Macroeconomic Policy in Resource-Rich Economies

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Abstract

This thesis considers how fiscal and monetary policy should be conducted in resource-rich economies. It consists of three papers addressing: whether governments should spend, save or invest volatile oil income; the assets they should save in; and how monetary policy should respond.

The first, *"Eight principles for managing resource wealth"*, shows that capital-scarce countries should save relatively less against oil price volatility, and invest more in domestic capital. They also should prepare for volatility in advance, and treat savings as a source of income rather than a temporary buffer. To show this the paper develops a framework that nests a variety of existing results, which are presented in eight principles.

The second, *"The Elephant in the Ground: Oil extraction and asset allocation in sovereign wealth funds"*, shows that governments should use sovereign wealth funds to offset oil price risk, extract oil faster if its price is pro-cyclical, and use precautionary savings to manage any residual volatility. To do this it combines three strands of literature for the first time: on continuous-time portfolio theory, oil extraction and precautionary savings.

The third, *"Optimal monetary responses to oil discoveries"*, addresses the anticipation effects around an oil discovery. It shows that the terms of trade will need to appreciate twice: once when oil is discovered and consumers anticipate future revenues; and again when the government begins spending the revenues. Oil wealth will give the monetary authority an incentive to appreciate the terms of trade, in addition to stabilising domestic inflation and the output gap. Optimal policy is well-approximated by a standard monetary rule that also responds to expected changes in the natural level of output.

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Contents

Chapter 1

Introduction

How should fiscal and monetary policy be conducted in resource-rich economies? This thesis addresses that question in three stages. The first considers whether governments should spend, save or invest their resource wealth. The second considers what types of assets they should save in. The third considers how monetary policy should respond.

Non-renewable natural resources dominate nearly a quarter of the world's economies.[1] Of the approximately 200 sovereign states in the world, 130 are endowed with natural resources and 47 are resource-dependent (by the IMF's definition, USGS, 2013 and Baunsgaard et al., 2012). Sixty per cent of resource-dependent economies have a sovereign wealth fund, though they are more common in wealthier nations (SWF Institute, 2013). Seventy-five per cent have some form of exchange rate targeting regime, split evenly between developing and developed countries (IMF, 2009). Some resource-dependent economies have benefited greatly from their natural wealth, including Norway, Qatar and the UAE. However many have not, fifty-five per cent of these countries have a human development index of "low" or "medium", including Angola, Chad and Nigeria (Baunsgaard et al., 2012).

These experiences have led natural resources to be described as a "curse", though

[1] This thesis focuses on non-renewable natural resources such as minerals, oil and gas, rather than renewable resources such as agriculture, forestry, fisheries, water and renewable energy. While the major challenge for non-renewable resources is to make the most of a temporary windfall, the challenge for renewable resources is to ensure that they are sustainable.

they may be more appropriately described as a “challenge”. In a seminal study, Sachs and Warner (1997) find that natural resource dependence has a strong, negative effect on the growth of GDP per capita, controlling for a host of characteristics. This effect has been attributed to de-industrialization (or Dutch disease, see Corden and Neary, 1982), oil price volatility (Poelhekke and van der Ploeg, 2009), political instability and corruption (Sala-i-Martin and Subramanian, 2003 and Acemoglu et al., 2004), environmental degradation (see review by van der Ploeg, 2011), and absorption constraints (van der Ploeg and Venables, 2013).

The challenge posed by natural resources is compounded by their transience. This leads to a practical, if not moral, imperative to make the most of their fleeting benediction. The Pacific island nation of Nauru is a case in point. At 10,000 people and 8 square miles Nauru is the second smallest sovereign state in the world, after Vatican City. For a brief period in the late 1980s they were also one of the richest. For millenia, migrating birds rested on its coral outcrops, forming dense and valuable reserves of phosphate. At their independence from Australia in 1968 these reserves were valued at AU$ 500,000 per person (1968 currency, Hughes, 2004). By 2004 this was whittled to only AU$ 3,000 per person, after a succession of failed investments including West-End musicals, a cruise ship that did not leave port and a yellow Lamborghini for the police chief (despite a single coastal road with 25mph speed limit, McDaniel and Gowdy, 1999). Mining has also left much of the island uninhabitable. I witnessed this transformation first-hand as a child living on Nauru. A desire not to see these mistakes repeated motivates this work.

Addressing the resource challenge involves everything from initial exploration to the transition to a post-resource economy. Topics include: extracting resources - involving whether to explore and how quickly to extract; raising revenues - through auctions, taxes and royalty schemes; managing revenues - through spending, saving or investing and the associated monetary policy; and designing institutions - to avoid corruption, protect the environment and preserve wealth for future generations. An overview of current policy advice on these topics is given by the Natural Resource Charter, a non-government initiative by policymakers and academics which was adopted by the African Union Heads of State steering committee in 2012, and is now being implemented in Nigeria (Natural

Resource Charter, 2013).[2]

This thesis focuses on how resource revenues should be managed. Once revenues have been received, the government must strike an important balance. Spending revenues too quickly will deprive future generations of a national asset, and may lead to problems with absorption, corruption and volatility. Spending them too slowly may forgo the opportunity to accelerate development and boost consumption. Saving the revenues in poorly constructed portfolios will expose public finances to excessive risk. Poorly conducted monetary policy will see oil price fluctuations buffet the economy, through inflation and the output gap.

The thesis consists of three papers. The first begins broadly, considering whether the government should consume resource wealth, save it abroad in a sovereign wealth fund or invest it in domestic capital. The second steps deeper into the question of saving, by considering how sovereign wealth funds should coordinate their above-ground portfolio with below-ground resources. The third then considers how the central bank should respond to government policy.

The first paper, titled *"Eight principles for managing resource wealth"*, considers whether the government should consume, save or invest volatile resource wealth. It extends existing literature by showing that if capital is scarce, as in developing countries, then less should be saved against volatile oil prices and more should be invested domestically. In doing so the paper develops a framework that nests a variety of existing results as special cases, which are presented in eight principles.

Existing literature has found that oil wealth should be saved in a sovereign wealth fund: saving more if prices are volatile and less if capital is scarce. The argument for saving oil wealth in a sovereign wealth fund, and consuming only the permanent income, is based on sharing wealth across generations (the permanent income hypothesis by Friedman, 1957). If capital is abundant then the fund should hold foreign assets, because any profitable

[2] Also see overview by van der Ploeg (2011), Daniel et al. (2010) on raising revenues and van der Ploeg and Venables (2012) on managing revenues.

domestic investments should already be financed by debt (the separation theorem by Fisher, 1930). Oil prices are also volatile, so there should be additional precautionary saving to build up a Volatility Fund (see Kimball, 1990, the review by Carroll and Kimball, 2008, and in a resource context by van den Bremer and van der Ploeg, 2013). Finally, if capital is scarce - as in developing countries - then less should be saved because domestic investments will have a higher return (van der Ploeg and Venables, 2011).

This paper is the first to study the effects of volatility when capital is scarce. It shows that capital-scarce countries should save relatively less in a Volatility Fund, and invest relatively more domestically, than capital-abundant ones. The Volatility Fund should be smaller because capital-scarce countries should already be saving to promote development. The benefit of saving even more will be outweighed by the cost of consuming less. Part of the Volatility Fund should be invested domestically. This is because the Fund is a relatively cheap source of finance, and will have a higher realised (and social) rate of return at home than abroad. Domestic capital may be less liquid than foreign assets, but the Volatility Fund should be a source of income rather than a buffer against temporary oil price shocks, as they are very persistent.

To show this the paper develops an intuitive framework than nests a variety of existing results as special cases, which are presented in eight principles. Oil price volatility is modelled using stochastic control. Its effect on the economy is summarised in two equations describing the expected paths of consumption and total wealth: the Euler equation and budget constraint, respectively. This framework lends itself to explicit results, which can be neatly explained using phase diagrams. This framework nests as special cases: the permanent income hypothesis (Friedman, 1957); capital-abundant economies with volatile income (van den Bremer and van der Ploeg, 2013); and capital-scarce economies with certain income (van der Ploeg and Venables, 2011). These results, and some extensions, are presented in eight principles for use by policymakers, students and researchers.

The second paper, titled *"The Elephant in the Ground: Extracting oil and allocating assets in sovereign wealth funds"*, considers how sovereign wealth funds should coordinate their above-ground portfolio with below-ground resources. It combines three strands of literature for the first time: on portfolio allocation, oil extraction and precautionary savings. The paper shows that sovereign wealth funds should offset oil price risk, extract oil faster if it is positively correlated with the market, and use precautionary savings to manage any residual volatility.

Oil exporters have two large components of wealth, oil below the ground and sovereign wealth funds above it. Norway's Government Pension Fund Global is worth approximately US$ 700 billion, and its subsoil oil is worth approximately the same. Norway has also been described as a "model" for managing sovereign wealth fund assets (Chambers et al., 2012). However, like most resource exporters, Norway's allocation of assets above the ground does not take into account its assets below the ground.

This paper combines three strands of literature: on portfolio allocation, precautionary saving and oil extraction. There is a long and distinguished literature on how assets should be allocated in a portfolio (see for example Markowitz, 1952 and 1959, Tobin, 1958, and Merton, 1990). Recent papers have noted that resource wealth should affect sovereign wealth funds, though they ignore consumption and extraction, and rely on simulations.[3] Precautionary saving describes how volatile income should delay consumption to build up an additional stock of assets (see Leland, 1968, Kimball 1990 and the review by Carroll and Kimball, 2008). Oil extraction has been studied since at least Hotelling (1931). Pindyck (1980 and 1981) and van der Ploeg (2010) both consider extraction when oil prices are volatile and also find that volatility should hasten extraction. Oil investment has been studied by van den Bremer and van der Ploeg (2013) and the first paper in this thesis. However, these papers do not acknowledge that oil wealth will still be volatile after it has been extracted, because it may be invested in financial assets. This paper addresses that shortcoming.

This paper shows that sovereign wealth funds should offset oil price risk, extract oil faster if it is positively correlated with the market, and use precautionary savings to manage any residual volatility. Offsetting oil price risk has two effects on the assets in

[3]See the numerical work of Gintschel and Scherer (2008), Scherer (2009a, 2009b) and Balding and Yao (2011).

the fund: a wealth and a substitution effect. The wealth effect increases the amount of each risky asset, because there is other wealth outside the fund. The substitution effect generally increases (reduces) the holdings of assets that are negatively (positively) correlated with oil, though the exact allocation requires principle components analysis. Oil should be extracted faster when considered as part of a financial portfolio, to generate a risk premium on volatile subsoil assets. Precautionary savings should only manage residual volatility; because the government should be insulated from any idiosyncratic shocks by the sovereign wealth fund.

The third paper, titled *"Optimal monetary responses to oil discoveries"*, considers how monetary policy should respond to the government's management of oil revenues. This paper characterises oil discoveries, derives the central bank's objective function and shows that optimal monetary policy is well-approximated by a standard monetary rule that also responds to expected changes in the natural level of output.

There is little existing theory on oil exporters in a modern setting. Early literature found that anticipated oil revenue can cause a recession before production begins, though this was before the advances in dynamic, stochastic general equilibrium models (see Eastwood and Venables, 1982, Buiter and Purvis, 1983, van Wijnbergen, 1984, and Neary and van Wijnbergen, 1984). Recent literature has focused on resource importers (see Kilian, 2009, Bodenstein et al., 2012), or numerical simulation of resource exporters (Elekdag et al., 2007, Dagher et al., 2010). Romero (2008) considers optimal monetary policy for an oil exporter, but abstracts from anticipated oil revenues.

The paper makes three contributions. This first is to characterise an oil discovery. It will cause the real exchange rate to appreciate twice: once when it is discovered and consumers anticipate future revenues, and again when the government spends the revenues. If prices are sticky and the nominal exchange rate is fixed, as in 75% of resource exporters, then this will lead to an extended period of inflation: both forward- and backward-looking. The second is to derive the central bank's objective. This will be the standard goal of stabilising inflation and the output gap. In addition it will have an incentive to appreciate the terms of trade, to exploit oil wealth. The third is to show that optimal monetary policy will be well approximated by a typical monetary rule, with an additional term that responds to the expected change in the natural level of output. In practice it will cause a sharp, temporary loosening of policy when government spending begins. This allows the real appreciation to happen entirely through the nominal exchange rate (one flexible price), rather than firm's prices (many sticky prices).

Each paper emphasises a different technique in the economists toolkit. The first paper uses a simple application of stochastic control in continuous time, but focuses on drawing out policy implications. The second paper uses a more detailed treatment of stochastic control, as is common in the literature on continuous-time finance. The third paper uses a small dynamic stochastic general equilibrium model to derive explicit solutions, which is at the core of modern macroeconomics.

This work has direct policy implications, and has been used to inform policy in Ghana, Iraq and Norway. In December 2010 Ghana began producing oil for the first time, from the offshore Jubilee field. A project for the International Growth Centre and the government of Ghana (Stefanski, van der Ploeg and Wills, 2012) concluded that Ghana's priority should be investing oil revenues in domestic capital, to accelerate development. The remaining revenues should be saved in volatility and future generations funds. A parking fund, to temporarily alleviate absorption constraints, is optional given the modest size of the windfall. These recommendations were closely linked to the eight principles presented in Paper 1, and informed the design of Papers 1 and 2. They were also reflected in the final structure of Ghana's Petroleum Revenue Management Act (2011), described in Paper 1.

In December 2011, all remaining US troops were withdrawn from Iraq, and the country had signed oil extraction contracts amounting up to 12 million barrels per day for the coming decades. If filled, these will make Iraq the world's largest oil producer. A project for the World Bank and the government of Iraq (van der Ploeg, Venables and Wills, 2012), analysed a range of scenarios for government spending. It found that Iraq's priority should be to establish a temporary parking fund, as the size of the windfall would lead to significant absorption constraints. Investment should slowly be released into the economy to build up absorptive capacity. This work also draws on Paper 1, and informed the design of Papers 1 and 2. The project involved building a model for future use by Iraq's Ministry of Finance, and the report will form part of a Country Economic Memorandum on Iraq (World Bank, forthcoming). A similar project is now starting on Libya.

Case studies on Ghana and Iraq were also prepared for the Natural Resource Charter, which was since adopted by the Nigerian government in 2012.

Norway's Ministry of Finance is continuously reviewing the allocation of assets in its sovereign wealth fund. In 2008 they considered selling all oil and gas stocks, because of their subsoil exposure. However they have not considered offsetting their exposure to oil across a range of asset classes. In May 2013 I presented the results of Paper 2 to the Ministry of Finance's Asset Management Department, which raised the issue for the first time. Discussion is still ongoing.

Finally, the third paper was assisted by an additional project at the Bank of England in 2012. The Bank of England's conduct of quantitative easing since 2009 has posed a challenge for existing models. These models do not incorporate an explicit role for money, provide limited insight into sectoral phenomena and do not consider the feedback between money and lending. A project in 2012 (Cloyne, Thomas and Wills, forthcoming) addressed those shortcomings by developing and estimating a sectoral model of money and lending, which could be used to analyse recent policy. It found that monetary policy has a greater, though slower, effect than previously thought. While not directly linked, this project assisted the third paper by providing an insight into the practical conduct of monetary policy. The project is now being integrated into the Bank of England's suite of models.

The remainder of this thesis proceeds as follows. The first paper considers whether governments should spend resource wealth, save it in a sovereign wealth fund or invest it in domestic capital. The second paper considers the types of assets that the sovereign wealth fund should invest in, and how this should be coordinated with the extraction of oil. The third paper considers how monetary policy should respond to fiscal policy.

References

[1] Acemoglu, D., Verdier, T., and Robinson, J. A. Kleptocracy and divide-and-rule: A model of personal rule. *Journal of the European Economic Association 2*, 2-3 (2004), 162–192.

[2] Balding, C., and Yao, Y. Portfolio allocation for sovereign wealth funds in the shadow of commodity-based national wealth. *International Finance Review 12* (2011), 293–312.

[3] Baunsgaard, T., V. M. P.-R. M., and Richmond, C. Fiscal frameworks for resource rich developing countries. *IMF Staff Discussion Note* (2012).

[4] Bodenstein, M., Guerrieri, L., and Kilian, L. Monetary policy responses to oil price fluctuations. *IMF Economic Review 60*, 4 (2012), 470–504.

[5] Buiter, W., and Purvis, D. Oil, disinflation, and export competitiveness: A model of the "dutch disease".

[6] Carroll, C. D., and Kimball, M. S. Precautionary saving and precautionary wealth. *The New Palgrave Dictionary of Economics 6* (2008), 579–584.

[7] Chambers, D., Dimson, E., and Ilmanen, A. The norway model. *The Journal of Portfolio Management 38*, 2 (2012), 67–81.

[8] Cloyne, J., Thomas, R., and Wills, S. Unconventional monetary policy: A sectoral model. Tech. rep., Bank of England, forthcoming.

[9] Corden, W., and Neary, P. Booming sectors and deindustrialisation in small open economies. *Economic Journal 92* (1982), 825–848.

[10] Dagher, J., Gottschalk, J., and Portillo, R. Oil windfalls in ghana: A dsge approach. *IMF Working Paper WP/10/116* (2010).

[11] Daniel, P., Keen, M., and McPherson, C. *The taxation of petroleum and minerals: principles, problems and practice.* Routledge, 2010.

[12] Eastwood, R., and Venables, A. The macroeconomic implications of a resource discovery in an open economy. *The Economic Journal 92*, 366 (1982), 285–299.

[13] Elekdag, S., Lalonde, R., Laxton, D., Muir, D., and Pesenti, P. Oil price movements and the global economy: A model-based assessment. Working Papers 07-34, Bank of Canada, 2007.

[14] Fisher, I. The theory of interest. *New York 43* (1930).

[15] Friedman, M. *A Theory of the Consumption.* Princeton University Press Princeton, 1957.

[16] Gintschel, A., and Scherer, B. Optimal asset allocation for sovereign wealth funds. *Journal of Asset Management 9*, 3 (2008), 215–238.

[17] Hotelling, H. The economics of exhaustible resources. *The Journal of Political Economy 39*, 2 (1931), 137–175.

[18] Hughes, H. From rags to riches: what are nauru's options and how can australia help? *Issues Analysis: Centre for Independent Studies, Australia* (2004).

[19] IMF. De facto classification of exchange rate regimes and monetary policy frameworks. Tech. rep., International Monetary Fund, 2009.

[20] Kilian, L. Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *The American Economic Review 99*, 3 (2009), 1053–1069.

[21] Kimball, M. S. Precautionary saving in the small and in the large. *Econometrica 58*, 1 (1990), 53–73.

[22] Leland, H. E. Saving and uncertainty: The precautionary demand for saving. *The Quarterly Journal of Economics 82*, 3 (1968), 465–473.

[23] Markowitz, H. Portfolio selection*. *The journal of finance 7*, 1 (1952), 77–91.

[24] Markowitz, H. *Portfolio selection: efficient diversification of investments*. No. 16. Yale university press, 1959.

[25] McDaniel, C. N., and Gowdy, J. M. *Paradise for sale: a parable of nature*. Univ of California Press, 1999.

[26] Merton, R. C. *Continuous-Time Finance*. Basil Blackwell, 1990.

[27] Natural Resource Charter. http://naturalresourcecharter.org/, 2013.

[28] Neary, J. P., and Wijnbergen, S. v. Can an oil discovery lead to a recession? a comment on eastwood and venables. *The Economic Journal 94*, 374 (1984), 390–395.

[29] Pindyck, R. S. Uncertainty and exhaustible resource markets. *The Journal of Political Economy* (1980), 1203–1225.

[30] Pindyck, R. S. The optimal production of an exhaustible resource when price is exogenous and stochastic. *The Scandinavian Journal of Economics* (1981), 277–288.

[31] Poelhekke, S., and van der Ploeg, F. Volatility and the natural resource curse. *Oxford Economic Papers, Vol. 61, Issue 4, pp. 727-760, 2009* (2009).

[32] Romero, R. E. Monetary policy in oil-producing economies. Working Papers 1053, Princeton University, Department of Economics, Industrial Relations Section., Jan. 2008.

[33] Sachs, J. D., and Warner, A. M. The curse of natural resources. *European economic review 45*, 4 (2001), 827–838.

[34] Sala-i Martin, F. X., and Subramanian, A. Addressing the natural resource curse: An illustration from nigeria. *SSRN eLibrary Working Paper Series* (2003).

[35] Scherer, B. Portfolio choice for oil-based sovereign wealth funds. *EDHEC-Risk Publication, EDHEC-Risk Institute* (2009a).

[36] Scherer, B. A note on portfolio choice for sovereign wealth funds. *Financial Markets and Portfolio Management 23*, 3 (2009b), 315–327.

[37] Sovereign Wealth Fund Institute. Sovereign wealth fund rankings, 2013.

[38] Tobin, J. Liquidity preference as behavior towards risk. *The Review of Economic Studies 25*, 2 (1958), pp. 65–86.

[39] USGS. Mineral resources data system. Tech. rep., US Geological Survey, 2013.

[40] van den Bremer, T. S., and Van der Ploeg, F. Managing and harnessing volatile oil windfalls. *IMF Economic Review 61*, 1 (2013), 130–167.

[41] van der Ploeg, F. Aggressive oil extraction and precautionary saving: Coping with volatility. *Journal of Public Economics 94*, 5 (2010), 421–433.

[42] van der Ploeg, F. Natural resources: Curse or blessing? *Journal of Economic Literature 49*, 2 (2011), 366–420.

[43] van der Ploeg, F., and Venables, A. J. Harnessing windfall revenues: Optimal policies for resource-rich developing economies*. *The Economic Journal 121*, 551 (2011), 1–30.

[44] van der Ploeg, F., and Venables, A. J. Absorbing a windfall of foreign exchange: Dutch disease dynamics. *Journal of Development Economics* (2013).

[45] van der Ploeg, R., Stefanski, R., and Wills, S. Harnessing oil revenues in ghana. *International Growth Centre, Working Paper 12/0034 12* (2012).

[46] van der Ploeg, R., and Venables, A. Natural resource wealth: The challenge of managing a windfall. *Annual Review of Economics 4* (2012), 315–337.

[47] van der Ploeg, R., Venables, A., and Wills, S. Oil and fiscal policy in iraq. *World Bank Iraq Report and OxCarre Policy Paper 13* (2012).

[48] Van Wijnbergen, S. Inflation, employment, and the dutch disease in oil-exporting countries: A short-run disequilibrium analysis. *The Quarterly Journal of Economics 99*, 2 (1984), 233–250.

[49] World Bank. Country economic memorandum: Iraq. Tech. rep., World Bank, forthcoming.

Chapter 1

Eight principles for managing resource wealth

Eight principles for managing resource wealth

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Abstract

Should countries consume, save or invest their resource wealth? Existing literature has established that resource wealth should be saved in a sovereign wealth fund: more if commodity prices are volatile and less if capital is scarce. This paper considers how volatility affects capital-scarce countries for the first time. They should save relatively less in a Volatility Fund, and invest part of the fund domestically. The fund should be built in advance if the windfall is anticipated, and should be a source of income, not a buffer against temporary shocks. Policy should depend on the rate of extraction, as financial markets are not available to smooth income. The real exchange rate should appreciate if capital is scarce, but should be stabilised by investment if capital is abundant. To show this the paper develops a framework that nests a variety of existing results as special cases, which are presented in eight principles. The first three apply to developed, capital-abundant economies: i) Smooth consumption using a Future Generations Fund; ii) Build a Volatility Fund quickly, then leave it alone; and iii) Invest to stabilise the real exchange rate, if capital is abundant. The remaining five apply to developing, capital-scarce economies: iv) Finance consumption and investment with oil if capital is scarce; v) Save less if production is fast and capital is scarce; vi) Use a temporary Parking Fund to improve absorption, vii) Invest part of the Volatility Fund domestically if capital is scarce; and viii) Support private investment.

Keywords: Natural resources, oil, volatility, precautionary saving, capital scarcity, anticipation.

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1 Introduction

Should governments consume, save or invest their natural resource wealth? Existing work suggests that resource wealth should be saved in a sovereign wealth fund and consumed steadily. More should be saved if commodity prices are volatile and less if capital is scarce. This paper is the first to combine the stochastic literature on precautionary saving with recent deterministic literature on capital-scarcity. It shows that capital-scarce countries should save relatively less in a Volatility Fund, and invest more domestically, than capital-abundant ones. The fund should be built in advance if the windfall is anticipated, and be a source of income rather than a temporary buffer to smooth out oil shocks. The real exchange rate should appreciate steadily if capital is scarce, but be stabilised by investment if capital is abundant. To show this the paper develops an intuitive framework that nests a variety of existing results, which we present in eight principles.

Managing natural resources presents a major challenge for policymakers. Using global panel data from 1963 to 2003, Collier and Goderis (2009) estimate that for a country with commodity exports accounting for 35% of GDP, a 10% increase in commodity price leads to a 4 to 5% lower long run level of GDP per head.[1] The cause of this "resource curse" has been attributed to de-industrialization (or Dutch disease, see Corden and Neary, 1982), oil price volatility (Poelhekke and van der Ploeg, 2009), political instability and corruption (Sala-i-Martin and Subramanian, 2003 and Acemoglu et al., 2004), and environmental degradation (such as the case of Nauru in Hughes, 2004).

Addressing this challenge is at the forefront of the international policy debate. The International Monetary Fund (IMF) and World Bank regularly provide guidance to resource exporters (see Barnett and Ossowski, 2003, Baunsgaard et al., 2012, IMF, 2012 amongst others). Non-government organizations also advise policymakers on resource management, including the International Growth Centre and the Natural Resource Charter: recently adopted by the African Union Heads of State steering committee in 2012, and now being implemented in Nigeria.

[1] There is a large body of literature on the resource curse, including Sachs and Warner (1999, 2001), Gylfason et al. (1999), Sala-i-Martin and Subramanian (2003) and Collier and Goderis (2009), and reviewed by van der Ploeg (2011).

The decision to consume, save or invest resource wealth lies at the heart of this debate. For many years the policy consensus was based on Friedman's (1957) permanent income hypothesis: that a temporary income stream should be saved and only the permanent income from interest should be consumed. It assumes that capital is readily available, so domestic investment should not be affected by oil discoveries, as all profitable projects should already be financed by debt (the Fisher, 1930, separation theorem).[2] This underpins why 60% of resource-exporters save wealth abroad in a sovereign wealth fund (as advocated by Davis et al., 2001 and Barnett and Ossowski, 2003).[3]

Recent research is challenging the conventional wisdom. Oil prices are volatile, so there should be some precautionary savings in a Volatility Fund (see the review by Carroll and Kimball, 2008, and van den Bremer and van der Ploeg, 2013). Oil[4] wealth also provides an important source of finance, particularly for capital-scarce developing countries. The rate of return on domestic investment in these countries is likely to be much higher than on offshore government bonds. Thus, some oil wealth should be directed towards investment (van der Ploeg and Venables, 2011, van der Ploeg, 2012). However, the effects of volatile oil prices if capital is scarce have not been studied.

This paper extends past work by considering how volatile oil prices will affect capital-scarce economies.[5] If capital is scarce, then the government should save relatively less in a Volatility Fund. This is because such countries should already be saving a lot to overcome capital-scarcity. The benefit from additional saving will partly be outweighed by the costs of consuming less. Some of the Volatility Fund should also be invested domestically, as saving abroad (or repaying debt) will reduce borrowing costs and make more domestic investments viable. Investing some of the Volatility Fund domestically will equate the rate of return on saving abroad with that on investing at home. These results are presented

[2]Hartwick (1977) first argued that revenues from exhaustible resources should be invested in domestic capital, to offset the declining stock of reserves. In essence, assets below the ground should be replaced by assets above the ground. However, this ignores access to foreign capital.

[3]Rather than using the permanent income hypothesis to justify a sovereign wealth fund, Hsieh (2003) uses payments from Alaska's Permanent Fund to test the hypothesis, and confirms it by showing that households do smooth anticipated income.

[4]For convenience we will use "oil" as shorthand for resources more generally.

[5]"Capital-scarcity" is captured by assuming that there is a debt premium on interest rates. This mechanism is also used to "close" small open economies, inducing stationarity in general equilibrium models facing stochastic shocks (see Schmitt-Grohe and Uribe, 2003).

in Principle 7.

This paper also adds new emphasis to existing results that may have been overlooked. Principles 1,2 and 4 show that policymakers should start preparing for volatility as soon as oil is discovered, even if it is received in the future. Principle 2 shows that the Volatility Fund should be treated as an additional source of permanent income, rather than a buffer that can be consumed when oil prices are low. This stems from the high persistence of oil price shocks. Principle 3 emphasises that the real exchange rate should be stabilised during an oil boom by investment, rather than just accumulating foreign reserves. Principle 5 shows that the speed at which oil is extracted will alter how it is spent, if capital is scarce. This is because financial markets are not available to separate income from expenditure. Principle 8 emphasises that governments in developing countries should also support private capital if it is credit-constrained, to improve the productivity of public capital.

To do this the paper develops an intuitive framework that nests a variety of existing results, which we present in eight principles. Oil price volatility is captured using stochastic control, and its effect on the economy is summarised in two equations describing the expected path of consumption (the Euler equation) and total wealth (the budget constraint). This permits explicit results, and can be neatly explained using phase diagrams. The framework nests as special cases: the permanent income rule (Principle 1, Friedman, 1957); precautionary savings if capital is abundant and prices are volatile (Principle 2, van den Bremer and van der Ploeg, 2013); and saving for development if capital is scarce and prices are certain (Principle 4, van der Ploeg and Venables, 2011). Simple extensions capture the real exchange rate (Principle 3, van der Ploeg and Venables, 2013), and absorption constraints (Principle 6, van der Ploeg, 2012).

1.1 Characteristics of resource windfalls

The eight principles respond to the particular challenge of managing resource revenues. These revenues can be treated as a windfall of foreign currency. They are volatile, though the volatility diminishes as oil is extracted. Oil revenues are also usually anticipated, because of delays between discovery and production. Finally, the majority of resource-dependent economies are developing.

Resource revenues can be treated as a windfall of foreign currency. Commodity transactions, particularly oil, tend to be conducted in US dollars. Resource extraction is not particularly labour intensive, Norway's petroleum sector accounts for 21% of GDP but only 7% of employment, directly or indirectly (Statistics Norway, 2013). Extraction is more capital intensive, however in developing countries this tends to be imported by extraction firms, see Tullow Oil in Ghana. Resource wealth is therefore often modelled as an exogenous windfall (see review by van der Ploeg and Venables, 2012).

Resource revenues are also notoriously volatile, though this volatility is only temporary. Regnier (2007) finds that the prices of crude oil, refined petroleum, and natural gas are more volatile than 95% of products sold by US producers. Commodity price volatility has been used to explain the poor growth of resource dependent economies (see Ramey and Ramey, 1995, Gylfason et al., 1997, Blattman et al., 2007, Aghion et al., 2009, Poelhekke and Poelhekke, 2010 and van der Ploeg, 2010). However, this volatility is temporary, as the exposure to oil prices falls as oil is extracted. This is not particularly important for countries with extensive reserves: Venezuela, Canada and Iraq each have over 100 years left at current rates of production. However, if reserves will be exhausted in the near future then the change in exposure will be important: as in the next twenty years for Brazil, Oman and Brunei (BP, 2012). Policymakers face the challenge of what to do with a Volatility Fund if there is no more volatility.

Oil wealth is also usually anticipated. Once oil is discovered it can take years to build the rigs, pipelines and processing facilities necessary for production. For example, of the 400 offshore oil and gas fields discovered in the UK between 1957-2011, the mean time from discovery to production was 4.5 years (Wills, 2013). Policymakers face the challenge of what to do during the period of anticipation.

Finally, most resource-dependent economies are developing: 55% have a Human Development Index of low or medium (Baunsgaard et al., 2012). These countries face many challenges, in particular limited access to capital. Recent work has shown that some resource wealth should be invested domestically if capital is scarce, as the permanent income hypothesis does not apply (van der Ploeg and Venables, 2011). Policymakers in developing countries still face many challenges, including what to do before the windfall begins, whether to build a Volatility Fund, how big it should be, and what it should invest in.

1.2 The eight principles

This paper presents eight principles for managing natural resource wealth. The principles are presented in a single framework, and differ for developed and developing countries.

The first set of principles apply to developed, capital-abundant economies, such as Australia, Norway, the Netherlands and the UAE. The first priority of these countries should be to save oil wealth for future generations (principle one) and against volatility (principle two). Investment should then be used to stabilise the real exchange rate (principle three). These principles are consistent with current policy in some, but not all, developed resource exporters.

The first principle is to *"Smooth consumption using a Future Generations Fund"*. A Future Generations Fund should be used to borrow before-, save during-, and finance consumption after the period of oil production. The size of the windfall will matter: if it is large or long, then more should be borrowed beforehand and less saved during. The fund is based on the permanent income hypothesis, and should be invested abroad if capital is abundant, as discussed above. This principle underpinned policy advice over the past decade,[6] and provides a useful starting point for the analysis. The challenge of what assets these funds should invest in has been addressed by van den Bremer, van der Ploeg and Wills (2013).

The second principle is to *"Build a Volatility Fund quickly, then leave it alone"*, as oil price shocks are persistent. A Volatility Fund should be used as an additional source

[6]See Barnett and Ossowski, 2003, and IMF, 2012, which note that 39% of resource dependent economies were evaluated against the permanent income hypothesis on IMF visits.

of income if prices are volatile. This based on precautionary savings.[7] It has also been widely studied in a natural resource setting.[8] This principle extends previous work in three ways. First, it focuses on analytical results. Second, it considers anticipated income and finds that the Volatility Fund should be prioritised at the start of the windfall. This includes saving (or borrowing less) before the windfall begins. Third, it shows that the fund should be treated as a source of permanent income, rather than a temporarily buffer. This is because oil price shocks are persistent, so if prices are low then they are likely to stay that way. The Volatility Fund should therefore not be consumed after oil has been extracted, but combined with the Future Generations Fund.

The third principle is to *"Invest to stabilise the real exchange rate, if capital is abundant"*. When some goods are not traded, the extra demand from an oil discovery will drive up their price and attract workers from traded sectors.[9] Investing in the non-traded sector will meet higher demand with higher production, rather than higher prices. This principle extends previous work[10] in two ways. First, by emphasising that a stable real exchange rate is a useful goal for guiding investment policy. Second, by showing that investment can limit Dutch disease if the initial capital stock is low, as fewer workers will need to be attracted from the traded sector. This only applies when capital is abundant. When capital is scarce the real exchange rate should slowly appreciate as capital is accumulated, as found in section 4.4.

These principles support the policies of most, but not all, developed resource exporters. Nineteen of the twenty resource-dependent economies with a Human Development Index of "very high" or "high" have sovereign wealth funds (excluding Ecuador). These include measures to manage volatility, such as Norway's fixed-deposit/fixed-withdrawal rule[11].

[7] Leland (1968) gave one of the first formal treatments of precautionary savings which was followed by a number of papers including Sandmo (1970) and Zeldes (1986), and reviewed in Carroll and Kimball (2008). Kimball (1990a and 1990b) introduced "prudence" as a measure of precautionary savings. For a given utility function, $U(C(t))$, prudence is described by the coefficient of absolute prudence, $CAP(t) = -U'''(C(t))/U''(C(t))$, akin to that for risk aversion.

[8] In a natural resources setting recent work includes van der Ploeg (2010) in two periods, van den Bremer and Van der Ploeg (2013) in continuous time and with numerical simulations, and Bems and de Carvalho Filho (2011), Cherif and Hasanov (2011) and (2013), and Berg et al. (2012) in calibrated DSGE models.

[9] This is referred to as "Dutch disease", see Corden and Neary (1982). This overturns the argument for saving all resource wealth abroad based on the Fisher separation theorem.

[10] Such as van der Ploeg and Venables, 2011, who focus on absorption constraints.

[11] van den Bremer, van der Ploeg and Wills (2013) show that such a rule, with a withdrawal rate slightly

The majority (75%) of these countries have pegged exchange rates, supported by stable non-traded goods prices such as Saudi Arabia since 1990 (Darvas, 2012). Sound policies have allowed countries such as Norway and the UAE to retain manufacturing sectors despite large oil exports (see Fasano, 2002 and Larsen, 2005). Lessons could still be taken from this analysis by countries such as Australia and Canada. These do not have national sovereign wealth funds, relying instead on state-based vehicles in Western Australia and Alberta. They have also seen high inflation in resource-rich states (ABS, 2013), which could be managed by further investment in the non-traded sector.

The next set of principles apply to developing countries with limited access to capital, like Ghana, Iraq and Nigeria. The first priority of developing countries should be to invest domestically (principle four). As capital is scarce, the path of extraction will alter their policy decision (principle five). If investment cannot be absorbed, then a temporary Parking Fund should be used to hold revenues until they can be productively used (principle six). In these countries the Volatility Fund should not all be saved abroad, as domestic returns are high and the fund is an alternative source of income (principle seven). Finally, if private capital is also constrained, then some resource revenues should be used to boost it (principle eight).

The fourth principle is to *"Finance consumption and investment with oil if capital is scarce"*. Developing countries face far higher costs of borrowing than developed ones. Ghana's 2013 issue of ten-year, dollar denominated bonds yielded 8%, whilst ten-year US treasuries yielded 2.5%. At this cost of borrowing, many domestic investments will have a higher realised (as well as social) rate of return than foreign assets. Van der Ploeg and Venables (2012) show that oil revenues provide a cheaper source of finance for these countries, and so should be used to accelerate development. Consumption should also rise relatively more than capital-abundant economies, in the knowledge that future generations will be wealthier.

The fifth principle is to *"Save less if production is fast and capital is scarce"*. This extends previous work by showing that the rate of oil production in capital-scarce countries

below the market rate of return, is consistent with precautionary savings.

is important, as financial markets cannot be used to smooth revenues. Usually extracting a windfall quickly will increase savings. Fast extraction increases the present value, which needs to be shared with future generations. However, fast extraction in developing countries will quickly relax borrowing constraints and reduce the debt burden on future generations. Consumption should therefore jump further, anticipating higher wealth in the future. Policymakers in developing countries should thus consider their development needs when negotiating extraction agreements.

The sixth principle is to *"Use a temporary Parking Fund to improve absorption"*. An oil discovery should boost investment in capital-scarce countries. However, they may have difficulties absorbing a large influx of investment. This is because capital is needed to produce capital (as established by van der Ploeg, 2011 and van der Ploeg and Venables, 2011). A lot of investment may also lead to corruption and poor project choice, as in the island of Nauru (Hughes, 2004). Temporarily saving some funds abroad lets capital accumulate, so that future investment is more productive. As the Parking Fund is temporary, it should be managed separately from the more permanent Future Generations and Volatility Funds.

The seventh principle is to *"Invest part of the Volatility Fund domestically if capital is scarce"*. This shows how oil price volatility affects capital-scarce countries for the first time. Capital-scarce countries should build a smaller Volatility Fund than their capital-abundant neighbours. This is because the Future Generations Fund will be large, so there is less benefit from additional saving against volatility. The Volatility Fund should be treated as an alternative source of income, so some of it should be invested in high-yielding domestic capital. This keeps the rate of return on foreign assets and domestic capital in balance.[12] Ghana recently established a Volatility Fund with its Resource Revenue Management Act (2011). It has priority over the Future Generations Fund, but receives 70% of sovereign wealth payments and does not invest domestically (see Figure 4.1).

The eighth and final principle is to *"Support private investment"*. Some developing

[12]van den Bremer and van der Ploeg (2013) illustrate precautionary savings in a two-period model with capital scarcity. We extend this to many periods, and include investment.

resource exporters have a significant deficit in private capital. The highly centralised economies of Iraq and Libya both have capital stocks over 90% owned by the government (World Bank, 2012 and van der Ploeg, Venables and Wills, 2012). As there are diminishing returns to public capital, and private capital improves its productivity, some oil wealth should be used to alleviate private capital constraints. This can involve financing via the domestic banking sector to "top up" private capital, similar to the UK's Funding for Lending scheme (Churm et al., 2012).

The last five principles provide support for current policy advice, including that provided by the Natural Resource Charter (NRC, 2013). The NRC is a collection of twelve precepts designed to guide policymakers in resource-rich developing countries. Those covering revenue management cover from Precept 7: "promote economic development through high levels of investment" to Precept 10: "facilitate private sector investment". This paper adds further detail to how policy should be conducted during the anticipation phase of a windfall, and on how volatility should be managed in developing countries.

The rest of the paper establishes these principles by proceeding as follows. Section 2 introduces the small, partial equilibrium model at the heart of our analysis, and reduces it to two core equations: the Euler equation and the budget constraint. Section 3 then uses the model to investigate policy in a capital-abundant developed economy, yielding principles one, two and three. Section 4 extends this to the case of a capital-scarce developing economy, adding principles four to eight. Section 5 concludes with suggested extensions.

2 Model

This section introduces the framework underpinning our analysis. The model will be stochastic and in continuous time, to neatly capture the effects of volatility. To simplify the analysis it is partial equilibrium, so all consumption and output can be bought and sold at the world price.[13] The model is extended in sections 3.3 and 4.4 to general

[13] It is also a common simplification, see van der Ploeg 2011, van der Ploeg and Venables, 2012 and 2013, van den Bremer and van der Ploeg, 2013.

equilibrium, to present important intuition that otherwise would be missed. This section will use Ito calculus to reduce the problem to two simple equations: one governing the expected evolution of consumption (the Euler equation); and the other governing the evolution of total assets (the budget constraint). The following sections then use these two equations to present the principles.

Let us begin with a social planner that receives exogenous oil income, $P(t)O^i$, and non-oil income from production, $Y(K^i)$, and must choose how much to consume, $C^i(t)$, how much to invest, $I^i(t)$, in domestic capital, $K^i(t)$, and how much to save in foreign assets, $F^i(t)$. The superscript $i = [A, B, C]$ denotes the three phases the economy passes through after an oil discovery at time $t = 0$: Anticipation, where $O^A = 0$ for $0 \leq t < T_1$; Boom, where $O^B = O$ for $T_1 \leq t < T_2$; and Constant income, where $O^C = 0$ for $T_2 \leq t^C$. The planner's general problem can then be summarised as follows,

$$
\begin{aligned}
J(F^i, K^i, P, t) &= \max_{C(t)} \left[\int_t^\infty U(C^i(\tau)) e^{-\rho(\tau - t)} d\tau \right] && (2.1)\\
&\quad s.t.\\
dF^i(t) &= (r(F^i(t))F^i(t) + P(t)O^i + Y(K^i(t)) - C^i(t) - I^i(t))dt && (2.2)\\
dK^i(t) &= (I^i(t) - \delta K^i(t))dt && (2.3)\\
dP(t) &= \alpha P(t)dt + \sigma P(t)dZ(t) && (2.4)
\end{aligned}
$$

where $J(\cdot)$ is the value function, $U(\cdot)$ is the utility function, ρ is the rate of time preference, $r(F(t))$ is the interest rate faced by the planner and which may depend on the level of assets/debt, δ is the rate of depreciation on domestic capital and Z is a Wiener process. In addition we will make four further assumptions regarding utility, interest rates, output and oil prices.

We will assume that utility exhibits constant absolute risk aversion (CARA), $U(C) = \frac{1}{-a}e^{-aC}$. This utility function is useful because it makes the effect of volatility on consumption very clear (and is especially conducive to the explicit solutions we are after, as noted by Kimball and Mankiw, 1989, Merton, 1990, and Kimball, 1990b). As the absolute degree of risk aversion - and by extension prudence - is constant, the effect of a given level of volatility on consumption will also be constant. It will not depend on the level of consumption itself. This is useful because the level of volatility faced by the planner will change over time, as oil is extracted and less remains exposed to volatile oil prices. With CARA preferences we are able to isolate the effects of these changes in volatility.[14]

Interest rates will have a linear premium on debt, given in equation 2.5. This is a tractable way to capture the capital scarcity facing developing countries.[15] If the country is capital-scarce and indebted, $F(t) < 0$, then its cost of borrowing will rise according to $\omega > 0$. If the country is capital abundant or a net lender, $F(t) \geq 0$, then we assume that the country can borrow or lend freely at the world interest rate, $\omega = 0$. We assume that the world rate of interest is constant. In practice the prices of financial assets that sovereign wealth funds invest in are also volatile. However, they tend to be less volatile than oil prices, and so we make the simplifying assumption that the world rate of interest is constant.

$$r(F(t)) = \begin{cases} r - \omega F(t) & \text{for } F(t) < 0 \\ r & \text{otherwise} \end{cases} \tag{2.5}$$

Output will be produced using domestic capital and a fixed stock of labour, and sold at a constant world price. We assume that domestic firms produce a single internationally traded good using technology, capital and labour, $Y(K(t)) = AK(t)^{\alpha}\bar{L}^{1-\alpha}$, where technology is constant and the supply of labour is fixed, $\bar{L} = 1$. This good will be sold at

[14] In contrast, the common "constant relative risk aversion" (CRRA) utility function exhibits diminishing absolute risk aversion. So, the effect of oil price volatility in our model could fall for two reasons: because oil is being extracted, and because consumption is rising. This makes the results less clear, and is less amenable to explicit solutions. Another way to think of it is that that consumption is affected by the size of a volatile income stream under CARA, but by the share in total wealth of a volatile income stream under CRRA.

[15] A similar debt premium is used for resource-rich developing countries by van der Ploeg and Venables, 2011 and 2012, and van den Bremer and van der Ploeg, 2013. A debt premium on interest rates is also commonly used as a tool to remove the unit-root in models of a small-open economy (Schmitt-Grohe and Uribe, 2003).

the constant world price, $P^* = 1$. It is also the same as the single good consumed by the social planner, $C^i(t)$. Assuming a constant world price is a useful simplification, and it lets us begin by ignoring the general equilibrium effects of changes in the relative price of domestic goods. This also makes the analysis consistent with work by van der Ploeg and Venables (2012) and van den Bremer and van der Ploeg (2013). In sections 3.3 and 4.4 we will take brief detours into general equilibrium, introducing a traded and a non-traded good to see how the decision to consume, invest and save changes.

Oil prices will follow a geometric Brownian motion with zero drift, $\alpha = 0$. In practice oil price shocks are very persistent with little drift.[16] By setting the drift of oil prices to zero we make the analysis simpler and ensure the present value of the income stream is finite. As price shocks are persistent, a shock today will affect the price in all future periods. Positive or negative shocks are equally likely at any point in time. This means that the volatility of oil prices increases exponentially into the future.

Now we turn to solving the social planner's problem. Dropping superscripts for simplicity, the Hamilton-Jacobi-Bellman equation for the problem in 2.1 to 2.4 is,

$$\max_C \left[U(C(t))e^{-\rho t} + \frac{1}{dt} E_t[dJ(F, K, P, t)] \right] = 0 \tag{2.6}$$

where Ito's lemma can be used to express the right hand term as,

$$\begin{aligned} \frac{1}{dt} E_t[dJ(F, K, P, t)] &= J_F(r(F)F + PO + Y(K) - C - I) \\ &\quad + J_K(I - \delta K) + J_t + \tfrac{1}{2} J_{PP}\sigma^2 P^2 \end{aligned} \tag{2.7}$$

The first order conditions with respect to consumption, investment, foreign assets, and domestic capital can be summarised in the following two equations, derived in Appendix

[16] see Bems and de Carvalho Filho (2011) and van den Bremer and van der Ploeg (2013) who estimate $\alpha = 0.009$.

A, which describe the evolution of the marginal utility of foreign assets and equating the marginal product of capital with the interest paid on foreign borrowings,

$$\begin{aligned} \frac{E_t[dJ_F]/dt}{J_F} &= -(r - 2\omega F) &(2.8) \\ r - 2\omega F &= \alpha A K^{\alpha-1} - \delta &(2.9) \end{aligned}$$

Domestic capital and foreign assets will be tied together by an optimal asset allocation condition, equation 2.9. This condition intuitively states that the social planner should allocate total assets so that the marginal benefit of paying down foreign debt (reducing both the stock of debt and the interest that is paid on it), equals the marginal benefit of investing more in domestic capital (boosting output). So, if consumption differs from income at any time, both domestic capital and foreign assets should increase or decrease together. Overall it can be seen that domestic capital increases more rapidly than foreign assets, so that its share in total assets, $\bar{S} = F + K$, will rise.

The value function must satisfy the optimality condition in 2.10. This is derived from the first order conditions in Appendix A,

$$-\tfrac{1}{a}J_F + \{J_t + J_F(r(F)F + PO + Y(K) - C - I) + J_K(I - \delta K) + \tfrac{1}{2}J_{PP}\sigma^2 P^2\} = 0 \quad (2.10)$$

Finding an explicit solution for the value function is not straight-forward, for three reasons. The first is the risk premium on debt. The second is the non-linear production function. The third is that the exposure to oil price volatility varies as oil is extracted. It is relatively straightforward to find the value function when volatile income is constant in size (under CARA preferences), or is constant as a share of total wealth (under CRRA preferences as discussed in Merton, 1990 and Chang, 2004). However, it is much more challenging when volatile income changes over time. In the next section we arrive at an explicit solution for the value function when oil prices are certain. When oil prices are uncertain we will take a different tack.

Rather than finding an explicit solution for the value function, we can still gain a lot of intuition by examining the dynamics of consumption and total assets, $\bar{S} = F + K$. These are given in equations 2.11 and 2.12 and derived in Appendix A. They are expressed in terms of linear deviations from a steady state, $S = \bar{S} - S^*$, where capital scarcity is overcome and oil income is exhausted, $F^* = 0$ and $O^* = 0$,

$$\begin{aligned} \tfrac{1}{dt}E[dC^i(t)] &= (r - \rho - \tfrac{2\omega}{a} f S^i(t)) + \tfrac{1}{2} a P(t)^2 C_P^i(t)^2 \sigma^2 && (2.11) \\ \tfrac{1}{dt} dS^i(t) &= r S^i(t) + P(t) O^i - C^i(t) + C^* && (2.12) \end{aligned}$$

The expected dynamics of consumption are given by the Euler equation in 2.11. The first term describes the typical trade-off between consuming today, or saving at rate r for future consumption, which is discounted at rate ρ. In addition, the desire to save will depend on total assets through the risk premium on debt, $F^i(t) = fS(t)$. This is based on a linear approximation around the steady state, $f = (1 + \frac{\omega}{\alpha(1-\alpha)}(\frac{\alpha}{r+\delta})^2)^{-1}$. The second term describes how oil price volatility affects the expected change in consumption. Higher volatility, σ, will delay consumption from today until tomorrow, in line with precautionary savings. The severity of this effect depends on the marginal propensity to consume from a change in the oil price, $C_P^i \equiv \partial C^i(t)/\partial P(t)$. This will depend on the size of the remaining windfall, as seen in the next section.

The expected dynamics of total assets are given by the budget constraint in 2.12. This uses the relationship tying foreign assets to capital in equation 2.9, and is linearised around the steady state level of assets, S^*. The term $rS^i(t)$ describes the rate of return to total assets near this steady state. This can actually be split into two components, as seen in Appendix A. The first is the rate of return earned on foreign assets, rf. The second is the marginal product of capital less depreciation, $(Y'(K^*) - \delta)(1 - f) = r(1 - f)$. Note that we do not assume that the share of foreign assets and domestic capital is constant.

Rather, we assume that the share of capital in total assets increases linearly, rather than the non-linear relationship in equation 2.9. Also note that linearising does not remove the effect of the risk premium, ω. It will still appear in the Euler equation so the incentive to save, $\dot{C}^i(t) > 0$, will still increase with the level of debt, $F^i(t) = fS(t)$.

The rest of this paper will involve using these two equations to demonstrate the eight principles, both explicitly and with the use of diagrams. There will also be some short detours into general equilibrium in sections 3.3 and 4.4.

3 Oil discoveries in a developed country

This section considers how developed countries should manage their resource wealth. Of the forty-five countries identified by the IMF as resource-dependent, five have a Human Development Index (HDI) of "very high": Bahrain, Brunei, Norway, Qatar and the UAE (IMF, 2012). A further fifteen have an HDI of "high", while other developed countries have extensive natural resources but are not classed as dependent, including Australia, Canada, Israel, the Netherlands, the UK and the US. Despite their many differences these countries share a similar trait: ready access to international capital.

This section shows that the first consideration for developed, capital-abundant economies should be to share the windfall across generations. This will involve saving in offshore Future Generations and Volatility Funds, with the latter being prioritised in the early days of the windfall (principles one and two). There should also be investment in non-traded sectors like retail, health care and education, to limit inflation and stabilise the real exchange rate in the medium to long term (principle three).

These results are generally consistent with what we observe in practice. Of the developed countries classed as resource-dependent, all but Ecuador have established offshore sovereign wealth funds. These vary by size and use, from Norway's fixed-deposit/fixed-withdrawal rule to Russia's focus on consuming a moving average of past revenues. In addition, 75% of these economies have pegged exchange rates, which are supported by stable non-traded goods prices. The lessons from this section tend to apply more to non-dependent economies like Australia and Canada. These have not established national sovereign wealth funds and rely instead on state-based vehicles. Inflation in places such as Perth, Western Australia and Edmonton, Alberta may be limited by further localised investment in the non-traded sector.

The section proceeds as follows. Section 3.1 establishes the first principle in the base case when capital is abundant and oil prices are certain. Section 3.2 establishes the second principle by letting the oil price fluctuate. Section 3.3 presents the third principle by introducing a non-traded sector and general equilibrium. This extension to general equilibrium is revisited in section 4.4.

3.1 The case for a Future Generations Fund

This section presents the first principle: *"Smooth consumption using a Future Generations Fund"*. This is based on the permanent income hypothesis and underpins the use of sovereign wealth funds in almost all resource-dependent economies. Investment will not be affected by an oil discovery, as all profitable projects should already be undertaken. We assume capital is abundant, oil prices are constant and goods are freely traded as a baseline. The analysis is based on two dynamic equations that give an explicit solution, which will be used in the following sections.[17]

The economy is governed by the dynamics of consumption and assets, equations 3.1 and 3.2. These follow from equations 2.11 and 2.12, if capital is abundant, $\omega = 0$, and prices are certain, $\sigma = 0$. Capital will be constant if interest rates are constant and goods can be freely bought and sold at the world price, $K = K^* = (\frac{r+\delta}{\alpha A})^{1/(\alpha-1)}$ from equation 2.9. Any change in total assets will be caused by foreign assets, so $f = 1$ and $S^i(t) = F^i(t)$. Therefore we concentrate on foreign assets in equation 3.2, which holds without any need for approximation. Note the notation $\dot{C}^i(t)$ rather than $\frac{1}{dt}E[dC^i]$, to make clear that we are treating the expected change in each variable as deterministic.

