Topological moduli spaces, homological stability, and operads



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Abstract

This thesis consists of two relatively independent parts.

In the first part, we study operads with homological stability, which are topological operads that satisfy a homological stability condition. Despite the fact that these operads are in general very far from being E_∞-operads, their algebras give rise to infinite loop spaces upon group completion. We show that under a mild condition on an operad with homological stability, one can naturally associate topologically enriched symmetric strict monoidal categories to its algebras such that the group completion of the algebra is weakly equivalent to the loop space on the classifying space of the associated category, which is evidently an infinite loop space. We also show that this new infinite loop space structure coming from the associated category is equivalent to the previously known infinite loop space structure on the group completion of the algebra.

The second part is concerned with quotients of mapping class groups $\Gamma_{g,1}$ of oriented surfaces with one boundary component by the subgroups $\mathscr{I}_{g,1}(k)$ in their Johnson filtration. We show that the stable classifying spaces $\mathbb{Z} \times B\left(\Gamma_\infty / \mathscr{I}_\infty(k)\right)^+$ after Quillen's plus-construction are infinite loop spaces, fitting into a tower of infinite loop space maps that interpolates between the infinite loop spaces $\mathbb{Z} \times B\Gamma_\infty^+$ and $\mathbb{Z} \times B\left(\Gamma_\infty / \mathscr{I}_\infty(1)\right)^+ \simeq \mathbb{Z} \times B\mathrm{Sp}(\mathbb{Z})^+$. We also show that for each level k in the Johnson filtration, the homology of these quotients with suitable systems of twisted coefficients stabilises as the genus g of the surface tends to infinity.

Statement of Originality

I declare that the work presented in this thesis is, to the best of my knowledge, original except when indicated otherwise, cited from the literature, or when commonly known. This thesis (or parts thereof) has not been submitted for a degree at another university, or for another degree at the University of Oxford.

An earlier version of Chapter 2 has been uploaded to the arXiv. The chapter is currently under review for publication.

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Chapter 1

Operads with strong homological stability

1.1 Introduction

1.1.1 Operads with homological stability

In [3], Basterra, Bobkova, Ponto, Tillmann and Yeakel introduced the notion of an operad with homological stability, or OHS. This is a topological operad $\mathcal{O}$ with the following properties:

(1) the space $\mathcal{O}(0)$ of nullary operations is not required to be a point, but it does come with a choice of a basepoint $* \in \mathcal{O}(0)$;

(2) each space $\mathcal{O}(n)$ of n-ary operation is the disjoint union of spaces $\mathcal{O}_g(n)$ for $g \geqslant 0$ such that the action of the symmetric group Σ_n on $\mathcal{O}(n)$ restricts to an action on each $\mathcal{O}_g(n)$, and the grading is additive with respect to operad composition;

(3) the maps $\mathcal{O}_g(n) \to \mathcal{O}_g(0)$ induced by composing an n-ary operation on the right with $* \in \mathcal{O}(0)$ are integral homology isomorphisms in a range of degrees that tends to infinity as g tends to infinity;

(4) there is an operad map $\mu\colon \mathcal{A} \to \mathcal{O}$ from an A_∞-operad $\mathcal{A}$ that lands in grading 0 and such that $\mu(\mathcal{A}(2)) \subseteq \mathcal{O}(2)$ lies in a path-connected component.

We give a more detailed definition in Definition 1.3.1 and Remark 1.3.3.

The motivating example of an OHS is the surface operad $\mathcal{M}$, whose space $\mathcal{M}(n)$ of n-ary operations is a moduli space of connected oriented surfaces with $n+1$ parametrised

boundary components, one of which is marked as 'outgoing', and with an ordering of the remaining n 'incoming' boundary components. Composition in $\mathcal{M}$ is defined by gluing the outgoing boundary of one surface to an incoming boundary of another surface. Each $\mathcal{M}(n)$ is a disjoint union of spaces $\mathcal{M}_g(n)$ for $g \geqslant 0$, where $\mathcal{M}_g(n)$ consists precisely of those surfaces that have genus g. (We recall the construction of $\mathcal{M}$ in greater detail in Section 2.3.1.) Tillmann proved in [54] that $\mathcal{M}$ detects infinite loop spaces: an $\mathcal{M}$-algebra is in particular a framed E_2-algebra since the genus 0 suboperad of $\mathcal{M}$ is equivalent to the framed little discs operad. Hence one can speak of the group completion of an $\mathcal{M}$-algebra, and Tillmann proved that this group completion is always an infinite loop space. This is a somewhat surprising result, since $\mathcal{M}$ is very far from being an E_∞-operad.

The main result of [3] is a generalisation of Tillmann's theorem (*cf.* also the more precise statement of Theorem 1.3.8):

Theorem 1.1.1 ([3, Theorem 1.2])**.** *Let $\mathcal{O}$ be an OHS. Group completion lifts to a functor from the category of $\mathcal{O}$-algebras to the category of infinite loop spaces and infinite loop space maps.*

Thus in particular whenever $\mathcal{O}$ is an OHS, then the group completion of the initial $\mathcal{O}$-algebra $\mathcal{O}(0)$ is an infinite loop space. In the case of the surface operad $\mathcal{M}$, the group completion of $\mathcal{M}(0)$ is weakly equivalent to $\mathbb{Z} \times B\Gamma_\infty^+$, where $\Gamma_\infty = \operatorname{colim}_g \Gamma_{g,1}$ is the stable mapping class group and + denotes the Quillen plus-construction. Tillmann [55] showed that $\mathbb{Z} \times B\Gamma_\infty^+$ is weakly equivalent to $\Omega B\mathcal{S}$, where $\mathcal{S}$ is the 1+1 cobordism category: its objects are finite collections of circles, and its morphisms are oriented cobordisms. $\mathcal{S}$ is a symmetric monoidal category with respect to disjoint union of manifolds (provided one is careful in its construction, see [55]), so $B\mathcal{S}$ (and $\Omega B\mathcal{S}$) is an infinite loop space. It is a theorem of Wahl that these *a priori* potentially inequivalent deloopings of the group completion of $\mathcal{M}(0)$ do in fact agree:

Theorem 1.1.2 ([57])**.** *The two infinite loop space structures on $\mathbb{Z} \times B\Gamma_\infty^+$ — one coming from the identification $\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B\mathcal{M}(0)$, the other from the identification $\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B\mathcal{S}$ — are equivalent to one another.*

We remark that while the underlying idea of Wahl's proof is sound and can be generalised to other operads with homological stability (as we indeed do below), there are several technical issues in [57] which render the proof invalid. The main issue is a seemingly

trivial basepoint issue, *cf.* Remark 1.3.20. Nonetheless, it took us a non-trivial amount of work to fix it: the crucial result in this direction is Theorem 1.2.26, and it took us an entire section (Section 1.4) to sketch its proof. This chapter can therefore serve as a corrected proof of Theorem 1.1.2.

1.1.2 Operads of high-dimensional manifolds

Other interesting examples of OHSs arise as higher-dimensional analogs of $\mathcal{M}$. We recall their construction from [3]: fix an even dimension $2k$ and a tangential structure — this is a Serre fibration $\theta\colon B \to BO(2k)$ with B path-connected. We denote by $\theta^*\gamma_{2k}$ the pullback to B of the canonical $2k$-dimensional vector bundle γ_{2k} over $BO(2k)$. A θ-structure on a $2k$-dimensional manifold W is a bundle map $TW \to \theta^*\gamma_{2k}$. A θ-structure on a $(2k-1)$-dimensional manifold P is a bundle map $\underline{\mathbb{R}} \oplus TP \to \theta^*\gamma_{2k}$.

Remark 1.1.3. Following [23], by a *bundle map* we mean a morphism of vector bundles which is an isomorphism on each fibre of the domain. In particular, if $\ell\colon TW \to \theta^*\gamma_{2k}$ is a θ-structure on the $2k$-manifold W, then the left-hand square in the commutative diagram

$$\begin{array}{ccccc} TW & \xrightarrow{\ell} & \theta^*\gamma_{2k} & \longrightarrow & \gamma_{2k} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ W & \longrightarrow & B & \xrightarrow[\theta]{} & BO(2k) \end{array}$$

is a pullback square, and hence so is the boundary of the diagram. In particular, the map $W \to B$ that underlies ℓ is a lift of a classifying map $W \to BO(2k)$ of TW along θ.

Given a $2k$-dimensional manifold W and a θ-structure

$$\ell_{\partial W}\colon \underline{\mathbb{R}} \oplus T\partial W \cong TW|_{\partial W} \longrightarrow \theta^*\gamma_{2k}$$

let $\mathrm{Bun}^{\theta}_{k,\partial}(TW;\ell_{\partial W})$ be the space of all bundle maps $TW \to \theta^*\gamma_{2k}$ that restrict to $\ell_{\partial W}$ on the boundary and whose underlying map $W \to B$ is k-connected. Finally, given $\ell_W \in \mathrm{Bun}^{\theta}_{k,\partial}(TW;\ell_{\partial W})$, we let

$$\mathcal{M}^{\theta}_k(W;\ell_W) \subseteq \mathrm{Bun}^{\theta}_{k,\partial}(TW;\ell_{\partial W}) /\!\!/ \mathrm{Diff}(W,\partial)$$

be the connected component containing ℓ_W of the homotopy quotient of $\mathrm{Bun}^{\theta}_{k,\partial}(TW;\ell_{\partial W})$ in the action of the boundary-fixing diffeomorphism group $\mathrm{Diff}(W,\partial)$.

Let $W_{g,n}$ be a $2k$-dimensional manifold obtained by removing the interiors of n disjoint embedded $2k$-discs from a connected sum of g copies of $S^k \times S^k$. Assuming that one can find a θ-structure $\ell_{g,n}$ for each $W_{g,n}$ satisfying certain compatibility relations (namely, there should be a fixed θ-structure ℓ_S on S^{2k-1} such that each $\ell_{g,n}$ restricts to this θ-structure on every boundary component of $W_{g,n}$, and the different $\ell_{g,n}$ should be compatible with respect to gluing the manifolds $W_{g,n}$ along boundary components — see [3] for details), then there is an operad $\mathcal{W}$ with

$$\mathcal{W}(n) \simeq \coprod_{g \geqslant 0} \mathcal{M}_k^\theta(W_{g,1+n}, \ell_{g,1+n})$$

Operad composition is again by gluing manifolds along boundary components.

Example. When $2k = 2$ and θ is the orientation cover $BSO(2) \to BO(2)$, this construction yields the surface operad.

Theorem 1.1.4 ([3, Theorem 7.3])**.** *If $2k \geqslant 6$ and the space B is k-connected, then $\mathcal{W}$ is an OHS.*

Thus when the hypothesis of the above theorem is satisfied, then

$$\Omega B\mathcal{W}(0) \simeq \Omega B\left(\coprod_g \mathcal{M}_k^\theta(W_{g,1}, \ell_{g,1})\right) \simeq \mathbb{Z} \times \left(\operatorname*{hocolim}_g \mathcal{M}_k^\theta(W_{g,1}, \ell_{g,1})\right)^+$$

is an infinite loop space, where the homotopy colimit is taken over the stabilisation maps

$$\mathcal{M}_k^\theta(W_{g,1}, \ell_{g,1}) \longrightarrow \mathcal{M}_k^\theta(W_{g+1,1}, \ell_{g+1,1})$$

induced by attaching a copy of $W_{1,1}$ to a boundary component of $W_{g,1}$.

However, Galatius and Randal-Williams [23] prove that the identity component of this space is an infinite loop space by establishing the weak equivalence

$$\left(\operatorname*{hocolim}_g \mathcal{M}_k^\theta(W_{g,1}, \ell_{g,1})\right)^+ \simeq \Omega_0 BC_\theta \tag{1.1}$$

where C_θ is a topological category of embedded $2k$-dimensional cobordisms with θ-structures: its objects are smooth $(2k-1)$-dimensional submanifolds of $\mathbb{R}^\infty$ equipped with θ-structures. Its morphisms are $2k$-cobordisms embedded in $\mathbb{R}^\infty \times [0, s]$ for some $s > 0$, and equipped with θ-structures. This leads to the following question, raised in [3].

Question 1.1.5. Are these two infinite loop space structures on $\Omega B\mathcal{W}(0)$ equivalent?

Let $S_n \in \operatorname{ob} C_\theta$ for $n \geqslant 0$ be an object which is diffeomorphic to $\coprod^n S^{2k-1}$ with the θ-structure $\coprod^n \ell_S$. The space $\operatorname{mor} C_\theta$ is topologised in such a way that the connected component of $C_\theta(S_n, S_1)$ that contains an embedded $W_{g,n+1}$ with the θ-structure $\ell_{g,n+1}$ is a model for $\mathcal{M}_k^\theta\left(W_{g,n+1}, \ell_{g,n+1}\right)$. In particular, we obtain a map

$$\mathcal{M}_k^\theta\left(W_{g,1}, \ell_{g,1}\right) \to \Omega_{[S_0,S_1]} B C_\theta \tag{1.2}$$

into the space of paths from S_0 to S_1, and this is the map used to construct the weak equivalence (1.1).

Thus in trying to answer Question 1.1.5, we were naturally led to consider the associated PROP of an operad with homological stability.

1.1.3 Associated PROPs

Given an operad $\mathcal{O}$, its associated PROP $\underline{\mathcal{O}}$ is the topologically enriched category whose objects are finite sets; a morphism $S \to T$ in $\underline{\mathcal{O}}$ is a T-tuple of operations from $\mathcal{O}$ with total arity $|S|$. (See Definition 1.3.14 for more details.) Disjoint union of sets lifts to a symmetric strict monoidal structure on $\underline{\mathcal{O}}$, so the classifying space $B\underline{\mathcal{O}}$ is an infinite loop space, *cf.* [48].

In Definition 1.3.1 below, we introduce a mild strengthening of the OHS condition, which we call an *operad with strong homological stability*, or an OSHS. All of the aforementioned examples of operads with homological stability also satisfy this stronger condition. The condition allows us to prove the following generalisation of the main theorem of [55].

Theorem 1.1.6. *Let $\mathcal{O}$ be an OSHS. Then the group completion of $\mathcal{O}(0)$ is weakly equivalent to the infinite loop space $\Omega B\underline{\mathcal{O}}$.*

The greatest part of this chapter is dedicated to proving the following generalisation of Wahl's theorem 1.1.2.

Theorem 1.1.7. *Let $\mathcal{O}$ be an OSHS. The infinite loop space structures on $\Omega B\mathcal{O}(0)$ arising from Theorems 1.1.1 and 1.1.6 are equivalent to one another.*

It is natural to ask whether the infinite loop space structure on ΩBX of Theorem 1.1.1 for an arbitrary $\mathcal{O}$-algebra X may be presented in a similar fashion. We answer this question in the affirmative in Section 1.5 (subject to mild conditions on the monoid $\pi_0 X$): for an operad $\mathcal{P}$ and a $\mathcal{P}$-algebra A, we construct the 'homotopy enveloping operad' $\mathcal{P}\{A\}$ under

$\mathcal{P}$, for which $\mathcal{P}\{A\}(0)$ is equivalent as a $\mathcal{P}$-algebra to A. We prove the following result.

Theorem 1.1.8. *Let $\mathcal{O}$ be an O(S)HS and let X be an $\mathcal{O}$-algebra such that $\pi_0 X$ with the commutative monoid structure coming from the monoid map $\mu\colon \mathcal{A} \to \mathcal{O}$ is finitely generated. Then $\mathcal{O}\{X\}$ is also an O(S)HS.*

Thus for an OSHS $\mathcal{O}$, the group completion of a suitable $\mathcal{O}$-algebra X is equivalent as an infinite loop space to the space $\Omega B\underline{\mathcal{O}\{X\}}$.

For the operad $\mathcal{W}$ of high-dimensional manifolds recalled above, the associated PROP $\underline{\mathcal{W}}$ is a cobordism category of sorts: its objects may be thought of as finite disjoint unions of copies of S^{2k-1}, and its morphisms are $2k$-dimensional cobordisms whose connected components are diffeomorphic to some $W_{g,n+1}$ and have precisely one outgoing component. It seems reasonable to expect that the maps (1.2) should factor in a suitably coherent way through the category $\underline{\mathcal{W}}$.

However, the technical details of such a comparison have thus far eluded us. The issue is one of finding compatible models: the category $\mathcal{C}_\theta$ is not monoidal under disjoint union of cobordisms, but it is a Γ-category. (*Cf.* [37] where the Γ-category structure on the cobordism category of oriented surfaces is expounded in some detail, and [42] where the Γ-structure on the general cobordism is studied in great detail.) This means that perhaps the most obvious line of attack — *i.e.* taking $\mathcal{W}$ to be the endomorphism operad of a sphere in $\mathcal{C}_\theta$ — runs into difficulties. The paper [3] sidesteps this difficulty in constructing $\mathcal{W}$ by first constructing an operad in (discrete) groupoids: the objects of this operad are $2k$-cobordisms of the diffeomorphism type of $W_{g,n+1}$ with θ-structures *and* a decomposition into 'atomic pieces', which are fixed copies of $W_{0,1}$, $W_{1,2}$ and $W_{0,3}$. (The purpose of this extra structure is to ensure that the resulting groupoid is small and it has a geometric realisation.) The morphisms of the operad are diffeomorphisms that preserve the θ-structure (but not necessarily the atomic decomposition). The operad $\mathcal{W}$ is then precisely the geometric realisation of this operad in groupoids. *Cf.* Section 2.3.1 where we recall the construction of the surface operad $\mathcal{M}$ from [54]. Thus the category $\underline{\mathcal{W}}$ is a category of non-embedded cobordisms equipped with decompositions into the atomic pieces.

We intend to return to this rather frustrating issue in a future paper.

1.1.4 Outline of chapter

In Section 1.2, we briefly recall the definition of a (topological) operad and an operad algebra, and we review several variations of this definition that will be relevant in the later parts of the text, namely pointed operads, graded operads and coloured operads. We give several examples to illustrate these definitions. In the second half of Section 1.2, we review the work of Berger, Moerdijk and Spitzweck on the homotopy theory of coloured operad algebras, and of uncoloured operads. These results are crucial in our proofs of Theorems 1.1.6 and 1.1.7, where we use them to strictify structures that only exist up to homotopy. We also review the homotopy theory of simplicial operad algebras, and we state a set of conditions on an operad O and an O-algebra A that guarantee that the bar resolution of A as an O-algebra is suitably cofibrant. This is Theorem 1.2.26, whose proof is deferred until Section 1.4.

In Section 1.3, we define operads with strong homological stability and we contrast the definition with the original notion of operads with homological stability introduced in [3]. We show how the homotopical results of the preceding section enable a simplification of the O(S)HS condition: instead of assuming that an O(S)HS receives a morphism from an arbitrary A_∞-operad, we may assume without loss of generality that it receives a map from *the* associative operad Ass. Thus in particular all operad algebras in sight become strictly associative and unital monoids.

In Section 1.3.2, we give a more rigorous definition of the associated PROP of an operad. We also prove Theorem 1.1.6. Section 1.3.3 is devoted to the proof of Theorem 1.1.7.

Sections 1.4 and 1.5 are mostly technical, with the former section giving a (detailed) sketch of the proof of Theorem 1.2.26, and the latter section giving the definition of a homotopy enveloping operad and showing that under mild conditions, homotopy enveloping operads associated to operads with (strong) homological stability also satisfy the O(S)HS condition.

1.2 Definitions and some properties

This section serves to collect in one place the definitions and concepts which will be relevant in the later parts of the text.

1.2.1 Operads and operad algebras

Unless stated otherwise, in this chapter we work in the category Top of compactly generated weakly Hausdorff topological spaces. This category is closed symmetric monoidal with respect to the Cartesian product of spaces, and we shall write Y^X for the space of maps from X to Y.

1.2.1.1 Topological operads

The following definition is essentially taken from [39], except that we do not require the space of nullary operations to be a point.

Definition 1.2.1. A *(topological) operad* O consists of the following data:

(1) for each $n \geqslant 0$, a topological space $O(n)$ equipped with a right action of the symmetric group Σ_n,

(2) for each $n \geqslant 0$ and each n-tuple $m_1, \ldots, m_n \geqslant 0$ of integers, a map

$$\gamma : O(n) \times O(m_1) \times \cdots \times O(m_n) \longrightarrow O(m_1 + \cdots + m_n),$$

(3) an identity $1 \in O(1)$.

The operadic composition maps γ are required to satisfy the following properties:

(i) For all $a \in O(n)$, all $b_i \in O(m_i)$ for $1 \leqslant i \leqslant n$ and all $c_j \in O(k_j)$ for $1 \leqslant j \leqslant m_1 + \cdots + m_n$, we have

$$\gamma\left(a; b'_1, \ldots, b'_n\right) = \gamma\left(\gamma\left(a; b_1, \ldots, b_n\right); c_1, \ldots, c_{m_1+\cdots+m_n}\right)$$

where $b'_i \coloneqq \gamma(b_i; c_{m_1+\cdots+m_{i-1}+1}, \ldots, c_{m_1+\cdots+m_i})$.

(ii) For all $a \in O(n)$, we have $\gamma(1; a) = a = \gamma(a; 1, \ldots, 1)$.

(iii) For all $a \in O(n)$, all $b_i \in O(m_i)$ for $1 \leqslant i \leqslant n$, all $\sigma \in \Sigma_n$ and all $\tau_i \in \Sigma_{m_i}$ for $1 \leqslant i \leqslant n$, we have

$$\gamma\left(a \cdot \sigma; b_1, \ldots, b_n\right) = \gamma\left(a; b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(n)}\right) \cdot \sigma(m_1, \ldots, m_n)$$

and

$$\gamma\left(a; b_1 \cdot \tau_1, \ldots, b_n \cdot \tau_n\right) = \gamma\left(a; b_1, \ldots, b_n\right) \cdot (\tau_1 \oplus \cdots \oplus \tau_n)$$

where $\sigma(m_1, \ldots, m_n) \in \Sigma_{m_1+\cdots+m_n}$ permutes n blocks of sizes $m_1, \ldots, m_n$ according to σ, and $\tau_1 \oplus \cdots \oplus \tau_n$ denotes the image of $(\tau_1, \ldots, \tau_n)$ under the evident inclusion of $\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$ into $\Sigma_{m_1+\cdots+m_n}$.

A *morphism* $\phi\colon \mathcal{O} \to \mathcal{P}$ *of operads* is a collection of Σ_n-equivariant maps $\phi(n)\colon \mathcal{O}(n) \to \mathcal{P}(n)$ which commute with the operadic structure maps and which preserve the distinguished point 1.

Remark 1.2.2. Replacing the category Top with an arbitrary symmetric monoidal category $(\mathscr{C}, \otimes)$ (and reformulating the conditions on γ in a suitably diagrammatic way) yields the notion of an operad in $\mathscr{C}$. We recall an alternative definition of an operad in Section 1.4.1.

Standard examples of topological operads include the following:

(i) The commutative operad Com, given by $\mathrm{Com}(n) = *$.

(ii) The associative operad Ass, given by $\mathrm{Ass}(n) = \Sigma_n$ viewed as a discrete space with the obvious right Σ_n-action. Composition is by block sum.

Related to this operad are A_∞ operads. We say an operad $\mathcal{A}$ is A_∞ if it admits an operad map $\phi\colon \mathcal{A} \to \mathrm{Ass}$ such that each $\phi(n)\colon \mathcal{A}(n) \to \Sigma_n$ is a Σ_n-equivariant weak homotopy equivalence.

(iii) The Barratt–Eccles operad Γ, introduced in [2], given by $\Gamma(n) = E\Sigma_n$, the universal principal Σ_n-bundle.

Definition 1.2.3. An *algebra over an operad* $\mathcal{O}$ (or an $\mathcal{O}$*-algebra*) is a space X equipped with structure maps

$$\theta\colon \mathcal{O}(n) \times X^n \longrightarrow X$$

for all $n \geqslant 0$ such that the following are satisfied:

(i) For all $a \in \mathcal{O}(n)$, all $b_i \in \mathcal{O}(m_i)$ for $1 \leqslant i \leqslant n$ and all $x_j \in X$ for $1 \leqslant j \leqslant m_1 + \cdots + m_n$, we have

$$\theta\left(\gamma\left(a; b_1, \ldots, b_n\right); x_1, \ldots, x_{m_1+\cdots+m_n}\right) = \theta\left(a; x_1', \ldots, x_n'\right)$$

where $x_i' \coloneqq \theta(b_i; x_{m_1+\cdots+m_{i-1}+1}, \ldots, x_{m_1+\cdots+m_i})$.

(ii) $\theta(1; x) = x$ for all $x \in X$.

(iii) For all $a \in \mathcal{O}(n)$, all $x_i \in X$ for $1 \leqslant i \leqslant n$ and all $\sigma \in \Sigma_n$, we have

$$\theta\left(a \cdot \sigma; x_1, \ldots, x_n\right) = \theta\left(a; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right).$$

A map $X \to Y$ of $\mathcal{O}$-algebras is a map of spaces which intertwines with the structure maps θ_X and θ_Y in the obvious way. We write $\mathcal{O}$–Alg for the category of $\mathcal{O}$-algebras and $\mathcal{O}$-algebra maps.

Remark 1.2.4. For an operad $\mathcal{O}$ and an $\mathcal{O}$-algebra X, we shall often write

$$a(b_1, \ldots, b_n) \qquad \text{and} \qquad a(x_1, \ldots, x_n)$$

for

$$\gamma(a; b_1, \ldots, b_n) \in \mathcal{O}(m_1 + \cdots + m_n) \qquad \text{and} \qquad \theta(a; x_1, \ldots, x_n) \in X$$

where $a \in \mathcal{O}(n)$, $b_i \in \mathcal{O}(m_i)$ for $1 \leqslant i \leqslant n$, and $x_i \in X$ for $1 \leqslant i \leqslant m$.

Given an operad $\mathcal{O}$, there is a free–forgetful adjunction

$$F^{\mathcal{O}} \dashv U^{\mathcal{O}} \colon \mathcal{O}\text{–Alg} \longrightarrow \mathrm{Top}$$

whose induced monad on Top is given by the formula

$$\mathbb{O}(X) \coloneqq U^{\mathcal{O}} F^{\mathcal{O}}(X) = \coprod_{n \geqslant 0} \mathcal{O}(n) \times_{\Sigma_n} X^n.$$

This adjunction is monadic, see *e.g.* [39, Proposition 2.8]. Since for a space A and any $n \geqslant 0$, the diagonal map $A \to A^n$ is Σ_n-equivariant, where A is equipped with the trivial Σ_n-action, there are maps

$$A \times \mathcal{O}(n) \times_{\Sigma_n} X^n = \mathcal{O}(n) \times_{\Sigma_n} (A \times X^n) \longrightarrow \mathcal{O}(n) \times_{\Sigma_n} (A \times X)^n$$

which assemble to a natural (in A, X and $\mathcal{O}$) map

$$A \times \mathbb{O}(X) \longrightarrow \mathbb{O}\left(A \times X\right). \tag{1.3}$$

Proposition 1.2.5 ([22, Proposition 3.3.1])**.** *The forgetful functor* $U^{\mathcal{O}} \colon \mathcal{O}\text{–Alg} \to \mathrm{Top}$ *creates all limits, filtered colimits and coequalisers which are reflexive in* Top.

More generally, given a map $\phi \colon \mathcal{O} \to \mathcal{P}$ of operads, $\mathcal{O}$ acts on $\mathcal{P}$-algebras via ϕ, yielding a 'restriction of scalars' functor $\phi^* \colon \mathcal{P}\text{–Alg} \to \mathcal{O}\text{–Alg}$. This functor admits a left adjoint $\phi_! \colon \mathcal{O}\text{–Alg} \to \mathcal{P}\text{–Alg}$, *cf.* [22, Section 3.3.5].

1.2.1.2 Pointed operads

We will also make use of the concept of a pointed operad. The following definition is equivalent to the definition given in [3]:

Definition 1.2.6. Let $*$ be the topological operad with

$$*(n) = \begin{cases} * & \text{if } n = 0 \text{ or } n = 1, \\ \varnothing & \text{otherwise.} \end{cases}$$

A *pointed operad* is a map $* \longrightarrow \mathcal{O}$ of operads.

Note that a $*$-algebra is precisely a pointed topological space. If $\psi : * \longrightarrow \mathcal{O}$ is a pointed operad, then the aforementioned adjunction

$$\psi_! \dashv \psi^* : \mathcal{O}\text{–Alg} \longrightarrow \mathrm{Top}_*$$

is the free–forgetful adjunction of [39] between $\mathcal{O}$-algebras and pointed spaces. We will denote the functors in this adjunction by $F_*^{\mathcal{O}} := \psi_!$ and $U_*^{\mathcal{O}} := \psi^*$, suppressing the basepoint inclusion ψ from the notation. This should not cause confusion since we will never consider a topological operad with more than one choice of basepoint.

Definition 1.2.7. We denote the monad on Top_* associated to the adjunction $F_*^{\mathcal{O}} \dashv U_*^{\mathcal{O}}$ by $\mathbb{O}_* := U_*^{\mathcal{O}} \circ F_*^{\mathcal{O}}$.

A map $\phi : \mathcal{O} \longrightarrow \mathcal{P}$ of operads gives rise to a morphism between the associated monads.

1.2.1.3 Graded operads

Definition 1.2.8. A *graded operad* is an operad $\mathcal{O}$ along with a choice of $\mathbb{N}$-indexed decompositions

$$\mathcal{O}(n) = \coprod_{g \geqslant 0} \mathcal{O}_g(n)$$

of Σ_n-spaces for each $n \geqslant 0$ such that the operadic composition maps restrict to maps of the form

$$\mathcal{O}_g(n) \times \mathcal{O}_{h_1}(m_1) \times \cdots \times \mathcal{O}_{h_n}(m_n) \longrightarrow \mathcal{O}_{g+h_1+\cdots+h_n}(m_1 + \cdots + m_n).$$

Remark 1.2.9. In light of Remark 1.2.2, we may define a graded operad more succinctly as an operad in the category of non-negatively graded topological spaces $X = \coprod_{g \geqslant 0} X_g$ and grading-preserving maps. This is a symmetric monoidal category with the monoidal product given by

$$(X \times Y)_g = \coprod_{g_1+g_2=g} X_{g_1} \times Y_{g_2}.$$

1.2.1.4 Coloured operads

Operads have a vast generalisation in the concept of a coloured operad. We follow the notation of [8]:

Definition 1.2.10. Let C be a set, thought of as a set of 'colours'. A *C-coloured operad $\mathcal{O}$* consists of the following data:

(1) for each $n \geqslant 0$ and for each $(n+1)$-tuple $(c_1, \ldots, c_n; c)$ of colours, a topological space $\mathcal{O}(c_1, \ldots, c_n; c)$,

(2) for each $\sigma \in \Sigma_n$ and each $(n+1)$-tuple $(c_1, \ldots, c_n; c)$ of colours, a map

$$\sigma^* \colon \mathcal{O}(c_1, \ldots, c_n; c) \longrightarrow \mathcal{O}\left(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c\right),$$

(3) for each $(n+1)$-tuple $(c_1, \ldots, c_n; c)$ of colours, m-tuple $(d_1, \ldots, d_m)$ of colours and integers $m_1, \ldots, m_n \geqslant 0$ such that $m_1 + \cdots + m_n = m$, a map

$$\gamma \colon \mathcal{O}(c_1, \ldots, c_n; c) \times \mathcal{O}\left(d_1, \ldots, d_{m_1}; c_1\right) \times \cdots \times \mathcal{O}\left(d_{m_1+\cdots+m_{n-1}+1}, \ldots, d_m; c_n\right) \longrightarrow \mathcal{O}(d_1, \ldots, d_m; c),$$

(4) for each colour c, an identity $1_c \in \mathcal{O}(c; c)$.

This data is required to satisfy conditions analogous to the conditions put on an ordinary operad. Namely, the maps σ^* in (2) are required to compose as though they defined a right Σ_n-action, and the structure maps γ are required to be suitably associative, unital and equivariant.

Coloured operads are a 'multi-object' generalisation of ordinary operads: while an ordinary operad controls algebraic structures on individual spaces, an algebra over a coloured operad consists of a collection of spaces with algebraic structures, and structure-preserving maps between them:

Definition 1.2.11. An *algebra over a C-coloured operad $\mathcal{O}$* (or an *$\mathcal{O}$-algebra*) is a family $(X(c))_{c\in C}$ of spaces equipped with structure maps

$$\theta\colon \mathcal{O}(c_1,\dots,c_n;c)\times X(c_1)\times\cdots\times X(c_n)\longrightarrow X(c)$$

for each tuple $(c_1,\dots,c_n;c)$ of colours. These maps are required to satisfy appropriate conditions on associativity, unitality and equivariance. A morphism $f\colon X\to Y$ of $\mathcal{O}$-algebras is a C-indexed family $(f(c)\colon X(c)\to Y(c))_{c\in C}$ of maps which commute with the algebra maps in the obvious way. We write $\mathcal{O}$–Alg for the category of $\mathcal{O}$-algebras and $\mathcal{O}$-algebra maps.

Analogously to the situation for ordinary operads, a C-coloured operad gives rise to a free–forgetful adjunction

$$F^{\mathcal{O}}\dashv U^{\mathcal{O}}\colon \mathcal{O}\text{–Alg}\longrightarrow \mathrm{Top}^C$$

between the category of $\mathcal{O}$-algebras and the category Top^C of C-indexed families of spaces. Top^C is nothing else than the category of algebras over the initial C-coloured operad which only contains the identity elements 1_c and no other operations. For a general morphism $\phi\colon \mathcal{O}\to\mathcal{P}$ of C-coloured operads, there is an adjunction

$$\phi_!\dashv\phi^*\colon \mathcal{P}\text{–Alg}\longrightarrow\mathcal{O}\text{–Alg}$$

between the respective categories of algebras.

Remark 1.2.12. Just like in the case of ordinary operads, Definitions 1.2.10 and 1.2.11 generalise to an arbitrary symmetric monoidal category.

Here are some examples of coloured operads.

(i) When C is a singleton, the definition of a C-coloured operad reduces to that of an ordinary operad.

(ii) Given a small Top-enriched category $\mathscr{C}$, let $\mathcal{O}_{\mathscr{C}}$ be the $(\mathrm{ob}\,\mathscr{C})$-coloured operad given by

$$\mathcal{O}_{\mathscr{C}}(c_1,\dots,c_n;c)=\begin{cases}\varnothing & \text{if } n\neq 1,\\ \mathscr{C}(c_1,c) & \text{if } n=1.\end{cases}$$

Then $\mathcal{O}_{\mathscr{C}}$-algebras are precisely functors $\mathscr{C}\to\mathrm{Top}$.

(iii) There is an $\mathbb{N}$-coloured operad whose algebras are precisely ordinary operads. More generally, given any set C of colours, there is a coloured operad whose algebras are precisely C-coloured operads. The set of colours of this operad can be seen to be the set of finite tuples $(c_1, \ldots, c_n; c)$ of elements of C.

1.2.2 Homotopy theory of operad algebras

In this section we recall results on model category structures on categories of (coloured) operad algebras. The main references for this section are the articles [8, 5] of Berger and Moerdijk.

1.2.2.1 Projective and transferred model structures

Since our proof of the main theorem is rather model category-theoretic in nature, we recall here some results on model structures. We use the term *model category* in the sense of Quillen; *cf.* [28] for a modern treatment.

Recall that a model category $\mathscr{C}$ is cofibrantly generated if it admits sets of generating cofibrations and acyclic cofibrations with small domains. Fibrations (resp. acyclic fibrations) in a cofibrantly generated model category are precisely those morphisms that have the right lifting property against all generating acyclic cofibrations (resp. generating cofibrations). Our central example of a cofibrantly generated model category will be Top with the classical model structure, which is generated by the cofibrations $S^{n-1} = \partial D^n \hookrightarrow D^n$ and acyclic cofibrations $D^n \times \{0\} \hookrightarrow D^n \times [0, 1]$ for $n \geqslant 0$.

When $\mathscr{C}$ is a cofibrantly generated model category and $\mathscr{D}$ is any small category, then the category $\mathscr{C}^{\mathscr{D}}$ of functors from $\mathscr{D}$ to $\mathscr{C}$ admits the cofibrantly generated *projective* model structure [28, Theorem 11.6.1] whose fibrations and weak equivalences are 'pointwise': a morphism $f\colon F \to G$ in $\mathscr{C}^{\mathscr{D}}$ is a fibration (resp. a weak equivalence) if and only if $f_d\colon Fd \to Gd$ is a fibration (resp. weak equivalence) in $\mathscr{C}$ for every object $d \in \mathscr{D}$.

For us, the most important case of the above construction is that when $\mathscr{D} = G$ is a group, thought of as a groupoid with a unique object. In this case, $\mathscr{C}^G$ is the category of G-objects in $\mathscr{C}$ and G-equivariant morphisms. When $\mathscr{C} = \mathrm{Top}$, the generating (acyclic) cofibrations can be taken to be the images of the generating (acyclic) cofibrations in Top under the free G-space functor. Thus the cofibrations in Top^G are retracts of relative G-free G-CW-complexes, *cf.* [25, Theorem 5.1].

Under certain conditions, cofibrantly generated model structures transfer across left adjoint functors. Suppose that a cofibrantly generated model category $\mathscr{C}$ participates in an adjunction $F \dashv G\colon \mathscr{D} \to \mathscr{C}$. Under some mild conditions on the functor F (see *e.g.* [5, Section 2.5], [28, Theorem 11.3.2]), $\mathscr{D}$ admits a cofibrantly generated model structure which makes the functor $G\colon \mathscr{D} \to \mathscr{C}$ preserve and reflect fibrations and weak equivalences. (This model structure is necessarily unique, since this condition fully determines its fibrations and weak equivalences.) An example of a transferred model structure is the aforementioned projective model structure on G-objects in $\mathscr{C}$, which is transferred along the free–forgetful adjunction $\mathscr{C} \rightleftarrows \mathscr{C}^G$.

1.2.2.2 Model structures on categories of algebras

Let C be a set of colours, and let Top be equipped with the classical Quillen–Serre model structure. Then $\mathrm{Top}^C = \prod^C \mathrm{Top}$ admits the product model structure, and it turns out that for every C-coloured operad $\mathcal{O}$, this model structure transfers to $\mathcal{O}$–Alg along the free–forgetful adjunction. This was proved by Berger and Moerdijk, first in the case of uncoloured operads in [5, Proposition 4.1(c)], and later in the case of coloured operads in [8, Theorem 2.1]. We collect here some results about the existence and properties of these model structures.

Theorem 1.2.13 ([8, Theorem 2.1])**.** *Let $\mathcal{O}$ be a C-coloured operad. Then $\mathcal{O}$–*Alg *admits a cofibrantly generated model structure for which the forgetful functor $U^{\mathcal{O}}\colon \mathcal{O}$–Alg $\to \mathrm{Top}^C$ preserves and detects fibrations and weak equivalences.*

Thus in particular the category Oper of topological operads — which is the category of algebras over a suitable coloured operad — admits a model structure where a map $\phi\colon \mathcal{O} \to \mathcal{P}$ is a weak equivalence (resp. a fibration) if and only if each $\phi(n)\colon \mathcal{O}(n) \to \mathcal{P}(n)$ is a weak equivalence (resp. a fibration) of spaces. This model structure fails to be left proper[1] (see [26, Section 4] for an illuminating counterexample), but it does satisfy a weaker 'relative' left properness condition. To state it, we need the following definitions.

