ABSTRACT

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Let X be a projective non-singular algebraic variety over an algebraically closed field k. Let f: $X \longrightarrow X$ be an endomorphism of X and let F be a locally free sheaf on X. Then the vector spaces $H^q[X;F]$ are known to be of finite rank and to be zero except for finitely many integers q. Let ϕ: $f^*F \longrightarrow F$ be a morphism of sheaves. Then endomorphisms, e^q, of each $H^q[X;F]$ may be constructed as the composite: $H^q[X;F] \xrightarrow{f^*} H^q[X;f^*F] \xrightarrow{\phi^*} H^q[X;F]$. Set $L(f,F,\phi) = \sum (-)^q \mathrm{trace}(e^q)$.

If f has a non-singular fixed point set and subject to a slight restriction on the nature of the endomorphism induced by f on the normal bundle of each fixed component, Y, formulae are obtained of the form:

$$L(f,F,\phi) = \sum_Y \nu'(Y,\phi),$$

with the sum being taken over the set of fixed components, in the two special cases:

(α) f is periodic with period prime to the characteristic of k.

(β) each fixed component is a point or a curve.

The definition of $\nu'(Y;\phi)$ involves Chern characters and the like of the restriction of F and ϕ to Y, as well as the induced action on the normal bundle.

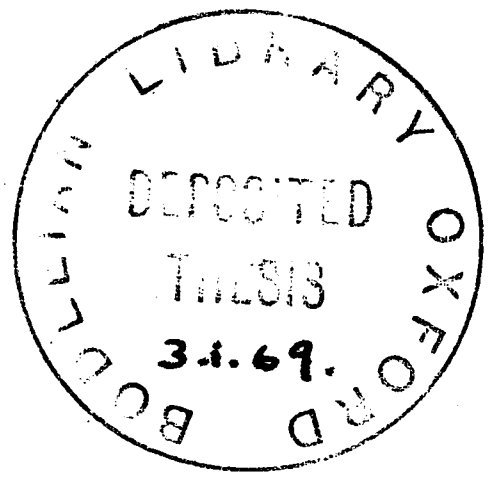
If k has characteristic $p \neq 0$, in case (α) a stronger p-adic lifting of the above formula is obtained.

The methods used are setting up an appropriate formalism and using the theory of derived categories in homological algebra. The algebraic geometry used is essentially that of the Borel-Serre paper (1958) vintage.

SOME PROBLEMS IN ALGEBRAIC GEOMETRY

by

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INTRODUCTION.

'Let X be a projective non-singular algebraic variety over an algebraically closed field k. Let f: $X \to X$ be an endomorphism of X, and let F be a locally free sheaf on X. Then the vector spaces $H^q[X;F]$ are known to be of finite rank and to be zero except for finitely many integers q. Let ϕ: $f^*F \to F$ be a morphism of sheaves. Then endomorphisms, e^q, of each $H^q[X;F]$ may be constructed as the composite: $H^q[X;F] \xrightarrow{f^*} H^q[X;f^*F] \xrightarrow{\phi_*} H^q[X;F]$. Set $L(f,F,\phi) = \sum(-)^q \text{trace}(e^q)$. Compute $L(f,F,\phi)$ as a sum of 'contributions' from the fixed components of f.'

Why should there be such a formula at all? A brief answer is to point out the existence of analogues: the Atiyah-Bott formulae in differential geometry (see [1] and [10]); the Eichler formula for function fields given in [3]; the Verdier formula in étale cohomology sketched in [17] and other Lefschetz type formulae, such as that of [11].

Atiyah and Bott discovered four years ago that if f has only finitely many fixed points and if each is unramified, the solution is (very) latent in [8]. They left it at that. Hartshorne, in [RD p. 18] mentions the problem and comments 'In particular, the determination of the local contribution at a non-simple fixed point presents an interesting topic for future investigation.'

Grothendieck, in [2, exp 3] 'Formule de Lefschetz-Verdier', sketches the general theory basic to this sort of problem, shows how one gets (in general) a sum of contributions from fixed components, and goes on (p. 19): 'La question se pose de calculer ces termes locaux dans divers cas particuliers, en fonction d'invariants connus. Il s'agit là d'une problème fort intéressant (et probablement difficile), auquel on ne s'est pas encore sérieusement attaqué. Les résultats dont on dispose pour le moment concernent seulement les deux cas

particuliers suivants.'

(These remarks deal with the étale case. He later remarks that the algebraic sheaves case is similar.)

The investigation has to start somewhere, and the best place is the case f = identity. Then, if ϕ = id: F$\rightarrow$F, the Riemann-Roch formula of [BS] gives a formula for the Euler-Poincare characteristic of F. The expression, which involves Chern classes, is formally an element of the field of rationals, Q. So, if k has characteristic 0, it is natural to replace Q by k as 'co-efficients' for the cohomology theory throughout [BS], and deduce trace formulas for arbitrary constant ϕ = c.id: F$\rightarrow$F, for c$\in$k and F perhaps not locally free but only coherent. It turns out that the formula for arbitrary ϕ is an easy consequence of this (c.f. 3.1.1) but one is forced to use such a formalism as my M (given, in 2.2-2.4, for general f.)

If k has characteristic p $\neq$ 0 a more subtle device must be used. The inspiration comes from the concept of a Brauer character in the modular representation theory of finite groups. The Witt ring, W(k), and its quotient field, FW(k), are introduced. It is then equally easy to convert the Riemann-Roch formula into a 'Brauer trace' formula, which can be reduced modulo p to give a formula for the classical trace.

We must now return to general endomorphisms f. How does one define M(f)? Unfortunately it is inadequate to simply take a Grothendieck group of covering homomorphisms ϕ: f*F$\rightarrow$F for F locally free. It is essential to work in an appropriate derived category.

The above three paragraphs are intended to persuade the reader that the rather heavy formalism introduced in Chapters 1 and 2 probably cannot be simplified substantially. He is now warned that although the algebraic geometry of ten years ago is adequate for the problem at hand, the homological algebra of [M] is not. The first chapter of [RD] gives all

the general theory of derived categories that is needed here, but, for the convenience of the reader who knows only the derived functor theory of [M], I give a brief summary of what extra is needed. It suffices at the moment to know that the objects of the derived category are, roughly, complexes of sheaves.

Section 1.2, whilst nominally concerned with justifying the restriction of attention to projective non-singular varieties, also reminds the reader of their basic properties. It shows, in particular, why fixed point sets have to be assumed to be non-singular.

Sections 1.3 and 1.4 are concerned with the question of which derived category is appropriate. This is equivalent to choosing finiteness conditions for complexes of sheaves on a non-singular projective variety; these turn out to be simply that the complex be bounded, have quasi-coherent objects and coherent cohomology.

Now the required derived category has been set up, it can be used to obtain a new, but equivalent, definition of the functor K of [BS]. So section 1.5 deals with K. Section 1.6, which depends on [RD] for proofs, lists the derived functors that will be needed and gives their required properties.

Section 1.7, which is needed only for Chapter 3, deals with a 'theorem of syzyzies'. It is the only part of Chapter 1 that is even slightly original.

In section 2.1, a category whose objects are endomorphisms of varieties is set up mostly to allow the use of functorial notation for the M of 2.2. M is a sort of 'Grothendieck group' on the set of covering homomorphisms (which includes the ψ's of the original problem).

In section 2.3, it is shown that there are simpler equivalent definitions of the M of 'equivariantly embeddable'

endomorphisms of varieties. This class turns out later (in 3.2) to include the periodic ones. This shows, incidentally, that Chapter 3 could easily be made independent of derived category theory. A study of the M of identity endomorphisms is needed for both Chapter 3 and Chapter 5.

In section 2.4 a useful modification of the above M, called M', is introduced. In 2.5, the Brauer-Chern trace that is required to formulate the solution to our problem is defined. The reader will appreciate that no useful purpose would be served by repeating it in this introduction, and so the two theorems of this thesis cannot be repeated here either.

In section 2.6 a study is made of local co-ordinates near a fixed point of an endomorphism of a variety. This is used in later chapters as well as in section 2.7, where the induced action on the normal bundle of a fixed component is discussed.

In chapter 3 a 'Lefschetz-Riemann-Roch' theorem for periodic endomorphisms of varieties is proved. It is formulated in section 3.1. A few essential facts about periodic endomorphisms are demonstrated in section 3.2. Then one suddenly discovers how favourable the situation is: the above formalism reduces the proof to fairly easy checks of three special cases, which are made in sections 3.3, 3.4 and 3.5.

Section 4.1 is concerned with 'dualising'. It is, in common with the rest of Chapter 4, preparatory to Chapter 5. It is followed by a few sections which set out what is needed of the 'duality' theorems of [RD]. These are essentially Serre duality heavily disguised. Next the relevant version of the Kunneth formula is given. Finally an important compatibility, the 'Verdier diagram', is investigated.

In section 5.1 the trace formula for endomorphisms whose fixed point set consists of points and non-singular curves with suitable induced action on the normal bundles is set out. The burden of the proof follows in 5.2 and 5.3. These do not

depend on the dimension of the fixed components, but do depend on their non-singularity and on the suitability of the induced action on the normal bundle.

In section 5.4 the proof is completed painlessly for fixed points and with some difficulty for fixed curves. In section 5.5 it is shown that a plausible conjecture implies the fixed point formula without this restriction on the dimension of the fixed components.

The last chapter lists some ideas which are not yet sufficiently well developed to appear in the main body of the thesis.

Chapters 1 and 4. These comprise essentially standard material that cannot be given by simple reference to the literature. Despite their length, much of the matter in them is sketched rather than proved. If the reader is familiar with the literature, he will be able to read them very quickly.

If the reader is interested only in the periodic case, he should skim through Chapter 1 and then read Chapters 2 and 3.

Chapter 5 is independent of 3.2, 3.3, 3.4 and 3.5. If the reader of this chapter has omitted 4.3, 4.5, 4.6, 4.7 and 4.8, he will be able to pick up the theorem proved in these sections in 5.1.

Sections 3.1 and 5.1. These sections state the results of this thesis. They are the key sections.

CHAPTER 1. THE DERIVED CATEGORY.

1.1. INTRODUCTION.

The bulk of this section is a thumbnail sketch of derived category theory. Some familiarity with this theory is essential for Chapters 4 and 5. Section 1.2 explains why attention is restricted to non-singular projective varieties; it is also intended to remind the reader of their basic properties.

Section 1.3 demonstrates the properties of quasi-coherent sheaves on a variety that make them into a more workable category than that of all algebraic sheaves. Section 1.4 sets out the basic properties that make the bounded derived category with coherent cohomology of the category of quasi-coherent sheaves the appropriate one for our purposes. In other words, the appropriate ~~bound~~ finiteness conditions for complexes of sheaves on a variety are that they be bounded, have coherent cohomology and have quasi-coherent objects. Section 1.6 lists all the derived functors that will be needed.

Section 1.5 gives certain properties of the functor K of [BS]; the reader may find it useful to reread this before reading about the analogous functor M in Chapter 2. Section 1.7 has nothing to do with derived categories but is preparatory to 3.3.

The notation has been kept as standard as possible. It is mostly that of [RD] and [EGA]. 1.6 specifies much of it; the appendix to this section gives more. Note that in this thesis (unlike [EGA]) all complexes are of degree +1; that h^t is the symbol for the t^{th} cohomology of a complex; and $F^\bullet[N]$ is the complex obtained by shifting the complex $F^\bullet$ N places to the left (as in [RD]). An index to the notation is given at the end of the thesis.

Appendix: Derived Category Theory.

This appendix is intended to give those who have read [M] a sketch of derived category theory sufficient to enable them to read this thesis.

Homological algebra involves the study of derived functors. For example, if $F: \mathcal{A} \to \mathcal{B}$ is a left exact functor from one abelian category to another, under certain conditions there are associated 'right derived functors of F', $R^qF: \mathcal{A} \to \mathcal{B}$, $q \geq 0$, with $R^0F = F$ and with the usual properties. Given $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$, then there is, under appropriate conditions, a spectral sequence: $E_2^{p,q} = R^pG.R^qF(-) \Rightarrow R^*(GF)(-)$. But this is inadequate for many purposes including this thesis.

Another theme is that of the last chapter of [M]. The hyperderived functors, $\mathcal{R}^*F$, are defined. They take bounded below complexes of $\mathcal{A}$ into objects of $\mathcal{B}$. This is a natural extension of the above derived functors, but in fact is a more fundamental concept.

The basic method used to construct these functors is:

(α) 'Replace' the object (or complex) concerned by a complex called a 'resolution'.

(β) Apply the functor F to the resolution.

(γ) Take the cohomology of this complex in $\mathcal{B}$.

(δ) Show that this cohomology is independent (up to canonical isomorphism) of the choice of resolution.

Verdier, as described in [RD ch. I], found a more powerful approach. Suppose we perform only (α) and (β). Then our $\mathcal{A}$-complex yields a $\mathcal{B}$-complex whose cohomology is determined up to isomorphism but which itself is not quite so well determined. So he forces the $\mathcal{B}$-complex to be determined up to isomorphism by adjoining inverses to certain morphisms in the category of $\mathcal{B}$-complexes.

It turns out that one should consider morphisms of complexes up to homotopy only (this does not affect the induced

morphisms of their cohomology) and then adjoin inverses to all homotopy classes of morphisms of complexes which induce isomorphisms on cohomology ('quasi-isomorphisms'). This turns out to make the above $\mathcal{B}$-complex independent of the choice of resolution, up to canonical isomorphism, in the 'derived category', $D(\mathcal{B})$, which has just been constructed.

What sort of a category results? Although it need not be abelian, it is always additive and 'triangulated'. A triangulated category [RD p. 20] is one with an automorphism T, called translation, (shifting a complex one place to the left) and a class of distinguished triples of morphisms: $A \to B$, $B \to C$ and $C \to TA$, satisfying certain axioms. Rather than list these I will state that stable homotopy theory provides a good example. In our case a long exact sequence of the cohomology of A, B and C is induced. In [RD p. 22] a covariant ∂-functor from one triangulated category to another is defined to be an additive functor which takes triangles to triangles and commutes with the translations. A contravariant ∂-functor is likewise defined.

Now return to our functor $F: \mathcal{A} \to \mathcal{B}$. Under appropriate conditions a derived functor $RF: D(\mathcal{A}) \to D(\mathcal{B})$ can be defined as in [RD p. 49] to be a ∂-functor with a certain universal property. In this thesis RF will be defined by the analogues of (α) and (β): (α') An object in $D(\mathcal{A})$ may be represented by an $\mathcal{A}$-complex. Replace this one by a quasi-isomorphic one with F-acyclic objects. (β') Take the image of F of the second complex in $D(\mathcal{B})$.

In the above I have not specified what sort of complexes are allowed: bounded below? bounded? finiteness conditions? Each gives rise to a different derived category. In view of the results of 1.4 the only complexes needed in the sequel are bounded. The F-acyclic objects will be either injective or locally free as appropriate. Right exact functors can be similarly treated; often we will have both together.

For example, let X be a variety and $0 \to A \to B \to C \to 0$ be an exact sequence of coherent sheaves. Then the images of these three sheaves (considered as trivial complexes) in the derived category form a triangle. We wish to calculated $R\Gamma_X$ of them. Using the theory of resolutions of sequences of [M p. 78], we construct (bounded) injective resolutions $I^\bullet$, $J^\bullet$ and $K^\bullet$ of A, B and C (i.e. there are quasi-isomorphisms $A \to I^\bullet$ etc.) and an exact sequence of complexes $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$. As the images of $I^\bullet$, $J^\bullet$ and $K^\bullet$ form a triangle, $\Gamma_X I^\bullet$, $\Gamma_X J^\bullet$ and $\Gamma_X K^\bullet$ have images which form a triangle in D(pt). The long exact cohomology sequence of this triangle should be familiar.

We are not quite back where we started: the spectral sequences like that of the second paragraph have dropped out: under the same appropriate conditions $RG.RF = R(GF)$ up to canonical isomorphism.

In the sequel I tend to abuse notation and identify complexes with their images in the derived category. E.g.: 'Represent $F^\bullet \in D(X)$ by a locally free complex also called $F^\bullet$'. Also, to save space, I shall be rather careless about canonical isomorphisms; an apologia for this is in [RD p. 118-119].

Note that the derived category of sheaves on a variety is 'not a local object' ([RD p. 10]). Hence it it valid to work on the open sets of an open cover only to calculate things such as the cohomology of objects in the derived category (which, being sheaves, are 'local objects'). The derived category of sheaves on affine varieties, which is not treated explicitly, follows the same lines as that of projective varieties.

1.2. THE ASSUMPTIONS.

This section is an attempt to justify the restriction of consideration to projective non-singular varieties over an algebraically closed field.

Irreducibility is not taken as part of the definition of a variety, though in this thesis several constructions have to be done component by component. In 2.1 it becomes convenient to formally adjoin an empty variety to the category; the special conventions required for it are fairly obvious.

As closed graph properties are required for chs. 4 and 5, 'pre-varieties' must be excluded.

The algebraic closure of the field is assumed to have a good eigenvalue theory for endomorphisms of finite rank vector spaces and to make subvarieties more manageable. Although it is perhaps not essential to have this restriction, non-perfect fields must be excluded to allow the construction of an adequate Witt ring.

Non-singularity is assumed partly to have the theory of regular local rings available and partly to be able to obtain theorems expressable in terms of the theory of [BS]. If this assumption were dropped, $D_c^b(\mathrm{Qco}\ X)$ would no longer be suitable. See [2] for Riemann-Roch theorems for singular varieties.

Projectivity is assumed partly to have the duality theorem reasonably manageable (in [RD] 70 pages are needed for the projective case and 200 for the non-projective case) and partly to have locally free resolutions of coherent sheaves. As the principal results of this thesis require completeness for their formulation, quasi-projective varieties must be excluded.

In the next two sections it will be demonstrated that the appropriate finiteness restriction for complexes of sheaves is that they be bounded, quasi-coherent and have coherent cohomology.

Hence it is convenient to impose the following:-

Restriction 1.2.1. 'k' is an algebraically closed field that is fixed once and for all.

Except when stated otherwise, 'variety' will mean 'projective non-singular k-variety' (perhaps reducible).

'Sheaf' will mean 'algebraic sheaf', and, after 1.3, 'quasi-coherent algebraic sheaf'.

1.3. QUASI-COHERENT SHEAVES.

Although the main objects of study are coherent sheaves, it is necessary to consider bounded complexes of sheaves with coherent cohomology. For reasons more technical than conceptual it is necessary that these complexes should at least be quasi-coherent, i.e. have quasi-coherent objects. For the convenience of the reader the definition of [EGA 0 5.1.3] is repeated below:

Definition 1.3.0. Let $(X,\mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-Module F is said to be quasi-coherent if, for each $x \in X$, there exists an open U containing x such that $F|U$ is isomorphic to the cokernel of a homomorphism of the form $\mathcal{O}_X^{(I)}|U \longrightarrow \mathcal{O}_X^{(J)}|U$, where the indexing sets I and J are arbitrary.

So, unlike Hartshorne [RD p. 133], I adopt the third of the restrictions 1.2.1 throughout and consider only quasi-coherent sheaves. It is possible to apply the relevant parts of [RD] because:

Quote 1.3.1. 1. The category Qco(X) of quasi-coherent sheaves on a variety X has sufficient injectives.

2. If $f: Y \longrightarrow X$ is a morphism of varieties, then the following three functors may be defined by restriction from Mod(X) and Mod(Y) (the categories of (algebraic) sheaves):

(i) $f^*: Qco(X) \longrightarrow Qco(Y)$.

(ii) $f_*: Qco(Y) \longrightarrow Qco(X)$.

(iii) $\otimes_X: Qco(X) \times Qco(X) \longrightarrow Qco(X)$.

3. If U is an affine variety, the category Qco(U) is equivalent to that of modules over the corresponding algebra.

References. For 1., see [RD p. 85 (and errata)] and [RD p. 133, Thm. 13.3]. For 2.(ii), see [EGA III 1.4.10] and [EGA III 6.2.3]. For 3., see [EGA I 1.4.1] and [EGA I 1.3.7]. The other two parts of 2. follow from 3 in view of the local nature and right exactness of these functors.

The parts of [RD] that will be quoted in this thesis

could now be rewritten with Qco(X) replacing Mod(X); in view of their length this will not be done here.

Lemma 1.3.2. Let U be an irreducible non-singular affine variety of dimension n with affine algebra A. Then, for any pair of A-modules M and N, $\mathrm{Ext}^q_A(M,N) = 0$ for $q>n$.

Proof. The concept of global dimension of a commutative ring with unit is developed in [M p. 111]. It is required to prove that $\mathrm{gl.dim}(A) \leqslant n$. By the definition of non-singularity, for each maximal ideal m of A, $\mathrm{gl.dim}(A_m) \leqslant n$, where A_m is the local ring. Exercise 11 of [M p. 142] now shows that $\mathrm{Ext}^q_A(M,-) = 0$ for each finitely generated A-module M and each $q>n$. The result for general M now follows by Thm. 1 of [19, p. 68].

Lemma 1.3.3. Let Y be an irreducible variety of dimension n. Then, for any pair of quasi-coherent sheaves F and G on Y, $\mathrm{Ext}^q_Y(F,G) = 0$ for $q>n$.

Proof. In view of [7 p. 187] it suffices to prove the lemma for an affine variety. But in this case 1.3.1.3 shows that it is equivalent to 1.3.2, which has just been proved.

1.4. CERTAIN BASIC FACTS.

In this section I demonstrate that a certain derived category, defined by certain finiteness conditions on complexes of sheaves on a variety, has good properties. In 1.6 it will be shown that the [RD] constructions of certain derived functors can be 'restricted' to this category. Y will denote an irreducible variety of dimension n, but each result below can be extended trivially to reducible varieties.

Quote 1.4.1. If $0 \to A \to B \to C \to 0$ is a short exact sequence of sheaves on Y, then the coherence of any two of A, B and C implies the coherence of the third.

The categories of locally free (always of finite rank) sheaves on Y and of vector bundles over Y are canonically equivalent and will be identified. (See [FAC] for these facts.)

In [FAC sn. 54] an invertible sheaf $\mathcal{O}(1)$ on the projective space P is defined. It is often referred to as the Hopf sheaf or the Hopf line bundle; the notation $\mathcal{O}(N)$ or $\mathcal{O}_P(N)$ will be used for its N^{th} tensor power. Frequent use will be made of the cohomology of this sheaf; in particular $H^0[P;\mathcal{O}_P(1)]$, which is of rank dimP+1 over k, will often be identified with the space of linear forms in the dimP+1 projective co-ordinates of P. If F is a sheaf on P, F(N) is sometimes written for $F \otimes \mathcal{O}(N)$.

A great deal of theory about projective fibrations, including that of the last paragraph, is given in [EGA II 2-5] and [EGA III 2]. As only trivial fibrations are used in this thesis the reader does not need to know much of the [EGA] presentation. It is occasionally referred to.

If $i: Y \to P$ is a projective embedding, $i^*\mathcal{O}_P(1)$ is 'very ample' ([EGA II 4.4]) sheaf on Y. Sometimes it is called the Hopf sheaf and the notation F(N) used, but this 'Hopf sheaf' depends on the choice of the embedding.

Quote 1.4.2. 1. If F is a coherent sheaf on Y and if a projective embedding of Y is chosen and fixed, there is an integer n_o such that for all $N>n_o$ $F(N)$ is generated by its global sections. Hence there exists a locally free sheaf L and a surjection $L \to F$.

2. Let $f: X\times P \to X$ and $\pi: X\times P \to P$ be the projections, where X is a variety and P a projective space. Let $H = \pi^*\mathcal{O}_P(1)$. Then if F is a coherent sheaf on XxP, there exists an n_o such that for all $N>n_o$ $R^qf_*(F\otimes H^{\otimes N}) = 0$ for $q>0$ and the adjunction morphism $f^*f_*(F\otimes H^{\otimes N}) \longrightarrow F\otimes H^{\otimes N}$ is surjective.

Reference. The first may be deduced from the second in the well-known way. The second is a special case of [EGA III 2.2.1] which generalises the result of [FAC].

Lemma 1.4.3. If A and B are (quasi-coherent) sheaves on Y, $\mathrm{Ext}^r[Y;A,B] = 0$ for $r>2n$.

Proof. In the last section it was proved that $\underline{\mathrm{Ext}}^q_Y(A,B) = 0$ for $q>n$. Secondly, by [5 p. 198], $H^p[Y;-] = 0$ for $p>n$. The result now follows from the composite functor spectral sequence: $H^p[Y;\underline{\mathrm{Ext}}^q_Y(A,B)] \Rightarrow \mathrm{Ext}^*[Y;A,B]$.

Lemma 1.4.4. Let $I^\bullet$ be a complex of sheaves on Y such that I^t is injective and $h^tI^\bullet = 0$ for $t>s$. Then $\mathrm{Im}(I^{s+2n+1} \longrightarrow I^{s+2n+2})$ is injective and so is a direct summand of I^{s+2n+2}.

Proof. Set $J^t = I^{s+t+1}/\mathrm{Im}(I^{s+t})$. Then if A is a sheaf, the short exact sequences: $0 \to J^t \to I^{s+t+2} \to J^{t+1} \to 0$ for $t = -1, 0, \ldots, 2n-1$ enable one to show that $\mathrm{Ext}^1[Y;A,\mathrm{Im}] \approx \mathrm{Ext}^{2n+1}[Y;A,J^\bullet]$ which is zero by 1.4.3. This gives the result.

Lemma 1.4.5. Let $A^\bullet$ be a bounded compex of sheaves on Y. Then there is an injective complex $I^\bullet$ (i.e. one with each I^t injective) and an injective morphism of complexes ~~such~~ $i: A^\bullet \longrightarrow I^\bullet$ such that $I^\bullet$ is bounded and i is a quasi-isomorphism.

Proof. The construction of [RD p. 103] or [M p. 362-365] may be used to find such an $I^\bullet$ which is bounded below but perhaps

not bounded above. However 1.4.4 may be used to truncate this $I^\bullet$ and obtain another $I^\bullet$ which satisfies all the requirements.

Notation 1.4.6. It is now time to recall from [RD] the notation which will be used. $K^b(\mathcal{A})$, for $\mathcal{A}$ an additive category, is the category whose objects are the bounded (i.e. all of the objects of the complex are zero except for a finite number of them) complexes of $\mathcal{A}$, and whose morphisms are homotopy classes of complex morphisms. $K^b_c(Y)$ is the full subcategory of $K^b(Qco(Y))$ ($Qco(Y)$ is the category of quasi-coherent sheaves on Y) whose objects are those complexes with coherent cohomology. $D^b_c(Y)$ is the derived category obtained from $K^b_c(Y)$ by the construction sketched in 1.1 and given in full in [RD]. The following result is given to show what $D^b_c(Y)$ 'looks like'.

Lemma 1.4.7. Let Inj(Y) denote the additive category of injective (quasi-coherent) sheaves on Y. Then, with the obvious notation, the canonical functor $K^b_c(Inj(Y)) \to D^b_c(Y)$ is an equivalence of categories.

Proof. Half of the proof is contained in Lemma 1.4.5. The other half is proved by a virtually verbatim repetition of the proof of an analogue in [RD p. 46].

The following two lemmas are rather artificial but seem necessary to obtain 1.4.11. There should be an elementary proof for 1.4.8.

Lemma 1.4.8. Let C be a coherent sheaf and Q a sheaf on Y. Let π: $Q \to C$ be surjective. Then there is a coherent subsheaf Q_o of Q such that the composite $Q_o \longrightarrow C$ is surjective.

Proof. Note that the corresponding result for modules is clear. Choose and fix a projective embedding for Y. Then the theory of [EGA II 2.7.5], which sets up an equivalence between the category of quasi-coherent sheaves on Y and a quotient of a certain category of graded modules, may be used to supply a proof. This theory is too long to sketch here, but the reader

will have no difficulty in reconstructing the proof if he bears in mind the exactness ([EGA II 2.5.]) of the functor $\sim$.

Lemma 1.4.9. Let $A^\cdot$ be a bounded complex of sheaves on Y with coherent cohomology. Then there exists a coherent subcomplex $B^\cdot \xrightarrow{i} A^\cdot$ with i a quasi-isomorphism.

Proof. This is by induction on the distance between the two extreme non-zero objects of $A^\cdot$.

First consider the case when this distance is 1. So, with the obvious notation, consider an exact sequence:

$$0 \to \mathrm{Ker} \to A^0 \xrightarrow{\psi} A^1 \to \mathrm{Coker} \to 0,$$

with Ker and Coker coherent. Choose a coherent $B^1 \subset A^1$ with $B^1 \to \mathrm{Coker}$ surjective. Note that $K = \psi A^0 \cap B^1$ is coherent. Set $B^0 = \psi^{-1}K$, which is coherent by 1.4.1. Then this subcomplex will do.

Now I deal with the general case. Without loss of generality assume that $A^t = 0$ except for $m \geq t \geq 0$. Then, by the inductive hypothesis, there exists a $P \subset A^1$ and coherent subsheaves:

$$\begin{array}{ccccccc} 0 \to P/d^0A^0 & \longrightarrow & B^2 & \longrightarrow & B^3 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to A^1/d^0A^0 & \longrightarrow & A^2 & \longrightarrow & A^3 & \longrightarrow & \cdots \end{array},$$

such that isomorphisms are induced on cohomology. Now apply the distance 1 case to the complex $0 \to A^0 \to P \to 0$ to get B^0 and B^1. I omit the tedious proof that these B^t do.

Lemma 1.4.10. Let $L^\cdot$ be a bounded above coherent complex of sheaves on Y such that L^t is locally free and $h^tL^\cdot = 0$ for $t \leq r$. Then it has a subcomplex $L_0^\cdot$ with these properties, with $L_0^t = L^t$ for $t \geq r-n$, $L_0^t = 0$ for $t < r-n-1$ and such that the inclusion is a quasi-isomorphism.

Proof. I claim the kernel of $L^{r-n-1} \to L^{r-n}$ is locally free. For by an argument analogous to that of the proof of 1.4.4,

working with Ext* over the local rings of each point, one shows that the localisation of this kernel at each point is a free module over the local ring of the point. The 'free at each point' criterion for local freeness ([FAC p. 242]) gives the claim. (The dimension theory of regular local rings has been used as in [BS p. 106].) Set L_o^{r-n-1} = this kernel for the result.

Lemma 1.4.11. Let $A^\bullet$ be a bounded complex of sheaves on Y with coherent cohomology. Then there exists a bounded locally free complex $L^\bullet$ and a quasi-isomorphism π: $L^\bullet \longrightarrow A^\bullet$. If $A^\bullet$ has coherent objects, π may be chosen to be surjective.

Proof. In view of 1.4.9 it is sufficient to deal with the case when $A^\bullet$ is coherent. Now 1.4.2.1 and the construction of [RD p. 103] for constructing resolutions of complexes may be used to obtain such an $L^\bullet$ that need not be bounded below. 1.4.10 may be used to truncate it if necessary.

The following theorem summarises these results:

Theorem 1.4.12. Let $A^\bullet$ be a bounded complex of sheaves on Y with coherent cohomology. Then there exists a bounded locally free complex $L^\bullet$ and a bounded injective complex $I^\bullet$ and quasi-isomorphisms: $L^\bullet \xrightarrow{\pi} A^\bullet \xrightarrow{i} I^\bullet$.

Notation 1.4.13. Henceforth I write, for X a variety, D(X) for $D_c^b(Qco(X))$. Of course if X is reducible D(X) is the direct sum of D(components of X).

Remark 1.4.13. In 1.6 many derived functors will be presented, mostly by reference to [RD]. The [RD] presentation uses larger derived categories, but the theory given in 1.3 and 1.4 shows that restriction to this derived category is valid. (Strictly this should be checked for each construction, but this task is omitted.) For example, a locally free complex is flat, so his flat resolutions may be obtained in this context.

1.5. THE FUNCTOR K.

This section recalls some of the basic properties of the functor K of [BS sn. 4]. Two other properties are then presented. The category of irreducible varieties is used and trivial extensions to the category of varieties will not be made explicitly. [BS sn 5] establishes cofunctoriality, functoriality, ring structure and the projection formula for K; the first two will be sketched after 1.5.1.

In [BS p. 105] K(X) is defined to be the Grothendieck group of the category of coherent sheaves on the variety X, $K_1(X)$ is defined to be the Grothendieck group of the category of locally free sheaves on X, and the following basic theorem is proved:

Quote 1.5.1. The canonical homomorphism $\epsilon : K_1(X) \to K(X)$ is an isomorphism.

