DEPARTMENT OF ECONOMICS
OxCarre (Oxford Centre for the Analysis of Resource Rich Economies)

Manor Road Building, Manor Road, Oxford OX1 3UQ
Tel: +44(0)1865 281281 Fax: +44(0)1865 281163
reception@economics.ox.ac.uk www.economics.ox.ac.uk

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Pricing sovereign debt in resource rich economies?

Thomas McGregor
OxCarre

Direct tel: +44(0) 1865 281281 E-mail: antonella.surdi@economics.ox.ac.uk

Pricing sovereign debt in resource rich economies

Thomas McGregor *

Oxford Centre for the Analysis of Resource Rich Economies (OxCarre)
St Hugh's College, University of Oxford
Department of Economics, University of Oxford

Abstract

This paper investigates the link between commodity price movements and risk premiums in resource-dependent, developing economies. I develop a stochastic general equilibrium model of a small open economy that receives a stream of resource revenues. The government sells bonds to foreign investors which it can renege on in the future, at some cost, whilst international investors form expectations on the likelihood of sovereign default. This delivers an endogenous risk premium which is inversely related to the price of oil. The model is able to explain a large proportion of the business cycle fluctuations in interest-rate spreads in resource dependent developing economies. I then ask how specific structural features of developing economies affect the relationship between commodity prices and the optimal price of sovereign debt, including: a higher dependence on natural resource revenues, impatient consumers and governments, a higher degree of risk-aversion, and a lower ability to substitute consumption inter-temporally. Including them in the model significantly improves the ability of the model to explain the key macroeconomic co-movements in a resource rich, developing economy context. Model simulations reveal an interesting policy insight. An endogenous risk premium that is driven by falling oil prices, provides an additional rationale for a *volatility* fund in which liquidity buffers are accumulated to manage debt repayments. These buffers should be larger the stronger the link between oil prices and the domestic economy is, the more impatient policy makers are and the more willing they are to substitute current for future consumption.

JEL codes: E13, E32, E44, F34, O11, O13, O16, H63
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*Correspondence: Department of Economics, Manor Road Building, Oxford OX1 3UQ, UK; Email: thomas.mcgregor@economics.ox.ac.uk

1 Introduction

After a series of high profile defaults in the late-90s, including by Russia (1998), Ecuador (1999) and Argentina (2001), sovereign default issues moved to the forefront of economic policy discussions. Interest was again renewed by the, still ongoing, debt crisis in Greece and other peripheral eurozone countries following the financial crisis of 2007/08. The collapse in commodity prices in 2014/15 has sparked a new wave of debt distress across the resource-dependent developing world[1] leading to the de-facto default by Mozambique earlier this year. Given the continued slump in commodity prices and the sharp rise in sovereign debt levels in many developing economies, the question of how countries can best manage volatile oil revenues is as pertinent as ever.

The conventional wisdom for countries managing oil and gas windfalls relies on the permanent income hypothesis (PIH) of Friedman (1957). Governments should smooth consumption by borrowing before a windfall and building up foreign assets in a sovereign wealth fund, or repaying existing debt, during the windfall. Whilst this is often good advice for advanced economies, it can be dangerous for developing countries. First, the inherently volatile nature of commodity prices has led some to propose the use of *volatility* funds, which would be larger the more permanent the windfall is (Bems & de Carvalho Filho (2011)). Second, if a country faces high borrowing costs or restricted access to international finance, investment will be too low and oil revenues may be better used to invest in the domestic economy. Van der Ploeg & Venables (2011b) show that an *investment* fund, which prioritises domestic asset accumulation, may be useful in this respect. In addition, given that developing countries often face significant absorptive constraints, this fund could be used to manage the timing of these investments.[2]

This paper investigates a specific source of volatility in resource rich economies - the link between the oil prices and sovereign risk premiums. Empirically, there is a strong negative relationship between oil prices and interest rates in resource rich developing economies. I develop a stochastic general equilibrium model of a small open economy that receives a stream of income from production and a separate stream of resource revenues. The government trades in sovereign bonds with risk-neutral, competitive investors and chooses the level of consumption and debt position, as well as whether or not to default on its existing debt, where defaulting carries some cost. International investors form expectations on the likelihood of default in any given period and charge a risk premium on sovereign borrowing to account for this. This endogenous risk premium, which is affected by the stock of debt, as well as the stochastic processes in the model, then feeds back into the government's optimal debt position. The result is a negative correlation between oil prices and interest rates. The model is able to explain a large proportion of the business cycle fluctuations in interest-rate spreads in resource dependent, developing economies, particularly the strong countercyclical movements of interest rate spreads with oil prices. I also show that a standard model without oil revenues is unable to explain a significant proportion

[1]This includes ongoing debt crises in Nigeria, Gabon and Azerbaijan, and worryingly high debt levels in Angola, Ghana, Kenya and South Africa

[2]See Collier et al. (2010); Van der Ploeg & Venables (2011b); Wills (2015) for a review of the theoretical underpinnings of these different 'types' of funds

of the variability in the data.

Using the model, I attempt to answer two important questions. How do commodity price movements affect the pricing of sovereign debt in resource rich economies? And, what does this mean for the optimal management of volatile oil revenues in developing countries? Falling oil prices present an important source of default risk for resource dependent sovereigns. Managing this risk is important for developing countries, particularly those that are capital scare. The model delivers an endogenous link between oil prices and country risk premiums and is, to the best of my knowledge, new to the literature[3]. This mechanism provides an additional rationale for a *volatility* fund in which liquidity buffers are accumulated to manage debt repayments during negative oil price shocks.

This work is closely related to a large literature on sudden capital flow reversals in emerging markets, which emerged following the Latin American and East Asian financial crises of the 1990s. Original work by Calvo (1998) investigates the mechanisms of sudden capital reversals in developing economies, arguing that concerns of bankruptcy and financial disruption can often become self-fulfilling with large deleterious effects on the domestic economy. More recent work has highlighted the importance of debt-inflation dynamics and endogenous credit constraints in explaining sudden reversals in emerging markets. Calvo & Mendoza (2000) show that when the collection of information on developing countries is costly and imperfect, globalisation can reduce the expected-utility gain of paying information costs, thus leading to actions by a small number of informed investors causing large swings in portfolio rebalancing by uninformed investors. Mendoza (2006) develops a quantitative analysis of this mechanism in a dynamic, stochastic general equilibrium (DSGE) framework. He shows that these episodes, which are sparked by shocks to fundamentals, can result in credit constraints that bind when countries are highly indebted, causing significant output declines. Mendoza & Yue (2012) extend this framework further to include working capital constraints. Here, default results in capital market exclusions and triggers an efficiency loss as these inputs are replaced by imperfect local substitutes.

There is also a growing literature on the empirics of default premiums in emerging economies. Volatile interest rate spreads have been found to be an important driver of output volatility in developing economies. Neumeyer & Perri (2005) and Uribe & Yue (2006) provide evidence that interest rate spreads can explain a sizeable proportion of output volatility and vice versa. Shocks to trend output are also empirically important in emerging markets. Aguiar & Gopinath (2006) develop a model in which the likelihood of default is affected more after a trend shock than compared to a transitory one.

[3]Work by Van der Ploeg & Venables (2011a) shows the importance that an exogenous risk premium on international borrowing has on the saving and investment decisions for developing economies. Whilst some recent work investigates the empirical nature of this question (Arezki & Brückner (2012)), to the best of my knowledge, this paper is the first to address this in a comprehensive macroeconomic context. Work by Van der Ploeg & Venables (2011a) shows the importance that an exogenous risk premium on international borrowing has on the saving and investment decisions for developing economies.

The seminal works of Eaton & Gersovitz (1981) and Grossman & Van Huyck (1988) were the first to study the link between default risks and optimal borrowing behaviour in a dynamic macroeconomic framework. Eaton & Gersovitz (1981) develop a model of sovereign debt in which the possibility of permanent market exclusion following default is sufficient to generate an endogenous debt limit; below which it is always optimal to service outstanding debt but above which default is optimal for the sovereign. Grossman & Van Huyck (1988) show that even non-permanent market exclusion can present a sufficient risk to the sovereign, yielding a similar result (see also Kletzer & Wright (2000)). Finally, Park (2017) shows that in a model with capital accumulation default can also occur in 'good' times[4].

Following this work, the literature on sovereign default has expanded to tackle a number of different questions. Arellano (2008) develops a small open economy model of default under limited commitment. Their model delivers a strongly countercyclical process for interest rate spreads and output. Default is more likely in recessions because this is when it is more costly for a risk averse borrower to repay non-contingent debt. Adam & Grill (2017) extend the existing Ramsey policy literature to a setting with non-contingent sovereign debt and continuous default costs. They find that, when government bond markets are incomplete, and under certain parameterisations of default costs, partial default is Ramsey optimal when a country's wealth is sufficiently low.[5]

There also exists a large body of research that explores the link between commodity prices and macroeconomic performance. Hamilton (2003) and Barsky & Kilian (2004) find that whilst there does exist a strong negative link between oil prices and growth in the US, the relationship is complex, often subject to reverse causality and almost certainly non-linear. For commodity exporters, higher commodity prices have positive short-term effects on output and growth but adverse long-term effects (see Deaton et al. (1995); Raddatz (2007); McGregor (2017)). This is particularly true in countries with poor governance (Collier & Goderis (2012) Mehlum et al. (2006)). An extensive literature on the 'resource curse' (Sachs & Warner (1999)) in developing economies also exists, in which resource booms lead to worsening economic and development performance over the long run (see Van der Ploeg (2011) for a review of the literature).

The remainder of the papr is organised as follows. Section 2 describes the evolution of interest rate spreads in resource rich, developing economies and presents some evidence for a counter-cyclical relationship between spreads and oil prices. Section 3 sets out the stochastic debt model. Section 4 solves the model under a baseline calibration and discusses some of it's key properties. Section 5 derives an analytic expression for the price of sovereign debt and estimates a numerical approximation to this under the baseline model. Section 6 asses the model's performance along some key dimensions. Section 7 concludes.

[4]The model delivers a U-shape in the capital stock: at both low and high levels of capital, the economy has an incentive to default on it's debt. Default in good times occurs when the economy has over-invested in capital during booms

[5]Others include the inclusion of long-term debt (Chatterjee & Eyigungor (2012)), CRRA lenders Lizaro (0173), misspecified default probabilities (Pouzo & Presno (2016)) and bailout risk Fink & Scholl (2016).

2 Emerging market spreads

There is growing evidence that emerging market output volatility may be related to the ability for emerging markets to access international financial markets in times of need. In particular, a large proportion of the volatility in output is due to highly countercyclical interest rates and countercyclical default risk. Work by Neumeyer & Perri (2005) presents evidence that in contrast to advanced economies, emerging market business cycles are more volatile and interest rates are countercyclical and lead the economic cycle. Uribe & Yue (2006) show that the majority of the movements in emerging market spreads are due directly to changes in the country specific spread, or the risk-premium. To date, the literature has remained relatively silent on these issues in a resource rich setting.

These relationships are presented in figure 1. I plot quarterly real GDP in logs and the interest-rate spread over four oil exporting, developing economies: Colombia, Indonesia, Mexico and Russia.[6] The data are seasonally-adjusted and de-trended.[7] They cover the period 1995q1-2016q3. Interest rate spreads in these four economies have been volatile, as has GDP growth. The various economic downturns are visible for each country, with an upward spike in the interest rate spread coinciding with, and often leading, the economic slump. All experienced sharp downturns between 2009 and 2010 in the aftermath of the financial crisis, apart from Indonesia which recovered relatively quickly.

Critically, these countries are highly resource-dependent who rely heavily on oil exports as a source of foreign exchange. An empirical regularity in these economies is a negative correlation between oil prices and interest rate spreads. Spreads remain low during the 'good' times when oil prices are high, and rise in the 'bad'. This negative relationship can be seen clearly in figure 2, which plots the real world oil price against the interest rate spreads for these four economies. The price of crude oil has been highly volatile, with a pronounced boom and bust cycle during the 2007/08 financial crisis and a continued downward spiral since the end of 2014. It has fluctuated between a low of around $US 25 in 2001q4 and a peak of $US 124 in 2008q2. Spreads have also been volatile and negatively correlated with the oil price. This negative relationship is confirmed in the data where the correlation coefficient between the mean interest rate spread and the oil price is $-.14$[8]. Clearly, oil price volatility is an important contributor to the pricing of risk and debt in these economies.