Definition 1.2.14. Given an n-tuple $(c_1, \ldots, c_n) \in C^n$ of colours, let $\Sigma_{c_1,\ldots,c_n} \leqslant \Sigma_n$ be the stabiliser of this tuple in the obvious action of Σ_n on C^n.

[1] Recall that a model category is called *left proper* if pushouts along cofibrations preserve weak equivalences.

Thus for instance if $c_1 = \ldots = c_n$, then $\Sigma_{c_1,\ldots,c_n} = \Sigma_n$. If the c_i are pairwise distinct, then $\Sigma_{c_1,\ldots,c_n} = 1$.

Remark 1.2.15. If $\mathcal{O}$ is a C-coloured operad, then each $\mathcal{O}(c_1, \ldots, c_n; c)$ is evidently a right $\Sigma_{c_1,\ldots,c_n}$-space.

Definition 1.2.16 ([8, Section 3.1])**.** A C-coloured operad $\mathcal{O}$ is called Σ*-cofibrant* if each $\mathcal{O}(c_1, \ldots, c_n; c)$ is cofibrant in the projective model structure on $\mathrm{Top}^{\Sigma_{c_1,\ldots,c_n}}$.

It is *well pointed* if for every $c \in C$, the inclusion $\{1_c\} \hookrightarrow \mathcal{O}(c; c)$ is a cofibration in Top.

These notions are related to cofibrancy in the model structure on Oper in the following way.

Proposition 1.2.17. *A cofibrant operad is* Σ*-cofibrant and well-pointed.*

Proof. This is essentially [5, Proposition 4.3] — but see also [7, Appendix] for a correction of a small mistake. □

With this definition we can now formulate the weakened notion of left properness that holds in Oper.

Theorem 1.2.18 ([51, Theorem 3])**.** *Suppose that*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{w} & \mathcal{P} \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ \mathcal{Q} & \xrightarrow[w']{} & \mathcal{R} \end{array}$$

is a pushout square in Oper*, f is a cofibration and $\mathcal{O}$ and $\mathcal{P}$ are well-pointed Σ-cofibrant operads. If w is a weak equivalence, then so is w'.*

Proof. It is an easy check that Spitzweck's 'J-semi model structure' on Oper agrees with the model structure of Berger and Moerdijk when the underlying category is Top. □

Let $\phi\colon \mathcal{O} \to \mathcal{P}$ be a morphism of C-coloured operads. The pullback functor

$$\phi^*\colon \mathcal{P}\text{–Alg} \to \mathcal{O}\text{–Alg}$$

evidently preserves fibrations and weak equivalences, so the induced adjunction $\phi_! \dashv \phi^*$ is a Quillen adjunction between $\mathcal{O}$–Alg and $\mathcal{P}$–Alg. It turns out that if ϕ is a weak equivalence, then this adjunction is a Quillen equivalence under mild conditions on the operads $\mathcal{O}, \mathcal{P}$.

Theorem 1.2.19 ([8, Theorem 4.1]). *If $\phi\colon \mathcal{O} \to \mathcal{P}$ is a weak equivalence of Σ-cofibrant well-pointed C-coloured operads, then the adjunction $\phi_! \dashv \phi^*\colon \mathcal{P}\text{–Alg} \to \mathcal{O}\text{–Alg}$ is a Quillen equivalence.*

1.2.2.3 Cofibrancy of bar constructions

A key role in our proof of Theorem 1.1.7 is played by bar constructions. We therefore take the time here to recapitulate their construction, and place them within the context of the model structures recalled in the preceding section. We follow the treatment in [39].

Let us first record several observations about simplicial $\mathcal{O}$-algebras.

Definition 1.2.20. Given a category $\mathscr{C}$, we write $\mathrm{s}\mathscr{C}$ for the category of simplicial objects in $\mathscr{C}$. This is the functor category $\mathscr{C}^{\mathbb{\Delta}^{\mathrm{op}}}$ where $\mathbb{\Delta}$ is the simplex category. We likewise write $\mathrm{s}F\colon \mathrm{s}\mathscr{C} \to \mathrm{s}\mathscr{D}$ for the functor induced by a functor $F\colon \mathscr{C} \to \mathscr{D}$.

Let $\Delta^\bullet\colon \mathbb{\Delta} \to \mathrm{Top}$ be the realisation of the standard cosimplicial simplicial set

$$[n] \mapsto \mathbb{\Delta}(-, [n]) \in \mathrm{sSet}.$$

For every uncoloured operad $\mathcal{O}$, the category $\mathcal{O}$–Alg is powered over Top; for pointed spaces, this is proved in [39, Lemma 1.5]. Hence powering by the cosimplicial space $\Delta^\bullet$ yields a functor $(-)^{\Delta^\bullet} : \mathcal{O}\text{–Alg} \to \mathrm{s}\mathcal{O}\text{–Alg}$.

Lemma 1.2.21. *For every uncoloured topological operad $\mathcal{O}$, the functor $(-)^{\Delta^\bullet}$ admits a left adjoint $|-| : \mathrm{s}\mathcal{O}\text{–Alg} \to \mathcal{O}\text{–Alg}$, which is given by ordinary geometric realisation on the underlying spaces.*

Proof. Since geometric realisation of simplicial spaces preserves finite products, the geometric realisation of the underlying simplicial space of a simplicial $\mathcal{O}$-algebra comes with a natural $\mathcal{O}$-algebra structure. It is an easy check that this lift of the geometric realisation of simplicial spaces to simplicial $\mathcal{O}$-algebras is left adjoint to the singular simplicial $\mathcal{O}$-algebra functor. □

Lemma 1.2.22. *Let $\phi\colon \mathcal{O} \to \mathcal{P}$ be a morphism of uncoloured operads. Then the square*

$$\begin{array}{ccc} \mathrm{s}\mathcal{O}\text{–Alg} & \xrightarrow{\mathrm{s}\phi_!} & \mathrm{s}\mathcal{P}\text{–Alg} \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathcal{O}\text{–Alg} & \xrightarrow[\phi_!]{} & \mathcal{P}\text{–Alg} \end{array}$$

commutes (up to natural equivalence).

Proof. Note that all arrows in the square admit right adjoints, and the corresponding square of right adjoints commutes on the nose. The claim follows by essential uniqueness of adjoints. □

Definition 1.2.23 ([39, Definition 9.4])**.** Let (T, η, μ) be a monad on a category $\mathscr{C}$. A *T-functor* is a pair (F, λ) of a functor $F\colon \mathscr{C} \to \mathscr{D}$ and a natural transformation $\lambda\colon F \circ T \to F$ which makes the diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FT \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \lambda} \\ & & F \end{array} \qquad \text{and} \qquad \begin{array}{ccc} FTT & \xrightarrow{F\mu} & FT \\ {\scriptstyle \lambda T}\downarrow & & \downarrow{\scriptstyle \lambda} \\ FT & \xrightarrow[\lambda]{} & F \end{array}$$

commute.

If in addition (X, θ) is a T-algebra, where $X \in \mathscr{C}$ is the underlying object and $\theta\colon TX \to X$ is the structure map, we write $B_\bullet(F, T, X)$ for the associated bar construction; this is the simplicial object in $\mathscr{D}$ whose object of n-simplices is $B_n(F, T, X) \coloneqq FT^nX$. The face maps $d_i\colon B_n(F, T, X) \to B_{n-1}(F, T, X)$ are given by

$$d_i = \begin{cases} \lambda T^{n-1}X & \text{if } i = 0, \\ FT^{i-1}\mu T^{n-i-1}X & \text{if } 0 < i < n, \\ FT^{n-1}\theta & \text{if } i = n, \end{cases} \tag{1.4}$$

and the degeneracy maps are simply $s_i = FT^i\eta T^{n-i}X\colon FT^nX \to FT^{n+1}X$.

Thus for instance if $\mathcal{O}$ is an (uncoloured) operad and $\mathbb{O}$ is the associated monad on Top, then $\mathbb{O}$ itself and the free $\mathcal{O}$-algebra functor $F^{\mathcal{O}}\colon \mathrm{Top} \to \mathcal{O}\text{–Alg}$ are $\mathbb{O}$-functors. For an $\mathcal{O}$-algebra X, the resulting simplicial $\mathcal{O}$-algebra admits an augmentation

$$\epsilon\colon B_\bullet\left(F^{\mathcal{O}}, \mathbb{O}, X\right) \to X \tag{1.5}$$

coming from the $\mathcal{O}$-algebra map $F^{\mathcal{O}}(X) \to X$.

Lemma 1.2.24. *Upon realisation, the map $|\epsilon| : \left|B_\bullet\left(F^{\mathcal{O}}, \mathbb{O}, X\right)\right| \to X$ is a weak equivalence in $\mathcal{O}$–Alg.*

Proof. We can equivalently think of the augmentation η as a simplicial map from the bar construction to the constant simplicial $\mathcal{O}$-algebra at X. This map is a simplicial homotopy equivalence in sTop by [39, Proposition 9.8]. Hence the underlying map of spaces of $|\epsilon|$ is a weak equivalence. □

Remark 1.2.25. An analogous result holds for the bar construction $B_\bullet\left(F^{\mathcal{O}}_*, \mathbb{O}_*, X\right)$ when $\mathcal{O}$ comes with a choice of a basepoint $* \to \mathcal{O}$.

Now suppose that $* \to \mathcal{O}$ is a pointed operad. Then we have the adjunctions

$$F^{\mathcal{O}} \dashv U^{\mathcal{O}} \colon \mathcal{O}\text{–Alg} \to \mathrm{Top} \qquad \text{and} \qquad F^{\mathcal{O}}_* \dashv U^{\mathcal{O}}_* \colon \mathcal{O}\text{–Alg} \to \mathrm{Top}_*$$

as well as the adjunction $(-)_+ \dashv U \colon \mathrm{Top}_* \to \mathrm{Top}$ where U is the functor which forgets the basepoint, and $(-)_+$ freely adjoins a basepoint to an unpointed space. (This is nothing but the free–forgetful adjunction of the initial pointed operad $*$.) Clearly $U^{\mathcal{O}} = UU^{\mathcal{O}}_*$, so there is a natural isomorphism $F^{\mathcal{O}} \cong F^{\mathcal{O}}_* (-)_+$, and we obtain a natural transformation

$$F^{\mathcal{O}}U \cong F^{\mathcal{O}}_* (-)_+ U \longrightarrow F^{\mathcal{O}}_*$$

which induces a simplicial map

$$\pi_\bullet \colon B_\bullet\left(F^{\mathcal{O}}, \mathbb{O}, X\right) \longrightarrow B_\bullet\left(F^{\mathcal{O}}_*, \mathbb{O}_*, X\right)$$

for every $X \in \mathcal{O}$–Alg. It is easy to see that the augmentation (1.5) factors as the composition of $\pi_\bullet$ with the augmentation of the pointed bar construction. Hence by Lemma 1.2.24 and the remark immediately following it (and the 2-out-of-3 property), the realisation of $\pi_\bullet$ is a weak equivalence of $\mathcal{O}$-algebras.

Suppose now that $\phi \colon \mathcal{O} \to \mathcal{P}$ is a morphism of pointed operads. Since $U^{\mathcal{P}} = U^{\mathcal{O}}\phi^*$ and $U^{\mathcal{P}}_* = U^{\mathcal{O}}_*\phi^*$, there are natural isomorphisms $F^{\mathcal{P}} \cong \phi_! F^{\mathcal{O}}$ and $F^{\mathcal{P}}_* \cong \phi_! F^{\mathcal{O}}_*$. Hence by Lemma 1.2.22, $\phi_!$ carries $|\pi_\bullet|$ to the $\mathcal{P}$-algebra morphism

$$\left|B_\bullet\left(F^{\mathcal{P}}, \mathbb{O}, X\right)\right| \longrightarrow \left|B_\bullet\left(F^{\mathcal{P}}_*, \mathbb{O}_*, X\right)\right| \tag{1.6}$$

and we would like to claim that this morphism is a weak equivalence.

Theorem 1.2.26.

(i) *Let $\mathcal{O}$ be a Σ-cofibrant well-pointed uncoloured operad and let X be an $\mathcal{O}$-algebra such that $U^{\mathcal{O}}(X) \in \mathrm{Top}$ is cofibrant. Then the bar construction $\left|B_\bullet\left(F^{\mathcal{O}}, \mathbb{O}, X\right)\right|$ is a cofibrant $\mathcal{O}$-algebra.*

(ii) *Let $\mathcal{O}$ and X be as before. Suppose that $\mathcal{O}$ is pointed, and the pointed spaces $U^{\mathcal{O}}_*(\mathcal{O}(0))$ and $U^{\mathcal{O}}_*(X)$ are cofibrant in Top_*. Then the bar construction $\left|B_\bullet\left(F^{\mathcal{O}}_*, \mathbb{O}_*, X\right)\right|$ is a cofibrant $\mathcal{O}$-algebra.*

Assuming the theorem, the following corollary is an immediate consequence of Ken Brown's lemma [28, Lemma 7.7.1].

Corollary 1.2.27. *Let O and X be as in Theorem 1.2.26*(ii)*. If $\phi\colon O \to \mathcal{P}$ is a morphism of pointed operads, then the $\mathcal{P}$-algebra morphism* (1.6) *is a weak equivalence.*

While part (i) of the theorem has been addressed by various authors in varying levels of generality (see [47, Corollary 3.7.4] which deals with simplicial sets and simplicial modules over a ring, or [44] which deals with symmetric spectra), we have not been able to locate any results on the cofibrancy of the pointed bar constructions. We are of the firm opinion that this question must have been addressed in the literature and that our search has simply not been thorough enough. Nonetheless, for the sake of completeness, we indicate in Section 1.4 how one might modify the methods of [44] to prove Theorem 1.2.26(ii).

1.3 Operads with homological stability

Let O be a graded operad and fix an element $s \in O_1(1)$. This choice gives rise to maps $s_* := \gamma(s;-)\colon O(n) \to O(n)$ which raise the grading by 1, and we write $O_\infty(n)$ for the homotopy colimit

$$O_\infty(n) := \operatorname{hocolim}\left(O(n) \xrightarrow{s_*} O(n) \xrightarrow{s_*} \cdots\right)$$

Given an n-tuple $\mathbf{v} = (v_1, \ldots, v_n) \in O(0)^n$, we write

$$D^{\mathbf{v}} := \gamma\left(-; v_1, \ldots, v_n\right) : O(n) \to O(0).$$

This map commutes with the map $s_*\colon O(n) \to O(n)$, and so it induces a 'stabilised' map $D^{\mathbf{v}}_\infty\colon O_\infty(n) \to O_\infty(0)$.

Definition 1.3.1. An *operad with strong homological stability* (or an *OSHS*) is a triple (O, μ, s) of

(1) a graded operad O,

(2) a map of graded operads $\mu\colon \mathcal{A} \to O$ from an A_∞-operad $\mathcal{A}$ concentrated in grading 0,

(3) an element $s \in O_1(1)$,

such that

(i) for all n-tuples $\mathbf{v} \in \mathcal{O}(0)^n$, the map $D^{\mathbf{v}}_\infty \colon \mathcal{O}_\infty(n) \to \mathcal{O}_\infty(0)$ is an integral homology isomorphism,

(ii) $\mu(\mathcal{A}(2)) \subseteq \mathcal{O}(2)$ is contained in a path-component.

We will generally suppress the map μ and the element s from the notation.

Remark 1.3.2. Just like [3], we shall assume throughout that the stabilising element $s \in \mathcal{O}(1)$ is of the form $s = \mu_2(s_0, 1)$ for some $s_0 \in \mathcal{O}_1(0)$, where $\mu_2 \in \mathcal{O}(2)$ is the image under μ of some binary operation in $\mathcal{A}$.

Remark 1.3.3. This definition is a variation of that given in [3]. There, an operad with homological stability (or OHS for short) is a graded *pointed* operad $\mathcal{O}$ such that the basepoint $* \in \mathcal{O}(0)$ lies in grading 0. Furthermore, condition (i) is weakened by only requiring that only the maps $D^*_\infty \colon \mathcal{O}_\infty(n) \to \mathcal{O}_\infty(0)$ induced by the n-tuples $(*, \dots, *) \in \mathcal{O}(0)^n$ be integral homology isomorphisms. In particular, if $\mathcal{O}$ is an OSHS, then for any choice of basepoint $* \in \mathcal{O}_0(0)$ we obtain an OHS.

Remark 1.3.4. Note that Definition 1.3.1 and Remark 1.3.3 are subtly different from the informal discussion of an OHS at the beginning of the Introduction. Namely, the 'homological stability' condition from the Introduction, requiring that the maps $D^* \colon \mathcal{O}_g(n) \to \mathcal{O}_g(0)$ be isomorphisms on homology in a range of degrees that tends to ∞ with g, has been replaced with a weaker 'homological equivalence' condition, requiring that the maps D^* (or more generally $D^{\mathbf{v}}$ in the case of Definition 1.3.1) only give rise to a homology equivalence D^*_∞ (or $D^{\mathbf{v}}_\infty$) *after* stabilisation. It is easily seen that the first condition implies the second. The weaker condition is sufficient for our proofs to go through, but all interesting examples of OHS satisfy the more familiar stronger condition.

Remark 1.3.5. In [3], an OHS is allowed to be graded by any commutative finitely generated monoid I. The stabilising element s is then required to come from $\mathcal{O}_g(1)$ where g is the product of some finite generating set of I. We have chosen to assume $I = \mathbb{N}$ in the definition, as this is the case for all interesting examples of operads with homological stability, and it avoids the notational difficulty of the general case. All our results also go through in the general case, however.

Thus if $\mu\colon \mathcal{A} \to \mathcal{O}$ is an O(S)HS, then every $\mathcal{O}$-algebra is in particular an A_∞-algebra through the action of $\mathcal{A}$, and condition (ii) ensures that it is homotopy commutative (although perhaps not an E_2-algebra). Hence we can make sense of the group completion of an $\mathcal{O}$-algebra.

Definition 1.3.6 ([3])**.** For an A_∞ operad $\mathcal{A}$, let

$$\mathcal{G}\colon \mathcal{A}\text{–Alg} \longrightarrow \text{Top}$$

be the functor that takes X to

$$\Omega B\,|B_\bullet(\mathbb{A}\text{ss}, \mathbb{A}, X)|\,,$$

where $\mathbb{A}$ss is an $\mathbb{A}$-functor via the operad equivalence $\mathcal{A} \to \text{Ass}$. The bar construction is an Ass-algebra and hence a topological monoid.

Remark 1.3.7. There is a zig-zag

$$X \longleftarrow |B_\bullet(\mathbb{A}, \mathbb{A}, X)| \longrightarrow |B_\bullet(\mathbb{A}\text{ss}, \mathbb{A}, X)|$$

of $\mathcal{A}$-algebra equivalences, so the above definition is a reasonable construction of the group completion functor on the category $\mathcal{A}$–Alg.

The motivation for the OHS condition is the following theorem of Basterra, Bobkova, Ponto, Tillmann and Yeakel.

Theorem 1.3.8 ([3])**.** *Let $\mathcal{O}$ be an OHS which admits a map $\pi\colon \mathcal{O} \to \mathcal{C}$ to an E_∞-operad $\mathcal{C}$. Then for every $\mathcal{O}$-algebra X, the group completion $\mathcal{G}X$ is weakly equivalent to the infinite loop space*

$$\mathcal{G}\,|B_\bullet(\mathbb{C}_*, \mathbb{O}_*, \mathcal{O}(0) \times X)|$$

where $\mathcal{O}(0) \times X$ is equipped with the diagonal $\mathcal{O}$-algebra structure and $\mathbb{C}_*$ is an $\mathbb{O}_*$-functor via the operad map π.

We also record the following lemma from [3] which is a crucial step in the proof of Theorem 1.3.8.

Lemma 1.3.9 ([3, Lemma 6.2])**.** *Let $\mathcal{O}$, π and X be as in the preceding theorem. The sequence*

$$\mathcal{O}(0) \longrightarrow |B_\bullet(\mathbb{O}_*, \mathbb{O}_*, X)| \longrightarrow |B_\bullet(\mathbb{C}_*, \mathbb{O}_*, X)|$$

of $\mathcal{O}$-algebra maps, where the second map is constructed using $\pi\colon \mathcal{O} \to \mathcal{C}$, becomes a homotopy fibre sequence after group completion.

1.3.1 Rectifying operads under an A_∞-operad

We show here that every O(S)HS $(\mathcal{O}, \mu, s)$ is equivalent to an O(S)HS of a particularly nice form.

Definition 1.3.10. An O(S)HS $(\mathcal{O}, \mu\colon \mathcal{A} \to \mathcal{O}, s \in \mathcal{O}_1(1))$ is *superb* if $\mathcal{A} = \mathrm{Ass}$ is the associative operad, and $\mathcal{O}$ admits an operad map $\pi\colon \mathcal{O} \to \Gamma$ to the Barratt–Eccles operad such that $\pi \circ \mu\colon \mathrm{Ass} \to \Gamma$ is the standard inclusion.

Proposition 1.3.11. *Let $\mathcal{A}$ be an A_∞ operad and $\mu\colon \mathcal{A} \to \mathcal{O}$ an operad morphism. There is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Ass} & \xleftarrow{\simeq} & \widetilde{\mathcal{A}} & \xrightarrow{\simeq} & \mathcal{A} \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \widetilde{\mu}} & & \downarrow{\scriptstyle \mu} \\ \mathcal{O}' & \xleftarrow[\simeq]{} & \widetilde{\mathcal{O}} & \xrightarrow[\simeq]{} & \mathcal{O} \end{array}$$

of operad morphisms where all horizontal morphisms are weak equivalences, and $\widetilde{\mathcal{O}}$ and $\mathcal{O}'$ (and $\widetilde{\mathcal{A}}$) are Σ-cofibrant and well pointed.

There is also an identical commutative diagram with the same properties such that moreover $\mathcal{O}'$ admits a morphism $\pi\colon \mathcal{O}' \to \Gamma$ to the Barratt–Eccles operad for which $\pi \circ \mu'\colon \mathrm{Ass} \to \Gamma$ is the inclusion of vertices in the simplicial structure on Γ.

Proof. Let $\widetilde{\mu}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{O}}$ be a cofibrant replacement of μ in Oper. Then the composite $\widetilde{\mathcal{A}} \to \mathcal{A} \to \mathrm{Ass}$ is a weak equivalence. Both $\widetilde{\mathcal{A}}$ and Ass are Σ-cofibrant and well pointed — the first by Proposition 1.2.17, the second by inspection — and so by Theorem 1.2.18, the pushout of this composition along $\widetilde{\mu}$ is a weak equivalence. The pushout $\mu'\colon \mathrm{Ass} \to \mathcal{O}'$ of $\widetilde{\mu}$ along the composition is certainly a cofibration, so $\mathcal{O}'$ is also Σ-cofibrant and well pointed.

To prove the stronger claim, first replace μ with $\mu \times p\colon \mathcal{A} \to \mathcal{O} \times \Gamma$ where p is the composite $\mathcal{A} \to \mathrm{Ass} \hookrightarrow \Gamma$, and then run the above argument on the modified map. □

In particular, if $\mu\colon \mathcal{A} \to \mathcal{O}$ is an O(S)HS, then so is its replacement $\mu'\colon \mathrm{Ass} \to \mathcal{O}'$. The following result shows that the deloopings given by Theorem 1.3.8 for an $\mathcal{O}$-algebra X and for a suitable $\mathcal{O}'$-algebra X' are equivalent. Hence we may assume without loss of generality that every O(S)HS is superb.

Proposition 1.3.12. *Let $\mu\colon \mathcal{A} \to \mathcal{O}$ be an OHS with a map $\pi\colon \mathcal{O} \to \mathcal{C}$ to an E_∞-operad, and let $\widetilde{\mu}$ and μ' be as in Proposition 1.3.11. Let X be an $\mathcal{O}$-algebra with cofibrant underlying pointed space, and let*

$$X' \coloneqq \left|B_\bullet\left(\mathbb{O}'_*, \widetilde{\mathbb{O}}_*, X\right)\right|$$

with the evident $\mathcal{O}'$-structure. Then

$$\mathcal{G}\,|B_\bullet(\mathbb{C}_*, \mathbb{O}_*, X)| \qquad \textit{and} \qquad \mathcal{G}\left|B_\bullet\left(\mathbb{\Gamma}_*, \mathbb{O}'_*, X'\right)\right|$$

are equivalent infinite loop spaces.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}(0) & \longrightarrow & |B_\bullet(\mathbb{O}_*, \mathbb{O}_*, X)| & \longrightarrow & |B_\bullet(\mathbb{C}_*, \mathbb{O}_*, X)| \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{O}(0)\times\Gamma(0) & \longrightarrow & |B_\bullet((\mathbb{O}\times\mathbb{\Gamma})_*, (\mathbb{O}\times\mathbb{\Gamma})_*, X)| & \longrightarrow & |B_\bullet((\mathbb{C}\times\mathbb{\Gamma})_*, (\mathbb{O}\times\mathbb{\Gamma})_*, X)| \\
\| & & \| & & \downarrow \\
\mathcal{O}(0)\times\Gamma(0) & \longrightarrow & |B_\bullet((\mathbb{O}\times\mathbb{\Gamma})_*, (\mathbb{O}\times\mathbb{\Gamma})_*, X)| & \longrightarrow & |B_\bullet(\mathbb{\Gamma}_*, (\mathbb{O}\times\mathbb{\Gamma})_*, X)| \\
\uparrow & & \uparrow & & \uparrow \\
\widetilde{\mathcal{O}}(0) & \longrightarrow & \left|B_\bullet\left(\widetilde{\mathbb{O}}_*, \widetilde{\mathbb{O}}_*, X\right)\right| & \longrightarrow & \left|B_\bullet\left(\mathbb{\Gamma}_*, \widetilde{\mathbb{O}}_*, X\right)\right| \\
\| & & \uparrow & & \uparrow \\
\widetilde{\mathcal{O}}(0) & \longrightarrow & \left|B_\bullet\left(\widetilde{\mathbb{O}}_*, \widetilde{\mathbb{O}}_*, \widetilde{X}\right)\right| & \longrightarrow & \left|B_\bullet\left(\mathbb{\Gamma}_*, \widetilde{\mathbb{O}}_*, \widetilde{X}\right)\right| \\
\| & & \downarrow & & \downarrow \\
\widetilde{\mathcal{O}}(0) & \longrightarrow & \left|B_\bullet\left(\widetilde{\mathbb{O}}_*, \widetilde{\mathbb{O}}_*, X'\right)\right| & \longrightarrow & \left|B_\bullet\left(\mathbb{\Gamma}_*, \widetilde{\mathbb{O}}_*, X'\right)\right| \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}'(0) & \longrightarrow & \left|B_\bullet\left(\mathbb{O}'_*, \mathbb{O}'_*, X'\right)\right| & \longrightarrow & \left|B_\bullet\left(\mathbb{\Gamma}_*, \mathbb{O}'_*, X'\right)\right|
\end{array}$$

of $\widetilde{\mathcal{A}}$-algebras, where the top three rows and the bottom row of vertical arrows come from the commutative diagrams

$$\begin{array}{ccccccc}
\widetilde{\mathcal{O}} & \longrightarrow & \mathcal{O}\times\Gamma & = & \mathcal{O}\times\Gamma & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma & = & \Gamma & \longleftarrow & \mathcal{C}\times\Gamma & \longrightarrow & \mathcal{C}
\end{array} \qquad \text{and} \qquad
\begin{array}{ccc}
\widetilde{\mathcal{O}} & \longrightarrow & \mathcal{O}' \\
\downarrow & & \downarrow \\
\Gamma & = & \Gamma
\end{array}$$

of operads, and the middle two rows are induced by the zig-zag

$$X \longleftarrow \widetilde{X} \coloneqq \left|B_\bullet\left(\widetilde{\mathbb{O}}_*, \widetilde{\mathbb{O}}_*, X\right)\right| \longrightarrow X'$$

of $\widetilde{\mathcal{O}}$-algebra equivalences.

All vertical arrows in the first column are weak equivalences of $\widetilde{\mathcal{A}}$-algebras by construction. In the middle column, the first four spaces admit compatible $\widetilde{\mathcal{A}}$-algebra equivalences to X, so the arrows between them are weak equivalences by 2-out-of-3. The bottom middle vertical arrow is a weak equivalence for the same reason. The second and third arrow from the bottom are weak equivalences because they are realisations of Reedy equivalences between Reedy cofibrant simplicial $\widetilde{\mathcal{O}}$-algebras by Theorem 1.2.26(ii).

Upon group-completion of the entire diagram, all rows become homotopy fibre sequences by Lemma 1.3.9, and so the arrows in the last column become weak equivalences, completing the proof. □

We conclude this section with a particular model of a group-completion map

$$X \to \mathcal{G}\,|B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, \mathcal{O}(0) \times X)|$$

for an algebra X over a superb OHS $\mathrm{Ass} \xrightarrow{\mu} \mathcal{O} \xrightarrow{\pi} \Gamma$.

Proposition 1.3.13. *Let X be an algebra over the superb OHS $\mathrm{Ass} \xrightarrow{\mu} \mathcal{O} \xrightarrow{\pi} \Gamma$. Then the monoid map*

$$|B_\bullet(\pi\mu, \pi, * \times \mathrm{id})| : \; |B_\bullet(\mathbb{A}\mathrm{ss}_*, \mathbb{A}\mathrm{ss}_*, X)| \longrightarrow |B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, \mathcal{O}(0) \times X)|\,,$$

where $* \times \mathrm{id} \colon X \to \mathcal{O}(0) \times X$ maps X to the basepoint of $\mathcal{O}(0)$, group-completes to a weak equivalence.

Proof. Consider the square

$$\begin{array}{ccc} B_\bullet(\mathbb{A}\mathrm{ss}_*, \mathbb{A}\mathrm{ss}_*, \mathcal{O}(0)) & \xrightarrow{B_\bullet(\mathrm{id}, \mathrm{id}, \mathrm{id} \times \theta_0)} & B_\bullet(\mathbb{A}\mathrm{ss}_*, \mathbb{A}\mathrm{ss}_*, \mathcal{O}(0) \times X) \\ \downarrow & & \downarrow{\scriptstyle B_\bullet(\mu, \mu, \mathrm{id})} \\ \mathcal{O}(0) & \longrightarrow & B_\bullet(\mathbb{O}_*, \mathbb{O}_*, \mathcal{O}(0) \times X) \end{array} \tag{1.7}$$

in sAss–Alg, where $\mathcal{O}(0)$ is the constant simplicial monoid at $\mathcal{O}(0)$, the map $\theta_0 \colon \mathcal{O}(0) \to X$ is the unique $\mathcal{O}$-algebra map out of the initial $\mathcal{O}$-algebra $\mathcal{O}(0)$, the left vertical arrow is levelwise evaluation, and the bottom horizontal arrow is levelwise the unique $\mathcal{O}$-algebra map.

The square fails to commute on the nose, but it does commute up to simplicial homotopy in the category of simplicial monoids. Indeed, for $i \geqslant 0$, consider the composite monoid homomorphism

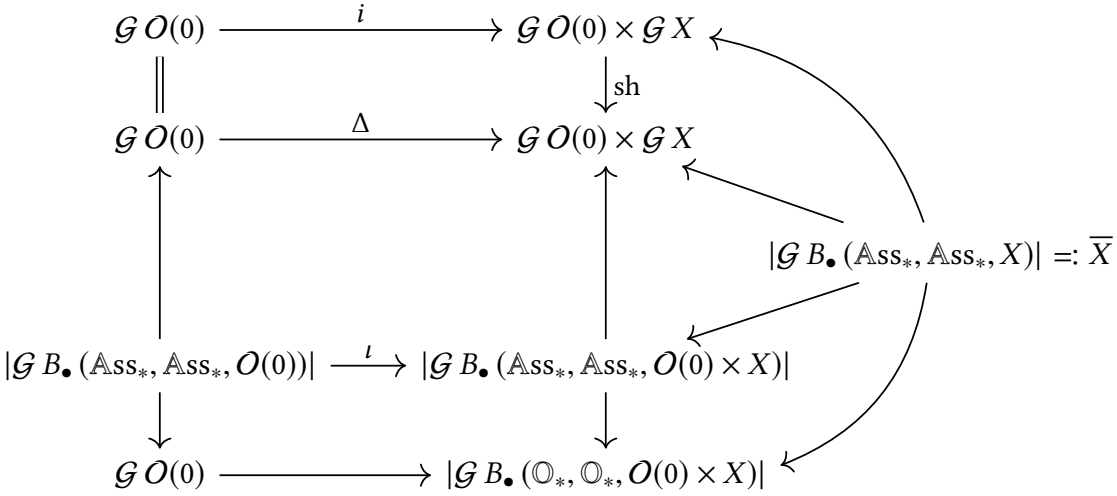
$$t_i \colon \mathbb{A}\mathrm{ss}_*^i(\mathcal{O}(0)) \longrightarrow \mathcal{O}(0) \longrightarrow \mathbb{O}_*^{i+1}(\mathcal{O}(0) \times X)$$

where the first arrow is defined using the monoid structure on $\mathcal{O}(0)$, and the second arrow is the unique $\mathcal{O}$-algebra homomorphism out of the initial $\mathcal{O}$-algebra $\mathcal{O}(0)$. The components of the homotopy are then the monoid homomorphisms

$$h_i^n \colon \mathbb{A}\mathrm{ss}_*^{n+1}(\mathcal{O}(0)) = \mathbb{A}\mathrm{ss}_*^{i+1}\mathbb{A}\mathrm{ss}_*^{n-i}(\mathcal{O}(0)) \xrightarrow{\mu^{i+1}(t_{n-i})} \mathbb{O}_*^{n+2}(\mathcal{O}(0) \times X)$$

for $0 \leqslant i \leqslant n$. It is an easy if tedious check that these maps do satisfy the axioms of a simplicial homotopy — see *e.g.* [40, Definition 5.1].

Consider now the diagram



where i is the inclusion of the fibre at the basepoint of $\mathcal{G}\,X$, and Δ and sh are as in [3, Proposition 6.4]: Δ is the group completion of the universal $\mathcal{O}$-algebra morphism $\mathcal{O}(0) \to \mathcal{O}(0) \times X$, remembering that the group completion functor $\mathcal{G}$ preserves products (*cf.* [3, Theorem 3.5]). The map sh is the identity on the first factor, and the action map

$$\mathcal{G}\,\mathcal{O}(0) \times \mathcal{G}\,X \longrightarrow \mathcal{G}\,X \times \mathcal{G}\,X \xrightarrow{\text{loop sum}} \mathcal{G}\,X$$

on the second factor. The top square commutes up to homotopy, the middle square commutes on the nose, and the bottom square commutes up to homotopy, being the image of (1.7) under group completion and realisation. The space $\overline{X}$ maps into the top two spaces via the evaluation map $B_\bullet(\mathbb{A}\mathrm{ss}_*, \mathbb{A}\mathrm{ss}_*, X) \to X$, and to the bottom two spaces using the

inclusion $X \hookrightarrow \mathcal{O}(0) \times X$ at the unit of $\mathcal{O}(0)$, and the map $\mu\colon \mathrm{Ass} \to \mathcal{O}$. The three triangles involving $\overline{X}$ all commute up to homotopy.

The four horizontal arrows are all infinite loop space maps by Theorem 1.3.8, so they can be extended to the right to form homotopy fibre sequences by taking homotopy cofibres in the category of infinite loop spaces. The vertical arrows in the above diagram are weak equivalences, so the induces maps between the cofibres are also. But the cofibre on the top row is evidently $\mathcal{G}X$, and the composite map from $\overline{X}$ to it is an equivalence. Hence the maps from $\overline{X}$ into all the cofibres are equivalences. But the cofibre on the bottom row is $|\mathcal{G}B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, \mathcal{O} \times X)|$ by Lemma 1.3.9, and the map into it from $\overline{X}$ is precisely the map from the statement of the proposition, concluding the proof. □

1.3.2 Associated PROPs

Recall that for a topological operad $\mathcal{O}$, its associated PROP (or *pro*duct and *p*ermutation category, [35]) is the free symmetric strict monoidal category generated by $\mathcal{O}$:

Definition 1.3.14. Given a topological operad $\mathcal{O}$, let $\underline{\mathcal{O}}$ be the Top-enriched category with objects the sets $\mathbf{n} = \{1, 2, \ldots, n\}$ for $n \geqslant 0$ and with morphism spaces

$$\underline{\mathcal{O}}(\mathbf{n}, \mathbf{m}) \coloneqq \coprod_{f\colon \mathbf{n} \to \mathbf{m}} \prod_{t \in \mathbf{m}} \mathcal{O}\left(\left|f^{-1}(t)\right|\right)$$

where the coproduct runs over all set maps. Composition is induced from the composition of $\mathcal{O}$.

In words, an arrow in $\underline{\mathcal{O}}$ from $\mathbf{n}$ to $\mathbf{m}$ is a partition $\mathbf{n} = \coprod_{t \in \mathbf{m}} S_t$ and a choice of an $|S_t|$-ary operation in $\mathcal{O}$ for each $t \in \mathbf{m}$.

Disjoint union of finite sets lifts to a symmetric strict monoidal structure on $\underline{\mathcal{O}}$ with unit $\mathbf{0}$, which we denote with $\sqcup$. Hence the classifying space $B\underline{\mathcal{O}}$ is naturally a Γ-algebra. When $\mathcal{O}$ is an OSHS, it has operations of all arities (since it receives a map from an A_∞-operad), so $B\underline{\mathcal{O}}$ is connected and hence weakly equivalent to an infinite loop space.

1.3.2.1 PROPs of OSHSs

When $\mathcal{O}$ is an OSHS, the group completion of its initial algebra $\mathcal{O}(0)$ is weakly equivalent to the loop space $\Omega B\underline{\mathcal{O}}$. Indeed, the following proof of this statement is a straightforward

generalisation of [55, Proposition 4.1]. We first recall the following convenient formulation of the generalised group completion theorem:

Let $\mathscr{C}$ be a topologically enriched category and let $F\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Top}$ be an enriched functor. Let $F \wr \mathscr{C}$ be the category of elements of F — this is the category internal to Top with spaces of objects and morphisms

$$\mathrm{ob}\,(F \wr \mathscr{C}) = \coprod_{c \in \mathrm{ob}\,\mathscr{C}} F(c)$$
$$\mathrm{mor}\,(F \wr \mathscr{C}) = \coprod_{c,d \in \mathrm{ob}\,\mathscr{C}} F(c) \times \mathscr{C}(d,c)$$

This category comes with an evident enriched functor $F \wr \mathscr{C} \to \mathscr{C}$. The space of k-simplices of the nerve of $F \wr \mathscr{C}$ is

$$\coprod_{c_0,\dots,c_k \in \mathrm{ob}\,\mathscr{C}} \mathscr{C}(c_0,c_1) \times \cdots \times \mathscr{C}(c_{k-1},c_k) \times F(c_k).$$

From this formula it is easily seen that the functor $N_\bullet(- \wr \mathscr{C}) : \mathrm{Top}^{\mathscr{C}^{\mathrm{op}}} \to \mathrm{sTop}$ preserves colimits.