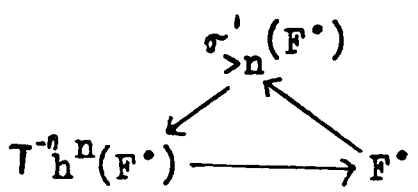
If F is a coherent sheaf on X, [F] will be used to denote its image in K(X). The same use of [...] will be made for other Grothendieck groups later. If $f: Y \to X$ is a morphism of irreducible varieties, a morphism $f^! : K(X) \to K(Y)$ of abelian groups (indeed of rings) is defined by linearity from the rule: $f^![F] = \sum_t [L^t f^* F].(-)^t$. A morphism $f_! : K(Y) \to K(X)$ of abelian groups (but not of rings) is defined by linearity from the rule: $f_![G] = \sum_t (-)^t.[R^t f_* G]$ for each coherent sheaf G on Y. It turns out that this defines cofunctoriality and functoriality for K.

1.5.1 may be supplemented with:

Proposition 1.5.2. Consider the free abelian group on all (isomorphism classes of) objects of D(X). Define $K_2(X)$ to be the factor group defined by the relations $[F_1^\bullet] - [F_2^\bullet] + [F_3^\bullet] = 0$ whenever $F_1^\bullet$, $F_2^\bullet$ and $F_3^\bullet$ in that order are the vertices of a triangle in D(X). Then $K_2(X)$ is canonically isomorphic to K(X) and hence also to $K_1(X)$.

Proof. Define $\alpha: K_2(X) \to K(X)$ by setting $\alpha[F^\bullet] = \sum_t (-)^t.[h^t F^\bullet]$. The sum is finite and the long exact sequence of cohomology for

triangles in D(X) shows that α induces an abelian group morphism. Now define β: $K(X) \to K_2(X)$ by $\beta[F] = [F]$ where on the right hand side F is considered as the image in D(X) of the complex whose only non-zero object is F in the zero position. An obvious triangle shows that, for $G^\cdot \in D(X)$, $[G^\cdot]+[TG^\cdot] = 0$. Using this fact, the triangle defined by a short exact sequence of coherent sheaves shows that β induces an abelian group morphism. It is clear that $\alpha\beta = id: K(X) \to K(X)$. To prove that $\beta\alpha = id$ it suffices to exhibit triangles in D(X) which break up an object $F^\cdot$ into its cohomology sheaves. If $F^\cdot$ is the image of a complex $F^\cdot$ with $F^t = 0$ for $t<n$, the triangle (4) of [RD p. 70] drawn below:



where $\sigma'_{>n}(F^\cdot)$ is the complex $\ldots 0 \to F^{n+1}/\mathrm{Im} \to F^{n+2} \to F^{n+3} \to \ldots$ and the triangle's morphisms need not be specified here, shows that in $K_2(X)$ $[F^\cdot] = [\sigma'_{>n}F^\cdot]+[T^n h^n F^\cdot]$. Repeated application of this shows that $[F^\cdot] = \sum_t (-)^t [h^t F^\cdot]$. From this one may read off $\beta\alpha = id$, completing the proof of 1.5.2.

The reader may check that, for f: $Y \to X$, $f^!$ and $f_!$ may be defined using K_2 by setting, for $F^\cdot \in D(X)$, $f^![F^\cdot] = [Lf^*F^\cdot]$ etc., and that modulo the above isomorphism the original definitions are recovered. (This fact will not be needed.)

Proposition 1.5.3. Let X be an irreducible variety. Let i: $pt \to X$ be the embedding of a point and let c: $X \to pt$. Then $K(pt) \xleftarrow{i^!} K(X) \xleftarrow{c^!} K(pt) = Z$ shows that Z (the integers) is a direct summand of K(X) (independently of i): $K(X) \approx Z + \tilde{K}(X)$. Then the augmentation ideal $\tilde{K}(X)$ is nilpotent.

Reference. In [2 X, 1] the filtrations of the ring K(X) are described and the proposition follows from Thm. 1.3.2 of the loc. cit. As this theory is long and used only here, I will not discuss it further.

1.6. REMARKS ON CERTAIN DERIVED FUNCTORS.

This topic, which is treated in [RD ch. II] at length, will be given only a sketch treatment here. His definitions simplify to these with the simplification from preschemes to varieties, D^+ or D^- to $D = D^b_c(\mathrm{Qco})$. Another useful reference, which does not use derived categories, is [EGA 0 12].

Construction 1.6.1. The functor Rf_*. For $f: Z \to Y$ a morphism of varieties, the functor $Rf_*: D(Z) \to D(Y)$ is defined by: take an object of $D(Z)$. Represent it by a bounded injective complex $I^\cdot$. Then set $Rf_*(\text{the object}) = f_*I^\cdot$. Likewise define Rf_* of a morphism in $D(Z)$. As f is proper, it is an immediate consequence of [BS p. 103, Thm. 1] (note the misprint there of > for $\geqslant$) or [EGA III 3.2.1] that $f_*I^\cdot$ has coherent cohomology.

If $Z \xrightarrow{f} Y \xrightarrow{g} X$, there is a natural canonical transformation $\alpha: R(gf)_* \longrightarrow Rg_*.Rf_*$ which is defined and shown to be an isomorphism in [RD p. 100].

Construction 1.6.2. In particular, if $c: Z \to \mathrm{pt}$, Rc_* is defined. It is often written $R\Gamma_Z$ or $R\Gamma$. Its t^{th} cohomology k-vector space is often written $H^t[Z;-]$, and is of finite rank.

Construction 1.6.3. The functor $R\underline{\mathrm{Hom}}^\cdot_Z$. This functor, $D(Z)^{\mathrm{op}} \times D(Z) \to D(Z)$ (op means 'opposed category of') is defined by the sign conventions of [RD p. 90] and constructed by representing either the first variable by a locally free bounded complex or the second variable by an injective bounded complex. The 'local hyperext', $\underline{\mathrm{Ext}}^t_Z(-,-)$ is defined to be $h^t R\underline{\mathrm{Hom}}^\cdot_Z(-,-)$.

Construction 1.6.4. The functor $\mathrm{RHom}^\cdot[Z;-,-]$. This is defined by $\mathrm{RHom}^\cdot[Z;-,-] = R\Gamma_Z.R\underline{\mathrm{Hom}}^\cdot_Z(-,-)$. It is easily proved that $h^0\mathrm{RHom}^\cdot[Z;-,-]$ '=' $\mathrm{Hom}_{D(Z)}(-,-)$. The 'global hyperext', $\mathrm{Ext}^t[Z;-,-]$, is defined to be $h^t\mathrm{RHom}^\cdot[Z;-,-]$.

Construction 1.6.5. The functor $\overset{L}{\otimes}_Z$. This functor, called $\underset{=Z}{\otimes}$ in [RD], is the left derived functor of $\otimes_Z$, and is usually written $\overset{L}{\otimes}$. It is defined by the sign convention of [RD p. 93]

and calculated by representing either (or both) variable by a bounded locally free complex. The 'local hypertor', $\underline{\mathrm{Tor}}^Z_t(-,-)$, is defined to be $h^{-t}(-\ -)$.

Construction 1.6.6. The functor Lf*. For $f: Z \to Y$, $Lf^*: D(Y) \to D(Z)$ is defined to be the left derived functor of f*. It is constructed by representing the object in D(Y) by a bounded locally free complex and applying f* to this complex.

As an example of derived category technique I show how one computes Lf* of a morphism of D(Y). Let $\beta: F^\cdot \to G^\cdot$ be such a morphism. Represent $F^\cdot$ and $G^\cdot$ by locally free bounded complexes which (by the usual abuse of language) are also called $F^\cdot$ and $G^\cdot$. Then, by 1.4.11 again, β may be represented by $F^\cdot \xleftarrow{\alpha} K^\cdot \xrightarrow{\gamma} G^\cdot$, where $K^\cdot$ is a bounded locally free complex and α and γ are morphisms of complexes and α is a quasi-isomorphism. One finds that $f^*\alpha$ is also a quasi-isomorphism, and so $Lf^*\beta$ is represented by $f^*F^\cdot \xleftarrow{f^*\alpha} f^*K^\cdot \xrightarrow{f^*\gamma} f^*G^\cdot$.

The above derived functors have certain relationships among themselves which Hartshorne investigates in [RD ch. II, sn. 5]. Most of these are well known in the form of spectral sequences, for example that given at the end of 1.6.1. I give below statements of those relationships which are most relevant to this thesis.

Lemma 1.6.7. Let $f: Z \to Y$. Then there is a canonical natural isomorphism $\alpha: R\Gamma_Z \xrightarrow{\sim} R\Gamma_Y \cdot Rf_*$.

Proof. Put X = point in the last paragraph of 1.6.1. (Note that this is usually stated in the weaker form of the 'Leray spectral sequence': for F a coherent sheaf on Z, there is a spectral sequence $H^p[Y;R^qf_*F] \Rightarrow H^*[Z;F]$.)

Lemma 1.6.8. Let $Z \xrightarrow{f} Y \xrightarrow{g} X$. Then there is a canonical natural isomorphism $\alpha: L(gf)^* \xrightarrow{\sim} Lf^* \cdot Lg^*: D(X) \to D(Z)$.

Proof. See [RD p. 102, Prop. 5.4]

Lemma 1.6.9. Let $Z \xrightarrow{f} Y$. Then there is a canonical natural

transformation: $Rf_* \underline{RHom}^\cdot_Z(-,-) \xrightarrow{\theta_f} \underline{RHom}^\cdot_Y(Rf_*-,Rf_*-)$. Further, there is a commutative diagram:-

$$\begin{array}{ccc}
h^0 R\Gamma_Y Rf_* \underline{RHom}^\cdot_Z(-,-) & \xrightarrow{\theta} & h^0 R\Gamma_Y \underline{RHom}^\cdot_Y(Rf_*-,Rf_*-) \\
\downarrow 1.6.7 & & \\
h^0 R\Gamma_Z \underline{RHom}^\cdot_Z(-,-) & & \downarrow 1.6.4 \\
\downarrow 1.6.4 & & \\
Hom_{D(Z)}(-,-) & \xrightarrow{Rf_*} & Hom_{D(Y)}(Rf_*-,Rf_*-).
\end{array}$$

Proof. The transformation is constructed in [RD p. 103-105] by representing both variables by bounded injective complexes. I omit the proof of the compatibility to save space. It is a long but mechanical grind from the construction of [RD].

I also omit the proof of the following compatibility for the same reason. The reader is referred to [RD ch. 2, sn.6] for general comments on such compatibilities.

Lemma 1.6.10. Let $Z \to Y \to X$. Let α be the natural isomorphism of 1.6.1. Then, in the obvious sense, $\alpha.\theta_{gf} = \theta_g.\theta_f.\alpha$: $R(gf)_* \underline{RHom}^\cdot_Z(-,-) \to \underline{RHom}^\cdot_X(Rg_*Rf_*-,Rg_*Rf_*-)$.

Lemma 1.6.11. If in 1.6.9 the first variable is taken to be $\mathcal{O}_Z$, then the following diagram commutes:

$$\begin{array}{ccc}
Rf_* \underline{RHom}^\cdot_Z(\mathcal{O}_Z,-) & \xrightarrow{\theta_f} & \underline{RHom}^\cdot_Y(Rf_*\mathcal{O}_Z,-) \\
\| & & \downarrow \\
Rf_*- & = & \underline{RHom}^\cdot_Y(\mathcal{O}_Y,-).
\end{array}$$

Proof. The right vertical morphism is induced by functoriality in the first variable from the canonical $\mathcal{O}_Y \to Rf_*\mathcal{O}_Z$. The best proof is to use 1.6.10 to reduce the proof to the special cases of f a closed embedding and f: $X \times P \to X$ the projection, with P a projective space. But in both special cases the lemma is an immediate consequence of the construction of [RD p. 104].

Lemma 1.6.12. Let f: $Z \to Y$, $F^\cdot \in D(Z)$ and $G^\cdot \in D(Y)$. Then there is a natural isomorphism: $Rf_*F^\cdot \overset{L}{\otimes}_Y G^\cdot \approx Rf_*(F^\cdot \overset{L}{\otimes}_Z Lf^*G^\cdot)$.

Proof. See [RD p. 106, prop. 5.6]. This is traditionally called the 'projection formula'.

Lemma 1.6.13. Let $f: Y \longrightarrow X$. Then there is a morphism, natural in $F^\cdot \in D(X)$: $\varrho: F^\cdot \longrightarrow Rf_*Lf^*F^\cdot$, which, using the morphism of 1.6.9, yields a natural isomorphism:

$$\tau: Rf_*\underline{RHom}_Y^\cdot(Lf^*F^\cdot, G^\cdot) \longrightarrow \underline{RHom}_X^\cdot(F^\cdot, Rf_*G^\cdot)$$

(natural in $G^\cdot \in D(Y)$ too) and $h^0R\Gamma_X\tau$ gives an adjointness between Lf^* and Rf_*.

Reference. See [RD p. 110-111], where this ϱ is constructed from the adjunction morphism of the pair of adjoint functors f^*, f_*.

If $Z \xrightarrow{g} Y \xrightarrow{f} X$, and if F is a sheaf on X, the following square of adjunction morphisms commutes:

$$\begin{array}{ccc} F & \longrightarrow & f_*f^*F \\ \downarrow & & \downarrow \\ (fg)_*(fg)^*F & = & f_*g_*g^*f^*F \end{array}$$

(see [EGA 0 4.4] and especially [EGA 0 4.4.8].)

It is possible to deduce the corresponding compatibility for the derived functors from this, but I omit the details.

Lemma 1.6.14. Let $f: Y \longrightarrow X$. Then there is an isomorphism, natural in $F^\cdot$ and $G^\cdot \in D(X)$:

$$\gamma: Lf^*\underline{RHom}_X^\cdot(F^\cdot, G^\cdot) \longrightarrow \underline{RHom}_Y^\cdot(Lf^*F^\cdot, Lf^*G^\cdot).$$

Proof. Represented each variable by a bounded locally free complex also called $F^\cdot$ and $G^\cdot$. It is then required to construct a quasi-isomorphism: $f^*\underline{Hom}_X^\cdot(F^\cdot, G^\cdot) \longrightarrow \underline{Hom}_Y^\cdot(f^*F^\cdot, f^*G^\cdot)$. Such an isomorphism is well-known when $F^\cdot$ and $G^\cdot$ are locally free sheaves. The extension to general locally free complexes is, therefore, also an isomorphism.

1.7. THE THEOREM OF SYZYGIES.

The 'theorem of syzygies' (1.7.7) given here is a generalisation of Hilbert's theorem [1 p. 217, thm. 6.1]. As there does not seem to be a proof written down anywhere, I give the following modification of MacLane's proof. It could easily be adapted, if necessary, to give a version of 1.7.7 in which the product is replaced by a general projectivised vector bundle over the variety X.

Lemma 1.7.1. Let A be a noetherian local ring. Then finitely generated projective A-modules are free.

Proof. If M is such a module, m the maximal ideal of A, then $M \otimes A/m$ is a finitely generated A/m-vector space, with basis a finite set of elements of the form $\gamma_t \otimes 1$, $t \in T$. Set $F = A^T$ and consider the map $F \longrightarrow M$ defined by (t^{th} basis element) $\longmapsto \gamma_t$. There is an exact sequence of A-modules: $0 \longrightarrow \mathrm{Ker} \longrightarrow F \longrightarrow M \longrightarrow \mathrm{Coker} \longrightarrow 0$. Tensoring with A/m shows that $\mathrm{Coker} \otimes A/m = 0$, and then Nakayama's lemma shows that Coker = 0. Then, in view of the fact that $\mathrm{Tor}_1^A(M, A/m) = 0$, tensoring with A/m shows that $\mathrm{Ker} \otimes A/m = 0$, whence Ker = 0. Q.E.D.

Set $P = A[x_1, \ldots, x_n]$, which is a graded ring. We will consider finitely generated Z-graded P-modules, but $\otimes_P$ and Tor^P will always be taken neglecting the gradings. A is canonically a graded P-module. **Here Z is the group of integers.**

Lemma 1.7.2. If M is a finitely generated Z-graded P-module such that $M \otimes_P A = 0$, then M = 0.

Proof. Let $J = \langle x_1, \ldots, x_n \rangle$ be the augmentation ideal of P. Then the exact sequence $0 \longrightarrow J \longrightarrow P \longrightarrow A \longrightarrow 0$ induces the exact sequence $M \otimes J \longrightarrow M \otimes P \longrightarrow M \otimes A = 0$. But if M is non-zero, it has a homogeneous element γ of least grading. But such a γ cannot be in the image of $M \otimes J$, contradicting the exactness of the last sequence.

Lemma 1.7.3. If M is a finitely generated Z-graded P-module

which is projective as an ungraded P-module then it is free.

Proof. Since M is graded, $M\otimes_P A$ is a graded A-module which must be projective. As it has only finitely many non-zero graded summands, each of these is projective and of course finitely generated. As in the proof of 1.7.1 choose a finite set m_t, $t \in T$, of homogeneous elements of M such that $m_t \otimes 1$ freely generate $M\otimes_P A$. Consider the map $P^T \longrightarrow M$ defined by the set m_t. Then form the exact sequence of finitely generated Z-graded P-modules: $0 \to \mathrm{Ker} \to P^T \longrightarrow M \to \mathrm{Coker} \to 0$. Tensoring with A shows that $\mathrm{Coker}\otimes_P A = 0$, whence Coker = 0. Then, in view of the fact that $\mathrm{Tor}_1^P(M,A) = 0$, tensoring with A shows that $\mathrm{Ker}\otimes_P A = 0$, whence Ker = 0. Q.E.D.

Lemma 1.7.4. Let A_a be a regular local ring of dimension d. Set $P_a = A_a[x_1,\ldots,x_n]$ considered as a graded ring. Then if an exact sequence of finitely generated Z-graded P_a-modules: $0 \to \mathrm{Ker} \to X^{-n-d} \to \ldots \to X^{-1} \longrightarrow X^0$ is such that each X^t is free, then Ker is free.

Proof. As P_a has dimension n+d (see MacLane, [12, p. 210]), Ker is projective as an ungraded P_a-module. The result now follows from 1.7.3.

Lemma 1.7.5. Let U be an affine irreducible non-singular k-variety of dimension d and affine algebra A(U). Set $P(U) = A(U)[x_1,\ldots,x_n]$ considered as a graded ring. Consider an exact sequence of finitely generated Z-graded P(U)-modules: $0 \to \mathrm{Ker} \longrightarrow X^{-n-d} \to \ldots \to X^{-1} \longrightarrow X^0$, with each X^t free. Then for each $a \in U$ there is an open affine U', with $a \in U' \subset U$, such that $\mathrm{Ker}\otimes_{A(U)} A(U')$ is free as a graded P(U') module.

Proof. By 1.7.4. there is a finite set T and an exact sequence of finitely generated Z-graded P(U)-modules: $0 \to \mathrm{Ker}' \to P(U)^T \to \mathrm{Ker} \longrightarrow \mathrm{Coker} \longrightarrow 0$ such that the localisations (recall this functor is exact) of Ker' and Coker at a are zero. As these are both finitely generated, the result follows.

Lemma 1.7.6. Let X be an irreducible variety of dimension d and let P be a projective space of dimension n-1. Let F be a coherent sheaf on XxP such that there exists a finite open affine cover U_i, $i \in I$, of X such that the restriction of F to each U_ixP is isomorphic to a finite direct sum of sheaves of the form (locally free on U_i)$\otimes \mathcal{O}_P(t)$, where the second factor is the t^{th} tensor power of the Hopf sheaf on P. Then F is isomorphic to a finite direct sum of sheaves of the form (locally free on X)$\otimes \mathcal{O}_P(t)$.

Proof. Let π: XxP $\to$ X and π_2:XxP $\to$ P be the projections. If only one t, without loss of generality t = 0, occurs, the adjunction map $\pi^*\pi_* F \to F$ sets up, as may be checked on each U_ixP, the required isomorphism. If several different t's occur, decompose F as $\oplus F_t$ in the obvious way and apply the above argument to each $F_t \otimes \pi_2^* \mathcal{O}_P(-t)$ to obtain the general result.

Proposition 1.7.7. With X and P as in 1.7.6, if an exact sequence of sheaves on XxP: $0 \to \text{Ker} \to F^{-n-d} \to \ldots \to F^0$ has each F^s of the form of the 'F' of 1.7.6, then Ker is also of this form.

Proof. This is an immediate consequence of [EGA II 3.3.4], 1.7.5 and 1.7.6.

CHAPTER 2. EQUIVALENCE CLASSES OF COVERING HOMOMORPHISMS.

2.1. INTRODUCTION.

Definition 2.1.1. Let y be an endomorphism of the perhaps reducible variety Y. Then a covering homomorphism is an element of $\mathrm{Hom}_{D(Y)}(Ly^*F^\bullet, F^\bullet)$ for some $F^\bullet \in D(Y)$. In particular, if F is a locally free sheaf on Y, an element of Hom[Y;y*F,F] is one.

This chapter sets up certain 'Grothendieck groups' of these things which play a part in this thesis analogous to that played by the functor K in [BS]. It is convenient to set up a category, End, thus: The objects of End are pairs (Y,y), where y is an endomorphism of the variety Y such that its fixed point set, Y^y, is non-singular. (In 2.7 attention will be restricted to a subcategory of 'admissible' endomorphisms.) An 'empty variety' is adjoined to allow for the case when Y^y is empty. It is an initial object of End. The morphisms $f: (Y,y) \to (X,x)$ of End are the $f: Y \to X$ such that $fy = xf: Y \to X$. Thus the category of varieties is canonically a full subcategory of End. In the sequel many of the constructions and lemmas will need trivial modification for the special case of the 'empty variety', but the details are usually omitted (quite safely).

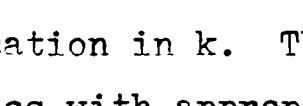
The basic functor, M, is defined in 2.2 on End. It should be compared with the K of 1.5. In 2.3 a class of 'equivariantly embeddable' endomorphisms is introduced. For them M is easier to handle; indeed it can be defined without use of derived category theory for this class. The name should perhaps be 'projective endomorphisms'. It includes the identity endomorphisms.

2.4 introduces a useful modification, M', of M. 2.5 gives what is needed to formulate the theorems of Chapters 3 and 5. 2.6 sets out what will be needed about local co-ordinates. 2.7 deals with the embedding of a fixed component of an endomorphism in a variety and brings in the 'admissibility' restriction.

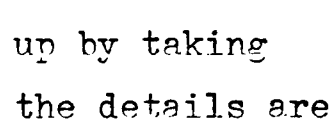
2.2. THE FUNCTOR M.

M is a cofunctor from End to the category of commutative rings with unit. It is simultaneously a functor to the category of abelian groups. It is defined thus:

Definition 2.2.1. For $(Y,y) \in$ End, $M(y)$ is the quotient of the free abelian group on the set of all (isomorphism classes of) covering homomorphisms ϕ: $Ly^*K^\bullet \longrightarrow K^\bullet \in D(Y)$ by the subgroup generated by all elements of the form $[\phi_1]-[\phi_2]+[\phi_3]$ whenever the ϕ_i 'form a triangle', i.e.



and therefore



are triangles and the ϕ_i form a morphism from the second triangle to the first.

Example 2.2.2. If Y = pt, $M(pt) \approx Z[k]$, the free abelian group on k. Note that this has a ring structure induced from the multiplication in k. The isomorphism is set up by taking eigenvalues with appropriate multiplicities; the details are trivial linear algebra.

Definition 2.2.3. Let p: $(Y,y) \longrightarrow (X,x) \in$ End. Define $p^!$: $M(y) \leftarrow M(x)$ thus: As $Ly^*.Lp^* \text{ '=' } L(py)^* = L(xp)^* \text{ '=' } Lp^*.Lx^*$, where '=' means 'is canonically isomorphic to', for ϕ: $Lx^*K^\bullet \rightarrow K^\bullet$ define $p^!\phi$ to be the composite of $Ly^*(Lp^*K^\bullet) \text{ '=' } Lp^*Lx^*K^\bullet \xrightarrow{Lx^*\phi} Lp^*K^\bullet$. Then set $p^![\phi] = [p^!\phi] \in M(y)$.

As this respects the relations of definition 2.2.1, it indeed defines an abelian group morphism $p^!$. A check of the canonical isomorphisms used shows that $(pq)^! = q^!p^!$ in the obvious notation. Whence the cofunctorial structure has been fully defined.

Lemma 2.2.4. There is a natural commutative and associative ring structure (with unit) on M.

Proof. Firstly I define the product in M(y) of the images of

two covering homomorphisms, ϕ_1 and ϕ_2. Let ϕ be the composite of $Ly^*(K_1^{\cdot} \overset{L}{\otimes} K_2^{\cdot}) \text{ '=' } Ly^*K_1^{\cdot} \overset{L}{\otimes} Ly^*K_2^{\cdot} \xrightarrow{\phi_1 \overset{L}{\otimes} \phi_2} K_1^{\cdot} \overset{L}{\otimes} K_2^{\cdot}$, where the '=' is the canonical isomorphism of [RD p. 110]. Then set $[\phi_1].[\phi_2] = [\phi] \in M(y)$. A long but obvious check shows that this is compatible with the relations and so induces a commutative and associative ring structure. Again another long check of compatibilities shows that the $p^!$ maps are ring morphisms. The unit is the canonical homomorphism $y^*\mathcal{O}_Y \to \mathcal{O}_Y$. (As the ring structure is used mostly in the case y = identity, when the compatibilities are easier to check, I have left out the details above.)

Lemma 2.2.5. M is also a covariant functor to the category of abelian groups. For $p: (Y,y) \to (X,x)$, write $p_!: M(y) \to M(x)$.

Proof. I will use the adjointness relation of 1.6.13 twice to define a covering morphism $p_!\phi$, for $\phi: Ly^*K^{\cdot} \to K^{\cdot}$. By 1.6.13 there is a natural transformation $Rp_* \to Rp_*.Ry_*.Ly^* \text{ '=' } Rx_*.Rp_*.Ly^*$, and the adjoint of this (now using 1.6.13 for x) is the first morphism in the composite $Lx^*.Rp_*K^{\cdot} \to Rp_*.Ly^*K^{\cdot} \xrightarrow{Rp_*\phi} Rp_*K^{\cdot}$. This composite is defined to be $p_!\phi$. As this construction respects triangles, it defines an abelian group morphism also called $p_!$ as required. Instead of directly proving that with the obvious notation, $p_!q_! = (pq)_!$ etc., it is convenient to use remark 2.2.6. to make the proof as simple as the proof for the contravariant maps.

Remark 2.2.6. The above definition of $p_!\phi$ may be rewritten as follows: In the adjointness isomorphisms of 1.6.13,

$$\begin{array}{ccc} \mathrm{Hom}_{D(Y)}(Ly^*K^{\cdot}, K^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{D(Y)}(K^{\cdot}, Ry_*K^{\cdot}) \\ \phi & \longmapsto & \phi' \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{D(X)}(Lx^*Rp_*K^{\cdot}, Rp_*K^{\cdot}) & \xrightarrow{\sim} & \mathrm{Hom}_{D(X)}(Rp_*K^{\cdot}, Rx_*Rp_*K^{\cdot}) \\ \psi & \longmapsto & \psi' \end{array}$$

$\psi = p_!\phi$ if and only if $\psi' = Rp_*\phi'$ modulo the identification $Rp_*Ry_* = Rx_*Rp_*$.

2.3. EQUIVARIANTLY EMBEDDABLE ENDOMORPHISMS.

If y is an automorphism of the variety Y, y* is an equivalence of categories. Hence it is exact and takes injectives to injectives. So, given a covering homomorphism ϕ: $y^*K^\bullet \longrightarrow K^\bullet \in D(Y)$, we may represent $K^\bullet$ by an injective bounded complex and use 1.4.6 to deduce that ϕ may be represented by a morphism of complexes. Then the argument of the proof of 1.5.2 shows that, in M(y), $[\phi] = \Sigma(-)^t.[h^t\phi: y^*h^tK^\bullet = h^ty^*K^\bullet \longrightarrow h^tK^\bullet]$.

Definition 2.3.1. If y is an automorphism of the variety Y, define $M_1(y)$ to be the quotient of the free abelian group on the set of all (isomorphism classes of) ϕ: $y^*F \to F$, for F a coherent sheaf on Y, by the subgroup generated by all elements of the form $[\phi_1]-[\phi_2]+[\phi_3]$ whenever the ϕ_i form 'an exact sequence', i.e. there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & y^*F_1 & \xrightarrow{y^*\alpha} & y^*F_2 & \xrightarrow{y^*\beta} & y^*F_3 & \longrightarrow & 0 \\ & & \downarrow\phi_1 & & \downarrow\phi_2 & & \downarrow\phi_3 & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{\alpha} & F_2 & \xrightarrow{\beta} & F_3 & \longrightarrow & 0. \end{array}$$

Proposition 2.3.2. There is a natural isomorphism: γ: $M(y) \xrightarrow{\sim} M_1(y)$.

Proof. With the notation of the first paragraph of this section, set $\gamma[\phi] = \Sigma(-)^t.[h^t\phi] \in M_1(y)$. The proof is now formally almost identical with that of 1.5.2, and so can be omitted.

Definition 2.3.3. An automorphism y of a variety Y is said to be equivariantly embeddable if there exists a closed embedding i: $Y \hookrightarrow P$, with P a projective space with a diagonalisable linear automorphism, p, such that pi = iy. (Note that if y is the identity then such embeddings exist.)

Lemma 2.3.4.1. Let y be equivariantly embeddable. Let ϕ: $y^*F \to F$ be a morphism of coherent sheaves on Y. Then there exists a locally free sheaf L on Y, a ψ: $y^*L \to L$ and a surjective π: $L \to F$ such that the following square commutes:

$$\begin{array}{ccc} y^*L & \xrightarrow{y^*\pi} & y^*F \\ \psi\downarrow & & \downarrow\phi \\ L & \xrightarrow{\pi} & F. \end{array}$$

Lemma 2.3.4.2. If, further, $y = id$ and, for some $c \in k$, $\phi - c$ is nilpotent, then ψ may also be chosen such that $\psi - c$ is nilpotent.

Proof. Choose a projective embedding as in 2.3.3. If H is the restriction of the Hopf sheaf to Y, then there is an isomorphism β: $y^*H \xrightarrow{\sim} H$. (It is determined up to multiplication by non-zero elements of k, and is induced by the β obtained in the obvious way for p.) Then, by 1.4.2, for suitably large n, the canonical adjunction morphism (notation: c: Y $\longrightarrow$ pt.) $c^*c_*(F \otimes H^{\otimes n}) \longrightarrow F \otimes H^{\otimes n}$ is surjective. I claim that this implies that $c_*c^*c_*(F \otimes H^{\otimes n}) \longrightarrow c_*(F \otimes H^{\otimes n})$ is surjective. (Proof: recall the general fact [4 p. 338] that for a pair of adjoint functors c_* and c^* the composite $c_* \longrightarrow c_*c^*c_* \longrightarrow c_*$ is the identity.) For such an n consider the diagram:

$$\begin{array}{ccccc} \mathcal{O}_Y^N & \approx & y^*\mathcal{O}_Y^N & \xrightarrow{y^*\pi'} & y^*F \otimes y^*H^{\otimes n} \\ & \searrow & \downarrow & & \downarrow \phi\otimes\beta^{\otimes n} \\ & & \mathcal{O}_Y^N & \xrightarrow{\pi'} & F \otimes H^{\otimes n}, \end{array}$$

which has surjective π'. But as $c_*\pi'$ is surjective, the broken arrows may be filled in. Tensoring with $H^{\otimes -n}$ now gives 2.3.4.1.

To prove 2.3.4.2, find L, π and ψ as above, with $\psi - c$ not necessarily nilpotent. Then for all $d \in k$ and for all integers M, $\pi(\psi - d)^M = (\phi - d)^M\pi$. By the eigenspace argument of 2.4.3, which does not depend on this lemma, $L = \bigoplus_d L_d$, $\psi = \bigoplus_d \psi_d$, where ψ_d: $L_d \longrightarrow L_d$ is such that $\psi_d - d$ is nilpotent. But the image of L_d in F is annihilated by $(\phi - d)^M$ and $(\phi - c)^M$ for all large M and so is zero for $d = c$. Whence $L_c \longrightarrow F$ is surjective as required.

Lemma 2.3.5. Let F be a coherent sheaf on Y and let $c \in k$. Let ϕ: $F \longrightarrow F$ be such that $\phi - c$ is nilpotent. Then in $M(Y) = M(id_Y)$ $[\phi] = [c.id_F]$.