[17]An alternative solution based on explicitly solving the value function in equation 2.6 is presented in Appendix B.

$$
\begin{aligned}
\dot{C}^i(t) &= \tfrac{1}{a}(r-\rho) & (3.1)\\
\dot{F}^i(t) &= rF^i(t) + PO^i - C^i(t) + C^* & (3.2)
\end{aligned}
$$

The path of consumption and assets is found by solving this linear system. For simplicity we set the time rate of preference to equal the risk-free rate, $r = \rho$. Taking the time derivative of 3.2 and substituting in 3.1 gives the single differential equation, $\ddot{F}^i(t) - r\dot{F}^i(t) = 0$. This has the general solution,

$$
\begin{aligned}
C^i(t) &= ra_2^i + PO^i + C^* & (3.3)\\
F^i(t) &= a_1^i e^{rt^i} + a_2^i & (3.4)
\end{aligned}
$$

The parameters a_1^i and a_2^i will depend on the phase the oil boom is in: $i = [A, B, C]$, Anticipation, Boom or Constant income. They are found by imposing conditions at the start and end of each phase. The first condition is that each phase begins with the assets at the end of the last phase, $F(0) = a_1^A + a_2^A$, $F^A(T_1) = a_1^B + a_2^B$ and $F^B(T_2) = a_1^C + a_2^C$. The second condition is that consumption is chosen to move smoothly between phases. This requires a constant level of assets at the end of the Boom, to satisfy the transversality condition. Moving recursively through each phase these conditions give,

$$
\begin{aligned}
a_1^C &= 0 &;\quad a_2^C &= F(T_2)\\
a_1^B &= POe^{-r(T_2-T_1)}/r &;\quad a_2^B &= F(T_1) - a_1^B \qquad (3.5)\\
a_1^A &= -PO(e^{-rT_1} - e^{-rT_2})/r &;\quad a_2^A &= F(0) - a_1^A
\end{aligned}
$$

The effect of oil prices on consumption will depend on the remaining value of oil, given in equation 3.6.[18] This is found by combining equations 3.3 and 3.5. The marginal propensity to consume from a change in oil prices, $C_p^i(t)$, will be important in the following

[18]This is verified by solving the value function directly in Appendix B.

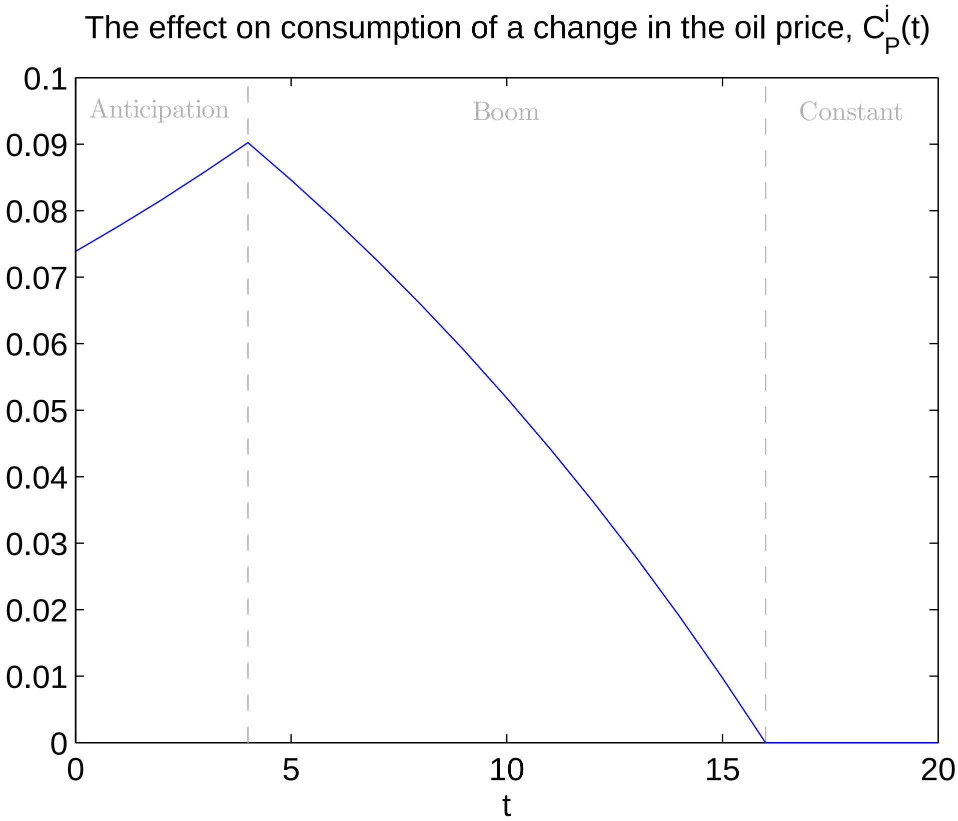

Figure 3.1: Consumption is most affected by oil price changes at the start of the Boom, when the present value of future revenues is highest. Consumption is less affected as oil is extracted and less remains exposed to price changes.

sections. It dictates how oil price volatility affects consumption. It is difficult to derive in the stochastic model, so will be approximated with the deterministic case (see also van den Bremer and van der Ploeg, 2013). It captures the changing effect of oil price volatility, increasing during the Anticipation phase and decreasing during the Boom, as illustrated in Figure 3.1 using the calibration in Appendix C.

$$\begin{aligned} C_P^C(t) &= 0 && \text{for} \quad T_2 \leq t \\ C_P^B(t) &= O(1 - e^{-r(T_2 - t)}) && \text{for} \quad T_1 \leq t < T_2 \\ C_P^A(t) &= O(e^{-r(T_1 - t)} - e^{-r(T_2 - t)}) && \text{for} \quad 0 \leq t < T_1 \end{aligned} \tag{3.6}$$

We can now present the first principle of managing resource windfalls,

Principle 1. ***"Smooth consumption using a Future Generations Fund"***

i) A social planner facing a certain but temporary windfall, which can freely borrow *at the world rate of interest, should smooth consumption based on the rates of return and time preference, $\dot{C}^i(t) = \frac{1}{a}(r-\rho)$ for all t.*

ii) Consumption will be a constant proportion of total wealth: the sum of foreign assets, oil wealth and steady state consumption, $C^i(t) = rW(t)$ where $W(t) = F(t) + V(t) + \frac{1}{r}C^*$ and $r = \rho$.

iii) This will involve borrowing before the windfall, saving during the windfall in a Future Generations Fund, and consuming the interest earned after the windfall, $\dot{F}^A(t) < 0, \dot{F}^B(t) > 0, \dot{F}^C(t) = 0$.

iv) Less will be saved from a long windfall, and more borrowed beforehand, $\frac{\partial}{\partial T_2}\dot{F}^i(t) < 0$ for $t < T_2$.

v) Investment will be independent of any oil discovery if capital is abundant and goods are freely traded, consistent with the Fisher separation theorem, $K = K^*$.

Proof. See Appendix 1. □

The first principle makes the case for establishing a sovereign wealth fund, to replace the assets below the ground with assets above it, see the blue lines in Figure 3.2. The Hartwick (1977) rule is based on similar reasoning, replacing oil reserves with productive capital. However, this principle (part v) shows that capital should be constant, as all profitable investments can already be financed with abundant capital. Thus, oil income should be saved abroad in a sovereign wealth fund, and borrowing should smooth consumption before production begins.

Consumption will be smooth, but may not be constant. Part i) shows that if the rate of interest is higher than the discount rate then consumption will grow, $r > \rho$. The amount depends on the degree of intertemporal substitution, a, which is also the coefficient of absolute risk aversion.[19] In contrast if the policymaker is impatient (the discount rate is high), then more will be consumed in the early years of the windfall. This may happen

[19]This is a result of the constant absolute risk aversion utility function. An alternative is to use Epstein-Zin preferences, which separate intertemporal substitution from risk aversion.

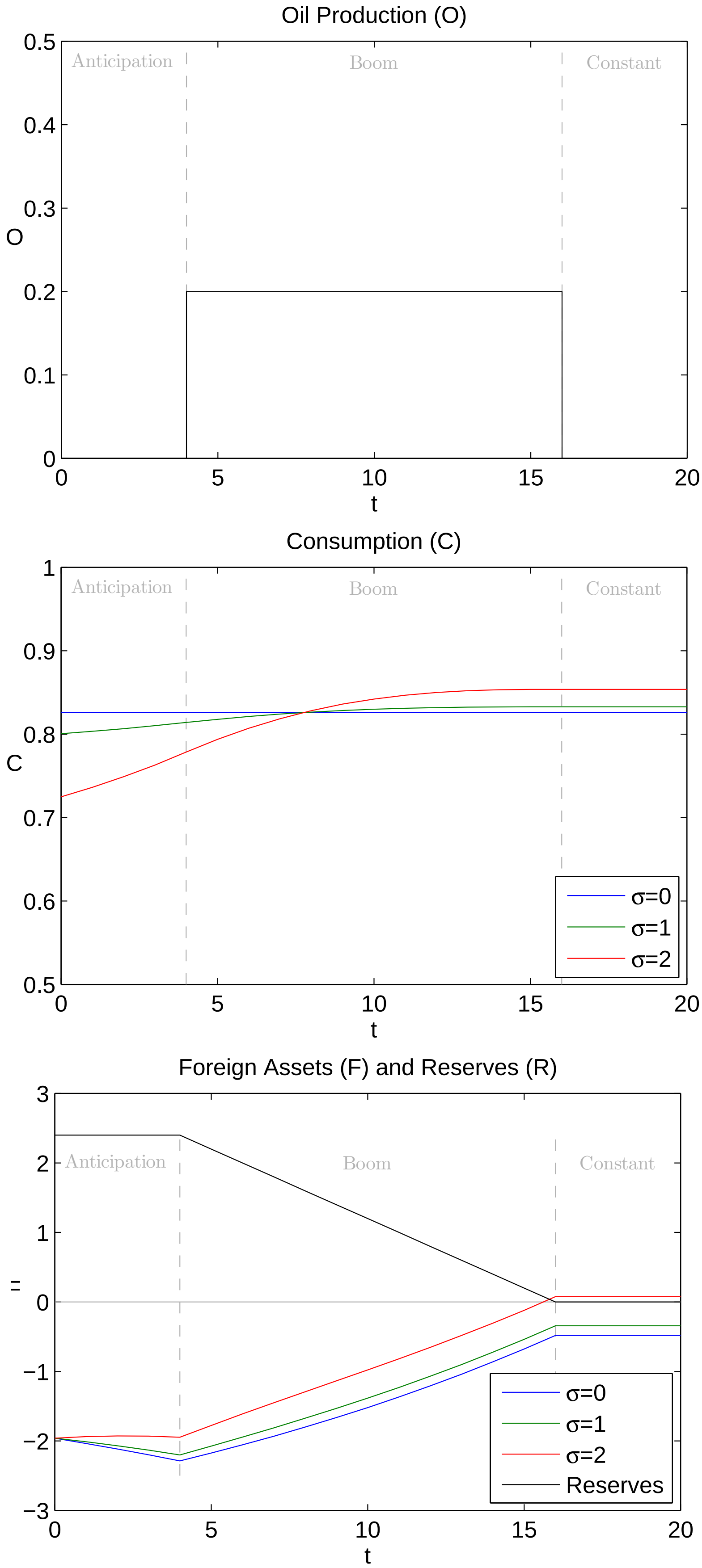

Figure 3.2: An oil boom should lead to borrowing before it begins, saving during, and consuming interest after it ends. Volatility will lead to additional precautionary savings (from equations, 3.3, 3.4, 3.9, and 3.10).

if they value future generations less than current generations, are concerned with the possibility of being removed from office, or expect output to grow.

Longer windfalls will involve less saving. This is because the income produced by a long windfall will be closer to its permanent income. In the limit, if a constant windfall becomes permanent then it makes sense to consume all income as it is received.

The effects of an oil discovery can also be illustrated using the phase diagram in Figure 3.3. The blue lines are the same as those in Figure 3.2, but directly compare the movement of consumption and foreign assets. The solid line illustrates where consumption exactly equals non-oil income, so assets are constant before and after the windfall, $\dot{F}^{A,C} = 0$ and $O^{A,C} = 0$ in equation 3.2. The economy starts on this line, at X_0. On discovering oil consumption jumps up to X_0^A as households become wealthier. Consumption is higher than non-oil income, so assets are consumed and the economy moves to point X_1at the beginning of the boom, $t = T_1$. During the boom households also receive oil income, so the solid line moves up to the dotted line, $\dot{F}^B = 0$ and $O^B > 0$ in equation 3.2. Consumption is below this level, so assets are accumulated until the end of the boom, $t = T_2$. Consumption will have been chosen so that the economy lands on the initial steady state line at this time, point X_2.

3.2 The case for a Volatility Fund

This section presents the second principle, *"Build a Volatility Fund quickly and then leave it alone"*. The Volatility Fund is motivated by precautionary savings, and has been established in previous literature on natural resources.[20] It should be prioritized at the start of the windfall, when oil price risk is greatest. The fund will be larger when the windfall is long or prices are more volatile. The Volatility Fund should be treated as a source of income, rather than as a "shock absorber", as oil price shocks are persistent. An example is Norway's fixed withdrawal rule, which is consistent with precautionary savings for certain withdrawal rates (see van den Bremer, van der Ploeg and Wills, 2013).

[20]see Kimball (1990), van der Ploeg (2010), van den Bremer and Van der Ploeg (2013), Bems and de Carvalho Filho (2011), Cherif and Hasanov (2011) and (2013), and Berg et al. (2012).

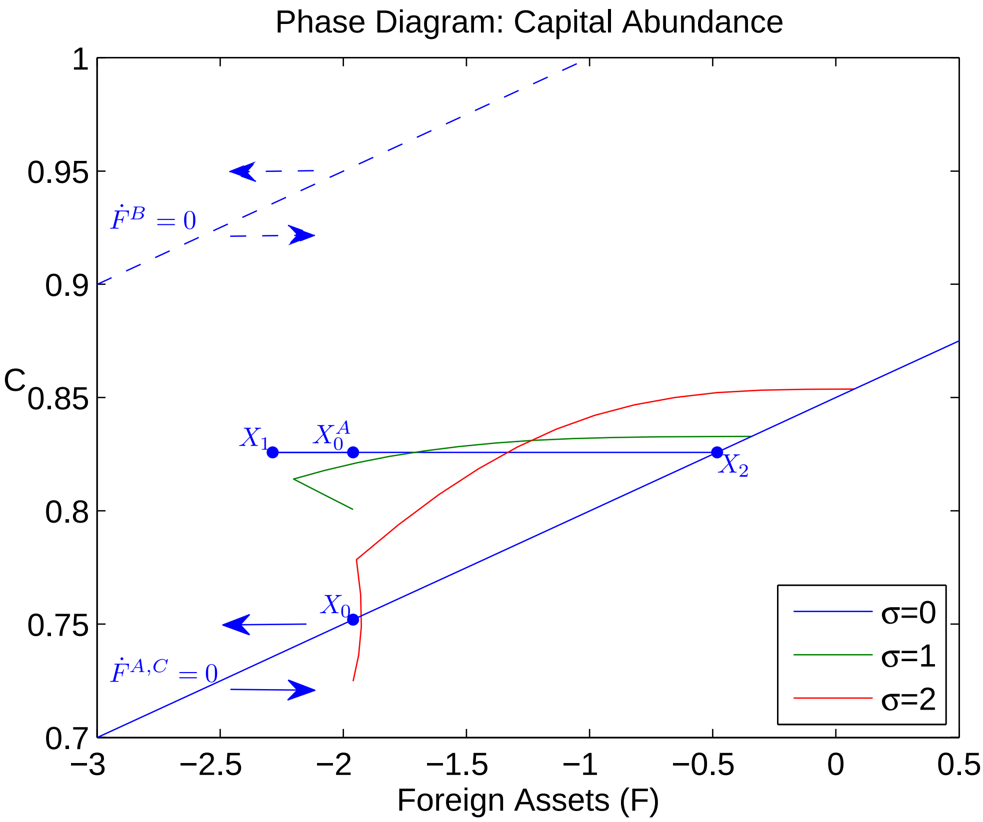

Figure 3.3: An anticipated windfall in a capital-abundant economy. Before and after the windfall the movement of consumption and assets is dictated by the solid blue line and arrows, $\dot{F}^{A,C} = 0$. During the windfall it is dictated by the dashed blue line and arrows, $\dot{F}^B = 0$. If prices are certain the economy will follow the path $[X_0, X_0^A, X_1, X_2]$. If prices are volatile (green and red), then consumption will also rise over time.

Consumption and assets will satisfy the two equations 2.11 and 2.12, when capital is abundant and oil prices are volatile. This follows from equations 2.11 and 2.12 with $\omega = 0$, $f = 1$ as capital remains constant, $i = [A, B, C]$ and $C_P^i(t)$ as defined in equation 3.6,

$$\dot{C}^i(t) = \tfrac{1}{a}(r - \rho) + \tfrac{1}{2}aP^2 C_P^i(t)^2\sigma^2 \tag{3.7}$$

$$\dot{F}^i(t) = rF^i(t) + PO^i - C^i(t) + C^* \tag{3.8}$$

These two equations describe the expected dynamics of consumption and assets, and so we treat them as deterministic. The effects of volatility are still captured in the second term of the Euler equation, 3.7. Volatility will increase the change in consumption, all else being equal, giving rise to more savings in the near term, to fund more consumption in the future. The adjustment for volatility will decrease with time, as $C_P^i(t) \to 0$ in equation 3.6.

The paths of consumption and assets are found by solving this system of equations. The two equations can be summarised by a single differential equation $\ddot{F}^i(t) - r\dot{F}^i(t) = \frac{1}{2}aP^2\sigma^2 C_P^i(t)^2$, as before.[21] The general solution to this is,

$$F^i(t) = a_1^i e^{rt^i} + a_2^i + F_V^i(t) \tag{3.9}$$

$$C^i(t) = ra_2^i + PO + Y + C_V^i(t) \tag{3.10}$$

where $C_V^i(t) = rF_V^i(t) - \dot{F}_V^i(t)$. To aid our interpretation we choose a "particular solution", $F_V^i(t)$, so that a_1^i, a_2^i are the same as before, in equation 3.5. This lets us split the assets accumulated by the social planner into two funds. The Future Generations

[21] This is a is a non-homogeneous, second-order linear differential equation. If $F^i(t)$ and $F_V^i(t)$ are both solutions, then $\ddot{f}^i(t) - r\dot{f}^i(t) = 0$ for $f^i(t) \equiv F^i(t) - F_V^i(t)$ and the problem collapses to that discussed in section 3.1. The "particular solution" $F_V^i(t)$ can be found by the methods of undetermined coefficients or variation of parameters (see Robinson, 2004).

Fund, $F^i_{FG}(t) = a^i_1 e^{rt^i} + a^i_2$, holds the funds that are used to smooth consumption over time, in accordance with principle one. It need not be positive, such as borrowing during the Anticipation phase of a windfall. The Volatility Fund, $F^i_V(t)$, then captures all the funds that are accumulated in response to oil price volatility. The volatility fund will begin at zero before the oil discovery is announced, $F^A_V(0) = 0$, and will always be positive in expectation, $F^i_V(t) \geq 0$ for all t. This brings us to the second principle of managing resource wealth,

Principle 2. ***"Build a Volatility Fund quickly and then leave it alone"***

Build a Volatility Fund quickly:

i) A social planner facing a volatile temporary windfall, who can freely borrow at the world rate of interest, should engage in precautionary savings and build up a "Volatility Fund", in addition to the "Future Generations" fund in principle 1, $F^i_V(t) > 0$ *for all* $t > 0$.

ii) The Volatility Fund should receive relatively more savings during the early years of the windfall, including during the Anticipation phase, $\dot{F}^A_V(t), \dot{F}^B_V(t) > 0$ *and* $\ddot{F}^A_V(t), \ddot{F}^B_V(t) < 0$ *for* $t < T_2$.

Then leave it alone:

iii) The assets in the Volatility Fund should not be consumed in the presence of a persistent negative shock to the oil price. Only interest should be consumed, $\frac{\partial}{\partial P_0} F^i_V(0) = 0$.

iv) As oil is extracted, the need for a Volatility Fund will diminish, $\dot{C}^B(t) \to 0$ *as* $t \to T_2$.

v) Any funds remaining in the Volatility Fund should be saved, and only the permanent income consumed, $\dot{F}^i_V(t) \geq 0$ *for all* $t \leq T_2$.

Proof. See Appendix 2. □

Principle two begins by emphasizing that the Volatility Fund should be prioritised at the start of the windfall, when oil price exposure is greatest. Consumption from an oil windfall will depend on two effects: the wealth effect and the precautionary effect. The wealth effect increases consumption, as an oil discovery increases lifetime wealth. The precautionary effect reduces consumption, as oil also makes income more volatile.

The Volatility Fund should be prepared before oil production even begins. During the Anticipation phase the Future Generations Fund will go into debt, borrowing against future income because of the wealth effect. This will be offset by the Volatility Fund, built because of the precautionary effect as illustrated in Figure 3.4.[22] The volatility effect is strongest near the start of the windfall when its present value is highest, as seen in Figure 3.1. If the windfall is particularly long or volatile then CARA preferences permit the counter-intuitive result that consumption can fall below its pre-oil level, as seen in red in Figure 3.3. This follows from two particular characteristics of CARA preferences. First, under these preferences the effect of a given level of volatility on consumption is constant, regardless of the level of consumption. Second, CARA preferences also do not exclude the (nonsensical) possibility of consumption falling below zero, $C < 0$.[23] So, if the oil price is very volatile then the volatility effect may outweigh the wealth effect and marginal consumption from an oil discovery can be negative. This would not occur under other utility functions (such as CRRA). However, the assumption of CARA preferences remains crucial in making the model tractable, as discussed in Section 2.

Both the Volatility and the Future Generations Funds should accumulate during oil production. The Volatility Fund will receive relatively more at the start of the windfall, when the exposure to oil price volatility is highest. If the windfall is long or volatile then the Volatility Fund may receive more than the Future Generations Fund in absolute value. The need to save against future shocks will then fall as oil is extracted. After the windfall both the Volatility and Future Generations Funds can be combined, and the interest they earn should finance consumption in perpetuity.

Principle two also emphasises that the Volatility Fund should be treated as a source of income, rather than as a "shock absorber". It may be tempting for policymakers to

[22]In practice this may involve simply borrowing less in the Future Generations Fund, rather than borrowing in one fund, whilst saving in another.

[23]This can only be prevented by choosing an appropriate values for the parameters and the starting level of consumption.

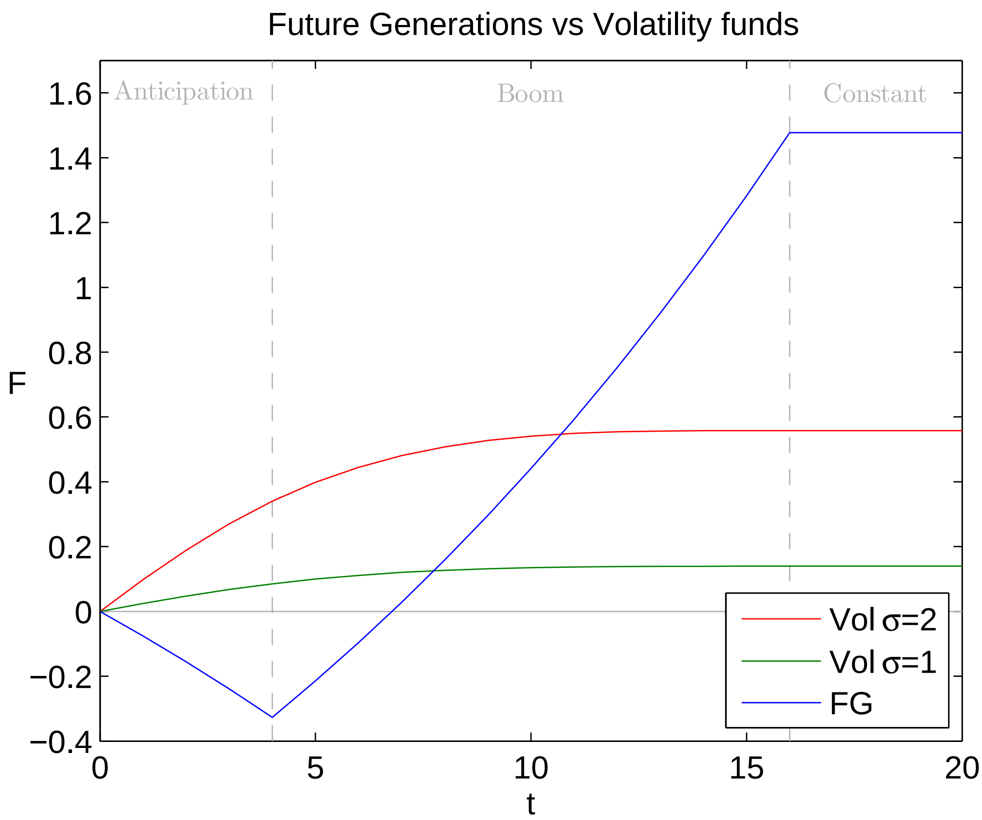

Figure 3.4: The Future Generations and Volatility Funds from the beginning of each stage of the windfall. The Volatility Fund will receive relatively more savings during the early years of the windfall, and may receive more savings in absolute value if the windfall is particularly long or volatile.

dip into the Volatility Fund when oil prices are low. However, oil price shocks are very persistent, so if prices are low today then they are also likely to be low in the future.[24] Instead, when oil prices fall the social planner should re-evaluate the planned path for future consumption. In this case the income from a Volatility Fund will finance more consumption than would be possible otherwise. This is illustrated in the example in Figure 3.5, which compares consumption and assets with and without a Volatility Fund, after a negative shock to the oil price.

The need for a Volatility Fund will diminish over time, but its capital should not be consumed. The exposure to oil price volatility falls as oil is extracted. Policymakers may see this as an opportunity to consume the fund, once the threat of volatility has passed. However, principle two shows that the income from the Volatility Fund compensates future generations for the risk they have had to bear.[25] The Volatility and Future Generations Funds therefore behave the same way after the windfall ends, suggesting that they can be combined.

The role of the Volatility Fund is also illustrated in the green and red lines of the phase diagram in Figure 3.3, and the time paths in Figure 3.2. Volatility causes consumption to jump less when oil is discovered ($\sigma = 1$) and can even fall ($\sigma = 2$), because of the precautionary effect. Consumption will then rise steadily over time, both before and during the windfall, according to equation 3.7. The Future Generations Fund still goes into debt before the windfall, and accumulates during it. This is offset by the Volatility Fund, which grows throughout. At the end of the windfall the economy ends on the steady state line, $\dot{F}^{A,C} = 0$, above X_2. The extra assets from the Volatility Fund will then finance higher consumption in perpetuity.

3.3 The case for investing to stabilise the real exchange rate

This section presents the third principle, *"Invest to stabilise the real exchange rate, if capital is abundant"*. Policy advice has been influenced by the separation result, that

[24]see Bems and de Carvalho Filho (2011) and van den Bremer and van der Ploeg (2013) who estimate $\alpha = 0.009$.

[25]The social planner prefers certain to uncertain income according to Jensen's inequality.

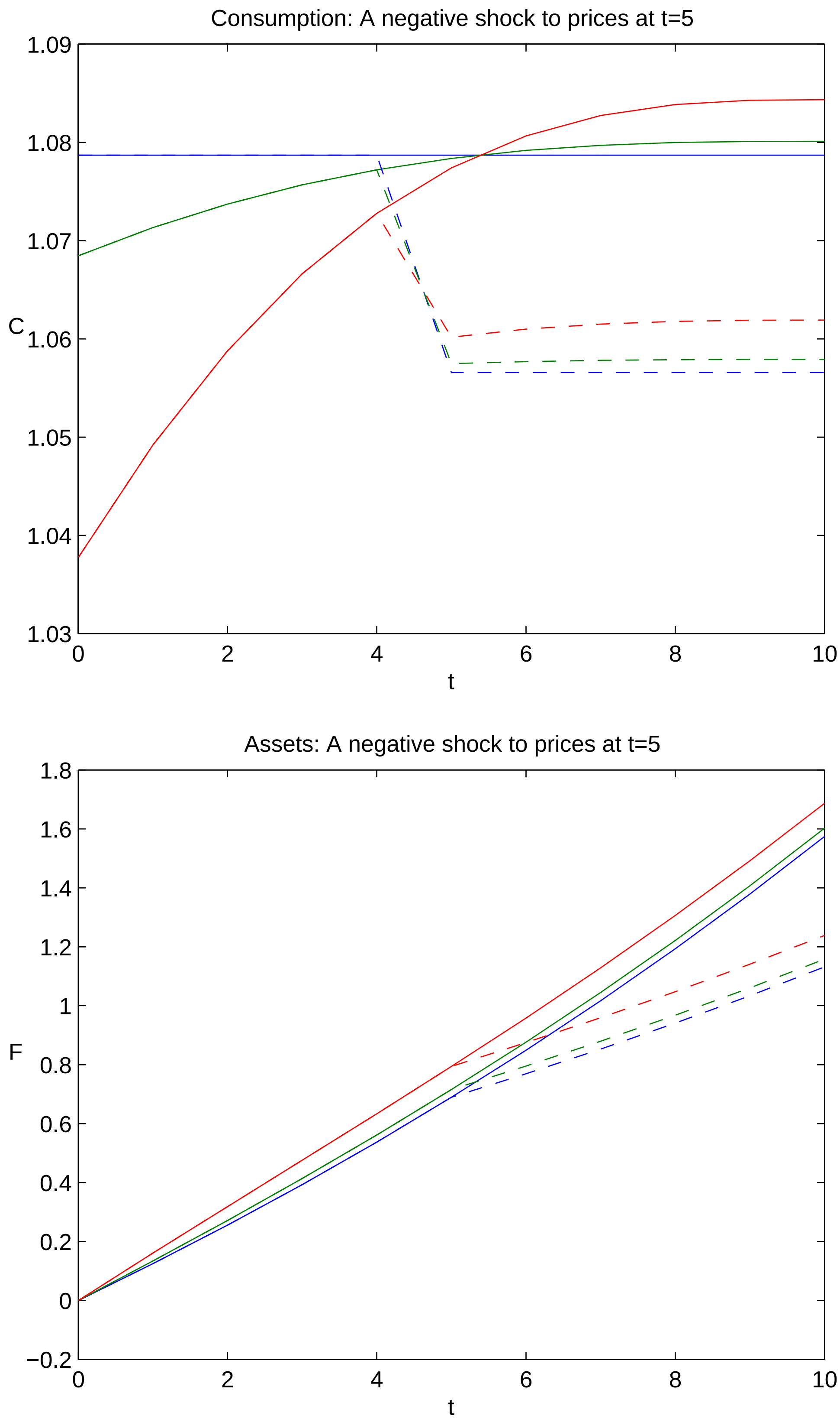

Figure 3.5: Building up a Volatility Fund buffers consumption against negative price shocks. The expected path of consumption and assets before (solid) and after (dotted) an unanticipated shock to the oil price, from $P = 1$ to $P = 0.5$ at $t = 5$.

an oil discovery should not affect investment in capital abundant economies (for example Barnett and Ossowski, 2003, and IMF, 2012). However, this will not be true if some goods are non-traded. Demand for non-traded goods will drive up their relative price (a real appreciation),[26] and attract workers from the traded sector - causing it to shrink (Dutch disease). Policymakers should stabilise the real exchange rate by investing in the non-traded sector. While this may prevent a real appreciation, it will not necessarily prevent Dutch disease. Some resource exporters have maintained very stable real exchange rates for decades, such as Saudi Arabia since 1990 (Darvas, 2012).

To more accurately capture investment in capital-abundant economies we take a short detour into general equilibrium. Previous sections focused on the allocation of resources over time. This section focuses on the allocation at any given point in time. The social planner will now split consumption between goods to minimise expenditure,

$$\begin{aligned} \min_{C_N, C_T} P(t)C(t) &= P_N(t)C_N(t) + P_T(t)C_T(t) \qquad (3.11) \\ \text{subject to,} & \\ C(t) &= C_N^{a_N}(t)C_T^{a_T}(t) \cdot a_N^{-a_N} a_T^{-a_T} \qquad (3.12) \\ P(t) &= P_N^{a_N}(t)P_T^{a_T}(t) \qquad (3.13) \end{aligned}$$

Spending on each good will be a constant share of total consumption, $P_j(t)C_j(t) = a_j P(t)C(t)$ for $j = N, T$. The traded good will be freely bought and sold at the world price, $P_T = 1$. The non-traded good's price (which is also the real exchange rate, $P_N = e$) will adjust to equate domestic supply and demand, $C_N(t) = Y_N(t)$. Each good will be produced competitively using technology, public and private capital, and labour. Public and private capital will be freely available at the world rate of interest, r, and the wage will adjust to clear the labour market, $\bar{L} = L_N(t) + L_T(t)$. They are chosen to maximise profits,

[26] Such as the rapid appreciation of Australia's real exchange rate from 2002-2008 (Darvas, 2012).

$$\max_{K_{Gj}, K_{Pj}, L_j} \pi_j(t) = P_j(t)Y_j(t) - rK_{G,j} - rK_{P,j} - wL_j \tag{3.14}$$

where

$$Y_j(t) = AK_{G,j}^{\gamma_G} K_{P,j}^{\gamma_P} L_j^{1-\gamma_G-\gamma_P} \tag{3.15}$$

This gives rise to the third principle for managing resource windfalls,

Principle 3. ***"Invest to stabilise the real exchange rate, if capital is abundant"***

i) When there is both a traded and a non-traded sector, an oil discovery should increase investment in the non-traded sector, and decrease investment in the traded sector, with the aim of stabilising the real exchange rate, $e = e^*$.

ii) This is not just a reallocation of capital. The total capital stock can increase or decrease, depending on the production technology of each good.

iii) During the Anticipation phase this will involve borrowing to finance both consumption and investment in the non-traded sector.

iv) Investing in the non-traded sector may limit the decline of the traded sector (Dutch disease) when the initial capital stock is low, but immigration is more effective.

Proof. See Appendix 3 □

Dutch disease describes the shrinking of the non-oil traded sector after an oil discovery, and is often linked to a real appreciation. When oil is discovered there will be an increase in demand, as noted in principle one. This will be spread across both traded and non-traded goods. Traded goods can be imported, but non-traded goods must be produced domestically. Corden and Neary (1982) attribute the shrinking of the traded sector to both a resource movement and a spending effect. The resource movement effect describes capital and labour moving from the traded to the non-traded sector. The spending effect describes the increase in the price of non-traded goods to meet demand.

Principle three emphasises that investment should prevent the real exchange rate appreciating when oil is discovered. An oil discovery will create extra demand for non-traded goods, which can be met by buying more goods, or the same amount of goods at higher prices. The former is preferred. As capital is freely available on international markets, any rise in demand should be met by an inflow of capital and movement in labour from the traded sector. The amount will depend on the demand for non-traded goods and their capital intensity. The capital inflow should keep the price of non-traded goods, and in turn the real exchange rate, constant. This is consistent with van der Ploeg and Venables (2011), who also show that the real exchange rate will appreciate if capital is slow to adjust.

Investing in the non-traded sector may prevent a real appreciation, but it will not necessarily prevent Dutch disease. Since Corden and Neary (1982) Dutch disease has been closely associated with a real exchange rate appreciation. As principle three states that investment should stabilise the real exchange rate, it is tempting to think that it will also prevent Dutch disease. This is only possible if the initial capital stock is low. In that case, labour in both sectors will be very unproductive. A large amount of labour would need to move from the traded to the non-traded sector. So, by increasing the stock of capital in the traded sector, the labour that moves there becomes more productive and less will be required. However, promoting immigration would be more successful as it relaxes the constraint on labour at its source. This has been widely used in many Gulf states, including the UAE and Saudi Arabia, which have not seen a major appreciation of their real effective exchange rate as a result (Darvas, 2012 and Espinoza et al., 2013).

In practice, 75% of the world's developed resource exporters have fixed nominal exchange rates (IMF, 2012). Of the five with a Human Development Index of "Very High", Qatar, Bahrain and the UAE have an explicit US dollar peg, Brunei is pegged to the Singaporean dollar and only Norway follows an inflation target. Of the further fifteen with an HDI of "High", eleven have currency pegs. In maintaining these pegs most resource exporters have accumulated large reserves of foreign assets. In our framework this means accumulating foreign assets, F, which limits the rise in consumption, C, and in turn the appreciation of the real exchange rate. Principle three highlights that investment in the non-traded sector, K_N, should complement foreign saving to stabilise the real exchange rate. While a constant nominal exchange rate does not require a constant real exchange rate, it is easier to defend if investment is being used to stabilise prices in the non-traded sector.

These results are easily reconciled with the previous sections. Some of the oil windfall should be set aside for investment. The rest should be consumed according to the permanent income hypothesis. The investment will be chosen to stabilise the real exchange rate at this level of consumption. Formally, it involves choosing additional capital, K', so that the price of non-traded goods stays constant, $P_N = P_N^*$, at some level of aggregate consumption, C,

$$a_N C = P_N^{*1-a_N} Y_N(K_N^* + K_N')$$

This aggregate level of consumption will be based on the permanent income hypothesis below, where total output is defined as, $Y(K^*+K') \equiv P_N Y_N(K_N^*+K_N')+P_T Y_T(K_T^*+K_T')$ and capital is allocated efficiently between each sector.

$$C = rF(0) + r(PO(1 - e^{-rT})\frac{1}{r} - K') + Y(K^* + K') - \delta(K^* + K')$$

4 Oil discoveries in a developing country

We now consider how policymakers in developing countries should manage resource wealth. Of the world's forty-five resource-dependent economies, the majority have an HDI of "low" or "medium". They range from the Democratic Republic of Congo, with a GDP per capita (PPP) of less than US$500, to Botswana with GDP per capita of nearly US$17,000 (World Bank, 2012). To borrow from Tolstoy's *Anna Karenina*, while developed countries are alike, every developing country is developing in its own way. This analysis focuses on the shared challenge of capital scarcity. While developed countries can readily borrow at the world rate of interest, developing countries can not. Ghana is a case in point. Ghana is in the midst of an oil boom, having discovered it in 2007 and started production in 2010. In July 2013 Ghana sold US$750 million of ten-year, dollar-denominated bonds at a yield of 8%, six years after their previous international issue. US government bonds were yielding 2.5% at the time. To capture capital scarcity we assume that developing countries face a debt premium on borrowing.[27]

The first priority for developing countries should be to invest domestically (principle four). As capital is scarce, the path of extraction will alter their policy decision (principle five). If investment cannot be absorbed, then a temporary Parking Fund should be used to hold revenues until they can be productively used (principle six). In these countries the Volatility Fund should not all be saved abroad, as domestic returns are high and the fund is an alternative source of income (principle seven). Finally, if private capital is also constrained, then some resource revenues should be used to boost it (principle eight).

There remains scope for these principles to improve policy. Genuine savings rates[28] remain low in many developing resource-exporters, suggesting that more resource wealth needs to be saved. Most egregiously, eight resource-dependent countries have negative genuine savings rates.[29] Many developing countries also have a severe deficit in private capital. It is thought that less than 10% of the capital stock in Iraq and Libya is privately owned (van der Ploeg, Venables and Wills, 2012 and World Bank, 2012). There has been recent progress in establishing sovereign wealth funds. Eight of the twenty-five developing resource-dependent economies now have funds, with five being established in the past three years.[30] Ghana is a recent example of this: its Petroleum Revenue Management Act (2011) directs 70% of the windfall to the annual budget and 30% to offshore "Stabilization" and "Heritage" funds, see Figure 4.1.

[27]This follows van der Ploeg and Venables (2011) who find empirical evidence that the log of annual average bond spreads is increasing in the ratio of public and publicly guaranteed debt to gross national income, PPG/GNI. They also do not find evidence for natural resource discoveries alone reducing the cost of borrowing.

[28]Gross savings plus education spending less capital depreciation and resource depletion, as calculated by the World Bank (2008).

[29]Angola, Bolivia, Chad, Equatorial Guinea, Guinea, Saudi Arabia, Sudan and Zambia.

[30]Ghana in 2011, Mongolia in 2011, Nigeria in 2011, Papua New Guinea in 2011 and Angola in 2012.

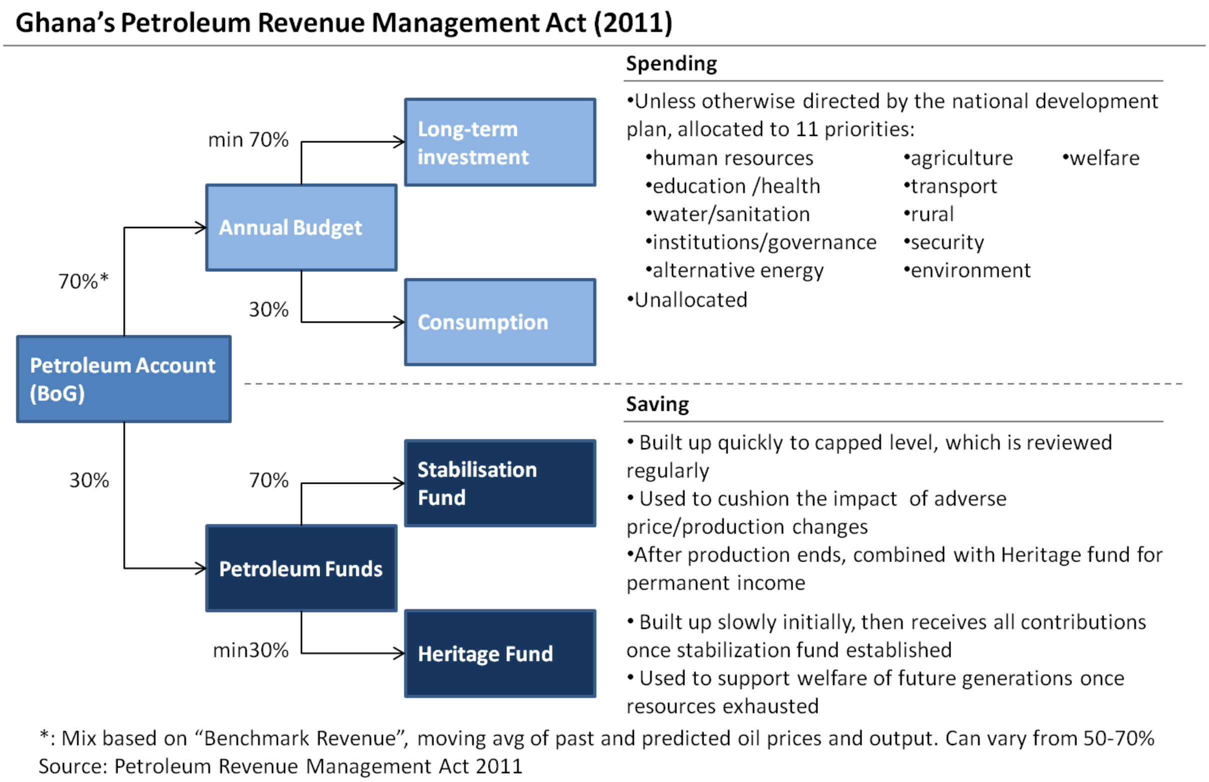

Figure 4.1: Ghana's Petroleum Revenue Management Act (2011) established one of the world's newest sovereign wealth funds.

The principles are consistent with the Natural Resource Charter (NRC, 2013). The NRC is a set of twelve precepts for managing resource wealth, adopted by the African Union Heads of State steering committee in 2012. Principle four supports the NRC's Precept 7: "promote economic development through high levels of investment". Principles six and seven add detail to Precept 8: "build up investment gradually and smooth revenue volatility". Principle eight is consistent with Precepts 9: "increase efficiency of public spending", and 10: "facilitate private sector investment". We extend the NRC by analysing the anticipation phase of a windfall, and how volatility affects capital-scarce economies.

The section proceeds as follows. Section 4.1 establishes principles four and five, by extending the analysis in section 3.1 to the case when capital is scarce and oil prices are certain. Section 4.2 establishes principle six by briefly introducing absorption constraints. Section 4.3 presents principle seven by extending previous literature to the case when capital is scarce and oil prices are volatile. Finally section 4.4 establishes principle eight by returning to the simple general equilibrium setting in section 3.3.

4.1 The case for using oil to accelerate development

This section presents principles four and five. Principle four is to *"Finance consumption and investment with oil if capital is scarce"*. Saving oil wealth for future generations will naturally be met with skepticism if current generations are suffering. If capital is scarce then the realised (and social) return on domestic investments will be higher than on foreign bonds. Investing oil wealth will accelerate development, justifying higher consumption in the short-term (see van der Ploeg and Venables, 2011). This principle is consistent with the impressive development experiences of Botswana, Bahrain, Qatar and the UAE, which combined foreign savings and domestic investment.

Principle five is to *"Save less if production is fast and capital is scarce"*. The rate of oil production will affect capital-scarce countries more than capital-abundant ones, as financial markets cannot be used to smooth revenues. In developed countries, rapid oil production will increase its present value, and in turn the amount saved. In developing countries this will quickly relax borrowing constraints. Future generations will be better off, justifying less saving in the present. A development timeline should be therefore be considered when planning extraction in capital-scarce countries.

We consider two types of windfall: small and large. Small windfalls will not overcome capital scarcity before they are exhausted (following the characterisation in van der Ploeg and Venables, 2011). Large booms will, so the economy will begin to behave like those discussed in section 3 during the period of extraction.

4.1.1 Small oil discoveries

If the oil discovery is small then the social planner will have to contend with capital scarcity both during and after the windfall. Consumption and assets will follow equations 4.1 and 4.2, from equations 2.11 and 2.12 with $\sigma = 0$ and $\omega > 0$, for the duration

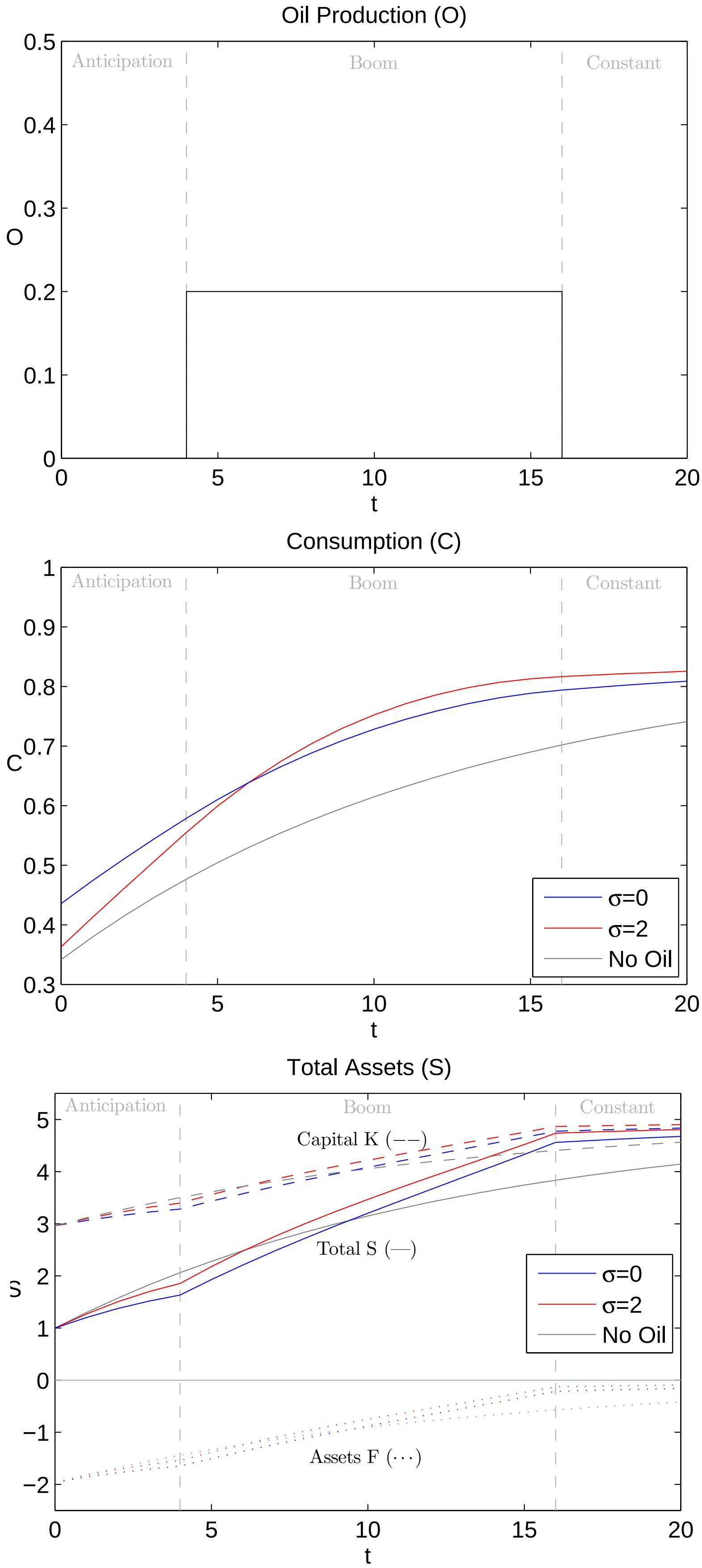

Figure 4.2: An oil boom will accelerate development if capital is scarce. Before the boom both capital and assets will fall, to smooth consumption. During the boom both accumulate quickly, financing an increase in consumption. Volatility leads to more savings (from equations 4.3, 4.4, 4.13, and 4.14).

of the windfall. When capital is scarce, repaying debt will reduce the cost of borrowing. Domestic capital will therefore be linked to foreign assets by the asset allocation condition, 2.9. This means that the share of capital in total assets will grow over time, so $f \neq 1$ and $F(t) = \frac{f}{1-f}(K(t) - K^*)$ as illustrated in Figure 4.2.

$$\dot{C}^i(t) = r - \rho - \tfrac{2\omega}{a} f S^i(t) \tag{4.1}$$

$$\dot{S}^i(t) = rS^i(t) + PO^i - C^i(t) + C^* \tag{4.2}$$

The paths of consumption and assets are found by solving equations 4.1 and 4.2. Extending the approach in section 3.1, these equations are summarised by a single differential equation, $\ddot{S}^i(t) - r\dot{S}^i(t) - \frac{2\omega}{a} f S^i(t) = 0$. The general solution is as follows, where t^i is measured from the beginning of each phase and, $\lambda_j = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 + 8\omega f/a}$, are the roots of the characteristic equation with $\lambda_1 < 0$ and $\lambda_2 > 0$,

$$S^i(t^i) = c_1^i e^{\lambda_1 t^i} + c_2^i e^{\lambda_2 t^i} \tag{4.3}$$

$$C^i(t^i) = c_1^i (r - \lambda_1) e^{\lambda_1 t^i} + c_2^i (r - \lambda_2) e_2^{\lambda_2 t^i} + PO^i + C^* \tag{4.4}$$

The coefficients, c_1^i and c_2^i, are found by imposing conditions at the beginning and end of each phase, $i = [A, B, C]$: Anticipation, Boom and Constant income. Each phase will begin with the assets left from the last one, $S(0) = c_1^A + c_2^A$, $S^A(T_1) = c_1^B + c_2^B$ and $S^B(T_2) = c_1^C + c_2^C$. During the Constant income phase, consumption and assets must stay on the stable saddle path, so $c_2^C = 0$. During the Boom phase consumption must be chosen so that the windfall ends with consumption and assets on the stable saddle path, $C^B(T_2) = (r - \lambda_1)S^B(T_2) + Y$. Recursively, consumption during the Anticipation phase will be chosen to smoothly enter the boom phase, so $C^A(T_1) = (r - \lambda_1)S^A(T_1) + PO(1 - e^{-\lambda_2(T_2 - T_1)}) + C^*$. Together these give the parameter values,

$$
\begin{array}{rclcrl}
c_1^C & = & S(T_2) & ; & c_2^C & = 0 \\
c_1^B & = & S(T_1) - c_2^B & ; & c_2^B & = PO(\lambda_2 - \lambda_1)^{-1} e^{-\lambda_2 (T_2 - T_1)} \\
c_1^A & = & S(0) - c_2^A & ; & c_2^A & = PO(\lambda_2 - \lambda_1)^{-1} (e^{-\lambda_2 T_2} - e^{-\lambda_2 T_1})
\end{array} \tag{4.5}
$$

The permanent income hypothesis will not hold in a capital-scarce country. Instead, consumption will change over time and depend on the level of total assets, seen directly from 4.1. Before oil is discovered consumption will begin far below its permanent level, C^*. This is because the planner has an incentive to repay initial debts, F_0, and reduce the rate of interest they face, $r(F(t))$. This brings us to our fourth and fifth principles,

Principle 4. ***"Finance consumption and investment with oil if capital is scarce"***

i) Consumption in capital-scarce countries should begin lower than in capital abundant economies, $C_L^C(0) < C_H^C(0)$, but jump further when oil is discovered, $C_L^A(0) - C_L^C(0) > C_H^A(0) - C_H^C(0)$. It should still remain lower in absolute level, $C_L^A(0) < C_H^A(0)$.

ii) Borrowing in capital-scarce countries before a boom should initially be lower than capital abundant countries, $\dot{F}_L^A(0) > \dot{F}_H^A(0)$, and if they begin to borrow it will happen near the date of extraction, $\ddot{F}_L^A(t) < 0$ for $t < T_1$.

iii) Domestic capital in capital-scarce economies should grow over time, $\dot{K}_L^C(t) > 0$, be accelerated during an oil boom, $\dot{K}_L^B(t) > \dot{K}_L^C(t)$, and comprise an increasing share of the total capital stock over time, $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$.

Proof. See Appendix 4 □

This principle shows that capital-scarcity changes the way policymakers should respond to an oil discovery, illustrated in Figure 4.2.

Principle four highlights how capital-scarcity changes the trade-off between consuming today and consuming tomorrow. Capital-scarcity makes debt particularly costly. The first part of the principle states that before oil is discovered capital-scarce countries (L) should prioritise repaying debt and investing in capital: they should save more. Consumption should therefore be lower than a similar country with ready access to capital (H). It is tempting to extend this intuition to oil discoveries: that the poor should save more from marginal oil revenues. However, this is not the case.

Principle four also states that a capital-scarce country should consume relatively more from an oil boom than a capital abundant one. This happens because some oil revenues will be used to repay debt and accumulate capital. Future generations will benefit from this, which justifies higher consumption in the short term in the interests of sharing the benefits. It should be noted that the initial jump in consumption is dictated by the debt burden, ω, rather than the level of debt itself, $\partial(C_j^A(0) - C_j^C(0))/\partial F(0) = 0$ as shown in Appendix 4.