Proposition 1.3.15 ([24, Proposition 7.1]). *Let $\mathscr{C}$ and F be as above. Suppose that every $f \in \mathscr{C}(c,d)$ induces an isomorphism*

$$F(f)_*\colon H_*(F(d)) \longrightarrow H_*(F(c))$$

on integral homology. Then for every $c \in \mathrm{ob}\,\mathscr{C}$, the pullback square

$$\begin{array}{ccc} F(c) & \longrightarrow & B(F \wr \mathscr{C}) \\ \downarrow & & \downarrow \\ \{c\} & \longrightarrow & B\mathscr{C} \end{array}$$

is homology cartesian, i.e. *the map from $F(c)$ to the homotopy fibre of the right-hand map at c is a homology equivalence.*

Let $\mu\colon \mathrm{Ass} \to \mathcal{O}$ be a superb OSHS, and write $\mu_2 \in \mathcal{O}(2)$ for the image of $\mathrm{id} \in \Sigma_2 = \mathrm{Ass}(2)$ under μ. As in Remark 1.3.2, we assume that the propagator is of the form $s = \mu_2(s_0, 1)$ for some $s_0 \in \mathcal{O}_1(0)$.

The map μ_2 equips $\mathcal{O}(0)$ with a strictly associative monoid structure. $\mathcal{O}(1)$ is a monoid with respect to composition of unary operations, and the morphism

$$\mathcal{O}(0) \longrightarrow \mathcal{O}(1)$$
$$x \longmapsto \mu_2(x,1)$$

is a monoid homomorphism. Let $\alpha\colon \mathcal{O}(0) \to \underline{\mathcal{O}}$ be the functor

$$\mathcal{O}(0) \xrightarrow{\mu_2(-,1)} \mathcal{O}(1) \hookrightarrow \underline{\mathcal{O}}$$

where the two monoids are viewed as categories with a single object, and the second functor is the inclusion of the full subcategory of $\underline{\mathcal{O}}$ spanned by the object $\mathbf{1}$.

Proposition 1.3.16. *The map $\Omega B\alpha\colon \Omega B\mathcal{O}(0) \to \Omega B\underline{\mathcal{O}}$ is a weak equivalence.*

Proof. Consider the contravariant functor $\mathcal{O}_\infty\colon \underline{\mathcal{O}}^{\mathrm{op}} \to \mathrm{Top}$ given by

$$\mathbf{n} \longmapsto \mathrm{Tel}\left(\underline{\mathcal{O}}(\mathbf{n},\mathbf{1}) \xrightarrow{s_*} \underline{\mathcal{O}}(\mathbf{n},\mathbf{1}) \xrightarrow{s_*} \cdots\right)$$

where Tel denotes the mapping telescope construction and s_* is the postcomposition with the propagator $s \in \mathcal{O}_1(1) \subset \underline{\mathcal{O}}(\mathbf{1},\mathbf{1})$.

Note that $\mathcal{O}_\infty$ takes all morphisms in $\underline{\mathcal{O}}^{\mathrm{op}}$ to integral homology equivalences: for every $\mathbf{v} \in \underline{\mathcal{O}}(\mathbf{0},\mathbf{n})$, we have $\mathcal{O}_\infty(\mathbf{v}) = D^{\mathbf{v}}_\infty$, where the latter is the map from Definition 1.3.1, and hence a homology equivalence by assumption. Given $f \in \underline{\mathcal{O}}(\mathbf{n},\mathbf{m})$, pick $\mathbf{v} \in \underline{\mathcal{O}}(\mathbf{0},\mathbf{n})$. Then $\mathcal{O}_\infty(\mathbf{v})$ and $\mathcal{O}_\infty(f\mathbf{v})$ are homology equivalences, whence $\mathcal{O}_\infty(f)$ is also. Hence by Proposition 1.3.15, the sequence

$$\mathcal{O}_\infty(\mathbf{1}) \longrightarrow B\left(\mathcal{O}_\infty \wr \underline{\mathcal{O}}\right) \longrightarrow B\underline{\mathcal{O}}$$

is a homology fibration.

Note that

$$\mathcal{O}_\infty(\mathbf{0}) = \mathrm{Tel}\left(\mathcal{O}(0) \xrightarrow{s_0\cdot(-)} \mathcal{O}(0) \xrightarrow{s_0\cdot(-)} \cdots\right)$$

is a right $\mathcal{O}(0)$-module, and the monoid map $\mu_2(-,1)\colon \mathcal{O}(0) \to \mathcal{O}(1)$ gives rise to a right $\mathcal{O}(0)$-module map $m_\infty\colon \mathcal{O}_\infty(\mathbf{0}) \to \mathcal{O}_\infty(\mathbf{1})$, where $\mathcal{O}_\infty(\mathbf{1})$ is an $\mathcal{O}(0)$-module by restriction along α. $\mathcal{O}(0)$ acts on $\mathcal{O}_\infty(\mathbf{0})$ by homology equivalences, and so by Proposition 1.3.15 again, we obtain a map of homology fibrations

$$\begin{array}{ccccc}
\mathcal{O}_\infty(\mathbf{0}) & \longrightarrow & B\left(\mathcal{O}_\infty(\mathbf{0}) \wr \mathcal{O}(0)\right) & \longrightarrow & B\mathcal{O}(0) \\
\downarrow{\scriptstyle m_\infty} & & \downarrow & & \downarrow{\scriptstyle B\alpha} \\
\mathcal{O}_\infty(\mathbf{1}) & \longrightarrow & B\left(\mathcal{O}_\infty \wr \underline{\mathcal{O}}\right) & \longrightarrow & B\underline{\mathcal{O}}
\end{array}$$

The middle spaces are weakly contractible. Indeed, since $B\left(- \wr \underline{\mathcal{O}}\right) : \mathrm{Top}^{\underline{\mathcal{O}}^{\mathrm{op}}} \to \mathrm{Top}$ preserves colimits, we have the equivalence

$$B\left(\mathcal{O}_\infty \wr \underline{\mathcal{O}}\right) \simeq \mathrm{Tel}\left(B\left(\underline{\mathcal{O}}(-,\mathbf{1}) \wr \underline{\mathcal{O}}\right) \xrightarrow{s_*} B\left(\underline{\mathcal{O}}(-,\mathbf{1}) \wr \underline{\mathcal{O}}\right) \xrightarrow{s_*} \cdots\right).$$

But $\underline{O}(-,\mathbf{1})\wr\underline{O}$ is the slice category of $\underline{O}$ over $\mathbf{1} \in \operatorname{ob}\underline{O}$, so its classifying space is contractible. We also have that $B\,(O_\infty(\mathbf{0}) \wr O(0)) \simeq *$ using the same argument.

Hence $\Omega B\alpha$ is a homology equivalence, because m_∞ is. But $\Omega B\alpha$ is a map of simple spaces, so by Whitehead's theorem, it is a weak equivalence. □

We conclude this section with the following observation, which is a direct analogue of Wahl's [57, Proposition 4.2].

Lemma 1.3.17. *The square*

$$\begin{array}{ccc} O(0) & \xrightarrow{\psi} & \Omega B\underline{O} \\ \downarrow & & \uparrow\simeq \\ \Omega BO(0) & \xrightarrow[\Omega B\alpha]{} & \Omega_{\mathbf{1}} B\underline{O} \end{array}$$

is homotopy commutative, where ψ takes $x \in O(0) = \underline{O}(\mathbf{0},\mathbf{1})$ to the loop that follows the 1-simplex x from $\mathbf{0}$ to $\mathbf{1}$, and then returns from $\mathbf{1}$ to $\mathbf{0}$ along the 1-simplex determined by the basepoint $* \in O(0)$, and the right-hand vertical map conjugates a loop based at $\mathbf{1}$ by the path from $\mathbf{0}$ to $\mathbf{1}$ determined by the basepoint of $O(0)$.

Proof. It is easily seen from the definitions that the lower boundary of the square takes a point $a \in O(0)$ to the loop in $B\underline{O}$ that goes from $\mathbf{0}$ to $\mathbf{1}$ along $* \in O(0)$, then from $\mathbf{1}$ to itself along $\mu_2(a,1) \in O(1)$, and then back to $\mathbf{0}$ along $* \in O(0)$. The 2-simplex $\mathbf{0} \xrightarrow{*} \mathbf{1} \xrightarrow{\mu_2(a,1)} \mathbf{1}$ of $B\underline{O}$ provides a homotopy from this loop to $\psi(a)$. The assignment

$$\begin{aligned} O(0) &\longrightarrow N_2\underline{O} \\ a &\longmapsto \left(\mathbf{0} \xrightarrow{*} \mathbf{1} \xrightarrow{\mu_2(a,1)} \mathbf{1}\right) \end{aligned}$$

is clearly continuous, yielding the desired homotopy. □

1.3.3 Comparison of structures

We have seen that when $\mathrm{Ass} \xrightarrow{\mu} O \xrightarrow{\pi} \Gamma$ is a superb OSHS, then $\mathcal{G}\,O(0)$ is weakly equivalent (as a space) to two infinite loop spaces, namely the bar construction

$$|\mathcal{G}\,B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, O(0) \times O(0))|$$

of [3], and the loop space $\Omega B\underline{O}$. The purpose of this section is to show that these two infinite deloopings of $\mathcal{G}\,O(0)$ are equivalent. Explicitly, we prove the following theorem.

Theorem 1.3.18. *There is a map of* Γ*-algebras* $\phi' \colon X' \to Y'$ *such that* X' *is equivalent*[2] *as a* Γ*-algebra to* $|B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, \mathcal{O}(0) \times \mathcal{O}(0))|$*,* Y' *is equivalent as a* Γ*-algebra to* $\Omega B\underline{\mathcal{O}}$*, and the group completion* $\mathcal{G}\,\phi' \colon \mathcal{G}\,X' \to \mathcal{G}\,Y'$ *is a weak equivalence.*

Here and throughout this section, $\Omega B\underline{\mathcal{O}}$ is the space of loops in $B\underline{\mathcal{O}}$ based at $\mathbf{0}$. The action of Γ on $B\underline{\mathcal{O}}$ fixes $\mathbf{0}$, so $\Omega B\underline{\mathcal{O}}$ is a sub-Γ-algebra of the 'pointwise' Γ-algebra $B\underline{\mathcal{O}}^{S^1}$. This is the Γ-algebra structure referred to in the statement of the theorem, and used throughout this section.

1.3.3.1 Models for infinite loop spaces

The proof of Theorem 1.3.18 consists of constructing suitable replacements of the two Γ-algebras involved until the comparison map ϕ' can be written down. The first replacements we make are as follows.

Definition 1.3.19. Let $X_\bullet$ be the simplicial Γ-algebra

$$X_\bullet \coloneqq B_\bullet(\mathbb{\Gamma}, \mathbb{O}, \mathcal{O}(0) \times \mathcal{O}(0)).$$

Let $Y_\bullet$ be the simplicial Γ-algebra

$$Y_\bullet \coloneqq B_\bullet\left(\mathbb{\Gamma}, \mathbb{\Gamma}, \Omega B\underline{\mathcal{O}}\right).$$

Then $|X_\bullet|$ admits a weak equivalence of Γ-algebras to $|B_\bullet(\mathbb{\Gamma}_*, \mathbb{O}_*, \mathcal{O}(0) \times \mathcal{O}(0))|$ by Corollary 1.2.27, while the basepoint collapse map $|Y_\bullet| \to \left|B_\bullet\left(\mathbb{\Gamma}_*, \mathbb{\Gamma}_*, \Omega B\underline{\mathcal{O}}\right)\right|$ is a weak equivalence by abstract nonsense, *cf.* [39, Section 9].

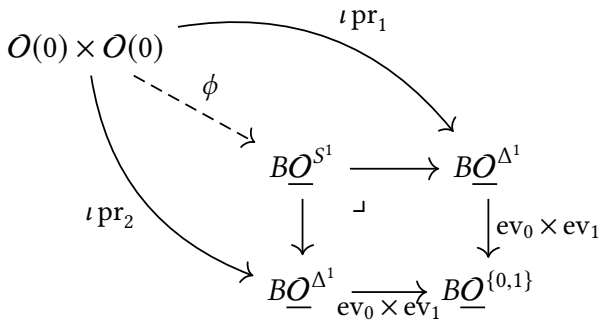
We start by constructing a map $\phi \colon \mathcal{O}(0) \times \mathcal{O}(0) \to \Omega B\underline{\mathcal{O}}$. Let $\iota \colon \mathcal{O}(0) \to B\underline{\mathcal{O}}^{\Delta^1}$ be the adjoint of the characteristic map

$$\Delta^1 \times \mathcal{O}(0) = \Delta^1 \times \underline{\mathcal{O}}(\mathbf{0}, \mathbf{1}) \longrightarrow B\underline{\mathcal{O}};$$

The map ι takes a point of $\mathcal{O}(0)$, interpreted as a morphism in $\underline{\mathcal{O}}$, to the path in $B\underline{\mathcal{O}}$ that traverses the corresponding 1-simplex. The image of ι lies in the subspace of paths from $\mathbf{0}$

[2] We say two Γ-algebras are equivalent if there is a zig-zag of weak Γ-algebra equivalences that links them.

to **1**, so we get the following universal map



where $\mathrm{pr}_1, \mathrm{pr}_2 \colon \mathcal{O}(0)\times\mathcal{O}(0) \to \mathcal{O}(0)$ are the projections onto the first and second factor, $\mathrm{ev}_0, \mathrm{ev}_1$ are the evaluation maps at $0, 1 \in \Delta^1$, and the pullback square is the image under $B\underline{\mathcal{O}}^{(-)}$ of the square which identifies S^1 as a pushout of two copies of Δ^1 along the inclusions $\partial\Delta^1 \hookrightarrow \Delta^1$. It is easily seen that ϕ corestricts to a map $\phi \colon \mathcal{O}(0)\times\mathcal{O}(0) \to \Omega B\underline{\mathcal{O}}$. It sends a pair $(x, y) \in \mathcal{O}(0)\times\mathcal{O}(0)$ to the loop in $B\underline{\mathcal{O}}$ that goes from $\mathbf{0}$ to $\mathbf{1}$ along the 1-simplex x and returns to $\mathbf{0}$ along the 1-simplex y.

We would like to use ϕ, along with the operad map $\mathcal{O} \to \Gamma$, to write down a simplicial comparison map $\phi \colon X_\bullet \to Y_\bullet$. We do obtain maps

$$\phi_d \coloneqq \mathbb{\Gamma}\pi^d(\phi) \colon \mathbb{\Gamma}\mathbb{O}^d\left(\mathcal{O}(0)\times\mathcal{O}(0)\right) \longrightarrow \mathbb{\Gamma}^{d+1}\left(\Omega B\underline{\mathcal{O}}\right)$$

on the spaces of simplices, but since ϕ fails to be a map of $\mathcal{O}$-algebras, these maps do not commute with the last face maps $d_n \colon X_n \to X_{n-1}$ (and the same for Y), *i.e.* the face maps in $X_\bullet$ and $Y_\bullet$ that come from the action of $\mathcal{O}$ on $\mathcal{O}(0)\times\mathcal{O}(0)$, and the action of Γ on $\Omega B\underline{\mathcal{O}}$, respectively. They do, however, commute up to coherent homotopies.

Remark 1.3.20. The map ϕ is not even a map of pointed spaces, so it does not induce a map between $\mathbb{O}_*(\mathcal{O}(0)\times\mathcal{O}(0))$ and $\mathbb{\Gamma}_*\left(\Omega B\underline{\mathcal{O}}\right)$. This is why we replaced $\mathbb{O}_*$ and $\mathbb{\Gamma}_*$ with their unpointed analogues in the definition of $X_\bullet$ and $Y_\bullet$.

Wahl attempted to solve this problem in the case of the surface operad by freely adjoining a basepoint to $\mathcal{O}(0)\times\mathcal{O}(0)$ and extending the $\mathcal{O}$-algebra structure on it by letting the new basepoint act as an additional unit, *cf.* [57, Proposition 5.1]. Such an extended $\mathcal{O}$-algebra structure cannot exist, however. Indeed, if it did, then $(\mathcal{O}(0)\times\mathcal{O}(0))_+$ would in particular be a monoid with two distinct units, which is impossible.

There is a way to extend the $\mathcal{O}$-algebra structure on $\mathcal{O}(0)\times\mathcal{O}(0)$ to $(\mathcal{O}(0)\times\mathcal{O}(0))_+$ by making the new basepoint † 'absorbing': $a(x_1, \ldots, x_n) = \dagger$ for all $a \in \mathcal{O}(n)$ and $x_i \in (\mathcal{O}(0)\times\mathcal{O}(0))_+$ as soon as $x_i = \dagger$ for at least one i. But the group completion of the underlying monoid is then contractible.

1.3.3.2 The category $\mathscr{T}$

A map of simplicial spaces can equivalently be thought of as a functor $\mathbb{\Delta}^{\mathrm{op}} \times 2 \to \mathrm{Top}$ where 2 is the 'walking arrow' — this is the poset $[1] = \{0, 1\}$ considered as a category. We use the idea of Wahl [57] to 'fatten' the category $\mathbb{\Delta}^{\mathrm{op}} \times 2$ to a topologically enriched category $\mathscr{T}$ in such a way that the homotopies witnessing the commutativity-up-to-homotopy of the maps ϕ_n with the last face maps give rise to a functor $\mathscr{T} \to \Gamma\text{–Alg}$.

We shall write $[n]_i \coloneqq ([n], i) \in \mathbb{\Delta}^{\mathrm{op}} \times 2$ for the objects of the product category. Recall [40] that every map $f\colon [n] \to [m]$ in $\mathbb{\Delta}^{\mathrm{op}}$ factors uniquely as

$$f = s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_1} \tag{1.8}$$

for some $0 \leqslant i_s < \cdots < i_1 \leqslant n$ and $0 \leqslant j_1 < \cdots < j_t < m$, where $d_i\colon [k] \to [k-1] \in \mathbb{\Delta}^{\mathrm{op}}$ is the opposite of the injective monotone map that misses $i \in [k]$ and $s_j\colon [k] \to [k+1] \in \mathbb{\Delta}^{\mathrm{op}}$ is the opposite of the surjective monotone map that hits $j \in [k]$ twice.

Definition 1.3.21 ([57, Section 5.2])**.** Let $\mathscr{T}$ be the topologically enriched category with the same objects as $\mathbb{\Delta}^{\mathrm{op}} \times 2$. The hom spaces of $\mathscr{T}$ are given by

$$\mathscr{T}([n]_i, [m]_j) = \begin{cases} \mathbb{\Delta}^{\mathrm{op}}([n], [m]) & \text{if } i = j \\ \coprod_{f \in \mathbb{\Delta}^{\mathrm{op}}([n],[m])} \Delta^{n-f(m)} & \text{if } i < j \\ \varnothing & \text{otherwise} \end{cases} \tag{1.9}$$

In particular, for each n there is a unique morphism $\phi_n\colon [n]_0 \to [n]_1$ indexed by the identity on $[n] \in \mathbb{\Delta}^{\mathrm{op}}$. Supposing $f \in \mathbb{\Delta}^{\mathrm{op}}([n], [m])$ factors as in (1.8), note that $n - f(m)$ is precisely the degeneracy degree $d(f)$ of Wahl, *i.e.* the number of 'last face maps' $d_k\colon [k] \to [k-1]$ occurring in (1.8). We label the vertices of the $d(f)$-simplex in (1.9) indexed by f with the $d(f) + 1$ formal expressions

$$f^{(l)} \coloneqq s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_{l+1}} \phi_{n-l} d_{i_l} \cdots d_{i_1}$$

for $0 \leqslant l \leqslant d(f)$. Composition in $\mathscr{T}$ is now defined on the vertices of the simplices, using the facts that the $s_\bullet$ and the $d_\bullet$ satisfy the usual simplicial identities, the $\phi_\bullet$ commute with all degeneracy maps, and with all face maps with the exception of the last face maps $d_k\colon [k] \to [k-1] \in \mathbb{\Delta}^{\mathrm{op}}$. It is extended linearly to the rest of the simplices.

Remark 1.3.22. We write down the composition in $\mathscr{T}$ explicitly, as it will be useful later when constructing a functor out of $\mathscr{T}$. We do not give explicit derivations; all the formulae

below are easily worked out using simplicial identities. As in the above definition, let $f = s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_1} \colon [n] \to [m]$ be a morphism in $\mathbb{\Delta}^{\mathrm{op}}$ with degeneracy degree $d(f) = n - f(m)$. Thus $i_k = n - k + 1$ for $k \leqslant d(f)$ and $i_k < n - k + 1$ for $k > d(f)$.

(i) For $d_k \colon [n+1] \to [n]$ and $0 \leqslant l \leqslant d(f)$,

$$f^{(l)} d_k = \begin{cases} (f d_k)^{(l)} & \text{if } l < n - k + 1 \\ (f d_k)^{(l+1)} & \text{otherwise} \end{cases}$$

If $k \geqslant i_{d(f)} = n - d(f) + 1 = f(m) + 1$, then $d(f d_k) = d(f) + 1$ and precomposition with d_k takes the $d(f)$-simplex indexed by f to the $(d(f)+1)$-simplex indexed by $f d_k$ by the face inclusion $\delta_{n-k+1} \colon \Delta^{d(f)} \to \Delta^{d(f)+1}$. If $k < i_{d(f)}$, then $d(f d_k) = d(f)$ and precomposition with d_k is the identity $\Delta^{d(f)} \to \Delta^{d(f)}$.

(ii) For $d_k \colon [m] \to [m-1]$ and $0 \leqslant l \leqslant d(f)$, $d_k f^{(l)} = (d_k f)^{(l)}$, so postcomposition with d_k induces the 'front face' inclusion $\Delta^{d(f)} \to \Delta^{d(d_k f)}$. Note that

$$d(d_k f) = \begin{cases} d(f) & \text{if } k < m \\ n - f(m-1) & \text{if } k = m \end{cases}$$

(iii) For $s_k \colon [n-1] \to [n]$ and $0 \leqslant l \leqslant d(f)$,

$$f^{(l)} s_k = \begin{cases} (f s_k)^{(l)} & \text{if } l < n - k \\ (f s_k)^{(l-1)} & \text{otherwise} \end{cases}$$

If $k \geqslant i_{d(f)} - 1 = f(m)$, then $d(f s_k) = d(f) - 1$ and precomposition with s_k takes the $d(f)$-simplex indexed by f to the $(d(f)-1)$-simplex indexed by $f s_k$ by the degeneracy map $\sigma_{n-k-1} \colon \Delta^{d(f)} \to \Delta^{d(f)-1}$. If $k < f(m)$, then $d(f s_k) = d(f)$ and precomposition with s_k is the identity $\Delta^{d(f)} \to \Delta^{d(f)}$.

(iv) Postcomposition with a degeneracy map has no effect on the degeneracy degree, and it maps the corresponding simplices by the identity.

1.3.3.3 The action of Γ on $B\underline{O}$

We briefly recall how the Barratt–Eccles operad acts on the classifying space $B\underline{O}$.

Let $\Sigma_n /\!\!/ \Sigma_n$ be the translation groupoid of the left Σ_n-action on itself: the objects are the elements of Σ_n, and for each $\sigma, \tau \in \Sigma_n$, there is a unique morphism $\tau\sigma^{-1} \colon \sigma \to \tau$. The

groupoid $\Sigma_n /\!\!/ \Sigma_n$ admits a right Σ_n-action which preserves the arrow labels. If $\mathscr{C}$ is a symmetric strict monoidal category, there are functors

$$\begin{aligned} \Sigma_n /\!\!/ \Sigma_n \times \mathscr{C}^n &\longrightarrow \mathscr{C} \\ (\sigma; X_1, \dots, X_n) &\longmapsto X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)} \end{aligned} \tag{1.10}$$

which are evidently Σ_n-equivariant. (We take the empty monoidal product to be the monoidal identity.)

The groupoids $\Sigma_n /\!\!/ \Sigma_n$ for $n \geqslant 0$ fit into an operad in categories, and every symmetric strict monoidal category is then an algebra over this operad via the functors (1.10). (In fact one can use the requirement that the functors (1.10) be the structure functors of an operad algebra to infer what the operad structure ought to be.) The Barratt–Eccles operad Γ is precisely the geometric realisation of this operad in categories, and the action of Γ on $B\mathscr{C}$ comes from realising the functors (1.10).

Lemma 1.3.23. *Let $a \in \Gamma(n)$ lie in the image of a (possibly degenerate) m-simplex*

$$\sigma_0 \to \cdots \to \sigma_m \in N_m(\Sigma_n /\!\!/ \Sigma_n)$$

of the nerve of $\Sigma_n /\!\!/ \Sigma_n$. Let $x_1, \dots, x_n \in \underline{O}(\mathbf{0}, \mathbf{1})$ be morphisms in $\underline{O}$, interpreted as points of the path space $B\underline{O}^{\Delta^1}$ with its natural Γ-algebra structure. The path $a(x_1, \dots, x_n) \in B\underline{O}^{\Delta^1}$ is contained in the image of the simplex

$$\mathbf{0} \xrightarrow{x_1 \sqcup \cdots \sqcup x_n} \mathbf{n} \xrightarrow{\sigma_0} \mathbf{n} \xrightarrow{\sigma_1\sigma_0^{-1}} \mathbf{n} \xrightarrow{\sigma_2\sigma_1^{-1}} \cdots \xrightarrow{\sigma_m\sigma_{m-1}^{-1}} \mathbf{n} \xrightarrow{\sigma_m^{-1}} \mathbf{n} \tag{1.11}$$

of the nerve of $\underline{O}$.

Proof. This claim is an exercise in the prism decomposition of products of simplices. The path consists of straight line segments contained in the simplices

$$\begin{array}{ccccccccc} \mathbf{0} & & & & \mathbf{0} & & \mathbf{1} & & \mathbf{1} \\ \sqcup & & & & \sqcup & & \sqcup & & \sqcup \\ \vdots & \xrightarrow{\sigma_1\sigma_0^{-1}} & \cdots & \xrightarrow{\sigma_i\sigma_{i-1}^{-1}} & \vdots & \xrightarrow{\widetilde{x}_i} & \vdots & \xrightarrow{\sigma_{i+1}\sigma_i^{-1}} & \cdots & \xrightarrow{\sigma_m\sigma_{m-1}^{-1}} & \vdots \\ \sqcup & & & & \sqcup & & \sqcup & & & & \sqcup \\ \mathbf{0} & & & & \mathbf{0} & & \mathbf{1} & & & & \mathbf{1} \end{array}$$

for $0 \leqslant i \leqslant m$, where $\widetilde{x}_i \coloneqq x_{\sigma_i^{-1}(1)} \sqcup \cdots \sqcup x_{\sigma_i^{-1}(n)} \colon \mathbf{0} \to \mathbf{n}$. But these are all degeneracies of faces of (1.11). Hence the path $a(x_1, \dots, x_n)$ in $B\underline{O}$ is the image of a piecewise linear path in the simplex (1.11). □

1.3.3.4 The weak simplicial map $X_\bullet \to Y_\bullet$

In this section we show that the maps $\phi_d \colon X_d \to Y_d$ extend to a Top-enriched functor $\mathscr{T} \to \Gamma\text{–Alg}$.

For $n \geqslant 0$ and $0 \leqslant l \leqslant n$, let $\lambda_n^{(l)} \colon \mathbb{O}^n(\mathcal{O}(0)) \to B\underline{\mathcal{O}}^{\Delta^1}$ be defined as the composite

$$\mathbb{O}^n(\mathcal{O}(0)) \xrightarrow{\mathbb{O}^{n-l}(\varepsilon_l)} \mathbb{O}^{n-l}(\mathcal{O}(0)) \xrightarrow{\pi^{n-l}(\iota)} \mathbb{\Gamma}^{n-l}\left(B\underline{\mathcal{O}}^{\Delta^1}\right) \xrightarrow{\varepsilon_{n-l}} B\underline{\mathcal{O}}^{\Delta^1}$$

where for an operad $\mathcal{P}$ and $\mathcal{P}$-algebra A, $\varepsilon_i \colon \mathbb{P}^i(A) \to A$ denotes the iterated evaluation map. For each $0 \leqslant l \leqslant n$, the morphisms

$$\mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) \underset{\mathbb{O}^n(\mathrm{pr}_2)}{\overset{\mathbb{O}^n(\mathrm{pr}_1)}{\rightrightarrows}} \mathbb{O}^n(\mathcal{O}(0)) \xrightarrow{\lambda_n^{(l)}} B\underline{\mathcal{O}}^{\Delta^1}$$

are coequalised by the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon B\underline{\mathcal{O}}^{\Delta^1} \to B\underline{\mathcal{O}}$, so they glue to give the morphism $g_n^{(l)} \colon \mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) \to \Omega B\underline{\mathcal{O}}$ which is equal to the composite

$$\begin{array}{ccc} \mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) & \xrightarrow{\mathbb{O}^{n-l}(\varepsilon_l)} & \mathbb{O}^{n-l}(\mathcal{O}(0) \times \mathcal{O}(0)) \\ & & \downarrow \pi^{n-l}(\phi) \\ & & \mathbb{\Gamma}^{n-l}\left(\Omega B\underline{\mathcal{O}}\right) \xrightarrow{\varepsilon_{n-l}} \Omega B\underline{\mathcal{O}} \end{array}$$

Proposition 1.3.24. *There are morphisms*

$$g_n \colon \Delta^n \times \mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) \longrightarrow \Omega B\underline{\mathcal{O}}$$

for all $n \geqslant 0$ such that g_n restricts to $g_n^{(l)}$ at the l-th vertex of Δ^n for all $0 \leqslant l \leqslant n$.

Proof. We show that the maps $\lambda_n^{(l)}$ extend to a map $\Delta^n \times \mathbb{O}^n(\mathcal{O}(0)) \to B\underline{\mathcal{O}}^{\Delta^1}$.

By inspecting the formula

$$\mathbb{O}(X) = \coprod_{n \geqslant 0} \underline{\mathcal{O}}(\mathbf{n}, \mathbf{1}) \times_{\Sigma_n} X^n,$$

it is easy to see that a point of $\mathbb{O}^n(\mathcal{O}(0))$ is represented by an n-simplex

$$\mathbf{1} = \mathbf{m}_n \xleftarrow{a_n} \mathbf{m}_{n-1} \xleftarrow{a_{n-1}} \cdots \xleftarrow{a_1} \mathbf{m}_0$$

in the nerve of $\underline{\mathcal{O}}$, along with an m_0-tuple $x_i \in \mathcal{O}(0)$ for $1 \leqslant i \leqslant m_0$. Such an m_0-tuple can equivalently be thought of as the morphism $x \coloneqq x_1 \sqcup \cdots \sqcup x_{m_0} \colon \mathbf{0} \to \mathbf{m}_0$ in $\underline{\mathcal{O}}$, so the above data can also be repackaged as an $(n+1)$-simplex in $N_\bullet \underline{\mathcal{O}}$ of the form

$$\mathbf{1} = \mathbf{m}_n \xleftarrow{a_n} \mathbf{m}_{n-1} \xleftarrow{a_{n-1}} \cdots \xleftarrow{a_1} \mathbf{m}_0 \xleftarrow{x} \mathbf{0} \tag{1.12}$$

(This representation is not quite unique: if $\sigma \in \Sigma_{m_j}$, then precomposing a_{j+1} with σ and postcomposing a_j, where $a_0 \coloneqq x$, with σ^{-1} yields data that represents the same point of $\mathbb{O}^n(\mathcal{O}(0))$.)

Given the data of (1.12) representing a point $P \in \mathbb{O}^n(\mathcal{O}(0))$, let

$$u_i \coloneqq a_i \cdots a_1 \colon \mathbf{m}_0 \longrightarrow \mathbf{m}_i$$
$$v_i \coloneqq a_n \cdots a_{i+1} \colon \mathbf{m}_i \longrightarrow \mathbf{m}_n = \mathbf{1}$$

for $0 \leqslant i \leqslant n$, where u_0 and v_n are the respective identities in $\underline{\mathcal{O}}$. Now $\lambda_n^{(l)}(P)$ is the path

$$\pi(v_l)(u_l x) \in B\underline{\mathcal{O}}^{\Delta^1}$$

where $v_l \in \underline{\mathcal{O}}(\mathbf{m_l}, \mathbf{1}) = \mathcal{O}(m_l)$, $\pi(v_l) \in \Gamma(m_l)$ is its image under $\pi \colon \mathcal{O} \to \Gamma$, and $u_l x \colon \mathbf{0} \to \mathbf{m_l}$ is thought of as an m_l-tuple of 1-simplices of $B\underline{\mathcal{O}}$ from $\mathbf{0}$ to $\mathbf{1}$. By Lemma 1.3.23, this path is contained in a simplex

$$\mathbf{0} \xrightarrow{u_l x} \mathbf{m}_l \xrightarrow{\sigma_0^{(l)}} \mathbf{m}_l \xrightarrow{\sigma_1^{(l)}(\sigma_0^{(l)})^{-1}} \cdots \xrightarrow{\sigma_{p_l}^{(l)}(\sigma_{p_l-1}^{(l)})^{-1}} \mathbf{m}_l \xrightarrow{(\sigma_{p_l}^{(l)})^{-1}} \mathbf{m}_l \tag{1.13}$$

where

$$\sigma_0^{(l)} \to \cdots \to \sigma_{p_l}^{(l)} \in N_{p_l}\left(\Sigma_{m_l} /\!\!/ \Sigma_{m_l}\right)$$

is some p_l-simplex of $\Gamma(m_l)$ that contains $\pi(v_l)$.

But the simplices (1.13) for varying l are all faces of the simplex

$$\mathbf{0} \xrightarrow{x} \mathbf{m}_0 \xrightarrow{\sigma_0^{(0)}} \cdots \xrightarrow{(\sigma_{p_0}^{(0)})^{-1}} \mathbf{m}_0 \xrightarrow{a_1} \mathbf{m}_1 \xrightarrow{\sigma_0^{(1)}} \cdots \xrightarrow{(\sigma_{p_1}^{(1)})^{-1}} \mathbf{m}_1 \xrightarrow{a_2} \cdots \xrightarrow{(\sigma_{p_n}^{(n)})^{-1}} \mathbf{m}_n \tag{1.14}$$

Thus we have shown that the $\lambda_n^{(l)}$-images of P, for $0 \leqslant l \leqslant n$, are all paths contained in a single simplex of $B\underline{\mathcal{O}}$. Hence we can define a map

$$\lambda_n \colon \Delta^n \times \mathbb{O}^n(\mathcal{O}(0)) \to B\underline{\mathcal{O}}^{\Delta^1}$$

by setting it to be $\lambda_n^{(l)}$ at the l-th vertex of Δ^n, and extending the definition to the rest of Δ^n linearly. It is easy to see that even though we've made several choices along the way

(namely, picking a representative of a point of $\mathbb{O}^n(\mathcal{O}(0))$ of the form (1.12), and picking simplices of $\Gamma(m_l)$ that contain $\pi(v_l)$), the resulting map λ_n is independent of these choices, and hence well defined. It is also continuous, because the $\lambda_n^{(l)}$ are.

The morphisms

$$\Delta^n \times \mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) \underset{\mathrm{id} \times \mathbb{O}^n(\mathrm{pr}_2)}{\overset{\mathrm{id} \times \mathbb{O}^n(\mathrm{pr}_1)}{\rightrightarrows}} \Delta^n \times \mathbb{O}^n(\mathcal{O}(0)) \xrightarrow{\lambda_n} B\underline{\mathcal{O}}^{\Delta^1}$$

are again coequalised by $\mathrm{ev}_0, \mathrm{ev}_1 \colon B\underline{\mathcal{O}}^{\Delta^1} \to B\underline{\mathcal{O}}$, and they glue to give the advertised map $g_n \colon \Delta^n \times \mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) \to \Omega B\underline{\mathcal{O}}$. □

Proposition 1.3.25. *The maps $\phi_d \colon X_d \to Y_d$ assemble to an enriched functor $\Phi \colon \mathscr{T} \to \Gamma$–Alg.*

Proof. The functor Φ is already determined on the two copies of $\mathbb{\Delta}^{\mathrm{op}}$ inside $\mathscr{T}$ — sending one of them to $X_\bullet$ and the other to $Y_\bullet$ — and on the vertices of the simplices that make up the spaces $\mathscr{T}([n]_0, [m]_1)$: for $f \colon [n] \to [m]$ in $\mathbb{\Delta}^{\mathrm{op}}$ and $0 \leqslant l \leqslant n - f(m)$, the vertex labelled by $f^{(l)}$ is taken to the composite

$$\begin{array}{ccccc} X_n & = & B_n(\mathbb{\Gamma}, \mathbb{O}, \mathcal{O}(0) \times \mathcal{O}(0)) & \xrightarrow{d_{i_l} \cdots d_{i_1}} & B_{n-l}(\mathbb{\Gamma}, \mathbb{O}, \mathcal{O}(0) \times \mathcal{O}(0)) \\ {\scriptstyle f^{(l)}}\downarrow & & & & \downarrow{\scriptstyle \phi_{n-l}} \\ Y_m & = & B_m(\mathbb{\Gamma}, \mathbb{\Gamma}, \Omega B\underline{\mathcal{O}}) & \xleftarrow[s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_{l+1}}]{} & B_{n-l}(\mathbb{\Gamma}, \mathbb{\Gamma}, \Omega B\underline{\mathcal{O}}) \end{array}$$

We need to show that these assignments can be continuously extended to the entire mapping spaces $\mathscr{T}([n]_0, [m]_1)$ in a functorial fashion.

Note that the map $f^{(l)} \colon X_n \to Y_m$ can also be expressed in terms of the map $g^{(l)}_{d(f)}$ introduced at the beginning of this section: it is equal to the composite

$$\begin{array}{ccc} X_n = \mathbb{\Gamma}\mathbb{O}^n(\mathcal{O}(0) \times \mathcal{O}(0)) & \xrightarrow{\mathbb{\Gamma}\pi^{n-d(f)}(g^{(l)}_{d(f)})} & \mathbb{\Gamma}^{n-d(f)+1}(\Omega B\underline{\mathcal{O}}) = Y_{n-d(f)} \\ & & \downarrow{\scriptstyle s_{j_t} \cdots s_{j_1} d_{i_s} \cdots d_{i_{d(f)+1}}} \\ & & \mathbb{\Gamma}^{m+1}(\Omega B\underline{\mathcal{O}}) = Y_m \end{array}$$

Hence in light of Proposition 1.3.24, we define the map $\widetilde{f}\colon \Delta^{d(f)} \times X_n \to Y_m$ to be the composite

$$\begin{array}{c}
\Delta^{d(f)} \times X_n \;=\; \Delta^{d(f)} \times \mathbb{\Gamma}\mathbb{O}^n\left(O(0) \times O(0)\right) \\
\downarrow \\
\mathbb{\Gamma}\mathbb{O}^{n-d(f)}\left(\Delta^{d(f)} \times \mathbb{O}^{d(f)}\left(O(0) \times O(0)\right)\right) \\
\downarrow{\scriptstyle \mathbb{\Gamma}\pi^{n-d(f)}(g_{d(f)})} \\
\mathbb{\Gamma}^{n-d(f)+1}\left(\Omega B\underline{O}\right) \\
\downarrow{\scriptstyle s_{j_t}\cdots s_{j_1} d_{i_s}\cdots d_{i_{d(f)+1}}} \\
\mathbb{\Gamma}^{m+1}\left(\Omega B\underline{O}\right) \;=\; Y_m
\end{array}$$

where the first arrow is constructed by repeated application of the map (1.3) of Section 1.2.1.1.

