see Figure 5.2). Finally, we deform the contour onto the imaginary axis, where along this line $\text{Re } \phi(p)$ has a maximum at $p = 0$ (the contribution from $p = -\hat{p}$ is exponentially small).

Since $f(p)$ has a double pole at $p = 0$, we consider the regularised integral

$$\tilde{u} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \tilde{f}(p) e^{\phi(p)\tau} dp \quad (5.2.30)$$

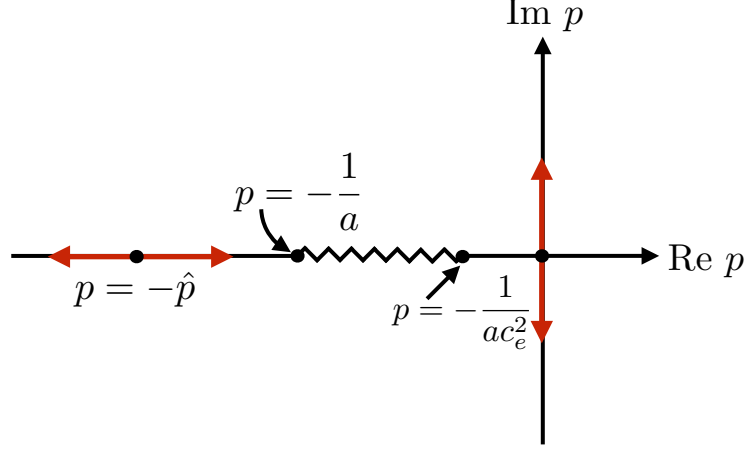


Figure 5.2: Directions of steepest descent at the saddle points $p = 0$ and $p = -\hat{p}$.

where $\tilde{f} = \frac{1}{(p+\epsilon)^2} \exp\left(-p\sqrt{\frac{ap+1}{ac_e^2p+1}}z\right)$ and $\epsilon \ll 1$; that is, we shift the location of the pole by an amount ϵ and study the limiting behaviour of the integral as $\epsilon \rightarrow 0$.

To evaluate the integral (5.2.30) we set $p = ir$ and then the local contribution from the saddle point is obtained by Taylor expanding the functions $\tilde{f}$ and ϕ about $r = 0$:

$$\phi \sim -a \frac{c_e^2 - 1}{2} r^2, \quad \tilde{f} \sim \frac{1}{(ir + \epsilon)^2} \exp[-irz]. \quad (5.2.31)$$

Hence

$$\tilde{u} \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\epsilon + ir)^2} \exp\left[-ar^2 \frac{(c_e^2 - 1)}{2} \tau - irz\right] dr, \quad (5.2.32)$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\exp\left[-ar^2 \frac{(c_e^2 - 1)}{2} \tau\right]}{(\epsilon^2 + r^2)^2} [(\epsilon^2 - r^2) \cos(rz) - 2\epsilon r \sin(rz)] dr, \quad (5.2.33)$$

$$= \frac{1}{\pi} (I_1 + I_2), \quad (5.2.34)$$

where

$$I_1 = \int_0^{\infty} \frac{\exp\left[-ar^2 \frac{(c_e^2 - 1)}{2} \tau\right]}{(\epsilon^2 + r^2)^2} (\epsilon^2 - r^2) \cos(rz) dr, \quad (5.2.35)$$

$$\text{and } I_2 = - \int_0^{\infty} \frac{\exp\left[-ar^2 \frac{(c_e^2 - 1)}{2} \tau\right]}{(\epsilon^2 + r^2)^2} 2\epsilon r \sin(rz) dr. \quad (5.2.36)$$

To evaluate I_1 we split the range of integration as follows

$$\int_0^{\infty} \cdot dr = \left(\int_{\mu}^{\infty} + \int_0^{\mu} \right) \cdot dr, \quad \epsilon \ll \mu \ll 1 \quad (5.2.37)$$

and in the second of these integrals we rescale $r = \epsilon s$. Therefore we obtain

$$I_1 \sim - \int_{\mu}^{\infty} \frac{\exp \left[-ar^2 \frac{c_e^2 - 1}{2} \tau \right] \cos(rz)}{r^2} dr + \frac{1}{\epsilon} \int_0^{\mu/\epsilon} \frac{1 - s^2}{(1 + s^2)^2} ds + O(\epsilon^2) \quad \text{as } \epsilon \rightarrow 0. \quad (5.2.38)$$

Evaluating the second integral we see that

$$\frac{1}{\epsilon} \int_0^{\mu/\epsilon} \frac{1 - s^2}{(1 + s^2)^2} ds \sim \frac{1}{\mu} = \int_{\mu}^{\infty} \frac{1}{r^2} dr. \quad (5.2.39)$$

Thus we can write

$$I_1 \, dr \sim \int_{\mu}^{\infty} \frac{1}{r^2} \left[1 - \cos(rz) \exp \left(-ar^2 \frac{c_e^2 - 1}{2} \tau \right) \right] dr \quad (5.2.40)$$

which converges as $\mu \rightarrow 0$ to

$$I_1 \, dr \sim \sqrt{a \frac{\pi(c_e^2 - 1)\tau}{2}} \exp \left(-\frac{z^2}{2a(c_e^2 - 1)\tau} \right) + \frac{\pi z}{2} \operatorname{erf} \left(\frac{z}{\sqrt{2a(c_e^2 - 1)\tau}} \right) \quad (5.2.41)$$

For I_2 we set $r = \epsilon s$ to obtain

$$\begin{aligned} I_2 &= \int_0^{\infty} \exp \left[-a\epsilon^2 s^2 \frac{(c_e^2 - 1)}{2} \tau \right] \frac{s \sin(\epsilon s z)}{\epsilon(1 + s^2)^2} ds \\ &\sim z \int_0^{\infty} \frac{s^2}{(1 + s^2)^2} ds = \frac{\pi z}{4} \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

Combining I_1 and I_2 we obtain

$$u \sim \sqrt{a \frac{(c_e^2 - 1)\tau}{2\pi}} \exp \left(-\frac{z^2}{2a(c_e^2 - 1)\tau} \right) - \frac{z}{2} \operatorname{erfc} \left(\frac{z}{\sqrt{2a(c_e^2 - 1)\tau}} \right) + \frac{3c_e}{2} \quad \text{as } \tau \rightarrow \infty. \quad (5.2.42)$$

In Figure 5.1(b) we plot the solution obtained from the numerical inversion of the Laplace transform (5.2.27) and observe that the solutions obtained from (5.2.17) indeed collapse onto (5.2.42) as $\tau \rightarrow \infty$. The self-similar solution (5.2.42) will provide matching conditions for the outer problem analysed below in §5.3.

Without resorting to the method of steepest descents, we note that further analytical progress can be made in the nearly-rate-independent limit $a \ll 1$. Referring to equations (5.2.12)–(5.2.14), the leading-order outer solution with $a = 0$ is

$$\tilde{u} = T - \left(1 - \frac{1}{c_e} \right) x, \quad \tilde{\nu} = \left(1 - \frac{1}{c_e} \right) x - T. \quad (5.2.43)$$

To examine the region close to the front of the propagating plastic wave we make the substitution

$$\sqrt{a}x' = x - \mu T, \quad T = T' \quad (5.2.44)$$

where $\mu = c_e/(c_e - 1)$, and also set

$$\tilde{u} = \sqrt{a}U, \quad \tilde{\nu} = \sqrt{a}\chi, \quad \tilde{\varepsilon} = aE \quad (5.2.45)$$

before taking the limit $a \rightarrow 0$.

The governing equations (5.2.12)–(5.2.14) become

$$\mu\chi_{x'} + \mu U_{x'} - \sqrt{a} \left(\chi_{T'} + \frac{1}{c_e} U_{T'} \right) + O(a) = 0, \quad (5.2.46)$$

$$\mu U_{x'} + \mu\chi_{x'} - \sqrt{a} \left(U_{T'} + \frac{1}{c_e} \chi_{T'} - \frac{3\mu}{2}(c_e^2 - 1)E_{x'} \right) + O(a) = 0, \quad (5.2.47)$$

$$\left(-\frac{2}{3}\mu\chi_{x'} + \sqrt{a} \left(\frac{2}{3}\chi_{T'} + \mu E_{x'} \right) \right) + O(a) = E. \quad (5.2.48)$$

At leading order, Equations (5.2.46) and (5.2.47) provide the same equation twice, while Equation (5.2.48) is an equation for E . A closure condition is obtained by subtracting (5.2.46) from (5.2.47), and using Equation (5.2.48) to eliminate E we obtain

$$U_{T'} = \gamma U_{x'x'} \quad \text{and} \quad \chi + U = 0 \quad (5.2.49)$$

where $\gamma = \mu^3(c_e^2 - 1)/2$. That is, near the propagating front, the dynamics is governed by the diffusion equation with diffusion constant γ , subject to the matching conditions

$$U \sim -\frac{x'}{\mu} \quad \text{as} \quad x' \rightarrow -\infty, \quad U \rightarrow 0 \quad \text{as} \quad x' \rightarrow \infty. \quad (5.2.50)$$

We look for a similarity solution of the form

$$-\chi = U = \sqrt{T'} F(s), \quad s = \frac{x'}{\sqrt{T'}}. \quad (5.2.51)$$

Upon scaling $s = \sqrt{\gamma}\hat{s}$ and $F = \sqrt{\gamma}\hat{F}/\mu$, we arrive at the following o.d.e problem:

$$\hat{F}'' + \frac{\hat{s}}{2}\hat{F}' - \frac{1}{2}\hat{F} = 0, \quad (5.2.52)$$

$$\hat{F} \rightarrow \hat{s} \quad \text{as} \quad \hat{s} \rightarrow -\infty, \quad \hat{F} \rightarrow 0 \quad \text{as} \quad \hat{s} \rightarrow \infty \quad (5.2.53)$$

It may be easily shown that

$$\hat{F} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\hat{s}^2}{4}\right) - \frac{\hat{s}}{2} \operatorname{erfc}\left(\frac{\hat{s}}{2}\right) \quad (5.2.54)$$

and hence

$$U \sim \sqrt{\frac{\gamma T'}{\mu^2 \pi}} \exp\left(-\frac{x'^2}{4\gamma T'}\right) - \frac{x'}{2\mu} \operatorname{erfc}\left(\frac{x'}{2\sqrt{\gamma T'}}\right) \quad \text{as} \quad a \rightarrow 0. \quad (5.2.55)$$

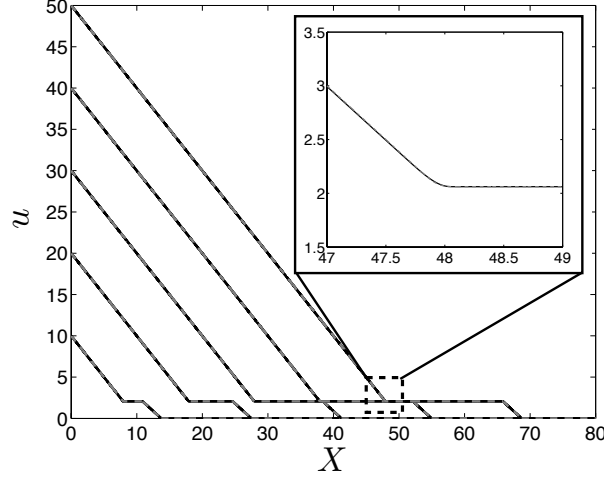


Figure 5.3: Numerical solution of the problem (5.2.2)–(5.2.4) subject to (5.2.5)–(5.2.6) (solid black) is plotted against the solution (5.2.55) plus the elastic solutions (5.2.7) (dashed grey) for the velocity u . Here $t = 0, 10, \dots, 50$, $a = 0.001 \ll 1$, and $k_0 = 2$.