Proof. Use the isomorphic functor $M_1(Y)$ instead of $M(Y)$. The proof is an induction on the least integer N such that $(\phi - c)^N$

$= 0$. For $N = 1$ the result is the hypothesis, for $N = 2$ the two diagrams superimposed below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F & \xrightarrow{\phi - c} & F & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0 \\ & & \downarrow c & & \downarrow c & \swarrow \phi & \downarrow c & & \downarrow c & & \\ 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F & \xrightarrow[\phi - c]{} & F & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0 \end{array}$$

(where Ker and Coker are defined by the row's exactness) show $[\phi] = [c.\mathrm{id}_{\mathrm{Ker}}] + [c.\mathrm{id}_F] - [c.\mathrm{id}_{\mathrm{Coker}}] = [c.\mathrm{id}_F]$. For other N, assume the result for N-1 and use similarly:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F & \xrightarrow{(\phi - c)^{N-1}} & F & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0 \\ & & \downarrow c & \swarrow \phi|\mathrm{Ker} & \downarrow c & \swarrow \phi & \downarrow c & & \downarrow c & & \\ 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F & \xrightarrow[(\phi - c)^{N-1}]{} & F & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0. \end{array}$$

Definition 2.3.6. If y is an equivariable embeddable automorphism of the variety Y, define $M_2(y)$ to be the quotient of the free abelian group on the set of all (isomorphism classes of) ϕ: $y^*L \longrightarrow L$, for L a locally free sheaf on Y, by the subgroup generated by all elements of the form $[\phi_1] - [\phi_2] + [\phi_3]$ whenever the ϕ_i form an exact sequence in the sense of 2.3.1.

Theorem 2.3.7. The natural morphism ϵ: $M_2(y) \longrightarrow M_1(y)$, where y is equivariantly embeddable, is an isomorphism. Hence, by 2.3.2, $M_2(y) \approx M_1(y) \approx M(y)$.

Proof. As the proof is formally identical with that of [BS p. 105-108] which proves their thm. 2 (quoted here as 1.5.1), I omit this proof.

Corolary 2.3.8. If Y is a variety, M(Y) is generated by the images of the constant covering homomorphisms $c.\mathrm{id}_F$: $F \longrightarrow F$, with F locally free and $c \in k$.

Proof. Use 2.3.7 and 2.3.5.

Remark 2.3.9. If the functor M is restricted to the subcategory of equivariantly embeddable automorphisms of End, then the definitions of 2.2 may be simplified. In particular, derived category theory is not needed for ch. 3. However the general case is needed in ch. 5.

Appendix: Chapter 3 does not need derived categories.

This demonstrates how, if only equivariantly embeddable endomorphisms are being studied (as is the case in Chapter 3), derived category theory may be eliminated from the text.

Firstly define $M_1(y)$ and $M_2(y)$ as in 2.3.1 and 2.3.6. Then prove 2.3.7, i.e. show $M_1 \approx M_2$, and repeat all of 2.3 except 2.3.2. Thirdly, consider a morphism $p: (Y,y) \to (X,x)$ of equivariantly embeddable endomorphisms. In view of the exactness of y_*, y^*, x_* and x^*, $p^!: M_1(x) \to M_1(y)$ and $p_!: M_1(y) \to M_1(x)$ may be defined directly as follows:

If $\phi: x^*F \to F$ is a covering homomorphism on X with F locally free, let $p^!\phi: y^*p^*F \to p^*F$ be the composite $y^*p^*F = p^*x^*F \xrightarrow{p^*\phi} p^*F$. $p^!: M_2(x) \to M_2(y)$ is induced by linearity from the rule $p^![\phi] = [p^!\phi]$. Now use 2.3.7 to cross to M_1.

If $\psi: y^*G \to G$ is a covering homomorphism on Y, with G coherent, define $R^tp_!\psi: x^*R^tp_*G \to R^tp_*G$ as follows: The adjointness of y^* and y_* and of x^* and x_* gives rise to a natural transformation of functors $x^*p_* \to x^*p_*y_*y^* = x^*x_*p_*y^* \to p_*y^*$, and so, taking derived functors, for each integer t there is a natural transformation ${}_tx^*R^tp_* \to R^tp_*y^*$. $R^tp_!\psi$ is the composite of $x^*R^tp_*G \to R^tp_*y^*G \xrightarrow{R^tp_*\psi} R^tp_*G$. Then $p_!: M_1(y) \to M_1(x)$ is induced by linearity from the rule $p_![\psi] = \Sigma(-)^t.[R^tp_!\psi]$.

In view of the origins of the vital adjointness 1.6.13, it is easy to show that modulo 2.3.2 this is the required specialisation of the original definition of M.

Fourthly, discard 1.3, 1.4 and 1.6 but retain 2.3, 2.4, 2.5 and 2.6 virtually without change. 2.7 does not simplify much in the equivariantly embeddable case and needs slight changes to work with the functor M_1 instead of M. Chapter 3 may now be read with M_1 replacing M.

The above comments may help the reader understand 2.2. They should make Chapter 3 more accessible.

2.4. THE FUNCTOR M'.

This functor is in some ways analogous to localised equivariant K-theory (c.f. [15]). I begin by recalling some standard facts about the Witt ring.

Definition 2.4.1.1. If k has characteristic $p > 0$, then set $W(k)$ = its Witt ring in the sense of Serre ([16 p. 45 sqq.]). There it is shown that $W(k)$ is a complete discrete valuation ring which is absolutely non-ramified and has k as its residue field. Indeed it is characterised by these properties.

Quote 2.4.1.2. According to Mumford ([13 p. 174]), there is a unique multiplicative (but not additive) homomorphism $w: k \longrightarrow W(k)$ which sections the projection $W(k) \rightarrow k$.

Quote 2.4.1.3. According to Serre (loc. cit.) the field of fractions, $FW(k)$, of $W(k)$ is of characteristic 0. Denote also by $w: k \rightarrow FW(k)$ the composite of the above w with the inclusion.

Convention 2.4.1.4. If k has characteristic 0, set $W(k) = k = FW(k)$ and $w = id$.

Definition 2.4.2. By 2.4.1.2, $W(k)$ is a $Z[k]$-algebra. By cofunctoriality, for $(Y,y) \in$ End, $M(y)$ is naturally a $M(pt) = Z[k]$-algebra. Define $M'(y) = M(y) \otimes_{Z[k]} W(k)$.

Construction 2.4.3. Let Y be an irreducible variety, L a locally free sheaf on Y and $\phi: L \rightarrow L$. Then the characteristic equation of ϕ, which is first constructed locally, has coefficients which are global functions on Y and hence constant. Hence the roots of this equation are constant. Let c be one of them. Then $L_c = \mathrm{Ker}\ (\phi - c)^N$, for $N >$ rank L, is a subsheaf of L independent of N. As in the theory of endomorphisms of finite rank k-vector spaces, one shows that $L = \bigoplus_c L_c$ and that $\phi = \bigoplus_c \phi_c$, where each $\phi_c: L_c \rightarrow L_c$ is such that $(\phi_c - c)$ is nilpotent. As each L_c is a direct summand of L, it is locally free. If Y is a reducible variety, the same argument may be applied to each component and the same conclusion drawn.

Proposition 2.4.4. Let Y be a perhaps reducible variety. Then $M(Y) \approx K(Y) \otimes_Z Z[k]$ and $M'(Y) \approx K(Y) \otimes_Z W(k)$.

Proof. Define a morphism $M(Y) \approx M_2(Y) \xrightarrow{\alpha} K(Y) \otimes Z[k]$ by, with the notation of 2.4.3, $[\phi: L \to L] = \sum_c [\phi_c: L_c \to L_c] \longmapsto \sum_c [L_c] \otimes [c]$. (These square brackets mean the image of the covering homomorphism or sheaf in the appropriate Grothendieck group.) Note that as the construction 2.4.3 respects short exact sequences of ϕ's, the above rule induces an abelian group, indeed a ring, homomorphism. Define a morphism $\beta: K(Y) \otimes Z[k] \to M_2(Y)$ by $\beta([L] \otimes [c]) = [c.id_L]$. Then 2.3.5 shows that $\beta\alpha = id$, whilst trivially $\alpha\beta = id$. This gives the first result which in turn gives the second.

Proposition 2.4.5. M' is a cofunctor from End to the category of W(k)-algebras. It is also a functor from End to the category of W(k)-modules. (Use the same notation, i.e. $p^!$, $p_!$, for the induced morphisms for M' as for M.)

Proof. This follows immediately from the corresponding result for M given in 2.2. Just tensor each object with W(k).

Remark 2.4.6. A further 'localisation' is available in the functor L defined thus: For $(Y,y) \in$ End, defined L(y) to be the quotient of the free k-vector space on the set of all (isomorphism classes of) covering homomorphisms by the subspace generated by all elements of the forms (i) $[\phi_1] - [\phi_2] + [\phi_3]$ whenever the ϕ_i form a triangle; (ii) $[\phi_1 + \phi_2] - [\phi_1] - [\phi_2]$ whenever the sum is defined; and (iii) $[c.\phi] - c.[\phi]$ for $c \in k$. The functorial properties of L then follow the same pattern as those of M. In this thesis, however, I use M and M' only.

2.5. BRAUER TRACE AND CHERN CHARACTER.

Definition 2.5.1. Let ϕ be an endomorphism of a finite rank k-vector space. Define the Brauer trace (B.tr.) of ϕ by: $B.tr.\phi = \sum w(c_i) \in W(k) \subset \hat{W}(k)$, the sum being taken over the eigenvalues c_i of ϕ, taken with the appropriate multiplicity.

Remarks 2.5.2. If the characteristic of k is 0, this is the classical trace. If it is $p \neq 0$, the classical trace is the reduction mod p of the Brauer trace. It is well-known from linear algebra that B.tr. is additive for short exact sequences. For $\phi: L^{\cdot} \to L^{\cdot} \in D(pt)$, set $B.tr.\phi = \Sigma(-)^t B.tr.(h^t\phi)$. Adopt a similar sign convention for the classical trace.

Next certain properties of a cofunctor A, the Chow ring, from the category of irreducible varieties to the category of commutative rings with unit are recalled from [BS p. 111-112].

Quotes 2.5.3. A is a cofunctor from the category of irreducible varieties to the category of finitely graded commutative rings with unit. The zero-graded component is always the integers.

It is also a functor to the category of abelian groups.

For $f: Y \to X$, write $f^*: A(X) \to A(Y)$ and $f_*: A(Y) \to A(X)$; the latter is only additively a homomorphism. When X = pt, $f_*(\eta)$ is usually written $[\eta][Y]$. Then the 'projection formula' holds: for $\eta \in A(Y), \xi \in A(X)$ $f_*(\eta . f^*\xi) = f_*\eta . \xi$.

There is a ring homomorphism, Chern character: $ch: K(Y) \to A(Y) \otimes Q$, such that $f^* ch \xi = ch f^! \xi$ for $\xi \in K(X)$.

There is a canonical unit, T(Y), the Todd class, in $A(Y) \otimes Q$, described in [BS p. 112-113]. Note also that T(Y) and ch can be explicitly written down when Y is a projective space, see [BS p. 119].

Remarks 2.5.4. If the variety Y is reducible with components Y_t, set $A(Y) = \prod A(Y_t)$ and extend 2.5.3 to the category of varieties in the obvious way.

In 2.5.3, Q may be replaced by any other characteristic zero field, in particular $FW(k)$. I shall do this, but still use the symbols f*, ch etc.

Definition 2.5.5. For Y a not necessarily irreducible variety, define a ring morphism, Chern trace, written ct, as the composite of: $M(Y) \xrightarrow{\sim} K(Y) \otimes Z[k] \xrightarrow{ch \otimes W} A(Y) \otimes FW(k)$. Similarly define a ring morphism also called Chern trace: ct: $M'(Y) \longrightarrow A(Y) \otimes FW(k)$.

Definition 2.5.6. For $(Y,y) \in End$, let i: $(Y^y,id) \longrightarrow (Y,y)$ be the injection of the fixed point set. Define A(y) to be $A(Y^y)$. Define ct: M(y) (or M'(y)) $\longrightarrow A(y) \otimes FW(k)$ as the composite of $M(y) \xrightarrow{i^*} M(Y^y) \xrightarrow{ct} A(Y^y) \otimes FW(k) = A(y) \otimes FW(k)$ (similarly for M').

Lemma 2.5.7. A is seen to be a cofunctor from End to rings. It is also a functor from End to the category of abelian groups. For f: $(Y,y) \longrightarrow (X,x) \in End$, write f* and f_* respectively for the induced morphisms. Then the 'projection formula' still holds, and for $\xi \in M(x)$ $f^* ct \xi = ct f' \xi$.

Proof. There is a commutative square in End:

$$\begin{array}{ccc} (Y^y,id) & \xrightarrow{f_0} & (X^x,id) \\ {\scriptstyle i_y}\downarrow & & \downarrow{\scriptstyle i_x} \\ (Y,y) & \xrightarrow{f} & (X,x). \end{array}$$

Of course one defines f* to be f_0^* etc. The projection formula is obvious. To prove the final equality, one first notes that it suffices to prove it for f_0, and then applies 2.3.8 to $M(X^x)$ to reduce it to the fourth quote of 2.5.3.

2.6. LOCAL CO-ORDINATES.

This section gathers together most of what we need to know about the behaviour of an endomorphism of a ~~==~~variety near a fixed point.

Firstly, I write down certain facts about local algebra. A noetherian regular local k-algebra, A, with residue field k, may be characterised in several ways as in [ALM passim]. The following lemma is taken from [ALM p. IV-40] and Zariski-Samuel [18 p. 303]:

Lemma 2.6.1. If $\bar{x}_1,\ldots,\bar{x}_r$ are r elements of the maximal ideal m of A (as above), the following three statements are equivalent: (i) $\bar{x}_1,\ldots,\bar{x}_r$ form part of a regular system of parameters (see [ALM p. IV-40]) of A.

(ii) The images of the $\bar{x}_t$ in the finite rank k-vector space m/m^2 are linearly independent.

(iii) Setting $\bar{I} = \langle\bar{x}_1,\ldots,\bar{x}_r\rangle$ (the ideal generated by these elements), the local ring $A/\bar{I}$ is regular of dimension dimA - r. In particular, $\bar{I}$ is a prime ideal.

Further, if $\bar{I}$ is an ideal of A such that $A/\bar{I}$ is regular, it is of the above form.

The above results may be interpreted in the usual way for varieties. Recall that a variety X is non-singular if and only if for each $a \in X$, the local ring of X at a, written $\mathcal{O}_{a,X}$ or A_a, is regular. Let Y be a nonsingular closed subvariety of the nonsingular variety X. Let $a \in Y \subset X$. Then $\mathcal{O}_{a,X}$ has an ideal $\bar{I}$ such that $\mathcal{O}_{a,Y} = \mathcal{O}_{a,X}/\bar{I}$. So choose elements $\bar{x}_1,\ldots,\bar{x}_r$ which generate $\bar{I}$ as in 2.6.1. If U is an affine open subset of X containing a, write A(U) for the affine algebra of U. Then there certainly exists a U such that the $\bar{x}_t$ are the images of elements x_t of A(U) under the canonical map $A(U) \longrightarrow \mathcal{O}_{a,X}$. But $A(Y \cap U) = A(U)/I$ for a suitable ideal I which contains the x_t. By considering the images of a finite set of generators of I in $\bar{I}$, we see that, perhaps after a smaller U has been chosen, we

may assume that the x_t generate I.

In fact, as is stated in [BS p. 120], by replacing this U by a smaller U if necessary, we may require that the x_t have the following properties:

Definition 2.6.2. With the above notation, $x_1,\ldots,x_r$ are called a set of local co-ordinates for Y in U if:

(α) $Y \cap U$ is defined by $x_1 = \ldots = x_r = 0$ (i.e. $x_1,\ldots,x_r \in A(U)$ generate the ideal I of $Y \cap U$).

(β) $dx_1,\ldots,dx_r$ are linearly independent at each $b \in Y \cap U$.

(γ) The images of the x_t in $\mathcal{O}_{b,X}$ generate $\mathcal{I}(Y)_b = I_b$ and satisfy the conditions of 2.6.1 for each $b \in Y \cap U$.

Remark 2.6.3. For $i: Y \hookrightarrow X$, the normal bundle, N, of Y in X is defined by the exactness of the sequence of vector bundles on Y: $0 \to T_Y \longrightarrow i^*T_X \longrightarrow N \to 0$, where T denotes 'tangent vector bundle'. As is noted in [BS p. 120], the above local co-ordinates show that the map $D: \mathcal{I}(Y)/\mathcal{I}(Y)^2 \to \mathcal{O}_Y(N^\vee)$ is an isomorphism of locally free sheaves on Y, where $N^\vee$ is the dual vector bundle of N.

Notation 2.6.4. Consider the following circumstance: $x: X \to X$ is an endomorphism of the irreducible variety X. The fixed component (i.e. component of the fixed point set)^Y is assumed to be nonsingular. $a \in Y \xhookrightarrow{i} X$. Set $n = \dim X$, $n-q = \dim Y$. In 2.6.5 I shall set out the facts that are needed about local co-ordinates near a, with sketch proofs given concurrently. It may be necessary to shrink the open affine neighbourhoods for some steps, but this will not affect the properties given in the previous steps. At the end of 2.6.5 the existence of U', U, the f'_t and the f_t with all these properties will have been proved.

Construction 2.6.5.1. Choose an open affine $a \in U \subset X$ such that there exist $f_1,\ldots,f_q,\ldots,f_n \in A(U)$ which are a set of local co-ordinates for a in U such that $f_1,\ldots,f_q$ are one for Y in U.

2.6.5.2. We may assume that U has been chosen so that it does not intersect any other fixed component.

2.6.5.3. If $a \in U' \subset U \cap x'U$ is another open affine set, then by 2.6.2 the restrictions $f'_t \in A(U')$ are local co-ordinates in U' as in 2.6.5.1. We also denote by $x: U' \to U$ the restriction of $x: X \to X$, and so $f_t x \in A(U')$. Since $f_1 x, \ldots, f_q x$ vanish on $Y \cap U'$, there exist $c_{\alpha\beta} \in A(U')$ $(1 \leqslant \alpha, \beta \leqslant q)$ such that $f_t x = \sum c_{t\beta} f'_\beta$ for $1 \leqslant t \leqslant q$.

2.6.5.4. By [EGA I.7.3], x induces a local (i.e. respecting the maximal ideal m) endomorphism of $A_a = \mathcal{O}_{a,X}$. Call it $\tilde{x}$. $\tilde{x}$ induces an endomorphism of m/m^2, which is also called $\tilde{x}$. The images $\bar{f}_t \in m$ of the f_t are such that the elements $(\bar{f}_t + m^2)$ form a basis for m/m^2. By 2.6.5.3, if $\bar{c}_{\alpha\beta} \in A_a/m = k$ are the images of the $c_{\alpha\beta}$, $\tilde{x}(\bar{f}_t + m^2) = \sum \bar{c}_{t\beta}(f_\beta + m^2)$, $t \leqslant q$.

2.6.5.5. Of course by 2.6.1 any set of r local co-ordinates may be replaced by any r k-linear expressions in themselves provided that the determinant of the transformation is not 0. Thus we may assume that the qxq matrix $(\bar{c}_{\alpha\beta})$ of 2.6.5.4. is upper triangular. As Y is fixed, the endomorphism induced by x on the quotient vector space $(m/m^2)/\langle \bar{f}_1 + m^2, \ldots, \bar{f}_q + m^2 \rangle$ is the identity.

2.6.5.6. By 2.6.1, the elements $f'_1 - f_1 x, \ldots, f'_q - f_q x$ of $A(U')$ form a set of local co-ordinates for Y in U' (for suitably small U') if and only if $\det(I - (\bar{c}_{\alpha\beta})) \neq 0$.

2.6.5.7. Now consider the 'square' XxX of X and its closed subvarieties $\gamma X = (x, id)X$ (the 'graph') and $\delta X = (id, id)X$ (the 'diagonal'). Each has dimension and codimension n. Then $(a,a) \in \gamma X \cap \delta X$ has UxU' as an open affine neighbourhood. By definition, $A(UxU') \approx A(U) \otimes A(U')$. After suitably shrinking U' and U we may assume that the elements $f_t \otimes 1 - 1 \otimes f_t x$ form a set of local co-ordinates for γX in UxU' and that the elements $f_t \otimes 1 - 1 \otimes f'_t$ form a set of local co-ordinates for δX in UxU'.

2.6.5.8. $\delta_i Y = \gamma_i Y = (i,i)Y \subset XxX$ is a closed subvariety of

dimension n-q and codimension n+q. After again suitably shrinking U and U', local co-ordinates for it in UxU' may be taken to be $f_t \otimes 1$ $(1 \leq t \leq q)$, $1 \otimes f'_t$ $(1 \leq t \leq q)$ and $f_t \otimes 1 - 1 \otimes f'_t$ $(q+1 \leq t \leq n)$.

2.6.5.9. Local co-ordinates for (a,a) in UxU' (again suitably shrunk) may be taken to be $f_t \otimes 1$ $(1 \leq t \leq n)$ and $1 \otimes f'_t$ $(1 \leq t \leq n)$.

2.6.5.10. Let I be the ideal of A(U) generated by $f_1, \ldots, f_q$. Similarly define the ideal I' of A(U'). Then $A(Y \cap U) = A(U)/I$, $A(Y \cap U') = A(U')/I'$. For $t > q$, let f^*_t be the image of f_t in this quotient ring, likewise define f'^*_t. Then local co-ordinates for the diagonal of YxY in $(Y \cap U) \times (Y \cap U')$ may be taken to be $f^*_t \otimes 1 - 1 \otimes f'^*_t$.

2.7. ADMISSIBLE ENDOMORPHISMS.

Let $x: X \longrightarrow X$ have non-singular fixed point set. In this section the concept of admissibility for x is defined and a canonical element, λ_x or λ_X, of $M(X^x)$ is constructed. The section is an adaptation of [BS sn. 10].

To simplify the notation assume that X is irreducible of dimension n. Let Y be a fixed component of dimension n-q. The notation of 2.6.4 and 2.6.5 will be used, sometimes without reference. An element $\lambda_Y \in M(Y)$ will be found (and $\lambda_x = \bigoplus_Y \lambda_Y \in M(X^x)$). In 2.2 the following two morphisms were defined:

$$M(Y) \xrightarrow{i_!} M(x) \xrightarrow{i^!} M(Y).$$

Proposition 2.7.1.1. There is a canonically induced endomorphism, ϕ, of the dual, $N^\vee$, of the normal bundle of Y in X. Hence there is an endomorphism, $\lambda^t\phi$, of $\lambda^t N^\vee$ for $0 \leqslant t \leqslant q$.

Proof. It is possible to obtain ϕ from an endomorphism of the exact sequence of 2.6.3. However it is more convenient to use the isomorphism D of 2.6.3 and construct instead an endomorphism of the locally free sheaf $\mathcal{I}(Y)/\mathcal{I}(Y)^2$ of rank q on Y.

For each $a \in Y$, using the open affine neighbourhood U' of 2.6.5, an endomorphism $\phi|U': \mathcal{I}(Y)/\mathcal{I}(Y)^2|_{Y\cap U'} \longrightarrow \mathcal{I}(Y)/\mathcal{I}(Y)^2|_{Y\cap U'}$ will be constructed. Instead of working with the category of sheaves on $Y\cap U'$ it is convenient to work with the equivalent one of $A(Y\cap U')$-modules. Note $A(Y\cap U') = A(U')/I'$, where $I' = \langle f'_1, \ldots, f'_q \rangle$ (the ideal generated by these elements). $\mathcal{I}(Y)/\mathcal{I}(Y)^2|_{Y\cap U'}$ corresponds to the $A(Y\cap U')$-module I'/I'^2, which is freely generated by the q elements $f'_1+I'^2, \ldots, f'_q+I'^2$.

Define an endomorphism $\phi_o(U'): I'/I'^2 \longrightarrow I'/I'^2$ by

$$f'_t+I'^2 \longmapsto \sum_\beta c^*_{t\beta}(f'_\beta+I'^2),$$

where the $c^*_{t\beta} \in A(Y\cap U')$ are the images of the $c_{t\beta} \in A(U')$. (Crudely, the endomorphism is induced by $f'_t \longmapsto f_t x$.) Let $\phi|U': \mathcal{I}(Y)/\mathcal{I}(Y)^2|_{Y\cap U'} \longrightarrow \mathcal{I}(Y)/\mathcal{I}(Y)^2|_{Y\cap U'}$ be the corresponding endomorphism. These endomorphisms clearly 'patch together' to

define the required global ϕ: $\mathcal{O}(Y)/\mathcal{O}(Y)^2 \longrightarrow \mathcal{O}(Y)/\mathcal{O}(Y)^2$.

Theorem 2.7.1.2. If $\lambda_Y = \sum(-)^t.[\lambda^t\phi] \in M(Y)$, then, for $\eta \in M(Y)$,

$$i^!.i_!\eta = \eta.\lambda_Y.$$

Proof. By linearity and the fact (2.3.8) that $M(Y)$ is generated by the images of the constant covering homomorphisms $c.id_F$: $F \longrightarrow F$ (where $c \in k$ and F is a locally free sheaf on Y), it is sufficient to prove the theorem for η of this form.

In this case $i^!.i_!\eta$ is the image in $M(Y)$ of a certain endomorphism of $Li^*.i_*F$. In view of 2.3.2 it is therefore the alternating sum of the images in $M(Y)$ of the induced endomorphisms of the $h^{-t}Li^*.i_*F$. But, by [BS p. 122], $h^{-t}Li^*.i_*F \approx F \otimes_Y \lambda^t N^\vee$.

This last isomorphism arises as follows: For each $a \in Y \subset X$, we have a set of local co-ordinates, $f_1, \ldots, f_q$ for Y in U. (Likewise $f'_1, \ldots, f'_q$ form one for Y in U'.) We may assume that in 2.6.5 U was chosen sufficiently small such that $F|U\cap Y$ (and, therefore, also $F|U'\cap Y$) is free of rank e. Again it is convenient to work with $A(U)$-modules rather than with sheaves on U. So $i_*F|U$ is represented by the direct sum of e copies of the Koszul complex over $A(U)$ on the q elements $f_1, \ldots, f_q$. Recall that $A(Y\cap U) = A(U)/I$, where I is the ideal $\langle f_1, \ldots, f_q\rangle$. Hence $Li^*.i_*F|Y\cap U$ is represented by the direct sum of e copies of the Koszul complex over $A(Y\cap U)$ on q elements, all 0.

This shows that $h^{-t}Li^*.i_*F$ is a locally free sheaf of rank $e.\binom{q}{t}$ on Y; it may be shown to be isomorphic to $F \otimes_Y \lambda^t N^\vee$ by either a direct check of overlaps of co-ordinate patches or use of the algebra structure of the Koszul complex as in [BS].

So it remains to identify the endomorphism of $h^{-t}Li^*.i_*F$. The definitions of 2.2 show that the required endomorphism of $Li^*.i_*F$ is the image of $c \in Hom[Y;F,F]$ under the composite of the following sequence of morphisms, whose definitions are recalled on the right of the page:

$$
\begin{array}{lll}
c \in \mathrm{Hom}[Y;F,F] & & \\
\downarrow & & i_* \\
\mathrm{Hom}[X;i_*F,i_*F] & & \\
\downarrow\wr & & i_* = Rx_*.i_* \\
\mathrm{Hom}[X;i_*F,Rx_*.i_*F] & & \\
\downarrow & & Lx^* \\
\mathrm{Hom}[X;Lx^*i_*F,Lx^*Rx_*i_*F] & & \\
\downarrow & & Lx^*Rx_* \longrightarrow \mathrm{id}_{D(X)} \\
\mathrm{Hom}[X;Lx^*i_*F,i_*F] & & \\
\downarrow & & Li^* \\
\mathrm{Hom}[Y;Li^*Lx^*i_*F,Li^*i_*F] & & \\
\downarrow\wr & & Li^*Lx^* = Lx^* \\
\mathrm{Hom}[Y;Li^*i_*F,Li^*i_*F]. & &
\end{array}
$$

For each $a \in Y \subset X$ these morphisms will be studied in the open affine neighbourhood U' by using the corresponding $A(U')$-modules. Set $A' = A(U')$.

Consider the Koszul complexes over A', $K^\cdot(f_t x)$ and $K^\cdot(f'_t)$, on the sets of q elements $f_1 x, \ldots, f_q x$ and $f'_1, \ldots, f'_q$ respectively. For $0 \geq s \geq -q$, $K^s(f'_t)$ and $K^s(f_t x)$ are defined to be $\lambda^{-s}(A'^q)$; they are 0 for other values of s. The differentials of these complexes are the usual ones, specified (e.g.) in [BS p. 120]. By 2.6.5.3, $f_t x = \sum_\beta c_{t\beta} f'_\beta$. The endomorphism $(c_{\alpha\beta})$ of A'^q induces a morphism $\psi^s = \bar{\lambda}^s(c_{\alpha\beta}): K^s(f_t x) \to K^s(f'_t)$. It may be checked that these ψ^s form a morphism of complexes: $\psi: K^\cdot(f_t x) \longrightarrow K^\cdot(f'_t)$.

Now $i_*F|U$ and therefore also $Rx_*i_*F|U = x_*i_*F|U$ may be represented by the direct sum of e copies of the Koszul complex over $A(U)$ on the elements $f_1, \ldots, f_q$. Hence $Lx^*i_*F|U'$ and $Lx^*Rx_*i_*F|U'$ may be represented by the A'-complex $K^\cdot(f_t x)^e$. Also $i_*F|U'$ may be represented by the A'-complex $K^\cdot(f'_t)^e$. As these two A'-complexes are free, any morphism between their images in the derived category may be represented by a morphism of complexes, which is determined up to homotopy. (c.f. [RD p. 48].)

I claim that the image of c in $\mathrm{Hom}[U';Lx^*i_*F|U', i_*F|U']$ is represented by $c.\psi^e$. This will give the result, as it shows that the induced morphisms $h^sLi^*i_*F|Y\cap U' \longrightarrow h^sLi^*i_*F|Y\cap U'$ are represented by $c.\lambda^{-s}(c^*_{\alpha\beta})^e$, and the result then follows from the construction of the isomorphism $h^sLi^*i_*F \approx F\otimes_Y \lambda^{-s}N^\vee$.

The claim is seen to be equivalent to identifying ψ^e with the adjunction morphism $Lx^*Rx_*i_*F|U' \longrightarrow i_*F|U'$.

Unfortunately, in 1.6.13 the adjointness was defined by the other adjunction morphism ϱ: $G^\cdot \longrightarrow Rx_*Lx^*G^\cdot$. ϱ was given by reference to [RD p. 110-111], where it is constructed from the corresponding adjunction morphism $\mathrm{id} \longrightarrow x_*x^*$. However the adjunction used here, $\bar{\varrho}$: $Lx^*Rx_*K^\cdot \longrightarrow K^\cdot$, may be constructed in a way similar to that used in [RD] for ϱ: Represent $K^\cdot$ by an x_*-acyclic complex $K^\cdot_o$. Then use the composite $Lx^*x_*K^\cdot_o \xrightarrow{\xi} x^*x_*K^\cdot_o \longrightarrow K^\cdot_o$, where ξ is obtained from the universal property of the derived functor (see the appendix to 1.1 and then [RD p. 50]) and the second morphism is the ordinary adjunction. (Of course it should be proved that this construction does give the required adjunction: one method is to establish the compatibility with composition of variety morphisms and then check directly for closed embeddings and projections $PxX \longrightarrow X$.)

Note that $h^ri_*F = 0$ for $r \neq 0$. In view of the fact that $Rx_*i_*F = x_*i_*F$, the above paragraph shows that it is sufficient to identify $h^o\psi^e$ with the adjunction $x^*x_*i_*F|U' \longrightarrow i_*F|U'$. But this follows from the definition of this adjunction.

The above notation is retained for the next two lemmas, which are proved together on the next page:

<u>Lemma 2.7.1.3.</u> The image, λ'_Y, of λ_Y in $M'(Y)$ is a unit if and only if ϕ has no eigenvalues 1. (See 2.4.3 for 'eigenvalues'.)

<u>Lemma 2.7.1.4.</u> $ct(\lambda_Y)$ is a unit in $A(Y)\otimes FW(k)$ if and only if ϕ has no eigenvalues 1.