It is clear from the construction of the map $g_{d(f)}$ that the map $\widetilde{f}$ restricts to the map $f^{(l)}\colon X_n \to Y_m$ at the l-th vertex of $\Delta^{d(f)}$. For every $t \in \Delta^{d(f)}$, the map $\widetilde{f}\,(t,-) : X_n \to Y_m$ is a morphism of Γ-algebras, so $\widetilde{f}$ is adjoint to a continuous map $\Delta^{d(f)} \to \Gamma\text{–Alg}\,(X_n, Y_m)$, and we use these maps to define the functor $\Phi\colon \mathscr{T} \to \Gamma\text{–Alg}$ on morphisms.

It remains to check that what we have written down is indeed functorial. The category $\mathscr{T}$ is constructed so that $\Phi\colon \mathscr{T} \to \Gamma\text{–Alg}$ respects composition of pairs of morphisms where one of the morphisms is a vertex of a simplex in (1.9), or they are both contained in one of the two copies of $\mathbb{\Delta}^{\mathrm{op}}$ in $\mathscr{T}$. To check that Φ also preserves compositions of pairs of morphisms when one of them is a non-vertex point of a simplex in (1.9), we have four cases to verify, corresponding to the four cases discussed in Remark 1.3.22. We discuss one of the cases here, using the notation of the Remark. The remaining three, which can be verified in an analogous fashion, are left to the diligent reader.

Let $d_k\colon [n+1] \to [n]$ be a face map in $\mathbb{\Delta}^{\mathrm{op}} \cong \mathbb{\Delta}^{\mathrm{op}} \times \{0\} \subseteq \mathscr{T}$. We wish to prove the commutativity of the square

$$\begin{array}{ccc}
\Delta^{d(f)} \times X_{n+1} & \xrightarrow{\mathrm{id}\times d_k} & \Delta^{d(f)} \times X_n \\
{\scriptstyle d_k^* \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\
\Delta^{d(fd_k)} \times X_{n+1} & \xrightarrow[\widetilde{fd_k}]{} & Y_m
\end{array} \tag{1.15}$$

where d_k^* denotes precomposition in $\mathscr{T}$ with d_k.

If $k \leqslant n - d(f) = f(m)$, then $d(fd_k) = d(f)$ and $d_k^* \colon \Delta^{d(f)} \to \Delta^{d(fd_k)}$ is the identity, and it follows by inspection of the construction of $\widetilde{f}$ that the square (1.15) commutes.

If $k > n - d(f)$, then $d(fd_k) = d(f)+1$ and $d_k^* = \delta_{n-k+1} \colon \Delta^{d(f)} \to \Delta^{d(fd_k)}$ is the inclusion of the face opposite to the vertex labelled by $(fd_k)^{(n-k+1)}$. The commutativity of (1.15) then follows from the commutativity of the square

$$\begin{array}{ccc} \Delta^{d(f)} \times \mathbb{O}^{d(f)+1}(O(0) \times O(0)) & \xrightarrow{\mathrm{id} \times d_{d(f)-n+k-1}} & \Delta^{d(f)} \times \mathbb{O}^{d(f)}(O(0) \times O(0)) \\ \downarrow{\scriptstyle \delta_{n-k+1} \times \mathrm{id}} & & \downarrow{\scriptstyle g_{d(f)}} \\ \Delta^{d(f)+1} \times \mathbb{O}^{d(f)+1}(O(0) \times O(0)) & \xrightarrow[g_{d(f)+1}]{} & \Omega B\underline{O} \end{array} \tag{1.16}$$

where the top horizontal arrow is indexed as in (1.4). By inspection of the proof of Proposition 1.3.24, the square (1.16) commutes as soon as it commutes at the vertices of $\Delta^{d(f)}$, *i.e.* as soon as we have equalities

$$g^{(l)}_{d(f)} d_{k-n+d(f)-1} = \left\{ \begin{array}{ll} g^{(l)}_{d(f)+1} & \text{if } l < n-k+1 \\ g^{(l+1)}_{d(f)+1} & \text{otherwise} \end{array} \right\} : \mathbb{O}^{d(f)+1}(O(0) \times O(0)) \longrightarrow \Omega B\underline{O}$$

for all $0 \leqslant l \leqslant d(f)$. But these equalities are immediate from the construction of the maps $g^{(\bullet)}_\bullet$. □

Remark 1.3.26. It appears counterintuitive that the face map and face inclusion in (1.16) should have the grading they do. This is simply an artefact of our grading conventions. We number the face maps of a bar construction $B_\bullet(F, T, X)$ from left to right: the 0th face map

$$B_d(F, T, X) = FT^d(X) \longrightarrow FT^{d-1}(X) = B_{d-1}(F, T, X)$$

involves the leftmost (or outermost) instance of the monad T, *etc.* The vertices of the simplices $\Delta^{d(f)}$ of (1.9) are numbered in the opposite fashion: $f^{(0)}$ is that vertex of $\Delta^{d(f)}$ in which $\phi_\bullet$ appears in the rightmost admissible position, *etc.*

1.3.3.5 Rectifying Φ to a simplicial map

In this section we replace the functor $\Phi \colon \mathscr{T} \to \Gamma\text{–Alg}$ constructed in the preceding section with a genuine simplicial map between simplicial Γ-algebras that are equivalent to $X_\bullet$ and $Y_\bullet$, respectively.

Let us first briefly recall the Boardman–Vogt tensor product of coloured operads [9]: if $\mathcal{P}$ is a C-coloured operad and $\mathcal{Q}$ is a D-coloured operad, then their Boardman–Vogt tensor

product $\mathcal{P} \otimes \mathcal{Q}$ is a $(C \times D)$-coloured operad whose algebras are precisely the $\mathcal{P}$-algebras in $\mathcal{Q}$–Alg, or equivalently the $\mathcal{Q}$-algebras in $\mathcal{P}$–Alg. It is generated by operations of the form

$$p \otimes d \in \mathcal{P} \otimes \mathcal{Q}\,((c_1, d), \ldots, (c_n, d); (c, d))$$

for $p \in \mathcal{P}\,(c_1, \ldots, c_n; c)$ and $d \in D$, and

$$c \otimes q \in \mathcal{P} \otimes \mathcal{Q}\,((c, d_1), \ldots, (c, d_n); (c, d))$$

for $q \in \mathcal{Q}\,(d_1, \ldots, d_n; d)$ and $c \in C$, subject to the following relations

(i) for a fixed $d \in D$, operations of the form $p \otimes d$ for $p \in \mathcal{P}$ compose as in $\mathcal{P}$, and likewise for operations of the form $c \otimes q$ for a fixed $c \in C$,

(ii) for every $p \in \mathcal{P}\,(c_1, \ldots, c_n; c)$ and $q \in \mathcal{Q}\,(d_1, \ldots, d_m; d)$, we have

$$\sigma^*_{n,m}\,(p \otimes d)\,(c_1 \otimes q, \ldots, c_n \otimes q) = (c \otimes q)\,(p \otimes d_1, \ldots, p \otimes d_m)$$

where $\sigma_{n,m} \in \Sigma_{nm}$ is the order-preserving isomorphism from the lexicographic order to the reverse lexicographic order on $\mathbf{n} \times \mathbf{m}$.

Considering the category $\mathscr{T}$ as a coloured operad with colours the objects of $\mathscr{T}$, unary operations the morphisms of $\mathscr{T}$, and no operations of any other arity, the functor $\Phi\colon \mathscr{T} \to \Gamma$–Alg of Proposition 1.3.25 can equivalently be thought of as an algebra over the coloured operad $\mathscr{T} \otimes \Gamma$.

The Boardman–Vogt product is rather ill-behaved in general, but in our application it turns out to be amenable to a very simple analysis. The following lemma will allow us to employ the results of Berger and Moerdijk on the homotopy theory of algebras of coloured operads.

Lemma 1.3.27. *The evident projection $p\colon \mathscr{T} \to \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{2}$ gives rise to a weak equivalence*

$$p \otimes \mathrm{id}\colon \mathscr{T} \otimes \Gamma \longrightarrow (\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{2}) \otimes \Gamma$$

of well-pointed Σ-cofibrant coloured operads.

Proof. Observe that since $\mathscr{T}$ only has unary operations and Γ only has one colour, the Boardman–Vogt product $\mathscr{T} \otimes \Gamma$ has a particularly simple description: its colours are the colours of $\mathscr{T}$, and the operations are given by

$$(\mathscr{T} \otimes \Gamma)\left([n_1]_{i_1}, \dots, [n_m]_{i_m}; [n]_i\right) = \Gamma(m) \times \prod_{j=1}^{m} \mathscr{T}([n_j]_{i_j}, [n]_i) \tag{1.17}$$

This is the geometric realisation of a simplicial set in an evident way, and the action of the group $\Sigma_{[n_1]_{i_1},\dots,[n_m]_{i_m}} \leqslant \Sigma_m$ of Definition 1.2.14 is simplicial and free (since the action of Σ_m on $\Gamma(m)$ is free), so the operad is Σ-cofibrant. It is also well pointed, since the identity morphisms are vertices in the simplicial structures on the constituent spaces.

The operad $(\mathbb{\Delta}^{\mathrm{op}} \times 2) \otimes \Gamma$ can be analysed analogously. The functor $p\colon \mathscr{T} \to \mathbb{\Delta}^{\mathrm{op}} \times 2$ induces weak equivalences on hom spaces. It is clear from the description (1.17) that the components of the operad morphism $p \otimes \mathrm{id}_\Gamma$ are also weak equivalences, as required. □

Let $s, t\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathscr{T}$ be the inclusions of the two copies of $\mathbb{\Delta}^{\mathrm{op}}$, *i.e.* $s\colon [n] \mapsto [n]_0$ and $t\colon [n] \mapsto [n]_1$.

Lemma 1.3.28. *The functor*

$$\begin{aligned} s^*\colon (\mathscr{T} \otimes \Gamma)\text{–Alg} \cong \Gamma\text{–Alg}^{\mathscr{T}} &\longrightarrow \mathrm{s}\Gamma\text{–Alg} \\ \Theta &\longmapsto \Theta \circ s \end{aligned}$$

is left Quillen relative to the projective model structure on $\mathrm{s}\Gamma$–Alg. *The functor*

$$t^*\colon \Gamma\text{–Alg}^{\mathscr{T}} \longrightarrow \mathrm{s}\Gamma\text{–Alg}$$

is left Quillen relative to the Reedy model structure on $\mathrm{s}\Gamma$–Alg.

The same result holds when we replace $\mathscr{T}$ *with the category* $\mathbb{\Delta}^{\mathrm{op}} \times 2$.

Proof. We remark that the model structure on $\Gamma\text{–Alg}^{\mathscr{T}} \cong (\mathscr{T} \otimes \Gamma)\text{–Alg}$ of Theorem 1.2.13 coincides with the projective model structure (or rather its Top-enriched version): a natural transformation between functors $\mathscr{T} \to \Gamma$–Alg is a weak equivalence or a fibration if it is so componentwise on the underlying spaces, or equivalently if it is so componentwise on the Γ-algebras.

The functors s^*, t^* do admit right adjoints $s_*, t_* \colon \mathrm{s}\Gamma\text{–Alg} \to \Gamma\text{–Alg}^{\mathscr{T}}$, which send a simplicial Γ-algebra $A_\bullet \colon \mathbb{\Delta}^{\mathrm{op}} \to \Gamma\text{–Alg}$ to its right Kan extensions along s and t, respectively. It is easy to write down the right adjoint of s^* explicitly: s_* takes $A_\bullet$ to the composite

$$\mathscr{T} \xrightarrow{p} \mathbb{\Delta}^{\mathrm{op}} \times 2 \xrightarrow{A_\bullet \to *} \Gamma\text{–Alg}$$

where the second functor represents the simplicial morphism $A_\bullet \to *$ to the terminal simplicial Γ-algebra. This is patently a right Quillen adjoint with respect to the projective model structure on $\mathrm{s}\Gamma\text{–Alg}$.

The case of t_* is a little more involved. Since $\Gamma\text{–Alg}$ is complete, the right Kan extension of $A_\bullet \colon \mathbb{\Delta}^{\mathrm{op}} \to \Gamma\text{–Alg}$ along $t \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathscr{T}$ is given by the end formula

$$t_*(A_\bullet) : [n]_i \longmapsto \int_{[m] \in \mathbb{\Delta}^{\mathrm{op}}} A_m^{\mathscr{T}([n]_i, t([m]))}$$

We need to prove that when $A_\bullet \to B_\bullet$ is an (acyclic) Reedy fibration of simplicial Γ-algebras, then for all objects $[n]_i$ of $\mathscr{T}$, the induced morphism

$$t_*(A_\bullet)([n]_i) \to t_*(B_\bullet)([n]_i) \tag{1.18}$$

is an (acyclic) fibration of Γ-algebras. But since the forgetful functor $U^\Gamma \colon \Gamma\text{–Alg} \to \mathrm{Top}$ creates all small limits and preserves and reflects (acyclic) fibrations (and $\mathrm{s}U^\Gamma$ preserves and reflects (acyclic) Reedy fibrations), we may as well forget all Γ-algebra structures in sight and work with (simplicial) spaces.

Since $t \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathscr{T}$ is fully faithful, [34, Proposition 2.1] applies to show that there are natural (in $A_\bullet \in \mathrm{sTop}$ and $[n] \in \mathbb{\Delta}^{\mathrm{op}}$) isomorphisms $t_*(A_\bullet)(t([n])) \cong A_n$. Hence (1.18) is an (acyclic) fibration for all $[n]_i$ with $i = 1$ as soon as $A_\bullet \to B_\bullet$ is an (acyclic) Reedy fibration.

To show that (1.18) is an (acyclic) fibration also for $i = 0$, consider the lifting problem

$$\begin{array}{ccc} Z & \longrightarrow & \int_{[m] \in \mathbb{\Delta}^{\mathrm{op}}} A_m^{\mathscr{T}([n]_0, [m]_1)} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow \\ W & \longrightarrow & \int_{[m] \in \mathbb{\Delta}^{\mathrm{op}}} B_m^{\mathscr{T}([n]_0, [m]_1)} \end{array}$$

in Top. This can be seen to be equivalent to the lifting problem

$$\begin{array}{ccc} Z \times \mathscr{T}([n]_0, [-]_1) & \longrightarrow & A_\bullet \\ {\scriptstyle j \times \mathrm{id}}\downarrow & \nearrow & \downarrow \\ W \times \mathscr{T}([n]_0, [-]_1) & \longrightarrow & B_\bullet \end{array} \tag{1.19}$$

in sTop. But by Remark 1.3.22(iv), the inclusion of the m-th latching object of the simplicial space $\mathscr{T}([n]_0, [-]_1)$ is precisely

$$\coprod_{\substack{f\in\mathbb{\Delta}^{\mathrm{op}}([n],[m])\\ f=s_k f'\\ \text{for some } 0\leqslant k<m}} \Delta^{d(f)} \hookrightarrow \coprod_{f\in\mathbb{\Delta}^{\mathrm{op}}([n],[m])} \Delta^{d(f)}$$

Hence the left-hand vertical arrow in (1.19) is an (acyclic) Reedy cofibration as soon as $j\colon Z \to W$ is an (acyclic) cofibration of spaces, concluding the proof.

The proof for $\mathbb{\Delta}^{\mathrm{op}} \times \mathcal{2}$ in place of $\mathscr{T}$ is identical. □

Before we put the pieces together, we recall the classical formulation of the group completion theorem.

Theorem 1.3.29 ([41, Proposition 1])**.** *Let M be a topological monoid. If $\pi_0(M)$ is in the centre of the Pontryagin ring $H_*(M)$, then the group completion map $M \to \mathcal{G}M = \Omega BM$ induces the localisation homomorphism*

$$H_*(M) \longrightarrow H_*(M)[\pi_0(M)^{-1}] \cong H_*(\mathcal{G}M) \tag{1.20}$$

on homology.

We are now ready to prove Theorem 1.3.18. We repeat its statement below to maintain the continuity of the notation we've established so far.

Theorem 1.3.30. *There is a map $\phi'_\bullet\colon X'_\bullet \to Y'_\bullet$ of simplicial Γ-algebras such that*

(i) $\left|X'_\bullet\right| \simeq |X_\bullet|$ *and* $\left|Y'_\bullet\right| \simeq |Y_\bullet|$ *as Γ-algebras,*

(ii) *the group-completion $\mathcal{G}\left|\phi'_\bullet\right| : \mathcal{G}\left|X'_\bullet\right| \to \mathcal{G}\left|Y'_\bullet\right|$ is a weak equivalence.*

Proof. Let $q\colon \widetilde{\Phi} \xrightarrow{\simeq} \Phi$ be a cofibrant replacement of Φ in the model structure on the category $(\mathscr{T} \otimes \Gamma)$–Alg of Theorem 1.2.13, and let

$$\Phi' := (p \otimes \mathrm{id}_\Gamma)_!\left(\widetilde{\Phi}\right) \in ((\mathbb{\Delta}^{\mathrm{op}} \times \mathcal{2}) \otimes \Gamma)\text{–Alg}$$

be its pushforward. This is equivalently a functor $\Phi'\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathcal{2} \to \Gamma$–Alg, or a simplicial map between simplicial Γ-algebras, and we denote this map by $\phi'_\bullet\colon X'_\bullet \to Y'_\bullet$.

Since $\widetilde{\Phi}$ is cofibrant and $(p \otimes \mathrm{id}_\Gamma)_! \dashv (p \otimes \mathrm{id}_\Gamma)^*$ is a Quillen equivalence by Theorem 1.2.19 and Lemma 1.3.27, the canonical morphism $\eta\colon \widetilde{\Phi} \to (p \otimes \mathrm{id}_\Gamma)^*(\Phi')$ is a weak equivalence. Thus we obtain a zig-zag

$$X_\bullet = s^*(\Phi) \xleftarrow{s^*(q)} s^*\left(\widetilde{\Phi}\right) \xrightarrow{s^*(\eta)} s^*(\Phi') = X'_\bullet \tag{1.21}$$

of levelwise weak equivalences of simplicial Γ-algebras. $X_\bullet$ is Reedy cofibrant by Theorem 1.2.26, and $s^*\left(\widetilde{\Phi}\right)$ and $X'_\bullet$ are projectively cofibrant (hence Reedy cofibrant) by Lemma 1.3.28. Hence the geometric realisation of (1.21) also consists of weak equivalences. The argument for $Y_\bullet$ is analogous, establishing (i).

To prove (ii), consider the following diagram in sAss–Alg:

$$\begin{array}{ccccc}
X_\bullet & \xleftarrow{s^*(q)} & s^*\left(\widetilde{\Phi}\right) & \xrightarrow{s^*(\eta)} & X'_\bullet \\
\uparrow{\scriptstyle \iota_\bullet} & & \uparrow{\scriptstyle r^s_\bullet} & & \\
B_\bullet(\mathbb{A}\mathrm{ss}, \mathbb{A}\mathrm{ss}, \mathcal{O}(0)) & \xleftarrow{r_\bullet} & Z_\bullet & & \downarrow{\scriptstyle \phi'_\bullet} \\
 & & \downarrow{\scriptstyle r^t_\bullet} & & \\
Y_\bullet & \xleftarrow[t^*(q)]{} & t^*\left(\widetilde{\Phi}\right) & \xrightarrow[t^*(\eta)]{} & Y'_\bullet
\end{array}$$

where

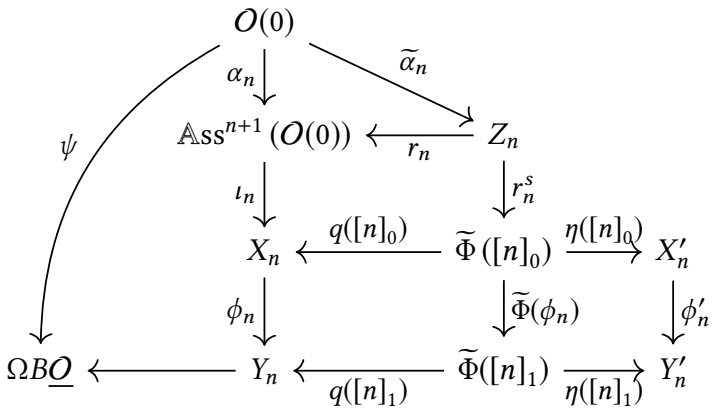
$$\iota_\bullet \coloneqq B_\bullet(\pi\mu, \mu, * \times \mathrm{id}) : B_\bullet(\mathbb{A}\mathrm{ss}, \mathbb{A}\mathrm{ss}, \mathcal{O}(0)) \longrightarrow B_\bullet(\mathbb{\Gamma}, \mathbb{O}, \mathcal{O}(0) \times \mathcal{O}(0)) = X_\bullet,$$

$Z_\bullet$ is a projectively cofibrant replacement of $B_\bullet(\pi\mu, \mu, * \times \mathrm{id})$, $r^s_\bullet$ is a lift of $\iota_\bullet r_\bullet$ across the projective acyclic fibration $s^*(q)$, and $r^t_\bullet$ is a lift of $\phi'_\bullet s^*(\eta) r^s_\bullet$ across the projective acyclic fibration $t^*(\eta)$. By construction, the diagram is commutative.

All objects in the diagram are Reedy cofibrant, and all horizontal arrows are Reedy weak equivalences, so they realise to weak equivalences of monoids. Moreover, by Proposition 1.3.13, the group-completed map $\mathcal{G}\,|\iota_\bullet|$ is a weak equivalence. Hence if we can show that $\mathcal{G}\left|r^t_\bullet\right|$ is a weak equivalence, then so is $\mathcal{G}\left|\phi'_\bullet\right|$, concluding the proof.

To this end, let $\alpha_\bullet\colon \mathcal{O}(0) \to B_\bullet(\mathbb{A}\mathrm{ss}, \mathbb{A}\mathrm{ss}, \mathcal{O}(0))$ be the usual simplicial homotopy inverse of the evaluation map $B_\bullet(\mathbb{A}\mathrm{ss}, \mathbb{A}\mathrm{ss}, \mathcal{O}(0)) \to \mathcal{O}(0)$, *cf.* [39, Section 9]. Let $\widetilde{\alpha}_\bullet\colon \mathcal{O}(0) \to Z_\bullet$ be a lift of $\alpha_\bullet$ across $r_\bullet$ — this exists because $\mathcal{O}(0)$ is a cofibrant space. Now for each

$n \geqslant 0$ there is the commutative diagram



where all maps except those emanating from $\mathcal{O}(0)$ are monoid homomorphisms, and ψ is as in Lemma 1.3.17. It is not necessarily the case that $r_n^t = \widetilde{\Phi}(\phi_n) \circ r_n^s$, but both these maps are lifts of $\phi'_n \circ \eta([n]_0) \circ r_n^s \colon Z_n \to Y'_n$ across the acyclic fibration $\eta([n]_1)$ of monoids. Since Z_n is a cofibrant monoid by construction, it follows that r_n^t and $\widetilde{\Phi}(\phi_n) \circ r_n^s$ are homotopic through monoid homomorphisms. Hence $\mathcal{G}\, r_n^t$ and $\mathcal{G}\, \widetilde{\Phi}(\phi_n) \circ \mathcal{G}\, r_n^s$ induce the same map on homology.

Given a (semi)simplicial space $S_\bullet$, the integral homology of the fat geometric realisation $\|S_\bullet\|$ is computed by the natural spectral sequence

$$E^1_{pq} \cong H_p(S_q) \Longrightarrow H_{p+q}(\|S_\bullet\|);$$

cf. [49, Section 5], [20, Section 1.4]. (We shall not make a distinction between the fat and ordinary geometric realisations, as these are equivalent for all the simplicial spaces in sight.) Consider the map

$$\mathcal{O}(0) \xrightarrow{|\widetilde{\alpha}_\bullet|} |Z_\bullet| \to |\mathcal{G}\, Z_\bullet| \xrightarrow{\left|\mathcal{G}\, r_\bullet^t\right|} \left|\mathcal{G}\, t^* \left(\widetilde{\Phi}\right)\right| \xrightarrow{|\mathcal{G}\, t^*(q)|} |\mathcal{G}\, Y_\bullet| \to \mathcal{G}\left(\Omega B\underline{\mathcal{O}}\right)$$

By computing its effect on homology using the above spectral sequence, it can be seen that it induces the same map on homology as

$$\mathcal{O}(0) \xrightarrow{\psi} \Omega B\underline{\mathcal{O}} \to \mathcal{G}\left(\Omega B\underline{\mathcal{O}}\right)$$

But by Lemma 1.3.17, ψ induces the group completion map (1.20) on homology (since all $\mathcal{O}$-algebras are in particular homotopy commutative monoids), and the usual map $\Omega B\underline{\mathcal{O}} \to \mathcal{G}\left(\Omega B\underline{\mathcal{O}}\right)$ is an equivalence because $\Omega B\underline{\mathcal{O}}$ is a grouplike monoid. Likewise,

$O(0) \to |Z_\bullet| \to |\mathcal{G}\, Z_\bullet|$ induces the group completion map (1.20) on homology. Since ring localisations are unique, it follows that the map

$$|\mathcal{G}\, Z_\bullet| \xrightarrow{|\mathcal{G}\, r_\bullet^t|} \left|\mathcal{G}\, t^*\left(\widetilde{\Phi}\right)\right| \xrightarrow{|\mathcal{G}\, t^*(q)|} |\mathcal{G}\, Y_\bullet| \to \mathcal{G}\left(\Omega B \underline{O}\right)$$

induces an isomorphism on homology. But the second and third maps are weak equivalences by construction. Thus $\left|\mathcal{G}\, r_\bullet^t\right|$ is a homology equivalence. By Whitehead's theorem for simple spaces, we conclude that it is a weak equivalence. □

1.4 Proof of Theorem 1.2.26

In this section we show that for a 'sufficiently nice' (pointed) operad O and O-algebra X, the bar resolutions

$$\left|B_\bullet\left(F^O, \mathbb{O}, X\right)\right| \qquad \text{and} \qquad \left|B_\bullet\left(F^O_*, \mathbb{O}_*, X\right)\right|$$

are cofibrant replacements of X in the model structure on O–Alg of Theorem 1.2.13.

We work in a closed symmetric monoidal category with a compatible cofibrantly generated model category structure that satisfies some technical conditions; see Definition 1.4.33. The reader who is not interested in this level of generality will be safe in substituting throughout the category Top of compactly generated weakly Hausdorff spaces with the Cartesian monoidal structure and the usual model structure consisting of weak homotopy equivalences and Serre fibrations.

This section is very closely modelled on the paper of Pereira [44] who analysed the cofibrancy of bar constructions for operads in symmetric spectra. His arguments generalise immediately, but we haven't found a better way of convincing the reader of this without essentially repeating his proof, which explains the length of this section. Our contribution consists of finding a suitable framework for dealing with pointed operads which allows for an easy analysis of the interaction of the algebraic structures with model category structures.

1.4.1 Operads as monoids

Let $(\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$ be a bicomplete closed symmetric monoidal category in the sense of [36], with internal hom-objects denoted by Y^X for $X, Y \in \mathscr{C}$:

$$\mathscr{C}\left(X \otimes Z, Y\right) \cong \mathscr{C}\left(Z, Y^X\right)$$

Since every functor $X \otimes (-) : \mathscr{C} \to \mathscr{C}$ for $X \in \mathscr{C}$ is a left adjoint, it preserves all colimits.

Since $\mathscr{C}$ is cocomplete, it is tensored over Set via the functor

$$\begin{aligned}(-) \cdot (-) : \mathscr{C} \times \mathrm{Set} &\longrightarrow \mathscr{C} \\ (X, S) &\longmapsto X \cdot S \coloneqq \coprod_S X\end{aligned}$$

For any group G, the trivial G-action functor $\mathscr{C} \to \mathscr{C}^G$ admits a left adjoint

$$(-)_G : \mathscr{C}^G \to \mathscr{C},$$

which is given on a G-object — thought of as a functor $G \to \mathscr{C}$ — by its colimit. Given a right G-object $X \in \mathscr{C}^{G^{\mathrm{op}}}$ and a left G-object $Y \in \mathscr{C}^G$, let $X \otimes Y$ be their monoidal product equipped with the diagonal G-action, and write

$$X \otimes_G Y \coloneqq (X \otimes Y)_G .$$

If $Z \in \mathrm{Set}^G$ is a left G-set, we write $X \cdot_G Z \coloneqq (X \cdot Z)_G$.

1.4.1.1 Ordinary collections

One of the standard ways of defining operads in $\mathscr{C}$ is via *collections* (or *symmetric sequences*), *cf.* [33], [22]. We briefly recall the framework here in order to motivate our definition of a pointed collection.

Definition 1.4.1. Let Σ be the category of sets of the form $\mathbf{n} \coloneqq \{1, 2, \ldots, n\}$, and set bijections.

Σ admits a strict symmetric monoidal product coming from the disjoint union of sets: $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$, and an obvious definition of $\sqcup$ on morphisms.

Definition 1.4.2. The category of *collections* in $\mathscr{C}$ is the functor category $\mathrm{Coll}(\mathscr{C}) \coloneqq \mathscr{C}^{\Sigma^{\mathrm{op}}}$.

The monoidal product $\sqcup$ gives rise to a Day convolution product $\otimes$ on $\mathrm{Coll}(\mathscr{C})$ [19]. Explicitly, this product is given by the formula [22, Section 2.1.7]:

$$(\mathcal{F} \otimes \mathcal{G}) : \mathbf{n} \longmapsto \coprod_{n=n_1+n_2} (\mathcal{F}(\mathbf{n_1}) \otimes \mathcal{G}(\mathbf{n_2})) \cdot_{\Sigma_{n_1} \times \Sigma_{n_2}} \Sigma_n$$

This can also be expressed somewhat more transparently in the language of coends [36, Section IX.6], [34, Definition 6.2]:

$$(\mathcal{F} \otimes \mathcal{G}) : \mathbf{n} \longmapsto \int^{(\mathbf{n}_1,\mathbf{n}_2)\in\Sigma\times\Sigma} (\mathcal{F}(\mathbf{n}_1) \otimes \mathcal{G}(\mathbf{n}_2)) \cdot \Sigma(\mathbf{n}, \mathbf{n}_1 \sqcup \mathbf{n}_2)\,.$$

(The benefits of this more abstract language shall be reaped when we introduce pointed collections.)

The monoidal unit of $\otimes$ is given by $\mathbb{1}_{\otimes} : \mathbf{n} \mapsto \mathbb{1}_{\mathscr{C}} \cdot \Sigma(\mathbf{n}, \mathbf{0})$. Explicitly, this is the collection

$$\mathbf{n} \longmapsto \begin{cases} \mathbb{1} & \text{if } n = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

where $\varnothing \in \mathscr{C}$ is an initial object, equipped with the unique Σ_n-action.

Definition 1.4.3 (*cf. e.g.* [22, Section 2.2.2])**.** The *composition product* $\bar{\circ}$ on $\mathrm{Coll}(\mathscr{C})$ is defined by the formula

$$\mathcal{F} \,\bar{\circ}\, \mathcal{G} : \mathbf{n} \longmapsto \coprod_{k\geqslant 0} \mathcal{F}(\mathbf{k}) \otimes_{\Sigma_k} \left(\mathcal{G}^{\otimes k}(\mathbf{n})\right)$$

where Σ_k acts on the functor $\mathcal{G}^{\otimes k} : \Sigma^{\mathrm{op}} \to \mathscr{C}$ by permuting the factors.

The composition product makes $\mathrm{Coll}(\mathscr{C})$ into a non-symmetric monoidal category with unit $\mathbb{1}_{\bar{\circ}} : \mathbf{n} \mapsto \mathbb{1}_{\mathscr{C}} \cdot \Sigma(\mathbf{n}, \mathbf{1})$.

It can again be expressed using coends (in the category $\mathrm{Coll}(\mathscr{C})$) as follows:

$$\mathcal{F} \,\bar{\circ}\, \mathcal{G} = \int^{\mathbf{k}\in\Sigma} \mathcal{F}(\mathbf{k}) \otimes \mathcal{G}^{\otimes \mathbf{k}}$$

where $\mathcal{G}^{(-)} : \Sigma \to \mathrm{Coll}(\mathscr{C})$ is the evident functor defined using the symmetry of the Day convolution product on $\mathrm{Coll}(\mathscr{C})$.

Now we can make the following very succinct definition.

Definition 1.4.4. An operad $\mathcal{O}$ in $\mathscr{C}$ is a monoid object in $(\mathrm{Coll}(\mathscr{C}), \bar{\circ}, \mathbb{1}_{\bar{\circ}})$. An $\mathcal{O}$-algebra in $\mathscr{C}$ is a left $\mathcal{O}$-module in $(\mathrm{Coll}(\mathscr{C}), \bar{\circ}, \mathbb{1}_{\bar{\circ}})$ which is concentrated in degree 0.[3]

When $(\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$ is the category Top with the Cartesian monoidal structure, this definition agrees with Definitions 1.2.1 and 1.2.3.

[3] A collection $\mathcal{F}$ is *concentrated in degree* 0 if $\mathcal{F}(\mathbf{n}) = \varnothing$ for $n \neq 0$

1.4.1.2 Pointed collections

Definition 1.4.5. Let $\mathbb{I}$ be the category of sets of the form $\mathbf{n} = \{1, 2, \ldots, n\}$ for $n \geqslant 0$, and set injections. This is a strict symmetric monoidal category under the 'disjoint union' functor $\sqcup$, with unit the empty set $\mathbf{0}$.

Definition 1.4.6. The category of *contracting collections* in $\mathscr{C}$ is the functor category $\mathrm{CColl}(\mathscr{C}) \coloneqq \mathscr{C}^{\mathbb{I}^{\mathrm{op}}}$. This is a symmetric monoidal category when equipped with the Day convolution product:

$$(\mathcal{F} \otimes \mathcal{G}) : \mathbf{n} \longmapsto \int^{(\mathbf{n}_1, \mathbf{n}_2) \in \mathbb{I} \times \mathbb{I}} (\mathcal{F}(\mathbf{n}_1) \otimes \mathcal{G}(\mathbf{n}_2)) \cdot \mathbb{I}(\mathbf{n}, \mathbf{n}_1 \sqcup \mathbf{n}_2)\,. \tag{1.22}$$

The monoidal unit is given by $\mathbb{1}_{\otimes} \colon \mathbf{n} \mapsto \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(\mathbf{n}, \mathbf{0})$.

Remark 1.4.7. This definition is inspired by the concept of *preoperads* of Berger [4], which in turn are a modification of *coefficient systems* of Cohen, May and Taylor [18].

The following is well known; see *e.g.* [34, Proof of 6.2].

Lemma 1.4.8. *The Day convolution product on* $\mathrm{CColl}(\mathscr{C})$ *is closed.*

Remark 1.4.9. Note that the functor

$$\begin{aligned} \mathscr{C} &\longrightarrow \mathrm{CColl}(\mathscr{C}) \\ X &\longmapsto (\mathbf{n} \mapsto X \cdot \mathbb{I}(\mathbf{n}, \mathbf{0})) \end{aligned}$$

is symmetric monoidal with respect to the convolution product, making $\mathrm{CColl}(\mathscr{C})$ tensored over $\mathscr{C}$. Moreover we have

$$\mathcal{F} \otimes (\mathbf{n} \mapsto X \cdot \mathbb{I}(\mathbf{n}, \mathbf{0})) \cong (\mathbf{n} \mapsto \mathcal{F}(\mathbf{n}) \otimes X)\,.$$

for $\mathcal{F} \in \mathrm{CColl}(\mathscr{C})$, so this tensoring is 'pointwise'.

Definition 1.4.10. The category of *pointed collections* is the under-category

$$\mathrm{Coll}_*(\mathscr{C}) \coloneqq \mathrm{CColl}(\mathscr{C})_{\mathbb{1}_{\otimes}/}$$

of morphisms $\mathbb{1}_{\otimes} \to \mathcal{F}$, and commutative triangles.

Remark 1.4.11. While technically the objects of $\mathrm{Coll}_*(\mathscr{C})$ are morphisms of the form $p \colon \mathbb{1}_{\otimes} \to \mathcal{F}$ in $\mathrm{CColl}(\mathscr{C})$, we shall generally refer to them by their underlying collapsing collection $\mathcal{F}$ and leave the 'basepoint morphism' p implicit.

Remark 1.4.12. The canonical comonoid structure on $\mathbb{1}_\otimes$ allows us to lift $\otimes$ to a symmetric monoidal product on $\mathrm{Coll}_*(\mathscr{C})$, with the product of $p\colon \mathbb{1}_\otimes \to \mathcal{F}$ with $q\colon \mathbb{1}_\otimes \to \mathcal{G}$ being the pointed collection

$$\mathbb{1}_\otimes \cong \mathbb{1}_\otimes \otimes \mathbb{1}_\otimes \xrightarrow{p\otimes q} \mathcal{F} \otimes \mathcal{G}.$$

In analogy to Remark 1.4.9, the functor

$$\begin{aligned} \mathscr{C}_{\mathbb{1}_\mathscr{C}/} &\longrightarrow \mathrm{Coll}_*(\mathscr{C}) \\ (\mathbb{1}_\mathscr{C} \to X) &\longmapsto (\mathbb{1}_\mathscr{C} \cdot \mathbb{I}(-,\mathbf{0}) \to X \cdot \mathbb{I}(-,\mathbf{0})) \end{aligned}$$

is symmetric monoidal.

Definition 1.4.13. Given a pointed collection $p\colon \mathbb{1}_\otimes \to \mathcal{F}$, let $Z(\mathcal{F})\colon \mathbb{I} \to \mathrm{CColl}(\mathscr{C})$ be the functor defined by $\mathbf{n} \mapsto \mathcal{F}^{\otimes n}$ on objects and by the composites

$$\mathcal{F}^{\otimes n} \xrightarrow{\simeq} \mathcal{F}^{\otimes n} \otimes \mathbb{1}_\otimes^{\otimes m-n} \xrightarrow{\mathrm{id}\otimes p^{\otimes m-n}} \mathcal{F}^{\otimes m}$$

on the inclusions $\mathbf{n} \hookrightarrow \mathbf{m}$ for $n \leqslant m$. Automorphisms of $\mathbf{n}$ act on $\mathcal{F}^{\otimes n}$ in the evident way.

Remark 1.4.14. The functor $Z(\mathcal{F})$ is the unique (up to equivalence) symmetric monoidal functor $(\mathbb{I}, \sqcup, \mathbf{0}) \to (\mathrm{CColl}(\mathscr{C}), \otimes, \mathbb{1}_\otimes)$ sending the unique morphism $\mathbf{0} \to \mathbf{1}$ to the morphism $\mathbb{1}_\otimes \to \mathcal{F}$.