Comparing (5.2.55) with (5.2.42), we see that the asymptotic behaviour of (5.2.12)–(5.2.14) as $a \rightarrow 0$ is analogous to the solution as $t \rightarrow \infty$. Formally, this is a consequence of the fact that by setting $(x, T) = a(\hat{x}, \hat{T})$ and $(u, \nu, \varepsilon) = a(\hat{u}, \hat{\nu}, \hat{\varepsilon})$, the parameter a may be scaled out of the problem entirely. Thus, the limit $a \rightarrow 0$ is equivalent to studying the large-time asymptotics of the unscaled problem.

In Figure 5.3, we compare the solution (5.2.55) (dashed grey) (in original unscaled variables u and X) with a full numerical solution of the problem (5.2.2)–(5.2.4) subject to (5.2.5)–(5.2.6) (solid black) for $a \ll 1$. These plots show that the solutions are well described by the rate-independent approximation detailed in §4.5, apart from in a region $O(\sqrt{a})$ in the vicinity of the plastic wave front $X = t - t_*$, where the solution is captured by (5.2.55).

5.3 Weakly nonlinear behaviour

As $t' \rightarrow \infty$ the solutions obtained from the linearised system (4.3.2)–(4.3.4) start to break down and give way to nonlinear terms of higher order present in the governing equations (2.3.54)–(2.3.56). In this section we perform a weakly nonlinear analysis, the details of which depend on how the rate parameter a scales with δ as $\delta \rightarrow 0$. We focus our attention on three subcases: (i) $a = 0$, (ii) $a = O(\delta)$, and (iii) $a = O(1)$, as $\delta \rightarrow 0$.

In case (i), we begin by introducing the moving space variable $X = \tau + \delta \tilde{X}$, $\tau = t - \delta t^*$

and expand the dependent variables in powers of δ as follows

$$\nu_i = 1 + \delta \nu_i^{(1)} + \delta^2 \nu_i^{(2)} + O(\delta^3), \quad u_i = \delta u_i^{(1)} + \delta^2 u_i^{(2)} + O(\delta^3), \quad \varepsilon = \varepsilon_i^{(0)} + \delta \varepsilon_i^{(1)} + O(\delta^2). \quad (5.3.1)$$

At leading order (dropping the tilde notation), the governing equations (2.3.54)–(2.3.56) become

$$\frac{\partial \nu_i^{(1)}}{\partial X} + \frac{\partial u_i^{(1)}}{\partial X} = 0, \quad (5.3.2)$$

$$\frac{\partial \nu_i^{(1)}}{\partial X} + \frac{\partial u_i^{(1)}}{\partial X} = 0, \quad (5.3.3)$$

$$\varepsilon_i^{(0)} = -1; \quad (5.3.4)$$

that is, we obtain the same equation twice in (5.3.2) and (5.3.3). A closure condition is then obtained at next order

$$\frac{\partial \nu_i^{(2)}}{\partial X} + \frac{\partial u_i^{(2)}}{\partial X} = \frac{\partial \nu_i^{(1)}}{\partial \tau}, \quad (5.3.5)$$

$$\frac{\partial u_i^{(2)}}{\partial X} + \frac{\partial \nu_i^{(2)}}{\partial X} = \frac{\partial u_i^{(1)}}{\partial \tau} - \left(\frac{2k_0}{3} - \frac{2k'(1)}{3} - \frac{2k_0}{3} \varepsilon'_c(1) + h'''(1) \nu_i^{(1)} \right) \frac{\partial \nu_i^{(1)}}{\partial X}, \quad (5.3.6)$$

$$\varepsilon_i^{(1)} = -\varepsilon'_c(1) \nu_i^{(1)}. \quad (5.3.7)$$

Subtracting (5.3.5) from (5.3.6) and using the leading-order equation (5.3.2), we find a single equation for the velocity perturbation $u_i^{(1)}$, namely the inviscid Burgers' equation

$$u_{i\tau}^{(1)} + \left(p + q u_i^{(1)} \right) u_{iX}^{(1)} = 0 \quad (5.3.8)$$

where

$$p = \frac{1}{2} \left(\frac{2k_0}{3} - \frac{2k'(1)}{3} - \frac{2k_0}{3} \varepsilon'_c(1) \right), \quad q = -\frac{1}{2} h'''(1) \quad (5.3.9)$$

and $h'''(1) < 0$. We note that Equation (5.3.8) predicts finite time blow-up (and hence shock formation) provided the initial acceleration at $X = 0$ is positive.

An analogous approach may be followed for case (ii), hence we expand as follows:

$$\nu_i = 1 + \delta \nu_i^{(1)} + \delta^2 \nu_i^{(2)} + O(\delta^3), \quad u_i = \delta u_i^{(1)} + \delta^2 u_i^{(2)} + O(\delta^3), \quad \varepsilon = -1 + \delta \varepsilon_i^{(1)} + O(\delta^2). \quad (5.3.10)$$

We also set $a = \delta a'$ where $a' = O(1)$ as $\delta \rightarrow 0$. At leading order we once more obtain (5.3.2) and (5.3.3), while (5.3.7) is replaced by

$$-\frac{2a'}{3} \frac{\partial \nu_i^{(1)}}{\partial X} = \varepsilon_i^{(1)} + \varepsilon'_c(1) \nu_i^{(1)} \quad (5.3.11)$$

from (2.3.56). Note that we have retained a derivative of $\nu_i^{(1)}$ on the left-hand side. At next order we get

$$\frac{\partial \nu_i^{(2)}}{\partial X} + \frac{\partial u_i^{(2)}}{\partial X} = \frac{\partial \nu_i^{(1)}}{\partial \tau}, \quad (5.3.12)$$

$$\frac{\partial u_i^{(2)}}{\partial X} + \frac{\partial \nu_i^{(2)}}{\partial X} = \frac{\partial u_i^{(1)}}{\partial \tau} - \frac{2k_0}{3} \frac{\partial \varepsilon_i^{(1)}}{\partial X} - \left(\frac{2k_0}{3} - \frac{2k'(1)}{3} + h'''(1)\nu_i^{(1)} \right) \frac{\partial \nu_i^{(1)}}{\partial X}. \quad (5.3.13)$$

Following the same procedure of subtracting and using the leading order equations as in case (i), we find an equation for the velocity perturbation $u^{(1)}$, namely the *viscous* Burgers' equation

$$u_{i\tau}^{(1)} + \left(p + qu_i^{(1)} \right) u_{iX}^{(1)} = ru_{iXX}^{(1)} \quad (5.3.14)$$

where

$$p = \frac{1}{2} \left(\frac{2k_0}{3} - \frac{2k'(1)}{3} - \frac{2k_0}{3} \varepsilon'_c(1) \right), \quad q = -\frac{1}{2} h'''(1) \quad \text{and} \quad r = \frac{a'}{2} \frac{4k_0}{9}. \quad (5.3.15)$$

Our final case requires the scaling $X = \tau + \sqrt{\delta} \tilde{X}$, $\tau = t - \delta t_*$, and we expand the dependent variables as follows

$$\nu = 1 + \sqrt{\delta} \nu_i^{(1)} + \delta \nu_i^{(2)}, \quad u = \sqrt{\delta} u_i^{(1)} + \delta u_i^{(2)}, \quad \varepsilon = -1 + \varepsilon_i^{(1)}. \quad (5.3.16)$$

Following a similar approach to cases (i) and (ii) leads to the slightly modified Burgers' equation

$$u_{i\tau}^{(1)} + qu_i^{(1)} u_{iX}^{(1)} = ru_{iXX}^{(1)} \quad (5.3.17)$$

where $r = \frac{a}{2} \frac{4k_0}{9}$.

In summary, we see that case (ii) is the distinguished limit as $\delta \rightarrow 0$ (since we retain as many terms as possible in the amplitude equation (5.3.14)), while other scalings for a result in simpler special cases of (5.3.14).

Equations (5.3.8), (5.3.14), and (5.3.17) require the specification of one initial condition and two boundary conditions. To derive the upstream far-field condition, we first expand ν and u in the outer region in powers of δ :

$$\nu_o = \nu_o^{(0)} + \delta \nu_o^{(1)} + O(\delta^2), \quad u_o \sim u_o^{(0)} + \delta u_o^{(1)} + O(\delta^2). \quad (5.3.18)$$

At leading order, the equations (2.3.54)–(2.3.55) reduce to the equations of barotropic flow

$$\frac{\partial \nu_o^{(0)}}{\partial t} - \frac{\partial u_o^{(0)}}{\partial X} = 0, \quad (5.3.19)$$

$$\frac{\partial u_o^{(0)}}{\partial t} - h''(\nu_o^{(0)}) \frac{\partial \nu_o^{(0)}}{\partial X} = 0. \quad (5.3.20)$$

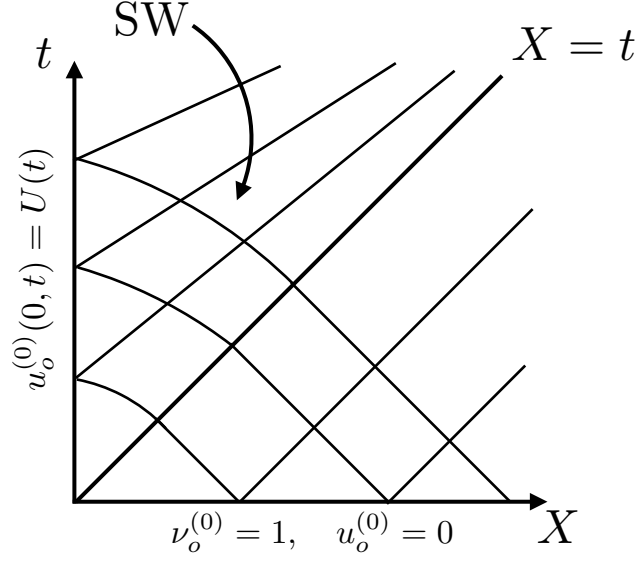


Figure 5.4: Characteristic diagram for the problem (5.3.19)–(5.3.22). As the velocity ramps up at $X = 0$, a simple wave labelled SW propagates behind the leading characteristic $X = t$.

This system may be solved subject to the initial and boundary conditions

$$\nu_o^{(0)} = 1, \quad u_o^{(0)} = 0 \quad \text{at} \quad t = 0, \quad X > 0, \quad (5.3.21)$$

$$u_o^{(0)} = U(t) \quad \text{at} \quad X = 0, \quad t > 0. \quad (5.3.22)$$

The problem (5.3.19)–(5.3.22) is analogous to a compressive piston problem encountered in gas dynamics; a sketch of the characteristic plane is shown in Figure 5.4.