Proof. By 2.4.3, $N^{\vee} = \bigoplus_c L_c$ and $\phi = \bigoplus_c \phi_c$, with $\phi_c : L_c \to L_c$ such that $\phi_c - c$ is nilpotent, for a finite set of $c \in k$. By 2.3.5, $[\lambda^t \phi_c] = [c^t . id_{\lambda^t L}] \in M(Y)$. So $ct[\lambda^t \phi_c] = w(c)^t . ch\lambda^t L_c$. So $ch(\lambda_Y) = \prod_c (1 - w(c).chL_c + w(c)^2 . ch\lambda^2 L_c - \dots)$. As $A(Y) \otimes FW(k)$ is a finitely graded ring (2.5.3), $ct(\lambda_Y)$ is a unit if and only if its zero-graded component, $\prod_c (1-w(c))^{rank(L_c)}$, is in $FW(k)$. This happens if and only if ϕ has no eigenvalues 1. (a unit) This is 2.7.1.4.

Now, by 1.5.3, $K(Y) \approx Z \oplus \tilde{K}(Y)$, where the augmentation ideal is nilpotent. Hence, in view of 2.4.4, $M'(Y) \approx W(k) \oplus \tilde{M}'(Y)$, where the augmentation ideal $\tilde{M}'(Y)$ (defined analogously to $\tilde{K}(Y)$) is nilpotent. So, for L a locally free sheaf of rank r on Y and $c \in k$, $(1 - w(c)[id_L] + w(c)^2[id_{\lambda^2 L}] - \dots)$ is a unit in $M'(Y)$ if and only if its augmentation is a unit in $W(k)$, i.e. if and only if $(1-\binom{r}{1}w(c)+\binom{r}{2}w(c)^2 - \dots)$ is a unit in $W(k)$, i.e. if and only if $c \neq 1$. 2.7.1.3 now follows from the first two sentences of the last paragraph.

Definition 2.7.2. The endomorphism x is said to be admissible if and only if each fixed component, Y, is non-singular and the equivalent conditions of 2.7.1.3 are satisfied for each Y. Then $\lambda'_X = \lambda'_x = \bigoplus_Y \lambda'_Y$ is a unit in $M'(X^X)$.

Restriction 2.7.3. In Chapters 3, 4, and 5 End will mean the full subcategory of the End of 2.1 whose objects are the admissible endomorphisms. M, M' and A will denote the restriction of the original M, M' and A to this subcategory.

Remark 2.7.4. Non-admissible endomorphisms exist. The reader may check that a linear automorphism of a projective space is admissible if and only if it is diagonalisable. (This is why non-diagonalisable ones were excluded from the definition of equivariantly embeddable endomorphisms.) It also explains the restriction imposed on the word 'periodic' on the next page.

CHAPTER 3. THE FIXED POINT FORMULA FOR PERIODIC ENDOMORPHISMS.

3.1. INTRODUCTION.

This chapter can easily be made independant of derived category theory, as is explained later in this section. The formulation of the main theorem (3.1.5) is rather long and is done in this section. The formulation, but not the result, will be needed in Chapter 5. Otherwise the next two chapters are independent of this one.

'Periodic' will always be taken as meaning 'periodic with period prime to the characteristic of k if this is not zero'. Periodic endomorphisms are particularly easy to handle, no doubt because equivariant K-theory (Segal [15]) dominates the scene. It will, however, become apparent that equivariantly embeddable (2.3.3) endomorphisms form a larger class of endomorphisms which can be treated by this approach. This extra generality is avoided mostly to keep 3.3 reasonably simple.

Recall now the fundamental result of [BS]:

Quote 3.1.0. Let $f: Y \longrightarrow X$ be a morphism of irreducible projective varieties. Let $\eta \in K(Y)$. Then:

$$f_*(ch\eta.T(Y)) = ch(f_!\eta).T(X) \in A(X) \otimes Q.$$

The notation has been partly explained in 1.5 and 2.5. In [BS] it is proved in greater generality (for proper morphisms of quasi-projective varieties). It easily yields:

Corolary 3.1.1. Let $f: Y \longrightarrow X$ be a morphism of varieties. Let $\eta \in M(Y)$ (or $M'(Y)$). Then:

$$f_*(ct\eta.T(Y)) = ct(f_!\eta).T(X) \in A(X) \otimes FW(k).$$

Proof. As usual it is sufficient to work componentwise, so we may assume additionally that Y and X are irreducible. As (2.3.8) the images of the constant covering homomorphisms $c.id_F: F \longrightarrow F$, where $c \in k$ and F is a locally free sheaf on Y, generate the abelian group M(Y) and the W(k)-module M'(Y), it suffices to

prove 3.1.1 for η of this form. But then $ct\eta = w(c)ch[F]$ whilst $ct(f_!\eta) = \Sigma(-)^t w(c)ch[R^t f_* F] = w(c)ch(f_![F])$ ([F] means the image of F in K(Y) and similarly $[R^t f_* F]$) and so 3.1.1 has been reduced to 3.1.0.

It is convenient to deduce an 'integrality theorem' from 3.1.1. As it will be used mostly in Chapter 5, I do not restrict it to periodic endomorphisms.

Corolary 3.1.2. Let the characteristic of k be $p \neq 0$. Let $x: X \longrightarrow X$ be an admissible endomorphism of the variety X. Let $\psi: Lx^*F^\bullet \longrightarrow F^\bullet$ be a covering homomorphism. Let $\xi \in M(X)$ (or $M'(X)$). Let $i: Y \longrightarrow X$ be the embedding of the fixed component Y of x and let $\lambda_Y \in M(Y)$ be the canonical element of 2.7.1.2. Then the elements $\nu(Y,\psi) = [ct(i^![\psi]).T(Y).(ct\lambda_Y)^{-1}][Y] \in FW(k)$ ($[\psi]$ means the image of ψ in $M'(x)$) and $\nu(Y,\xi) = [ct(i^!\xi).T(Y).(ct\lambda_Y)^{-1}][Y]$ of FW(k) are in fact in W(k).

If k has characteristic 0, I still define $\nu(Y,\psi)$ etc. in this way but there is no analogous statement.

Proof. Apply 3.1.1 to the morphism $c: Y \longrightarrow pt$. It shows that for $\eta \in M'(Y)$ $[ct\eta.T(Y)][Y] \in W(k) \subset FW(k)$. Let $\lambda_Y^!$ be the element of 2.7.1.3. Set $\eta = (\lambda_Y^!)^{-1}.i^![\psi]$ etc. to obtain the results of 3.1.2.

The following definition makes the formulation of 3.1.5 more elegant:

Definition 3.1.3. Let x be an admissible endomorphism of the variety X. Define the Todd class, T(x), of x by $T(x) = \bigoplus_Y T(Y).(ct\lambda_Y)^{-1} \in A(X^x) \otimes FW(k)$, the sum being taken over the set of fixed components, Y, of x.

Notes 3.1.4. Recall the definition of the functor A on the category End given in 2.5. Note that 3.1.5 is not asserted for non-periodic endomorphisms; indeed a non-periodic counter-example is given in Chapter 6. 3.1.6 is the important special case of 3.1.5 when X = pt, so it does not need an individual proof.

Theorem 3.1.5. Let $y: Y \longrightarrow Y$ and $x: X \longrightarrow X$ be periodic endomorphisms of varieties. Then they are admissible. For $\eta \in M(y)$ and for $f: (Y,y) \longrightarrow (X,x)$:

$$f_*(ct_\eta.T(y)) = ct(f_!\eta).T(x) \in A(X^X) \otimes FW(k).$$

Corolary 3.1.6. If $y: Y \longrightarrow Y$ is periodic and has fixed components Y_t, then for $c: Y \longrightarrow pt$ and for $\eta \in M(y)$:

$$B.tr.(c_!\eta) = \sum_t \nu(Y_t,\eta) \in W(k).$$

Remark 3.1.7. Let G be a finite group acting on the variety Y. It is possible to define (c.f. [15]) an 'equivariant K-theory' $K_G(Y)$. This is, briefly, a 'Grothendieck group' of the category of coherent G-sheaves on Y. A G-sheaf on Y is a sheaf F on Y together with for each $g \in G$ a morphism $\psi_g: g^*F \longrightarrow F$ satisfying an appropriate compatibility condition. I will not write out the formalism in detail here. There is a natural way of giving each $H^t[Y;F]$ the structure of a G-module. 3.1.6 gives a formula for the alternating sum of the Brauer characters of these representations. In 6.2 the possibility of doing this for an algebraic group G is raised.

Guide to the proof of 3.1.5. In 3.2 the admissibility of periodic endomorphisms is proved. It is also shown that they are equivariantly embeddable. Hence the derived category theory used to define M is irrelevant; it suffices to work with the simpler definitions of 2.3. 2.7.1 can also be made independent of derived category theory in the periodic case.

The proof splits up into examining three special cases, as opposed to the two used in [BS] to prove 3.1.0. In 3.3 the case when f is an embedding is dealt with. In 3.4 a rather weak 'localisation theorem' is used to deal with the case when $f: (PxX, pxid_X) \longrightarrow (X,id)$ is the projection and p a periodic linear endomorphism of the projective space P. In 3.5 the case of 3.4 is generalised to non-trivial endomorphisms of the X factor and it is shown there how to put these special cases together for the general result.

3.2. GENERALITIES ON PERIODIC ENDOMORPHISMS.

Proposition 3.2.1. Let y be a periodic endomorphism of the variety Y. Then there exists a projective space P and a diagonalisable linear endomorphism p: $P \to P$ with the same period and an embedding i: $Y \to P$ such that iy = pi. Hence y is equivariantly embeddable.

Proof. Let G be the cyclic group of order n generated by y acting on Y. If H is a very ample invertible sheaf on Y (i.e. $Y \to P((\Gamma_Y H)')$ is a projective embedding; see [EGA II.4] for the general theory used), consider $H' = \bigotimes_{t<n} (y^t)^*H$. By [EGA II.4.4.9] H' is also very ample. The isomorphism θ: $y^*H' \xrightarrow{\sim} H'$ induced by cyclic permutation of the factors of the tensor product induces a linear automorphism of period n, denoted by p, of $P = P((\Gamma_Y H')')$ which is such that iy = pi, where i is the canonical embedding. Now p^n = identity and n is assumed to be prime to the characteristic of k if this is not zero. The Jordan canonical form of an endomorphism of period n of a finite rank k-vector space shows that p is diagonalisable.

Note 3.2.2. The 'diagonalisable' part of 3.2.1 certainly fails in the (already excluded) case of automorphisms of period equal to the characteristic of k. The same proof gives the analogue of 3.2.1 for G-varieties, for any finite group G.

Proposition 3.2.3. (part of Theorem 3.1.5). Let y be a periodic endomorphism of the variety Y. Then each component of Y^y is non-singular. Further, y is admissible (2.7.2).

Proof. Without loss of generality assume additionally that Y is irreducible of dimension d. Let $a \in Y$ be such that ya = a. Recall the general theory of 2.6.

Choose an open affine neighbourhood U of a in Y and local co-ordinates $f_1, \ldots, f_d \in A(U)$ for a in U. By replacing U by $\bigcap_t y^t U$ if necessary, we may assume yU = U. Write y also for $y|U$: $U \to U$. Let m_a be the maximal ideal of the regular local ring $A_a = \mathcal{O}_{a,Y}$. y induces an automorphism $\bar{y}$ of period n (the period

of y) on the rank d k-vector space m_a/m_a^2. This vector space has a basis consisting of the images, $\bar{f}_t$, of the f_t. Hence, by replacing the original f_t by d k-linear combinations of them, we may assume additionally that $\overline{yf}_t = \alpha_t \bar{f}_t$, where the α_t are n^{th} roots of unity. Renumber them so that $\alpha_t \neq 1$ for $1 \leqslant t \leqslant q$ only, for some $0 \leqslant q \leqslant d$. By further replacing each of these f_t by $n^{-1} \cdot \sum_{e=1}^{n} \alpha_t^{-e} f_t y^e$, we may further assume (perhaps after shrinking U again) that $f_t y = \alpha_t f_t$ for each t.

I claim that, perhaps for a smaller U, $f_1, \ldots, f_q$ form a set of local co-ordinates for Y^y in U. This will prove both that Y^y is non-singular and that none of the eigenvalues, which are of course $\alpha_1, \ldots, \alpha_q$ of the induced action on the normal bundle of the fixed component containing a are 1, i.e. it establishes 3.2.3.

Proof of the claim: for perhaps a smaller U (still with $yU = U$) $f_1, \ldots, f_q$ form a set of local co-ordinates for a non-singular irreducible closed subvariety Y_o of U, such that $a \in Y_o$ and that Y_o has codimension q. Clearly $yY_o = Y_o$. If $b \in U$ is such that $yb = b$, then $f_t(b) = \alpha_t f_t(b) = 0$ for $t \leqslant q$, i.e. $b \in Y_o$. So $Y^y \cap U \subset Y_o$. It remains to show that Y_o is left pointwise fixed by y. This follows from the following lemma in local algebra: Let A $(= A_a/\langle f_1, \ldots, f_q \rangle)$ be a regular local k-algebra with residue field k. Let $x_1, \ldots, x_s$ generate the maximal ideal, m, of A. Let y be a local k-endomorphism of A leaving fixed each x_t. Then y is the identity.

Proof of the lemma: Let $f \in A$. Then for each integer r there exists, for each multi-index $\underline{i} = (i_1, \ldots, i_s)$ with $|\underline{i}| = i_1 + \ldots + i_s \leqslant r$, $c_{\underline{i}} \in k$ such that $f \equiv \sum_{\underline{i}} c_{\underline{i}} . x_1^{i_1} \ldots x_s^{i_s}$ modulo m^r. So $f - yf \in m^r$ for each r. Hence, by [ALM II-11], $f - yf = 0$ as required.

The proof of 3.2.3 is complete.

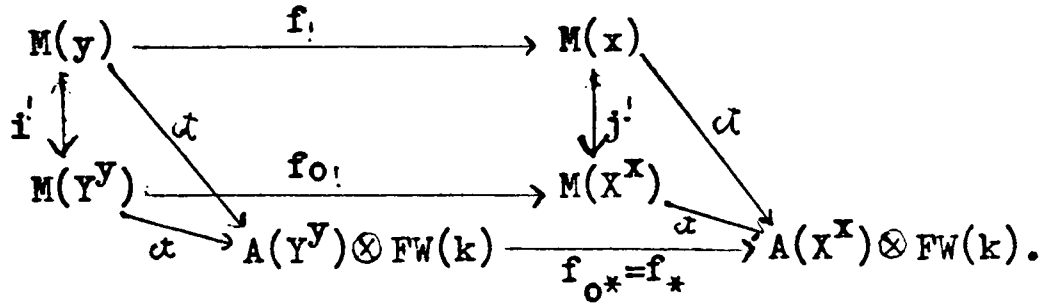
3.3. THEOREM 3.1.5 FOR EMBEDDINGS.

Proposition 3.3.1. Let $y: Y \to Y$ and $x: X \to X$ be periodic (and therefore admissible) endomorphisms of varieties. Let $f: Y \to X$ be a closed embedding with $fy = xf$. Then for $\eta \in M(y)$ (or $M'(y)$) $f_*(ct\eta.T(y)) = ctf_!\eta.T(x) \in A(X^X) \otimes FW(k)$.

Proof. As usual it is sufficient to prove this under the additional hypothesis that both Y and X are irreducible. So assume this. Use the notation defined by the commutative square of embeddings:

$$\begin{array}{ccc} Y^y & \xrightarrow{f_o} & X^x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{f} & X \end{array} .$$

There is a non-commutative diagram (the triangles commute by definition; the rest does not commute in general):



Let λ_X, λ_Y be the canonical elements of 2.7.1 in $M(X^X)$, $M(Y^y)$. Assume temporarily the following:

Proposition 3.3.2. There is an element $\gamma \in M(Y^y)$ with the properties: $\lambda_Y.\gamma = f_o^! \lambda_X$.

$$j^! f_! \eta = f_{o!}(i^! \eta . \gamma) \text{ for all } \eta \in M(y).$$

Then:

$$\begin{aligned} ctf_!\eta.T(x) &= ctj^! f_!\eta.T(X^X).(ct\lambda_X)^{-1} && \text{by definition} \\ &= ctf_{o!}(i^!\eta.\gamma).T(X^X).(ct\lambda_X)^{-1} && \text{by 3.3.2,} \\ &= f_{o*}(ct(i^!\eta.\gamma).T(Y^y)).(ct\lambda_X)^{-1} && \text{by 3.1.1,} \\ &= f_{o*}(ct\eta.ct\gamma.T(Y^y).f_o^*(ct\lambda_X)^{-1}) && \text{see below,} \\ &= f_*(ct\eta.T(Y^y).(ct\lambda_Y)^{-1}) && \text{by 3.3.2,} \\ &= f_*(ct\eta.T(y)) && \text{by definition.} \end{aligned}$$

This is 3.3.1. The 'see below' line uses the projection formula in Chow ring theory given in [BS p. 112, l. 14]. So it remains to prove 3.3.2.

Proof of 3.3.2. Let $X_1, X_2, \ldots$ be the fixed components of x. For notational convenience whenever a typical one has to be studied it will be taken to be X_1. Let $Y_1, Y_2, \ldots$ be the fixed components of y; each is contained in some fixed component of x. For notational convenience assume that only $Y_1, Y_2, \ldots, Y_r$ are contained in X_1. Henceforth t will denote a typical element of the indexing set 1, 2, ... ,r. Use the notation defined by the commutative square of embeddings:

$$\begin{array}{ccc} Y_t & \xrightarrow{f_t} & X_1 \\ {\scriptstyle i_t}\downarrow & & \downarrow{\scriptstyle j_1} \\ Y & \xrightarrow{f} & X. \end{array}$$

After $M(Y^y)$ and $M(X^x)$ have been expressed as direct sums over the fixed components it becomes apparent that it is required to find elements $\gamma_t \in M(Y_t)$ such that:

$$\lambda_{Y_t} \cdot \gamma_t = f_t^! \lambda_{X_1} \qquad \text{and}$$

$$j_1^! f_! \eta = \sum_{t=1}^{r} f_{t!} i_t^! \eta \in M(X_1) \text{ for all } \eta \in M(y).$$

These elements will be found by examining the various normal bundles of the embeddings of the diagram.

$$\begin{array}{ccccccccc} \text{In:} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tan Y_t & \longrightarrow & i_t{}^*\tan Y & \longrightarrow & N(Y_t \text{ in } Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & f_t{}^*\tan X_1 & \longrightarrow & f_t{}^* j_1{}^* \tan X & \longrightarrow & f_t{}^* N(X_1 \text{ in } X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & N(Y_t \text{ in } X_1) & \dashrightarrow & i_t{}^* N(Y \text{ in } X) & \dashrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

'tan' means 'tangent bundle', 'N' means 'normal bundle', the first two rows and columns are the usual exact sequences and the rest of the diagram is defined by requiring commutativity and exactness. The local co-ordinates worked out below will show that $0 \longrightarrow N(Y_t \text{ in } X_1) \dashrightarrow i_t^*N(Y \text{ in } X)$ locally splits and so will show that M is a vector bundle ('=' locally free sheaf) on Y_t.

Now let $a \in Y_t \subset X$. The investigations of local co-ordinates of 2.6 and 3.2 will be continued. Firstly, use the theory of 2.6.5.1 to select an open affine $U \subset X$ containing a such that:

(1) $xU = U$.
$(\text{Write } x \text{ for } x|U\colon U \longrightarrow U.)$
U intersects no other Y_s.

and such that there exist $f_1, \dots, f_q, g_1, \dots, g_d \in A(U)$, where:

(2) $f_1, \dots, f_q$ form a set of local co-ordinates for Y in U.
$f_1, \dots, f_q, g_1, \dots, g_d$ form a set of local co-ordinates for a in U.

Secondly, using a slight modification of the second paragraph of the proof of 3.2.3, we may assume additionally (perhaps for a smaller U) that:

(3) $f_\nu x = \alpha^{(\nu)} f_\nu$.
$g_\nu x = \beta^{(\nu)} g_\nu$.

where $\alpha^{(\nu)}, \beta^{(\nu)} \in k$ are n^{th} roots of unity, where n is the period of x. It is convenient to arrange the numbering so that:

(4) $\alpha^{(\nu)} \neq 1$ precisely for $1 \leqslant \nu \leqslant p$, for some $p \leqslant q$.
$\beta^{(\nu)} \neq 1$ precisely for $1 \leqslant \nu \leqslant c$, for some $c \leqslant d$.

Thirdly, by the usual considerations of the local ring of a, we may assume additionally (perhaps for a smaller U) that:

(5) $f_1, \dots, f_q, g_1, \dots, g_c$ form a set of local co-ordinates for Y_t in U. Y_t has dimension d-c.

(6) $f_1, \dots, f_p, g_1, \dots, g_c$ form a set of local co-ordinates for X_1 in U. X_1 has dimension q-p+d-c.

(7) Y has dimension d. X has dimension q+d.

Fourthly, $A(Y \cap U) = A(U)/I(Y)$, where $I(Y)$ is the ideal $<f_1, \ldots, f_q>$, contains elements $g_1^*, \ldots, g_d^*$ (the images of $g_1, \ldots, g_d$). Likewise, $A(X_1 \cap U) = A(U)/I(X_1)$, where $I(X_1)$ is the ideal $<f_1, \ldots, f_p, g_1, \ldots, g_c>$, contains elements $f''_{p+1}, \ldots, f''_q, g''_{c+1}, \ldots, g''_d$ (the images of the corresponding f_r and g_r). Likewise $A(Y_t \cap U) = A(U)/I(Y_t)$, where $I(Y_t)$ is the ideal $<f_1, \ldots, f_q, g_1, \ldots, g_c>$, contains elements $g'_{c+1}, \ldots, g'_d$ (the images of $g_{c+1}, \ldots, g_d$). Then:

(8) $f''_{p+1}, \ldots, f''_q$ form a set of local co-ordinates for Y_t in $X_1 \cap U$.

(9) $g_1^*, \ldots, g_c^*$ form a set of local co-ordinates for Y_t in $Y \cap U$.

Consider now the restriction to $Y_t \cap U$ of the morphism $i_t^* N(Y \text{ in } X)^\vee \longrightarrow N(Y_t \text{ in } X_1)^\vee$, the dual of that in the last diagram. By (8), $N(Y_t \text{ in } X_1)^\vee | Y_t \cap U$ may be identified with the free sheaf on $Y_t \cap U$ on $q-p$ elements corresponding to $df''_{p+1}, \ldots, df''_q$. By (2), $N(Y \text{ in } X)^\vee | Y \cap U$ may be identified with the free sheaf on $Y \cap U$ on q elements corresponding to $df_1, \ldots, df_q$. With the obvious notation, the restriction of the morphism under consideration is seen to map $i_t^* df_1$ to $0, \ldots, i_t^* df_p$ to 0, $i_t^* df_{p+1}$ to $df''_{p+1}, \ldots$ and $i_t^* df_q$ to df''_q. Hence $i_t^* N(Y \text{ in } X)^\vee \longrightarrow N(Y_t \text{ in } X_1)^\vee$ is surjective and locally splits. So its kernel, $M^\vee$, is locally free on Y_t and may be identified with the the dual of the M defined by the diagram. So M is a vector bundle on Y_t.

Next a canonical endomorphism, ψ, of $M^\vee$ will be constructed. As in the last paragraph, $M^\vee | Y_t \cap U$ may be identified with the free sheaf on $Y_t \cap U$ on p elements corresponding to $i_t^* df_1, \ldots, i_t^* df_p$. With the obvious notation, define $\psi | Y_t \cap U : M^\vee | Y_t \cap U \to M^\vee | Y_t \cap U$ by $i_t^* df_\nu \longmapsto \alpha^{(\nu)} . i_t^* df_\nu$ for $\nu = 1, \ldots, p$. These $\psi | Y_t \cap U$, defined for an open neighbourhood of each point of $Y_t \cap U$, 'patch together' to give the required ψ.

Next consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & f_t{}^*N(X_1 \text{ in } X) & \longrightarrow & N(Y_t \text{ in } Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle f_t{}^*\phi_x} & & \downarrow{\scriptstyle \phi_y} & & \\ 0 & \longrightarrow & M & \longrightarrow & f_t{}^*N(X_1 \text{ in } X) & \longrightarrow & N(Y_t \text{ in } Y) & \longrightarrow & 0, \end{array}$$

where the rows are the dual of the third column of the diagram which defined M, ϕ_x is the endomorphism of 2.7.1.1 for the fixed component X_1 of x and ϕ_y is the endomorphism of 2.7.1.1 for the fixed component Y_t of y. By (6), $N(X_1 \text{ in } X)^\vee|X_1\cap U$ may be identified with the free sheaf on $X_1\cap U$ on p+c ~~general~~ elements corresponding to $df_1, \dots, df_p, dg_1, \dots, dg_c$. With the obvious notation, $\phi_x|X_1\cap U$ is given by $df_1 \longmapsto \alpha^{(1)}.df_1$, ... , $df_p \longmapsto \alpha^{(p)}.df_p$, $dg_1 \longmapsto \beta^{(1)}.dg_1$, ... , $dg_c \longmapsto \beta^{(c)}.dg_c$. By (9), $N(Y_t \text{ in } Y)^\vee|Y_t\cap U$ may be identified with the free sheaf on $Y_t\cap U$ on c elements corresponding to $dg_1^*, \dots, dg_c^*$. With the obvious notation, $\phi_y|Y_t\cap U$ is given by $dg_1^* \longrightarrow \beta^{(1)}.dg_1^*$, ... , $dg_c^* \longrightarrow \beta^{(c)}.dg_c^*$. The reader may now check that the restriction of the diagram to $Y_t\cap U$ commutes. As it commutes when restricted to an open neighbourhood of each point, a, of Y_t, the diagram commutes.

Now define $\gamma_t = \sum_{u=1}^{p} (-)^u.[\lambda^u\psi] \in M(Y_t)$. In view of 2.3.8, 2.4.3 and the well-known filtration of the exterior algebra of the middle sheaf of a short exact sequence of locally free sheaves ([BS p. 109]), the last diagram shows that:

$$\lambda_{Y_t}.\gamma_t = f_t^!\lambda_{X_1}.$$

To avoid confusion, I henceforth write $M_t^\vee$ for $M^\vee$ and ψ_t for ψ.

Now let F be a locally free sheaf of rank e on Y. We may assume that U was chosen such that $F|Y\cap U \approx \mathcal{O}_Y^{\,e}|Y\cap U$. As F is $i_t{}^*$-acyclic and f_{t*} is exact, $f_{t*}Li_t{}^*F \in D(X_1)$ is represented by the sheaf $f_{t*}i_t{}^*F$.

Consider $Lj_1{}^*f_*F$. Clearly the $h^uLj_1{}^*f_*F$ are coherent sheaves on X_1 with support in $Y \cap X_1 = \bigcup_{t=1}^{r} Y_t$. As the closed Y_t are disjoint, $h^uLj_1{}^*f_*F$ is the direct sum over t of its Y_t

components, which will be denoted by $h^u Lj_1{}^* f_* F|_t$. I claim that:

(A) $$h^{-u} Lj_1{}^* f_* F|_t \approx f_{t*}(i_t{}^* F \otimes_{Y_t} \lambda^u M_t{}^\vee).$$

This claim is proved as follows. Take $a \in Y_t$. First consider the open affine neighbourhoods $Y \cap U$, U and $X_1 \cap U$ of a, and work with modules over the algebras $A(Y \cap U)$, $A(U)$ and $A(X_1 \cap U)$ rather than with sheaves on these affine neighbourhoods. Let I' be the ideal $\langle f_{p+1}, \ldots, f_q \rangle$ of $A(U)$. Let $K^\cdot(f_1, \ldots, f_p)$ be the Koszul complex over $A(U)$ on the p elements $f_1, \ldots, f_p$. Form the complex of $A(U)$-modules $K^\cdot(f_1, \ldots, f_p) \otimes A(U)/I'$. Each object in this last complex is a direct sum of copies of the $A(U)$-module $A(U)/I'$, which is seen to correspond to a $j_1{}^*$-acyclic sheaf on U. Now $f_* F|U$ may be represented by the direct sum of e copies of the $A(U)$-complex $K^\cdot(f_1, \ldots, f_p) \otimes A(U)/I'$. Hence $Lj_1{}^* f_* F|X_1 \cap U$ may be represented by the direct sum of e copies of the Koszul complex over $A(X_1 \cap U)$ on p elements, all zero, tensored with (c.f. (8)) $A(X_1 \cap U)/I(Y_t)$. So there is an obvious local isomorphism:

$$h^{-u} Lj_1{}^* f_* F|_t | X_1 \cap U \approx f_{t*}(i_t{}^* F \otimes \lambda^u M_t) | X_1 \cap U.$$

But these local isomorphisms, defined on an open neighbourhood of each point, a, of the support, Y_t, of these sheaves on X_1, are compatible on the overlaps of such neighbourhoods. So they define the required isomorphism (A).

It has already been proved that $\lambda_{Y_t} \cdot \gamma_t = f_t{}^! \lambda_{X_1}$. It remains to show that:

(B) $$j_1{}^! f_! \eta = \sum_{t=1}^{r} f_{t!} i_t{}^! \eta \in M(X_1) \text{ for all } \eta \in M(y).$$

By 3.2.1, 2.3.7 and additivity, it suffices to prove (B) in the case when η is the image of a covering homomorphism ϕ: $y^* F \to F$, where F is a locally free sheaf of rank e on Y.

Now $j_1{}^! f_! \eta$ is the image in $M(X_1)$ of a certain covering homomorphism $j_1{}^! f_! \phi$: $Lj_1{}^* f_* F \to Lj_1{}^* f_* F$; in view of (A) and 2.3.2 it is equal to the sum over t of the alternating sum over u of the images in $M(X_1)$ of certain covering homomorphisms :

$$\phi_{t,u}\colon f_{t*}(i_t{}^*F \otimes \lambda^u H_t^\vee) \longrightarrow f_{t*}(i_t{}^*F \otimes \lambda^u H_t^\vee).$$

It remains to prove that $\phi_{t,u} = f_{t*}(i_t{}^!\phi \otimes \lambda^u \psi)$.

For $a \in Y_t$, there are open affine neighbourhoods $Y_t \cap U$, $Y \cap U$, $X_1 \cap U$ and U of a in Y_t, Y, X_1 and X respectively. We will work with modules over $A(Y_t \cap U)$, $A(Y \cap U)$, $A(X_1 \cap U)$ and $A(U)$ instead of the corresponding sheaves over the four affine neighbourhoods. The proof is now completed by examining the construction of the isomorphism (A); the reader will have no difficulty in doing this by inspection if he remembers that once $f_*F|U$ has been represented by the $A(U)$-complex $K^\bullet(f_1,\ldots,f_p) \otimes A(U)/I'$ or rather by the direct sum of e copies of it, $x^*f_*F|U$ must be represented by the direct sum of e copies of the $A(U)$-complex $K^\bullet(a^{(1)}f_1,\ldots,a^{(p)}f_p) \otimes A(U)/I'$.

3.4. A LOCALISATION THEOREM.

This section calculates $K(X\times P_n)$ in terms of $K(X)$ and then gives a 'localisation theorem' (3.4.3) of very restricted generality, which is used to establish a second special case of 3.1.5. This section could easily be generalised to projectivisations of arbitrary vector bundles over the variety X, but this will not be needed in the sequel.

Proposition 3.4.1. Let X be an irreducible variety. Let P_n be the n-dimensional projective space, and set $P = X\times P_n$. Let π_2: $P\to P_n$ and π: $P\longrightarrow X$ be the projections. Set $H = \pi_2^*$(Hopf sheaf on P_n). Then $K(X)[\eta]/(\eta-1)^{n+1} \xrightarrow{\theta} K(P)$, induced by $\theta(\eta) = [H]$ is an isomorphism of $K(X)$-algebras.

Proof. $K(P)$ is a $K(X)$-algebra under $\pi^!$. To show that $[H]$ satisfies the relation, it suffices to prove that if H' is the Hopf sheaf on P_n $([H']^{-1}-1)^{n+1} = 0 \in K(P_n)$. Take n+1 hyperplanes $P^{(t)}$, $t = 1,\ldots,n+1$, with no common point in P_n. Then there are exact sequences: $0\to H'\longrightarrow \mathcal{O}_{P_n}\longrightarrow \mathcal{O}_{P^{(t)}}\to 0$, and (as the Tor's vanish, c.f. [BS p.121]) taking the tensor product of the $\mathcal{O}_{P^{(t)}}$ yields the relation. Hence θ is well defined.

