The Algebraic Construction of Invariant Differential Operators

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by

ert James Baston

Magdalen College

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ABSTRACT

Let G be a complex semisimple Lie Group with parabolic subgroup P, so that G/P is a generalized flag manifold. An algebraic construction of invariant differential operators between sections of homogeneous bundles over such spaces is given and it is shown how this leads to the classification of all such operators. As an example of a process which naturally generates such operators, the algebraic Penrose transform between generalized flag manifolds is given and computed for several cases, extending standard results in Twistor Theory to higher dimensions. It is then shown how to adapt the homogeneous construction to manifolds with a certain class of tangent bundle structure, including conformal manifolds. This leads to a natural definition of invariant differential operators on such manifolds, and an algebraic method for their construction. A curved analogue of the Penrose transform is given.

Foreword

It is convenient in this Thesis to work with complex Lie algebras and groups for the simplifications afforded by algebraic closure. Similar results on invariant differential operators will be true over $\mathbb{R}$. In Chapter Four, manifolds need not be holomorphic, except in section (4.8) on the curved Penrose Transform.

A note on organisation : chapters two, three and four are divided into sections. Within each section, theorems, lemmas and examples are labelled consecutively by Roman capitals and referred to by means of these. When it is necessary to refer to them from outside their section, the section number is given also.

References are indicated by [name].

I should like to thank my supervisors, Prof Roger Penrose and Dr Michael Eastwood, for their encouragement and the fascinating mathematics they have taught me, and the Rhodes Trustees for financial support. Support and encouragement from my wife and parents is happily acknowledged.

R J Baston
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1 : Introduction

The aim of this thesis is to give an algebraic construction of invariant differential operators on homogeneous bundles over complex generalized flag manifolds, then seeking a natural extension of the construction to more general curved manifolds, which admit a finite order osculation by a suitable generalized flag manifold. The resulting class of differential operators is a natural invariant of the osculation structure, and may reasonably be taken to define what invariant differential operators on such manifolds should be.

Examples of invariant differential operators on homogeneous bundles over generalized flag manifolds have long been studied as physically interesting - in the abstract index notation of [Penrose and Rindler I] (which will be used throughout the thesis) noted examples are those occuring in the *zero rest mass field* equations on complex compact Minkowski space M^4:

$$\nabla_{A}^{A_1'} \varphi_{A_1' \ldots A_n'} = 0 \qquad (1.1)$$

or

$$\nabla_{A'}^{A_1} \psi_{A_1 \ldots A_n} = 0 \qquad (1.2)$$

whose solutions represent, for example, self-dual electromagnetic fields ((1.1) with n = 2) or linearized anti-self dual gravity ((1.2) with n = 4). Another example is the wave operator $\Box$ acting on functions with conformal weight [-1] to functions with conformal weight [-3]. These operators are invariant under the conformal motions of Minkowski space : if $f : M^4 \longrightarrow M^4$ is such a motion, and $\mathcal{D}$ is one of the above operators, then $\mathcal{D} \circ f^* = f^* \circ \mathcal{D}$.

They are also invariant in a somewhat different sense : as written, they appear to depend not on the conformal geometry of M^4 alone, but on a choice, locally, of a metric in the allowed conformal class on M^4, defining the Levi-Civita connection on spinors which appears. Nonetheless, if a choice g_{ab} of metric is altered by rescaling - $g_{ab} \longrightarrow \hat{g}_{ab} = \Omega^2 g_{ab}$ - so that $\nabla_{AA'}$ is replaced by the new Levi Civita connection $\hat{\nabla}_{AA'}$, and if, for example, $\varphi_{A'}$ is replaced by $\hat{\varphi}_{A'} = \Omega^{-1}\varphi_{A'}$ then

$$\begin{aligned} \hat{\nabla}_{BB'} \hat{\varphi}_{A'} &= \Omega^{-1} \{\nabla_{BB'} \varphi_{A'} - \Upsilon_{BA'} \varphi_{B'} - \Upsilon_{BB'} \varphi_{A'}\} \\ &= \Omega^{-1} \{\nabla_{BB'} \varphi_{A'} - 2\, \Upsilon_{B(A'}\varphi_{B')}\}, \end{aligned} \tag{1.3}$$

where $\Upsilon_{AA'} = \Omega^{-1} \nabla_{AA'}\Omega$, whence

$$\hat{\nabla}_A^{A'} \hat{\varphi}_{A'} = \Omega^{-2} \nabla_A^{A'} \varphi_{A'}$$

In other words, the operator

$$\nabla_A^{A'} : \mathcal{O}_{A'}[-1] \longrightarrow \mathcal{O}_A[-2],$$

which acts between sections of bundles of conformally weighted spinors, is independent of any choice of metric within the allowed conformal class, and is thus an operator invariant of the conformal structure of M^4. (The same is true of the other operators in (1.1) and (1.2) and the wave operator $\Box$).

This second form of invariance extends to more general manifolds which may be curved, not admitting any global symmetries. In particular, if X is a conformal four dimensional complex manifold (with a spin structure), then the first order operators in (1.1), (1.2) with $\nabla_A^{A'}$ the Levi-Civita connection of a (curved) metric in the conformal class, are, by the same calculation (1.3), invariants of the conformal structure of X. A less trivial example is a modification of the wave operator $\Box = \nabla^a\nabla_a$ - consider instead the operator

$$\Box + R/6 \quad : \quad \mathcal{O}[-1] \longrightarrow \mathcal{O}[-3]$$

(where R is the scalar curvature of the locally chosen metric) This is also independent of the choice of local metric.

The first definition of invariance is clear. Complex compact

Minkowski space is a homogeneous space under the action of its group of conformal motions, which is covered by $SO(6,\mathbb{C})$, or, more conveniently for Twistorial purposes, by $G = SL(4,\mathbb{C})$. The stabilizer P of a point is a covering of the conformal Poincare group $CO(4,\mathbb{C}) \ltimes \mathbb{C}^4$, where $CO(4,\mathbb{C}) = SO(4,\mathbb{C}) \times \mathbb{C}$. P sits inside $SL(4,\mathbb{C})$ as the subgroup of tracefree complex matrices of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \qquad (1.4)$$

(where * denotes an arbitrary entry), realizing M^4 as the Grassmanian of 2-planes in $\mathbb{C}^4$. The natural principle bundle $SL(4,\mathbb{C}) \xrightarrow{P} M^4$ induces homogeneous bundles over M^4 (from representations of P) between which differential operators may act. One of these is invariant if it commutes with the conformal translations of M^4 induced by the left translations of $SL(4,\mathbb{C})$. It is determined by its action on jets of sections at a base point. Jet bundles of homogeneous bundles are homogeneous also, so that invariant differential operators are equivalent to homomorphisms between representations of P.

This is ofcourse true for any G and P. In the case of Minkowski space, and indeed all of the common spaces of Twistor Theory, G and P have further properties - G is a *complex semisimple* Lie group, and P is a *parabolic subgroup*. As we shall see, this means that the homomorphisms equivalent to invariant differential operators promote to homomorphisms of representations of $\mathfrak{g}$ (the Lie algebra of G) induced from representations of $\mathfrak{p}$ (the Lie algebra of P).

There is a rich body of theory concerning such parabolically induced representations. Using it, [Eastwood & Rice] were able to completely classify all invariant differential operators between homogeneous bundles on Minkowski space. Their ideas, which apply to any complex semisimple G and parabolic P, are developed in the sequel.

We begin, in Chapter Two, by giving an account of the geometry of the quotients G/P which are called *Generalized Flag Manifolds*. To do this, we review the structure and representation theory of complex semisimple Lie algebras and their parabolic subalgebras, introducing a useful notation for parabolics, weights, representations and homogeneous bundles. There is a discussion of the natural fibrations of generalized flag manifolds, and inverse images of homogeneous bundles under such fibrations (in preparation for work on an algebraic form of the Penrose Transform). A section is devoted to the Weyl group of a semisimple Lie algebra $\mathfrak{g}$ which introduces several important technical tools. By definition the Weyl group acts on weights for $\mathfrak{g}$; we give the action explicitly in our notation for weights in preparation for the computation of classes of homogeneous bundles between which invariant operators may exist and the computation of the algebraic Penrose Transform. We examine the directed graph structure of the Weyl group, and single out a subgraph associated to any given parabolic subalgebra $\mathfrak{p}$. This is important in proving results concerning the existence of invariant operators, and gives the topological structure of the associated generalized flag manifolds. The last section of chapter two is something of a digression - we include for completeness the elegant realization of generalized flag manifolds as projective algebraic varieties and a discussion of their cell structure. It would appear from the work of [Kempff] and others that these structures are closely related to the composition series structure of induced modules; equally, since they provide simple affine coverings of the spaces, we might hope that they will be useful in classifying vector bundles on complex homogeneous spaces - the investigation of these matters is for the future, however.

With a theoretical and notational foundation laid, we are able, in Chapter Three, to review and develop the ideas of [Eastwood & Rice]. We give a general discussion of Jet Bundles realizing them as homogeneous bundles by means of an inducing functor $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ from the category of finite dimensional $\mathfrak{p}$ modules to the category of $\mathfrak{g}$ modules. We use this to formulate the classification problem for invariant differential operators on generalized flag manifolds in algebraic terms, and following [Eastwood and Rice] give a *symbol principle* as the rudimentary method of determining invariant operators. The notion of *infinitesmal character* is introduced, which divides bundles into (small finite) classes, within which, only, invariant differential operators may act. We employ a theorem of [Harish Chandra] and our notation for weights to compute the elements of any given class. The simplest situation arises when the parabolic $\mathfrak{p}$ is minimal - that is when it is a maximal solvable or *Borel* subalgebra of $\mathfrak{g}$. In this case, it is possible to give a complete theory of the existence and uniqueness (up to scale) of invariant operators. By taking direct images, many, but not all, invariant operators may be deduced for the more general case. It is perfectly possible, for example, for the direct image of an operator to be zero, and for there yet to be a non-standard invariant operator on the direct image bundles. Indeed, the wave operator $\Box$ above is an example of this phenomenon. Following work of [Lepowsky I], we discuss this and place some limitations on its occurence. In particular, it is shown how to compute when a direct image of an invariant operator vanishes. The final construction is the powerful *Translation Principle*, which is a means of generating new invariant operators from old - in effect, it reduces the classification problem to the checking of finitely many special cases. An important set of homogeneous bundles - those of *non-singular infinitesmal character* - together with all their invariant operators may be generated from (a subsequence of) the de Rham resolution and its invariant operators.

The structures of generalized flag manifolds are ideally suited to the execution of an *Algebraic Penrose Transform*. We give this in section (3.8), generalizing to higher dimensions the standard results of four dimensional Twistor Theory. The transform is the major source of invariant operators.

The second notion of invariance is the topic of Chapter Four. The idea is quite simple. Given a manifold X, with some appropriate local structure, which may be curved, we seek a naturally defined P-principal bundle $\mathcal{G}$ over X, together with an osculation (a *Cartan connection*) of $\mathcal{G}$ by G along the fibres to X. This structure is sufficient to realise the Jet bundles of induced bundles as as induced bundles. The constructions of the previous chapter then apply to give a distinguished family of differential operators on X. Members of this family are in one to one correspondence with those of the homogeneous case.

Section (4.2) is devoted to the definition of a Cartan connection ω and the deduction of the family of differential operators. This presupposes the existence of $\mathcal{G}$ and the Cartan connection ω. To construct these, following work of [Tanaka] and [Ochiai], we introduce locally $|1|$-graded manifolds in section (4.3). These are manifolds modelled to finite order at each point on a generalized flag manifold whose tangent bundle admits an irreducible structure. Conformal manifolds are an example. For these manifolds, given the satisfaction of certain conditions on the *Spencer cohomology* of the Lie algebra $\mathfrak{g}$, we show that $\mathcal{G}$ and ω are uniquely defined. Using the details of the construction, we show how one of the distinguished differential operators may be written in terms of any local connection preserving the structure on the tangent bundle of X. It is manifestly apparent that the operator is independent of the choice of local connection and hence an invariant of the structure of X. It is in this sense that the differential operators are called *invariant*.

The theory is illustrated, in section (4.7), by the construction of invariant differential operators on four dimensional conformal manifolds in terms of a Levi Civita connection for a metric in the conformal class. We reconstruct the modified wave operator and give several other examples.

Finally, a curved version of the Penrose Transform is given. The transform is used to regain results of [Hitchin] on massless fields on half conformally flat manifolds and to clarify work of [Le Brun II] concerning manifolds conformal to Einstein manifolds : in particular a criterion is given for conformality to a vacuum solution of Einstein's equations.

2 : The Geometry of Generalized Flag Manifolds

The common spaces of Twistor Theory are examples of a general class of spaces, the complex homogeneous spaces X = G/P where P is a connected parabolic subgroup of the complex semisimple Lie Group G. These are now called *generalized flag manifolds* (in recognition of the case G = SL(n,$\mathbb{C}$)). They form the basic spaces of this and the next chapter, whilst a subclass will provide models for the local geometry of curved spaces in the final chapter. This chapter reviews their geometry, and the structure and representation theory of complex semi-simple Lie Algebras. The general references for terminology and unsubstantiated assertions are [Humphreys], [Vogan] (for Lie Algebras and representations) and [Bernstein et al I] (for the geometry of G/P).

2.1 Structure Theory of Lie Algebras

Let G be a complex semi-simple Lie Group, with Lie Algebra g. Choose a Cartan subalgebra $\mathfrak{h}$ of g and denote by $\Delta(g,\mathfrak{h})$ (or just Δ) the set of roots of g with respect to $\mathfrak{h}$. Given a choice $\mathcal{S} \subset \Delta(g,\mathfrak{h})$ of simple roots for g, $\Delta^+(g,\mathfrak{h})$ will denote the positive roots of g with respect to the usual lexicographic ordering. Root spaces of g will be denoted E_α, $\alpha \in \Delta$, so that g is the direct sum

$$g = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} E_\alpha) \qquad \text{with} \quad [E_\alpha, E_\beta] \subset E_{\alpha+\beta}$$

where we set $E_{\alpha+\beta} = 0$ if $\alpha+\beta$ is not in Δ. By definition, $\Delta(\mathfrak{g},\mathfrak{h})$ is an integral lattice in $\mathfrak{h}^*$, and $\mathfrak{h}^*_{\mathbb{R}}$ will denote the real span of Δ.

A maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called a *Borel subalgebra* of $\mathfrak{g}$. Given $\mathfrak{h}$ and $\mathcal{S}$, such an algebra is defined by setting $\Delta(\mathfrak{b},\mathfrak{h})$ (the roots of $\mathfrak{b}$) = $\Delta^+(\mathfrak{g},\mathfrak{h})$, defining

$$\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta(\mathfrak{b},\mathfrak{h})} E_\alpha).$$

A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra. Given $\mathfrak{h}$, $\mathcal{S}$, a *standard parabolic* subalgebra with defining set $\Psi(\mathfrak{p}) \subset \mathcal{S}$ of simple roots is defined as follows : let $\Delta(\mathfrak{l},\mathfrak{h})$ be the span of Ψ in $\Delta(\mathfrak{g},\mathfrak{h})$, $\Delta(\mathfrak{u},\mathfrak{h})$ the complement of $\Delta(\mathfrak{l},\mathfrak{h})$ in $\Delta^+(\mathfrak{g},\mathfrak{h})$ and set $\Delta(\mathfrak{u}_-,\mathfrak{h}) = - \Delta(\mathfrak{u},\mathfrak{h})$. Thus

$$\mathfrak{l} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta(\mathfrak{l})} E_\alpha) \ ; \qquad \mathfrak{u} = \oplus_{\alpha \in \Delta(\mathfrak{u})} E_\alpha \ ;$$
$$\mathfrak{u}_- = \oplus_{\alpha \in \Delta(\mathfrak{u}_-)} E_\alpha ,$$

and we set $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ (a Levi decomposition of $\mathfrak{p}$). $\mathfrak{l}$ is a reductive subalgebra in $\mathfrak{g}$ (hence a direct sum of a semi-simple part $\mathfrak{l}^s$ and a centre $\mathfrak{l}_Z$) and $\mathfrak{u}$ is nilpotent. $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_-$. Up to conjugation, all parabolic subalgebras of $\mathfrak{g}$ are of the standard form.

We now introduce a useful *notation for parabolic subalgebras*.

Recall that semisimple Lie Algebras are classified by their Dynkin diagrams. For chosen $\mathfrak{h}$ and $\mathcal{S}$, each node of the diagram corresponds to a simple root α_i of g. A standard parabolic of g is specified by crossing through those nodes of the diagram for g corresponding to simple roots $\alpha_i \in \mathcal{S} \setminus \Psi$. The Dynkin diagram for $\mathfrak{l}^s$ is obtained by deleting crossed through nodes and their incident edges. The centre $\mathfrak{l}_Z$ has $|\mathcal{S}\setminus\Psi|$ generators.

<u>Examples</u>

i) A standard Borel subalgebra of g is obtained by crossing through all nodes in the Dynkin diagram of g

ii) For standard twistor theory, g = sl(4,$\mathbb{C}$). $\mathfrak{h}$ consists of diagonal trace-free matrices, and the root spaces are the matrices with single non-zero off diagonal entries. We label generators by x_i, X_i, Y_i, y_i, h_i, indicating the position of a single unit non-zero entry thus :

$$\begin{bmatrix} * & X_1 & x_2 & x_4 \\ Y_1 & * & x_1 & x_3 \\ y_2 & y_1 & * & X_2 \\ y_4 & y_3 & Y_2 & * \end{bmatrix}$$

with h_1 = diag (0,1,-1,0), h_2 = (1,-1,0,0) and h_3 = (0,0,1,-1), spanning the Cartan subalgebra $\mathfrak{h}$.

The Dynkin diagram for g is $\overset{\alpha_2}{\bullet}\!\!-\!\!\overset{\alpha_1}{\bullet}\!\!-\!\!\overset{\alpha_3}{\bullet}$ (with simple root ordering to emphasise the isomorphism sl(4,$\mathbb{C}$) ≅ so(6,$\mathbb{C}$) ≅ $\alpha_1 \bullet\!\!<^{\bullet\,\alpha_2}_{\bullet\,\alpha_3}$). The standard Borel subalgebra of upper triangular matrices is given by $\times\!\!-\!\!\times\!\!-\!\!\times$. The parabolic $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$ is:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \in \mathrm{sl}(4,\mathbb{C})$$

(where * denotes an arbitrary entry), and ×—•—• is:

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \in sl(4,\mathbb{C})$$

Parabolics •—•—×, ×—•—×, ×—×—•, etc are similar.

2.2 Generalized Flag Manifolds

Let $P \subset G$ be the normalizer of a parabolic $\mathfrak{p}$ in $\mathfrak{g}$ under the Adjoint action of G. By taking a cover of G, if necessary, we may assume that P is connected, and the space $X = G/P$ is called a generalized flag manifold. X is compact, simply connected, Kähler and complex homogeneous, and [Wang] has shown that all such spaces have this form (cf also [Warner]). We will denote the space X by the Dynkin Diagram for $\mathfrak{p}$, also.

Examples

i) The simplest example is X = ×. G is $SL(2,\mathbb{C})$ with P, the upper triangular matrices, stabilizing the line $(\mu,0)$ in $\mathbb{C}^2$, so X is $\mathbb{C}P^1$.

ii) In Twistor Theory, complexified compactified Minkowski space M is the Grassmanian of 2-planes in $\mathbb{C}^4$ (cf [Penrose & Ward]. Under the action of $SL(4,\mathbb{C})$ on $\mathbb{C}^4$, the 2-plane $(\mu,\nu,0,0)$ is stabilized by P corresponding to •—×—•, so that M = •—×—• . Similarly:

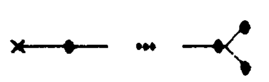
PT = $\mathbb{C}P_3$ = ×—•—• ;

PT* = $\mathbb{C}P_3^*$ = •—•—× ;

A (ambitwistor space) = ;

F (correspondence space between M, PT) = etc.
(Ofcourse these have always been thought of as flag spaces in the traditional sense [Eastwood I])

iii) To generalize the construction of M to higher dimensions, recall that a particular method of realizing M as a homogeneous space is to embed M into the light-cone of the origin in $\mathbb{C}^6-\{0\}$ and projectify. The projective action of SO(6,$\mathbb{C}$) then covers the action of the conformal motions on M. This generalizes to all dimensions, and one obtains higher dimensional *complexified spheres* :

M^{2n} =  (n+1 nodes) ;

M^{2n+1} =  (n+1 nodes) .

Here, G may be taken to be either SO(2n+2,$\mathbb{C}$), SO(2n+3,$\mathbb{C}$) or their simply connected covering groups Spin(2n+2,$\mathbb{C}$), Spin(2n+3,$\mathbb{C}$). The stabilizer subgroup is accordingly either $CO(m,\mathbb{C}) \ltimes \mathbb{C}^m$ or $CSpin(m,\mathbb{C}) \ltimes \mathbb{C}^m$, its double cover.

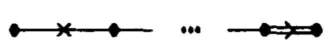
Twistor space generalizations are :

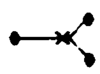
PT^{2n} =  ; PT^{2n*} =  ;

PT^{2n+1} =  . (all n+1 nodes).

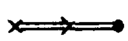
G is taken simply connected here. These spaces will turn out to be the spaces of pure spinors of Spin(m,$\mathbb{C}$). Ambitwistor spaces are obviously defined in even dimensions - but for $n \neq 2$, they are not the spaces of null rays N^m in M^m. These are readily identified by noting that they are complex homogeneous spaces (any null ray may be taken into any other by a translation and a rotation), admitting a contact structure [Le Brun I]. These have been classified by [Boothby], there being one for each complex semi-simple Lie Group. So

N^{2n} =  $(n \geq 4)$;

N^{2n+1} =  $(n \geq 3)$;

N^6 =  ;

N^4 = ;

N^3 =  .

2.3 Fibrations of Generalized Flag Manifolds

If $\mathfrak{p} \subset \mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$, then there is a natural fibration $G/P \longrightarrow G/P'$, with fibre P'/P, which is itself a generalized flag manifold (whose Dynkin diagram is obtained by deleting from the Dynkin diagram for $\mathfrak{p}$ all crossed through nodes and incident edges *shared* with $\mathfrak{p}'$, and then ignoring all connected components with no crossed through nodes). For example, the following fibration is well known in Twistor Theory :

 $\xrightarrow{\mu}$ (F_{123} $\xrightarrow{\mu}$ M in [Eastwood I]) whose fibre is × × = $\mathbb{C}P_1 \times \mathbb{C}P_1$.

The fibration

G^{2n} = ×—×— ⋯ —•< ⟶ ×—•— ⋯ —•< = M^{2n} (n+1 nodes)

has as fibre M^{2n-2} = ×—•— ⋯ —•< (n nodes), so that G^{2n} is the fibre bundle of projectified null cones on M^{2n}. Combined with the fibration

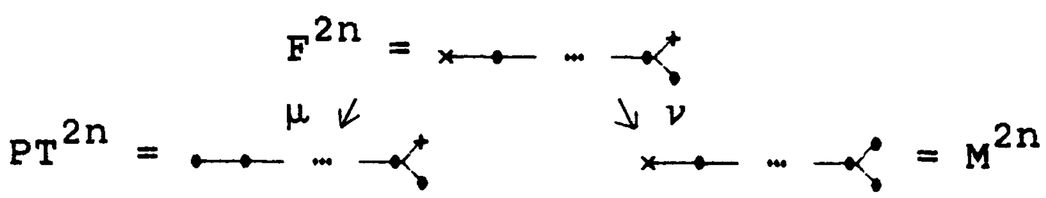
G^{2n} = ×—×— ⋯ —•< ⟶ •—×— ⋯ —•< = N^{2n}

with fibre CP_1 = × , a point of N^{2n} clearly corresponds to a null line in M^{2n} as claimed.

There is always a *double fibration* [Eastwood et al] associated to any pair of generalized flag spaces: consider two standard parabolics P, P', and let P" = P∩P'. Then, taking a covering of G if necessary, P" is a connected standard parabolic, and there is a double fibration :

$$\begin{array}{ccc} & G/P'' & \\ \mu \swarrow & & \searrow \nu \\ G/P & & G/P' \end{array}$$

Well known examples of these are ×—•—• ← ×—×—• → •—×—• ($PT^4 \leftarrow F^4 \rightarrow M^4$) and ×—•—× ← ×—×—× → •—×—• ($A \leftarrow G^4 \rightarrow M^4$), and more generally we have higher dimensional twistor correspondences :



Accepting that the ν-fibre  (n nodes) is the space of pure spinors for $so(2n,\mathbb{C})$, F^{2n} is the bundle of totally null α-planes at each point of M^{2n}. The μ-fibre is $\mathbb{C}P^n$, so that a point in T^{2n} corresponds to an α-plane in M^{2n}. A similar construction applies to PT^{2n*} and PT^{2n+1}.

Double fibrations such as this form the basic geometry of the *Generalized Penrose Transform.*

2.4 Bundles on Homogeneous Spaces

The P-principal bundle $G \to G/P = X$ together with a representation $\rho : P \to End(V)$ for some $\mathbb{C}$-vector space V induces the vector bundle $G \times_P V = \mathcal{O}(V)$, where, as often in Twistor Theory, we blur the distinction between locally free sheaves and vector bundles. A section of $\mathcal{O}(V)$ corresponds to a V-valued function f on G satisfying $f(gp) = \rho(p^{-1})f$, for $g \in G$ and $p \in P$.

2.4.1 Representations of $\mathfrak{g}$ and $\mathfrak{p}$

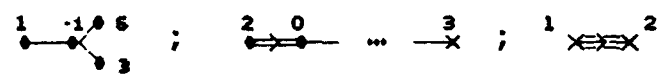
Our basic representations of P will be either irreducible or a restriction of an irreducible representation of G. Since we may take G and P to be simply connected we may work with representations of $\mathfrak{g}$ and $\mathfrak{p}$ whose irreducible representations are classified by means of the theory of weights [Humphreys].

Let a Cartan subalgebra $\mathfrak{h}$ and a set $\mathcal{S}$ of simple roots for $\mathfrak{g}$ be fixed. Recall that a *weight* for $\mathfrak{g}$ (or $\mathfrak{p}$, since $\mathfrak{h} \subset \mathfrak{p}$) is an element of $\mathfrak{h}^*$ and that a vector v in a representation V for $\mathfrak{g}$ or $\mathfrak{p}$ is a *weight vector*, of weight λ, if $\forall\ h \in \mathfrak{h}$, $h \cdot v = \lambda(h)v$. λ is then a *weight of* V and the set of such is denoted $\Delta(V)$. The set of all vectors of weight λ is called a *weight space* of V, and its dimension is the *multiplicity* of λ. A weight vector is called *maximal* if it is annihilated by the maximal nilpotent subalgebra $\mathfrak{n} = \oplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})} E_\alpha$ of *raising operators* of $\mathfrak{g}$, $\mathfrak{p}$. Next, the lexicographical ordering of $\Delta(\mathfrak{g},\mathfrak{h})$, determined by $\mathcal{S}$, extends to (a partial ordering of) all of $\mathfrak{h}^*$. Thus, if $\lambda,\lambda' \in \mathfrak{h}^*$, we say $\lambda' \leq \lambda$ if $\lambda-\lambda'$ is a sum of non-negative integral multiples of elements of $\mathcal{S}$. Then a maximal vector of weight λ is *highest* in V (and λ is a *highest weight* of V) if $\lambda' \leq \lambda$ for all $\lambda' \in \Delta(V)$.

Let us specialize the discussion to those weights in $\mathfrak{h}^*_{\mathbb{R}}$, the real span of $\Delta(\mathfrak{g},\mathfrak{h})$ in $\mathfrak{h}^*$. The Killing form on $\mathfrak{g}$ induces a bilinear positive definite form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*_{\mathbb{R}}$. For each $\alpha \in \Delta$, let the wall W_α be the hyperplane perpendicular to α in $\mathfrak{h}^*_{\mathbb{R}}$. The walls partition $\mathfrak{h}^*_{\mathbb{R}}$ into distinct open regions or *Weyl chambers*. An unique chamber - the *fundamental chamber* - is distinguished requiring its points λ to satisfy $\langle\lambda,\alpha\rangle > 0$ for all $\alpha \in \Delta^+$ (equivalently, all $\alpha \in \mathcal{S}$). A weight in the closure of the fundamental chamber is called *dominant for* $\mathfrak{g}$. A weight is *singular* if it lies on a wall, and *regular* otherwise.

Let $\mathscr{S} = \{\alpha_j\}$, and define a dual set of weights $\{\lambda_i\}$ by requiring that $\langle\lambda_i, \alpha_j^\vee\rangle = \delta_{ij}$, where for any $\alpha \in \Delta$, the *co-root* $\alpha^\vee = 2\alpha / \langle\alpha,\alpha\rangle$. Since $\{\alpha_j\}$ is a basis for $\mathfrak{h}^*$, so is $\{\lambda_i\}$, and any $\lambda \in \mathfrak{h}^*$ may be written as $\lambda = \Sigma_i \langle\lambda,\alpha_i^\vee\rangle \lambda_i$ and is *integral* if all $\langle\lambda,\alpha_i^\vee\rangle$ are integers. Clearly λ is dominant iff these coefficients are non-negative. This basis representation of weights gives the following:

Notation for weights: Represent a weight λ for $\mathfrak{g}$ (or $\mathfrak{p}$) by inscribing the coefficient $\langle\lambda,\alpha_j^\vee\rangle$ over the j'th node in the Dynkin diagram for $\mathfrak{g}$ (or $\mathfrak{p}$).

Examples:  etc.

Note that we will always use the basic diagram to indicate whether we are considering a weight for $\mathfrak{g}$ or a parabolic $\mathfrak{p}$.

The fundamental theorem of irreducible finite dimensional representations of $\mathfrak{g}$ is :

Theorem A :

A finite dimensional irreducible representation of $\mathfrak{g}$ has a unique highest weight λ which is dominant integral for $\mathfrak{g}$, and this induces a one to one correspondence between the finite dimensional irreducible representations of $\mathfrak{g}$ and the dominant integral weights for $\mathfrak{g}$ □

The irreducible $\mathfrak{g}$-module of highest weight λ will be denoted F^{λ}.

Notice that if $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of $\mathfrak{p}$, then $\mathfrak{u}$ acts trivially on finite dimensional irreducible representations V of $\mathfrak{p}$. For $\mathfrak{u} \subset [\mathfrak{b},\mathfrak{b}]$ so, by Lie's Theorem [Humphreys] $\mathfrak{u}$ acts by nilpotent endomorphisms in any finite dimensioanl representation of $\mathfrak{p}$. Thus $\mathfrak{u}\cdot V$ is properly contained in V. But $\mathfrak{p}\cdot(\mathfrak{u}\cdot V) \subseteq (\mathfrak{u}\cdot V)$, since $\mathfrak{l}$ is reductive, and $V = \{0\}$. So an irreducible representation of $\mathfrak{p}$ corresponds to an irreducible representation of $\mathfrak{l}$ and hence of the semi-simple $\mathfrak{l}^{s}$. Define

$$\mathfrak{h}_{\mathfrak{p}}^{*} = \{ \lambda \in \mathfrak{h}^{*} \mid \langle\lambda,\alpha^{\vee}\rangle \in \mathbb{R},\ \forall\ \alpha \in \Psi(\mathfrak{p}) \}$$

and let $\tau : \mathfrak{h}_{\mathfrak{p}}^{*} \longrightarrow (\mathfrak{h}\cap\mathfrak{l}^{s})_{\mathbb{R}}^{*}$ (space of real weights for $\mathfrak{l}^{s}$). A weight $\lambda_{\mathfrak{p}} \in \mathfrak{h}_{\mathfrak{p}}^{*}$ is *dominant or integral for* $\mathfrak{p}$ according as $\tau\lambda_{\mathfrak{p}}$ is dominant or integral for $\mathfrak{l}^{s}$.

<u>Theorem B</u>

Irreducible representations of $\mathfrak{p}$ are in one to one correspondence with $\lambda_{\mathfrak{p}} \in \mathfrak{h}_{\mathfrak{p}}^{*}$ dominant integral for $\mathfrak{p}$. □

(Where confusion may arise, we will indicate that a weight is associated to an irreducible $\mathfrak{p}$ module by a suffix $\mathfrak{p}$) The $\mathfrak{p}$ representation corresponding to a dominant integral $\lambda_{\mathfrak{p}}$ will be denoted $F_{\mathfrak{p}}^{\lambda}$, or, if D is the Dynkin diagram for $\lambda_{\mathfrak{p}}$, by F(D). Having given the theorem in generality, we will restrict further considerations to those representations of $\mathfrak{p}$ arising from weights which are also integral for $\mathfrak{g}$. Notice that regular/singular is *always* meant with respect to $\mathfrak{g}$.