As discussed in [49, 53], the p.d.e (5.3.19)–(5.3.20) admit the Riemann invariants

$$u_o^{(0)} \mp \phi(\nu_o^{(0)}) = \text{const.} \quad \text{on} \quad \frac{dX}{dt} = \pm c(\nu_o^{(0)}) \quad (5.3.23)$$

where

$$\phi(\nu_o^{(0)}) = \int_1^{\nu_o^{(0)}} c(\nu') \, d\nu', \quad c(\nu_o^{(0)}) = \sqrt{h''(\nu_o^{(0)})}. \quad (5.3.24)$$

This allows the solution of (5.3.19)–(5.3.22) to be written parametrically as

$$X = c(\nu_o^{(0)})(t - T), \quad (5.3.25)$$

$$u_o^{(0)} = U(T), \quad (5.3.26)$$

$$U(T) + \phi(\nu_o^{(0)}) = 0. \quad (5.3.27)$$

To proceed with matching we appeal to van Dyke's matching principle [80]. Matching the 2-term inner expansion with the 1-term outer expansion, we find the appropriate

matching condition is

$$u_i^{(1)} \sim -\frac{\tilde{X}U'(0)}{1 - U'(0)q\tau} \quad \text{as } \tilde{X} \rightarrow -\infty. \quad (5.3.28)$$

For case (i), the initial condition is obtained by matching with the early time solution discussed in §5.2 when $a = 0$, and we require that $u_i^{(1)} = u_* = 3c_e/2$ on an as yet unknown free-boundary $X = X_f$ (recall that u_* is the magnitude of the velocity perturbation in the “plateau” region where $\varepsilon \equiv -1$). In summary we have

$$u_{i\tau}^{(1)} + (p + qu_i^{(1)})u_{iX}^{(1)} = 0, \quad (5.3.29)$$

$$u_i^{(1)} = u_* \quad \text{on } X = X_f, \quad (5.3.30)$$

$$u_i^{(1)} \sim -\frac{U'(0)X}{1 - qU'(0)\tau} \quad \text{as } X \rightarrow -\infty, \quad (5.3.31)$$

$$u_i^{(1)} \sim -U'(0)X \quad \text{as } \tau \rightarrow 0. \quad (5.3.32)$$

It is straightforward to solve (5.3.29)–(5.3.32) by the method of characteristics, and the solution in the plastic region is thus found to be

$$u_i^{(1)} = -\frac{U'(0)(X - (p + 3q/2)\tau)}{1 - qU'(0)\tau} + u_* \quad \text{for } X \leq (p + 3q/2)\tau. \quad (5.3.33)$$

Thus we find that the plastic wave front propagates along the line

$$X = c_p(t - \delta t_*) \quad (5.3.34)$$

where $c_p = 1 + \delta(p + \frac{3}{2}q)$ defines the speed of the plastic wave front. Note that this is unlike the linear model of chapter 4, where the plastic wave propagated with speed unity. By retaining higher order terms (namely p and q) we find that the plastic wave propagates slightly *faster* than the prediction of the linear model.

For cases (ii) and (iii), the remaining initial and boundary conditions are obtained by matching with the early time solution (5.2.42) and having the solution attenuate to u_* in the downstream far-field; thus we have

$$(5.3.14)/(5.3.17) \quad \text{for } \tilde{X} \in (-\infty, \infty), \tau > 0 \quad (5.3.35)$$

$$(5.2.42) \quad \text{as } \tau \rightarrow 0, \quad (5.3.36)$$

$$u_i^{(1)} \rightarrow u_* \quad \text{as } \tilde{X} \rightarrow \infty, \quad (5.3.37)$$

$$(5.3.28) \quad \text{as } \tilde{X} \rightarrow -\infty. \quad (5.3.38)$$

In Figure 5.5 we compare the rate-independent solution (5.3.33) (dashed grey), and the corresponding solution for ν , with the solutions of the elastic-model (2.3.54)–(2.3.56)

(black) for small a , and the leading order barotropic flow model where $\delta = 0$ (5.3.19)–(5.3.20). The elastic-plastic solutions follow those in §3.2, and hence the constitutive functions and parameter values are the same other than the magnitude of a . These solutions show that in regions where ν and u are $O(1)$, the elastic-plastic solutions are well-described by those of the barotropic flow model. In regions where ν and u are $O(\delta)$, the elastic-plastic solution approaches the plastic wave front with finite slope, which is well described by the solution (5.3.33).

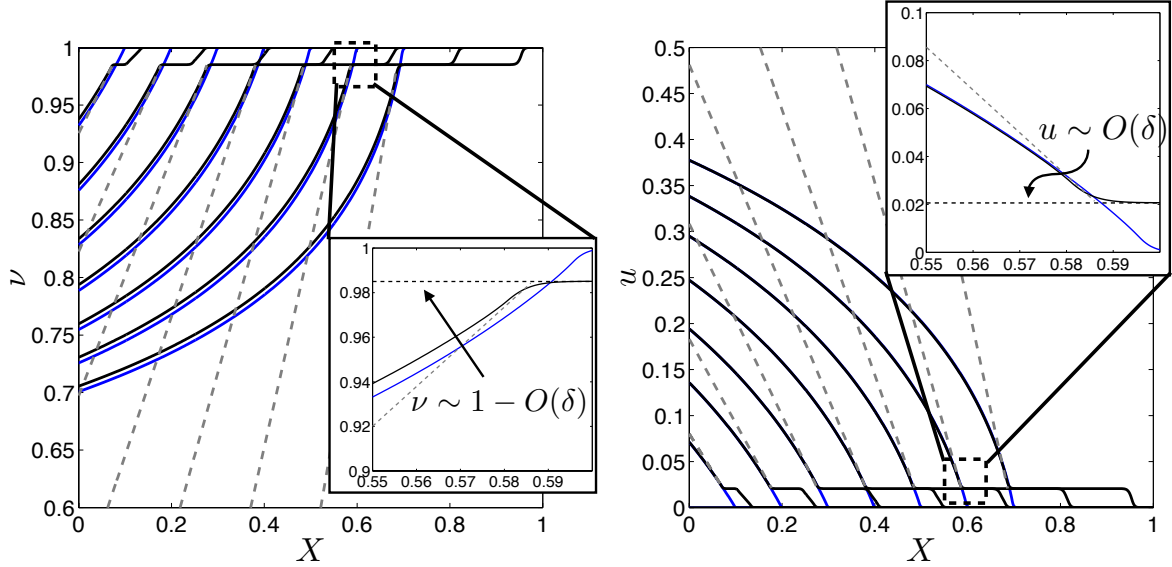


Figure 5.5: A plot showing the comparison between the elastic-plastic model (2.3.54)–(2.3.56) (solid black), the barotropic flow model (5.3.19)–(5.3.20) (solid blue), and the solution (5.3.33) (grey dashed) using the same parameter values as in §3.2. Here $t = 0.1, 0.2, \dots, 0.7$ and $a = 0.001$. The numerical solution is terminated before a shock has formed.

In cases (ii) and (iii), we compute the solution of the problem (5.3.35)–(5.3.38) numerically using the method of lines. The spatial derivatives are discretised using standard central finite differencing and we use the MATLAB function `ode15s` for the temporal integration. The finite difference scheme for this solution uses a grid spacing of $\Delta X = 0.01$.

Figures 5.6 (a)–(c) show a comparison of the solutions of the elastic-plastic model (2.3.54)–(2.3.56) (black), Burgers' equation (5.3.14) in (grey-dashed) and the leading order barotropic flow model where $\delta = 0$ (5.3.19)–(5.3.20) (blue) for ν , u , and ε when $a = O(\delta)$. Figures 5.6 (d)–(f) show the same plots for $a = O(1)$. These plots again show good agreement between the elastic-plastic solutions and the barotropic flow model in regions where the strain is $O(1)$. However, in regions where u and ν are $O(\delta)$ the elastic-plastic model and barotropic flow model diverge as the elastic precursor propagates ahead

of the main plastic wave. It is in this region where we observe that the front of the plastic wave is accurately described by the respective solutions of (5.3.14) and (5.3.17), and again exhibits local smoothing in regions where $\nu = 1 + O(\delta)$ and $u = O(\delta)$.

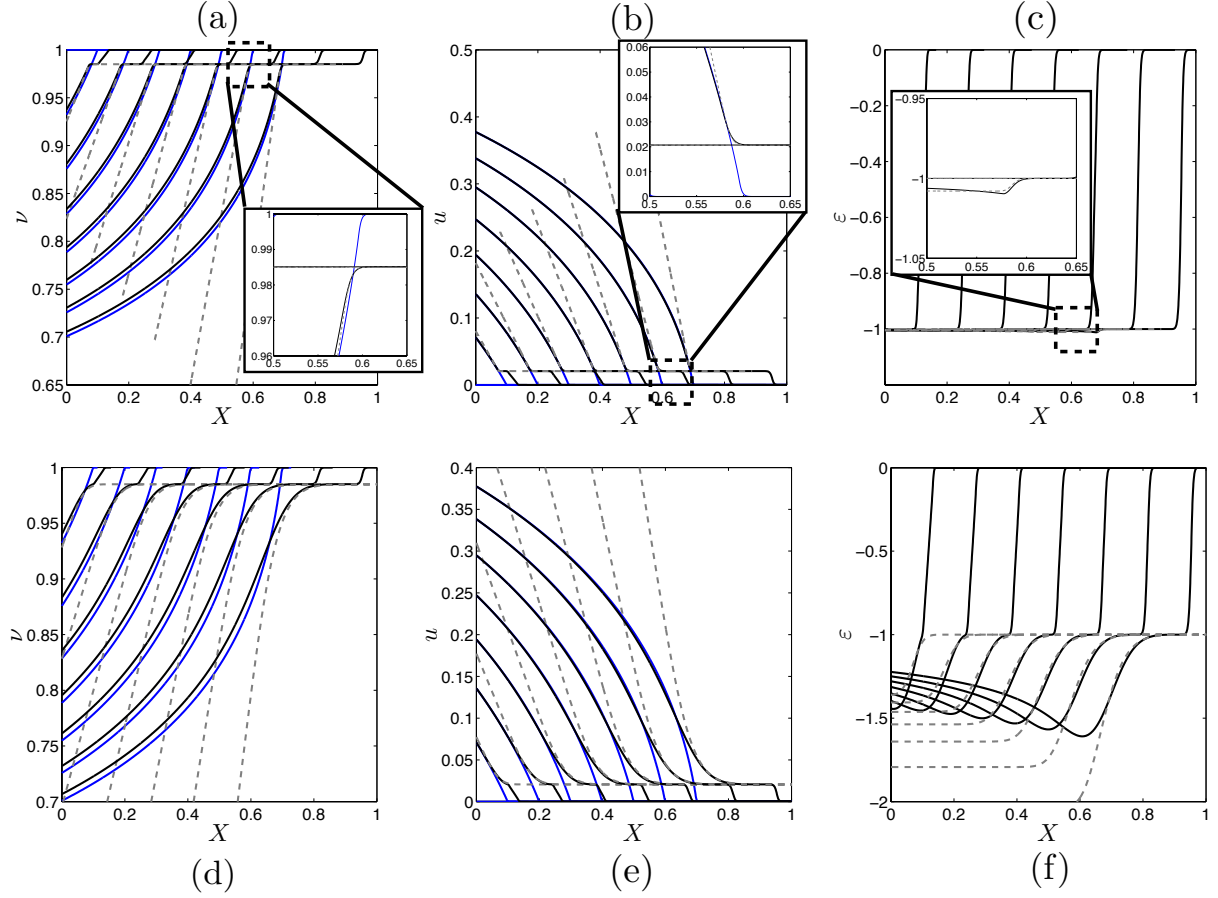


Figure 5.6: A plot showing the comparison between the elastic-plastic model (2.3.54)–(2.3.56) (solid black), the barotropic flow model (2.3.54)–(2.3.55) with $\delta = 0$ (solid blue), and the solution of (5.3.35)–(5.3.38) (grey dashed) using the same parameter values as §3.2 and $t = 0.1, 0.2, \dots, 0.7$. In (a)–(c) $a = \delta = 0.01$ and in (d)–(f) $a = 1$. In regions where the velocity u is $O(1)$ the plastic wave is well described by the solution of the barotropic flow model (5.3.19)–(5.3.20), while in regions where $u = O(\delta)$ the solution is more accurately described by Burgers' equation.