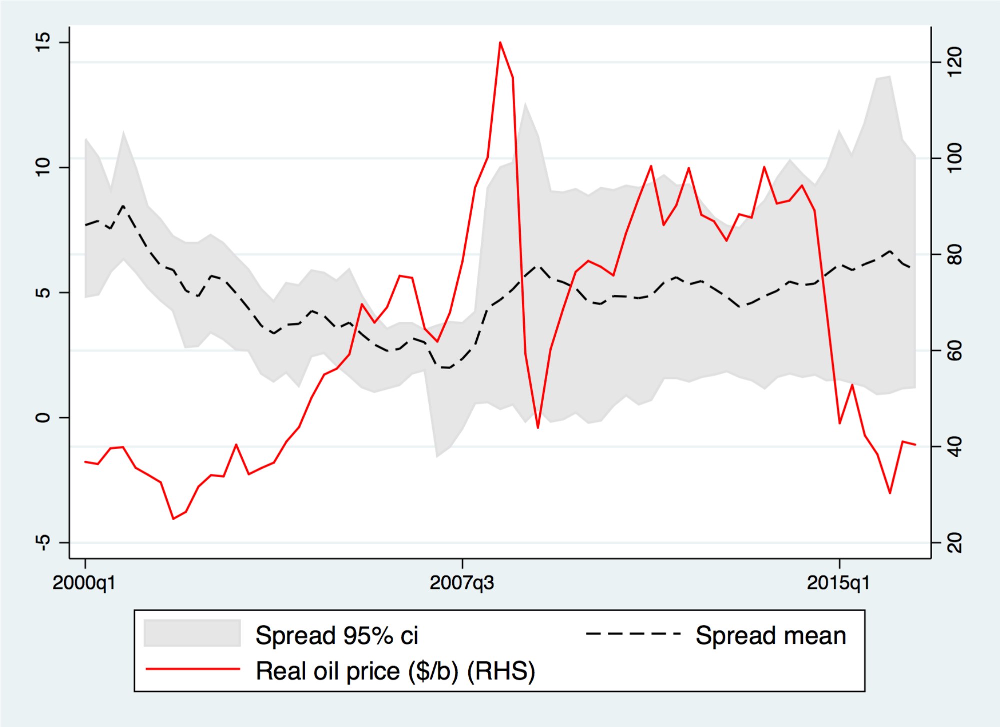
[6]These four countries comprise the full set of resource rich, developing economies who are net exporters of oil and for which we have quarterly GDP and interest-rate data.

[7]I decompose the real GDP and spread series into a trend and cyclical component, using a simple HP filter with smoothing parameter of 1600. The interest rate spread is the difference between the yield on 10-year government bonds in each emerging economy and the US

[8]The individual country correlations are: Colombia = $-.33$, Indonesia = $-.03$, Mexico = $-.09$ and Russia = $-.12$

Figure 1: Emerging market economic performance

Figure 2: World oil price



3 The model

The model here is based on original work by Eaton & Gersovitz (1981) on international lending which has been extended by Arellano (2008) to study the default experience of Argentina. Consider a small open economy that receives two stochastic streams of income; one from output, y, and one from tax revenues from the oil sector, p, as follows:

$$\begin{aligned} y_t &= \mu_y + z_t \\ z_t &= \rho_z z_{t-1} + \epsilon_t^z \\ \epsilon_t^z &\sim N(0, \sigma_z^2) \end{aligned} \tag{1}$$

$$\begin{aligned} p_t &= \mu_p + h_t \\ h_t &= \rho_h h_{t-1} + \epsilon_t^h \\ \epsilon_t^s &\sim N(0, \sigma_h^2) \end{aligned} \tag{2}$$

where μ_y and μ_p are the means of output and the oil price respectively, and ϵ_t^z and ϵ_t^h are the i.i.d shocks to each series. The representative agent seeks to maximise CES utility as follows:

$$U_t = \left[c_t^{1-\sigma} + \beta\left(\mathbb{E}_t U_{t+1}^{1-\sigma}\right)\right]^{\frac{1}{1-\sigma}} \tag{3}$$

where, as is standard in this class of utility function, the coefficient of relative risk aversion is given by $\sigma \geq 0$, $1/\sigma$ is the inter-temporal elasticity of substitution (IES), and $\beta = \frac{1}{1+\rho}$ is the discount factor.

In every period the government chooses to either default on existing debt, D, or to not default and continue servicing the debt stock, N. This is different to the proportional default approach taken in some of the literature. In the default state consumption is simply equal to income plus oil revenues under default, $c^D = y^D + o^D$, where y^D and o^D are as follows:

$$y_t^D = \begin{cases} y_t, & \text{if } y_t < \overline{y} \\ \hat{y} = \psi\overline{y}, & \text{if } y_t > \overline{y} \end{cases} \tag{4}$$

$$o_t^D = \begin{cases} p_t\bar{s}, & \text{if } p_t < \overline{p} \\ \hat{o} = \psi\left(\overline{p}\overline{s}\right), & \text{if } p_t > \overline{p} \end{cases} \tag{5}$$

where $\psi \in (0, 1)$ represents the (common) output and oil revenues cost of default (higher ψ means lower cost of default), $\overline{y} = \mathbb{E}\{y\}$ is expected output and $\overline{p} = \mathbb{E}\{p\}$ is the expected oil price, $\bar{s}$ is the annual flow of extracted oil, assumed to be fixed, and p is the stochastic world oil price. The default cost is assumed to be a fixed proportion of income for sufficiently high levels of income.

If the government defaults it is then assumed to be in autarky (i.e. the country is excluded from capital markets) with an exogenous probability of re-entry in the future. If either output or the oil price is above their respective means, then actual output and oil revenues under default are truncated down by a fixed proportion of the expected variables, as governed by the cost parameter, ψ. The result is an asymmetric cost of default schedule. The output and oil revenue processes are truncated because it is assumed that default entails some direct cost. For output these costs may arise due to the inability of firms to access credit for investment or imports, reduced foreign investment, costly legal processes, political upheaval and civil protest, etc. For oil revenues, defaulting may simply result in a punitive default procedure in which the taxation of oil revenue, which are an immediate source of foreign exchange earnings for the government, are used to pay preferred creditors immediately.[9] The main motivation for doing this is to bring the default probability implied by the model in line with the data (see Arellano (2008)).

In the no-default state consumption, c^N is equal to stochastic income plus oil plus new borrowing, net of interest payment on existing borrowing as follows:

$$c_t^N = y_t + p_t\bar{s} - q_t(d_{t+1}, y_t, p_t)d_{t+1} + d_t \tag{6}$$

where, d_{t+1} represents bonds sold at price $q(\cdot)$, and d_t is the repayment due to previous borrowing (for indebted countries, $d < 0$).

Creditors are assumed to be risk-neutral. They lend to sovereigns in the current period by buying bonds, d_{t+1}, at price $q_t(\cdot)$. In the following period, the creditors receive the face value of the bond if the sovereign does not default, and nothing if the sovereign defaults. The probability of default is given by:

$$\delta(d_{t+1}, y_t, p_t) = Pr\big[V^D(y_t, p_t) > V^N(d_{t+1}, y_t, p_t)\big] \tag{7}$$

so the probability of the creditor getting paid (i.e. no default) is $1 - \delta(d_{t+1}, y_t, p_t)$.

The bond pricing schedule, $q_t(d_{t+1}, y_t, p_t)$, is given by:

$$q_t(d_{t+1}, y_t, p_t) = \frac{1 - \mathbb{E}_t\{\delta(d_{t+1}, y_t, p_t)\}}{1 + r_t} \tag{8}$$

where r_t is the risk-free interest rate and $\mathbb{E}$ is the expectations operator. The bond price depends on the default probability, $\delta(d_{t+1}, y_t, p_t)$, the current output state, y_t and the asset holdings chosen by the sovereign in the following period, d_{t+1}.

Every period the sovereign's default decision can be represented by:

$$V(d_t, y_t, p_t) = \max_{D,N}\{V^D(y_t, p_t), V^N(d_t, y_t, p_t)\} \tag{9}$$

[9] See figure 11 in the appendix for graphical exposition of this output truncation process simulated over 1000 periods. As we can see, the output realisation under default has a ceiling imposed; high output realisations are truncated. A similar process applies to oil revenues under default.

where output, y_t follows the stochastic process defined above, and d_t denotes the asset stock of the sovereign ($d_t < 0$ signifies indebtedness). Default is chosen by the sovereign if $V^D > V^N$.

The present value of being in the default state, V^D, is given by:

$$V^D(y_t, p_t) = \left[u(y_t^D + o^D) + \beta \mathbb{E}_t V^D(y_{t+1}, p_{t+1})^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{10}$$

whilst the present value of being in the non-default state, V^N, is given by:

$$V^N(d_t, y_t, p_t) = \max_{d_{t+1}} \left[u(c_t^N) + \beta \mathbb{E}_t V^N(d_{t+1}, y_{t+1}, p_{t+1})^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{11}$$

where $\beta = \frac{1}{1+\rho}$ is the discount factor and $\sigma \geq 0$ is the inverse of the IES. The expression for the expected future value will differ depending on whether the government chooses to default or not. These are given by:

$$\mathbb{E}_t V^D(y_{t+1}, p_{t+1}) = \sum_{y=1}^{yn} \sum_{p=1}^{pn} \left[\theta V(0, y_{t+1}, p_{t+1}) + (1-\theta) V^D(y_{t+1}, p_{1+1})\right] \Pi\left(y_{t+1}, p_{t+1} | y_t, p_t\right) \tag{12}$$

and

$$\mathbb{E}_t V^N(d_{t+1}, y_{t+1}, p_{t+1}) = \sum_{y=1}^{yn} \sum_{p=1}^{pn} \left[V(d_{t+1}, y_{t+1}, p_{t+1})\right] \Pi\left(y_{t+1}, p_{t+1} | y_t, p_t\right) \tag{13}$$

where θ is the exogenously determined probability of re-entry into credit markets after default. Finally, $\Pi(\cdot)$ is the transition probably matrix associated with each stochastic process and is given by:

$$\Pi\left(y_{t+1}, p_{t+1} | y_t, p_t\right) = \Pr\left(y_{t+1}, p_{t+1} | y_t, p_t\right) \tag{14}$$

Under the assumption of independence between the shocks series to output and the oil price I define the transition probability of each series following Tauchen (1986). When the shocks are allowed to have a non-zero covariance, the transition matrix is different as it needs to take these cross-correlations into account. In this case the derivation of the transition probability is described more fully in section 8.2.

4 Quantitative analysis

4.1 Solving the Model

The above equations are sufficient to characterise a simple default model. Due to the non-linear nature of the model, it is necessary to use numerical methods to solve for the optimal policy functions. Here I use a value-function iteration approach. This is implemented by discretising the state-space for debt, output and oil revenues[10], using a Markov chain for the stochastic

[10] I use 11 output states, 125 debt states and 5 oil price states

output and oil price processes, and solving for the policy functions using a guess and verify procedure.

The recursive solution to the model consists of a set of policy functions for the representative agents' choice of consumption, $c(d_t, y_t, p_t)$, as well as default decision, (D, N), and optimal asset holdings in the following period, d_{t+1}; and a bond pricing function, $q(d_{t+1}, y_t, p_t)$, which reflects the sovereign's default probabilities and are consistent with the creditors' expected zero profit condition. For a given choice of parameters, the numerical solution method involves: taking an initial guess for the bond price schedule and the value functions, under default and no-default, and solving for the optimal policy functions and probability of default. The bond pricing schedule is then updated using the probability of default and the preceding steps repeated until the problem converges.[11]

A simplified RBC version of the model in section 3 is described in section 8.1 of the technical appendix 8 in which I derive expressions for a number of analytical properties of the model. I show in proof 1 that the Euler equation of the RBC debt model collapses to that of a standard RBC model when default is not an option for the soveriegn. Further, I show in proof 2 that there is a single parameterisation of the fixed default cost and risk-free interest rate such that the optimal debt choice collapses to that of a simple RBC model with perfect capital markets.

The optimal choice of sovereign debt depends crucially on whether we allow for default. Without default the level of debt is decreasing in the elasticity of the sovereign's interest rate with respect to the level of indebtedness. That is, the more the interest rate charged on borrowing, or conversely the bond price, responds to the level of outstanding debt, the less the sovereign chooses to borrow. This is described in lemma 1 in the technical appendix 8. When default is possible, the sovereign may choose to borrow over and above that required for pure consumption smoothing. In particular, consumption tilting may occur when the sovereign is impatient relative to the risk-free interest rate, or when the debt-elasticity of the bond price is low. This relationship is described in lemma 2 in the technical appendix 8.

4.2 Base calibration

The model includes the following nine base parameters: $\rho, \beta, \sigma, \theta, \psi, r, r^*, \hat{r}, \bar{s}$. Table 1 lists the choice of parameters along with their baseline calibration.

[11] See section 9.1 in the appendix for a description of the algorithm.