2.4.2 Bundles on homogeneous spaces

We can now give a simple notation for homogeneous bundles on generalized flag manifolds. Because of the nature of subsequent work, it is convenient to signify bundles by the *dual* of the inducing representation. Accordingly, if $\lambda_{\mathfrak{p}}$ is a dominant integral weight for $\mathfrak{p}$, corresponding to the irreducible representation $F_{\mathfrak{p}}^{\lambda}$ for $\mathfrak{p}$, denote:

$$\mathcal{O}(\lambda_{\mathfrak{p}}) \cong \mathcal{O}(F_{\mathfrak{p}}^{\lambda^*})$$

Where no confusion can arise, the Dynkin diagram for $\lambda_{\mathfrak{p}}$ will also indicate $\mathcal{O}(\lambda_{\mathfrak{p}})$. We now give examples which will familiarise the reader with the important bundles occuring in the sequel:

Examples:

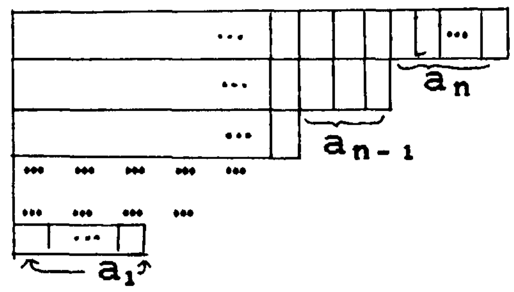
i) If $\mathfrak{p} = \mathfrak{g}$, the Dynkin diagram of λ indicates the representation $(F^{\lambda})^*$ of $\mathfrak{g}$, being a vector bundle over a point.

ii) The case $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$ has been dealt with in [Eastwood I], being expanded there to the case $\mathfrak{g} = sl(n+1,\mathbb{C})$. To make contact with his notation, let:

$$\lambda = \overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\cdots\!\!-\!\!\overset{0}{\bullet} \quad ; \quad \mu = \overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet}\!\!-\!\!\cdots\!\!-\!\!\overset{a_n}{\bullet}$$

Then F^{λ} is the self representation of $sl(n+1,\mathbb{C})$, and F^{μ} is the representation $T \cdot F^{\lambda}$, where T is the Young Tableaux

[Whybourne]



Accordingly, the bundle $\mathcal{O}(\mu)$ is given in Eastwood's notation by

$$(0, a_1, a_1+a_2, \dots, a_1+\dots+a_n).$$

Crossed through nodes in the present notation gives rise to bars in the notation of Eastwood :

$$\overset{a_1}{\bullet}\!-\!\overset{a_2}{\times}\!-\!\overset{a_3}{\bullet}\!-\!\overset{a_4}{\times}\!-\; \dots \;-\!\overset{a_n}{\bullet} \;=\; (0, a_1 \,|\, a_1+a_2, a_1+a_2+a_3 \,|\, a_1+\dots+a_4, \dots)$$

for example. Using this identification, or simply working things out, we arrive at the following series of bundles, which we identify in the abstract index notation of [Penrose & Rindler I]:

<u>Spin Bundles on M^4</u> $= \bullet\!-\!\times\!-\!\bullet$

$\mathcal{O}^{A'} = \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}$ $\qquad\qquad$ $\mathcal{O}^{A} = \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}$

$\mathcal{O}_{A'} = \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}$ $\qquad\qquad$ $\mathcal{O}_{A} = \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet}$

$\mathcal{O}[n] = \overset{0}{\bullet}\!-\!\overset{n}{\times}\!-\!\overset{0}{\bullet}$ (the conformal weight bundle)

and taking tensor products:

$$\mathcal{O}^{(A'_1 \dots A'_p)(A_1 \dots A_r)}[q] = \mathcal{O}_{(A'_1 \dots A'_p)(A_1 \dots A_r)}[p+q+r]$$

$$= \overset{p}{\bullet}-\overset{q}{\times}-\overset{r}{\bullet}$$

<u>Tangent Bundle on M</u> $\quad T(M) = \overset{1}{\bullet}-\overset{0}{\times}-\overset{1}{\bullet} = \mathcal{O}^{A'A}$

<u>Forms on M</u>

$\Omega^1 M = \mathcal{O}_{A'A} = \overset{1}{\bullet}-\overset{-2}{\times}-\overset{1}{\bullet}$; $\qquad \Omega^2 = \overset{2}{\bullet}-\overset{-3}{\times}-\overset{0}{\bullet} \oplus \overset{0}{\bullet}-\overset{-3}{\times}-\overset{2}{\bullet}$

$\Omega^3 = \overset{1}{\bullet}-\overset{-4}{\times}-\overset{1}{\bullet}$; $\qquad \Omega^4 = \overset{0}{\bullet}-\overset{-4}{\times}-\overset{0}{\bullet}$

iii) Generalizing to M^{2n}, M^{2n+1}, it is natural to take the following definitions :

<u>Spin bundles on M^{2n}</u>

The spinor bundles will be taken to be:

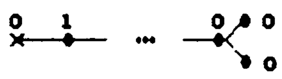
$\mathcal{O}^{\alpha'} = \overset{0}{\times}-\overset{0}{\bullet}-\cdots-\overset{0}{\bullet}\!<\!\begin{array}{l}\bullet^{1}\\ \bullet_{0}\end{array}$ and $\mathcal{O}^{\alpha} = \overset{0}{\times}-\overset{0}{\bullet}-\cdots-\overset{0}{\bullet}\!<\!\begin{array}{l}\bullet^{0}\\ \bullet_{1}\end{array}$;

these are dual to each other if n is odd (this is familiar in twistor theory where twistors are spinors for so(8,ℂ)). If n is even, their duals are :

$\mathcal{O}_{\alpha'} = \overset{-1}{\times}-\overset{0}{\bullet}-\cdots-\overset{0}{\bullet}\!<\!\begin{array}{l}\bullet^{1}\\ \bullet_{0}\end{array}$ and $\mathcal{O}_{\alpha} = \overset{-1}{\times}-\overset{0}{\bullet}-\cdots-\overset{0}{\bullet}\!<\!\begin{array}{l}\bullet^{0}\\ \bullet_{1}\end{array}$

<u>Tangent bundle on M^q</u>

The tangent bundle of any homogeneous space X = G/P is induced by the Adjoint representation of P on $\mathfrak{g}/\mathfrak{p}$. For M^{2n} (and a subclass of generalized flag manifolds, studied in Chapter 4), this representation is irreducible. Writing $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, the dual representation is the adjoint representation on $\mathfrak{u}$, whose highest weight is just the highest weight of $\mathfrak{g}$. So :

$T(M^{2n})$ =  and $T(M^{2n+1})$ = 

<u>Forms on M^q</u>

Dualizing the previous observation, Ω^1 is dually induced by the Adjoint representation of P on $g/\mathfrak{p}$. The highest weight of this is clearly $-\alpha$, where α is the simple root of g corresponding to the crossed through node in the Dynkin diagram for g. Thus:

$\Omega^1(M^{2n})$ =  ; $\Omega(M^{2n+1})$ =  .

At this point, one may take exterior products and decompose them to obtain higher forms. We will see (in section 3.8) how to obtain these bundles far more quickly using a resolution of [Bernstein et al II]. The result is displayed in Table 2.4 below □

<u>Note</u> Fixing a conformal factor (i.e. choosing a metric, locally) corresponds to restriction to the semi-simple part $\mathfrak{l}^s$ of $\mathfrak{l}$, which is $so(q,\mathbb{C})$. Under this restriction (which deletes crossed through nodes and incident edges) the isomorphism (Hodge dual) $* : \Omega^p \to \Omega^{r-p}$ is clearly apparant. Re-introducing the conformal factor, note that * may be thought of as an operator of conformal weight 2(p-n) (even case) or 2(p-n)-1 (odd case).

<u>TABLE 2.4 Form Bundles on M^r</u>

n	$\Omega^n(M^{2p})$	n	$\Omega^n(M^{2p+1})$
0<n<p-1	−n−1 0 … 0 1 0 … 0 (0, 0) n+1'th ↑ node	0<n<p	−n−1 0 … 0 1 0 … 0 0 n+1'th ↑ node
p-1	−p 0 0 … 0 (1, 1)	p	−p−1 0 … 0 2
p	−p−1 0 0 … 0 (2, 0) ⊕ −p−1 0 0 … 0 (0, 2)	p+1	−p−1 0 … 0 2
p+1	−p−2 0 0 … 0 (1, 1)	p+1<n<2p+1	−2n−1 0 … 0 1 0 … 0 0 2p+2−n'th ↑ node
p+1<n<2p	−n−1 0 … 0 1 0 … 0 (0, 0) 2p+1−n'th ↑ node	2p+1	−2p−1 0 … 0 0
2p	−2p 0 0 … 0 (0, 0)		

2.5 Inverse images of bundles

Suppose $\mathfrak{p} \subset \mathfrak{p}'$ are standard parabolic subalgebras of $\mathfrak{g}$. Then a dominant integral weight λ for $\mathfrak{p}'$ is dominant integral for $\mathfrak{p}$ also, and there is a natural injection of $\mathfrak{p}$ modules: $F^{\lambda}_{\mathfrak{p}} \longrightarrow F^{\lambda}_{\mathfrak{p}'}$. So if $\mu : G/P \longrightarrow G/P'$, there is an injection of vector bundles :

$$0 \longrightarrow \mu^{-1}\,\mathcal{O}(\lambda_{\mathfrak{p}'}) \longrightarrow \mathcal{O}(\lambda_{\mathfrak{p}}) \qquad (2.5.1)$$

(where μ^{-1} is the topological inverse image functor) which comes from:

$$\mu^* \mathcal{O}(\lambda_{\mathfrak{p}'}) \longrightarrow \mathcal{O}(\lambda_{\mathfrak{p}})$$

(2.5.1) is the start of the important (dual) Bernstein-Gelfand-Gelfand resolution, which will form part of the algebraic Penrose Transform below (section 2.8).

2.6 The Weyl Group W(g)

We now introduce the Weyl group W(g) of a complex semisimple Lie Algebra, together with its action on weights; we show how to compute this action on our notation for weight, which will be one of the fundamental computational devices needed in the thesis. From it, we will be able to compute the existence of invariant operators and compute both the cohomology of homogeneous bundles (in the generalized Penrose transform below) and the Spencer cohomology of nilpotent Lie algebras, in Chapter Four.

Recall that on $\mathfrak{h}^*_{\mathbb{R}}$, the bilinear form $\langle\cdot,\cdot\rangle$ is positive definite, and for any $\alpha \in \Delta(g,\mathfrak{h})$, the wall W_α is the hyperplane perpendicular to α. Reflection in W_α is denoted σ_α and the Weyl group W(g) of g is the group generated by the identity, id, and the σ_α. In fact, if $\mathcal{S}$ is a set of simple roots for g, then W(g) is generated by id and $\{\sigma_\alpha | \alpha \in \mathcal{S}\}$. Given any $w \in W(g)$, there exists a minimal integer $l(w)$, called the length of w, such that w admits expression as:

$$w = \sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_{l(w)}}$$

where $\alpha_i \in \mathcal{S}$. Such an expression is not usually unique - any such is called a *reduced expression* for w. There is an unique element in W(g) of maximal length.

We shall need the explicit action of a simple reflection σ_α on a weight, in our notation for weights. If $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, then:

$$\sigma_\alpha(\lambda) = \lambda - \langle\lambda,\alpha^\vee\rangle\cdot\alpha$$

and so the node co-efficients of $\sigma_\alpha(\lambda)$ are given by:

$$\langle\sigma_\alpha(\lambda),\alpha_i^\vee\rangle = \langle\lambda,\alpha_i^\vee\rangle - \langle\lambda,\alpha^\vee\rangle\langle\alpha,\alpha_i^\vee\rangle$$

for $\{\alpha_i\} = \mathcal{S}$. $\langle\lambda,\alpha_i^\vee\rangle$ are the node co-efficients of λ, whilst $\langle\alpha,\alpha_i^\vee\rangle$ are the Cartan integers of g, determined from the Dynkin diagram as follows :

$\langle\alpha,\alpha_i^\vee\rangle$	=	2	if $\alpha = \alpha_i$
	=	0	if nodes α, α_i are not connected.
	=	-i	if there is a directed edge from α to α_i nodes, of multiplicity i
	=	-1	otherwise

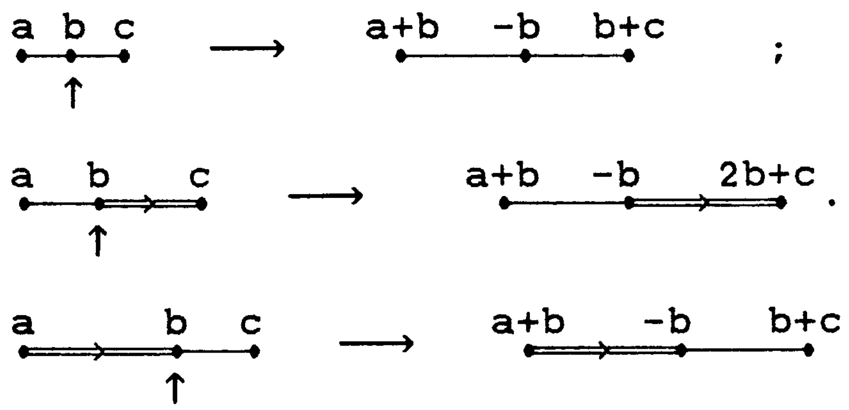
Thus we have:

Recipe for action of simple reflections

To compute $\sigma_\alpha(\lambda)$, let c be the co-efficient of the node associated to α. Add c to adjacent coefficients, with multiplicity if there is a multiple edge directed towards the adjacent nodes, and replace c over the node α by $-c$.

Examples A

Reflect in the node indicated by ↑:



□

It is known that $W(g)$ acts transitively on the Weyl chambers - the orbit of any regular weight λ has precisely one element in each chamber. In particular, any regular weight is uniquely conjugate to a dominant weight. If λ is singular, then λ is conjugate to at most one dominant weight. If F^λ is an irreducible representation of g, then $W(g)$ fixes $\Delta(F^\lambda)$; the weights of the orbit of the highest weight are called extremal, and extremal weight spaces are 1-dimensional. To specify F^λ it clearly suffices to specify an extremal weight, μ say; the resulting module will be denoted $F(\mu)$. In any F^λ, there is a unique *lowest* weight, μ' say, which is extremal with $-\mu'$ dominant. The representation $F(-\mu')$ is contragredient to $F(\mu)$.

Similar comments apply to irreducible representations of $\mathfrak{p}$, with $W(\mathfrak{g})$ replaced by $W(\mathfrak{l}^s) \subseteq W(\mathfrak{g})$.

$W(\mathfrak{g})$ admits the structure of a *directed graph*. We set:

$$w \longrightarrow w' \quad \text{if} \quad l(w') = l(w)+1 \text{ and } w' = \sigma_\alpha w,\ \alpha \in \Delta(\mathfrak{g})$$

Then, for $w, w' \in W(\mathfrak{g})$, set:

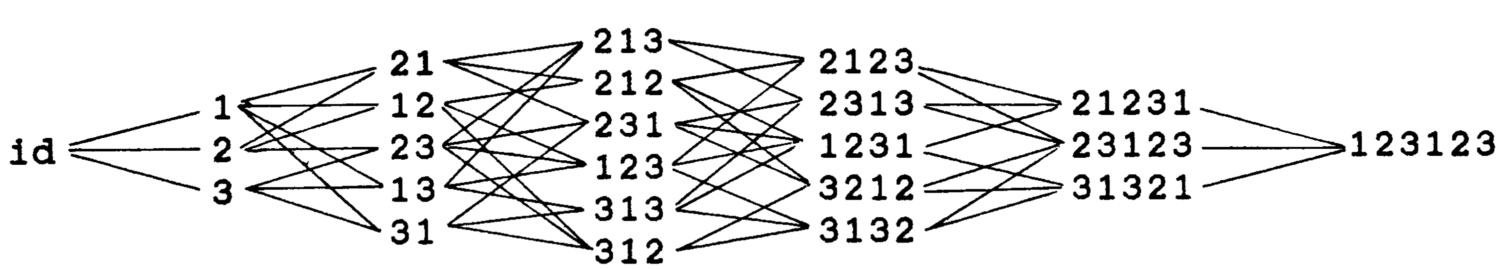
$$w \leq w' \text{ if } w = w' \text{ or } \exists \text{ directed path from } w \text{ to } w'$$

(cf [Bernstein et al I]).

<u>Example B</u>

Let $\mathfrak{g} = sl(4,\mathbb{C}) = \overset{\alpha_2}{\bullet}\!-\!\overset{\alpha_1}{\bullet}\!-\!\overset{\alpha_3}{\bullet}$. Let $ij \ldots k$ denote $\sigma_{\alpha_i}\sigma_{\alpha_j}\ldots\sigma_{\alpha_k}$. Then, $W(\mathfrak{g})$ is given in the Table 2.6 below.

<u>TABLE 2.6 : Weyl group of sl(4,$\mathbb{C}$)</u>



For completeness, we include the following lemma, due to [Bernstein et al I], which aids the computation of the directed graph structure on W(g):

<u>Lemma A</u>

If $w \longrightarrow w'$ and $w' = \sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_n}$ is a reduced expression, then a reduced expression is obtained for w by omitting precisely one simple reflection. More generally, $w \le w'$ iff w can be obtained from a reduced expression for w' by omitting several simple reflections. □

<u>Example</u> In Table 2.6, $12 \le 2123 \le 23123$.

By definition, W(g) acts on $\mathfrak{h}^*_{\mathbb{R}}$; it permutes the roots in $\Delta(\mathfrak{g},\mathfrak{h})$. With a lexicographic ordering fixed, let

$$\Delta(w) = \{ \alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}) \mid w^{-1}\alpha \in -\Delta^+(\mathfrak{g},\mathfrak{h}) \}$$

(i.e. $\Delta(w)$ is the set of positive roots mapped to negative roots by w^{-1}). It is known that $|\Delta(w)| = l(w)$.

A subgraph of W(g) may be defined if a parabolic $\mathfrak{p}$ of $\mathfrak{g}$ is chosen. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition of $\mathfrak{p}$ with respect to $\mathfrak{h}$, and set

$$W^1(\mathfrak{p}) = \{ w \in W(\mathfrak{g}) \mid \Delta(w) \subseteq \Delta(\mathfrak{u},\mathfrak{h}) \}$$

<u>Examples D</u>

i) Take $\mathfrak{g} = sl(4,\mathbb{C})$ and $\mathfrak{p}$ = •—×—•; then $W^1(\mathfrak{p})$ is:

$$\mathrm{id} \longrightarrow 1 \begin{array}{c}\nearrow\\ \searrow\end{array} \begin{array}{c}12\\ \\ 13\end{array} \begin{array}{c}\searrow\\ \nearrow\end{array} 123 \longrightarrow 1231$$

(in the notation above).

ii) More generally, let $g = so(2n+2,\mathbb{C})$ $n \geq 3$, and suppose that in the Dynkin diagram for g, the nodes are numbered left to right, top to bottom. Setting $\mathfrak{p}$ = , and using the above notation, $W^1(\mathfrak{p})$ is

$$\mathrm{id} \longrightarrow 1 \longrightarrow 12 \longrightarrow \dots \longrightarrow 12\dots(n-1) \begin{array}{c}\nearrow\\ \searrow\end{array} \begin{array}{c}12\dots(n-1)n\\ \\ 12\dots(n-1)(n+1)\end{array} \begin{array}{c}\searrow\\ \nearrow\end{array} 12\dots n(n+1)$$

$$\longrightarrow 12\dots n(n+1)(n-1) \longrightarrow \dots \longrightarrow 12\dots n(n+1)(n-1)\dots 21$$

For example, if $n = 4$, $W^1(\mathfrak{p})$ is

$$\mathrm{id} \rightarrow 1 \rightarrow 12 \rightarrow 123 \begin{array}{c}\nearrow\\ \searrow\end{array} \begin{array}{c}1234\\ \\ 1235\end{array} \begin{array}{c}\searrow\\ \nearrow\end{array} 12345 \rightarrow 123453 \rightarrow 1234532$$

$$\rightarrow 12345321$$

□

To aid the reader in computation, and for later use, we give two lemmas

<u>Lemma B</u> [Lepowsky I]

If $w = \sigma_{\alpha_1}\sigma_{\alpha_2}\dots\sigma_{\alpha_p}$ is a reduced decomposition of $w \in W^1(\mathfrak{p})$, then for $1 \leq n \leq p$, $w' = \sigma_{\alpha_1}\sigma_{\alpha_2}\dots\sigma_{\alpha_n} \in W^1(\mathfrak{p})$ □

Lemma C (Decomposition Lemma) [Kostant]

Every element w of W(g) can be uniquely written in the form

$$w = w_0 \, w^1$$

where $w_0 \in W(\mathfrak{t}^S) \subseteq W(\mathfrak{g})$ and $w^1 \in W^1(\mathfrak{p})$. Further,

$l(w) = l(w_0) + l(w^1)$ □

$W^1(\mathfrak{p})$ will be used in the Bernstein-Gelfand-Gelfand resolution below; the reason for the similarity of the above diagrams to the de Rham resolution of forms on M^{2n} will then be apparent.

2.7 Projective realizations of generalized flag manifolds

To end this chapter, we give a discussion of the projective geometry and cell structure of generalized flag manifolds. Although somewhat incidental to the thesis, it is elegant and gives a useful insight into their geometry. Certain important twistor structures such as "infinity" (in Minkowki space or Twistor space) bear a special relationship to the cell structure. We would hope in the future that a better understanding of the cell structure especially would lead, for example, to the classification of vector bundles over generalized flag manifolds.

2.7.1 Projective Realization of G/P

Let $\mathfrak{p} \subset \mathfrak{g}$ be a standard parabolic. To $\mathfrak{p}$, associate a weight λ

for $\mathfrak{g}$ by setting:

$$\langle \lambda, \alpha_i^\vee \rangle = \begin{cases} 0 & \alpha_i \in \Psi(\mathfrak{p}) \\ 1 & \alpha_i \in \mathcal{S} \setminus \Psi(\mathfrak{p}) \end{cases}$$

(where $\mathcal{S}$, $\Psi(\mathfrak{p})$ are as in section (2.1)). Thus the diagram for λ is obtained from the diagram for $\mathfrak{p}$ by replacing each crossed through node by a node with coefficient 1, and setting zeros over all other nodes. Taking G to be simply connected, if necessary, consider the $\mathfrak{g}$-representation F^λ as a G-representation. Let f be a highest weight vector in F^λ (unique up to scale, once $\mathfrak{h}$ and $\mathcal{S}$ have been given) and consider the action of G on f. f, being maximal, is annihilated by the positive root spaces of $\mathfrak{g}$; maximality and $\langle\lambda,\alpha^\vee\rangle = 0$ for $\alpha \in \Psi(\mathfrak{p})$, hence for $\alpha \in \Delta(\mathfrak{l},\mathfrak{h})$, implies that f is annihilated by the root spaces of $\mathfrak{l}^s \subset \mathfrak{p}$ also. $\mathfrak{h}$ preserves f up to scale, and so it follows that P is contained in the stabilizer of the line [f] in the projective representation of G on $P(F^\lambda)$. It is easy to see that if $y \in \mathfrak{u}_-$, say $y \in E_\alpha$ for $\alpha \in \Delta(\mathfrak{u}_-)$, then $\langle\lambda,\alpha^\vee\rangle > 0$, so that y acts non trivially on f. Thus P is precisely the stabilizer of the line [f], and

$$G/P \simeq G\cdot[f] \subset P(F^\lambda)$$

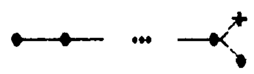
realising G/P projectively.

<u>Examples:</u>

i) Consider M^4 = ; the associated weight is λ = .

Then F^λ is just $\Lambda^2\mathbb{C}^4$ with $sl(4,\mathbb{C})$ acting on $\mathbb{C}^4$ by the self representation. Alternatively, F^λ is the self representation of $so(6,\mathbb{C})$. In Twistor Theory, $\mathbb{C}^4$ is usually called T^α (using the abstract index notation of [Penrose & Rindler I]), and then $\Lambda^2\mathbb{C}^4 = T^{[\alpha\beta]}$. It is easy to see that a highest weight vector for F^λ is *simple* as a vector of $T^{[\alpha\beta]}$, equivalently, *null* as a vector of the self representation of $SO(6,\mathbb{C})$, and that its orbit under G simply consists of all such. The argument above finds $M^4 \hookrightarrow \mathbb{CP}_5 = P(T^{[\alpha\beta]})$ and $M^4 \cong$ null directions at the origin of $\mathbb{C}^6$ as is well known.

ii) A similar construction for higher M^p realizes these as the projective light cone of the origin in $\mathbb{C}^{p+2}$, that is as a quadric in $\mathbb{CP}_{p+1}$.

iii) The associated representation of X =  is a spinor representation of $Spin(2n+2,\mathbb{C})$. A highest weight vector is easily seen to be pure, and $Spin(2n+2,\mathbb{C})$ acts transitively on all pure spinors, whence X is a space of pure spinors as claimed above. Similar comments apply to the other higher dimensional twistor spaces. □

2.7.2 The Cell Structure of G/P

The subgraph $W^1(\mathfrak{p})$ introduced above actually characterizes the topology of G/P - precisely, it gives its cell structure. To see this, pick f^λ to be a highest weight vector in the associated representation of $\mathfrak{p}$, and let $f^{w^{-1}\lambda}$ be a weight vector of weight $w^{-1}\lambda$ in F^{λ}, defined up to scale, so the projective point $[f^{w^{-1}\lambda}]$ is well defined. Then

Lemma

The extremal weights of F^{λ} are in one to one correspondence with the elements of $W^1(\mathfrak{p})$ □

Proof

Note that the stabilizer of λ in $W(\mathfrak{g})$ is $W(\mathfrak{l}^s)$ and use the decomposition of Lemma (2.6) C. □

Thus we obtain, for each $w \in W^1(\mathfrak{p})$ a distinct point in the projective realization of G/P. Pick a weight vector basis $\{f^i\}$ of F^{λ}, and let $\{f_i\}$ be the dual basis of $F(-\lambda)$. If U_- corresponds to $\mathfrak{u}_-$ in G, define the affine set

$$X_w = \{ [f] \in P(F^{\lambda}) \mid f \in U_- \cdot f^{w^{-1}} \text{ and } f_{w^{-1}\lambda}(f) \neq 0 \}$$

The X_w are the *Schubert cells* of X = G/P. They are open affine subspaces of codimension l(w) [Bernstein et al I] For instance, X_{id} may be co-ordinatized by $u_-(\mathfrak{p})$, since on f^{λ} the action of U_- reduces to that of exp u_-. A theorem of [Steinberg] shows that

i) $X = \cup_{w \in W^1(\mathfrak{p})} X_w$

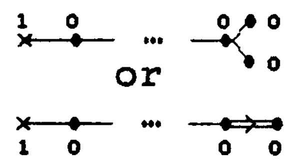
ii) $X_w \subset \overline{X}_{w'}$ iff $w' \leq w$

so that the directed graph structure of $W^1(\mathfrak{p})$ records the cell structure of X.

2.7.3 Note on the Infinity Twistor

Suppose given a point $\iota \in M^p$ (= $\mathbb{C}S^p$), which is to be thought of as the *point at infinity*, hence a line [I] in $\mathbb{C}P_{p+1}$. Fix a representative $I^{я} \in$ [I] and use the metric of SO(p+2,$\mathbb{C}$) on $\mathbb{C}^{p+2}$ to obtain an element $I_{я}$ of $F(-\lambda)$, the dual representation associated to M^p. Employing the sequence

$$0 \longrightarrow F(-\lambda) \xrightarrow{\mu} \mathcal{O}(\lambda_{\mathfrak{p}}) =$$



one obtains a section $\mu(I_{я})$ of the conformal weight line bundle $\mathcal{O}[1]$ of M^p which fixes a metric on M\{zero divisor of section}. The zero divisor of the section is the light cone in M^p of ι.

Equally, $I_{я}$ gives a section of the bundles , , on the indicated spaces, which have direct interpretations in Twistor Theory [Penrose, I], [Le Brun II]. We will indicate in Chapter Four how this construction may be partially generalized to the curved case, giving a means of determining when a conformal manifold admits a (local) metric, which makes it Einstein or Vacuum.

3 : Invariant Differential Operators on Generalized Flag Manifolds

This Chapter reviews and develops the theory of invariant differential operators on generalized flag manifolds as conceived (for Minkowski Space) by [Eastwood and Rice]. We define invariant differential operators and show that the existence of such an operator between homogeneous bundles over the flag manifold is equivalent to the existence of a homomorphism between certain natural induced modules. With this algebraic form of the problem established we investigate some of the standard theory of these homomorphisms. We establish strict limitations on the bundles between which invariant differential operators may exist (using *infinitesmal character*) and investigate the case when the stabilizer group is Borel, showing how many, but not all, invariant differential operators may be deduced by taking direct images. Criteria are given for when such direct images yield zero. We give an exposition of the *translation principle* which is a powerful means of generating new differential operators from old. Finally, we give an algebraic form of the Penrose transform, between holomorphic objects on generalized flag manifolds - this computes, for example, higher dimensional analogues of the the standard results of twistor theory on the representation of zero rest mass fields by means of cohomology classes on twistor spaces, but its main value in the future will be in investigating homomorphisms of induced modules.

3.1 Jet Bundles and Differential Operators

Let $\mathcal{E}$ be a locally free sheaf on a complex holomorphic manifold X. At each $x \in X$ form the stalk

$$\mathcal{I}^r_x(\mathcal{E}) = \text{set of germs in } \mathcal{E}_x \text{ which vanish to order } r \text{ at } x.$$

For instance, if $\mathcal{E} = \mathcal{O}_X$, the sheaf of holomorphic functions on X, then

$$\mathcal{I}_x(\mathcal{O}_X) = \mathcal{I}^1_x(\mathcal{O}_X) = \text{ideal sheaf at } x$$
$$= \text{germs of functions vanishing a } x$$

Thus one has

$$\mathcal{I}^r_x(\mathcal{E}) \cong \mathcal{E}_x \otimes \mathcal{I}^r_x \qquad (3.1.1)$$

Define the stalk of the r^{th} Jet sheaf at x by

$$\mathcal{J}^r_x(\mathcal{E}) = \mathcal{E}_x \,/\, \mathcal{I}^{r+1}_x(\mathcal{E})$$

and let $\mathcal{J}^r(\mathcal{E})$ denote the resulting sheaf of germs of r-jets of $\mathcal{E}$.

Then there is clearly a map - a universal r^{th} order differential -

$$\mathcal{E} \xrightarrow{d^r} \mathcal{J}^r(\mathcal{E})$$

and (employing (3.1.1) cf [Spencer]) a jet exact sequence

$$0 \longrightarrow \odot^r \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathcal{J}^r(\mathcal{E}) \xrightarrow{\phi^r} \mathcal{J}^{r-1}(\mathcal{E}) \longrightarrow 0 \tag{3.1.2}$$

Taking a projective limit,

$$\mathcal{J}^\infty(\mathcal{E}) = \varprojlim \mathcal{J}^r(\mathcal{E})$$

(under the maps ϕ^r), and there is a universal differential

$$\mathcal{J}^r(\mathcal{E}) \xrightarrow{d} \mathcal{J}^\infty(\mathcal{E}) \tag{3.1.3}$$

for $r \geq 0$, with $\mathcal{J}^\circ(\mathcal{E}) = \mathcal{E}$.

This gives the definition of a differential operator :

<u>Def</u>

A (linear) differential operator $\mathcal{D}$ between two locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ of order r is a map of locally free sheaves

$\tilde{\mathfrak{D}} : \mathfrak{J}^r(\mathcal{E}) \to \mathcal{F}$. □

(Note : although it is pedantic, we could say that $\tilde{\mathcal{D}} : \mathfrak{J}^\infty(\mathcal{E}) \to \mathcal{F}$, factoring through $\mathfrak{J}^r(\mathcal{E})$. It will be convenient to think of it in this way. Clearly $\tilde{\mathcal{D}}$ extends to map $\mathfrak{J}^{r+k}(\mathcal{E}) \longrightarrow \mathfrak{J}^r(\mathcal{F})$).

Def

If $\mathcal{D}$ is a differential operator of order r, the (principle) symbol of $\mathcal{D}$ is the composition

$$\sigma_{\mathcal{D}} \quad : \quad \odot^r \, \Omega^1_X \otimes \mathcal{E} \longrightarrow \mathfrak{J}^r(\mathcal{E}) \longrightarrow \mathcal{F}$$ □

The idea now is to interpret these standard differential structures in algebraic terms.

3.2 Jet Bundles as Homogeneous Bundles - the functors $\mathcal{J}$ and $\frac{\mathfrak{g}}{\mathfrak{p}}$

3.2.1 $\mathfrak{J}^\infty(\mathcal{E})$ as a homogeneous bundle

To begin with, consider $\mathcal{O}_G$. Recall that g may be thought of as the Lie Algebra of left invariant vector fields on G. Composing these and taking linear combinations, obviously:

$$\mathfrak{J}^\infty_e(\mathcal{O}_G) \cong \mathfrak{A}(g)^*$$

i.e. the differential operators at the identity e of G are essentially elements of the universal enveloping algebra $\mathfrak{A}(g)$

of $\mathfrak{g}$. Next, if $\mu : G \to G/P = X$ is the coset projection, consider the topological inverse image sheaf $\mu^{-1}\, \mathcal{O}_X$ on G. At the identity, the left ideal in $\mathfrak{U}(\mathfrak{g})$ generated by $\mathfrak{U}(\mathfrak{p})$, $\mathfrak{I}(\mathfrak{p})$ say, is the annihilator of this sheaf in $\mathfrak{U}(\mathfrak{g})$, thought of as acting differentially and so if $\circ$ is the origin of X (i.e the identity coset) then :

$$\mathfrak{F}^{\infty}_{\circ}(\mathcal{O}_X) \cong [\ \mathfrak{U}(\mathfrak{g}) \ / \ \mathfrak{I}(\mathfrak{p})\ ]^{*} .$$

Letting $\mathfrak{p}$ act trivially on $\mathbb{C}$, this is

$$\mathfrak{F}^{\infty}_{\circ}(\mathcal{O}_X) \cong [\ \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{C}\ ]^{*}$$
$$\cong \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\, \mathbb{C}$$

where the Inducing functor

$$\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,(\ \cdot\ ) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} (\ \cdot\ )$$

is defined on (finite dimensional) $\mathfrak{p}$ modules. Thus $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mathbb{C}$ with the natural left action of $\mathfrak{p}$, hence P, dually induces the sheaf $\mathfrak{F}^{\infty}(\mathcal{O}_X)$.

A similar argument shows that for F a finite dimensional $\mathfrak{p}$ (i.e. P) module the sheaf $\mathfrak{F}^{\infty}(F^{*})$ is dually induced by the left $\mathfrak{p}$-module $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,F$. (One considers $\mathfrak{U}(\mathfrak{g})\otimes F$ as dual to $\mathfrak{F}^{\infty}_{e}(\mathcal{O}_G \otimes F^{*})$ at the identity e of G, and notes that the left $\mathfrak{U}(\mathfrak{g})$-sub-module generated by elements of the form

$x \otimes f - 1 \otimes d\pi(f)$

for $d\pi$ the representation of $\mathfrak{p}$ on F, annihilates $\mu^{-1}\,\mathcal{O}(F^*)$.) Thus $\mathcal{J}^\infty(F^*)$ is dually induced by $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,F$, and we have identified $\mathcal{J}^\infty(F^*)$ as a homogeneous bundle. We will use this algebraic identification below to define and classify invariant differential operators.

Before we do that, however, this is a good point to revise some of the useful properties of the algebraic inducing functor $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ which we will need in the sequel, and to relate this abstract algebraic construction to the jet exact sequence (3.1.2) above. We will denote $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,F^{\lambda}_{\mathfrak{p}}$ by $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ in future, where it is implicit in the notation that λ is to be thought of as a weight for $\mathfrak{p}$, inducing an irreducible representation for $\mathfrak{p}$ which might vary. λ may be replaced by its Dynkin diagram.

<u>3.2.2 Functorial properties and facts about $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$.</u>

a) If $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, then the modules $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,F^{\lambda}_{\mathfrak{b}}$ (for $F^{\lambda}_{\mathfrak{b}}$ a one dimensional irreducible representation of $\mathfrak{b}$) are classically known as *Verma modules* [Verma], [Dixmier]. [Lepowsky I] applies a similar terminology to $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ for a more general parabolic $\mathfrak{p}$.

b) In both these cases, the surjection

$$\mathfrak{U}(\mathfrak{g}) \otimes_{\mathbb{C}} F^{\lambda}_{\mathfrak{b}} \to \mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$$

sends the vector $1 \otimes 1$ to a highest weight vector which generates $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$ under the action of $\mathfrak{U}(\mathfrak{p})$. [Humphreys] calls any $\mathfrak{U}(\mathfrak{g})$ module generated under the action of $\mathfrak{U}(\mathfrak{g})$ from a highest weight vector a *standard cyclic module*. He observes that :

i) any such module is a direct sum of weight spaces of finite dimension,

ii) the highest weight space has dimension 1,

iii) the homomorphic image of any such module is again standard cyclic. Indeed $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ (λ any weight of $\mathfrak{g}$) is a universal standard cyclic module, surjecting onto any other generated by a highest weight vector of weight λ,

iv) each standard cyclic module has a unique proper maximal $\mathfrak{U}(\mathfrak{g})$ sub-module, and hence a unique associated irreducible quotient. The maximal submodule of $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$ will be denoted $\mathcal{S}ub\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$ and the irreducible quotient by $\mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$. Thus :

$$0 \longrightarrow \mathcal{S}ub\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda \longrightarrow \mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda \longrightarrow \mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda \longrightarrow 0$$

$\mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ is also *universal* - it is a quotient of every standard cyclic module generated from a highest weight vector of weight λ.

c) We will be interested in studying g-module homomorphisms between induced modules, $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu \to \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$, say. Such a homomorphism must take a highest weight vector in $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu$ to a maximal vector in $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$. Conversely, since a maximal vector of weight μ generates, under $\mathfrak{U}(\mathfrak{p})$, a copy of $F^{\mu}_{\mathfrak{p}}$, it induces such a homomorphism. Since $\mathfrak{p}$ and g share all their positive root spaces, so that maximal has, as observed, the same meaning for $\mathfrak{p}$ as for g, such g-module homomorphisms are in $1 \leftrightarrow 1$ correspondence with $\mathfrak{p}$-module homomorphisms :

$$F^{\mu}_{\mathfrak{p}} \to \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda.$$

d) If it is possible to pick a complement to $\mathfrak{p}$ in g, i.e. a subalgebra u_- of g with $\mathfrak{g} = u_- \oplus \mathfrak{p}$ (as it is, for example when $\mathfrak{p}$ is parabolic), then $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda \cong \mathfrak{U}(u_-) \otimes_{\mathbb{C}} F^{\lambda}_{\mathfrak{p}}$ by virtue of the Poincare-Birkhoff-Witt theorem (cf [Humphreys].) It will be convenient to denote elements of u_- by the generic y_i, and an element of $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ is specified by a linear combination of expressions of the form $P(y) \otimes f$, for some polynomial $P(y)$ and $f \in F$.

e) As a functor on the category of finite dimensional $\mathfrak{p}$ modules (indeed more generally), $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ is exact [Vogan]. Thus, given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of $\mathfrak{p}$-modules, there is an exact sequence

$$0 \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} A \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} B \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} C \longrightarrow 0.$$

This will be the basis of the *translation principle* below.