The second part of principle four states that policymakers should borrow less before a windfall when capital is scarce. This happens for two reasons. The first is that borrowing is more costly, so less should be done. The second is that the policymaker also can use the capital stock. It may be better to let some capital depreciate, rather than access expensive foreign borrowing. This follows from the desire to keep the marginal product of capital in line with the cost of borrowing, in equation 2.9. It is interesting that there is any borrowing before a windfall at all. If the debt burden, ω, is sufficiently large - or the windfall small - then the capital-scarce country will not borrow. Consumption should begin so low that the entire jump can be financed directly from permanent income, Y. The need to borrow should also be highest just before production begins, as consumption should be steadily increasing in capital-scarce countries, $\dot{C}_L^i(t) > 0$.

The third part of this principle states that policymakers should accelerate investment during an oil boom. Part of an oil boom should be saved in foreign assets to relax the debt premium on interest rates, $r(F) = r - \omega F$. This means that capital should also grow. At every point in time, the policymaker should direct their savings to where it will earn the highest return: foreign assets or domestic capital. The marginal product of capital will therefore equal the cost of borrowing, 2.9. Domestic capital will also increase its share in total assets, as debt is repaid and capital accumulates.

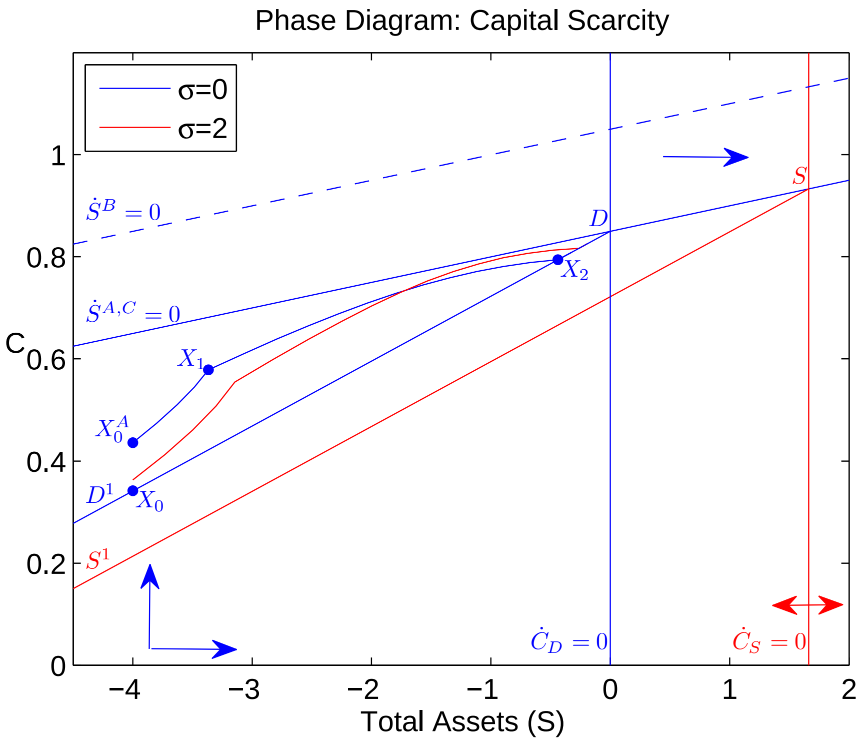

Figure 4.3: An anticipated windfall in a capital-scarce economy. Before and after the windfall the movement of consumption and assets is dictated by the solid blue lines and arrows. During the windfall the line $\dot{S}^{A,C} = 0$ shifts up to $\dot{S}^B = 0$. If oil prices are certain then the dynamics are dictated by the blue arrows and the economy follows the path $[X_0, X_0^A, X_1, X_2]$. If prices are volatile then the steady state line $\dot{C}_S = 0$ will move to the right before-, and return to $\dot{C}_D = 0$ during the windfall. This changes the path of consumption and assets.

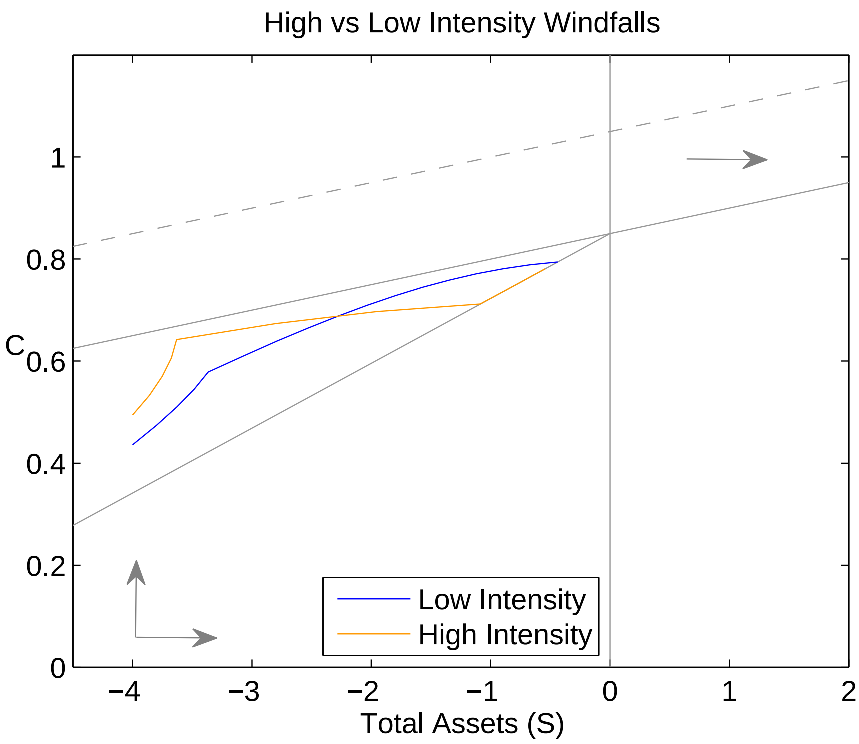

Figure 4.4: "Save less if production is fast and capital is scarce". More is consumed from a high intensity windfall, resulting in fewer assets.

The phase diagram in Figure 3.3 illustrates the effects of an oil discovery when capital is scarce. When the country is a net borrower, $S(t) < 0$, the economy will be governed by equations 4.1 and 4.2. These are summarised by the two blue steady state lines, $\dot{C}_D = 0$ and $\dot{F}^{A,C} = 0$ when prices are deterministic (D). These intersect at the point where all foreign debt is repaid, so $F^i = 0$ and $\bar{S}^i = K^i = S^* = (\frac{\alpha}{r+\delta})$ from Appendix A. In the absence of any shocks the social planner will choose to be on the stable (deterministic) saddle path, DD^1, which ensures the economy will move towards the steady state. This path is a line with slope $(r - \lambda_1)$, and converges to the line $\dot{S}^{A,C} = 0$ as $\omega \to 0$. When the country is a net saver, $S(t) > 0$, the economy behaves as in Figure 3.3 and anywhere on the line $\dot{S}^{A,C} = 0$ above zero is a steady state.

Before oil is discovered the economy will begin on the stable saddle path, X_0. When oil is discovered, consumption will immediately jump and then continue to grow, X_0^A. If the windfall is small, then consumption will remain below $\dot{S}^{A,C} = 0$ and the economy will continue to accumulate assets by repaying foreign debt and investing in domestic capital, F and K. Alternatively, if the windfall is large or the debt burden small, then consumption may jump above the line $\dot{S}^{A,C} = 0$ and the planner will deplete assets: both foreign and domestic.

When the Boom begins the planner will save, X_1. The line $\dot{S}^{A,C} = 0$ will shift up, to $\dot{S}^{B} = 0$. The dynamics of the economy will now be dictated by a new steady state. Consumption will be below this, so total assets will grow: repaying debt and investing in capital. If the windfall is small, then the planner will have chosen consumption to arrive on the line DD^1 at the end of the Boom, X_2. If the windfall is large, then the planner will repay all the debt and begin to accumulate assets in a sovereign wealth fund, ending the boom somewhere on $\dot{S}^{A,C} = 0$ above zero. This is discussed in the next section.

After the boom, consumption will eventually exceed that in a capital abundant economy for the same initial level of debt. In a capital abundant economy there is no incentive to repay debt. In a capital-scarce economy, the incentive to reduce the debt premium will mean that eventually the economy will converge to the steady state, D.

The next principle shows that the policy response to an oil discovery should also depend on the extraction path, when capital is scarce.

Principle 5. ***"Save less if production is fast and capital is scarce"***

i) All else being equal (including extraction costs), a shorter, high intensity windfall (HI) is always preferred to a long, low intensity one (LI), $\int_0^\infty U(C_{HI}(\tau))e^{-\rho\tau}d\tau > \int_0^\infty U(C_{LI}(\tau))e^{-\rho\tau}d\tau$.

ii) However, if capital is sufficiently scarce then fewer assets will be accumulated from a short, high intensity windfall than a long, low intensity one after an equal amount of time has passed, $S_{HI}(t) < S_{LI}(t)$ *for* $t > T_2$.

Proof. See Appendix 5 □

Principle five states that the extraction path matters, particularly in capital-scarce countries. The extraction path refers to the way a given level of oil reserves $R = O(T_2 - T_1)$ is extracted at a certain intensity nO over a period of time $(T_2 - T_1)/n$. The first component of this principle states that, all else equal, a short, high-intensity windfall is preferred to a long, low-intensity one. This is because it will have a higher present value. The preference will only be exacerbated when oil prices are volatile, with longer windfalls being exposed to more volatility.

This principle also states that, for a given stock of reserves, capital-scarce countries should save less if they are extracted quickly. This is slightly less intuitive. If capital is abundant then the opposite is true, because rapidly extracting oil increases its present value. More should therefore be saved for the future. However, if capital is scarce then a fast windfall will rapidly reduce the debt burden, benefiting future generations. Consumption should be higher in the present, to ensure it grows smoothly as in Figure 4.4. This result is exacerbated when oil prices are volatile: shorter windfalls are less exposed and so require less precautionary savings.

4.1.2 Large oil discoveries

If the discovery is large then all debt should be repaid before accumulating a Future Generations Fund. Large windfalls will repay all debt before they are exhausted.[31] As the economy will no longer be capital-constrained it will behave as in section 3.1. Consumption will stop growing and assets will accumulate. The economy will end the boom with a positive Future Generations Fund, on the steady state line $\dot{F}^{A,C} = 0$ above zero in Figure 4.3. The planner's behaviour will be a combination of the analysis in section 4.1.1 and section 3.1.

4.2 The case for a Parking Fund

This section presents principle six, *"Use a temporary Parking Fund to improve absorption"*. Oil wealth should increase investment if capital is scarce. However, this investment may be difficult to absorb. A major challenge is the need to use non-traded capital to build

[31] Formally, this will happen when $F(0)(\lambda_2 - \lambda_1) + PO(e^{-\lambda_1 T_2} - e^{-\lambda_1 T_1} + e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) > 0$, which is satisfied for larger, longer windfalls and lower initial levels of debt.

capital. Roads are built more efficiently with a motor grader (physical capital), and it requires teachers (human capital) to teach new teachers (see van der Ploeg, 2012). A lot of investment at once may also lead to corruption and poor project choice. An example is Nauru, an island nation of 10,000 people and 8 square miles in the central Pacific. At independence from Australia in 1968 Nauru had phosphate reserves of AU$ 500,000 per person (1968 currency, Hughes, 2004). In 2004 communal net assets were estimated to be AU$ 3,000 per person, after a succession of failed projects including an unsuccessful West-End musical, a cruise ship that did not leave port and a yellow Lamborghini for the police chief (despite a single coastal road with 25mph speed limit, McDaniel and Gowdy, 1999).

The Parking Fund will alleviate absorption constraints. In practice absorption constraints are measured by the IMF using the Public Investment Management Index (PIMI) (Dabla-Norris et al., 2011). We follow van der Ploeg (2012) and van der Ploeg and Venables (2013) and assume that absorption depends on the amount of investment and the stock of capital. For every I bricks of investment laid there will be $I + \frac{1}{2}\phi I^2/K$ bricks purchased. The social planner's budget constraint in equation 2.2 becomes,

$$dF^i = (r(F^i)F^i + PO^i + Y(K^i) - C^i - (I^i + \tfrac{1}{2}\phi I^{i2}/K^i))dt \tag{4.6}$$

Optimal investment will depend on the ratio of the marginal utilities of capital and foreign assets, $q = J_K/J_F$. When absorption is not constrained this ratio equals one, $q = 1$ from equation 5.2 in Appendix A. When absorption is constrained the ratio satisfies the following conditions, derived in Appendix 6 and giving rise to principle six,

$$q = 1 + \phi\frac{I}{K} \tag{4.7}$$

$$\dot{q} = q(r - \omega F + \delta) - Y_K - 0.5\tfrac{1}{\phi}(q-1)^2 \tag{4.8}$$

Principle 6. ***"If capital is scarce and absorption is constrained, use a temporary Parking Fund abroad"***

i) In capital scarce countries with absorption constraints, part of an oil discovery should be accumulated in a temporary offshore Parking Fund while absorption constraints are relaxed, $F_P^i(t) > 0$ *if* $\phi > 0$.

ii) In capital abundant countries an oil discovery will not affect the size of any Parking Fund.

Proof. See Appendix 6. □

Principle six introduces a third type of sovereign wealth fund, the Parking Fund. Oil discoveries should stimulate investment if capital is scarce. However, if absorption is constrained then investment may not be productive, seen in the ratio $q = 1 + \phi \frac{I}{K}$. If there are absorption constraints and investment is positive, $\phi, I > 0$, then this ratio will be greater than one, $J_K > J_F$. Absorption constraints will increase the optimal amount of foreign assets relative to capital. This should be achieved with an offshore Parking Fund, to hold oil wealth while capital is built and absorption constraints are relaxed, as found in van der Ploeg (2012) and van der Ploeg and Venables (2013). The need for the Parking Fund will diminish as capital is accumulated, so it is inherently temporary. It will therefore require a different legal structure to the Future Generations and Volatility Funds, which are intended to be permanent.

Oil discoveries will not affect the Parking Fund when capital is abundant. The analysis in section 3.1 showed that an oil discovery should not affect investment, if capital is abundant and all goods are traded. All positive net present value projects should be developed regardless of the source of funding. The investment path may be altered by absorption constraints, but the discovery of oil will not change it further. Of course, if some goods are non-traded then an oil discovery should boost investment in that sector, as discussed in principle three. This would require an additional Parking Fund, as discussed in Van der Ploeg (2011).

4.3 The case for using the Volatility Fund for investment

This section presents the seventh principle, *"Invest part of the Volatility Fund domestically if capital is scarce"*. This extends previous work by considering how capital-scarce economies should respond to oil price volatility. The Volatility Fund will be an alternative source of income if the commodity price falls, from principle two. If the Volatility Fund is only held abroad then the cost of borrowing will fall, but there will be unfunded profitable investments at home. It is better to invest part of fund domestically, and keep the rates of return on foreign saving and domestic investment in line. The illiquidity of domestic investment should not be a concern as the Volatility Fund is a source of income. It is not a buffer to temporarily support consumption when oil prices are low, because oil price shocks are persistent.

4.3.1 Small oil discoveries

We start by concentrating on a small windfall, so capital will still be scarce when it is exhausted. If the windfall is volatile, then consumption and assets will evolve according to the two equations 2.11 and 2.12, reproduced below. Again let $r = \rho$ for simplicity.

$$\dot{C}^i(t) = r - \rho - \tfrac{2\omega}{a} f S^i(t) + \tfrac{1}{2} a P^2 C_P^i(t)^2 \sigma^2 \tag{4.9}$$

$$\dot{S}^i(t) = r S^i(t) + P O^i - C^i(t) + C^* \tag{4.10}$$

It will be useful to define the instantaneous "steady state" of the system. At any point in time this state will govern the behaviour of consumption and total assets, analogous to the steady state in section 4.1. However, it will move over time, as the remaining exposure to volatility changes, through $C_P^i(t)$. It is given by the two equations,

$$
\begin{aligned}
S_S^i(t) &= \tfrac{1}{4\omega f} a^2 P^2 C_P^i(t)^2 \sigma^2 && (4.11)\\
C_S^i(t) &= r S_S^i(t) + PO^i + C^* && (4.12)
\end{aligned}
$$

The dynamics of consumption and assets are found by solving equations 4.9 and 4.10. The dynamics around the steady state are summarised by a single differential equation, $\ddot{S}^i(t) - r\dot{S}^i(t) - \frac{2\omega f}{a} S^i(t) = \frac{1}{2} a P^2 C_P^i(t)^2 \sigma^2$. As this is a non-homogeneous second-order linear differential equation, it must be solved relative to a reference point. The solution is of the form:

$$
\begin{aligned}
S^i(t) &= c_1^i e^{\lambda_1 t} + c_2^i e^{\lambda_2 t} + S_V^i(t) && (4.13)\\
C^i(t) &= c_1^i (r - \lambda_1) e^{\lambda_1 t} + c_2^i (r - \lambda_2) e^{\lambda_2 t} + C_V^i(t) && (4.14)
\end{aligned}
$$

where $C_V^i(t) = r S_V^i(t) - \dot{S}_V^i(t)$. We choose the coefficients, c_1^i, c_2^i, to equal those in equation 4.5, following the approach in section 3.2. This allows total assets to be separated into those accumulated for Future Generations, and those to manage Volatility, $S^i(t) = S_{FG}^i(t) + S_V^i(t)$. Some assets would be accumulated without an oil windfall, to overcome capital scarcity. These are included in the Future Generations assets. The Volatility assets then capture the adjustment to manage oil price volatility. Both sets of assets should be allocated between a foreign fund and domestic capital, $S_{FG}^i(t) = F_{FG}^i(t) + K_{FG}^i(t)$ and $S_V^i(t) = F_V^i(t) + K_V^i(t)$. This ensures that the asset allocation condition in equation 2.9 is satisfied, and gives rise to the next principle,

Principle 7. *"When capital is scarce, direct some precautionary savings to investment"*

i) Capital-scarce countries should accumulate less in an offshore Volatility Fund than capital abundant economies, $F_{V,L}^i < F_{V,H}^i$, despite the fund accelerating the expected rate of development.

ii) Capital-scarce countries should also direct some precautionary savings from an oil boom to domestic investment, $K_V^i(t) > 0$.

Proof. See Appendix 7. $\square$

The seventh principle states that capital-scarcity reduces the precautionary motive, as illustrated in 4.2. This is because policymakers in these countries should already be saving a great deal, to overcome capital scarcity. We include this in the Future Generations assets, which grow much more quickly when capital is scarce as stated in principle four. There will be less additional benefit from saving more in a Volatility Fund.[32] Capital-scarce countries should therefore have a large Future Generations Fund, and a relatively small Volatility Fund.

Capital-scarce countries should have less precautionary savings, despite the additional benefits of reducing capital scarcity. Saving will accelerate the pace of development if capital is scarce, reducing the debt premium on interest rates. Policymakers may see this as an additional incentive to save against volatility. However, together the precautionary and development incentives to save are less than the sum of their parts. This is because utility is concave in consumption, and the benefits to saving are realised in the future, whilst the costs are borne in the present.

Principle seven also states that part of the Volatility Fund should be invested domestically, if capital is scarce. Holding all of the Volatility assets in an offshore sovereign wealth fund, $F_V > 0$ and $K_V = 0$, will reduce the cost of capital, $r(F) = r - \omega(F_{FG} + F_V)$. More domestic investment will be profitable at this lower cost of capital. Thus, some of the Volatility Funds should be invested domestically to keep the marginal products of capital and foreign assets equal, from equation 2.9. Domestic capital is less liquid than foreign assets. This should not be a major concern as the Volatility Fund should be treated as a source of permanent income, rather than a "shock absorber". This is because oil price shocks are persistent, as argued in principle two.

[32]Note that this analysis assumes prudence is constant, regardless of consumption. This makes the effects of volatility more clear. If prudence is higher at low levels of consumption (such as if relative risk aversion is constant), then higher prudence may outweigh the costs of lower consumption.

The phase diagram in Figure 3.3 also illustrates the effects of oil price volatility. The stochastic case (red, subscript S) illustrates the dynamics in equations 4.14 and 4.13. Volatility introduces a crucial difference to the deterministic case in section 4.1. The vertical "steady state" line will move, $\dot{C}_S = 0$ in equation 4.11. This happens because the exposure to oil price volatility changes over time as oil is extracted, $C_P^i(t)$. Therefore, the level of assets the social planner is "targeting" also changes, illustrated in the stable saddle path SS^1. If the windfall is immediate and permanent, then this line will immediately jump to its new level and stay there.

When oil is discovered, the policymaker becomes exposed to oil price volatility. The line $\dot{C}_S = 0$ will jump to the right as the policymaker targets a higher level of total assets, because of precautionary savings. The line continues to move right until oil production begins, when the exposure to oil price volatility is highest. The stable saddle path, SS^1, moves to the right as well. This results in consumption being less than the deterministic case, because of precautionary savings.

When oil production begins, the policymaker's exposure to volatility will fall. The line $\dot{C}_S = 0$ will move to the left throughout the Boom phase, as the amount of subsoil oil exposed to volatile prices falls. At the same time the line $\dot{S}^{A,C} = 0$ shifts up to $\dot{S}^B = 0$. At the end of the Boom the line will return to meet the deterministic steady state line, $\dot{C}_D = 0$ at $t = T_2$, as there is no further exposure to volatility. Higher savings throughout the boom means that the economy meets the line DD^1 at a higher level of both consumption and assets.

4.3.2 Large oil discoveries

If the discovery is large and volatile, then all debt should be repaid and the desired capital stock reached, before accumulating a Future Generations Fund. A large discovery will start in capital scarcity, and finish in capital abundance. At the start of the windfall the economy will behave as in section 4.3.1. At some point during the windfall total assets will accumulate far enough that debt is repaid and capital-scarcity is relaxed, $F^B(t^*) = 0$ at some point $T_1 < t^* < T_2$. For the rest of the windfall the economy will behave as if it were capital-abundant, from section 3.2. The vertical steady state line, $\dot{C}_S = 0$, will disappear as capital is no longer scarce, $\omega = 0$ in equation 4.11. Thus, the social planner never actually reaches this "target". Consumption will be chosen to grow smoothly throughout the windfall and finish on the steady state line at the end of the boom, $\dot{S}^{A,C} = 0$ at $t = T_2$ in equation 4.12.

4.4 The case for investing in private capital

This section presents the final principle, *"Support private investment"*. Capital-scarcity does not just apply to public investment; the private sector may also require capital. Iraq and Libya are a case in point. After decades of centralised government, over 90% of capital is publicly owned (van der Ploeg, Venables and Wills, 2012). Private access to capital will vary by country, and by sector. While large agricultural businesses may have ready access to foreign direct investment, small service industries will usually not.[33] An oil discovery is a source of cheap finance for the government. If the government's cost of borrowing is less than the private sector's, then it should direct some investment to private capital. This in turn boosts the productivity of public capital. Public financing of private capital should be done indirectly and transparently. This may be done by financing the domestic banking sector as in the UK's Funding for Lending scheme.

To establish this principle we consider three cases of private capital scarcity: no domestic market in capital; a domestic market with limited access to foreign capital; and a domestic market with ready access to foreign capital. This is done in the general equilibrium setting introduced in section 3.3, which includes both public and private capital. We now assume that the government is capital scarce, so $r_G(F) = r - \omega_G F$. If there is no domestic market in capital then the marginal product of capital will differ in the non-traded and traded sectors, $Y_{K,N} \neq Y_{K,T}$. If there is a domestic market in capital then firms in both sectors can borrow and lend from domestic banks at the same rate of

[33] The lack of access to finance in small service industries and smallholder agriculture has been partly addressed in some countries by the expansion of micro-finance institutions.

interest, r_P.[34] If the banks have limited access to foreign capital we assume the private sector is more capital scarce than the government, $r_P(F) = r - \omega_P F$ where $\omega_P > \omega_G$. If there is ready access to foreign capital then $r_P = r$. This leads us to the final principle,

Principle 8. ***"Support private investment"***

i) If a capital scarce country has no domestic private capital market, the government should promote its development to increase the productivity of public investment.

ii) If the country has a private capital market with limited access to foreign capital, then the government should "top-up" private capital by lending through the domestic banking sector.

iii) If the country has a private capital market and ready access to foreign investment, then the government should invest relatively more in public capital.

iv) Investment in capital scarce countries should allow the real exchange rate to steadily appreciate as borrowing constraints are relaxed and capital is accumulated, according to the Rybczynski theorem.

Proof. See Appendix 8 □

Principle eight begins by showing that the productivity of public capital depends on an efficient allocation of private capital. The marginal product of public capital in a sector depends on its amount of technology, private capital and labour. The most efficient allocation of these factors across sectors equates their marginal products. If there is no domestic market for private capital, then there is no mechanism to ensure this will happen. As a result the overall productivity of public capital will suffer.

The second part of principle eight shows that an oil windfall should be used to increase investment in private capital, if the government has better access to finance than the private sector. An oil windfall should promote additional investment if capital is scarce,

[34] We abstract from any monitoring costs that would introduce a spread between borrowing and lending (see for example Curdia and Woodford, 2010).

as stated in principle four. If the government has a given amount of capital to allocate between different uses, it is best to equate their marginal products. An increase in public investment will also promote “co-investment” by the private sector, as the productivity of private capital will have improved. If the government can then “top-up” private investment at a cheaper cost of finance, this will lift the productivity of public capital and boost output. This is similar to the UK’s Funding for Lending scheme, where the Bank of England offered discounted financing to private banks, if they could demonstrate it was being used to increase lending (Churm et al., 2012).

The third part of principle eight shows that ready access to capital in the private sector should boost investment in the public sector. The productivity of public capital depends on the amount of private capital, and vice versa. If the government installs public capital, it will encourage private investment. This in turn increases the productivity of public capital, encouraging further investment.

The final part of principle eight shows that the real exchange rate in capital-scarce countries should rise as capital is accumulated. This is a counterpart to principle three, which stated that the real exchange rate should be stable in capital-abundant economies. An oil discovery will help alleviate capital scarcity, reducing the cost of borrowing over time. The amount of capital in the economy will rise accordingly. If we assume that the traded sector is more capital intensive than the non-traded sector, then an increase in capital will reduce the relative price of the traded good, according to the Rybczynski theorem. This is achieved by an increase in the relative price of the non-traded good, which is a real appreciation.

5 Conclusion

This paper considers whether policymakers should spend, save or invest an oil windfall. It extends existing literature by showing that capital-scarce countries should do relatively less precautionary saving in a Volatility Fund, and invest some of it domestically. This Volatility Fund should be built in advance if the windfall is anticipated, and treated as a source of income, rather than a temporary buffer for smoothing out oil shocks. Capital-scarce countries should also invest so that the real exchange rate appreciates slowly, and support private capital if it is also constrained. In contrast to current policy advice, we show that even capital-abundant economies should invest part of their windfall, to stabilise the real exchange rate. To do this the paper develops a framework that nests a variety of existing results, which we present in eight principles.

1. Smooth consumption using a Future Generations Fund
2. Build a Volatility Fund quickly, then leave it alone
3. Invest to stabilise the real exchange rate, if capital is abundant
4. Finance consumption and investment with oil if capital is scarce
5. Save less if production is fast and capital is scarce
6. Use a temporary Parking Fund to improve absorption
7. Invest part of the Volatility Fund domestically if capital is scarce
8. Support private investment

These principles also suggest a number of extensions: on extraction paths, investment volatility and political economy. Principle five shows that the extraction path matters if capital is scarce, as income can not be smoothed with financial assets. Thus, the optimal extraction path will differ from capital abundant economies. An extension could consider how optimal extraction differs when capital is scarce and when it is abundant (see van den Bremer, van der Ploeg and Wills, 2013).

A second extension could add additional sources of volatility. Principle seven shows that capital scarcity reduces the precautionary motive and encourages investment in domestic capital. Further work should also allow for volatile investment returns and absorption constraints.

A third extension could introduce political economy. Some capital-abundant democratic countries (such as Australia and Canada) do not have national sovereign wealth funds, while non-democratic (Saudi Arabia and the UAE), and capital-scarce democratic countries (such as Ghana and Botswana) do. Norway is a conspicuous exception. The uncertainty surrounding election may shorten the government's investment horizon, and reduce its incentive to save. Developing countries may still have an incentive to save because of the desire to overcome capital-scarcity. A formal treatment of this could be the subject of future work.

References

[1] Acemoglu, D., Verdier, T., and Robinson, J. A. Kleptocracy and divide-and-rule: A model of personal rule. *Journal of the European Economic Association 2*, 2-3 (2004), 162–192.

[2] Aghion, P., Bacchetta, P., Rancière, R., and Rogoff, K. Exchange rate volatility and productivity growth: The role of financial development. *Journal of Monetary Economics 56*, 4 (2009), 494 – 513.

[3] Australian Bureau of Statistics. www.abs.gov.au, 2013.

[4] Barnett, S., and Ossowski, R. *Operational aspects of fiscal policy in oil-producing countries.* International Monetary Fund, 2003.

[5] Baunsgaard, T., V. M. P.-R. M., and Richmond, C. Fiscal frameworks for resource rich developing countries. *IMF Staff Discussion Note* (2012).

[6] Bems, R., and de Carvalho Filho, I. The current account and precautionary savings for exporters of exhaustible resources. *Journal of International Economics 84*, 1 (2011), 48–64.

[7] Berg, A., Portillo, R., Yang, S.-C. S., and Zanna, L.-F. Public investment in resource-abundant developing countries. *IMF Economic Review 61*, 1 (2013), 92–129.

[8] Blattman, C., Hwang, J., and Williamson, J. G. Winners and losers in the commodity lottery: The impact of terms of trade growth and volatility in the periphery 1870-1939. *Journal of Development Economics 82*, 1 (2007), 156 – 179.

[9] BP. Statistical review of world energy. Tech. rep., BP, 2012.

[10] Carroll, C. D., and Kimball, M. S. Precautionary saving and precautionary wealth. *The New Palgrave Dictionary of Economics 6* (2008), 579–584.

[11] Chang, F.-R. *Stochastic optimization in continuous time.* Cambridge University Press, 2004.

[12] Cherif, R., and Hasanov, F. Oil exporters dilemma: How much to save and how much to invest. *World Development 52* (2013), 120–131.

[13] Churm, R., Radia, A., Leake, J., Srinivasan, S., and Whisker, R. The funding for lending scheme. *Bank of England Quarterly Bulletin* (2012), Q4.

[14] Collier, P., and Goderis, B. Commodity prices, growth, and the natural resource curse: Reconciling a conundrum. *Growth, and the Natural Resource Curse: Reconciling a Conundrum (June 5, 2008)* (2009).

[15] Corden, W., and Neary, P. Booming sectors and deindustrialisation in small open economies. *Economic Journal 92* (1982), 825–848.

[16] Cúrdia, V., and Woodford, M. Credit spreads and monetary policy. *Journal of Money, Credit and Banking 42*, s1 (2010), 3–35.

[17] Dabla-Norris, E., Brumby, J., Kyobe, A., Mills, Z., and Papageorgiou, C. Investing in public investment: an index of public investment efficiency. *Journal of Economic Growth 17*, 3 (2012), 235–266.

[18] Darvas, Z. Real effective exchange rates for 178 countries: A new database. *Working Paper 2012/06, Bruegel* (2012).

[19] Davis, J. M. *Stabilization and savings funds for nonrenewable resources: Experience and fiscal policy implications*, vol. 205. International monetary fund, 2001.

[20] Epstein, L. G., and Zin, S. E. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*, 4 (1989), 937–969.

[21] Espinoza, R., Fayad, G., and Prasad, A. *The Macroeconomics of the Arab States of the Gulf*. Oxford University Press, 2013.

[22] Fasano, U. With open economy and sound policies, uae has turned oil curse into a blessing, imfsurvey. *October 21* (2002), 330–332.

[23] Fisher, I. The theory of interest. *New York 43* (1930).

[24] Friedman, M. *A Theory of the Consumption*. Princeton University Press Princeton, 1957.

[25] Gylfason, T., Herbertsson, T. T., and Zoega, G. A mixed blessing. *Macroeconomic Dynamics 3*, 02 (1999), 204–225.

[26] Hartwick, J. M. Intergenerational equity and the investing of rents from exhaustible resources. *The american economic review 67*, 5 (1977), 972–974.

[27] Hsieh, C.-T. Do consumers react to anticipated income changes? evidence from the alaska permanent fund. *The American Economic Review 93*, 1 (2003), 397–405.

[28] Hughes, H. From rags to riches: what are nauru's options and how can australia help? *Issues Analysis: Centre for Independent Studies, Australia* (2004).

[29] IMF. Macroeconomic policy frameworks for resource-rich developing countries. Tech. rep., International Monetary Fund, 2012.

[30] Kimball, M. S. Precautionary saving in the small and in the large. *Econometrica 58*, 1 (1990), 53–73.

[31] Kimball, M. S., and Mankiw, N. G. Precautionary saving and the timing of taxes, 1989.

[32] Larsen, E. R. Are rich countries immune to the resource curse? evidence from norway's management of its oil riches. *Resources Policy 30*, 2 (2005), 75–86.

[33] Leland, H. E. Saving and uncertainty: The precautionary demand for saving. *The Quarterly Journal of Economics 82*, 3 (1968), 465–473.

[34] McDaniel, C. N., and Gowdy, J. M. *Paradise for sale: a parable of nature.* Univ of California Press, 1999.

[35] Merton, R. C. Continuous-time finance.

[36] Natural Resource Charter. http://naturalresourcecharter.org/, 2013.

[37] Poelhekke, S., and van der Ploeg, F. Volatility and the natural resource curse. *Oxford Economic Papers, Vol. 61, Issue 4, pp. 727-760, 2009* (2009).

[38] R. Espinoza, G. F., and Prasad, A. *The Macroeconomics of the Arab States of the Gulf.* Oxford University Press, 2013.

[39] Ramey, G., and Ramey, V. A. Cross-country evidence on the link between volatility and growth. *The American Economic Review 85*, 5 (1995), 1138–1151.

[40] Ramsey, F. P. A mathematical theory of saving. *The Economic Journal 38*, 152 (1928), pp. 543–559.

[41] Regnier, E. Oil and energy price volatility. *Energy Economics 29*, 3 (2007), 405–427.

[42] Robinson, J. C. *An introduction to ordinary differential equations.* Cambridge University Press, 2004.

[43] Sachs, J. D., and Warner, A. M. The curse of natural resources. *European economic review 45*, 4 (2001), 827–838.

[44] Sala-i Martin, F. X., and Subramanian, A. Addressing the natural resource curse: An illustration from nigeria. *SSRN eLibrary Working Paper Series* (2003).

[45] Samuelson, P. A. International trade and the equalisation of factor prices. *The Economic Journal 58*, 230 (1948), 163–184.

[46] Sandmo, A. The effect of uncertainty on saving decisions. *The Review of Economic Studies 37*, 3 (1970), 353–360.

[47] Schmitt-Grohe, S., and Uribe, M. Closing small open economy models. *Journal of International Economics 61*, 1 (2003), 163 – 185.

[48] Statistics Norway. http://www.ssb.no/, 2013.

[49] van den Bremer, T., van der Ploeg, R., and Wills, S. The elephant in the ground: Extracting oil and allocating assets in sovereign wealth funds. *OxCarre Working Paper, University of Oxford* (2013).

[50] van den Bremer, T. S., and Van der Ploeg, F. Managing and harnessing volatile oil windfalls. *IMF Economic Review 61*, 1 (2013), 130–167.

[51] van der Ploeg, F. Aggressive oil extraction and precautionary saving: Coping with volatility. *Journal of Public Economics 94*, 5-6 (2010), 421 – 433.

[52] van der Ploeg, F. Natural resources: Curse or blessing? *Journal of Economic Literature 49*, 2 (2011), 366–420.

[53] van der Ploeg, F. Bottlenecks in ramping up public investment. *International Tax and Public Finance 19*, 4 (2012), 509–538.

[54] van der Ploeg, F., and Venables, A. J. Harnessing windfall revenues: Optimal policies for resource-rich developing economies*. *The Economic Journal 121*, 551 (2011), 1–30.

[55] van der Ploeg, F., and Venables, A. J. Absorbing a windfall of foreign exchange: Dutch disease dynamics. *Journal of Development Economics* (2013).

[56] van der Ploeg, R., and Venables, A. Natural resource wealth: The challenge of managing a windfall. *Annual Review of Economics 4* (2012), 315–337.

[57] van der Ploeg, R., Venables, A., and Wills, S. Oil and fiscal policy in iraq. *OxCarre Policy Paper 13* (2012).

[58] Wills, S. Optimal monetary responses to oil discoveries. *OxCarre Research Paper 121* (2013).

[59] World Bank. Country economic memorandum: Iraq. Tech. rep., The World Bank Group, 2012.

[60] Zeldes, S. P. Optimal consumption with stochastic income: Deviations from certainty equivalence. *The Quarterly Journal of Economics 104*, 2 (1989), 275–298.

Appendix A: Consumption and Total Assets

This appendix derives the equations governing the expected dynamics of consumption and total assets. We begin by taking the first-order conditions of the Hamilton-Jacobi-Bellman equation in 2.6 and 2.7. This is done with respect to consumption, investment, foreign assets, and domestic capital, where 5.3 and 5.4 make use of the envelope condition,

$$\begin{aligned}
U'(C)e^{-\rho t} - J_F &= 0 && (5.1)\\
-J_F + J_K &= 0 && (5.2)\\
0 + J_{tF} + J_{FF}(r(F)F + PO + Y(K) - C - I) + J_F(r + r_F F) &= 0 && (5.3)\\
+J_{KF}(I - \delta K) + \tfrac{1}{2}J_{PPF}\sigma^2 P^2 & && \\
J_{tK} + J_{FK}(r(F)F + PO + Y(K) - C - I) + Y_K J_F &= 0 && (5.4)\\
+J_{KK}(I - \delta K) - \delta J_K + \tfrac{1}{2}J_{PPK}\sigma^2 P^2 & &&
\end{aligned}$$

The results in equations 2.8 and 2.9 come from combining equations 5.2, 5.3 and 5.4 with $dJ_s = J_{sF}dF + J_{sK}dK + J_{sP}dP + J_{st}dt + \frac{1}{2}J_{sPP}dP^2$ for $s = [F, K]$.

The optimality condition in equation 2.10 comes from substituting 5.1 and 2.7 into 2.6, and using $U(C) = \frac{1}{-a}e^{-aC}$.

The social planner's problem can be summarised in terms of the overall evolution of total assets, $\bar{S} = F + K$, as foreign assets and capital are linked by equation 2.9. The evolution of total assets, equation 5.5 below, comes from substituting 2.3 into 2.2. Total assets will be constant at the steady state, S^*, when capital scarcity is overcome and oil income is exhausted, $F^* = 0$ and $O^* = 0$. We define technology so that output is standardised at this point, $A = (\frac{r+\delta}{\alpha})^\alpha$ so $Y^* = 1$ and$K^* = (\frac{\alpha}{r+\delta})$.

$$d\bar{S}^i(t) = (r(F^i(t))F^i(t) + P(t)O^i + Y(K^i(t)) - C^i(t) - \delta K(t))dt \tag{5.5}$$

To analyse the dynamics of consumption and total assets we will need to linearise this equation. We do this around S^* to give the following, where $S = \bar{S} - S^*$.

$$dS^i(t) = \Big(rF^i(t) + P(t)O^i + Y'(K^*)(K^i(t) - K^*) - (C^i(t) - C^*) - \delta(K^i(t) - K^*)\Big)\, dt \tag{5.6}$$

Both foreign assets and domestic capital can be expressed as a share of total assets. At any point in time the optimal level of domestic capital can be linked to the level of foreign assets, $K(F) = (\frac{r+\delta-\omega F}{\alpha})^{1/(\alpha-1)}A^{-1/(\alpha-1)}$ from equation 2.9. Therefore, we can express total assets as a non-linear function of foreign assets, $S = S(F)$. However, we need an expression for foreign assets as a function of total assets, $F = F(S)$. The only way to proceed with a tractable solution is to linearise $S(F)$, which we do around the deterministic steady state S^*. Linearisation yields $S = f^{-1}(F - F^*)$ where $f^{-1} = (1 + \frac{\omega}{\alpha(1-\alpha)}(\frac{\alpha}{r+\delta})^2)$. Using this, $\bar{S} = F + K$ and 5.6 gives the relationship,

$$\frac{1}{dt}dS^i(t) \quad = \quad \theta S^i(t) + P(t)O^i - C^i(t) + C^* \tag{5.7}$$

where $\theta = rf + (Y'(K^*) - \delta)(1 - f)$ is the overall rate of return on total assets, and $C^* = Y^* - \delta K^*$ is the steady state level of consumption. Finally, making use of equation 2.9 at the steady state gives $r = Y'(K^*) - \delta$, so $\theta = r$ and we have the budget constraint in equation 2.12. This simple expression is possible because we have linearised the relationship between foreign assets and domestic capital in equation 2.9 to give, $F = f/(1-f)(K-K^*)$.

The Euler equation in 2.11 is found by taking the expected rate of change with respect to time of 5.1, given in 5.8 below, and combining it with expressions 2.8, $U(C) = \frac{1}{-a}e^{-aC}$, and $S = f^{-1}(F - F^*)$.

$$\frac{E[dJ_F]/dt}{J_F} = \frac{U''(C)}{U'(C)}E[dC]/dt - \rho + \tfrac{1}{2}\frac{U'''(C)}{U'(C)}C_P^2\sigma^2P^2dt \tag{5.8}$$

Appendix B: The Permanent Income Hypothesis

This appendix derives an explicit expression for the value function, $J(F, K, P, t)$ in equation 2.1. This is done in the special case when preferences are CARA, $U(C) = \frac{1}{-a}\exp(-aC)$, oil prices are certain, $\sigma = 0$, and the interest rate is constant, $\omega = 0$. The value function must satisfy the optimality condition in equation 2.10, which reduces to,

$$-\tfrac{1}{a}J_F + \{J_t dt + J_F(rF + PO + Y - C)\} = 0 \tag{5.9}$$

Let us define total wealth at time t, $W(t)$, to be the sum of foreign assets, $F(t)$, the present value of oil income $V^i(t)$, and the present value of permanent non-oil income less depreciation $C^*/r = (Y^* - \delta K^*)/r$. The present value of oil income is found by discounting at the risk-free rate, and will depend on whether the economy is in the Anticipation, Boom or Constant income stage of the oil windfall,

$$\begin{array}{rcll} V^C(t) & = & 0 & \text{for} \quad T_2 \le t \\ V^B(t) & = & PO(1 - e^{-r(T_2-t)})/r & \text{for} \quad T_1 \le t < T_2 \\ V^A(t) & = & PO(e^{-r(T_1-t)} - e^{-r(T_2-t)})/r & \text{for} \quad 0 \le t < T_1 \end{array} \tag{5.10}$$

The explicit form of the value function will be as follows, which can be verified by substitution into the partial differential equation 5.9,

Parameter	Value	Parameter	Value
a	1.00	f	0.49
ρ	0.05	$F_j(0)$	-1.96
$Y(K^*)$	1.00	P_0	1.00
α	0.40	O	0.2
δ	0.03	T_1	4
ω	0.01	T_2	16
$S_L(0)$	-4.00	σ	Various

Table 5.1: Calibration

$$\begin{aligned} J(F,P,t) &= \frac{-A}{a} e^{-ra(W(t))-\rho t} \qquad (5.11) \\ \text{where} \quad A &= \tfrac{1}{r}\exp((r-\rho)/r) \end{aligned}$$

Consumption will satisfy the permanent income hypothesis. From equation 5.1, consumption will be a fixed proportion of total wealth, $C(t) = rW(t)$ when $r = \rho$. This means that as oil wealth $V(t)$ is extracted, it is converted into foreign assets $F(t)$, and only the permanent income is consumed. The partial derivatives $\partial C^i(t)/\partial P(t)$ are the same as those in equation 3.6.

Appendix C: Calibration

The following values were used in the calibrated simulations:

Appendix 1: Proof of Principle 1

i) A social planner facing a certain but temporary windfall, which can freely borrow at the world rate of interest, should smooth consumption based on the rates of return and time preference, $\dot{C}^i(t) = \frac{1}{a}(r-\rho)$ for all t.

This follows directly from equation 3.1.

ii) Consumption will be a constant proportion of total wealth: the sum of foreign assets, oil wealth and steady state consumption, $C^i(t) = rW(t)$ where $W(t) = F(t) + V(t) + \frac{1}{r}C^*$ and $r = \rho$.

The present value of oil income is given in equation 5.10. Combining this with equations 3.3 and 3.5 gives the result.

iii) This will involve borrowing before the windfall, saving during the windfall in a "Future Generations" fund, and consuming the interest earned after the windfall, $\dot{F}^A(t) < 0, \dot{F}^B(t) > 0, \dot{F}^C(t) = 0$.

From $F^i(t) = a_1^i e^{rt^i} + a_2^i$ and equation 3.5, assets will evolve according to,

$$\begin{aligned}
F^C(t) &= F(T_2) && \text{for} \quad T_2 \le t \\
F^B(t) &= F(T_1) + PO(e^{rt} - e^{rT_1})e^{-rT_2}/r && \text{for} \quad T_1 \le t < T_2 \\
F^A(t) &= F(0) + PO(e^{rt} - 1)(e^{-rT_2} - e^{-rT_1})/r && \text{for} \quad 0 \le t < T_1
\end{aligned} \tag{5.12}$$

Therefore, $\dot{F}^A(t) = POe^{rt}(e^{-rT_2} - e^{-rT_1}) < 0$, $\dot{F}^B(t) = POe^{-r(T_2-t)} > 0$ and $\dot{F}^C(t) = 0$.

iv) Less will be saved from a long windfall, and more borrowed beforehand, $\frac{\partial}{\partial T_2}\dot{F}^i(t) < 0$ for $t < T_2$.

$\dot{F}^A(t) < 0$ is the amount borrowed in period $0 \le t < T_1$. From the results in ii), $d\dot{F}^A(t)/dT_2 = -rPOe^{-r(T_2-t)} < 0$, so a longer windfall will cause the social planner to borrow more before the windfall. $\dot{F}^B(t) > 0$ is the amount saved in period $T_1 \le t < T_2$. Also from part ii), $d\dot{F}^B(t)/dT_2 = -rPOe^{-r(T_2-t)} < 0$, so a longer windfall reduces the amount saved each period during the windfall.

v) Investment will be independent of any oil discovery if capital is abundant and goods are freely traded, consistent with the Fisher separation theorem, $K = K^*$.

When interest rates are constant and goods can be freely bought and sold at the world price, then $\omega = 0$ in equation 2.9. Capital will therefore be constant, $K = K^* = (\frac{r+\delta}{\alpha A})^{1/(\alpha-1)}$. Technology is defined to normalise $Y(K^*) = 1$ so $K^* = \frac{\alpha}{r+\delta}$.

Appendix 2: Proof of Principle 2

The particular solution to equation 3.9, so that a_1^i, a_2^i are defined in equation 3.5, is,

$$\begin{aligned} F_V^C(t) &= F_V^B(T_2) && (5.13)\\ F_V^B(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(t^B B_1 - B_2(1 + r(T_2 - T_1 - t^B))(e^{rt^B} - 1) \\ &\quad -B_3(e^{2rt^B} - 1)\Big) + F_V^A(T_1) && (5.14)\\ F_V^A(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1(e^{rt} - 1) - A_2(e^{2rt} - 1)\Big) && (5.15) \end{aligned}$$

where $t^B = t - T_1$, $B_1 = r(1 + 2e^{-r(T_2-T_1)})$, $B_2 = 2e^{-r(T_2-T_1)}$, $B_3 = \frac{1}{2}e^{-2r(T_2-T_1)}$ $A_1 = 2\Big((e^{-rT_1} - e^{-rT_2}) - (T_2 - T_1)re^{-rT_2}\Big)$, $A_2 = \frac{1}{2}(e^{-rT_1} - e^{-rT_2})^2$, and all $A_j, B_j \geq 0$. This can be verified by substitution into $\ddot{F}^i(t) - r\dot{F}^i(t) = \frac{1}{2}aP^2\sigma^2C_P^i(t)^2$.

The proof of this principle relies on the following two Lemmas,

Lemma 1. *The balance of the Volatility Fund never decreases in expectation, $\dot{F}_V^i(t) > 0$ for all $t > 0$.*

Proof. The time derivative of the fund during each phase is,

$$\begin{aligned} \dot{F}_V^C(t) &= 0 && (5.16)\\ \dot{F}_V^B(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(B_1 - rB_2(1 + (T_2 - T_1 - t^B)re^{rt^B}) - B_3 2re^{2rt^B}\Big) && (5.17)\\ \dot{F}_V^A(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1re^{rt} - A_2 2re^{2rt}\Big) && (5.18) \end{aligned}$$

During the Anticipation phase we begin by showing that $\dot{F}_V^A(T_1) > 0$,

$$
\begin{aligned}
\dot{F}_V^A(T_1) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1re^{rT_1} - A_2 2re^{2rT_1}\Big) \\
&= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r}\Big(1 - (2r(T_2 - T_1) + e^{-r(T_2-T_1)})e^{-r(T_2-T_1)}\Big)
\end{aligned}
$$

Now, when $T_2 = T_1$ then $\dot{F}_V^A(T_1) = 0$. Also, by l'Hopital's rule, $\lim_{(T_2-T_1)\to\infty} \dot{F}_V^A(T_1) = \frac{1}{2}aP^2O^2\sigma^2\frac{1}{r} > 0$. As $\ddot{F}_V^A(T_1) < 0$ for all T_1 then $\dot{F}_V^A(T_1) > 0$. It is also easy to see that $\dot{F}_V^A(t) > \dot{F}_V^A(T_1)$ for all $t < T_1$. Thus, $\dot{F}_V^A(t) > 0$.

During the Boom phase it is straightforward to show that $\dot{F}_V^B(T_2) > 0$ by substitution (that is when $t^B = T_2 - T_1$). Furthermore, $\dot{F}_V^B(t) > \dot{F}_V^B(T_2)$ for all $t < T_2$. Thus, $\dot{F}_V^B(t) > 0$. □

Lemma 2. *The rate of increase of the Volatility Fund slows over time, $\ddot{F}_V^A(t), \ddot{F}_V^B(t) < 0$ for $t < T_2$.*

Proof. The second derivative of $F_V^i(t)$ with respect to time is,

$$
\begin{aligned}
\ddot{F}_V^C(t) &= 0 && (5.19) \\
\ddot{F}_V^B(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\Big(B_2(1 - r(T_2 - T_1 - t^B))e^{rt^B} - B_3 4e^{2rt^B}\Big) && (5.20) \\
\ddot{F}_V^A(t) &= \tfrac{1}{2}aP^2O^2\sigma^2\Big(A_1e^{rt} - A_2 4e^{2rt}\Big) && (5.21)
\end{aligned}
$$

During the Anticipation phase we begin by showing that $\ddot{F}_V^A(0) < 0$. When $t = 0$,

$$
\ddot{F}_V^A(0) = aP^2O^2\sigma^2\Big((e^{-rT_1} - e^{-rT_2}) - (T_2 - T_1)re^{-rT_2} - (e^{-rT_1} - e^{-rT_2})^2\Big)
$$

If $T_2 = T_1$then $\ddot{F}_V^A(0) = 0$. So, if $T_2 > T_1$ then $\ddot{F}_V^A(0) < 0$. Now, we also know from inspection of equation 5.21 that $\ddot{F}_V^A(t) < \ddot{F}_V^A(0)$ for all $0 < t < T_1$. Therefore $\ddot{F}_V^A(t) < 0$ for all $0 < t < T_1$.

During the Boom phase we begin by showing that $\ddot{F}_V^B(T_1) < 0$ for $r < \frac{2}{3}$ (that is, when $t^B = 0$),

$$\ddot{F}_V^B(T_1) \;=\; \tfrac{1}{2}aP^2O^2\sigma^2\Big(r(1+2e^{-r(T_2-T_1)})(1-r(T_2-T_1)) - 2e^{-2r(T_2-T_1)}\Big)$$

If $T_2 = T_1$ then $\ddot{F}_V^B(T_1) < 0$ if $r < \frac{2}{3}$. If $T_2 > T_1$ then $\ddot{F}_V^B(T_1)$ will become more negative. We can also see from inspection of equation 5.20 that $\ddot{F}_V^B(t) < \ddot{F}_V^B(T_1)$ for all $T_1 < t < T_2$. Therefore $\ddot{F}_V^B(t) < 0$ for all $T_1 < t < T_2$. □

Turning now to the parts of principle two:

Build a Volatility Fund quickly:

i) A social planner facing a volatile temporary windfall, who can freely borrow at the world rate of interest, should engage in precautionary savings and build up a "Volatility Fund", in addition to the "Future Generations" fund in principle 1, $F_V^i(t) > 0$ for all $t > 0$.

The Volatility Fund begins at $F_V^A(0) = 0$, as prior to discovering oil there is no exposure to volatility. The expected balance of the fund is also always increasing, $\dot{F}_V^i(t) > 0$ for all $0 < t < T_2$ by Lemma 1. Therefore $F_V^i(t) > 0$ for all $t > 0$.

ii) The Volatility Fund should receive relatively more savings during the early years of the windfall, including during the Anticipation phase, $\dot{F}_V^A(t), \dot{F}_V^B(t) > 0$ and $\ddot{F}_V^A(t), \ddot{F}_V^B(t) < 0$ for $t < T_2$.

The balance of the Volatility Fund increases for all $0 < t < T_2$, $\dot{F}_V^A(t), \dot{F}_V^B(t) > 0$ by Lemma 1. However, the rate it increases will diminish over time, $\ddot{F}_V^A(t), \ddot{F}_V^B(t) < 0$ for $t < T_2$ by Lemma 2. Thus, it will receive relatively more savings in the early years of the windfall.

Then leave it alone:

iii) The assets in the Volatility Fund should not be consumed in the presence of a persistent negative shock to the oil price. Only interest should be consumed, $\frac{\partial}{\partial P_0} F_V^i(0) = 0$.

We know that $F_V^i(0) = 0$. Thus, a contemporaneous oil price shock will not affect the stock of assets in the Volatility Fund, only the subsequent path of consumption.

iv) As oil is extracted, the need for a Volatility Fund will diminish, $\dot{C}^B(t) \to 0$ as $t \to T_2$.

The marginal effect of a change in oil prices on consumption falls as oil is extracted, $C_P^B(t) = O(1 - e^{-r(T_2 - t)}) \to 0$ as $t \to T_2$ from equation 3.6. The result then follows directly from equation 3.7, capturing a diminishing incentive for precautionary savings.

v) Any funds remaining in the Volatility Fund should be saved, and only the permanent income consumed, $\dot{F}_V^i(t) \geq 0 \quad \text{for all} \quad t \leq T_2$.

This follows from Lemma 1.

Appendix 3: Proof of Principle 3

i) When there is both a traded and a non-traded sector, an oil discovery should increase investment in the non-traded sector, and decrease investment in the traded sector, with the aim of stabilising the real exchange rate, $e = e^*$.