Table 1: Model Parameters - base

Parameter	Value	Description
ρ	.020	Discount rate (set for consumption smoothing)
$\beta = \frac{1}{1+\rho}$	.980	Discount factor
σ	2.0	Coefficient of relative risk aversion
θ	.25	Probability of re-entry into capital market
ψ	.95	1 - output cost of default
$\bar{s}$	.060	Annual production of oil (oil share in GDP)
r^*	.004	Risk free interest rate
$\hat{r}$	.016	Risk premium (exog)
r	$r^* + \hat{r} = .020$	Default-risk-free rate

The risk free interest rate is set to $r^* = .004$ per quarter. When compounded over four quarters, this is equivalent to the 1.6% mean annual real yield on 10-year US government bonds between 1995q1 and 2016q3. I include an exogenous risk premium for emerging market 10-year bonds of $\hat{r} = .016$, which is equivalent to the 6.5% per year mean annual interest rate spread of the four economies in our sample over the US. The total interest rate facing emerging markets, without default risk, is therefore set to $r = r^* + \bar{r} = 0.020$ per quarter, or 8.2% per year.

The rate of time preference is calibrated to yield consumption smoothing behaviour in our model, that is $\rho = r = .020$. This ensures the model has a well-defined steady state. The coefficient of relative risk aversion is set to $\sigma = 2$ as is standard in the RBC literature. The asymmetric default cost parameter is set to $\psi = .95$.

The probability of re-entry following default is set to $\theta = .25$ every quarter. Using a monte carlo simulation approach, I find that, over 1000 runs, the average number of years that a country would be excluded from financial markets using this re-entry probability, is just over 2 years. This is somewhat lower than some of the evidence suggests. For example, Richmond & Dias (2009) find that countries are typically excluded for around 5.7 years following a sovereign default episode.

How dependent the domestic economy is on oil revenues is a key parameter in the model. Given the model set-up, oil revenues are important as they provide vital foreign exchange earnings which can then be used to pay down foreign denominated debt. One way to calibrate this parameter would be to estimate the share of oil production in GDP. Using data from the World Bank's WDI database, I find the average share of oil and gas rents in total GDP across our sample of countries over the period 1995q1 to 2016q3 to be 6%. Another way to calibrate this parameter would be to estimate the share of total export earnings which are due to oil exports. Using UN Comtrade data, I find the average share of petroleum in total exports to be 35%. Table 2 gives these share estimates for each country. What is clear from the data is the large variation in oil shares (GDP or exports) across the four countries.[12]

I calibrate the oil share parameter using the GDP share of $\bar{s} = .060$ but present simulation results for calibrating this parameter using the oil export share in the appendix (see section 9.2). In general a higher oil share increases the spread persistence as well as the strength of the negative correlation between interest rate spreads and the oil price.

Table 2: Oil and gas shares

country	GDP share	Export share
Colombia	.039	.445
Indonesia	.037	.265
Mexico	.036	.114
Russia	.128	.579
All	.060	.351

5 Pricing sovereign debt

The model described in section 3 and solved in section 4.1 presents a theoretical framework in which the ability of a resource rich economy to borrow is dependent on the cyclical properties of the commodity price. The result is that international capital markets, when setting the price of a sovereign's debt, internalise the effect that commodity price fluctuations might have on the likelihood of default.

We can be more specific about this relationship. In the simplified RBC model setup described in section 8.1 in the technical appendix 8 I show that with no opportunity of default and a stochastic income process, the price of sovereign debt is simply inversely related to the interest rate and the amount of debt outstanding. When default is open to the sovereign, and a fixed default cost associated with this decision, the relationship between the debt price and the level of debt becomes non-linearly related to the level of debt as well as the default cost and the likelihood of the sovereign to default in any given period. Using this simple model, I derive an expression for the steady-state price of sovereign debt in expression 44 in the technical appendix 8, which I reproduce here:

$$q = \frac{1}{1+R} = \frac{1-\lambda}{dR\left(\delta'(\lambda - d) - \delta + 1\right)} + \frac{1}{d\delta' R} \tag{15}$$

The steady-state price of sovereign debt is a complex, non-linear function of the fixed cost of default, λ, the interest rate charged to the sovereign, R, the level of debt, d, the choice of the sovereign to default, δ, and the sensitivity of this choice to the level of debt, δ' (i.e. it's first

[12] It is worth noting that while the mean GDP share has remained relatively stable over time, the export share has increased steadily. See figure 8 in the appendix

derivative).

When default is not possible for the sovereign we set $\delta = \delta' = 0$ and $\lambda = 0$, and the bond pricing expression collapses to $q = \frac{1}{dR}$. This simple expression states that the steady state bond price of sovereign debt is inversely related to the level of debt, d and the interest rate facing the sovereign, R. When default is possible, this inverse relationship is augmented by the cost of default, λ, and the optimal default decision by the sovereign, δ. The more sensitive the default decision of the sovereign is with respect the level of debt (that is, the larger is δ'), the lower the price investors are willing to pay to hold this debt, and the higher the effective interest rate. This relationship is described in lemma 4 in the technical appendix 8. Note that the interest rate, R, is itself a function of the level of debt as well as the output and oil price states as described in the expression 25 in the technical appendix 8.

Crucially, the bond price is inversely related to the level of debt and positively correlated with output and the oil price, but in a non-linear way. Given that we do not have explicit functional form representations of the interest rate function, $R(\cdot)$, and the default decision, $\delta(\cdot)$, the shape of this non linear bond pricing schedule can only be approximated using the numerical solution algorithm. For the base calibration of our model, the bond pricing schedule is given in figure 3. As we can see from the figure, the bond price is a downward sloping, non-linear function of the stochastic output and oil price process.

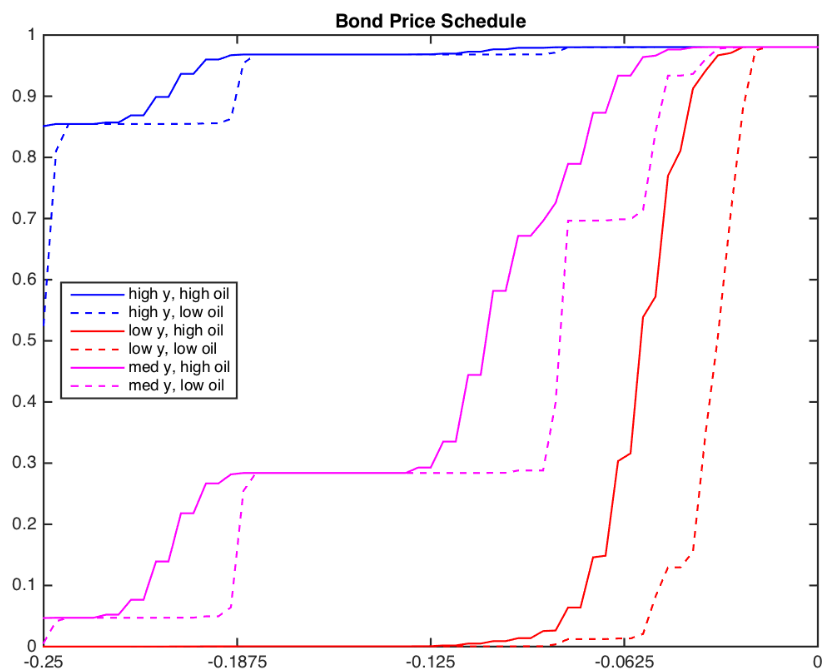
Figure 3 shows how the equilibrium bond price (vertical axis) varies with the level of indebtedness (horizontal axis) for different output and oil price states. The solid lines represent the schedule from Eq. 8 for 'high', 'medium' and a 'low' output states as a function of the current debt level given a 'high' oil price[13]. The bond price is an increasing function of foreign assets (negative assets signify a net debt position). The sovereign has a choice in any single period: (i) default on debt today, increasing consumption and so utility today, but being excluded from asset markets and facing a default cost, or (ii) do not default today, lowering consumption and utility today due to interest payments, but maintaining the ability to borrow and avoiding the negative consequences of default. When setting the bond price, international investors take the sovereigns default decision into account, decreasing the amount they would be willing to pay for this debt the more likely it is that the government will choose to default. The larger the outstanding debt (i.e. a more negative asset position), the more likely the sovereign is to default and so the lower is the bond price. The pricing schedule for the 'high' output state is above that of the 'low' output state simply because higher output realisations mean a lower chance of default. This in turn means investors are more willing to hold government debt and so are willing to pay a higher price for it.

The dashed lines represent the same bond pricing schedules but for the 'low' oil price state. When oil prices are low the pricing schedules shift down. This is due to the oil revenue entering additively into the sovereign's budget constraint: a lower oil price reduces revenues, increasing

[13] I use the 3rd, 6th and 9th positions for the high, medium and low states respectively.

the likelihood of default and lowering the equilibrium bond price. We get a similarly shaped bond pricing schedule in a model without oil. That is, the bond price is still a downward sloping, non-linear function of the output states. However, without oil in the model the bond pricing schedule has significantly larger steps in it. See figure 12 in the graphical appendix.

Figure 3: Bond pricing schedule



6 Model performance

6.1 Independent shocks

The model includes two stochastic processes: innovations to output and the oil price. I calibrate the output series, y, using quarterly data from the four developing economies and the oil price series, p, using quarterly data on on the real crude oil price. I deflate the nominal GDP series for each country in local currency using the national GDP deflator, and the crude oil price using US CPI data. I then take the natural log of these real GDP series and the real oil price. All data are quarterly. Finally, I pass the data through a Hodrick-Prescott (HP) filter with a smoothing parameter of 1600.

Data on nominal GDP, national deflators and US CPI all come from the Global Financial Database. The data on oil prices is the quarterly average of the daily West Texas oil price in USD per barrel taken from the IMF's IFS database.

To retrieve the structural parameters required for the model, I estimate the following auto-regressive process for the cyclical components of output and the oil price:

$$x_t = \rho x_{t-1} + u_t \tag{16}$$

where x_t is the cyclical component of the series in question at time t and u_t is assumed to be an i.i.d error. The coefficient estimates for ρ are used to calibrate parameters ρ_z and ρ_h in equations 1 and 2 respectively, while the sample standard errors of the residuals, u_t, are used to calibrate the volatility parameters σ_z and σ_h.

Table 3 gives the calibration of the four parameters governing the stochastic processes of the model: $\sigma_z, \sigma_h, \rho_z, \rho_h$. For the output process, the parameters are the mean of the estimated individual country parameters. The quarterly output process has a persistence term of 0.841 and is subject to an i.i.d shocks process with standard deviation of 0.015. Oil prices have a persistence term of 0.725 and shock standard deviation of 0.136.[14]

Table 3: Model parameters - stochastics

Parameter	Value	Description
ρ_z	.841	Persistence of productivity
ρ_h	.725	Persistence of oil price
σ_z	.015	Std. dev of output shocks
σ_h	.136	Std. dev of oil price shock

Using the above calibration I solve the model for the sovereign's policy functions using the value function iteration algorithm. I then simulate the model over 1000 periods and assess the ability of the model to reproduce the moments observed in our data. Tables 4 and 5 compare the moments of the simulated data to the data moments for our four resource rich economies between 1995q1 and 2016q3.

The first two columns in table 4 present the mean AR(1) coefficients and standard deviations for output, the oil price, consumption and the interest rate spread from our panel dataset. Consumption is the most persistent series, followed by output, then the oil price and finally the interest rate spread. The oil price is the most variable. The next two columns present these same data moments from the simulated model. The first thing to note is that the model successfully reproduces the correct signs on all of the moments. In addition, the ordering of the persistence and standard deviation terms between the four series are also correctly predicted. These are promising results.

[14]It is worth noting that the shock processes to these series are not very persistent. In fact the half-life of these shocks can be calculated using: $h = T\log(2)/\log\left(\frac{x_t}{x_T}\right)$. This gives a half-life of 4.1 quarters for innovations to output and 2.2 for innovations to the oil price.

Table 4: AR(1) model: autocorrelations and standard dev

	Data		Model	
	AR(1)	std	AR(1)	std
output (HP filt)	.841	.032	.717	.025
oil price (HP filt)	.725	.198	.650	.195
cons (HP filt)	.898	.026	.834	.018
spread	.631	.020	.188	.003

Table 5: AR(1) model: cross correlations with real interest rate spread

	Data	Model
output dev	-.138	-.106
oil price dev	-.279	-.094

The model correctly reproduces the autocorrelations and standard deviations of output and the oil price. This is to be expected as the relevant model parameters were chosen to match these two processes. It also does a good job on consumption, which is highly persistent and has a low variance. However the model substantially under predicts the persistence and standard deviation in the interest rate spread.

The first column in table 5 presents the average correlation between the real interest rate spread and the deviations of output and the oil price in the data respectively. Both output and the oil price are negatively correlated with the spread in these economies. That is, periods of above trend output and oil prices are associated with lower interest rate spreads. A key feature of spreads in resource rich, emerging economies, is strongly countercyclical movements with oil prices. The second column presents the same moments from the model. The first thing to note is that the model produces the correct signs on these correlations, which is reassuring. However, whilst the model gets quite close to the output correlation, it fails to generate a sufficient oil price correlation.