3.2.3 Finite Jet Bundles

The realisation of the universal enveloping algebra $\mathfrak{A}(\mathfrak{g})$ of $\mathfrak{g}$ as a quotient of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ provides $\mathfrak{A}(\mathfrak{g})$ with the structure of a *filtered algebra*, i.e. a sequence

$$\begin{aligned}\mathbb{C} = \mathfrak{A}(\mathfrak{g})^0 \hookrightarrow \mathfrak{A}(\mathfrak{g})^1 \hookrightarrow \ldots \hookrightarrow \mathfrak{A}(\mathfrak{g})^i \\ \hookrightarrow \mathfrak{A}(\mathfrak{g})^{i+1} \hookrightarrow \ldots \hookrightarrow \mathfrak{A}^{\infty}(\mathfrak{g}) = \mathfrak{A}(\mathfrak{g})\end{aligned} \tag{3.2.1}$$

The Poincare-Birkhoff-Witt Theorem [Humphreys] states that if

$$\mathrm{Gr}\ \mathfrak{A}(\mathfrak{g}) \cong \oplus_i\ \mathfrak{A}(\mathfrak{g})^i/\mathfrak{A}(\mathfrak{g})^{i-1}$$

is the graded algebra associated to $\mathfrak{A}(\mathfrak{g})$, then

$$\mathrm{Gr}\ \mathfrak{A}(\mathfrak{g}) \cong \mathfrak{S}(\mathfrak{g})$$

where $\mathfrak{S}(\mathfrak{g})$ is the symmetric algebra on $\mathfrak{g}$. Thus $\mathfrak{A}(\mathfrak{g})^i$ may be thought of as the algebra of differential operators of order $r \le i$ at the identity of G with

$$\mathfrak{F}_e^{\infty}(\mathcal{O}_G) \longrightarrow \dots \longrightarrow \mathfrak{F}_e^{i+1}(\mathcal{O}_G) \longrightarrow \mathfrak{F}^{i}(\mathcal{O}_G) \longrightarrow \dots \longrightarrow \mathbb{C}$$

dual to the sequence (3.2.1).

All of this descends to the induced modules $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,F$. These become filtered $\mathfrak{U}(\mathfrak{p})$-modules, and if there is a direct sum of subalgebras $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_-$, then

$$0 \longrightarrow (\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,F)^{i-1} \longrightarrow (\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,F)^{i} \longrightarrow \odot^{i}\mathfrak{u}_- \otimes F \longrightarrow 0 \tag{3.2.2}$$

is an exact sequence of $\mathfrak{U}(\mathfrak{p})$-modules. $(\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,F)^{r}$ dually induces $\mathfrak{F}^{r}(F^{*})$ and (3.2.2) dually induces the Jet exact sequence (3.1.2).

3.3 Invariant Differential Operators

3.3.1 Definition and algebraic formulation

For the moment, let P be any closed (connected) subgroup of G. To define invariant differential operators between homogeneous bundles, recall the left action of G on sections of a homogeneous bundle $\mathcal{O}(F^{*})$ represented by F^{*} valued functions f on G : for $g, g' \in G$,

$$L_g^{*} \circ f\,(g') = f\,(g^{-1}g').$$

Def An Invariant Differential Operator $\mathcal{D} : \mathcal{O}(F^*) \to \mathcal{O}(E^*)$ is a differential operator between homogeneous bundles satisfying $\mathcal{D} \circ L_g^* = L_g^* \circ \mathcal{D}$ for all $g \in G$. □

It follows that $\mathcal{D}$ is entirely determined by the action of $\tilde{\mathcal{D}}$ on $\mathcal{J}_\circ^\infty(\mathcal{O}(F^*)) \longrightarrow \mathcal{O}_\circ(E^*) \cong E^*$ which necessarily intertwines the action of $\mathfrak{p}$ on the stalks. Any such intertwining map which factors through $\mathcal{J}_\circ^r(\mathcal{O}(F^*))$ for some finite r will yield an invariant differential operator. Thus, dualizing we have :

Theorem cf [Eastwood & Rice]

The invariant differential operators on a (complex) homogeneous space G/P between homogenous bundles $\mathcal{O}(F^*)$ and $\mathcal{O}(E^*)$ are in $1 \leftrightarrow 1$ correspondence with the $\mathfrak{p}$-module maps $E \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} F$. □

When $\mathfrak{p}$ is parabolic, using (3.2.2(c)) above, we get a

Corollary

The invariant differential operators between irreducible homogeneous bundles $\mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(\mu_{\mathfrak{p}})$ are classified by the $\mathfrak{U}(\mathfrak{g})$-module maps $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \mu \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \lambda$. □

3.3.2 Symbols of Invariant Operators : Examples

The composition of the map $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \mu \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \lambda$ (defining an invariant differential operator by the corollary) with $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \lambda \longrightarrow \odot^i u_- \otimes F_{\mathfrak{p}}^{\lambda}$ gives (left translating over X) the dual of the symbol of the operator. Notice that this implies

Symbol Principle

There $\exists$ an invariant differential operator from $\mathcal{O}(\lambda_{\mathfrak{p}})$ to $\mathcal{O}(\mu_{\mathfrak{p}})$ if and only if $F^{\mu}_{\mathfrak{p}}$ is a $\mathfrak{p}$-submodule of $\odot^{i}\mathfrak{u}_{-} \otimes F^{\lambda}_{\mathfrak{p}}$. Thus a necessary condition for the existence of such an operator is that $F^{\mu}_{\mathfrak{p}}$ be a factor in the decomposition of $\odot^{i}\mathfrak{u}_{-} \otimes F^{\lambda}_{\mathfrak{p}}$ into irreducible $\mathfrak{l}^{s}$-submodules, where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition and $\mathfrak{l}^{s}$ is the semisimple part of $\mathfrak{l}$. □

Thus, as a first step in constructing invariant differential operators acting on a bundle $\mathcal{O}(\lambda_{\mathfrak{p}})$, one computes the composition factors (irreducible under the action of $\mathfrak{l}^{s}$) in the tensor products $(\odot^{i}\mathfrak{u}_{-}) \otimes F^{\lambda}_{\mathfrak{p}}$ finding maximal weights for $\mathfrak{l}^{s}$ in the resulting modules. One may then test each of these in turn, to see if it is maximal for $\mathfrak{p}$. If it is, an invariant differential operator has been described, and all may be found in this way.

To concretely describe the operator is simple. Above a point $x \in X$, choose a $g \in G$. If U_{-} is the nilpotent subgroup of G normalizing u_{-}, then $U_{-}\cdot g$ is an affine space in G which affinely co-ordinatises a neighbourhood of x in X (called a "big cell" [Bernstein et al I]), providing a trivialization of $\mathcal{O}(\lambda_{\mathfrak{p}})$ and a realization of a section as a function $\tilde{f}$ on G, with values in $F^{\lambda}_{\mathfrak{p}}$. If $v \in \mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}\lambda$ is a maximal vector of weight μ, pick any lift to $\mathfrak{U}(\mathfrak{g}) \otimes F^{\lambda}_{\mathfrak{p}}$ and act with $\mathfrak{U}(\mathfrak{p})$ to generate a differential operator on $F^{\lambda}_{\mathfrak{p}}$ valued functions, with values in $F^{\mu}_{\mathfrak{p}}$. Acting with this on $\tilde{f}$ produces an $F^{\mu}_{\mathfrak{p}}$ valued function, representing the image of the invariant differential operator on f in $\mathcal{O}(\mu_{\mathfrak{p}})$.

Often the invariant operator is identifiable in terms of some known operator, such as a Levi-Civita connection. Over the space M^{2n} = ×—•— ⋯ —•< , for instance, the orbit $U_{-}\cdot g$ has a natural flat metric. Indeed, acting on $U_{-}\cdot g$ with L^{s} (corresponding to $\mathfrak{l}^{s}$) which is (a covering of) $SO(2n,\mathbb{C})$, yields (a covering of) an $SO(2n,\mathbb{C})$-principle subbundle of the frame bundle of M^{2n}, and the orbits $U_{-}\cdot g$ correspond to the horizontal lifts of an affine neighbourhood of the point x. This enables invariant differential operators on bundles on M^{2n} (and equally, M^{2n+1}) to be expressed in terms of the Levi Civita connection of a locally chosen metric in the allowed conformal class. Indeed they are given by the projection of the operator $\nabla_{a_1}\nabla_{a_2}\nabla_{a_3}\cdots\ \nabla_{a_n}$ on $\mathcal{O}(\lambda)$ in the appropriate composition factor of $\odot^{i}\Omega^{1}_{X}\otimes\mathcal{O}(\lambda)$.

3.3.2 Examples of Invariant Differential Operators

Example A ($\mathbb{C}P_1$): The simplest of invariant differential operators are those on $\mathbb{C}P_1$ with its homogeneous structure $\times$. Powers of the Hopf Bundle may be written as $\mathcal{O}(n) = \overset{n}{\times}$. One checks that this has the right sign by employing the injection of bundles (for $n \geq 0$ and μ: $G/P \to G/G = \{pt\}$):

$$0 \longrightarrow \mu^{-1}\overset{n}{\bullet} \longrightarrow \overset{n}{\times}$$

so that $\mathcal{O}(n)$ has global sections if $n \geq 0$. Generate $sl(2,\mathbb{C})$ by x,y,h satisfying

$$[x,y] = h \quad ; \quad [h,x] = 2x \quad ; \quad [h,y] = -2y$$

with other commutators zero. Let $\mathfrak{p}$ be the span of x and h, and let h span a Cartan sub-algebra of $sl(2,\mathbb{C})$.

Let α, β, be the highest weight vectors in $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ and $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu$ respectively, where $\lambda = \overset{-1+n}{\times}$ and $\mu = \overset{-1-n}{\times}$, $n \geq 1$. Then :

$$\begin{aligned} x\, y^{n}\, \alpha &= [x,y]\, y^{n-1} + y\,(x\, y^{n-2})\, \alpha \\ &= h\, y^{n-1}\, \alpha + y\, h\, y^{n-2}\, \alpha + \ldots + y^{n-1}\, h\, \alpha \\ &= \{(1-n) + (3-n) + \ldots + (n-1)\}\, y^{n}\, \alpha \\ &= 0 \end{aligned}$$

and

$$h\, y^{n}\, \alpha = (-1-n)\, y^{n}\, \alpha$$

so that y^n is maximal, of weight $\overset{-1-n}{\times} = \mu$; there are thus invariant differential operators (of order $n \geq 1$)

$$\tilde{\eth}^{n} \; : \; \mathcal{O}(-1+n) \longrightarrow \mathcal{O}(-1-n)$$

on $\mathbb{C}P_1$. These are the *conformally invariant* powers of edth described in [Eastwood & Tod], which may be easily described:

let $[\pi^A]$ be homogeneous co-ordinates on $\mathbb{C}P_1$ and represent a section of $\mathcal{O}(n)$ by a function f homogeneous of degree n. Then $\tilde{\partial}^n$ is defined by

$$\pi^{A_1}\pi^{A_2}\ldots\pi^{A_n}\ \tilde{\partial}^n f \quad = \quad \partial^n/\partial\pi^{A_1}\partial\pi^{A_2}\ldots\partial\pi^{A_n}\ f$$

<u>Example B   Conformally Invariant Differential Operators on Minkowski Space</u>

Consider $M^4 = \bullet{-}\times{-}\bullet$; $\Omega^1 = \overset{1}{\bullet}{-}\overset{-2}{\times}{-}\overset{1}{\bullet}$ and we easily compute

$$\begin{aligned} \odot^2\Omega^1 &= \overset{2}{\bullet}{-}\overset{-4}{\times}{-}\overset{2}{\bullet} \oplus \overset{0}{\bullet}{-}\overset{-2}{\times}{-}\overset{0}{\bullet} \\ \odot^3\Omega^1 &= \overset{3}{\bullet}{-}\overset{-6}{\times}{-}\overset{3}{\bullet} \oplus \overset{1}{\bullet}{-}\overset{-4}{\times}{-}\overset{1}{\bullet} \\ \odot^4\Omega^1 &= \overset{4}{\bullet}{-}\overset{-8}{\times}{-}\overset{4}{\bullet} \oplus \overset{2}{\bullet}{-}\overset{-6}{\times}{-}\overset{2}{\bullet} \oplus \overset{0}{\bullet}{-}\overset{-4}{\times}{-}\overset{0}{\bullet} \end{aligned} \qquad (3.3.2)$$

Let us first attempt to compute differential operators on $\mathcal{O} = \mathcal{O}(\overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet})$. The symbol principle above suggests an invariant differential operator for each term of (3.3.2); we try the following possibilities :

i) $\overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet}{-}\overset{-2}{\times}{-}\overset{1}{\bullet} = \Omega^1$

ii) $\overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet}{-}\overset{-4}{\times}{-}\overset{1}{\bullet} = \Omega^3$

iii) $\overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet} \longrightarrow \overset{0}{\bullet}{-}\overset{-4}{\times}{-}\overset{0}{\bullet} = \Omega^4$

For α a highest weight of $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet}$ we compute vectors of weights $\overset{1}{\bullet}{-}\overset{-2}{\times}{-}\overset{1}{\bullet}$, $\overset{1}{\bullet}{-}\overset{-4}{\times}{-}\overset{1}{\bullet}$, $\overset{0}{\bullet}{-}\overset{-4}{\times}{-}\overset{0}{\bullet}$ in $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{0}{\bullet}{-}\overset{0}{\times}{-}\overset{0}{\bullet}$ which are maximal for $\mathfrak{t}^S$ (thus annihilated by X_1, X_2 of example (2.1) (ii)). We find, respectively

i) $y_1 \alpha$

ii) $y_1(y_1 y_4 - y_2 y_3)\alpha$

iii) $(y_1 y_4 + y_4 y_1 - y_2 y_3 - y_3 y_2)^2 \alpha$

which generate the image of $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}$ etc. under a $\mathfrak{U}(\mathfrak{l}^S)$ homomorphism.

The next step is to check that these $\mathfrak{U}(\mathfrak{l}^S)$ homomorphisms promote to $\mathfrak{U}(\mathfrak{g})$ homomorphisms, by checking that x_1 (hence all x_i) annihilate the indicated vectors. One easily finds this to be true for (i) and (iii) but not for (ii). So we conclude the existence of two invariant differential operators

$$d \ : \mathcal{O} \longrightarrow \Omega^1 \quad \text{(the usual exterior derivative)}$$
$$\Box^2 \ : \mathcal{O} \longrightarrow \Omega^4$$

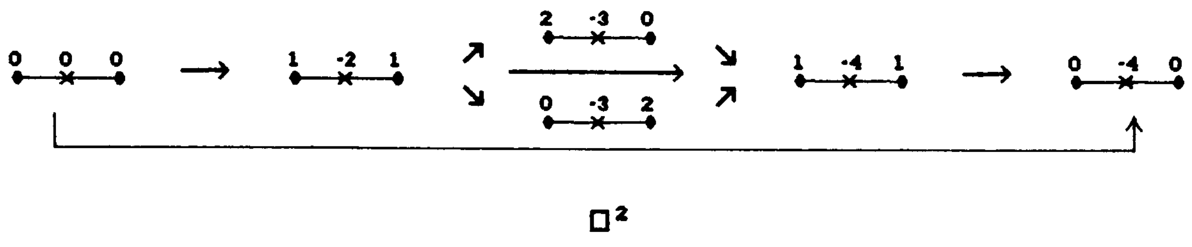
It is easy, though tedious, to check that the remaining invariant operators between bundles of forms are the usual exterior differentials together with the composition of

$$\Omega^1 \xrightarrow{\ d^+\ } \Omega^2_+ \xrightarrow{\ d\ } \Omega^3$$

For example, if β is a highest weight vector of $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}$, then a maximal vector of weight $\overset{2}{\bullet}\!\!-\!\!\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}$ in $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}$, given by $(-y_3 + y_1 Y_3)\beta$ induces the operator d^+, whilst for γ a highest weight vector of $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{1}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{1}{\bullet}$, a maximal vector $(y_1 Y_1 Y_2 - y_3 Y_1 + y_3 Y_2 - y_4)\gamma$ of weight $\overset{0}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}$ in $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{1}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{1}{\bullet}$

induces d : $\Omega^3 \longrightarrow \Omega^4$.

In summary, we have the following invariant differential operators between the form bundles on M^4 :



(3.3.3)

Indeed, we shall see in example (3.4) B(i) that any invariant differential operator on forms must occur in (3.3.3). In section (3.7), we shall see how these give rise to a whole family of invariant differential operators on other bundles, by means of the *translation principle*.

Some other examples of invariant differential operator which may be similarly constructed are the following : (here α denotes an appropriate highest weight vector)

$$\lambda^{A'}_{(ABC)} \in \overset{1\ \ -2\ \ 3}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \longrightarrow \nabla^{A}_{(A'}\nabla^{B}_{B'}\nabla^{C}_{C'}\lambda_{D')ABC} \in \overset{4\ \ -5\ \ 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}$$

induced by $(-y_3 + y_1Y_2)(-2y_3 + y_1Y_2)(-3y_3 + y_1Y_2)\alpha$

$$f \in \overset{0\ \ -1\ \ 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} \longrightarrow \Box f \in \overset{0\ \ -3\ \ 0}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet}$$

induced by $(y_1Y_4 + Y_4y_1 - Y_2Y_3 - Y_3Y_2)\alpha$ $\Box$

3.4 Infinitesmal Character

As observed, the modules $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \lambda$ are highest weight modules, generated under the action of $\mathfrak{U}(\mathfrak{g})$ from a single highest weight vector α. Consider the action of the centre $\mathfrak{Z}(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ on this vector. If $z \in \mathfrak{Z}(\mathfrak{g})$, $z\alpha$ is still highest, hence a multiple of α, $\xi(z)$ say. Then the element z must act by the scalar $\xi(z)$ on all elements of $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \lambda$. $\xi : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ is dependent only on the weight λ [Humphreys]; that is, there is a mapping

$$\begin{array}{lll} \mathfrak{h}_{\mathbb{R}}^{*} & \longrightarrow & (\mathfrak{Z}(\mathfrak{g}))^{*} \\ \lambda & \longrightarrow & \xi_{\lambda} \quad \text{(the \textit{infinitesmal character} of } \lambda) \end{array}$$

If there is to be a $\mathfrak{U}(\mathfrak{g})$-module homomorphism $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \lambda \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \mu$, then it is clear that $\mathfrak{Z}(\mathfrak{g})$ must act by the same scalars on both modules; that is $\xi_{\lambda} = \xi_{\mu}$. It is a classical theorem of [Harish Chandra] that $\xi_{\lambda} = \xi_{\mu}$ if and only if $\lambda+\rho$ and $\mu+\rho$ are conjugate under the action of $W(\mathfrak{g})$, where ρ is the weight

$$\rho \quad = \quad (\Sigma_{\alpha \in \Delta^{+}(\mathfrak{g},\mathfrak{h})}\ \alpha)/2$$

($\langle\rho,\alpha^{\vee}\rangle = 1\ \forall\ \alpha \in \mathcal{S}(\mathfrak{g},\mathfrak{h})$, so ρ is represented by a Dynkin diagram with a "1" over each node). Denote by $w\cdot\lambda$, for $w \in W(\mathfrak{g})$, $\lambda \in \mathfrak{h}_{\mathbb{R}}^{*}$, the affine action of $W(\mathfrak{g})$:

$$w\cdot\lambda \quad = \quad w(\lambda + \rho) - \rho$$

Lemma A [infinitesmal character]

An invariant differential operator $\mathcal{O}(\lambda) \to \mathcal{O}(\mu)$ may exist only if $\mu = w\cdot\lambda$ for some $w \in W(\mathfrak{g})$. □

Example A

In example 3.3 (B) above there seemed the possibility, based on the symbol principle of a second order invariant operator $\mathcal{O} \longrightarrow \overset{2}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{2}{\bullet}$ (for instance). The reader may check however that there does not exist an element w of $W^1(\mathfrak{p})$ with $w\cdot\overset{0}{\bullet}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet} = \overset{2}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{2}{\bullet}$ □

The easiest situation in which to apply the Lemma is when λ is dominant for $\mathfrak{g}$, and hence for $\mathfrak{p}$. Notice first that $\lambda + \rho$ is necessarily non-singular, hence not fixed by any element of $W^1(\mathfrak{p})$ save the identity. A weight μ with infinitesmal character ξ_λ is said to have non-singular infinitesmal character and the bundle $\mathcal{O}(\mu)$ is called non-singular.

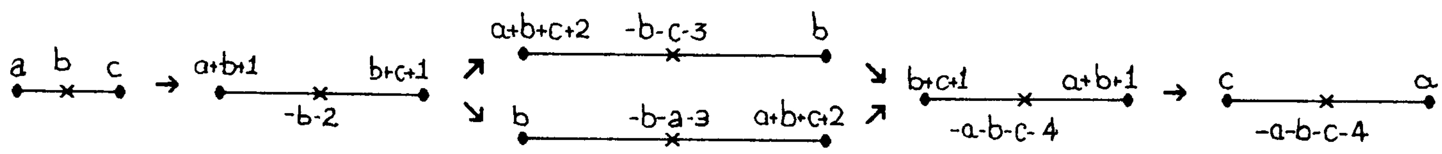
Lemma B [non-singular infinitesmal character]

Let λ be dominant for $\mathfrak{g}$; a complete list of weights μ dominant for $\mathfrak{p}$ with $\xi_\lambda = \xi_\mu$ is given by $\{w\cdot\lambda \mid w \in W^1(\mathfrak{p})\}$ □

This gives an efficient means of computing a list of non-singular bundles which may be related by an invariant differential operator.

Examples B

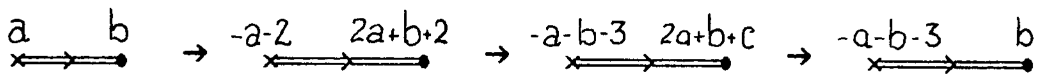
i) Consider Minkowski space M^4 = •—×—•, setting λ = $\overset{a\ b\ c}{\bullet\!-\!\times\!-\!\bullet}$ with a,b,c ≥ 0 so λ is dominant for $\mathfrak{g}$. Applying the elements of $W^1(\mathfrak{p})$ to λ and retaining the directed graph structure of $W^1(\mathfrak{p})$, obtain the following list of non-singular bundles with infinitesmal character ξ_λ :



(3.4.1)

We will see that the arrows represent invariant differential operators; for a=b=c=0, this is essentially the de Rham sequence on M.

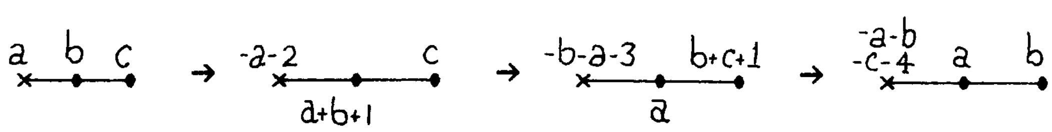
ii) A similar pattern arises for X = $\mathbb{C}S^{2n}$ or $\mathbb{C}S^{2n+1}$. For instance :



(3.4.2)

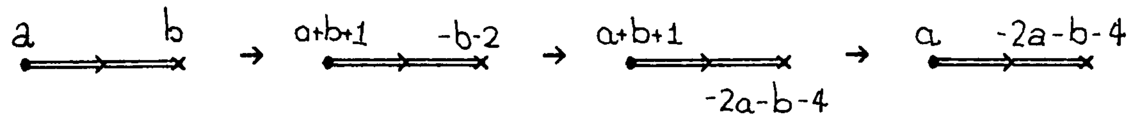
Again arrows will represent invariant differential operators, and for a=b=c=0 this is essentially the de Rham sequence.

iii) On $\mathbb{C}P_n$ = ×—•— ⋯ —•, a "single row" pattern (as in (ii)) exists, since there is precisely one element of length l, with 0 ≤ l ≤ n, in W^1(×—•— ⋯ —•). On Twistor space, $\mathbb{C}P_3$:



(3.4.3)

iv) By way of a variant, consider X = •⇒×; the pattern is



(3.4.4)

Actually, as a topological space, X is $\mathbb{C}P_3$, by its cell structure, but with the indicated complex homogeneous structure, the tangent bundle of X is not irreducible, and the above sequence is not the de Rham sequence for a=b=c=o.

Comment

We will see below how lists of bundles of identical infinitesmal character ξ_λ (λ dominant for $\mathfrak{g}$) are in fact resolutions of the representation F^λ of $\mathfrak{g}$ - this observation, made first by [Buchdahl N P] in the case $\mathfrak{g}$ = sl(4,$\mathbb{C}$) and then called the *generalized de Rham sequence*, will be crucial to the generalization of the Penrose transform to generalized flag manifolds.

3.5 The Borel Case

The simplest case to study is when $\mathfrak{p} = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. In the case $\mathfrak{g}$ = sl(4,$\mathbb{C}$), this corresponds to studying invariant differential operators on the space G^4 which is the intermediate space between Minkowski space and Ambitwistor space. The Borel case has been extensively studied, first by D.-N. Verma in his thesis [Verma] and then by Bernstein, I Gelfand and S Gelfand [Bernstein et al II], who completed the classification of the homomorphisms of

Verma modules. Our exposition is adapted from [Lepowsky I], and a complete exposition may be found in [Dixmier].

The Verma modules $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ are fundamental to the theory of all induced modules, since any $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$ is a quotient of $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$. We need to understand something of their structure; a most useful result is the following proposition of [Humphreys]. Because it gives something of the flavour and mechanics of the subject, we reproduce its proof:

Proposition

Let $\lambda \in \mathfrak{h}^{*}_{\mathbb{R}}$; then $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ has a finite composition series, each of whose terms has the form $\mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu$, where $\xi_{\mu} = \xi_{\lambda}$ and $\mu \leq \lambda$ (in the ordering on weights) □

Proof

If $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ is irreducible, there is nothing to prove. Otherwise, $\mathcal{S}ub\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ is non-zero. Since $\mathcal{S}ub\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$ is a sum of weight spaces, and since λ bounds these weights above, there is a maximal weight μ amoungst them. Let v be a vector in the corresponding weight space; necessarily v is maximal, and it follows, by the universality property of $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu$, that the submodule V generated by the action of $\mathfrak{U}(g)$ on v is a quotient of $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu$. Thus μ satisfies $\xi_{\mu} = \xi_{\lambda}$. The irreducible quotient of V is just $\mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu$ (cf subsection (3.2.2) (b iv)). This gives the first composition factor of $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$. Turn attention now to the unique maximal submodule of V, and to $\mathcal{S}ub\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu$ / V and repeat the process. Since there are only finitely many μ satisfying $\xi_\mu = \xi_\lambda$, the process ends finitely. □

From this proposition is deduced the fact that

<u>Theorem A [Verma]</u>

$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{U}(g)}(\mathcal{I}nd\,_b^g\, \mu\ ,\ \mathcal{I}nd\,_b^g\, \lambda\ ) \le 1$ □

(cf [Lepowsky II] also), and

<u>Theorem B [Verma]</u>

Let $\alpha \in \Delta^+(g,h)$ and suppose that $< \lambda+\rho,\ \alpha^\vee > \ \ge 0$. Then

i) $\exists$ non-zero homomorphism $\mathcal{I}nd\,_b^g\, (\sigma_\alpha\cdot\lambda) \to \mathcal{I}nd\,_b^g\, \lambda$

ii) $\mathcal{Q}uo\,_b^g\, (\sigma_\alpha\cdot\lambda)$ is a subquotient of $\mathcal{I}nd\,_b^g\, \lambda$ □

Iterating the theorem obtain

<u>Corollary</u>

Let β_i be a sequence of positive roots of g. Suppose that

i) $< \lambda+\rho,\ \beta_1{}^\vee > \ \ge 0$ and

ii) for each p = 1,2,…,n, $<(\sigma_{\beta_{p-1}}\sigma_{\beta_{p-2}}\cdots\sigma_{\beta_1}\cdot\lambda)+\rho,\beta_p^\vee> \ \ge 0$.

Then setting $\mu = \sigma_{\beta_n}\sigma_{\beta_{n-1}}\cdots\sigma_{\beta_1}\cdot\lambda$ there is a non-zero homomorphism $\mathcal{I}nd\,_b^g\, \mu \to \mathcal{I}nd\,_b^g\, \lambda$. □

(The reader should think of this corollary as developing (part of) the composition series of $\mathcal{I}nd\,_b^g\, \lambda$. The following result shows that this process obtains *all* μ such that $\mathcal{Q}uo\,_b^g\, \mu$ is a composition factor of $\mathcal{I}nd\,_b^g\, \lambda$. Such factors may occur with multiplicity greater than one, however).

[Bernstein et al II] observed that the form of μ given in the corollary is sufficient:

<u>Theorem C [Bernstein et al II]</u>

The following are equivalent statements

i) $\exists$ non-zero homomorphism $\mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,\mu \to \mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$

ii) $\mathcal{Q}uo_{\mathfrak{b}}^{\mathfrak{g}}\,\mu$ is a subquotient of $\mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$

iii) μ has the form of the previous corollary. □

<u>Example A</u>

Consider $\lambda = \overset{0}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\times}$, labelling the simple roots as $\overset{\alpha_2}{\times}\!-\!\overset{\alpha_1}{\times}\!-\!\overset{\alpha_3}{\times}$ as before. Then $\sigma_2\cdot\lambda = \overset{-2}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times}$ $\sigma_3\sigma_2\cdot\lambda = \overset{-2}{\times}\!-\!\overset{1}{\times}\!-\!\overset{-2}{\times}$ $\sigma_1\sigma_2\sigma_3\cdot\lambda = \overset{0}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\times}$. Thus deduce that on $\times\!-\!\times\!-\!\times = G^4$ there is an invariant differential operator $\overset{0}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\times} \to \overset{0}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\times}$ (pull backs of $\mathcal{O}[-1]$ and $\mathcal{O}[-3]$ on Minkowski space). The homomorphism of induced modules is given by $\beta \to (y_1)^2\, Y_2 Y_1\, \alpha$, where α,β are highest weight vectors. The associated invariant differential operator is $D = \nabla^2\partial\tilde{\partial}$ where $\nabla = \pi^{A'}\tilde{\pi}^{A}\,\nabla_{AA'}$ is the relative differential along the fibre to Ambitwistor space and $\partial,\tilde{\partial}$ are the derivatives along the fibre to Minkowski space. D annihilates pull backs of sections of $\mathcal{O}[-1]$ on M^4, and so does *not* descend to a non-trivial invariant operator on M^4.

□

When in the realm of *non-singular* infinitesmal character, theorem C may be stated in a more concise and practical way;

we give it in a form directly applicable to differential operators:

<u>Theorem D</u> [differential operators in non-singular infinitesmal character]

Let $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ be dominant for $\mathfrak{g}$. With the ordering of $W(\mathfrak{g})$ given in section (2.6) above, $\exists$ an invariant differential operator $\mathcal{D} : \mathcal{O}(w\cdot\lambda) \longrightarrow \mathcal{O}(w'\cdot\lambda)$ on G/B iff $w \le w'$ in $W(\mathfrak{g})$. □

<u>Examples B</u>

i) Again on $\times\!-\!\times\!-\!\times$, notice that if $w = \sigma_1$ and $w' = \sigma_1\sigma_2\sigma_3\sigma_1$ then $w \le w'$ and there is an invariant differential operator on $\overset{1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\times} = \sigma_1\cdot\overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times} \longrightarrow \sigma_1\sigma_2\sigma_3\sigma_1\cdot\overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times} = \overset{0}{\times}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\times}$. This does not yield a non-zero invariant differential operator $\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \longrightarrow$ $\overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}$, as will be apparent in the next section.

ii) Another interesting example concerns again $\overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times}$; there is an invariant operator $\overset{p}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{-2}{\bullet} = \sigma_3\cdot\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{p+1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{-3}{\bullet} =$ $\sigma_3\sigma_1\cdot\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}$. This operator will induce the antitwistor operator of [Eastwood and Josza].

iii) Now take $\lambda = \overset{0}{\times}\!-\!\overset{1}{\times}\!-\!\overset{0}{\times}$; $\sigma_1\cdot\lambda = \overset{2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\times}$ and deduce an invariant operator $\nabla^2 : \overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times} \longrightarrow \overset{2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\times}$. This operator, in its curved extension, plays an important role in the Penrose Transform of the Einstein bundle of [Le Brun II]. □

The next step is to allow $\mathfrak{p}$ to be more than Borel, and see to what extent differential operators defined in the Borel case induce operators in the more general case.

3.6 Standard Maps

Let us return to the case of a more general parabolic $\mathfrak{p}$, and attempt to exploit something of the Borel case by considering the fibration

$$G/B \xrightarrow{\nu} G/P$$

where B is a Borel subgroup of G contained in P. Recall equation (2.5.1) :

$$0 \longrightarrow \nu^{-1}\,\mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(\lambda_{\mathfrak{b}})$$

where λ is a dominant weight for $\mathfrak{p}$. Suppose now that μ is also dominant for $\mathfrak{p}$ and that there is an invariant differential operator

$$\mathcal{D}_B : \mathcal{O}(\lambda_{\mathfrak{b}}) \longrightarrow \mathcal{O}(\mu_{\mathfrak{b}})$$

Recall also the Borel-Weil Theorem [Warner], [Bott], to deduce that sections of $\mathcal{O}(\mu_{\mathfrak{b}})$ global on the fibres of ν push down to sections of $\mathcal{O}(\mu_{\mathfrak{p}})$ and compose to obtain

$$\mathcal{D}_P = \nu_* \circ \mathcal{D} \circ \nu^{-1} \quad : \quad \mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(\mu_{\mathfrak{p}})$$

which will be referred to as the *standard* differential induced by $\mathcal{D}_B$. It is clear that $\mathcal{D}_P$ may vanish because $\mathcal{D}_B$ annihilates $\nu^{-1}\,\mathcal{O}(\lambda_{\mathfrak{p}})$ - we have seen an example of this in the previous section. This necessarily happens if μ is not dominant for $\mathfrak{p}$.