Definition 1.4.15. The *composition product* $\circ$ on $\mathrm{Coll}_*(\mathscr{C})$ is defined by the formula

$$\mathcal{F} \circ \mathcal{G} \coloneqq \int^{\mathbf{k}\in\mathbb{I}} \mathcal{F}(\mathbf{k}) \otimes Z(\mathcal{G})(\mathbf{k}). \tag{1.23}$$

Before we show that the composition product makes $\mathrm{Coll}_*(\mathscr{C})$ into a (non-symmetric) monoidal category, we make the following observations.

Lemma 1.4.16. *Let $\mathcal{F}$, $\mathcal{G}$ be pointed collections.*

(i) *If $\mathcal{F}$ is concentrated in degree* 0, i.e. $\mathcal{F}(\mathbf{n}) \cong \mathcal{F}(\mathbf{0}) \cdot \mathbb{I}(\mathbf{n}, \mathbf{0})$, *then*

$$\mathcal{F} \circ \mathcal{G} \cong \mathcal{F}.$$

(ii) *There is a natural (in $(\mathbf{k}, \mathbf{n}) \in \mathbb{I} \times \mathbb{I}^{\mathrm{op}}$ and $\mathcal{F}, \mathcal{G} \in \mathrm{Coll}_*(\mathscr{C})$) isomorphism*

$$Z(\mathcal{F} \circ \mathcal{G})(\mathbf{k}, \mathbf{n}) \cong \int^{\mathbf{m}\in\mathbb{I}} Z(\mathcal{F})(\mathbf{k}, \mathbf{m}) \otimes Z(\mathcal{G})(\mathbf{m}, \mathbf{n}). \tag{1.24}$$

Proof. (i) By properties of coends,

$$\begin{aligned}\mathcal{F} \circ \mathcal{G} &\cong \int^{\mathbf{k}\in\mathbb{I}} (\mathcal{F}(\mathbf{0}) \cdot \mathbb{I}(\mathbf{k}, \mathbf{0})) \otimes Z(\mathcal{G})(\mathbf{k}) \\ &\cong \int^{\mathbf{k}\in\mathbb{I}} (\mathcal{F}(\mathbf{0}) \otimes Z(\mathcal{G})(\mathbf{k})) \cdot \mathbb{I}(\mathbf{k}, \mathbf{0}) \\ &\cong \mathcal{F}(\mathbf{0}) \otimes Z(\mathcal{G})(\mathbf{0}).\end{aligned}$$

But $Z(\mathcal{G})(\mathbf{0}) = \mathbb{1}_\otimes$, and the result follows.

(ii) By the first part, $\mathcal{F} \circ \mathcal{G}$ is naturally pointed by the morphism

$$\mathbb{1}_\otimes \cong \mathbb{1}_\otimes \circ \mathbb{1}_\otimes \longrightarrow \mathcal{F} \circ \mathcal{G},$$

and so $\circ$ does define a functor $\mathrm{Coll}_*(\mathscr{C}) \times \mathrm{Coll}_*(\mathscr{C}) \to \mathrm{Coll}_*(\mathscr{C})$. Hence the left-hand side of (1.24) is well defined. In light of Remark 1.4.14, it suffices to check that the functor

$$\begin{aligned} X \colon \mathbb{I} &\longrightarrow \mathrm{CColl}(\mathscr{C}) \\ \mathbf{k} &\longmapsto \int^{\mathbf{m}\in\mathbb{I}} Z(\mathcal{F})(\mathbf{k}, \mathbf{m}) \otimes Z(\mathcal{G})(\mathbf{m}) \end{aligned}$$

is symmetric monoidal and that it agrees with $\mathbf{k} \mapsto Z(\mathcal{F} \circ \mathcal{G})(\mathbf{k})$ on the morphism $\mathbf{0} \to \mathbf{1}$ of $\mathbb{I}$. The second claim is immediate, since $Z(\mathcal{F})$ takes this morphism to the basepoint morphism $\mathbb{1}_\otimes \to \mathcal{F}$ by definition.

To make the notation slightly less unwieldy, we shall use the Einstein notation, writing covariant functor arguments as subscripts and contravariant arguments as superscripts, so for instance $\mathbb{I}^{\mathbf{m}}_{\mathbf{m}_1 \sqcup \mathbf{m}_2} \coloneqq \mathbb{I}(\mathbf{m}, \mathbf{m}_1 \sqcup \mathbf{m}_2)$. With this convention, we have the following sequence of natural isomorphisms.

$$\begin{aligned} X(\mathbf{k}_1 \sqcup \mathbf{k}_2) &\cong \int^{\mathbf{m}\in\mathbb{I}} \left(Z(\mathcal{F})_{\mathbf{k}_1} \otimes Z(\mathcal{F})_{\mathbf{k}_2}\right)^{\mathbf{m}} \otimes Z(\mathcal{G})_{\mathbf{m}} \\ &\cong \int^{\mathbf{m}\in\mathbb{I}} \left(\int^{(\mathbf{m}_1,\mathbf{m}_2)\in\mathbb{I}\times\mathbb{I}} Z(\mathcal{F})^{\mathbf{m}_1}_{\mathbf{k}_1} \otimes Z(\mathcal{F})^{\mathbf{m}_2}_{\mathbf{k}_2} \cdot \mathbb{I}^{\mathbf{m}}_{\mathbf{m}_1 \sqcup \mathbf{m}_2} \right) \otimes Z(\mathcal{G})_{\mathbf{m}} \\ &\cong \int^{(\mathbf{m}_1,\mathbf{m}_2)\in\mathbb{I}\times\mathbb{I}} Z(\mathcal{F})^{\mathbf{m}_1}_{\mathbf{k}_1} \otimes Z(\mathcal{F})^{\mathbf{m}_2}_{\mathbf{k}_2} \otimes \left(\int^{\mathbf{m}\in\mathbb{I}} Z(\mathcal{G})_{\mathbf{m}} \cdot \mathbb{I}^{\mathbf{m}}_{\mathbf{m}_1 \sqcup \mathbf{m}_2} \right) \\ &\cong \int^{(\mathbf{m}_1,\mathbf{m}_2)\in\mathbb{I}\times\mathbb{I}} Z(\mathcal{F})^{\mathbf{m}_1}_{\mathbf{k}_1} \otimes Z(\mathcal{F})^{\mathbf{m}_2}_{\mathbf{k}_2} \otimes Z(\mathcal{G})_{\mathbf{m}_1 \sqcup \mathbf{m}_2} \\ &\cong \left(\int^{\mathbf{m}_1\in\mathbb{I}} Z(\mathcal{F})^{\mathbf{m}_1}_{\mathbf{k}_1} \otimes Z(\mathcal{G})_{\mathbf{m}_1} \right) \otimes \left(\int^{\mathbf{m}_2\in\mathbb{I}} Z(\mathcal{F})^{\mathbf{m}_2}_{\mathbf{k}_2} \otimes Z(\mathcal{G})_{\mathbf{m}_2} \right) \\ &\cong X(\mathbf{k}_1) \otimes X(\mathbf{k}_2) \end{aligned}$$

Hence X is symmetric monoidal, as required.

□

Proposition 1.4.17. *The composition product makes* $\mathrm{Coll}_*(\mathscr{C})$ *into a monoidal category with monoidal unit* $\mathbb{1}_\circ\colon \mathbf{n} \mapsto \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(\mathbf{n}, \mathbf{1})$ *(pointed by the morphism* $\mathbb{1}_\otimes = \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(-, \mathbf{0}) \to \mathbb{1}_\circ$ *induced by the unique map* $\mathbf{0} \to \mathbf{1}$ *in* $\mathbb{I}$*).*

Proof. Associativity follows from Lemma 1.4.16(ii):

$$\begin{aligned}
\mathcal{F} \circ (\mathcal{G} \circ \mathcal{H}) &= \int^{\mathbf{k}\in\mathbb{I}} \mathcal{F}(\mathbf{k}) \otimes Z(\mathcal{G} \circ \mathcal{H})(\mathbf{k}) \\
&\cong \int^{\mathbf{k},\mathbf{m}\in\mathbb{I}} \mathcal{F}(\mathbf{k}) \otimes Z(\mathcal{G})(\mathbf{k}, \mathbf{m}) \otimes Z(\mathcal{H})(\mathbf{m}) \\
&\cong \int^{\mathbf{m}\in\mathbb{I}} (\mathcal{F} \circ \mathcal{G})(\mathbf{m}) \otimes Z(\mathcal{H})(\mathbf{m}) \\
&= (\mathcal{F} \circ \mathcal{G}) \circ \mathcal{H}
\end{aligned}$$

Note that $Z(\mathbb{1}_\circ)(\mathbf{k}, \mathbf{n}) = \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(\mathbf{n}, \mathbf{k})$, and hence

$$\mathcal{F} \circ \mathbb{1}_\circ \cong \int^{\mathbf{k}\in\mathbb{I}} \mathcal{F}(\mathbf{k}) \cdot \mathbb{I}(-, \mathbf{k}) \cong \mathcal{F}.$$

Also

$$\mathbb{1}_\circ \circ \mathcal{F} \cong \int^{\mathbf{k}\in\mathbb{I}} Z(\mathcal{F})(\mathbf{k}) \cdot \mathbb{I}(\mathbf{k}, \mathbf{1}) \cong Z(\mathcal{F})(\mathbf{1}) \cong \mathcal{F}.$$

The monoidal axioms are inherited from $(\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$. □

The following proposition is the pointed version of a well-known property of the usual composition product.

Proposition 1.4.18. *The composition product preserves colimits in the first variable, and sifted colimits in the second variable.*

Proof. Let $\mathcal{F}\colon \mathscr{I} \to \mathrm{Coll}_*(\mathscr{C})$ be a diagram, and let $\mathscr{I}_+$ be the category $\mathscr{I}$ with a freely adjoined initial object i. The diagram $\mathcal{F}$ naturally gives rise to a diagram $\mathcal{F}_+\colon \mathscr{I}_+ \to \mathrm{CColl}(\mathscr{C})$ which sends $i \in \mathscr{I}_+$ to $\mathbb{1}_\otimes$, and the colimit of $\mathcal{F}$ in $\mathrm{Coll}_*(\mathscr{C})$ can be computed by finding the colimit of $\mathcal{F}_+$ in $\mathrm{CColl}(\mathscr{C})$ and pointing it by the universal map from $\mathcal{F}_+(i) = \mathbb{1}_\otimes$. But colimits in $\mathrm{CColl}(\mathscr{C})$ are computed pointwise, and the first claim follows by inspection of (1.23).

The second claim is established analogously, once we observe that the functor

$$Z\colon \mathrm{Coll}_*(\mathscr{C}) \to \mathrm{CColl}(\mathscr{C})^{\mathbb{I}}$$

preserves sifted colimits. □

Remark 1.4.19. It is relatively straightforward to show that the structure of a contracting collection is a right $*$-module structure in the category of ordinary collections equipped with the usual composition product, where $*$ is the initial pointed operad: $*(\mathbf{n}) = \mathbb{1}_{\mathscr{C}}$ if $n = 0, 1$, and $*(\mathbf{n}) = \varnothing$ otherwise. Likewise, basepoints correspond to left $*$-module structures, so $\mathrm{Coll}_*(\mathscr{C})$ is isomorphic to the category of $*$-bimodules in $\mathrm{Coll}(\mathscr{C})$. There is also a way of constructing the pointed composition product in this context: if $\mathcal{F}$ and $\mathcal{G}$ are pointed collections viewed as $*$-bimodules, then their composition product fits into the reflexive coequaliser diagram

$$\mathcal{F} \circ * \circ \mathcal{G} \rightrightarrows \mathcal{F} \circ \mathcal{G} \longrightarrow \mathcal{F} \circ \mathcal{G}$$

in $*$-bimodules, where $\circ$ is the composition product of ordinary collections, the two parallel arrows are defined using the right action of $*$ on $\mathcal{F}$ and left action of $*$ on $\mathcal{G}$, and their simultaneous splitting comes from the unit map $\mathbb{1}_{\circ} \to *$. We have nonetheless opted for the present setup, because we believe that it makes the analysis of the interaction of the monoidal structure with a model category structure somewhat easier.

1.4.1.3 Iterated collections

Since $(\mathrm{CColl}(\mathscr{C}), \otimes, \mathbb{1}_{\otimes})$ is a bicomplete closed symmetric monoidal category, we can consider contracting collections valued in this category.

Definition 1.4.20. The category of *double (contracting) collections* is denoted by

$$2\mathrm{Coll}(\mathscr{C}) \coloneqq \mathrm{CColl}(\mathrm{CColl}(\mathscr{C})).$$

The category of *double pointed collections* is denoted by

$$2\mathrm{Coll}_*(\mathscr{C}) \coloneqq \mathrm{Coll}_*(\mathrm{CColl}(\mathscr{C})).$$

As before, $2\mathrm{Coll}(\mathscr{C}) = \mathscr{C}^{\mathbb{I}^{\mathrm{op}} \times \mathbb{I}^{\mathrm{op}}}$ is a bicomplete closed symmetric monoidal category when equipped with the Day convolution product $\widehat{\otimes}$, which takes the form

$$\left(\mathcal{F} \mathbin{\widehat{\otimes}} \mathcal{G}\right)(\mathbf{m}, \mathbf{n}) = \int^{\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{I}} \mathcal{F}(\mathbf{m}_1, \mathbf{n}_1) \otimes \mathcal{G}(\mathbf{m}_2, \mathbf{n}_2) \cdot \mathbb{I}(\mathbf{m}, \mathbf{m}_1 \sqcup \mathbf{m}_2) \times \mathbb{I}(\mathbf{n}, \mathbf{n}_1 \sqcup \mathbf{n}_2)$$

The monoidal unit for $\widehat{\otimes}$ shall be denoted $\mathbb{1}_{\widehat{\otimes}} \colon (\mathbf{m}, \mathbf{n}) \mapsto \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(\mathbf{m}, \mathbf{0}) \times \mathbb{I}(\mathbf{n}, \mathbf{0})$.

Similarly, $2\mathrm{Coll}_*(\mathscr{C}) = 2\mathrm{Coll}(\mathscr{C})_{\mathbb{1}_{\widehat{\otimes}}/}$ is a monoidal category with respect to the composition product $\widehat{\circ}$.

In analogy to Remark 1.4.12, we have a functor $\mathrm{Coll}_*(\mathscr{C}) \to 2\mathrm{Coll}_*(\mathscr{C})$ which is symmetric monoidal with respect to the convolution products. However, in this setting we obtain an extra functor.

Definition 1.4.21. The *vertical inclusion functor* $(-)^v : \mathrm{Coll}_*(\mathscr{C}) \to 2\mathrm{Coll}_*(\mathscr{C})$ is given by

$$\mathcal{F}^v(\mathbf{m}, \mathbf{n}) = \mathcal{F}(\mathbf{n}) \cdot \mathbb{I}(\mathbf{m}, \mathbf{0}).$$

The *horizontal inclusion functor* $(-)^h : \mathrm{Coll}_*(\mathscr{C}) \to 2\mathrm{Coll}_*(\mathscr{C})$ is given by

$$\mathcal{F}^h(\mathbf{m}, \mathbf{n}) = \mathcal{F}(\mathbf{m}) \cdot \mathbb{I}(\mathbf{n}, \mathbf{0}).$$

It is easy to see that both these inclusions are symmetric monoidal with respect to the convolution products. We also have the following result.

Lemma 1.4.22. *The horizontal inclusion functor is monoidal with respect to the composition products. There is a natural isomorphism*

$$\mathcal{F}^h \mathbin{\widehat{\circ}} \mathcal{G}^v \cong (\mathcal{F} \circ \mathcal{G})^v .$$

Proof. Since the horizontal and vertical inclusions are symmetric monoidal with respect to the convolution products, there are natural isomorphisms

$$Z(\mathcal{F}^{h/v})(\mathbf{k}) \cong (Z(\mathcal{F})(\mathbf{k}))^{h/v}$$

where 'h/v' indicates a consistent choice of either h or v throughout. Hence we have

$$\begin{aligned} \mathcal{F}^h \mathbin{\widehat{\circ}} \mathcal{G}^{h/v} &\cong \int^{\mathbf{k} \in \mathbb{I}} \mathcal{F}^h(\mathbf{k}) \otimes (Z(\mathcal{G})(\mathbf{k}))^{h/v} \\ &\cong \int^{\mathbf{k} \in \mathbb{I}} (\mathcal{F}(\mathbf{k}) \otimes Z(\mathcal{G})(\mathbf{k}))^{h/v} \\ &\cong (\mathcal{F} \circ \mathcal{G})^{h/v} \end{aligned}$$

since each $\mathcal{F}^h(\mathbf{k}) \in \mathrm{CColl}(\mathscr{C})$ is concentrated in degree 0. □

1.4.2 Exquisite categories

Let $\mathscr{C}$ be a cofibrantly generated model category, with generating cofibrations I and generating acyclic cofibrations J.

1.4.2.1 G-equivariant model structures

Let G be an abstract group. We recalled in Section 1.2.2.1 that $\mathscr{C}^G$ admits the projective model structure, where a morphism is a weak equivalence or a fibration if its underlying morphism in $\mathscr{C}$ is a weak equivalence or a fibration. Under some conditions on $\mathscr{C}$, $\mathscr{C}^G$ admits another model structure, akin to the model structure of G-equivariant homotopy theory on Top^G. The following is after [25].

Given a subgroup $H \leqslant G$, G/H is a left G-set, so the functor $(-) \cdot G/H \colon \mathscr{C} \to \mathscr{C}$ naturally factors through $\mathscr{C}^G$.

Definition 1.4.23. The right adjoint of $(-)\cdot G/H \colon \mathscr{C} \to \mathscr{C}^G$ is denoted by $(-)^H : \mathscr{C}^G \to \mathscr{C}$ and called the *H-fixed point functor*.

Note that for $X \in \mathscr{C}^G$, X^H is the limit of the diagram $H \hookrightarrow G \xrightarrow{X} \mathscr{C}$. Since $\mathscr{C}$ is complete, fixed point functors always exist.

Definition 1.4.24 ([25, Definition 3.7])**.** A fixed point functor $(-)^H$ is *cellular* if

(i) it preserves pushouts along morphisms of the form

$$f \cdot G/K \colon X \cdot G/K \longrightarrow Y \cdot G/K$$

where $K \leqslant G$ and $f \colon X \to Y$ is a morphism in $\mathscr{C}^G$ such that X and Y are trivial H-objects and the underlying morphism f in $\mathscr{C}$ is a cofibration, and

(ii) it preserves filtered colimits in $\mathscr{C}^G$ where each morphism is a cofibration in $\mathscr{C}$.

We say *$\mathscr{C}^G$ has cellular fixed point functors* if $(-)^H : \mathscr{C}^G \to \mathscr{C}$ is cellular for each $H \leqslant G$.

Theorem 1.4.25 ([25, Theorem 3.12])**.** *If $\mathscr{C}^G$ has cellular fixed point functors, then $\mathscr{C}^G$ admits a cofibrantly generated model structure, called the* G-equivariant model structure*, with generating cofibrations $\{i \cdot G/H\}_{i\in I, H\leqslant G}$ and acyclic cofibrations $\{j \cdot G/H\}_{j\in J, H\leqslant G}$.*

1.4.2.2 Generalised Reedy model structures

In [6], Berger and Moerdijk present a generalisation of the notion of Reedy category which allows for non-trivial automorphisms. We shall not recall their exact statements here. Instead we remark that the category $\mathbb{I}^{\mathrm{op}}$ is a generalised Reedy category in the sense of [6, Definition 1.1], with $(\mathbb{I}^{\mathrm{op}})^- = \mathbb{I}^{\mathrm{op}}$ and $(\mathbb{I}^{\mathrm{op}})^+$ being the maximal subgroupoid of $\mathbb{I}^{\mathrm{op}}$. Thus we obtain the following consequence of [6, Theorem 1.6].

Theorem 1.4.26. *The category* $\mathrm{CColl}(\mathscr{C}) = \mathscr{C}^{\mathbb{I}^{\mathrm{op}}}$ *admits a model structure where a morphism* $\mathcal{F} \to \mathcal{G}$ *is an (acyclic) cofibration if and only if* $\mathcal{F}(\mathbf{n}) \to \mathcal{G}(\mathbf{n})$ *is an (acyclic) cofibration in* $\mathscr{C}^{\Sigma_n^{\mathrm{op}}}$ *with the projective model structure for all* $\mathbf{n} \in \mathbb{I}^{\mathrm{op}}$.

We claim that this model structure is cofibrantly generated.

Definition 1.4.27.

(i) For $n \geqslant 0$, let $\mathbb{I}_{\mathbf{n}} \coloneqq \mathbb{I}(-, \mathbf{n}) : \mathbb{I}^{\mathrm{op}} \to \mathrm{Set}$. For $n, m \geqslant 0$, let $\mathbb{I}_{(\mathbf{n}),\mathbf{m}}$ be the subfunctor of $\mathbb{I}_{\mathbf{n} \sqcup \mathbf{m}}$ given by

$$\mathbb{I}_{(\mathbf{n}),\mathbf{m}}(\mathbf{k}) = \left\{ f \in \mathbb{I}_{\mathbf{n}(\mathbf{k}) \sqcup \mathbf{m}} \mid \mathbf{n} \nsubseteq \operatorname{im} f \right\}$$

We write $\mathbb{I}_{(\mathbf{n})} \coloneqq \mathbb{I}_{(\mathbf{n}),\mathbf{0}}$. We shall abuse notation slightly and also use the symbol $\mathbb{I}_{\mathbf{n}}$ for the functor $\mathbf{k} \mapsto \mathbb{1}_{\mathscr{C}} \cdot \mathbb{I}(\mathbf{k}, \mathbf{m}) \colon \mathbb{I}^{\mathrm{op}} \to \mathrm{CColl}(\mathscr{C})$, *etc.*

(ii) For $f : X \to Y$ in $\mathscr{C}$ and $n \geqslant 0$, let $P_n(f)$ be the universal morphism in $\mathrm{CColl}(\mathscr{C})$

$$P_n(f) \colon X \cdot \mathbb{I}_{\mathbf{n}} \cup_{X \cdot \mathbb{I}_{(\mathbf{n})}} Y \cdot \mathbb{I}_{(\mathbf{n})} \longrightarrow Y \cdot \mathbb{I}_{\mathbf{n}}$$

induced by f and the inclusion $\mathbb{I}_{(\mathbf{n})} \hookrightarrow \mathbb{I}_{\mathbf{n}}$.

Remark 1.4.28. Note that $P_n(f)(\mathbf{k})$ is an isomorphism in $\mathscr{C}$ for all $k \neq n$, while

$$P_n(f)(\mathbf{n}) = \widehat{f_n}(\mathbf{n}) = f \cdot \Sigma_n \colon X \cdot \Sigma_n \to Y \cdot \Sigma_n.$$

Hence if f is a cofibration in $\mathscr{C}$, then $P_n(f)$ is a cofibration in the model structure of Theorem 1.4.26.

Lemma 1.4.29. *Let* $f : \mathcal{F} \to \mathcal{G}$ *be a morphism in* $\mathrm{CColl}(\mathscr{C})$ *and suppose that for each* $\mathbf{n} \in \mathbb{I}$, $f(\mathbf{n})$ *is a retract in* $\mathscr{C}^{\Sigma_n}$ *of a morphism* $\widetilde{f_n} \colon \widetilde{\mathcal{F}_n} \to \widetilde{\mathcal{G}_n}$. *Then there is a morphism* $\widetilde{f} \colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ *in* $\mathrm{CColl}(\mathscr{C})$ *such that* $\widetilde{f}(\mathbf{n}) = \widetilde{f_n}$, *and* f *is a retract of* $\widetilde{f}$.

Proof. Suppose that the retractions in $\mathscr{C}^{\Sigma_n}$ are witnessed by the commutative diagrams

$$\begin{array}{ccccc}
 & & \overset{\mathrm{id}}{\frown} & & \\
\mathcal{F}(\mathbf{n}) & \xrightarrow{\iota_n} & \widetilde{\mathcal{F}}_n & \xrightarrow{\pi_n} & \mathcal{F}(\mathbf{n}) \\
{\scriptstyle f(\mathbf{n})}\downarrow & & {\scriptstyle \widetilde{f}_n}\downarrow & & {\scriptstyle f(\mathbf{n})}\downarrow \\
\mathcal{G}(\mathbf{n}) & \xrightarrow{\iota'_n} & \widetilde{\mathcal{G}}_n & \xrightarrow{\pi'_n} & \mathcal{G}(\mathbf{n}) \\
 & & \underset{\mathrm{id}}{\smile} & &
\end{array}$$

Let $\widetilde{\mathcal{F}} \in \mathrm{CColl}(\mathscr{C})$ be given by $\mathbf{n} \mapsto \widetilde{\mathcal{F}}_n$ on objects, and by

$$\widetilde{\mathcal{F}}(\phi\colon \mathbf{k} \to \mathbf{n}) = \begin{cases} \phi^* & \text{if } k = n \\ \iota_k \mathcal{F}(\phi)\pi_n & \text{if } k < n \end{cases}$$

on morphisms, where $\phi^*\colon \widetilde{\mathcal{F}}_n \to \widetilde{\mathcal{F}}_n$ is the right action of $\phi \in \Sigma_n$ on $\widetilde{\mathcal{F}}_n$. Let $\widetilde{\mathcal{G}}$ be defined analogously. Then $\iota_n, \pi_n, \iota'_n, \pi'_n$ and $\widetilde{f}_n$ all give rise to morphisms in $\mathrm{CColl}(\mathscr{C})$ which exhibit f as a retract of $\widetilde{f}$. □

Proposition 1.4.30. *The model structure on* $\mathrm{CColl}(\mathscr{C})$ *of Theorem 1.4.26 is cofibrantly generated with generating cofibrations* $\{P_n(i)\}_{n\geqslant 0, i\in I}$ *and generating acyclic cofibrations* $\{P_n(j)\}_{n\geqslant 0, j\in J}$.

Proof. It is evident from Remark 1.4.28 that transfinite compositions of pushouts of morphisms of the form $P_n(i)$ for $n \geqslant 0$ and $i \in I$ are precisely those morphisms $f\colon \mathcal{F} \to \mathcal{G}$ such that each $f(\mathbf{n})$ is a cellular Σ_n-projective cofibration. But by the preceding lemma, every cofibration in $\mathrm{CColl}(\mathscr{C})$ is a retract of such a morphism f. The argument for acyclic cofibrations is identical. □

1.4.2.3 Exquisite categories

Suppose that $\mathscr{C}$ is a cofibrantly generated model category and also a closed symmetric monoidal category with monoidal product $\otimes$ and unit $\mathbb{1}_{\mathscr{C}}$.

Definition 1.4.31. Let $\mathbb{2}$ be the category of shape $\bullet \to \bullet$. Let $T_n \subseteq \mathbb{2}^n$ be the full subcategory containing all but the terminal object.

Thus $\mathscr{C}^{\mathbb{2}}$ is precisely the arrow category of $\mathscr{C}$.

Definition 1.4.32. Given a morphism $f\colon X \to Y$ in $\mathscr{C}$, let $f^n_\otimes$ be the functor

$$f^n_\otimes\colon 2^n \xrightarrow{f^n} \mathscr{C}^n \xrightarrow{\otimes} \mathscr{C}.$$

The *n-th pushout-power* of f is the morphism

$$f^{\square n}\colon \operatorname*{colim}_{T_n} f^n_\otimes \longrightarrow \operatorname*{colim}_{2^n} f^n_\otimes$$

in $\mathscr{C}^{\Sigma_n}$. We write $Q_n(f) \coloneqq \operatorname{colim}_{T_n} f^n_\otimes$ for the domain of $f^{\square n}$.

We will require pushout-powers of cofibrations to be cofibrant in a suitable sense. Requiring them to be cofibrant in the projective model structure on $\mathscr{C}^{\Sigma_n}$ is too strong, however. Indeed, consider the case $\mathscr{C} = \mathrm{Top}$ and let f be the inclusion of the endpoints into a finite closed line segment. Then $f^{\square 2}$ is the inclusion of the boundary into a square, and the Σ_2-action is by reflecting the square in a diagonal. The complement of the image of $f^{\square 2}$ is not Σ_2-free, so $f^{\square 2}$ has no hope of being projectively cofibrant. It is, however, a Σ_2-equivariant cofibration in the sense of Theorem 1.4.25. This motivates the following definition.

Definition 1.4.33. An *exquisite category* is a cofibrantly generated symmetric monoidal model category $(\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$ such that

(i) $\mathscr{C}^G$ admits cellular fixed point functors for all finite groups G,

(ii) for every generating cofibration i of $\mathscr{C}$ and every $n \geqslant 1$, the n-th pushout-power $i^{\square n}$ is a Σ_n-equivariant cofibration,

(iii) the monoidal unit $\mathbb{1}_{\mathscr{C}}$ is cofibrant.

Remark 1.4.34. We use the term *symmetric monoidal model category* in the sense of [29, Definition 4.2.6]: it is a closed symmetric monoidal model category equipped with a model structure such that the pushout-product

$$f \square g\colon (X \otimes Y') \cup_{X \otimes Y} (X' \otimes Y) \longrightarrow X' \otimes Y'$$

is a cofibration whenever $f\colon X \to X'$ and $g\colon Y \to Y'$ both are, and it is an acyclic cofibration whenever in addition one of f, g is acyclic. Hovey [29] also requires that $q \otimes X\colon Q\mathbb{1} \otimes X \to \mathbb{1} \otimes X$ be a weak equivalence for all cofibrant X, where $q\colon Q\mathbb{1} \to \mathbb{1}$ is the (functorial) cofibrant replacement of the monoidal unit. This condition becomes vacuous in the presence of our condition (iii).

Lemma 1.4.35. *There are natural isomorphisms*

$$P_n(f)\square P_m(g) \cong P_{n+m}(f\square g)$$

of functors $\mathscr{C}^2\times\mathscr{C}^2 \to \mathrm{CColl}(\mathscr{C})^2$*, where the pushout-product on the left-hand side is relative to the convolution product on* $\mathrm{CColl}(\mathscr{C})$*.*

Proof. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. Then $P_n(f) = f\square\left(\mathbb{I}_{(\mathbf{n})} \hookrightarrow \mathbb{I}_{\mathbf{n}}\right)$ is a pushout-product with respect to the convolution product. Since the convolution product is symmetric, we have that

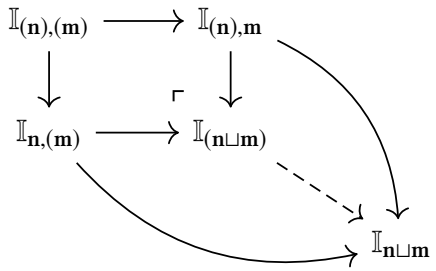
$$P_n(f)\square P_m(g) \cong (f\square g)\square\left(\left(\mathbb{I}_{(\mathbf{n})} \hookrightarrow \mathbb{I}_{\mathbf{n}}\right)\square\left(\mathbb{I}_{(\mathbf{m})} \hookrightarrow \mathbb{I}_{\mathbf{m}}\right)\right)$$

and it suffices to prove that the second factor is the inclusion $\mathbb{I}_{(\mathbf{n}\sqcup\mathbf{m})} \hookrightarrow \mathbb{I}_{\mathbf{n}\sqcup\mathbf{m}}$.

This is an exercise in coend manipulation. Certainly $\mathbb{I}_{\mathbf{n}}\otimes\mathbb{I}_{\mathbf{m}} \cong \mathbb{I}_{\mathbf{n}\sqcup\mathbf{m}}$ by [34, Proposition 2.1]. Likewise,

$$\begin{aligned}\left(\mathbb{I}_{(\mathbf{n})}\otimes\mathbb{I}_{\mathbf{m}}\right)(\mathbf{k}) &= \int^{\mathbf{k}_1,\mathbf{k}_2\in\mathbb{I}^{\mathrm{op}}} \mathbb{I}_{(\mathbf{n})}(\mathbf{k}_1)\times\mathbb{I}(\mathbf{k}_2,\mathbf{m})\times\mathbb{I}(\mathbf{k},\mathbf{k}_1\sqcup\mathbf{k}_2)\\ &\cong \int^{\mathbf{k}_1\in\mathbb{I}^{\mathrm{op}}} \mathbb{I}_{(\mathbf{n})}(\mathbf{k}_1)\times\mathbb{I}(\mathbf{k},\mathbf{k}_1\sqcup\mathbf{m})\\ &\cong \mathbb{I}_{(\mathbf{n}),\mathbf{m}}(\mathbf{k})\end{aligned}$$

where the last isomorphism is given by the composition $\mathbb{I}_{(\mathbf{n})}(\mathbf{k}_1)\times\mathbb{I}(\mathbf{k},\mathbf{k}_1\sqcup\mathbf{m}) \to \mathbb{I}(\mathbf{k},\mathbf{n}\sqcup\mathbf{m})$, which only hits elements of $\mathbb{I}_{(\mathbf{n}),\mathbf{m}}$. By a similar argument, $\mathbb{I}_{(\mathbf{n})}\otimes\mathbb{I}_{(\mathbf{m})} \cong \mathbb{I}_{(\mathbf{n}),(\mathbf{m})}$ is the subfunctor of $\mathbb{I}_{\mathbf{n}\sqcup\mathbf{m}}$ consisting of injections $i\colon\mathbf{k}\hookrightarrow\mathbf{n}\sqcup\mathbf{m}$ such that neither $\mathbf{n}$ nor $\mathbf{m}$ is a subset of the image of i. Hence $\left(\mathbb{I}_{(\mathbf{n})} \hookrightarrow \mathbb{I}_{\mathbf{n}}\right)\square\left(\mathbb{I}_{(\mathbf{m})} \hookrightarrow \mathbb{I}_{\mathbf{m}}\right)$ fits into the square



as the dashed universal map, concluding the proof. □

Proposition 1.4.36. *Let* $(\mathscr{C},\otimes,\mathbb{1}_{\mathscr{C}})$ *be an exquisite category. Then* $(\mathrm{CColl}(\mathscr{C}),\otimes,\mathbb{1}_{\otimes})$ *is also exquisite when equipped with the model structure from Theorem 1.4.26.*

Proof. The Day convolution product is closed, see *e.g.* [34, Proof of 6.2].

To check compatibility with the model structure, we only need to check the pushout-product axiom on generating (acyclic) cofibrations by [29, Corollary 4.2.5] — but that is immediate by the preceding lemma and Proposition 1.4.30.

For each finite group G, $\mathrm{CColl}(\mathscr{C})^G$ clearly admits cellular fixed point functors because $\mathscr{C}^G$ does, and the colimits in Definition 1.4.24 are computed pointwise.

Finally, by comparing generating (acyclic) fibrations, it can be seen that $\mathrm{CColl}(\mathscr{C})^G$ and $\mathrm{CColl}(\mathscr{C}^G)$ with the G-equivariant model structures are isomorphic model categories. Hence for a generating cofibration $P_n(i)$ of $\mathrm{CColl}(\mathscr{C})$, the pushout-power $P_n(i)^{\square k} \cong P_{kn}(i^{\square k})$ is a Σ_k-equivariant cofibration. □

Lemma 1.4.37. *Let $(\mathscr{C}, \otimes, \mathbb{1}_{\mathscr{C}})$ be an exquisite category. Then for every finite group, the functor $\otimes_G \colon \mathscr{C}^{G^{\mathrm{op}}} \times \mathscr{C}^G \to \mathscr{C}$ is a Quillen bifunctor with respect to the G-equivariant model structures on the factors of the domain.*

Proof. For every $X \in \mathscr{C}^{G^{\mathrm{op}}}$, the functor $X \otimes_G (-) : \mathscr{C}^G \to \mathscr{C}$ has a right adjoint, given by $Y \mapsto Y^X$. By [29, Corollary 4.2.5], it suffices to test the pushout-product condition for generating (acyclic) cofibrations. But for morphisms i, j of $\mathscr{C}$ and subgroups $H, K \leqslant G$, the pushout-product of $i \cdot H\backslash G \in \mathscr{C}^{G^{\mathrm{op}}}$ and $j \cdot G/K \in \mathscr{C}^G$ with respect to $\otimes_G$ is $i \square j \cdot H\backslash G/K$, where $i \square j$ is computed in $\mathscr{C}$ and $H\backslash G/K$ denotes the set of double cosets of H and K in G. This morphism is a cofibration whenever both i, j are, and it is acyclic as soon as in addition one of i, j is. □

Definition 1.4.38. We say that a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in a model category has the *pushout corner property* if the induced morphism $B \cup_A C \to D$ is a cofibration.

Proposition 1.4.39. *Let $\mathscr{C}$ be an exquisite category and let $f \colon \mathcal{F} \to \mathcal{F}'$, $g \colon \mathcal{G} \to \mathcal{G}'$ be cofibrations in $\mathrm{Coll}_*(\mathscr{C})$ with $\mathcal{G}$ cofibrant. Then the pushout-product with respect to the composition product*

$$f \square g \colon \mathcal{F} \circ \mathcal{G}' \cup_{\mathcal{F} \circ \mathcal{G}} \mathcal{F}' \circ \mathcal{G} \longrightarrow \mathcal{F}' \circ \mathcal{G}'$$

is a cofibration.

Proof. The proof is virtually identical to that of [44, Theorem 1.1], and we give only a brief sketch.

The functor $(-)^v : \mathrm{Coll}_*(\mathscr{C}) \to 2\mathrm{Coll}_*(\mathscr{C})$ is cocontinuous, so by Lemma 1.4.22,

$$(f \square g)^v \cong f^h \square g^v.$$

The functors $(-)^v$ and $(-)^h$ preserve and reflect cofibrations. Hence by Proposition 1.4.36, we may reduce to the case when $g\colon G \to G'$ is a cofibration between cofibrant objects of $\mathscr{C}_{\mathbb{1}_{\mathscr{C}}/}$.

Since the pushout-product construction preserves retracts, we may assume that g is a relative cell complex, *i.e.* a transfinite composition of pushouts of generating cofibrations. Since the composition product preserves sifted colimits in the second variable, expressing g as a composition of a transfinite sequence $\widetilde{g}\colon \alpha \to \mathscr{C}_{\mathbb{1}_{\mathscr{C}}/}$ for an ordinal α yields the diagram

$$\begin{array}{ccccc}
\mathcal{F} \circ \widetilde{g}(0) & \longrightarrow & \mathcal{F} \circ \widetilde{g}(1) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathcal{F}' \circ \widetilde{g}(0) & \longrightarrow & \mathcal{F}' \circ \widetilde{g}(1) & \longrightarrow & \cdots
\end{array}$$

whose 'boundary square' is precisely the defining square of $f \square g$. The diagram is a projective cofibration of α-diagrams in $\mathscr{C}$ precisely when each small square has the pushout corner property. The transfinite composition functor $\mathscr{C}^\alpha \to \mathscr{C}^{2}$ is a left Quillen adjoint with respect to the projective model structures, and so if the above diagram is a projective cofibration of α-diagrams, then also the boundary square has the pushout corner property, and $f \square g$ is a cofibration.