5.4 Elastic-plastic wave interactions

5.4.1 Rate-independent case

We now consider an analogous problem to that considered in §4.6.2, in which an elastic wave which has reflected from the stress-free surface at $X = 1$ subsequently interacts with the incoming plastic wave. We begin with the simplest case of rate-independent

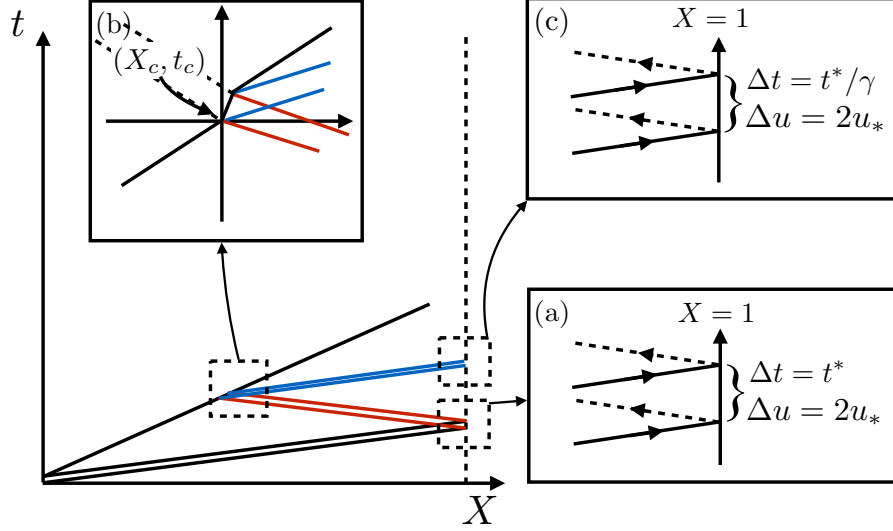


Figure 5.7: A schematic showing the progress of an elastic wave as it reflects from the free surface at $X = 1$ and interacts with the incoming plastic wave. In region (a), the elastic wave reflects from the free surface back toward the incoming plastic wave, eventually interacting with it in region (b). In (b) the red lines demarcate the incoming elastic characteristics while the blue are the outgoing characteristics. The solid black line is the position of the plastic wave front. The interaction of these two waves produces an elastic wave which propagates back toward the free surface and can lead to further reverberation as shown in (c), with the velocity increasing in a series of small steps at $X = 1$.

plasticity, i.e. $a \equiv 0$, and then say what we can about the interaction when a is non-zero in §5.4.2.

Referring to Figure 5.7, after reflecting from the free surface at event (a) (and causing an $O(\delta)$ increase in the velocity there), the elastic and plastic waves will first collide at the point where the leading plastic characteristic and the first reflected elastic characteristic meet as shown in Figure 5.7(b). This point is

$$t_i = \frac{2 + \delta c_p t^*}{c_p + c_e}, \quad X_i = -c_e t_i + 2 = c_p(t_i - \delta t_*). \quad (5.4.1)$$

We recall the analogous result from the linear theory of §4.6.2, and note that inclusion of (weak) non-linearity results in a slightly modified interaction time. We recover (4.6.17) as $\delta \rightarrow 0$ since $c_p \sim 1$.

We now move into a frame centered at the point of interaction $X = X_i + \delta X'$, $t = t_i + \delta t'$, and linearise about $\nu = 1 + \delta \nu'$, $u = \delta u'$. Figure 5.8 is a close-up of region (b) in Figure 5.7. The incoming elastic and plastic waves in regions EI and PI are given by

and the elastic and plastic solutions in EII and PII are

$$\nu_{\text{EII}} = \frac{1}{c_e} U'(0) \left(t + \frac{X}{c_e} \right) - \frac{3}{2} - \frac{\gamma}{c_e} U'(0) \left(t - \frac{X}{c_e} \right), \quad (5.4.8)$$

$$u_{\text{EII}} = U'(0) \left(t + \frac{X}{c_e} \right) + \frac{3c_e}{2} + \gamma U'(0) \left(t - \frac{X}{c_e} \right), \quad (5.4.9)$$

$$\varepsilon_{\text{EII}} = \frac{2}{3} \nu_{\text{EII}} \quad (5.4.10)$$

and

$$\nu_{\text{PII}} = -U'(0)\beta(t - X) - \frac{3}{2} + U'(0) \frac{\beta(c_e + 1)}{c_e(2\beta - 1) + 1} (t + X), \quad (5.4.11)$$

$$u_{\text{PII}} = U'(0)\beta(t - X) + \frac{3c_e}{2} + U'(0) \frac{\beta(c_e + 1)}{c_e(2\beta - 1) + 1} (t + X), \quad (5.4.12)$$

$$\varepsilon_{\text{PII}} = -1 \quad (5.4.13)$$

where $\gamma = (c_e + c_i)/(c_e - c_i) > 1$. We note that this result reduces to the linear model of §4.6.2 since in that case $q = -\frac{1}{2}h'''(1) = 0$, and hence $\beta = 1$ and $c_i = 0$ i.e. the free-boundary velocity is zero is the absence of non-linearity.

The two waves cease to interact at time

$$t_f = \frac{t^*}{1 + c_i/c_e}. \quad (5.4.14)$$

and we are left with the reflected elastic wave EIII

$$\nu_{\text{EIII}} = -\frac{\gamma}{c_e} U'(0) \left(t - \frac{X}{c_e} \right), \quad (5.4.15)$$

$$u_{\text{EIII}} = 3c_e + \gamma U'(0) \left(t - \frac{X}{c_e} \right), \quad (5.4.16)$$

$$\varepsilon_{\text{EIII}} = \frac{2}{3} \nu_{\text{EIII}} \quad (5.4.17)$$

which propagates between the characteristics

$$X_e^{(1)} = c_e t \quad \text{and} \quad X_e^{(2)} = c_e \left(t - \frac{t^*}{\gamma} \right). \quad (5.4.18)$$

We note that (5.4.18) tells us that the width of the reflected elastic wave is $\Delta X = c_e t^*/\gamma$, in comparison with the incident wave which had width $\Delta X = c_e t^*$. Therefore the process of interaction with the plastic wave results in the reflected elastic wave steepening by a factor $1/\gamma$. This phenomenon is not predicted by the linear problem of 4.6.2, where imposing a constant acceleration resulted in $\gamma = 1$, and hence no steepening of the outgoing elastic wave. This behaviour is summarised in Figure 5.7(c).

In region CII we have $\nu = 0$, $u = 2u_*$, and $\varepsilon = 0$, while in CIII $\nu = -3/2$, $u = 3u_*$, and $\varepsilon = -1$. Finally, the front of the plastic wave is along the line

$$X_p = t - \frac{t^*}{\beta} \quad (5.4.19)$$

and the outgoing plastic wave in region PIII is of the form

$$\nu_{\text{PIII}} = -U'(0)\beta(t - X) + \frac{3}{2}(c_e - 1), \quad (5.4.20)$$

$$u_{\text{PIII}} = U'(0)\beta(t - X) + 3c_e, \quad (5.4.21)$$

$$\varepsilon_{\text{PIII}} = -1. \quad (5.4.22)$$

The outgoing elastic wave in region EIII will eventually reach the free surface and reflect from it, leading to a further increase in the velocity at $X = 1$ as shown in Figure 5.7(c). Another interaction can therefore occur between this elastic wave and the incoming plastic wave. However, we note that in contrast to the linear problem considered in chapter 4, by the time of the next interaction, not only has the elastic wave steepened by a factor of γ , the compressive plastic wave has also steepened as a result of nonlinearity. Therefore, the next interaction will be analogous to the scenario depicted in Figure 5.8, however we will now have

$$u_{\text{PI}} = U'(0)\beta'(t - X) + 3u_*, \quad \nu_{\text{PI}} = -U'(0)\beta'(t - X) - \frac{3}{2}, \quad \varepsilon_{\text{PI}} = -1 \quad (5.4.23)$$

and

$$u_{\text{EI}} = U'(0)\gamma \left(t + \frac{X}{c_e} \right) + 3u_*, \quad \nu_{\text{EI}} = U'(0)\frac{\gamma}{c_e} \left(t + \frac{X}{c_e} \right) - \frac{3}{2}, \quad \varepsilon_{\text{EI}} = \frac{2}{3}\nu_{\text{EI}} \quad (5.4.24)$$

where $\beta' > \beta$. This leads to

$$\xi = \frac{\beta' - \gamma}{\beta' + \gamma/c_e} t = c'_i t. \quad (5.4.25)$$

In principle, each subsequent interaction can be computed in a similar fashion, however this quickly becomes an arduous task in the absence of a general solution such as the functional equation (4.7.30), and hence will require further investigation.

5.4.2 Rate-dependent case

The case when $a \neq 0$ is a little more complicated, as we are left to solve

$$\nu_t - u_X = 0, \quad (5.4.26)$$

$$u_t - \nu_X - \frac{2k_0}{3}\varepsilon_X = 0, \quad (5.4.27)$$

$$a \left(\frac{2}{3}\nu_t - \varepsilon_t \right) = G(\varepsilon) \quad (5.4.28)$$

where the initial conditions are obtained by evaluating the solution of (4.6.13)–(4.6.15) and (5.3.35)–(5.3.38) at $t = t_i$. The far-field conditions are (5.3.28) as $X \rightarrow -\infty$ and matching with the incoming elastic wave as $X \rightarrow +\infty$. In the absence of an exact solution, we can nevertheless solve the rate-dependent interaction problem numerically, once again using the method in reference [39]. We do this in the case $a = O(\delta)$, and compare it with the analytical solution obtained in the rate-independent case.

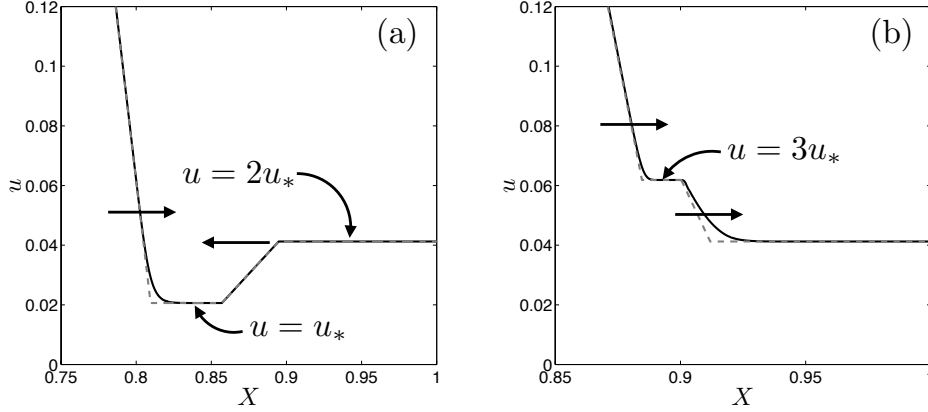


Figure 5.9: Plots showing (a) the incoming elastic and plastic waves pre-interaction and (b) the outgoing elastic and plastic waves post-interaction for the case $a = 0$ (grey-dashed) and $a = \delta\alpha$ (solid black) where $\delta = 0.01$, $\alpha = 1$. All other parameter values remain unchanged.

Figure 5.9 plots the pre- and post-interaction profiles for the velocity u , comparing the solutions when $a = 0$ and $a = O(\delta)$. In black is the numerical solution of (5.4.26)–(5.4.28), with the solution of (5.3.35)–(5.3.38) approaching from $X = -\infty$ and the elastic wave from $X = +\infty$, while the dashed grey line indicates the analytical solution derived for the case $a = 0$ in §5.4.1. Before the interaction occurs, the two solutions are almost identical except near the front of the plastic wave where the rate-dependent case exhibits a degree of local smoothing. Post-interaction we observe that in both cases the velocity has increased by an amount u_* downstream of the outgoing elastic wave propagating from left to right. However, the rate-dependent solution has inherited some of the smoothness of the plastic wave, while the rate-independent solution maintains a sharp profile with the elastic wave steepening by a factor γ . Eventually the outgoing elastic wave will again interact with and reflect from the free surface at $X = 1$, causing a further small increase in the velocity, before interacting with the plastic wave again, and so on. However, computing the result of the n -th interaction of the two wave types is a non-trivial task and will require further exploration.