Now let us see this operator induction in algebraic terms; recall the surjection of $\mathfrak{U}(\mathfrak{g})$ modules

$$\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda \longrightarrow \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda \longrightarrow 0$$

Suppose there is a non-zero homomorphism f $:\mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,\mu \to \mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$. Let v be a highest weight vector of the former, and consider the image of f(v) in $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$. Clearly it is either zero or maximal, of weight μ, in which case it defines a homomorphism $\tilde{f} : \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\mu \to \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$. The resulting map, which may be zero, is called the *standard map* induced by f, and it induces the standard operator.

The standard map is clearly zero iff f(v) is in the kernel of the projection of $\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$ onto $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$. The following lemma and theorem identify this kernel, and so determine when the standard map is zero:

<u>Lemma A</u>

Let λ be dominant for $\mathfrak{p}$; then the following is a resolution :

$$\oplus_{\alpha \in \Psi(\mathfrak{p})} \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,(\sigma_\alpha \cdot \lambda) \xrightarrow{r} \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda \xrightarrow{p} \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda \longrightarrow 0 \qquad (3.6.1)$$

where, recall, $\Psi(\mathfrak{p}) \subseteq \mathcal{S}(\mathfrak{g},\mathfrak{h})$ is a defining set of simple roots for $\mathfrak{p}$. □

Proof

The lemma is proved as a "relative" version of Theorem 21.4 of [Humphreys]. The kernel of p is the submodule generated by vectors of the form

$$v_n = y_n^{\langle\lambda,\alpha_n^\vee\rangle + 1} \cdot v_0$$

for $\alpha_n \in \Psi(\mathfrak{p})$, $\{y_n\}$ a root space basis for $u\_(\mathfrak{p})$ (with $y_n \in E_{\alpha_n}$), and v_0 a highest weight vector of $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$. These are maximal, of weight $\sigma_{\alpha_n} \cdot \lambda$. □

Note The generation of ker p is not free - the image of r in $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$ is not a direct sum of the individual images. We will see in section (3.6) below how to complete (3.6.1) into a resolution of [Bernstein et al II].

Theorem A [on zero standard maps]

Let μ, λ be dominant for $\mathfrak{p}$ such that a non-zero homomorphism $f : \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\mu \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$ exists. The associated standard map $\tilde{f} : \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ is zero iff $\exists$ non-zero homomorphism from $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\mu \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\,\sigma_\alpha \cdot \lambda$, for some $\alpha \in \Psi(\mathfrak{p})$. □

Proof (after [Lepowsky I])

Suppose a non-zero homomorphism $\mathcal{I}nd\,_{b}^{g}\,\mu \longrightarrow \mathcal{I}nd\,_{b}^{g}\,\sigma_{\alpha}\cdot\lambda$ exists. Then by Theorem (3.5) A, the homomorphism $\mathcal{I}nd\,_{b}^{g}\,\mu \to \mathcal{I}nd\,_{b}^{g}\,\lambda$ factors through it, whence the standard map is zero, by Lemma A. On the other hand, in (3.6.1) write $\operatorname{Ker} p = \Sigma_{\alpha \in \Psi(\mathfrak{p})} \mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha}\cdot\lambda)$ (but the sum is not direct). Order $\Psi(\mathfrak{p}) = \{\alpha_1, \dots, \alpha_l\}$. For some n,

$$\mathcal{I}nd\,_{b}^{g}\,\mu \subseteq \Sigma_{1}^{n+1}\, \mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha_i}\cdot\lambda) \subset \mathcal{I}nd\,_{b}^{g}\,\lambda$$

but

$$\mathcal{I}nd\,_{b}^{g}\,\mu \not\subseteq \Sigma_{1}^{n}\, \mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha_i}\cdot\lambda) \subset \mathcal{I}nd\,_{b}^{g}\,\lambda$$

It follows that

$$\mathcal{I}nd\,_{b}^{g}\,\mu \longrightarrow \Sigma_{1}^{n+1}\, \mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha_i}\cdot\lambda) \;/\; \Sigma_{1}^{n}\, \mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha_i}\cdot\lambda)$$

is a surjection, so that necessarily $\mathcal{Q}uo\,_{b}^{g}\,\mu$ is a subquotient of $\mathcal{I}nd\,_{b}^{g}\,(\sigma_{\alpha_{n+1}}\cdot\lambda)$. But then Theorem (3.5) C implies the existence of a non-zero homomorphism $\mathcal{I}nd\,_{b}^{g}\,\mu \to \mathcal{I}nd\,_{b}^{g}\,\sigma_{\alpha_{n+1}}\cdot\lambda$.

□

<u>Examples A</u>

i) The standard invariant differential operator $\mathcal{O}[-1] \to \mathcal{O}[-3]$ ($\overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet}$) is zero, since σ_3 corresponds to α_3, a defining simple root for $\bullet\!-\!\times\!-\!\bullet$; (cf example (3.5) A). Nonetheless $(y_1 y_4 + y_4 y_1 - y_2 y_3 - y_3 y_2)\alpha$ induces the non-zero wave operator $\square : \mathcal{O}[-1] \longrightarrow \mathcal{O}[-3]$.

ii) The invariant differential operator $\overset{p}{\times}\!-\!\overset{1}{\times}\!-\!\overset{-2}{\times} \longrightarrow \overset{p+1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{-3}{\times}$ descends to a non-zero standard operator $\overset{p}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{-2}{\times} \longrightarrow \overset{p+1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{-3}{\times}$ for $p \geq 0$ (cf example (3.5) B (ii)) which is the anti-twistor operator of [Eastwood and Josza]. □

We have seen that it is perfectly possible for the standard homomorphism to be zero. Example A (i) above shows however that even when this happens, a non-zero *non-standard* homomorphism may exist. It is important to place some limitation on this phenomenon, in particular to know that a non-standard homomorphism cannot arise between induced modules unless there is a homomorphism of (Borel) Verma modules. To see this, first a

<u>Lemma B</u>

Suppose there is a non-zero homomorphism $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\,\mu \to \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$, with μ, λ dominant for $\mathfrak{p}$. Then $\mathcal{Q}uo\,^{\mathfrak{g}}_{\mathfrak{b}}\,\mu$ is a composition factor of $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$ □

<u>Proof</u>

Let $\tilde{v}$ be the image in $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda$ of a highest weight vector of $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\,\mu$ and let v be a representative of the coset of $\tilde{v}$ under the projection, p, from $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$. Then as in the proof of proposition (3.5), v is in the image V of some $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{b}}\,\mu'$ in $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{b}}\,\lambda$. Clearly (V ∩ Ker p) is contained in the maximal proper submodule of V and hence the image of v in the irreducible quotient $\mathcal{Q}uo\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu'$ is maximal. This proves $\mu = \mu'$ and the lemma. □

Applying Theorem (3.5) C, obtain

Theorem B

A non-zero homomorphism $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \mu \longrightarrow \mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{p}}\ \lambda$ implies the existence of a non-zero homomorphism $\mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \mu \longrightarrow \mathcal{I}nd\ {}^{\mathfrak{g}}_{\mathfrak{b}}\ \lambda$. Thus an invariant differential operator

$$\tilde{\mathcal{D}} : \mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(\mu_{\mathfrak{p}})$$

can exist only if there is an invariant operator

$$\mathcal{D} : \mathcal{O}(\lambda_{\mathfrak{b}}) \longrightarrow \mathcal{O}(\mu_{\mathfrak{b}})$$

on G/B. □

Ofcourse, $\tilde{\mathcal{D}}$ need not be the standard operator induced by $\mathcal{D}$. There is one occasion on which this is necessarily so:

Theorem C

Let λ be dominant for $\mathfrak{g}$. Suppose $w, w' \in W^1(\mathfrak{p})$ with $l(w') = l(w) + 1$. Then the standard operator

$$\tilde{\mathcal{D}} : \mathcal{O}(w\cdot\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(w'\cdot\lambda_{\mathfrak{p}})$$

exists and is non-zero iff $w \longrightarrow w'$ in the ordering of the Weyl group $W(\mathfrak{g})$ (section (2.6)). □

(Thus " $\longrightarrow$ " in the Weyl group may be thought of as indicating a differential operator).

Proof

Working with induced modules, if there exists a non-zero homomorphism $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\, w'\cdot\lambda \longrightarrow \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\, w\cdot\lambda$ for $w, w' \in W^1(\mathfrak{p})$, with $l(w') = l(w) + 1$, Lemma B implies in Theorem (3.5) C that $w' = \sigma_\alpha w$ for some $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$, whence $w \longrightarrow w'$ by definition. On the other hand, if $w \longrightarrow w'$, then by Verma's theorem (3.5) B, there is a non-zero homomorphism

$$\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\, w'\cdot\lambda \longrightarrow \mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\, w\cdot\lambda .$$

Suppose that the associated standard map is zero. Then for some $\alpha \in \Psi(\mathfrak{p})$ $\exists$ non-zero homomorphism

$$\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\, w'\cdot\lambda \longrightarrow \mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\, \sigma_\alpha w\cdot\lambda$$

Thus, $l(w') < l(\sigma_\alpha w) = l(w) + 1 = l(w')$, by Lemma (2.6) C, a contradiction. □

To conclude this section, we remark on certain structures associated to Weyl groups, with their directed graph structure. The reader will have noticed already that many of the results given above concerning homomorphisms of induced modules (at least for non-singular infinitesmal character) depend more on the structure of $W(\mathfrak{g})$, $W^1(\mathfrak{p})$, than on the infinitesmal character itself. We will make this rigorous in the next section, but some observations are in order here. These will tighten and summarise the theory given above, and perhaps make it more transparent. First:

Theorem D

The following are equivalent statements, with λ dominant for $\mathfrak{g}$:

i) The standard map $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ w'\cdot\lambda \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ w\cdot\lambda$ is zero;

ii) The standard Invariant Differential Operator $\mathcal{O}(w\cdot\lambda) \longrightarrow \mathcal{O}(w'\cdot\lambda)$ is zero

iii) There is a path from w to w' in $W(\mathfrak{g})$ passing through $\sigma_\alpha w$ for some $w \in \Psi(\mathfrak{p})$

iv) There is a path from w to w' in $W(\mathfrak{g})$ passing through w" with w" not in $W^1(\mathfrak{p})$. □

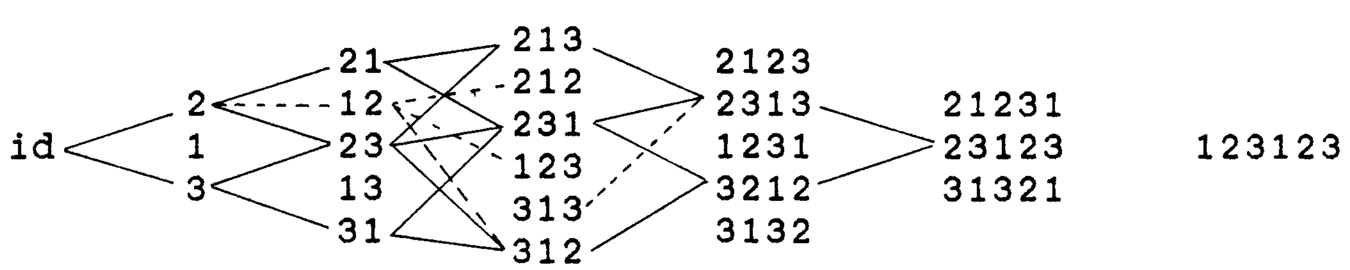
(It is instructive to vizualize (iii) & (iv) graphically, thinking of paths exiting $W^1(\mathfrak{p})$ in $W(\mathfrak{g})$).

Proof

(i) $\Leftrightarrow$ (ii) is Theorem (3.3); (i) $\Rightarrow$ (iii) is Theorem A above; (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are trivial. □

Example B

Consider invariant differential operators on Ambitwistor space ×—•—×. The following diagram displays (using solid lines) the subgraph $W^1(\mathfrak{p}) \subset W(\mathfrak{g})$. (Notation as in section (2.6)).



There is, for λ dominant for $\mathfrak{g}$, a map

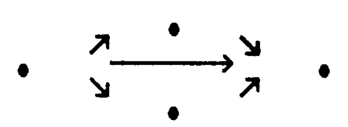
$$\mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,(2313)\cdot\lambda \longrightarrow \mathcal{I}nd_{\mathfrak{b}}^{\mathfrak{g}}\,(2)\cdot\lambda\ .$$

If this is to induce the zero standard map, there must be a path from (2) to (2313) in $W(\mathfrak{g})$ exiting $W^1(\mathfrak{p})$. Using the theory of section 2.6, the reader will confirm that such a path would have to pass through (12); edges to the right of (12) and the left of (2313) not in $W^1(\mathfrak{p})$ are indicated as broken lines in the diagram above. Clearly no such path exists. Accordingly, there is a non-zero standard map

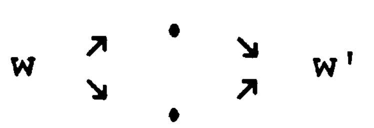
$$\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\,(2313)\cdot\lambda \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\,(2)\cdot\lambda$$

for all λ dominant for $\mathfrak{g}$. In particular there is an invariant differential operator $\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times} \longrightarrow \overset{-4}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}$. □

[Bernstein et al II] observe that the following diagram can never occur in $W(\mathfrak{g})$



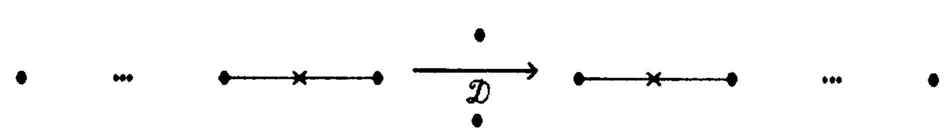
It follows immediately that whenever a diagram (a square)



occurs in $W^1(\mathfrak{p})$, the standard map $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, w'\cdot\lambda \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, w\cdot\lambda$ and the associated invariant differential operator are non-zero.

<u>Example C</u>

In example (3.4) B (i) there exists a standard differential operator $\mathcal{D}$ as indicated :



3.7 The Translation Principle

We have already observed, at the end of section (3.6) that at least in the realm of non-singular infinitesmal character, questions concerning the existence of homomorphisms of induced modules were apparently a function of the structure of $W^1(\mathfrak{p})$ in $W(\mathfrak{g})$ rather than the actual infinitesmal character. This section is a formalization of that observation in what is called the *Translation Principle* of [Jantzen] and [Zuckerman]. It is extremely simple in concept: we want a functor on induced modules (which works for more general modules, actually) which allows us to translate between categories $\mathfrak{M}(\xi_\lambda)$ of modules of infinitesmal character ξ_λ. To do this, we simply tensor modules through with a fixed finite dimensional representation of $\mathfrak{g}$, and select from the result the part with the desired infinitesmal character. This sort of construction, or at any rate the tensoring part of it, has been used in Twistor Theory for some time. It is used in the Penrose and Ward Transforms [Eastwood et al], where the topological inverse image of a bundle over Twistor space is resolved by means of the relative de Rham sequence, tensored through with the pull-back bundle. If the pull back bundle is homogeneous, taking parts of the reolution of appropriate infinitesmal character yields a maximally efficient resolution, as we shall see in section (3.8).

We introduce the translation functors and demonstrate their action in some individual cases. We then show (Theorem (A)) that as functors between modules of non-singular infinitesmal character, they are isomorphisms, which establishes differential operators on non-singular bundles from the "de Rham" case of zero infinitesmal character.

3.7.1 The Translation Functors

Let M be any $\mathfrak{U}(g)$-module. We say that M is $\mathcal{Z}(g)$-*finite* if there is an ideal $\mathcal{I}(M) \subset \mathcal{Z}(g)$, of finite codimension, which annihilates M. Clearly a finite dimensional module and an induced module are $\mathcal{Z}(g)$-finite. Given an ideal $\mathcal{I}$ of $\mathcal{Z}(g)$, we say that $\mathcal{I}$ is primary if there is a $\chi \in (\mathcal{Z}(g))^*$ and an integer n with $\mathcal{I} \subset (\ker \chi)^n$. When this is true for $\mathcal{I}(M)$, we say that M has *generalized infinitesmal character* χ. χ is said to be a *character* of $\mathcal{Z}(g)$.

So now define a functor P_χ on the category of $\mathcal{Z}(g)$-finite modules to the category of modules of generalized infinitesmal character χ, $\mathfrak{M}(\chi)$, by

$$P_\chi(M) = \{ m \in M \mid \exists n \in \mathbb{Z}^+ \text{ such that } \forall z \in Z(g), (z - \chi(z))^n m = 0 \}$$

Applying P_χ is called "taking χ-primary part". Then

Lemma A

P_χ has the following properties :

i) $P_\chi(M)$ is a direct summand of M;

indeed $M = \oplus_{\chi \in \text{Spec}\, \mathcal{Z}(g)} P_\chi(M)$.

ii) P_χ is an exact functor. □

(the proof is a straight forward excercise in the localization theory of commutative algebras - cf [Zuckerman]).

<u>Note:</u> If $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is a weight for g, then $\xi_\lambda \in (\mathcal{Z}(g))^*$. We will abuse notation, and write P_λ for P_{ξ_λ}. A similar abuse of notation will occur on the translation functors below. Notice that for $w \in W(g)$, $P_{w\cdot\lambda} = P_\lambda$. Also we denote $\mathfrak{M}(\lambda) = \mathfrak{M}(\xi_\lambda)$.

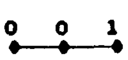
□

Next recall (section (2.6)) that if $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is any weight, then $F(\lambda)$ denotes the unique finite dimensional irreducible representation of g of extremal weight λ. Define the *translation functors* by :

$$\Psi_{\lambda_2}^{\lambda_1} (\cdot) = P_{\lambda_2} (F(\lambda_2-\lambda_1) \otimes_{\mathbb{C}} P_{\lambda_1}(\cdot)) \qquad (3.7.1)$$

from the category of $\mathcal{Z}(g)$-finite modules to $\mathfrak{M}(\lambda_2)$. Here, λ_1, $\lambda_2 \in \mathfrak{h}_{\mathbb{R}}^*$. Before investigating the theory of these functors, let us take some concrete examples.

<u>Examples A</u>

i) Let $\lambda =$  and consider the g module F^λ in terms of its composition series as a $\mathfrak{p}$ module :

$$0 \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}) \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}) \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}) \longrightarrow 0 \tag{3.7.2}$$

Applying the exact functor $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}$, obtain

$$0 \longrightarrow \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \longrightarrow \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow 0 \tag{3.7.3}$$

This splits, by Lemma A(i) and $P_\lambda(\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \lambda) = \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \lambda$:

$$\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet} \cong \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \oplus \mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}$$

This can be shown explicitly; let $F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet})$ be composed of weight spaces spanned by the following vectors :

$$\alpha \quad (\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}) \quad ; \quad \beta = Y_2\alpha \quad (\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{-1}{\bullet}) \quad ;$$

$$\gamma = Y_1\beta \quad (\overset{1}{\bullet}\!-\!\overset{-1}{\bullet}\!-\!\overset{0}{\bullet}) \quad ; \quad \delta = Y_1\gamma \quad (\overset{-1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \quad .$$

Let τ be a highest weight vector in $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}$; split (3.7.3) by mapping τ to the maximal vector $\gamma - (Y_1\beta - Y_3\alpha)/2$ in $\mathcal{I}nd\,^{\mathfrak{g}}_{\mathfrak{p}}\, \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$.

Dually, this means that there is a conformally invariant differential splitting of the sequence of sheaves

$$0 \longrightarrow \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \mathcal{T}^{\alpha} \longrightarrow \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \longrightarrow 0$$

where $\mathcal{T}^{\alpha}$ is the local twistor sheaf, induced by the representation of $\mathfrak{p}$ on $(F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}))^{*} = F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) = T^{\alpha}$, i.e. twistors. Recalling that $\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \cong \mathcal{O}_{A'}$ and $\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \cong \mathcal{O}^{A}$, this splitting is (relative to the splitting given by a local choice of flat metric):

$$\mathcal{O}^{A} \longrightarrow \mathcal{T}^{\alpha} \quad \text{by} \quad \omega^{A} \longrightarrow (\omega^{A}, \nabla_{AA'}\omega^{A})$$
$$\mathcal{T}^{\alpha} \longrightarrow \mathcal{O}_{A'} \quad \text{by} \quad (\omega^{A}, \pi_{A'}) \longrightarrow \pi_{A'} - \nabla_{AA'}\omega^{A}$$

ii) Take $\lambda_1 = \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$, $\lambda_2 = \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$ and apply the translation functor to $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\; \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}$. $F(\lambda_2-\lambda_1)$ has the following composition series as a $\mathfrak{p}$-module:

$$0 \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\times}) \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{-1}{\times}) \longrightarrow 0$$

Tensoring with $F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times})$, obtain

$$0 \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}) \longrightarrow \begin{matrix} F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \\ \otimes \\ F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}) \end{matrix} \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\times}) \longrightarrow 0 \tag{3.7.4}$$

We shall shortly see that as $\mathfrak{U}(\mathfrak{g})$-modules,

$$\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,(F(\lambda) \otimes F^{\mu}_{\mathfrak{p}}) \cong F(\lambda) \otimes \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu$$

(for μ dominant for $\mathfrak{p}$) so that (3.7.4) gives

$$\Psi^{\lambda_1}_{\lambda_2}\left(\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\; \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times} \right) \cong \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\; \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times} \qquad \square$$

3.7.2 The Translation Functors on Induced Modules

We now seek to compute the translation functors on induced modules. Essentially, we generalize example (ii) above, so the first step is to verify the following

Lemma B

Suppose F is a finite dimensional $\mathfrak{g}$-module, and V is a $\mathfrak{p}$-module. Then $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(F \otimes V) \cong F \otimes_{\mathbb{C}} \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} V$. □

Proof

An element of degree one in $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(F \otimes V)$ may be written as $y(f \otimes v)$ for $y \in \mathfrak{u}_-$, $f \in F$ and $v \in V$. Simply map this to $(y \cdot f) \otimes v + f \otimes yv$, and extend in the obvious manner to elements of higher degree. □

Observe that $P_\lambda(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \lambda) = \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} \lambda$ and so reduce the computation of the translation functors to the evaluation of the functors P_{λ_2} on $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(F(\lambda_2-\lambda_1) \otimes F_{\mathfrak{p}}^{\lambda_1})$. To do this, compute a composition series of $F(\lambda_2-\lambda_1) \otimes F_{\mathfrak{p}}^{\lambda_1}$ as a finite dimensional $\mathfrak{p}$ module. Each factor in this will be of the form $F_{\mathfrak{p}}^{\mu}$ for some $\mu \in \mathfrak{h}_{\mathbb{R}}^*$, dominant for $\mathfrak{p}$. To aid in this computation, here is a lemma of [Klimyk] (cf [Humphreys], excercise 24.9)

Lemma C

Let ch_λ denote the formal character (i.e. the formal sum of weights, with multiplicity) of a $\mathfrak{p}$-module $F_\mathfrak{p}^\lambda$, λ dominant integral for $\mathfrak{p}$. Let $ch_{\lambda\otimes\lambda'}$ denote the formal character of $F_\mathfrak{p}^\lambda \otimes F_\mathfrak{p}^{\lambda'}$. Suppose

i) $\Delta(\lambda') = \Delta(F_\mathfrak{p}^{\lambda'})$ is the set of weights in $F_\mathfrak{p}^{\lambda'}$

ii) $m_{\lambda'}(\mu)$ is the multiplicity of μ in $\Delta(\lambda')$

iii) $\rho_\mathfrak{l} = (\Sigma_{\alpha \in \Delta^+(\mathfrak{l})} \alpha) / 2$ and

iv) if $\lambda'+\mu+\rho_\mathfrak{l}$ is $W(\mathfrak{l})$ conjugate (by $w(\mu)$, say) to a regular weight, dominant for $\mathfrak{p}$, set $t(\mu) = (-1)^{l(w(\mu))}$, otherwise take $t(\mu) = 0$.

Then

$$ch_{\lambda\otimes\lambda'} = \Sigma_{\mu \in \Delta(\lambda')} m_{\lambda'}(\mu)\ t(\mu)\ ch_{w(\mu)\cdot(\mu+\lambda)} \qquad \square$$

Here, $\cdot$ signifies the affine action of $W(\mathfrak{l}^s)$ on weights for $\mathfrak{l}^s$, via $\rho_\mathfrak{l}$. Actually, as the reader will easily verify, it makes no difference to the formula if we work with ρ (section 3.4) and the affine action of $W(\mathfrak{l}^s) \subset W(g)$ instead, and we do this below.

Corollary

If F is a finite dimensional representation of g, and λ is dominant for $\mathfrak{p}$ then the $\mathfrak{p}$-irreducible composition factors of $F \otimes F_\mathfrak{p}^\lambda$ are all of the form $F_\mathfrak{p}^{\lambda+\mu}$ for some $\mu \in \Delta(F)$. $\square$

Thus we have some idea of the terms which can occur in the composition series we are computing. We want some control on the infinitesmal characters of the composing modules. The following lemma of [Vogan] gives us what we need:

Lemma D

Let λ be any weight, of non-singular infinitesmal character, and let μ be an extremal weight of a finite dimensional irreducible representation $F(\mu)$ of g. Let $W_0 \subset W(g)$ stabilize $\lambda+\rho+\mu$. If this sum is dominant, then for $w \in W(g)$, $\theta \in \Delta(F(\mu))$,

$$w \cdot (\lambda + \mu) = \lambda + \theta$$

if and only if

$$w \in W_0 \quad \text{and} \quad \theta = \mu .$$

More generally, if $w_1 \in W^1(\mathfrak{p})$,

$$w \cdot (w_1 \cdot (\lambda + \mu)) = w_1 \cdot \lambda + \theta$$

if and only if

$$w \in W_0 \quad \text{and} \quad \theta = w_1 \mu . \qquad \square$$

With this, we can compute the action of the translation functors on induced modules:

Theorem A (Translation Principle for induced modules)

Let $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{R}}$ with

a) λ dominant for g

b) $\mu + \rho$ dominant for g

and let $w \in W^1(\mathfrak{p})$, where $\mathfrak{p}$ is any parabolic subalgebra of g and $\rho = (\Sigma_{\alpha \in \Delta^+(g,\mathfrak{h})} \alpha) / 2$. Then

i) $\Psi^{\lambda}_{\mu} (\mathcal{I}nd^{g}_{\mathfrak{p}} \, w \cdot \lambda) = \mathcal{I}nd^{g}_{\mathfrak{p}} (w \cdot \mu)$.

If, *additionally*, μ is dominant, then

ii) Ψ_{λ}^{μ} is a natural inverse of Ψ_{μ}^{λ} on the categories of induced modules of infinitesmal character ξ_{μ}, ξ_{λ}, so that Ψ_{μ}^{λ} is an equivalence of these categories. □

Proof

We take this in three short stages.

I $\Psi_{\mu}^{\lambda} = \Psi_{w\cdot\mu}^{w\cdot\lambda}$ for $w \in W(g)$. For $P_{w\cdot\theta} = P_{\theta}\ \forall\ \theta \in \mathfrak{h}_{\mathbb{R}}^{*}$ and $F(w\cdot\mu - w\cdot\lambda) = F(w(\mu-\lambda)) = F(\mu-\lambda)$.

II
$$\begin{aligned}\Psi_{\mu}^{\lambda}\left(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\lambda)\right) &= \Psi_{w\cdot\mu}^{w\cdot\lambda}\left(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\lambda)\right)\\ &= P_{w\cdot\mu}\left(F(w(\mu-\lambda)) \otimes \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\lambda)\right)\\ &= P_{w\cdot\mu}\left(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(F(w(\mu-\lambda)) \otimes F_{\mathfrak{p}}^{w\cdot\lambda})\right)\\ &= \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\mu)\end{aligned}$$
using I and Lemma B and applying part (i) of Lemma D, noting that $\mu+\rho$ dominant sets $W_0 = \{id\}$ therein. Thus we have part (i) of the Theorem.

III To prove part (ii), note that by (i)
$$\Psi_{\lambda}^{\mu}\Psi_{\mu}^{\lambda}\,\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\lambda) \cong \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}(w\cdot\lambda)$$
and it is easy to check that this isomorphism is natural. □

Remark Theorem A is concerned with translation from non-singular infinitesmal character; the target category may be of modules of singular infinitesmal character (although then part (ii) of the theorem does not hold). In fact, part (ii) of the theorem is true for translation between any infinitesmal characters provided the stabilizer (cf Lemma D)

$W_0(\mu+\rho) = W_0(\lambda+\rho)$ (cf [Zuckerman]). We incorporate this in the theorem for invariant differential operators below.

<u>Theorem B</u> (Translation Principle for Differential Operators)

Let μ, λ be weights for $\mathfrak{g}$, $w, w' \in W(\mathfrak{g})$ with $w\cdot\mu$, $w'\cdot\mu$, $w\cdot\lambda$, and $w'\cdot\lambda$ dominant for $\mathfrak{p}$. Suppose that the stabilizers $W_0(\mu+\rho)$, $W_0(\lambda+\rho)$ of $\mu+\rho$, $\lambda+\rho$ in $W(\mathfrak{g})$ are equal. Then on G/P, there exists an invariant differential operator

$$\mathcal{D} : \mathcal{O}(w\cdot\lambda) \longrightarrow \mathcal{O}(w'\cdot\lambda)$$

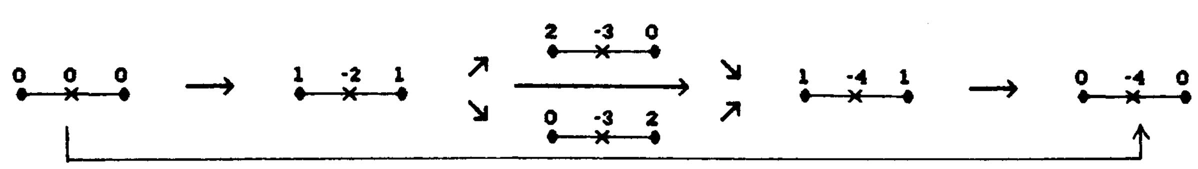
if, and only if, there exists an invariant differential operator

$$\tilde{\mathcal{D}} : \mathcal{O}(w\cdot\mu) \longrightarrow \mathcal{O}(w'\cdot\mu). \qquad \square$$

By way of a concrete example of Theorem B, consider

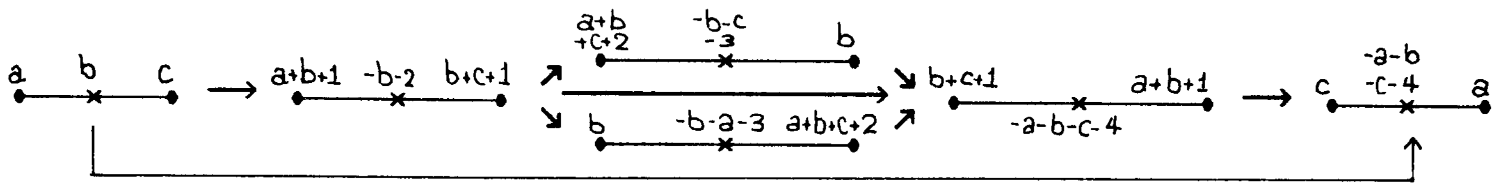
<u>Example B Invariant operators on M^4</u>

We have already concluded, in example (3.3) B that the invariant differential operators on bundles of infinitesmal character ξ_0 on Minkowki space M^4 are represented by arrows in the following diagram

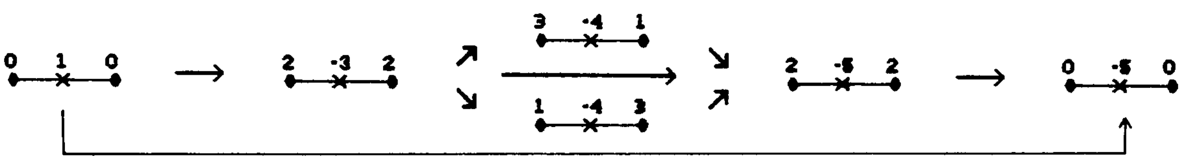


(3.7.5)

Application of the translation principle for differential operators yields the following exhaustive list of differential operators on bundles of non-singular infinitesmal character on Minkowski space [Eastwood and Rice]



for integers a,b,c ≥ 0. These are readily identified by means of the symbol technique. (cf also [Buchdahl N P] and [Penrose and Rindler II]). We consider two examples. First:

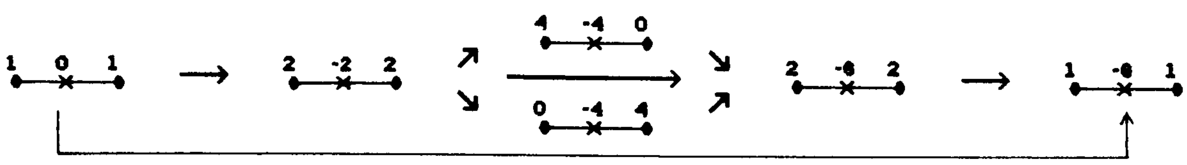


(3.7.6)

The map $\overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}$ is $\varphi \longrightarrow \nabla^{(A'}_{(A}\nabla^{B')}_{B)}\varphi \in \mathcal{O}^{(A'B')}_{(A\ B)}[-1]$

and $\overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{1}{\bullet}$ is $\kappa^{A'B'}_{AB} \in \mathcal{O}^{(A'B')}_{(A\ B)}[-1]$ to $\nabla^{A}_{(A'}\kappa_{B'C')AB}$

A second example of much importance in the sequel is



(3.7.7)

This sequence comes, as we shall see, from a resolution of the coadjoint representation of sl(4,ℂ). We will find it useful in Chapter Four in computing certain *Spencer cohomology groups*, and below we will relate it to certain infinitesmal deformations of Ambitwistor space. The bundles appearing in it are the natural home for certain important geometric objects (we will see why in the next chapter). For instance $\overset{4}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}$ is the home for (linearized) self dual Weyl curvature $\Psi_{A'B'C'D'}$ with $\overset{0}{\bullet}\!\!-\!\!\overset{-4}{\times}\!\!-\!\!\overset{4}{\bullet}$ home for the anti-self dual part $\tilde{\Psi}_{ABCD}$. $\overset{2}{\bullet}\!\!-\!\!\overset{-6}{\times}\!\!-\!\!\overset{2}{\bullet}$ is home for the (linearized) Bach Tensor B_{ab} [Penrose and Rindler I], which is the image of $\Psi_{A'B'C'D'}$ or $\tilde{\Psi}_{ABCD}$ under invariant differential operators:

$$B_{A'B'AB} = \nabla^{C'}_{(A}\nabla^{D'}_{B)}\Psi_{A'B'C'D'} = -\nabla^{C}_{(A'}\nabla^{D}_{B')}\tilde{\Psi}_{ABCD} \qquad \square$$

3.7.3 Differential Splittings

A final application of Lemmas A and D is as follows. Consider the category of $\mathfrak{p}$-modules which admit a finite composition series in terms of irreducible $\mathfrak{p}$-modules. We are particularly interested in finite dimensional $\mathfrak{p}$-modules which are neither irreducible nor reducible, such as those resulting from a restriction of an irreducible $\mathfrak{g}$-module. Let F be such an object, with composition factors $F_i = F_{\mathfrak{p}}^{\lambda_i}$ $i \in I$, say, where λ_i are dominant for $\mathfrak{p}$. Apply the inducing functor $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ to obtain a composition series for $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ F. Then apply the functors P_χ as χ runs over the infinitesmal characters present in $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ F. This produces in general a non-trivial composition series for the parts of $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ F of infinitesmal character χ, with $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ F a direct sum as in Lemma A(ii). The situation of most interest to us is when the factors of the composition series have distinct infinitesmal character, so that $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$ F is a direct sum of $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\,\lambda_i$. When this happens, there are invariant differential operators from the induced bundles $\mathcal{O}(\lambda_i)$ on G/P into the bundle $\mathcal{O}(F^*)$ (on G/P) and visa versa. This is a generalization of example A(i) above.