6.2 Covarying shocks

Up to now I have assumed independent output and oil price shocks. It seems plausible that for resource rich developing economies, positive terms-of-trade shocks (that is, increases in the price of oil relative to imports) might lead to increases in domestic output. In the model of section 3, output, y, is defined as total non-oil output while the resource sector is defined as, $p\bar{s}$, where the oil price, p is subject to shocks. So positive co-movements in this setting would be between oil prices and non-oil output.

Devaragan et al. (1990) provide a parsimonious model for a single economy producing two goods (exports and domestic non-tradables) in a three good world (which includes imports).

Their model is useful for thinking about the effects of a change in the export price (oil in this case) on the domestic economy[15]. In the model, a positive terms-of-trade shock has two effects: an *income* effect and a *substitution* effect. The fall in the relative price of imports means that domestic consumers face a positive income effect as their real income's rise, thus increasing the demand for non-tradable. The substitution effect reduces demand for non-tradables as the their price relative to imports has risen. The overall effect on domestic output, excluding imports, is ambiguous. If consumers cannot easily substitute between imports and non-tradables, a likely assumption in developing economies, then the income effect will dominate and domestic output will increase. The result is that increases in domestic output are likely to coincide with positive oil price shocks.

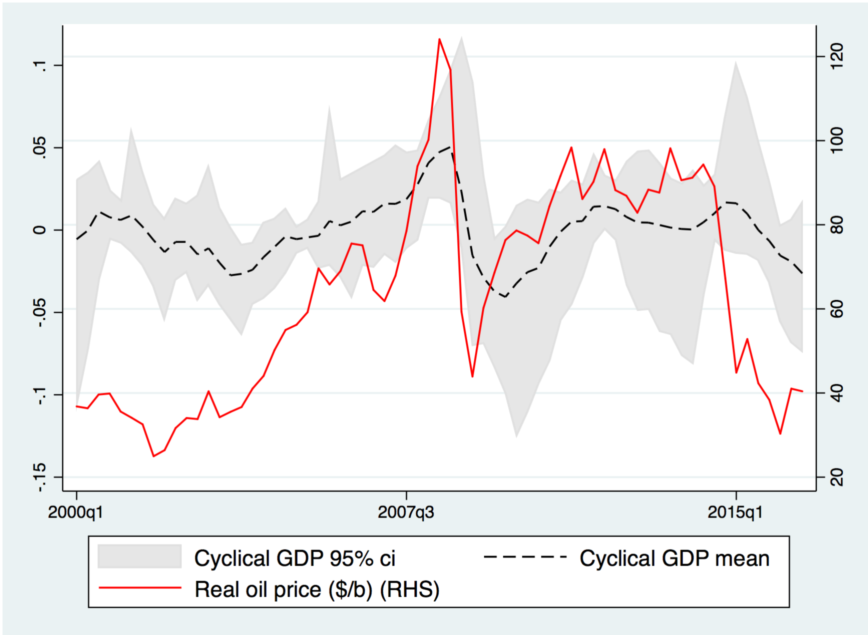
So far, this simple analysis has focused only on the immediate intersectional adjustment of the economy following a terms-of-trade shock, and has hinged on the low elasticity of substitution assumption. Other channels however are likely to operate, all of which would strengthen this positive correlation between oil prices and domestic output. These include: cheaper imported inputs, increased inward investment flows and larger foreign exchange earnings. These effects will be stronger the more important are exports relative to total domestic output, which is typically the case for resource dependent economies.

In resource rich economies this effect typically manifests itself as a positive co-movement of commodity prices and growth. This co-movement can be seen in figure 4 which plots the oil price against the cyclical component of GDP in our four countries. The mean correlation coefficient between real GDP and the oil price is in our four economies is .20[16].

[15]Some structure may be useful here. On the demand side, consumers substitute (imperfectly) between imports, M, and domestic non-tradbales, D, while on the production side the economy produces, D, and exports, X, which are imperfectly transformed between. A positive terms-of-trade shock implies a fall in the price of imports, P_M, relative to exports, P_X.

[16]The individual country correlations are: Colombia = .41, Indonesia = $-.06$, Mexico = .14 and Russia = .29

Figure 4: Covarying oil price and growth



I model these covarying output and oil price series in a simple 2-variable VAR model, over the period 1995q1 - 2016q3 for each country, as follows:

$$\mathbf{x_t} = \mu + \rho \mathbf{x_{t-1}} + \epsilon_\mathbf{t}$$

$$\begin{bmatrix} Y_t \\ P_t \end{bmatrix} = \begin{bmatrix} \mu^Y \\ \mu^P \end{bmatrix} + \begin{bmatrix} \rho^Y & \rho^{YP} \\ ... & \rho^P \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ P_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_t^Y \\ \epsilon_t^P \end{bmatrix} \tag{17}$$

$$\text{where } \Sigma_{YP} = \text{Var}(\epsilon^Y, \epsilon^P) = \begin{bmatrix} \sigma_Y^2 & ... \\ \sigma_{YP}^2 & \sigma_P^2 \end{bmatrix} \text{ and } [\epsilon^Y, \epsilon^Y]' \sim N(0, \Sigma_{YP}).$$

where Y_t and P_t are the cyclical components from the HP filter applied to real GDP of each country and the real oil price respectively. The parameter estimates for ρ^{YP} for each country are given in table 6. As we can see, there is a significant variation in these estimates.

Table 6: Cross AR(1) estimates - ρ^{YP}

	Coef.	Std. Err.
Colombia	$.035^{**}$	.014
Indonesia	$.077^{***}$	.019
Mexico	$-.007$	.008
Russia	$.074^{***}$	.010
Mean	.045	n/a
Range	$[-.007, .077]$	n/a

Given this lack of tight priors about the strength of the correlation between output and oil price shocks, I explore the dynamics of the model for a reasonable range of values around the central estimate of $\rho_{zh} = .045$[17]. I choose upper and lower values for the covariance parameter, $\rho_{zh} \in [.01, .10]$, and solve the model for 50 points in between these bounds. In each case, the model is then simulated over 1000 periods and the various data moments retrieved. I focus on the estimates of the spread persistence and the its correlation with the oil price from the simulated data, as these are the key data moments we are trying to explain.

Figure 5a presents the AR(1) parameter for the interest rate spread across the range of parameters, while figure 5b presents the spread's correlation with the oil price[18]. The auto-regressive properties of the interest rate spread is reduced for very low and very high correlations of output and oil price shocks. For high and low correlations, the model displays a low spread auto-correlation, whilst for moderate correlations the spread is relatively persistent. The reason for this inverse-u shape is that there are two competing effects influencing interest rate spreads as the two shocks become more correlated. The first is an increased risk of default which drives up spreads. When the shocks are independent they act as partial insurance mechanisms against each other, and so default is less likely, ceteris paribus, and spreads remain low. As the shocks become more correlated, periods of low output combined with low oil prices become more frequent, thus increasing the likelihood of default and raising the risk premium. The second effect is the rising cost associated with default as default becomes more frequent. As the shocks become more correlated and the risk of default increases, the sovereign finds a lower level of indebtedness optimal at a given point in time. This in turn reduces the likelihood of default and so too the spread persistence.

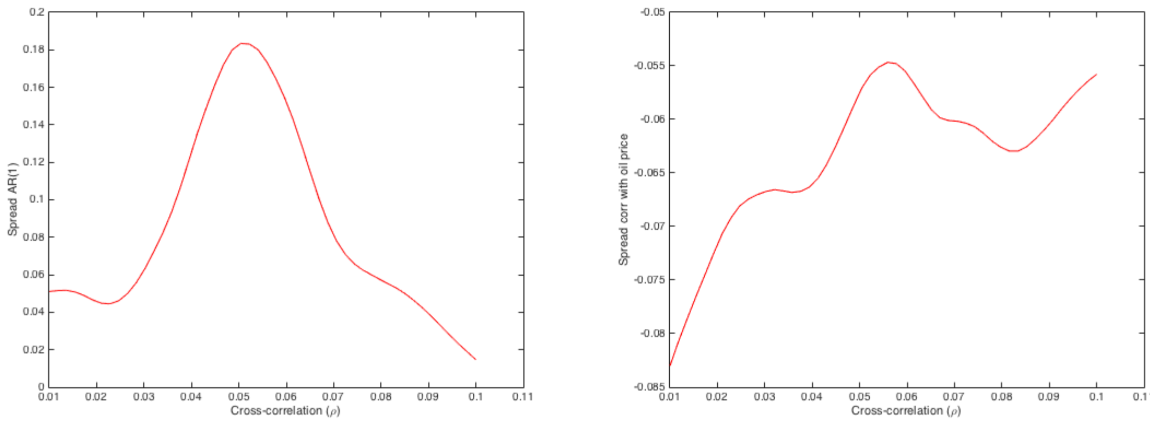
The correlation of the spread with the oil price is broadly increasing with the correlation of output and oil price shocks. The intuition here is relatively straightforward. The more correlated are the shocks, the more likely low oil prices coincide with low output, and so the more likely default becomes. As the default probability increases so too does the interest rate spread and

[17]cross-correlation, ρ^{YP}, is estimated to be 0.045 in the data, the range is from $-.007$ to .077

[18]Given the discretised state-space and the numerical solution method, the data moments estimated from the simulations display significant estimation noise. I therefore display a smoothed version of these estimates in the figures.

so the higher the correlation of the interest rate spread with oil price shocks.[19]

Figure 5: Sensitivity to correlation - $\rho_{zh} \in [.01, .10]$



(a) AR(1) parameter (b) Correlations

6.3 Impatience and oil dependence

The baseline model assumed pure consumption smoothing. That is, the impatience parameter is set equal to the risk-free real interest rate, $\rho = r = .020$, or 8.2% per year. While this assumption may hold for developed economies, it is unlikely to be the case in developing economies where current generations discount the future more heavily due to stronger growth and greater uncertainty.

The permanent income theory of consumption yields the basic inter-temporal approach to the current account: in a small open economy, under perfect consumption smoothing, temporary income shocks will be offset by changes in national savings via the current account (Obstfeld & Rogoff (1995); Sachs (1982)). Consumption tilting, which arises due to differences between the subjective discount rate of the domestic agent and the prevailing world interest rate, yields behaviour in which a country shifts its consumption toward the present or the future, independently to the balance of prevailing shocks to income. In general, agents in developed economies are observed to be relatively patient. Braeu (2010) finds that households in Canada tilt consumption toward the future, while Cashin & McDermott (2002) find that the dynamics of international capital flows to Australia during the 1990s, when the country was a net capital importer, were broadly consistent with utility maximisation under consumption smoothing.

More recent work has examined the degree of impatience in consumption in developing country settings. Zhuang et al. (2007) and Harrison (2010) for example, find social discount rates

[19] A similarly upward sloping relationship exists between the correlation of the shocks and the correlation of the spread with output shocks.

that are in excess of prevailing market interest rates in poor countries. These high social discount rates are applied in practice by development organisations, such as the World Bank and the Asian Development Bank, who typically use annual discount rates in the range of 10-12% when evaluating projects in developing countries. Some developing country governments apply discount rates as high as 15% in their project appraisals.

Another consideration, is the degree to which governments attempt to smooth changes in fiscal policy. The basic Ricardian view, as described by Barro (1979), is that given the amount of government spending, the balance of government financing between taxes and debt should have no real effects. If taxes are distorting however, it will be optimal to smooth tax rates over time, thus implying that temporary shocks to government spending will be debt financed. The basic theory also implies that government deficits should move independently from the outstanding debt-income ratio as well as with the level of government spending. This inter-temporal tax-smoothing model appears to hold in developed economies such as the US.

A growing literature explores tax smoothing behaviour in emerging and developing country settings. These studies typically find evidence of both tax smoothing and tax tilting. Tax tilting is when a government chooses fiscal deficits that are either larger or smaller than those generated from the trade-off between tax smoothing and debt-sustainability objectives. Cashin et al. (2003) find that taxes in Pakistan remained fairly unresponsive to anticipated changes in expenditure, but that deficits were systematically larger over the 1970-90 period than would be expected from a series of optimal tax smoothing fiscal decisions. Tax tilting in developing countries may also be due to increased political risk. Pastén & Cover (2015) study the case of Latin America between 1984 and 2009, and find that a higher risk of losing power in the future increases the rate at which government's discount the future resulting in higher deficits today.

The second issue is the degree to which countries depend on oil revenues as a source of foreign exchange. Our baseline model is calibrated to the mean hydrocarbon share, $\bar{s} = 0.060$. However, the level of dependence on oil of each of the four economies in our sample is highly variable. Mexico has the lowest dependence, with the share of GDP coming from the oil and gas sector standing at around 3.6% on average. In Russia, this figure stands at just under 13%.