Secondly, I show θ is surjective. By 1.4.2.2, if F is a coherent sheaf on P, there exists a finite direct sum S of sheaves of the form $\pi^*G\otimes H^{\otimes e}$, with G coherent on X, and a surjection $S\longrightarrow F$. Using 1.4.2.1 again, one can find such a surjection with each G locally free on X. Hence there exists a left resolution of F by sheaves of this form; by the 'theorem of syzyzies' (1.7.7) such a resolution may be taken of finite length. So θ is indeed surjective.

Thirdly, by [EGA III 2.1.12], $\pi_!\pi^! = \mathrm{id}$, i.e. $K(X)$ is a direct summand of $K(P)$ under $\pi^!$. Assume that there is a relation of the form $c_0+\ldots+c_n[H]^{-n} = 0$ in $K(P)$. Then, using [EGA III 2.1.12] and the 'projection formula' for K-theory of [BS p.110], application of $\pi_!$ shows that $c_0 = 0$. Multiply the

relation by [H] and repeat the argument to show that $c_1 = 0$ and so on. Hence the first n+1 non-positive powers of [H] are independent over K(X), i.e. θ is injective. This completes the proof.

Corolary 3.4.2. With the notation of 3.4.1, $M'(X)[\eta]/(\eta-1)^{n+1} \xrightarrow{\sim} M'(P)$.

Proof. By 2.4.4, $M'(X) \approx K(X) \otimes W(k)$ etc., and the various $\pi^!$ morphisms are compatible with this isomorphism, 3.4.2 is an immediate consequence of 3.4.1.

Proposition 3.4.3. Let X be an irreducible variety. Let $a_1,\ldots,a_q$ be q distinct non-zero elements of k. Let $V_1,\ldots,V_q$ be q non-zero k-vector spaces of finite ranks $n_1,\ldots,n_q$. Let $V = \oplus V_t$ have rank $n = \Sigma n_t$; let $a: V \to V$ be the automorphism $\oplus a_t.id_V$. Set $P = X \times P(V)$, $p = id_X \times P(a): P \to P$, $P_t = X \times P(V_t)$ and let $i_t: P_t \hookrightarrow P$ be the canonical embedding. Then:

(α) p is equivariantly embeddable.

(β) $\amalg P_t$ is the fixed point set of p.

(γ) $\oplus i_t^!: M'(p) \to \oplus M'(P_t)$ is an isomorphism.

(δ) $\oplus i_{t!}: \oplus M'(P_t) \to M'(p)$ is an isomorphism.

Further, if $H = \pi_2^*$(Hopf sheaf on P(V)), where π_2 is the projection, there is a canonical covering homomorphism $\alpha: p^*H \to H$ and a polynomial, $g(\eta)$, of degree n in $W(k)[\eta]$ such that $M'(X)[\eta]/g(\eta) \xrightarrow{\theta} M'(p)$, induced by $\theta(\eta) = [\alpha]$, is an isomorphism of M'(X)-algebras.

Proof: Note that (α) and (β) are obvious. They were included for emphasis. I first prove two lemmas, then deduce the last sentence from them, and then prove (γ) and (δ). The covering homomorphism α is specified thus: Hom[P; p*H, H] is a rank 1 k-vector space; by naturality in X it is sufficient to specify an α when X = pt; in this case it is specified by the requirement that it induces the endomorphism $\oplus a_t.id_{V_t'}$ on $\Gamma_P(H) \approx V'$.

Lemma 3.4.3.1. Let F be a coherent sheaf on P and $\phi: p^*F \to F$ a covering homomorphism. Then there exists a commutative

square:

$$\begin{array}{ccc} p^*\pi^*L \otimes p^*H^{\otimes e} & \xrightarrow{\quad p^*\theta \quad} & p^*F \\ \downarrow{\scriptstyle \pi^!\psi \otimes \alpha^{\otimes e}} & & \downarrow{\scriptstyle \phi} \\ \pi^*L \otimes H^{\otimes e} & \xrightarrow{\quad \theta \quad} & F, \end{array}$$

where θ (and therefore $p^*\theta$) are surjective, L is a locally free sheaf on X, ψ: L $\to$L, and $\pi^!\psi$ is the composite of $p^*\pi^*L$ $= \pi^*L \xrightarrow{\pi^*\psi} \pi^*L$. ($\pi$: P $\to$X is still the projection.)

Proof of 3.4.3.1. By 1.4.2.1 there is, for large N, a commutative diagram with the horizontal arrows surjective:

$$\begin{array}{ccccc} \pi^*\pi_*p^*(F \otimes H^{\otimes N}) & \longrightarrow & p^*(F \otimes H^{\otimes N}) & \approx & p^*F \otimes p^*H^{\otimes N} \\ \downarrow{\scriptstyle \pi^*\pi_*\phi'} & & \downarrow{\scriptstyle \phi'} & \swarrow{\scriptstyle \phi \otimes \alpha^{\otimes N}} & \\ \pi^*\pi_*(F \otimes H^{\otimes N}) & \longrightarrow & F \otimes H^{\otimes N}. & & \end{array}$$

Tensoring with $H^{\otimes -N}$ gives, once certain canonical identifications have been made, such an L except that L need not be locally free. However now 2.3.4.1 may be applied to L to obtain another L which satisfies all the requirements. (I have omitted some of the details here.)

Lemma 3.4.3.2. The map $M'(X)[\eta] \to M'(p)$ induced by $\eta \mapsto [\alpha]$ is surjective.

Proof. As p is equivariantly embeddable, M'(p) is generated by covering homomorphisms of the form ϕ: $p^*F \to F$. Now 3.4.3.2 may be repeatedly applied to obtain a left resolution of length n + dimX of ϕ by covering homomorphisms of the form of that of 3.4.3.1. By the 'theorem of syzygies' (1.7.7 kernel covering homomorphism of such a resolution is of form ψ: $p^*(\bigoplus_e \pi^*L_e \otimes H^{\otimes e}) \to \bigoplus_e \pi^*L_e \otimes H^{\otimes e}$, where the L_e are l free sheaves on X, zero for all but finitely many intege

Now, by an easy application of [EGA III 2.1.12], for d>e $\mathrm{Hom}[P; p^*(\pi^*L_d \otimes H^{\otimes d}), (\pi^*L_e \otimes H^{\otimes e})] = 0$. So ψ is 'upper triangular' under the given direct sum decomposition, so its image in M'(p) is the sum of the images of the 'diagonal' covering morphisms ψ_e: $p^*(\pi^*L_e \otimes H^{\otimes e}) \to (\pi^*L_e \otimes H^{\otimes e})$. Another applicat-

ion of [EGA III 2.1.12] shows that this ψ_e may be written in form of Lemma 3.4.3.1. This gives the result.

Proof of 3.4.3 resumed. The relation $g(\eta)$ is $\prod_1^q (\eta - w(a_t))^{n_t}$ and is shown to be satisfied in the same way as in the corresponding part of the proof of 3.4.1. (The choice of n hyperplanes is obvious.)

Assume that θ is not injective. Then there is a relation: $m_{n-1}[\alpha]^{n-1} + \ldots + m_o = 0$, with $m_i \in \pi'M'(X)$, not all zero. Apply each $i_t^!$ to this relation. Note that $i_t^![\alpha] = w(a_t)[H \text{ for } P_t]$ and that $M(P_t)$ is already known by 3.4.2. Whence $(\eta - w(a_t))^{n_t}$ divides $m_{n-1}\eta^{n-1} + \ldots + m_o$; this holds for each t, so $m_{n-1} = \ldots = m_o = 0$, contradicting the assumption. So the second 'further' part of 3.4.3 has been shown.

(γ) follows immediately from the explicit calculation of both range and domain of the morphism which has to be proved to be an isomorphism.

(δ) follows formally from (γ) and the fact (2.7.1.) that $i_t^! . i_{t!} = (\text{unit}).\text{id}: M'(P_t) \to M'(P_t)$. For then, up to units, $\oplus\, i_{t!}$ is a right inverse of the isomorphism $\oplus\, i_t^!$ and so is an isomorphism.

Proposition 3.4.4. Let P, p, π and X be as above. Then for $\eta \in M(p)$ or $M'(p)$:

$$\pi_*(ct\eta . T(p)) = ct\pi_!\eta . T(X).$$

(If all the a_t's are roots of unity this is a special case of 3.1.5.)

Proof. In view of 3.4.3(δ) it suffices to prove this for η of the form $i_{s!}\xi$ for $\xi \in M'(P_s)$ for each $1 \leq s \leq q$.

As, for $t \neq s$, $i_t^! . i_s \xi = 0 \in M'(P_t)$, $ct\eta$ has nonzero component in $A(P_s) \otimes FW(k)$ only. This component is $cti_s^! i_{s!}\xi = ct\lambda_{P_s} . ct\xi$ by 2.7.1. $T(p)$ has $A(P_s) \otimes FW(k)$ component $T(P_s).(ct\lambda_{P_s})^{-1}$. Writing $\pi_s: P_s \to X$ for the projection, it has now been shown

that $\pi_*(ct_\eta.T(p)) = \pi_{S*}(ct_\xi.T(P_S))$. But by 3.1.1 $\pi_{S*}(ct_\xi.T(P_S))$ $= ct_{\pi_{S!}\xi}.T(X)$ and trivially $ct_{\pi_{S!}\xi}.T(X) = ct_{\pi_{!}i_{S!}\xi}.T(X) =$ $ct_{\pi_!\eta}.T(X)$. So 3.4.4 is proved.

Alternative proof. One may use 3.4.3.2 directly to reduce the proof of 3.4.4 to a ghastly explicit check of fixed point formulas for covering homomorphisms of tensor powers of the Hopf sheaf on projective spaces with diagonalisable linear automorphisms.

3.5. COMPLETION OF THE PROOF OF 3.1.5.

Remark 3.5.1. Let $f: (Y,y) \longrightarrow (X,x) \in \mathrm{End}$, where y and x are periodic. Then if $i: (Y,y) \longrightarrow (P,p)$ is an embedding, where p is a periodic automorphism of the projective space P, f may be factorised:

$$\begin{array}{ccccc} Y & \xrightarrow{(i,f)} & PxX & \xrightarrow{\pi} & X \\ {\scriptstyle y}\downarrow & & {\scriptstyle pxx}\downarrow & & {\scriptstyle x}\downarrow \\ Y & \xrightarrow{(i,f)} & PxX & \xrightarrow{\pi} & X, \end{array}$$

where (i,f) is also an embedding and where π is the projection. By the usual formal arguments (c.f. [BS p. 113]) to prove Theorem 3.1.5 for f it suffices to prove it for (i,f) and for π. Now embeddings where dealt with in 3.3.1. In the special case of $x = \mathrm{id}$ (and X irreducible) π was dealt with in 3.4.4. So 3.1.5 has already been proved in the case $x = \mathrm{id}$; it remains to prove it for the above π: $(PxX,pxx) \longrightarrow (X,x)$. As usual there is no loss of generality in assuming X irreducible and this will be done.

Notation 3.5.2. The fixed components of the periodic $x: X \longrightarrow X$ will be denoted by X_α, $\alpha = 1, 2, \ldots$. Write $j_\alpha: X_\alpha \hookrightarrow X$. $a_1, \ldots, a_q$ are q distinct roots of unity. $V_1, \ldots, V_q$ are q non-zero vector spaces of finite ranks $n_1, \ldots, n_q$. $V = \bigoplus_t V_t$ has rank $n = \sum_t n_t$ and $a: V \longrightarrow V$ is the endomorphism $\bigoplus_t a_t.\mathrm{id}_{V_t}$. Set $P = P(V)$, $p = P(a)$ and $i_t: P_t = P(V_t) \hookrightarrow P(V)$. Set $j'_\alpha = {}^t\mathrm{id} x j_\alpha: PxX_\alpha \hookrightarrow PxX$. Let $\pi: PxX \longrightarrow X$, $\pi_\alpha: PxX_\alpha \longrightarrow X_\alpha$ and $\pi_{t\alpha}: P_t x X_\alpha \longrightarrow X_\alpha$ be the projections. So the following square in End commutes:

$$\begin{array}{ccc} (PxX_\alpha, pxid) & \xrightarrow{j'_\alpha} & (PxX,pxx) \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi} \\ (X_\alpha, \mathrm{id}) & \xrightarrow{j_\alpha} & (X,x). \end{array}$$

Lemma 3.5.3. With the above notation, for $\eta \in M(pxx)$:

$$\pi_{\alpha!} \cdot j'^{!}_\alpha \eta = j^{!}_\alpha \cdot \pi_! \eta \in M(X_\alpha).$$

Proof. This will be given after 3.5.4.

Proposition 3.5.4. With the above notation, for $\eta \in K(pxx)$:

$$\pi_*(ct\eta.T(pxx)) = ct\pi_!\eta.T(x) \in A(X^X)\otimes FW(k).$$

Proof. It suffices to prove that for each α the $A(X_\alpha)\otimes FW(k)$ components of each side are equal. Denote them by $(\ \dots\ )|_\alpha$.

$$\begin{aligned}
ct\pi_!\eta.T(x)|_\alpha &= ctj_\alpha^!\pi_!\eta.T(X_\alpha).(ct\lambda_{X_\alpha})^{-1} && \text{by definition,}\\
&= ct\pi_{\alpha!}j_\alpha'^!\eta.T(X_\alpha).(ct\lambda_{X_\alpha})^{-1} && \text{by 3.5.3,}\\
&= \pi_{\alpha*}(ctj_\alpha'\eta.T(pxid_{X_\alpha})).(ct\lambda_{X_\alpha})^{-1} && \text{by 3.4.4,}\\
&= \pi_{\alpha*}(ctj_\alpha'\eta.T(pxid_{X_\alpha}).\pi_\alpha{}^*(ct\lambda_{X_\alpha})^{-1}) && \text{see below,}\\
&= \pi_*(ct\eta.T(pxx))|_\alpha && \text{see below.}
\end{aligned}$$

The first 'see below' is another use of the projection formula in Chow ring theory (see [BS p. 112, l. 14]). The second comes about as follows: the expression within the outer brackets of the fourth line is in $\oplus_t A(P_t x X_\alpha)\otimes FW(k)$; by an easy examination of the normal bundles $T(pxx)|_{t\alpha} = T(pxid_X)|_t.\pi_{t\alpha}{}^*(ct\lambda_{X_\alpha})^{-1}$ where $|_{t\alpha}$ and $|_t$ are the analogues of $|_\alpha$; the result follows from summation over t.

So 3.5.4, and hence 3.1.5 will be proved as soon as 3.5.3 is proved. The following proof consists of establishing an isomorphism and demonstrating a complicated but essentially trivial compatibility.

Proof of 3.5.3. There is a natural transformation of functors: $Lj_\alpha{}^*R\pi_* \xrightarrow{\Psi} R\pi_{\alpha*}Lj_\alpha'{}^*: D(PxX) \to D(X_\alpha)$. It is defined by $Lj_\alpha{}^*R\pi_*F^\cdot \longrightarrow Lj_\alpha{}^*R\pi_*Rj'_{\alpha*}Lj_\alpha'{}^*F^\cdot = Lj_\alpha{}^*Rj_{\alpha*}R\pi_{\alpha*}Lj_\alpha'{}^*F^\cdot \longrightarrow R\pi_{\alpha*}Lj'^*F^\cdot$. (Here and for the rest of this proof the adjunction morphisms of the adjointness situation of the type of 1.6.13 are used without comment and '=' will be used for identifying $Lf^*.Lg^*$ and $L(g.f)^*$ (etc.) also without comment.)

I claim that this transformation is an isomorphism for all $F^\cdot \in D(PxX)$. It suffices to check this claim for $F^\cdot$ drawn from a set whose images generate $K(PxX)$ by the 5-lemma. By 3.4.1 such a set is made up of all sheaves of the form $\mathcal{O}_p(t)\boxtimes L$ with

$0 \geq t > -n$ and L locally free on X. (See 4.5.4 for $\bar{\pi}$.) For $0 > t > -n$ [EGA III 2...12] shows that both range and domain vanish. So it remains to show isomorphism for $F^\cdot = \pi^*L$. As this case is local on X and X_α it suffices to examine the case $L = \mathcal{O}_X$ and it certainly suffices to show that h^0 of the transformation is an isomorphism. But when $F^\cdot = \mathcal{O}_{PxX}$ h^0 of each of the morphisms used to define the transformation is seen to be an isomorphism.

It suffices to prove 3.5.3 in the case when η is the image of a covering homomorphism ϕ: $(pxx)^*F^\cdot \longrightarrow F^\cdot$ on PxX. By the definitions of 2.2 $j_\alpha^! \cdot \pi_! \eta$ is the image of the covering homomorphism $j_\alpha^! \pi_! \phi$ on X_α defined by:

$$\begin{array}{ccccc}
Lj_\alpha^* R\pi_* F^\cdot & = & Lj_\alpha^* x^* R\pi_* F^\cdot & \longrightarrow & Lj_\alpha^* x^* R\pi_* (pxx)_* (pxx)^* F^\cdot \\
\downarrow & & & & \| \\
Lj_\alpha^* R\pi_* F^\cdot & \xleftarrow{\phi} & Lj_\alpha^* R\pi_* (pxx)^* F^\cdot & \longleftarrow & Lj_\alpha^* x^* x_* R\pi_* (pxx)^* F^\cdot,
\end{array}$$

and $\pi_{\alpha!} \cdot j_\alpha'^! \eta$ is the image of the covering homomorphism $\pi_{\alpha!} j_\alpha'^! \phi$ on X_α defined by:

$$\begin{array}{ccccc}
R\pi_{\alpha*} Lj_\alpha'^* F^\cdot & \longrightarrow & R\pi_{\alpha*} (pxid)_* (pxid)^* Lj_\alpha'^* F^\cdot \\
\downarrow & & \| \\
R\pi_{\alpha*} Lj_\alpha'^* F^\cdot & \xleftarrow{\phi} & R\pi_{\alpha*} Lj_\alpha'^* (pxx)^* F^\cdot.
\end{array}$$

So it now suffices to prove that the following square:

$$\begin{array}{ccc}
Lj_\alpha^* R\pi_* F^\cdot & \xrightarrow{j_\alpha^! \pi_! \phi} & Lj_\alpha^* R\pi_* F^\cdot \\
\psi \downarrow s & & s \downarrow \psi \\
R\pi_{\alpha*} Lj_\alpha'^* F^\cdot & \xrightarrow{\pi_{\alpha!} j_\alpha'^! \phi} & R\pi_{\alpha*} Lj_\alpha'^* F^\cdot
\end{array} \qquad \text{commutes.}$$

This compatibility is broken down into 18 smaller compatibilities on the next page. These will not be treated explicitly, but use is made of the compatibility of 1.6.13 with composition of variety morphisms in many of them.

For typographical reasons set $\alpha = \pi_\alpha$, $i = j'_\alpha$, $j = j_\alpha$, $f = pxid$, $g = pxx$ and $F = F^\bullet$. Write π^* for $L\pi^*$ etc. and π_* for $R\pi_*$ etc. Then the square of 3.5.2 becomes:

$$\begin{array}{ccc} (PxX_\alpha, f) & \xrightarrow{i} & (PxX, g) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\pi} \\ (X_\alpha, id) & \xrightarrow{j} & (X,x). \end{array}$$

$$\begin{array}{ccccccccc}
j^*\pi_*F & = & j^*x^*\pi_*F & \longrightarrow & j^*x^*\pi_*g_*g^*F & = & j^*x^*x_*\pi_*g^*F & \longrightarrow & j^*\pi_*g^*F \longrightarrow j^*\pi_*F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
j^*\pi_*i_*i^*F & = & j^*x^*\pi_*i_*i^*F & \longrightarrow & j^*x^*\pi_*i_*f_*f^*i^*F & = & j^*x^*\pi_*g_*i_*i^*g^*F & = & j^*x^*x_*\pi_*i_*i^*g^*F \rightarrow j^*\pi_*i_*i^*g^*F \rightarrow \cdots\cdots \\
\| & & \| & & \| & & \| & & \| \\
j^*j_*\alpha_*i^*F & = & j^*x^*j_*\alpha_*i^*F & \rightarrow & j^*x^*j_*\alpha_*f_*f^*i^*F & = & j^*x^*j_*\alpha_*f_*i^*g^*F & = & j^*x^*x_*j_*\alpha_*i^*g^*F \rightarrow j^*j_*\alpha_*i^*g^*F \rightarrow \cdots\cdots \\
\downarrow & & \| & & \| & & \| & & \downarrow \\
 & & j^*j_*\alpha_*i^*F & \longrightarrow & j^*j_*\alpha_*f_*f^*i^*F & = & j^*j_*\alpha_*i^*g^*F & & \downarrow \\
 & & & & \downarrow & & \searrow & & \downarrow \\
\alpha_*i^*F & & \longrightarrow & & \alpha_*f_*f^*i^*F & = & & & \alpha_*i^*g^*F \longrightarrow \alpha_*i^*F.
\end{array}$$

The two omitted objects of the right hand column are $j^*\pi_*i_*i^*F$ and $j^*j_*\alpha_*i^*F$ and each morphism fror the sixth to the right hand column is that induced by ϕ. All other morphisms are, as stated at the start of this proof, either adjunctions or identifications. The proof of the commutativity of each part of this diagram is fairly easy and will be omitted; the perimeter is the required compatibility.

CHAPTER 4. DUALITY.

4.1. DUALISING.

Although a far more general treatment of this topic is available in [RD ch. V], this section gives a self-sufficient treatment of what is needed for this thesis.

Lemma 4.1.1. Let $F^\cdot$, $G^\cdot \in D(X)$, where X is a variety. Then there is a natural isomorphism:

$$\underline{\mathrm{RHom}}_X^\cdot(F^\cdot,G^\cdot) \xleftarrow{\sim} \underline{\mathrm{RHom}}_X^\cdot(F^\cdot,\mathcal{O}_X) \overset{L}{\otimes} G^\cdot.$$

Proof. Note that the sign conventions used in defining $\underline{\mathrm{Hom}}_X^\cdot$ and $\otimes$ correspond. The required morphism is the obvious composition. To show that it is an isomorphism, represent $F^\cdot$ and $G^\cdot$ by locally free bounded complexes also called $F^\cdot$ and $G^\cdot$. Then the composition induces an isomorphism of complexes: $\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,G^\cdot) \xleftarrow{\sim} \underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\mathcal{O}_X^\cdot) \otimes G^\cdot$ which yields the required isomorphism in D(X).

Definition 4.1.2. For $F^\cdot \in D(X)$ set $F^{\cdot\vee} = \underline{\mathrm{RHom}}_X^\cdot(F^\cdot,\mathcal{O}_X)$. Note that in the same way as in 4.1.1 one may find a natural isomorphism $F^\cdot \longrightarrow F^{\cdot\vee\vee}$. (This will be used without comment for invertible sheaves. For general $F^\cdot$ the $^\vee$ will be used as an abbreviation only.)

Lemma 4.1.3. Let $f: Y \longrightarrow X$ be a morphism of varieties. Then there is a canonical morphism, for $F^\cdot \in D(X)$:

$$\mathrm{tr}: \ \underline{\mathrm{RHom}}_X^\cdot(F^\cdot,F^\cdot) \longrightarrow \mathcal{O}_X,$$

such that the following square commutes:

$$\begin{array}{ccc} Lf^*\underline{\mathrm{RHom}}_X^\cdot(F^\cdot,F^\cdot) & \xrightarrow{Lf^*\mathrm{tr}} & Lf^*\mathcal{O}_X \\ {\scriptstyle 1.6.14}\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ \underline{\mathrm{RHom}}_Y^\cdot(Lf^*F^\cdot,Lf^*F^\cdot) & \xrightarrow{\mathrm{tr}} & \mathcal{O}_Y. \end{array}$$

(It is regretable that another 'trace' morphism, also called 'tr', is brought in in the next section. However no confusion should result. When X is a point this tr corresponds to taking the trace of an endomorphism of a k-complex.)

Proof. For a locally free sheaf, G, on X there is a well-known morphism tr: $\underline{\mathrm{Hom}}_X(G,G) \longrightarrow \mathcal{O}_X$ constructed thus: If $G = \mathcal{O}_X$, use the obvious isomorphism. If G is the direct sum of n copies of $\mathcal{O}_X$, say $G = \bigoplus_{t=1}^{n} G_t$, use the composite of $\underline{\mathrm{Hom}}_X(G,G) \approx \bigoplus_{s,t} \underline{\mathrm{Hom}}_X(G_s,G_t) \longrightarrow \bigoplus_t \underline{\mathrm{Hom}}_X(G_t,G_t) \to \mathcal{O}_X$, where the second morphism is the projection and the third is the direct sum of the trace for each G_t. If G is only locally free, then the above construction may be made locally and then is easily shown to fit together to define the required global morphism. Also, for locally free G, the compatibility with f* is trivial.

Now represent the F· of the lemma by a bounded locally free complex also called F·. So $R\underline{\mathrm{Hom}}_X^{\cdot}(F^{\cdot},F^{\cdot})$ is represented by the bounded locally free complex with n^{th} object $\bigoplus_{q-p=n} \underline{\mathrm{Hom}}_X(F^p,F^q)$ and with differentials induced from those of F· according to the sign convention of [RD p. 90]. In particular its 0^{th} object is $\bigoplus_q \underline{\mathrm{Hom}}_X(F^q,F^q)$. I claim that the morphism $\bigoplus_q (-)^q \mathrm{tr}_{F^q} : \bigoplus_q \underline{\mathrm{Hom}}_X(F^q,F^q) \to \mathcal{O}_X$ defines a morphism of locally free complexes which induces the required morphism in the derived category.

This claim is easily established by a consideration of complexes of free modules over affine algebras. The same local considerations show that tr is independent of the choice of locally free representative and the compatibility with Lf*. The details may safely be omitted.

4.2. REMARKS ON DUALITY.

The category under consideration in these three sections on the duality theorem will be that of irreducible varieties. The notational conventions which trivially generalise these results to reducible varieties will not be needed. The basic result that is needed is:

Quote 4.2.1. For every morphism $f: Y \longrightarrow X$ of irreducible varieties there is a functor:

$$f^!: \quad D(X) \longrightarrow D(Y),$$

and a natural transformation of functors, called the trace morphism:

$$Tr_f: \quad Rf_* f^! \longrightarrow id_{D(Y)},$$

such that, if $g: Z \longrightarrow Y$ is another morphism, $(fg)^! = g^! f^!$ and $Tr_{fg} = Tr_f . Tr_g$, and such that the composite θ_f:

$$\begin{array}{ccc} RHom^\cdot[Y;F^\cdot, f^! G^\cdot] & \longrightarrow & RHom^\cdot[X;Rf_*F^\cdot, Rf_* f^! G^\cdot] \\ & \searrow^{\theta_f} & \downarrow Tr_f \\ & & RHom^\cdot[X;Rf_*F^\cdot, G^\cdot] \end{array}$$

is an isomorphism for all $F^\cdot \in D(Y)$ and $G^\cdot \in D(X)$.

Remark 4.2.2.1. This is a specialisation of the main theorem of [RD ch. III]. To save a considerable amount of space complications arising from canonical isomorphisms not being equalities have been ignored. (e.g., strictly $(fg)^! \xrightarrow{\sim} g^! f^!$.)

Remark 4.2.2.2. Application of h^0 to θ_f shows that $f^!$ is the right adjoint of Rf_* and so it is unique if it exists.

Remark 4.2.2.3. If we assume that $(fg)^! = g^! f^!$ and $Tr_{fg} = Tr_f . Tr_g$, then, by [BS p. 104, Cor. 5], it is sufficient to investigate $f^!$ and Tr_f in the two special cases of f a closed embedding and $f: PxX \longrightarrow X$ the projection, with P a projective space. In view of the use I shall make of 4.2.1, I shall examine closed embeddings in detail in 4.4, consider $f: P \longrightarrow pt$ in 4.3, and leave the rest of the proof to [RD].

Remark 4.2.2.4. In fact $f^!$ is of the form:

$$f^!G^\cdot = Lf^*G^\cdot \otimes \omega_{Y/X}[\dim Y - \dim X],$$

whilst Tr_f is made up of:

$$Rf_*(Lf^*G^\cdot \otimes \omega_{Y/X})[\ ] \xrightarrow[\sim]{1.6.12} G^\cdot \otimes Rf_*\omega_{Y/X}[\ ]$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$G^\cdot$$

where $\omega_{Y/X}$ is an invertible (= locally free rank 1) sheaf on Y. Hence the first step is to construct $\omega_{Y/X}$. In [RD] this is done on pp. 137-144 using [EGA IV 16]. Hence I remark here that it suffices to do it for smooth f (which includes $P_X X \to X$) and closed embeddings (which are always locally complete intersections); that Hartshorne deals with each case explicitly and shows that there is an isomorphism (when $Z \xrightarrow{g} Y \xrightarrow{f} X$, and both f and g are either smooth or closed embeddings):

$$\zeta_{g,f}: \quad \omega_{Z/X} \xrightarrow{\sim} g^*\omega_{Y/X} \otimes \omega_{Z/Y},$$

satisfying [RD p. 143] the obvious compatibility property. It formally follows that for any f one can find an ω and that there exists a compatible system of ζ's.

Notation 4.2.2.5. For a variety X, 'ω_X' will always mean $\omega_{X/pt}$. As all varieties here are non-singular, $X \to pt$ is always smooth.

4.3. DUALITY FOR PROJECTIVE SPACE.

In this section I will sketch the proof of the duality theorem 4.2.1 for f: $P \longrightarrow pt$, where P is a projective space of dimension n. For any irreducible variety of dimension n, X, ω_X is defined to be $\bigwedge^n \Omega^1_X$; when X = P this invertible sheaf turns out to be $\mathcal{O}_P(-n-1)$ in the [FAC] notation. So, by [EGA III 2.1.12], $H^t[P;\omega_P] = 0$ for $t \neq n$ and $H^n[P;\omega_P] \approx k$ (canonically, see below for the basis element.) This shows:

$$Rf_*\omega_P[n] \approx k.$$

This isomorphism, coupled with the projection formula, enables Tr_f to be set up as in 4.2.2.4.

A quick proof of 4.2.1 for this f runs as follows: By an obvious use of the 5-lemma it suffices to prove that θ_f is an isomorphism for a set of $F^\cdot$'s whose images generate K(P). By 3.4.1 such a set is the $\mathcal{O}_P(t)$ for t = -n,...,-1,0. For t = -n, ...,-1 both the range and domain of θ_f are zero; for t = 0 we are back at the above isomorphism.

A fuller proof, valid for f: $P \times X \longrightarrow X$, is given in [RD ch. III, sn. 5]. Another is given in [8].

It remains to define the canonical basis element of the rank 1 k-vector space $H^n[P,\omega_P]$. Hartshorne does it by choosing a co-ordinate system for P and using the associated open affine cover and Cech cohomology to write down a cycle explicitly. In [RD p. 153] he claims that it is invariantly defined. In [8] Grothendieck uses a different device, which works for an arbitrary irreducible variety. He selects a point of P (i: $pt \longrightarrow P$) and considers $\underline{Ext}^t_P(i_*\mathcal{O}_{pt},\omega_P)$. By arguments about embeddings which are, in effect, repeated in the next section, these are zero except when t = n, and there is an isomorphism $H^0[P;i_*\mathcal{O}_{pt}] \xrightarrow{\sim} Ext^n[P;i_*\mathcal{O}_{pt},\omega_P]$. He shows that the image of the '1' which generates this first vector space under $Ext^n[P;i_*\mathcal{O}_{pt},\omega_P] \longrightarrow Ext^n[P;\mathcal{O}_P,\omega_P] = H^n[P;\omega_P]$ is independent of the point selected and is not zero.

It is not obvious that these two canonical elements coincide. Assuming Ha tshorne's proof of 4.2.1, one considers $pt \xrightarrow{i} P \xrightarrow{f} pt$ and applies the compatibility with composition of variety morphisms part of 4.2.1 to show that in fact they do.

4.4. DUALITY FOR CLOSED EMBEDDINGS.

This section deals with the special case of the duality theorem (4.2.1) when $f: Y \longrightarrow X$ is taken to be a closed embedding of codimension q. Hence f_* is exact (and so Rf_* may be written f_*) but f^* need not be exact. The arguments of this section are essentially those of [8], but this is a special case of that treated in [RD ch. III, sn. 6-7].

Definition 4.4.1.1. (This is from [RD p. 141].) Define $\omega = \omega_{Y/X} = \Lambda^q(\mathcal{I}(Y)/\mathcal{I}(Y)^2)^\vee$, where $\mathcal{I}(Y)$ is the sheaf of ideals of Y and $\mathcal{I}(Y)/\mathcal{I}(Y)^2$ is shown by the local co-ordinates of 2.6 to be locally free of rank q on Y.