Irrespective of the multiplicity of the infinitesmal characters of the composition factors, if F^λ, λ dominant for $\mathfrak{g}$, is decomposed into $\mathfrak{p}$-irreducibles, there is a single factor in the composition series for $\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, F$ of infinitesmal character ξ_λ, by Lemma D. Thus on the bundle level, over G/P, there is always an invariant differential splitting

$$\mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(F(-\lambda))$$

<u>Example C</u>

Let $T^{[\alpha\beta]} = F(\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet})$ be the $\bullet\!-\!\times\!-\!\bullet$-module restricted from the $\mathfrak{g}$-module. Then $T^{[\alpha\beta]}$ has composition series

$$0 \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet}) \longrightarrow T^{[\alpha\beta]} \longrightarrow F' \longrightarrow 0$$

$$0 \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet}) \longrightarrow F' \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}) \longrightarrow 0$$

or in bundle form

$$0 \longrightarrow \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \mathcal{O}(F'^*) \longrightarrow \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(F'^*) \longrightarrow \mathcal{T}_{[\alpha\beta]} \longrightarrow \overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow 0$$

(where $\mathcal{T}_{[\alpha\beta]}$ is induced by $T^{[\alpha\beta]*} = T_{[\alpha\beta]}$ restricted to a $\mathfrak{p}$ module). In terms of a local splitting of these sequences by choice of a flat metric, the invariant differential splittings are

$$\overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \mathcal{T}^{[\alpha\beta]} \quad \text{by} \quad f\ (\in \mathcal{O}[1]) \longrightarrow (f, \nabla_A^{A'} f, \Box f) \quad (3.7.8)$$

$$\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \mathcal{T}^{[\alpha\beta]} \quad \text{by} \quad v_A^{A'} \longrightarrow (0, v_A^{A'}, \nabla_{A'}^{A} v_A^{A'})$$

These will persist in the presence of curvature : (3.7.8) will isolate conformal manifolds which are conformal to a vacuum solution of Einstein's equations. □

3.7.4 Translation near singular infinitesmal character : Invariant Differential Operators on $\mathbb{C}S^{2n}$

To conclude our discussion of the Translation Principle we consider invariant differential operators on $\mathbb{C}S^{2n}$; in deriving certain non-standard operators, it will be necessary to translate between singular and non-singular infinitesmal character.

As in four dimensions, the basic pattern of invariant differential operators in infinitesmal character χ_0 involves the bundles Ω^p, with standard operators either the exterior derivative d, or d^+ and d^- : $\Omega^{n-1} \longrightarrow \Omega^n_+$ and Ω^n_- together with the composition d o d^+ : $\Omega^{n-1} \longrightarrow \Omega^{n+1}$ which is the square diagonal map given at the end of the last section :

$$\mathcal{O} \longrightarrow \Omega^1 \longrightarrow \dots \longrightarrow \Omega^{n-1} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \Omega^n_+ \\ \longrightarrow \\ \Omega^n_- \end{matrix} \begin{matrix} \searrow \\ \nearrow \end{matrix} \Omega^{n+1} \longrightarrow \dots \longrightarrow \Omega^{2n} \longrightarrow 0$$

We may ask what non-standard operators, if any, exist. To investigate this, we will translate by tensoring with the module $F = F(\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0})$, which has the following composition series:

$$0 \longrightarrow F(\overset{1}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow F \longrightarrow F' \longrightarrow 0$$

$$0 \longrightarrow F(\overset{-1}{\times}\!-\!\overset{1}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow F' \longrightarrow F(\overset{-1}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow 0$$

The first step is to verify the existence of a non-standard invariant differential operator (higher dimensional wave operator)

$$\square : \mathcal{O}[-n+1] \longrightarrow \mathcal{O}[-n-1]$$

(which is easy to do directly, mimicing the construction in the four dimensional case). Repeated application of the translation principle with $F(\mu-\lambda) = F$ yields a series of non-standard operators

$$\square^{p} : \mathcal{O}[-n+p] \longrightarrow \mathcal{O}[-n-p] \qquad \text{for} \quad 1 \le p \le n-1.$$

The next translation, for $p = n-1$, is from singular to non-singular infinitesmal character : on tensoring $F(\overset{-1}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0})$, $F(\overset{-2n+1}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0})$ with F, we obtain the following composition series:

$$0 \longrightarrow F(\overset{0}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow F_1 \longrightarrow Q_1 \longrightarrow 0$$

$$0 \longrightarrow F(\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow Q_1 \longrightarrow F(\overset{-2}{\times}\!-\!\overset{0}{\bullet}\!- \cdots -\overset{0}{\bullet}\!<^{0}_{0}) \longrightarrow 0$$

and

$$0 \longrightarrow F\left(\overset{-2n+2}{\times}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}\right) \longrightarrow F_1 \longrightarrow Q_1 \longrightarrow 0$$

$$0 \longrightarrow F\left(\overset{-2n}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}\right) \longrightarrow Q_1 \longrightarrow F\left(\overset{-2n}{\times}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}\right) \longrightarrow 0$$

Applying $\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}$, and P_{ξ_0} obtain

$$0 \to \mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \to P_{\xi_0}(\mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}(F_1)) \to \mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \to 0$$

$$\uparrow$$

$$0 \to \mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{-2n}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \to P_{\xi_0}(\mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}(F_2)) \to \mathcal{I}nd^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{-2n}{\times}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \to 0$$

The composition

$$\mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{-2n}{\times}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \qquad (3.7.9)$$

is non-zero, providing a non-standard invariant differential operator

$$\Omega^1 \longrightarrow \Omega^{2n-1}$$

Translate the homomorphism (3.7.9) by tensoring with F again. Four distinct infinitesmal characters result leading to invariant operators

$$\overset{-1}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \longrightarrow \overset{-2n-1}{\times}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}$$

$$\overset{-3}{\times}\!-\!\overset{2}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\ \ \end{smallmatrix} \longrightarrow \overset{-2n-1}{\times}\!-\!\overset{2}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}$$

$$\overset{-2}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix} \longrightarrow \overset{-2n}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\!<\!\begin{smallmatrix}0\\0\end{smallmatrix}$$

$$\overset{-3}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \longrightarrow \overset{-2n+1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0}$$

Translating the homomorphism of the last of these to infinitesmal character ξ_0, obtain

$$0 \to \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \to P_{\xi_0}(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ (F_1)) \to \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \to 0$$

$$\uparrow$$

$$0 \to \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{-2n+1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \to P_{\xi_0}(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ (F_2)) \to \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\ \overset{-2n}{\times}\!-\!\overset{1}{\bullet}\!-\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \to 0$$

and again obtain a non-zero invariant differential operator

$$\Omega^2 = \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} \longrightarrow \overset{-2n+1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\cdots\!-\!\overset{0}{\bullet}\!<\!{}^{\bullet\,0}_{\bullet\,0} = \Omega^{2n-2}$$

Repeating this process generates non-standard invariant differential operators

$$\Omega^p \longrightarrow \Omega^{2n-p} \qquad \text{for} \quad 1 \le p \le n-2$$

Work of [Boe and Collingwood I & II] which generalizes work of [Lepowsky III] shows that these, with the standard operators of (3.7.8) above, constitute a complete list of invariant operators in infinitesmal character ξ_0. Further application of the translation principle will generate all invariant operators between bundles of non-singular infinitesmal character, ofcourse.

3.8 The Algebraic Penrose Transform

In the last part of chapter three, we sketch the ingredients of the *Algebraic Penrose Transform* and give some applications of the results given along the way. For material on the Penrose Transform, the reader may consult [Eastwood I], [Wells], and [Eastwood et al]. and the many references therein.

To begin, we sketch the machinery of the transform. The basic situation is a double fibration of generalized flag manifolds:

$$\begin{array}{ccccc} & & F = G/P\cap P' & & \\ & \mu \swarrow & & \searrow \nu & \\ N = G/P & & & & G/P' = M \end{array}$$

(strictly speaking, we consider related subspaces of this double fibration). Given a homogeneous vector bundle $\mathcal{F}$ on N, the transform interprets the cohomology $H^*(N,\mathcal{F})$ in terms of solutions of differential equations on M. It proceeds in four steps : (i) compare the cohomology $H^*(N,\mathcal{F})$ and $H^*(F,\mu^{-1}\mathcal{F})$ where $\mu^{-1}\mathcal{F}$ is the topological inverse image of $\mathcal{F}$ on F; (ii) compute a resolution of $\mu^{-1}\mathcal{F}$ on F; (iii) use the *hypercohomology spectral sequence* to compute direct images $\nu_*^q(\mu^{-1}\mathcal{F})$ from direct images of the resolution, and hence compute, via the *Leray spectral sequence* $H^*(F,\mu^{-1}\mathcal{F})$, and (iv) interpreting the result as realizing the cohomology $H^*(N,E)$ in terms of cohomology (normally just sections) of homogeneous bundles on M, with operators (normally invariant differential operators) between them. The reader will find the details below, if he is not already familiar with them.

In principle, since the spaces and fibres under consideration are projective algebraic, the entire transform should be realized as an excercise in algebraic geometry. We will not do this but rather supply an algebraic form of (ii) and (iii), the latter (the Borel-Weil-Bott theorem) being already well known in Twistor Theory.

In the final part of Chapter Four, we will show how some, at least, of the transform can be constructed over curved manifolds.

3.8.1 The Bernstein-Gelfand-Gelfand Resolution

Recall the following two exact sequences, for λ dominant for $\mathfrak{g}$, $\Psi(\mathfrak{p})\subset\mathcal{S}$ the defining set of simple roots for $\mathfrak{p}$ and $F(\lambda)$ $\mathfrak{g}$-irreducible.

$$\oplus_{\alpha\,\in\,\mathcal{S}\backslash\Psi(\mathfrak{p})}\ \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\ \sigma_\alpha\cdot\lambda \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\lambda \xrightarrow{\ \epsilon\ } F(\lambda) \longrightarrow 0 \qquad (3.8.1)$$

$$0 \longrightarrow F(-\lambda) \longrightarrow \mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \oplus_{\alpha\,\in\,\mathcal{S}\backslash\Psi(\mathfrak{p})}\ \mathcal{O}(\sigma_\alpha\cdot\lambda_{\mathfrak{p}}) \qquad (3.8.2)$$

on G/P. We have encountered these, in slightly less general form, in section (2.5) and Lemma (3.6) A. Indeed that lemma provides a relative example of the acorn of a resolution in prospect :

$$\oplus_{\alpha \in \Psi(\mathfrak{p})} \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\, \sigma_\alpha \cdot \lambda \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{b}}\lambda \longrightarrow \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}\, \lambda_{\mathfrak{p}} \longrightarrow 0 \qquad (3.8.3)$$

for $\lambda_{\mathfrak{p}}$ $\mathfrak{p}$-dominant, which is dual to

$$0 \longrightarrow \mu^{-1}\, \mathcal{O}(\lambda_{\mathfrak{p}}) \longrightarrow \mathcal{O}(\lambda_{\mathfrak{b}}) \longrightarrow \oplus_{\alpha \in \Psi(\mathfrak{p})} \mathcal{O}(\sigma_\alpha \cdot \lambda_{\mathfrak{b}}) \qquad (3.8.4)$$

on $G/B \xrightarrow{\mu} G/P$. (note that "dualizing" (3.8.1), (3.8.3) porves only the formal exactness of (3.8.2) and (3.8.4). Convergence questions are settled in [Kempff]).

The fibres of a projection $G/P \cap P' \longrightarrow G/P$ being themselves generalized flag manifolds, it is sufficient to complete the resolution (3.8.1), hence (3.8.2), the relative case then following easily. This has been done by [Lepowsky I], following earlier work in [Bernstein et al I] in the Borel case. The result may be stated as follows :

Theorem A (Bernstein-Gelfand-Gelfand Resolution)

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra. Denote $W^1(\mathfrak{p})$ as in section (2.6) and let λ be a dominant weight for $\mathfrak{g}$, with $F(\lambda)$ the corresponding irreducible representation of $\mathfrak{g}$. Define

$$C_j = \oplus_{w \in W^1(\mathfrak{p})\ ;\ l(w) = j}\ \mathcal{I}nd\,{}^{\mathfrak{g}}_{\mathfrak{p}}(w \cdot \lambda)$$

where $j \in \{0,1,\dots,\dim_{\mathbb{C}} G/P\}$. Then there exist maps

$$d_j : C_j \longrightarrow C_{j+1} \qquad \text{with} \quad d_{j+1} \circ d_j = 0$$

composed of (± 1) multiples of the induced standard homomorphisms, so that the following is a resolution:

$$C_* \xrightarrow{\ \epsilon\ } F(\lambda) \longrightarrow 0$$

with $\epsilon : \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, \lambda_{\mathfrak{p}} \longrightarrow F(\lambda)$ the projection onto the unique irreducible quotient. □

<u>Corollary A</u>

Let $\mathfrak{g}$, $\mathfrak{p}$, λ be as in Theorem A. Set

$$\mathcal{R}_j = \oplus_{w \in W^1(\mathfrak{p})\ ;\ l(w)=j}\ \mathcal{O}(w\cdot\lambda_{\mathfrak{p}})$$

with $j \in \{0,1,\dots,\dim_{\mathbb{C}} G/P\}$. Then

$$0 \longrightarrow F(-\lambda) \xrightarrow{\ \epsilon^*\ } \mathcal{R}_* \tag{3.8.5}$$

is a resolution. □

<u>Examples A</u>

(i) Take $\mathfrak{g} = \bullet\!-\!\bullet\!-\!\bullet$ and $\mathfrak{p} = \bullet\!-\!\times\!-\!\bullet$; applying lemma A with $\lambda = 0$, so that $F(\lambda) = \mathbb{C}$, yields the de Rham resolution :

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \longrightarrow \Omega^1 \longrightarrow \begin{matrix}\Omega^2_+\\ \oplus \\ \Omega^2_-\end{matrix} \longrightarrow \Omega^3 \longrightarrow \Omega^4 \longrightarrow 0$$

This typical of the situation in which the tangent bundle $\tau(G/P)$ is irreducible (table 4.3 indicates which spaces these are). For other spaces, the resolution (3.8.5) is very much more efficient. For example on Ambitwistor space $\times\!\!-\!\!\bullet\!\!-\!\!\times$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \longrightarrow \begin{matrix}\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\times}\\ \oplus \\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}\end{matrix} \longrightarrow \begin{matrix}\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{-3}{\times}\\ \oplus \\ \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{-2}{\times}\\ \oplus \\ \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}\end{matrix} \longrightarrow \begin{matrix}\overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\times}\\ \oplus \\ \overset{-3}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{-3}{\times}\\ \oplus \\ \overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{-4}{\times}\end{matrix} \longrightarrow \begin{matrix}\overset{-4}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-2}{\times}\\ \oplus \\ \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-4}{\times}\end{matrix} \longrightarrow \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{-3}{\times} \longrightarrow 0$$

is a resolution with successive bundle dimensions 1,4,5,5,4, 1, compared with the de Rham resolution's 1,5,10,10,5,1.

ii) There is no need to restrict to the trivial representation of g. Consider for example $\lambda = \overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$. Then, as indicated before, $F(-\lambda)$ is traditionally denoted T^α and there is a resolution:

$$0 \longrightarrow T^\alpha \longrightarrow \underset{\mathcal{O}^{A}}{\underset{\|}{\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}}} \longrightarrow \underset{\mathcal{O}^{A'}_{(AB)}}{\underset{\|}{\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}}} \longrightarrow \begin{matrix}\overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}\\ \oplus \\ \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{3}{\bullet}\end{matrix} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{1}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{0}{\bullet} \longrightarrow 0 \,. \qquad (3.8.6)$$

with $\omega^A \in \mathcal{O}^A \longrightarrow \nabla^{A'}_{(A}\omega_{B)} \in \mathcal{O}^{A'}_{(AB)}$. An element of T^α (a "twistor") gives rise to a spinor field ω^A (a "twistor field") satisfying the twistor equation $\nabla^{A'}_{(A}\omega_{B)} = 0$. Clearly, a general choice of λ will give rise to spinor fields on M^4 representing elements of any irreducible tensor power of T^α.

Taking $\lambda = \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}$, for instance, gives $sl(4,\mathbb{C}) \cong$ conformal killing vector fields on M^4.

Another important aspect of (3.6.8) concerns the central square :

$$\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \begin{matrix} \overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \\ \oplus \\ \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{3}{\bullet} \end{matrix} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet}$$

Exactness near the central square implies that

$$\mathrm{Ker}\ \overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet} \quad \cong \quad \frac{\mathrm{Ker}\ \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{3}{\bullet}}{\mathrm{Im}\ \overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}}$$

This is the usual identification [Eastwood et al] of zero rest mass fields (elements of the left hand space) and potentials for these fields, modulo gauge (elements of the right hand space). Both of these spaces are products of the Penrose transform, on PT^4 and PT^{4*} respectively, and the isomorphism constitutes the *Twistor Transform* (cf e.g. [Singer] and [Eastwood and Pilato]). This construction motivated Buchdahl when he introduced these sequences (for general λ on M^4) which he called the *Generalized de Rham sequences* [Buchdahl N.P]. It is a construction which the Bernstein-Gelfand-Gelfand resolution generalizes to higher dimensions.

iii) As observed at the end of section (2.7), elements of $T_{[\alpha\beta]} = F(\overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet})$ correspond, if simple (or null, thinking of

$T_{[\alpha\beta]}$ as the self-representation of so(6,$\mathbb{C}$)) to choices of a point at "infinity" together with a conformal scale. The sections of $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}$ and $\overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet}$, ι and f say, corresponding to such elements under

$$0 \longrightarrow T_{[\alpha\beta]} \longrightarrow \overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet}$$

$$0 \longrightarrow T_{[\alpha\beta]} \longrightarrow \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times} \longrightarrow \begin{array}{c}\overset{0}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{-2}{\times}\\ \oplus \\ \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\times}\end{array}$$

obey the differential equations

$$\nabla_{(A}^{(A'}\nabla_{B)}^{B')} f = 0 \qquad \text{and} \qquad d^{+}\iota = d^{-}\iota = 0$$

where d^{+} and d^{-} correspond to the exterior differential d composed with projection onto the factors $\overset{0}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{-2}{\times} \oplus \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\times}$ of $\Omega^1 \otimes \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}$. The equation for f, in its curved form, is the starting point of the Le Brun construction to be outlined below. □

Theorem A above generalizes to a relative version in the obvious manner:

<u>Corollary B</u>

Let $\mathfrak{g}$ be a complex semi-simple Lie Algebra, $\mathfrak{p} \subset \mathfrak{p}'$ both parabolic subalgebras (sharing a common Borel subalgebra of $\mathfrak{g}$). Let G, P, P' be the corresponding groups (taken sufficiently connected). Then $\hat{\mathfrak{p}} = \mathfrak{p} \cap \mathfrak{l}^{s}_{\mathfrak{p}'}$ is parabolic in $\mathfrak{l}^{s}_{\mathfrak{p}'}$, where $\mathfrak{l}^{s}_{\mathfrak{p}'}$ is the semi-simple part of the reductive part of a Levi decomposition of $\mathfrak{p}'$. Define $W^1(\hat{\mathfrak{p}})$ as in section (2.6) and set

$$\hat{\mathcal{R}}_i = \oplus_{w \in W^1(\hat{\mathfrak{p}})\ ;\ l(w) = i}\ \mathcal{O}(w \cdot \lambda_{\mathfrak{p}}) .$$

Then

$$0 \longrightarrow \mu^{-1}\mathcal{O}(\lambda_{\mathfrak{p}'}) \longrightarrow \hat{\mathcal{R}}_*$$

is a resolution, for $\mu: G/P \longrightarrow G/P'$. □

In this version, the Bernstein-Gelfand-Gelfand resolution provides step (ii) of the algebraic Penrose transform. The first example below is an old favourite:

<u>Example B</u>

i) Let $\mathfrak{p} = \times\!-\!\bullet\!-\!\bullet$, $\mathfrak{p}' = \bullet\!-\!\times\!-\!\bullet$, $\mathfrak{p} \cap \mathfrak{p}' = \times\!-\!\times\!-\!\bullet$ and apply the corollary, to obtain a resolution up the fibres of μ in :

$$\begin{array}{ccccc} & & F = \times\!-\!\times\!-\!\bullet & & \\ & \mu \swarrow & & \searrow \nu & \\ PT^4 = \times\!-\!\bullet\!-\!\bullet & & & & M^4 = \bullet\!-\!\times\!-\!\bullet \end{array}$$

For instance, if $\lambda = \overset{n}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$, so $\mathcal{O}(\lambda_{\mathfrak{p}}) = \mathcal{O}(n)$, then the resolution is

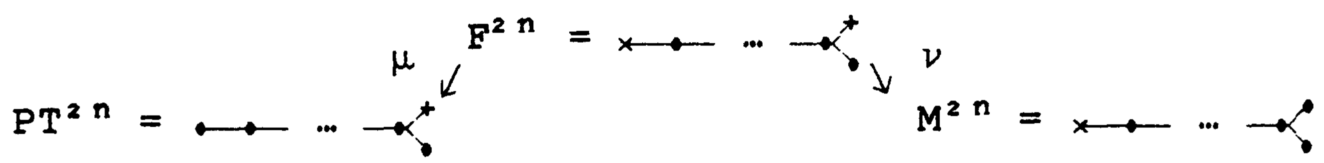
$$0 \longrightarrow \mu^{-1}\ \overset{n}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \longrightarrow \overset{n}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{n+1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{n+2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet} \longrightarrow 0 \qquad (3.8.7)$$

On the other hand, take λ = ; obtain

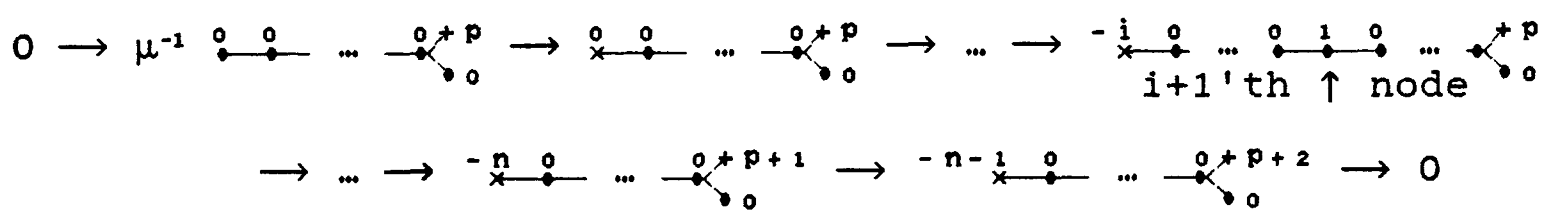
(3.8.8)

(the reader may wish to compare this with (3.7.6).

ii) We may begin the Penrose Transform on higher dimensional twistor spaces:



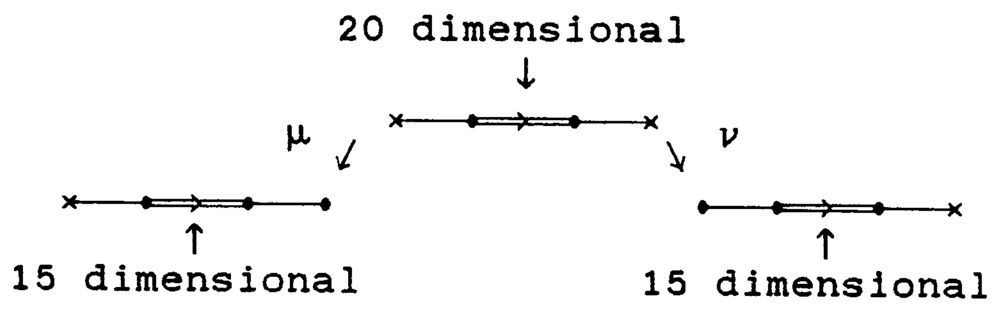
The analogue of (3.8.7) is

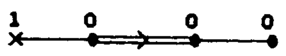


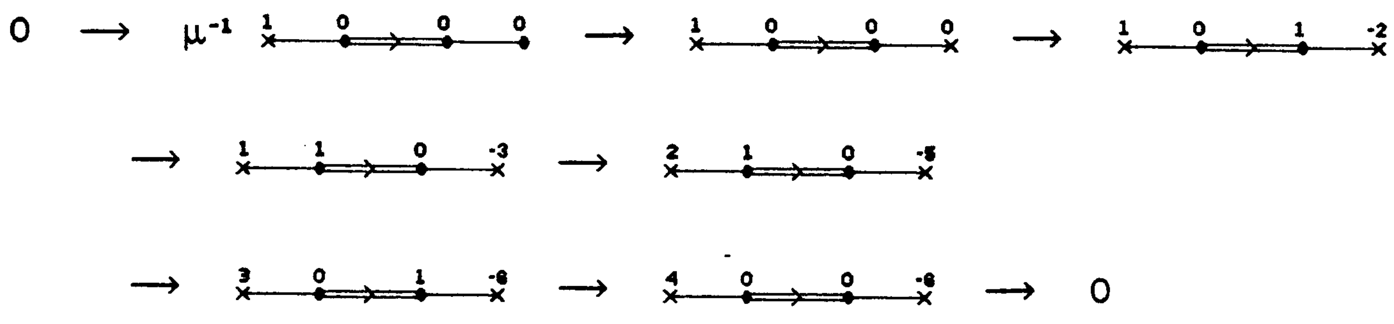
(3.8.9)

A similar construction works in odd dimensions.

iii) As a curiosity, we may begin a Penrose transform involving the exceptional group F_4.



$\lambda =$  gives :



(the μ fibre  is, topologically, $\mathbb{C}P_5$). □

3.8.2 The Borel-Weil-Bott theorem, Hypercohomology and the completion of the Penrose transform.

To compute direct images of the resolution just taken, we employ the celebrated Borel-Weil-Bott theorem ([Bott], [Kostant]). This is ofcourse no innovation in Twistor Theory. For an algebraic proof of Kostant's version of the theorem, see [Vogan].

Theorem B (Borel-Weil-Bott)

Let $\mathfrak{g}$ be a complex semi-simple Lie Algebra with $\mathfrak{p} \subset \mathfrak{p}'$ both parabolic (with a common Borel subalgebra). Take corresponding Lie Groups (sufficiently connected) with $G/P \xrightarrow{\nu} G/P'$. Let $\lambda_{\mathfrak{p}}$ be dominant for $\mathfrak{p}$; define $\hat{\mathfrak{p}}$ as in

Corollary B above. Then :

$$\nu_*^i\, \mathcal{O}(\lambda_{\mathfrak{p}}) = 0 \qquad \forall\ i$$

unless $\exists\ w \in W^1(\hat{\mathfrak{p}})$ of length q such that $w\cdot\lambda_{\mathfrak{p}}$ is $\mathfrak{p}'$-dominant *when*

$$\begin{aligned} \nu_*^i\, \mathcal{O}(\lambda_{\mathfrak{p}}) &= \mathcal{O}(\ (w\cdot\lambda)_{\mathfrak{p}'}\ ) && i = q \\ &= 0 && \text{otherwise} \end{aligned}$$

(ofcourse, at most one such w exists). □

<u>Examples C</u>

i) Consider the resolution (3.8.7). There are three cases to consider, namely $p < -2$, $p = -2$ and $p > -2$. For instance (with other direct images vanishing) :

$p < -2$

$$\nu_*^1\ \overset{p}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} = \overset{-p-2}{\bullet}\!-\!\underset{p+1}{\times}\!-\!\overset{0}{\bullet} \cong \mathcal{O}_{(A_1^! A_2^! \ldots A_{-p-2}^!)}[-1]$$

$$\nu_*^1\ \overset{p+1}{\times}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} = \overset{-p-3}{\bullet}\!-\!\overset{p}{\times}\!-\!\overset{1}{\bullet} \cong \mathcal{O}_{(A_1^! A_2^! \ldots A_{-p-3}^!)A}[-2]$$

$$\nu_*^1\ \overset{p+2}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet} = 0 \text{ if } p = -3 \text{ and } \overset{-p-4}{\bullet}\!-\!\overset{p}{\times}\!-\!\overset{0}{\bullet} \text{ for } p \le -4.$$

ii) Similar formuli hold in higher dimensions. We will summarise some results below, but, for example, if $p \le -4$:

$$\nu_*^3\ \overset{0}{\times}\!-\!\overset{0}{\bullet}\!<\!{}^{\overset{p}{\times}}_{\underset{0}{\bullet}} = \overset{p+2}{\times}\!-\!\overset{0}{\bullet}\!<\!{}^{\overset{0}{\bullet}}_{\underset{-p-4}{\bullet}}$$

□

The Penrose Transform is completed by using the direct images of elements of the resolution to compute direct images $\nu^q_* \mu^{-1} E$ and using these to compute $H^*(N,\mu^{-1}E)$. The first of these tasks is accomplished using the hypercohomology spectral sequence :

<u>Theorem C</u> (Hypercohomology spectral sequence)
Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{R}_*$ be a resolution of $\mathcal{F}$ on G/P, with G, P etc. as in Theorem B. Then then is a spectral sequence

$$E_1^{p,q} = \nu^q_* \mathcal{R}^p \longrightarrow E_\infty^{p+q} = \nu^{p+q}_* \mathcal{F}. \qquad \square$$

The second is accomplished by means of

<u>Theorem D</u> (Leray spectral sequence)
With ν, G, P, $\mathcal{F}$ as in Theorem B, let $U_{P'} \subset G/P'$ and set $U_P = \nu^{-1} U_{P'}$. Then there is a spectral sequence

$$E_2^{p,q} = H^p(U_{P'}, \nu^q_* \mathcal{F}) \longrightarrow E_\infty^{p,q} = H^{p+q}(U_P, \mathcal{F}) \qquad \square$$

(cf [Godement]) In applications, $U_{P'}$ is usually affine, so the sequence collapses to the series of isomorphisms

$$H^q(U_P, \mathcal{F}) \cong \Gamma(U_{P'}, \nu^q_*)$$

These last two results are standard in Twistor Theory. We employ them without further ado.

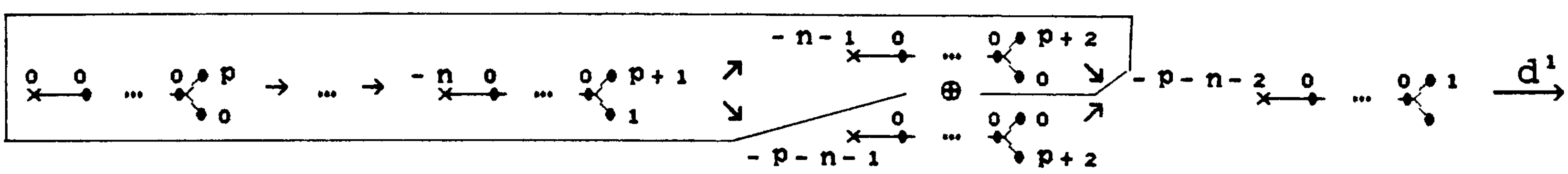
3.8.3 Higher dimensional Twistors

We complete example B(ii) above. Taking direct images of (3.8.9) obtains several distinct answers, based on p and n.

Case Regular $p \leq -2n$ or $p \geq 0$

When either inequality holds, the weights $\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots -\!\bullet\!\!<^{p}_{0}$ (hence the weights of (3.8.9) are of non-singular infinitesmal character. For $p \leq -2n$, the unique dominant weight conjugate under the affine action of $W(g)$ to these weights is either $\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots -\!\bullet\!\!<^{-p-2n}_{0}$ (n odd) or $\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots -\!\bullet\!\!<^{0}_{-p-2n}$ (n even). One expects therefore to find the direct image of (3.8.9) in the Bernstein-Gelfand-Gelfand resolutions (on M^{2n}) of these weights ($p \leq -2n$) or $\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots -\!\bullet\!\!<^{p}_{0}$ ($p \geq 0$); direct calculation verifies this fact: below we give the relevant Bernstein-Gelfand-Gelfand resolutions, encircling indicating where the direct images fall.

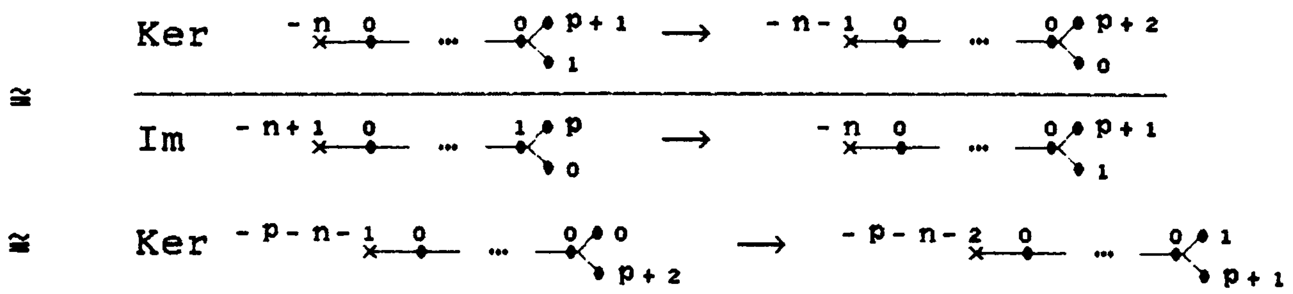
$p \geq 0$: ν^{0}_{*} (3.8.9) only non-zero



(3.8.10)

From this, the hypercohomology spectral sequence, and the Leray spectral sequence, if $U_M \subset M^4$ is affine, $U_F = \nu^{-1} U_M$ and $U_T = \mu U_F$, it follows that

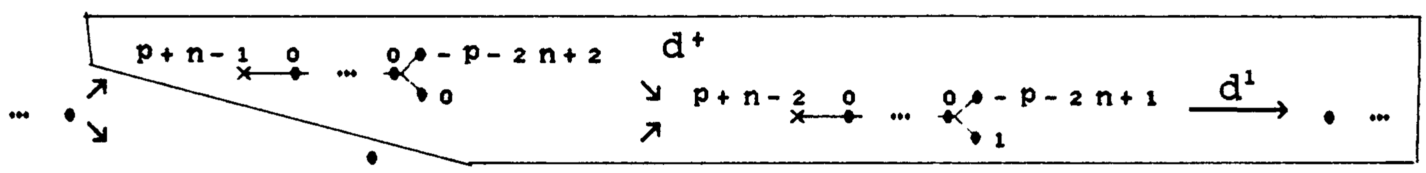
$$H^{n-1}(U_T, \overset{0}{\bullet}\!-\!\overset{0}{\bullet} \cdots -\!\bullet\!\!<^{p}_{0}) \cong H^{n-1}(U_T, \mathcal{O}(p))$$



on U_M . This last may be interpretted as the higher dimensional analogues of the *negative helicity zero rest mass Fields* on M^4. (cf [Penrose and Rindler I & II])

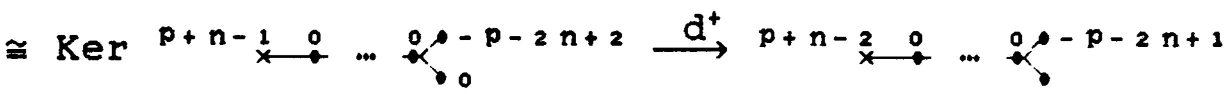
<u>$p \leq -2n$: $\nu_*^{(\frac{n}{2})}$ (3.8.9) only non-zero</u>

n even



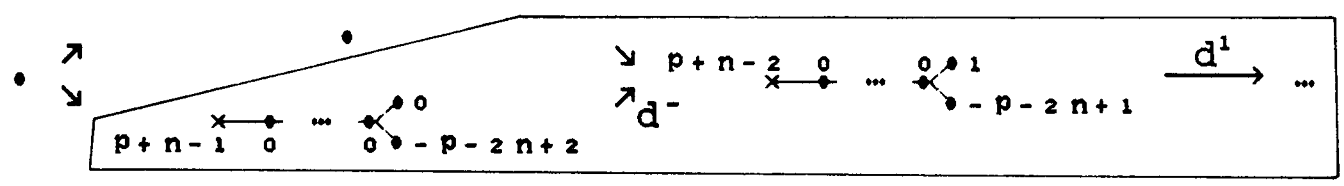
(3.8.11)

so $H^{(\frac{n}{2})}(U_T, \mathcal{O}(p))$



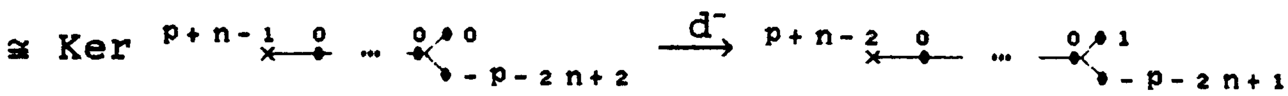
which may again be interpretted as higher dimensional analogues of positive helicity zero rest mass fields.

n odd



(3.8.12)

so $H^{(\frac{n}{2})}(U_T, \mathcal{O}(p))$



which are the negative helicity zero rest mass fields again.