Firms in this model choose the mix of capital and labour to maximise profit, from equation 3.14. This gives rise to the following first order conditions, and the following Lemma,

$$\gamma_G P_j Y_j / K_{G,j} = r_G \tag{5.22}$$

$$\gamma_P P_j Y_j / K_{P,j} = r_P \tag{5.23}$$

$$(1 - \gamma_G - \gamma_P) P_j Y_j / L_j = w \tag{5.24}$$

Lemma 3. *When the traded good is sold on international markets for a constant price, capital is freely allocated at constant interest rates, and there are constant returns to scale; both the wage and the price of non-traded goods will also be constant, $w = w^*$ and $P_N = P_N^*$. They are also independent of the aggregate level of consumption, C. This follows from the factor-price equalization theory.*

Proof. Production in each sector is given by $Y_j = AK_{G,j}^{\gamma_G} K_{P,j}^{\gamma_P} L_j^{1-\gamma_G-\gamma_P}$. When capital is freely traded on world markets, the only scarce factor of production is labour. If a unit of labour moves into a sector, it will be accompanied by both private and public capital, which have been freely borrowed at the world rate of interest. This ensures that the marginal product of each factor equals its price. The amount of capital in each sector can therefore be expressed in terms of labour, using equations 5.22 to 5.24,

$$K_{G,j} = wL_j/P_j \frac{\gamma_G/r}{(1-\gamma_G-\gamma_P)} \tag{5.25}$$

$$K_{P,j} = wL_j/P_j \frac{\gamma_P/r}{(1-\gamma_G-\gamma_P)} \tag{5.26}$$

Output in each sector can also be expressed in terms of labour, making use of the constant returns to scale assumption and substituting 5.25 and 5.26 into the production function,

$$Y_j = A_j L_j b \left(w/P_j\right)^{\gamma_1+\gamma_2} \tag{5.27}$$

where $b = \gamma_1^{\gamma_G} \gamma_2^{\gamma_P} (r(1-\gamma_G-\gamma_P))^{-\gamma_G-\gamma_P}$. The marginal product of labour is essentially constant, as it will be accompanied by capital. The marginal product of labour also equals the wage, $(1-\gamma_G-\gamma_P)P_j Y_j/L_j = w$. Combining this with equation 5.27 gives,

$$w/P_j \;=\; c_j \tag{5.28}$$

where $c_j = (A_j(\gamma_G/r)^{\gamma_G}(\gamma_P/r)^{\gamma_P})^{1/(1-\gamma_G-\gamma_P)}(1-\gamma_G-\gamma_P)$.

In the traded sector, the price is set on world markets, $P_T = 1$, so the wage will be dictated by the price of the traded good, the world rate of interest and the production technology in the traded sector $w = w^* = c_T$.

In the non-traded sector, the price will be set to equate demand and supply for the non-traded good. As the non-traded sector must compete with the traded sector for labour, the price of non-traded goods will also be constant - even if the production technology in each sector differs, $P_N = c_T/c_N$. These results stem from Samuelson's (1948) factor-price equalization theory. $\square$

The social planner will increase aggregate consumption when oil is discovered, as stated in principle one. Equilibrium in the non-traded sector requires that $P_N^{1-a_N}Y_N = a_N C$. An increase in consumption must be met by either a rise in prices or a rise in output. The price of non-traded goods should be constant, from lemma 3. Higher aggregate consumption should therefore be met by higher output. This can only be achieved with a movement of labour from the traded sector, and investment in both public and private capital. The reduction in labour in the traded sector will also cause the sector's capital stock to fall, according to equations 5.25 and 5.26. We assume that this faces no restrictions. Thus, discovering oil will cause investment in the traded sector, with the aim of maintaining a constant real exchange rate, $e = e^* = c_N/c_T$.

ii) This is not just a reallocation of capital. The total capital stock can increase or decrease, depending on the production technology of each good.

The total stock of public and private capital in the economy is given by, $K_k = K_{k,N} + K_{k,T}$ for $k = G, P$. From equations 5.25 and 5.26 we have,

$$K_k = w \sum_{j=N,T} L_j/P_j \frac{\gamma_{k,j}/r}{(1-\gamma_{G,j}-\gamma_{P,j})}$$

So, the allocation of labour, $\bar{L} = L_N + L_T$ will affect the total stock of both public and private capital, unless both sectors use exactly the same production technologies (so that $P_N = P_T = 1$).

iii) During the Anticipation phase this will involve borrowing to finance both consumption and investment in the non-traded sector.

On discovering oil the social planner will immediately increase consumption as stated in principle one. Consumption will remain at this new level in perpetuity, in the absence of volatility. Higher consumption will immediately increase demand for both traded and non-traded goods. To prevent prices rising this will require an immediate investment in the non-traded sector, I_N, as stated in part i). If this exceeds the amount of capital that can be released by the traded sector then it will reduce the stock of foreign assets, F, according to the budget constraint in equation 2.2.

iv) Investing in the non-traded sector may limit the decline of the traded sector (Dutch disease) when the initial capital stock is low, but immigration is more effective.

To show this we compare how an increase in consumption affects the labour market, when capital is abundant and when it is fixed.

When capital is abundant, an increase in consumption will have a constant effect on labour. In this case investment will ensure that the real exchange rate is constant, $P_N = P_N^*$ from Lemma 1. Therefore, an increase in consumption must be met by an increase in output of the non-traded good, $C = P_N^{*1-a_N} Y_N / a_N$. Using this, equation 5.27 and $\frac{dL_N}{dC} \cdot \frac{dC}{dL_N} = 1$ we get,

$$\frac{dL_N}{dC} = \frac{a_N}{A_N b} w^{*(-\gamma_G-\gamma_P)} P_N^{*(1-a_N-\gamma_G-\gamma_P)} \tag{5.29}$$

So when capital is abundant, an exogenous increase in consumption will have a constant effect on labour (unless all labour is already employed in the non-traded sector, which we ignore).

When capital is fixed, an increase in consumption will have a varying effect on labour, depending on the initial level of consumption. The wage will still be driven by the marginal product of labour in the traded sector, $w = (1 - \gamma_G - \gamma_P)k_T(1 - L_N)^{-\gamma_G - \gamma_P}$, using the constant, $k_T = A_T K_{G,T}^{\gamma_G} K_{P,T}^{\gamma_P}$, for simplicity. In this case an increase in aggregate consumption will be met by both a higher price and quantity of the non-traded good, $C = P_N^{1-a} Y_N / a$. To isolate the effect on the labour market we combine this with the marginal product of labour in the non-traded and the traded sectors, equation 5.24, to give,

$$\begin{aligned} w &= (1 - \gamma_G - \gamma_P)(aC/Y_N)^{1/(1-a)}(Y_N/L_N) \\ C &= \tfrac{1}{a} k_N^a k_T^{(1-a)} L_N^{(1-a\gamma_G - a\gamma_P)}(1 - L_N)^{-(1-a)(\gamma_G+\gamma_P)} \end{aligned}$$

The effect of a change in consumption on the labour market when capital is fixed, $L_N^{'}$, will be,

$$\frac{dL_N^{'}}{dC} = C^{-1} \frac{(1 - L_N^{'})L_N^{'}}{(1 - a\gamma_G - a\gamma_P) - L_N^{'}(1 - \gamma_G - \gamma_P)} \tag{5.30}$$

Thus, the effect of a change in consumption on the labour market when capital is fixed will depend on the initial level of consumption. This in turn is a function of the initial capital stock, and the associated allocation of labour between the traded and non-traded sectors. If the initial capital stock is sufficiently low, then an increase in consumption will require a large movement of labour from the traded to the non-traded sector when capital is fixed. Investing in the non-traded sector may then reduce the marginal effect of consumption on labour, $\frac{dL_N^{'}}{dC} > \frac{dL_N}{dC}$. Output in each sector will be driven by the reallocation of labour. We have $\frac{dY_T'}{dC} = \frac{dY_T'}{dL_N'} \cdot \frac{dL_N'}{dC} < \frac{dY_T}{dC} = \frac{dY_T}{dL_N} \cdot \frac{dL_N}{dC}$ if the initial capital stock is sufficiently low, as $\frac{dY_T'}{dL_N'}, \frac{dY_T}{dL_N} < 0$. So, under some parametrisations, investing in the non-traded sector can limit the decline of the traded sector.

Immigration is more effective at limiting the decline of the traded sector. The contraction of the traded sector is driven by the limited availability of labour. Consider instead the case of ready access to immigrant labour. If the social planner sets the wage at $w = w^*$, such as a reference wage paid to government employees, total labour will freely adjust to ensure that its marginal product equals the wage in each sector. The decline in the traded sector described in part i), driven by the reallocation of labour from the traded to the non-traded sector, would be avoided as immigrant workers fill the demand for labour in the non-traded sector.

Appendix 4: Proof of Principle 4

i) Consumption in capital-scarce countries should begin lower than in capital abundant economies, $C_L^C(0) < C_H^C(0)$, but jump further when oil is discovered, $C_L^A(0) - C_L^C(0) > C_H^A(0) - C_H^C(0)$. It should still remain lower in absolute level though, $C_L^A(0) < C_H^A(0)$.

Consumption in capital scarce economies should begin lower, $C_L^C(0) < C_H^C(0)$. Phase C describes the economy in the absence of oil. If capital is scarce (L), consumption and assets will lie on the stable saddle path during this phase, $C_L^C(t) = (r - \lambda_1)S_L^C(t) + C^*$, from equations 4.3, 4.4 and 4.5. If capital is abundant (H) they will lie on the steady state line, $C_H^C(t) = rS_H^C(t) + C^*$, from equation 3.2 with $\dot{F} = O = 0$. As $\lambda_1 < 0$ and $S_L^C(0) < S_H^C(0) < 0$ (as the capital stock will also be lower when capital is scarce), $C_L^C(0) < C_H^C(0)$ for any given level of C^*.

Consumption in capital scarce economies should jump further when oil is discovered, $C_L^A(0) - C_L^C(0) > C_H^A(0) - C_H^C(0)$. First we note that $C_j^C(0)$ is the level of consumption given the starting level of assets, $S_j(0)$, during the Constant income phase for $j = [L, H]$. This is the level of consumption that would hold at time $t = 0$ if there is no windfall. Now, we have,

$$\begin{aligned}
C_L^A(0) - C_L^C(0) &= (r-\lambda_1)S_L(0) + PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) + C^* \\
&\quad - S_L(0)(r-\lambda_1) - C^* \\
&= PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2})
\end{aligned}$$

$$\begin{aligned}
C_H^A(0) - C_H^C(0) &= rS_H(0) + PO(e^{-rT_1} - e^{-rT_2}) + C^* - rS_H(0) - C^* \\
&= PO(e^{-rT_1} - e^{-rT_2})
\end{aligned}$$

Now, $(e^{-rT_1} - e^{-rT_2}) - (e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) < 0$, for $\omega > 0$ as $\lambda_2 = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 + 8\omega f/a}$. So on discovering oil, consumption will jump further in a capital-scarce than a capital abundant economy.

However, consumption in capital scarce economies will remain lower in absolute level, $C_L^A(0) < C_H^A(0)$. Let the difference between consumption in capital abundant and capital-scarce economies be, $c^i(t) = C_H^i(t) - C_L^i(t)$. The associated difference in assets is $s^i(t) = S_H^i(t) - S_L^i(t)$. The difference in consumption will satisfy,

$$\begin{aligned}
\dot{c}^i(t) &= \dot{C}_H^i(t) - \dot{C}_L^i(t) \\
&= \tfrac{2\omega}{a} f S_L^i(t) \\
&= -\tfrac{2\omega}{a} f s^i(t) + \tfrac{2\omega}{a} f S_H^i(t) \\
&< 0
\end{aligned}$$

So, consumption in the capital-scarce economy grows more quickly than consumption in the capital abundant economy. The difference in assets will satisfy $\dot{s}^i(t) = rs^i(t) - c^i(t)$. These dynamics are summarised by the differential equation $\ddot{s}^i(t) - r\dot{s}^i(t) - \frac{2\omega}{a}s^i(t) = -\frac{2\omega}{a} f S_H^i(t)$. The explicit solution to this is,

$$
\begin{aligned}
s^i(t) &= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + a_1 e^{rt} + a_2 && (5.31)\\
&= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + S_H^A(t) \\
c^i(t) &= (r-\lambda_1) k_1 e^{\lambda_1 t} + (r-\lambda_2) k_2 e^{\lambda_2 t} + r a_2 && (5.32)
\end{aligned}
$$

To find the coefficients k_1, k_2 we impose two constraints. First, at $t = 0$, $0 = k_1 + k_2 + S_L(0)$ and $s(0) > 0$ as the initial domestic capital stock will be lower when capital is scarce. Second, the difference between the capital-scarce and capital abundant economies will disappear if foreign assets ever reach zero. So, if for any point in time $t = t^* > 0$, $S_H^i(t^*) = S_L^i(t^*) = s^i(t^*) = 0$, then the difference in consumption will also be zero, $c^i(t^*) = 0$. This gives $k_1 = -S_L(0) - k_2$ and $k_2 = ra_2(\lambda_2 - \lambda_1)^{-1} e^{-\lambda_2 t^*}$. Substituting this into equation 5.32 gives $c^A(0) = \lambda_1 S_L(0) + PO(e^{-rT_1} - e^{-rT_2})(1 - e^{-\lambda_2 t^*}) > 0$. Thus, the absolute level of consumption on discovering oil in a capital-scarce economy will be lower than in a capital abundant economy, $C_H^A(0) > C_L^A(0)$.

ii) Borrowing in capital-scarce countries before a boom should initially be lower than capital abundant countries, $\dot{F}_L^A(0) > \dot{F}_H^A(0)$, and if they begin to borrow it will happen near the date of extraction, $\ddot{F}_L^A(t) < 0$ for $t < T_1$.

From part i) we have $C_H^A(0) > C_L^A(0)$. As $\dot{S}_j^A(t) = rS_j^A(t) + C^* - C_j^A(t)$ for $j = [L, H]$, so $\dot{S}_H^A(0) < \dot{S}_L^A(0)$. We also know that $F_H^i(t) = S_H^i(t)$ and $F_L^i(t) = fS_L^i(t)$ where $f < 1$. So, $\dot{F}_L^A(0) > \dot{F}_H^A(0)$ and capital scarce countries will initially borrow less.

As the Anticipation phase proceeds, the amount saved by a capital-scarce economy will decrease (or the rate of borrowing will increase). From equation 4.3 we have $S^i(t^i) = c_1^i e^{\lambda_1 t^i} + c_2^i e^{\lambda_2 t^i}$ and $\ddot{S}^A(t) = c_1^A \lambda_1^2 e^{\lambda_1 t} + c_2^A \lambda_2^2 e^{\lambda_2 t}$. We know that $c_1^A = S(0) - c_2^A$ and $c_2^A = -PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2})(\lambda_2 - \lambda_1)^{-1} < 0$ from equation 4.5. If a windfall is small, its present value will be much less than is needed to overcome capital scarcity, $PO(e^{-rT_1} - e^{-rT_2})/r \ll \mid S(0) \mid$. This is because the country will also be saving from income Y and we assume that $S(T_2) < 0$. As $c_2^A \approx -PO(e^{-rT_1} - e^{-rT_2})/r$ and $S(0) < 0$ we have,

$c_1^A = S(0) - c_2^A < 0$ for a small windfall. So, $\ddot{S}^A(t) = c_1^A\lambda_1^2 e^{\lambda_1 t} + c_2^A\lambda_2^2 e^{\lambda_2 t} < 0$ and in turn $\ddot{F}^A(t) < 0$.

iii) Domestic capital in capital-scarce economies should grow over time, $\dot{K}_L^C(t) > 0$, be accelerated during an oil boom, $\dot{K}_L^B(t) > \dot{K}_L^C(t)$, and comprise an increasing share of the total capital stock over time, $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$.

Domestic capital should grow over time, $\dot{K}_L^C(t) > 0$. In equation 4.3 we see that $\dot{S}^C(t) > 0$ for all t. We also know that $S^C(t) = (1-f)(K^C(t) - K^*)$. As $f < 1$ we have $\dot{K}_L^C(t) > 0$.

The growth of domestic capital will be accelerated during an oil boom, $\dot{K}_L^B(t) > \dot{K}_L^C(t)$. From equation 4.3 we have $\dot{S}^i(t) = c_1^i\lambda_1 e^{\lambda_1 t^i} + c_2^i\lambda_2 e^{\lambda_2 t^i}$. Let us consider how total assets evolve with or without an oil boom, starting at the same initial level of assets S_0. From equation 4.5 we would have $c_1^C = S_0$, $c_2^C = 0$, $c_1^B = S_0 - c_2^B$ and $c_2^B > 0$. So, $\dot{S}^B(t) > \dot{S}^C(t)$ as $c_1^B < c_1^C$ and $\lambda_1 < 0$, and $c_2^B > c_2^C$ and $\lambda_2 > 0$. Therefore $\dot{K}^B(t) > \dot{K}^C(t)$.

Domestic capital will also comprise an increasing share of the total capital stock over time, $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$. We know that $\bar{S}(t) = F(t) + K(t)$. We also can see that $\dot{F}(t) = \frac{\alpha(1-\alpha)}{2\omega}AK(t)^{\alpha-2}\dot{K}(t)$ from equation 2.9. At $K(t) = K^* = \frac{\alpha}{r+\delta}$ and using $A = (\frac{r+\delta}{\alpha})^\alpha$ from Appendix A, $\frac{\alpha(1-\alpha)}{2\omega}AK(t)^{\alpha-2} = \frac{(1-\alpha)}{2\omega}\frac{(r+\delta)^2}{\alpha}$. For $\alpha \approx (1-\alpha)$ and r, δ and ω of similar order of magnitude, $\frac{\alpha(1-\alpha)}{2\omega}AK^{\alpha-2} < 1$. So, near K^* domestic capital grows more quickly than foreign assets, $\dot{F}(t) < \dot{K}(t)$. Domestic capital will therefore increase its share in total assets, $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$. This will not necessarily hold if $K(t)$ is significantly below K^*, which we abstract from in our linearised analysis.

Appendix 5: Proof of Principle 5

i) All else being equal (including extraction costs), a shorter, high intensity windfall (HI) is always preferred to a long, low intensity one (LI), $\int_0^\infty U(C_{HI}(\tau))e^{-\rho\tau}d\tau > \int_0^\infty U(C_{LI}(\tau))e^{-\rho\tau}d\tau$.

This follows from examining the budget constraint, 4.2. The present value of a long, low intensity windfall will be lower than a short, high intensity one. Assuming that production begins at $T_1 = 0$ for simplicity, the present value of a windfall of intensity nO, for period T_n/n is $PV(n) = PnO(1 - e^{-rT_2/n})/r$. As $\partial PV/\partial n > 0$, more intense windfalls have a higher present value. Windfalls with a higher present value relax the budget constraint further. So, they must increase utility as any consumption path under a long, mild windfall can be replicated under a short, intense one.

ii) However, if capital is sufficiently scarce then fewer assets will be accumulated from a short, high intensity windfall than a long, low intensity one after an equal amount of time has passed, $S_{HI}(t) < S_{LI}(t)$ for $t > T_2$.

For simplicity assume $T_1 = 0$. The value of assets at the end of a low intensity windfall is given by $S^B_{LI}(T_2) = S(0)e^{\lambda_1 T_2} + PO(\lambda_2 - \lambda_1)^{-1}(1 - e^{(\lambda_1-\lambda_2)T_2})$. The value of assets at the end of a high intensity windfall, of nO units of oil per period for T_2/n periods ($n > 1$), is given by $S^B_{HI}(T_2/n) = S(0)e^{\lambda_1 T_2/n} + P(nO)(\lambda_2 - \lambda_1)^{-1}(1 - e^{(\lambda_1-\lambda_2)T_2/n})$. By time $t = T_2$this will have grown to $S^B_{HI}(T_2) = S^B_{HI}(T_2/n)e^{\lambda_1 T_2(1-1/n)}$. The difference in assets at time $t = T_2$ is given by,

$$\begin{aligned} S^B_{LI}(T_2) - S^B_{HI}(T_2) &= PO(\lambda_2 - \lambda_1)^{-1} \\ &\times\Big((1 - e^{(\lambda_1-\lambda_2)T_2}) - n(e^{\lambda_1 T_2(1-1/s)} - e^{T_2(\lambda_1-\lambda_2/n)})\Big) \quad (5.33) \end{aligned}$$

If $\omega = 0$ then $S^B_{LI}(T_2) - S^B_{HI}(T_2) = PO(r)^{-1}\Big((1 - e^{-rT_2}) - n(1 - e^{-rT_2/n})\Big)$, which is unambiguously negative. Thus, in a capital abundant economy a high intensity windfall will lead to a larger Future Generations Fund.

However, if ω is sufficiently large, or s sufficiently high, then equation 5.33 will be positive and a low intensity windfall will lead to a larger Future Generations Fund. This is easily seen in the illustration in Figure 4.4.

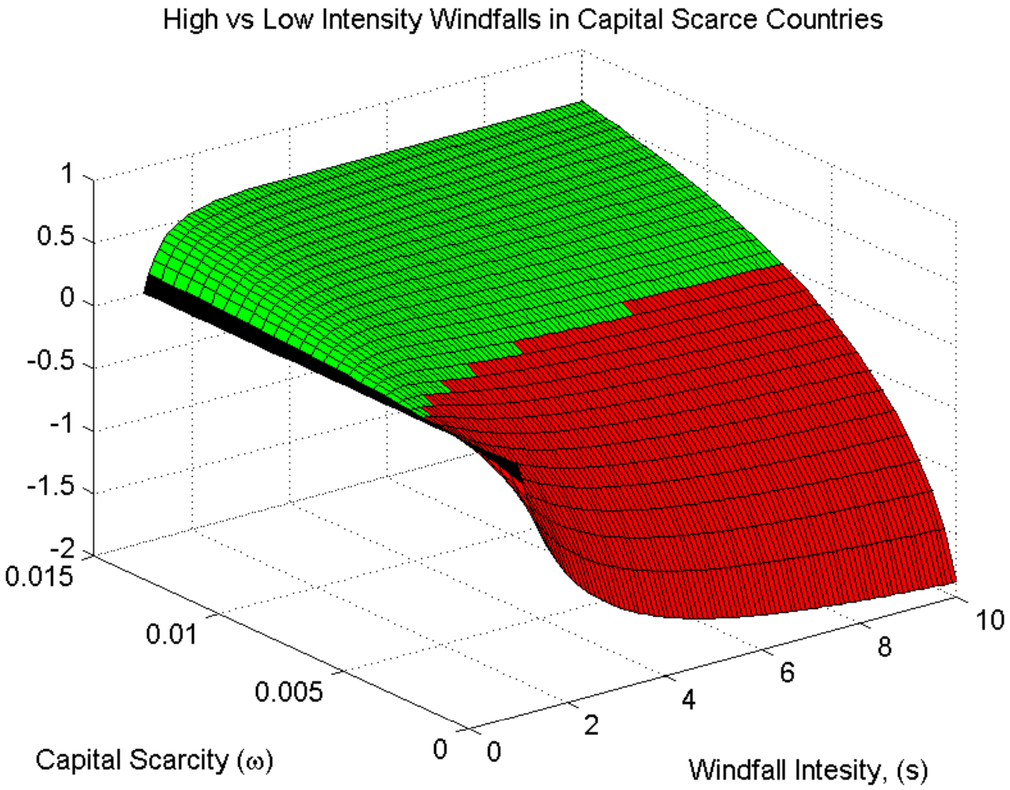

Figure 5.1: A comparison of assets accumulated after High and Low intensity windfalls. This is a plot of equation 5.33, where $S^B_{LI}(T_2) - S^B_{HI}(T_2) > 0$ is illustrated in green. In the green region more assets will be accumulated under a low intensity than a high intensity windfall.

Appendix 6: Proof of Principle 6

i) In capital scarce countries with absorption constraints, part of an oil discovery should be accumulated in a temporary offshore Parking Fund while absorption constraints are relaxed, $F_P^i(t) > 0$ if $\phi > 0$.

We begin by replacing the social planner's budget constraint in equation 2.2 with equation 4.6, and finding the first order conditions to the social planner's problem, replacing those in equations 5.1 to 5.4,

$$\begin{aligned} U'(C)e^{-\rho t} - J_F &= 0 && (5.34)\\ -J_F(1+\phi I/K) + J_K &= 0 && (5.35)\\ -J_F(Y_K + 0.5\phi I^2 K^{-2}) + \delta J_K &= \tfrac{1}{dt}E[dJ_K(F,P,K,t)] && (5.36)\\ -(r-2\omega F)J_F &= \tfrac{1}{dt}E[dJ_F(F,P,K,t)] && (5.37) \end{aligned}$$

Now let us introduce the ratio of the marginal utility from an extra unit of capital to that of foreign assets, $q = J_K/J_F$. Using this new variable we can summarise equations 5.35 to 5.37 in equations 4.7 and 4.8, reproduced below,

$$\begin{aligned} q &= 1 + \phi\frac{I}{K}\\ \dot{q} &= q(r - \omega F + \delta) - Y_K - 0.5\tfrac{1}{\phi}(q-1)^2 \end{aligned}$$

The first equation shows that the marginal utility from an extra unit of installed capital will be greater than for an extra unit of foreign assets, $q > 1$ as $\phi I/K > 0$. In the absence of absorption constraints the marginal utilities are equal, $q = 1$ when $\phi = 0$ from equation 5.2. Marginal utility is diminishing in both capital and assets, $J_{KK}, J_{FF} < 0$. This is also consistent with equations 5.36 and 5.37, where the marginal utilities fall over time as capital and foreign assets are accumulated, $\frac{1}{dt}E[dJ_K], \frac{1}{dt}E[dJ_F] < 0$. So, for a given stock of assets, $\bar{S} = F + K$, when $\phi > 0$ then $J_K > J_F$ and there will be relatively more foreign assets and less domestic capital than when $\phi = 0$ and $J_F = J_K$. We call the additional foreign assets in the presence of absorption constraints a Parking Fund, which is the difference between foreign assets when $\phi > 0$ and $\phi = 0$, $F_P^i(t) = F_\phi^i(t) - F^i(t) > 0$. Illustrations of the size of this fund are provided in van der Ploeg (2012).

The parking fund will be temporary. As capital is accumulated absorption constraints will disappear, the marginal utilities of foreign assets and capital will converge and the Parking Fund will disappear, as $K \to \infty$, $q \to 1$ and $F_P^i(t) \to 0$.

ii) In capital abundant countries an oil discovery will not affect the size of any Parking Fund.

In capital abundant economies there is no debt premium on interest, $\omega = 0$. In this case the equations 4.7, 4.8 and 2.3 form a system of three dynamic equations in three variables, I, K and q. This system can be solved without any reference to consumption, foreign assets or oil. Thus, the separation theorem in principle 1 holds and an oil discovery will not alter the path of investment in a capital abundant economy.

Appendix 7: Proof of Principle 7

We begin with the following expressions for $S_V^i(t)$:

$$S_V^C(t) = S_V^B(T_2) \tag{5.38}$$

$$\begin{aligned} S_V^B(t) &= \tfrac{1}{4\omega f} a^2 P^2 O^2 \sigma^2 \Big(B_1(e^{2rt^b} - e^{\lambda_1 t^b}) - B_2(e^{rt^b} - e^{\lambda_1 t^b}) \\ &\quad - B_3(e^{\lambda_2 t^b} - e^{\lambda_1 t^b}) + (1 - e^{\lambda_1 t^b}) \Big) \end{aligned} \tag{5.39}$$

$$S_V^A(t) = \tfrac{1}{4\omega f} a^2 P^2 O^2 \sigma^2 \Big(A_1(e^{2rt} - e^{\lambda_1 t}) - A_2(e^{\lambda_2 t} - e^{\lambda_1 t}) \Big) \tag{5.40}$$

Where $t^b = t - T_1$, $B_1 = \frac{\omega f/a}{\omega f/a - r^2} e^{-2r(T_2 - T_1)}$, $B_2 = 2e^{-r(T_2-T_1)}$, $B_3 = \frac{r^2}{\omega f/a - r^2} \frac{2r - \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2(T_2-T_1)}$, $A_1 = \frac{\omega f/a}{\omega f/a - r^2}(e^{-rT_1} - e^{-rT_2})^2$ and $A_2 = \frac{e^{-\lambda_2 T_1}}{(\lambda_2 - \lambda_1)} \left\{ \frac{(2r-\lambda_1)}{\omega f/a - r^2} r^2 \left(1 - 2e^{-r(T_2-T_1)} + e^{-\lambda_2(T_2-T_1)}\right) + 2r(1 - e^{-r(T_2-T_1)}) \right\}$, where all $B_j, A_k > 0$ for $\omega f/a - r^2 > 0$. Also, we have, $C^i_V(t) = rS^i_V(t) - \dot{S}^i_V(t)$. This can be verified by substitution into equations 4.9 and 4.10.

i) Capital-scarce countries should accumulate less in an offshore Volatility Fund than capital abundant economies, $F^i_{V,L} < F^i_{V,H}$, despite the fund accelerating the expected rate of development.

The Volatility Fund is the difference between the total level of foreign assets in a stochastic and a deterministic setting (the Future Generations Fund), $F^i_{V,j} = F^i_j - F^i_{FG,j}$, for both capital-scarce (L) and capital-abundant (H) economies, $j = [L, H]$. This Volatility Fund will be a fraction of the total level of assets accumulated to manage volatility, $F^i_{V,j} = fS^i_{V,j}$, which satisfies the dynamic equation, $\dot{S}^i_{V,j} = rS^i_{V,j} + PO^i + C^* - C^i_{V,j}$ for both capital abundant and capital-scarce economies. The volatility effect on consumption, $C^i_{V,j} = C^i_j - C^i_{FG,j}$, will satisfy the following equations, for capital abundant and capital-scarce economies respectively,

$$\begin{aligned} \dot{C}^i_{V,H} &= \dot{C}^i_H - \dot{C}^i_{FG,H} \\ &= \tfrac{1}{2} a P^2 C^i_P(t)^2 \sigma^2 \qquad (5.41) \\ \dot{C}^i_{V,L} &= \dot{C}^i_L - \dot{C}^i_{FG,L} \\ &= \{-\tfrac{2\omega f}{a} S^i_L + \tfrac{1}{2} a P^2 C^i_P(t)^2 \sigma^2\} - \{-\tfrac{2\omega f}{a} S^i_{FG,L}\} \\ &= -\tfrac{2\omega f}{a} S^i_{V,L} + \tfrac{1}{2} a P^2 C^i_P(t)^2 \sigma^2 \qquad (5.42) \end{aligned}$$

The rate of change of the volatility adjustment to consumption is higher in capital-abundant economies, $\dot{C}^i_{V,L}(t) < \dot{C}^i_{V,H}(t)$ for all $t > 0$. This follows from equations 5.38 to 5.40, which show that $S^i_{V,j}(t) > 0$ for all $t > 0$, and equation 5.42. This is consistent with precautionary savings building up a positive Volatility Fund.

Therefore, the adjustment to consumption from volatility will be greater in capital-abundant economies during the early years of the windfall, $C^i_{V,H}(t) < C^i_{V,L}(t) < 0$ for t less than some t^V. The volatility effect on total assets is zero before oil is discovered, for both capital-scarce and capital-abundant economies, $S_{V,j}(0) = 0$ for $j = [L, H]$. The "budget constraint" tying together the volatility adjustments to consumption and total assets is also the same in both capital-abundant and capital-scarce economies, $\dot{S}^i_{V,j} = rS^i_{V,j} + PO^i + C^* - C^i_{V,j}$ for $j = [L, H]$. If both the initial state and the budget constraint are the same in both countries, but the consumption adjustment grows more quickly in the capital-abundant economy, then its initial level must be lower. So, $C^i_{V,H}(t) < C^i_{V,L}(t) < 0$.

As a result, capital-abundant economies will accumulate a larger volatility fund than capital-scarce economies, $F^i_{V,H} > F^i_{V,L}$. The overall amount of assets accumulated to manage volatility will grow faster in a capital-abundant country, $\dot{S}^i_{V,H} > \dot{S}^i_{V,L}$. All of these assets in a capital-abundant economy are accumulated in a Volatility Fund, $F_{V,H} = S_{V,H}$, while only a fraction are in a capital-scarce economy, $F_{V,L} = fS_{V,L}$ where $f \in [0, 1]$. Thus, capital-abundant economies will accumulate a larger Volatility Fund than capital-scarce economies. Over time, the larger Volatility Fund in the capital-abundant country will generate interest payments, which will finance a larger volatility effect on consumption, $C^i_{V,H}(t) > C^i_{V,L}(t)$ for $t > t^V$.

This holds despite the Volatility Fund accelerating the pace of development in expectation. This follows directly from equation 4.9, where volatility increases the rate of change of consumption $\frac{\partial}{\partial\sigma}\dot{C}^i(t) > 0$. This means that consumption will be lower in the short-run, to finance precautionary savings. These precautionary savings lead to the accumulation of a Volatility Fund, $F^i_V(t) > 0$ from equations 5.38 to 5.40, in addition to the future generations fund. This additional Volatility Fund will reduce capital scarcity, and generate income that can finance a higher level of consumption in the future.

ii) Capital-scarce countries should also direct some precautionary savings from an oil boom to domestic investment, $K^i_V(t) > 0$.

From equations 5.40 to 5.38 we have that $S^i_V(t) > 0$ for all $t > 0$. We also know that $K^i_V(t) - K^{*i}_V = (1 - f)S^i_V(t)$ where $f \in [0, 1]$. Finally, we know that the volatility adjustment to the capital stock in the deterministic steady state will be zero, as we assume that there is no volatile income at this point, $K_V^{*i} = 0$. So, a volatile oil windfall should increase the stock of capital, $K_V^i(t) > 0$.

Appendix 8: Proof of Principle 8

i) If a capital scarce country has no domestic private capital market, the government should promote its development to increase the productivity of public investment.

The optimal allocation of a fixed quantity of any factor of production between two sectors is to equate its marginal product in each sector. Consider the problem of allocating a fixed amount of public and private capital and labour across two sectors, $j = N, T$, to maximise revenues,

$$\begin{aligned}
\max_{K_{G,j}, K_{P,j}, L_j} \quad & \textstyle\sum_{j=N,T} \quad P_j A_j K_{G,j}^{\gamma_G} K_{P,j}^{\gamma_P} L_j^{(1-\gamma_G-\gamma_P)} \\
& s.t. \\
1 \quad & = \quad K_{G,N} + K_{G,T} \\
1 \quad & = \quad K_{P,N} + K_{P,T} \\
1 \quad & = \quad L_N + L_T
\end{aligned}$$

The optimal allocation of every factor will equate its marginal product in each sector. For private capital this gives us $\gamma_P P_N Y_N / K_{P,N} = \gamma_P P_T Y_T / K_{P,T}$. This maximises revenues, and in turn the productivity of public investment.

If a country does not have a domestic capital market, then there is nothing to guarantee that the marginal product of private capital in each sector will be equal. Firms in each sector will only be able to invest from retained earnings. The decision to invest or save would be based on equating the marginal product of capital with the marginal utility of consumption by the owner of the firm, based on a condition similar to $U'(C)e^{-\rho t} = J_K$

from equations 5.1 and 5.2. The amount of investment would therefore be based on the existing capital stock in each sector.

However, if a domestic market for private capital can be established, then private capital will be allocated efficiently. Consider the profit maximisation problem of each firm in the traded and non-traded sector, assuming that the government is allocating public capital efficiently, $\gamma_G P_j Y_j / K_{G,j} = r_G$

$$\max_{K_{P,j}, L_j} \pi_j = P_j Y_j - r_G K_{G,j} - r_P K_{P,j} - w L_j$$

This gives the first order condition for each firm, $\gamma_P P_j Y_j / K_{P,j} = r_P$. If there is a well-functioning domestic capital market then both firms will face the same r_P, so capital will be allocated to equate its marginal product in each sector.

ii) If the country has a private capital market with limited access to foreign capital, then the government should "top-up" private capital by lending through the domestic banking sector.

We begin with production before oil is discovered, and assume that public and private capital are allocated efficiently. Their marginal product will equal their respective costs of borrowing, for $j = N, T$,

$$\begin{aligned} \gamma_G P_j Y_j / K_{G,j} &= r - \omega_G F \\ \gamma_P P_j Y_j / K_{P,j} &= r - \omega_P F \end{aligned}$$

Now consider production after oil is discovered. As noted in principle 4, at any point in time the government will have more capital to allocate between each sector, $K' = K'_N + K'_T$, and will seek to maximise output,

$$
\begin{aligned}
\max_{K'_{G,j},K'_{P,j}} \quad & \textstyle\sum_{j=N,T} \quad P_j A_j (K_{G,j} + K'_{G,j})^{\gamma_G} (K_{P,j} + K'_{P,j} + K''_{P,j})^{\gamma_P} L_j^{(1-\gamma_G-\gamma_P)} \\
& s.t. \\
K' \quad & = \quad K'_{G,N} + K'_{G,T} + K'_{P,N} + K'_{P,T}
\end{aligned}
$$

Investment by the public sector will increase the productivity of private capital, so there will be some co-investment by the private sector, $K''_{P,j}$. The private sector will allocate its capital, $K''_{P,j}$, so that its marginal product equals the cost of borrowing. The government will then "top-up" private capital to equate its marginal product with that of public capital. So,

$$
\begin{aligned}
\gamma_P P_j Y_j/(K_{P,j} + K''_{P,j}) &= r_P \\
\gamma_P P_j Y_j/(K_{P,j} + K''_{P,j} + K'_{P,j}) &= r_G \\
\gamma_G P_j Y_j/(K_{G,j} + (K'_j - K'_{P,j})) &= r_G
\end{aligned}
$$

Combining the first and last gives $K''_{P,j} = \frac{r_G/\gamma_G}{r_P/\gamma_P}(K_{G,j} + K'_{G,j}) - K_{P,j}$, so private co-investment is increasing in public investment. Combining all three gives,

$$
K'_{P,j} = \frac{(r_P - r_G)\gamma_P}{(r_P - r_G)\gamma_P + r_P\gamma_G}(K_{G,j} + K'_j)
$$

So, if $r_P = r_G$ then the additional investment by the government in the private sector is zero. As capital scarcity in the private sector increases relative to the public sector, $r_P > r_G$, the optimal government "top-up" to the private sector increases.

iii) If the country has a private capital market and ready access to foreign investment, then the government should invest relatively more in public capital.

Let us now assume that the resource exporter has a well functioning domestic capital market, with foreign direct investment that is financed at the world rate of interest, r.

In this setting there will be more investment in private capital. As the cost of private capital is lower, $r < r - \omega_G F$ for $F < 0$, its marginal product in each sector will also be lower. As there are diminishing returns to private capital, this requires the level of private capital to be higher. Thus, there will be more private capital if it can be financed cheaply at the world interest rate.

More private capital increases the marginal product of public capital. The marginal product of public capital is, $MPK_{G,j} = \gamma_G P_j A_j K_{G,j}^{\gamma_G - 1} K_{P,j}^{\gamma_P} L_j^{1-\gamma_G-\gamma_P}$. Thus, a higher stock of private capital, $K'_{P,j} > K_{P,j}$, will increase the marginal product of public capital, $MPK'_{G,j} > MPK_{G,j}$.

If the government can borrow at a particular rate of interest, then a higher marginal product of public capital will require more investment in public capital. The government can borrow at the rate of interest, $r_G = r - \omega_G F$. For a given rate of interest and two different levels of private capital, $\gamma_G P_j A_j K'^{\gamma_G-1}_{G,j} K'^{\gamma_P}_{P,j} L_j^{1-\gamma_G-\gamma_P} = \gamma_G P_j A_j K_{G,j}^{\gamma_G-1} K_{P,j}^{\gamma_P} L_j^{1-\gamma_G-\gamma_P} = r_G$, then if $K'_{P,j} > K_{P,j}$, $K'_{G,j} > K_{G,j}$ as $\gamma_G \in [0, 1]$.

iv) Investment in capital scarce countries should allow the real exchange rate to steadily appreciate as borrowing constraints are relaxed and capital is accumulated, according to the Rybczynski theorem.

Let us return to the setting in part ii) of this principle, where public and private capital at any point in time are financed at the respective rates of interest, $r_G(F(t)) < r_P(F(t))$. Following the reasoning in Lemma 3, public and private capital will accompany labour as it moves between sectors,

$$K_{G,j} = wL_j/P_j \frac{\gamma_G/r_G(F(t))}{(1-\gamma_G-\gamma_P)} \tag{5.43}$$

$$K_{P,j} = wL_j/P_j \frac{\gamma_P/r_P(F(t))}{(1-\gamma_G-\gamma_P)} \tag{5.44}$$

Output in each sector will be,

$$Y_j \;=\; A_j L_j b(F(t))\,(w/P_j)^{\gamma_1+\gamma_2} \tag{5.45}$$

where $b(F(t)) = (\gamma_G/r_G(F(t)))^{\gamma_G}\,(\gamma_P/r_P(F(t)))^{\gamma_P}(1-\gamma_G-\gamma_P)^{-\gamma_G-\gamma_P}$. In addition we know that the marginal product of labour equals the wage, $(1-\gamma_G-\gamma_P)P_jY_j/L_j = w$. Combining this with equation 5.45 gives,

$$w/P_j \;=\; c_j(F(t)) \tag{5.46}$$

where $c_j(F(t)) = (A_j\,(\gamma_G/r_G(F(t)))^{\gamma_G}\,(\gamma_P/r_P(F(t)))^{\gamma_P})^{1/(1-\gamma_G-\gamma_P)}(1-\gamma_G-\gamma_P)$, and,

$$c_j'(F) \;=\; c_j(F)\left(\frac{-\gamma_G r_G'(F)}{(1-\gamma_G-\gamma_P)r_G(F)}+\frac{-\gamma_P r_P'(F)}{(1-\gamma_G-\gamma_P)r_P(F)}\right) \tag{5.47}$$

In the traded sector $P_T = 1$, so at any point in time the wage will be a function of the level of foreign assets, via the cost of borrowing, $w^*(F(t)) = c_T(F(t))$. Over time, as the cost of borrowing falls there will be more capital employed in the traded sector. The marginal product of labour will therefore rise, and the wage will rise with it.

In the non-traded sector, $P_N^*(F(t)) = c_T(F(t))/c_N(F(t))$. If the production technology in each sector is the same, then $P_N^* = 1$ for any value of $F(t)$, and the real exchange rate should stay constant. However, if the production technology differs then $P_N^*(F(t))$ will change as foreign debt is repaid, the cost of borrowing falls and capital accumulates. If the traded sector is more capital intensive, $\gamma_{j,T} > \gamma_{j,N}$ for $j = N, T$, then the price of the non-traded good will rise as debt is repaid. This is a real appreciation, and follows from the Rybczynski (1955) theorem.

Chapter 2

The Elephant in the Ground:

Oil extraction and asset allocation in sovereign wealth funds

The Elephant in the Ground:
Extracting oil and allocating assets in sovereign wealth funds

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Abstract

Most oil exporters have a large share of their national wealth beneath the ground. Many of these countries also establish sovereign wealth funds with the proceeds from oil extraction. However, the decisions of when to extract oil and what assets to hold are usually disconnected. This paper asks the question, how should subsoil oil change a country's asset allocation above the ground? We show that oil-rich countries should offset their subsoil exposure by changing their above-ground portfolio in two ways, relative to countries without oil. A wealth effect increases the holdings of all risky assets, because additional wealth exists outside the fund. A substitution effect invests less in assets that are positively correlated with oil, because of the subsoil exposure. Extraction will be faster than the standard Hotelling rule on average, if oil prices are pro-cyclical. This generates a risk premium on volatile subsoil assets. Consumption will manage any residual volatility through precautionary savings.

Keywords: Oil, asset allocation, extraction, sovereign wealth fund, portfolio theory, endogenous income

JEL Classification: E21, E62, F34, F65, G10, G11, G13, G15, G23, H63, O13, Q32, Q33

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1 Introduction

How should a government coordinate its sovereign wealth fund, its extraction of natural resources, and consume the proceeds? This paper shows that policymakers should offset the risk from subsoil oil[1] in their sovereign wealth fund, extract oil faster if its price is positively correlated with the market, and use precautionary savings to manage any residual volatility.

Sovereign wealth funds account for a large component of many commodity exporters' wealth. Azerbaijan's US$ 33 billion fund accounts for nearly 50% of GDP. Qatar's US$ 115 billion fund accounts for 62%, Saudi Arabia's 82%, Norway's 143% and the funds of both the UAE and the tiny island nation of Kiribati over 200% of GDP (SWF Institute, 2013 and IMF, 2013). Collectively commodity sovereign wealth funds hold over US$ 3 trillion in assets (SWF Institute, 2013). In broader terms they hold approximately 1.3% of the US$ 225 trillion in financial assets managed globally (MGI, 2013).

Of the nearly 30 countries with commodity funds, Norway has both the largest and one of the best managed. Established in 1990, Norway's Government Pension Fund Global was designed to smooth Norway's expenditure of its North Sea oil following a period of fiscal volatility in the 1970s and 1980s. At over US$ 700 billion it is the largest single fund in existence. It is also one of the best managed. Truman (2008) examines the management of sovereign wealth funds by constructing an index covering four dimensions: governance, accountability and transparency, structure and behavior. Norway's fund ranked first in two dimensions (governance, and accountability and transparency) and second overall behind the US$ 45 billion Alaska permanent fund. The Norwegian fund also receives the highest rating in the Sovereign Wealth Fund Institute's Linaburg-Maduell Transparency Index (SWF Institute, 2012). The Norwegian fund has thus been referred to as a "model" for managing sovereign wealth fund assets (Chambers et al, 2012; see also Larsen, 2005).

Norway's Government Pension Fund Global (GPFG) is required to allocate its assets across a diversified portfolio, and release approximately 4% of the fund for government

[1] We will use "oil" to refer to resources more generally, for convenience.

spending each year. The Ministry of Finance issues a management mandate to Norges Bank Investment Management (NBIM) that dictates the mix of equity, bonds and real estate, and provides benchmarks for the asset allocation in each category. Since 2009 the mix has been 60% equity, up to 40% bonds and up to 5% real estate. Equities use the highly diversified FTSE All Cap Index as a benchmark. Bonds are split between government (70%) and corporate (30%), and use a mix of Barclay's indices as a target, while real estate follows the Investment Property Databank's Global Property Benchmark – excluding Norway (NBIM, 2013). Since 2001 the fund has adopted a *handlingsregelen*, or budgetary rule, that each year the government receives no more than 4% of fund's value for general expenditure. As we will see in Section 2, this approach is consistent with a standard application of portfolio theory (such as in Merton, 1990).

However Norway, like most commodity exporters, also has a large and volatile stock of wealth beneath the soil - an "elephant in the ground". Current estimates suggest that Norway has over 5 billion barrels of proved oil, and 73 trillion cubic feet of proved natural gas reserves (EIA, 2012). At 2012 prices these are worth approximately US$ 590 billion and US$ 190 billion respectively, and collectively are comparable to the size of the GPFG (EIA, 2012). However, this value is also volatile, given the notorious volatility of commodity prices and resource discoveries.

Norway's allocation of assets above the ground does not take into consideration the assets below the ground. In 2008 the Ministry of Finance investigated the possibility of not holding oil and gas sector assets in the GPFG, but found that the effect would be trivial. However, they did not consider changing the weights of other assets based on their correlation with oil (Ministry of Finance, 2008). As it stands "oil" is not mentioned at all in NBIM's current management mandate (NBIM, 2012). Instead, the fund is focused on diversifying the assets above the ground, which leaves the government and people of Norway highly exposed to the volatility of oil prices and reserves below the ground.

This problem has begun to be identified in the empirical finance literature. Gintschel and Scherer (2008) consider the correlation of oil with other asset classes in a single period, and use numerical simulations to suggest that Norway's exposure to aggregate volatility can be reduced by 50% if oil wealth is hedged in the sovereign wealth fund. Scherer (2009a, 2009b) and Balding and Yao (2011) simulate a similar model over time and find that the fund should invest less aggressively as it ages. These works focus on studying asset allocation using numerical simulations, with little emphasis on theory. They also do not consider consumption or extraction, leaving scope for a theoretical foundation which addresses all three.

How below-ground assets should influence above-ground allocations gives rise to a host of questions, which we attempt to address in this paper. First, how should assets above the ground be allocated given the large and volatile stock of assets below the ground? How does this change if markets are incomplete? Second, will the above ground allocation influence how quickly subsoil assets are extracted? And third, how will asset allocation and extraction influence how wealth is consumed?

1.1 Results

In this paper we will show that the fund should offset subsoil oil exposure, extract resources more quickly when oil prices are positively correlated with the market, and use precautionary savings to manage any residual volatility.

Offsetting subsoil oil will have two effects on the sovereign wealth fund: a wealth effect and a substitution effect. The wealth effect increases the amount invested in all risky assets - and reduces the amount in the risk-free asset - relative to a case without oil. This is because some wealth is held outside the fund. The wealth effect makes the fund appear riskier in its early years, but reverses as oil is extracted and the fund matures.

The substitution effect offsets the exposure to oil, and will depend on whether markets are complete. In the simplest case - if markets are complete or traded assets are independent of one another - this involves holding fewer assets that are positively correlated with oil, or more assets that are negatively correlated. If markets are incomplete but the oil price is “spanned” by financial assets (they are affected by the same underlying shocks), then oil can be offset by changing the weights of many assets in the fund. The amount will depend on each asset's correlation with oil, and how uniquely it is affected by the underlying shocks. If the oil price is not spanned then the fund will not offset all risk; though evidence presented in Section 3 suggests that the market spans over 99% of oil price volatility.

Extraction should be faster - on average - when oil prices are positively correlated with the market. This extends the standard Hotelling (1931) rule to consider, for the first time, how other assets affect extraction. Extraction is faster on average because it raises the rate of return by rapdily reducing convex extraction costs. This generates a risk premium on subsoil assets.

Precautionary savings should manage any residual volatility. The sovereign wealth fund allows consumption to be smoothed in two ways: in expectation and in variance. In expectation, consumption will be a constant share of total wealth, regardless of when oil is extracted. This is similar to Friedman's (1957) permanent income hypothesis and Hartwick's (1977) rule for replacing below- with above-ground assets, and is made possible by borrowing and saving in the fund. In variance, consumption will be insulated from oil shocks, as the fund will offset them against other assets. The policymaker will thus only be exposed to residual volatility, which then must be managed by precautionary savings.

1.2 Existing literature

As well as being of practical interest, these questions combine three important strands of literature. Portfolio theory has considered how to invest in risky assets since the seminal work of Markowitz (1952 and 1959). The study of resource extraction began even earlier, with the work of Hotelling (1931). More recently the precautionary demand for savings was introduced by Leland (1968), to understand how consumption behaves under uncertainty. The following analysis combines these strands by jointly deriving the key equation of each. Asset allocation will be described by the portfolio condition, oil extraction by the Hotelling equation and consumption by the Euler equation.

Portfolio theory has generated an extensive literature on how to invest in risky assets.

It began in its current form with Markowitz's (1952 and 1959) fundamental theorem of mean-variance portfolio theory: that variance should be minimised for any given return. He introduced the idea that households should construct a diversified portfolio based on the co-movement between assets. Tobin (1958) formalised the conditions for mean-variance theory to be optimal, and introduced the separation theorem. This states that the optimal allocation within a portfolio can be separated from the decision to hold the portfolio or a risk-free asset. Sharpe (1964) introduced the capital asset pricing model, and the idea of a market portfolio that is held by all investors if they have equal information about the future. Merton (1990) extended the literature to continuous time, solving the portfolio allocation and consumption decisions simultaneously and confirming many previous results. We will draw a lot on Merton's elegant exposition of portfolio theory in continuous time.

Since the foundational studies, portfolio theory has been extended in many directions,[2] and remains an active area of research. Allocating assets with a non-tradable stream of income has been studied in the context of university endowments by Merton (1993), and Brown and Tiu (2012). Labour income, including endogenous labour effort has been studied by Bodie et al. (1992) and Wang et al. (2013). Non-traded and uninsurable income has been studied by Svensson and Werner (1993) and Koo (1998). Non-financial stores of wealth, such as housing, have been considered by Flavin and Yamashita (2002), Sinai and Souleles (2005), and Case, Quigley and Shiller (2005).

Natural resource extraction also has a distinguished history. Hotelling's (1931) landmark analysis showed that, in the absence of uncertainty, oil should be extracted so that its marginal rent rises at the rate of interest. This equates the rate of return on above-ground and below-ground assets. Since then the topic has proceeded in phases, but received sufficient attention for Robert Solow to dedicate the 1974 Ely lecture to it, and over thirty years later for Gerard Gaudet to do the same with the 2007 Canadian Economic Association's Presidential address. Both provide useful overviews of the literature in their respective generations.

[2]see review by Elton and Gruber (1997).

The resource literature has considered extracting and investing oil separately, but not together or in the presence of financial assets. Extracting oil when prices and reserves are uncertain was first addressed in Pindyck (1980 and 1981). This also finds that oil should be extracted more quickly when prices are volatile: but assumes it is free from risk once extracted. This is a form of "extractive prudence", driven by the convexity of marginal extraction costs. Gaudet and Khadr (1991) and Atewamba and Gaudet (2012) also consider extraction in the presence of physical capital. However, these papers do not acknowledge that oil wealth is still volatile above the ground, because financial assets are also risky. Investing oil revenues when its price is volatile has been considered by van den Bremer and van der Ploeg (2013) and Wills (2013). The former shows how much more Ghana, Iraq and Norway should save in their sovereign wealth funds when oil prices are risky (precautionary savings), using numerical simulations. The latter explicitly derives theoretical results and shows that there should be less precautionary savings when capital is scarce, some of which should be invested domestically. However, both of these papers assume that extraction is pre-determined and do not consider that oil wealth is still subject to risk after it is extracted.

The third component of our analysis is consumption under uncertainty. When a consumer faces a volatile income stream it is optimal to reduce consumption in the short-term, to build up a buffer stock of precautionary savings (see for example Leland, 1968; Sandmo, 1970; Zeldes, 1989; Carroll, 1994; and reviewed by Carroll and Kimball, 2008). The propensity to engage in precautionary savings can be described by the coefficient of relative prudence, which is driven by the convexity of marginal utility (Kimball, 1990). We will see that precautionary savings will be used to manage any residual volatility that cannot be managed by diversifying assets and extracting oil.

The rest of this paper proceeds as follows. Section 2 lays the foundation by revisiting standard portfolio theory without oil, following Merton (1990). Section 3 extends this to include a pre-determined oil windfall, and shows that asset allocations will change with subsoil oil, according to a wealth and a substitution effect. Section 4 lets oil extraction be chosen, and shows that it will happen more slowly if oil prices are correlated with the market. Section 6 concludes.