Definition 4.4.1.2. Define, for $G^\cdot \in D(X)$, $f^!G^\cdot = Lf^*G^\cdot \otimes \omega[-q]$.

Construction 4.4.2. By an easy local calculation with a Koszul complex, given in [8 p. 5-9], one proves that $\underline{Ext}^t_X(f_*\omega, \mathcal{O}_X) = 0$ for $t \neq q$ and finds a canonical isomorphism $\underline{Ext}^q_X(f_*\omega, \mathcal{O}_X) \xrightarrow{\sim} f_*\mathcal{O}_Y$. Hence, using the edge homomorphism of the collapsing composite functor spectral sequence of [8 p. 2, Prop. 1], one shows: $H^0[Y, \mathcal{O}_Y] \xrightarrow{\sim} H^0[X; f_*\mathcal{O}_Y] \xrightarrow{\sim} Ext^q[X; f_*\omega, \mathcal{O}_X] = Hom[X; f_*\omega[-q], \mathcal{O}_X]$. Hence a canonical morphism:

$$Tr_f: \quad f_*\omega_{Y/X}[-q] \longrightarrow \mathcal{O}_X$$

has been constructed. Tr_f is now constructed for a general $G^\cdot \in D(X)$ by means of the projection formula as in 4.2.2.4.

It is now necessary to prove that the θ_f of 4.2.1 is an isomorphism for all $F^\cdot \in D(Y)$ and all $G^\cdot \in D(X)$. In the usual way the task can be reduced to proving isomorphism when $F^\cdot = F$ is a locally free sheaf on Y and $G^\cdot = G$ is a locally free sheaf on X. Then:

$$\begin{array}{ccc} RHom^\cdot[Y; F, f^*G \otimes \omega[-q]] & \longrightarrow & RHom^\cdot[X; f_*F, f_*(f^*G \otimes \omega[-q])] \\ & \searrow^{\theta_f} & \downarrow Tr_f \\ & & RHom^\cdot[X; f_*F, G] \end{array}$$

is seen to have θ_f an isomorphism because the corresponding diagram of local Hom's gives an isomorphism. See [RD p. 150]

for a complete proof.

It is convenient to assemble here three facts about the duality isomorphism for closed embeddings. The first deals with a special embedding:

Lemma 4.4.3. Let $x: X \dashrightarrow X$ be an endomorphism of the irreducible variety X. Let $\gamma = (x,id): X \to XxX = W$ be the graph morphism. Then γ is a closed embedding of codimension $n = \dim X$, and $\omega_{X/W} \xrightarrow{\sim} (x^*\omega_X)^\vee$.

Proof. The 'closed graph' condition is part of the definition of a variety. The isomorphism is proved in [RD p. 141] as Lemma 1.4; the local co-ordinates of 2.6 may be used to give another proof.

Note 4.4.4. This second fact is based on the well-known theory of adjoint functors of, for example, [4 p.338]. The adjointness situation:

$$h^0\theta_f: \quad \mathrm{Hom}[Y;F^\bullet, f^!G^\bullet] \xrightarrow{\sim} \mathrm{Hom}[X;f_*F^\bullet, G^\bullet],$$
$$\text{for } F^\bullet \in D(Y),\ G^\bullet \in D(X),$$

(f is now an arbitrary embedding again) may be set up, as I have done, by constructing a suitable natural transformation: $Tr_f: \ f_*f^! \longrightarrow id_{D(X)}$ or by a suitable natural transformation: $cotr_f: \ id_{D(Y)} \longrightarrow f^!f_*$. Of course any one of $h^0\theta_f$, Tr_f and $cotr_f$ determines the other two. It is required to exhibit $cotr_f: \ F^\bullet \longrightarrow f^!f_*F^\bullet$ when $F^\bullet = \mathcal{O}_Y$. (Fortunately this is the only case that is needed.)

Choose and fix a finite locally free left resolution, $M^\bullet$, of $f_*\mathcal{O}_Y$ over X. Then, appealing to local calculations like those of 4.4.2 and 2.7, it is known that the locally free complex on Y $f^*M^\bullet \otimes \omega[-q]$ has cohomology in positions $0,1,\ldots,q$ only and in particular $h^0(f^*M^\bullet \otimes \omega[-q]) \approx \mathcal{O}_Y$. By definition this complex represents $f^!f_*\mathcal{O}_Y$ in D(Y). There is a 'truncation' morphism of complexes:

$$\begin{array}{ccccccccccc}
\cdots & \to & G^{-1} & \longrightarrow & \mathrm{Ker}\beta & \to & 0 & \longrightarrow & 0 & \to & \cdots \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & G^{-1} & \longrightarrow & G^{0} & \xrightarrow{\beta} & G^{1} & \longrightarrow & G^{2} & \to & \cdots ,
\end{array}$$

where $G^{\cdot} = f^{*}M^{\cdot} \otimes \omega[-q]$, and so a morphism $\alpha\colon \mathcal{O}_Y \to f^{!}f_{*}\mathcal{O}_Y$ has been exhibited by this 'truncation' morphism. I claim that this α is the cotr defined above.

By elementary adjointness theory, it is required to prove that the image of α under the composite:

$$\begin{array}{ccc}
\mathrm{Hom}[Y;\mathcal{O}_Y, f^{!}f_{*}\mathcal{O}_Y] & \xrightarrow{\quad f_{*} \quad} & \mathrm{Hom}[X;f_{*}\mathcal{O}_Y, f_{*}f^{!}f_{*}\mathcal{O}_Y] \\
 & & \downarrow \text{s 1.6.12} \\
\mathrm{Hom}[X;f_{*}\mathcal{O}_Y, f_{*}\mathcal{O}_Y] & \xleftarrow{\quad \mathrm{Tr} \quad} & \mathrm{Hom}[X;f_{*}\mathcal{O}_Y, f_{*}\mathcal{O}_Y \overset{\cdot}{\otimes} f_{*}\omega[-q]]
\end{array}$$

is the identity. ($\mathrm{Tr}_f\colon f_{*}f^{!}f_{*}\mathcal{O}_Y \to f_{*}\mathcal{O}_Y$ is given as the composite of the two morphisms used to define it.) This is the same thing as being required to prove that the composite:

$f_{*}\mathcal{O}_Y \xrightarrow{f_{*}\alpha} f_{*}f^{!}f_{*}\mathcal{O}_Y \xrightarrow{\sim} f_{*}\mathcal{O}_Y \overset{\cdot}{\otimes} f_{*}\omega[-q] \xrightarrow{\mathrm{Tr}} f_{*}\mathcal{O}_Y$ is the identity.

This statement is purely local. It is proved by taking, for each point $a \in Y \subset X$, a set of local co-ordinates for Y in a suitable open affine neighbourhood U of a. These give free resolutions for $f_{*}\mathcal{O}_Y|U$ and $f_{*}\omega|U$. The double complex obtained by tensoring these two resolutions represents $f_{*}\mathcal{O}_Y \overset{\cdot}{\otimes} f_{*}\omega|U$; the result follows from the usual double complex tecnique.

Note 4.4.5. (The following purely formal result will be needed in 5.2, where $n = \dim X$ and $n-q = \dim Y$.) The k-dual of $\mathrm{Ext}^r[X;G^\cdot,\omega_X] \xrightarrow{i^!} \mathrm{Ext}^r[Y;Li^*G^\cdot \otimes \omega_{Y/X},\omega_Y]$ is the composite of $H^{n-q-r}[Y;Li^*G^\cdot \otimes \omega_{Y/X}] = H^{n-q-r}[X;i_*(Li^*G^\cdot \otimes \omega_{Y/X})] \xrightarrow{\mathrm{Tr}} H^{n-r}[X;G^\cdot]$, where $G^\cdot \in D(X)$ and r is an integer.

Proof. As $G^\cdot$ may be shifted to either the right or the left, there is no loss of generality in assuming that $r = 0$. By the naturality of $\mathrm{Tr} = \mathrm{Tr}_i$, for $\psi\colon G^\cdot \to \omega_X$ the following square commutes:

$$\begin{array}{ccc} i_*i^!G^\cdot & \xrightarrow{i_*i^!\psi} & i_*i^!\omega_X \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ G^\cdot & \xrightarrow{\psi} & \omega_X. \end{array}$$

Now consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Hom}[X;G^\cdot,\omega_X] & & \xrightarrow{\quad i^! \quad} & & \mathrm{Hom}[Y;i^!G^\cdot,i^!\omega_X] \\ \downarrow\wr & \overset{\mathrm{Tr}}{\searrow} & & & i_*\downarrow \\ & \mathrm{Hom}[X;i_*i^!G^\cdot,\omega_X] & \xleftarrow{\mathrm{Tr}} & \mathrm{Hom}[X;i_*i^!G^\cdot,i_*i^!\omega_X] & \downarrow\wr \\ \mathrm{Hom}[pt;R\Gamma_X G^\cdot[n],k] & \downarrow\wr & & & \\ & \overset{\mathrm{Tr}}{\searrow} & & & \\ & \mathrm{Hom}[pt;R\Gamma_X i_*i^!G^\cdot[n],k] & = & \mathrm{Hom}[pt;R\Gamma_Y i^!G^\cdot[n],k]. & \end{array}$$

The three vertical morphisms not yet defined are the duality isomorphisms for $X \to pt$, $X \to pt$ and $Y \to pt$ respectively; the horizontal identification comes from $R\Gamma_X \cdot i_* = R\Gamma_Y$.

The left hand square commutes by the naturality of duality; the other square by the remark given immediately above the big diagram; and the remaining pentagon commutes in view of the compatibility of duality with composition of variety morphisms.

The perimeter of the diagram now translates into the required fact.

4.5. THE KUNNETH FORMULA.

This section is required mostly because the references do not use derived categories. Thus the theory outlined here is standard up to notation. It is drawn from [FAC], [RD p. 148-150], [5 p. 203-259] and especially [14], [EGA III 1.2] and [EGA III 6].

Recall that all complexes used here have quasi-coherent sheaves as objects, are bounded (this is why only finite open affine coverings are used below) and have degree +1 (unlike those of [EGA III 6]). Also remember that separation conditions are part of the definition of a variety.

The basic facts about Cech hypercohomology that are needed are as follows:

Construction 4.5.1.1. Let X be an irreducible variety. Let $\underline{U} = \{U_i\}_{i\in I}$ be a finite open cover of X, where the indexing set I is taken to be the first N positive integers. Let $F^\cdot$ be a bounded complex of quasi-coherent sheaves on X. Following [RD p. 148] a double complex $C^{\cdot\cdot}(\underline{U},F^\cdot)$ is constructed thus: For each (p+1)-tuple of indices $\underline{i} = i_0<i_1< \ldots <i_p$ set $U_{\underline{i}} = \bigcap_{i\in\underline{i}} U_i$. The sheaf $C^{p,q}(\underline{U},F^\cdot)$ is defined by setting, for each open $V\subset X$, $C^{p,q}(\underline{U},F^\cdot)(V) = \bigoplus_{\underline{i}} F^q(V\cap U_{\underline{i}})$. According to [RD p. 148] this indeed defines a sheaf and according to [EGA I 9.4.2] it is quasi-coherent. A 'differential' $d_1^{p,q}: C^{p,q}(\underline{U},F^\cdot) \longrightarrow C^{p+1,q}(\underline{U},F^\cdot)$ may be constructed in the usual way; see [RD p. 149] for the sign convention adopted here. A 'differential' $d_2^{p,q}: C^{p,q}(\underline{U},F^\cdot)\to C^{p,q+1}(\underline{U},F^\cdot)$ is defined to be that induced by $(-)^p d_F^q.: F^q\longrightarrow F^{q+1}$. This gives a double complex in the sense of [M p. 60] and the construction is natural in $F^\cdot$. Denote by $C^\cdot(\underline{U},F^\cdot)$ the associated single complex. There are natural augmentations:

$$\epsilon: F^\cdot \longrightarrow C^{0\cdot}(\underline{U},F^\cdot) \longrightarrow C^\cdot(\underline{U},F^\cdot),$$

and by [5 p. 206] or by a trivial direct calculation ϵ is known to be a quasi-isomorphism $F^\cdot \to C^\cdot(\underline{U},F^\cdot)$.

Construction 4.5.1.2. If two (or more) finite open covers of the variety X, say $\underline{U}$ and $\underline{V}$, the above construction may be modified to give a triple (or multiple) complex $C^{\cdot\cdot\cdot}(\underline{U},\underline{V},F^{\cdot})$ which is again functorial in the bounded quasi-coherent complex $F^{\cdot}$. As this construction is well known, and set out in detail in [14 sn. 4], I will not give it explicitly here. Denote by $C^{\cdot}(\underline{U},\underline{V},F^{\cdot})$ the associated single complex.

If $\underline{U}$, $\underline{V}$ are two finite open covers of the irreducible variety X, a third finite open cover $\underline{W}$ may be constructed by $\underline{W} = \{U_i \cap V_j\}$ suitably ordered. Then in [14 sn. 4] it is shown that:

Lemma 4.5.1.3. There are quasi-isomorphisms:

$$F^{\cdot} \xrightarrow{\epsilon} C^{\cdot}(\underline{W},F^{\cdot}) \xrightarrow{i_*} C^{\cdot}(\underline{U},\underline{V},\underline{W},F^{\cdot}) \xleftarrow{i_*} C^{\cdot}(\underline{U},\underline{V},F^{\cdot})$$

which are natural in the bounded quasi-coherent complex $F^{\cdot}$.

Lemma 4.5.1.4. If $\underline{U}$ is a finite open cover of the variety X, and if $\psi\colon F^{\cdot} \longrightarrow G^{\cdot}$ is a quasi-isomorphism of bounded quasi-coherent complexes, the induced morphism $C^{\cdot}(\underline{U},F^{\cdot}) \longrightarrow C^{\cdot}(\underline{U},G^{\cdot})$ is a quasi-isomorphism. Similarly for two or more finite open covers.

Proof. This follows immediately from the spectral sequence for the double complex $C^{\cdot\cdot}(\underline{U},-)$.

4.5.1.4 will be used later to show that the Kunneth formula makes sense in a derived category notation. However we need first the following result, taken from [RD p. 149] (where ξ is defined). It is, in effect, [EGA III 6.2.3].

Lemma 4.5.2. If $\underline{U}$ is a finite open affine cover of the irreducible variety X and $F^{\cdot}$ is a bounded quasi-coherent complex of sheaves on X with ~~quasi~~ coherent cohomology, then:

$$\Gamma_X(C^{\cdot}(\underline{U},F^{\cdot})) \xrightarrow{\xi} R\Gamma_X(C^{\cdot}(\underline{U},F^{\cdot})) \xleftarrow{R\Gamma_X\epsilon} R\Gamma_X F^{\cdot}$$

are quasi-isomorphisms. Similar results hold for pairs (etc.) of open covers $\underline{U}$, $\underline{V}$ such that each $U_i \cap V_j$ is affine.

Remark 4.5.3. If $\underline{U}$ is a finite open affine cover, then if $F^\cdot$ is a bounded quasi-coherent sheaf with coherent cohomology, the spectral sequence of the double complex $\Gamma_X C^{\cdot\cdot}(\underline{U},F^\cdot)$ is $E_2^{p,q} = H^p[X;h^qF^\cdot] \Longrightarrow H^*[X;F^\cdot]$. This spectral sequence may of course be obtained in other ways.

Remark 4.5.4. The construction 4.5.1.2 may easily be repeated for bounded quasi-coherent double complexes $F^{\cdot\cdot}$. This will not be done explicitly here.

Now let X and Y be irreducible varieties with finite open affine coverings $\underline{U}$ and $\underline{V}$ respectively. Let $F^\cdot$ and $G^\cdot$ be bounded quasi-coherent complexes on X and Y respectively with coherent cohomology. By replacing them by quasi-isomorphic locally free complexes if necessary, we may assume $F^\cdot$ and $G^\cdot$ are locally free. Let $\pi_X\colon XxY \to X$ and $\pi_Y\colon XxY \to Y$ be the projections.

A functor $\overset{L}{\boxtimes}$ is now defined: $\overset{L}{\boxtimes}\colon D(X)xD(Y) \to D(XxY)$. For bounded locally free complexes $F^\cdot$, $G^\cdot$ on X, Y let $K^{\cdot\cdot}$ be the double complex on XxY given by $K^{p,q} = \pi_X{}^*F^\cdot \otimes \pi_Y{}^*G^\cdot$, with differentials induced from those of $F^\cdot$ and $G^\cdot$ subject to the usual sign conventions. Define $F^\cdot \overset{L}{\otimes} G^\cdot \in D(XxY)$ to be the image of the associated single complex. As this depends only on the quasi-isomorphism class of $F^\cdot$ and $G^\cdot$, this extends to define $\overset{L}{\boxtimes}\colon D(X)xD(Y) \to D(XxY)$.

In [14] it is in effect shown that there is, with the above notation, an isomorphism of quadruple complexes which interchanges the second and third indices:

$$\Gamma_{XxY} C^{\cdot\cdot\cdot\cdot}(\underline{U}x\underline{V}, Xx\underline{V}, K^{\cdot\cdot}) \xrightarrow{\sim} \Gamma_X C^{\cdot\cdot}(\underline{U},F^\cdot) \otimes \Gamma_Y C^{\cdot\cdot}(\underline{V},G^\cdot)$$

from which the following may be deduced:

Theorem 4.5.5. (The Kunneth formula.) With the above notation, for $F^\cdot \in D(X)$ and $G^\cdot \in D(Y)$, there are natural isomorphisms:

$$R\Gamma_{XxY}(F^\cdot \overset{L}{\boxtimes} G^\cdot) \xrightarrow{\sim} R\Gamma_X F^\cdot \otimes R\Gamma_Y G^\cdot.$$

$$H^n[XxY;F^\cdot \overset{L}{\otimes} G^\cdot] \xrightarrow{\sim} \bigoplus_t H^t[X;F^\cdot] \otimes H^{n-t}[Y;G^\cdot].$$

Unfortunately rather more than this result is needed for the details of the proof of parts of the next two sections. The following remarks will be even more sketchy than the above ones.

Proposition 4.5.6. Let $f: Y \to X$ and $f': Y' \to X'$ be morphisms of irreducible varieties. Let $F^{\cdot} \in D(Y)$ and $F'^{\cdot} \in D(Y')$. Then there is a canonical natural isomorphism:

$$R(fxf')_*(F^{\cdot} \overset{L}{\boxtimes} F'^{\cdot}) \xrightarrow{\sim} Rf_*F^{\cdot} \overset{L}{\otimes} Rf'_*F'^{\cdot}.$$

Reference. Note that if $X = X' = pt$ this reduces to 4.5.5. The proof of a more general result (involving fibre products) is to be found in spectral sequence form in [EGA III 6]. It would not be worth while to write down here the way in which open covers of all four varieties have to be chosen, but the reader is assurred that the proof follows the same lines as the one sketched for 4.5.5.

4.6. THE VERDIER DIAGRAM - I.

Diagram 4.6.1. In this diagram:

X is an irreducible variety of dimension n.

$V = X \times X$; $\delta = (id,id): X \to V$.

$\pi_1, \pi_2: V \to X$ are the projections.

$F^\cdot, G^\cdot \in D(X)$.

ω_X is as in 4.2.2.5.

The diagram is in $D(pt)$.

$$
\begin{array}{ccc}
R\Gamma_X R\underline{Hom}_X^\cdot(F^\cdot,G^\cdot) & \xrightarrow{\quad (5) \quad} & R\Gamma_X(R\underline{Hom}_X^\cdot(F^\cdot,\mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \\
(1)\downarrow & & \wr\downarrow (6) \\
Hom_{pt}^\cdot(R\Gamma_X F^\cdot, R\Gamma_X G^\cdot) & & R\Gamma_V R\underline{Hom}_V^\cdot(\delta_*\mathcal{O}_X, R\underline{Hom}_X^\cdot(F^\cdot,\omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
(2)\downarrow \wr & & \downarrow (7) \\
Hom_{pt}^\cdot(R\Gamma_X F^\cdot,k) \otimes R\Gamma_X G^\cdot & & R\Gamma_V R\underline{Hom}_V^\cdot(\mathcal{O}_V, R\underline{Hom}_X^\cdot(F^\cdot,\omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
(3)\downarrow \wr & & \wr\downarrow (8) \\
R\Gamma_X R\underline{Hom}_X^\cdot(F^\cdot,\omega_X) \otimes R\Gamma_X G^\cdot[n] & \xrightarrow[\sim]{(4)} & R\Gamma_V(R\underline{Hom}_X^\cdot(F^\cdot,\omega_X) \overset{L}{\boxtimes} G^\cdot)[n].
\end{array}
$$

The numbered morphisms are defined as follows:

1 is functoriality for $R\Gamma_X$.

2 is dualising over a point, see 4.1.1.

3 is the duality isomorphism for $c: X \to pt$.

4 is the Kunneth formula of 4.5.

5 is dualising over X, see 4.1.1.

6 is the duality isomorphism for the closed embedding δ, modulo the identification: $R\Gamma_X(R\underline{Hom}_X^\cdot(F^\cdot,\mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \xrightarrow{\sim} R\Gamma_V R\delta_* R\underline{Hom}_X^\cdot(\mathcal{O}_X, L\delta^*(R\underline{Hom}_X^\cdot(F^\cdot,\omega_X) \overset{L}{\boxtimes} G^\cdot) \otimes \omega_X^\nu)$. See 4.4.3.

7 is induced by cofunctoriality from the canonical $\mathcal{O}_V \to \delta_*\mathcal{O}_X$.

8 is obtained by dropping out the trivial factor $R\underline{Hom}_V^\cdot(\mathcal{O}_V, \cdot$

Lemma 4.6.2. If $A^\cdot, B^\cdot \in D(pt)$ and $\psi \in Hom_{pt}(A^\cdot, B^\cdot)$, then $\psi = 0$ if and only if each $h^t\psi = 0$. If 4.6.1 commutes for a fixed $F^\cdot$ and for any two of $G_1^\cdot$, $G_2^\cdot$ and $G_3^\cdot$, where $G_1^\cdot - G_2^\cdot \to G_3^\cdot \to TG_1^\cdot$ is a triangle in $D(X)$ it commutes for the thir[illegible] If 4.6.1

commutes for a fixed $G^\bullet$ and for any two of $F_1^\bullet$, $F_2^\bullet$ and $F_3^\bullet$ where $F_1^\bullet \longrightarrow F_2^\bullet \longrightarrow F_3^\bullet \longrightarrow TF_1^\bullet$ is a triangle in $D(X)$ then it commutes for the third.

Proof. The first sentence is trivial linear algebra. To prove the second sentence, first note that each of the numbered morphisms commutes with those of the triangle. Next consider the difference of the two composites from the upper left object to the lower right object in 4.6.1 for $G^\bullet = G_1^\bullet$, $G_2^\bullet$ and $G_3^\bullet$. The first sentence enables the result to be read off from the induced morphism of the long exact cohomology sequence. The third sentence is proved in the same way.

Lemma 4.6.3. Diagram 4.6.2 commutes when $X = P$, the n-dimensional projective space.

Proof. In view of 4.6.2 and the formula for $K(P)$ given in 3.4.1, it suffices to prove this for $F^\bullet$ and $G^\bullet$ drawn from the set $\mathcal{O}_P(t)$, $0 \geqslant t \geqslant -n$. However by [EGA III 2.1.12] for $0 > t \geqslant -n$ $R\Gamma_P \mathcal{O}_P(t) = 0$ and so commutativity is automatic in this case.

So it remains to prove commutativity when $F^\bullet = G^\bullet = \mathcal{O}_P$. This is done by considering the endomorphism $x: P \longrightarrow pt \longrightarrow P$ of P. In this special case (only) the arguments of the next section may be reversed to show that it is sufficient to prove that 4.7.1 commutes when $X = P$, and for this x, $F^\bullet$ and $G^\bullet$. But this commutativity is simply the statement that the two canonical elements (see the end of 4.4) of $H^n[P, \omega_P]$ coincide, which is known to be the case.

Proposition 4.6.4. Diagram 4.6.1 commutes.

Sketch Proof. By 4.6.2 it suffices to prove commutativity for locally free sheaves $F = F^\bullet$ and $G = G^\bullet$. Choose an embedding $i: X \hookrightarrow P$, where P is a projective space of dimension n+q. The result is known for the sheaves i_*F and i_*G on P. The result for the sheaves F and G on X may be deduced by drawing a suitable cylindrical diagram analogous to that of 4.7. As this task is long but mechanical it will be omitted.

4.7. THE VERDIER DIAGRAM - II.

Diagram 4.7.1. In this diagram:

X is an irreducible variety of dimension n.

$x: X \longrightarrow X$ is an endomorphism of X.

$V = X \times X$; $\delta = (id,id)$, $\gamma = (x,id): X \longrightarrow V$.

$\pi_1, \pi_2: V \longrightarrow X$ are the projections.

$F^\cdot, G^\cdot \in D(X)$.

ω_X is as in 4.2.2.5.

The diagram is in D(pt).

$$
\begin{array}{ccc}
R\Gamma_X R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, G^\cdot) & \xrightarrow[\sim]{\textcircled{5}} & R\Gamma_X(R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \\
\textcircled{1}\downarrow & & \downarrow\textcircled{6} \\
\mathrm{Hom}^\cdot_{pt}(R\Gamma_X F^\cdot, R\Gamma_X G^\cdot) & & R\Gamma_V R\underline{\mathrm{Hom}}^\cdot_V(\gamma_* \mathcal{O}_X, R\underline{\mathrm{Hom}}^\cdot_X(F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
\textcircled{2}\downarrow & & \downarrow\textcircled{7} \\
\mathrm{Hom}^\cdot_{pt}(R\Gamma_X F^\cdot, k) \otimes R\Gamma_X G^\cdot & & R\Gamma_V R\underline{\mathrm{Hom}}^\cdot_V(\mathcal{O}_V, R\underline{\mathrm{Hom}}^\cdot_X(F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
\textcircled{3}\downarrow & & \downarrow\textcircled{8} \\
R\Gamma_X R\underline{\mathrm{Hom}}^\cdot_X(F^\cdot, \omega_X) \otimes R\Gamma_X G^\cdot[n] & \xrightarrow[\sim]{\textcircled{4}} & R\Gamma_V(R\underline{\mathrm{Hom}}^\cdot_X(F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n].
\end{array}
$$

The numbered morphisms are defined as follows:

1 is the composite of $R\Gamma_X R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, G^\cdot) \to \mathrm{Hom}^\cdot_{pt}(R\Gamma_X Lx^*F^\cdot, R\Gamma_X G^\cdot)$ $\to \mathrm{Hom}^\cdot_{pt}(R\Gamma_X Rx_* Lx^*F^\cdot, R\Gamma_X G^\cdot) \to \mathrm{Hom}^\cdot_{pt}(R\Gamma_X F^\cdot, R\Gamma_X G^\cdot)$ where the first of these morphisms is functoriality for $R\Gamma_X$, the second comes from $R\Gamma_X = R\Gamma_X . Rx_*$ and the third comes from the adjointness of 1.6.13.

2 is dualising over a point, see 4.1.1.

3 is the duality isomorphism for $c: X \to pt$.

4 is the Kunneth formula of 4.5.

5 is dualising over X, see 4.1.1.

6 is the duality isomorphism for the closed embedding γ, modulo the identification: $R\Gamma_X(R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \xrightarrow{\sim}$ $R\Gamma_V R\delta_* R\underline{\mathrm{Hom}}^\cdot_X(\mathcal{O}_X, L\gamma^*(R\underline{\mathrm{Hom}}^\cdot_X(F^\cdot, \omega_X) \boxtimes G^\cdot) \otimes x^*\omega_X^\vee)$. See 4.4.3.

7 is induced by cofunctoriality from the canonical $\mathcal{O}_V \to \gamma_* \mathcal{O}_X$.

8 is obtained by dropping out the trivial factor $R\underline{\mathrm{Hom}}^\cdot_V(\mathcal{O}_V, .$

Proposition 4.7.2. Diagram 4.7.1 commutes.

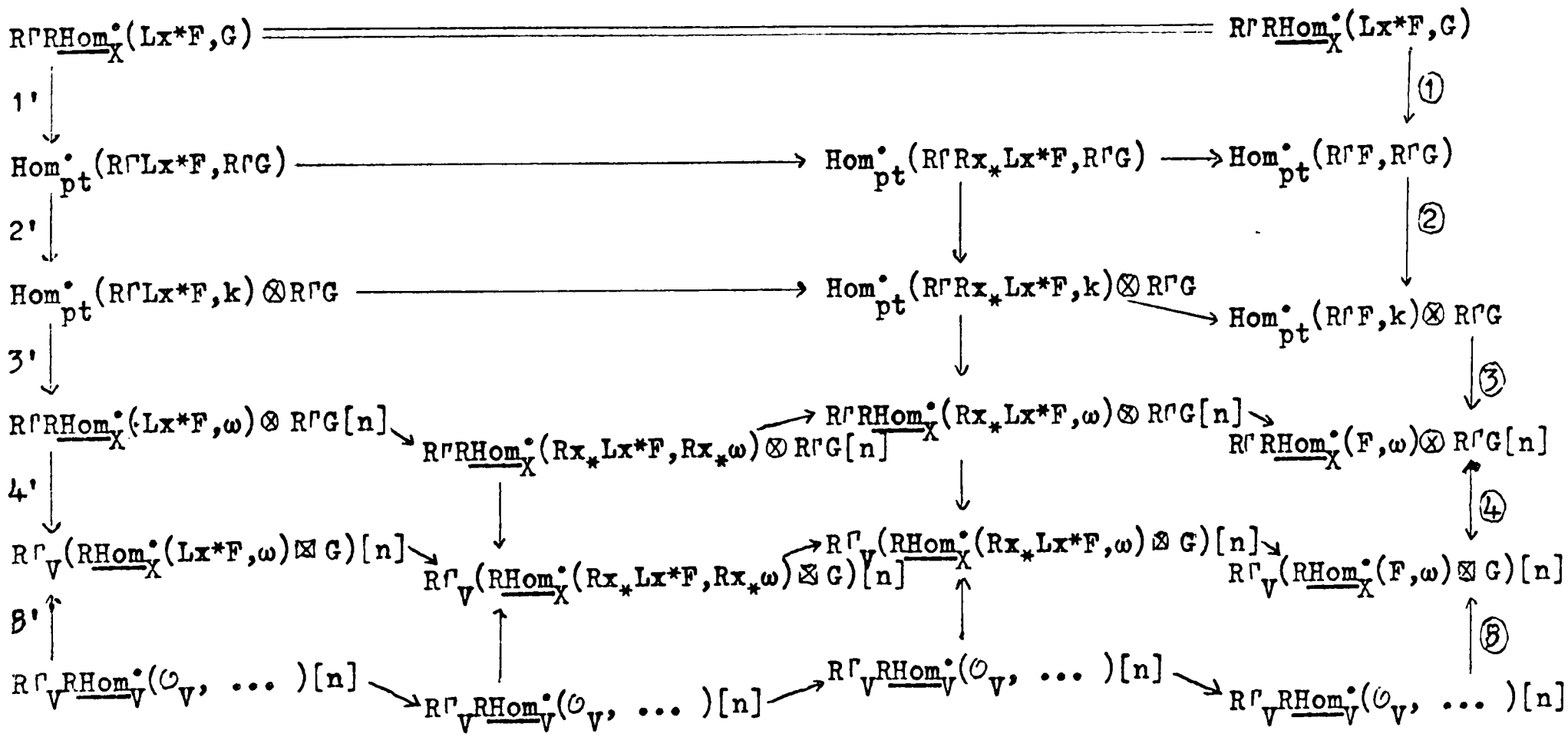
Proof. It is convenient to write $v = x \times id: V \longrightarrow V$.

The proof is a reduction to the case of Diagram 4.6.1. It requires the proof of the commutativity of many squares, most of which are trivial. These are arranged in two large diagrams.