Case Irregular

Two pairs of irregular case exist : the first of pair one is $p = -2n+2$. For this, as the reader may verify, only two terms of the resolution (3.8.9) have non-zero direct images :

$$\mu_*^{\binom{n}{2}}\ \overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!-\ \cdots\ -\overset{0}{\bullet}\!\!<^{-2n+2}_{0} \;=\; \overset{1-n}{\times}\!\!-\!\!\overset{0}{\bullet}\!-\ \cdots\ -\overset{0}{\bullet}\!\!<^{0}_{0} \;\cong\; \mathcal{O}[-n+1]$$

(first term of 3.8.9)

$$\mu_*^{\binom{n}{2}}\ \overset{-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}\ \cdots\ -\overset{0}{\bullet}\!\!<^{-2n+2}_{0} \;=\; \overset{-n-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!-\ \cdots\ -\overset{0}{\bullet}\!\!<^{0}_{0} \;\cong\; \mathcal{O}[-n-1]$$

(second term of 3.8.9)

The hypercohomology spectral sequence has $E_1^{p,q} = E_2^{p,q} = E_\infty^{p,q}$ and we obtain:

$$H^{\binom{n}{2}}(U_T,\mathcal{O}(2-2n)) \cong \text{Ker}\ \Box : \mathcal{O}[-n+1] \longrightarrow \mathcal{O}[-n-1]$$

where $\Box$ is the (higher dimensional) wave operator.

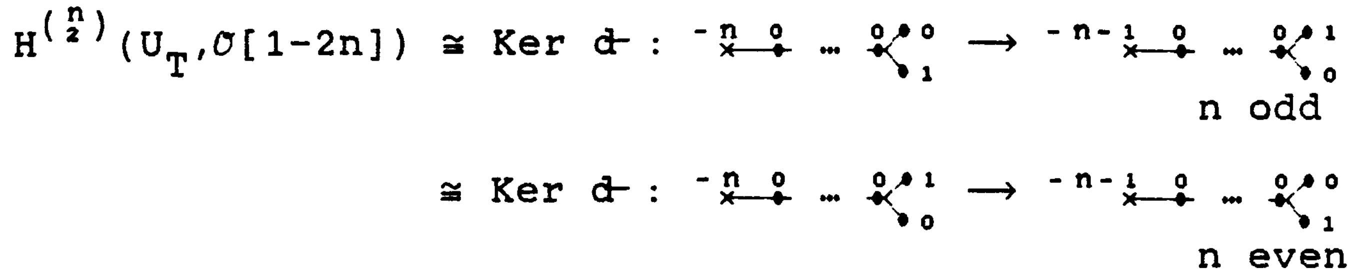
The second of pair one, for $p = -2$ gives

$$H^{n-1}(U_T,\mathcal{O}(-2)) \cong \text{Ker}\ \Box : \mathcal{O}[-n+1] \longrightarrow \mathcal{O}[-n-1].$$

The second pair of the singular case gives higher dimensional massless neutrino fields: we have for $p = -1$

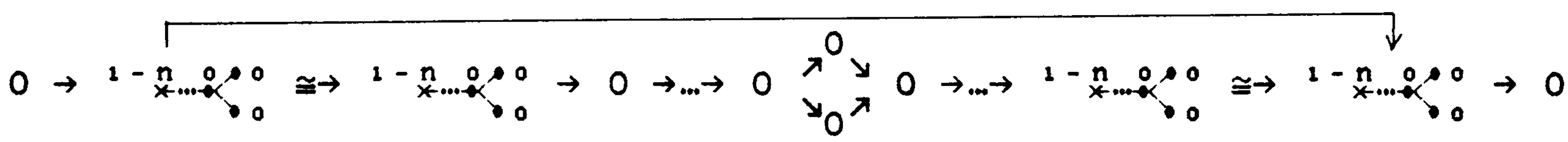
$$H^{n-1}(U_T,\mathcal{O}[-1]) \cong \text{Ker}\ \not{d} : \overset{-n}{\times}\!\!-\!\!\overset{0}{\bullet}\!-\ \cdots\ -\overset{0}{\bullet}\!\!<^{0}_{1} \longrightarrow \overset{-n-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!-\ \cdots\ -\overset{0}{\bullet}\!\!<^{1}_{0}$$

whilst for $p = 1-2n$ we have :



The singular direct images may be realized as subsequences of *singular* Bernstein-Gelfand-Gelfand sequences (which are not resolutions but are exact) These contain all bundles on M^{2n} of the given singular character :

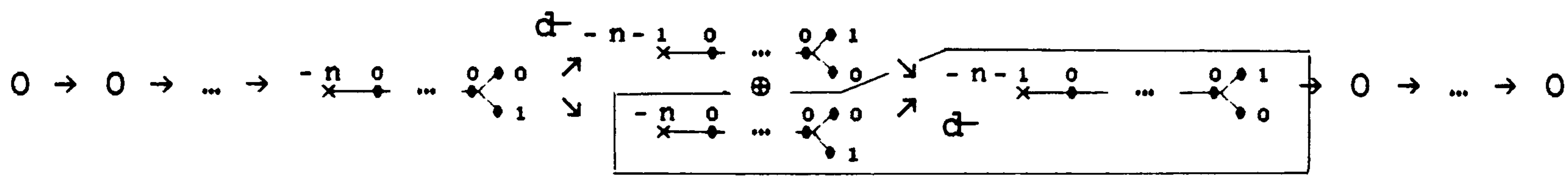
Pair One



(3.8.13)

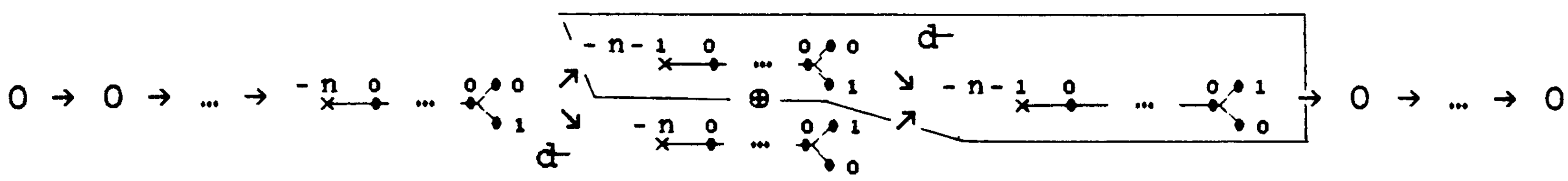
Pair Two

Either



(3.8.14¹)

or



(3.8.14²)

Note

The reader should note the differences which occur in the above depending on the parity of n - if n is odd, it is clear that twistor transforms will operate from PT^{2n} to itself (cf [Baston & Eastwood].)

3.8.4 The Fierz-Pauli conditions

As an illuminating by product of the considerations of this chapter, we mention the Fierz-Pauli conditions. These arise when one modifies the zero rest mass equations by coupling them to a background electromagnetic field : strictly, one is attempting to replace the ambient connection ($\nabla_{AA'}$ for M^4) by one associated to an electromagnetic field. More informally, one chooses a "potential" for the field, that is a $\Phi \in \Omega^1$. Let $\psi \in \overset{p+n-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!- \cdots -\!\overset{0}{\bullet}\!\!<^{\bullet\, -p-2n+2}_{\bullet\, 0}$ (for $p \leq -2n+2$ and working in even dimensions for definiteness, though a similar construction will work in odd dimensions). Denote by $\tilde{\Phi}\cdot\psi$ the image of $\Phi \otimes \psi$ under the projection

$$\Omega^1 \otimes \overset{p+n-1}{\times}\!\!-\!\!\overset{0}{\bullet}\!- \cdots -\!\overset{0}{\bullet}\!\!<^{\bullet\, -p-2n+2}_{\bullet\, 0} \longrightarrow \overset{p+n-2}{\times}\!\!-\!\!\overset{0}{\bullet}\!- \cdots -\!\overset{0}{\bullet}\!\!<^{\bullet\, -p-2n+1}_{\bullet\, 1}$$

Then seek solutions of equations such as

$$(d^+ + \tilde{\Phi}\cdot)\,\psi = 0 \qquad (p\leq -2n+1) \qquad (3.8.15)$$

$$(\Box + \Phi\cdot\nabla + (\nabla\cdot\Phi) + \Phi\cdot\Phi)\,\psi = 0 \qquad (p=-2n+2) \qquad (3.8.16^1)$$

$$(\Box + \Phi\cdot\Phi)\,\psi = 0 \qquad (p=-2n+2) \qquad (3.8.16^2)$$

where in the last two expressions "·" means contraction in some local flat metric. But notice that the exactness of (3.8.11) imposes a consistency condition on (3.8.15) for

$p \le -2n$; that is, *in the non-singular cases* :

$$d^1\ (\Phi \cdot \psi) \ = \ 0$$

These are the conditions of [Fierz & Pauli], which *must* be satisfied if (3.8.15) is to admit solutions.

In the *singular cases*, there are *no* immediately apparent consistency conditions (there is no "d^1" in (3.8.13), ($3.8.14^1$) or ($3.8.14^2$)). The work of [Spencer] will show that indeed these equations are integrable.

3.8.5 Deformations of Ambitwistor Space and the vanishing of the linearized Bach Tensor

As a final example of the machinery of this chapter, we consider infinitesmal deformations of Ambitwistor space, $\times\!\!-\!\!\bullet\!\!-\!\!\times$ = A, which preserve its contact structure [Le Brun I].

The contact structure of A is the projection to $\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}$ of the composition series for the tangent bundle τA :

$$0 \longrightarrow \begin{matrix} \overset{1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times} \\ \oplus \\ \overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{1}{\times} \end{matrix} \longrightarrow \tau A \longrightarrow \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times} \longrightarrow 0$$

which, recall, is split by an invariant differential operator $\mathcal{D}_H$, say. Deformations preserving this structure correspond to elements of $H^1(U_A, \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times})$ under the map $H^1(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\times}) \longrightarrow H^1(\tau A)$

induced by $\mathcal{D}_H$ (for U_A a neighbourhood of a quadric in A corresponding to a point of M^4). [Mason].

Using the double fibration $\times\!-\!\bullet\!-\!\times \leftarrow \times\!-\!\times\!-\!\times \rightarrow \bullet\!-\!\times\!-\!\bullet$, one obtains an exact sequence on M^4 :

$$\begin{array}{ccccccccc} & & & & & & \text{deformations} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} & \longrightarrow & \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} & \longrightarrow & H^1(U_A, \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \text{vectors} & & \text{trace-free metric perturbations} & & & & \end{array} \qquad (3.8.17)$$

(3.8.17) fits into the Bernstein-Gelfand-Gelfand resolution:

$$0 \rightarrow sl(4,\mathbb{C}) \longrightarrow \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \begin{array}{c} \overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \\ \oplus \\ \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet} \end{array} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{1}{\bullet} \rightarrow 0 \qquad (3.8.18)$$

(cf (3.7.7)). Exactness implies that an element of $H^1(U_A, \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet})$ is a pair $\Psi_{A'B'C'D'} \oplus \tilde{\Psi}_{ABCD}$ in Ker $\overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{2}{\bullet}$, which should be thought of as perturbations of the Weyl curvature in the associated infinitesmal deformation of M^4. The element $\nabla^{C'}_{(A}\nabla^{D'}_{B)}\Psi_{A'B'C'D'} = B_{A'B'AB} \in \overset{2}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{2}{\bullet}$ is the perturbation of the Bach Tensor. If this vanishes, then the exactness of (3.8.18) implies that there exist h^+_{ab} and h^-_{ab} in $\overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}$ with

$$\Psi_{A'B'C'D'} = \nabla^A_{(A'}\nabla^B_{B'}h^+_{C'D')AB} \quad ; \quad \tilde{\Psi}_{ABCD} = \nabla^{A'}_{(A}\nabla^{B'}_{B}h^-_{CD)A'B'}$$

so that the perturbation splits into self- and and anti-self- dual parts.

We may be more explicit about this, since (3.8.18) contains information about the Penrose transforms of $H^1(U_{PT}, \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet})$ and $H^1(U_{PT}{}^*, \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times})$. Here $\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$ and $\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}$ are the tangent bundles of PT and PT^* respectively and U_{PT} and $U_{PT}{}^*$ correspond to U_A under the natural projections. Elements of these groups correspond to infinitesmal deformations of U_{PT} and $U_{PT}{}^*$:

$H^1(U_{PT}, \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}) \cong \text{Ker } \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \to \overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}$ modulo $\text{Im } \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \to \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}$

$H^1(U_{PT}, \overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\times}) \cong \text{Ker } \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \to \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet}$ modulo $\text{Im } \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet} \to \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}$

Thus a contact preserving infinitesmal deformation of Ambitwistor space gives a perturbation of Minkowski space with vanishing Bach Tensor iff it is the restriction to $A \hookrightarrow PT \times PT^*$ of a pair of infinitesmal perturbations of PT and PT^* (cf [Baston and Mason]).

4 : Invariant Operators on Curved Spaces

4.1 Introduction

The aim of this Chapter is to extend the calculations of the previous chapter to manifolds X which are not homogeneous. We will see that given a tangent bundle structure of a certain type, such as a conformal structure, it is possible in an unique and manifestly invariant fashion to produce analogues of the operators in the previous chapter.

To do this, in section (4.2), we first make a study of the structure needed for the constructions of the previous chapter; it is sufficient to have a P-principle bundle $\mathcal{G}$ over X together with an identification of the tangent spaces $\tau_x\mathcal{G}$ for $x \in X$ with $\mathfrak{g}$, which gives an osculation of $\mathcal{G}$ by G along the fibres to X. This structure is called a *Cartan connection* ω. Given such a structure, we deduce a distinguished family of differential operators on induced bundles over X.

In section (4.3) we consider those generalized flag manifolds whose tangent bundles are irreducible. We model a class of manifolds - the *locally |1|-graded manifolds* - on them. Following work of [Tanaka] and [Ochiai], inspired by work of [Cartan I & II], we show that, given the satisfaction of a certain condition on the *Spencer cohomology* of $\mathfrak{g}$, there exists an uniquely defined $\tilde{P}$-principal bundle over X (where P covers $\tilde{P}$), $\mathcal{G}$. We compute the cohomology in section (4.4). Section (4.5) shows, again assuming some cohomological conditions, that an unique *normal* Cartan connection ω exists on $\mathcal{G}$. We give the construction of ω in full and use this in (4.6) to show how any one of the distinguished operators of (4.2) may be computed in terms of any torsion free connection which preserves the tangent bundle structure of X. The resulting operators are manifestly invariants of the structure of X, independent of any choice made in their construction. Accordingly, we call them *invariant*.

To illustrate the theory, we show in (4.7) how to compute differential operators which are invariants of conformal four manifolds. We compute, for example, the modified wave operator ($\Box$ + R/6) and give some general techniques to ease calculation.

Finally, we give a curved version of the Penrose Transform.

4.2 Cartan Connections and a Class of Differential Operators

The aim of this section is to understand precisely the structures needed over some space X (for the moment assumed merely a differentiable complex manifold) in order to carry out the construction of differential operators of the last chapter. G will be, as before, a complex semi-simple Lie Group, and P a closed connected subgroup (not necessarily parabolic).

To begin with, simply to be able to define the homogeneous bundles $\mathcal{E}$, $\mathcal{F}$ on G/P, we need the P-principle bundle structure $G \longrightarrow G/P$. Next, to identify the Jet bundles as homogeneous bundles, it is necessary to identify $\mathfrak{g}$ with the Lie Algebra of left invariant vector fields on G; formally, this is accomplished by the *Maurer-Cartan form* ω, which is a distinguished section of $\Omega^1_G \otimes \mathfrak{g}$ - for each $g \in G$, ω_g is an isomorphism of $\tau_g G$ with $\mathfrak{g}$. From this, $\mathfrak{U}(\mathfrak{g})$ is recognised as the algebra of left invariant differential operators on G, admitting a $\mathfrak{U}(\mathfrak{p})$-bimodule structure, whence the homogeneous structure of $J^\infty(\mathcal{E})$.

The idea is to replace G, G/P by a P-principle bundle $\mathcal{G} \longrightarrow X$, naturally defined in terms of some structure on X, equipped with a distinguished section of $\Omega^1_{\mathcal{G}} \otimes \mathfrak{g}$ - a *Cartan Connection* ω - designed to mimic the properties of the Maurer-Cartan form, as follows :

i) $\omega \in \Gamma(\Omega^1_{\mathcal{G}} \otimes g)$

ii) $\omega_g : \tau\mathcal{G}_g \longrightarrow g$ is an isomorphism $\forall\, g \in \mathcal{G}$.

iii) If $v \in \mathfrak{p}$, v^* the corresponding fibrewise vector fields, then $v^* \lrcorner \omega = v$.

iv) For $p \in P$, and R_p the right translation by p on $\mathcal{G}$, $R_p^* \, \omega = Ad(p^{-1}) \, \omega$.

(cf [Kobayashi]). ω^{-1} provides a $\mathbb{C}$-module map

$$\omega^{-1} : g \longrightarrow \Gamma(\tau\mathcal{G}).$$

As a $\mathbb{C}$-module (not, ofcourse, as an algebra), $\mathfrak{A}(g) \cong Gr\, \mathfrak{A}(g) \cong \mathfrak{S}\,(g)$ (the symmetric algebra on g), by the Poincare-Birkhoff-Witt theorem [Humphreys]. Thus, given a homogeneous element $u_1 \cdot u_2 \dots u_p$ of $\mathfrak{A}(g)$, we may form the differential operator on $\mathcal{G}$ which is the symmetric composition of the vector fields $u_i^* = \omega^{-1}(u_i)$, thus extending the $\mathbb{C}$-module map :

$$\omega^{-1} : \mathfrak{A}(g) \longrightarrow DO(\mathcal{G}) = \text{Differential Operators on } \mathcal{G}$$

($\omega^{-1}(\mathfrak{A}(g))$ spans $DO(\mathcal{G})$ as a left $\mathcal{O}_{\mathcal{G}}$-module). Then (iii) implies that ω^{-1} restricted to ($\mathfrak{p}$), $\mathfrak{A}(\mathfrak{p})$ is a homomorphism of (Lie) algebras. By (iv), if $\pounds_{v^*}$ denotes the Lie derivative with respect to a vector field v^* and if for $v \in \mathfrak{p}$, $u \in g$, we define v^*, u^* by $v^* \lrcorner \omega = v$, $u^* \lrcorner \omega = u$, then

$$[v^*,u^*] \lrcorner\, \omega = (\pounds_{v^*} u^*)\lrcorner\, \omega = \pounds_{v^*}(u^*\lrcorner\omega) - u^*\lrcorner(\pounds_{v^*} \omega)$$
$$= ad(v)\cdot(u^*\lrcorner\omega)$$
$$= [v,u]$$

so that ω^{-1} is a is a homomorphism of $\mathfrak{A}(\mathfrak{p})$-bimodules, if we identify $\mathfrak{A}(\mathfrak{p})$ with $\omega^{-1}(\mathfrak{A}(\mathfrak{p}))$.

ω^{-1} is not generally a morphism of (Lie) algebras. The extent of its failure is its *curvature* Ω defined by

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]$$

where, if θ, κ are $\mathfrak{g}$-valued one forms on $\mathcal{G}$ then

$$[\theta,\kappa](X,Y) = [X\lrcorner\theta, Y\lrcorner\kappa] - [Y\lrcorner\theta, X\lrcorner\kappa]$$

(for vectors X,Y). Thus, if $u^*\lrcorner\omega$, $v^*\lrcorner\omega$ are as above, then

$$\Omega(u^*,v^*) = [u,v] - [u^*,v^*] \lrcorner\, \omega$$

Notice, however, that in (3.2) it was not necessary to use the algebra structure of $\mathfrak{A}(\mathfrak{g})$ at all, so we have

<u>Theorem A</u>

Let $\mathcal{G} \longrightarrow X$ be a P-principle bundle, equipped with a Cartan connection ω. Suppose that E is a representation of P, inducing a vector bundle $\mathcal{E}$ over X. Then ω gives a realization of the Jet bundles $J^\infty(\mathcal{E})$ as induced by the exponentiation of the $\mathfrak{p}$-module $(\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} E^*)^*$ to a P-module. □

(Note that this $\mathfrak{p}$-module certainly exponentiates to a P module, since exp o ad = Ad o exp). Applying Theorem 3.3, obtain

Theorem B

Let $\mathcal{G}$, P, X, ω be as in Theorem A, with induced bundles $\mathcal{E}$, $\mathcal{F}$ arising from P-modules E,F. Then to each $\mathfrak{U}(\mathfrak{g})$-homomorphism

$$\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} F^* \longrightarrow \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}} E^*$$

there corresponds a differential operator $\mathcal{E} \longrightarrow \mathcal{F}$. □

Notes

i) The construction relies on the global parallelism of $\tau\mathcal{G}$ afforded by the Cartan connection. We will see that for suitable structure on X, $\mathcal{G}$ and ω are uniquely defined, so that the operators of the theorem are *invariants* of the *structure* of X.

ii) Once given the global parallelism of τG, the operators are given by global choice and will be invariant under any structure preserving translations of X which may exist - in particular, if X is complex homogeneous, G = $\mathcal{G}$ and ω = Maurer Cartan form may be taken, so that the operators are those of the previous chapter. It is therefore reasonable to take Theorem B as a *definition* of invariant operator, given $\mathcal{G}$, ω.

□

We now seek to construct $\mathcal{G}$, ω, for suitable X.

4.3 Local |1|-grading, higher order Frame Bundles and $\mathcal{G}$.

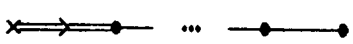
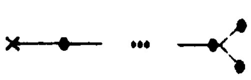
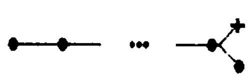
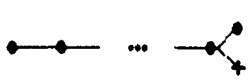
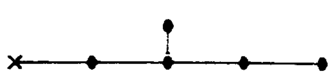
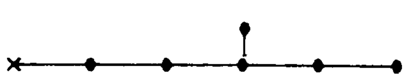
4.3.1 |1|-graded Lie Algebras and Generalized Flag Manifolds

We will construct the P-principle bundle $\mathcal{G}$ with Cartan connection ω over X in several steps, following [Ochiai]. X is to be modelled locally (to some finite order) by a subclass of generalized flag manifolds defined by the requirement that their tangent bundles be irreducible homogeneous bundles. In this section we classify this subclass and construct $\mathcal{G}$ for such X, provided a cohomological condition on $\mathfrak{g}$ is satisfied.

Fix $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, a system of simple roots $\mathcal{S}$ and a parabolic subalgebra $\mathfrak{p}$. If $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition and $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{p}$, set $\mathfrak{g}_{-1} = \mathfrak{u}_-$, $\mathfrak{g}_0 = \mathfrak{l}$ and $\mathfrak{g}_1 = \mathfrak{u}$ and $\mathfrak{g}_j = 0$ otherwise. Suppose $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (so that $\mathfrak{g}_{-1}$, $\mathfrak{g}_1$ are abelian). Such a structure is a |1|-grading of $\mathfrak{g}$. [Kobayashi & Nagano] classify all possible |1|-gradings for semi-simple $\mathfrak{g}$ as follows: the centre $\mathfrak{l}_Z$ of $\mathfrak{l} = \mathfrak{g}_0$ is necessarily one dimensional and contained within $\mathfrak{h}$ and there exists an unique generator, e, of $\mathfrak{l}_S$ whose weights on $\mathfrak{g}$ are -1, 0, 1. Then $\mathfrak{g}_j = \{ x \in \mathfrak{g} \mid [e,x] = jx \}$. For all $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$, $\alpha(e) \in \{-1,0,1\}$; $\exists$ unique $\alpha' \in \mathcal{S}$ with $\alpha'(e) = 1$ and $\Psi(\mathfrak{p}) = \{\alpha \in \mathcal{S} \mid \alpha(e) = 0\}$. If θ is the highest root of $\mathfrak{g}$, then $\theta(e) = 1$ and α' occurs precisely once in θ. Conversely, given α' with this property, e, and hence a $|1|$-grading of $\mathfrak{g}$, is clearly defined. This observation, and a table of the highest roots of simple Lie algebras (such as table (12.2) of [Humphreys]), classifies all $|1|$-graded semi-simple Lie algebras.

A generalized flag manifold will be called $|1|$-graded if it arises from a $|1|$-graded Lie algebra. Since $\mathfrak{g}_1$ acts trivially on $\mathfrak{g}/\mathfrak{p}$, the tangent bundles of these manifolds are irreducible. Conversely, a generalized flag manifold with irreducible tangent bundle is $|1|$-graded. We list these, for $\mathfrak{g}$ simple, in Table 4.3 below.

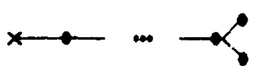
<u>Table 4.3 $|1|$-graded Generalized Flag Manifolds</u>

$\mathfrak{g}$		Generalized Flag Manifolds	
A_n	i	 i'th ↑ node	$1 \leq i \leq (n+1)/2$
B_n			
C_n			
D_n	I	 $= M^{2n-2}$	
	II	 $= PT^{2n-2}$	
	III	 $= PT^{*2n-2}$	
E_6			
E_7			

Notice that whereas the tangent bundle of PT^{2n} is irreducible, the tangent bundle of PT^{2n-1} is not.

Now, given a $|1|$-grading on $\mathfrak{g}$, with associated parabolic $\mathfrak{p}$, let L be the normalizer of $\mathfrak{l}$ in P. Then $P = L \ltimes \exp \mathfrak{g}_1$. Factoring L, if necessary, by the action of a finite group, obtaining $\tilde{L}$, the irreducibility of $\tau(G/P)$ under P gives an $\tilde{L}$-structure on the frame bundle of G/P.

Example

Consider X =  = $\mathbb{C}S^{2n}$; if we take Spin$(2n+2,\mathbb{C})$ for G, then L is Spin$(2n,\mathbb{C})\times\mathbb{C}$ = CSpin$(2n,\mathbb{C})$ (cf (2.2)). To obtain a principle structure on the frames of $\mathbb{C}S^{2n}$ it is necessary to factor CSpin$(2n,\mathbb{C})$ by $\mathbb{Z}_2$ to obtain $\tilde{L} = CO(2n,\mathbb{C})$. □

4.3.2 Locally |1|-graded manifolds

Consider now the category of complex manifolds which admit an $\tilde{L}$-structure for $\tilde{L}$ as above. Such a manifold will be called *locally |1|-graded*. From the point of view of Twistor Theory, the most important examples of such manifolds are the conformal manifolds, where $\tilde{L}$ is $CO(n,\mathbb{C})$. We now seek an extension of the $\tilde{L}$-principle bundle, in a natural manner, to a $\tilde{P}$-principle bundle, where $\tilde{P} = \tilde{L} \ltimes \mathfrak{g}_1$. In order to do this, we need some natural structure on X on which $\mathfrak{g}_1$ acts non-trivially. Such a structure is provided by the second order frame bundle of X, which will be seen to admit a unique

$\widetilde{P}$-principle subbundle under cohomological conditions on g (with its $|1|$ grading).

We need, therefore, some definitions and theory of higher order frame bundles. It appears that these were first introduce systematically by [Ehresmann], and studied extensively by [Kobayashi] and others.

To define higher frame bundles, consider the diffeomorphisms of an open neighbourhood of the origin of g_{-1} with an open neighbourhood of a point $x \in X$ (we assume X locally $|1|$-graded) which take the origin $\underline{0}$ to x. Define an equivalence relation $\sim_r$ on such diffeomorphisms f,f' so that $f \sim_r f'$ if f and f agree to order r at x (that is, in some, hence any, co-ordinate system centred on x, they have the same partial derivatives at $\underline{0}$). The *r-frames* at x are the equivalence classes of $\sim_r$. Denote the space of such by $F^r_x(X)$. Then the disjoint union $\sqcup_{x \in X} F^r_x(X) = F^r(X)$ is a fibre bundle over X, the *r'th order frame bundle*. Clearly the equivalence class $[f] \in F^r_x(X)$ may be thought of as the r-jet of the mapping f, in the obvious extension of the earlier discussion of jets. The map $f \longrightarrow [f]$ will be denoted j^r_x.

$F^1(X)$ is the usual bundle of frames of $\tau(X)$. $F^0(X) \cong X$.

$F^r(X)$ is a principle bundle. To describe the group of the fibre, denote by $G^r(n)$ the r-frames at the origin of g_{-1} (where $n = \dim_{\mathbb{C}} g_{-1}$). If $[f]$, $[f'] \in G^r(n)$, then so is

$[f \circ f']$, and this makes $G^r(n)$ a Lie group. $G^r(n)$ acts on $F^r(X)$ on the right by composition making $F^r(X)$ a $G^r(n)$-principle bundle. The natural projection $\pi : F^{r+1}(X) \longrightarrow F^r(X)$ coincides with $G^{r+1}(n) \longrightarrow G^r(n)$. Indeed $G^{r+1}(n)$ is a semi-direct product :

$$\{1\} \longrightarrow \odot^r(g_{-1})^* \longrightarrow G^{r+1}(n) \longrightarrow G^r(n) \longrightarrow \{1\} \qquad (4.3.1)$$

$G^1(n) \cong GL(g_{-1})$ in its usual action on g. Because such linear transformations are distinguished amounst the diffeomorphisms of g_{-1}, the surjection

$$G^r(n) \longrightarrow GL(g_{-1}) \longrightarrow \{1\}$$

splits - this splitting is of considerable importance in the construction which follows.

If $\varphi : X \longrightarrow Y$ is a diffeomorphism, then φ *prolongs* to a diffeomorphism $\varphi^r : F^r(X) \longrightarrow F^r(Y)$, by composition.

$F^r(X)$, $r \geq 1$, comes equipped with a natural one form $\Theta^{(r)}$ - the *canonical* or *soldering form* - taking values in the tangent space to the identity of $G^{r-1}(n)$. For if $[f] \in F^r_x(X)$, then f induces an isomorphism f_* of the tangent space $\tau_{id}G^{r-1}(n)$ with the tangent space to $\pi[f] \in F^{r-1}(X)$. Then for $u \in \tau_{[f]}G^r(X)$,

$$u \lrcorner \Theta^{(r)} = (f_*)^{-1} \pi_* u$$

The cases of interest here are for $r = 1$ - when $\Theta^{(1)}$, denoted θ, is the ordinary soldering form - and $r = 2$. Notice that $\tau_{id}(G^1(n)) \cong g_{-1} \oplus gl(n,\mathbb{C})$, and we will accordingly denote $\Theta^{(2)} = \Theta_{-1} \oplus \Theta_0$.

The following lemma summarises the elementary properties of $\Theta^{(r)}$ (cf [Kobayashi]) :

Lemma A

i) Let v^* be a vector field on $F^r(X)$ generated by the action of $G^r(n)$, corresponding to $v \in g^r(n)$ (the Lie algebra of $G^r(n)$). Then

$$v^* \lrcorner \Theta^{(r)} = \pi_* v \in g^{r-1}(n).$$

ii) If $g \in G^r(n)$, and R_g denotes right translation on $F^r(X)$, then

$$R_g^* \Theta = \mathrm{Ad}\ (\ \pi(g)^{-1}\ )\ \Theta$$

iii) For $r = 2$, $\pi^* \theta = \Theta_{-1}$ and there is a *structure equation*

$$d\,\Theta_{-1} + [\Theta_0 , \Theta_{-1}] = 0 \qquad (4.3.2)$$

□

Example

By way of interest and as a motivation for Kobayashi's theorem which follows, suppose that Γ is a connection on $F^1(X)$. Define $\tilde{\Gamma} = \theta + \Gamma$. Then it is easy to see that $\tilde{\Gamma}$ is a Cartan connection on $F^1(X)$ with values in $\mathfrak{g}_{-1} \oplus \mathfrak{gl}(\mathfrak{g}_{-1})$. Writing $\tilde{\mathcal{R}}$ for the curvature of $\tilde{\Gamma}$ one has a decomposition $\tilde{\mathcal{R}} = \mathcal{R}_{-1} \oplus \mathcal{R}$ where $\mathcal{R}$ is the ordinary curvature of Γ and

$$\mathcal{R}_{-1} = d\theta + [\Gamma,\theta] \qquad (4.3.3)$$

is the torsion. $\mathcal{R}_{-1}$ bears a formal similarity to the left hand side of (4.3.2) - Kobayashi's theorem below makes that precise. □

We are now in a position to begin the construction of $\mathcal{G}$. Let X be locally |1|-graded, as above; the first step is to find local liftings of the $\tilde{L}$-principle structure subbundle $\mathcal{Q}$, say to $F^2(X)$. We will say that a local section

$$\sigma : F^1(X) \longrightarrow F^2(X)$$

is *admissible* if, for $g \in GL(\mathfrak{g}_{-1})$, $f \in F^1(X)$,

$$\sigma(f\cdot g) = \sigma(f)\cdot g$$

(using the splitting $GL(\mathfrak{g}_{-1}) \longrightarrow G^2(n)$). [Kobayashi] has shown

<u>Theorem A</u> (Kobayashi's Theorem)

There is a one to one correspondence between local torsion free connections Γ on $F^1(X)$ and local admissible sections σ_Γ, by $\Gamma = \sigma_\Gamma^* \Theta_0$. □

Since locally we can find a connection Γ on $F^1(X)$ which is torsion free, and which preserves $\mathcal{Q}$, we may locally lift $\mathcal{Q}$ to $\mathcal{Q}(\Gamma) = \sigma_\Gamma \mathcal{Q}$. which, by virtue of the admissibility of σ_Γ, is an $\tilde{L}$-principle subbundle of $F^2(X)$. We want to "thicken it out" to a $\tilde{P}$-principle subbundle. To do this, we need the following

Lemma B

$\tilde{P}$ is a subgroup of $G^2(n)$. □

Proof

Define a mapping $Exp : g_{-1} \longrightarrow \tilde{G}/\tilde{P}$ by $u \longrightarrow \exp(u) / \tilde{P}$ ($\tilde{G} = \tilde{P} \ltimes g_{-1}$). Then Exp is a diffeomorphism; for $p \in \tilde{P}$, define

$$g_p = j^2\ (Exp^{-1} \circ L_p \circ Exp) \quad \in \quad G^2(n)$$

where L_p is left translation on $\tilde{G}$. The mapping $p \longrightarrow g_p$ is clearly a homomorphism, agreeing with the injective composition

$$\tilde{L} \longrightarrow GL(g_{-1}) \longrightarrow G^2(n)$$

To prove the lemma, therefore, it remains to show that $p \rightarrow g_p$ is injective on $\exp g_1$. So let $p = \exp v_p$ and $q = \exp v_q$ for v_p, $v_q \in g_1$ and note that $g_p = g_q$ is equivalent (using the usual expansion of exp $v_p \cdot \exp v_q$) to the condition that $\forall\ v \in \mathfrak{g}_{-1}$,

$$[[v_p,v],v] - [[v_q,v],v] \quad \in \quad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

But this easily implies $v_p = v_q$. □

It follows, acting on $\mathcal{Q}(\Gamma)$ with $\tilde{P} \subset G^2(n)$, that locally we obtain a $\tilde{P}$-principle subbundle $\mathcal{G}(\Gamma)$ of $F^2(X)$.