Hence it suffices to prove the claim in the case when g is a pushout of a generating cofibration. [44, Proposition 5.20] specialises readily to show that if

$$\begin{array}{ccc}
 & & \mathbb{1}_{\mathscr{C}} \\
 & & \downarrow \\
X & \longrightarrow & G \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \longrightarrow & G'
\end{array}$$

is a pushout square in $\mathscr{C}$ (and a choice of a basepoint for G), then for every pointed collection $\mathcal{F}$, the map $\mathcal{F} \circ G \to \mathcal{F} \circ G'$ is a transfinite composition of the form

$$G_0^{\mathcal{F}} \longrightarrow G_1^{\mathcal{F}} \longrightarrow \cdots$$

where $G_0^{\mathcal{F}} \coloneqq \mathcal{F} \circ G$ and the $G_i^{\mathcal{F}}$ are defined inductively by the pushout squares

$$\begin{array}{ccc} \mathcal{F}_G(i) \otimes_{\Sigma_i} Q_i(g') & \longrightarrow & G_{i-1}^{\mathcal{F}} \\ \downarrow & & \downarrow \\ \mathcal{F}_G(i) \otimes_{\Sigma_i} Y^{\otimes i} & \longrightarrow & G_i^{\mathcal{F}} \end{array}$$

in $\mathscr{C}$, where $\mathcal{F}_G \coloneqq \mathcal{F} \circ (\mathbb{1}_\circ \sqcup G)$. Thus if $g\colon G \to G'$ is the pushout in $\mathscr{C}$ of a generating cofibration $g'\colon X \to Y$, then $f \,\square\, g$ is a cofibration as soon as each square in the diagram

$$\begin{array}{ccccc} G_0^{\mathcal{F}} & \longrightarrow & G_1^{\mathcal{F}} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ G_0^{\mathcal{F}'} & \longrightarrow & G_1^{\mathcal{F}'} & \longrightarrow & \cdots \end{array}$$

has the pushout corner property. By the above characterisation of the $G_i^{\mathcal{F}}$, this is the case if the squares

$$\begin{array}{ccc} \mathcal{F}_G(i) \otimes_{\Sigma_i} Q_i(g') & \longrightarrow & \mathcal{F}'_G(i) \otimes_{\Sigma_i} Q_i(g') \\ \downarrow & & \downarrow \\ \mathcal{F}_G(i) \otimes_{\Sigma_i} Y^{\otimes i} & \longrightarrow & \mathcal{F}'_G(i) \otimes_{\Sigma_i} Y^{\otimes i} \end{array}$$

have the pushout corner property, which by Lemma 1.4.37 is the case as soon as $\mathcal{F}_G(i) \to \mathcal{F}'_G(i)$ is a Σ_i-equivariant cofibration, since $(g')^{\square i} : Q_i(g') \to Y^{\otimes i}$ is a Σ_i-equivariant cofibration by assumption on $\mathscr{C}$.

But it turns out that $\mathcal{F}_G \to \mathcal{F}'_G$ is a cofibration whenever f is a cofibration and G is cofibrant. This is proved by picking a cellular filtration $\widetilde{p}\colon \beta \to \mathscr{C}$ of the basepoint map $p\colon \mathbb{1}_{\mathscr{C}} \to G$ (which exists up to retracts, since G was assumed to be cofibrant) and analysing the diagram

$$\begin{array}{ccccccccc} \mathcal{F} = \mathcal{F} \circ \big(\mathbb{1}_\circ \sqcup \widetilde{p}(0)\big) & \longrightarrow & \mathcal{F} \circ \big(\mathbb{1}_\circ \sqcup \widetilde{p}(1)\big) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F} \circ (\mathbb{1}_\circ \sqcup G) = \mathcal{F}_G \\ f\downarrow \quad\downarrow & & \downarrow & & & & \downarrow \quad\downarrow \\ \mathcal{F}' = \mathcal{F}' \circ \big(\mathbb{1}_\circ \sqcup \widetilde{p}(0)\big) & \longrightarrow & \mathcal{F}' \circ \big(\mathbb{1}_\circ \sqcup \widetilde{p}(1)\big) & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}' \circ (\mathbb{1}_\circ \sqcup G) = \mathcal{F}'_G \end{array}$$

A transfinite induction argument on the little squares in the diagram shows that each has the pushout corner property. Hence every vertical morphism in the diagram is a cofibration since f is, finishing the proof. □

Corollary 1.4.40. *The composition product of two cofibrant pointed collections is again cofibrant. For a cofibrant pointed collection $\mathcal{F}$, $\mathcal{F} \circ (-) : \mathrm{Coll}_*(\mathscr{C}) \to \mathrm{Coll}_*(\mathscr{C})$ preserves cofibrations between cofibrant objects, whereas $(-) \circ \mathcal{F}$ preserves cofibrations.*

Proof. Indeed, suppose that $p\colon \mathbb{1}_\otimes \to \mathcal{F}$ and $q\colon \mathbb{1}_\otimes \to \mathcal{G}$ are cofibrations in $\mathrm{CColl}(\mathscr{C})$. Then $p\square q$ fits into the diagram

$$\begin{array}{ccccc} \mathbb{1}_\otimes & = & \mathbb{1}_\otimes & & \\ {\scriptstyle p(\mathbf{0})}\downarrow & & \downarrow{\scriptstyle p(\mathbf{0})} & \searrow^{p\circ q} & \\ \mathcal{F}(\mathbf{0}) & = & \mathcal{F}(\mathbf{0}) & \xrightarrow[p\square q]{} & \mathcal{F}\circ\mathcal{G} \end{array}$$

where we used the fact that the functor $\mathbb{1}_\otimes \circ (-) : \mathrm{Coll}_*(\mathscr{C}) \to \mathrm{Coll}_*(\mathscr{C})$ is constant at $\mathbb{1}_\otimes$, while the functor $(-)\circ\mathbb{1}_\otimes$ is the composition

$$\mathrm{Coll}_*(\mathscr{C}) \xrightarrow{\mathrm{eval}_{\mathbf{0}}} \mathscr{C}_{\mathbb{1}_{\mathscr{C}}/} \hookrightarrow \mathrm{Coll}_*(\mathscr{C}).$$

Thus $p\circ q = (p\square q)\, p(\mathbf{0})$ is a composition of two cofibrations.

The remaining claims are established in much the same way. □

1.4.3 Proof of Theorem 1.2.26(ii)

For the rest of the section, let $\mathscr{C} = \mathrm{Top}$ with the standard exquisite category structure:

Lemma 1.4.41. *The category* Top *with the cartesian closed monoidal structure and the Quillen–Serre model structure is exquisite.*

Proof. The category Top is certainly symmetric monoidal model, *cf.* [29, Proposition 4.2.11], and the monoidal unit $*$ is cofibrant. The cellularity condition holds by [25, Example 4.1].

The n-th pushout-power of the generating cofibration $S^{k-1} = \partial D^k \hookrightarrow D^k$ is the inclusion of the boundary into $\left(D^k\right)^n$. But $\left(D^k\right)^n \cong \left(\left(D^1\right)^k\right)^n \cong \left(D^1\right)^{nk}$ as Σ_n-spaces. Hence if we exhibit the inclusion $\partial\left(D^1\right)^{nk} \hookrightarrow \left(D^1\right)^{nk}$ as a relative Σ_{nk}-CW-complex, then by restriction it will also be a relative Σ_n-CW-complex, completing the proof.

But for each n, the n-cube $\left(D^1\right)^n$ admits the structure of the geometric realisation of a semi-simplicial Σ_n-set in such a way that the boundary is the realisation of a subset. Indeed, recall that the indexing category $\mathbb{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$ of semi-simplicial objects is isomorphic to the category of interval surjections: the objects are *intervals*, or finite linearly ordered sets of the form $\{-\infty\} \sqcup \mathbf{k} \sqcup \{\infty\}$ with distinct top and bottom elements $\pm\infty$, and morphisms are monotone surjections. Let $C_n\colon \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathrm{Set}$ be the functor taking the interval $\mathbf{k}\sqcup\{\pm\infty\}$ to the set of functions $f\colon \mathbf{n} \to \mathbf{k}\sqcup\{\pm\infty\}$ such that $\mathbf{k} \subseteq \operatorname{im} f$; on morphisms, C_n is defined by post-composition. Then C_n is a semi-simplicial Σ_n-set, where $\sigma \in \Sigma_n$ takes $f\colon \mathbf{n} \to \mathbf{k}\sqcup\{\pm\infty\}$ to $f\sigma^{-1}$.

It is now easy to write down a Σ_n-equivariant homeomorphism $\|C_n\| \to \left(D^1\right)^n$, where $D^1 = [0,1] \subset \mathbb{R}$ is the unit interval. On the k-simplex indexed by $f : \mathbf{n} \to \mathbf{k} \sqcup \{\pm\infty\}$, the homeomorphism is given by

$$\begin{aligned} \Delta^k \cong \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid 0 \leqslant x_1 \leqslant \cdots \leqslant x_k \leqslant 1\right\} &\longrightarrow \left(D^1\right)^n \\ (x_1, \dots, x_k) &\longmapsto \left(x_{f(1)}, \dots, x_{f(n)}\right) \end{aligned}$$

where $x_{-\infty} \coloneqq 0$ and $x_\infty \coloneqq 1$. Under this homeomorphism, the boundary of $\left(D^1\right)^n$ corresponds to the subcomplex of maps $f : \mathbf{n} \to \mathbf{k} \sqcup \{\pm\infty\}$ such that $f^{-1}(\pm\infty) \neq \varnothing$. □

It is an easy check from the definitions that if $\mathcal{O}$ is an operad and X is a pointed space, then $\mathcal{O} \circ X$ (where X is considered as a pointed collection in the usual way) can be canonically identified with $\mathbb{O}_*(X)$ as defined in Section 1.2.1.2.

Proposition 1.4.42. *Let $\mathcal{O}$ be a pointed operad with a cofibrant underlying pointed collection, and let X be an $\mathcal{O}$-algebra whose underlying pointed space is cofibrant. Then the associated bar construction $B_\bullet(\mathbb{O}_*, \mathbb{O}_*, X) = \mathcal{O}^{\circ(\bullet+1)} \circ X$ is a Reedy cofibrant simplicial $\mathcal{O}$-algebra.*

Proof. We write $B_d = B_d(\mathbb{O}_*, \mathbb{O}_*, X)$ for short. We need to prove that the inclusions of the latching objects $L_d \to B_d$ are cofibrations of $\mathcal{O}$-algebras for all d.

Recall that $L_0 = \varnothing$, $L_1 = B_0$ and for $d > 1$, the d-th latching object is given by the coequaliser

$$\coprod_{0 \leqslant i < j \leqslant d-1} B_{d-2} \rightrightarrows \coprod_{0 \leqslant i \leqslant d-1} B_{d-1} \longrightarrow L_d$$

where the top arrow takes the summand indexed by $i < j$ to the summand indexed by j by the i-th degeneracy $s_i : B_{d-2} \to B_{d-1}$, while the bottom arrow takes the $(i < j)$-summand to the i-th summand by s_{j-1}. Since $B_d = \mathcal{O}^{\circ(d+1)} \circ X$ and the degeneracy maps are given by the unit map $\eta : \mathbb{1}_\circ \to \mathcal{O}$, it can be seen by comparison with Definition 1.4.32 that the inclusion $L_d \to B_d$ can be identified with the morphism $\mathcal{O} \circ \eta^{\square d} \circ X$. η is a cofibration with a cofibrant domain, so by repeated application of Proposition 1.4.39, $\eta^{\square d} \cong \eta^{\square(d-1)} \square \eta$ is a cofibration. Hence by Corollary 1.4.40, $\eta^{\square d} \circ X$ is a cofibration in $\mathrm{Coll}_*(\mathrm{Top})$. Since it is concentrated in degree 0, we can equivalently think of it as a cofibration in Top_*. The latching map is now its image under the free $\mathcal{O}$-algebra functor $F^{\mathcal{O}}_* = \mathcal{O} \circ (-) : \mathrm{Top}_* \to \mathcal{O}\text{–Alg}$ which is a left Quillen adjoint. This concludes the proof. □

It remains to observe that the adjunction

$$|-| \dashv (-)^{\Delta^\bullet} : \mathcal{O}\text{–Alg} \longrightarrow s\mathcal{O}\text{–Alg}$$

of Lemma 1.2.21 is Quillen. But this is immediate, since the (acyclic) fibrations on either side are preserved and reflected by the forgetful functors $U^{\mathcal{O}}$ and $sU^{\mathcal{O}}$, and the functor $(-)^{\Delta^\bullet} : \mathrm{Top} \to \mathrm{sTop}$ carries (acyclic) fibrations to Reedy (acyclic) fibrations.

This concludes the proof of Theorem 1.2.26(ii) and the section.

1.5 Homotopy enveloping operads and homology stability

Given a topological operad $\mathcal{O}$ and an $\mathcal{O}$-algebra A, one can define the *enveloping operad* $\mathcal{O}[A]$ which controls $\mathcal{O}$-algebras under A. Equivalently, $\mathcal{O}[A]$ is the initial operad under $\mathcal{O}$ satisfying $\mathcal{O}[A](0) = A$, *cf.* [22, Section 4.1].

In this section we describe a 'homotopically correct' version of this construction, which will allow us to analyse the homology of the constituent spaces of the enveloping operad, and conclude in certain cases that it is an OSHS.

Remark 1.5.1. The strict enveloping operad is constructed as a coequaliser of a pair of operad maps between enveloping operads of free algebras, introduced in the next section. This coequaliser is computed on the underlying collections, but it is nonetheless not as easy to analyse from a homological perspective as the homotopy enveloping operad we introduce below. In fact, it can be deduced from Remark 1.5.8 that the strict construction yields an equivalent operad to the homotopy construction for cofibrant algebras.

1.5.1 Enveloping operads of free algebras

We begin by recalling the construction of the strict enveloping operad of a free algebra. This construction is by no means original; *cf.* [22, Section 4.1]. The main difference from the reference is that we work with pointed operads.

Definition 1.5.2. Given a pointed topological operad $\mathcal{O}$, viewed as a monoid in the category $(\mathrm{Coll}_*(\mathrm{Top}), \circ, \mathbb{1}_\circ)$, and a pointed space $X \in \mathrm{Top}_*$, viewed as a pointed collection concentrated in degree 0, let $\mathcal{O}_X$ be the pointed collection

$$\mathcal{O}_X := \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)\,.$$

Lemma 1.5.3. *The pointed collection $\mathcal{O}_X$ is naturally a pointed operad.*

Proof. Indeed, note first that $\mathbb{1}_\circ \sqcup X$ is an operad, with multiplication given by

$$(\mathbb{1}_\circ \sqcup X) \circ (\mathbb{1}_\circ \sqcup X) \cong (\mathbb{1}_\circ \sqcup X) \sqcup X \longrightarrow \mathbb{1}_\circ \sqcup X$$

where the isomorphism used the fact that $\circ$ preserves colimits in the first variable, and Lemma 1.4.16(i), and the last map is the codiagonal $X \sqcup X \to X$. (Explicitly, $\mathbb{1}_\circ \sqcup X$ is the operad controlling spaces under X.)

Now consider the morphism

$$\tau\colon\ (\mathbb{1}_\circ \sqcup X) \circ \mathcal{O} \cong \mathcal{O} \sqcup X \cong (\mathcal{O} \circ \mathbb{1}_\circ) \sqcup (\mathbb{1}_\circ \circ X) \longrightarrow \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)$$

of pointed collections, where the last arrow is defined on the factors of the coproduct using the unit $\mathbb{1}_\circ \to \mathcal{O}$ and the inclusions $\mathbb{1}_\circ, X \to \mathbb{1}_\circ \sqcup X$. The multiplication on $\mathcal{O}_X$ can now be defined by

$$\mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) \circ \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) \xrightarrow{\mathcal{O} \circ \tau \circ (\mathbb{1}_\circ \sqcup X)} \mathcal{O} \circ \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) \circ (\mathbb{1}_\circ \sqcup X) \longrightarrow \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)$$

This is associative and unital as soon as the diagram

$$\begin{array}{ccc}
(\mathbb{1}_\circ \sqcup X) \circ \mathcal{O} \circ \mathcal{O} & \xrightarrow{\mathrm{id} \circ \gamma} & (\mathbb{1}_\circ \sqcup X) \circ \mathcal{O} \\
{\scriptstyle \tau \circ \mathrm{id}}\downarrow & & \\
\mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) \circ \mathcal{O} & & \Big\downarrow {\scriptstyle \tau} \\
{\scriptstyle \mathrm{id} \circ \tau}\downarrow & & \\
\mathcal{O} \circ \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) & \xrightarrow[\gamma \circ \mathrm{id}]{} & \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)
\end{array}$$

and its analogue, witnessing the compatibility of τ with the multiplication on $\mathbb{1}_\circ \sqcup X$, commute. This is a simple diagram chase. □

It is easy to work out from the definitions that

$$\mathcal{O}_X(n) = \left(\coprod_{k \geqslant 0} \mathcal{O}(n+k) \times_{\Sigma_k} X^k \right) \Big/ \sim$$

where Σ_k acts on $\mathcal{O}(n+k)$ via the inclusion $\Sigma_k \hookrightarrow \Sigma_{n+k}$ of permutations that only act on the last k letters, and the basepoint equivalence relation $\sim$ is generated by

$$(a; x_1, \ldots, x_{k-1}, *) \sim \left(a(1^{n+k-1}, *); x_1, \ldots, x_{k-1} \right)$$

for $a \in \mathcal{O}(n+k)$ and $x_1, \dots, x_{k-1} \in X$. In particular, $\mathcal{O}_X(0) = \mathbb{O}_*(X)$ is the free $\mathcal{O}$-algebra on the pointed space X. Also, each $\mathcal{O}_X(n)$ is filtered by the spaces

$$F_i\mathcal{O}_X(n) \coloneqq \left(\coprod_{k=0}^{i} \mathcal{O}(n+k) \times_{\Sigma_k} X^k \right) \Big/ \sim \tag{1.25}$$

The following result is essentially proved by an 'operadification' of the proof of [3, Theorem 5.4].

Proposition 1.5.4. *Let X be a path-connected pointed space, and let $\mathrm{Ass} \xrightarrow{\mu} \mathcal{O} \xrightarrow{\pi} \Gamma$ be a superb OSHS with stabilising element $s \in \mathcal{O}_1(1)$. Then $\mathcal{O}_X$ is also an OSHS when stabilised by the image $s_X \in \mathcal{O}_X(1)$ of s under the canonical map $\mathcal{O} \to \mathcal{O}_X$.*

Proof. The operad $\mathcal{O}_X$ certainly inherits a grading and a map out of Ass from $\mathcal{O}$. Since the map $\mathrm{Ass} \to \mathcal{O}_X$ factors through $\mathcal{O}$, the connectedness condition Definition 1.3.1(ii) is certainly satisfied. Hence we only need to verify that for all $\mathbf{v} = (v_1, \dots, v_n) \in \mathcal{O}_X(0)^n$, the map

$$D^{\mathbf{v}}_{\infty} \colon (\mathcal{O}_X)_\infty(n) \to (\mathcal{O}_X)_\infty(0)$$

of Definition 1.3.1 is a homology equivalence. But since X is connected, the map $\mathcal{O}(0) \to \mathbb{O}_*(X) = \mathcal{O}_X(0)$ is a π_0-bijection, and so we may assume that $\mathbf{v} \in \mathcal{O}(0)^n \subseteq \mathcal{O}_X(0)^n$.

The maps $X \to *$ and $\pi \colon \mathcal{O} \to \Gamma$ induce maps $\mathcal{O}_X \to \mathcal{O}_* \cong \mathcal{O}$ and $\pi_X \colon \mathcal{O}_X \to \Gamma_X$ of operads, and hence maps

$$(\mathcal{O}_X)_\infty(n) \longrightarrow \mathcal{O}_\infty(n) \times (\Gamma_X)_\infty(n) \tag{1.26}$$

for all n, where $(\Gamma_X)_\infty(n)$ is stabilised by $\pi_X(s_X) \in \Gamma_X(1)$. Evidently this stabilising element lies in $\Gamma(1) \subseteq \Gamma_X(1)$ which only contains the identity, so canonically $(\Gamma_X)_\infty(n) = \Gamma_X(n)$. These maps fit into the commutative diagrams

$$\begin{array}{ccc} (\mathcal{O}_X)_\infty(n) & \longrightarrow & \mathcal{O}_\infty(n) \times \Gamma_X(n) \\ \Big\downarrow{\scriptstyle D^{\mathbf{v}}_\infty} & & \Big\downarrow{\scriptstyle D^{\mathbf{v}}_\infty \times D^*} \\ (\mathcal{O}_X)_\infty(0) & \longrightarrow & \mathcal{O}_\infty(0) \times \Gamma_X(0) \end{array}$$

for all $\mathbf{v}$, where $D^* \colon \Gamma_X(n) \to \Gamma_X(0)$ is induced by the unique n-tuple $(*, \cdots, *) \in \Gamma(0)^n \subseteq \Gamma_X(0)^n$. We claim that the horizontal arrows and the right-hand vertical arrow are homology equivalences.

The map $D^{\mathbf{v}}_{\infty}\colon \mathcal{O}_\infty(n) \to \mathcal{O}_\infty(0)$ is a homology equivalence by assumption. To show that the horizontal arrows are homology equivalences, note that the spaces in (1.26) are filtered as in (1.25), and the map respects these filtrations (where the filtration on $\mathcal{O}_\infty(n)$ is constant). On the subquotients of the filtrations, it induces the maps

$$\frac{F_k(\mathcal{O}_X)_\infty(n)}{F_{k-1}(\mathcal{O}_X)_\infty(n)} = \mathcal{O}_\infty(n+k) \ltimes_{\Sigma_k} X^{\wedge k} \longrightarrow \mathcal{O}_\infty(n) \ltimes \left(\Gamma(n+k) \ltimes_{\Sigma_k} X^{\wedge k}\right) \tag{1.27}$$

where $\ltimes$ denotes the 'half-smash product' $Y \ltimes Z \coloneqq (Y \times Z)/(Y \times *)$. These fit into the maps

$$\begin{array}{ccccc}
\mathcal{O}_\infty(n+k)/\Sigma_k & \longrightarrow & \mathcal{O}_\infty(n+k) \times_{\Sigma_k} X^{\wedge k} & \longrightarrow & \mathcal{O}_\infty(n+k) \ltimes_{\Sigma_k} X^{\wedge k} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_\infty(n) \times \Gamma(n+k)/\Sigma_k & \to & \mathcal{O}_\infty(n) \times \left(\Gamma(n+k) \times_{\Sigma_k} X^{\wedge k}\right) & \to & \mathcal{O}_\infty(n) \ltimes \left(\Gamma(n+k) \ltimes_{\Sigma_k} X^{\wedge k}\right)
\end{array}$$

of cofibre sequences. The first two vertical arrows are homology equivalences by [3, Lemma 5.2]. Hence the last vertical arrow is a homology equivalence. The filtrations of the spaces in (1.26) induce a pair of spectral sequences that converge to the homologies of the spaces, and the map (1.26) induces a map of these spectral sequences. But the map between the second pages of these spectral sequences is induced by the subquotient maps (1.27), and thus it is an isomorphism. We conclude that the maps (1.26) are homology isomorphisms for all n.

An identical filtration argument can be employed to prove that $D^*\colon \Gamma_X(n) \to \Gamma_X(0)$ is a homology equivalence, concluding the proof. □

Remark 1.5.5. When X is not path-connected, the above argument still goes through to show that $\mathcal{O}_X$ is an OHS, but when one tries to prove the stronger OSHS condition, the proof flounders on the fact that the map

$$D^{\mathbf{x}}\colon \Gamma_X(n) \to \Gamma_X(0)$$

will not be a homology isomorphism for $\mathbf{x} = (x_1, \ldots, x_n) \in X^n \subset \Gamma_X(0)^n$ whenever at least one x_i doesn't belong to the basepoint component of X. Indeed, both $\Gamma_X(n)$ and $\Gamma_X(0)$ are equivalent to $\mathbb{\Gamma}_*(X)$ which is a commutative monoid, and the above map corresponds to multiplication by the product $x_1 \cdots x_n$. This is not even a π_0-isomorphism in general, since $\pi_0\mathbb{\Gamma}_*(X)$ is the free commutative monoid generated by the pointed set $\pi_0 X$ (*i.e.* the basepoint component of X acts as the monoid unit).

If $\pi_0 X$ is finite, the proof can be salvaged by changing the stabilising element (and allowing the more general definition of an OSHS in the sense of Remark 1.3.5). Indeed, $\mathcal{O}_X$ is graded by the free commutative monoid on $\pi_0 X$ in an obvious way. Let $x_1, \dots, x_k$ be representatives of the non-basepoint components of X, and let $x \in \mathcal{O}_X(0)$ be their product in some order (under the Ass-action on $\mathcal{O}_X$). Then the stabilising element $s' \coloneqq \mu_2(s_X, x) \in \mathcal{O}_X(1)$ makes $\mathcal{O}_X$ into an OSHS. We leave the details to the reader.

1.5.2 General homotopy enveloping operads

Let again $\mathcal{O}$ be a pointed operad and X a pointed space. There is a natural operad map $\mathcal{O}_{\mathbb{O}_*(X)} \to \mathcal{O}_X$ given by

$$\mathcal{O}_{\mathbb{O}_*(X)} = \mathcal{O} \circ (\mathbb{1}_\circ \sqcup (\mathcal{O} \circ X)) \longrightarrow \mathcal{O} \circ \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) \longrightarrow \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X) = \mathcal{O}_X \tag{1.28}$$

where the first arrow is induced by the obvious maps $\mathbb{1}_\circ \to \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)$ and $\mathcal{O} \circ X \to \mathcal{O} \circ (\mathbb{1}_\circ \sqcup X)$, and the second map is the multiplication on $\mathcal{O}$. The following is a simple diagram chase.

Lemma 1.5.6. *The functor*

$$\mathcal{O}_{(-)} \colon \mathrm{Top}_* \longrightarrow \mathrm{Oper}_*$$

to the category of pointed operads is an $\mathbb{O}_*$-functor in the sense of Definition 1.2.23.

Hence if X is an $\mathcal{O}$-algebra, we can form the following simplicial operad.

Definition 1.5.7. If $\mathcal{O}$ is a pointed operad and $X \in \mathcal{O}\text{–Alg}$, we write $\mathcal{O}\{A\}$ for the simplicial object

$$\mathcal{O}\{A\} \coloneqq B_\bullet \left(\mathcal{O}_{(-)}, \mathbb{O}_*, A\right)$$

in Oper_*.

Remark 1.5.8. Equivalently, $\mathcal{O}\{A\}$ is the (genuine) enveloping operad of the $\mathcal{O}$-algebra $|B_\bullet (\mathbb{O}_*, \mathbb{O}_*, A)|$.

It is easily checked from the definitions that $\mathrm{sColl}_*(\mathrm{Top}) \cong \mathrm{Coll}_*(\mathrm{sTop})$, and that the componentwise composition product on the left-hand side agrees with the composition product on the right-hand side that comes from the categorical product on sTop. Hence $\mathcal{O}\{A\}$ can equivalently be viewed as an operad in simplicial spaces. We shall abuse notation slightly and use $\mathcal{O}\{A\}$ to also denote the realisation of this operad in simplicial spaces, which is an operad in spaces.

Theorem 1.5.9. *Let* $\mathrm{Ass} \xrightarrow{\mu} \mathcal{O} \xrightarrow{\pi} \Gamma$ *be a superb OSHS, and let* A *be a connected* $\mathcal{O}$*-algebra. Then* $\mathcal{O}\{A\}$ *is also an OSHS, with the map from an* A_∞*-operad and the stabilising element inherited from* $\mathcal{O}$ *along the canonical map* $\mathcal{O} \to \mathcal{O}\{A\}$.

Proof. As remarked at the beginning of the proof of Proposition 1.5.4, $\mathcal{O}_X$ is naturally graded, but the map (1.28) doesn't preserve this grading. Instead, we define each $\mathcal{O}_{\mathbb{O}^n_*(A)}$ with the following grading: let $\mathbb{O}_*(A)$ be graded in the evident way, and for each $i \geqslant 1$, let $\mathbb{O}^{i+1}_*(A)$ be inductively graded in such a way that the point

$$(a; x_1, \ldots, x_n) \in \mathbb{O}^{i+1}_*(A),$$

where $a \in \mathcal{O}_g(n)$ and $x_j \in \left(\mathbb{O}^i_*(A)\right)_{h_j}$, lies in grading $g + h_1 + \cdots + h_n$. We let $\mathcal{O}_{\mathbb{O}^n_*(A)}$ be graded similarly. With this grading, $\mathcal{O}\{A\}$ is a simplicial object in *graded* operads, and hence its realisation is a graded operad.

Hence as in the proof of Proposition 1.5.4, it suffices to check that for all n-tuples $\mathbf{v} \in \mathcal{O}\{A\}(0)^n$, the stabilised map

$$D^{\mathbf{v}}_\infty \colon \mathcal{O}\{A\}_\infty(n) \longrightarrow \mathcal{O}\{A\}_\infty(0) \tag{1.29}$$

is a homology equivalence. But since A is connected, so is

$$\mathcal{O}\{A\}(0) \cong |B_\bullet(\mathbb{O}_*, \mathbb{O}_*, A)|$$

and hence we may again assume that $\mathbf{v}$ consists of n copies of the basepoint.

The simplicial operad $\mathcal{O}\{A\}$ receives a simplicial operad map from the constant simplicial operad at $\mathcal{O}$. Thus in particular each simplicial level of $\mathcal{O}\{A\}$ comes with a compatible choice of stabilising element, and we have

$$\mathcal{O}\{A\}_\infty(n) \cong \left|B_\bullet\left(\left(\mathcal{O}_{(-)}\right)_\infty(n), \mathbb{O}_*, A\right)\right|$$

The map (1.29) is also simplicial with respect to this simplicial structure (at least when $\mathbf{v} = (*, \ldots, *) \in \mathcal{O}\{A\}(0)$), and so we may compute its effect on homology using the spectral sequence for the homology of a (semi)simplicial space. But the component of (1.29) on the space of d-simplices is

$$D^{\mathbf{v}}_\infty \colon \left(\mathcal{O}_{\mathbb{O}^d_*(A)}\right)_\infty(n) \longrightarrow \left(\mathcal{O}_{\mathbb{O}^d_*(A)}\right)_\infty(0)$$

which is a homology equivalence since $\mathbf{v} = (*, \dots, *)$ and $\mathcal{O}_{\mathbb{O}^d_*(A)}$ is an OHS (though not in general an OSHS, *cf.* Remark 1.5.5) by Proposition 1.5.4. Hence (1.29) induces isomorphisms on the second pages of the respective spectral sequences, and so it is itself a homology isomorphism, concluding the proof. □

Remark 1.5.10. We can again drop the assumption that A be connected — and replace it with the assumption that $\pi_0 A$ be finitely generated as a monoid — if we relax the definition of an O(S)HS to allow operads that are graded by other commutative monoids than $\mathbb{N}$.

Since A is an $\mathcal{O}$-algebra, $\pi_0 A$ is in particular a commutative monoid. If it is finitely generated, we can write down a stabilising element for $\mathcal{O}_A$ just like in Remark 1.5.5 such that stabilisation with it inverts $s \in \mathcal{O}(1)$ and the path-components of $A \hookrightarrow \mathbb{O}_*(A) = \mathcal{O}_A(0)$ that correspond to a finite generating set of $\pi_0 A$.

This stabilising element will *not* make the d-simplex operad $\mathcal{O}_{\mathbb{O}^d_*(A)}$ of $\mathcal{O}\{A\}$ into an OSHS, but it will ensure that the map

$$D^{\mathbf{v}}_\infty \colon \left(\mathcal{O}_{\mathbb{O}^d_*(A)}\right)_\infty (n) \longrightarrow \left(\mathcal{O}_{\mathbb{O}^d_*(A)}\right)_\infty (0)$$

is a homology equivalence whenever $\mathbf{v} \in X^n \subset \mathbb{O}^{d+1}_*(A)^n = \mathcal{O}_{\mathbb{O}^d_*(A)}(0)^n$ consists of points that lie in path-components corresponding to the finite generating set of $\pi_0 A$. This is sufficient for the spectral sequence argument in the proof of the above theorem to go through for these n-tuples $\mathbf{v}$. Since every n-tuple of elements of $\mathcal{O}\{A\}(0) = |B_\bullet(\mathbb{O}_*, \mathbb{O}_*, A)|$ may be replaced up to homotopy with an m-tuple of elements of A that all lie in generating path-components of $\pi_0 A$ (for some $m \geqslant n$), we conclude that $\mathcal{O}\{A\}$ is an OSHS.

Chapter 2

On the quotients of mapping class groups of surfaces by the Johnson subgroups

2.1 Introduction and statement of results

Let $S_{g,n}$ be an oriented surface of genus g with n boundary components, and let $\Gamma_{g,n}$ be its mapping class group, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ that fix a neighbourhood of the boundary pointwise.

If we fix a base-point $* \in \partial S_{g,1}$, the fundamental group $\pi := \pi_1(S_{g,1}, *)$ enjoys an action of $\Gamma_{g,1}$. This action preserves the lower central series $\{\gamma_k(\pi)\}_{k\geqslant 1}$ of π, where $\gamma_1(\pi) = \pi$ and $\gamma_{k+1}(\pi) = [\gamma_k(\pi), \pi]$. The *Johnson filtration* $\{\mathscr{I}_{g,1}(k)\}_{k\geqslant 1}$ of $\Gamma_{g,1}$ consists of kernels of the induced actions of $\Gamma_{g,1}$ on the quotients $\pi/\gamma_{k+1}(\pi)$. First defined by Johnson in [30] to study the Torelli group, this filtration has since become a subject of active research in its own right.

2.1.1 Infinite loop space structures

The pair-of-pants product endows the space $\coprod_{g\geqslant 0} B\Gamma_{g,1}$ with the structure of a framed E_2-algebra. The group-completion of this algebra is (weakly equivalent to) $\mathbb{Z} \times B\Gamma_\infty^+$ by the group-completion theorem [41, Proposition 1], where $\Gamma_\infty = \operatorname{colim}_g \Gamma_{g,1}$ is the stable mapping class group, and X^+ denotes the Quillen plus-construction on X with respect to the maximal perfect subgroup of $\pi_1(X)$. Tillmann [55, 54] showed that $\mathbb{Z} \times B\Gamma_\infty^+$ is in fact an infinite loop space, and that the map $\mathbb{Z} \times B\Gamma_\infty^+ \to K_{\mathrm{Sp}}(\mathbb{Z})$ to the Hermitian K-theory of $\mathbb{Z}$ arising from considering the middle homology groups of the surfaces is

an infinite loop space map. We show that analogous constructions for the quotients $\Gamma_{g,1}(k) \coloneqq \Gamma_{g,1}/\mathscr{I}_{g,1}(k)$ also yield infinite loop spaces and that we obtain a tower of infinite loop spaces interpolating between $\mathbb{Z} \times B\Gamma_\infty^+$ and $K_{\mathrm{Sp}}(\mathbb{Z})$.

Theorem 2.1.1. *The obvious quotient homomorphisms of groups give rise to a commutative diagram*

$$\begin{array}{ccccccc} & & \mathbb{Z} \times B\Gamma_\infty^+ & & & & \\ & \swarrow & & \searrow & & \searrow & \\ \cdots & \longrightarrow & \mathbb{Z} \times B\Gamma_\infty(3)^+ & \longrightarrow & \mathbb{Z} \times B\Gamma_\infty(2)^+ & \longrightarrow & \mathbb{Z} \times B\Gamma_\infty(1)^+ \simeq K_{\mathrm{Sp}}(\mathbb{Z}) \end{array}$$

of infinite loop spaces and infinite loop space maps.

2.1.2 Homological stability

Harer [27] proved that the groups $\Gamma_{g,1}$ satisfy homological stability in the sense that the 'stabilisation' homomorphisms $\Gamma_{g-1,1} \to \Gamma_{g,1}$ — coming from attaching a copy of $S_{1,2}$ to the boundary of $S_{g-1,1}$ and extending mapping classes by the identity — induce isomorphisms on integral homology in a range of degrees that tends to infinity as the genus g tends to infinity. Since the lower central series of π consists of characteristic subgroups, these stabilisation homomorphisms preserve the Johnson filtration and descend to the quotients $\Gamma_{g,1}(k) = \Gamma_{g,1}/\mathscr{I}_{g,1}(k)$. Hence a natural question arises whether these quotients also exhibit homological stability.

Theorem 2.1.2. *Let $k \geqslant 1$, and let $F\colon \mathrm{SI} \to \mathrm{Ab}$ be an* SI*-module of finite degree. Then for every i, the homomorphisms in the sequence*

$$\cdots \to H_i(\Gamma_{g,1}/\mathscr{I}_{g,1}(k); F_g) \to H_i(\Gamma_{g+1,1}/\mathscr{I}_{g+1,1}(k); F_{g+1}) \to \cdots$$

are eventually (for large g) isomorphisms.

See Section 2.4 for the relevant definitions of SI-modules and finite degree; in essence, these are well-behaved sequences of representations of the symplectic groups $\mathrm{Sp}(2g, \mathbb{Z})$. In particular, for any abelian group A, the trivial representations of $\mathrm{Sp}(2g, \mathbb{Z})$ on A constitute an SI-module of finite degree, so Theorem 2.1.2 holds with arbitrary constant coefficients.

Our proof is by induction, bootstrapping known results about homological stability of the symplectic groups $\mathrm{Sp}(2g,\mathbb{Z}) \cong \Gamma_{g,1}(1)$ to the higher quotients $\Gamma_{g,1}(k)$, $k > 1$, by means of a spectral sequence argument applied to the extensions

$$0 \to \mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1) \to \Gamma_{g,1}(k+1) \to \Gamma_{g,1}(k) \to 1.$$

Our approach has a drawback in that we get no hold on the stable range: the heart of our argument is Theorem 2.5.7 which says that certain FI-modules which encapsulate the successive quotients $\mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1)$ are finitely generated. Its proof relies on the noetherian property of the category of FI-modules over $\mathbb{Z}$ which was proved by Church, Ellenberg, Farb and Nagpal in [17]. This property is non-constructive in the sense that it guarantees the existence of a finite generating set of a sub-FI-module of a finitely generated FI-module, but it offers no information on the size or degrees of some finite set of generators.

The inspiration to use the above bootstrapping method came from a paper of Szymik [53] where this method is applied to the closely related automorphism groups of free nilpotent groups. Szymik considers the extensions

$$0 \to K_n(k) \to \mathrm{Aut}\, N_n(k) \to \mathrm{Aut}\, N_n(k-1) \to 1$$

where $N_n(k)$ is the free nilpotent group on n generators of nilpotency class k: if we write $\mathbb{F}_n$ for the free group on n generators, then $N_n(k) \coloneqq \mathbb{F}_n/\gamma_{k+1}(\mathbb{F}_n)$. The kernels $K_n(k)$ in the above extensions can be characterised completely. While the quotients $\mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1)$ become subgroups of $K_{2g}(k+1)$ upon fixing an isomorphism $\pi_1(S_{g,1},*) \cong \mathbb{F}_{2g}$, it is a difficult problem to determine *which* subgroups they are. It is for this reason that we use the techniques of representation stability and consequently lose information about the stable range.

2.1.3 Outline of chapter

We begin in Section 2.2 where we recall the main objects of this chapter, namely automorphism groups of free groups, mapping class groups of surfaces, and the Johnson filtration.

Section 2.3 is dedicated to recalling the surface operad of [54] and showing carefully that it acts on (suitable models of) the spaces $\mathbb{Z} \times B\left(\Gamma_\infty/\mathscr{I}_\infty(k)\right)^+$, establishing Theorem 2.1.1.