5.5 Conclusions

In this chapter, we explored the consequences of incorporating the effects of nonlinearity and rate-dependence into our analysis of elastic-plastic wave propagation during violent impact. This gave rise to new phenomena not predicted by the linear formulation considered in chapter 4. The early time problem and subsequent weakly nonlinear analysis carried out in §5.2 and §5.3 revealed the absence of a well-defined “plateau” region when the rate-parameter a is finite. However, close to the plastic wave front, incorporating rate-dependence led to a local smoothing of the solution as a direct consequence of the material residing close to, but not precisely on, the yield surface. This in turn complicated the analysis of the interaction between the elastic and plastic waves, where analytical progress was only possible in the rate-independent case. However we found that, for small a , the solution was closely approximated by the solution of the rate-independent problem, apart from regions close to the front of propagating waves where the rate-dependent solution is smoothed out.

The computation of successive interactions is a non-trivial task and will require further investigation. In the rate-independent problem, we observed two competing effects at play which determine the perturbation to the free-boundary upon each interaction, namely the steepening of the plastic wave as a result of non-linearity, and the steepening of each successive elastic wave as a result of the previous interaction. It quickly becomes an arduous task to compute the result of the n -th interaction between the elastic and plastic wave. Furthermore, at this stage, we have been unable to shed further light on the mechanism by which the plastic wave releases from the free surface after multiple reverberations of the elastic wave, and hence this will require further work.

Chapter 6

Non-uniaxial deformations

6.1 Synopsis

Motivated by the study of isentropic compression experiments, our primary focus thus far has been on uniaxial deformations. However, the three-dimensional model formulated in chapter 2 allows us to explore other geometries of interest. In this chapter we consider the problem of the radial deformation of a cylindrical cavity in an elastic-plastic medium. After writing down our model in cylindrical polar coordinates, we compute numerical solutions in which the cavity is subject to a monotonically increasing velocity drive on its inner surface. The axisymmetric problem presented here is an elementary step into more sophisticated modelling relating to the highly non-trivial problem of the penetration of a solid object by a shaped charge device.

6.2 Motivation and modelling philosophy

A long-standing problem in extreme mechanics and ballistics research is to gain an understanding of how a high velocity projectile [73, 74, 77], or plasticised jet [4, 7], penetrates and deforms a metal target. The latter scenario is found in a device known as a shaped charge, the efficacy of which relies on the rapid plasticisation of a metal liner, which produces a high velocity jet travelling at tens of kilometres per second. Once incident on the intended target, the jet exerts a pressure on the pristine material which is typically several orders of magnitude greater than the yield stress of the material. Motivated by this single observation, the natural question to ask is whether the three-dimensional model proposed in chapter 2 can be applied to the modelling of shaped charge jet penetration.

Recent work in this area is due to Novokshanov and Ockendon [48], who note that a *post hoc* analysis of a deformed metal target, such as that shown in Figure 6.1, reveals two striking features; first, the cavity is approximately cylindrical away from the boundaries of the target, and second, the target has undergone significant irreversible deformation

at several radii away from the cavity, manifested by “bulging” at the extremities of the sample. Once the jet has ceased impacting on the target, and penetration has terminated, the applied pressure must decrease to zero. As shown by Poole [61, Chp. 4], a model of the deforming target based on linear elasticity predicts the cavity radius decreasing to zero as the pressure returns to ambient conditions, which is clearly insufficient. Both Poole and Ockendon [48, 61] then attribute the finite cavity radius to severe plastic deformation of the target, with residual stresses and displacements “locked in” to the material after impact. We note that melting of the metal target is not predicted to play an important role in the deformation process, and that very little target material is ejected from the point of entry [61]; hence we identify plastic deformation as the principal mechanism by which the target deforms.

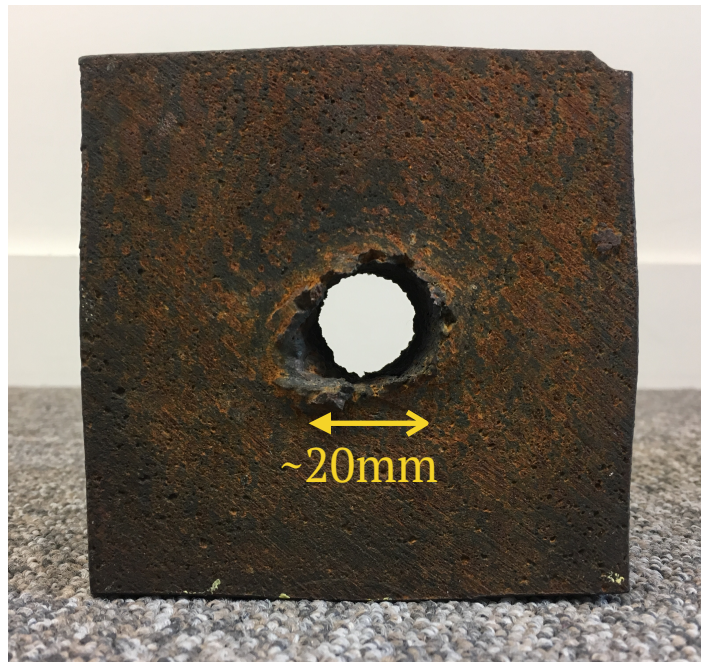


Figure 6.1: Block of steel penetrated by a shaped charge jet. The cavity left behind by the impact is approximately cylindrical, with diameter of approximately 20 mm. There is noticeable protrusions at the boundaries of the target, thought be due to irreversible plastic deformation at several radii greater than the cavity radius.

Historically, there are numerous models in the literature pertaining to the penetration of a highly plasticised jet into a metal target, the most rudimentary of which are due to Birkhoff et al. [7, 8] who treat both the jet and plastically deforming target as ideal fluids. Various attempts have since been made to incorporate the effects of elastic strength [4, 73, 74], including the recent work by Poole [61] and Novokshanov and Ockendon [48] who propose elastic-plastic models which account for residual stress and displacement. While these models have been somewhat successful in predicting some of the basic observations

of shaped charge jet penetration, they often require the solution of complex elastic-plastic free boundary problems. The model presented in chapter 2, however, is posed as a system of conservation laws, encompassing many of the salient phenomena at play during violent elastic-plastic deformation. Thus, we are presented with an opportunity to consolidate existing theoretical work on the key phenomena driving the penetration of the target, namely large deformations due to plasticity, superimposed with small deformations due to elastic strength.

A significant challenge in the modelling of shaped charge jet penetration is reconciling the deformation at the tip of the advancing jet with the deformation of material “far-away” from the tip region. The deformation and dynamics near the tip are governed by fully three-dimensional elastoplasticity, with significant flow thought to occur in the direction parallel to that of the advancing jet. Modelling the deformation in this region is a highly non-trivial task and thus, as a first step, we consider the simpler, but related, scenario depicted in Figure 6.2 in which a cavity in an elastic-plastic material expands under the action of an applied stress on the inner boundary $R = R_0$. This scenario is analogous to the so-called “gun-barrel” problem from solid mechanics [32, Chp. 8], however unlike classical treatments of this problem, inertia may not be neglected and we now need to accommodate for large stresses and strains. We note that recent work pertaining to cavitation in an elastic-plastic medium has been performed by Cohen et al. [13, 15, 16, 43] in the context of so-called J_2 plasticity theory. However, even in this recent work, the deformation is considered to be quasi-static, and hence the transient time-dependent deformation of the radially expanding cavity appears to have received little attention.

6.3 Axisymmetric model

6.3.1 Governing equations

We consider the case of axisymmetric radial expansion of a cavity, in which a particle initially at Lagrangian position $\mathbf{X} = (R, \Theta, Z)$ is displaced to Eulerian position $\mathbf{x} = (r(R, t), \Theta, Z)$ after time t . Guided by the numerical solutions of chapter 3, we hypothesise that the structure of the deformation, shown in Figure 6.2, will consist of five distinct regions, namely a small amplitude elastic wave propagating into quiescent material, followed by a large amplitude plastic wave, with the wave fronts separated by a plateau where the material remains close to the yield surface. An analogous structure was also described by Taylor in [77].

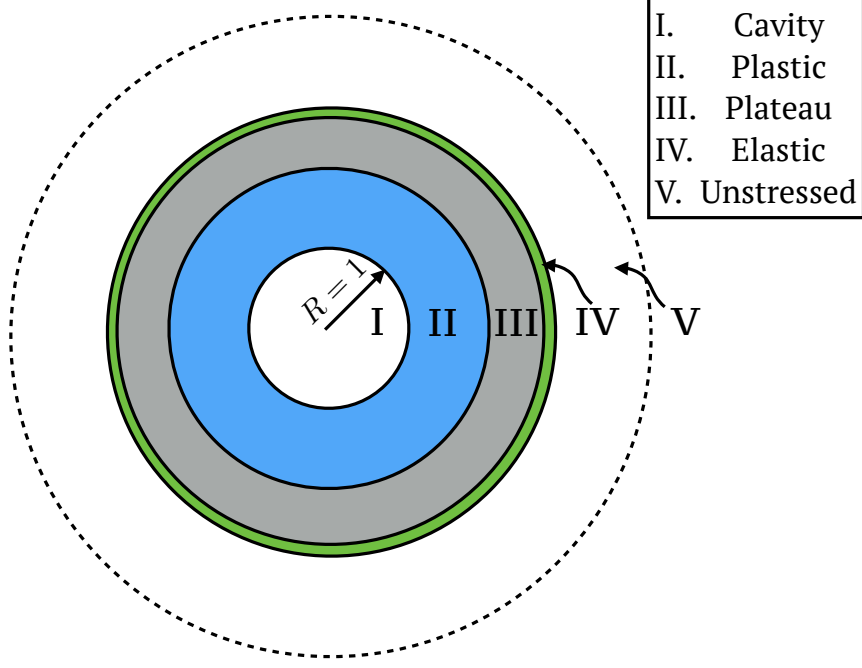


Figure 6.2: Schematic of the anticipated elastic-plastic structure of an axisymmetrically expanding cavity.

The deformation gradient tensor in curvilinear coordinates is [55]

$$F_{ij} = \frac{h_i}{H_j} \frac{\partial x_i}{\partial X_j} \quad (6.3.1)$$

where the scale factors

$$h_i = \left| \frac{\partial \mathbf{x}}{\partial x_i} \right|, \quad H_i = \left| \frac{\partial \mathbf{X}}{\partial X_i} \right|. \quad (6.3.2)$$

In the current radially axisymmetric set-up, the deformation gradient and Cauchy stress tensor are

$$\mathbf{F} = \text{diag} \left(\frac{\partial r}{\partial R}, \frac{r}{R}, 1 \right), \quad \boldsymbol{\sigma} = \text{diag} (\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}) \quad (6.3.3)$$

where σ_{rr} , the radial stress, and $\sigma_{\theta\theta}$, the hoop stress, are the only non-zero in-plane stresses. The Jacobian of $\mathbf{F}$ is therefore

$$\det \mathbf{F} = J = \frac{\partial r}{\partial R} \frac{r}{R} = \nu. \quad (6.3.4)$$

In an analogous way to §2.4.5, it can be easily verified that conservation of mass (2.4.85) and conservation of momentum (2.4.86) (after using (2.4.88)) lead to

$$\frac{\partial \nu}{\partial t} = \frac{r}{R} \frac{\partial u_r}{\partial R} + \frac{\partial r}{\partial R} \frac{u_r}{R}, \quad (6.3.5)$$

$$\rho_0 \frac{\partial u_r}{\partial t} = \frac{r}{R} \frac{\partial}{\partial R} \left(\frac{\partial \mathcal{W}_0}{\partial \nu} + \frac{\Omega(\nu)}{\nu} \mathcal{E}_{rr} \right) + \frac{3\Omega}{2r} \mathcal{E}_{rr}, \quad (6.3.6)$$

where

$$u_r = \left. \frac{\partial r}{\partial t} \right|_R \quad (6.3.7)$$

is the radial velocity, $\mathcal{E}_{rr}$ is the deviatoric strain in the R direction, and the function Ω is as defined in (2.4.70).