Two questions remain to be settled - to what extent does $\mathcal{G}(\Gamma)$ depend on the choice of Γ, and can $\mathcal{G}(\Gamma_i)$'s be globally patched in a consistent fashion (if, for example, we are working in the holomorphic category, so that globally defined connections preserving $\mathcal{Q}$ may not be defined)? For most $\mathfrak{g}$, we will show that $\mathcal{G}(\Gamma)$ is *independent* of Γ, so the second question is superfluous.

To obtain this result, we need to introduce some chain complexes which were in fact implicit in the algebraic construction of the Penrose Transform given at the end of the previous chapter (in the Borel-Weil-Bott theorem). [Kostant] and [Vogan] are good references for what follows. Define

$$C^{p,q} \;=\; \mathfrak{g}_{p-1} \;\otimes\; \Lambda^q(\mathfrak{g}_{-1})^*$$

and

$$\partial : C^{p,q} \longrightarrow C^{p-1,q+1}$$

satisfying $\partial^2 = 0$ by the formula, for $c \in C^{p,q}$

$$(\partial c)(u_0, u_1, \dots, u_p) = \Sigma_{i=1}^{p} (-1)^i [u_i, c(u_0, \dots, \hat{u}_i, \dots, u_p)] \tag{4.3.4}$$

where the caret $\wedge$ means "omit". Form the cohomology $H^*(C)$. H^* is the bigraded *Spencer cohomology* of g_{-1} with coefficients in the adjoint representation of g_{-1} on g. We will easily compute $H^{p,q}(g_{-1}, g)$ below, using the translation functor. Note that since the adjoint action of g_0 preserves g_i, and since ∂ commutes with this action, $H^{p,q}$ admits an action of g_0.

This gives us the technical tool to formulate

<u>Theorem B</u> ([Nagano],[Ochiai])
Locally on X, equivalence classes of $\mathscr{G}(\Gamma)$ are parametrized by sections of the bundle induced over X by the $\tilde{L}$-principle bundle $\mathcal{Q}$ and the representation of g_0 on $H^{1,1}$. □

($\mathscr{G}(\Gamma)$ and $\mathscr{G}(\Gamma')$ are equivalent if there exists a g_1-valued function f on $\mathcal{Q}$ with $\omega_\Gamma = \omega_{\Gamma'} \cdot \exp f$).

<u>Corollary</u>
If $H^{1,1}(g_{-1}, g) = 0$, then $\mathscr{G} = \mathscr{G}(\Gamma)$ is defined independently of Γ and globally over X. □

Proof of Theorem

We proceed in two steps; let $\tilde{u}$, $\tilde{v}$, be vector fields on $\mathcal{Q}$ with $u = \tilde{u} \lrcorner \theta$ and $v = \tilde{v} \lrcorner \theta$ fixed elements of $\mathfrak{g}_{-1}$. Γ, Γ' are torsionless connections preserving $\mathcal{Q}$.

i) Define a $C^{1,1}$ valued function F on $\mathcal{Q}$ by $u \lrcorner F_q = \tilde{u}_q \lrcorner (\Gamma_q - \Gamma'_q)$ for $q \in \mathcal{Q}$. Under right translations by $l \in \tilde{L}$, F transforms according to the representation of $\tilde{L}$ on $C^{1,1}$, and defines a section of the bundle on X induced by $C^{1,1}$. We claim that F is ∂-closed and hence its Spencer cohomology class yields a section of the bundle induced on X by the irreducible representation of $\mathfrak{l}^s$ on $H^{1,1}(\mathfrak{g}_{-1},\mathfrak{g})$. For

$$\begin{aligned} \partial F\ (u,v) &= [u, v \lrcorner F] - [v, u \lrcorner F] \\ &= [u, \tilde{v} \lrcorner (\Gamma - \Gamma')] - [v, \tilde{u} \lrcorner (\Gamma - \Gamma')] \end{aligned}$$

But

$$[u, \tilde{v} \lrcorner \Gamma] - [v, \tilde{u} \lrcorner \Gamma] = d\Theta_{-1}\ (s_* \tilde{u}, s_* \tilde{v}) = d\theta(\tilde{u}, \tilde{v})$$

by the structure equation (4.3.2). The same is true of Γ' and so $\partial F = 0$

ii) We now show that if $\mathcal{G}(\Gamma)$ and $G(\Gamma')$ are equivalent, then the cohomology class of F is zero : thus local sections of the bundle induced by $H^{1,1}$ parameterize local equivalence classes of $\mathcal{G}(\Gamma)$'s.

If $\mathcal{G}(\Gamma)$ is equivalent to $\mathcal{G}(\Gamma')$ then $\sigma_\Gamma = \sigma_{\Gamma'}\cdot\exp f$ where f is a g_1 valued function on $\mathcal{Q}$. Then (abusing notation)

$$F = \sigma_\Gamma^* \Theta_0 \; - \; \sigma_{\Gamma'}^* \Theta_0 \; = \; \sigma_{\Gamma'}^* \; (R^*_{\exp f} \Theta_0 - \Theta_0)$$

Using lemma A (ii), the expression in parenthesis is $-\mathrm{ad}f\cdot\Theta_{-1}$ so $F = -\,\mathrm{ad}\,(f)\cdot\theta$, equivalently $F = \partial f$.

On the other hand, if $F = \partial f$ for some $C^{1,1}$ valued function f on $\mathcal{Q}$ (transforming as a section of the bundle $C^{1,1}$) then we may take $\sigma_\Gamma = \sigma_{\Gamma'}\cdot\exp f$ and this is the only choice by Theorem A. □

The next section shows how to compute the Spencer cohomology. We shall see that for most locally |1|-graded manifolds X, $\mathcal{G}$ is uniquely defined.

As a passing remark, if $\mathcal{Q}$ lifts to an L-principle bundle (assuming $L \neq \tilde{L}$) then evidently $\mathcal{G}$ lifts to a P-principle bundle. We will assume that this has been done where-ever necessary and denote the result by $\mathcal{G}$ also. Clearly Θ pulls back to such a lift.

4.4 Computing Spencer Cohomology

The cohomology groups $H^q(g_{-1}, g)$ are particular cases of the cohomology of g_{-1} with coefficients in a representation of g restricted to g_{-1}. These are finite dimensional analogues of sheaf cohomology and are computed by means of Kostant's theorem, an algebraic form of the Borel-Weil-Bott theorem. [Kostant] [Vogan]. Define, for λ dominant for $\mathfrak{g}$

$$C^q(F(-\lambda)) = F(-\lambda) \otimes \Lambda^q (\mathfrak{g}_{-1})^*$$

Then ∂ (as in formula (4.3.4) extends obviously, and the cohomology groups $H^*(\mathfrak{g}_{-1}, F(-\lambda))$ arise. These are computed by

<u>Theorem A</u> [Kostant]

$$H^q(\mathfrak{g}_{-1}, F(-\lambda)) = \oplus_{w \in W^1(\mathfrak{p})\ ;\ l(w) = q} F_{\mathfrak{p}}(-w\cdot\lambda)$$

as irreducible $\mathfrak{l}$ modules, where $W^1(\mathfrak{p})$ is defined in section (2.6) □

Thus $H^*(\mathfrak{g}_{-1}, F(-\lambda))$ may be computed as $\Psi_{-\lambda}^{0} H^*(\mathfrak{g}_{-1}, \mathbb{C})$ where $\Psi_{-\lambda}^{0}$ is a translation functor of section (3.7) and $\mathbb{C}$ is the trivial representation. This gives the Spencer Cohomology $H^{p,q}(\mathfrak{g}_{-1}, \mathfrak{g})$ for we know that

$$F_{\mathfrak{p}}(-w\cdot\lambda) = \Psi_{-\lambda}^{0} F_{\mathfrak{p}}(-w\cdot 0) \subset F_{\mathfrak{p}}(-w\lambda) \otimes F_{\mathfrak{p}}(-w\cdot 0)$$

and if λ is the highest weight of $\mathfrak{g}$, then $F_{\mathfrak{p}}(-w\lambda)$ may be identified with (a subalgebra, possibly, of) one of the $\mathfrak{g}_i$'s whilst $F_{\mathfrak{p}}(-w\cdot 0)$ is contained in $\Lambda^q(\mathfrak{g}_{-1})^*$, for q = length(w). With this it is easy to compute the following

Proposition

Let g be a complex simple $|1|$-graded Lie algebra. Then

i) If $g = sl(2,\mathbb{C})$, then $H^{2,1} = F(\overset{-2}{\times})$ and $H^{1,1} = H^{0,1} = 0$

For all other g, irrespective of grading, $H^{2,1} = 0$

ii) If $g \neq A_n$, $(n \geq 2)$ with $|1|$ grading $\times\!-\!\bullet\cdots\bullet\!-\!\bullet$ or $\bullet\!-\!\bullet\cdots\bullet\!-\!\times$,

and $g \neq so(3,\mathbb{C})$, then $H^{1,1}(g_{-1},g) = 0$.

iii) In the exceptional cases of (ii), we have $H^{1,1}(g_{-1},g)$:

A_2 : $\overset{-3}{\times}\!-\!\!-\!\overset{3}{\bullet}$ or $\overset{3}{\bullet}\!-\!\!-\!\overset{-3}{\times}$;

A_n $(n \geq 3)$: $\overset{-3}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!-\!\overset{1}{\bullet}$ or the reverse

$B_2 = so(3,\mathbb{C})$: $\overset{-3}{\bullet}\!\Longrightarrow\!\overset{4}{\bullet}$. □

In particular,

Corollary

If X is a conformal manifold, $\mathcal{G}$ is uniquely defined, independently of any choice. □

Example

Equation (3.7.7) of Example (3.7) B gives that for $g = sl(4,\mathbb{C})$, with $|1|$-grading $\bullet\!-\!\times\!-\!\bullet$, the Spencer cohomology is

$$H^{0,0} = F(-\overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}) \quad ; \quad H^{0,1} = F(-\overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet})$$

$$H^{1,2} = F(-\overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}) \oplus F(-\overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet})$$

$$H^{2,3} = F(-\overset{2}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{2}{\bullet}) \quad ; \quad H^{2,4} = F(-\overset{1}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{1}{\bullet})$$ □

There is an important operator ∂^* on the chain complex C^*

defined as follows : let $\{y_i\}$ be any basis of $\mathfrak{g}_{-1}$; under the Killing form of $\mathfrak{g}$, $\mathfrak{g}_1$ is naturally dual to $\mathfrak{g}_{-1}$, so we may pick a basis of $\mathfrak{g}_1$, $\{x^i\}$, say, dual to $\{y_i\}$. Then for $c \in C^{p,q}$, $u_1,\dots,u_{q-1} \in \mathfrak{g}_{-1}$ define

$$\partial^* c\ (u_1,\dots,u_{q-1}) \quad = \quad \Sigma_j\ [x^j, c(y_j, u_1,\dots,u_{q-1})]$$

(∂^* is ofcourse independent of the choice of basis $\{y_i\}$) It is possible to equip $\mathfrak{g}$ with an hermitian form so that ∂^* is the adjoint of ∂ (cf [Kostant] and [Nagano]). As may be expected, this leads to a Hodge theory for $H^*(\mathfrak{g}_{-1},\mathfrak{g})$: define

$$\Box \quad = \quad \partial^*\partial \quad + \quad \partial\partial^*$$

Then there is an unique harmonic representative f, satisfying $\Box f = 0$, in each cohomology class of $H^{p,q}(\mathfrak{g}_{-1},\mathfrak{g})$. In fact, this is how [Kostant] computes his theorem - he shows that $\Box$ acts by scalars on irreducible representations of $\mathfrak{p}$. Specifically, if $\mu_{\mathfrak{p}}$ is $\mathfrak{p}$-dominant, $\Box$ acts on $F_{\mathfrak{p}}(-\mu_{\mathfrak{p}})$ by the scalar

$$\tfrac{1}{2}\ (\ \langle\lambda+\rho,\lambda+\rho\rangle\ -\ \langle\mu+\rho,\mu+\rho\rangle\ ) \qquad (4.4.1)$$

where ρ is the semi-sum of the positive roots of $\mathfrak{g}$, and λ is the highest root of $\mathfrak{g}$.

4.5 The Cartan Connection on $\mathcal{G}$

We now consider the computation of a Cartan connection on $\mathcal{G}$, assuming that $\mathcal{G}$ is defined and unique. To do this, we first introduce the concept of an admissible Cartan connection, generalizing the affine connections of $F^1(X)$, and show how a generalization of the Weyl tensor for conformal manifolds is defined. Imposing a *harmonicity* condition on this leads to a Cartan connection dependent only on the local |1|-graded structure of X, provided $\mathfrak{g}$ and its |1|-grading satisfy the cohomology condition $H^{1,2}(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$.

With respect to the decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, any Cartan connection ω on $\mathcal{G}$, and its curvature Ω, may be decomposed as

$$\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1 \qquad ; \qquad \Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$$

Notice that the decomposition is not $\mathfrak{p}$ invariant (but $\mathfrak{l}$ invariant) and that we have, for $p \in P$,

$$R_p^* \omega = \mathrm{Ad}(p^{-1}) \omega \qquad ; \qquad R_p^* \Omega = \mathrm{Ad}(p^{-1}) \Omega$$

by virtue of condition (iv) on ω. Recall that if $u^* \lrcorner \Theta_{-1} = 0$, then $u^* \lrcorner \Omega = 0$ also, where u^* is a vector on $\mathcal{G}$.

The admissible Cartan connections are those of the form

$$\omega = \Theta_{-1} \oplus \Theta_0 \oplus \omega_1$$

(Θ_0 now takes values in $\mathfrak{g}_0$) where ω_1 is a $\mathfrak{g}_1$-valued one form on $\mathcal{G}$. For such ω, by virtue of the structure equation (4.3.2), Ω_{-1} vanishes, and Ω_0 now obeys the transformation law

$$R_p^* \Omega_0 = \mathrm{Ad}(p^{-1})\, \Omega_0 \quad \mathrm{mod} \quad \mathfrak{g}_1$$

Now if $u, v \in \mathfrak{g}_{-1}$, define u^*, v^* by $u^* \lrcorner \omega = u$, $v^* \lrcorner \omega = v$. Then we may define a $C^{1,2}$-valued function on $\mathcal{G}$ by

$$\mathscr{W}(u,v) = \Omega_0(u^*, v^*)$$

and $\mathscr{W}$ transforms under R_p^* by

$$R_p^* \mathscr{W} = \tilde{\mu}(p^{-1})\, \mathscr{W}$$

where $\tilde{\mu}$ is the irreducible representation of $\mathfrak{l}$, extended trivially to all of $\mathfrak{p}$, on $C^{1,2}$. Thus $\mathscr{W}$ defines a section of the bundle on X induced by $\tilde{\mu}$; we may think of this as a generalization of the Weyl tensor of conformal manifolds, associated to ω.

Now consider $\partial \mathscr{W}$. Taking $y^* \lrcorner \omega = y \in \mathfrak{g}_{-1}$, obtain

$$\partial \mathscr{W}(u,v,y) = [\Theta_{-1}, \Omega_0]\, (u^*, v^*, y^*) \qquad (4.5.1)$$

But then,

$$[\Theta_{-1}, \Omega_0] = [\ \Theta_{-1},\ d\Theta_0 + \tfrac{1}{2}[\Theta_0 . \Theta_0] + [\Theta_{-1}, \omega_1]\ ]$$

The only piece of this which might make a contribution to the right-hand side of (4.5.1) is $[\Theta_{-1}, d\Theta_0]$. Differentiating the structure equation (4.3.2) shows this to be $[d\Theta_{-1}, \Theta_0]$ so that it too makes no contribution. That is

$$\partial \mathscr{w} = 0$$

so that $\mathscr{w}$ defines a class in $H^{1,2}$. The scheme now is to show that this class is independent of ω_1, and then adjust ω_1 so that $\mathscr{w}$ becomes the unique *harmonic* representative in its class.

<u>Theorem</u> [Tanaka]

Let $\mathfrak{g}$ be a |1|-graded semi-simple complex Lie algebra, with $H^{2,1}(\mathfrak{g}_{-1},,\mathfrak{g}) = 0$, and let X be a locally |1|-graded manifold (based on $\mathfrak{g}$) with a $\tilde{P}$-principle sub-bundle, $\mathscr{G}$, of $F^2(X)$, constructed as above. Then, if ω' is any admissible Cartan connection on $\mathscr{G}$, the class of $\mathscr{w}(\omega')$ in $H^{1,2}(\mathfrak{g}_{-1},\mathfrak{g})$ is independent of ω', and there exists an unique admissible Cartan connection ω with $\mathscr{w}(\omega)$ harmonic. □

The unique connection of the theorem is called *normal*.

<u>Proof</u>

We proceed in two steps

i) Uniqueness of class

Let ω', ω'' be two admissible Cartan connections on $\mathscr{G}$; $(\omega')^{-1}$ and $(\omega'')^{-1}$ differ only on $\mathfrak{g}_{-1}$, and then by an element of $(\omega')^{-1}\ \mathfrak{g}_1$. Thus $\exists$ a $C^{2,1}$-valued function f on $\mathscr{G}$ defined by

$$u' - u'' \quad = \quad (\omega')^{-1}\ f(u)$$

where $u' \lrcorner \omega' = u'' \lrcorner \omega'' = u \in \mathfrak{g}_{-1}$. Since Ω_0' and Ω_0'' are zero on $(\omega')^{-1}\ \mathfrak{p}$, if v, v', v'' are similarly defined,

$$\{\ w(\omega') - w(\omega'')\ \}\ (u,v)$$
$$= \ \Omega_0'\ (u'v') - \Omega_0''\ (u'',v'')$$
$$= \ (\Omega_0' - \Omega_0'')\ (u'',v'')$$
$$= \ [\Theta_{-1}, \omega_1' - \omega_1'']\ (u'',v'')$$
$$= \ \partial f(u,v)$$

ii) Existence and Uniqueness

We first establish the local existence of an admissible Cartan connection. Let $\mathcal{Q}$ be, as before, the $\tilde{L}$-principle subbundle of $F^1(X)$ and pick a connection Γ on $\mathcal{Q}$ so that $\mathscr{G} = \mathscr{G}(\Gamma)$, at least locally. Let $g = qa \in \mathscr{G}(\Gamma)$, where $q \in \mathscr{s}_\Gamma\mathcal{Q}$ and $a \in \exp(\mathfrak{g}_{-1})$. If $Y \in \tau_g(\mathscr{G})$, then Y has the unique form:

$$Y = (R_a)_* \ X + v^*$$

where $X \in \tau_q(\mathscr{s}_\Gamma\mathcal{Q})$ and v^* corresponds to a vector $v \in \mathfrak{g}_{-1}$. Define admissible ω' by

$$X \lrcorner \omega_1' = 0$$

$$Y \lrcorner \omega' = Ad(a^{-1})(X \lrcorner \omega) + v$$

To check that ω transforms correctly under the action of $\tilde{P}$, let $b \in \exp(g_{-1})$, $l \in \tilde{L}$. Then, since g_1 is abelian, $(R_b)_* v^* = v^*$ whence

$$(R_b)_* Y = (R_b)_*(R_a)_* X + v^* = (R_{ab})_* X + v^*$$

so that

$$\begin{aligned} Y \lrcorner R_b^* \omega' &= Ad(b^{-1}a^{-1}) X \lrcorner \omega' + v \\ &= Ad(b^{-1}) \{ Ad(a^{-1}) X \lrcorner \omega' + v \} \\ &= Ad(b^{-1}) Y \lrcorner w' \end{aligned} \qquad (4.5.2)$$

and

$$\{(R_l)_* (R_a)_* X\} \lrcorner \omega' = \{(R_l)_* (R_a)_* (R_{l^{-1}})_* (R_l)_* X\} \lrcorner \omega'$$

$$= \{(R_{l^{-1}al})_* (R_l)_* X\} \lrcorner \omega'$$

$$= Ad(l^{-1}a^{-1}l) \{(R_l)_* X\} \lrcorner \omega' \qquad \text{by (4.5.2) since } l^{-1}a^{-1}l \in \exp g_{-1}$$

$$= Ad(l^{-1}a^{-1})(X \lrcorner w')$$

whilst $(R_l)_* v^* = (Ad(l^{-1}) v)^*$ so that

$$Y \lrcorner (R_l)_* \omega' = Ad(l^{-1})(Y \lrcorner \omega')$$

as required.

Second, establish the existence and uniqueness of a local normal Cartan connection. Recall that if ω, ω' are admissible, we may write

$$\mathscr{K}(\omega) - \mathscr{K}(\omega') = \partial f$$

for f as above; thus

$$\partial^* \mathscr{K}(\omega) = \square f + \partial^* \mathscr{K}(\omega').$$

Given ω', set

$$\square f = - \partial^* \mathscr{K}(\omega') \tag{4.5.3}$$

Since $H^{2,1}(g_{-1}, g) = 0$, this has an unique solution for f. It is easy to check that if we use this f to define ω, then ω is an admissible Cartan connection, and $\partial^*\omega = \partial\omega = 0$ so $\square\ \omega = 0$ and ω is normal. Indeed since the solution of (4.5.3) is unique, so is ω, and therefore globally defined also. □

Remarks

i) Suppose that X = G/P is |1|-graded. The with *Exp* as in lemma (4.3) B, define

$$\Phi : G \longrightarrow F^2(X) \quad ; \qquad g \longrightarrow j^2 (L_g \circ Exp)$$

Φ is a local embedding, for it intertwines the action of P on G and on $F^2(X)$ (via $P \longrightarrow \tilde{P} \subset G^2(n)$) Thus G is a covering space of $\mathcal{G}$ and a normal Cartan connection is the Maurer-Cartan form. Indeed, if $H^{1,1}(\mathfrak{g}_{-1},\mathfrak{g}) = H^{2,1}(\mathfrak{g}_{-1},\mathfrak{g}) = 0$, this is the only possibility.

ii) Suppose X is a conformal (complex) manifold, of dimension $n \geq 4$. Then X admits an unique $CO(n,\mathbb{C})\ltimes\mathbb{C}^n$ - principle subbundle $\mathcal{G}$ of $F^2(X)$ with unique normal Cartan connection - the conformal Cartan connection [Cartan I]. If X is spin, then this lifts to a $CSpin(n,\mathbb{C})\ltimes\mathbb{C}^n$ - principle bundle, which is the *local twistor bundle* [Penrose II] in dimension four.

iii) Let $\mathfrak{g} = A_n$, with $|1|$-graded structure $\times\!\!-\!\!\bullet\!\!-\cdots-\!\!\bullet$, X locally $|1|$-graded. A $\tilde{P}$-principal subbundle of $F^2(X)$ is called a *projective* structure on X, and admits an unique normal Catran connection - this is the *projective* connection of [Cartan II].

4.6 Computing Curved Invariant Differential Operators

Theorems (4.2) B, (4.4) B and (4.5) now enable us to compute a series of differential operators on induced bundles over a locally $|1|$-graded manifold X, extending those of the previous chapter. If Γ is a torsion free connection on X, respecting its $\tilde{L}$-structure, then we may use Tanaka's Theorem (4.5) to compute the normal Cartan connection in terms of Γ

and so express the differential operators in terms of the covariant derivative on induced bundles associated to Γ.

So let ω' denote the Cartan connection constructed in part (ii) of Theorem (4.5) from Γ. (We assume that $H^{1,1}(\mathfrak{g}_{-1},\mathfrak{g}) = H^{2,1}(\mathfrak{g}_{-1},\mathfrak{g}) = 0$ so that $\mathcal{G}$ and the normal Cartan connection ω are uniquely defined). Let $y \in \mathfrak{g}_{-1}$ and take $y^* \lrcorner \omega = y$ and $y' \lrcorner \omega' = y$. Then, with f as in part (ii) of the theorem, $y \lrcorner f \in \mathfrak{g}_1$ and

$$y^* = y' + \omega^{-1} (y \lrcorner f) \qquad (4.6.1)$$

If $\varphi : \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\mu \longrightarrow \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$ is a homomorphism of induced modules, then φ is given by the mapping of a highest weight vector γ of $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\mu$ to a maximal vector in $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$ of weight μ and of form $\Sigma p^i(y_j)\kappa_i$, where $\{\kappa_i\}$ is a weight basis of $F_{\mathfrak{p}}^{\lambda}$ and the p^i are polynomials in the y_j (which form a root space basis of $\mathfrak{g}_{-1}$). To obtain a differential operator, each $p^i(y_j)$ must be replaced by a polynomial $p^i(y_j^*)$ in the vector fields y_j^* defined by $y_j^* \lrcorner \omega = y_j$. (care must taken to use the mapping $\omega^{-1} : \mathfrak{A}(\mathfrak{g}_{-1}) \longrightarrow \mathfrak{A}(\Gamma(\tau\mathcal{G}))$ precisely - the resulting polynomials must be symmetric in the y_j^*, for otherwise, since the commutators $[y_i, y_j]^*$ differ from $[y_i^*, y_j^*]$, spurious curvature terms may arise).

To obtain differential operators expressed in terms of the covariant derivative of Γ, one simply substitutes for y_j^* in terms of y_j' (with $y_j' \lrcorner \omega' = y_j$) from (4.6.1), noting that the

y_j' will induce differential operators formally the same as in the previous chapter, but with the homogeneous differential replaced by the covariant derivative of Γ. The terms $\omega^{-1}(y \lrcorner f)$ will construct suitable correction terms needed to make the resulting operators invariant of the choice of Γ, and manifestly invariants of the local |1|-graded structure of X.

To compute f, let s_Γ be the (local) admissible section associated to Γ by Kobayashi's theorem. Let $\mathcal{R}(\Gamma)$ denote the curvature form of Γ on $\mathcal{Q}$, which is, as before, the $\tilde{L}$-principle subbundle of $F^1(X)$. Then, on $\mathcal{Q}$,

$$s_\Gamma \Omega_0 = \mathcal{R}(\Gamma) + [\theta, \overset{*}{s_\Gamma}\omega_1]$$

Thus if $\tilde{u}$, $\tilde{v}$ are horizontal vector fields on $\mathcal{Q}$, with $\tilde{u} \lrcorner \theta = u$, $\tilde{v} \lrcorner \theta = v$,

$$\varpi(u,v) = \overset{*}{s_\Gamma} \Omega_0 (\tilde{u}, \tilde{v})$$

Using the fact that $\overset{*}{s_\Gamma} \omega_1^{!} = 0$,

$$\varpi(u,v) = \mathcal{R}(\Gamma)(\tilde{u}, \tilde{v}) + \partial f(u,v).$$

Writing $r(\Gamma)(u,v) = \mathcal{R}(\tilde{u}, \tilde{v})$ and using the normality of ω together with $H^{2,1}(g_{-1}, g) = 0$,

$$f = - \square^{-1} \partial^* r(\Gamma)$$

Now $\partial^* \mathcal{r}(\Gamma)$ represents the *Ricci* curvature of Γ [Ochiai], and f is easily computed using the scalars of equation (4.4.1) above.

Remarks

i) The correction terms computed from f involve only the (derivatives of the) Ricci curvature of Γ.

ii) The resulting operators are evidently expressed as the same combination of covariant derivatives and Ricci curvature correction terms irrespective of Γ. They are thus invariant in the sense most often taken in the Physics literature (cf [Penrose and Rindler I & II] for the case of conformally invariant operators).

iii) The highest order part of these operators is formally as in the homogeneous case (with the covariant derivative of Γ replacing the homogeneous differential). If we consider expanding an expression of the form

$$\overset{*}{Y}_{i_1} \overset{*}{Y}_{i_2} \cdots \overset{*}{Y}_{i_n} (Y_{j_1} Y_{j_2} \cdots Y_{j_p} \gamma)$$

(where Y_{i_j} are the lowering operators of g_0) in terms of $y_i{}'$, recalling that $[g_1, g_{-1}] \subset g_0$ and $[g_1, g_0] \subset g_1$, then it is clear that

(a) $y^*_{i_n}$ gives rise to no curvature correction terms

(b) for each occurence of a curvature correction term in the expansion, there are two fewer occurences of y_i' than y_i^*.

It follows from (a) that all first order invariant differential operators can include no curvature correction terms, and are formally as in the homogeneous case. From (b), clearly, the first curvature correction term is associated to a differential operator of order at least two less than that of the leading term. □

It is easiest to display further properties of this construction in the context of an actual example.

4.7 Conformally Invariant Differential Operators in Dimension Four

Let X be a conformal four-manifold. Locally reduce the $CO(n,\mathbb{C})$-structure to an $SO(n,\mathbb{C})$ structure - i.e. pick a metric in the conformal class. Let Γ be the associated Levi-Civita connection, which clearly extends to a $CO(n,\mathbb{C})$ connection. The aim is to compute the invariant differential operators (called conformally invariant, now) in terms of this connection.

Now $\tau(\Gamma)$ is a $\mathfrak{g}_0 \otimes \Lambda^2(\mathfrak{g}_{-1})^* = C^{1,2}$ valued function on the $CO(n,\mathbb{C})$-principle subbundle $\mathcal{Q}$ of $F^1(X)$; it represents a section of the bundle

$$(\overset{2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{2}{\bullet}) \otimes (\overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet})$$

on X (we assume X spin where necessary). Because the Levi-Civita form is torsion free, $r(\Gamma)$ is actually a section of

$$\overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet} \oplus \overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}$$

(comprising self and anti-self dual Weyl, tracefree Ricci and scalar curvature). Similarly, $\partial^{*} r(\Gamma)$ gives a section of

$$\overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}$$

(i.e. Ricci curvature, as claimed, since ∂^{*} intertwines the irreducible action of $\mathfrak{p}$). We write this as

$$\begin{aligned} R_{ab} &= (R_{ab} - \tfrac{1}{4} R\, g_{ab}) \oplus \tfrac{1}{4} R\, g_{ab} \\ &= -2\, \Phi_{ab} \oplus 6\, \Lambda\, g_{ab} \end{aligned}$$

in the abstract index notation of [Penrose and Rindler I], with Φ_{ab} the tracefree Ricci spinor, and $\Lambda = R/24$ as in [op cit]. $\Box^{-1}$ acts by 1/2 on $\overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet}$ and by 1/6 on $\overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}$, so that $f = -\Box^{-1} \partial^{*} r(\Gamma)$ induces the tensor

$$\rho_{ab} = \Phi_{ab} - \Lambda\, g_{ab}$$

(P is capital rho - cf [Penrose II], [Eastwood & Rice], [Penrose and Rindler II])

Recall that if $d\rho$ is the irreducible representation of $\mathfrak{p}$ on $C^{2,1}$, then $u \in \mathfrak{p}$ implies

$$\pounds_u{}^* f = - d\rho\,(u)\, f$$

so that we may think of the function f as an element of the *dual representation* $(C^{2,1})^*$ under the *differential action* of $\mathfrak{p}$ (this is one of the reasons for working with the dual representation throughout this thesis). Split f into its irreducible parts, and let Φ, $-\Lambda$ denote the highest weight parts under the differential action. If we use the notation of example (2.1) (ii) for the generators of sl(4,$\mathbb{C}$), and write

$$y_i \lrcorner f = \mathrm{P}_{ij}\, x^j \quad \text{so} \quad y_i^* = y_i' + \Sigma\, \mathrm{P}_{ij}\, (x_j)^* \qquad (4.7.1)$$

then P_{ij} is as in Table 4.7 below.

TABLE 4.7 Curvature correction terms P_{ij}

Φ	$-\frac{1}{2} Y_1 \Phi$	$\frac{1}{2} Y_2 \Phi$	$-\frac{1}{4} Y_1 Y_2 \Phi - \Lambda$
$-\frac{1}{2} Y_1 \Phi$	$\frac{1}{2} Y_1{}^2 \Phi$	$\Lambda - \frac{1}{4} Y_1 Y_2 \Phi$	$\frac{1}{4} Y_1{}^2 Y_2 \Phi$
$\frac{1}{2} Y_2 \Phi$	$\Lambda - \frac{1}{4} Y_1 Y_2 \Phi$	$\frac{1}{2} Y_2{}^2 \Phi$	$-\frac{1}{4} Y_1 Y_2{}^2 \Phi$
$-\frac{1}{4} Y_1 Y_2 \Phi - \Lambda$	$\frac{1}{4} Y_1{}^2 Y_2 \Phi$	$-\frac{1}{4} Y_1 Y_2{}^2 \Phi$	$\frac{1}{4} Y_1{}^2 Y_2{}^2 \Phi$

(To form ρ_{ij}, think of $\Sigma\, \rho_{ij}\, x_j$ as an element of $F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet})$ in the tensor product $F(\overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet}) \otimes F(\overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet})$ and compute $y_1 \lrcorner f = \Sigma\, \rho_{ij}\, x_j$ as the highest weight. Then lower to find $y_j \lrcorner f$).

It is now time for several concrete examples.

Local Twistor Transport

The first example relates our construction to *local twistor transport* [Penrose II]. Let $\mathcal{T}^{\alpha}$ (the local twistor bundle over X) be induced by the representation $F(\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}) = T_{\alpha}$ restricted to a representation of $\mathfrak{p} = \bullet\!-\!\times\!-\!\bullet$. Then there is a homomorphism of induced modules ($\mathbb{C}$ denoting the trivial module)

$$
\begin{array}{llll}
\mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, (F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}) \otimes T_{\alpha}) & \cong\!\longrightarrow & \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \otimes T_{\alpha} & \\
 & \varphi \otimes \mathrm{id} \downarrow & & \\
 & & \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, \mathbb{C} \otimes T_{\alpha} \quad \cong\!\longrightarrow & \mathcal{I}nd_{\mathfrak{p}}^{\mathfrak{g}}\, (\mathbb{C} \otimes T_{\alpha})
\end{array} \tag{4.7.2}
$$

where the isomorphisms $\cong$ are as in Lemma (3.7) B and φ is the map generated by $\tau \longrightarrow y_1\ \underline{1}$ for τ, $\underline{1}$ highest weight vectors in $F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet})$ and $\mathbb{C}$ respectively.