In Section 2.4, we recall some definitions and constructions from the theory of functors valued in abelian groups. Following [46], we define the degree of an Ab-valued functor on a PROP, we recall the categories FI and SI in Section 2.4.3, and we give a full characterisation of FI-modules of finite degree in Section 2.4.4.

In Section 2.5, we construct specific FI-modules out of automorphisms of free nilpotent groups, and SI-modules out of the Johnson filtration of mapping class groups. We show that these objects have finite degree in the sense of Section 2.4.

This becomes crucial in Section 2.6, where we prove Theorem 2.1.2. In Section 2.6.1, we make some remarks on the stable rational cohomology of the quotients $\Gamma_{g,1}/\mathscr{I}_{g,1}(k)$.

2.2 Background

In this section we fix our notation pertaining to automorphism groups of free nilpotent groups and to mapping class groups, and we recall some facts that will be useful in our proof of Theorem 2.1.2.

2.2.1 Automorphisms of free nilpotent groups

We write $\{\gamma_k(G)\}_{k\geqslant 1}$ for the lower central series of a group G. This is defined inductively by setting $\gamma_1(G) = G$ and $\gamma_k(G) = [G, \gamma_{k-1}(G)]$ for $k > 1$. Let $\mathbb{F}_n$ be the free group on the letters $x_1, \dots, x_n$, and let $N_n(k) \coloneqq \mathbb{F}_n/\gamma_{k+1}(\mathbb{F}_n)$ be the free nilpotent group of rank n and nilpotency class k. Thus for instance $N_n(1) \simeq \mathbb{Z}^n$, whereas $N_2(2)$ is the Heisenberg group.

The lower central series of a group consists of characteristic subgroups, so there are homomorphisms $\operatorname{Aut}\mathbb{F}_n \to \operatorname{Aut} N_n(k)$ for every n and every k. Fixing n, the kernels

$$IA_n(k) = \left\{\phi \in \operatorname{Aut}\mathbb{F}_n \mid \phi(x)x^{-1} \in \gamma_{k+1}(\mathbb{F}_n) \text{ for all } x \in \mathbb{F}_n\right\}$$

of these homomorphisms form a filtration of $\operatorname{Aut}\mathbb{F}_n$, introduced and studied by Andreadakis [1]. We state the following theorem, recalling that a descending series $G = G_1 \supseteq G_2 \supseteq \cdots$ of a group G is *central* if $\left[G_i, G_j\right] \subseteq G_{i+j}$ for all $i, j \geqslant 1$.

Theorem 2.2.1 ([1, Theorem 1.1(ii)]). *The filtration $\{IA_n(k)\}_{k\geqslant 1}$ of $IA_n(1)$ is central.*

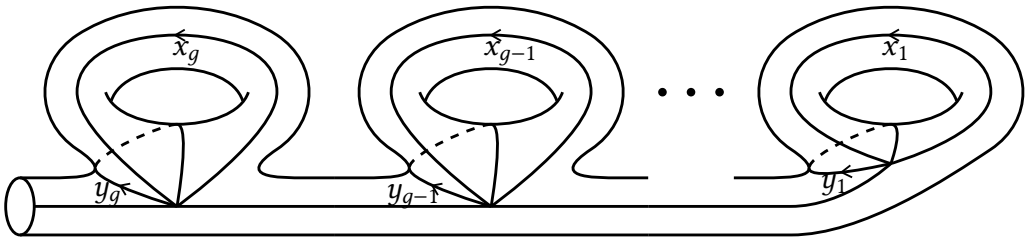


Figure 2.1: $S_{g,1}$

2.2.2 Mapping class groups

Let $S_{g,n}$ be an oriented surface of genus g with n boundary components. Fix collars $[0,\epsilon)\times S^1 \hookrightarrow S_{g,n}$ of the boundary components.

Let $\mathrm{Diff}^+(S_{g,1},\partial)$ be the group of orientation-preserving diffeomorphisms of the surface which fix the collar pointwise. The mapping class group $\Gamma_{g,1}$ of $S_{g,1}$ is the group of isotopy classes of such diffeomorphisms:

$$\Gamma_{g,1} \coloneqq \mathrm{Diff}^+(S_{g,1},\partial)\big/\mathrm{Diff}^+_0(S_{g,1},\partial)$$

where $\mathrm{Diff}^+_0(S_{g,1},\partial)$ is the path-component of the identity in the diffeomorphism group.

We fix a point $*\in\partial S_{g,1}$ and we choose generators $x_1,y_1,\dots,x_g,y_g$ of $\pi \coloneqq \pi_1(S_{g,1},*)\cong \mathbb{F}_{2g}$ as in Figure 2.1. Then the boundary word $\zeta\in\pi$ is equal to

$$\prod_i [x_i,y_i]\,.$$

The group π admits an action of $\Gamma_{g,1}$, yielding a homomorphims $\Gamma_{g,1}\to \mathrm{Aut}\,\pi$. By the Dehn–Nielsen–Baer theorem, this homomorphism is injective with image precisely the stabiliser of $\zeta\in\pi$:

$$\Gamma_{g,1}\cong \left\{\phi\in \mathrm{Aut}\,\mathbb{F}_{2g} \mid \phi(\zeta)=\zeta\right\}.$$

Just like in the case of the automorphism group of the free group, the mapping class group $\Gamma_{g,1}$ can be filtered by the kernels of its action on the quotients $\pi/\gamma_{k+1}(\pi)$. The Johnson filtration [30] $\left\{\mathscr{I}_{g,1}(k)\right\}_{k\geqslant 0}$ of $\Gamma_{g,1}$ is given by

$$\mathscr{I}_{g,1}(k)=\left\{\phi\in\Gamma_{g,1}\mid \phi(x)x^{-1}\in\gamma_{k+1}(\mathbb{F}_{2g}) \text{ for all } x\in\mathbb{F}_{2g}\right\}=\Gamma_{g,1}\cap IA_{2g}(k)$$

We write

$$\Gamma_{g,1}(k)\coloneqq \Gamma_{g,1}\big/\mathscr{I}_{g,1}(k)$$

for the quotients. When $k = 1$, $\pi/\gamma_2(\pi) = \pi^{\mathrm{ab}}$ is isomorphic to $H_g \coloneqq H_1(S_{g,1};\mathbb{Z}) \cong \mathbb{Z}^{2g}$. The action of $\Gamma_{g,1}$ on H_g preserves the intersection pairing, so the image of $\Gamma_{g,1}$ in $\mathrm{Aut}\, N_{2g}(1) \cong \mathrm{GL}(H_g)$ lies inside $\mathrm{Sp}(H_g)$. In fact, this image is the whole of $\mathrm{Sp}(H_g)$ ([21, Theorem 6.4]).

Lemma 2.2.2. *The filtration $\left\{\mathscr{I}_{g,1}(k)\right\}_{k\geqslant 1}$ of $\mathscr{I}_{g,1}$ is central.*

Proof. Immediate from Theorem 2.2.1. □

2.3 The infinite loop space structure

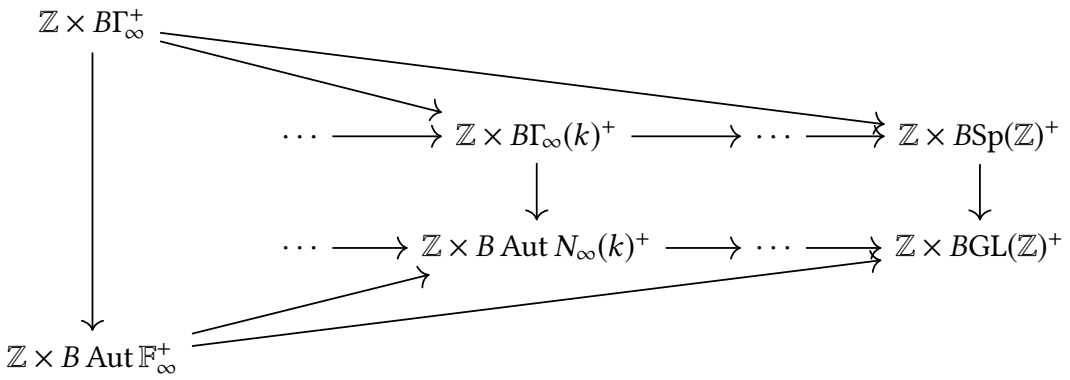
The pair-of-pants products $\Gamma_{g,1} \times \Gamma_{h,1} \to \Gamma_{g+h,1}$ can be described as follows: given a pair of elements of $\Gamma_{g,1}$ and $\Gamma_{h,1}$, represent these by diffeomorphisms of $S_{g,1}$ and $S_{h,1}$, respectively. Gluing these surfaces to two connected components of the boundary of the pair of pants $S_{0,3}$ yields a copy of $S_{g+h,1}$, and extending the diffeomorphisms of $S_{g,1}$ and $S_{h,1}$ by the identity on $S_{0,3}$ then gives a diffeomorphism of $S_{g+h,1}$. By choosing a suitable model of $B\Gamma_{g,1}$ — see [55, Section 4] — one can ensure that these homomorphisms endow the space $X \coloneqq \coprod_{g\geqslant 0} B\Gamma_{g,1}$ with the structure of a strictly associative monoid. Tillmann also shows that the group completion ΩBX of this topological monoid can be identified with $\mathbb{Z} \times B\Gamma_\infty^+$ and that this space is an infinite loop space ([55, Theorem 3.1]). Here $\Gamma_\infty \coloneqq \mathrm{colim}_g\, \Gamma_{g,1}$ is the stable mapping class group and + denotes the Quillen plus construction.

In [54], Tillmann showed that the monoid structure on X extends to the structure of an algebra over the surface operad. In [54, Theorem 4.4], she showed that the group completion of an algebra over the surface operad is an infinite loop space. (We briefly recalled a definition of group completion in Definition 1.3.6, but we direct the reader to [3, Section 3] for a very thorough explanation and a list of properties, as well as for a recollection of the group-completion theorem [41, Proposition 1]; see also Theorem 1.3.29 above.)

We show below that the spaces $X_k \coloneqq \coprod_{g\geqslant 0} B\Gamma_{g,1}(k)$ for $1 \leqslant k \leqslant \infty$, and the quotient maps $X_k \to X_{k'}$ for $k' \leqslant k$, can likewise be made into algebras, and maps of algebras, over the surface operad. (Here $\Gamma_{g,1}(\infty) \coloneqq \Gamma_{g,1}$.) Then [54, Theorem 4.4] gives us the following result.

Theorem 2.3.1. *For every $k \geqslant 1$, $\Omega BX_k \simeq \mathbb{Z} \times B\Gamma_\infty(k)^+$ is an infinite loop space. For every $1 \leqslant k' \leqslant k \leqslant \infty$, the quotient map $X_k \to X_{k'}$ group-completes to a map of infinite loop spaces.*

Remark 2.3.2. It is straightforward to verify that the inclusions $\Gamma_{g,1}(k) \hookrightarrow \operatorname{Aut} N_{2g}(k)$ also give rise to infinite loop space maps for all k (the case $k = \infty$ having been treated by Wahl [56, Theorem 1.1]), so there is a ladder of infinite loop space maps



Despite the formal similarities between the two rows, the relationship between them appears to be rather subtle. We remark that the squares

$$\begin{array}{ccc} \mathbb{Z} \times B\Gamma_\infty(k)^+ & \longrightarrow & \mathbb{Z} \times B\mathrm{Sp}(\mathbb{Z})^+ \\ \downarrow & & \downarrow \\ \mathbb{Z} \times B \operatorname{Aut} N_\infty(k)^+ & \longrightarrow & \mathbb{Z} \times B\mathrm{GL}(\mathbb{Z})^+ \end{array}$$

are not homotopy pullbacks. Indeed, Szymik shows that the bottom horizontal map is a rational homology equivalence ([52, Corollary 1.2]). Since the map comes from a map of spectra, it follows that the homotopy fibres of the bottom horizontal map are rationally acyclic. We show in Section 2.6.1 below that the top horizontal arrow is far from being a rational homology isomorphism, so its homotopy fibre must have non-trivial rational homology.

2.3.1 The surface operad

We briefly recall the construction of the operad $\mathcal{M}$ from [54] as it will be useful in the proof of Theorem 2.1.1. Recall that our surfaces $S_{g,n}$ come with a choice of parametrisations of collars around the boundary circles. From now on, we will only consider surfaces with non-empty boundary, and we will assume that each such surface comes with a choice of a marked boundary component. We fix three 'atomic' surfaces: a disc $D \coloneqq S_{0,1}$, a pair of pants $P \coloneqq S_{0,3}$ and a torus with two boundary components $T \coloneqq S_{1,2}$.

Let $\mathscr{E}_{g,n}$ be the groupoid whose objects are pairs (S, σ) where $S \cong S_{g,n+1}$ is a surface that is built from the atomic surfaces by gluing the marked boundary circle of one surface

to an unmarked boundary circle of another surface, and σ is an ordering of the unmarked boundary components of S. A morphism $(S,\sigma) \to (S',\sigma')$ in $\mathscr{E}_{g,n}$ is an isotopy class of diffeomorphisms $S \to S'$ that are compatible with the fixed collars, take the marked boundary circle to the marked boundary circle and preserve the ordering of the unmarked boundary circles. We will generally suppress σ from the notation.

Gluing of surfaces along boundary components yields functors

$$\gamma \colon \mathscr{E}_{g,n} \times \mathscr{E}_{h_1,i_1} \times \cdots \times \mathscr{E}_{h_n,i_n} \longrightarrow \mathscr{E}_{g+h,i}$$

where $h = \sum_j h_j$ and $i = \sum_j i_j$. These functors make the sequence $\left\{\coprod_{g\geqslant 0} \mathscr{E}_{g,n}\right\}_{n\geqslant 0}$ into an operad in groupoids, and the surface operad $\mathcal{M}$ is obtained from this operad by geometric realisation. The main result of [54] can now be stated as follows.

Theorem 2.3.3 ([54, Main Theorem], *cf.* also [3, Theorem 1.2])**.** *Group completion is a functor from the category of $\mathcal{M}$-algebras to the category of infinite loop spaces and infinite loop space maps.*

Remark 2.3.4. The papers [54], [57] first replace the groupoids $\mathscr{E}_{g,n}$ with equivalent quotient groupoids. The purpose of this is to ensure that the pair of pants gives rise to a strictly associative and unital product on algebras of the resulting topological operad. By the time of [3], it has been realised that this is unnecessary: one only needs a homotopy commutative E_1-structure for the above theorem to go through.

2.3.2 The $\mathcal{M}$-algebras $B\mathscr{E}_0(k)$

In this section we construct models of $\coprod_{g\geqslant 0} B\Gamma_{g,1}(k)$ with an $\mathcal{M}$-algebra structure.

Fix a basepoint on S^1; since our surfaces come with parametrisations of the boundary circles, this yields a choice of a basepoint in each boundary circle of a surface. We take the basepoint of $S \in \mathscr{E}_n \coloneqq \coprod_{g\geqslant 0} \mathscr{E}_{g,n}$ to be the basepoint of the marked boundary circle, and $\pi_1 S$ will always be understood as the fundamental group at this basepoint.

Consider the groupoid $\mathscr{E}_0$ of surfaces with a unique boundary component. All boundary-fixing diffeomorphisms $S \to S'$ of $S, S' \in \mathscr{E}_0$ in particular preserve the basepoints, yielding a well-defined set map

$$\mathscr{E}_0(S,S') \longrightarrow \mathrm{Iso}(\pi_1 S, \pi_1 S'). \tag{2.1}$$

This function is injective by the Dehn–Nielsen–Baer theorem, *c.f.* Section 2.2.2.

Definition 2.3.5. For $k \geqslant 1$, let $\sim_k$ be the equivalence relation on the hom-sets of $\mathscr{E}_0$ which is defined as follows: $\phi, \phi' \in \mathscr{E}_0(S, S')$ have $\phi \sim_k \phi'$ if and only if the images of ϕ and ϕ' under (2.1) induce the same isomorphism

$$\pi_1 S/\gamma_{k+1}(\pi_1 S) \longrightarrow \pi_1 S'/\gamma_{k+1}(\pi_1 S').$$

Let $\mathscr{E}_0(k) \coloneqq \mathscr{E}_0/\sim_k$ be the quotient groupoid.

Note that for every k and g, the quotient $\mathscr{E}_{g,0}/\sim_k$ is equivalent to $\Gamma_{g,1}(k)$. Hence we have

$$B\mathscr{E}_0(k) \simeq \coprod_{g \geqslant 0} B\Gamma_{g,1}(k).$$

We claim that $B\mathscr{E}_0(k)$ is an $\mathcal{M}$-algebra. This is a consequence of the following result.

Proposition 2.3.6. *Suppose that $\phi\colon A \to A'$ is a morphism in $\mathscr{E}_n$ and $\psi_i \sim_k \psi_i'\colon B_i \to B_i'$ are pairs of $\sim_k$-equivalent morphisms in $\mathscr{E}_0$. Then*

$$\gamma(\phi; \psi_1, \dots, \psi_n) \sim_k \gamma(\phi; \psi_1', \dots, \psi_n').$$

Proof. Let us first assume that $n = 1$, $\phi = \mathrm{id}_A$ and $B_1 = B_1'$. We drop the subscripts from B and ψ, ψ'.

By the van Kampen theorem, $\pi_1(\gamma(A; B)) \cong \pi_1 A *_\partial \pi$ where π is a subgroup of $\pi_1(\gamma(A; B))$ isomorphic to $\pi_1 B$ and $*_\partial$ denotes amalgamation over the identified boundary circle of A and B. Likewise $\pi_1(\gamma(A; B')) \cong \pi_1 A *_\partial \pi'$. The isomorphisms $\pi \cong \pi_1 B$ and $\pi' \cong \pi_1 B'$ are not canonical, but depend instead on a choice of paths in A from the basepoint of A to the basepoint on its unmarked boundary. If we let both of these isomorphisms be induced by the same path, we obtain the diagram

$$\begin{array}{ccc} \pi_1 A *_\partial \pi_1 B & \xrightarrow{\sim} & \pi_1(\gamma(A; B)) \\ {\scriptstyle \mathrm{id} *_\partial \psi_*}\downarrow & & \downarrow{\scriptstyle \gamma(A;\psi)_*} \\ \pi_1 A *_\partial \pi_1 B' & \xrightarrow{\sim} & \pi_1(\gamma(A; B')) \end{array}$$

which commutes since the class $\gamma(A; \psi)$ of diffeomorphisms $\gamma(A; B) \to \gamma(A; B')$ can be represented by a diffeomorphism which restricts to the identity on A.

The diagram remains commutative when ψ is replaced with ψ'. Now consider the effect of applying the functor $G \mapsto G/\gamma_{k+1}(G)\colon \mathrm{Grp} \to \mathrm{Grp}$ to both diagrams. The left vertical

arrows become equal to one another since $\psi \sim_k \psi'$. The horizontal arrows also agree, and so $\gamma(A;\psi) \sim_k \gamma(A;\psi')$.

Now suppose that $B \neq B'$. The arrows $\psi, \psi' \colon B \to B'$ are $\sim_k$-equivalent if and only if $\mathrm{id}_B, \psi^{-1}\psi' \colon B \to B$ are. Hence by the above, $\gamma(A;\mathrm{id}_B) \sim_k \gamma(A;\psi^{-1}\psi')$ if $\psi \sim_k \psi'$. Clearly $\gamma(A;\mathrm{id}_B) = \mathrm{id}_{\gamma(A;B)}$ and $\gamma(A;\psi^{-1}\psi') = \gamma(A;\psi)^{-1}\gamma(A;\psi')$, so $\gamma(A;\psi) \sim_k \gamma(A;\psi')$ in this case.

For the most general case, note that

$$\gamma(\phi;\psi_1,\dots,\psi_n) = \gamma(\phi;B_1',\dots,B_n') \prod_{i=1}^{n} \gamma(A;B_1,\dots,B_{i-1},\psi_i,B_{i+1}',\dots,B_n')$$

and

$$\gamma(A;B_1,\dots,B_{i-1},\psi_i,B_{i+1}',\dots,B_n') = \gamma(A_i,\psi_i)$$

where $A_i \coloneqq \gamma(A;B_1,\dots,B_{i-1},1,B_{i+1}',\dots,B_n') \in \mathscr{E}_1$. By the above,

$$\gamma(A_i;\psi_i) \sim_k \gamma(A_i;\psi_i'),$$

finishing the proof. □

Now we are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. By Proposition 2.3.6, we certainly have a tower $\cdots \to B\mathscr{E}_0(k) \to \cdots \to B\mathscr{E}_0(2) \to B\mathscr{E}_0(1)$ of $\mathcal{M}$-algebra maps which filters the projection $B\mathscr{E}_0 \to B\mathscr{E}_0(1)$. Hence it remains to argue that the group completion of $B\mathscr{E}_0(k)$ is precisely $\mathbb{Z} \times B\Gamma_\infty(k)^+$ for every k. But this is an easy consequence of the group completion theorem — *cf.* [3, Corollary 3.10] for a convenient formulation. □

2.4 $\mathscr{C}$-modules of finite degree

This section serves mainly to recall some definitions and constructions from the theory of functors valued in abelian groups.

Perhaps best developed is the theory of FI-modules, *i.e.* functors from the category FI to abelian groups. We turn to these in Section 2.4.4, where we prove Proposition 2.4.14 and deduce from it Corollary 2.4.20; these two results will be crucial in the proof of Theorem 2.1.2. Proposition 2.4.14 provides the converse implication to [46, Proposition 4.18].

2.4.1 PROP-modules

Let $\mathscr{C}$ be a small category and k a commutative unital ring. By a $\mathscr{C}$-module over k we will mean a functor $\mathscr{C} \to k\text{–Mod}$ to the category of k-modules; by a map of $\mathscr{C}$-modules, a natural transformation of functors $\mathscr{C} \to k\text{–Mod}$.

Remark. When k is omitted, it will be understood to be the ring of integers $\mathbb{Z}$.

We write $\mathscr{C}\text{–Mod}_k$ for the category of $\mathscr{C}$-modules over k. This is an abelian category, with kernels and cokernels computed pointwise. Hence it makes good sense to talk about sub-$\mathscr{C}$-modules, quotient $\mathscr{C}$-modules *etc.* See [58, A.3.3, A.4.3].

Definition 2.4.1 ([35, Section 24])**.** A *product and permutation category* (or *PROP* for short) is a small strict symmetric monoidal category $(\mathscr{C}, \oplus, 0)$ with objects the natural numbers such that the object function of $\oplus$ is $n \oplus m = n + m$.

Remark 2.4.2. PROPs also play a crucial part in the first chapter of this thesis; here they will, however, be used for an entirely different purpose. While in the first chapter we associated a PROP to every operad of interest, in the present chapter we shall restrict ourselves to two particularly simple PROPs, to be introduced in Section 2.4.3, and the interesting data shall instead be contained in functors out of them into an abelian category.

Let $\mathscr{C}$ be a PROP, suppose that $0 \in \mathscr{C}$ is initial, and consider the functor $R_{\mathscr{C}} \coloneqq 1 \oplus (-) : \mathscr{C} \to \mathscr{C}$. Since 0 is initial, the unique arrow $0 \to 1$ induces a natural transformation $\rho_{\mathscr{C}} : 0 \oplus (-) \Rightarrow 1 \oplus (-) = R_{\mathscr{C}}$ from the identity functor on $\mathscr{C}$.

Definition 2.4.3. Let F be a $\mathscr{C}$-module.

(i) The *suspension* of F is $\Sigma F \coloneqq F \circ R_{\mathscr{C}}$.

(ii) The *kernel* and *cokernel* of F, denoted $\operatorname{Ker} F$ and $\operatorname{Coker} F$ respectively, are the kernel and cokernel in the abelian category $\mathscr{C}\text{–Mod}$ of the morphism $F\rho_{\mathscr{C}} : F \to \Sigma F$.

Notation 2.4.4. To avoid confusion, ker and coker with small initials will always refer to the kernel and cokernel of a *morphism* in an abelian category, while Ker and Coker with capital initials will be reserved for the above construction.

Example 2.4.5. The poset $\mathbb{N}$, viewed as a category, is a PROP under integer addition. An $\mathbb{N}$-module F is a composable sequence $F_0 \to F_1 \to \cdots$ of homomorphisms. The morphism $F\rho_{\mathbb{N}} \colon F \to \Sigma F$ is

$$\begin{array}{ccccc} F_0 & \longrightarrow & F_1 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots \end{array}$$

The $\mathbb{N}$-module $\operatorname{Ker} F$ then consists of the kernels of the homomorphisms in F and zero homomorphisms, with a similar description of $\operatorname{Coker} F$.

Lemma 2.4.6. *A short exact sequence $0 \to F' \xrightarrow{\iota} F \xrightarrow{\pi} F'' \to 0$ of $\mathscr{C}$-modules gives rise to an exact sequence*

$$\begin{aligned} 0 \to \operatorname{Ker} F' \xrightarrow{\operatorname{Ker}\iota} \operatorname{Ker} F \xrightarrow{\operatorname{Ker}\pi} \operatorname{Ker} F'' \to \\ \to \operatorname{Coker} F' \xrightarrow{\operatorname{Coker}\iota} \operatorname{Coker} F \xrightarrow{\operatorname{Coker}\pi} \operatorname{Coker} F'' \to 0 \end{aligned}$$

Proof. This is a special case of the snake lemma. □

Let $\psi \colon \mathscr{D} \to \mathscr{C}$ be a morphism of PROPs, *i.e.* a symmetric monoidal functor which is the identity on objects. Let $\psi^* \colon \mathscr{C}\text{–Mod} \to \mathscr{D}\text{–Mod}$ be the functor that precomposes a given $\mathscr{C}$-module with ψ.

Lemma 2.4.7. *If $\psi \colon \mathscr{D} \to \mathscr{C}$ is a morphism of PROPs, then $\psi^* \colon \mathscr{C}\text{–Mod} \to \mathscr{D}\text{–Mod}$ commutes with Σ,* Ker *and* Coker.

Proof. Immediate since $R_{\mathscr{C}} \circ \psi = \psi \circ R_{\mathscr{D}}$ and $\rho_{\mathscr{C}}\psi = \psi\rho_{\mathscr{D}}$, and kernels and cokernels are computed pointwise. □

2.4.2 The degree of a $\mathscr{C}$-module

Let $\mathscr{C}$ be a PROP such that $0 \in \mathscr{C}$ is initial.

Definition 2.4.8 ([46, Definition 4.10])**.** A $\mathscr{C}$-module F has degree $d \leqslant -1$ if $F_n = 0$ for all sufficiently large n. For $d \geqslant 0$, F has degree d if $\operatorname{Ker} F$ has degree -1 and $\operatorname{Coker} F$ has degree $d - 1$.

Lemma 2.4.9. *Let $\psi \colon \mathscr{D} \to \mathscr{C}$ be a morphism of PROPs. Then a $\mathscr{C}$-module F has degree d if and only if the $\mathscr{D}$-module $\psi^* F$ has degree d.*

Proof. When $d < 0$, this is immediate since the object function of ψ is the identity. The case $d \geqslant 0$ follows by induction on d, using Lemma 2.4.7. □

2.4.3 The categories FI and SI

In this section we recall two more examples of PROPs, namely the category FI of finite sets and injections, and the category SI of free finitely generated symplectic $\mathbb{Z}$-modules and symplectic homomorphisms.

The category FI was introduced by Church, Ellenberg and Farb in [15] to study the representation theory of symmetric groups. It will be advantageous to consider the following skeleton of FI.

Definition 2.4.10. Let FI be the category of sets of the form $\underline{n} \coloneqq \{1, 2, \dots, n\}$ for $n \geqslant 0$, and set injections.

Disjoint union of sets makes FI into a PROP: the product of morphisms $f : \underline{n} \to \underline{m}$ and $g : \underline{k} \to \underline{l}$ is given by

$$(f \sqcup g)(i) = \begin{cases} f(i) & \text{if } i \leqslant n, \\ g(i-n) + m & \text{if } i > n. \end{cases}$$

The category SI was introduced by Putman and Sam in [45, Section 1.3] to study the representation theory of finite linear groups. We will again benefit from considering a specific skeleton rather than the entire category as it is defined in [45].

Definition 2.4.11. Given a commutative unital ring k, let V_n for $n \geqslant 0$ be the free k-module with basis $a_1, b_1, \dots, a_n, b_n$, equipped with the symplectic form $(\cdot, \cdot)$ given by

$$\begin{aligned} \left(a_i, a_j\right) &= 0 = \left(b_i, b_j\right) \\ \left(a_i, b_j\right) &= \delta_{ij} = -\left(b_j, a_i\right) \end{aligned}$$

for all $1 \leqslant i, j \leqslant n$, where δ_{ij} is the Kronecker delta. Let $\mathrm{SI}(k)$ be the category with objects the V_n and arrows the form-preserving k-linear maps. Let $\mathrm{SI} \coloneqq \mathrm{SI}(\mathbb{Z})$.

Lemma 2.4.12. *The following hold in* SI.

(i) SI *is a PROP, with the monoidal product given by symplectic direct sum.*

(ii) *Every morphism is injective. Hence* $\mathrm{End}_{\mathrm{SI}}(V_n) = \mathrm{Aut}_{\mathrm{SI}}(V_n) = \mathrm{Sp}(2n, \mathbb{Z})$.

(iii) *For every* $f : V_n \to V_m$ *in* SI*, there is a unique symplectic submodule* $C \leqslant V_m$ *such that* $V_m \cong (\operatorname{im} f) \oplus C$*. Moreover,* $C \cong V_{m-n}$.

(iv) *If f and C are as above and $s \in \mathrm{Aut}_{\mathrm{SI}}(V_m)$ satisfies $f = s \circ f$, then s preserves the direct sum decomposition $V_m \cong (\mathrm{im}\, f) \oplus C$ and $s = \mathrm{id} \oplus s'$ for some $s' \in \mathrm{Aut}(C)$.*

Proof. The first three statements follow immediately from [45, Example 1.17] (with an easy check in (i) that the symmetric monoidal structure can be made strict on the skeleton). To prove the last statement, note that s restricts to the identity on $\mathrm{im}\, f$. Hence by the uniqueness part of (iii), $s(C) = C$ and $s = \mathrm{id} \oplus s'$ where $s' = s|_C$. □

We shall write $\Psi : \mathrm{FI} \to \mathrm{SI}$ for the morphism of PROPs that takes $\underline{n}$ to V_n and $f : \underline{n} \to \underline{m}$ to the symplectic homomorphism $\Psi(f) : V_n \to V_m$ given by $\Psi(f)(a_i) = a_{f(i)}$ and $\Psi(f)(b_i) = b_{f(i)}$.

Remark 2.4.13. Both FI and SI arise from symmetric monoidal groupoids by means of a construction given in [46, Section 1.1]: FI arises from the groupoid of finite sets and bijections, whereas $\mathrm{SI}(k)$ arises from the groupoid of finitely generated free symplectic k-modules and symplectic isomorphisms.

2.4.4 FI-modules

FI-modules were introduced in [15] to study stability phenomena in the representation theory of symmetric groups, and much work has been done with them in recent years. We briefly recall the relevant terms and results; we refer the reader to [17] for a more thorough introduction. We also prove the following characterisation of FI-modules of finite degree (in the sense of Definition 2.4.8).

Proposition 2.4.14. *An* FI*-module has finite degree if and only if it is generated and related in finite degrees.*

We write $P^n := \mathbb{Z}\left\langle \mathrm{FI}(\underline{n}, -) \right\rangle$ for the n-th principal projective FI-module, *i.e.* the FI-module that sends $\underline{m} \in \mathrm{FI}$ to the free $\mathbb{Z}$-module generated by the set $\mathrm{FI}(\underline{n}, \underline{m})$. Recall that an FI-module F is generated in finite degree if it admits a surjection $\bigoplus_{i \in I} P^{n_i} \to F$ for some (possibly infinite) bounded collection $(n_i)_{i \in I}$ of non-negative integers. F is generated and related in finite degrees if it fits into an exact sequence of the form

$$\bigoplus_{j \in J} P^{m_j} \longrightarrow \bigoplus_{i \in I} P^{n_i} \longrightarrow F \longrightarrow 0$$

for some (possibly infinite) bounded collections $(n_i)_{i \in I}$ and $\left(m_j\right)_{j \in J}$ of non-negative integers.

Randal-Williams and Wahl proved in [46, Proposition 4.18] that an FI-module which is generated and related in finite degrees has finite degree. Hence we only need to prove the converse implication. Church and Ellenberg [14] give a characterisation of FI-modules that are generated in finite degrees in terms of an iterated Coker-construction.

Lemma 2.4.15 ([14, Proposition 4.6])**.** *An* FI*-module F is generated in finite degree if and only if* $\mathrm{Coker}^k F$ *has degree* -1 *for sufficiently large* k*.*

Warning. The notation of [14] is very different from ours: the functor Coker is called the *derivative* and denoted by D. The *degree* of an FI-module F in the sense of [14] is the least integer $\deg F \geqslant -1$ such that $F_i = 0$ for all $i > \deg F$ (or ∞ if no such integer exists). In particular, an FI-module has degree -1 in the sense of Definition 2.4.8 if and only if it has finite degree in the sense of [14]. This is the reason why we elected to include the statement of the lemma instead of simply referring to [14].

Remark 2.4.16. The symbol Coker^k denotes the k-fold iteration of Coker: FI–Mod $\to$ FI–Mod. This should not be confused with the cokernel of the composition

$$F \xrightarrow{F\rho} FR \xrightarrow{FR\rho} FR^2 \longrightarrow \cdots \longrightarrow FR^k. \tag{2.2}$$

Indeed, $\mathrm{Coker}^k F$ is the cokernel of the map

$$\lambda^k \colon \bigoplus_{i=1}^{k} FR^{k-1} \longrightarrow FR^k$$

whose i-th component $\lambda_i^k \colon FR^{k-1} \to FR^k$ is the natural transformation

$$FR^{i-1}\rho R^{k-i} \colon FR^{i-1}R^{k-i} \longrightarrow FR^{i-1}RR^{k-i}.$$

This is can be deduced by induction on k: when $k = 1$, this is just the definition of Coker. If the formula holds for some k, then there is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^{k} FR^{k-1} & \xrightarrow{\lambda^k} & FR^k & \longrightarrow & \mathrm{Coker}^k F & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \lambda_{k+1}^{k+1}} & & \downarrow & & \\
\bigoplus_{i=1}^{k} FR^{k} & \xrightarrow{\lambda^k R} & FR^{k+1} & \longrightarrow & \left(\mathrm{Coker}^k F\right) R & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Coker}^{k+1} F & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

where the top vertical arrows are compositions of the functors on the top line with the natural transformation $\rho\colon \mathrm{Id} \Rightarrow R$ (see section 2.4.1) and we use the evident fact that precomposition with R (*i.e.* the functor $\Sigma\colon \mathrm{FI\text{–}Mod} \to \mathrm{FI\text{–}Mod}$) is exact — *c.f.* [14, Section 2.2]. Considering the left-hand square of the above diagram as a double complex, its associated spectral sequence computes the homology of the associated double complex. The spectral sequence collapses on the second page for shape reasons: there are no differentials between two non-zero groups. Hence the second-page entry in the place of FR^{k+1}, which is precisely $\mathrm{Coker}^{k+1} F$ by the exactness of the above diagram, is isomorphic to the corresponding homology group of the total complex, which is precisely the cokernel of the map

$$\lambda^{k+1} = \lambda^k R \oplus \lambda^{k+1}_{k+1}\colon \bigoplus_{i=1}^{k} FR^k \oplus FR^k \longrightarrow FR^{k+1},$$

as claimed.

Note that the composition (2.2) factors (non-uniquely) through λ, so the cokernel of (2.2) admits surjective maps to $\mathrm{Coker}^k F$.

Proof of Proposition 2.4.14. As mentioned above, in light of [46, Proposition 4.18] we only need to prove that a finite-degree FI-module F is generated and related in finite degrees. By the preceding lemma, F is generated in finite degree. Hence there is a short exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{i\in I} P^{n_i} \xrightarrow{\pi} F \longrightarrow 0$$

where the n_i are bounded. Hence it suffices to prove that K is generated in finite degree.

For $k \geqslant 0$, let $K^{(k)}$ be the kernel of $\mathrm{Coker}^k \pi\colon \mathrm{Coker}^k(\bigoplus_i P^{n_i}) \to \mathrm{Coker}^k F$. Since Coker is a right-exact functor, there are canonical surjections $u^{(k)}\colon \mathrm{Coker}^k K \to K^{(k)}$.

Claim. *For each k, there is an integer N_k such that $u^{(k)}_n\colon \left(\mathrm{Coker}^k K\right)_n \to K^{(k)}_n$ is an isomorphism for all $n \geqslant N_k$.*

Proof of claim. We proceed by induction on k. When $k = 0$, $u^{(0)}$ is the identity and there is nothing to prove. Suppose the claim holds for $k-1$. By Lemma 2.4.6 applied to the short exact sequence

$$0 \longrightarrow K^{(k-1)} \longrightarrow \mathrm{Coker}^{k-1}\left(\bigoplus_i P^{n_i}\right) \xrightarrow{\mathrm{Coker}^{k-1}\pi} \mathrm{Coker}^{k-1} F \longrightarrow 0,$$

there is an exact sequence

$$\operatorname{Ker}\operatorname{Coker}^{k-1} F \longrightarrow \operatorname{Coker} K^{(k-1)} \longrightarrow \operatorname{Coker}^{k}\left(\bigoplus_i P^{n_i}\right) \longrightarrow \operatorname{Coker}^{k} F \longrightarrow 0.$$

Since F has finite degree, $\operatorname{Ker}\operatorname{Coker}^{k-1} F$ has degree -1. In other words, the induced surjection $\operatorname{Coker} K^{(k-1)} \twoheadrightarrow K^{(k)}$ is an isomorphism in all but finitely many degrees. Hence by the induction hypothesis,

$$u^{(k)}\colon \operatorname{Coker}^{k} K \xrightarrow{\operatorname{Coker} u^{(k-1)}} \operatorname{Coker} K^{(k-1)} \longrightarrow K^{(k)}$$

is also an isomorphism in all but finitely many degrees. This concludes the proof of the claim.

To complete the proof of the proposition, note that for large enough k, $\operatorname{Coker}^{k}(\bigoplus_i P^{n_i})$ has degree -1 by Lemma 2.4.15. But then $K^{(k)}$ also has degree -1 since it is a submodule of $\operatorname{Coker}^{k}(\bigoplus_i P^{n_i})$. Hence by the claim, $\operatorname{Coker}^{k} K$ has degree -1, and we are done by Lemma 2.4.15. □

We deduce a corollary with the help of the following lemma.

Lemma 2.4.17. *For every $n, m \geqslant 0$, the* FI*-module*

$$P^n \otimes P^m\colon \mathrm{FI} \xrightarrow{P^n \times P^m} \mathrm{Ab} \times \mathrm{Ab} \xrightarrow{\otimes} \mathrm{Ab}$$

is a finite direct sum of principal projectives.