Turning now to the plastic flow rule, for simplicity and to compare results with the model in uniaxial strain, we adopt the Tresca yield criterion [32]

$$Y(\boldsymbol{\sigma}) = \max \{ |\sigma_{rr} - \sigma_{\theta\theta}|, |\sigma_{\theta\theta} - \sigma_{zz}|, |\sigma_{zz} - \sigma_{rr}| \} - \sigma_Y \leq 0. \quad (6.3.8)$$

Models pertaining to axisymmetric deformation of a cylindrical cavity, in which the cavity is subject to pressurised loading, assume that the stress components satisfy [30, 32, 61]

$$\sigma_{rr} \leq \sigma_{zz} \leq \sigma_{\theta\theta}, \quad (6.3.9)$$

whence the yield condition (6.3.8) becomes

$$|\sigma_{rr} - \sigma_{\theta\theta}| \leq \sigma_Y. \quad (6.3.10)$$

By analogy with (2.3.22), we define the flow rate

$$\Lambda = \frac{b}{\sigma_Y} [|\sigma_{rr} - \sigma_{\theta\theta}| - \sigma_Y]^+, \quad (6.3.11)$$

where b is a rate constant with units of inverse time, to avoid confusion with the Eulerian variable r . Then, after finding the rate-of-strain tensor $\boldsymbol{\mathcal{D}}$ and the Jaumann derivative $\dot{\hat{\mathcal{E}}}$, the plastic flow rule (2.4.87) becomes

$$\frac{2}{3\nu} \frac{\partial \nu}{\partial t} - \frac{\partial \mathcal{E}_{rr}}{\partial t} - \frac{u_r}{r} = bG \left(\frac{\mathcal{E}_{rr}}{\mathcal{E}_Y(\nu)} \right). \quad (6.3.12)$$

6.3.2 Non-dimensionalisation

We non-dimensionalise (6.3.5), (6.3.6), (6.3.7), and (6.3.12) with

$$(r, R) = R_0(r', R'), \quad t = \frac{R_0}{c_0} t', \quad u = c_0 u' \quad (6.3.13)$$

where R_0 is the initial cavity radius and c_0 is again given by (2.3.26). We also define

$$\mathcal{E}_{rr} = \delta \varepsilon_{rr} \quad (6.3.14)$$

in the same way as §2.3. In summary, the governing equations (6.3.5), (6.3.6), (6.3.7), (6.3.12) become

$$\frac{\partial \nu}{\partial t} = \frac{r}{R} \frac{\partial u_r}{\partial R} + \frac{\partial r}{\partial R} \frac{u_r}{R}, \quad (6.3.15)$$

$$\frac{\partial u_r}{\partial t} = \frac{r}{R} \frac{\partial}{\partial R} \left(h'(\nu) + \delta \frac{2k(\nu)}{3\nu} \varepsilon_{rr} \right) + \delta \frac{k(\nu)}{r} \varepsilon_{rr}, \quad (6.3.16)$$

$$a \left(\frac{2}{3\nu} \frac{\partial \nu}{\partial t} - \delta \frac{\partial \varepsilon_{rr}}{\partial t} - \frac{u_r}{r} \right) = G \left(\frac{\varepsilon_{rr}}{\varepsilon_c(\nu)} \right), \quad (6.3.17)$$

$$u_r = \frac{\partial r}{\partial t}, \quad (6.3.18)$$

where, as before in §2.3, we have

$$a = \frac{c_0}{bR_0}, \quad \varepsilon_c(\nu) = \frac{\sigma_Y(\nu)\nu}{k(\nu)\sigma_{Y0}}, \quad \frac{\Omega(\nu)}{K_b} = \frac{2}{3}k(\nu). \quad (6.3.19)$$

6.4 Compressive wave propagation

In this section we compute numerical solutions of the model (6.3.15)–(6.3.18), subject to the initial and boundary conditions

$$\nu = 1, \quad u_r = 0, \quad \varepsilon_{rr} = 0, \quad r = R \quad \text{at} \quad t = 0, \quad R \geq 1, \quad (6.4.1)$$

$$u_r = U_0(1 - e^{-t}) \quad \text{at} \quad R = 1, \quad t > 0. \quad (6.4.2)$$

We also employ the constitutive assumptions (3.2.3), and choose the same parameter values as in Figure 3.1.

6.4.1 Radial gas dynamics

Before proceeding to the solution of the full elastic-plastic model (6.3.15)–(6.3.18) subject to (6.4.1)–(6.4.2), we first compute solutions to the corresponding barotropic flow problem when $\delta = 0$.

With $\delta = 0$, the system (6.3.15), (6.3.16), and (6.3.18) is hyperbolic with characteristics

$$\frac{dX}{dt} = 0, \quad \frac{dX}{dt} = \pm \frac{r}{R} \sqrt{h''(\nu)}. \quad (6.4.3)$$

Note that, unlike the leading order model in uniaxial strain, namely (2.3.41) and (2.3.42), this system does not admit Riemann invariants, where following the usual procedure leads to

$$\frac{d}{dt} \left(u \mp \int^\nu \sqrt{h''(s)} \, ds \right) + \frac{u}{r} \frac{dr}{dt} = 0, \quad (6.4.4)$$

which cannot be integrated.

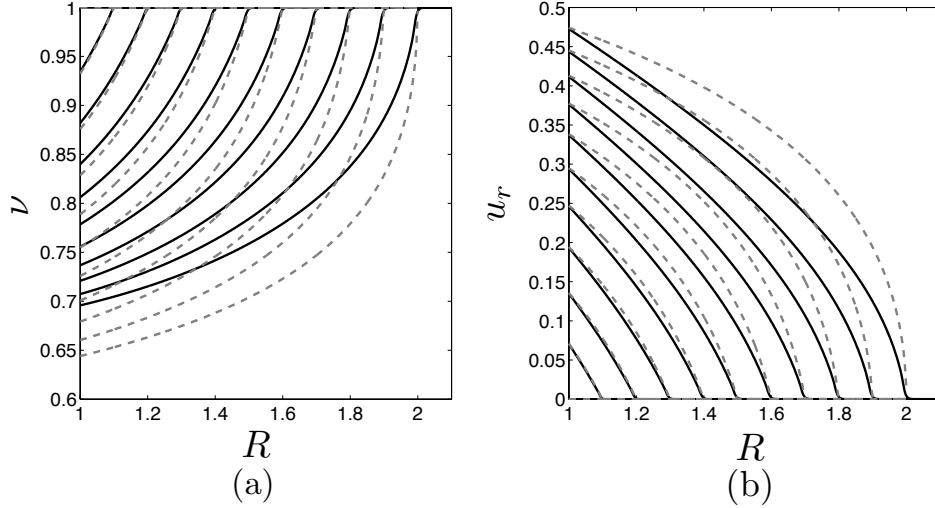


Figure 6.3: Plots of (a) the specific volume ν and (b) the velocity u_r for the leading order problem (6.3.15)–(6.3.16) with $\delta = 0$ (black), and the corresponding solutions for uniaxial compression (grey-dashed). The parameter values are identical to those used in §3.2. Figure 3.1. Profiles are shown at $t = 0, 0.1, \dots, 1$.

In Figure 6.3 we plot (a) the specific volume ν and (b) the radial velocity u_r for the problem (6.3.15), (6.3.16), and (6.3.18) in the case $\delta = 0$ (solid black). We also superimpose the solutions of the one-dimensional barotropic flow model (2.3.41)–(2.3.42) (dashed grey), subject to the same boundary conditions, but in the domain $X \in (1, \infty)$. For early times we observe that the two solutions are close, with the wave fronts propagating along the leading characteristic $R = t + 1$ (or $X = t + 1$ in the uniaxial case). However, for larger t the two models begin to diverge, with the solutions obtained from the uniaxial model steepening much more rapidly compared to the axisymmetric model. Furthermore, we note that the specific volume ν increases more slowly in the axisymmetric model, implying the density is higher close to $R = 1$. Recalling Figure 6.2, we note that as the compressive wave propagates, the arc length of the wave front increases. The foregoing observations pose the question of whether the effect of increasing R is to inhibit, and possibly prevent, the eventual formation of a shock at some finite time t .

On $R = t + 1$ we have $\nu = 1$ and $u = 0$, and therefore

$$\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial R} = 0, \quad \frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial R} = 0. \quad (6.4.5)$$

From this we infer that

$$\frac{dr}{dt} = u_r + \frac{\partial r}{\partial R} = \frac{t+1}{r} \quad \text{on} \quad R = t + 1, \quad (6.4.6)$$

and hence

$$r = t + 1. \quad (6.4.7)$$

Inserting (6.4.5) and (6.4.7) into (6.3.15) and (6.3.16) with $\delta = 0$, we get two copies of

$$\frac{\partial \nu}{\partial R} + \frac{\partial u_r}{\partial R} = 0. \quad (6.4.8)$$

A closure condition is then obtained by differentiating (6.3.15) and (6.3.16) with respect to R which, after some rearranging, leads to the following Bernoulli equation for the gradient $y = \partial u_r / \partial R$ at the front of the advancing compression wave:

$$\frac{dy}{dt} = \frac{1}{2} h'''(1) y^2 - \frac{1}{2(t+1)} y. \quad (6.4.9)$$

The general solution of this o.d.e is

$$y = \frac{y(0)}{\sqrt{t+1} [1 + 2\alpha(\sqrt{t+1} - 1)y(0)]} \quad (6.4.10)$$

where $\alpha = -\frac{1}{2} h'''(1) > 0$. This solution gives finite-time blow up, and hence implies shock formation is inevitable, provided the initial conditions are such that $y(0) < 0$, or equivalently, if $\partial u / \partial t > 0$ from (6.4.5) i.e. that the initial acceleration is positive, as in (6.4.2).

6.4.2 Radial elastic-plastic model

We now compute numerical solutions to the full elastic-plastic model (6.3.15)–(6.3.18) subject to the initial and boundary conditions (6.4.1) and (6.4.2). In Figure 6.4, we plot (a) the specific volume $\nu(R, t)$, (b) the radial velocity $u_r(R, t)$, and (c) the deviatoric strain $\varepsilon_{rr}(R, t)$. The solid black curves show the solution to the full elastic-plastic model (6.3.15)–(6.3.18), and as before we superimpose the corresponding results of the uniaxial model (2.3.54)–(2.3.56) (grey dashed). Referring to the nomenclature used in Figure 6.2, we see that in Figure 6.4(b) a large amplitude plastic wave propagates in region II, which is separated from a small amplitude elastic wave in region IV by a plateau (region III). Other than the rate of steepening of the plastic wave, another interesting difference between the two models is manifested in Figure 6.4(c). We see that, in the region that is the plateau in the uniaxial model, ε_{rr} (and hence the deviatoric stress $\sigma_{rr} - \sigma_{\theta\theta}$) is smaller in the axisymmetric model, and that there is a change in slope of ε_{rr} as it approaches the yield surface. These observations suggest that, while the material is deforming elastically, the hoop stress $\sigma_{\theta\theta}$ has more of a confining effect in the axisymmetric model, compared to the corresponding stress component σ_{22} in the uniaxial model.