This section tests the sensitivity of the model to the level of impatience and the dependence on oil. I vary the rate of time preference in the model in the range, $\rho \in [0.01, 0.05]$, per quarter, which works out at a discount rate roughly in the range of 4-20% of per annum. I also vary the oil share parameter in the range, $\bar{s} \in [0.02, 0.6]$.[20]

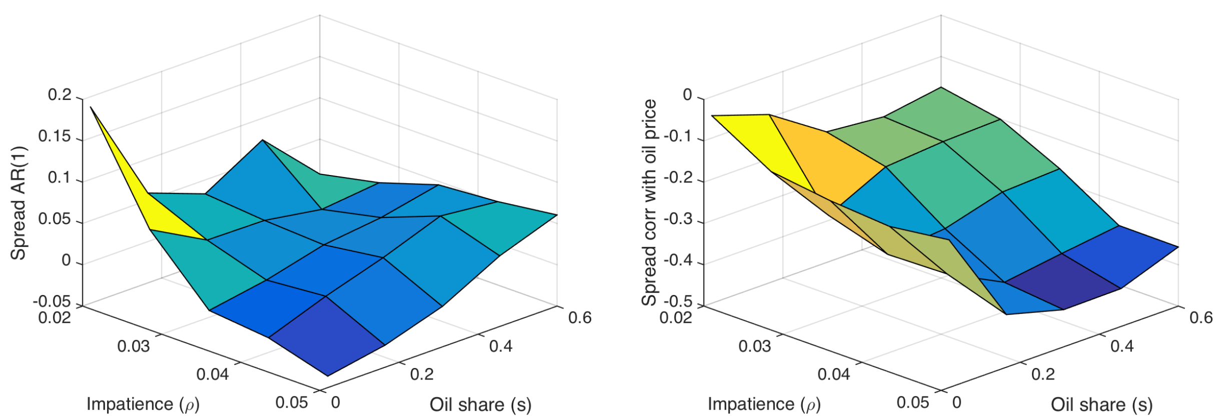
Figure 6a presents the AR(1) parameter for the interest rate spread for different combinations of impatience and oil dependence. The non-linear relationship between these deep parameters and the data moments is again clear. Broadly speaking, a higher level of impatience reduces the

[20] Given the computational weight of solving and simulating the model for different parameter sets jointly, I choose 5 evenly spaced points covering each parameter range.

persistence of the interest rate spread whilst a higher oil share increases it. The more impatient the sovereign is, the more they tilt consumption to the present and push debt repayments to the future. This is turn increases the probability of default, ceteris paribus, thus increasing the interest rate spread. The more important oil revenues are to the sovereign, the more likely oil price shocks are to have an impact on the the ability to meet debt repayments, and in turn on the likelihood of default.

The correlation between the spread and the oil price is given in figure 6b. There is a relatively strong positive relationship between increasing impatience and oil dependency and the strength of the negative correlation between interest rate spreads and movements in the oil price. Impatience and oil dependency have a similar effect here. They both act to bring consumption forward and push debt repayments back, thus increasing the default probability when oil prices (and output) are low and raising risk premiums.

Figure 6: Sensitivity to impatience, $\rho \in [.01, .05]$, and oil dependence, $\bar{s} \in [0.02, 0.6]$



(a) AR(1) parameter

(b) Correlations

6.4 Recursive preferences

How agents make inter-temporal trade-offs and how they treat risk are likely to be important in any analysis of sovereign debt dynamics in a stochastic setting. A substantial literature exists documenting the low degree of inter-temporal substitution in developing countries, as well as the relatively high risk-aversion, although the evidence here is more mixed.

Poorer households that face a subsistence consumption requirement will have a lower inter-temporal elasticity of substitution because a smaller portion of their budget is available. At the aggregate level, poor countries where budget shares of food are relatively high, the interest elasticity of saving is likely to be low, thus leading to a low inter-temporal elasticity (see Ogaki et al. (1996) for a discussion of these hypotheses). Atkeson & Ogaki (1996) use panel data in

India to show that the inter-temporal elasticity amongst poor households is substantially lower that rich households.

On risk-aversion, the evidence here is informative but far from conclusive. A number of studies conclude that agents in developing countries are highly risk averse (Binswanger et al. (1980); Yesuf & Bluffstone (2009); Akay et al. (2012)). More recent work however, has argued that these earlier studies may be biased towards overestimating risk-aversion. Vieider et al. (2013) finds evidence of risk-neutrality amongst poor farmers in Vietnam. Gandelman & Hernadez-Murillo (2014) estimate a coefficient of relative risk aversion close to unity in developing countries, although risk aversion in African economies may be higher than in Asia.

The challenge here is that, up to now, the sovereign's behaviour has been modelled using deterministic preferences in which risk aversion and inter-temporal preferences are inversely related. It is not possible in the current set-up to alter the degree of risk aversion separately from the elasticity of inter-temporal substitution. One option is to use recursive preferences in which the risk-aversion and substitution parameters are independent.[21] To this end, I incorporate Epstein-Zin preferences into the model. Specifically, the value function for the sovereign under default in 10 becomes:

$$V^D(y_t, p_t) = \left[u(y_t^D + o^D)^{1-\sigma} + \beta \left(\mathbb{E}_t V^D(y_{t+1}, p_{t+1})^{\frac{1-\sigma}{\eta}}\right)\right]^{\frac{\eta}{1-\sigma}} \tag{18}$$

whilst under no default the value function in 11 becomes:

$$V^N(d_t, y_t, p_t) = \max_{d_{t+1}} \left[u(c_t^N)^{1-\sigma} + \beta \left(\mathbb{E}_t V^N(y_{t+1}, p_{t+1})^{\frac{1-\sigma}{\eta}}\right)\right]^{\frac{\eta}{1-\sigma}} \tag{19}$$

where $\sigma \geq 0$ is the coefficient of relative risk aversion, $\xi \geq 0$ is the inter-temporal elasticity of substitution (IES), and:

$$\eta = \frac{1-\sigma}{1-\frac{1}{\xi}} \tag{20}$$

Under deterministic consumption, we set the IES equal to the inverse of the risk aversion parameter, i.e. $\sigma = 1/\xi$ and $\eta = 1$, so the recursive preferences collapse to the standard time-separable expected discounted utility with discount factor β.

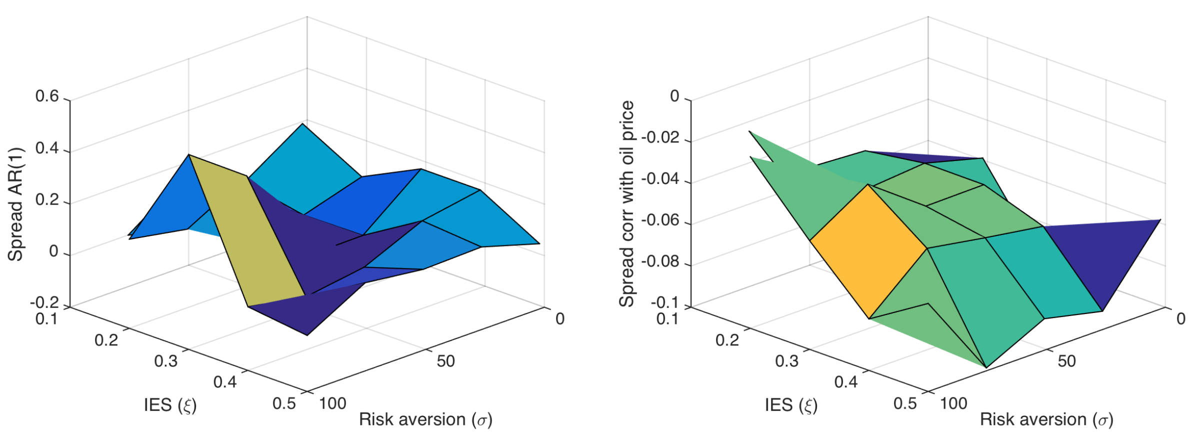
The use of recursive preferences allows us to move the IES independently from the risk-aversion parameter. Exactly, what values to choose for these parameters however, is unclear. The micro and macroeconomic evidence suggests a range of possible values for these parameters in developing economy settings. I solve and simulate the model, as before, using a range of plausible values for the risk-aversion and inter-temporal substitution parameters, $\sigma \in [2, 100]$ and $\xi \in [0.1, 0.5]$.

Once again, figures 7a and 7b present the spread persistence and the spread's correlation

[21] Another option would be to use some form of hyperbolic preferences following work by Laibson (1997)

with oil prices respectively for different combinations of risk aversion and inter-temporal substitutability. The persistence of the interest rate spread remains relatively unaffected by changes in either parameter. However, the same is not true for the correlation of the spread with the oil price. The easier it is for the sovereign to substitute consumption inter-temporally (that is a higher IES coefficient), the greater is the negative correlation between spreads and movements in the oil price. This is due to the trade-off between consumption and debt stability. The easier it is for the sovereign to substitute consumption between periods, the more volatile consumption will be. At the same time, the path for debt will be more stable. However, this immediately means that when the oil price rises again, the level of outstanding debt will be reduced relatively slowly, thus increasing the risk of default, ceteris paribus. International creditors know this, and so charge a higher risk premium when oil prices fall initially. A similar, albeit less pronounced, relationship exists with the risk aversion parameter.

Figure 7: Sensitivity to risk aversion, $\sigma \in [2, 100]$, and IES, $\xi \in [0.1, 0.5]$



(a) AR(1) parameter

(b) Correlations

6.5 Final model

The previous sections used comparative statics to investigate the behaviour of the model along key parameter dimensions. This section aims to present a sensibly parameterised model which best fits the observed data moments for the four resource rich developing countries in our sample. In particular we calibrate the model to reflect higher risk-aversion, more impatience, larger oil shares and a stronger correlation between domestic output and oil price shocks. Table 7 presents the full parameter set.

Table 7: Model parameters - best fit

Parameter	Value	Description
ρ	.025	Discount rate
$\beta = \frac{1}{1+\rho}$	.980	Discount factor
σ	100	Coefficient of relative risk aversion
ξ	0.5	Inter-temporal elasticity of substitution
θ	.25	Probability of re-entry into capital market
ψ	.95	1 - output cost of default
$\bar{s}$	.212	Annual production of oil (oil share in GDP)
r^*	.004	Risk free interest rate
$\hat{r}$	.016	Risk premium (exog)
r	$r^* + \hat{r} = .020$	Default-risk-free rate
ρ_z	.841	Persistence of productivity
ρ_h	.725	Persistence of oil price
σ_z	.015	Std. dev of output shocks
σ_h	.136	Std. dev of oil price shock
ρ_{zh}	.05	Cross correlation
σ_{hz}	0	Cross std dev

Table 8: AR(1) model: autocorrelations and standard dev

	Data		Model	
	AR(1)	std	AR(1)	std
output (HP filt)	.841	.032	.893	.043
oil price (HP filt)	.725	.198	.657	.195
cons (HP filt)	.898	.026	.977	.107
spread	.645	.022	.304	.010

Table 9: AR(1) model: cross correlations with real interest rate spread

	Data	Model
output dev	-.138	-.148
oil price dev	-.279	-.160

The model is then solved and simulated again over 1000 periods. Tables 8 and 9 compare the moments of the simulations data to actual data moments.

Overall, the final model calibration fits the data relatively well, and certainly better than any of the other models. The model is able to reproduce the correct ordering of autoregressive properties observed in the data. Consumption is the most persistent of the four key variables, followed by output, the oil price and then the interest rate spread. The model still under predicts the persistence and variability of the interest rate spread. However, this is substantially less than other models and is perhaps to be expected given the simplicity of the financial market structure in the model.

The biggest success of the model is that it comes much closer to matching the correlation of interest rate spreads and the oil price. The model produces a spread-output correlation which is very close to that observed in the data. Crucially however, it is able to produce a spread-oil price correlation that is larger that the spread-output one, albeit lower in magnitude than that observed in the data. Countercyclical interest rate spreads and commodity prices is a key feature of the data in resource rich developing economies.

Finally, I also solve and simulating this model with oil revenues 'turned off'. The result is a substantial worsening in the model's predictive power. In particular the spread persistence and the countercyclical correlation of the spread and the oil price are dramatically reduced. See tables 12 and 13 in the appendix.

7 Conclusion

How do commodity price movements affect the pricing of sovereign debt in resource rich economies? And, what does this mean for the optimal management of volatile oil revenues in developing countries? With debt distress across developing economies on the rise, this is a pertinent question for policy makers and investors alike. This paper sheds light on these two questions by investigating the link between commodity price movements and risk premiums in resource dependent developing economies.

I presented empirical evidence of a counter-cyclical relationship between oil prices and interest-rate spreads in these settings. I then developed a simple model that delivers this counter-cyclical relationship and showed that it comes relatively close to explaining some key macroeconomic co-movements in a sample of resource rich developing countries. The model generates an endogenous link between oil prices and interest rate risk premiums and is, to the best of my knowledge, new to the literature. This approach could be useful in a range of other applications involving optimal policy in resource rich settings going forward.