Proof. Let $X\colon \mathrm{FI} \to \mathrm{Set}$ be the set-valued functor given by $\underline{k} \mapsto \mathrm{FI}(\underline{n}, \underline{k}) \times \mathrm{FI}(\underline{m}, \underline{k})$. To each $(f, g) \in X_k$, we associate a partial injection $p(f, g)\colon \underline{n} \nrightarrow \underline{m}$ as follows. The domain of definition of $p(f, g)$ is $\operatorname{dom} p(f, g) \coloneqq f^{-1}(\operatorname{im} f \cap \operatorname{im} g) \subseteq \underline{n}$. On this domain, $p(f, g)$ is given by

$$p(f, g)\colon f^{-1}(\operatorname{im} f \cap \operatorname{im} g) \xrightarrow{f} \operatorname{im} f \cap \operatorname{im} g \xrightarrow{g^{-1}} g^{-1}(\operatorname{im} f \cap \operatorname{im} g) \subseteq \underline{m}.$$

It is easily seen that for any injection $h\colon \underline{k} \to \underline{l}$, the partial injections $p(f, g)$ and $p(h \circ f, h \circ g)$ are identical. Thus if we fix a partial injection $p\colon \underline{n} \nrightarrow \underline{m}$, then the subsets

$$X(p)_k \coloneqq \{(f, g) \in X_k \mid p(f, g) = p\} \subseteq X_k$$

form a sub-FI-set $X(p)$ of X, and moreover

$$X = \coprod_{p\colon \underline{n}\twoheadrightarrow\underline{m}} X(p) \tag{2.3}$$

where the disjoint union runs over all partial injections.

Given a partial injection $p\colon \underline{n} \twoheadrightarrow \underline{m}$, let $\underline{n} \cup_p \underline{m}$ be the pushout (in Set) of the span $\underline{n} \hookleftarrow \operatorname{dom} p \xrightarrow{p} \underline{m}$. Pairs $(f, g) \in X(p)_k$ biject naturally (in $\underline{k} \in \mathrm{FI}$) with injections $\underline{n} \cup_p \underline{m} \hookrightarrow \underline{k}$, so $X(p)$ is corepresented by $\underline{n} \cup_p \underline{m}$. If $\mathbb{Z}\langle - \rangle\colon \mathrm{Set} \to \mathrm{Ab}$ is the free $\mathbb{Z}$-module functor, then clearly $\mathbb{Z}\langle X \rangle \cong P^n \otimes P^m$. Hence

$$P^n \otimes P^m \cong \bigoplus_{p\colon \underline{n}\twoheadrightarrow\underline{m}} P^{|\underline{n}\cup_p\underline{m}|}$$

as required. □

Remark 2.4.18. The preceding lemma is a mild sharpening of [15, Proposition 2.3.6]. We present the above proof because we do need the stronger conclusion that the tensor product of two principal projectives is a direct sum of principal projectives, rather than just finitely generated, which is the statement in [15].

Remark 2.4.19. Note that $\left|\mathrm{FI}(\underline{n}, \underline{k})\right| = n!\binom{k}{n}$. Hence by taking cardinalities of both sides of (2.3) in degree k, we obtain the following rather pleasing identity:

$$\sum_i i!(n+m-i)!\binom{n}{i}\binom{m}{i}\binom{k}{n+m-i} = n!m!\binom{k}{n}\binom{k}{m}$$

where $i!\binom{n}{i}\binom{m}{i}$ is the number of partial injections $p\colon \underline{n} \twoheadrightarrow \underline{m}$ with $|\operatorname{dom} p| = i$.

Corollary 2.4.20. *If F, F' are* FI*-modules of finite degree, then so is $F \otimes F'$.*

Proof. By Proposition 2.4.14, there are exact sequences

$$L_1 \longrightarrow L_0 \longrightarrow F \longrightarrow 0$$
$$L_1' \longrightarrow L_0' \longrightarrow F' \longrightarrow 0$$

where L_i, L_i' are free and generated in finite degrees. Thus there is an exact sequence

$$(L_1 \otimes L_0') \oplus (L_0 \otimes L_1') \longrightarrow L_0 \otimes L_0' \longrightarrow F \otimes F' \longrightarrow 0.$$

By Lemma 2.4.17, the first two terms in the above sequence are free and generated in finite degrees. Hence we are done by Proposition 2.4.14. □

2.4.5 Duplication

Consider the functor

$$D\colon \mathrm{FI} \xrightarrow{\Delta} \mathrm{FI}\times\mathrm{FI} \xrightarrow{\sqcup} \mathrm{FI}$$

where Δ is the diagonal functor. D is clearly not a morphism of PROPs, so Lemma 2.4.9 doesn't apply, but we still have the following result.

Lemma 2.4.21. *If $F \in \mathrm{FI}\text{–}\mathrm{Mod}$ is finitely generated, then so is $D^*F = F \circ D$.*

Proof. The functor $D^*\colon \mathrm{FI}\text{–}\mathrm{Mod} \to \mathrm{FI}\text{–}\mathrm{Mod}$ is clearly exact. Hence it suffices to prove that D^*P^n is a finitely-generated FI-module for every n. Since every $(D^*P^n)_k = P^n_{2k}$ is a finitely generated abelian group, it suffices to prove that D^*P_n is generated in finite degree.

Let $f\colon \underline{n} \to \underline{2k}$ be a generator of $(D^*P^n)_k$ with $k > n$. Then there is some $i \in \underline{k}$ such that both $i, i+k \in \underline{2k}$ are *not* in the image of f. Let $g\colon \underline{k-1} \to \underline{k}$ be the unique order-preserving injection which misses i. Then f factors as $f = (g \sqcup g) \circ f'$ for a suitable injection $f'\colon \underline{n} \to \underline{2k-2}$, so D^*P^n is generated in degrees $\leqslant n$. □

2.5 $\mathscr{C}$-modules of automorphisms and mapping classes

In this section we give constructions of FI- and SI-modules which will play a role in our proof of Theorem 2.1.2.

2.5.1 FI-modules of automorphisms of free nilpotent groups

Suppose that G is an FI-group, *i.e.* a functor from FI to the category Grp of groups. Since the assignment $H \mapsto \gamma_k(H)$ is an endofunctor on Grp, we obtain a filtration of G by sub-FI-groups $\gamma_k(G)$, and a graded FI-Lie algebra $\mathrm{gr}\, G$ with

$$(\mathrm{gr}\, G)_n = \bigoplus_{k=1}^{\infty} \gamma_k(G_n)/\gamma_{k+1}(G_n).$$

We write $\mathrm{gr}_k G$ for the FI-module $\gamma_k(G)/\gamma_{k+1}(G)$.

Remark 2.5.1. By a graded Lie algebra over $\mathbb{Z}$ we mean a Lie algebra $\mathfrak{g}$ whose Lie bracket satisfies $[x,x] = 0$ for all $x \in \mathfrak{g}$, along with a direct sum decomposition $\mathfrak{g} = \bigoplus_{k=1}^{\infty} \mathfrak{g}_k$ such that $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$. In particular we do *not* require the Lie bracket to be graded-antisymmetric.

Let $\mathbb{F}\colon \mathrm{FI} \to \mathrm{Grp}$ be the restriction to FI of the free group functor $\mathrm{Set} \to \mathrm{Grp}$. Thus $\mathbb{F}_n$ is the free group generated by the set $\underline{n}$, although we will denote its generators $x_1, \ldots, x_n$ to avoid confusion.

Lemma 2.5.2. *Each* $\mathrm{gr}_k \mathbb{F}$ *is a finitely generated* FI*-module taking values in free abelian groups.*

Proof. It is a standard fact that $\left(\mathrm{gr}_k \mathbb{F}\right)_n = \gamma_k\left(\mathbb{F}_n\right)/\gamma_{k+1}\left(\mathbb{F}_n\right)$ is free abelian and of finite rank — see *e.g.* [50, Corollary I.IV.6.2] for a textbook account.

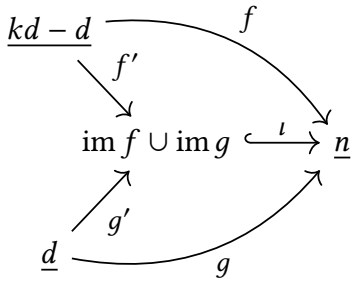
The rest of the lemma is a consequence of the following claim.

Claim. *Let* $L = \bigoplus_{k\geqslant 1} L(k)$ *be a graded* FI*-Lie algebra over* $\mathbb{Z}$, i.e. *for each* $\underline{n} \in \mathrm{FI}$, $L_n = \bigoplus_{k\geqslant 1} L(k)_n$ *is a graded Lie* $\mathbb{Z}$*-algebra. Assume that each* L_n *is generated (as a Lie algebra) by* $L(1)_n$*. If the* FI*-module* $L(1)$ *is generated in degrees* $\leqslant d$*, then each* $L(k)$ *is generated in degrees* $\leqslant kd$*.*

Proof of claim. We proceed by induction on k. Since each L_n is generated (as a Lie algebra) by $L(1)_n$, we have $L(k)_n = [L(k-1)_n, L(1)_n]$. Suppose that $n > kd$. By the induction hypothesis,

$$L(k)_n = \sum_{\substack{f\colon \underline{kd-d}\hookrightarrow \underline{n} \\ g\colon \underline{d}\hookrightarrow \underline{n}}} \left[f_*\left(L(k-1)_{kd-d}\right), g_*\left(L(1)_d\right)\right].$$

Every pair of injections $(f\colon \underline{kd-d} \to \underline{n}, g\colon \underline{d} \to \underline{n})$ factors as



where $|\mathrm{im}\, f \cup \mathrm{im}\, g| \leqslant kd$. Hence

$$L(k)_n = \sum_{\substack{f\colon \underline{kd-d}\hookrightarrow \underline{n} \\ g\colon \underline{d}\hookrightarrow \underline{n}}} \iota_*\left[f'_*\left(L(k-1)_{kd-d}\right), g'_*\left(L(1)_d\right)\right]$$

is generated in degree $\leqslant kd$. This concludes the proof of the claim.

Thus each $\mathrm{gr}_k \mathbb{F}$ is generated in degree $\leqslant k$. Since $\mathrm{gr}_k \mathbb{F}$ takes values in finitely generated abelian groups, it is necessarily finitely generated. □

2.5.1.1 Free nilpotent groups and their automorphisms

The groups $N_n(k) = \mathbb{F}_n/\gamma_{k+1}(\mathbb{F}_n)$ for fixed k and varying n fit into an FI-group $N(k) \coloneqq \mathbb{F}/\gamma_{k+1}(\mathbb{F})$. For every $k > 1$, there is an extension of FI-groups

$$0 \longrightarrow \operatorname{gr}_k \mathbb{F} \longrightarrow N(k) \longrightarrow N(k-1) \longrightarrow 1.$$

For every $i < k$, there is a canonical isomorphism $\gamma_i(\mathbb{F})/\gamma_{k+1}(\mathbb{F}) \cong \gamma_i(N(k))$ induced by the quotient morphism $\mathbb{F} \to N(k)$.

Consider the groups $\operatorname{Aut} N_n(k)$ of automorphisms of $N_n(k)$. For a fixed k, these also form an FI-group as follows: given $f\colon \underline{n} \hookrightarrow \underline{m}$ and $\phi \in \operatorname{Aut} N_n(k)$, the induced homomorphism $f_*(\phi)\colon N_m(k) \to N_m(k)$ is defined by

$$f_*(\phi)(x_i) = \begin{cases} x_i & \text{if } i \notin \operatorname{im} f, \\ (N(k)(f) \circ \phi)(x_j) & \text{if } i = f(j) \text{ for } j \in \underline{n}. \end{cases}$$

It is an easy check that $f_*(\phi)$ is an automorphism (indeed, $f_*(\phi^{-1})$ is its inverse) and that the assignment $\phi \mapsto f_*(\phi)\colon \operatorname{Aut} N_n(k) \to \operatorname{Aut} N_m(k)$ is a homomorphism.

There are morphisms of FI-groups $\rho(k)\colon \operatorname{Aut} N(k) \to \operatorname{Aut} N(k-1)$ defined as follows: an automorphism $\phi \in \operatorname{Aut} N_n(k)$ preserves $\gamma_k(N_n(k))$ and therefore descends to an endomorphism ϕ' of $N_n(k-1)$. The inverse ϕ^{-1} descends to the inverse of ϕ', so in fact $\phi' \in \operatorname{Aut} N_n(k-1)$. We define $\rho_n(k)$ by letting it send ϕ to ϕ'.

2.5.1.2 The kernel of $\rho(k)$

Suppose that $k > 1$. Consider the function

$$\begin{aligned} \chi_n(k) \;:\; \operatorname{Hom}(\mathbb{Z}^n, \operatorname{gr}_k \mathbb{F}_n) \;\cong\; \operatorname{Hom}(N_n(k), \gamma_k(N_n(k))) &\longrightarrow \operatorname{Aut} N_n(k) \\ f &\longmapsto \left(A_f\colon x \mapsto f(x)x\right) \end{aligned}$$

where the isomorphism holds since $\operatorname{gr}_k \mathbb{F}_n \cong \gamma_k(N_n(k))$ is abelian. This function is well defined: since each $f\colon N_n(k) \to \gamma_k(N_n(k))$ lands in the centre of $N_n(k)$, the assignment $A_f\colon x \mapsto f(x)x$ is an endomorphism of $N_n(k)$. Moreover, if $f'\colon N_n(k) \to \gamma_k(N_n(k))$ is another homomorphism, then $A_f \circ A_{f'} = A_{f+f'}$ since f vanishes on the image of f'. Thus in particular each A_f is an automorphism of $N_n(k)$, and $\chi_n(k)$ is a homomorphism. Clearly $\rho_n(k) \circ \chi_n(k) = 0$. In fact, $\chi_n(k)$ is an isomorphism of $\operatorname{Hom}(\mathbb{Z}^n, \operatorname{gr}_k \mathbb{F}_n)$ with $\ker \rho_n(k)$, with two-sided inverse given by

$$\begin{aligned} \psi_n(k) \;:\; \ker \rho_n(k) &\longrightarrow \operatorname{Hom}(N_n(k), \gamma_k(N_n(k))) \;\cong\; \operatorname{Hom}(\mathbb{Z}^n, \operatorname{gr}_k \mathbb{F}_n) \\ \phi &\longmapsto \left(x \mapsto \phi(x)x^{-1}\right) \end{aligned}$$

Let $\mathbb{Z}^\bullet\colon \mathrm{FI}^{\mathrm{op}} \to \mathrm{Ab}$ be the $\mathrm{FI}^{\mathrm{op}}$-module taking $\underline{n}$ to $\mathbb{Z}^n$ and $f\colon \underline{n} \hookrightarrow \underline{m}$ to the projection

$$\begin{aligned} f^*\colon \mathbb{Z}^m &\longrightarrow \mathbb{Z}^n \\ x_i &\longmapsto \begin{cases} 0 & \text{if } i \notin \operatorname{im} f \\ x_j & \text{if } i = f(j) \end{cases} \end{aligned}$$

Let $K(k)$ be the FI-module given by

$$K(k)\colon \mathrm{FI} \xrightarrow{(\mathbb{Z}^\bullet)\times \mathrm{gr}_k \mathbb{F}} \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \xrightarrow{\mathrm{Hom}(-,-)} \mathrm{Ab}.$$

We leave it to the reader to verify that the $\psi_n(k)$ are the components of a natural isomorphism of FI-modules $\psi(k)\colon \ker \rho(k) \to K(k)$.

Lemma 2.5.3. *For every $k \geqslant 1$, the* FI*-module* $\ker \rho(k)$ *is finitely generated and takes values in free abelian groups.*

Proof. There is an isomorphism (*cf.* [53, Proposition 3.1])

$$\ker \rho(k) \cong K(k) \cong (\mathbb{Z}^\bullet)^* \otimes \mathrm{gr}_k \mathbb{F} \cong \mathrm{gr}_1 \mathbb{F} \otimes \mathrm{gr}_k \mathbb{F}$$

where $(\mathbb{Z}^\bullet)^*$ is the dual FI-module, obtained by composing $\mathbb{Z}^\bullet\colon \mathrm{FI}^{\mathrm{op}} \to \mathrm{Ab}$ with the functor $\mathrm{Hom}(-,\mathbb{Z})\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$. It is easily seen that $(\mathbb{Z}^\bullet)^* \cong \mathrm{gr}_1 \mathbb{F}$. Thus $\ker \rho(k)$ certainly takes values in free abelian groups. Finite generation then follows from Lemma 2.5.2 and [15, Proposition 2.3.6]. □

2.5.2 SI-modules from the Johnson filtration

Suppose that $\phi\colon S_{g',1} \hookrightarrow S_{g,1}$ is an orientation-preserving embedding of surfaces. We can extend a given mapping class on $S_{g',1} \cong \operatorname{im}\phi$ by the identity on the complement $S_{g,1} \setminus \operatorname{im}\phi$ to obtain a well-defined mapping class on $S_{g,1}$. This construction produces a well-defined homomorphism $\phi_\flat\colon \Gamma_{g',1} \to \Gamma_{g,1}$.

There is an obvious embedding $T\colon S_{g',1} \hookrightarrow S_{g,1}$ for $g' \leqslant g$ as the subsurface consisting of the g' rightmost tori in Figure 2.1. The induced homomorphism $T_\flat\colon \Gamma_{g',1} \to \Gamma_{g,1}$ extends a given automorphism of $\mathbb{F}_{2g'}$ (representing a mapping class on $S_{g',1}$) to an automorphism of $\mathbb{F}_{2g}$ by the identity on the remaining generators x_i, y_i for $i > g'$. It is easy to see that $T_\flat$ preserves the Johnson filtration. In fact, this is true for any embedding:

Lemma 2.5.4. *Let $\phi\colon S_{g',1} \hookrightarrow S_{g,1}$ be an embedding. Then $\phi_\flat$ preserves the Johnson filtration.*

Proof. The homomorphism $\phi_\flat$ only depends on the isotopy class of ϕ. Hence we may assume that $\operatorname{im}\phi$ is disjoint from some collar of $\partial S_{g,1}$, and therefore that $S_{g,1} \setminus \operatorname{im}\phi$ is connected. Let S be the closure of $S_{g,1} \setminus \operatorname{im}\phi$. By the classification of oriented surfaces, $S \cong S_{g-g',2}$. Hence we can extend ϕ to a diffeomorphism $\widetilde{\phi}$ of $S_{g,1}$ in such a way that $\phi = \widetilde{\phi} \circ T$. But then $\phi_\flat$ and $T_\flat$ are conjugate by $\left[\widetilde{\phi}\right]$ in $\Gamma_{g,1}$. □

Thus $\phi_\flat$ descends to homomorphisms

$$\mathscr{I}_{g',1}(k)/\mathscr{I}_{g',1}(k+1) \longrightarrow \mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1)$$

which we will also denote $\phi_\flat$ without risk of confusion.

Recall from Section 2.2.2 that we write H_g for the middle homology group of the surface $S_{g,1}$. Equipped with the intersection pairing, H_g is isomorphic to V_g as a symplectic space.

Lemma 2.5.5.

(i) *Every symplectic homomorphism $f\colon H_{g'} \to H_g$ is induced by an orientation-preserving embedding $\phi\colon S_{g',1} \hookrightarrow S_{g,1}$ of surfaces.*

(ii) *If two orientation-preserving embeddings $\phi, \phi'\colon S_{g',1} \hookrightarrow S_{g,1}$ induce the same map on homology in dimension 1, then there is some $\psi \in \mathrm{Diff}^+(S_{g,1}, \partial)$ which represents a class in $\mathscr{I}_{g,1}$ and such that ϕ' and $\psi \circ \phi$ are isotopic.*

Proof.

(i) Recall the standard embedding $T\colon S_{g',1} \hookrightarrow S_{g,1}$ which induces the homomorphism $T_*\colon H_{g'} \to H_g$ that sends $[x_i]$ to $[x_i]$ and $[y_i]$ to $[y_i]$. By Lemma 2.4.12(iii), $\mathrm{Sp}(H_g)$ acts transitively on the set of symplectic maps $H_{g'} \to H_g$. Hence there is some $\psi \in \mathrm{Diff}^+(S_{g,1}, \partial)$ which takes T_* to f, *i.e.* ψ satisfies $(\psi \circ T)_* = f$. Then $\psi \circ T\colon S_{g',1} \hookrightarrow S_{g,1}$ is an embedding with the required property.

(ii) This is the content of [16, Lemma 4.1(ii)]. □

Write $Q_g(k) \coloneqq \mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1)$. We show that these groups fit into an SI-module $Q(k)$.

Proposition 2.5.6. *For every $k > 1$, there is an* SI*-module $Q(k)$ which takes H_g to $Q_g(k)$ and $f\colon H_{g'} \to H_g$ to $\phi_\flat\colon Q_{g'}(k) \to Q_g(k)$, where ϕ is as in Lemma 2.5.5*(i).

Proof. If it exists, $Q(k)$ is fully determined by the claim. We only need to check that the construction of $Q(k)$ is well defined on morphisms: indeed, suppose that two embeddings $\phi, \phi' \colon S_{g',1} \hookrightarrow S_{g,1}$ induce the same homomorphism $f \colon H_{g'} \to H_g$. Let ψ be as in Lemma 2.5.5(ii). Then $\phi'_\flat = (\psi \circ \phi)_\flat$. In other words, $\phi_\flat$ and $\phi'_\flat$ are conjugate by $[\psi]$ in $\Gamma_{g,1}$. Since $[\psi] \in \mathscr{I}_{g,1}$ and the Johnson filtration is central, it follows that $\phi_\flat$ and $\phi'_\flat$ induce the same homomorphism $Q_{g'}(k) \to Q_{g(k)}$. □

Recall the functor $\Psi \colon \mathrm{FI} \to \mathrm{SI}$ from Section 2.4.3. The following theorem is a consequence of [16, Theorem A], which was proved using the concept of central stability. We present a simple proof to make the present discussion more self-contained.

Theorem 2.5.7. *For every $k > 1$, $\Psi^* Q(k)$ is a finitely generated* FI*-module. It takes values in free abelian groups.*

Proof. For every g, there is an injection $\Gamma_{g,1}(k+1) \hookrightarrow \operatorname{Aut} N_{2g}(k+1)$. $Q_g(k)$ is mapped into $\ker \rho_{2g}(k+1)$ by this injection. We claim that these injections $j_g \colon \Psi^* Q_g(k) \hookrightarrow \ker \rho_{2g}(k+1)$ are the components of an injection

$$j \colon \Psi^* Q(k) \hookrightarrow D^* \ker \rho(k+1),$$

where $D \colon \mathrm{FI} \to \mathrm{FI}$ is the duplication functor of Section 2.4.5. Note that we only need to check naturality with respect to the inclusions $\underline{g-1} \hookrightarrow \underline{g}$ and automorphisms of $\underline{g}$, as these arrows generate all arrows of FI.

Naturality with respect to the inclusions $\underline{g-1} \hookrightarrow \underline{g}$ follows easily from the definitions. Given a permutation $\sigma \in \mathrm{Aut}_{\mathrm{FI}}(\underline{g})$, let $\widetilde{\sigma} \in \Gamma_{g,1}$ be a lift of $\Psi(\sigma) \in \mathrm{Aut}_{\mathrm{SI}}(H_g) = \mathrm{Sp}(H_g)$. Let also $s \in \operatorname{Aut} \mathbb{F}_{2g}$ be the obvious lift of $\Psi(\sigma)$, namely the automorphism sending $x_i \mapsto x_{\sigma(i)}$ and $y_i \mapsto y_{\sigma(i)}$. In general, $s \notin \Gamma_{g,1}$ because it does not preserve $\prod_i [x_i, y_i]$. We wish to prove that the diagram

$$\begin{array}{ccc} \mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1) & \xhookrightarrow{j_g} & \ker \rho_{2g}(k+1) \\ \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \sigma_*} \\ \mathscr{I}_{g,1}(k)/\mathscr{I}_{g,1}(k+1) & \xhookrightarrow{j_g} & \ker \rho_{2g}(k+1) \end{array}$$

commutes, where the left vertical arrow is conjugation by $\widetilde{\sigma}$, while the right vertical arrow is conjugation by s.

Since both s and $\widetilde{\sigma}$ induce the same automorphism of $\mathbb{F}_{2g}/\gamma_2(\mathbb{F}_{2g})$, $s^{-1}\widetilde{\sigma}$ is an element of $IA_{2g}(1)$. For every $\phi \in \mathscr{I}_{g,1}(k)$, the commutator $\left[s^{-1}\widetilde{\sigma}, \phi\right]$ lies in $IA_{2g}(k+1)$ by Theorem 2.2.1. Thence $s\phi s^{-1}$ and $\widetilde{\sigma}\phi\widetilde{\sigma}^{-1}$ induce the same automorphism of $N_{2g}(k+1)$ and j is natural with respect to σ.

$D^* \ker \rho(k+1)$ is finitely generated by Lemma 2.4.21. Hence $\Psi^* Q(k)$ is finitely generated by [17, Theorem A]. □

2.6 Homological stability for the groups $\Gamma_{g,1}(k)$

Fix embeddings $\phi\colon S_{g,1} \hookrightarrow S_{g+1,1}$ for all g. Each such embedding induces a homomorphism $\phi_\flat\colon \Gamma_{g,1} \to \Gamma_{g+1,1}$ which preserves the Johnson filtration. Moreover, we can think of $\phi_*\colon H_1(S_{g,1}) \to H_1(S_{g+1,1})$ as an arrow in SI. If F is an SI-module, we obtain a homomorphism $F(\phi_*)\colon F_g \to F_{g+1}$ which is equivariant with respect to $\phi_\flat\colon \Gamma_{g,1} \to \Gamma_{g+1,1}$. Thus for every k and i, we obtain a sequence

$$\cdots \longrightarrow H_i(\Gamma_{g,1}(k); F_g) \longrightarrow H_i(\Gamma_{g+1,1}(k); F_{g+1}) \longrightarrow \cdots$$

of homomorphisms. It can be seen that this sequence is in fact independent of the choice of embeddings ϕ. For if $\phi'\colon S_{g,1} \hookrightarrow S_{g+1,1}$ is another embedding, then for some $\psi \in \mathrm{Diff}(S_{g+1,1}, \partial)$, the embeddings ϕ and $\psi \circ \phi'$ are isotopic. Hence the homomorphisms

$$H_i(\Gamma_{g,1}; F_g) \longrightarrow H_i(\Gamma_{g+1,1}; F_{g+1})$$

induced by ϕ and by ϕ' differ by the conjugation action of ψ on the codomain, which is well-known to be trivial (*cf.* [12, Proposition III.8.1]). We need one final ingredient before we prove Theorem 2.1.2, and that is homological stability for symplectic groups. The first result in this direction was obtained by Charney [13], but we will use a version which is due to Randal-Williams and Wahl and which has much weaker hypotheses on the coefficient modules.

Theorem 2.6.1 ([46, Theorem 5.15])**.** *Let F be an* SI*-module of finite degree. Then for every $i \geqslant 0$, the homomorphism*

$$H_i(\mathrm{Sp}(2g, \mathbb{Z}); F_g) \longrightarrow H_i(\mathrm{Sp}(2g+2, \mathbb{Z}); F_{g+1}),$$

induced by the inclusion $(-) \oplus \mathrm{id}_{V_1}\colon \mathrm{Aut}_{\mathrm{SI}}(V_g) \hookrightarrow \mathrm{Aut}_{\mathrm{SI}}(V_{g+1})$, is an isomorphism for all sufficiently large g.

With this result in place, we can present a proof of Theorem 2.1.2. We use a spectral sequence argument which we learnt from [53, Theorem 1.9].

Theorem 2.6.2. *Let F be an* SI*-module of finite degree. Then for all $k \geqslant 1$ and all $i \geqslant 0$, the homomorphism*

$$H_i(\Gamma_{g,1}(k); F_g) \longrightarrow H_i(\Gamma_{g+1,1}(k); F_{g+1})$$

is an isomorphism for all sufficiently large g.

Proof. We proceed by induction on k, the case $k = 1$ being precisely Theorem 2.6.1. Suppose the result holds for some $k \geqslant 1$ and consider the short exact sequences

$$0 \longrightarrow Q_g(k) \longrightarrow \Gamma_{g,1}(k+1) \longrightarrow \Gamma_{g,1}(k) \longrightarrow 1. \tag{A_g}$$

The fixed embeddings $\phi\colon S_{g,1} \hookrightarrow S_{g+1,1}$ give rise to maps $(A_g) \to (A_{g+1})$ of short exact sequences, and hence to a composable sequence of maps of spectral sequences

$$\cdots \longrightarrow E^*_{**}(g) \longrightarrow E^*_{**}(g+1) \longrightarrow \cdots \tag{2.4}$$

Here $E^*_{**}(g)$ is the Lyndon–Hochschild–Serre spectral sequence in homology with coefficients in F_g associated to (A_g). Its second page is given by

$$E^2_{pq}(g) = H_p(\Gamma_{g,1}(k); H_q(Q_g(k); F_g)).$$

Let us now fix p and q and consider the sequence

$$\cdots \longrightarrow E^2_{pq}(g) \longrightarrow E^2_{pq}(g+1) \longrightarrow \cdots \tag{2.5}$$

of abelian groups and homomorphisms coming from the sequence (2.4) of spectral sequence morphisms. The action of the free abelian group $Q_g(k)$ on F_g is trivial since $Q_g(k) \leqslant \ker(\Gamma_{g,1}(k+1) \to \mathrm{Sp}(2g, \mathbb{Z}))$. Hence by the universal coefficient theorem,

$$H_q(Q_g(k); F_g) \cong F_g \otimes \Lambda^q Q_g(k).$$

It can be seen that the sequence (2.5) is precisely the sequence

$$\cdots \longrightarrow H_p(\Gamma_{g,1}(k); F'_g) \longrightarrow H_p(\Gamma_{g+1,1}(k); F'_{g+1}) \longrightarrow \cdots$$

where $F' \coloneqq F \otimes \Lambda^q Q(k) \in$ SI–Mod. The FI-module $\Psi^* F'$ is a quotient of the FI-module $\Psi^* F \otimes (\Psi^* Q(k))^{\otimes q}$ which has finite degree by Corollary 2.4.20 and Theorem 2.5.7. Thus by

the induction hypothesis, the sequence (2.5) eventually stabilises. Thus for every pair p, q, the sequence

$$\cdots \longrightarrow E^{\infty}_{pq}(g) \longrightarrow E^{\infty}_{pq}(g+1) \longrightarrow \cdots$$

also eventually stabilises, since $E^{\infty}_{pq}(g)$ only depends on the structure of a finite portion of $E^{*}_{**}(g)$. The result now follows from strong convergence of the spectral sequences $E^{*}_{**}(g)$. □

2.6.1 Towards a computation of the stable homology

In [52], Szymik shows that in the case of full automorphism groups of free nilpotent groups, the canonical homomorphisms $\mathrm{Aut}\, N_n(k) \to \mathrm{GL}(n, \mathbb{Z})$ are rational homology isomorphisms in the stable range. We remark that the corresponding maps $\Gamma_{g,1}(k) \to \mathrm{Sp}(2g, \mathbb{Z})$ in general are not. We will prove this by computing the stable rational cohomology of $\Gamma_{g,1}(2)$.

Recall from Definition 2.4.11 the category $\mathrm{SI}(\mathbb{Q})$ of finite-dimensional symplectic $\mathbb{Q}$-vector spaces and symplectic linear maps. Following Patzt [43], we call an $\mathrm{SI}(\mathbb{Q})$-module $F : \mathrm{SI}(\mathbb{Q}) \to \mathrm{Mod}_{\mathbb{Q}}$ over $\mathbb{Q}$ *rational* if each F_n is a rational representation of $\mathrm{Sp}(2n, \mathbb{Q})$, meaning that the action homomorphism $\mathrm{Sp}(2n, \mathbb{Q}) \to \mathrm{GL}(F_n)$ is a rational morphism of varieties.

Lemma 2.6.3. *Let F be a finitely generated rational $\mathrm{SI}(\mathbb{Q})$-module over $\mathbb{Q}$. Then for each p there is $n(p)$ such that*

$$H^p(\mathrm{Sp}(2n, \mathbb{Z}); F^*_n) \cong H^p(\mathrm{Sp}(2n, \mathbb{Z}); \mathbb{Q}) \otimes \left(F^*_n\right)^{\mathrm{Sp}(2n, \mathbb{Z})}$$

for all $n \geqslant n(p)$, where $F^*_n = \mathrm{Hom}_{\mathbb{Q}}(F_n, \mathbb{Q})$ denotes the $\mathbb{Q}$-linear dual.

Proof. By [43, Theorem B], F is uniformly representation stable. This means the following: let F_n be decomposed into irreducibles as

$$F_n \cong \bigoplus_{\lambda} V(2n, \lambda)^{\oplus c_{\lambda, n}}$$

where the sum runs over the dominant weights λ of the symplectic groups and $V(2n, \lambda)$ is the irreducible $\mathbb{Q}$-representation of $\mathrm{Sp}(2n, \mathbb{Q})$ with highest weight λ. Then there is some N such that for each λ, the multiplicity $c_{\lambda,n}$ is independent of n for $n \geqslant N$. Let $\lambda_1, \cdots, \lambda_k$ be those non-zero weights for which $c_{\lambda,n} \neq 0$ for $n > N$. (Note that for each n, $c_{\lambda,n} \neq 0$

for only finitely many λ, as each F_n is necessarily finite-dimensional.) It follows from [11, Paragraph 4.6] that for each $1 \leqslant i \leqslant k$ and each p, there is $n_i(p)$ such that

$$H^p(\mathrm{Sp}(2n,\mathbb{Z}); V(2n,\lambda_i)^*) = 0$$

for all $n \geqslant n_i(p)$. Thus if we take $n(p) \coloneqq \max_i n_i(p)$, then

$$H^p(\mathrm{Sp}(2n,\mathbb{Z}); F_n^*) \cong H^p(\mathrm{Sp}(2n,\mathbb{Z}); \mathbb{Q})^{\oplus c_{0,n}} \cong H^p(\mathrm{Sp}(2n,\mathbb{Z}); \mathbb{Q}) \otimes \left(F_n^*\right)^{\mathrm{Sp}(2n,\mathbb{Z})}$$

for all $n \geqslant n(p)$, where $\mathbb{Q} = V(2n,0)$ is the trivial representation. □

Proposition 2.6.4. *The stable cohomology ring* $\mathrm{colim}_g H^*(\Gamma_{g,1}(2);\mathbb{Q})$ *is isomorphic to the tensor product*

$$\mathbb{Q}\left[c_G \mid G \in \mathcal{G}\right] \otimes \operatorname*{colim}_g H^*(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q})$$

where $\mathcal{G}$ *is the set of isomorphism classes of finite non-empty connected trivalent graphs (with loops and multiple edges). The degree of a generator* c_G *is the order of the graph* $G \in \mathcal{G}$*.*

Proof. Consider the SI-module $Q(1) : \mathrm{SI} \to \mathrm{Ab}$ which takes V_g to the abelian group $\mathscr{I}_{g,1}(1)/\mathscr{I}_{g,1}(2)$. By [31, Theorem 5], there is an isomorphism of SI-modules $Q(1) \cong \Lambda^3 V$ where $V : \mathrm{SI} \to \mathrm{Ab}$ is the forgetful functor that forgets the symplectic structure on objects. We have isomorphisms

$$\begin{aligned} H^q(Q_g(1);\mathbb{Q}) &\cong \mathrm{Hom}(\Lambda^q Q_g(1), \mathbb{Q}) \\ &\cong \mathrm{Hom}(\Lambda^q(\Lambda^3 V_g), \mathbb{Q}) \\ &\cong \mathrm{Hom}_{\mathbb{Q}}(\Lambda^q(\Lambda^3 V_g^{\mathbb{Q}}), \mathbb{Q}) \end{aligned}$$

where the first isomorphism follows since $Q_g(1)$ is free abelian of finite rank, and the second isomorphism is a consequence of the aforementioned theorem of Johnson. The SI-module $V^{\mathbb{Q}} : \mathrm{SI} \to \mathrm{Mod}_{\mathbb{Q}}$ over $\mathbb{Q}$ is given by $V_g \mapsto V_g \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathrm{Hom}_{\mathbb{Q}}$ is the group of $\mathbb{Q}$-linear maps.

But $V^{\mathbb{Q}} : \mathrm{SI} \to \mathrm{Mod}_{\mathbb{Q}}$ is evidently the restriction of a finitely-generated rational $\mathrm{SI}(\mathbb{Q})$-module over $\mathbb{Q}$ along the functor $\mathrm{SI} \to \mathrm{SI}(\mathbb{Q})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Hence so is $\Lambda^q(\Lambda^3 V^{\mathbb{Q}})$, and the preceding lemma applies to give the isomorphism

$$H^p(\mathrm{Sp}(2g,\mathbb{Z}); H^q(Q_g(1);\mathbb{Q})) \cong H^p(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) \otimes \mathrm{Hom}_{\mathbb{Q}}(\Lambda^q(\Lambda^3 V_g^{\mathbb{Q}}), \mathbb{Q})^{\mathrm{Sp}(2g,\mathbb{Z})}$$

which holds in a stable range $g \gg p, q$. This gives us a stable description of the second page of the Lyndon–Hochschild–Serre spectral sequence in rational cohomology associated to the short exact sequence

$$0 \longrightarrow Q_g(1) \longrightarrow \Gamma_{g,1}(2) \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

Kawazumi and Morita [32, Section 3] showed that $\mathrm{Hom}_{\mathbb{Q}}(\Lambda^*(\Lambda^3 V_g^{\mathbb{Q}}), \mathbb{Q})^{\mathrm{Sp}(2g,\mathbb{Z})}$ is stably (in the colimit $g \to \infty$) isomorphic to the polynomial algebra $\mathbb{Q}[c_G \mid G \in \mathcal{G}]$. By [10, Paragraph 11.4], the stable rational cohomology of the integral symplectic groups is

$$\operatorname*{colim}_n H^*(\mathrm{Sp}(2n, \mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \ldots, c_i, \ldots \mid |c_i| = 4i - 2]$$

Thus in the stable limit, the second page of the aforementioned spectral sequence is concentrated in even bidegrees. Hence it stably collapses on the second page, establishing the claim. □

Remark 2.6.5. It is easily seen that the projection $\Gamma_{g,1}(2) \to \mathrm{Sp}(2g, \mathbb{Z})$ induces the obvious inclusion in stable rational cohomology. By [32, Theorem 4], the image of the homomorphism

$$\operatorname*{colim}_g H^*(\Gamma_{g,1}(2); \mathbb{Q}) \longrightarrow \operatorname*{colim}_g H^*(\Gamma_{g,1}; \mathbb{Q})$$

induced by the projection map $\Gamma_{g,1} \twoheadrightarrow \Gamma_{g,1}(2)$ is the subalgebra generated by the Miller–Morita–Mumford classes — hence by the Madsen–Weiss theorem [38], the homomorphism is in fact surjective.

Remark 2.6.6. By [43, Theorem 6.5], *each* $Q(k) \in$ SI–Mod extends to a rational $\mathrm{SI}(\mathbb{Q})$-module $Q(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Hence it is in theory possible to calculate the stable cohomology of $\Gamma_{g,1}(k)$ for higher k using the above method, but the actual computations become rather involved already for $k = 3$.

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