I redraw Diagram 4.6.1, whose commutativity is now known, below. $F^\cdot$ has been replaced by $Lx^*F^\cdot$ throughout and the naming of the morphisms has had to be amended to avoid confusion.

$$
\begin{array}{ccc}
R\Gamma_X R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, G^\cdot) & \xrightarrow{5'}_{\sim} & R\Gamma_X(R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \\
\downarrow 1' & & \downarrow 6' \\
\mathrm{Hom}^\cdot_{pt}(R\Gamma_X Lx^*F^\cdot, R\Gamma_X G^\cdot) & & R\Gamma_V R\underline{\mathrm{Hom}}^\cdot_V(\delta_*\mathcal{O}_X, R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
\downarrow 2' & & \downarrow 7' \\
\mathrm{Hom}^\cdot_{pt}(R\Gamma_X Lx^*F^\cdot, k) \otimes R\Gamma_X G^\cdot & & R\Gamma_V R\underline{\mathrm{Hom}}^\cdot_V(\mathcal{O}_V, R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n] \\
\downarrow 3' & & \downarrow 8' \\
R\Gamma_X R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \omega_X) \otimes R\Gamma_X G^\cdot[n] & \xrightarrow{4'}_{\sim} & R\Gamma_V(R\underline{\mathrm{Hom}}^\cdot_X(Lx^*F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n].
\end{array}
$$

Diagram 4.7.3. Here I write F for $F^\bullet$, G for $G^\bullet$, ω for ω_X and Γ for Γ_X.



The omitted second variables of the $R\underline{Hom}^\bullet_V$'s of the last row are the terms contained in the outer brackets of the objects immediately above. The morphisms of this diagram are defined on the next page.

The numbered morphisms have already been defined. The left morphism of the second row comes from $R\Gamma_X = R\Gamma_X Rx_*$; each of the morphisms from the third to the fourth column is induced by the adjointness $F^\bullet \longrightarrow Rx_* Lx^* F^\bullet$. So the uppermost pentagon commutes.

The left morphism of the third row comes from $R\Gamma_X = R\Gamma_X Rx_*$; the uppermost vertical morphism of the third column is dualising over a point; and so the two squares of the next layer commute.

The left morphism of the fourth row is induced by Rx_*; each morphism from the second to the third column is induced by Tr_x: $Rx_*\omega_X \longrightarrow \omega_X$ (note $\omega_X = x^!\omega_X$). The second morphism of the third column is the duality isomorphism for c: $X \longrightarrow pt$. The left hand pentagon of this layer commutes as $c = cx$ in view of the compatibility with composition of variety morphisms part of the duality theorem of 4.2. The right hand square commutes by naturality.

The left hand morphism of the fifth row is induced by $R\Gamma_V = R\Gamma_V . Rv_*$ and uses the strong form of the Kunneth formula of 4.5.6. The upper morphism of the second column is the Kunneth formula as is the third one of the third column. The left hand square of this layer commutes by the construction of the Kunneth formula. The central and right hand ones commute by naturality.

Each of the vertical morphisms from the sixth to th fifth row is induced by $R\underline{Hom}^\bullet_V(\mathcal{O}_V$, being dropped out; the left morphism of the lowest row is induced by $R\Gamma_V = R\Gamma_V . Rv_*$ and $\mathcal{O}_V \longrightarrow Rv_* \mathcal{O}_V$. The three squares of this layer trivially commute.

Diagram 4.7.4. Again I write F for $F^\cdot$, G for $G^\cdot$ and ω for ω_X. The fourth row is the same as the sixth row of Diagram 4.7.3.

$$
\begin{array}{ccccc}
R\Gamma_X R\underline{Hom}^\cdot_X(Lx^*F^\cdot, G^\cdot) & = & & = & R_X R\underline{Hom}^\cdot_X(Lx^*F^\cdot, G^\cdot) \\
\downarrow 5' & & & & \downarrow 5 \\
R\Gamma_X(R\underline{Hom}^\cdot_X(Lx^*F^\cdot, \mathcal{O}_X) \overset{L}{\otimes} G^\cdot) & = & & = & R\Gamma_X R\underline{Hom}^\cdot_X(Lx^*F^\cdot, \mathcal{O}_X) \overset{L}{\otimes} G^\cdot) \\
\downarrow 6' & & & & \downarrow 6 \\
R\Gamma_V R\underline{Hom}^\cdot_V(\delta_*\mathcal{O}_X, R\underline{Hom}^\cdot_X(Lx^*F^\cdot, \omega_X) \overset{L}{\boxtimes} G^\cdot)[n] & & R\Gamma_V R\underline{Hom}^\cdot_V(\gamma_*\mathcal{O}_X, R\underline{Hom}^\cdot_X(Rx_*Lx^*F^\cdot, \omega) \overset{L}{\boxtimes} G)[n] & & R\Gamma_V R\underline{Hom}^\cdot_V(\gamma_*\mathcal{O}_X, R\underline{Hom}^\cdot_X(F, \omega) \overset{L}{\boxtimes} G)[n] \\
 & \searrow & \downarrow & & \downarrow 7 \\
 & R\Gamma_V R\underline{Hom}^\cdot_V(\gamma_*\mathcal{O}_X, R\underline{Hom}^\cdot_X(Rx_*Lx^*F^\cdot, Rx_*\omega) \overset{L}{\boxtimes} G)[n] & & & \\
\downarrow 7' & & \downarrow & & \\
R\Gamma_V R\underline{Hom}^\cdot_V(\mathcal{O}_V, \dots)[n] & \longrightarrow & R\Gamma_V R\underline{Hom}^\cdot_V(\mathcal{O}_V, \dots)[n] & \longrightarrow & R\Gamma_V R\underline{Hom}^\cdot_V(\mathcal{O}_V, \dots)[n] \longrightarrow R\Gamma_V R\underline{Hom}^\cdot_V(\mathcal{O}_V, \dots)[n].
\end{array}
$$

The left morphism of the third row is induced by $R\Gamma_V = R\Gamma_V \cdot Rv_*$ and uses the strong form of the Kunneth formula of 4.5.6. The central morphism of the third row is induced by Tr_X: $Rx_*\omega_X \longrightarrow \omega_X$; the right one by the adjointness $F^\cdot \longrightarrow Rx_*Lx^*F^\cdot$. The morphisms of the second and third columns are induced by cofunctoriality from $\mathcal{O}_V \longrightarrow \gamma_*\mathcal{O}_X$. The three squares of the lowest layer are trivially commutative. The remaining heptagon will be investigated on the next page.

First note that for f: $Z \longrightarrow Y$, $G^\cdot \in D(Y)$ and $K^\cdot \in D(Z)$ the following square commutes:

$$\begin{array}{ccc} Rf_*\underline{RHom}_Z^\cdot(Lf^*G^\cdot, K^\cdot) & \longrightarrow & \underline{RHom}_Y^\cdot(Rf_*Lf^*G^\cdot, Rf_*K^\cdot) \\ \Big\downarrow{\scriptstyle 1.6.12}\wr & & \Big\downarrow{\scriptstyle G^\cdot \to Rf_*Lf^*G^\cdot} \\ G^\cdot \overset{\vee}{\otimes} Rf_*K^\cdot & \xrightarrow{4.1.1} & \underline{RHom}_Y^\cdot(G^\cdot, Rf_*K^\cdot). \end{array}$$

(This has to be checked from the definitions of [RD p. 110] which were not given in detail in 1.6; the proof consists of some two pages of assembling these definitions.)

The above general fact shows that the left hand square of:

$$\begin{array}{ccccc} Rx_*\underline{RHom}_X^\cdot(Lx^*F^\cdot, \omega_X) & \to & \underline{RHom}_X^\cdot(Rx_*Lx^*F^\cdot, Rx_*\omega_X) & \to & \underline{RHom}_X^\cdot(Rx_*Lx^*F^\cdot, \omega_X) \\ \Big\downarrow\wr & & \Big\downarrow & & \Big\downarrow \\ F^\cdot \overset{\vee}{\otimes} Rx_*\omega_X & \longrightarrow & \underline{RHom}_X^\cdot(F^\cdot, Rx_*\omega_X) & \longrightarrow & \underline{RHom}_X^\cdot(F^\cdot, \omega_X) \end{array}$$

commutes. The right hand square, in which the horizontal morphisms are induced by Tr_x: $Rx_*\omega_X \to \omega_X$ and the vertical ones by $F^\cdot \longrightarrow Rx_*Lx^*F^\cdot$ trivially commutes.

In view of the construction of the duality isomorphism and of the Kunneth formula there is a compatibility relationship between the Cartesian product of two duality isomorphisms and the duality isomorphism of the Cartesian product that will not be written down explicitly. This fact and the above show that the composite of the third row of 4.7.4 is the duality isomorphism for v: $V \to V$. But 6' is the duality isomorphism for δ: $X \to V$ and (6) that for γ: $X \to V$. As $\gamma = v\delta$ the compatibility of the duality isomorphism with composition of variety morphisms implies the commutativity of the pentagon and so of the diagram.

The reader may now fit together (on a cylinder) the three large diagrams of this proof and 4.7.1 and read off the commutativity of 4.7.1.

4.8. THE VERDIER DIAGRAM - III.

The notation of 4.7.1 is maintained except that now $F^\cdot = G^\cdot$ and the isomorphisms ②, ③ and ④ are replaced by their inverses. Now apply h^0 to Diagram 4.7.1; the following commutative diagram of k-vector spaces is obtained:

Diagram 4.8.1.

$$
\begin{array}{ccc}
\mathrm{Hom}[X;Lx^*F^\cdot,F^\cdot] & \xrightarrow[\sim]{⑤} & H^0[X;R\underline{\mathrm{Hom}}_X^\cdot(Lx^*F^\cdot,\mathcal{O}_X)\overset{L}{\otimes}F^\cdot] \\
① \downarrow & & \downarrow s\,⑥ \\
\mathrm{Hom}[pt;R\Gamma_X F^\cdot,R\Gamma_X F^\cdot] & & \mathrm{Ext}^n[V;\gamma_*\mathcal{O}_X,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\boxtimes}F^\cdot] \\
② \uparrow s & & \downarrow ⑦ \\
\bigoplus_p \mathrm{Hom}_k(H^p[X;F^\cdot],k)\otimes H^p[X;F^\cdot] & & \mathrm{Ext}^n[V;\mathcal{O}_V,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\boxtimes}F^\cdot] \\
③ \uparrow s & & \downarrow s\,⑧ \\
\bigoplus_p H^{n-p}[X;R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)]\otimes H^p[X;F^\cdot] & \xleftarrow[\sim]{④} & H^n[V;R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\boxtimes}F^\cdot].
\end{array}
$$

The next diagram is made up of five squares, each of which is easy to prove commutative.

Diagram 4.8.2.

$$
\begin{array}{ccc}
k & \xleftarrow{\text{Trace}} & \mathrm{Hom}[pt;R\Gamma_X F^\cdot,R\Gamma_X F^\cdot] \\
\| & & \uparrow ② \\
k & \xleftarrow{\text{Comp}} & \bigoplus_p \mathrm{Hom}_k(H^p[X;F^\cdot],k)\otimes H^p[X;F^\cdot] \\
\uparrow s & & \uparrow ③ \\
H^n[X;\omega_X] & \xleftarrow{\text{Cup}} & \bigoplus_p H^{n-p}[X;R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)]\otimes H^p[X;F^\cdot] \\
\mathrm{Tr} \uparrow & & s \uparrow ④ \\
H^n[X;R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot] & \xleftarrow{\alpha_1} & H^n[V;R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\boxtimes}F^\cdot] \\
\beta_1 \uparrow s & & s \uparrow ⑧ \\
\mathrm{Ext}^n[X;\mathcal{O}_X,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot] & \xleftarrow{\alpha_2} & \mathrm{Ext}^n[V;\mathcal{O}_V,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\boxtimes}F^\cdot] \\
\beta_2 \uparrow & & \uparrow ⑦ \\
\mathrm{Ext}^n[X;L\delta^*\gamma_*\mathcal{O}_X,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot] & \xleftarrow{⑥} & \mathrm{Ext}^n[V;\gamma_*\mathcal{O}_X,R\underline{\mathrm{Hom}}_X^\cdot(F^\cdot\ )\overset{L}{\boxtimes}F^\cdot]
\end{array}
$$

where: Trace is the trace of an endomorphism of an object of D(pt), taken with the alternating sign rule of 2.5.2.
Comp is the direct sum of the composition morphisms.
Cup is the cup product.
α_1 is induced by Lδ*.
Tr is the trace map of 4.1.3.
β_1 is obtained by identifying the functors $\mathrm{Ext}^n[X;\mathcal{O}_X,-]$ and $H^n[X;-]$.
α_2 and ⑨ are induced by Lδ*.
β_2 is induced by Lδ* of the $\mathcal{O}_V \longrightarrow \gamma_*\mathcal{O}_X$ morphism.

Note that a sign switch has come into the isomorphism ② in the translation from D(pt) to k-vector spaces; a sign $\bigoplus_p (-)^p$ is brought in by the sign convention used in defining ⊗.

Proof of commutativity. In view of the above remark the first square commutes by elementary linear algebra. The second commutes by the definition of duality. The third square commutes by the well-known relationship between cup and Cartesian products given in [5 p.258]. The fourth and the fifth squares trivially commute.

Lemma 4.8.3. The following diagram commutes:

$$\begin{array}{ccc}
H^n[X;\omega_X] & \xleftarrow{\ Tr\ } & H^n[X;R\underline{Hom}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot] \\
\uparrow & & \downarrow \beta_1 \\
\mathrm{Ext}^n[X;\mathcal{O}_X,\omega_X] & \xleftarrow{\ Tr\ } & \mathrm{Ext}^n[X;\mathcal{O}_X,R\underline{Hom}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot] \\
\uparrow & & \downarrow \beta_2 \\
\mathrm{Ext}^n[X;L\delta^*\gamma_*\mathcal{O}_X,\omega_X] & \xleftarrow{\ Tr\ } & \mathrm{Ext}^n[X;L\delta^*\gamma_*\mathcal{O}_X,R\underline{Hom}_X^\cdot(F^\cdot,\omega_X)\overset{L}{\otimes}F^\cdot]
\end{array}$$

Proof. The left hand vertical morphisms are defined analogously to the β's. Commutativity is trivial.

These three diagrams may be put together to prove:

Theorem 4.8.4. Diagram 5.1.4 (on the next page but one) commutes.

CHAPTER 5. THE TRACE FORMULA.

5.1. INTRODUCTION.

To formulate the result of this chapter, it is convenient to make the following definition:

Definition 5.1.1. Let $x: X \to X$ be an admissible (2.7.2) endomorphism of the variety X. Let $i: Y \to X$ be the embedding of a fixed component. Let $\phi: Lx^*F^\bullet \to F^\bullet$ be a covering homomorphism, let $\xi \in M(x)$ and let $\xi' \in M'(x)$. Let $\lambda_Y \in M(Y)$ be the canonical element of 2.7.1.2.

(α) If k has characteristic 0, define $\nu'(Y,\phi) \in k$ as: $[ct(i^![\phi]).T(Y).(ct\lambda_Y)^{-1}][Y]$.

(β) If k has characteristic $p \neq 0$, recall the definition of $\nu(Y,\phi) \in W(k)$ given in 3.1.2, and define $\nu'(Y,\phi)$ as its reduction mod p, which is an element of k.

Similarly define $\nu'(Y,\xi)$ and $\nu'(Y,\xi') \in k$.

This chapter will be devoted to proving the following theorem:

Theorem 5.1.2. With the notation of 5.1.1, and with $c: X \to pt$, trace $c_!\phi = \sum_Y \nu'(Y,\phi)$, the sum being taken over the fixed components, Y, of x, provided each such component is either a point or a curve.

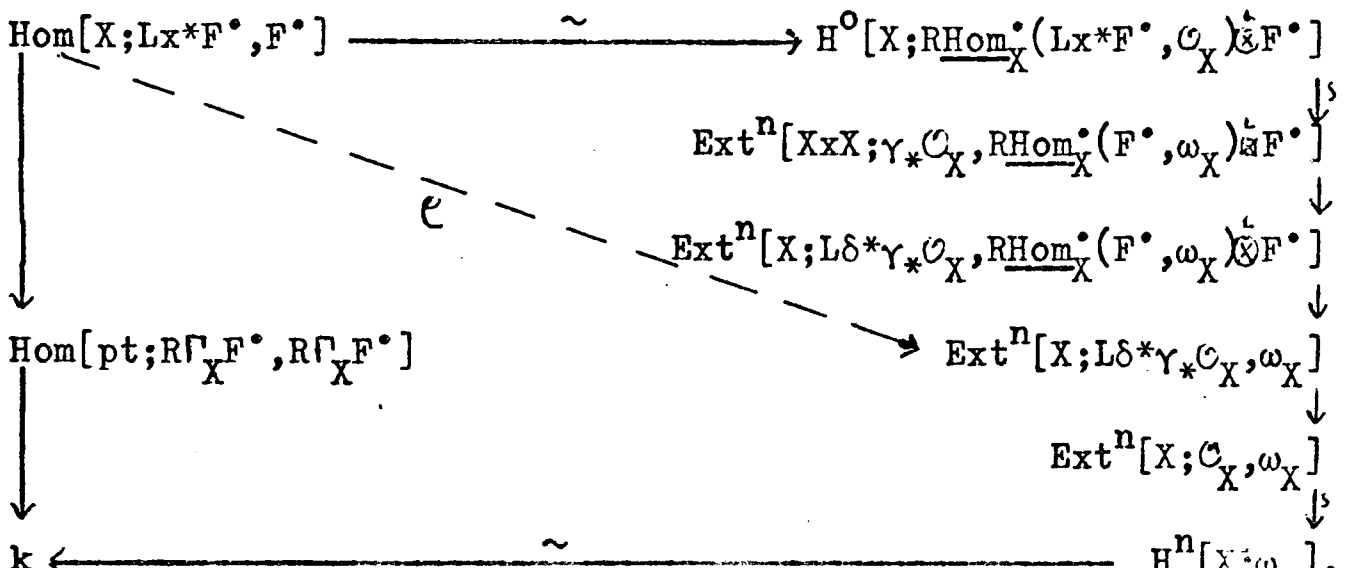
Remark 5.1.2.1. In section 5.5 I will present a plausible conjecture and show how to deduce from it Theorem 5.1.2 with the 'a point or a curve' restriction omitted.

Remark 5.1.2.2. If x is periodic, then 5.1.2 should be compared with 3.1.6. If the characteristic of k is $p \neq 0$, then 5.1.2 is obtainable from 3.1.6 by reduction mod p in the ring W(k). If k has characteristic zero, 5.1.2 and 3.1.6 coincide. If k has characteristic $p \neq 0$, the Brauer trace formula analogous to 3.1.6 is not valid for arbitrary endomorphisms x; this point is discussed in 6.9 and 6.10.

Note 5.1.3. As usual, it is sufficient to prove 5.1.2 under the additional hypothesis that X is irreducible. This is because a component of X which is not mapped into itself by x makes no contribution to either side of the equation in 5.1.2. So henceforth I will assume that X is irreducible of dimension n.

In 4.8.4 the following diagram was proved to commute:

Diagram 5.1.4.



The reader will recall that the morphisms of 5.1.4 were defined so that the left hand column gives the required trace. The other morphisms are, briefly, dualising on X; duality for the closed embedding $\gamma = (x,\mathrm{id})\colon X \to XxX$; that induced by $L\delta^*$, where $\delta = (\mathrm{id},\mathrm{id})\colon X \to XxX$; that induced by tr: $F^{\bullet\vee} \otimes^{\mathbb{L}} F^\bullet \to \mathcal{O}_X$; that induced by the canonical $\mathcal{O}_X \to L\delta^*\gamma_*\mathcal{O}_X$; and two canonical isomorphisms.

If the reader omitted 4.3, 4.5, 4.6, 4.7 and 4.8, he may pick up all that is now needed from them in the above diagram and in the paragraph explaining it.

Plan of the proof. A 'localisation theorem', proved in 5.2 and 5.3, will reduce the study of the composite ϱ defined by the broken arrow to the study of the ϱ obtained from the special case of 5.1.4 when x = identity. This result depends on x being admissible, but not on the dimension of the fixed components. In 5.4 the proof of 5.1.2 is completed.

5.2. THE LOCALISATION THEOREM - I.

The following notation will be used in 5.2 and 5.3:

X is an irreducible variety.

x: $X \to X$ is an admissible endomorphism.

$Y \xhookrightarrow{i} X$ is a fixed component of x.

ω_X, ω_Y as usual. Set $\omega = \omega_{Y/X}$.

$n = \dim X$. $n-q = \dim Y$.

$\delta_Y = (id,id)$: $Y \to YxY$.

$\delta = (id,id)$, $\gamma = (x,id)$: $X \to XxX$.

$j = ixi$: $YxY \to XxX$.

This section consists of a proof of the following proposition. In the next section the 'localisation theorem' will be deduced from it.

Proposition 5.2.1. There is a morphism, $i_{G^\cdot}$, natural in $G^\cdot \in D(X)$:

$$\mathrm{RHom}^\cdot[X; L\delta^*\gamma_*\mathcal{O}_X, G^\cdot \otimes \omega_X] \longrightarrow \mathrm{RHom}^\cdot[Y; L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, Li^*G^\cdot \otimes \omega_Y][-q].$$

Furthermore, $\bigoplus_Y i_{G^\cdot}$ (the direct sum being taken over all fixed components, Y, of x) is an isomorphism, at least when $G^\cdot = \mathcal{O}_X$.

(In fact $\bigoplus_Y i_{G^\cdot}$ is always an isomorphism. If this fact had to be proved, one would use the 5-lemma and 1.5.2 to reduce proving isomorphism to proving it in the special case when $G^\cdot$ is a locally free sheaf and then carry this sheaf through the local argument given below for $G^\cdot = \mathcal{O}_X$.)

Proof: $i_{G^\cdot}$ will be constructed thus:

$$\begin{aligned}
&\mathrm{RHom}^\cdot[X; L\delta^*\gamma_*\mathcal{O}_X, G^\cdot \otimes \omega_X] \\
\longrightarrow\ &\mathrm{RHom}^\cdot[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X, Li^*G^\cdot \otimes i^*\omega_X] \\
\longrightarrow\ &\mathrm{RHom}^\cdot[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega, Li^*G^\cdot \otimes \omega_Y] \\
\longrightarrow\ &\mathrm{RHom}^\cdot[Y; L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, Li^*G^\cdot \otimes \omega_Y][-q],
\end{aligned}$$

where the first morphism is induced by the functor Li^*, the second comes from the isomorphism $i^*\omega_X \xrightarrow{\sim} \omega_Y \otimes \omega^\vee$ of 4.2.2, and the third is induced by a fixed θ: $L\delta_Y^*\delta_{Y*}\mathcal{O}_Y[q] \longrightarrow Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega$ which will be defined below. Whatever θ is, $i_{G^\cdot}$ is now seen to be natural in $G^\cdot$.

Before θ can be defined, it is necessary to prove the following lemma. The reader should note that the following square commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\delta_Y} & YxY \\ \downarrow i & & \downarrow j \\ X & \underset{\delta}{\overset{\gamma}{\rightrightarrows}} & XxX \end{array}$$

Lemma 5.2.1.1. For $t>0$ and for $t<-q$, $h^t Lj^*\gamma_*\mathcal{O}_X = 0$. For other integers t, $h^t Lj^*\gamma_*\mathcal{O}_X \approx \delta_{Y*}\lambda^{-t}N^\vee$, where $N^\vee$ is the dual of the normal bundle of Y in X.

Proof of 5.2.1.1. Firstly, note that the support of such cohomology is contained in $\delta_Y Y \subset YxY$. So, for $a \in Y$, I procede to calculate this cohomology in a suitable affine open neighbourhood of axa in YxY. It is convenient to choose U, U', f_r etc. as in 2.6.5 and use the neighbourhood UxU' $\subset$ YxY of axa. The notation of 2.6.5 as well as its results will be used without further comment.

Local co-ordinates for γX in UxU' are $f_r \otimes 1 - 1 \otimes f_r x$, for $r = 1,\ldots,q,q+1,\ldots,n$. Form two Koszul complexes, $K_1^\bullet$ and $K_2^\bullet$, over the ring $A(U)\otimes A(U')$, the first with the first q of these elements, the second with the remaining n-q. Then the Koszul complex, $K^\bullet$, on all n of them is $K_1^\bullet \otimes K_2^\bullet$ and forms a free resolution of $\gamma_*\mathcal{O}_X|UxU'$. It is required to tensor $K_1^\bullet \otimes K_2^\bullet$ with $A(U)/I(Y)\otimes A(U')/I(Y)$. $K_2^\bullet$ tensored with this quotient ring is the Koszul complex over $A(Y\cap U)\otimes A(Y\cap U')$ on a set of local co-ordinates for $\delta_Y Y$ in $Y\cap U x Y\cap U'$. $K_1^\bullet$ tensored with this quotient ring is the Koszul complex over $A(Y\cap U)\otimes A(Y\cap U')$ on q elements, each of which is zero. So $h^t Lj^*\gamma_*\mathcal{O}_X$ is certainly δ_{Y*} of a locally free sheaf on Y of the desired rank. An easy check on the variance of this locally free sheaf (i.e. a consideration of the overlaps of 'co-ordinate patches') shows that the desired global isomorphism may be set up.

Proof of 5.2.1 continued. Now choose and fix a finite locally free resolution of $\gamma_*\mathcal{O}_X$ over XxX. Call it $F^\bullet$ and consider

$j^*F^\cdot$, which is a bounded locally free complex on YxY representing the object $Lj^*\gamma_*\mathcal{O}_X$ of D(YxY). So its cohomology has just been calculated. Let π_1: YxY $\to$Y be the first projection and set $M^\cdot = j^*F^\cdot \otimes \pi_1^*\omega$. So $M^\cdot$ is a locally free complex representing $Lj^*\gamma_*\mathcal{O}_X \otimes \pi_1^*\omega$; it is known that $h^t M^\cdot = 0$ for $t>0$ and $t<-q$ whilst $h^{-q}M^\cdot \approx \delta_{Y*}\mathcal{O}_Y$. So the 'truncation' morphism of complexes:

$$\begin{array}{ccccccccccccc}
\cdots & \to & M^{-N} & \to \cdots \to & M^{-q-1} & \to & \mathrm{Ker}\beta & \to & 0 & \to & 0 & \to & \cdots \\
 & & \| & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & M^{-N} & \to \cdots \to & M^{-q-1} & \to & M^{-q} & \xrightarrow{\beta} & M^{-q+1} & \to & M^{-q+2} & \to & \cdots
\end{array}$$

defines a morphism ψ: $\delta_{Y*}\mathcal{O}_Y[q] \to Lj^*\gamma_*\mathcal{O}_X \otimes \pi_1^*\omega$ in D(YxY). Now set $\Theta = L\delta_Y^*\psi$: $L\delta_Y^*\delta_{Y*}\mathcal{O}_Y[q] \to L\delta_Y^*Lj^*\gamma_*\mathcal{O}_X \otimes \omega = Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega$.

It now remains to be proved that for each integer r the composite of the following two morphisms is an isomorphism:

$$\begin{aligned}
&\mathrm{Ext}^r[X; L\delta^*\gamma_*\mathcal{O}_X, \omega_X] \\
&\to \bigoplus_Y \mathrm{Ext}^r[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega, \omega_Y] \\
&\to \bigoplus_Y \mathrm{Ext}^{r-q}[Y; L\delta_Y^*\delta_{Y*}\mathcal{O}_Y, \omega_Y].
\end{aligned}$$

It is of course equivalent to show that the composite of the duals of these two morphisms of k-vector spaces is an isomorphism. In view of the remarks of 4.4.5, we may identify the duals of the vector spaces and their morphisms as those in the next displayed column, where Tr_i: $i_*i^! \to \mathrm{id}_{D(X)}$ is the adjointness transformation and $i^!$ was defined in 4.4.1 by $i^!G^\cdot = Li^*G^\cdot \otimes \omega[-q]$:

$$\begin{aligned}
&\bigoplus_Y H^{n-r}[X; i_*L\delta_Y^*\delta_{Y*}\mathcal{O}_Y] \\
&\xrightarrow{\theta} \bigoplus_Y H^{n-q-r}[X; i_*(Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega)] \\
&\xrightarrow{\mathrm{Tr}_i} H^{n-r}[X; L\delta^*\gamma_*\mathcal{O}_X].
\end{aligned}$$

But to prove that this composite is an isomorphism, it suffices to show that the composite in D(X):

$$\bigoplus_Y i_*L\delta_Y^*\delta_{Y*}\mathcal{O}_Y \xrightarrow{\oplus i_*\theta} \bigoplus_Y i_*(Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega)[-q] \xrightarrow{\oplus \mathrm{Tr}_i} L\delta^*\gamma_*\mathcal{O}_X$$

is an isomorphism, and this is a purely local statement. The three objects of $D(X)$ concerned have cohomology with support on X^X. So it suffices to consider a suitable open neighbourhood of each point $a \in X^X$.

To fix the notation assume that a is in the component Y of X^X. To avoid confusion a typical component of X^X is now denoted by Z, with $g: Z \hookrightarrow X$ the embedding. The notation and results of 2.6.5 will now be used without explicit reference; U' is the open neighbourhood of a that will be considered.

For clarity I will first deal with the case when $q = n$, i.e. a is an isolated fixed point. In this case $\gamma_* \mathcal{O}_X | U \times U'$ is represented by the Koszul complex over $A(U) \otimes A(U')$ on the n elements $f_1 \otimes 1 - 1 \otimes f_1 x, \ldots, f_n \otimes 1 - 1 \otimes f_n x$. So $L\delta^* \gamma_* \mathcal{O}_X | U'$ is represented by the Koszul complex over $A(U')$ on the elements $f_1' - f_1 x, \ldots, f_n' - f_n x$, which are, by the assumption of admissibility, known to be a set of local co-ordinates for $a = Y$ in U'. Hence there is a (not canonical) isomorphism $L\delta^* \gamma_* \mathcal{O}_X | U' \approx i_* \mathcal{O}_Y | U'$.

Clearly $\bigoplus_Z g_* L\delta_Z^* \delta_{Z*} \mathcal{O}_Z | U' \approx i_* \mathcal{O}_Y | U'$. By an easy Koszul complex calculation $\bigoplus_Z g_* (Lg^* L\delta^* \gamma_* \mathcal{O}_X \otimes \omega_{Z/X})[-q_Z] | U'$ is shown to be represented by a complex with zero differentials and with t^{th} object (not canonically) isomorphic to the direct sum of $\binom{n}{t}$ copies of $i_* \mathcal{O}_Y | U'$. I claim that the two morphisms under consideration restricted to U' are now represented by:

$$
\begin{array}{ccccccc}
 & \cdots & 0 \to i_* \mathcal{O}_Y | U' & \longrightarrow & 0 & \longrightarrow & \cdots \\
i_* \theta & & \downarrow \qquad \downarrow s & & \downarrow & & \\
 & \cdots & 0 \to i_* \mathcal{O}_Y | U' & \xrightarrow{0} & \binom{n}{1} i_* \mathcal{O}_Y | U' & \to & \cdots \\
\mathrm{Tr} & & \downarrow \qquad \downarrow s & & \downarrow & & \\
 & \cdots & 0 \to i_* \mathcal{O}_Y | U' & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

$i_* \theta$ must be calculated by constructing $\psi | U \cap Y \times U' \cap Y$: $\delta_{Y*} \mathcal{O}_Y[q] | U \cap Y \times U' \cap Y \to Lj^* \gamma_* \mathcal{O}_X \otimes \pi_1^* \omega | U \cap Y \times U' \cap Y$. This second object is seen to be represented by a complex with zero differentials

and with t^{th} object isomorphic to the direct sum of $\binom{n}{-t}$ copies of $\delta_{Y*}\mathcal{O}_Y|U\cap Y\times U'\cap Y$. ψ is the obvious 'truncation' morphism. The construction of θ shows that $i_*\mathcal{E}|U'$ is the 'truncation' morphism given above. $Tr|U'$ may be identified with that of the claim by arguments similar to those of 4.4.4. So $Tr.\bigoplus_Z g_*\theta_Z|U'$ is a quasi-isomorphism as required.

The result is clear if q=0,Y=X. The general case now is (roughly) locally a product of the Y=X case and the isolated fixed point case.

The first step for dealing with the general case is determining $L\delta^*\gamma_*\mathcal{O}_X|U'$. $\gamma_*\mathcal{O}_X|U\times U'$ is represented by the tensor product of two free $A(U)\otimes A(U')$ complexes $K_1^\bullet$ and $K_2^\bullet$ defined three pages back. So $L\delta^*\gamma_*\mathcal{O}_X|U'$ is represented by the tensor product of the A(U') Koszul complex on the q elements $f_1'-f_1x$, ... , $f_q'-f_qx$ with the Koszul complex on the n-q elements $f_{q+1}'-f_{q+1}x$, ... ,$f_n'-f_nx$. Now the first q elements, by the assumption of admissibility, form a set of local co-ordinates for Y in U' and generate the ideal I(Y) of A(U') which contains the other n-q. Hence this complex is quasi-isomorphic to one with zero differentials and with t^{th} object (non-canonically) isomorphic to the direct sum of $\binom{n-q}{-t}$ copies of $i_*\mathcal{O}_Y$.