2 Portfolio allocation without oil

This section illustrates how sovereign wealth fund assets should be allocated in the absence of oil, laying the foundation for the subsequent sections. We revisit two key results, following the canonical exposition by Merton (1990). First, if asset prices are log-normally distributed then the asset allocation problem can be separated into two steps: i) finding the optimal risky portfolio, and ii) choosing the size of this portfolio. Second, the remaining undiversifiable risk in the portfolio will be managed by precautionary savings. We will then go on to see that this neatly describes the management of Norway's Government Pension Fund Global at present, although it ignores oil.

2.1 Baseline model

We begin with the policymaker's problem of choosing government consumption, C, and asset weights, w_i, to maximise the expected present value of utility subject to a budget constraint,

$$
\begin{aligned}
J(F,t) &= \max_{C,w_i} E_t\left[\int_t^\infty U\left(C(\tau)\right)e^{-\rho\tau}d\tau\right] && (2.1)\\
\text{s.t} & \\
dF &= \sum_{i=1}^m w_iF\left(\alpha_i-r\right)dt+\left(rF-C\right)dt+\sum_{i=1}^m w_iF\sigma_i dZ_i && (2.2)
\end{aligned}
$$

where the value function, $J(F,t)$, is a function of sovereign wealth fund assets, F, and time, t. The fund is a portfolio of m risky assets, $i=[1,\ldots,m]$, with drift α_i and volatility σ_i defined in equation 2.3, and one risk-free asset, $i=n$.

We will assume that utility exhibits constant relative risk aversion, $U(C)=C^{1-1/\theta}/(1-1/\theta)$. This utility function is part of the class of hyperbolic absolute risk aversion preferences, which readily permit explicit solutions to the asset allocation problem (Merton, 1971). While the coefficient of relative risk aversion is constant, $CRRA=-CU''(C)/U'(C)=$

$\frac{1}{\theta}$, the coefficient of absolute risk aversion will fall with consumption, $CARA = \frac{1}{\theta}C^{-1}$. This means that the absolute effect of volatility on consumption will diminish as the portfolio grows, which will be important in Proposition 2. These preferences also entwine risk aversion with the properties of intertemporal substitution ($IES = \theta$) and prudence ($CRP = -CU'''(C)/U''(C)= 1 + \frac{1}{\theta}$), both of which are important in our analysis.[3]

The sovereign wealth fund ("the fund") is a portfolio of N_i shares of assets $i = 1, \ldots, n$, each with price P_i, $F = \sum_{i=1}^{n} N_i P_i$. The model is partial equilibrium, treating the sovereign wealth fund as total wealth, and implicitly assuming that the government runs a non-fund balanced budget to abstract from taxes and non-fund income. The share of wealth in each asset is $w_i = N_i P_i / F$; and the price of each asset follows a Geometric Brownian Motion with constant coefficients,

$$dP_i = \alpha_i P_i dt + \sigma_i P_i dZ_i \tag{2.3}$$

The volatility of each asset price is driven by a Wiener process, dZ_i, where $dZ_i dZ_j = \rho_{ij} dt$ and $\rho_{ii} = 1$. As a result the n asset prices have a variance-covariance matrix, $\Sigma = [\sigma_{ij}]$, where $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$.

To make the analysis easier we assume that the nth asset is risk-free, so $\alpha_n = r$ and $\sigma_n = 0$. The risk-free asset allows us to treat the problem as an unconstrained one. This is because we assume markets are complete, so the weight of the risk-free asset in the fund can be greater or less-than zero, $w_n = 1 - \sum_{i=1}^{m} w_i \gtrless 0$. This is known as taking a "long" (saving, $w_n > 0$) or "short" (borrowing, $w_n < 0$) position in the risk-free asset. We call the combination of the remaining $m = (n-1)$ risky assets "the portfolio", $\sum_{i=1}^{m} w_i P_i$.

The model is solved by forming a Hamilton-Jacobi-Bellman equation and taking its first-order conditions, following the method of stochastic optimal control (see Merton, 1990) and derived in Appendix A.1. This gives rise to our first two results, concerning

[3]These properties can be separated in specifications such as Epstein-Zin (1989), though this reduces the tractability of the analysis. It has been done in similar continuous-time problems by Attanasio and Weber (1989) and Wang et al. (2013).

the portfolio of risky assets held by the sovereign wealth fund, and the rate these assets are consumed,

Proposition 1. Portfolio Allocation: Each asset will have a constant weight in the optimal portfolio, and the size of the portfolio will depend on preferences.

If markets are complete, asset prices follow a geometric Brownian motion, and preferences exhibit constant relative risk aversion then,

i) The weight of each risky asset in the fund, w_i, can be separated into its weight in the optimal risky portfolio: which is independent of preferences, δ_i, and the size of the optimal portfolio in the fund: which depends on preferences, w,

$$w_i = w\delta_i \tag{2.4}$$

where $w = \theta\nu$, $\delta_i = \nu^{-1}\sum_{j=1}^{m}\nu_{ij}(\alpha_j - r)$, $\nu_{ij} = [\Sigma^{-1}]_{ij}$ and $\nu = \sum_{i=1}^{m}\sum_{j=1}^{m}\nu_{ij}(\alpha_j - r)$ ensures $\sum_{i=1}^{m}\delta_i = 1$.

ii) This simplifies the allocation of assets to a choice between holding the optimally constructed risky portfolio, w, and the risk-free asset, $w_r = 1 - w$.

iii) The price-per share of the optimal risky portfolio, P, will follow,

$$dP = \alpha P dt + \sigma P dZ \tag{2.5}$$

where $\alpha = \sum_{i=1}^{m}\delta_i\alpha_i$, $\sigma^2 = \sum_{i=1}^{m}\sum_{j=1}^{m}\delta_i\delta_j\sigma_{ij}$ and $dZ = \frac{1}{\sigma}\sum_{i=1}^{m}\delta_i\sigma_i dZ_i$.

Proof. See Appendix A.1. □

This proposition illustrates the classical Tobin-Markowitz separation theorem: the optimal portfolio does not depend on preferences. The government's problem can therefore be separated into two steps: first, construct the optimal risky portfolio and second, choose the amount of this portfolio to hold in the sovereign wealth fund. The rest of the analysis can then be simplified by treating the optimal portfolio as a single risky asset.

The first step in designing the sovereign wealth fund involves deciding the weight of each asset in the optimal risky portfolio, δ_i. Proposition 1 states that this depends only on the returns on each asset and their covariance matrix. The portfolio will be diversified: any particular asset will have a higher weight in the portfolio if it is less correlated with other assets, or ν_{ij} is lower. Importantly, preferences do not affect the relative weight of any asset in the portfolio. This will be important in our analysis of oil.

The second step involves deciding how much of the optimal risky portfolio to hold, $w = \theta\nu$. The amount of the sovereign wealth fund invested in the risky portfolio, rather than the risk-free asset, depends on two elements. The first is the overall risk-adjusted return of the portfolio, ν. This is lower if the assets are correlated with one another, as there is less scope for fluctuations in one to offset, or "hedge", another. The second is the willingness to accept risk, measured by the coefficient of relative risk aversion, $CRRA = \frac{1}{\theta}$. If the portfolio has a lower risk-adjusted return, or the government is less willing to accept risk, then less of the sovereign wealth fund will be allocated to risky assets.

The separation theorem allows our analysis to be simplified by treating the optimal portfolio as a single asset, from part (ii) of Proposition 1. The problem in equation 2.1 becomes a two-asset problem: to hold the risky portfolio or the risk-free asset. The portfolio will follow a geometric Brownian motion if its assets do, from part (iii). We can use this to derive a closed form for the value function, in Appendix A.2, and characterise consumption and total assets in the following proposition,

Proposition 2. Consumption: Consumption will manage residual volatility through precautionary savings, and will be a constant proportion of the sovereign wealth fund.

Based on the same assumptions as Proposition 1,

i) The rate of change of consumption is described by the Euler equation,

$$\frac{\frac{1}{dt}E_t[dC]}{C} = \theta(r-\rho) + \tfrac{1}{2}(1+\tfrac{1}{\theta})w^2\sigma^2 \tag{2.6}$$

ii) Consumption will be a constant proportion of the fund,

$$C(t) = r^* F(t) \tag{2.7}$$

where $r^* = r + \theta(\rho - r) + \theta(1-\theta)\frac{(\alpha-r)^2}{2\sigma^2}$.

iii) The fund will follow a geometric Brownian motion,

$$\begin{aligned} dF &= \alpha^* F dt + w\sigma F dZ & (2.8) \\ F(t) &= F(0)\exp[(\alpha^* - \tfrac{1}{2}w^2\sigma^2)t + w\sigma Z(t)] & (2.9) \end{aligned}$$

where $\alpha^* = w(\alpha - r) + (r - r^*)$.

Proof. See Appendix A.2. □

This proposition shows that the government manages aggregate risk through precautionary saving. Precautionary saving describes how volatile income delays expected consumption, in order to build up a "buffer stock" of assets.[4] The buffer stock has two purposes: to smooth consumption and to compensate future periods for bearing additional risk. The buffer is used to smooth consumption if income shocks are not persistent, so negative shocks are expected to dissipate. However we assume that shocks are completely persistent as they are driven by a random walk, dZ. In our model the buffer stock therefore compensates future periods for bearing additional risk.

The amount of precautionary savings depends on aggregate risk and the degree of prudence. The government is only concerned with aggregate risk, $w\sigma$, as individual assets will hedge one another in the portfolio. This can be seen in equation 2.5, where the aggregate shock, dZ, is driven by the shocks to each asset and their covariances. If there is perfect positive or negative correlation between the assets, then the portfolio can be constructed so that all shocks offset and $dZ = 0$. There would be no need for precautionary savings. The degree of prudence describes the desire to engage in precautionary saving.

[4]See Leland (1968), Sandmo (1970) and Zeldes (1986) and the review by Carroll and Kimball (2008).

It is measured by the coefficient of relative prudence, described above $CRP = (1 + \frac{1}{\theta})$.[5] Precautionary saving is also apparent in more general Euler equations, as derived in equation 5.10,

$$\frac{\frac{1}{dt}E_t[dC]}{C} = CRRA^{-1}(r - \rho) + CRP\frac{C_F}{C^2/F^2}w^2\sigma^2 \tag{2.10}$$

where $CRRA = -CU''(C)/U'(C)$ and $CRP = -CU'''(C)/U''(C)$ as defined above

Precautionary savings can be implemented by a rule that ensures consumption is a fixed proportion of the fund, in 2.7. The total fund comprises both the risk-free asset and the risky portfolio. The risk of the fund is driven by the aggregate risk in the portfolio, dZ, from 2.8. Consumption will be a constant proportion of the total fund so that, as the fund grows through capital gains, consumption grows in a way that is consistent with precautionary savings. This result depends on preferences with constant relative risk aversion, so that absolute risk aversion falls as consumption rises.

The next section shows that these results accurately describe the way Norway's sovereign wealth fund is managed in practice.

2.2 Norway's sovereign wealth fund in practice

The results presented in Section 2.1 accurately describe the way that Norway's sovereign wealth fund is used in practice, although we have not yet considered oil. The results from standard portfolio theory make two key points. First, asset allocation can be separated into two steps: designing an optimal risky portfolio and choosing the size of that portfolio. This is consistent with NBIM's mandate from the Norwegian Ministry of Finance. Second, consumption should be a fixed proportion of wealth. The Norwegian government consumes 4% of the accumulated assets of the GPFG each year by releasing funds into the general budget.

[5] This measure was originally introduced by Kimball (1990).

Theory	**Norway's GPFG**
a. Asset allocation can be split into two steps	a. Assets are allocated in two stages, according to government mandate
i. Construct a diversified portfolio of all risky assets, independent of preferences (..ice cream and raincoats)	i. The FTSE All Cap index is the benchmark for asset shares in the Equity fund
ii. Find mix between the optimal risky portfolio and the risk free asset based on preferences	ii. The Equity/Bond mix is set by government, and can change with risk appetite (eg. 2009)
b. Consumption a linear function of wealth: $C(t) = s \cdot F(t)$	b. Fixed drawdown rule: $C(t) = 0.04 * F(t)$
Source: Merton (1990)	Source: www.nbim.no

Figure 2.1: The mandate for Norway's Government Pension Fund Global (GFPG) is consistent with standard portfolio theory, despite giving no consideration to oil wealth.

The first key point made in Section 2.1 is that asset allocation should be separated into two steps. The first step is to design the optimal risky portfolio by combining all assets in the market based only on their risk and return properties. If we assume that all investors in the market have the same information about future asset prices then we can follow Sharpe (1964) in interpreting this as the "market portfolio". This is consistent with Norway's use of the FTSE All Cap index as the benchmark for their equity investments.

The second step is to choose the size of the optimal risky portfolio based on the individual agent's preferences. The model suggests that the agent should choose the size of the risky portfolio based on their risk aversion. If the agent's risk/return preferences change, the size of the portfolio would change but the allocation of assets within it would not. In Norway's case we find that the Ministry of Finance dictates the mix between equity assets, which are risky, and bonds, which might crudely play the role of the risk-free asset. At present this mix is ~60% equity and ~40% bonds. However this was not always the case, being ~40% equity and ~60% bonds prior to 2009. It is important to note that when the Ministry of Finance changed their mandate they did not alter the benchmark within the equity portfolio, just the mix between equity and bonds.

The second key point is that consumption should be a fixed proportion of wealth. This result assumes log-normal asset prices and CRRA preferences, but is useful for finding an explicit rule for consumption. The proportion will be slightly higher than the risk-free rate, and be based on preferences and the risk and return of the market, from equation 2.7. Under Norway's budgetary rule, or *handlingsregelen*, the GPFG releases 4% of accumulated assets into the general government budget each year.

In summary, Norway's GPFG is currently managed in a way consistent with a standard treatment of the optimal portfolio allocation problem, as in Merton (1990). However this does not take into account subsoil oil: "the elephant in the ground". We turn our attention to this next.

3 Portfolio allocation for a given path of extraction

This section introduces oil, and illustrates that oil wealth should be offset within the sovereign wealth fund. Total wealth consists of oil and non-oil assets. The stock of oil wealth can not easily be traded. However, it can be treated as traded if it can be synthetically replicated by traded assets. The result should be to invest a constant share of total wealth in each asset, including the net exposure to oil. This will have two effects on the fund's portfolio, relative to the case without oil: a wealth and a substitution effect. The wealth effect increases the amount held of every risky asset, because some wealth is held outside the fund. The substitution effect offsets subsoil oil, roughly investing less in assets that are positively correlated with it. The net effect will depend on the covariance of each asset with oil, and with each other. Consumption will be insulated from oil shocks by the fund, should involve precautionary savings, and will be a constant share of total wealth.

The ability to offset, or "hedge", oil in the sovereign wealth fund will depend on whether markets are complete. If markets are complete then there will be a single asset that is perfectly correlated with oil. All hedging can be done with this single asset. However, if markets are incomplete then other assets will be involved: for example if the single asset does not exist, or if sovereign wealth funds cannot buy or sell enough of it without drastically changing its price. Oil can still be hedged if it is subject to the same shocks as assets in the market; such as demand, technology, war and weather. Hedging will be done with many assets. In practice the oil price and the market share over 99% of underlying shocks (Table 3.1), so if oil cannot be hedged with a single asset, it should be with many.

3.1 Complete markets

The policymaker's problem is now to choose consumption and asset weights while receiving an exogenous, temporary stream of oil income, P_oO, following the analysis of university endowments by Merton (1990),

$$\begin{aligned} J\left(F,P_o,t\right) &= \max_{C,w_i} E_t\left[\int_t^{\infty} U\left(C(\tau)\right)e^{-\rho\tau}d\tau\right] && (3.1)\\ \text{s.t} & \\ dF &= \sum_{i=1}^{m} w_iF\left(\alpha_i-r\right)dt+\left(rF-C+P_oO\right)dt+\sum_{i=1}^{m} w_iF\sigma_i dZ_i && (3.2)\\ dP_o &= \alpha_oP_odt+\sigma_oP_odZ_o && (3.3) \end{aligned}$$

Oil income is the product of the amount of oil extracted, O, and the oil price, P_o. Oil extraction is pre-determined, for example by geological or engineering constraints. We assume that extraction is constant at some level, O, for time, $t \in [0,T]$. The oil price is assumed to be exogenous and follows a geometric Brownian motion, as defined in equation 3.3. We are therefore focusing on small oil exporters that do not affect the market price. In practice the oil price has little drift, so we assume $\alpha_o < r$.[6] This also ensures that there is a solution to the extraction problem in Section 4.

The oil price will be perfectly correlated with a traded security if markets are complete, allowing oil to be hedged within the risky portfolio. The volatility of the oil price is driven

[6]See for example Bems and de Carvalho Filho (2011) and van den Bremer and van der Ploeg (2013) who estimate $\alpha_o = 0.009$.

by a Wiener process, dZ_o. Complete markets means that there will be a traded security, $k \in i$, whose instantaneous return is perfectly correlated with oil income,

$$dZ_o = dZ_k \tag{3.4}$$

A perfectly correlated asset allows oil to be replicated by a combination of traded assets, which we call a "bundle", as presented in the following proposition.

Proposition 3. Hedging Oil: Oil wealth can be replicated by a bundle of two traded assets if markets are complete, so it can be treated as a tradable part of total wealth.

If asset markets are complete and prices are log-normally distributed, then,

i) Oil revenues can be perfectly replicated by a particular bundle of the perfectly-correlated risky asset, k, and the risk-free asset, n. The value of this bundle is the capitalised value of oil revenues,

$$V(P_o, t) = P_o(t)O(1 - \exp[-\psi_k(T-t)])/\psi_k \tag{3.5}$$

where $\psi_k = r + \beta_k(\alpha_k - r) - \alpha_o$ and $\beta_k = \frac{\sigma_o}{\sigma_k}$.

ii) Total wealth is the sum of fund assets and subsoil oil, $W = F + V$, where,

$$dW = \sum_{i=1}^{m} \bar{w}_i W (\alpha_i - r)\, dt + (rW - C)\, dt + \sum_{i=1}^{m} \bar{w}_i W \sigma_i dZ_i \tag{3.6}$$

where

$$\bar{w}_i = \begin{cases} \frac{w_i F}{F+V} & \text{for } i \neq k \\ \frac{w_k F + \beta_k V}{F+V} & \text{for } i = k \end{cases} \tag{3.7}$$

Proof. Proved for a more general case in Appendix B.1. □

Proposition 3 begins by stating that oil revenues can be perfectly mimicked, or "hedged", by bundling together the risky asset, k, and the risk-free asset, n. This only requires oil and asset k to be perfectly correlated, not to have the same variance, $dZ_k = dZ_o$ but $\sigma_k \neq \sigma_o$. Intuitively, the bundle is constructed by buying a number of shares in the risky and the risk-free asset, and the price of this bundle is the value of oil wealth. The amount of the risky asset is chosen so that the bundle has the same variance as oil, and the risk-free asset is chosen so that the bundle has the same drift.

The price of constructing the replicating bundle is the value of oil wealth, 3.5. This follows from an arbitrage argument: oil wealth and the bundle behave identically, so must have the same value if markets are complete. Oil wealth in equation 3.5 can be thought of as the present value of oil revenues, discounted at the rate, ψ_k, which has been adjusted for the market price of risk.[7]

Oil revenues are pre-determined, but they can be treated as tradable because of the replicating bundle. The government owns the exogenous stream of oil revenue. We assume that claims to this oil cannot simply be packaged and sold off, because of political or practical constraints.[8] If they could, then the government would simply invest the proceeds in a diversified portfolio, as described in Section 2. However, if oil can be replicated then this is not necessary. Any net exposure to oil price risk, dZ_o, can be artificially constructed by combining oil revenues with an amount of the replicating bundle.

The government's problem can therefore be simplified into choosing the net weight of each asset, $\bar{w}_i$ for $i = 1 \ldots m$, in total wealth, $W(t) = F(t) + V(t)$. This is seen in equation 3.6, which reduces the government's original problem in equation 3.1 to choosing weights in total wealth, $\bar{w}_i W$, to maximise $J(W, t)$. This is analytically identical to the problem in Section 2. This brings us to our next result, about how oil affects the sovereign wealth fund portfolio,[9]

Proposition 4. Portfolio Allocation: The net exposure to each asset will be a constant

[7] Oil revenues can not simply be discounted at the risk-free rate, r, because individuals in the market are risk-averse. They require some compensation for bearing risk, which is incorporated by adjusting the discount rate. The value of an uncertain stream of income of this type can also be found by discounting at the risk-free rate, if the probability space is adjusted to a "risk-neutral measure" using the Cameron-Martin-Girsanov theorem.

[8] In practice there is little evidence of countries selling all rights to resource wealth up front. An initial payment for a well is often made by extracting firms as part of the auction process. However, extraction firms are risk averse, and rarely willing to take on all price and production risk.

[9] Again stemming from Merton's (1990) analysis of university endowments.

share of total wealth. This is achieved by varying the weight of each asset in the fund according to a wealth and a substitution effect.

If the government receives exogenous oil income, asset markets are complete, and prices are log-normally distributed then,

i) The net weight of every asset in total wealth will be constant,

$$\bar{w}_i = w\delta_i \tag{3.8}$$

where w and δ_i are defined in Proposition 1.

ii) The weight of each risky asset in the fund will be affected in two ways: a wealth effect, $\bar{w}_i V/F$ for all $i = 1, \dots m$, and a substitution effect, $-\beta_i V/F$ for the perfectly correlated asset $i = k$,

$$w_i = \begin{cases} \bar{w}_i + \bar{w}_i \frac{V}{F} & \textit{for } i \neq k \\ \bar{w}_k + (\bar{w}_k - \beta_k)\frac{V}{F} & \textit{for } i = k \end{cases} \tag{3.9}$$

Proof. By direct analogy to Proposition 1, and from equation 3.7. □

Proposition 4 begins by stating that the net exposure to each asset will be a constant share of total wealth. This is very closely related to the result in Proposition 1. If oil wealth can be replicated by a tradable asset, then together any net exposure can be chosen. The optimal exposure will be the same as if oil was treated as a tradable security in the portfolio.

The proposition then states that oil wealth should continuously change the weight of each asset in the fund, according to a wealth and a substitution effect. This happens relative to the reference weight of each asset, $\bar{w}_i$, from equation 2.4.

The wealth effect increases the weight of every risky asset in the portfolio, by an amount that depends on the relative value of oil, $\bar{w}_i V/F$. This effect causes the fund to appear riskier than it would be in the absence of oil, but only because the fund is a fraction of total wealth. The weight of the risk-free asset will in turn be less (or negative if

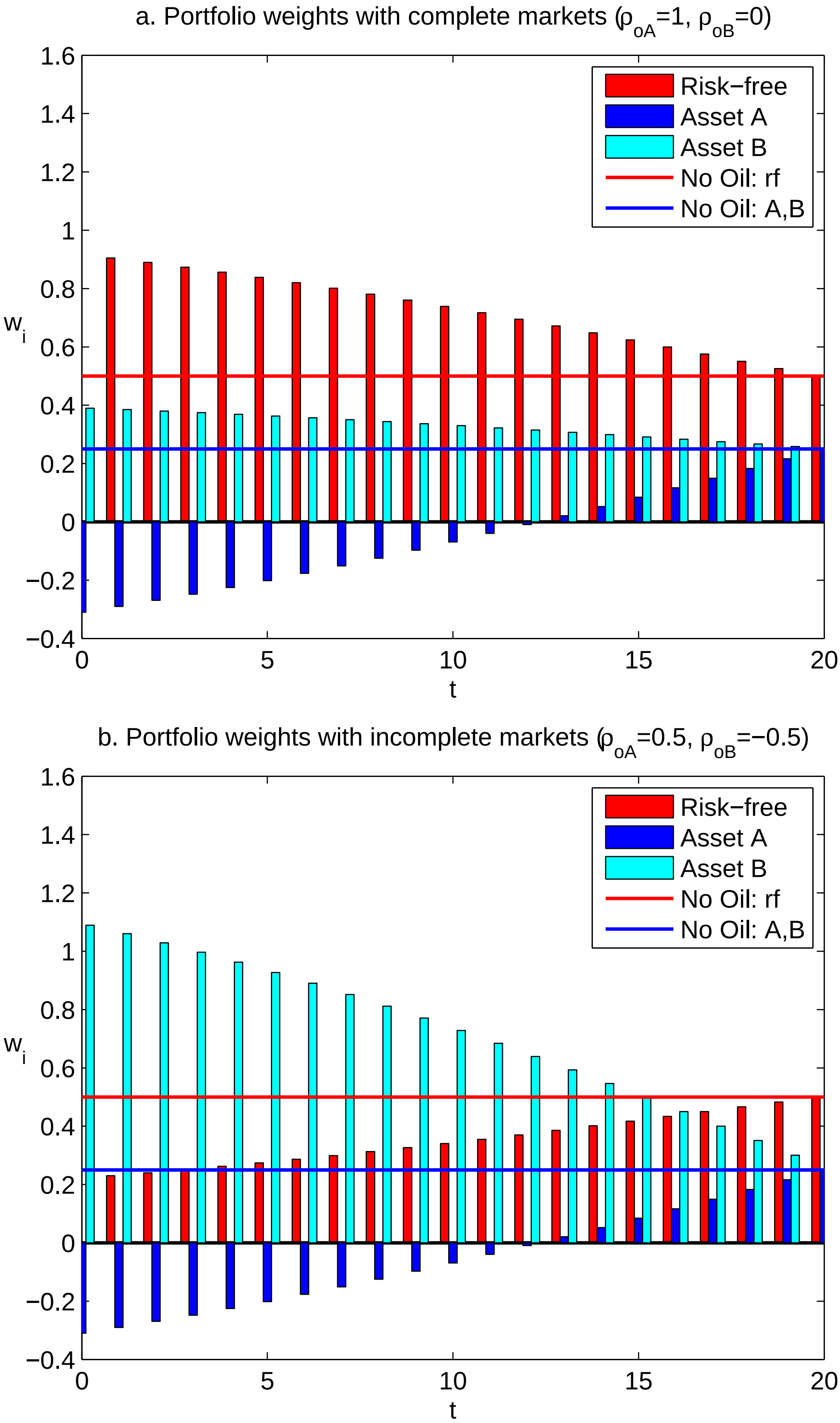

Figure 3.1: Oil's effect on the fund will depend on whether markets are complete. Assets A and B are independent. If markets are complete then oil can be hedged with only a short position in A. The wealth effect will increase the weight of B. If markets are incomplete then oil must be hedged by investing less in A (positive correlation), and more in B (negative correlation). These positions are reversed as oil is extracted.

a short position is required), $w_r = 1 - \sum_{i=1}^{m} w_i$. As oil is extracted and the fund matures, it should re-allocate away from risky and towards risk-free assets.[10]

The substitution effect reduces the relative amount of the correlated risky asset held by the fund, $-\beta_k V/F$. This is because the government has other exposure to oil prices in its reserves beneath the ground. The weight of the risk-free asset will be relatively higher as a result. The net effect on asset k in the fund will depend on how much exposure to oil price risk the government wants, $\bar{w}_i$, and on the relative volatility of oil and k, $\beta_k = \frac{\sigma_o}{\sigma_k}$. This is likely to involve a short position in k. Over time this position will also be reversed as V/F falls (see footnote 10).

These effects can be understood with a simple illustration. Figure 3.1a shows how oil wealth should affect three assets in a sovereign wealth fund over time: risky assets A and B and the risk-free asset. Markets are complete and asset A is perfectly correlated with oil. Asset B will be independent of both oil and asset A, for simplicity. The parameter values are given in Appendix B.2. Initially, oil should be offset by a short position in asset A, because of a strong substitution effect. Asset B will have a higher weight than it would without oil, because it is only subject to the wealth effect. More of the risk-free asset will also be held, so that the asset weights sum to one. Over time oil will be extracted, so less will need to be offset and there will be less wealth outside the fund. As a result all assets approach the weights they would have in the absence of oil. These are also the weights we would see if we illustrated each asset exposure as a share of total wealth.

In practice Norway is approaching the end of its oil reserves, with a reserve to production ratio of 9.2 (BP, 2012). These results suggest that Norway should previously have held a large offsetting position in oil securities (such as futures/forward contracts), and invested in other assets more aggressively. However, as subsoil oil is depleted over the coming decades it should re-weight the portfolio towards a more even allocation. In contrast, Kuwait can justify a higher proportion of risky assets in their fund, with a reserve to production ratio of nearly 100.

[10]This assumes that the government is not so rapacious (has such a high ρ), that it consumes so much that foreign assets fall faster than oil is extracted, so V/F rises over time. We would like to think that this is the case.

Our final proposition when markets are complete considers the effect of oil wealth on consumption, using the two-asset simplification described in Proposition 1,

Proposition 5. Consumption: Consumption will use precautionary savings to manage residual volatility, and will be a constant proportion of total wealth.

Based on the same assumptions as Proposition 4,

i) The rate of change of consumption is described by the Euler equation,

$$\frac{\frac{1}{dt}E_t[dC]}{C} = \theta(r-\rho) + \tfrac{1}{2}(1+\tfrac{1}{\theta})\bar{w}^2\sigma_W^2 \tag{3.10}$$

ii) Consumption will be a constant proportion of total wealth,

$$C(t) = r_W^* W(t) \tag{3.11}$$

where $r_W^* = r + \theta(\rho - r) + \theta(1-\theta)\frac{(\alpha_W - r)^2}{2\sigma_W^2}$ *and* α_W*,* σ_W *are defined as in Proposition 1.*

iii) Total wealth will follow a geometric Brownian motion,

$$\begin{aligned} dW &= \alpha_W^* W dt + \bar{w}\sigma_W W dZ_W & (3.12) \\ W(t) &= W(0)\exp[(\alpha_W^* - \tfrac{1}{2}\bar{w}^2\sigma_W^2)t + \bar{w}\sigma_W Z_W(t)] & (3.13) \end{aligned}$$

where $\alpha_W^* = \bar{w}(\alpha_W - r) + (r - r_W^*)$*.*

Proof. By direct analogy to Proposition 2. □

Proposition 5 states that consumption will be delayed to manage the residual volatility of all government assets, $\bar{w}^2\sigma_W^2$. Oil price shocks will be hedged within the sovereign wealth fund portfolio, so the government is only exposed to aggregate volatility, dZ_W. This can only be managed by precautionary savings, which is driven by the second term of the Euler equation, 3.10.

Consumption will also be a constant proportion of total wealth. This is intuitively similar to Friedman's (1957) permanent income hypothesis. This result, assuming ready

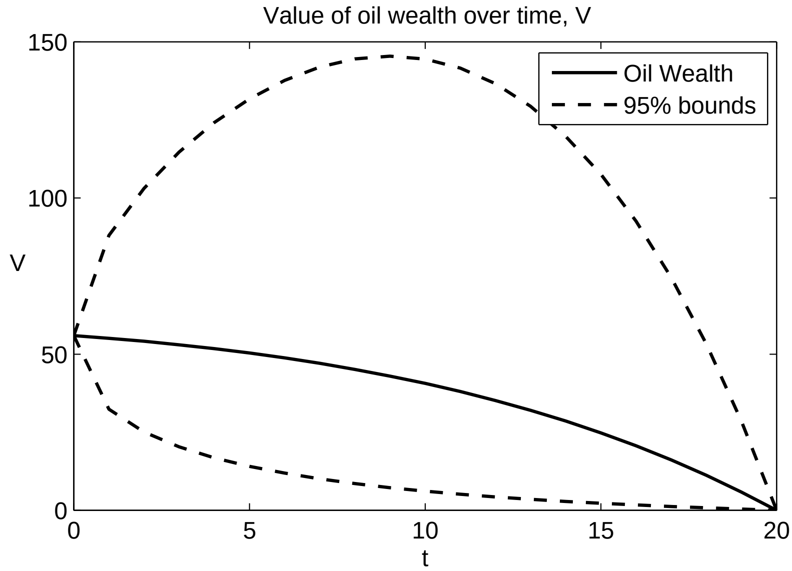

Figure 3.2: The value of subsoil oil will fall as it is extracted. After a point it will also become less volatile, as the amount remaining exposed to oil prices diminishes. Based on calibration in Appendix B.2.

access to borrowing and no uncertainty, states that consumption should equal the interest earned on the present value of lifetime income, $C = rPV(Income)$. The date income is received - or in our case oil is extracted - should not matter. The result in 3.11 adjusts this hypothesis for the risk and return of total wealth. In practice, if the sovereign wealth fund is constructed properly, then government consumption should not be directly affected by oil price shocks, only indirectly through their effect on total wealth.

The paths of oil, assets and consumption can also be understood with a simple illustration. Figures 3.2 and 3.3 also use the calibration in Appendix B.2 to show how each variable evolves over time. Oil rents are assumed to drift slowly ($\alpha_o < r$) and are more uncertain in the distant future. The value of oil will comprise a smaller share of wealth as it is extracted. The volatility of remaining oil wealth will also decline, as less remains exposed to volatile oil prices. Oil price uncertainty will be hedged within the portfolio, so that the policymaker is only exposed to the aggregate volatility of total wealth. If they follow the rule in equation 2.7 and consume a constant share of total wealth, then both wealth and consumption will steadily rise: incorporating precautionary savings.

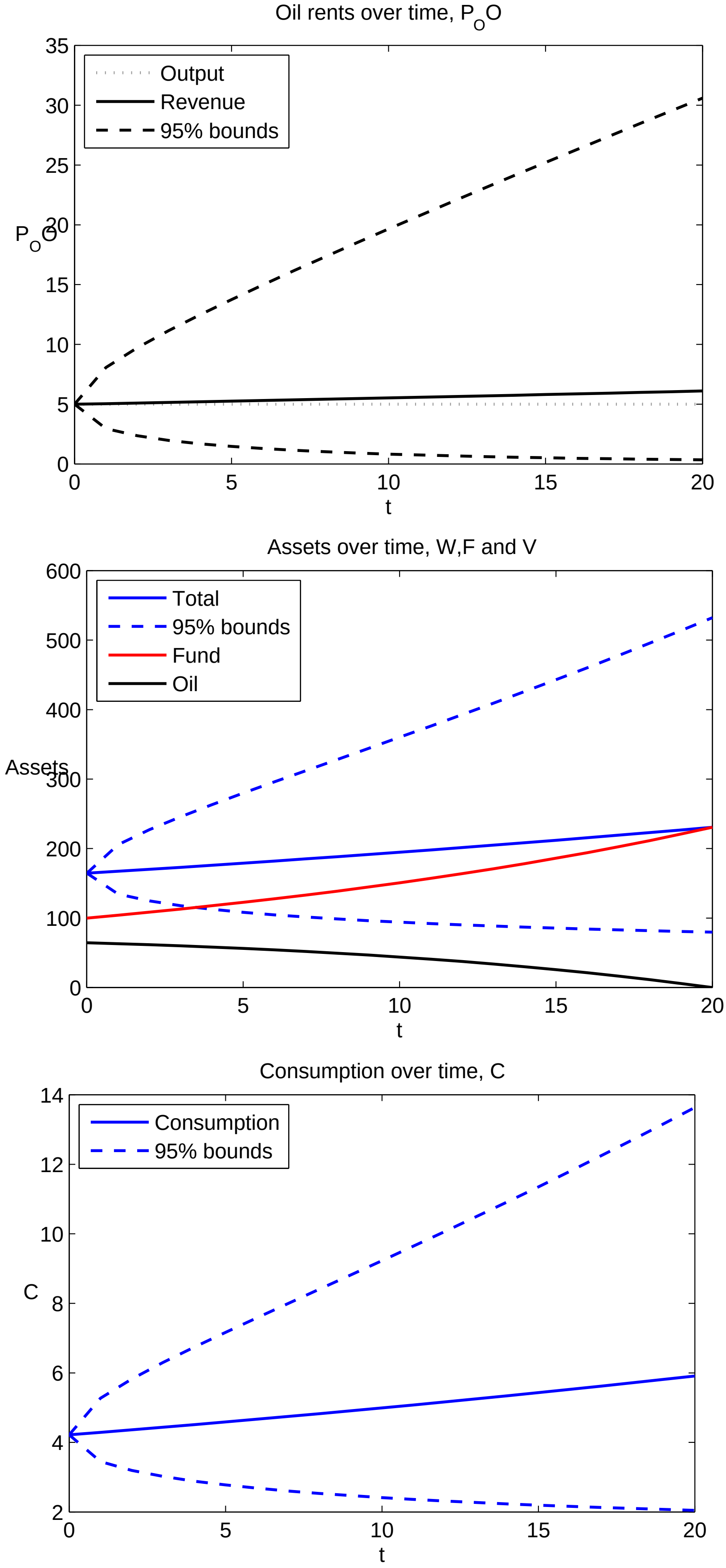

Figure 3.3: Price uncertainty will increase with time. The sovereign wealth fund will increase its share of total wealth as oil is extracted. Consumption should be a constant proportion of total wealth, being insulated from oil price shocks by the fund. Based on calibration in Appendix B.2.

Principle Component Analysis of the
Covariance of FTSE All Cap Industries with the Oil Price

Number of Eigenvectors	% Explained
1	77.13
2	84.59
3	89.09
4	92.35
5	94.18
6	95.82
7	97.27
8	98.25
9	98.97
10	99.66
11	100.00

Table 3.1: Oil prices are well spanned by the market. Ten underlying shocks explain 99.6% of the variation in oil and the ten industries in the FTSE All Cap Index. For details see Appendix B.2.

3.2 Incomplete markets

Oil wealth may still be hedged if markets are incomplete, by taking offsetting positions in many risky assets. The previous section assumed that asset markets are complete, which may not always be appropriate. There may not be a single asset that is perfectly correlated with the oil price. Sovereign wealth funds are also the largest individual investors in the world - for example Norway's Government Pension Fund Global holds 1% of global equities (NBIM, 2013). If a correlated asset does exist then taking a large enough position to offset subsoil oil may either be impossible, or move the oil price - violating our assumption of exogeneity. If markets are incomplete then every asset will be affected by the substitution effect. Consumption will still be a constant fraction of total wealth. This can be seen by introducing incomplete markets into the model in equations 3.1 to 3.3, generalising the results of the previous section.

Incomplete markets means that there is no single traded asset that is perfectly correlated with oil, $\nexists k \mid dZ_k = dZ_o$. Instead, we assume that oil returns are imperfectly correlated with all the assets in the market. We do this by letting all risky assets actually be driven by a common set of underlying shocks: to demand, supply, technology, weather, etc., $du \sim iid \quad N(0, dt)$. Each asset, and oil, is affected by the underlying shocks differently, which gives rise to correlation, $dZ = \Lambda du$. We also let some part of oil returns be subject to its own shocks, which are not "spanned"[11] by the market. This is summarised in the following expression, derived in Appendix B.1 and replacing equation 3.4,

$$dZ_o = \Lambda_o \Lambda^{-1} dZ + \lambda_{o,0} du_o \tag{3.14}$$

where $\Lambda_o = [\lambda_{o1}, \ldots, \lambda_{om}]$ is a $1 \times m$ vector of constants, $[\Lambda]_{ij} = \lambda_{ij}$ is an invertible $m \times m$ matrix of constants, $\lambda_{o,0}$ is a scalar constant, $dZ = [dZ_1, \ldots, dZ_m]'$ is the vector of (possibly correlated) Wiener processes driving asset returns in equation 2.3, and du_o is an independent Wiener process, $du_o dZ_i = 0$ for all $i = 1, \ldots, m$.

In practice the oil price is well-spanned by traded assets. The parameters Λ_o, Λ and $\lambda_{o,0}$ can be estimated using principal components analysis, as described in Appendix B.1. This allows us to see how much volatility in the oil price is shared with assets in the market. Using data on the 10 ICB sectors in the FSTE All Cap index,[12] table 3.1 illustrates that over 99% of volatility in the oil price is accounted for by ten underlying shocks in the market. As a result we can comfortably drop the final eleventh shock, and treat oil as being spanned by the market, $\lambda_{o,0} = 0$.

We now turn to the next proposition, on hedging oil in the sovereign wealth fund when markets are incomplete,

Proposition 6. Hedging Oil: Oil wealth can be replicated by a bundle of many traded assets when markets are incomplete, if oil prices are spanned by the market.

If asset markets are incomplete and prices are log-normally distributed, then,

i) Oil revenues can be replicated by a particular bundle of all risky assets in the market, $i = 1, \ldots, m$, and the risk-free asset, n. The value of this bundle is the capitalised value of oil revenues,

$$V(P_o, O, t) = P_o(t) O (1 - \exp[-\psi(T - t)])/\psi \tag{3.15}$$

[11] That is, "correlated in any way" with the market.

[12] The equity portfolio benchmark for Norway's Government Pension Fund Global.

where $\psi = r - \alpha_o + \sum_{i=1}^{m} \beta_i(\alpha_i - r)$.

ii) Total wealth is the sum of fund assets and subsoil oil, $W = F + V$*, and evolves according to,*

$$dW = \sum_{i=1}^{m} \bar{w}_i W \left(\alpha_i - r\right) dt + \left(rW - C\right) dt + \sum_{i=1}^{m} \bar{w}_i W \sigma_i dZ_i \tag{3.16}$$

where

$$\bar{w}_i = \frac{w_i F + \beta_i V}{F + V} \quad \textit{for all } i = 1, \dots m \tag{3.17}$$

and $\beta_i = \sigma_o M_i / \sigma_i$, $M_i \equiv [\Lambda_o \Lambda^{-1}]_i$.

Proof. See Appendix B.1. □

Oil wealth can still be hedged in a sovereign wealth fund, but it will involve a bundle of many assets. This is done by observing that the underlying shocks affecting the oil price will also affect all other assets in the market.[13] These underlying shocks will no affect each asset equally. So, using the assumption that the shocks combine linearly,[14] we can replicate the behaviour of the oil price by combining small exposures to many assets. These exposures will depend on the similarity of each asset to the oil price, and its uniqueness, β_i. We call this the "closest replicating bundle".

Every traded asset will now be affected by the substitution effect in addition to the wealth effect. The net exposure of each asset will still be a constant share of total wealth, $\bar{w}_i = w\delta_i$ from Proposition 4 and proved in Appendix B.1. However, each traded asset will now capture some of the underlying shocks that affect the oil price, du_i. To ensure the net exposure to each asset is constant, their weight in the fund will be 3.18, where β_i is defined in equation 3.17.

$$w_i = \bar{w}_i + (\bar{w}_i - \beta_i)\frac{V}{F} \quad \text{for all } i = 1, \dots, m \tag{3.18}$$

[13] Bodenstein et al. (2012) use a general equilibrium model to illustrate how oil price fluctuations are driven by more fundamental shocks.

[14] This stems directly from our assumption that the process driving each oil price, dZ_i, is correlated and normally distributed. It does not impose any further constraints on the model.

A simple rule-of-thumb for the substitution effect is to invest more in assets that are negatively correlated with oil, though this does not capture the whole story. The fund should simply invest more in assets that are negatively correlated with oil if all assets are also independent of each other; that is, if Λ is diagonal and Λ_o has positive and negative elements. This would mean investing more in assets that use oil as an input (so benefit from low oil prices), such as manufacturing and consumer goods, and less in assets that are substitutes for oil (benefiting from high oil prices), such as green energy. This is illustrated in Figure 3.1.

More precisely, the substitution effect will depend on the covariance of each asset with oil, and with each other. For example, consider an underlying shock, du_g, that affects oil and asset f but no others, $\lambda_{og}, \lambda_{fg} > 0$ and $\lambda_{ig} = 0$ for all $i = 1, \ldots, m$. Consider also that all other underlying shocks, du_j, affect oil and asset f in opposite ways, so $\lambda_{oj} > 0$ and $\lambda_{fj} < 0$ for all $j = 1, \ldots, m$. In this case oil and asset f may be negatively correlated, $\sum_{j=1}^{m} \lambda_{oj}\lambda_{fj} < 0$, but the fund should invest <u>less</u> in asset f, to offset the exposure to shock g. All other assets will adjust to hedge the effects of the remaining shocks, du_j for $j \neq g$. This is straightforward to achieve with principal components analysis, described in Appendix B.1.

These effects can also be understood with a simple illustration. Figure 3.1b shows that oil wealth will have a very different effect if markets are incomplete. We let asset A still be positively correlated with oil, but imperfectly so that $\rho_{oA} = 0.5$. We also let asset B be negatively correlated with oil, $\rho_{oB} = -0.5$, though the two assets remain independent. We can clearly see that oil wealth can be offset by investing less in assets that are positively correlated with oil (A), and more in those negatively correlated (B). Less of the risk-free asset must be held to make this possible. Over time the fund increases its share in total wealth, and the asset weights converge to where they would be without oil.

4 Portfolio allocation if oil extraction can be chosen

Now we will turn our attention to oil extraction. The previous section assumed that all extraction was pre-determined, such as by engineering constraints. This section lets extraction be chosen.[15] We find that extraction will be faster than the standard Hotelling (1931) rule on average, if oil prices are positively correlated with other assets. Fast extraction rapidly reduces extraction costs, which are convex. Lower extraction costs increases the rate of return on subsoil assets, acting as a risk premium. Oil wealth will again be hedged by the sovereign wealth fund. This will involve dynamically allocating assets to ensure that the net exposure to oil remains a constant share of wealth. Consumption should not be affected by the path of extraction, only by the way extraction changes the present value of oil wealth.

4.1 Optimal oil extraction

The policymaker now faces the problem of choosing consumption, C, extraction, O, and asset weights, w_i, to maximise welfare,

$$
\begin{aligned}
J(F, P_o, S, t) &= \max_{C, w_i, O} E_t \int_{\tau=0}^{\infty} U(C(\tau)) e^{\rho(\tau-t)} d\tau && (4.1) \\
& s.t. \\
dF &= \sum_{i=1}^{m} w_i F(\alpha_i - r)dt + (rF - C + \Omega(P_o, O))dt + \sum_{i=1}^{m} w_i F \sigma_i dZ_i && (4.2) \\
dP_o &= \alpha_o P_o dt + \sigma_o P_o dZ_o && (4.3) \\
dS &= -O dt && (4.4)
\end{aligned}
$$

where oil rents are revenues less extraction costs, $\Omega(P_o, O) = P_o O - G(O)$, extraction costs are convex,[16] $G'(O) > 0, G''(O) > 0$, and oil reserves, S, are depleted at the rate

[15] In practice extraction will change less quickly than asset portfolios, which can be adjusted almost instantly. However, for simplicity we assume that extraction can also adjust continuously.

[16] Convex extraction costs capture the difficulties of extracting oil from greater depths or less productive fields. It is also common in the literature, see Pindyck (1984).

of extraction, O. To simplify the analysis we return to the complete markets setting in Section 3.1, so there exists a traded asset k that is perfectly positively correlated with oil, $dZ_k = dZ_o$. Incomplete markets are an extension, as in Section 3.2. In practice, oil reserves also evolve stochastically as new fields are discovered and existing fields become more or less economical (see for example Pindyck, 1978). They are also endogenous to exploration effort, but we abstract from these possibilities for the moment.

Solving the policymaker's problem gives rise to a stochastic version of the Hotelling (1931) rule, presented in the following proposition,

Proposition 7. Extraction: Oil should be extracted more quickly on average if it is positively correlated with the assets in a nation's sovereign wealth fund, and more should be extracted when prices are high.

If asset markets are complete, prices are log-normally distributed, and the government can choose the path of oil extraction, then,

i) The expected path of oil extraction will satisfy the expected Hotelling rule,

$$\tfrac{1}{dt}E[d\Omega_O] = r\Omega_O - \frac{\tfrac{1}{dt}E[dJ_F d\Omega_O]}{J_F} \tag{4.5}$$

ii) The actual path of oil extraction will satisfy the following stochastic Hotelling rule when extraction costs are quadratic, $G(O) = \frac{1}{2}\gamma O^2$,

$$\begin{aligned} d\Omega_O &= r\Omega_o dt \\ &\quad -\gamma O_F \sum_{i=1}^{m} w_i F\left((\alpha_i - r)dt - \sigma_i dZ_i\right) \\ &\quad + (1-\gamma O_P)\, P_o \tfrac{\sigma_o}{\sigma_k}\left((\alpha_k - r)dt - \sigma_k dZ_o\right) \end{aligned} \tag{4.6}$$

iii) The stochastic extraction path can be approximated using the partial derivatives from the deterministic solution, O_F,O_P, and assuming $\alpha_O = 0$

$$dO \approx \left\{-\frac{1}{\gamma}\left(r+\frac{\sigma_O}{\sigma_k}(\alpha_k-r)\right)P_O+\left(r+\frac{1}{2}\frac{\sigma_O}{\sigma_k}(\alpha_k-r)\right)O\right\}dt+\frac{1}{2}O\sigma_O dZ_O \quad (4.7)$$

Proof. See Appendix C.1. □

Proposition 7 begins by stating that extraction will be faster than the original Hotelling (1931) rule if oil prices are pro-cyclical (see Ewing and Thompson, 2007). The original Hotelling rule states that oil should be extracted so that the marginal oil rent rises at the rate of interest if prices are deterministic, $\frac{d}{dt}\Omega_O = r\Omega_O$ as shown in Lemma 3 in Appendix C. Extraction will decline over time, ensured by assuming $\alpha_o < r$. Pindyck (1981) found that extraction will be faster if prices are stochastic, and costs are convex, as does van der Ploeg (2010). In both cases this is a type of "extractive prudence", driven by highly convex extraction costs, $G'''(O) > 0$.[17] They find that it is better to extract oil quickly because once it is out, the revenues are risk-free. However, our result acknowledges that oil wealth is still subject to risk after it has been extracted, as the revenues need to be invested somewhere. Extracting oil does not remove risk, it just changes it. Thus, the interaction between oil prices and the rest of the market will be crucial.[18]

Pro-cyclical oil prices will hasten extraction on average, as illustrated in Figure 4.1. This is again based on the calibration in Appendix B.2, assuming that oil is perfectly correlated with asset A. Faster extraction increases the rate of return earned on subsoil oil. This happens because fast extraction quickly reduces the quantity extracted at each time, and in turn reduces convex extraction costs. If oil and the market were independent, then the extraction path would simply depend on the expected path of oil prices. Only when oil and the market interact will there be an additional effect on extraction.

Intuitively this Hotelling rule stems from a simple marginal-benefit, marginal costs analysis. The marginal benefit of extracting oil is the rent it generates, multiplied by the

[17] This is like Kimball's (1990) prudence, which relies on the concavity or convexity of marginal utility, $U'''(C) \neq 0$.

[18] Our result does not rely at all on non-linear marginal extraction costs, as we have assumed they are quadratic, $G'''(O) = 0$.

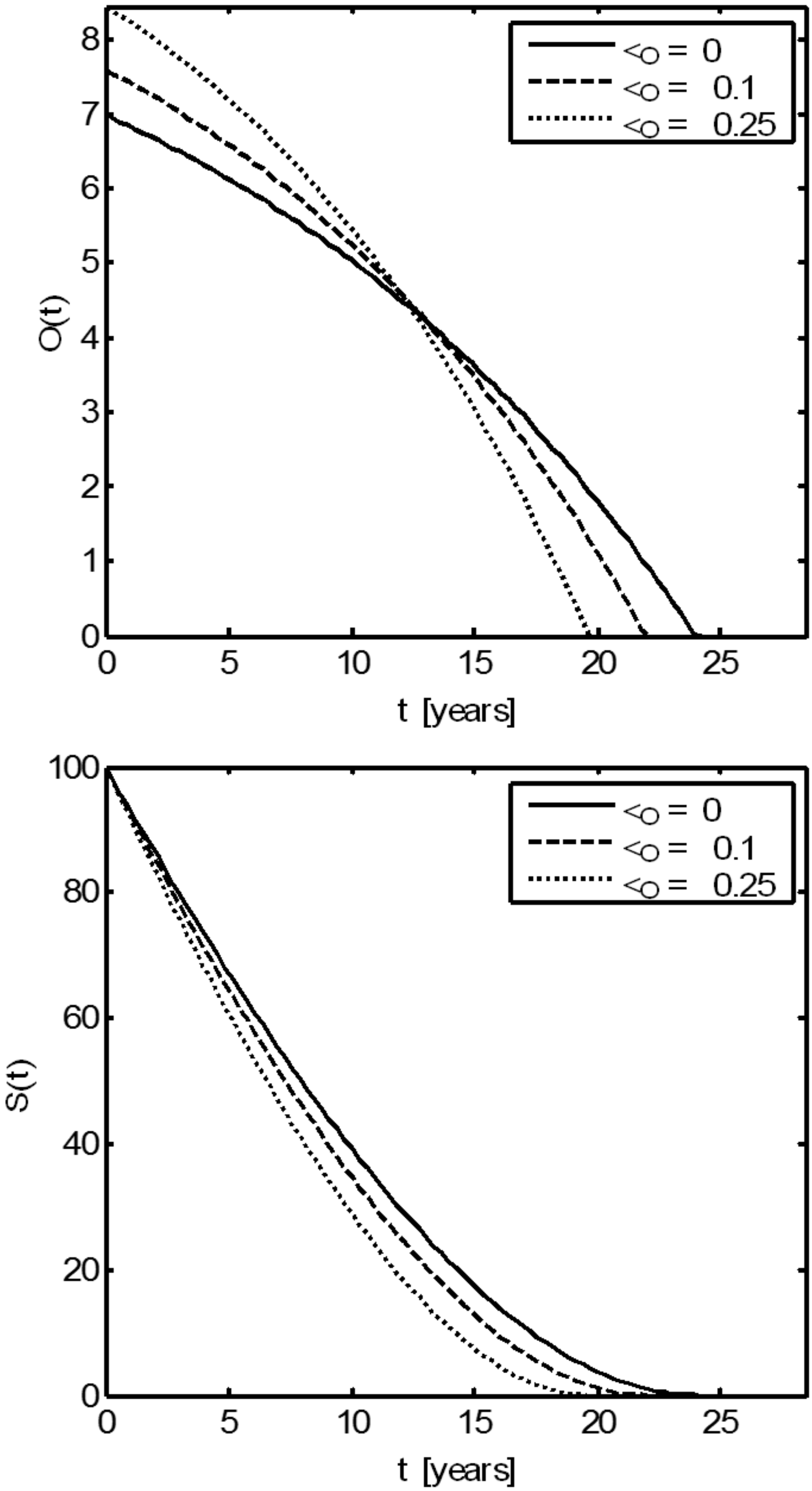

Figure 4.1: Extraction should be faster if the oil price is positively correlated with the assets in a sovereign wealth fund, especially if oil prices are very volatile. Based on calibration in Appendix B.2.

utility from investing the rents in the sovereign wealth fund,[19] $\Omega_O J_F$. This must equal the marginal cost, which is the utility from leaving the oil in the ground, J_S, shown in Appendix C.1. The cost of extracting oil should be shared over time, so $\frac{1}{dt}E[dJ_S] = 0$. The marginal utility from investing in the sovereign wealth fund will fall at the rate of interest, as the size of the fund grows, $\frac{1}{dt}E[dJ_F] = -rJ_F$. So the marginal oil rent must grow at the rate of interest, to keep everything in line, $J_S = \Omega_O J_F$. However, if oil rents and investments move together then an adjustment must be made, to hasten extraction. Otherwise, the benefit from extracting oil will fall behind the benefit from leaving it in the ground.