Let T_{α} be spanned by $\{\alpha, \beta, \gamma, \delta\}$ as in Example (3.7) A (i). Fixing a local metric in the conformal class is equivalent to restricting to the action of $\mathfrak{t}^{s}$ on T_{α}; do this and so split the sequence

$$
0 \longrightarrow F(\overset{0}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}) \longrightarrow T_{\alpha} \longrightarrow F(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet}) \longrightarrow 0 \tag{4.7.3}
$$

by thinking of the spans of

$$\{\ \alpha,\beta\ \}\ \longrightarrow\ \{\ \alpha,\beta,\gamma,\delta\ \}\ \longrightarrow\ \{\ \gamma,\delta\ \}$$

(This is ofcourse equivalent, dually, to the splitting of the sequence $0 \longrightarrow \mathcal{O}_{A'} \longrightarrow \mathcal{T}^{\alpha} \longrightarrow \mathcal{O}^{A} \longrightarrow 0$ by choosing a metric - cf [Penrose II]). Similarly splitting $T_{\alpha} \otimes F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet})$ obtains irreducibles $F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}) \oplus F(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\bullet})$ and $F(\overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}) \oplus F(\overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet})$, generated under $\mathfrak{U}(\mathfrak{t}^{S})$ by the following highest weight vectors, respectively

$$y_1 \otimes \alpha \qquad ; \qquad \tfrac{1}{2}(\ y_1 \otimes \beta\ -\ y_3 \otimes \alpha)$$

$$y_1 \otimes \gamma \qquad ; \qquad \tfrac{1}{2}(\ y_1 \otimes \delta\ -\ y_2 \otimes \gamma)$$

($F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}) \cong g_{-1}$ and y_i is the usual basis). Under (4.7.2), for instance,

$$y_1 \otimes \alpha \overset{\varphi\otimes id}{\longrightarrow} (y_1 \cdot \underline{1}) \otimes \alpha \cong\!\!\to y_1(\underline{1}\otimes\alpha) - \underline{1} \otimes y_1 \cdot \alpha = y_1(\underline{1}\otimes\alpha)$$

Replacing y_i by y_i^{*} written via (4.7.1) in terms of $y_i{}'$ using Table 4.7, (4.7.2) gives the mapping induced by the direct sum of

$$\begin{aligned}
y_1 \otimes \alpha &\longrightarrow y_1{}'\alpha \\
y_1 \otimes \beta - y_3 \otimes \alpha &\longrightarrow y_1{}'\beta - y_3{}'\alpha - 2\gamma \\
y_1 \otimes \gamma &\longrightarrow y_1{}'\gamma - \rho_{11}\beta + \rho_{13}\alpha \\
y_1 \otimes \delta - y_2 \otimes \gamma &\longrightarrow y_1{}'\delta - y_2{}'\gamma - 2(\rho_{14}\alpha - \rho_{12}\beta)
\end{aligned}$$

Overall this gives the conformally invariant

$$\nabla_{BB'} : (\omega^A, \pi_{A'}) \longrightarrow (\nabla_{BB'}\omega^A + \epsilon_B{}^A \pi_{B'} , \nabla_{BB'}\pi_{A'} - P_{BB'AA'}\omega^A)$$

that is, Penrose's local twistor transport. □

The Conformally Invariant Laplacian

Recall from Example (3.3) A (b) that

$$(y_1 y_4 + y_4 y_1 - y_2 y_3 - y_3 y_2)\,\alpha$$

generates the Laplacian $\Box : \mathcal{O}[-1] \longrightarrow \mathcal{O}[-3]$. Writing this in terms of y_i^* and expanding using (4.7.1) obtains

$$(y_1' y_4' + y_4' y_1' - y_2' y_3' - y_3' y_2')\,\alpha - (P_{14} + P_{41} - P_{23} - P_{32})\,\alpha$$

which, using Table 4.7 induces the conformally invariant Laplacian

$$(\Box + R/6) : \mathcal{O}[-1] \longrightarrow \mathcal{O}[-3]$$

where now $\Box = \nabla^a \nabla_a$ in the Levi-Civita connection of any metric in the allowed conformal class. □

Conformally Invariant Operator extending $\nabla_{(A'}{}^A \nabla_{B'}{}^B \lambda_{C')AB}$

The expression

$$(-y_3 + y_1 Y_2)(-2y_3 + y_1 Y_2)\, \alpha$$

induces the indicated operator on $\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}$ in the homogeneous case. Expanding the expression, one finds it to be symmetric in y_i; replacing y_i by y_i^* and substituting by (4.7.1), using Table 4.7, yields

$$(-y_3' + y_1' Y_2)(-2y_3' + y_1' Y_2)\alpha \;+\; \Phi\, Y_2{}^2\alpha - Y_2\Phi\, Y_2\alpha + Y_2{}^2\Phi\, \alpha$$

The correction terms come from the projection

$$F(\overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet}) \otimes F(\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{2}{\bullet}) \longrightarrow F(\overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet})$$

from which it follows that the operator

$$\lambda^{A'}_{AB} \longrightarrow (\nabla^{A}_{(A'}\nabla^{B}_{B'} + \Phi^{A\ B}_{(A'B'})\, \lambda_{C')AB}$$

is conformally invariant. □

The process just illustrated is certain to produce conformally invariant differential operators from homomorphisms of induced modules, but ofcourse as the order of the operator increases, it requires more effort. Observing remarks (iii) in section (4.6) we may be more efficient: any correction term in a conformally invariant operator $\mathcal{D} : \mathcal{O}(E^*) \longrightarrow \mathcal{O}(F^*)$ is a combination of (derivatives of) Φ_{ab} and (derivatives of) Λ, and differential operators and must arise from a projection of the tensor product

$$(\otimes^{i}\mathcal{O}_{(A'B')(AB)}) \otimes (\otimes^{j}\mathcal{O}[-2]) \otimes (\otimes^{p}\Omega^{1}) \otimes E^{*} \longrightarrow F^{*} \tag{4.7.4}$$

where $2i + 2j + p$ = order of $\mathcal{D}$. (Ofcourse, several correction terms, involving varying degrees of derivatives of Φ_{ab} and Λ, may arise from one such projection). This gives something like a *curved symbol principle*. It is, as before, a simple matter to compute highest weight vectors in such products. As in the homogeneous symbol principle, the resulting sum of correction terms must be annihilated by x_1: applying x_1 lowers orders of differentiation, and so imposes consistency relations between correction terms. After satisfying these by choosing appropriate proportions of correction factors, all that will remain is to relate the scale of the highest order term to the correction terms. This is most easily done by the judicious partial expansion of the inducing maximal vector, or by inspection of the terms generated by the highest order part of the operator under conformal rescaling (cf [Penrose and Rindler I & II] for appropriate formuli). This seldom requires much work, and the answer is more likely to be reliable.

We conclude this section with two examples of this in practice.

<u>Conformally Invariant Operator extending</u> $\nabla^{A}_{(A'}\nabla^{B}_{B'}\nabla^{C}_{C'}\,\lambda_{D')ABC}$

The invariant differential operator is induced by the expression

$$\varphi = (-y_3 + y_1 Y_2)(-2y_3 + y_1 Y_2)(-3y_3 + y_1 Y_2) \alpha \qquad (4.7.5)$$

It is easy to check that the only possibility in (4.7.6) with $i+j > 0$ is $i = 1$, $p = 1$; accordingly, correction terms must be of the form

$$(\nabla^{A}_{(A'} \Phi^{B\,C}_{B'C'}) \lambda_{D')ABC} \qquad \text{or} \qquad \Phi^{A\,B}_{(A'B'}\nabla^{C}_{C'} \lambda_{D')ABC}$$

corresponding to

$$\theta = (y_1'\rho_{11}) Y_2{}^3\alpha - 2(y_1'\rho_{13}) Y_2{}^2\alpha - (y_3'\rho_{11}) Y_2{}^2\alpha + 4(y_3'\rho_{13}) Y_2\alpha + 2(y_1'\rho_{33}) Y_2\alpha - 6(y_3'\rho_{33}) \alpha$$

$$\tau = \rho_{11}\, y_1'Y_2{}^3\alpha - 2\,\rho_{13}\, y_1'Y_2{}^2\alpha - \rho_{11}\, y_3'Y_2{}^2\alpha + 4\,\rho_{13}\, y_3'Y_2\alpha + 2\,\rho_{33}\, y_1'y_2\alpha - 6\,\rho_{33}\, y_3'\alpha$$

in the expansion of (4.7.5); requiring that x_1 annihilate correction terms easily shows that θ and τ must occur in the ratio scaling $(\theta + 2\tau)$ so if we let φ^*, φ' be φ with y_i^*, y_i' in place of y_i, then

$$\varphi^* = \varphi' + a(\theta + 2\tau)$$

To fix a, multiply φ^* out. $(y_3\rho_{33})\alpha$ can occur only in the expansions of $(y_3^*)^3\alpha$ and $(y_3^*)^2 y_1 Y_2\alpha$ in terms of y_i' and ρ_{ij}; this gives $-12(y_3\rho_{33})\alpha$ in all, so that $a = 2$. Thus the operator

$$\lambda_{ABC} \longrightarrow \nabla^{A}_{(A'}\nabla^{B}_{B'}\nabla^{C}_{C'}\lambda_{D')ABC} + 2\,(\nabla^{A}_{(A'}\Phi^{B}_{B'}{}^{C}_{C'})\,\lambda_{D')ABC} + 4\,\Phi^{A\,B}_{(A'B'}\,\nabla^{C}_{C'}\lambda_{D')ABC}$$

is conformally invariant.

Extension of $\Box^2$: $\mathcal{O} \longrightarrow \mathcal{O}[-4]$

Employing (4.7.4) again, the following curvature correction terms are possible:

$$(\nabla^a\nabla_a\,\Lambda)\,f \quad ; \qquad (\nabla^a\,\Lambda)\,\nabla_a f \quad ; \qquad \Lambda\,\Box f \qquad (4.7.6)$$

$$(\nabla^a\nabla_b\Phi^b_a)\,f \quad ; \qquad (\nabla^a\Phi^b_a)\,\nabla_b f \quad ; \qquad \Phi^{ab}\nabla_a\nabla_b f \qquad (4.7.7)$$

The terms in (4.7.6) correspond to inducing expressions

$$\theta_1 = \{\,(y_1'y_4' + y_4'y_1' - y_2'y_3' - y_3'y_2')\Lambda\,\}\,\alpha$$

$$\theta_2 = \{\,(y_1'\Lambda)y_4' + (y_4'\Lambda)y_1' - (y_2'\Lambda)y_3' - (y_3'\Lambda)y_2'\,\}\,\alpha$$

$$\theta_3 = \Lambda\,\{\,y_1'y_4' + y_4'y_1 - y_2'y_3' - y_3'y_2'\,\}\,\alpha$$

Then $x_1(\,a\theta_1 + b\theta_2 + c\theta_3\,) = 0$ iff $b = c$ and $a = 0$. A similar calculation for (4.7.7) reveals that the modified version of $\Box^2$ must have the form

$$\Box^2 f + d\,\{\,(\nabla^a\,\Lambda)\,\nabla_a f + \Lambda\,\Box f\,\} + e\,\{\,(\nabla^a\Phi^b_a)\,\nabla_b f + \Phi^a_b\,\nabla_a\nabla^b f\,\} \qquad (4.7.8)$$

To determine d and e, one simply computes how $\Box^2$ varies under the rescaling of the metric $g_{ab} \longrightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$. Formuli for this are given in [Penrose and Rindler I & II]. For instance if $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$, where ∇_a is the Levi-Civita connection of g_{ab}, then the Levi Civita connection $\tilde{\nabla}_a$ of $\tilde{g}_{ab}$ is given on a spinor field $\varphi_{(A_1'\dots A_i')(A_1\dots A_j)}$ by

$$
\begin{aligned}
&\tilde{\nabla}_{AA'} \varphi_{(A_1'\dots A_j')(A_1\dots A_i)} \\
&= \nabla_{AA'} \varphi_{(A_1'\dots A_j')(A_1\dots A_i)} - \Upsilon_{AA_1'}\varphi_{(A'A_2'\dots A_j')(A_1\dots A_i)} \\
&\quad - \dots - \Upsilon_{AA_n'}\varphi_{(A_1'A_2'\dots A_{n-1}'A'A_{n+1}'\dots A_j')(A_1\dots A_i)} \\
&\quad - \dots - \Upsilon_{A_1A'}\varphi_{(A_1'\dots A_j')(AA_2\dots A_i)} - \\
&\quad - \dots - \Upsilon_{A_nA'}\varphi_{(A_1'\dots A_j')(A_1A_2\dots A_{n-1}AA_{n+1}\dots A_i)} - \dots .
\end{aligned}
$$

whilst

$$
\begin{aligned}
\tilde{\Phi}_{A'B'}^{A\ B} &= \Phi_{A'B'}^{A\ B} - \nabla_{(A'}^{(A}\Upsilon_{B')}^{B)} + \Upsilon_{(A'}^{(A}\Upsilon_{B')}^{B)} \\
\tilde{\Lambda} &= \Omega^{-2} \{\Lambda + \tfrac{1}{4} \nabla^a \Upsilon_a + \tfrac{1}{4} \Upsilon^a\Upsilon_a
\end{aligned}
$$

Employing these in (4.7.8) and ignoring all terms but those of the form $\Upsilon \Upsilon \Upsilon \nabla f$, we find $d = e = 4$, so that

$$
f \longrightarrow \nabla_a (\nabla^a\nabla^b + 4 \Phi^{ab} + 4 g^{ab}\Lambda) \nabla_b f
$$

is comformally invariant. (cf [Eastwood and Singer]) □

4.8 The Curved Penrose Tranform

This final section attempts to extend the constructions of section (3.8) to locally |1|-graded spaces. We compute, where possible, a curved Penrose transform, clarifying work of [Le Brun II] and reobtaining results of [Hitchin] on massless fields on half conformally flat manifolds. Assume $\mathcal{G}$, ω given with $X = \mathcal{G}/P$.

In order to construct a curved version of the Penrose transform, it is clearly necessary that we require $\mathcal{G}$ to admit not only a right action of P, but of a second parabolic P' also so that we may form the double fibration

$$\begin{array}{ccc} & \mathcal{G}/P\cap P' & \\ \mu \swarrow & & \searrow \nu \\ \mathcal{G}/P' & & \mathcal{G}/P \end{array} \qquad (4.8.1)$$

The action of P' should be compatible with the Cartan connection ω on $\mathcal{G}$: if v^* is a vector field on $\mathcal{G}$ generated by $v \in \mathfrak{p}' \subset \mathfrak{g}$, then $v^* \lrcorner \omega = v$ is necessary, and so $\omega^{-1} : \mathfrak{g} \to \Gamma(\tau\mathcal{G})$ must restrict to a homomorphism of Lie algebras on $\mathfrak{p}'$. Conversely, if this is so, it is clear that (a neighbourhood of the identity of) P' acts on $\mathcal{G}$ and (4.8.1) is defined. The existence of (4.8.1) is thus equivalent to the requirement that the curvature Ω vanish on $\omega^{-1}(\mathfrak{p}'\cap\mathfrak{g}_{-1})$.

Examples A

i) Let $\mathfrak{p}'$ correspond to the parabolic defining the space of null rays in $\mathbb{C}S^n$; let X be an n dimensional complex conformal manifold. Then $\mathfrak{p}'\cap\mathfrak{p}$ is one dimensional and the curvature condition is trivially satisfied. (4.8.1) exists, and

$G/P' = \mathcal{N}$, the space of null geodesics in X.

ii) Let X be a four dimensional conformal manifold, and consider $\mathfrak{p}' = \times\!\!-\!\!\bullet\!\!-\!\!\bullet$; then (in the notation of Example (2.1) (ii)), $\mathfrak{p}'\cap\mathfrak{p}$ is spanned by y_1 and y_3 so that the condition on Ω for a P' action on $\mathcal{G}$ is

$$\Omega\ (\ y_1^{*},\ y_3^{*}\ ) = 0 \qquad (4.8.2)$$

To compute this condition, define a $C^{2,2}$-valued function ϖ_1 on $\mathcal{G}$ by $\varpi_1(u,v) = \Omega_1(u^{*},v^{*})$, for $u^{*}\lrcorner\omega = u$, $v^{*}\lrcorner\omega = v \in \mathfrak{g}_{-1}$. ϖ_1 is not a section of the bundle induced by the irreducible representation of $\mathfrak{p}$ on $C^{2,2}$, since the action of P on Ω mixes Ω_0 into Ω_1; it is well defined if we choose locally a metric in the conformal class, equivalently, if we restrict to an $\tilde{L}^{s}$-principle subbundle of $\mathcal{G}$. ($\tilde{L}^{s}$ is the semisimple part of $\tilde{L}$). Differentiating Ω_0, we obtain, on $\omega^{-1}(\mathfrak{g}_{-1})$,

$$d\Omega_0 \;=\; [\ \Theta_{-1},\Omega_1\ ] \qquad \text{i.e.} \qquad d\varpi = \partial\varpi_1 \qquad (4.8.3)$$

Now $H^{2,2}(\mathfrak{g}_{-1},\mathfrak{g}) = 0$, so $\Box$ is an isomorphism on $C^{2,2}$, and (4.8.3) gives

$$\varpi_1 \;=\; -\ \Box^{-1}\ \partial^{*}\ d\varpi \qquad (4.8.4)$$

Picking a local metric, $\partial^{*}\ d\varpi$ is the projection of $d\varpi$ to $\overset{3}{\bullet}\!\!-\!\!\overset{-5}{\times}\!\!-\!\!\overset{1}{\bullet} \oplus \overset{1}{\bullet}\!\!-\!\!\overset{-5}{\times}\!\!-\!\!\overset{3}{\bullet}$ and (4.8.2) is then simply the requirement that the projection

$$\mathcal{W} \in \overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet} \longrightarrow \Psi_{A'B'C'D'} \in \overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}$$

and hence the projection

$$\mathcal{W}_1 \in \overset{3}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet} \oplus \overset{1}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{3}{\bullet} \longrightarrow \nabla_A^{A'} \Psi_{A'B'C'D'} \in \overset{3}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet}$$

vanish - in other words, X must be half conformally flat. $G/P' = P\mathcal{T}$ is the curved twistor space of [Penrose I]. □

<u>Note</u> Example A (ii) relies heavily on the splitting $H^{1,2}(\mathfrak{g}_{-1},\mathfrak{g}) \cong \overset{4}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{4}{\bullet}$ which arises, ofcourse, because $\Omega^2(X)$ is reducible. In higher dimensions this is not the case, and no such simple minded analogue of the non-linear graviton will work - but cf [Eastwood II] and [Salamon]. □

The apparatus of section (3.8) may now be applied directly to transform the cohomology of induced bundles on $\mathcal{G}/P'$ to sections of induced bundles on G/P, together with curved invariant differential operators. Ofcourse, because of curvature, most of the resulting equations will not admit solutions, so from a twistorial point of view, the construction is an anti-climax. Algebraically, the curved version of the Bernstein-Gelfand-Gelfand sequence on X is no longer a complex, let alone exact; these properties depend on the global Lie algebra structure of $\mathfrak{g}$ and not just on $\mathfrak{p}$, which, under $w^{-1} : \mathfrak{g} \longrightarrow \Gamma(\tau\mathcal{G})$ are no longer available. For example, consider the composition of curved invariant differential operators

$$\overset{3}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{1}{\bullet} \longrightarrow \overset{1}{\bullet}\!-\!\overset{-5}{\times}\!-\!\overset{0}{\bullet}$$

(cf the resolution of $\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$: Example (3.8) A (ii)), which in the homogeneous case is the map

$$\lambda_{A'B'C'} \longrightarrow \nabla^{AA'} \nabla_{A}^{B'} \lambda_{A'B'C'}$$

This is induced by

$$3\,(Y_1{}^2[y_1{}^*, y_3{}^*])\,\alpha \;-\; 2\,(Y_1[y_1{}^*, y_3{}^*])\,Y_1\alpha \;+\; [y_1{}^*, y_3{}^*]\,Y_1{}^2\alpha \tag{4.8.5}$$

whose vanishing is the curvature restriction (4.8.2) (or, in rare circumstances, for specific α, a speciality condition on Ω - cf [Penrose and Rindler I]). (4.8.5) is

$$\lambda_{A'B'C'} \longrightarrow \Psi_{A'}{}^{B'C'D'} \lambda_{B'C'D'}$$

This construction is quite general, and leads to numerous tensors which are invariants of the local |1|-grading of X: we speculate that, in some sense, this is a complete list (cf [Fefferman and Graham]).

Notice however that singular infinitesmal character will lead again to the existence of solutions to equations which might otherwise have been expected to have followed the general pattern and been constrained by curvature (cf subsection (3.8.4)). The existence or not of such constraints is the Buchdahl conditions [Buchdahl H]; notice that the "first" non-singular Bernstein-Gelfand-Gelfand sequence on a locally |1|-graded manifold is always the de Rham sequence, which is ofcourse always exact.

Example B

Let $\mathcal{G}$, X, $P\mathcal{T}$ be as in example A (ii) above; we readily find

$$
\begin{aligned}
& H^1(P\mathcal{T}, \mathcal{O}(-n-2)) = H^1(P\mathcal{T}, \overset{-n-2}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \\
\cong\ & \operatorname{Ker} \nabla_A^{A_1'} : \mathcal{O}_{A_1' \ldots A_n'} \longrightarrow \mathcal{O}_{AA_2' \ldots A_n'} \qquad n \geq 1 \\
& \operatorname{Ker} (\Box + R/6) : \mathcal{O}[-1] \longrightarrow \mathcal{O}[-3] \qquad n = 0
\end{aligned}
$$

(and see that (4.8.2) implies that there are no curvature obstructions to the solution of these equations) (cf [Hitchin]). Notices that $n \leq -1$ produces certain spinor fields on X which, in the homogeneous case would be interpretted as potentials for massless fields of negative helicity; this is possible if $n = -1$ or -2, since no curvature obstructions can arise in the singular Bernstein-Gelfand-Gelfand sequence. The construction fails for $n \leq -3$ because the Bernstein-Gelfand-Gelfand sequence is no longer exact at its central square.

One might be more adventurous and attempt to transform other bundles on $P\mathcal{T}$, or, if this does not exist, on $\mathcal{N}$ = curved ambitwistor space. (cf Example A(i)). A natural bundle to consider (cf subsection (2.7.3)) is $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$ on $P\mathcal{T}$ or $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}$ on $\mathcal{N}$. [Le Brun II] calls this latter the Einstein Bundle on $\mathcal{N}$. We find :

$$H^0(\mathcal{N}, \times\!-\!\bullet\!-\!\times) \cong H^0(P\mathcal{T}, \times\!-\!\bullet\!-\!\bullet) \text{ (if it exists)} \tag{4.8.6}$$
$$\cong \text{ Ker } \nabla^{(A'}_{(A}\nabla^{B')}_{B)} + \Phi^{A'B'}_{A\,B} : \overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}$$

Suppose there exists a non-trivial element in the Kernel in (4.8.6), f, say, non-zero on some region of X. By definition, under $g_{ab} \longrightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$, $f \longrightarrow \tilde{f} = \Omega f$; in particular, taking $\Omega = f^{-1}$, $\tilde{f} = 1$ and $\Phi^{A'B'}_{A\,B}$ must vanish. In other words, as Le Brun observed, the existence of non-trivial elements of $H^0(\mathcal{N}, \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times})$ or $H^0(P\mathcal{T}, \overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet})$ (if it exists) is equivalent to X being conformally Einstein.

An element of either H^0 should be thought of (cf subsection (2.7.3)) as if it arose from an element of $F(\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet})^*$, $I_{\mathcal{A}}$; indeed, using the curved version of the differential splitting of $\mathcal{T}_{[\alpha\beta]}$ in example (3.7) C, f gives rise to a section of $\mathcal{T}_{[\alpha\beta]}$ on X by

$$f \longrightarrow (\ f\ ,\ \nabla^{A'}_{A} f\ ,\ (\Box + R/6)f\ )$$

in terms of the local splitting of $\mathcal{T}_{[\alpha\beta]}$ by choice of metric. Taking this to be the choice given by $f \equiv 1$, it is clear that the resulting section of $\mathcal{T}_{[\alpha\beta]}$ will be simple (equivalently null) iff $R = 0$. That is, manifolds X conformal to vacuum solutions of Einstein's equations are charaterized by the existence of simple sections of $H^0(\mathcal{N}, \overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times})$.

This generalizes to higher dimensions. □

5 : Conclusion

We have given an algebraic construction for invariant differential operators on homogeneous bundles over complex generalized flag manifolds. We have shown how they may be determined, via the *symbol principle*, and given several results concerning their existence, including results concerning the induction of invariant differential operators under direct images. We have studied the relationship of invariant differential operators to the directed graph structure of the Weyl group of a semisimple Lie algebra (and its subgraphs for parabolic subalgebras) and given the *Translation Principle*, an isomorphism of categories by means of which new invariant operators could be generated from old, enabling the classification problem to be reduced to a finite checking of cases.

Using the algebraic mechanisms to hand, with the *Bernstein - Gelfand - Gelfand* sequence, we have given an *Algebraic Penrose Transform* between holomorphic objects on generalized flag manifolds. The standard results of Twistor Theory in four dimensions have been generalized to higher dimensions.

By studying canonically defined principle bundles $\mathcal{G}$ over a class of curved manifolds - the *locally |1|-graded manifolds* - with a canonically defined *Cartan connection* (an osculation of $\mathscr{G}$ by a complex Lie group G), we have been able to extend the invariant operators of the homogeneous case to a class of differential operators on induced bundles over these curved spaces which are natural invariants of the local structure of these manifolds. We have shown how to construct these operators given a torsion free connection respecting the local structure. In particular we have given several examples for conformal four manifolds.

A *Curved Penrose Transform* has been constructed. We have employed it to encode zero rest mass fields of spin at least -1 in terms of cohomology classes on a curved twistor space for a half conformally flat spacetime, and to elucidate the *Einstein Bundle* of [Le Brun II].

There are several possibilities for the extension of some of the ideas used in the Thesis. The most promising would appear to be the use of an algebraic form of the Penrose transform to investigate the composition series of the modules $\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$ in terms of irreducible $\mathfrak{U}(\mathfrak{g})$-modules $\mathcal{Q}uo\,_{\mathfrak{b}}^{\mathfrak{g}}\,\mu$. We have seen that any element of this composition series gives rise to a homomorphism of induced modules, $\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\mu \longrightarrow \mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$ say. If λ, μ are dominant for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, this descends to a standard homomorphism $\mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\mu \longrightarrow \mathcal{I}nd\,_{\mathfrak{p}}^{\mathfrak{g}}\,\lambda$ which may be zero. If it is, there may nonetheless exist a non-zero non-standard homomorphism and a further factor $\mathcal{Q}uo\,_{\mathfrak{b}}^{\mathfrak{g}}\,\mu$ in the composition series for $\mathcal{I}nd\,_{\mathfrak{b}}^{\mathfrak{g}}\,\lambda$. The best result currently known concerning these composition series is the (proven)

Kazhdan - Lusztig conjecture [Vogan]; its proof relies on the Weil conjectures and uses Deligne-Goresky-MacPherson intersection homology. On the other hand, the *Algebraic Penrose Transform* gives a means, via the *hypercohomology spectral sequence* of constructing the non-standard homomorphisms from compositions of homomorphisms on modules induced from smaller parabolics. The results, expressed as invariant differential operators, are often extremely simple - the wave operator is an example. This suggests that a careful study of the *Algebraic Penrose Transform* may be quite fruitful.

Another avenue for investigation is the following : if $\mathfrak{g}$ is $sl(n,\mathbb{C})$ with $|1|$-grading ×—•— ⋯ —•, then the P-principle bundle $\mathcal{G}$ on a space X with such a local $|1|$-grading is not uniquely defined. It may be constructed, but its global possibilities will be parameterized by $H^1(X,\mathcal{H}^{1,1})$ where $\mathcal{H}^{1,1}$ is the bundle induced by the representation of P on the Spencer cohomology group $H^{1,1}(\mathfrak{g}_{-1},\mathfrak{g})$. There should then be an obstruction to the global construction of a holomorphic Cartan connection on $\mathcal{G}$, probably lying in $H^2(X,\mathcal{H}^{1,2})$. There is some hope that for n = 3 this may be related to the *googly problem* of encoding conformal curvature of the wrong handedness on a curved Twistor space.

Finally, in the curved case, the composition of invariant differential operators in the non-exact Bertstein-Gelfand-Gelfand sequences leads, as we have seen, to curvature invariants of the structure of X. This appears to be of some interest in complex analysis - cf [Fefferman & Graham]. It may be possible to show that these constitute all such invariants.

Notation

Below is some of the commonly used notation in the Thesis. It is listed by the section in which it is first defined.

Notation	Section	Brief Explanation
$\mathfrak{g}$	2.1	complex semisimple Lie Algebra
$\mathfrak{h}$	2.1	Cartan subalgebra
$\Delta(\mathfrak{g},\mathfrak{h})$	2.1	roots of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$
$\Delta^+(\mathfrak{g},\mathfrak{h})$, Δ^+	2.1	positive roots of $\mathfrak{g}$
$\mathcal{S}$	2.1	simple roots of $\mathfrak{g}$
E_α	2.1	α-root space of $\mathfrak{g}$
$\mathfrak{h}^*_{\mathbb{R}}$	2.1	real weights for $\mathfrak{g}$
$\mathfrak{b}$	2.1	Borel subalgebra of $\mathfrak{g}$
$\mathfrak{p}$	2.1	parabolic subalgebra of $\mathfrak{g}$
$\Psi(\mathfrak{p})$	2.1	simple defining roots for $\mathfrak{p}$
$\mathcal{u},\mathcal{u}_-$	2.1	nilpotent subalgebras of $\mathfrak{g}$
$\mathfrak{l}$	2.1	reductive Levi factor of $\mathfrak{p}$
$\mathfrak{l}^s$	2.1	semisimple part of $\mathfrak{l}$
$\mathfrak{l}_Z$	2.1	centre of $\mathfrak{l}$
x_i, X_i, y_i, Y_i h_i	2.1	generators of $sl(4,\mathbb{C})$
PT, PT^*, A, F	2.2	spaces of Twistor Theory
M^k, PT^k, N^k	2.2	higher dimensional Minkowski twistor and null geodesic spaces

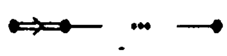
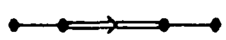
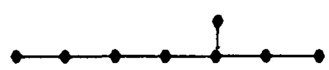
$\mathcal{O}(V)$	2.4	bundle induced by representation V
$\Delta(V)$	2.4	weights in V
$\lambda\leq\lambda'$	2.4	ordering on weights
$\langle\cdot,\cdot\rangle$	2.4	+ve def. forms on $h^*_{\mathbb{R}}$
W_α	2.4	wall perp to α
$\alpha^\vee$	2.4	co-root of α
λ_i	2.4	notation basis for weights
F^λ	2.4	g-rep of highest weight λ
$h^*_{\mathfrak{p}}$	2.4	space of weights for $\mathfrak{p}$
$\lambda_{\mathfrak{p}},\mu_{\mathfrak{p}}$	2.4	weights for $\mathfrak{p}$
$F^\lambda_{\mathfrak{p}}$	2.4	$\mathfrak{p}$-rep of highest weight $\lambda_{\mathfrak{p}}$
$O(\lambda_{\mathfrak{p}})$	2.4	bundle $\mathcal{O}(F^{\lambda^*}_{\mathfrak{p}})$
$\mathcal{O}^{A'},\mathcal{O}_A,\mathcal{O}[n]$	2.4	standard bundles on M^4
$W(g)$	2.6	Weyl group for g
σ_α	2.6	reflection in wall W_α
$l(w)$	2.6	length of $w\in W(g)$
$F(\lambda)$	2.6	g-rep of extremal weight λ
$F(\lambda_{\mathfrak{p}})$	2.6	$\mathfrak{p}$-rep of extremal weight λ wrt $W(\mathfrak{l}^S)$
$w\longrightarrow w'$ $w\leq w'$	2.6	directed graph structure of $W(g)$
$\Delta(w)$	2.6	positive roots sent negative by w^{-1}
$W^1(\mathfrak{p})$	2.6	subgraph associated to $\mathfrak{p}$
$\mathcal{J}^r(\mathcal{E})$, $\mathcal{J}^\infty(\mathcal{E})$	3.1	jet bundles
$\odot^r$	3.1	r'th symmetric tensor power
$\mathcal{I}nd^g_{\mathfrak{p}}$	3.2	inducing functor
$\mathcal{Q}uo^g_{\mathfrak{p}}$, $\mathcal{S}ub^g_{\mathfrak{p}}$	3.2	irreducible quotient, maximal proper submodule

ξ_λ	3.4	infinitesmal character assoc. to λ
ρ	3.4	semi-sum of positive roots of $\mathfrak{g}$
$w\cdot\lambda$	3.4	affine action of $W(\mathfrak{g})$
P_χ, P_λ	3.4	χ primary part functors
$\Psi_{\lambda_2}^{\lambda_1}$	3.4	translation functors
ω	4.2	Cartan connection
$\mathcal{G}$	4.2	P-principle bundle replacing G
$DO(\mathcal{G})$	4.2	linear differential operators on $\mathcal{G}$
$[\cdot,\cdot]$	4.2	commutator on $\mathfrak{g}$-valued forms
$\mathfrak{g}_i$	4.3	$\|1\|$-grading on $\mathfrak{g}$
$F^r(X)$	4.3	r'th frame bundle on X
$G^r(n)$	4.3	frame bundle fibre group in n dims
$\Theta^{(r)}$	4.3	canonical form on $F^r(X)$
θ	4.3	canonical form on $F^1(X)$
Θ_{-1}, Θ_0	4.3	$\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ components of $\Theta^{(1)}$
Γ	4.3	torsion free connection on $F^1(X)$
s_Γ	4.3	admissible section $F^1(X)\to F^2(X)$
$\tilde{L}$, $\tilde{P}$	4.3	groups covered by L, P
$\mathcal{Q}$	4.3	$\tilde{L}$-structure subbundle of $F^1(X)$
$\mathcal{G}(\Gamma)$	4.3	$\mathcal{G}$ constructed using Γ
∂	4.3	Spencer cohomology boundary operator
$C^{p,q}$	4.3	Spencer cohomology co-chains
$H^{p,q}$	4.3	Spencer cohomology

∂^*	4.4	adjoint of ∂
$\mathscr{w}(\omega)$	4.5	Weyl co-cycle of ω
f	4.5	difference in ω_1 and ω_1'
$\mathcal{R}(\Gamma)$	4.6	curvature two form of Γ
$r(\Gamma)$	4.6	curvature tensor of Γ
Φ_{ab}, Λ	4.7	tracefree and scalar parts of Ricci curvature in 4 dims
ρ_{ij}	4.7	matrix of correction terms

Dynkin Diagrams

For the convenience of the reader we list the Dynkin Diagrams of the complex simple Lie Algebras :

A_n		$sl(n+1,\mathbb{C})$
B_n		$so(2n+1,\mathbb{C})$
C_n		$sp(2n,\mathbb{C})$
D_n		$so(2n,\mathbb{C})$
F_4		
G_2		
E_6		
E_7		
E_8		

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