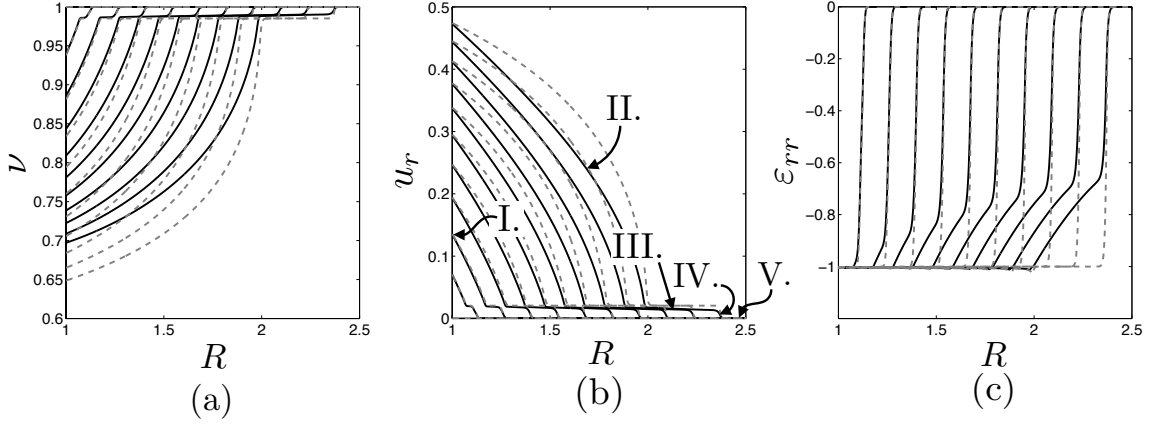


Figure 6.4: Plots of (a) the specific volume ν , (b) the velocity u , and (c) the deviatoric strain ε for the problem (6.3.15)–(6.3.18) (black) and the corresponding solutions for uniaxial compression (grey-dashed). The corresponding regions I.–V. from Figure 6.2 are shown in (b). The parameter values are identical to those used in §3.2. Figure 3.1. Profiles are shown at $t = 0, 0.1, \dots, 1$.

6.5 Conclusions

This chapter was a brief introduction into violent elastic-plastic deformation in geometries other than uniaxial strain. Using the three-dimensional model of §2.4, we were able to write down a radially axisymmetric formulation relevant to the elastic-plastic deformation of a cylindrical cavity in an infinite medium. This problem is an extension to the classical gun-barrel problem in the presence of inertia, and where the strain can no longer be considered small.

After writing down the relevant model in axisymmetric coordinates, we first computed solutions to the barotropic flow problem when $\delta = 0$. When compared to the analogous problem in uniaxial strain, there were two noticeable differences, namely that the density of the material close to the cavity boundary at $R = 1$ was higher than the density close to $X = 1$ in the uniaxial model, and also that the bulk wave appeared to steepen less rapidly in the axisymmetric model. This motivated the question of whether the “weakening” effect of increasing arc length of the wave front was sufficient to combat the steepening due to nonlinearity, in order to inhibit shock formation. By analysing the behaviour of (6.3.15), (6.3.16), and (6.3.18) close to the wave front $R = t + 1$, we derived a Bernoulli type o.d.e for the gradient $\partial u / \partial R$, which informed us that shock formation is inevitable provided the acceleration at $R = 1$ is positive. We then computed numerical solutions to the full elastic-plastic model (6.3.15)–(6.3.18), which corroborated the expected wave structure hypothesised in Figure 6.2. In comparison with the model in uniaxial strain, we found that while the material is deforming elastically, the deviatoric stress is lower

in the axisymmetric model, suggesting that the hoop stress $\sigma_{\theta\theta}$ has more of a confining effect than the corresponding stress component σ_{22} in the uniaxial model. These basic observations represent the start of a non-exhaustive list of interesting phenomena that we could investigate using our model, some of which we will suggest in the next chapter.

Looking to the future, this model is an elementary step into the wider problem of shaped charge jet penetration. This problem is significantly more challenging for a number of reasons, some of which we describe here. The first is coupling the flow in the jet with the deformation of the target. This requires a model for the highly plasticised jet, which some authors [7, 48, 61] approach from a hydrodynamic viewpoint. However, a more realistic model ought to treat both the jet and the target as deforming elastic-plastic media. This in itself raises difficult questions regarding the uniqueness of the mapping $\mathbf{F}$ as the jet impacts on the target, returns, and then begins to interact with itself. The second problem is writing down the appropriate elastic-plastic model at the tip of the jet. Some simplifying assumptions, outlined in [61], can be made to make this problem more tractable, namely by considering an axisymmetric jet where the components of the stress tensor $\sigma_{rz} = \sigma_{\theta z} = 0$. Our final remark concerns reconciling the dynamics at the tip of the jet with the deformation of the target “far away” from the tip region. Ockendon and Novokshanov [48] circumvent this issue by hypothesising that the jet simply provides a localised source of high pressure, while Poole [61] derives a more rigorous analysis based on slender body theory. Ultimately, it is inevitable that this highly complex problem will necessitate the use of numerical methods. However, it may be possible that the Eulerian model (2.4.90)–(2.4.92) can be implemented using novel computational methods in fluid-structure interaction, for example that described in Rycroft and Kamrin [37].

Chapter 7

Conclusions

In this final chapter, we summarise our main findings from chapters 2 to 6, followed by some suggestions for future work. We will conclude with some closing remarks regarding the implications of our research in the wider community of extreme mechanics.

7.1 Summary of results

Motivated by applications in the defence industry, in this thesis we have explored several problems relating to the violent elastic-plastic deformation of metal samples. By constructing, from first principles, a mathematical model applicable in regimes of stress far in excess of the yield stress, we systematically scrutinised many of the key phenomena underpinning elastic-plastic flow by employing techniques from asymptotic and numerical analysis.

In chapter 2, we first constructed a one-dimensional model for elastic-plastic deformation. This was motivated by a class of experiments known as isentropic compression experiments, the details of which allowed us to restrict our attention to uniaxial deformations. The smallness of the ratio of the yield stress to the applied stress is manifested through the non-dimensional parameter $\delta \ll 1$. Using the one-dimensional formulation and the work of Plohr [60] as a guide, we then generalised our model to three-dimensions. This greatly expands the utility of our model, both in its usefulness for AWE and in describing other high-rate phenomena, such as the penetration of a metal target by a shaped charge device.

In chapter 3, we employed state-of-the-art numerical techniques for solving hyperbolic p.d.e to compute numerical solutions of some problems relevant to violent impact testing. In particular, we found that

- compression of a semi-infinite slab of material gives rise to both classical elastic waves and plastic waves of much larger amplitude, confirming the prediction of

the one-dimensional model of chapter 2 that elastic effects should be noticeable in regions where the isotropic and deviatoric contributions to the strain balance;

- during the sequential loading and unloading of the sample, as the face $X = 0$ is decelerated, the material initially undergoes elastic deformation, before re-yielding to give rise to a second plastic region;
- by including a stress-free surface at $X = 1$, small amplitude elastic waves can reverberate between the plastic wave and the free surface, with the velocity increasing in a series of small steps at $X = 1$ before the arrival of the plastic wave;
- after a number of reverberations there is a region of unyielded material close to $X = 1$, and that, as the plastic wave releases from the boundary, the effect of the free surface is to force the material to re-yield over an outgoing elastic region propagating away from $X = 1$;
- our model compared well with the results of finite-element based hydrocodes used at AWE in the case of rate-independent plasticity when a constant yield strength and shear modulus were assumed.

On the final point, we note that further analysis is required to compare the models when more sophisticated functional forms of the yield strength and shear modulus are assumed.

In chapter 4, we considered the simplest problem of elastic-plastic wave propagation, namely rate-independent, linear elastoplasticity. Although these simplifying assumptions are a departure from the high regimes of stress which originally motivated our investigation, we found that even this ostensibly simplest problem gave rise to a fascinating series of previously unexplored phenomena. These include

- the presence of an alternating sequence of elastic and plastic regions, all separated by free boundaries propagating at non-characteristic wave speeds, as a semi-infinite material is sequentially loaded and then unloaded;
- the nature of the interaction between a reflected elastic wave and incoming plastic wave, which was found to give rise to an intricate free boundary problem, the details of which depend of the way in which the material is loaded at the face $X = 0$;
- an analysis of the effect of multiple reverberations of the elastic wave, in which we found an fascinating analogue to the pantograph problem;
- the observation that multiple interactions leads to a condensation of the elastic waves near $X = 1$, apart from when the face $X = 0$ is decelerated and the linear

problem predicts re-yielding of the material as the plastic wave releases from the boundary.

All of the above observations were corroborated with direct numerical solutions, which were shown to give excellent agreement with the analytical predictions.

Chapter 4 neglected the influence of rate-dependence and non-linearity, which motivated us in chapter 5 to study the effect of both of these phenomena when they are reinstated back into the model. We found that rate-dependence gave rise to local smoothing of the solutions obtained, which was a direct consequence of the fact that the stress may depart from the yield surface. This complicated the analysis of the wave interaction problem, where we had to resort to numerical solutions when a was made finite. Nevertheless, when compared to the corresponding rate-independent problem, we observed that the solutions for small- a were well described by the rate-independent solutions, apart from in regions close to the front of propagating waves where the solution was smoothed out. The effect of including nonlinearity causes the plastic compression wave to steepen as it propagates through the sample. This means that, upon interacting with the elastic wave, the plastic wave always “overcomes” the elastic wave, and no local unloading was found to occur as we showed could happen in chapter 4. In the absence of a rigorous analysis of the effect of multiple interactions, we were unable to shed any further light on the release of the plastic wave from the stress free surface. Nevertheless we hope that we have been able to highlight how sensitive the problem is to the inclusion of nonlinear effects.

Finally, in chapter 6, we considered a secondary problem relevant to extreme elastic-plastic deformation, namely the expansion of a circular cavity in an elastic-plastic medium. This was made possible by our three-dimensional formulation of chapter 2. Our elementary analysis was limited to computing numerical solutions of the model (6.3.15)–(6.3.18) when $\delta = 0$ and when δ is small but finite, which were compared to the corresponding problem in uniaxial flow. This analysis revealed that in axisymmetry, the plastic wave steepens less rapidly, prompting the question of whether shock wave formation could be inhibited in this geometry. We found that this could not happen, since a local analysis of the steepening of the wave front near $R = t + 1$ revealed blow-up of the solution in finite time. When elastic strength was reinstated into the model, we found that when the material was deforming elastically, the hoop stress $\sigma_{\theta\theta}$ appeared to have a greater confining effect than the corresponding stress component in uniaxial strain σ_{22} .

7.2 Suggestions for future work

7.2.1 The inverse problem

Our focus throughout this thesis has been on understanding the forward problem of investigating the deformation of elastic-plastic media when particular constitutive models are assumed. However, a major unsolved problem is developing a method by which to tackle the inverse problem for the equation of state in the presence of elastic strength. Hinch [31] and Platt et al. [51] describe an inverse problem by which to determine the equation of state of a highly plasticised metal in the absence of strength. The principal assumption in this work is that the highly plasticised metal may be treated as an inviscid compressible fluid. The problem then involves finding two unknown functions, namely the equation of state $p(\rho)$ and the velocity (or pressure) drive applied at $X = 0$.

In the limit $\delta \rightarrow 0$, the model (2.3.54)–(2.3.56) yields the equations of barotropic flow with an unknown equation of state $h(\nu)$. Thus, in principle, methods such as [31] and [51] may be used to determine the leading order contribution to the thermodynamic pressure. The task is then to find the correction to this equation of state in the presence of elastic strength. However, by taking into account higher order terms in (2.3.54)–(2.3.56) we incur additional unknown functions, specifically $k(\nu)$ and $\sigma_Y(\nu)$. It is unclear at this point how to then proceed along the same lines as Platt et al. and Hinch. An interesting avenue of investigation might therefore be to take alternative viewpoint and consider applying techniques from optimization theory to derive an approximation to $h(\nu)$ when higher order elastic terms are included. Since our model (2.3.54)–(2.3.56) is phrased as a system of nonlinear hyperbolic p.d.e., it may be more efficient to use in such an investigation rather than resorting to the vastly more complicated models found in computational hydrocodes. Furthermore, we hope that our analysis of the forward problem will be able to inform any future work as to the importance and influence of including elastic strength over a purely barotropic model.