Extraction will be positively correlated with the oil price, from part iii). If the oil price suddenly jumps, then extraction will also jump to make the most of it. More formally, increasing extraction will increase the marginal cost, $G''(O) > 0$, limiting the jump in marginal rents, $\Omega_O = P_o - G'(O)$. An oil price shock will most affect extraction at the start of a windfall, when the majority of reserves are still exposed to the oil price. As the date of exhaustion approaches, and thus $O \to 0$, the path of extraction gets closer to what it would be in the absence of volatility, from Lemma 3.

In practice, fluctuations in the oil price make reserves more or less economic. An example is the Athabasca oil sand reserves in Alberta, Canada. These reserves have been known since at least the 18th century. Production began very slowly in 1967, with 30,000 barrels per day. However, the rapid rise in oil prices since 2003 has made these reserves economically viable, and production increased to over 1.7 million barrels per day in 2011. During this period both oil prices and asset returns fell dramatically during the financial crisis of 2008-2009. Production also fell during this period (Alberta Energy Regulator, 2013).

The standard Hotelling rule suggests that Alberta's production should rise and fall with the oil price. However, our results suggest that the Albertan government should hasten production by more than a simple forecast of oil prices would suggest. This is because of Alberta's Heritage Savings Trust Fund. Bringing forward oil extraction on

[19] or consuming them, as $J_F = U'(C)e^{-\rho t}$.

average, and extracting more when oil prices are high, will generate a risk premium on subsoil assets.

4.2 Optimal portfolio allocation and consumption

This section shows that oil rents can be hedged by the sovereign wealth fund - regardless of the extraction path - if markets are complete. This will involve dynamically changing the asset allocation to ensure that the net exposure to oil risk at every instant remains a constant share of total wealth. The extraction path should not affect consumption directly, only through its effect on the expected present value of oil wealth.

Let oil extraction follow the optimised path in equation 4.7. Oil rents will follow a non-geometric Brownian motion, giving rise to the final proposition,

$$d\Omega = \mu_\Omega(P_o, S, t)dt + \sigma_\Omega(P_o, S, t)dZ_o \tag{4.8}$$

Proposition 8. Hedging Oil Wealth. Oil rents following the optimal extraction path can also be hedged in the sovereign wealth fund.

If markets are complete and so trading is continuous, then,

i) Oil wealth can be replicated using a bundle of value $X(t)$, comprising the perfectly correlated asset, k, and the risk-free asset. This bundle will evolve according to,

$$dX(t) + \Omega(t)dt = (\omega_k(O,t)X(t)(\alpha_k - r) + rX(t))\,dt + \omega_k(O,t)X(t)\sigma_k dZ_k(t) \tag{4.9}$$

where $\omega_k(O,t) = N_k P_k / X$ is the continuously adjusted share of asset k in the replicating bundle.

ii) The replicating bundle allows oil to be hedged within the sovereign wealth fund, regardless of the rate of extraction. Total wealth will evolve according to, $W = F + V(O)$,

$$dW = \sum_{i=1}^{m} \bar{w}_i W \left(\alpha_i - r\right) dt + \left(rW - C\right) dt + \sum_{i=1}^{m} \bar{w}_i W \sigma_i dZ_i \tag{4.10}$$

where

$$\bar{w}_i = \begin{cases} \frac{w_i F(t)}{F(t)+V(O,t)} & \textit{for } i \neq k \\ \frac{w_k(O,t)F(t)+\omega_k(O,t)V(O,t)}{F(t)+V(O,t)} & \textit{for } i = k \end{cases} \tag{4.11}$$

Proof. See Appendix C.2. □

This proposition begins by stating that oil wealth can be replicated by a bundle holding a perfectly correlated asset, k, and the risk-free asset, n. Oil rents no longer follow the geometric Brownian motion in Section 3, which made the analysis neat. However, they are still driven by two properties: the drift, $\mu_\Omega(P_o, S, t)dt$, and the volatility, $\sigma_\Omega(P_o, S, t)dZ_o$. We want to replicate these properties with a bundle of two assets, k and n.

Intuitively, the drift and volatility of oil rents can be replicated by changing the weights of the risk-free and the perfectly correlated assets in the bundle, respectively. The bundle's risky asset k is perfectly correlated with oil, $dZ_k = dZ_o$. The challenge is to continuously adjust the amount of k in the bundle so that the instantaneous change in the value of oil rents, $\sigma_\Omega(P_o, S, t)dZ_o$, is matched perfectly by the instantaneous change in the bundle, $\omega_k(O, t)X(t)\sigma_k dZ_k$. The portfolio's risk-less asset is then chosen so that the instantaneous drifts also match.

The sovereign wealth fund should be managed to ensure that the net exposure to each asset is a constant share of total wealth, $\bar{w}_i = w\delta_i$, from equation 4.10 and Proposition 4. As in previous sections, any given exposure to asset k that is embodied in oil, $\omega_k(O, t)$, can be offset by the asset's weight in the fund, w_k, to ensure its net weight in total wealth is constant, equation 4.11. The weight of each asset in the fund will need to continuously change, though this was also the case in Section 3. This gives rise to a wealth and substitution effect from rearranging 4.11, as described in Section 3.1,

$$w_i = \begin{cases} \bar{w}_i + \bar{w}_i \frac{V(O,t)}{F(t)} & \text{for } i \neq k \\ \bar{w}_k + (\bar{w}_k - \omega_k(O,t)) \frac{V(O,t)}{F(t)} & \text{for } i = k \end{cases} \tag{4.12}$$

Consumption can be partially separated from oil extraction when markets are complete. Consumption will be a constant share of total wealth, from equation 4.10 and Proposition 5,

$$C(t) = r_W^* W(t) \tag{4.13}$$

where r_W^* is defined in Proposition 5. Oil extraction will not affect consumption directly, only through its effect on total wealth. Extraction and consumption are separated by the sovereign wealth fund. The fund allows consumption to be smoothed, according to the permanent income hypothesis, by the ability to borrow and lend freely. The fund also buffers consumption from oil price volatility, by hedging it with traded assets. Only the residual volatility of total wealth will need to be managed by precautionary savings.

5 Conclusion

Resource exporters have two major types of national asset: resources below the ground, and a sovereign wealth fund above it. This paper makes the case for co-ordinating the two. To do this we adopt a continuous-time framework that combines three strands of literature: on portfolio allocation, consumption under uncertainty, and resource extraction. We find that the fund should invest in more risky assets, and hedge oil price fluctuations, while oil is beneath the ground. Extraction should be faster on average than the standard Hotelling rule if oil is positively correlated with the market, generating a risk premium on subsoil reserves. Consumption should be a constant share of total wealth and use precautionary savings to manage any residual volatility.

This paper argues that sovereign wealth funds should be coordinated with subsoil assets. In practice the management of sovereign wealth funds tends to be separated from oil extraction. Oil revenues are typically invested in well-diversified portfolios of international bonds and equities such as Norway's Government Pension Fund Global, and Chile's Economic and Social Stabilization fund. However, the changing value of subsoil oil - as the oil price fluctuates and reserves are depleted - is not considered. Some attempts at using financial instruments to hedge resource-price fluctuations have been attempted, from long-term forward agreements in iron-ore until 2010, to the purchase of oil price options by Mexico in 2008. However, there is no evidence of systematic coordination of below and above-ground evidence.

The sovereign wealth fund should invest more in risky assets, and hedge oil price fluctuations, while oil is beneath the ground. The fund should invest more in risky assets because of a wealth effect: only a fraction of wealth is included in the fund when there is oil beneath the ground. This means the fund should appear riskier at the start of the windfall, and allocate towards safer assets as oil is extracted and the fund matures. The fund should hedge oil price fluctuations because of a substitution effect: some exposure to traded assets is embodied in oil price fluctuations, and these should be taken into account when constructing the fund. Roughly the fund should invest more in assets that are negatively correlated with oil and vice versa, though the actual allocation should be determined by principal components analysis.

Extraction should be faster on average than the standard Hotelling rule, if oil is correlated with the market. The standard Hotelling rule states that marginal oil rents should rise at the rate of interest. Our results suggest that policymakers should extract faster that this, so rapidly falling convex extraction costs generate a risk premium on volatile oil reserves. Extraction should also respond to the oil price, extracting more when it is high.

Consumption should be a constant share of total wealth. Currently the Norwegian government employs a "bird-in-hand" rule: releasing a constant share of the sovereign wealth fund into the budget each year. This analysis suggests the rule should be redefined to consider total wealth. Consumption will be insulated from oil extraction by the fund. The ability to borrow and lend freely will smooth consumption, consistent with the permanent income hypothesis. The ability to hedge oil prices will reduce the aggregate volatility the government is exposed to, leaving only residual volatility to be managed by precautionary savings. Oil extraction should only affect consumption by altering the present value of total wealth.

Further work may consider exploration, domestic investment, more market imperfections and other price behaviour. Oil reserves are not simply known; they need to be discovered and exploration is costly. The amount of effort dedicated to exploration will naturally depend on the oil price, as noted by Pindyck (1978). It would be worth extending the current analysis to understand how hedging oil wealth in a sovereign wealth fund would affect exploration effort.

Subsoil oil and sovereign wealth funds are not the only assets that policymakers in resource-rich economies control. Domestic capital is a large, and largely non-traded, component of national wealth. Gaudet and Khadr (1991) and Atewamba and Gaudet (2012) consider how extraction will depend on domestic capital, but do not take the next step of considering tradable sovereign wealth fund assets. The ability to trade claims on shocks that affect the productivity of domestic capital may significantly change the way oil and capital interacts.

Introducing more market imperfections may address particular concerns of policymakers. In practice it may be infeasible to hedge the vast reserves of the world's resource exporters, even if the risk is spread across many assets as described in Section 3.2. National governments may also be averse to taking large short positions in oil, or selling it forward, as this means oil is sold at a lower price than the market when prices are high. Understanding how these constraints should affect investment and extraction would be a third area of interest for further work.

Finally, more closely modelling the observed behaviour of prices may make these recommendations more compelling. We have assumed that prices are log-normally distributed. This has lent the analysis to neat, closed-form solutions. In practice prices can exhibit mean reversion, large jumps and time-varying correlation. Incorporating these characteristics into a numerical analysis would make the analysis more accurate for implementation.

References

[1] Alberta Energy Regulator. Statistical report 3: Alberta energy resource industries monthly statistics. Tech. rep., Alberta Energy Regulator, www.aer.ca, 2013.

[2] Atewamba, C., and Gaudet, G. Prices of durable nonrenewable natural resources under stochastic investment opportunities. *Cahiers de recherche 2012-01, Centre interuniversitaire de recherche en économie quantitative, CIREQ.* (2012).

[3] Attanasio, O. P., and Weber, G. Intertemporal substitution, risk aversion and the euler equation for consumption. *The Economic Journal 99*, 395 (1989), pp. 59–73.

[4] Balding, C., and Yao, Y. Portfolio allocation for sovereign wealth funds in the shadow of commodity-based national wealth. *International Finance Review 12* (2011), 293–312.

[5] Bems, R., and de Carvalho Filho, I. The current account and precautionary savings for exporters of exhaustible resources. *Journal of International Economics 84*, 1 (2011), 48 – 64.

[6] Bodenstein, M., Guerrieri, L., and Kilian, L. Monetary policy responses to oil price fluctuations. *IMF Economic Review 60*, 4 (2012), 470–504.

[7] Bodie, Z., Merton, R. C., and Samuelson, W. F. Labor supply flexibility and portfolio choice in a life cycle model. *Journal of economic dynamics and control 16*, 3 (1992), 427–449.

[8] BP. Statistical review of world energy. Tech. rep., BP, 2012.

[9] Brown, K. C., and Tiu, C. I. The interaction of spending policies, asset allocation strategies, and investment performance at university endowment funds. In *How the Great Recession Affected Higher Education*. University of Chicago Press, 2012.

[10] Carroll, C. D. How does future income affect current consumption? *The Quarterly Journal of Economics 109*, 1 (1994), 111–147.

[11] Carroll, C. D., and Kimball, M. S. Precautionary saving and precautionary wealth. *The New Palgrave Dictionary of Economics 6* (2008), 579–584.

[12] Case, K. E., Quigley, J. M., Shiller, R. J., et al. Comparing wealth effects: the stock market versus the housing market. *Advances in Macroeconomics 5(1)* (2005), 1235–1235.

[13] Cass, D., and Stiglitz, J. E. The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. *Journal of Economic Theory 2*, 2 (1970), 122–160.

[14] Chambers, D., Dimson, E., and Ilmanen, A. The norway model. *The Journal of Portfolio Management 38*, 2 (2012), 67–81.

[15] EIA. Petroleum and other liquids database. Tech. rep., U.S. Energy Information Administration, 2013.

[16] Elton, E. J., and Gruber, M. J. Modern portfolio theory, 1950 to date. *Journal of Banking & Finance 21*, 11 (1997), 1743–1759.

[17] Ewing, B. T., and Thompson, M. A. Dynamic cyclical comovements of oil prices with industrial production, consumer prices, unemployment, and stock prices. *Energy Policy 35*, 11 (2007), 5535 – 5540.

[18] Flavin, M., and Yamashita, T. Owner-occupied housing and the composition of the household portfolio. *The American Economic Review 92*, 1 (2002), 345–362.

[19] Friedman, M. *A Theory of the Consumption Function.* Princeton University Press Princeton, 1957.

[20] Gaudet, G. Natural resource economics under the rule of hotelling. *Canadian Journal of Economics/Revue canadienne d'économique 40*, 4 (2007), 1033–1059.

[21] Gaudet, G., and Khadr, A. M. The evolution of natural resource prices under stochastic investment opportunities: An intertemporal asset-pricing approach. *International Economic Review* (1991), 441–455.

[22] Gintschel, A., and Scherer, B. Optimal asset allocation for sovereign wealth funds. *Journal of Asset Management 9*, 3 (2008), 215–238.

[23] Hartwick, J. M. Intergenerational equity and the investing of rents from exhaustible resources. *The american economic review 67*, 5 (1977), 972–974.

[24] Hotelling, H. The economics of exhaustible resources. *The Journal of Political Economy 39*, 2 (1931), 137–175.

[25] IMF. World economic outlook database. Tech. rep., International Monetary Fund, 2013.

[26] Kimball, M. S. Precautionary saving in the small and in the large. *Econometrica 58*, 1 (1990), 53–73.

[27] Koo, H. K. Consumption and portfolio selection with labor income: a continuous time approach. *Mathematical Finance 8*, 1 (1998), 49–65.

[28] Larsen, E. R. Are rich countries immune to the resource curse? evidence from norway's management of its oil riches. *Resources Policy 30*, 2 (2005), 75–86.

[29] Leland, H. E. Saving and uncertainty: The precautionary demand for saving. *The Quarterly Journal of Economics 82*, 3 (1968), 465–473.

[30] Markowitz, H. Portfolio selection. *The Journal of Finance 7*, 1 (1952), 77–91.

[31] Markowitz, H. *Portfolio selection: efficient diversification of investments*. No. 16. Yale university press, 1959.

[32] McKinsey Global Institute. Financial globalization: Retreat or reset? Tech. rep., McKinsey and Company, 2013.

[33] Merton, R. C. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory 3*, 4 (1971), 373–413.

[34] Merton, R. C. *Continuous-Time Finance*. Basil Blackwell, 1990.

[35] Merton, R. C. Optimal investment strategies for university endowment funds. In *Studies of supply and demand in higher education*. University of Chicago Press, 1993, pp. 211–242.

[36] Ministry of Finance. On the management of the government pension fund in 2008. Report number 20 (2008-2009) to the storting, Norwegian Ministry of Finance, 2008.

[37] NBIM. Management mandate. Website, Norges Bank Investment Management, http://www.nbim.no/en/About-us/governance-model/management-mandate/, 2013.

[38] Pindyck, R. S. The optimal exploration and production of nonrenewable resources. *Journal of Political Economy 86*, 5 (1978), pp. 841–861.

[39] Pindyck, R. S. Uncertainty and exhaustible resource markets. *The Journal of Political Economy* (1980), 1203–1225.

[40] Pindyck, R. S. The optimal production of an exhaustible resource when price is exogenous and stochastic. *The Scandinavian Journal of Economics* (1981), 277–288.

[41] Pindyck, R. S. Uncertainty in the theory of renewable resource markets. *The Review of Economic Studies 51*, 2 (1984), 289–303.

[42] Sandmo, A. The effect of uncertainty on saving decisions. *The Review of Economic Studies 37*, 3 (1970), 353–360.

[43] Scherer, B. Portfolio choice for oil-based sovereign wealth funds. *EDHEC-Risk Publication, EDHEC-Risk Institute* (2009a).

[44] Scherer, B. A note on portfolio choice for sovereign wealth funds. *Financial Markets and Portfolio Management 23*, 3 (2009b), 315–327.

[45] Sharpe, W. F. Capital asset prices: A theory of market equilibrium under conditions of risk*. *The journal of finance 19*, 3 (1964), 425–442.

[46] Sinai, T., and Souleles, N. S. Owner-occupied housing as a hedge against rent risk. *The Quarterly Journal of Economics 120*, 2 (2005), 763–789.

[47] Solow, R. M. The economics of resources or the resources of economics. *The American Economic Review 64*, 2 (1974), 1–14.

[48] Svensson, L. E., and Werner, I. M. Nontraded assets in incomplete markets: Pricing and portfolio choice. *European Economic Review 37*, 5 (1993), 1149–1168.

[49] SWF Institute. Sovereign wealth fund rankings. Website, Sovereign Wealth Fund Institute, www.swfinstitute.org, 2013.

[50] Tobin, J. Liquidity preference as behavior towards risk. *The Review of Economic Studies 25*, 2 (1958), pp. 65–86.

[51] Truman, E. M. A blueprint for sovereign wealth fund best practices. Policy brief, Peterson Institute for International Economics, April 2008.

[52] van den Bremer, T. S., and Van der Ploeg, F. Managing and harnessing volatile oil windfalls. *IMF Economic Review 61*, 1 (2013), 130–167.

[53] van der Ploeg, F. Aggressive oil extraction and precautionary saving: Coping with volatility. *Journal of Public Economics 94*, 5 (2010), 421–433.

[54] Wang, C., Wang, N., and Yang, J. Optimal consumption and savings with stochastic income. Working Paper 19319, National Bureau of Economic Research, August 2013.

[55] Wills, S. *Eight principles for managing natural resource wealth.* PhD thesis, University of Oxford, 2013.

[56] Zeldes, S. P. Optimal consumption with stochastic income: Deviations from certainty equivalence. *The Quarterly Journal of Economics 104*, 2 (1989), 275–298.

Appendix A: Portfolio allocation without oil

A.1 Proof of Proposition 1

This appendix derives the results on asset weights in the absence of oil, following Merton (1990, Ch 5). We begin with the Hamilton-Jacobi-Bellman equation for the government's problem in equation 2.1 and 2.8,

$$
\begin{aligned}
0 &= \max_{C,w_i} \left(U(C)e^{-\rho t} + \tfrac{1}{dt} E_t \left[dJ(F,t) \right] \right) \qquad (5.1)\\
&\text{where}\\
\tfrac{1}{dt} E_t \left[dJ(F,t) \right] &= J_F \left(\sum_{i=1}^{m} w_i F(\alpha_i - r) + rF - C \right) + J_t + \tfrac{1}{2} J_{FF} \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j F^2 \sigma_{ij} \qquad (5.2)
\end{aligned}
$$

where equation 5.2 follows from applying Ito's lemma to $J(F,t)$. For ease of notation we define $J_F \equiv \partial J(F,t)/\partial F$ and $J_{FF} \equiv \partial^2 J(F,t)/\partial F^2$. The first order conditions of equation 5.1 with respect to consumption and asset weights are,

$$
\begin{aligned}
U'(C^*)e^{-\rho t} - J_F &= 0 \qquad (5.3)\\
J_F F(\alpha_i - r) + J_{FF} F^2 \sum_{j=1}^{m} w_j^* \sigma_{ij} &= 0 \quad i = 1, \ldots, m \qquad (5.4)
\end{aligned}
$$

The optimal asset weight condition in equation 5.4 holds for all assets, $i = 1, \ldots, m$. Together this is a linear system of m equations, which can be solved explicitly for each w_i,

$$
w_i = -\frac{J_F}{J_{FF} F} \sum_{j=1}^{m} \nu_{ij} (\alpha_j - r) \qquad (5.5)
$$

where $\nu_{ij} = [\Sigma^{-1}]_{ij}$. This can be simplified using the following expression from equation 5.38, where $CRRA = -CU''(C)/U'(C)$,

$$-\frac{J_F}{J_{FF}F} = \frac{1}{CRRA}\frac{C/F}{C_F} \tag{5.6}$$

The result in part i) is found by substituting 5.6 into 5.5, and using $CRRA = 1/\theta$ and $C/F = C_F$ from equation 2.7. The inclusion of the term, ν, ensures that $\sum_{i=1}^{m} \delta_i = 1$.

The result in part ii) relies on the following lemma,

Lemma 1. *There exists an optimal risky portfolio such that, independent of preferences, the wealth distribution and time horizon, the government will be indifferent between choosing a linear combination of the optimal portfolio and the risk free asset, and the original n assets.*

Proof. The condition that all asset weights in the overall fund sum to one, represents a line in the n dimensional plane, $\sum_{i=1}^{n} w_i = 1$. There will exist two linearly independent vectors in that plane, the combination of which will give any point on that line (Cass and Stiglitz, 1970). These two vectors can be considered linearly independent portfolios of the n assets. Thus, regardless of preferences every individual would be indifferent between holding a combination of two unique portfolios, and the original portfolio of n assets. If one portfolio holds only the risk-free asset the other will be uniquely defined as the optimal risky portfolio. This follows Merton (1990, Ch 5). □

Let w be the proportion of the total fund invested in the optimal risky portfolio, so that $w_r = 1 - w$ is invested in the risk-free asset. From lemma 1, the government will be indifferent between holding these two portfolios, and the optimally chosen n assets, so $w_i^* = w\delta_i$ for $i \neq n$ where w and δ_i are defined in equation 2.4.

Finally, consider the value of the optimal risky portfolio, $V^* = \sum_{i=1}^{n} N_i^* P_i$. The overall portfolio is sold as a mutual fund, with N claims sold at price P, $V^* = NP$. This will evolve as follows, by Ito's lemma,

$$\begin{aligned} dV^* &= \sum_{i=1}^{n} N_i dP_i + \sum_{i=1}^{n} P_i dN_i + \sum_{i=1}^{n} dP_i dN_i \\ &= NdP + PdN + dPdN \end{aligned} \tag{5.7}$$

This can be separated into the net inflow of value that does and does not come from capital gains. The latter must equal the value of new shares issued,

$$\sum_{i=1}^{n} P_i dN_i + \sum_{i=1}^{n} dP_i dN_i = PdN + dPdN \tag{5.8}$$

Therefore, combining equations 5.7 and 5.8 gives $NdP = \sum_{i=1}^{n} N_i dP_i$. Using the weight of each asset in the optimal risky portfolio, $\delta_i = N_i P_i / P$, which is a constant, we get,

$$\begin{aligned} \frac{dP}{P} &= \sum_{i=1}^{n} \delta_i \frac{dP_i}{P_i} \\ &= \sum_{i=1}^{n} \delta_i \alpha_i dt + \sum_{i=1}^{n} \delta_i \sigma_i dZ_i \\ &= \alpha dt + \sigma dZ \end{aligned} \tag{5.9}$$

This gives the result in part iii), with $\alpha = \sum_{i=1}^{m} \delta_i \alpha_i$, $\sigma^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \delta_i \delta_j \sigma_{ij}$ and $dZ = \frac{1}{\sigma} \sum_{i=1}^{m} \delta_i \sigma_i dZ_i$.

A.2 Proof of Proposition 2

This appendix derives the results on consumption in the absence of oil, also following Merton (1990). The expected rate of change of consumption can be found by applying Ito's lemma to equation 5.3,

$$\frac{\frac{1}{dt} E_t[dJ_F]}{J_F} = \frac{CU''(C)}{U'(C)} \frac{\frac{1}{dt} E_t[dC]}{C} - \rho + \frac{CU'''(C)}{U'(C)} \frac{\frac{1}{dt} E_t[dC^2]}{C} \tag{5.10}$$

The first term in equation 5.10 is the expected rate of change of the marginal utility of assets, $\frac{1}{dt}E_t[dJ_F]/J_F$. Using Ito's lemma we have, $J_F(F,t) = J_{FF}dF + J_{Ft}dt + \frac{1}{2}J_{FFF}dF^2$. Combining this with the first-order condition of 5.2 with respect to assets gives,

$$\tfrac{1}{dt}E_t[dJ_F] + J_F(w(\alpha - r) + r) + J_{FF}Fw^2\sigma^2 = 0 \tag{5.11}$$

The expected rate of change of the marginal utility of assets will shrink at the rate of interest, equation 5.12 below. This follows from combining equation 5.48 with the optimal asset allocation condition in 5.4,

$$\frac{\frac{1}{dt}E_t[dJ_F]}{J_F} = -r \tag{5.12}$$

The final term in equation 5.10 can be found by applying Ito's lemma to $C(F,t)$, making use of the two-asset separation theorem in Proposition 1,

$$dC^2 = C_F w^2 F^2 \sigma^2 dt \tag{5.13}$$

The result in part i) comes from combining equations 5.10, 5.12, and 5.13, with the relationship $C/F = C_F$ from equation 2.7.

A closed form for the value function, $J(F,t)$, can be found given the assumptions in section 2.1. Substituting the first-order conditions in 5.3 and 5.4 into the Hamilton-Jacobi-Bellman condition in 5.1, and making use of the two-asset simplification from Proposition 1, gives the optimality condition for the value function,

$$0 = \tfrac{1}{\theta-1}\exp(-\rho t\theta)J_F^{1-\theta} + J_t + rFJ_F - \frac{J_F^2}{J_{FF}}\frac{(\alpha - r)^2}{2\sigma^2} \tag{5.14}$$

A closed form solution to this stochastic partial differential equation exists and takes the following form, which can be checked using Ito's lemma,

$$J(F,t) = \frac{\theta}{\theta-1}\exp(-\rho t)(\theta\rho - (\theta-1)\eta)^{-1/\theta}F^{(\theta-1)/\theta} \tag{5.15}$$

where $\eta = r + \theta(\alpha - r)^2/2\sigma^2$.

The result in part ii) follows from substituting equation 5.15 into the first order condition 5.3.

The result in part iii) follows from substituting equation 2.7 into 2.8.

Appendix B: Portfolio allocation for a given path of extraction

B.1 Proof of Propositions 3 and 6

This appendix extends the analysis of university endowments by Merton (1990, Ch 21), to the case of incomplete markets. We begin by proving the relationship in equation 3.14. Asset and oil prices follow the geometric Brownian motions in equations 2.3 and 2.5. Let the oil price be instantaneously imperfectly correlated with the return on each asset, $dZ_o dZ_i = \rho_{o,i}dt$ where $\rho_{o,i} \neq \pm 1 \forall i = 1,\ldots,m$. The returns on these $m+1$ prices (m assets plus oil) can be expressed as a linear combination of $m+1$ independent Wiener processes, $du_j \sim N(0,dt)$ for $j = 1,\ldots,m+1$,

$$\begin{pmatrix} dZ_o \\ dZ_1 \\ \vdots \\ dZ_m \end{pmatrix} = \begin{pmatrix} \lambda_{o,0} & \lambda_{o,1} & \cdots & \lambda_{o,m} \\ \tilde{\lambda}_{1,0} & \tilde{\lambda}_{1,1} & \cdots & \tilde{\lambda}_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\lambda}_{m,0} & \tilde{\lambda}_{m,1} & \cdots & \tilde{\lambda}_{m,m} \end{pmatrix} \begin{pmatrix} du_o \\ du_1 \\ \vdots \\ du_m \end{pmatrix} \tag{5.16}$$

Rearranging this allows the instantaneous return on oil to be separated from the other assets,

$$
\begin{pmatrix} dZ_o \\ dZ_1 \\ \vdots \\ dZ_m \end{pmatrix} = \begin{pmatrix} \lambda_{o,0} & \lambda_{o,1} & \cdots & \lambda_{o,m} \\ 0 & \lambda_{1,1} & \cdots & \lambda_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{m,1} & \cdots & \lambda_{m,m} \end{pmatrix} \begin{pmatrix} du_o \\ du_1 \\ \vdots \\ du_m \end{pmatrix} \quad (5.17)
$$

$$
dZ_o = \lambda_{o,0} du_o + \Lambda_o du \quad (5.18)
$$

$$
dZ = \Lambda du \quad (5.19)
$$

where $\Lambda_o = [\lambda_{o,1} \ldots \lambda_{o,m}]$, $[\Lambda]_{ij} = \lambda_{ij}$, $du = [du_1 \ldots du_m]'$ and $dZ = [dZ_1 \ldots dZ_m]'$. The result in equation 3.14 follows from equations 5.18 and 5.19 if Λ is invertible.

The parameters Λ_o, Λ and $\lambda_{o,0}$ can be estimated using principal components analysis. The instantaneous covariance between any two Wiener processes can be expressed as $cov(dZ_i, dZ_j) = \sum_{g=1}^{m} \lambda_{i,g}\lambda_{j,g} var(du_g)$. The covariance matrix for dZ is therefore, $\Sigma = (E_V E_A^{1/2} dt^{1/2})(E_V E_A^{1/2} dt^{1/2})'$, where E_V is the matrix of eigenvectors, $[E_A^{1/2}]_{ii} = e_i^{1/2}$ is the diagonal matrix of the square roots of eigenvalues, e_i, and $\Lambda = E_V E_A^{1/2}$. An estimate of the covariance matrix can be found after estimating the parameters $\hat{\alpha}_i, \hat{\sigma}_i$ in equations 3.1 to 3.3 by maximum-likelihood. Ranking each eigen-vector by the size of the eigenvalue, and discarding the smallest, gives an idea of how well the oil price is spanned by the market, as illustrated in Table 3.1.

Based on the results in Table 3.1 we will treat $\lambda_{0,o} = 0$, which significantly simplifies the analysis.

We now turn to the results in Propositions 3 and 6. If markets are incomplete then oil revenues can be expressed in terms of the prices of the m traded assets,

$$
P_o(t) = P_o(0) \exp(-\phi t) \prod_{i=1}^{m} [P_i(t)/P_i(0)]^{\beta_i} \quad (5.20)
$$

where $\beta_i = \sigma_o M_i/\sigma_i$, $M_i \equiv [\Lambda_o \Lambda^{-1}]_i$ and $\phi = -\alpha_o + \sum_{i=1}^{m} \beta_i(\alpha_i - \frac{1}{2}\sigma_i^2) + \frac{1}{2}\sum_{i=1}^{m}\sum_{i=1}^{m} \beta_i\beta_j\sigma_{ij}$. This can be found by applying Ito's lemma to 5.20 and comparing coefficients with 3.3.

If markets are complete then $M_k = 1$ and $M_i = 0$ for $i \neq k$.

The results in part i) of Propositions 3 and 6 establish the capitalised value of oil wealth, based on the following Lemma,

Lemma 2. *The capitalised value of oil income is as follows, when markets are incomplete,*

$$V(P_o, t) = P_o(t)O(1 - \exp[-\psi(T - t)])/\psi \tag{5.21}$$

where $\psi = r - \alpha_o + \sum_{i=1}^{m} \beta_i(\alpha_i - r)$.

Proof. First, we will construct a portfolio that will be identical to the capitalised value of oil. Second, we will construct another portfolio, consisting of the capitalised value of oil, the assets in the market and the risk free rate. Using an arbitrage argument we will find a condition that the capitalised value of oil must satisfy. Third, we will show that the posited capitalised value of oil satisfies this condition.

First, let us construct a portfolio with value V, consisting of traded and non-traded assets $i = 1, \ldots m$ and the risk-free asset, which distributes cash each period of $P_o(t)O$. This is equivalent to the capitalised value of oil. The value of the portfolio will evolve according to the dynamics,

$$V = F(P_1, \ldots P_m, t) \tag{5.22}$$

$$dV = (\mu_v V - P_o O)dt + \sigma_v V dZ_v \tag{5.23}$$

Using Ito's Lemma, the dynamics of the portfolio can also be written as,

$$\begin{aligned} dV &= \sum_{i=1}^{m} V_i dP_i + V_t dt + \tfrac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} V_{ij} dP_i dP_j \\ &= \left\{ \sum_{i=1}^{m} V_i \alpha_i P_i + V_t + \tfrac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} V_{ij} \sigma_i \sigma_j \rho_{ij} P_i P_j \right\} dt + \sum_{i=1}^{m} V_i \sigma_i P_i dZ_i \end{aligned}$$

Comparing coefficients with 5.23 gives,

$$\begin{aligned}
\mu_v V - P_o O &= \sum_{i=1}^{m} V_i \alpha_i P_i + V_t + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} V_{ij}\sigma_i\sigma_j\rho_{ij}P_iP_j && (5.24)\\
\sigma_v V dZ_v &= \sum_{i=1}^{m} V_i\sigma_i P_i dZ_i && (5.25)
\end{aligned}$$

Finally, let $dZ_v = \Lambda_v du$. This implies,

$$\sigma_v dZ_v = \sigma_v \Lambda_v du = \Gamma' \Lambda du \tag{5.26}$$

where $\Gamma = [V_1\sigma_1 P_1/V, \ldots, V_m\sigma_m P_m/V]'$.

Second, we make use of an arbitrage argument. Let us now create a second portfolio, with value $X(t)$, consisting of the original portfolio, $V(t)$, the risky assets $k = 1, \ldots, m$, and the risk free asset. Let this portfolio be dynamically constructed so that the short positions offset the long positions and the net value of the asset is always equal to zero. The return to this portfolio is given by the following, so that the weights $w_r = 0 - w_v - \sum_{k=1}^{m} w_k$,

$$\begin{aligned}
dX &= w_v \frac{dV + Y dt}{V} + \sum\nolimits_{i=1}^{m} w_i \frac{dP_i}{P_i} + w_r r dt\\
&= \Big\{ w_V(\mu_V - r) + \sum\nolimits_{i=1}^{m} w_i(\alpha_i - r) \Big\} dt + w_V \sigma_V dZ_V + \sum\nolimits_{i=1}^{m} w_i \sigma_i dZ_i\\
&= \Big\{ w_V(\mu_V - r) + \sum\nolimits_{i=1}^{m} w_i(\alpha_i - r) \Big\} dt + w_V \Gamma' \Lambda du + \Psi \Lambda du && (5.27)
\end{aligned}$$

where the second equality follows from 5.23, the third from 5.26 and $\Psi = [w_1\sigma_1 \ldots w_m\sigma_m]'$.

Now, suppose that the weights in this new portfolio are dynamically constructed so that there is no risk, $w_v^*\Gamma'\Lambda du + \Psi^*\Lambda du = 0$. The weights that would achieve this are $w_i^* = -(V_i/V)P_i w_v^* \forall i$. Arbitrage dictates that such a portfolio must have an expected excess return over the risk-free rate of zero,

$$
\begin{aligned}
0 &= w_v^*(\mu_v - r) + \textstyle\sum_{i=1}^m w_i^*(\alpha_i - r) \\
V(\mu_v - r) &= \textstyle\sum_{i=1}^m V_i P_i(\alpha_i - r)
\end{aligned} \tag{5.28}
$$

Combining this with 5.24 gives the following optimality condition, which must be satisfied by the portfolio $V(t)$,

$$
0 = \tfrac{1}{2}\sum_{i=1}^m \sum_{j=1}^m V_{ij}\sigma_i\sigma_j\rho_{ij}P_iP_j + \sum_{i=1}^m rP_iV_i - rV + V_t + P_oO \tag{5.29}
$$

Third, we test that the proposed capitalised value of oil income does in fact satisfy 5.29. The proposed value, V, and the associated partials, are as follows, which can be shown to satisfy 5.29 by substitution,

$$
V(P_o, t) = P_o(0)O\exp[-\phi t]\prod_{i=1}^m [P_i(t)/P_i(0)]^{\beta_i}(1 - \exp[-\psi(T-t)])/\psi \tag{5.30}
$$

$$
V_i = \frac{\beta_i}{P_i}V \tag{5.31}
$$

$$
V_t = -\phi V - P_o(t)O\exp[-\psi(T-t)] \tag{5.32}
$$

$$
V_{ij} = \begin{cases} \frac{\beta_i\beta_j}{P_iP_j}V & \text{if } i \neq j \\ \frac{\beta_i(\beta_i - 1)}{P_i^2}V & \text{if } i = j \end{cases} \tag{5.33}
$$

□

Lemma 2 gives the capitalised value of oil income. The instantaneous rate of change of this value can be found by applying Ito's lemma to 5.21,

$$
P_o(t)Odt + dV = (r + \sum_{i=1}^m \beta_i(\alpha_i - r))Vdt + \sigma_o V dZ_o \tag{5.34}
$$

The results in part ii) of Propositions 3 and 6 follow from substituting 5.34 and the evolution of fund assets in 3.2 into the expression for the evolution of total wealth, $dW = dF + dV$,

$$\begin{aligned} dW &= \sum_{i=1}^{m} w_i F\left(\alpha_i - r\right) dt + \sum_{i=1}^{m} \beta_i(\alpha_i - r) V dt + \left(rF + rV - C\right) dt \\ &\quad + \sum_{i=1}^{m} w_i F \sigma_i dZ_i + \sigma_o V dZ_o \end{aligned} \tag{5.35}$$

B.2 Data and Calibration

Data

The ability of traded assets to span the oil price is presented in Table 3.1. This is based on the following equity and oil prices.

Equity prices are based on the FTSE Global All Cap index (US dollars), which is the equity benchmark used by Norway's Government Pension Fund Global. It is provided directly by FTSE and consists of daily data from 01/01/2002 to 12/06/2013. The analysis is based on ten industry sub-indices based on the ICB classifications, covering ~7,400 equities. However this (very large) index remains only a small sub-sample of the entire market - ICB classifies over 75,000 equities. Industry sub-indices are used, rather than individual equities, to improve the power of our estimation of the covariance matrix. This approach is also taken by Gintschel and Scherer (2008).

Table 3.1 is calculated by estimating the covariance matrix of oil with the ten industries in the FSTE All Cap index, using maximum likelihood estimation. Principle components analysis is then applied to the matrix, ranking eigen-vectors by the size of their eigen-values. The results in the table use 1 year of daily data, but the results are robust to other time horizons: ten shocks explain 99.78%, 99.54% and 99.75% of covariance when calculated on 1, 6 and 24 months of data respectively.

Calibration

Parameter	Value	Parameter	Value
$F(0)$	100	$S(0)$	100
r, ρ	0.05	O	5.00
θ	0.50	α_o	0.01
$P_i(0)$	1.00	σ_o	0.25
α_i	0.07	Λ_o case 1	[1 0]
σ_i	0.20	Λ_o case 2	[1 -1]
ρ_{ij}	0	γ	0.20

Table 5.1: Calibration used in Figures 3.3, 3.1 and 4.1.

Calibration

The analysis in Figures 3.3, 3.1 and 4.1 are based on the simple illustrative calibration in Table 5.1.

Appendix C: Portfolio allocation if oil extraction can be chosen

C.1 Proof of Proposition 7

The policymaker's problem in equation 4.1 can be summarised by the Hamilton-Jacobi-Bellman (HJB) equation,

$$0 = \max_{C,w_i,O} \left(U(C)e^{-\rho t} + \tfrac{1}{dt} E_t \left[dJ(F, P_o, S, t) \right] \right) \tag{5.36}$$

where

$$\begin{aligned} \tfrac{1}{dt} E_t \left[dJ(F, P_o, S, t) \right] = & \; J_F \left(\sum_{i=1}^{m} w_i F(\alpha_i - r) + rF - C + P_o O - G(O) \right) \\ & + J_P \alpha_o P_o - J_S O + J_t \\ & + \tfrac{1}{2} J_{FF} \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j F^2 \sigma_{ij} + \tfrac{1}{2} J_{PP} \sigma_o^2 P_o^2 + J_{FP} \sum_{i=1}^{m} w_i F \sigma_i \sigma_o P_o \rho_{io} \end{aligned} \tag{5.37}$$

The first order conditions with respect to consumption, asset weights and extraction are respectively,

$$
\begin{aligned}
0 &= U'(C)e^{-\rho t} - J_F && (5.38)\\
0 &= J_F F(\alpha_i - r) + J_{FF}F^2\sum_{j=1}^{m} w_j\sigma_{ij} + J_{FP}F\sigma_i\sigma_o P\rho_{io} && (5.39)\\
0 &= J_F(P_o - G_O) - J_S && (5.40)
\end{aligned}
$$

We can also differentiate the HJB equation with respect to the state variables: foreign assets, F, oil reserves, S, and oil prices, P_O,

$$
\begin{aligned}
0 &= \tfrac{1}{dt}E[dJ_F] + J_F(\sum_{i=1}^{m} w_i(\alpha_i - r) + r) + J_{FF}F\sum_{i=1}^{m}\sum_{j=1}^{m} w_i w_j\sigma_{ij} && (5.41)\\
&\quad + J_{FP}\sum_{i=1}^{m} w_i\sigma_i\sigma_o P_o\rho_{io}\\
0 &= \tfrac{1}{dt}E[dJ_s] + 0 && (5.42)\\
0 &= \tfrac{1}{dt}E[dJ_P] + J_F O + J_P\alpha_o + J_{PP}\sigma^2 P_o + J_{FP}\sum_{i=1}^{m} w_i F\sigma_i\sigma_o\rho_{io} && (5.43)
\end{aligned}
$$

Substituting 5.39 into 5.41 gives,

$$
\frac{\frac{1}{dt}E[dJ_F]}{J_F} = -r \qquad (5.44)
$$

This gives two pieces of intuition. The first is that oil is extracted so that the marginal utility of an extra barrel of oil in the ground is always constant, $\frac{1}{dt}E[dJ_s] = 0$ in 5.42. The second is that the marginal utility of assets (or the marginal utility of consumption, from 5.38) will fall at the rate of interest, $\frac{1}{dt}E[dJ_F]/J_F = -r$ in 5.44. Combining 5.38 and 5.44 gives rise to the Euler equation,

$$\frac{\frac{1}{dt}E_t[dC]}{C} = -\frac{U'(C)}{CU''(C)}(r-\rho) - \frac{U'''(C)}{U''(C)}\frac{\frac{1}{dt}E_t[dC^2]}{C} \tag{5.45}$$

The Hotelling equation in part i) describes the expected path of marginal oil rents, $\Omega_O = P_o - G'(O)$. This can be found by applying Ito's lemma to equation 5.40, and dividing throughout by 5.40,

$$\frac{dJ_S}{J_S} = \frac{dJ_F}{J_F} + \frac{d\Omega_O}{\Omega_O} + \frac{dJ_F d\Omega_O}{J_F\Omega_O} \tag{5.46}$$

The result in part i) is found by combining 5.42, 5.44 and 5.46 to give the expected Hotelling rule,

$$\frac{\frac{1}{dt}E[d\Omega_O]}{\Omega_O} = r - \frac{\frac{1}{dt}E[dJ_F d\Omega_O]}{J_F\Omega_O} \tag{5.47}$$

The path of extraction is affected by the covariance between the marginal utility of wealth and marginal oil rents. Marginal oil rents are a function of oil prices and the rate of extraction, $\Omega_O = P_o - G'(O)$. Both the marginal utility of wealth and the rate of oil extraction will be a function of the four state variables, $J_F(F, P_o, S, t)$ and $O(F, P_o, S, t)$. Applying Ito's lemma to both gives,

$$dJ_F = -rJ_F dt + J_{FF}\sum_{i=1}^{m} w_i F\sigma_i dZ_i + J_{FP}\sigma_o P dZ_o \tag{5.48}$$

$$dO = \mu_o(F, P_o, S, t)dt + O_F\sum_{i=1}^{m} w_i F\sigma_i dZ_i + O_P P_o\sigma_o dZ_o \tag{5.49}$$

where we know $E[dJ_F] = -rJ_F dt$ from equation 5.44, and $\mu_o(F, P_o, S, t)$ is the as-yet-unknown expected rate of change of extraction. We derive an expression for $\mu_o(F, P_o, S, t) = \frac{1}{dt}E[dO]$ below. Applying Ito's lemma to $\Omega_O = P_o - G'(O)$ gives the following, where we make use of the assumption that $G(O) = \frac{1}{2}\gamma O^2$,

$$
\begin{aligned}
d\Omega_O &= dP_o - G''(O)dO - \tfrac{1}{2}G'''(O)dO^2 \\
&= (\alpha_o P_o - \gamma\mu_o(F, P_o, S, t))\, dt - \gamma O_F \sum_{i=1}^{m} w_i F \sigma_i dZ_i + (1 - \gamma O_P)\, P_o \sigma_o dZ_o \qquad (5.50)
\end{aligned}
$$

Multiplying equations 5.48 and 5.50 gives,

$$
\begin{aligned}
\frac{dJ_F d\Omega_O}{J_F \Omega_O} &= -\frac{\gamma O_F}{J_F \Omega_o}\left(\sum_{i=1}^{m} w_i F \sigma_i \left\{ J_{FF} F \sum_{j=1}^{m} w_j \sigma_j \rho_{ij} + J_{FP} \sigma_o P \rho_{io} \right\}\right) dt \\
&\quad + \frac{(1 - \gamma O_P)\, P_o \sigma_o}{J_F \Omega_o}\left(J_{FF} \sum_{i=1}^{m} w_i F \sigma_i \rho_{io} + J_{FP} P_o \sigma_o \right) dt
\end{aligned}
$$

This can be simplified by substituting in the optimal asset weight condition in equation 5.39 for all assets (O_F term), and for the perfectly correlated asset k (O_P term), to give,

$$
\frac{dJ_F d\Omega_O}{J_F \Omega_O} = \frac{1}{\Omega_O}\left(\gamma O_F \sum_{i=1}^{m} w_i F(\alpha_i - r) - (1 - \gamma O_P)\, \tfrac{\alpha_k - r}{\sigma_k} P_o \sigma_o \right) dt \qquad (5.51)
$$

The result in part ii) follows from combining the expected Hotelling rule in 5.47 with equations 5.50 and 5.51.

To gain further intuition we must approximate the partial derivatives, O_F and O_P. To do this we use the solution to the deterministic case - which can be found explicitly - to estimate the solution to the stochastic case. The deterministic solution is given in the following Lemma,

Lemma 3. Deterministic solution.

If all prices are deterministic then:

i) Marginal oil rents will rise at the rate of interest, the classical Hotelling (1931) rule,

$$\dot{\Omega}_O(t) = r\Omega_O(t) \tag{5.52}$$

ii) If the oil price is also without drift, $\alpha_O = 0$, then the date of exhaustion is,

$$T = -\tfrac{1}{r}\ln(\gamma O(0)/P_O(0)) \tag{5.53}$$

iii) The optimal rate of extraction (to a leading order approximation) is,

$$O(t) = \sqrt{2\tfrac{r}{\gamma}S(t)P_O(t)} \tag{5.54}$$

Proof. In the deterministic case the extraction problem depends only on extraction and reserves, O and S. This can be seen from equations 4.5 and 4.4 when $\sigma_o = 0$, using the notation $\frac{1}{dt}[d\Omega_O] = \dot{O}$ to emphasise that we are working in the deterministic case. This gives,

$$\begin{aligned} \dot{\Omega}_O(t) &= r\Omega_O(t) \\ \dot{S}(t) &= -O(t) \end{aligned}$$

If we assume that extraction costs are quadratic, $G(O) = \frac{1}{2}\gamma O^2$, so all oil will be depleted at some time T, that oil prices will evolve deterministically at $\dot{P}_o(t) = \alpha_o P_o(t)$, with $\alpha_o < r$ to ensure a solution exists, and marginal oil rents are $\Omega_O = P_o - G'(O)$, then extraction will follow the path,

$$\dot{O}(t) = rO(t) + \tfrac{1}{\gamma}P_o(0)(\alpha_o - r)e^{\alpha_o t} \tag{5.55}$$

The solution to this differential equation can be shown to be,

$$O(t) = O(0)e^{rt} + \tfrac{1}{\gamma}P_o(0)\left[e^{\alpha_O t} - e^{rt}\right] \tag{5.56}$$

We can never have $\alpha_O \geq r$ as price growth would delay extraction indefinitely. Provided initial marginal oil rents are positive, $\Omega_O(0) > 0$, and, $\alpha_O < r$, then the extraction rate remains finite. The optimal initial extraction rate must satisfy, $S(t) = \int_t^T O(\tau)d\tau$, and the date of exhaustion must satisfy, $O(T) = 0$. The date of exhaustion only has an explicit solution for $\alpha_O = 0$, which we assume with some empirical confidence. This gives,

$$T = -\tfrac{1}{r}\ln(1-R) \tag{5.57}$$

$$S(0) = -\tfrac{1}{r\gamma}P_O(0)(\ln(1-R)+R) \tag{5.58}$$

where $0 < R = \gamma O(0)/P_O(0) < 1$. As 5.58 only implicitly defines the rate of extraction, $O = f(S, P_O)$, we use asymptotic methods to find a series solution and study the leading order effect. Using $-\ln(1-R) = \sum_{n=1}^{\infty}\frac{1}{n}R^n$ we have,

$$\frac{r\gamma S(0)}{P_O(0)} = \sum_{n=2}^{\infty}\frac{R^n}{n} \tag{5.59}$$

This can be inverted to give,

$$O(t) = S(t)\left[\sqrt{2}\xi(t)^{-1} - \tfrac{2}{3} + \tfrac{1}{9\sqrt{2}}\xi(t) + \tfrac{2}{135}\xi(t)^2 + \tfrac{1}{540\sqrt{2}}\xi(t)^3 + o(\xi(t)^4)\right] \tag{5.60}$$

where $\xi(t) = \sqrt{r\gamma S(t)/P_O(t)}$,and the coefficients stem from the series inversion and so are independent of parameters. To the leading order this yields the relationship in 5.54 $\square$

This Lemma defines the path of oil extraction when prices are exogenous, according to the classical Hotelling (1931) rule. If oil is very abundant, then oil supply will have a price elasticity equal to one, $O(t) = \frac{1}{\gamma}P(t)$. Oil supply will simply rise with the exogenous price of oil. However, in general scarcity means that oil supply will be less price elastic than this. Exhaustion will take longer if the initial stock of reserves is high, the oil price is low, or extraction costs are high.

The marginal effect of wealth and oil prices on extraction can be approximated from Lemma 3,

$$\begin{aligned} O_F &= 0 \\ O_P &= \frac{1}{2}\frac{O}{P_O} = \sqrt{\frac{rS}{2\gamma P_O}} \end{aligned}$$

The approximate stochastic Hotelling rule comes from substituting these partial derivatives into equation 4.6,

$$d\Omega_O \approx r\Omega_O dt + (P_O - \gamma\tfrac{1}{2}O)\frac{\sigma_O}{\sigma_k}\left[(\alpha_k - r)dt + \sigma_k dZ_O\right] \tag{5.61}$$

So, a positive shock to the oil price, $+dZ_O$, increases marginal oil rents. The expected path of oil extraction comes from combining equations 5.50 and 5.61, and setting $\alpha_O = 0$, to give,

$$\begin{aligned} \tfrac{1}{dt}E[dO] &= \mu_O(F, P_O, S, t) \\ &= -\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)P_o + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)O \end{aligned} \tag{5.62}$$

The path for dO then follows from substituting this into 5.49.

C.2 Proof of Proposition 8

Let there be a traded asset, k, which is perfectly instantaneously correlated with oil, $dZ_o = dZ_k$.

The result in part i) states that oil rents can be replicated using a bundle containing N_k shares of asset k and N_r shares of the risk-free asset, $X(t) = \sum_{i=k,r} N_i(t)P_i(t)$. This bundle must yield a continuous dividend, exactly equal to the optimal path for oil rents $\Omega(t)$.

The replicating bundle can be constructed as follows. For simplicity we begin in discrete time with period length, h, before moving to continuous time $h \to 0$ following Merton (1990, pp125). Construct a bundle so that at every time, t, the number of shares $N_k(t)$ and $N_r(t)$ are chosen and held until time $t + h$. At the same time a dividend is declared to exactly equal oil rents, $\Omega(t)$. This might be observed from production data or government accounts. The dividend is paid continuously throughout the period, h. The bundle will begin period t with value $X(t) = \sum_{i=k,r} N_i(t-h)P_i(t)$, as the number of shares in each asset will have been chosen in the previous period. At time t the dividend and new shares are chosen so that the value of the bundle is preserved, $-\Omega(t)h = \sum_{i=k,r}[N_i(t) - N_i(t-h)]P_i(t)$. The same thing will happen at $t + h$, so,

$$\begin{aligned}
-\Omega(t+h)h &= \sum_{i=k,r}[N_i(t+h) - N_i(t)]P_i(t+h) \\
&= \sum_{i=k,r}[N_i(t+h) - N_i(t)][P_i(t+h) - P_i(t)] \\
&\quad + \sum_{i=k,r}[N_i(t+h) - N_i(t)]P_i(t)
\end{aligned} \tag{5.63}$$

Taking the limit as $h \to 0$ we have that $-\Omega(t)dt = \sum_{i=k,r} dN_i(t)dP_i(t) + dN_i(t)P_i(t)$, assuming all variables are right-continuous. This describes the amount of shares sold every period to finance the dividend.

The result in part i) combines the expression for dividends with the path for the replicating bundle. By Ito's lemma the replicating bundle, $X(t) = \sum_{i=k,r} N_i P_i$, will satisfy,

$$
\begin{aligned}
dX &= \sum_{i=k,r} N_i dP_i + \sum_{i=k,r} dN_i dP_i + \sum_{i=k,r} dN_i P_i \\
dX &= \sum_{i=k,r} N_i dP_i - \Omega dt \\
dX(t) + \Omega(t)dt &= \omega_k(t)X(t)(\alpha_k - r)dt + rX(t)dt + \omega_k(t)X(t)\sigma_k dZ_k(t) \qquad (5.64)
\end{aligned}
$$

where $\omega_k(t) = N_i(t)P_i(t)/X(t)$ is the weight of the risky asset k in the replicating bundle. The weights $\omega_k(t)$ must be updated continuously, to match the stochastic path of oil rents described in equation 5.63. Note that we have focused on an expression for $dV(t) + \Omega(t)dt$. The explicit expression for $V(t)$ can be found using Contingent Claims Analysis (see Merton, 1990). This is also applicable for when oil rents follow the general Ito process $d\Omega(t) = a(.)\Omega dt + s(.)\Omega dZ_o$, when $a(.)$ and $s(.)$ are not constants.