For developing countries, managing resource revenues can be challenging. Conventional wisdom suggests that part of these windfalls should be saved in a sovereign wealth fund to smooth consumption and avoid the Dutch disease. Another reason to save is to manage these inherently volatile revenue streams. Economists agree that macroeconomic volatility is bad for growth and a growing literature suggests that this is is particularly true in resource rich economies. Recent research highlights the importance of building up precautionary savings in the face of oil price volatility in a *liquidity fund*, the size of which will depend on the level of volatility, the size of the windfall and how prudent policy makers are.

The paper presented another reason that policy makers in developing countries may choose to use a volatility fund - to manage debt-dynamics. Large movements in the oil price can present an important source of default risk for resource dependent sovereigns. Managing this risk may be important for a country's growth trajectory, particularly in capital scarce settings. Higher default risk translates into higher interest rate premiums and so a lower rate of capital accumulation, which in turn is likely to affect the rate of growth. Policy makers should respond by building up liquid buffers to manage downward movements in the oil price. Generally speaking, these buffers should be larger the stronger the link between oil prices and the domestic economy is, the more impatient policy makers are and the more willing they are to substitute current for future consumption.

Finally, developing country governments that depend significantly on resource revenues are likely to face times of substantial interest rate premiums. This can lead to sub-optimal investment and capital stocks, particularly if significant absorption constraints exist in the domestic economy. The importance of using resource revenues to set up an *investment* fund in developing countries has been emphasised in resent research. Future work could extend the model developed in this paper to incorporate domestic physical capital. This would allow policy questions around the trade-offs between precautionary savings and domestic investment to be answered within a framework of endogenous capital accumulation and interest rate premiums.

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8 Technical appendix

8.1 Ramsey representation

8.1.1 Model setup

The model described in section 3 can be expressed as a Ramsey policy problem by setting the re-entry probability to 1. The agent maximises the following utility function:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{21}$$

Domestic output is subject to an exogenous endowment shock, $y_t > 0$. In addition to this endowment, the country receives income from selling its annual production of oil, $\bar{s}$, abroad at a stochastic world oil price, $p_t > 0$. The total stochastic income stream in any given period is given by:

$$x_t = y_t + p_t \bar{s} \tag{22}$$

The government can save by accumulating safe foreign assets, $f_t \geq 0$. Foreign bonds yield a risk-free return of r in each period, so the price of foreign assets is given by $\frac{1}{1+r}$.

The government can also issue bonds, $d_t \geq 0$, where a positive amount denotes positive debt or negative assets. These are issued at time t and promise to pay d_t units of consumption in period $t+1$. The government can choose to default on debt in any given period subject to some fixed cost, $\lambda(y_t, p_t)$ which is a function of the output state, $y = \{y^1, ..., y^{ny}\}$, and oil price state, $p = \{p^1, ..., p^{np}\}$. We use $\delta = \{(y_t, p_t, d_{t-1}) \in [0,1]\mathbb{R}$ to denote an indicator for default where $\delta_{t-1} = 1$ signifies default and $\delta_{t-1} = 0$ signifies no default. The default decision is a function of the output and oil price states as different realisations of these states will result in different optimal default decisions. It is also a function of the level of inherited debt from the last period, d_{t-1}. This is important for two reasons. The higher is indebtedness the more incentive the government has to default and so avoid repayment, but this default cost is a function of output and the oil price. Finally, we impose that $\delta(\cdot, \cdot, d_{t-1}) = 0$ if $d_{t-1} \geq 0$. That is, the government cannot default if it has positive assets (or negative debt).

At the beginning of the period, the government's total wealth, ω_t is given by:

$$\omega_t = y_t + p_t \bar{s} + f_{t-1} - d_{t-1}(1 - \delta_{t-1}(y_t, p_t, d_{t-1})) - \lambda(y_t, p_t)\delta_{t-1}(y_t, p_t, d_{t-1}) \tag{23}$$

The resources available in the economy are on the right hand side of equation 23 and are made up of income, y_t, oil revenues, $p_t \bar{s}$, foreign assets, f_t, net of the obligations on domestic debt repayment from last period, d_{t-1}.

The final two terms of the right hand side denote the decision by the government to default on existing debt through the choice of δ_{t-1} and the cost under default through λ respectively.

When the government chooses to service the debt stock from the previous period (that is, no default) then $\delta_{t-1} = 0$, and these two terms collapse to $-d_{t-1}$. This means that wealth is simply given by: $\omega_t = y_t + p_t\bar{s} + f_{t-1} - d_{t-1}$. When the government chooses to default on the previous period's debt, $\delta_{t-1} = 1$, then these terms collapse to $-\lambda$, and wealth is given by: $\omega_t = y_t + p_t\bar{s} + f_{t-1} - \lambda$.

The economy's total budget constraint is given by:

$$\omega_t + \frac{d_t}{1 + R_t(y_t, p_t, \delta_t)} = c_t + \frac{f_t}{1+r} \tag{24}$$

where c_t denotes consumption, f_t denotes new foreign asset accumulation and d_t denotes new debt issuance.

The interest rate on government debt is given by R_t and depends on the default profile, δ, chosen by the government and on the current output and oil price states, y_t and p_t, as these will affect the likelihood of entering different states in $t+1$. The government takes into account in its optimisation problem the effect that future default decisions have on the bond price today. This is described in the function $R(\cdot,\cdot,\cdot)$ in the following bond pricing formula which, through a zero profits assumption, equates the return on domestic and foreign bonds:

$$q(y_t, p_t, \delta_t) = \frac{1}{1 + R_t(y_t, p_t, \delta_t)} = \frac{1}{1+r}\sum_{i=1}^{ny}\sum_{j=1}^{np}\big(1 - \delta_t(y^{ny}, p^{np}, d^{nd})\big).\pi\big(y^i|y_t, p^j|p_t\big) \tag{25}$$

where r denotes the risk free world interest rate and $q(\cdot,\cdot,\cdot)$ denotes the prevailing market bond price. The indices ny and np denote the possible realisations of output and oil prices in the future respectively, whilst the nd index denotes the debt choice corresponding to these possible states.

The Ramsey problem characterising optimal policy is:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{26}$$

s.t.:

$$c_t = \omega_t + \frac{d_t}{1 + R_t(y_t, p_t, \delta_t)} - \frac{f_t}{1+r} \tag{27}$$

$$\omega_t = y_t + p_t\bar{s} + f_{t-1} - d_{t-1}(1 - \delta_{t-1}(y_t, p_t, d_{t-1})) - \lambda(y_t, p_t)\delta_{t-1}(y_t, p_t, d_{t-1}) \tag{28}$$

$$d_t \leq 0 \text{ and } f_t \geq 0$$

$$d_0, y_0, p_0 \text{ : given}$$

8.1.2 Solution

We formulate the Lagrangian L, letting ξ_t denote the multiplier on the budget constraint from combining 27 and 28 in period t:

$$\begin{aligned} L = \mathbb{E}_t \Bigg[& \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & + \beta^t \xi_t \bigg(y_t + p_t \bar{s} + \frac{d_t}{1 + R_t(y_t, p_t, \delta_t)} - \frac{f_t}{1+r} - c_t + f_{t-1} \\ & - d_{t-1}\big(1 - \delta_{t-1}(y_t, p_t, d_{t-1})\big) - \lambda(y_t, p_t)\delta_{t-1}(y_t, p_t, d_{t-1}) \bigg) \Bigg] \end{aligned} \tag{29}$$

Differentiating the Lagrangian with respect to the choice variables $(c_t, f_t, d_t, \delta_t)$ gives:

$$\begin{aligned} c_t &: \quad u'(c_t) - \xi_t = 0 \\ f_t &: \; -\frac{\xi}{1+r} + \beta \xi_{t+1} = 0 \\ d_t &: \; \xi_t \Big[\frac{-d_t \delta'_t R_t + R_t + 1}{(1+R_t)^2} \Big] - \beta \xi_{t+1} \big(\delta'_t(\lambda - d_t) - \delta_t + 1 \big) = 0 \\ \delta_t &: \; -\xi_t \frac{d_t R'_t}{(1+R_t)^2} + \beta \xi_{t+1}(1 - \lambda) = 0 \end{aligned} \tag{30}$$

Using the FOC for consumption to substitute out for ξ_t and ξ_{t+1} in the remaining FOCs for assets, debt and default yields the following three conditions. The first is the standard Euler Equation on risk free assets given in equation 31. It states that the growth in the marginal utility of consumption is determined by the risk-free interest rate, r, and the discount rate, β. When $\beta(1 + r) = 1$, the ratio of marginal utilities is constant and so the agent expects pure consumption smoothing.

$$\begin{aligned} u'(c_t) \frac{1}{1+r} &= \beta u'(c_{t+1}) \\ \frac{u'(c_t)}{u'(c_{t+1})} &= \beta(1+r) \end{aligned} \tag{31}$$

The second is the Euler equation for debt which is the same as the standard Eurler equation except for the terms in square brackets on the left and right hand sides of equation 32. The feedback of debt to interest rates in the model drives a wedge between the standard Euler equation and the equation for optimal debt which is governed by four things: the level of debt, d_t, the default decision, δ_t and its marginal effect, δ'_t, and the cost of default, λ.

$$
\begin{aligned}
u'(c_t)\left[\frac{-d_t\delta_t' R_t + R_t + 1}{(1+R_t)^2}\right] &= \beta u'(c_{t+1})\left[\delta_t'(\lambda - d_t) - \delta_t + 1\right] \\
\frac{u'(c_t)}{u'(c_{t+1})}\frac{1}{1+R_t}\left[\frac{-d_t\delta_t' R_t + R_t + 1}{1+R_t}\right] &= \beta\left[\delta_t'(\lambda - d_t) - \delta_t + 1\right] \\
\beta(1+r)\frac{1}{1+R_t}\left[\frac{-d_t\delta_t' R_t + R_t + 1}{1+R_t}\right] &= \beta\left[\delta_t'(\lambda - d_t) - \delta_t + 1\right] \\
\frac{1+r}{1+R_t}\left[\frac{-d_t\delta_t' R_t + R_t + 1}{1+R_t}\right] &= \left[\delta_t'(\lambda - d_t) - \delta_t + 1\right]
\end{aligned}
\tag{32}
$$

When $\left[\frac{-d_t\delta_t' R_t + R_t + 1}{1+R_t}\right] = [\delta_t'(\lambda - d_t) - \delta_t + 1]$ then $\frac{1+r}{1+R_t} = 1$, and the interest rate on sovereign debt is equal to the risk-free interest rate, $R_t = r$. In this case, the default model converges to an RBC model without default. One case where this occurs is when there is no option of default for the government, such that $\delta = \delta' = 0$ and $\lambda = 0$ due to no default cost. In this case the terms in square brackets collapse to unity, and equations 31 and 32 are identical. This gives rise to the first proof:

Proof 1 *The default model described in the system of equations 30 collapses to a standard Ramsey model with no financial market frictions when $\delta = 0$ and $\lambda = 0$.*

We can also solve the above condition for d_t as follows:

$$
\begin{aligned}
\frac{-d_t\delta_t' R_t + R_t + 1}{1+R_t} &= \delta_t'(\lambda - d_t) - \delta_t + 1 \\
d_t &= (\lambda - 1)(R_t + 1)
\end{aligned}
\tag{33}
$$

Given that the fixed default cost, λ, and the risk-free interest rate, r, are exogenous parameters, there is no reason for the expression in equation 33 to hold. When it does however, we define this as the level of debt that yields a perfect Ramsey model solution to the model. This gives rise to the second proof:

Proof 2 *There is a single parameterisation of the fixed default cost, λ, and risk-free interest rate, r, such that $d_t = (\lambda - 1)(R_t + 1)$ holds and the optimal debt choice, $d_t = d_t^{Ramsey}$, is such that the default model described in the system of equations 30 approximates a standard Ramsey model with no financial market frictions.*

The third condition, which is obtained from substituting out for ξ_t and ξ_{t+1} in the final FOC of 30, is the Euler equation for default given in equation 34. Again, this is identical to the standard Euler equation but for the terms in square brackets. A wedge is driven between the standard Euler equation and the equation for optimal default, which changes with respect to the level of debt, d_t, the marginal effect on the interest rate of debt, R_t', and the cost of default, λ.

$$
\begin{aligned}
u'(c_t)\left[\frac{d_t R_t'}{(1+R_t)^2}\right] &= \beta u'(c_{t+1})\left[1-\lambda\right] \\
\frac{u'(c_t)}{u'(c_{t+1})}\frac{1}{1+R_t}\left[\frac{d_t R_t'}{1+R_t}\right] &= \beta\left[1-\lambda\right] \\
\beta(1+r)\frac{1}{1+R_t}\left[\frac{d_t R_t'}{1+R_t}\right] &= \beta\left[1-\lambda\right] \\
\frac{1+r}{1+R_t}\left[\frac{d_t R_t'}{1+R_t}\right] &= \left[1-\lambda\right]
\end{aligned} \tag{34}
$$

When $\left[\frac{d_t R_t'}{1+R_t}\right] = [1-\lambda]$, then again the default model converges to an RBC model without default. When default is not an option for the sovereign $\lambda = 0$ and this expression collapses to $d_t = \frac{1+R_t}{R_t'}$, then the expression in 34 collapses to the simple Ramsey Euler equation with risk free assets. Note that the function $R_t' \geq 0$ is the first derivative of the interest rate charged on sovereign debt with respect to the level of debt and is assumed to be positive - higher debt leads to a higher interest rate. This expression shows that when default is not available, the sovereign chooses to borrow less, the more responsive it's interest rate is to the level of debt. This is the first lemma:

Lemma 1 *Without default as an option for the sovereign, the level of debt is decreasing in the elasticity of the sovereign's interest rate with respect to indebtedness according to the following expression:* $d_t = \frac{1+R_t}{R_t'} \mid R_t' \geq 0$.