$\bigoplus_Z g_*(L\delta_Z^*\delta_{Z*}\mathcal{O}_Z)|U'$ is shown to be represented by such a complex by using the set of local co-ordinates for $\delta_Y Y$ in $U\cap Y\times U'\cap Y$ set up in 2.6.5.

The rest of the argument follows that for the isolated fixed point case and so need not be repeated.

5.3. THE LOCALISATION THEOREM - II.

In this section several rather complicated diagrams will be exhibited. Fortunately their commutativity is easy to prove. They will then be brought together to transform the rather difficult Proposition 5.2.1 into the important Theorem 5.3.6. The notation of the last section is still used.

Lemma 5.3.1. For $F^\cdot \in D(X)$, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Ext}^n[X; L\delta^*\gamma_*\mathcal{O}_X, \underline{\mathrm{RHom}}_X^\cdot(F^\cdot, \omega_X) \overset{L}{\otimes} F^\cdot] & \longrightarrow & \mathrm{Ext}^n[X; L\delta^*\gamma_*\mathcal{O}_X, \omega_X] \\
\downarrow & & \downarrow \\
\mathrm{Ext}^n[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, i^*\omega_X) \overset{L}{\otimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^n[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X, i^*\omega_X] \\
\downarrow & & \downarrow \\
\mathrm{Ext}^n[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, \omega_Y) \overset{L}{\otimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^n[Y; Li^*L\delta^*\gamma_*\mathcal{O}_X \otimes \omega, \omega_Y] \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{n-q}[Y; L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, \omega_Y) \overset{L}{\otimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^{n-q}[Y; L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, \omega_Y].
\end{array}$$

Proof. The vertical columns are those used in 5.2 to define i_G. for $G^\cdot = F^{\cdot\vee} \overset{L}{\otimes} F^\cdot$ and $G^\cdot = \mathcal{O}_X$ respectively. The horizontal morphisms are induced by the composition morphism of 4.1.3. In view of 4.1.3, commutativity is clear.

Lemma 5.3.2. The following diagram, whose (omitted) right hand column is the left hand column of the diagram of 5.3.1, commutes:

$$\begin{array}{ccc}
\mathrm{Ext}^n[X\times X; \gamma_*\mathcal{O}_X, \underline{\mathrm{RHom}}_X^\cdot(F^\cdot, \omega_X) \overset{L}{\boxtimes} F^\cdot] & \longrightarrow & \mathrm{Ext}^n \\
\downarrow & & \downarrow \\
\mathrm{Ext}^n[Y\times Y; Lj^*\gamma_*\mathcal{O}_X, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, i^*\omega_X) \overset{L}{\boxtimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^n \\
\downarrow & & \downarrow \\
\mathrm{Ext}^n[Y\times Y; Lj^*\gamma_*\mathcal{O}_X \boxtimes \pi_1{}^*\omega, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, \omega_Y) \overset{L}{\boxtimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^n \\
\downarrow & & \downarrow \\
\mathrm{Ext}^{n-q}[Y\times Y; \delta_{Y*}\mathcal{O}_Y, \underline{\mathrm{RHom}}_Y^\cdot(Li^*F^\cdot, \omega_Y) \overset{L}{\boxtimes} Li^*F^\cdot] & \longrightarrow & \mathrm{Ext}^{n-q}.
\end{array}$$

Proof. The uppermost horizontal morphism is induced by $L\delta^*$, whilst the other three are induced by $L\delta_Y{}^*$. The uppermost left

vertical morphism is induced by Lj*, the second left vertical morphism is the obvious identification and the third is induced by cofunctoriality in the first variable from the morphism $\hat{\Psi}$ used to construct Θ in 5.1. Commutativity is clear.

Lemma 5.3.3. The following diagram, whose (omitted) left hand column is the left hand column of the diagram of 5.3.2, commutes:

$$\begin{array}{ccc}
\grave{E}xt^n & \longrightarrow & Ext^n[X;L\gamma^*\gamma_*\mathcal{O}_X, R\underline{Hom}_X^\bullet(Lx^*F^\bullet, x^*\omega_X)\dot{\otimes}F^\bullet] \\
\downarrow & & \downarrow \\
Ext^n & \longrightarrow & Ext^n[Y;Li^*L\gamma^*\gamma_*\mathcal{O}_X, R\underline{Hom}_Y^\bullet(Li^*F^\bullet, i^*\omega_X)\dot{\otimes}Li^*F^\bullet] \\
\downarrow & & \downarrow \\
Ext^n & \longrightarrow & Ext^n[Y;Li^*L\gamma^*\gamma_*\mathcal{O}_X\otimes\omega, R\underline{Hom}_Y^\bullet(Li^*F^\bullet, \omega_Y)\dot{\otimes}Li^*F^\bullet] \\
\downarrow & & \downarrow \\
Ext^{n-q} & \longrightarrow & Ext^{n-q}[Y;L\delta_Y^*\delta_{Y*}\mathcal{O}_Y, R\underline{Hom}_Y^\bullet(Li^*F^\bullet, \omega_Y)\dot{\otimes}Li^*F^\bullet].
\end{array}$$

Proof. The uppermost horizontal morphism is induced by $L\gamma^*$ whilst the second, third and fourth horizontal morphisms are induced by $L\delta_Y^*$. The uppermost right vertical morphism is induced by Li* and so the upper square commutes. The second right vertical morphism is the obvious identification and so the middle square commutes. The third right vertical morphism is induced by cofunctoriality in the first variable by $L\delta_Y^*$ of the (truncation) morphism $\delta_{Y*}\mathcal{O}_Y[q] \longrightarrow Lj^*\gamma_*\mathcal{O}_X\otimes\pi_1^*\omega$ used to construct the lowest left vertical morphism; so the third square also commutes.

Lemma 5.3.4. The following diagram, in which $G^\bullet = R\underline{Hom}_X^\bullet(Lx^*F^\bullet, \mathcal{O}_X)\dot{\otimes}F^\bullet$ and whose (omitted) left hand column is the right hand column of the diagram of 5.3.3, commutes:

$$\begin{array}{ccccc}
Ex \xrightarrow{\sim} Ext^n[X;L\gamma^*\gamma_*\mathcal{O}_X\otimes x^*\omega_X^\vee, G^\bullet] & & \longrightarrow & & Ext^0[X;\mathcal{O}_X, G^\bullet] \\
\downarrow \quad \downarrow & & & & \downarrow \\
Ex \xrightarrow{\sim} Ext^n[Y;Li^*L\gamma^*\gamma_*\mathcal{O}_X\otimes i^*\omega_X^\vee, Li^*G^\bullet] & & \longrightarrow & & Ext^0[Y;\mathcal{O}_Y, Li^*G^\bullet] \\
\downarrow \quad \downarrow & & & & \| \\
Ex \xrightarrow{\sim} Ext^n[Y;Li^*L\gamma^*\gamma_*\mathcal{O}_X\otimes\omega\otimes\omega_Y^\vee, Li^*G^\bullet] & & & & \| \\
\downarrow \quad \downarrow & & & & \| \\
Ex \xrightarrow{\sim} Ext^{n-q}[Y;L\delta_Y^*\delta_{Y*}\mathcal{O}_Y\otimes\omega_Y^\vee, Li^*G^\bullet] & & \longrightarrow & & Ext^0[Y;\mathcal{O}_Y, Li^*G^\bullet].
\end{array}$$

Proof. The four left hand horizontal morphisms are trivial isomorphisms obtained by moving various ω terms from the second variable of the Ext's to the first. The uppermost central vertical morphism is induced by Li*, the second central vertical morphism is the obvious identification and the lowest central vertical morphism is induced by cofunctoriality in the first variable by $L\delta_Y^*$ of the (truncation) morphism $\delta_{Y*}\mathcal{O}_Y[q] \longrightarrow Lj^*\gamma_* \mathcal{O}_X \otimes \pi_1^*\omega$. So the three left hand squares trivially commute.

The uppermost right horizontal morphism is induced by cofunctoriality in the first variable by $\mathrm{cotr}_\gamma : \mathcal{O}_X \to L\gamma^*\gamma_*\mathcal{O}_X \otimes x^*\omega_X\,[-n]$. Recall the remarks of 4.4.4, and the fact that $\omega_{X/X\times X}$ (for the embedding γ) is $x^*\omega_X$, which was proved in [RD p. 141] and mentioned in 4.4.3. It is of course a truncation morphism.

The upper right vertical morphism is induced by Li*. The second right horizontal morphism is induced by cofunctoriality in the first variable by $Li^*\mathrm{cotr}_\gamma$. Hence the upper right hand square commutes.

The lowest right horizontal morphism is induced by cofunctoriality in the first variable by $\mathrm{cotr}_{\delta_Y} : \mathcal{O}_Y \longrightarrow L\delta_Y^*\delta_{Y*}\mathcal{O}_Y \otimes \omega_Y^\vee\,[-(n-q)]$. So, to prove the pentagon commutative, it suffices to show that the following square commutes:

$$\begin{array}{ccc} Li^*L\gamma^*\gamma_*\mathcal{O}_X \otimes \omega \otimes \omega_Y^\vee\,[-n] & \xrightarrow{\ \sim\ } & Li^*L\gamma^*\gamma_*\mathcal{O}_X \otimes i^*\omega^\vee\,[-n] \\ \uparrow & & \uparrow{\scriptstyle Li^*\mathrm{cotr}_\gamma} \\ L\delta_Y^*\delta_{Y*}\mathcal{O}_Y \otimes \omega_Y^\vee\,[-(n-q)] & \xleftarrow[\mathrm{cotr}_{\delta_Y}]{} & \mathcal{O}_Y. \end{array}$$

However this diagram represents two equivalent methods of truncating the complex $\delta_Y^*(j^*F^\cdot \otimes \pi_1^*\omega)$ of 5.2, and so it commutes. So 5.3.4 is proved.

Lemma 5.3.5. The following diagram, in which the columns are pieces of Diagram 5.1.4 for x and id_Y respectively, the uppermost horizontal morphism is induced by Li* and the lowest horizontal morphism is the composite of the left hand

column of the diagram of 5.3.2, commutes:

$$\begin{array}{ccc}
\mathrm{Hom}[X;Lx^*F^\cdot,F^\cdot] & \longrightarrow & \mathrm{Hom}[Y;Li^*F^\cdot,Li^*F^\cdot] \\
\downarrow & & \downarrow \\
H^0[X;R\underline{\mathrm{Hom}}_X^\cdot(Lx^*F^\cdot,\mathcal{O}_X)\dot{\otimes}F^\cdot] & \longrightarrow & H^0[Y;R\underline{\mathrm{Hom}}_Y^\cdot(Li^*F^\cdot,\mathcal{O}_Y)\dot{\otimes}Li^*F^\cdot] \\
\wr\downarrow & & \wr\downarrow \\
\mathrm{Ext}^n[X\times X;\gamma_*\mathcal{O}_X,\ \ldots\ ] & \longrightarrow & \mathrm{Ext}^{n-q}[Y\times Y;\delta_{Y*}\mathcal{O}_Y,\ \ldots\ ].
\end{array}$$

<u>Proof.</u> The central horizontal morphism is induced by Li* and so the upper square commutes. Now note that there are trivial identifications $H^0[X;\ \ldots\ ] = \mathrm{Ext}^0[X;\mathcal{O}_X,\ \ldots]$ and $H^0[Y;\ \ldots\ ] = \mathrm{Ext}^0[Y;\mathcal{O}_Y,\ \ldots\ ]$. Modulo these identifications and in view of 4.4.4, 5.3.3 and 5.3.4 combine to prove the commutativity of a square with the same horizontal morphisms as the lower square, but with the vertical isomorphisms of the lower square replaced by their inverses. So 5.3.5 has already been proved.

The following theorem summarises all that will be needed in the sequel of 5.2 and the above part of 5.3. The reader will note that it may be proved by fitting together the diagrams of 5.2.1, 5.3.1, 5.3.2, and 5.3.5.

<u>Theorem 5.3.6.</u> (The 'localisation theorem'.) The following square, in which the vertical morphisms are the composites e of Diagram 5.1.4 for x and id_Y respectively, the upper horizontal morphism is induced by Li* and i" is the i_{ω_Y} of 5.2.1, commutes:

$$\begin{array}{ccc}
\mathrm{Hom}[X;Lx^*F^\cdot,F^\cdot] & \longrightarrow & \mathrm{Hom}[Y;Li^*F^\cdot,Li^*F^\cdot] \\
\downarrow e & & \downarrow e \\
\mathrm{Ext}^n[X;L\delta^*\gamma_*\mathcal{O}_X,\omega_X] & \xrightarrow{\ i''\ } & \mathrm{Ext}^{n-q}[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y].
\end{array}$$

Further $\bigoplus_Y i''$, the direct sum being taken over all fixed components, Y, of x, is an isomorphism.

<u>Remark 5.3.7.</u> Diagram 5.1.4 gives a linear functional: $\mathrm{Ext}^n[X;L\delta^*\gamma_*\mathcal{O}_X,\omega_X] \to k$, which is independent of $F^\cdot$. As the i" morphisms are also independent of $F^\cdot$, this linear functional induces linear functionals on the direct summands: $\mathrm{Ext}^{n-q}[Y;\ L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y] \to k$, which are also independent of $F^\cdot$. Hence it

is convenient to define, for $\phi \in \mathrm{Hom}[X; Lx^*F^\cdot, F^\cdot]$ a covering homomorphism, $\nu_0(Y,\phi)$ = (induced linear functional on $\mathrm{Ext}^{n-q}[Y; L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, \omega_Y]$).(image of ϕ in $\mathrm{Ext}^{n-q}[\ \ldots\ ]) \in k$. Then, by 5.1.4 and 5.3.6, trace $c_!\phi = \sum_Y \nu_0(Y,\phi)$.

Lemma 5.3.8. Let $G^\cdot \in D(Y)$, and let $\psi\colon G^\cdot \to G^\cdot$ be a covering morphism. Let $\phi = i_!\psi\colon Lx^*i_*G^\cdot \to i_*G^\cdot$ be the induced covering homomorphism on X in the sense of 2.2.6.

Then if Y' is a fixed component of x other than Y, $\nu_0(Y',\phi) = 0 = \nu'(Y',\phi)$, whilst $\nu_0(Y,\phi) = \nu'(Y,\phi)$ in the sense of Definition 5.1.1.

Proof. Let $i'\colon Y' \to X$ be the embedding. Then $Li'^*i_*G^\cdot$ has zero cohomology in each dimension and so is zero. Whence $\nu'(Y',\phi) = 0$. 5.3.6 shows that $\nu_0(Y',\phi) = 0$. Now 5.3.7 shows that $\nu_0(Y,\phi) = \text{trace } c_!\phi = \text{trace } (ic)_!\psi$. By 3.1.1, trace $(ic)_!\psi$ = $[ct\psi.T(Y)][Y]$ reduced modulo the characteristic of k. 2.7.1.2 shows that $ct_!\psi = ct\lambda_Y^{-1}.cti'^!i_!\psi$. The result now follows from Definition 5.1.1.

5.4. POINTS AND CURVES.

The notation of 5.1, 5.2 and 5.3 is used.

Proposition 5.4.1. If the fixed component Y is a point, $\nu_0(Y,\phi) = \nu'(Y,\phi) = (\mathrm{tr}\lambda_Y)^{-1}.\mathrm{tr}(i'\phi)$ for $\phi: Lx^*F^\cdot \to F^\cdot$.

Proof. The image of ϕ in $\mathrm{Ext}^0[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y] = k$ is, by 5.3.6, $\mathrm{tr}(i'\phi)$. The linear form on this vector space must be multiplication by a constant element, a_Y, of k. So $\nu_0(Y,\phi) = a_Y.\mathrm{tr}(i'\phi)$. It remains to determine $a_Y \in k$. Apply 5.3.8 to the covering homomorphism $\psi = \mathrm{id}: \mathcal{O}_Y \to \mathcal{O}_Y$. This yields $a_Y = (\mathrm{tr}\lambda_Y)^{-1}$ as required.

Before calculating $\nu_0(Y,\phi)$ when Y is a curve, it is necessary to gather together some facts about $H^0[Y;\mathcal{O}_Y] \oplus H^1[Y;\omega_Y]$. Each summand has rank 1; the first is isomorphic to k whilst the second has a canonical generator ('the class of a point'). If F is a locally free sheaf on Y, it is possible (c.f. Grothendieck [6]) to define a 'first Chern class', $c_1'(F) \in H^1[Y;\omega_Y]$ and even a 'Chern character', $\mathrm{ch}'(F) = \mathrm{rank}F \oplus c_1'(F) \in H^0[Y;\mathcal{O}_Y] \oplus H^1[Y;\omega_Y]$. (If k has characteristic 0, it is of course standard to take Chern character in the Hodge cohomology, but if k has characteristic $p \neq 0$ this can be done only for varieties of low dimension. In this second case my ch' cannot be considered to be the ordinary Chern character.)

Again it is possible to define a 'Todd class' in this rank 2 vector space by $T'(Y) = 1 \oplus (1 - \mathrm{genus}).(\mathrm{class\ of\ a\ point})$. Note that the rank 2 vector space is an algebra, being a subalgebra of the Hodge cohomology. If $\chi(F)$ is the Euler-Poincare characteristic of F, then the Riemann-Roch theorem (3.1.0), reduced modulo the characteristic of k if necessary, becomes: $[\mathrm{ch}'(F).T'(Y)][Y] = \chi(F)$ modulo the characteristic, where $[\ ][Y]$ denotes the projection onto the H^1 summand.

Now extend this notation in the obvious way and define a 'Chern trace', $\mathrm{ct}': M(Y) \to H^0[Y;\mathcal{O}_Y] \oplus H^1[Y;\omega_Y]$. Then, as in the

proof of 3.1.1, one may deduce from this last weak Riemann-Roch theorem a trace formula: let ψ: $G^{\cdot} \longrightarrow G^{\cdot}$ be a covering morphism on Y. Then trace $(ci)_! \psi = [ct'(\psi).T'(Y)][Y] \in k$.

Lemma 5.4.2. If Y is a curve, $L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y \approx \mathcal{O}_Y \oplus \omega_Y[1]$.

Proof. δY is a divisor in YxY. Hence there is a locally free left resolution: $0 \to M \to \mathcal{O}_{YxY} \longrightarrow \delta_{Y*}\mathcal{O}_Y \to 0$ of $\delta_{Y*}\mathcal{O}_Y$ over YxY. Hence $L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y$ is represented by the complex: $0 \to \delta_Y{}^*M \longrightarrow \mathcal{O}_Y \to 0$; by a standard local calculation with a Koszul complex the differential of this complex is 0 and $\delta_Y{}^*M \approx \omega_Y$.

Lemma 5.4.3. Let $\psi \in \mathrm{Hom}[Y;F',F']$. If ϱ is as in Diagram 5.1.4 the image of ψ under: $\mathrm{Hom}[Y;F^{\cdot},F^{\cdot}] \xrightarrow{\varrho} \mathrm{Ext}^1[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y] \xrightarrow{\sim} H^0[Y;\mathcal{O}_Y] \oplus H^1[Y;\omega_Y]$ is $ct'(\psi).T'(Y)$.

Proof. One may show that the H^1 factors of the image of ψ and of $ct'(\psi).T'(Y)$ coincide by using Diagram 5.1.4 (with (X,x) replaced by (Y,id)) and the trace formula mentioned just before 5.4.2.

The H^0 factor of the image of ψ has to be calculated directly. Consider the following diagram:

$$
\begin{array}{ccc}
\psi \in \mathrm{Hom}[Y;F^{\cdot},F^{\cdot}] & & \\
\downarrow \wr & & \\
H^0[Y;F^{\cdot\vee} \overset{L}{\otimes} F^{\cdot}] & \xrightarrow{\sim} & \mathrm{Ext}^1[YxY;\delta_{Y*}\mathcal{O}_Y, F^{\cdot\vee} \overset{L}{\boxtimes} F^{\cdot} \otimes \pi_1{}^*\omega_Y] \\
\| & & \downarrow \\
 & & \mathrm{Ext}^1[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, F^{\cdot\vee} \overset{L}{\otimes} F^{\cdot} \otimes \omega_Y] \\
 & & \wr \downarrow \\
H^0[Y;F^{\cdot\vee} \overset{L}{\otimes} F^{\cdot}] & \longleftarrow & H^0[Y;F^{\cdot\vee} \overset{L}{\otimes} F^{\cdot}] \oplus H^1[Y;F^{\cdot\vee} \overset{L}{\otimes} F^{\cdot} \otimes \omega_Y] \\
\downarrow & & \downarrow \\
H^0[Y;\mathcal{O}_Y] & \longleftarrow & H^0[Y;\mathcal{O}_Y] \quad \oplus H^1[Y;\omega_Y]
\end{array}
\qquad .
$$

The uppermost left vertical isomorphism is that of 5.1.4; the uppermost horizontal isomorphism is likewise that of 5.1.4, i.e. the duality isomorphism for δ_Y; the uppermost right vertical morphism is likewise that of 5.1.4, i.e. it is induced by $L\delta_Y{}^*$; the second ~~left~~ right vertical isomorphism is induced

by that of Lemma 5.4.2; the left and right lowest morphisms are induced by $F^{\cdot\vee}\otimes F^{\cdot}\to\mathcal{O}_Y$. The two lower horizontal morphisms are the projections onto the direct summands. So the lower square trivially commutes.

It is required to compute the projection into H^0 of the image of ψ in $H^0\oplus H^1$. To do this, I show that the pentagon commutes by noting that the composite: $\mathrm{Ext}^1[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y, F^{\cdot\vee}\otimes F^{\cdot}\otimes\omega_Y]\to H^0[Y;F^{\cdot\vee}\otimes F^{\cdot}]$ is the same as that induced by the 'cotr' morphism of 4.4.4, and so the commutativity of the pentagon follows from the theory given there.

Now the H^0 component of $ct'\psi.T'(Y)$ is of course, the trace of the restriction of ψ to any point of Y. So let e: pt$\hookrightarrow$Y be the inclusion of such a point. Then the required result may be read off from the commutative square:

$$\begin{array}{ccc} \psi\in \mathrm{Hom}[Y;F^{\cdot},F^{\cdot}] & \xrightarrow{\quad Le^*\quad} & \mathrm{Hom}[pt;Le^*F^{\cdot},Le^*F^{\cdot}] \\ \downarrow & & \downarrow \\ H^0[Y;\mathcal{O}_Y] & \xrightarrow{\quad\sim\quad} & H^0[pt;\mathcal{O}_{pt}]. \end{array}$$

<u>Proposition 5.4.4.</u> If Y is a fixed curve of the endomorphism x, $\nu_0(Y,\phi) = [ct'(i^!\phi).T'(Y).(ct'\lambda_Y)^{-1}][Y] = \nu'(Y,\phi)$.

<u>Proof.</u> The second equality is an immediate consequence of the definitions. In view of 5.3.7 and 5.4.3 it suffices to check the result for any two ϕ's, ϕ_1 and ϕ_2, which are such that $ct'(\phi_1)$ and $ct'(\phi_2)$ generate $H^0\oplus H^1$.

Let L be an invertible sheaf on Y such that $c_1'(L)\neq 0$. Set $\psi_1 = \mathrm{id}: \mathcal{O}_Y\to\mathcal{O}_Y$ and $\psi_2 = \mathrm{id}: L\to L$. Now put $\phi_1 = i_!\psi_1$ and $\phi_2 = i_!\psi_2$. Then these two ϕ's have the required property and the formula was proved for them in 5.3.8.

<u>Note.</u> Theorem 5.1.2 is now proved.

5.5. A CONJECTURE.

This section demonstrates that Theorem 5.1.2 without any restriction on the dimension of the fixed components of the endomorphism x is a consequence of the following conjecture. The notation of the previous sections of this chapter are maintained.

Conjecture 5.5.1. Let Y be an irreducible variety of dimension d. For each $G^\bullet \in D(Y)$, 5.1.4 defines a morphism ϱ: $\mathrm{Hom}[Y;G^\bullet,G^\bullet] \longrightarrow \mathrm{Ext}^d[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y]$. Suppose ψ_1: $G_1^\bullet \longrightarrow G_1^\bullet$, ψ_2: $G_2^\bullet \longrightarrow G_2^\bullet$ and ψ_3: $G_3^\bullet \longrightarrow G_3^\bullet$ form a triangle in the sense of 2.2.1. Then $\varrho(\psi_1) - \varrho(\psi_2) + \varrho(\psi_3) = 0$.

Evidence. In the last section this was, in effect, proved for d = 0 and d = 1. I will not attempt to write down the lengthy complication of the investigation of the case of curves which establishes the conjecture in the special case of Y being a product of d curves.

If, conversely, Theorem 5.1.2 is valid for all endomorphisms x which have a fixed component isomorphic to Y, then the argument given below may be reversed to give evidence for, but not a proof of, 5.5.1.

Proposition 5.5.2. Theorem 5.1.2 without any restriction on the dimension of the fixed components is a consequence of the above conjecture.

Proof-Case I. Assume that k has characteristic 0.

In this case, define a k-linear map α: $M'(Y) \longrightarrow \mathrm{Ext}^d[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y]$ thus: M'(Y) is generated additively by images of covering morphisms of the form ψ: $G^\bullet \longrightarrow G^\bullet$. Set $\alpha([\psi]) = \varrho(\psi)$. The conjecture shows that this indeed induces by linearity the required α. Let β: $\mathrm{Ext}^d[Y;L\delta_Y{}^*\delta_{Y*}\mathcal{O}_Y,\omega_Y] \longrightarrow \mathrm{Ext}^n[X;L\delta^*\delta_*\mathcal{O}_X,\omega_X] \longrightarrow \mathrm{Ext}^n[X;\omega_X] \longrightarrow k$ be the linear functional of 5.3.7.

Hence if ϕ: $Lx^*F^\bullet \longrightarrow F^\bullet$ is a covering homomorphism on X, $\nu_o(Y,\phi) = \beta\alpha(i^![\phi])$. But, by 2.7.1, $i^![\phi] = i^!i_!((\lambda_Y^!)^{-1}.i^![\phi])$.

But, for a covering homomorphism $\psi: G^{\cdot} \longrightarrow G^{\cdot}$ on Y, 5.3.8 shows that $\beta\alpha(i^{!}i_{!}[\psi]) = [ct[\psi].T(Y)][Y]$. This implies by linearity that for $\eta \in M'(Y)$ $\beta\alpha(i^{!}i_{!}\eta) = [ct\eta.T(Y)][Y]$. As this holds in particular when $\eta = ((\lambda'_Y)^{-1}.i^{!}[\phi])$, I have shown that $v_o(Y,\phi) = [(ct\lambda_Y)^{-1}.ct(i^{!}[\phi]).T(Y)][Y] = v'(Y,\phi)$ as required.

Proof-Case II. Assume that k has characteristic $p \neq 0$.

The proof given for Case I needs slight modification thus: Firstly, the various k-vector spaces must be considered as W(k)-modules via the canonical $W(k) \longrightarrow k$. So α and β are W(k)-homomorphisms. Secondly, M'(Y) is now generated as a W(k)-module, but not as an abelian group, by the images of covering homomorphisms. So a trivial check of W(k)-linearity must be made to ensure that α is well defined.

CHAPTER 6. UNRESOLVED QUESTIONS AND NOTES.

The following points are offered, without any guarantee, to any reader who wishes to pursue this investigation further.

6.1. Prove Conjecture 5.5.1.

6.2. Let G be a linear algebraic group acting on a variety X. When is an element of G equivariantly embeddable? Is there a representation module, V, of G and a closed embedding $i: X \hookrightarrow P(V)$ commuting with the action of G? If so, the methods of Chapter 3 give a Brauer character formula for the representation of G induced on the cohomology of a G-vector bundle on X. (This should not be difficult.)

6.3. If X is an irreducible variety of dimension n over the field k, there is a hypercohomology spectral sequence:

$$H^p[X;\Omega^{n-q}] \Rightarrow \mathrm{Ext}^*[X;L\delta^*\delta_*\mathcal{O}_X,\omega_X].$$

If k is the field of complex numbers, does the sequence collapse? If so, does there exist a 'canonical' isomorphism: $\mathrm{Ext}^n[X;L\delta^*\delta_*\mathcal{O}_X,\omega_X] \approx \bigoplus_p H^p[X;\Omega^p]$?

6.4. Whether k is the complex numbers or not, I can make $X \longmapsto \mathrm{Ext}^*[X; \ldots]$ a cofunctor from the category of varieties to the category of k-algebras.

6.5. The morphism φ of 5.1.4: $\mathrm{Hom}[X;F,F] \to \mathrm{Ext}^n[X; \ldots]$ provides a definition of a 'characteristic class', $\pi(F)$, for each locally free sheaf F by $\pi(F) = \varphi(\mathrm{id}_F)$. Is there an analogue of the Grothendieck Riemann-Roch theorem (3.1.0) for these characteristic classes?

6.6. When the 'canonical' isomorphism of 6.3 has been found, can $\pi(F)$ and $\mathrm{ch}(F).T(X)$ be identified? Is there a new proof of the Hirzebruch formula along these lines?

6.7. Even when all the above questions are settled, it would be extremely difficult to obtain by the methods of Chapter 5 fixed point formulae over a base variety analogous to those of Chapter 3.

6.8. In [RD p.195] the concept of a residue is given. If the fixed point set of an endomorphism consists of isolated points, perhaps not satisfying the admissibility criterion of 2.7.2, then this may be used to define a reasonable 'contribution' from each fixed point which gives a trace formula. Is there an analogue of this for a non-admissible non-singular fixed component of higher dimension?

6.9. Let k have characteristic $p \neq 0$. Let $f: P \to P$ be the Frobenius endomorphism of the projective line. Let $\phi: f^*\mathcal{O}_P \to \mathcal{O}_P$ be the identification. Then one may calculate that

$$B.tr.(c_1\phi) = 1 \neq 1+p = \sum_{fY=Y} \nu(Y,\phi).$$

Hence the analogues of the theorems of Chapter 3 are false for 'bad' non-periodic endomorphisms.

6.10. Continuing from the last sentence of 6.9, does there exist a large class of 'good' endomorphisms which have a Brauer trace formula?

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Key words. (The references are to pages.)

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of Lf^*, Rf_* 27.
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* For the Koszul complex see [EGA III 1] or [BS p. 120] or [ALM IV-A]. See also 48.

Symbols. (The references are to pages; with some a brief hint is given that may save the reader from looking further.)

A	Chow ring	40.	$Ext^t[Z;$	global	24.
B.tr.	Brauer trace	40.	FW(k)	field	38.
ch	Chern char.	40.	h^t	cohomology	9.
cotr	adjunction	79.	$H^t[$	hypercoh.	24.
ct	Chern trace	41.	9	ideals	78.
D	derived cat.	21.	Inj	category	19.
End	category	31.	k	field	14.
	(modified)	50.	K^b	category	19.
$\underline{Ext}_X^t($	local	24.	K	ring	22.

K_1, K_2		22.	$R\underline{Hom}_Z^\bullet($	local	24.
L		11.	RHom$^\bullet$[	global	24.
Lf*		25.	T	Todd class	40.
Λ,λ	ext. powers	46.		(of endo.)	52.
λ_x, λ_X	can. element	46.	T	left shift	11.
M	ring	32.	tr, Tr		72.
M_1		34.	Tr	adjunction	74.
M_2		36.	W(k)	Witt ring	38.
M'	localised	38.	w		38.
v	p-adic	52.	ω, $\omega_{Z/Y}$		75.
v'	in k	96.	Z[k]	ring	32.
v_o	ad hoc	107.	<...>	ideal	42.
$\mathcal{C}(N)$		17.	$\overset{\bullet}{\otimes}$		24.
Qco	category	15.	[ ]	image	22.
R		11.	[ ][Y]		40.
Rf_*		24.	$^\vee$	dual	72.
$R\Gamma$		24.	$\boxtimes$		84.

Overworked symbols.

* and $_*$ are used in the [EGA] sense for the 'pullback' and 'pushdown' of sheaves under a morphism. They are also used for the cofunctorially and functorially induced morphisms for A( ), A( )$\otimes$Q and A( )$\otimes$FW(k). See 41 and 40.

! and $_!$ are used in the [BS] sense for the induced morphisms for K; see 22. They are also used to denote induced covering homomorphisms; see 32 and 33. They are also used to denote the cofunctorially and functorially induced morphisms for M (see 32 and 33) and M' (see 38). This is in accordance with tradition.

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