The value of oil rents must equal that of the replicating bundle, $V(t) = X(t)$. This is because both share exactly the same properties, made possible by the perfect correlation of asset k and oil.

The result in part ii) states that the policymaker's problem can be summarised in terms of total wealth. Total wealth is given by $W(t) = F(t) + V(t)$. Combining equations 4.2 and 5.64 gives the result in equation 5.35.

Chapter 3

Optimal monetary responses to oil discoveries

Optimal Monetary Responses to Oil Discoveries

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Abstract

Monetary policy can play an important role in managing oil discoveries. Ideally governments will use fiscal policy to smooth consumption of oil income. In practice this often does not happen, as governments delay spending until oil revenues are received. This induces changes in the economy, both at discovery and when spending begins. In this paper we consider how monetary policy should respond. The paper makes three contributions. The first is to show that an oil discovery causes the real exchange rate to appreciate twice: when forward-looking households and then the government increase their consumption. This can cause a recession under standard monetary regimes, as firms anticipate the second appreciation. The second contribution is to micro-found the objective of monetary policy. The central bank should stabilise inflation, the output gap and the fiscal gap. It will also try to appreciate the non-oil terms of trade, to exploit the asymmetry from owning oil wealth. The third is to derive a closed form for optimal monetary policy, which will respond in advance to expected changes in government demand. This will delay the second real appreciation until the government can take up the slack left by private demand. Optimal policy significantly improves welfare relative to standard monetary regimes, and is well approximated by a simple Taylor rule that responds to expected changes in the natural level of output.

Keywords: Natural resources, oil, optimal monetary policy, small open economy, anticipated windfall.

JEL Classification: E52, E62, F41, O13, Q30, Q33

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1 Introduction

How should monetary policy respond to an oil discovery? With 1.4 billion people living in the forty-seven countries economically dominated by natural resources, this question is of broad interest (Baunsgaard et al., 2012). It will continue to be as new technologies, such as fracking, become widespread. Academic literature has recently focused on medium- to long-term fiscal policy. However, the question remains of how monetary policy can best accommodate these policies in the short-term. A particular challenge is presented by the time lag that exists between discovering oil, and spending the associated revenues. Previous literature has suggested that this period can experience recession under a variety of monetary regimes. This paper begins by revisiting the earlier literature in a fully rational expectations framework and shows that oil discoveries are likely to have effects both on announcement, and when production begins. It then shows that optimally designed monetary policy can overcome many of the distortions caused by these effects, and improve on standard regimes such as currency pegs and Taylor rules.

Recent academic work on natural resources has focused on medium- to long-term fiscal policy, as surveyed by van der Ploeg and Venables (2012). Fiscal policy is important in regulating the pass-through of oil income to the economy (Pieschacon, 2012). The benchmark recommendation for oil-rich governments has been to smooth expenditure as soon as oil is discovered. In practice this means spending only the permanent income from assets that are held offshore in a sovereign wealth fund (see for example the case of Norway, Chambers et al., 2011). Expenditure smoothing has been extended to account for specific distortions a resource-exporter may face, such as bottlenecks in investment (van der Ploeg, 2012) or scarce access to capital (van der Ploeg and Venables, 2011).

However, in practice there is usually a lag between discovering oil and starting production. During this period contracts are negotiated, permits are obtained and capital is imported or built. This is seen in the data. For oil and gas discoveries, of the 400 offshore fields discovered in the UK between 1957-2011, the mean time between discovery and production was 4.5 years (DECC, 2013). For mineral projects, of the 82 recently announced new projects in Western Australia at October 2012 the mean expected time to production was approximately 3 years (BREE, 2012).[1] While the length will vary with the type of commodity and the nature of the field, most resource discoveries will involve some delay.

Governments also often delay spending oil revenues at least until production begins. This happens because early income is recouped by extraction firms to cover their initial investment, or governments face borrowing constraints and make political commitments to balanced budgets. While clean illustrations of this are harder to come by, Azerbaijan in particular stands out. In 2002 Azerbaijan's proved reserves rose from 1 billion to 7 billion barrels of oil (EIA, 2013). However, production did not increase until the Baku-Tbilisi-Ceyhan pipeline was opened in 2006. In 2006 resource rents jumped by 42% (World Bank, 2013). Over the same year government spending jumped by 45%. While this example is starker than most, it gives an idea of the types of challenges that monetary policymakers might face.

Earlier literature has suggested that the lag between discovery and spending can cause a recession. Eastwood and Venables (1982) find that an oil discovery immediately causes the exchange rate to appreciate under the influence of expectations, in a Dornbusch-style framework. If the deflationary effect of this appreciation is not offset by increased demand, there may be a period of recession. The recession would only be reversed when oil-generated demand begins.[2] However, their analysis was performed without the benefit of today's fully-forward looking DSGE models, and assumed that households cannot borrow against future wealth. In what follows we show that such recessions can still occur, even if households smooth their consumption.

The majority of recent work on oil shocks has not focused on resource exporters. One strand of literature has emphasized that the source of oil price shocks matters for policy in resource importers (see Kilian, 2009, Nakov and Pescatori 2010, Bodenstein et

[1] Delays are still common even when development is certain. In Western Australia, of the 35 new projects that were "committed but not yet complete" in October 2012, only 6% expected to begin production within 3 months, and only 60% expected to begin production within 15 months.

[2] In commenting on that paper, Neary and van Wijnbergen (1984) find the results to be "relatively optimistic". By including a direct impact of oil wealth on money demand they show that the recession can continue even after government spending rises. Other papers at the time linking oil shocks to unemployment include Buiter and Purvis (1983) and van Wijnbergen (1984).

al, 2012). Another has considered the roles of monetary policy and oil prices in the US recessions of the 1970s and 80s (see Kilian and Lewis, 2011). The IMF has conducted a number of numerical analyses of oil shocks in multi-country frameworks (see for example Elekdag et al., 2007), and in specific countries such as with Ghana's 2007 discovery of oil (Dagher et al., 2010). There has been little work studying resource exporters from an optimal perspective. Perhaps one of the closest pieces of work is by Romero (2008), which investigates optimal monetary response of an oil exporter subject to oil price shocks. However, this abstracts from the anticipation effects around oil discoveries.

This paper attempts to fill the gap in the literature. In doing so it makes three contributions. The first is to characterise the effects of an oil discovery in a fully rational expectations framework. The second is to derive a micro-founded loss function for the monetary authority in an oil exporter. The third is to derive in closed form the optimal monetary response to an oil discovery.

The paper's first contribution is to characterise the effects of an oil discovery in a fully rational expectations framework. Oil discoveries will affect the economy in two stages. First, on discovery private consumption will jump in anticipation of future income. Second, when production begins government spending will add to demand for domestic goods. Each stage will require a separate terms of trade appreciation because of increased demand for domestic goods. Between discovery and spending firms will raise their prices for two reasons: to retrospectively account for the first appreciation, while prospectively preparing for the second. Under a currency peg, as used by 75% of resource-dependent economies (Baunsgaard et al., 2012), this will cause large fluctuations in the output gap and inflation and lead to recession.

The paper's second contribution is to derive a micro-founded loss function for the monetary authority in an oil exporter. The authority's objective will be to stabilise domestic inflation, the output gap and the fiscal gap; as in models of symmetric small open economies with a government (see Gali and Monacelli, 2005 and 2008). In addition, the monetary authority will have an incentive to appreciate the non-oil terms of trade, and exploit the asymmetry introduced by oil wealth. Appreciating the terms of trade will make imports cheaper, but reduce non-oil exports. Income from non-oil exports is less important for countries with oil than those without. Note that we abstract from any externalities in the traded sector that would make this central-bank induced "Dutch disease" sub-optimal (See Corden and Neary, 1982).

The paper's third contribution is to derive in closed form the optimal monetary response to an oil discovery. The optimal response to the first effect is to let the currency appreciate. This allows the terms of trade to appreciate through a single flexible price (the exchange rate), rather than many sticky prices (domestic goods). The optimal response to the second effect is to sharply loosen interest rates in the period before the windfall, and tighten in the period the windfall is received. This delays the terms of trade appreciation until government spending can take up the slack that is left by private demand. It also avoids the need for firms to raise prices individually, with the associated distortions caused by price dispersion. The loosening of policy in the future does not require the central bank to be credible: such policy is also optimal for a policymaker that can not commit. Optimal policy is also well approximated by a simple Taylor rule where interest rates respond to inflation, the output gap and expected changes in the natural level of output.

This paper builds on recent literature concerning optimal monetary policy in small open economies. We adopt a small open economy framework following Gali and Monacelli (2005, 2008) as our starting point. These works display a "divine coincidence" in which stabilising domestic inflation is equivalent to stabilising the welfare relevant output gap (Blanchard and Gali, 2007). In a closed economy, divine coincidence holds because the only distortion in the economy is sticky prices. In an open economy, divine coincidence holds only if all countries are symmetric and the elasticity of substitution between home and foreign goods is one (Monacelli, 2012). Otherwise the central bank can improve on the flexible price outcome by moving the terms of trade, as the income and substitution effects will not offset (De Paoli, 2009). In our paper divine coincidence breaks down, despite a unit elasticity of substitution. This happens because of the asymmetry introduced by oil wealth, mentioned above.

The discovery of oil can also be thought of as a type of news shock. There is an extensive recent literature on news about future productivity, while we focus on news about future income. The literature on news about productivity has attempted to replicate the dynamics of Pigou cycles (see Pigou, 1927). Standard models of the business cycle imply that good news about future productivity increases expected wealth, increasing consumption but reducing hours worked. In contrast, Pigou cycles suggest that good news about the future leads to positive co-movement in consumption, investment and hours worked today. Recent theory has successfully been able to generate Pigou cycles, though it has proven to be relatively challenging (see Beaudry and Portier, 2004, Den Haan and Kaltenbrunner, 2006 and Jaimovich and Rebelo, 2009). However, there remains some debate over the empirical evidence for these cycles.[3] This paper focuses on news about future income, rather than productivity. In recent work Mertens and Ravn (2011) find that anticipated future income, in the form of announced tax cuts, causes output to contract prior to implementation which is consistent with our findings. This paper contributes to the news shock literature by shedding some light on the optimal monetary response to news about anticipated fiscal policy.

Our results also support recent work on forward guidance. Woodford (2012) argues that central banks should provide guidance about future policy rules in the context of the zero lower bound. Guidance about future policy will also be necessary in responding to an oil discovery. The optimal response to an oil discovery requires loosening rates in the period before the windfall and tightening them again once the windfall is received. This delays the terms of trade appreciation until the government can replace private demand. The ability to delay the appreciation relies on households and firms correctly anticipating the behaviour of the central bank, as the standard Taylor rule will not achieve this outcome. We implicitly assume this anticipation by the rational expectations in our model, though in practice it would require clear communication from the central bank.

The remainder of this paper proceeds as follows. Section 2 presents a DSGE model of

[3]Beaudry and Portier (2006) and Beaudry and Lucke (2010) find evidence supporting Pigou cycles, while Barksy and Sims (2011, 2012) find evidence that positive news about the future reduces output today, consistent with standard models of the business cycle. The latter find that their identified news shock explains a significant fraction of variation in TFP at business cycle frequencies, whereas the shocks identified by the former do not.

a small open economy with a government that receives an exogenous oil windfall. Section 3 characterises the steady state and analyses dynamics around this state. Section 4 introduces the central bank's micro-founded loss function, and derives optimal monetary policy under discretion and commitment. Section 5 calibrates the model, and compares optimal policy to a flexible price benchmark and a currency peg. Section 6 concludes.

2 Model

This section develops a rational expectations model of a small open economy with a government that receives an exogenous oil windfall. It fits in the framework of models developed by Gali and Monacelli (2005, 2008) and Gali (2008), allowing us to compare our results with existing work. We make as few additions as possible, to keep the model simple and permit a closed-form solution. This paper therefore extends these models in two ways. The first is by introducing a government that receives an oil windfall and spends it according to a simple fiscal rule, which will be the focus of our analysis. The second is by allowing wealth to differ between home and foreign households, and to change over time. This allows us to model the wealth effects of an oil windfall, and will relax the "divine coincidence" that characterises similar models.

2.1 Households

The representative household will maximise utility subject to a per-period budget constraint,

$$\max_{C_t,N_t,D_t} U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left[(1-\chi)\ln C_t + \chi \ln G_t - \frac{N_t^{1+\varphi}}{1+\varphi}\right] \tag{2.1}$$

$$s.t. \qquad P_t C_t \leq W_t N_t - P_{H,t} T_t - E_t[M_{t,t+1} D_{t+1}] + D_t \tag{2.2}$$

where C_t is a domestic consumption bundle with consumer price index (CPI), P_t; G_t

is the government spending bundle which is partly funded by taxes, T, levied in domestic currency, $P_{H,t}$; N_t is hours worked for wages, W_t; and D_{t+1} is the nominal payoff in domestic currency at time $t+1$ of the household's portfolio of internationally traded financial assets (including shares in firms), which has stochastic discount factor, $M_{t,t+1}$. Utility will depend on three key parameters: the discount factor, β, the utility weight of government spending, χ, and the elasticity of labour supply, φ. This optimisation problem yields two first order conditions, describing labour supply and the stochastic Euler equation,

$$
\begin{aligned}
C_t N_t^{\varphi} &= (1-\chi) W_t P_t^{-1} && (2.3)\\
\beta C_t P_t &= E_t[\; M_{t,t+1} P_{t+1} C_{t+1}] && (2.4)
\end{aligned}
$$

If we use lower case letters to denote the log of each variable, define the time discount rate as, $\rho \equiv \beta^{-1} - 1$, define the nominal interest rate as, $i_t = ln(E_t[M_{t,t+1}]^{-1})$, and define CPI inflation as, $\pi_t \equiv p_t - p_{t-1}$, then the first order conditions can be expressed in log-linear form as,

$$
\begin{aligned}
w_t - p_t &= c_t + \varphi n_t - \ln(1-\chi) && (2.5)\\
c_t &= E_t[c_{t+1}] - (i_t - E_t[\pi_{t+1}] - \rho) && (2.6)
\end{aligned}
$$

In addition to choosing the total amount it works, consumes and saves, the representative household will also choose the types of goods it consumes. Total consumption will be a bundle of home and foreign goods, both of which are bundles of individual varieties that are produced in all countries.

Total consumption will be a Cobb-Douglas bundle of home (H) and foreign (F) goods, $C_t \equiv C_{H,t}^{1-\alpha} C_{F,t}^{\alpha} \cdot (1-\alpha)^{-(1-\alpha)} \alpha^{-\alpha}$. The preference for foreign goods is described by an index of openness, $\alpha \in [0,1]$. Home and foreign goods will be allocated optimally to minimise, $P_t C_t = P_{H,t} C_{H,t} + P_{F,t} C_{F,t}$, where CPI is an index of home and foreign prices, $P_t = P_{H,t}^{1-\alpha} P_{F,t}^{\alpha}$. Note that the elasticity of substitution between home and foreign goods will equal one. If home and foreign consumers are also perfectly symmetric, then there will be a "divine coincidence" in the objectives of a central bank in a closed and an open economy. This happens because any movement in the terms of trade will see the increase in purchasing power over imports perfectly offset by lower demand for exports. This breaks down in our model because of asymmetric wealth, as discussed in Section 2.4.

Home and foreign goods will both be CES bundles of individual varieties that are produced in all countries. The foreign good will be a bundle of goods produced by a continuum of foreign countries, $f \in [0,1]$, with an elasticity of substitution of one. In logs this will be, $c_{F,t} = \int_0^1 c_{f,t} df$, with price index, $p_{F,t} = \int_0^1 p_{f,t} df$. The consumption good produced at home and in every foreign country will be a CES bundle of individual varieties, $i \in [0,1]$, $C_{j,t} \equiv \left(\int_0^1 C_{j,t}(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}$ for $j = [H, f \in [0,1]]$. Each variety will be produced in every country and have an elasticity of substitution of ϵ, allowing for monopolistic price setting in Section 2.5.2. The associated price indices will be, $P_{j,t} \equiv \left(\int_0^1 P_{j,t}(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$ for $j = [H, f \in [0,1]]$.

2.2 Terms of Trade and Exchange Rates

The terms of trade will play a central role in the model, allocating global household demand between home and foreign goods. The terms of trade will link the price of home goods to the CPI, and to the nominal and the real exchange rate.

The effective non-oil terms of trade is defined as the price of the foreign bundle in terms of home goods. In logs this is, $s_t \equiv p_{F,t} - p_{H,t}$, where $s_t \equiv \ln S_t$ is the (log) non-oil terms of trade, and $p_{F,t}$ and $p_{H,t}$ are the (log) price indices for foreign and home goods as defined above. The effective non-oil terms of trade is an index of the bilateral non-oil terms of trade between the home country and each country f, $s_t = \int_0^1 s_{f,t} df$ where $s_{f,t} \equiv p_{f,t} - p_{H,t}$. For ease we will refer to the effective non-oil terms of trade as simply the "terms of trade" from now on.

The terms of trade links the CPI to the domestic price level. Using the (log) definitions of CPI and the terms of trade we have, $p_t = p_{H,t} + \alpha s_t$. CPI inflation can therefore be expressed as a function of domestic inflation, $\pi_t = \pi_{H,t} + \alpha \Delta s_t$.

The terms of trade is also linked to the effective nominal exchange rate. Exchange rates are defined by assuming the law of one price always holds for every variety. This implies that the price in home currency of variety i, produced in country f is (in logs), $p_{f,t}(i) = e_t^f + p_{f,t}^f(i)\ \forall i, f$, where e_t^f is the (log) bilateral nominal exchange rate between home and f, and $p_{f,t}^f(i)$ is the price of variety i, produced in country f (subscript) in currency f (superscript). The price index for all goods produced in f expressed in the home currency will be, $p_{f,t} = e_t^f + p_{f,t}^f$, where $p_{f,t}^f$ is the price index of goods produced in f expressed in currency f. The price index for all goods produced in any foreign country expressed in the home currency will be, $p_{F,t} = e_t + p_t^*$, where $e_t \equiv \int_0^1 e_t^f df$ is the effective nominal exchange rate and, $p_t^* \equiv \int_0^1 p_{f,t}^f df$, is the world price index. The terms of trade is thus related to the nominal exchange rate by, $s_t = e_t + p_t^* - p_{H,t}$.

Finally, the terms of trade will also move with the real exchange rate. The bilateral real exchange rate describes the CPI in country f as a proportion of the CPI at home, $q_{f,t} \equiv \ln \mathcal{Q}_{f,t} = e_t^f + p_t^f - p_t$, where p_t^f is the (log) CPI in country f. The effective real exchange rate is an index of the bilateral real exchange rates, $q_t \equiv \int_0^1 q_{f,t} df$. Linking the effective real exchange rate and the terms of trade gives, $q_t = \int_0^1 (e_t^f + p_t^f - p_t) df = e_t + p_t^* - p_t = (1 - \alpha) s_t$.

2.3 Government Spending

The government will receive all oil income and levy lump-sum taxes, to be spent on a public good and a small transfer to firms. We assume that all oil income will be consumed by the government according to a simple spending rule. This abstracts from any role for distortionary taxes.

Government spending is summarised by its per-period budget constraint. This is given below, where F_t is the portfolio of risk-free foreign assets held at time t, T_t are lump sum taxes, τ_t is a subsidy to firms and $\varepsilon_t P_{O,t} O_t$ are oil revenues.

$$P_{H,t} G_t + \tau_t + E_t[M_{t,t+1} F_{t+1}] \leq F_t + \varepsilon_t P_{O,t} O_t + P_{H,t} T_t \tag{2.7}$$

Oil revenue in each period is given by $\varepsilon_t P_{O,t} O_t$, for $t = 0, 1, \ldots$. Oil production O_t is exogenously determined, as in practice geological factors determine production over the time scales in which monetary policy is conducted. We will focus on a case where oil is discovered at time 0, but is extracted at some time in the future. The oil price $P_{O,t}$ is expressed in units of the global price index and numeraire $P_t^* = 1$, so that fluctuations in the exchange rate will alter the relative value of oil income.

The government spends its income on a public good and a small transfer to firms. The public good is a CES bundle of home-produced goods, covering categories such as health care, education and justice. We let the elasticity of substitution between goods be the same as for households, so that $G_t \equiv \left(\int_0^1 G_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$. The quantity of each variety is chosen optimally, yielding the government demand for each good, $G_t(i) = \left(P_{H,t}(i)/P_{H,t}\right)^{-\epsilon} G_t$. The government also makes a small transfer to firms, τ_t, to offset the marginal cost distortion from monopolistic competition, following Gali and Monacelli (2005).

We will summarise the amount of oil revenues spent by the government each period by introducing the "resource balance", RB_t. The resource balance is the net amount of resource revenues released into the government budget each period, which we express in units of world currency as, $RB_t = P_{O,t} O_t + F_t - E_t[M_{t,t+1} F_{t+1}]$. Substituting this into 2.7 gives, $P_{H,t} G_t + \tau_t = \varepsilon_t RB_t + T_t$, which in log-linear terms is, $\hat{g}_t = \hat{s}_t - \hat{p}_t^* + \widehat{rb}_t - \hat{t}_t$, where $t_t = -\ln(1 - \frac{T_t}{G_t})$. We assume that foreign prices and lump sum taxes as a share of government spending are constant, so $\hat{g}_t = \hat{s}_t + \widehat{rb}_t$. Assuming that the share, rather than the level, of lump sum taxes is constant makes the analysis more tractable, which is discussed in Appendix 14.

Fiscal policy is modelled by assuming the government follows one of three ad hoc rules for the resource balance: the Permanent Income, the Spend All or the Bird in Hand rule.

2.3.1 Permanent Income Rule

The Permanent Income rule involves perfectly smoothing the expenditure from a windfall. The government will borrow before the windfall begins, and save during it to finance spending afterwards. It is the first-best policy if the government does not face any borrowing constraints, as discussed in Van der Ploeg and Venables (2012). In this case the resource balance is, $RB_t = r^* E_t[\sum_0^\infty M_{t,t+s} P_{O,t+s} O_{t+s}]$.

2.3.2 Spend All Rule

The Spend All rule involves spending all oil income as it is received. In practice this characterises a number of countries to varying extents, including Angola, Yemen, and Norway prior to 1990. If we assume that there are no foreign assets initially and none are accumulated, $F_t = 0 \forall t$, then $RB_t = P_{O,t} O_t$. In log-linear terms this means $\hat{g}_t = \hat{s}_t + \hat{p}_{O,t} + \hat{o}_t$.

2.3.3 Bird in Hand Rule

The Bird in Hand rule involves saving all oil revenues in a sovereign wealth fund and consuming only a fixed proportion of accumulated assets. This is consistent with Norway's policy post-1990. In this case the resource balance is, $RB_t = \rho_{BH} F_t$. Foreign assets will change according to, $E_t[M_{t,t+1} F_{t+1}] = (1 - \rho_{BH}) F_t + P_{O,t} O_t$. In log-linear terms this means $\hat{g}_t = \hat{s}_t + \hat{\mathfrak{f}}_t$.

2.4 International Risk Sharing

The international risk sharing condition allows us to express domestic consumption as a function of world consumption. We assume that international markets in financial assets are partially complete. Claims on all shocks in the economy can be traded, except for claims on a nation's oil wealth. This means that an oil discovery will asymmetrically increase the wealth of home households relative to foreigners, which is the experiment we are interested in. Using this assumption and identical preferences across countries, then from equation 2.4 we have the following for every country, f,

$$E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)\left(\frac{P_t}{P_{t+1}}\right)\right] = E_t\left[M_{t,t+1}\right] = E_t\left[\beta\left(\frac{C_t^f}{C_{t+1}^f}\right)\left(\frac{P_t^f}{P_{t+1}^f}\right)\left(\frac{\varepsilon_t^f}{\varepsilon_{t+1}^f}\right)\right] \tag{2.8}$$

Households in each country also face a transversality condition. Everything they earn must eventually be consumed, so $\lim_{T\to\infty} M_{0,T}D_T = 0$ and from summing 2.2 over an infinite horizon, $E_t[\sum_{s=0}^{\infty} M_{t,t+s}P_{t+s}C_{t+s}] = E_t[\sum_{s=0}^{\infty} M_{t,t+s}\left(W_{t+s}N_{t+s} - P_{H,t+s}T_{t+s}\right)] + D_t$. Combining this with 2.4 gives,

$$P_tC_t \;=\; E_t\left[\sum_{s=0}^{\infty} M_{t,t+s}\left(W_{t+s}N_{t+s} - P_{H,t+s}T_{t+s}\right) + D_t\right] / E_t\left[\sum_{s=0}^{\infty}\beta^s\right] \tag{2.9}$$

A similar condition will hold for each country, f. Combining 2.9 with a similar condition for country f gives the following expressions, tying domestic consumption, C_t, to foreign consumption, C_t^f, adjusted for relative household wealth, Θ_t^f, and the real exchange rate, $\mathcal{Q}_{f,t}$.

$$C_t \;=\; \Theta_t^f C_t^f \mathcal{Q}_{f,t} \tag{2.10}$$

$$\text{where } \Theta_t^f \;=\; \frac{E_t[\sum_{s=0}^{\infty} M_{t,t+s}\left(W_{t+s}N_{t+s} - P_{H,t+s}T_{t+s}\right)] + D_t}{E_t[\sum_{s=0}^{\infty} M_{t,t+s}^f\left(W_{t+s}^fN_{t+s}^f - P_{H,t+s}^fT_{t+s}^f\right)] + D_t^f} \tag{2.11}$$

At time $t = 0$ we assume that $D_0 = D_0^f = 0$, so $\Theta_0^f = \frac{E_0[\sum_{s=0}^{\infty} M_{0,s}\left(W_sN_s - P_{H,s}T_s\right)]}{E_0[\sum_{s=0}^{\infty} M_{0,s}^f\left(W_s^fN_s^f - P_{H,s}^fT_s^f\right)]}$. Relative wealth is also not expected to change in the future, $E_t[\Theta_{t+s}] = \Theta_t \forall s \geq 0$. Taking logs, integrating over all countries f, and using $c_t^* = \int_0^1 c_t^f df$, gives the following, where $\vartheta_t \equiv \ln\Theta_t$,

$$c_t = \vartheta_t + c_t^* + (1-\alpha)s_t \tag{2.12}$$

A depreciation in the terms of trade ($s_t \uparrow$), means domestic goods become relatively cheap compared to foreign goods, and will boost domestic consumption if all else is equal. Relative household wealth describes the expected present value of a domestic household's future income (net taxes), relative to a household abroad. On discovering oil this will play an important role. Domestic consumers will anticipate higher lifetime income when oil is discovered, even if it is not immediately received. Their consumption will jump accordingly. If no further surprises are received about oil income, consumption will simply evolve according to the Euler equation in 2.4.

The change in relative household wealth can be expressed in terms of the government's spending rule. Let us assume that households calculate their lifetime wealth assuming flexible prices, so $\hat{\vartheta}_t = \hat{\vartheta}_t^n$. This makes the problem far more tractable, and means that short-term nominal rigidities do not have permanent effects. In this case relative household wealth is given in the following lemma,

Lemma 1. *If prices are flexible then the (log) wealth of domestic households relative to foreign households will be proportional to the present value of the government's resource-balance,*

$$\hat{\vartheta}_0 = \tfrac{(1-\beta)(\gamma_G-\chi)}{(1-\chi)-(1-\gamma_G)(1-\alpha)} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \widehat{rb}_t \right] \tag{2.13}$$

Proof. See Appendix 7. □

In this framework uncovered interest parity will also hold, $E_t[q_{t+1}] - q_t = r_t - r^*$. This follows from 2.6 at home and abroad, $E[\Delta c_{t+1}] = E[\Delta c_{t+1}^*] + (1-\alpha)E[\Delta s_{t+1}]$ (from equation 2.8), and the definition of the real interest rate, $r_t \equiv i_t - E[\pi_{t+1}]$.

2.5 Firms

2.5.1 Production

Domestic firms will produce differentiated goods using labour and a common technology. Oil is not included as a factor of production, to focus on the windfall effects of a resource discovery. Capital is also not included for tractability and to focus on short-term dynamics. The production function is $Y_t(i) = A_t N_t(i)$.

Real marginal cost will be common across domestic firms. It is given by, $mc_t = -\nu + w_t - p_{H,t} - a_t$, where $\nu \equiv -\ln(1-\tau)$ and τ is an employment subsidy which is used to remove the marginal cost distortion of monopolistic competition.

Up to a first-order approximation aggregate output is, $y_t = a_t + n_t$. Aggregate domestic output is given by the index, $Y_t \equiv \left[\int_0^1 Y_t(i)^{(\epsilon-1)/\epsilon}\right]^{\epsilon/(\epsilon-1)}$, similar to consumption. Aggregate employment is given by $N_t \equiv \int_0^1 N_t(i)di = \frac{Y_t Z_t}{A_t}$, where $Z_t \equiv \int_0^1 \frac{Y_t(i)}{Y_t}di = \int_0^1 \left(\frac{P_{H.,t}(i)}{P_{H,t}}\right)^{-\epsilon}$ is a measure of price dispersion. In Appendix 9 we follow Gali and Monacelli (2008) and show that equilibrium variations in $z_t \equiv \ln Z_t$ around the perfect foresight steady state are of second order. Thus, up to a first-order approximation we have the aggregate relationship, $y_t = a_t + n_t$.

2.5.2 Price-setting

We assume that firms set prices in a staggered way according to Calvo (1983). A measure of $(1-\varrho)$ randomly selected firms set new prices in each period, with an individual firm's probability of re-optimizing in any given period being independent of the time elapsed since it last reset its price. In a standard result, following Gali (2008, Chapter 5), the optimal price-setting strategy for the typical firm resetting its price in period t can be approximated by the log-linear rule,

$$\bar{p}_{H,t} = \mu + (1-\beta\varrho)\sum_{k=0}^{\infty}(\beta\varrho)^k E_t[mc_{t+k} + p_{H,t}] \tag{2.14}$$

where $\bar{p}_{H,t}$ denotes the log of newly set domestic prices, and $\mu \equiv \ln\left(\frac{\epsilon}{\epsilon-1}\right)$ is the log of the optimal markup in the flexible price economy. The pricing decision is thus forward-looking, as firms recognise that the price they set will last for a random number of periods. In the flexible price limit (as $\varrho \to 0$) we recover the markup rule $\bar{p}_{H,t} = \mu + mc_t^n + p_{H,t}$, so that the marginal cost in the flexible price state is $mc_t^n = -\mu$.

3 Equilibrium Dynamics

This section characterises the equilibrium of the model, and the log-linearised dynamics around it. Changes in government spending that are anticipated will immediately induce dynamics in the economy. These will have both an immediate effect on consumers anticipating future income, and a future effect when government spending begins. The dynamics will cause real distortions because of the rigidity of domestic prices. This will provide a role for the central bank in managing the distortion, as addressed in Section 4.

Before we begin it is important to clarify some notation. Let us assume that the economy begins at the steady state (log) level of output, $y^s \equiv \ln Y^s$. Discovering oil will induce changes in the economy that cause the actual level of output, y_t, to deviate from the steady state, $\hat{y}_t \equiv y_t - y^s$. These changes will also cause the "natural" (flexible-price) level of output to deviate from the steady state, $\hat{y}_t^n \equiv y_t^n - y^s$. The difference between the actual and the natural level of output will be the "output gap", $\tilde{y}_t \equiv y_t - y_t^n$. In the analysis below we will be interested in changes to both the actual and the natural level of output, and so will be keeping track of both $\hat{y}_t$ and $\hat{y}_t^n$.

3.1 Aggregate Supply

Firms set prices after observing the demand schedule of households and the government, which is derived in Section 3.2. In a standard result, following Gali and Monacelli (2005), domestic inflation is given by,

$$\pi_{H,t} = \beta E_t[\pi_{H,t+1}] + \lambda \widetilde{mc}_t \tag{3.1}$$

where $\lambda \equiv \frac{(1-\beta\varrho)(1-\varrho)}{\varrho}$ and $\widetilde{mc}_t \equiv mc_t - mc_t^n = \widehat{mc}_t - \widehat{mc}_t^n$ is the deviation of real marginal costs in time t from their flexible-price level, mc_t^n.

Real marginal costs are given by, $mc_t = -\nu + w_t - p_{H,t} - a_t$, where ν is a small wage subsidy from the government. The deviation of real marginal costs from the steady state can be expressed as a function of domestic output, $\widehat{mc}_t = (\frac{1}{1-\gamma_G} + \varphi)\hat{y}_t - \frac{\gamma_G}{1-\gamma_G}\hat{g}_t + \alpha\hat{\vartheta}_t + (1-\varphi)\hat{a}_t$, using 2.5, the goods market equilibrium 3.5 below, and $y_t = a_t + n_t$. A similar result holds when prices are flexible, $\widehat{mc}_t^n = (\frac{1}{1-\gamma_G} + \varphi)\hat{y}_t^n - \frac{\gamma_G}{1-\gamma_G}\hat{g}_t^n + \alpha\hat{\vartheta}_t^n + (1-\varphi)\hat{a}_t^n$. Subtracting the second from the first gives, $\widetilde{mc}_t = (\frac{1}{1-\gamma_G} + \varphi)\tilde{y}_t - \frac{\gamma_G}{1-\gamma_G}\tilde{g}_t$, as $\tilde{\vartheta}_t = \tilde{a}_t = 0$. Note that $\hat{g}_t \neq \hat{g}_t^n$ because the government will be spending revenues received in foreign currency. Therefore, the nominal exchange rate will affect the government's domestic purchasing power if prices are sticky.

The New Keynesian Phillips Curve will be a function of the output and government spending gaps. We separate these gaps to monitor the effect of an oil discovery on both the actual and natural levels of output and government spending, e.g. $\tilde{y}_t \equiv \hat{y}_t - \hat{y}_t^n$. The Phillips Curve is found by combining 3.1 and the expression for $\widetilde{mc}_t$ to give,

$$\pi_{H,t} = \beta E_t[\pi_{H,t+1}] + \lambda(\tfrac{1}{1-\gamma_G} + \varphi)(\hat{y}_t - \hat{y}_t^n) - \tfrac{\lambda\gamma_G}{1-\gamma_G}(\hat{g}_t - \hat{g}_t^n) \tag{3.2}$$

When prices are flexible, marginal costs will be $mc_t^n = -\mu$. This follows from taking the flexible price limit of 2.14. To remove the steady state distortion caused by monopolistic competition we choose a wage subsidy such that, $mc_t^n - mc^s = -\mu + \nu = 0$. Using this relationship the natural level of output is,

$$\hat{y}_t^n = \tfrac{\gamma_G}{1+\varphi(1-\gamma_G)}\hat{g}_t^n - \tfrac{(1-\gamma_G)\alpha}{1+\varphi(1-\gamma_G)}\hat{\vartheta}_t^n - \tfrac{(1-\gamma_G)(1-\varphi)}{1+\varphi(1-\gamma_G)}\hat{a}_t^n \tag{3.3}$$

The natural level of output is affected by the level of government spending, relative household wealth and technology. When the government alters the amount of resource revenues released into the budget, it will change the natural level of output. This happens both in the current period, and in all future periods due to changing household wealth. A change in the natural level of output will affect the marginal costs of firms, which in turn affects aggregate supply. In a sense, government spending and wealth shocks behave like negative cost-push shocks (see Gali, 2008 Ch5).

If we express government spending in terms of the resource balance, then $\tilde{g}_t = \tilde{s}_t = \tilde{y}_t$. This follows from $\hat{g}_t = \hat{s}_t + \widehat{rb}_t$, equation 3.6 below and $\widehat{rb}_t = \widehat{rb}_t^n$. In this case the Phillips Curve will be $\pi_{H,t} = \beta E_t[\pi_{H,t+1}] + \lambda(1+\varphi)(\hat{y}_t - \hat{y}_t^n)$, and the natural level of output will be $\hat{y}_t^n = \frac{\gamma_G}{1+\varphi}\widehat{rb}_t - \frac{\alpha+\gamma_G(1-\alpha)}{1+\varphi}\hat{\vartheta}_t$.

3.2 Aggregate Demand

The market clearing condition for each domestically produced good i, at each time t, requires output to equal demand from domestic consumption, government purchases and consumption from each foreign country f,

$$\begin{aligned} Y_t(i) &= C_{H,t}(i) + \int_0^1 C_{H,t}^f(i)df + G_t(i) \\ &= \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon} \left\{(1-\alpha)\left(\frac{P_t C_t}{P_{H,t}}\right) + \alpha \int_0^1 \left(\frac{\varepsilon_{f,t} P_t^f C_t^f}{P_{H,t}}\right) df + G_t\right\} \end{aligned}$$

The second equality above assumes symmetric preferences across countries implying, $C_{H,t}^f(i) = \alpha \left(P_{H,t}(i)/P_{H,t}\right)^{-\epsilon} \left(\varepsilon_{f,t} P_t^f / P_{H,t}\right) C_t^f$. Aggregating across goods gives,

$$\begin{aligned} Y_t &= (1-\alpha)\left(\frac{P_t C_t}{P_{H,t}}\right) + \alpha \int_0^1 \left(\frac{\varepsilon_{f,t} P_t^f C_t^f}{P_{H,t}}\right) df + G_t \\ &= S_t^\alpha \left[(1-\alpha)C_t + \alpha \int_0^1 \mathcal{Q}_{f,t} C_t^f df\right] + G_t \\ &= C_t S_t^\alpha \left[(1-\alpha) + \alpha \Theta_t^{-1}\right] + G_t \end{aligned} \tag{3.4}$$

Log-linearising this to the first order gives,

$$\hat{y}_t = (1-\gamma_G)\left(\hat{c}_t + \alpha\hat{s}_t - \alpha\hat{\vartheta}_t\right) + \gamma_G\hat{g}_t \tag{3.5}$$

where $\gamma_G \equiv G^s/Y^s$ and $-\alpha\hat{\vartheta}_t \approx \ln\left[(1-\alpha)+\alpha\Theta_t^{-1}\right] - \ln\left[(1-\alpha)+\alpha(\Theta^s)^{-1}\right]$ when $\Theta^s \approx 1$. This approximation makes the analysis far more tractable without a major loss in accuracy, as discussed in Appendix 14. Combining this with 2.12 and $\hat{c}_t^* = 0$ gives an expression for the terms of trade,

$$\hat{s}_t = \tfrac{1}{1-\gamma_G}\hat{y}_t - \tfrac{\gamma_G}{1-\gamma_G}\hat{g}_t - (1-\alpha)\hat{\vartheta}_t \tag{3.6}$$

The IS curve will be a function of government spending and expected future household income. This can be found by combining 2.6 with 3.5 and 3.3,

$$\begin{aligned}\hat{y}_t &= E_t[\hat{y}_{t+1}] - (1-\gamma_G)(i_t - E_t[\pi_{H,t+1}] - \rho) \\ &\quad - E_t[(1+\varphi(1-\gamma_G))\Delta\hat{y}_{t+1}^n + \gamma_G\Delta(\hat{g}_t - \hat{g}_t^n)]\end{aligned} \tag{3.7}$$

If we assume that the government follows a spending rule on its resource balance, $\hat{g}_t = \hat{s}_t + \widehat{rb}_t$, then the IS curve becomes, $\hat{y}_t = E_t[\hat{y}_{t+1}] - (i_t - E_t[\pi_{H,t+1}] - \rho) - (1+\varphi)E_t[\Delta\hat{y}_{t+1}^n]$ and $\hat{y}_t^n = \frac{\gamma_G}{1+\varphi}\widehat{rb}_t - \frac{\alpha+\gamma_G(1-\alpha)}{1+\varphi}\hat{\vartheta}_t$.

3.3 The Steady State

Here we define two types of steady state, the symmetric steady state in the case without oil, and the asymmetric steady state when the home government receives oil income. The symmetric steady state is defined from the perspective of a benevolent social planner, choosing the optimal levels of output, government spending and consumption. The asymmetric steady state allows the government to choose its level of spending and funding mix exogenously, deviating from this symmetric steady state.

3.3.1 Benchmark: No Oil

When the home country receives no resource revenues then the steady state will be symmetric and we define it from the perspective of a benevolent social planner. As home and foreign countries are assumed to be completely symmetric we have $\Theta = S = 1$. The social planner solves the following problem,

$$\begin{aligned}\max_{C_i^i, C_f^i, G^i, N^i} \mathcal{L} &= E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ [(1-\chi)\ln C^i + \chi_t \ln G^i - \frac{N^{i(1+\varphi)}}{1+\varphi}] \\ &\quad -\lambda_1 [A^i N^i - G^i - C_i^i - \int_0^1 C_i^f df] - \lambda_2 [(C_i^i)^{1-\alpha} (C_f^i)^{\alpha} \tfrac{1}{(1-\alpha)\alpha} - C^i]\end{aligned}$$

The solution to this problem is given by, $N^i = 1$, $Y^i = A^i$,$C^i = (1-\chi)(A^i)^{1-\alpha}(\int_0^1 A^f df)^{\alpha}$, $C_i^i = (1-\alpha)(1-\chi)A^i$, $C_i^f = \alpha(1-\chi)A^i$ and $G^i = \chi A^i$. Steady state output is not affected by the share of government spending, χ, or the degree of openness, α, because of symmetry. A change in either would affect both domestic and foreign demand for domestic production, which will offset one another. The optimal allocation for the world as a whole can be found by setting $\alpha \to 0$.

3.3.2 Scenarios: Oil

When the home government receives oil income the steady state will depend on fiscal policy, and will no longer be symmetric with the rest of the world. The steady state allocation is presented in the following lemma,

Lemma 2. *The steady state allocation in the home country when the home government receives oil revenues and foreign governments do not, and all countries are otherwise symmetric with $A^i = 1$, is,*

$$\begin{aligned} N^s &= \left(\tfrac{1-\chi}{1-\gamma_T}\right)^{\frac{1}{1+\varphi}} \quad ; \quad Y^s = N^s \quad ; \quad \Theta^s = \tfrac{\alpha(1-\gamma_T)}{(1-\gamma_G-(1-\alpha)(1-\gamma_T))} \\ S^s &= (N^s)^{-\varphi}(\Theta^s)^{-1} \quad ; \quad G^s = \gamma_G Y^s \quad ; \quad C^s = \Theta^s(1-\chi)(S^s)^{1-\alpha} \end{aligned} \tag{3.8}$$

where $\gamma_G \equiv G^s/Y^s$, $\gamma_T \equiv T^s/Y^s$ and $(\gamma_G - \gamma_T)$ is the steady state share of government spending financed by oil revenues, expressed as a proportion of GDP.

Proof. See Appendix 8. □

The steady state accounts for permanent differences in wealth stemming from oil income. The household wealth ratio will be greater than one, $\Theta^s > 1$, if $\gamma_G > \gamma_T$. This is the same as saying that home government spending is partly financed by oil revenues, while the rest of the world is not. This will also lead to an appreciated steady state terms of trade, $S < 1$, and relatively higher home consumption, $C^s > C^*$.

This definition of the steady state will collapse to the symmetric case under certain fiscal policies. If taxes are chosen optimally, $\gamma_T = \chi$, then output will be the same as in the symmetric case, $N^s = 1$. If the government receives no oil revenues, $\gamma_G = \gamma_T$, then we have the symmetric case with $\Theta^s = S^s = 1$. We will proceed assuming $\gamma_T = \chi$ for tractability, and $\gamma_G > \gamma_T$. We will define the change in household wealth on discovering oil by comparing Θ in the steady states before and after discovery.

4 Optimal Monetary Policy

In this section we derive a closed form for the optimal monetary response to an oil discovery, under discretion and commitment. Optimal policy is found by minimising the central bank's loss function, which is derived from household welfare. The loss function will have standard terms for a small open economy with government spending. In addition it will have a systematic bias to appreciate the terms of trade and lower government spending, because oil wealth makes the home country asymmetric with the rest of the world. We then derive a closed form for the optimal policy rule under both discretion and commitment. Under discretion the optimal policy rule responds to domestic inflation and anticipated changes in the natural level of output, driven by the way the government spends oil revenues. There will also be a small deflationary bias, which attempts to systematically appreciate the terms of trade. In contrast, under commitment the optimal policy rule responds to the domestic price level, and anticipated changes in the natural level of output. This overcomes the deflationary bias that exists under discretion.

Optimal policy is determined by minimising the central bank's loss function, which is derived from household welfare. This is given in Proposition 3.

Proposition 3. *If all incremental oil income is allocated to increasing government spending, then the central bank's loss function is given by,*

$$\mathbb{L} = E_0 \sum_{t}^{\infty} \beta^t \Big\{ \tfrac{1}{2} \left[\phi_\pi \pi_{H,t}^2 + \phi_{yy} \hat{y}_t^2 + \phi_f (\hat{g}_t - \hat{y}_t)^2 \right] + \phi_s \hat{s}_t + \phi_g \hat{g}_t \Big\} + t.i.p + o(\|\hat{g}_t\|^3) \quad (4.1)$$

where $\phi_\pi = \epsilon/\lambda$, $\phi_{yy} = (1+\varphi)+(1-\chi)(1-\alpha)/(1-\gamma_G)$, $\phi_f = (1-\chi)(1-\alpha)\gamma_G/(1-\gamma_G)^2$, $\phi_s = (1-\gamma_G) - (1-\chi)(1-\alpha)$, $\phi_g = \gamma_G - \chi$ *and t.i.p. are terms independent of policy.*

Proof. See Appendix 9. □

This proposition states that the central bank's objective is to stabilise domestic inflation, the output gap, and the fiscal gap, and has biases on the level of the terms of trade and government spending.

The objective of stabilising domestic inflation and the output gap is consistent with symmetric models of a small open economy without a government (see Gali and Monacelli, 2005). Such models display a "divine coincidence" in the objectives of closed and open economy central banks. This happens because the income and substitution effects of terms of trade movements perfectly offset one another (Monacelli, 2012).

The objective of stabilising the fiscal gap ($\hat{g}_t - \hat{y}_t$) - deviations in government spending as a share of GDP - is consistent with symmetric models of a small open economy with a government (see Gali and Monacelli, 2008). This happens because governments consume only home goods, while households consume home and foreign goods. Deviations in the share of GDP allocated to government spending will therefore affect the optimal allocation of household spending.

The objective of systematically appreciating the terms of trade comes from oil wealth breaking the symmetry between home and foreign consumers. Different starting levels of wealth means that changes in the terms of trade will affect consumption, and output, in the home country relative to abroad. The central bank will want to systematically appreciate the terms of trade ($\hat{s}_t \downarrow$), to boost domestic consumers' purchasing power abroad. The associated reduction in export income is less important because domestic consumers have an additional source of income: oil. Note that the approximation of this term requires deviations in the terms of trade to be relatively "small" (see Gali, 2008).

The objective of systematically lowering government spending comes from government spending taking a disproportionate share of output in the steady state $\gamma_G > \chi$. If government spending is exogenous then this would be a term "independent of policy". However, we are interested in oil exporters who receive an oil windfall in foreign currency. Thus, the central bank will be able to affect the level of government spending via the terms of trade.

If we assume that the government follows a rule for spending its oil revenues, then the central bank's objective can be simplified. In this case government spending is of the form, $\hat{g}_t = \hat{s}_t + \widehat{rb}_t$, and the central bank's objective becomes,[4]

$$\mathbb{L} = \sum_{t}^{\infty} \beta^t \left\{ \frac{1}{2} \left[\phi_\pi \pi_{H,t}^2 + \phi_{yy} \hat{y}_t^2 \right] + \phi_y \hat{y}_t \right\} + t.i.p + o(\|\hat{g}_t\|^3) \tag{4.2}$$

where $\phi_\pi = \epsilon/\lambda$, $\phi_{yy} = (1+\varphi) + (1-\chi)(1-\alpha)/(1-\gamma_G)$ and $\phi_y = \alpha(1-\chi)$. This is derived in Appendix 9. In this case government spending and the terms of trade can be expressed in terms of output using 3.6 and the government spending rule. The central bank's incentive to systematically lower output is simply a re-expression of its desire to appreciate the terms of trade.

[4]Following Gali (2008, p107), we assume that the oil discovery is such that the steady state distortion is "small", so that the linear term in the loss function does not bias our results. If the distortion is large then this loss function can be reduced to a quadratic form by taking a second order approximation of the relationship in equation 3.1, following Benigno and Woodford (2005).

4.1 Optimal Policy under Discretion

The optimal response to an oil discovery by a discretionary central bank, if the government follows an oil spending rule, is found by minimising the loss function subject to the Phillips curve,

$$\begin{aligned}\min_{\pi_{H,t},\hat{y}_t} \mathcal{L}_{\mathcal{D}} &= E_0 \sum_t^{\infty} \beta^t \Big\{ \tfrac{1}{2} \Big[\phi_\pi \pi_{H,t}^2 + \phi_{yy} \hat{y}_t^2 \Big] + \phi_y \hat{y}_t \\ &\quad - l_{D,t}(\lambda(1+\varphi)\hat{y}_t - \pi_{H,t}) + t.i.p. \end{aligned} \tag{4.3}$$

where we assume that $\widehat{rb}_t$ and $\hat{y}_t^n$ are independent of monetary policy, and the discretionary central bank can not choose $E_t[\pi_{H,t+1}]$. Minimising this Lagrangian yields paths for inflation, the output gap and interest rates, which are summarised in the following Proposition,

Proposition 4. *For a monetary authority acting with discretion and a government following an oil spending rule, the optimal paths for $\pi_{H,t}$, $\hat{y}_t$ and i_t are,*

$$\pi_{H,t} = -a_D \lambda(1+\varphi) \left\{ \frac{\phi_y \phi_{yy}^{-1}}{1 - a_D \beta} + E\left[\sum_0^{\infty} (a_D \beta)^s \hat{y}_{t+s}^n \right] \right\} \tag{4.4}$$

$$\hat{y}_t = -\left(\lambda(1+\varphi)\phi_\pi \phi_{yy}^{-1} \right) \pi_{H,t} - \phi_y \phi_{yy}^{-1} \tag{4.5}$$

$$(i_t - \rho) = d_1 \pi_{H,t} - d_2 E_t[\Delta \hat{y}_{t+1}^n] + d_3 E\left[\sum_0^{\infty} (a_D \beta)^s \hat{y}_{t+1+s}^n \right] + d_4 \tag{4.6}$$

where $a_D = (1 + \lambda^2(1+\varphi)^2 \phi_\pi/\phi_{yy})^{-1}$, $d_1 = \lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}$, $d_2 = 1+\varphi$, $d_3 = \Big(\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}} - 1\Big) a_D \lambda(1+\varphi)$ *and* $d_4 = \Big(\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}} - 1\Big) a_D \lambda(1+\varphi) \frac{\phi_y/\phi_{yy}}{1 - a_D\beta}$.

Proof. See Appendix 10. □

A discretionary central bank in an oil exporter should optimally respond to domestic inflation, and both current and future changes in the natural level of output caused by the release of oil revenues into the economy. There may also be a deflationary bias, which attempts to systematically appreciate the terms of trade.

Optimal discretionary policy in an oil exporter will first respond to domestic inflation. This is consistent with standard models of the small open economy (see Gali and Monacelli, 2005 and 2008). A number of interest rate paths are consistent with the optimal $\pi_{H,t}$ and $\hat{y}_t$. However, we are interested in an interest rate rule which is not only consistent with the optimal paths for $\pi_{H,t}$ and $\hat{y}_t$, but which will produce them uniquely. The rule in equation 4.6 will give a unique and determinate solution when $d_1 > 1$ or $\lambda(1+\varphi)\phi_\pi > \phi_{yy}$.

Optimal policy will also respond to today's natural level of output. Consider an unanticipated temporary increase in the oil price, which the government spends immediately according to a Spend All rule. This increase in demand will drive up the natural level of output, $\hat{y}_t^n = \frac{\gamma_G}{1+\varphi}\widehat{rb}_t - \frac{\alpha+\gamma_G(1-\alpha)}{1+\varphi}\hat{\vartheta}_t$ where $\widehat{rb}_t = \hat{p}_{O,t} + \hat{o}_t$. As prices are slow to adjust, the actual level of output will overshoot its natural level, requiring the monetary authority to tighten and offset this spike in demand $(+d_2\hat{y}_t^n)$.

The third component of optimal policy, and the most important in our context, is to respond to the expected change in the natural level of output. This will require the central bank to be forward looking, and respond to anticipated changes to the economy in advance. Consider an anticipated increase in the government's consumption of oil revenues, such as production beginning in the next period. It is optimal to sharply loosen monetary policy in the period before the windfall is received, to delay the terms of trade appreciation. In addition, there will be a slight tightening each period in response to expected future increases in oil production, to offset forward-looking price rises by firms.

The final component of optimal policy is a small but systematic deflationary bias. This is captured in the constant term (d_4) which gives the bank an incentive to systematically tighten rates to appreciate the terms of trade. It can be seen in the uncovered interest parity condition, $r_t - r^* = (1-\alpha)E_t[s_{t+1} - s_t]$. If the policymaker cannot commit to future policy it will continually raise the real interest rate $(r_t \uparrow)$ to depreciate the current terms of trade $(s_t \downarrow)$. This follows from domestic households having more wealth than foreign households. Appreciating the terms of trade thus increases household buying power abroad without an offsetting reduction in relative income, because they have more wealth to begin with.

4.2 Optimal Policy under Commitment

Now we consider the optimal response to an oil discovery by a credible central bank. If the government follows an oil spending rule, the optimal response is found by minimising the loss function subject to the Phillips curve,

$$\begin{aligned}\min_{\pi_{H,t},\hat{y}_t} \mathcal{L}_{\mathcal{D}} &= E_0 \sum_t^{\infty} \beta^t \Big\{ \tfrac{1}{2} \Big[\phi_\pi \pi_{H,t}^2 + \phi_{yy} \hat{y}_t^2 \Big] + \phi_y \hat{y}_t \\ &\quad - l_{c,t}(\beta \pi_{H,t+1} + \lambda(1+\varphi)\hat{y}_t - \pi_{H,t}) + t.i.p. \qquad (4.7)\end{aligned}$$

Minimising this Lagrangian yields paths for the price level, the output gap and interest rates, which are summarised in the following Proposition,

Proposition 5. *For a monetary authority acting under commitment, and a government following an oil spending rule, the optimal paths for $\hat{p}_{H,t}$, $\hat{y}_t$ and i_t are,*

$$\hat{p}_{H,t} = \delta \hat{p}_{H,t-1} - \lambda(1+\varphi) \left\{ \frac{\phi_y}{\phi_{yy}} \frac{\delta}{(1-\beta\delta)} + \delta E_t \left[\sum_{s=0}^{\infty} (\beta\delta)^s \hat{y}_{t+s}^n \right] \right\} \qquad (4.8)$$

$$\hat{y}_t = -\left(\lambda(1+\varphi)\phi_\pi \phi_{yy}^{-1} \right) \hat{p}_{H,t} - \phi_y \phi_{yy}^{-1} \qquad (4.9)$$

$$(i_t - \rho) = c_1 \hat{p}_{H,t} - c_2 E_t[\Delta \hat{y}_{t+1}^n] + c_3 E_t \left[\sum_{s=0}^{\infty} (\beta\delta)^s \hat{y}_{t+1+s}^n \right] + c_4 \qquad (4.10)$$

where$\delta = (1 - \sqrt{1-4a^2\beta})/(2a\beta)$,$a = (1+\beta+\lambda^2(1+\varphi)^2\phi_\pi/\phi_{yy})^{-1}$,$b = \lambda\gamma_G$,$\kappa = \lambda(1+\varphi)\phi_y/\phi_{yy}$, *and the terms on the interest rate rule are defined in Appendix 10.*

Proof. See Appendix 10. □

A credible central bank in an oil exporting economy should optimally respond to the domestic price level and expected changes in the natural level of output. The small deflationary bias that exists under discretion can be overcome under commitment.