7.2.2 Further suggestions

With regards to the modelling work of chapter 2, it would be useful to incorporate more sophisticated phenomena thought to be important in the elastic-plastic deformation of metals, in particular work hardening. This would require the further assumption that the yield stress is an increasing function of the plastic strain, to model the increasing strength of the material in response to plastic work. The resulting model could then be compared against AWE hydrocodes, as suggested in chapters 3 and 4, which would give us more confidence that the two approaches are consistent in the regimes of stress in which we are interested.

Another class of experiments of interest to AWE are so-called ramp-release experiments [9], which are designed to gain insight into the deviatoric stress response of the metal under study, and ultimately to obtain a measure of the “strength” of the material. Like ICEs, the free surface velocity data obtained from these experiments is perturbed by interactions between reflected elastic waves and the incoming plastic wave. Analysis is complicated further by the fact that the material is under tension and compression in different regions of the (X, t) plane (see, for example, Figure 3.4). Our linear analysis gave some insight into the loading and unloading of a semi-infinite sample of elastic-plastic material, but we never considered the influence of including a stress-free boundary in this case. Studying this problem could be useful in informing more sophisticated techniques of analysis as discussed in [9].

The three-dimensional model opens up an exciting avenue in the direction of shaped charge jet penetration. We have shown that our modelling philosophy encapsulates the salient phenomena involved in high strain-rate deformation, and furthermore consolidates many of the existing assumptions made about the mechanics of the highly plasticised jet and the deforming target. An interesting problem would be to consider the impacting jet as slender (as assumed by Poole [61]), and investigate whether an analogy can be made to the phenomenon of hypersonic similitude in compressible flow [50]. The Eulerian model (2.4.90)–(2.4.92) also makes our formulation amenable to the use of numerical methods, for example modern computational fluid dynamics software. We note that the aforementioned suggestions form the basis of a major academic undertaking, however the potential impact of such an investigation could be great.

7.3 Closing remarks

In this work, we have explored a range of fascinating phenomena which we hope will inform the way researchers think about elastic-plastic deformation, in particular with regard to isentropic compression experiments. While we appear to be reaching the limits of the capabilities of traditional mathematical techniques in this field, there is nevertheless a huge amount of value to be found in studying prototypical problems of the type presented in this thesis. For evidence of this, one need look no further than the remark of Davis et al. [19] highlighted in chapter 1. Thus, we hope that our analysis has identified some of the key phenomena at play in ICEs, which, until now, appear to have received little attention by the defence community. There is also an active interest in incorporating the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ into existing hydrocodes. We hope that our three-dimensional model demonstrates how, from first principles, one can rigorously

derive the type of “hydrodynamic-plus-strength” models currently employed in industry standard computational hydrocodes.

Appendix A

Centred Kurganov-Tadmor scheme

The numerical method described below was used by the author to compute numerical solutions of the one-dimensional elastic-plastic model (2.3.54)–(2.3.56) (and the corresponding barotropic flow model when $\delta = 0$) in chapters 3, 4, and 6. The method was implemented using the Fortran 90 programming language.

A.1 Summary of method

The centred Kurganov-Tadmor (KT) scheme [39] is a contemporary numerical method for solving systems of hyperbolic p.d.e. of the form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f} = \mathbf{S} \quad (\text{A.1.1})$$

where $\mathbf{u}$ is the solution vector, $\mathbf{f}$ is a vector of fluxes, and $\mathbf{S}$ is a source term. In comparison to its ancestors, namely Godunov upwinding and the Lax-Friedrichs scheme, the KT formulation is less dissipative (that is, it suffers from reduced numerical viscosity), and also permits a semi-discrete formulation, analogous to a method of lines discretisation.

Integrating the p.d.e (A.1.1) over a cell of width $x \in [x_j - \frac{1}{2}\Delta x, x_j + \frac{1}{2}\Delta x]$ leads to the following finite-volume [26] discretisation

$$\frac{du_j}{dt} + \frac{H_{j+1/2} - H_{j-1/2}}{\Delta x} = S_j, \quad (\text{A.1.2})$$

where the numerical fluxes are defined by

$$2H_{j+1/2} = f(u_{j+1/2}^+) + f(u_{j+1/2}^-) - a_{j+1/2} (u_{j+1/2}^+ - u_{j+1/2}^-). \quad (\text{A.1.3})$$

Second order accuracy is obtained using the interpolated values

$$u_{j+1/2}^+ = u_{j+1} - \frac{1}{2}\Delta x (u_x)_{j+1}, \quad u_{j+1/2}^- = u_j + \frac{1}{2}\Delta x (u_x)_j. \quad (\text{A.1.4})$$

The quantities $a_{j+1/2}$ in (A.1.3) denote the maximum local wave speed, which are defined as the largest eigenvalue of the flux Jacobian matrix when evaluated at the grid points u_j and u_{j+1} . From this, we define the local Courant number

$$\nu_{\text{local}} = a_{j+1/2} \frac{\Delta t}{\Delta x}. \quad (\text{A.1.5})$$

The maximum wave speed λ_{max} should always be greater than or equal to the largest value of $a_{j+1/2}$. The time-step for this scheme is then defined as

$$\Delta t = \frac{\Delta x}{\max_j a_{j+1/2}} \nu_{\text{max}}, \quad (\text{A.1.6})$$

where ν_{max} is the maximum Courant number, which is an adjustable parameter between the values 0 and 1.

The numerical derivatives $(u_x)_j$ are computed using a total variation diminishing (TVD) reconstruction, via the minmod limiter [26, 39]

$$(u_x)_j^n = \text{minmod} \left(\theta \frac{u_{j+1} - u_j}{x_{j+1} - x_j}, \frac{u_{j+1} - u_{j-1}}{x_{j+1} - x_{j-1}}, \theta \frac{u_j - u_{j-1}}{x_j - x_{j-1}} \right), \quad (\text{A.1.7})$$

where the minmod flux limit function is defined by

$$\text{minmod}(y_1, y_2, \dots) = \begin{cases} \min\{y_j\} & \text{if } y_j > 0 \quad \forall j, \\ \max\{y_j\} & \text{if } y_j < 0 \quad \forall j, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.1.8})$$

The adjustable parameter $\theta \in (1, 2)$ controls the amount of dissipation of the scheme; the limit $\theta = 1$ ensures strict monotonicity, while $\theta = 2$ amounts to the least dissipation. We note that setting $(u_x)_j^n = 0$ returns the scheme to a first-order Lax-Friedrichs scheme.

The temporal integration is handled using a second order TVD Runge-Kutta scheme proposed by Shu and Osher [69]. Writing the scheme (A.1.2) schematically as

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) \quad (\text{A.1.9})$$

the Shu and Osher scheme is

$$\mathbf{u}^* = \mathbf{u}(t) + \Delta t \mathbf{F}(\mathbf{u}(t)), \quad (\text{A.1.10})$$

$$\mathbf{u}^{**} = \mathbf{u}^* + \Delta t \mathbf{F}(\mathbf{u}^*), \quad (\text{A.1.11})$$

$$\mathbf{u}(t + \Delta t) = \frac{1}{2}(\mathbf{u} + \mathbf{u}^{**}), \quad (\text{A.1.12})$$

where the time-step Δt should be chosen according to the CFL condition (A.1.6).

A.2 Convergence study

As verification of the convergence of the numerical scheme employed throughout chapter 3, we compute solutions to the problem described in §3.2 at fixed $t = 0.5$ for different values of the rate parameter a , in each case reducing the grid-spacing until no further improvement in resolution is observed. In Figures (a)–(c) $a = 0.01$, (d)–(f) $a = 0.1$, and (g)–(i) $a = 1$. In all three cases, we observe excessive smearing of the solutions at the largest grid spacing $\Delta X = 0.01$, indicated by the black line. Reducing the grid spacing to $\Delta X = 0.001$ (employed throughout chapter 3) results in significant refinement of the solution, with the sharp features of the elastic precursor being well captured by this scheme. Reducing the grid spacing further beyond $\Delta X = 0.001$ offers no further significant improvement of the solution, and thus we can be confident that the solution has converged satisfactorily.

We note problems containing singular perturbations i.e. $\delta \rightarrow 0$ or $a \rightarrow 0$ often result in oscillatory numerical solutions as the relevant parameter value is decreased to zero [41]. If this is found to occur, the oscillations can be tamed by setting the time-step to be smaller than the CFL limit (A.1.6). However, this was not found to be a major issue in our computations, although this eventuality could arise at smaller values of a and δ .

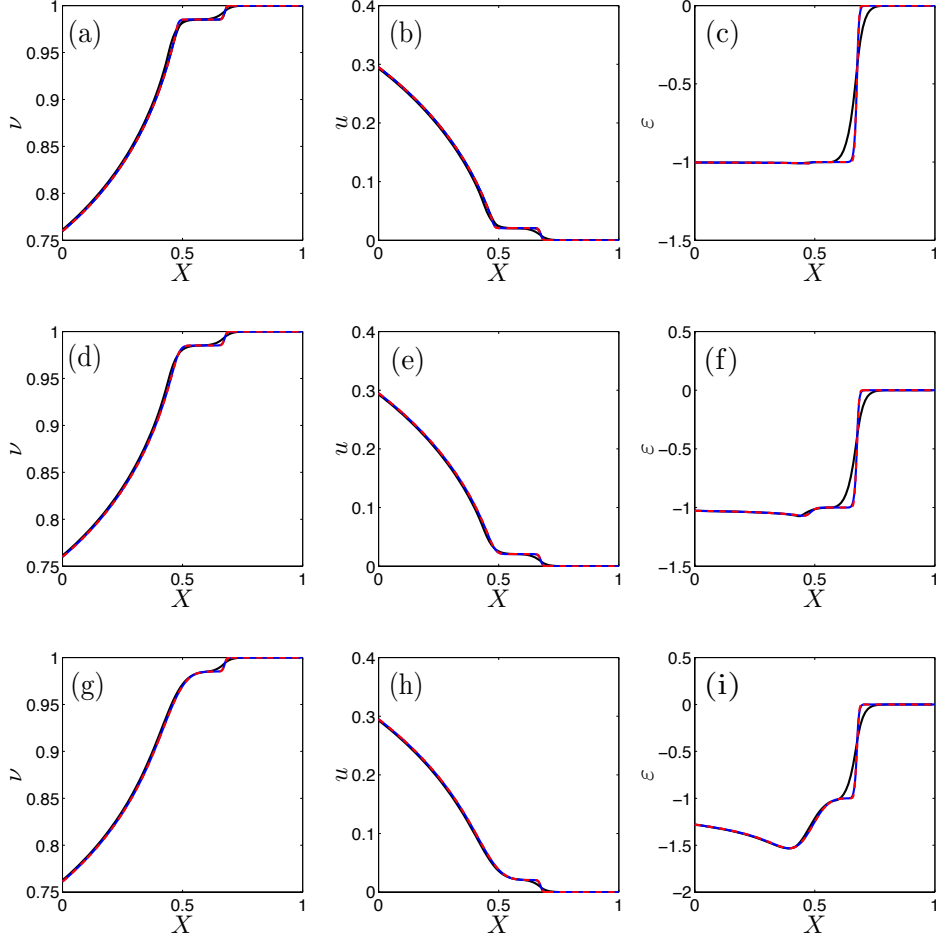


Figure A.1: Numerical solution of the problem presented in §3.2 at $t = 0.5$ where in (a)–(c) $a = 0.01$, (d)–(f) $a = 0.1$, and (g)–(i) $a = 1$. In all of the plots, the black line indicates a grid-spacing of $\Delta X = 0.01$, while the solid blue and dashed red lines represent $\Delta X = 0.001$ (used throughout chapter 3) and $\Delta X = 0.0005$ respectively. No significant improvement of the solution is observed beyond $\Delta X = 0.001$.

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