We can also use the expression in 34 to derive an expression for the optimal level of debt when default is available to the sovereign. By simply re-arranging we get:

$$
\begin{aligned}
u'(c_t)\frac{1}{1+R_t}\left[\frac{d_t R_t'}{1+R_t}\right] &= \beta u'(c_{t+1})\left[1-\lambda\right] \\
\frac{u'(c_t)}{u'(c_{t+1})}\left[\frac{d_t R_t'}{(1+R_t)(1+\lambda)}\right] &= \beta(1+R_t)
\end{aligned} \tag{35}
$$

When $\beta(1+r) = 1$ we get pure consumption smoothing only when the term in square brackets, $\left[\frac{d_t R_t'}{(1+R_t)(1+\lambda)}\right]$, is equal to unity. If this term is less than unity then the ratio of marginal utilities is required to be greater than unity. This means that the marginal utility of consumption is higher today that it is tomorrow and thus consumption is lower today that is is tomorrow. This requires that $d_t \leq \frac{(1+R_t)(1+\lambda)}{R_t'}$ or, in other words, the elasticity of the sovereign's interest rate is sufficiently low relative to the prevailing marked rate and the cost of default. This makes sense as when the sovereign's interest rate does not responds strongly to the level of debt (low debt elasticity of the interest rate) then the sovereign will increase indebtedness to bring forward consumption. This gives rise to the second lemma:

Lemma 2 *When sovereign default is possible the degree of consumption tilting depends on the level of impatience,* β*, the prevailing market interest rate,* R_t*, and its elasticity with respect to the level of debt,* R_t'*, relative to the cost of default,* λ*. Under the usual consumption smoothing* *condition of $\beta = (1+r)$, consumption will be brought forward, via debt issuance, when the condition $d_t \leq \frac{(1+R_t)(1+\lambda)}{R'_t}$ holds.*

8.1.3 Steady state

We now solve for the steady-state of the model. In the steady state, consumption, debt and assets are all constant, and output and the oil price are untty. We can represent the full system of equations which include the three Euler equations, 31, 32, 34, and the combined budget constraint from 27 and 28 as follows, where we have substituted out for $1 = \beta(1+r)$ in equations 37 and 38:

$$\begin{aligned} \frac{u'(c_t)}{u'(c_{t+1})} &= \beta(1+r) \\ \frac{u'(c)}{u'(c)} &= \beta(1+r) \\ 1 &= \beta(1+r) \end{aligned} \tag{36}$$

$$\begin{aligned} u'(c_t)\frac{1}{1+R_t}\left[\frac{-d_t\delta'_t R_t + R_t + 1}{1+r}\right] &= \beta u'(c_{t+1})\left[\delta'_t(\lambda - d_t) - \delta_t + 1\right] \\ u'(c)\frac{1}{1+R}\left[\frac{-d\delta' R + R + 1}{1+r}\right] &= \beta u'(c)\left[\delta'(\lambda - d) - \delta + 1\right] \\ \left[\frac{-d\delta' R + R + 1}{1+R}\right] &= \left[\delta'(\lambda - d) - \delta + 1\right] \end{aligned} \tag{37}$$

$$\begin{aligned} u'(c_t)\frac{1}{1+R_t}\left[\frac{d_t R'_t}{1+R_t}\right] &= \beta u'(c_{t+1})\left[1-\lambda\right] \\ u'(c)\frac{1}{1+R}\left[\frac{dR'}{1+R}\right] &= \beta u'(c)\left[1-\lambda\right] \\ \left[\frac{dR'}{1+R}\right] &= \left[1-\lambda\right] \end{aligned} \tag{38}$$

$$\begin{aligned} c_t &= y_t + p_t\bar{s} + f_{t-1} - d_{t-1}\left(1-\delta_{t-1}\right) - \lambda\delta_{t-1} + \frac{d_t}{1+R_t} - \frac{f_t}{1+r} \\ c &= \theta + f - d\left(1-\delta\right) - \lambda\delta + \frac{d}{1+R} - \frac{f}{1+r} \end{aligned} \tag{39}$$

where variables without time subscripts denote steady-state values and $\theta = y + p\bar{s}$ denotes the steady state level of income which depends on the share of oil in the domestic economy, $\bar{s}$.

8.1.4 Optimal debt choice

We can now solve for the optimal choice of debt in the steady-state by combining the three Euler equations in 36, 37 and 38 with the budget constraint in 39.

$$
\begin{aligned}
1 &= \beta(1+r) \\
\left[\frac{-d\delta' R + R + 1}{1+R}\right] &= [\delta'(\lambda - d) - \delta + 1] \\
\left[\frac{dR'}{1+R}\right] &= [1-\lambda] \\
c &= \theta - d\,(1-\delta) - \lambda\delta + \frac{d}{1+R}
\end{aligned} \tag{40}
$$

Substituting out for $1/1+R$ from the 2nd equation of 40 into the 4th equation, and solving for optimal debt in the steady-state, d^*, yields:

$$
d^* = \frac{(R+1)(c+\theta+\delta\lambda)}{(\delta-1)R} \tag{41}
$$

Noting that:

$$
\begin{aligned}
\frac{\partial d^*}{\partial c} &= \frac{R+1}{(\delta-1)R} \\
\frac{\partial d^*}{\partial R} &= \frac{c+\delta\lambda+\theta}{(\delta-1)R^2}
\end{aligned} \tag{42}
$$

where $(R+1)/R > 0$ and $\delta - 1 < 0$, leads to the third lemma

Lemma 3 *The optimal level of debt for a resource rich sovereign is a non-linear function of the demands on revenue, the prevailing interest rate and the government's likelihood of default. Optimal debt is increasing in the the steady state level of consumption, c, but at a rate that is decreasing in the default decision, δ. It is also decreasing in the interest rate, R, but at an increasing rate*

8.1.5 Pricing sovereign debt

Dividing equation 37 by 38 gives:

$$
\frac{-d\delta' R + R + 1}{dR'} = \frac{\delta'(\lambda - d) - \delta + 1}{1-\lambda} \tag{43}
$$

Now substituting in for the bond price, $q = \frac{1}{1+R}$, from 25 gives:

$$
q = \frac{1-\lambda}{dR\,(\delta'(\lambda-d) - \delta + 1)} + \frac{1}{d\delta' R} \tag{44}
$$

When default is not possible for the sovereign, $\delta = \delta' = 0$ and $\lambda = 0$, so the bond pricing expression collapses to $q = \frac{1}{dR}$. This simple expression states that the steady state bond price of sovereign debt is inversely related to the level of debt, d and the interest rate facing the sovereign, R. When default is possible, this inverse relationship is augmented by the cost of default, λ, and the marginal effect of the level of debt on the optimal default decision by the sovereign, δ'. The more sensitive the default decision of the sovereign is with respect the level of debt (that is, the larger is δ'), the lower the price investors are willing to pay to hold this debt (and the higher the effective interest rate). This is the fourth lemma:

Lemma 4 *Without default, the the steady state bond price of sovereign debt, q, is inversely related to the level of debt, d, and the interest rate facing the sovereign, R, according to the expression $q = \frac{1}{dR}$. When default is possible, this inverse relationship is augmented by the sensitivity of the sovereign's default decision with respect to the level of debt, δ'; the more sensitive this choice is, the less investors are willing to pay to hold the sovereign's debt. The bond pricing expression in the steady state is given by $q = \frac{1-\lambda}{dR(\delta'(\lambda-d)-\delta+1)} + \frac{1}{d\delta' R}$.*

8.2 Bivariate Markov shocks

I follow Tauchen (1986) in the exposition of a discretised state-space representation of a Markov shock process. Instead of using a single variable process however, I assume a bivariate autoregressive shock process consisting of two correlated variables, x_t and y_t. Consider the following shock process in matrix form:

$$\begin{aligned} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} &= \begin{bmatrix} \rho_x & \rho_{xy} \\ \rho_{yx} & \rho_y \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \epsilon_{x,t+1} \\ \epsilon_{y,t+1} \end{bmatrix} \\ \boldsymbol{z}_{t+1} &= \boldsymbol{\rho} z_t + \boldsymbol{\epsilon}_{t+1} \end{aligned} \tag{45}$$

where $\boldsymbol{\epsilon}_{t+1} \sim N(\mu, \Sigma)$, $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{\Sigma} = \mathbb{E}(\boldsymbol{\epsilon\epsilon}') = \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{yx} & \sigma_y \end{bmatrix}$. Then we can express the expected value of $\boldsymbol{z}_{t+1}$ and it's variance as:

$$\mathbb{E}(\boldsymbol{z}_{t+1}) = 0 \tag{46}$$

$$\text{Var}(\boldsymbol{z}_{t+1}) = \boldsymbol{\Omega} = \begin{bmatrix} \omega_x & \omega_{xy} \\ \omega_{yx} & \omega_y \end{bmatrix} = \boldsymbol{\Sigma} [\text{I} - \boldsymbol{\rho}]^{-1} \tag{47}$$

We define a state space grid for the correlated variables as:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} < \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} < \ldots < \begin{bmatrix} x_m \\ y_n \end{bmatrix}$$

where

$$\begin{aligned} \boldsymbol{z}_1 &= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -r\omega_x \\ -r\omega_y \end{bmatrix}, \\ \boldsymbol{z}_{mn} &= \begin{bmatrix} x_m \\ y_n \end{bmatrix} = \begin{bmatrix} r\omega_x \\ r\omega_y \end{bmatrix} \text{ and} \\ \boldsymbol{d} &= \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} x_i - x_{i-1} \\ y_k - y_{k-1} \end{bmatrix} \end{aligned} \tag{48}$$

where r is the state-space factor, which we set to 3, and m and n denote the respective number of state spaces. Since $\Pr(\epsilon_{t+1} \leq \boldsymbol{u}) = F\left([\boldsymbol{u} - 0]\,\boldsymbol{\Sigma}^{-1}\right)$, we can define the transition probability matrix $\boldsymbol{\Pi}_{ij,kl}$ as:

$$\begin{aligned} \boldsymbol{\Pi}_{ij,kl} &= \Pr\left\{\boldsymbol{z}_{jl} - \boldsymbol{d} \setminus 2 \leq \boldsymbol{\rho}\boldsymbol{z}_{ik} + \boldsymbol{\epsilon}_{t+1} \leq \boldsymbol{z}_{jl} + \boldsymbol{d} \setminus 2\right\} \\ &= F\left([\boldsymbol{z}_{jl} - \boldsymbol{\rho}\boldsymbol{z}_{ik} + \boldsymbol{d} \setminus 2]\,\boldsymbol{\Sigma}^{-1}\right) - F\left([\boldsymbol{z}_{jl} - \boldsymbol{\rho}\boldsymbol{z}_{ik} - \boldsymbol{d} \setminus 2]\,\boldsymbol{\Sigma}^{-1}\right) \end{aligned} \tag{49}$$

where $\boldsymbol{\Pi}$ is a matrix of dimensions $(m \times n \times m \times n)$, and $F(\cdot)$ denotes the CDF of the bivariate normal distribution. Subscript ij denotes the probability of moving from the ith position to the jth position of variable x, and kl denotes the probability of moving from the kth position to the lth position of variable y. The boundary transition probabilities are given by:

$$\boldsymbol{\Pi}_{i1,k1} = F\left([\boldsymbol{z}_{11} - \boldsymbol{\rho}\boldsymbol{z}_{ik} + \boldsymbol{d} \setminus 2]\,\boldsymbol{\Sigma}^{-1}\right) \tag{50}$$

and

$$\boldsymbol{\Pi}_{i\mathrm{m},k\mathrm{n}} = F\left([\boldsymbol{z}_{mn} - \boldsymbol{\rho}\boldsymbol{z}_{ik} - \boldsymbol{d} \setminus 2]\,\boldsymbol{\Sigma}^{-1}\right) \tag{51}$$

I have developed Matlab code that estimates the transition matrix for the bivariate VAR process described here, and another that simulates a Markov chain using this bivariate transition matrix. These codes allow the user to set the deep parameters of the bivariate process as well as the state space for the model.[22]

[22] Please visit `https://thomasmjmcgregor.wordpress.com/data/`

9 Appendix

9.1 Solution algorithm

The numerical solution method involves the following steps:

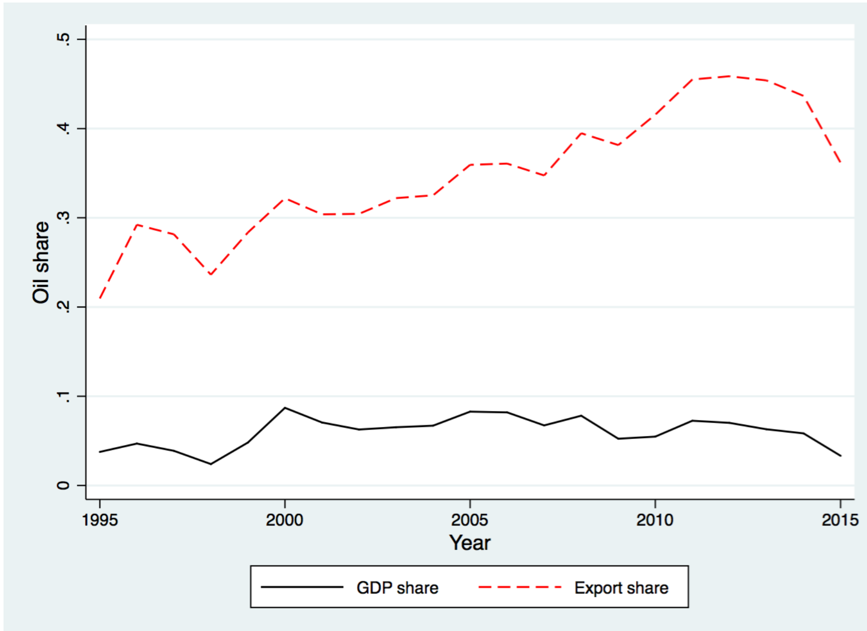
1. Choose parameter values for: $(\rho, \theta, \sigma, r, \bar{s}, \psi, \rho_z, \rho_h, \sigma_z^2, \sigma_h^2)$.
2. Discretise the state spaces for y, d, and p. I use 11 output and oil price states, and 125 debt states.
3. Start with a guess of the bond price schedule $q^0(d, y, p) = 1/(1+r)$ and use this to solve the sovereign's problem using value function iteration, obtaining the optimal policy functions for consumption, asset holdings and default choice.
4. Given these policy functions, compute the probability of default, $\delta(d', y, p)$.
5. Update the bond pricing schedule given this probability of default and use this updated schedule to repeat steps 3-4 until the the convergence criterion has been reached, namely $q^0(d, y, p) - q^1(d, y, p) < \Delta$, where Δ is some very small number.

9.2 Oil export share calibration

This appendix discusses the performance of the model with a substantially higher oil share parameter, $\bar{s}$. It turns out that the model is fairly sensitive to the calibration of this parameter. In general a higher oil share increases the spread persistence as well as the strength of the negative correlation between interest rate spreads and the oil price. Increasing the oil share parameter does, however, present challenges for the numerical solution algorithm. A higher share, means that the sovereign also has more resources at it's disposal on average. Relying on the same solution algorithm, with a pre-defined debt state space for example, often results in 'non-convergence' problems.

As discussed in section 4.2, an alternative way to calibrate the oil share parameter would be to use data on oil export shares. Figure 8 plots the average share of oil in total GDP and the oil export share in total exports for our four economies, between1995 and 2015. The export share has been broadly rising, whilst the output share has remained relatively constant.

Figure 8: Oil and gas shares over time



Here I present model simulation results setting with $\bar{s} = .35$. Table 10 presents the mean AR(1) coefficients and standard deviations for output, the oil price, consumption and the interest rate spread from simulating the model over 1000 periods. The key difference between these simulations and those in section sec:indepshocks is that the spread persistence is increases substantially. Table 11 presents the mean correlation between spreads and deviations of output and the oil price in the data respectively. The negative correlations are stronger in both cases.

Table 10: AR(1) model: autocorrelations and standard dev ($\bar{s} = .35$)

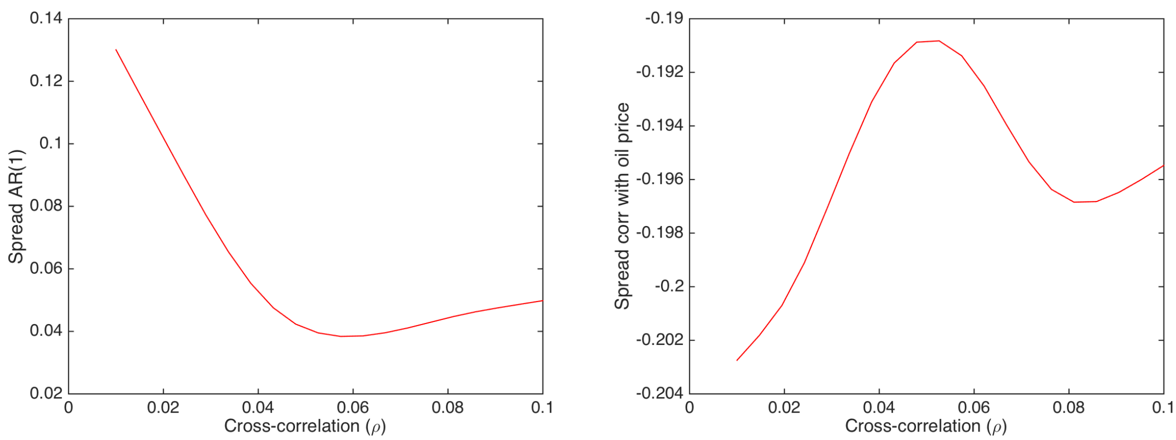
	Data		Model	
	AR(1)	std	AR(1)	std
output (HP filt)	.841	.032	.893	.043
oil price (HP filt)	.725	.198	.657	.195
cons (HP filt)	.898	.026	.977	.107
spread	.645	.022	.304	.010

Table 11: AR(1) model: cross correlations with real interest rate spread ($\bar{s} = .35$)

	Data	Model
	corr	corr
output dev	-.138	-.148
oil price dev	-.279	-.160

Figures 9a and 9b present the AR(1) parameter for the spread and the spread's correlation with oil prices respectively, for different levels of the correlation between output and oil price shocks. The shape of the spread persistence is similar to the model with $\bar{s} = .06$ for strongly correlated shocks, but very different for less correlated shocks. The shape of the cross correlation between spreads and oil prices is very similar to the model with $\bar{s} = .06$, but with a more negative correlation on average.

Figure 9: Sensitivity to correlation - $\rho_{zh} \in [.01, .10]$ ($\bar{s} = .35$)

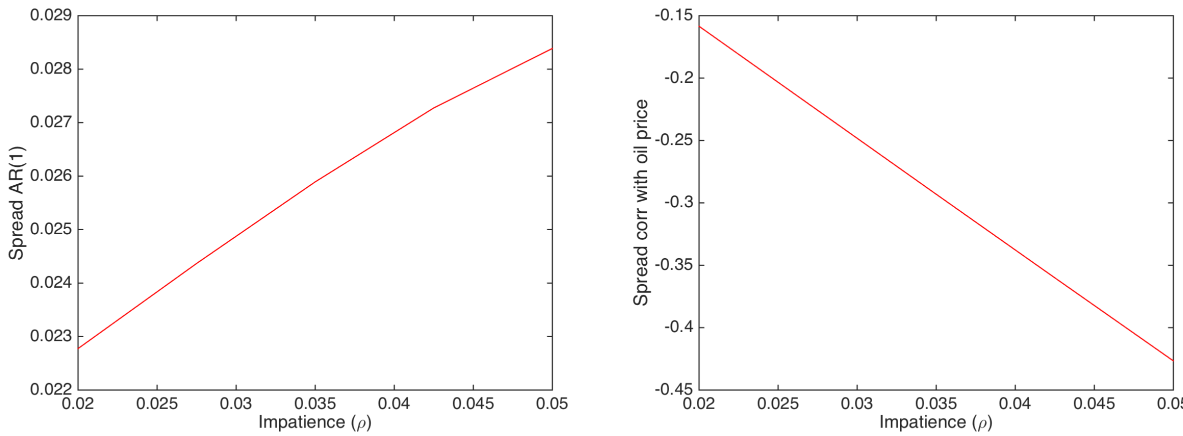


(a) AR(1) parameter

(b) Correlations

Figures 10a and 10b present the same figures, but for different levels of impatience. I was forced to limit the solution algorithm to solve the model for only 3 intermediate points on the impatience range due to non-convergence problems. In general the spread persistence responds in a similar way to impatience as in the model with $\bar{s} = .06$, that is, it is increasing with impatience. The main difference between the two models is the increase in persistence on average. The correlation between spreads and the oil price also responds in a similarly negative way to impatience, albeit with the negative correlation substantially increased on average.

Figure 10: Sensitivity to impatience, $\rho \in [.01, .05]$ ($\bar{s} = .35$)



(a) AR(1) parameter (b) Correlations

9.3 Model sensitivity

Figure 11: Output truncation under default

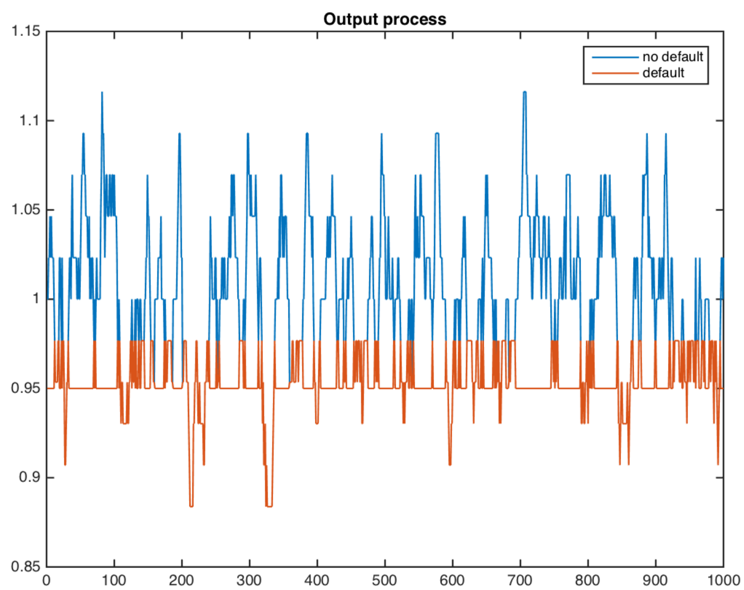


Figure 12: Bond pricing schedule (no oil)

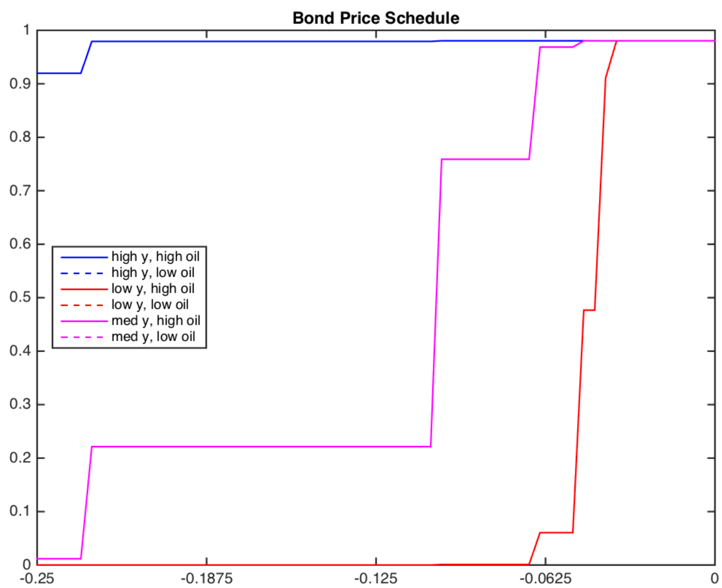


Table 12: AR(1) model (no oil): autocorrelations and standard dev

	Data		Model	
	AR(1)	std	AR(1)	std
output (HP filt)	.841	.032	.716	.024
oil price (HP filt)	.725	.198	.650	.195
cons (HP filt)	.898	.026	.839	.016
spread	.645	.022	.001	.005

Table 13: AR(1) model (no oil): cross correlations with real interest rate spread

	Data	Model
	corr	corr
output dev	-.138	-.116
oil price dev	-.279	-.129