The optimal monetary rule under commitment will first respond to the domestic price level. This is consistent with standard small open economy literature (see Gali, 2008). The rule will give a unique and determinate solution under the same conditions as the discretionary case, when $c_1 = \left(\lambda(1+\varphi)\phi_\pi\phi_{yy}^{-1} - 1\right)(1-\delta) > 0$ or $\lambda(1+\varphi)\phi_\pi > \phi_{yy}$. By committing to target the level rather than the change in prices, policy has more persistence. This allows the credible central bank to overcome the deflationary bias.

As with the discretionary case, the key component of optimal policy under commitment is responding to the expected release of oil income into the economy. Changes in oil income affect the natural level of output through aggregate demand. The main response is a sharp loosening in policy if demand is expected to increase in the next period. This sustains output in the current period and delays the terms of trade appreciating. Optimal policy will also slightly tighten against anticipated future oil output, to lean against firms raising prices in anticipation. This is discussed further in the next section.

Finally, the optimal rule under commitment will have an incentive to systematically tighten interest rates, to appreciate the terms of trade. This will increase consumption by relatively more than it will reduce domestic incomes, because the home country begins with some stock of oil wealth. However, as the target is the level rather than the change in prices, the deflationary bias under discretion is overcome under commitment.

4.3 Monetary Rules

The optimal policies will be compared to two ad-hoc monetary rules, to aid our intuition. The first rule is typical in the literature and will respond to both inflation and the output gap, see equation 4.11 (Taylor, 1993 and Woodford, 2001). The second makes a small addition as it also responds expected changes in the natural level of output, see equation 4.12. The addition is based on a similar term in the optimal interest rate paths in equations

4.6 and 4.10. Intuitively, if the rest of the economy is responding to anticipated changes in the economy, the monetary authority should as well. In the next section we will see that the second rule closely approximates optimal policy.

$$(i_t - \rho) = \phi_\pi \pi_{H,t} + \phi_y \hat{y}_t \tag{4.11}$$

$$(i_t - \rho) = \phi_\pi \pi_{H,t} + \phi_y \hat{y}_t + \phi_n E_t[\Delta \hat{y}^n_{t+1}] \tag{4.12}$$

5 Results

The effects of an oil discovery on an economy will depend on how the government manages it. In this section we study three illustrative scenarios, where monetary policy responds to different fiscal policies. The first is a base case, where the government perfectly smooths spending from the date oil is discovered, according to a Permanent Income rule. This requires the terms of trade to appreciate immediately, which is easily achieved with a floating exchange rate. The second involves no smoothing, as the government spends all oil revenues as they are received. This requires the terms of trade to appreciate twice: when oil is discovered and again when production begins. Standard monetary policies perform poorly[5], but are greatly improved if the central bank responds to expected future changes in fiscal policy. The third scenario has the government partially smooth spending, according to a Bird in Hand rule. This introduces continuous changes in the economy, as government spending rises and falls with the size of the sovereign wealth fund. Standard policies will again be greatly improved if the central bank responds to anticipated changes in fiscal policy.

[5] In this section we will be comparing optimal policy to a nominal exchange rate peg and two Taylor rules. The results for these policies are also based on closed form solutions, which are derived in Appendix 12 and Appendix 13 respectively.

5.1 Permanent Income Rule

Our first scenario is a base case, where the government perfectly smooths spending from the date oil is discovered. All oil revenues will be saved in a sovereign wealth fund, and the government will only consume their permanent income. There will be an immediate, permanent shock to the demand for home goods so the terms of trade must appreciate, which is easily achieved with a floating currency. Optimal monetary policy will try to improve on the flexible price outcome by appreciating the terms of trade, increasing households' buying power abroad.

An immediate jump in government spending will require the terms of trade to appreciate. This is best illustrated when prices are flexible, as seen in the "*Flex*" case in Figure 5.1. To smooth spending the government will begin borrowing as soon as oil is discovered, and then save during production. The government only consumes home goods, so spending accrues as wages to domestic households. Households spend these wages on leisure and on consumption goods, both home and foreign. The extra demand for home goods will drive up their relative price, so the terms of trade must appreciate. The net effect on output will depend on the elasticities of labour supply, and substitution between home and foreign goods.

Under a currency peg the only way the terms of trade can appreciate is through domestic inflation. This will happen slowly, because prices are sticky. While the price of home goods is slowly rising, the consumption and production of home goods will be distorted and overshoot their natural level (see "*Peg*" in Figure 5.1). These distortions only disappear once all prices have had time to adjust.

If the nominal exchange rate is floating then the flexible-price case is easily replicated. The nominal exchange rate is a single, flexible price, so can adjust far quicker than the many sticky prices set by domestic firms. The distortions from domestic inflation can therefore be avoided by letting the relative price of home and foreign goods adjust through the nominal exchange rate.

Optimal policy looks to improve on the flexible price outcome, by further appreciating

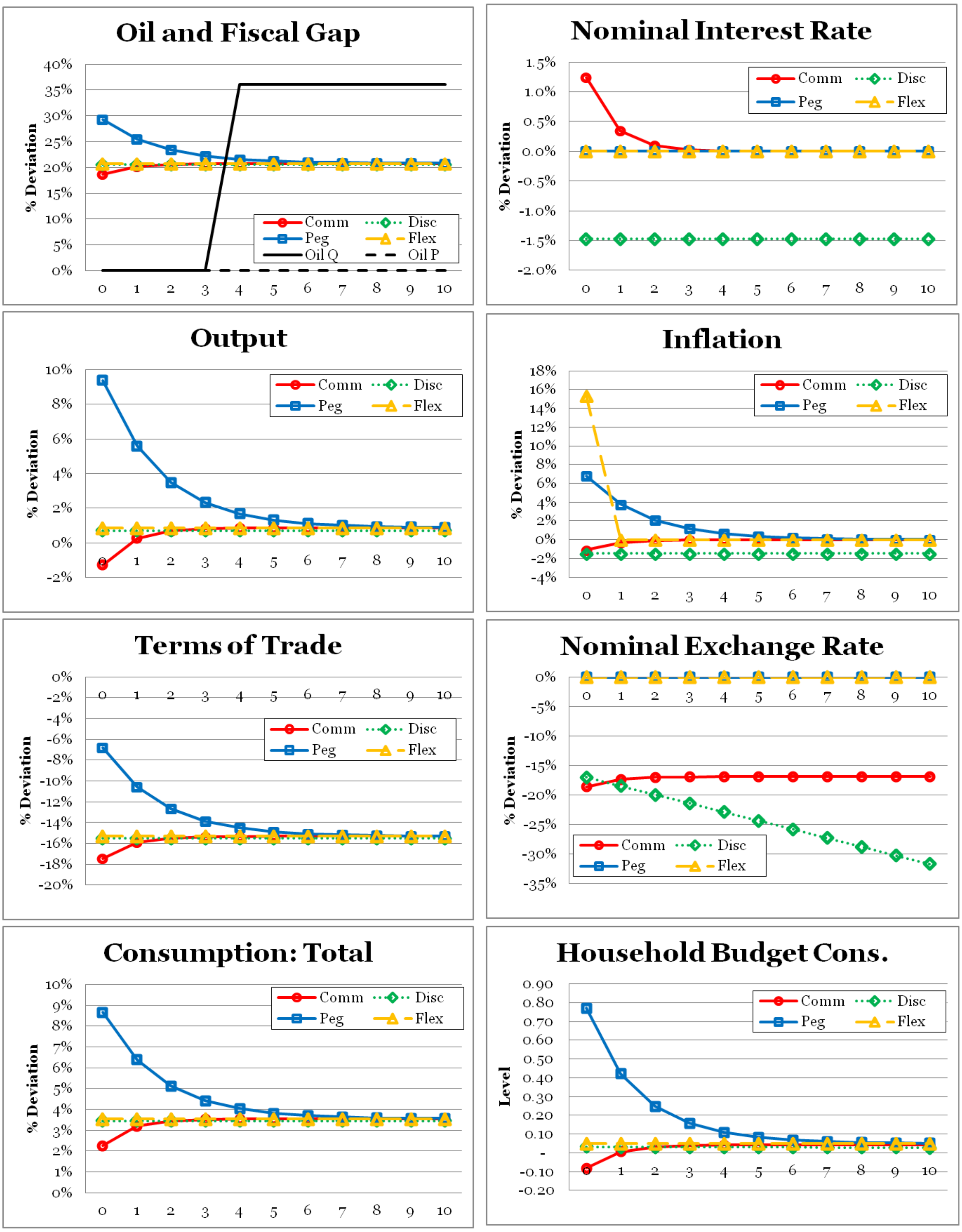

Figure 5.1: Forecast mean responses of key variables to an anticipated oil windfall under the Permanent Income rule. Four different scenarios are illustrated: optimal commitment (Comm), optimal discretion (Disc), a pegged currency (Peg) and flexible prices (Flex).

Welfare Loss relative to Commitment

	Comm	Disc	Peg	Flexible	TR	TR*
Welfare Loss ($\hat{c}_0$ units)	-	8.86%	11.45%	0.23%	3.01%	0.17%

Table 1: The welfare loss for each monetary regime is expressed in units of additional consumption needed at $t = 0$, $\hat{c}_0$, to replicate welfare under commitment.

the terms of trade. This additional appreciation would increase the buying power of home consumers abroad, at the cost of lost income from foreign sales of domestic goods. In standard models with a unit elasticity of substitution between home and foreign goods (Cobb-Douglas preferences), these two effects offset. However, our households also have oil wealth, so the lost income from other exports is less important. Under discretion this leads to a deflationary bias, which is overcome under commitment. Of course, it is important to note that in our simple framework there are no externalities from domestic production that would make this central bank-induced Dutch disease suboptimal.

5.2 Spend All Rule

Our second scenario assumes that the government does not smooth spending at all. All oil revenues are consumed by the government as soon as they are received. We will look at changes in both the quantity and the price of oil. Changes in the quantity of oil will have effects in two stages: immediately when it is discovered, and in the future when it is extracted. Each stage will require a separate terms of trade appreciation as demand for home goods rises from consumers, and then the government. The latter will be anticipated by households and firms. Under a standard Taylor rule or a currency peg (as used by 75% of resource-dependent countries) this will lead to an extended period of inflation and recession. Optimal policy avoids the recession, and is well approximated by a Taylor rule that also responds to expected changes in the natural level of output. These results are illustrated in Figure 5.2. Changes in the price of oil will also be transmitted directly to aggregate demand, and are poorly managed by a standard Taylor rule or a currency peg as well, as seen in Figure 5.4.

5.2.1 Flexible Prices

When oil is discovered, the Spend All rule will cause two major periods of change in the economy. The first is a jump in consumption when the discovery is announced. We illustrate this using the flexible price case ("$Flex$") in Figure 5.2. When oil is discovered, households will immediately anticipate higher future government spending, and the associated higher wage income. Domestic consumption will jump immediately, because of higher lifetime wealth. Higher consumption will cause inflation and appreciate the terms of trade, reducing demand for domestically produced goods. The household will finance this consumption by borrowing from abroad, as seen in the negative change in their budget constraint.

The second major change caused by the Spend All rule happens when production and government spending eventually begin. When the government begins to spend the oil income it will cause a second period of inflation, further appreciating the terms of trade (see "$Flex$" in Figure 5.2, at $t = 4$). Domestic inflation will affect both domestic and foreign households. Domestic households will face a tighter budget constraint, and so overall consumption will fall. They will also face a change in relative prices, and will switch from home to foreign goods. The net effect is that domestic households repay their borrowing from the first few periods. Foreign households will also respond to the appreciated terms of trade, and reduce their consumption of home goods. However, the net effect is an increase in output because the government only consumes home goods - such as education, health care and justice, while households consume a bundle of internationally produced goods.

5.2.2 Currency Peg

Under a currency peg, an oil discovery will induce large fluctuations in inflation, the output gap and the terms of trade. In practice, currency pegs are used by 75% of resource

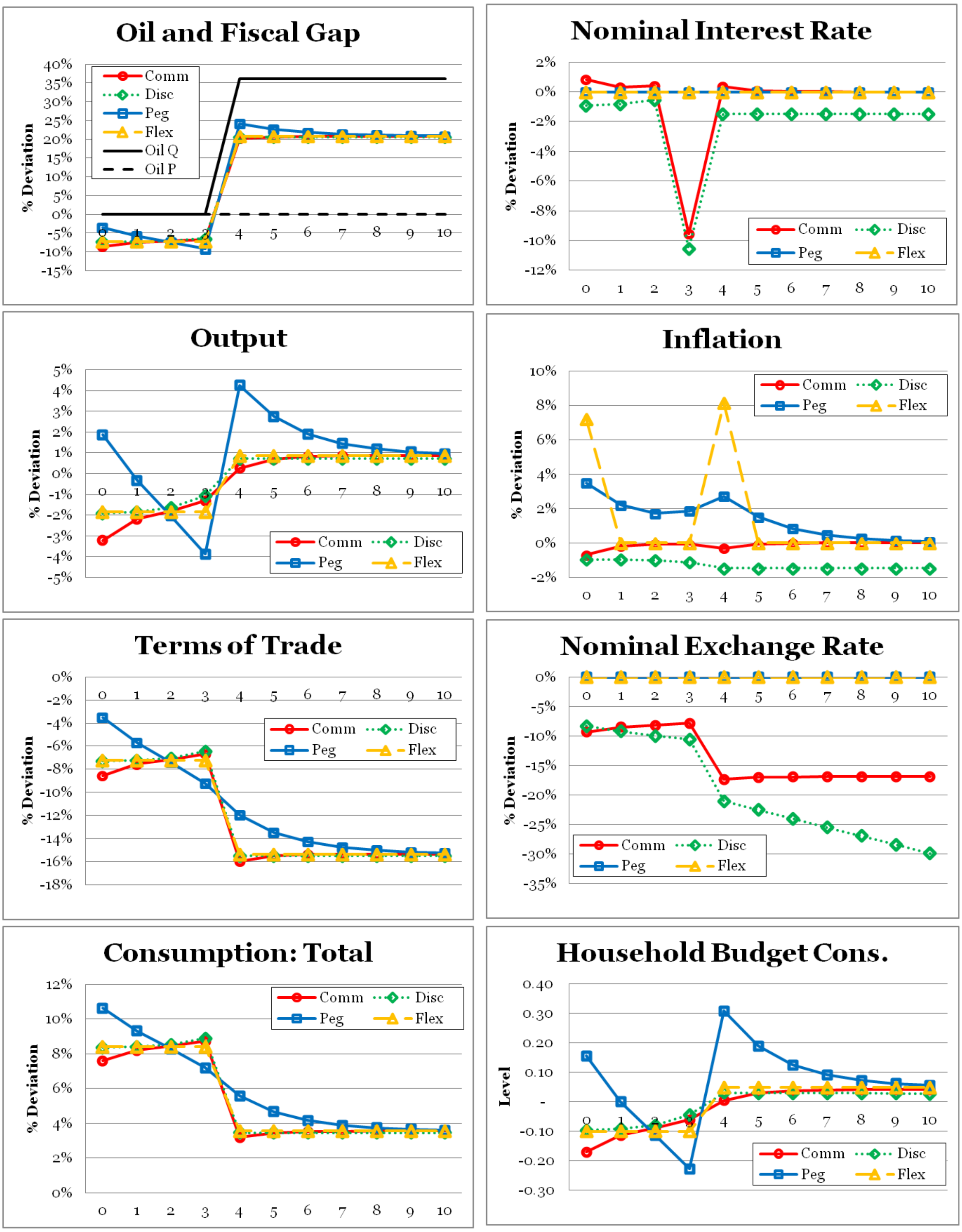

Figure 5.2: Forecast mean responses of key variables to an anticipated oil windfall under the Spend All rule. Four different scenarios are illustrated: optimal commitment (Comm), optimal discretion (Disc), a pegged currency (Peg) and flexible prices (Flex).

dependent economies (Baunsgaard et al., 2012). This analysis shows that a fixed exchange rate removes an important "shock absorber" for oil-exporters. This can have particularly adverse effects because of the heavy exposure to oil price shocks. The effect of an oil discovery under a peg is illustrated in the "*Peg*" case in Figure 5.2 (for derivation see Appendix 12). The effect of an oil price shock is illustrated in Figure 5.4.

On discovering oil the terms of trade will need to appreciate for the first time. This cannot happen via the currency, so must happen entirely through domestic inflation. As prices are slow to adjust, output, consumption and foreign borrowing will initially overshoot their natural levels. Nominal rigidities will also cause prices to disperse between varieties of goods, leading to real distortions.

When government spending begins, the terms of trade will need to appreciate for the second time. Again, this can only happen through domestic inflation. In contrast to the appreciation when oil is discovered, this appreciation is anticipated by firms. Firms can only set prices randomly due to Calvo pricing (see Section 2.5.2). Any firm that has an opportunity to raise prices before government spending begins will do so. Thus, between discovering and producing oil, firms will have two incentives to raise prices. The first is to raise prices retrospectively to deal with the jump in household consumption. The second is to raise prices prospectively to anticipate future government demand. The result is an extended period of high inflation, and a rapid decline in output after the euphoria at discovery.

Output only recovers when government spending begins. It will again jump too far, as all firms will not have had an opportunity to raise prices in advance. An oil discovery under a currency peg will therefore lead to an extended period of inflation and large fluctuations in output.

5.2.3 Optimal Policy:Discretion and Commitment

Optimal policy can dampen the sharp fluctuations caused by an oil discovery under a currency peg. This will involve letting the nominal exchange rate appreciate on discovery, and the key feature for our analysis: loosening policy before spending begins. There will also be a small incentive to appreciate the terms of trade, leading to a deflationary bias under discretion which is overcome under commitment.

When oil is discovered, the central bank will optimally let the terms of trade appreciate via the nominal exchange rate. On discovering oil, domestic consumption will jump because of an increase in household wealth. This requires the terms of trade to appreciate. Allowing them to appreciate via the currency, rather than domestic prices, avoids the sticky price distortions in the latter. This takes care of much of the initial shock to the economy that comes from higher household wealth.

When government spending begins, the central bank will optimally loosen interest rates in the period before to delay the second terms of trade appreciation. Between oil discovery and production, forward looking firms will have an incentive to raise prices in anticipation of future demand. This will lead to distortions in the economy as prices of different varieties of goods become dispersed. The central bank can avoid this by following a rule that will lead to a sharp loosening in policy in the period before spending begins. Loose policy in the period immediately before the windfall will cause firms to expect the currency to appreciate when the windfall begins (consistent with uncovered interest parity). Thus, firms need not raise their prices in anticipation, because they know the central bank will achieve this for them. In essence, this policy allows the central bank to take responsibility for appreciating the terms of trade from firms, using a single, flexible price rather than many sticky ones.

It is important to note that the effectiveness of loose policy in the future does not rely on the ability of the central bank to commit. Loose policy will be optimal in the period before the windfall, even for a discretionary central bank. However, the policy's effectiveness does rely on firms understanding the monetary rule that is being followed. In the model this is taken care of using rational expectations. In practice it may require communication by the central bank about the nature of the rule, and that it is optimal even for a discretionary central bank.

After government spending begins, a discretionary central bank will suffer from a small deflationary bias. This stems from incentive for the central bank to appreciate the terms of trade each period, and exploit the asymmetry in wealth between home and foreign consumers (as described in ϕ_y term in the loss function in equation 4.3). However, the associated inflation will lead to price dispersion in the economy, and in doing so reduce welfare.

In contrast, a credible central bank will overcome this deflationary bias through commitment. It is seen in the initial tightening of policy in the first period. This tightening stems from the bank's incentive to further appreciate the terms of trade and exploit the asymmetry in wealth from the home country's oil. By exploiting this terms of trade externality the central bank can improve on the flexible price outcome, as seen in Table 1.

5.2.4 Monetary Rules

Optimal policy can be well approximated by making a simple change to a standard monetary, or "Taylor", rule. The standard Taylor rule leads to a recession between discovery and spending. However, if we let it also respond to expected changes in the natural level of output it comes close to approximating optimal policy.

On discovering oil, the standard Taylor rule leads to a recession between discovery and production. The standard Taylor rule is assumed to take the form in equation 4.11 where $\phi_\pi = 1.5$ and $\phi_y = 0.5$. When oil is discovered, the terms of trade immediately start to appreciate, as forward looking firms anticipate future demand. As the terms of trade appreciate, consumer demand for domestic goods falls, leading to a negative output gap which we interpret as a recession. The central bank responds to the negative output gap by continually loosening policy between discovery and production. This causes the nominal exchange rate to appreciate, compensating for lower domestic returns as dictated by uncovered interest parity. Thus, the terms of trade appreciate not by firms raising prices, rather by firms failing to lower prices as quickly as the nominal exchange rate appreciates.

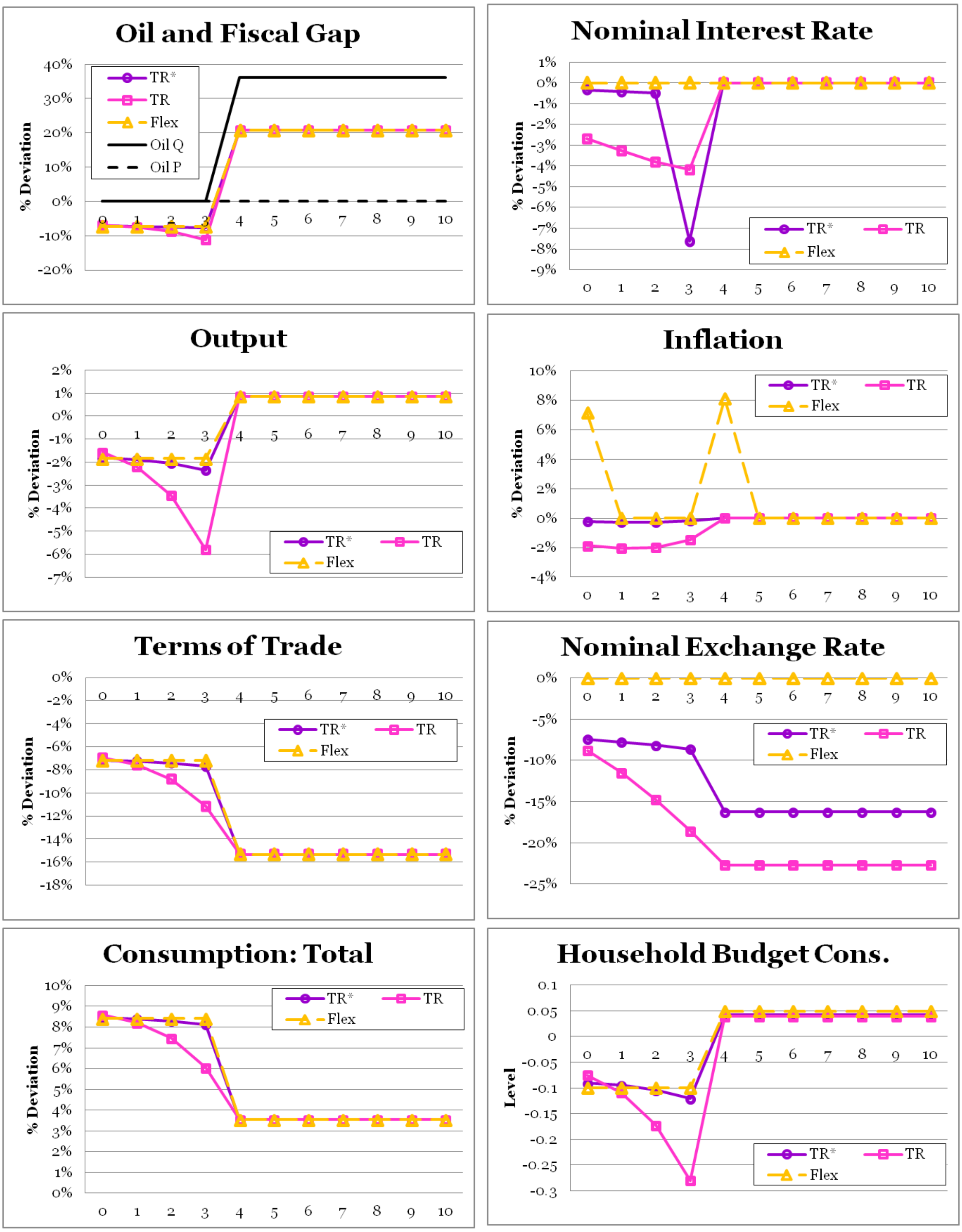

Figure 5.3: Forecast mean responses of key variables to an anticipated oil windfall under the Spend All rule. Three different scenarios are illustrated: a standard Taylor rule (TR), an optimised Taylor Rule (TR*) and flexible prices (Flex).

Standard Deviation of Key Variables

Variable	Comm	Disc	Peg	TR	TR*
$\hat{p}_{O,t}$	1.00	1.00	1.00	1.00	1.00
$\hat{g}_t$	0.75	0.76	0.92	0.81	0.78
$\pi_{H,t}$	0.01	0.01	0.07	0.04	0.01
$\hat{y}_t$	0.05	0.06	0.23	0.11	0.08
$\hat{s}_t$	0.25	0.24	0.07	0.19	0.22
$\hat{e}_t$	0.26	0.25	0	0.14	0.21
$\hat{c}_t$	0.15	0.14	0.04	0.11	0.13

Table 2: Standard deviation is calculated one period ahead, based on 10,000 Monte Carlo simulations of a N(0,1) shock to the oil price.

The standard Taylor rule can be improved by including a term that captures the key feature of optimal policy. As discussed above, the key feature of optimal policy is the ability to loosen interest rates in the period before the windfall hits. We can approximate this by including an additional term in the Taylor rule, as given in equation 4.12. Again, we let $\phi_\pi = 1.5$ and $\phi_y = 0.5$, and choose $\phi_{yn} = -2.6$ to minimise welfare losses. By responding to expected changes in the natural level of output, which will be driven by changes in government spending, the central bank can delay the second terms of trade adjustment until government spending begins. Firms realise this and so will not raise prices in anticipation, avoiding the negative output gaps that characterise the standard Taylor rule. Welfare under this rule is as high as the flexible price case.

So far this section has focused on anticipated changes in the mean path of government spending.[6] We now turn to the response of each monetary rule to the stochastic shock in our model - the oil price.

5.2.5 Price Shock

[6]Figures 5.2 and 5.3 illustrate forecasts for the mean path of each variable, rather than impulse responses. The forecasts do not return to the initial steady state, though this does not imply the model yields non-stationary responses to stochastic shocks (or alternatively unit roots in the paths of the variables). Instead, they illustrate a change in the mean level of each variable in response to an anticipated, deterministic change in oil output.

If the government follows a Spend All rule then it will also transmit stochastic oil price shocks directly to aggregate demand. In this section we compare how these shocks affect the volatility of key variables under each monetary regime. The impulse responses of each variable are illustrated in Figure 5.4, and can be interpreted as stationary fluctuations around a deterministic trend. Optimal discretion will respond to both the oil price and domestic inflation, while optimal commitment responds to the oil price and the domestic price level. Key variables are significantly more volatile under a currency peg than under optimal policy.

The impulse responses in Figure 5.4 can be interpreted as stationary fluctuations around a deterministic trend. The deterministic trend is based on the expected path of current and future oil production, as illustrated for the Spend All case in Figure 5.2. Once this trend is removed, we are left with the response of each variable to unanticipated, stochastic shocks to the oil price. The responses of the real variables are stationary.

Optimal policy responds to oil price shocks through their effect on prices and the natural level of output. If there is a positive shock to the oil price, interest rates will initially tighten to offset the slow response from domestic prices. If the central bank can commit then it will loosen interest rates in subsequent periods, boosting output above its flexible price level and smoothing the adjustment of the economy to the oil price shock. Discretionary policy is again affected by a deflationary bias, as discussed above.

Inflation and the output gap are significantly more volatile under a standard Taylor rule or a currency peg than under optimal policy. Table 2 illustrates the relative volatility of key variables under each monetary regime. A standard Taylor rule or a currency peg cause large fluctuations in output and inflation. In contrast, the optimal rules reduce the volatility of these key variables by letting the nominal exchange rate absorb much of the shock. The stochastic properties of these optimal policies are also well-approximated by our augmented Taylor rule ("TR^*").

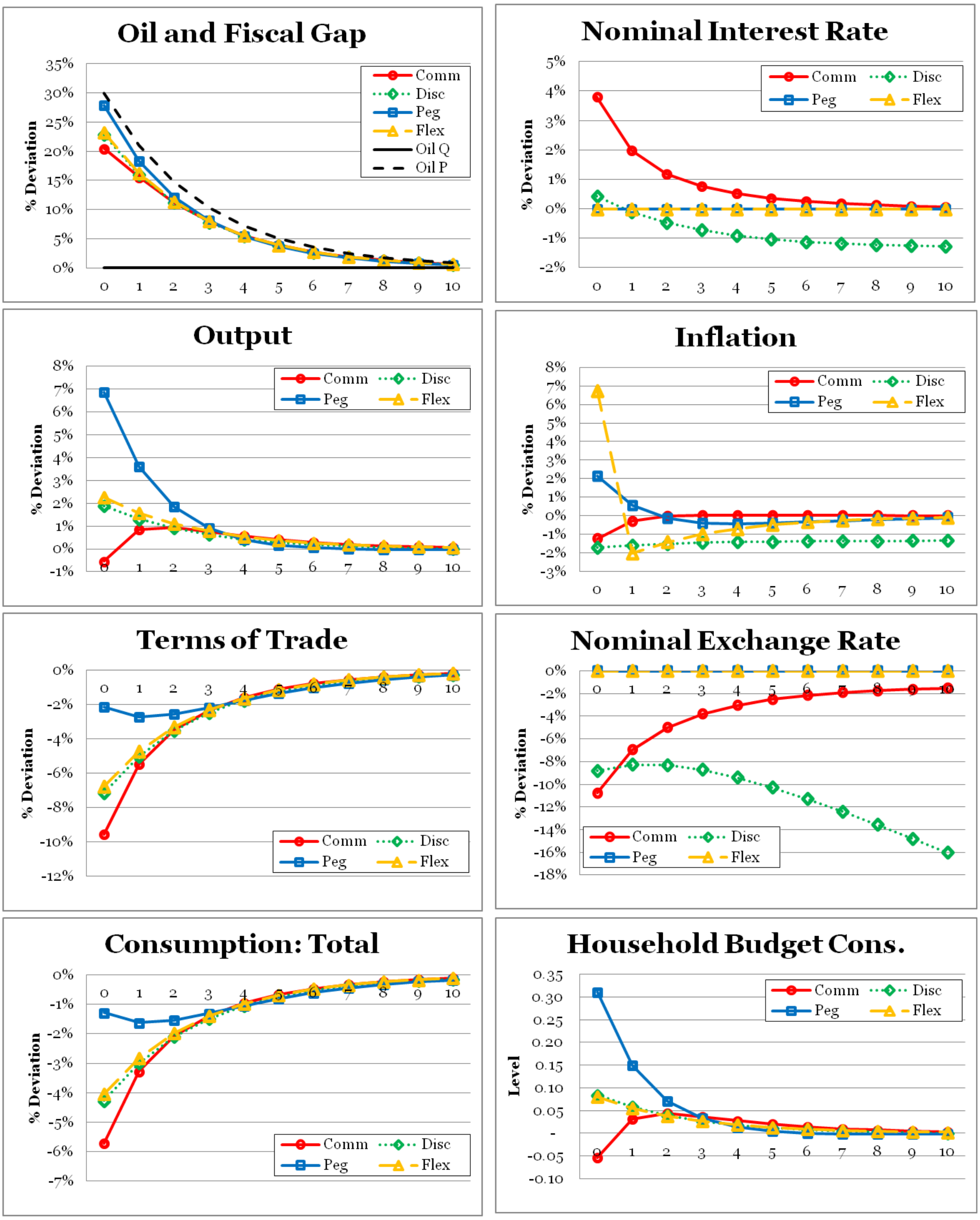

Figure 5.4: Impulse responses to a temporary oil price shock of 30% with persistence $\rho_o = 0.7$. Four different scenarios are illustrated: optimal commitment (Comm), optimal discretion (Disc), a pegged currency (Peg) and flexible prices (Flex).

5.3 Bird in Hand Rule

Our third scenario assumes that the government partially smooths its spending of a temporary windfall, according to a Bird in Hand rule. This involves the government saving all oil revenues in a sovereign wealth fund, and spending only a fixed proportion of the fund's assets each period. Since 1990 Norway has adopted a similar rule. The Bird in Hand rule will affect the economy in three phases. On discovery, consumption will immediately jump and cause the terms of trade to appreciate. During production, government spending will gradually rise which further appreciates the terms of trade and crowds out consumption. After production ends, government spending will slowly decline as foreign assets are consumed, letting the terms of trade depreciate to their new steady state. Under a currency peg or a standard Taylor rule the rise and fall of government spending will create inflationary and deflationary pressure, respectively. Optimal monetary policy can overcome this, and is well approximated by our simply augmented Taylor rule.

5.3.1 Flexible Prices

If the government manages an oil windfall with a Bird in Hand rule then the economy will go through three distinct phases. It is best to illustrate these phases when prices are flexible ("*Flex*" in Figure 5.5). The first phase begins at discovery ($t = 0$ in our example), when foresighted households immediately raise their consumption on learning of their new-found wealth. As domestic households prefer home to foreign goods, the relative price of home goods (the terms of trade) must rise, meaning the terms of trade must appreciate. The effects are less dramatic than in the Spend All scenario, because the windfall is only temporary.

The second phase happens during production ($t = 4, \ldots, 12$). Government spending will steadily rise, as it consumes a fixed share of the oil wealth accumulating in the sovereign wealth fund. The government only consumes home goods, so their relative price must rise even further. Consumers, both domestic and foreign, will substitute away from home goods during this period.

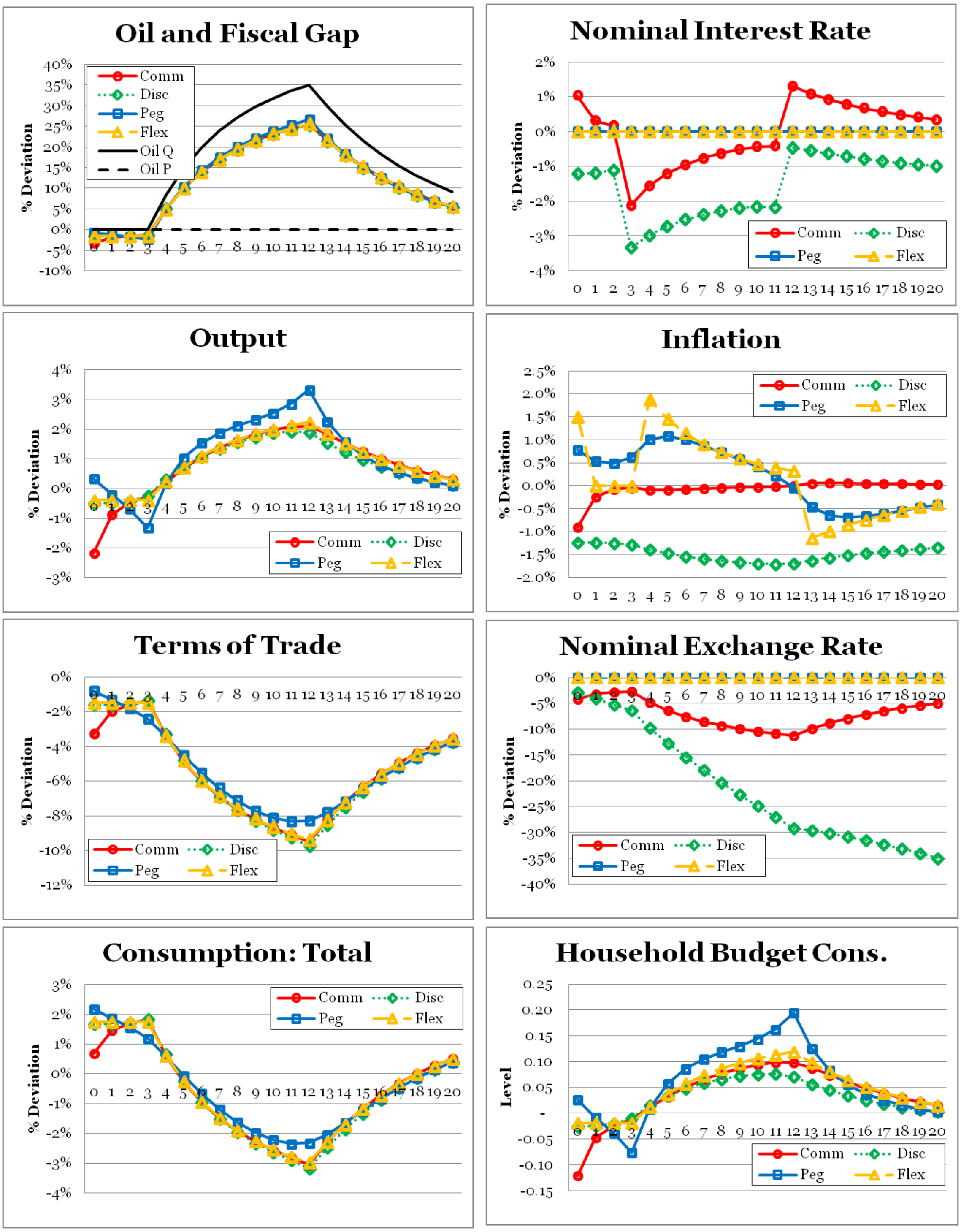

Figure 5.5: Forecast mean responses of key variables to an anticipated oil windfall under the Bird in Hand rule. Four different scenarios are illustrated: optimal commitment (Comm), optimal discretion (Disc), a pegged currency (Peg) and flexible prices (Flex).

The third phase occurs once oil production ends ($t > 12$). After this time government spending will fall, as the accumulated foreign assets are run down. The relative price of home goods will duly fall as well, and household consumption will rise back to its new steady-state. Consumption in this steady state will be the same as immediately after the oil was discovered. Foresighted households will have chosen to borrow initially, before saving during the boom years, to sustain a permanently higher level of consumption once they end.

5.3.2 Currency peg

Under a currency peg, a Bird in Hand rule will lead to sustained periods of inflation and deflation. The constantly changing level of government spending requires continuous changes in the price of home goods. As these prices are sticky this will lead to large welfare losses.

On discovering oil, the terms of trade will need to appreciate. The nominal exchange rate is fixed, so this can only happen through domestic inflation. Domestic prices are sticky, so there will be an extended period of inflation between discovery and spending as firms raise prices both retrospectively and prospectively. Output will initially overshoot its natural level, as prices are slow to adjust. However, it will quickly fall as firms raise prices in anticipation of government demand. Once again, the effect will be less extreme than the Spend All case because government spending - and in turn prices- will only rise gradually.

Once oil production begins, government spending will gradually rise for the life of the oil field. The price of home goods will also have to rise. However, nominal rigidities mean that prices will consistently be below, and output above their natural levels. When the end of the boom approaches, forward-looking firms will stop raising their prices as they anticipate the decline of government demand. This causes output to rise even further beyond its natural rate, just as the boom comes to an end.

After oil production ends, government spending will decline as assets in the sovereign wealth fund are consumed. During this period of falling government spending the price of domestic goods will also need to fall. However, as prices are sticky they will remain consistently too high. As a result, during the period of decline households will respond to the sub-optimally high prices by consuming too little, causing output to be consistently below its natural level. If we assume that the negative output gap is associated with involuntary unemployment then this could have a major effect on household welfare.

5.3.3 Optimal Policy: Discretion and Commitment

Optimal policy is able to avoid the extended period of positive and negative output gaps associated with changing government spending. As with the Spend All rule this involves letting the nominal exchange rate appreciate on discovering oil, and sharply loosening policy immediately before spending begins. During oil production monetary policy will slowly and continually tighten, offseting sub-optimally low prices and preventing output from overshooting its natural level. Once production ends policy will reverse, loosening to offset the sub-optimally high prices.

When oil is discovered optimal policy will let the nominal exchange rate appreciate immediately, and loosen just before spending begins. As in the Spend All case, households will immediately consume more on learning about their oil wealth. Home bias means that the price of home goods must appreciate relative to the price of foreign goods, which is easily achieved by a floating exchange rate. Loosening policy immediately before spending begins will remove the incentive for forward-looking firms to raise their prices, and delay the terms of trade appreciation. This loosening will be less dramatic than in the Spend All case, as government spending only rises gradually.

During oil production, optimal monetary policy will slowly and continually tighten. Over this period government spending will rise continually, as oil revenues accumulate in the offshore sovereign wealth fund. Increasing government demand will require the price of home goods to rise, though they are sticky. To prevent output continually exceeding its natural level, due to sticky prices which are sub-optimally low, optimal monetary policy will need to tighten slightly each period.

When production ends government spending will begin to decline. In this case optimal policy will sharply tighten policy immediately before government spending falls. In the opposite case to the start of government spending, this delays the terms of trade depreciation until it is necessary - preventing output jumping too far above its natural rate. During the period of declining government spending, optimal policy will be to gradually loosen interest rates each period, offsetting sub-optimally high sticky prices. This prevents output falling below its natural level.

Once again, the broad scope of the optimal policy does not depend on the central bank's ability to commit. The policy of loosening before the windfall begins, gradually tightening during the period of rising government spending, and gradually loosening again as government spending declines is optimal under both discretion and commitment. It relies only on the rest of the economy understanding that this is optimal for the central bank. The difference between commitment and discretion comes down to the deflationary bias under discretion, as described above.

5.3.4 Taylor Rules

The optimal monetary response to a Bird in Hand rule is also well approximated by the slightly augmented Taylor rule described in Section 5.2.4.

The standard Taylor rule causes large fluctuations in the output gap, if the government follows a Bird in Hand rule. Between discovery and spending there will be a similar recession to the Spend All case. Once oil production begins and the government starts consuming oil wealth, output will be consistently below its natural level. However, as the windfall comes to an end, forward looking firms and asset markets will begin to depreciate the terms of trade. Output will overshoot its natural level, and be above it as the remaining oil wealth is consumed.

In contrast, optimal policy is well approximated by a Taylor rule that responds to expected changes in the natural level of output. We again consider a Taylor rule of the form in equation 4.12, where $\phi_\pi = 1.5$ and $\phi_y = 0.5$. As in the previous section, numerically

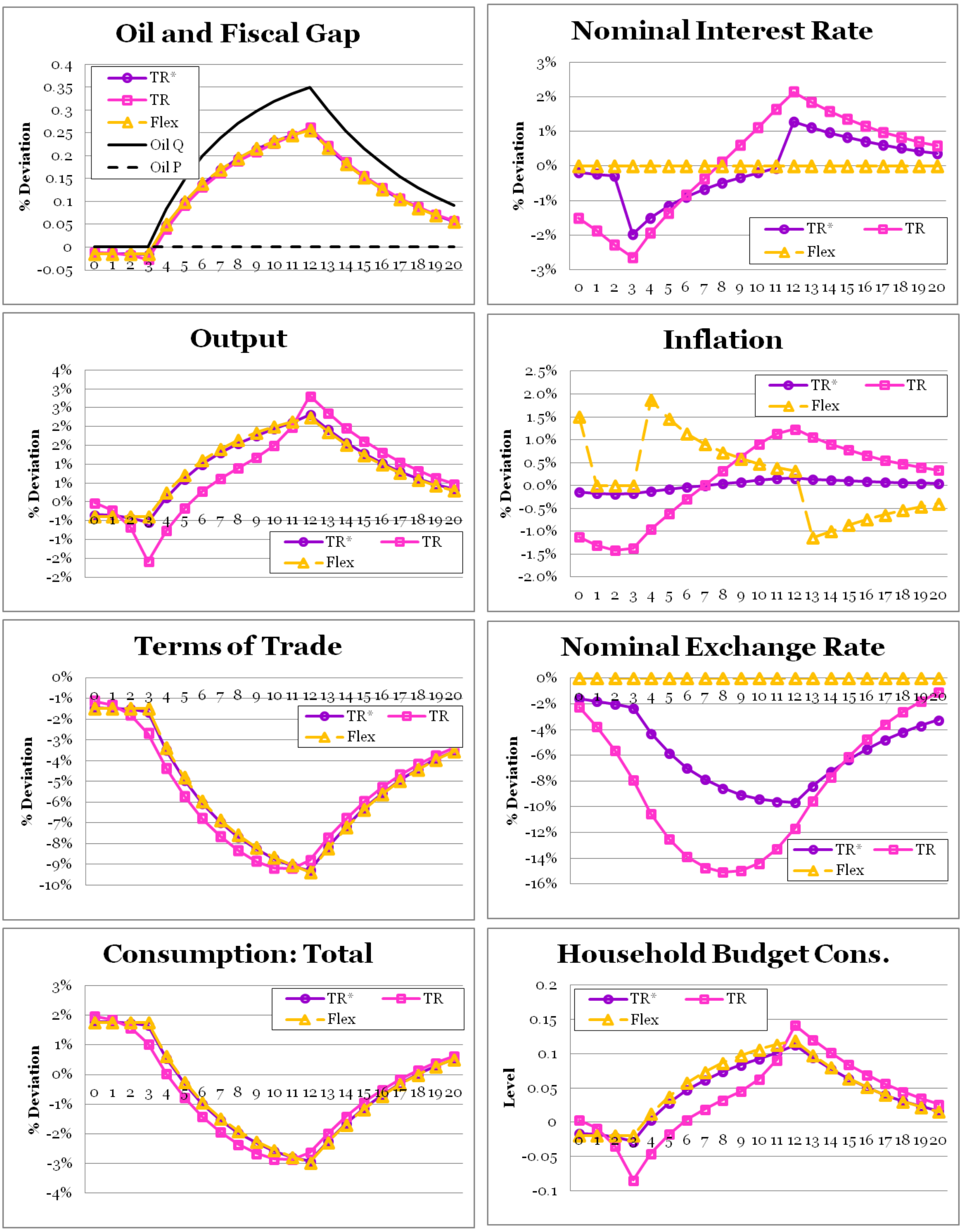

Figure 5.6: Forecast mean responses of key variables to an anticipated oil windfall under the Bird in Hand rule. Three different scenarios are illustrated: a standard Taylor rule (TR), an optimised Taylor Rule (TR*) and flexible prices (Flex).

optimising the third parameter gives $\phi_{yn} = -2.6$. This gives us some confidence in the stability of the estimate, despite a large difference in the intensity and timing of the oil windfall compared to the Spend All case. Examining Figure 5.6 again shows that this augmented Taylor rule closely tracks the flexible price outcome.

6 Conclusion

This paper considers how monetary policy should optimally respond to an oil discovery. In practice we find that there is often a significant delay between discovering oil, and beginning production. Governments also often delay spending until they receive oil revenues. Thus, oil discoveries can be thought of as anticipated shocks to government spending. These shocks can induce changes in the economy that create a number of distortions. We have been interested in how monetary policy should manage these distortions.

The paper's first contribution is to characterise the effects of an oil discovery under different fiscal rules, and rational expectations. To do this we develop a small, tractable model of a small open economy with oil. Regardless of the spending rule, discovering oil will require an immediate terms of trade appreciation as private or public spending jumps. If government consumption does not rise for some time, then there will also need to be a second appreciation. Firms will anticipate this and may begin raising prices in advance. This can lead to an extended period of inflation, and a recession, under monetary regimes such as a nominal currency peg or standard Taylor rule.

The paper's second contribution is to derive a closed form for the central bank's loss function. We show that the central bank's objective will be to stabilise domestic inflation, the output gap and the fiscal gap, which is consistent with standard models of a small open economy with a government. In addition there will be an incentive to appreciate the terms of trade to exploit the asymmetry introduced by the home country's extra oil wealth.

The paper's third contribution is to derive the optimal policy response to an oil discovery. On discovering oil, optimal policy will let the currency appreciate immediately.

This allows the terms of trade to appreciate without the distortions of sticky domestic prices. To prevent firms raising prices in anticipation of future demand, optimal policy will sharply loosen rates before the windfall and tighten them when it is received. This delays the terms of trade appreciation until government spending can take up the slack left by private demand.

Extensions to this work may consider other price setting assumptions and adding investment. Price setting plays an important role in this model, driving the initial period of recession. We have assumed Calvo pricing. An extension would investigate whether this result holds when firms set prices based on their state (such as Gertler and Leahy, 2008 and Golosov and Lucas, 2007). If there are concave menu costs then there may be an incentive to delay price rises as long as possible, overcoming distortions caused by firms raising prices in advance. Investment is also likely to have an important effect between the discovery and production of oil. Investment may reduce the necessary changes in prices, and boost employment between discovery and production.

While this analysis was conducted with an oil discovery in mind, it may also be of interest to other applications. Aid windfalls have similar characteristics to oil windfalls. If anticipating a windfall leads to recession, then it can be argued that aid grants should only be announced when government spending begins. Anticipated increases in government demand may also happen between an election and the implementation of a budget, though this would require a closer investigation of taxation.

References

[1] Barsky, R. B., and Sims, E. R. News shocks and business cycles. *Journal of Monetary Economics 58*, 3 (2011), 273–289.

[2] Barsky, R. B., and Sims, E. R. Information, animal spirits, and the meaning of innovations in consumer confidence. *American Economic Review 102*, 4 (June 2012), 1343–77.

[3] Baunsgaard, T., V. M. P.-R. M., and Richmond, C. Fiscal frameworks for resource rich developing countries. *IMF Staff Discussion Note* (2012).

[4] Beaudry, P., and Lucke, B. Letting different views about business cycles compete. In *NBER Macroeconomics Annual 2009, Volume 24*, NBER Chapters. National Bureau of Economic Research, Inc, Summer 2010, pp. 413–455.

[5] Beaudry, P., and Portier, F. An exploration into pigou's theory of cycles. *Journal of Monetary Economics 51*, 6 (September 2004), 1183–1216.

[6] Beaudry, P., and Portier, F. Stock prices, news, and economic fluctuations. *American Economic Review 96*, 4 (September 2006), 1293–1307.

[7] Beaudry, P., and Portier, F. When can changes in expectations cause business cycle fluctuations in neo-classical settings? *Journal of Economic Theory 135*, 1 (July 2007), 458–477.

[8] Benigno, P., and Woodford, M. Inflation stabilization and welfare: The case of a distorted steady state. *Journal of the European Economic Association 3*, 6 (2005), 1185–1236.

[9] Blanchard, O., and Gali, J. The macroeconomic effects of oil shocks: Why are the 2000s so different from the 1970s? *NBER Working Paper 13368* (2007).

[10] Blanchard, O., and Kahn, C. The solution of linear difference models under rational expectations. *Econometrica: Journal of the Econometric Society* (1980), 1305–1311.

[11] Bodenstein, M., Guerrieri, L., and Kilian, L. Monetary policy responses to oil price fluctuations. *IMF Economic Review 60*, 4 (2012), 470–504.

[12] Buiter, W., and Purvis, D. Oil, disinflation, and export competitiveness: A model of the "dutch disease".

[13] Calvo, G. Staggered prices in a utility-maximizing framework. *Journal of monetary Economics 12*, 3 (1983), 383–398.

[14] Corden, W., and Neary, J. Booming sector and de-industrialisation in a small open economy. *The Economic Journal 92*, 368 (1982), 825–848.

[15] Dagher, J., Gottschalk, J., and Portillo, R. Oil windfalls in ghana: A dsge approach. *IMF Working Paper WP/10/116* (2010).

[16] De Paoli, B. Monetary policy and welfare in a small open economy. *Journal of International Economics 77*, 1 (2009), 11–22.

[17] Den Haan, W., and Kaltenbrunner, G. Anticipated growth and business cycles in matching models. *Journal of Monetary Economics 56*, 3 (2009), 309–327.

[18] Eastwood, R., and Venables, A. The macroeconomic implications of a resource discovery in an open economy. *The Economic Journal 92*, 366 (1982), 285–299.

[19] Elekdag, S., Lalonde, R., Laxton, D., Muir, D., and Pesenti, P. Oil price movements and the global economy: A model-based assessment. Working Papers 07-34, Bank of Canada, 2007.

[20] Galí, J. *Monetary Policy, inflation, and the Business Cycle: An introduction to the new Keynesian Framework*. Princeton University Press, 2008.

[21] Gali, J., and Monacelli, T. Monetary policy and exchange rate volatility in a small open economy. *The Review of Economic Studies 72*, 3 (2005), 707–734.

[22] Gali, J., and Monacelli, T. Optimal monetary and fiscal policy in a currency union. *Journal of International Economics 76*, 1 (2008), 116–132.

[23] Gertler, M., and Leahy, J. A phillips curve with an ss foundation. *Journal of Political Economy 116*, 3 (2008).

[24] Golosov, M., and Lucas, R. Menu costs and phillips curves. *Journal of Political Economy 115*, 2 (2007).

[25] Jaimovich, N., and Rebelo, S. Can news about the future drive the business cycle? *American Economic Review 99*, 4 (September 2009), 1097–1118.

[26] Kilian, L. Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *The American Economic Review 99*, 3 (2009), 1053–1069.

[27] Kilian, L., and Lewis, L. T. Does the fed respond to oil price shocks?*. *The Economic Journal 121*, 555 (2011), 1047–1072.

[28] Lorenzoni, G. News shocks and optimal monetary policy. Tech. rep., National Bureau of Economic Research, 2007.

[29] Mertens, K., and Ravn, M. O. Understanding the aggregate effects of anticipated and unanticipated tax policy shocks. *Review of Economic Dynamics 14*, 1 (January 2011), 27–54.

[30] Monacelli, T. Is monetary policy in an open economy fundamentally different? *IMF Economic Review forthcoming* (2013).

[31] Nakov, A., and Pescatori, A. Monetary policy trade-offs with a dominant oil producer. *Journal of Money, Credit and Banking 42*, 1 (2010), 1–32.

[32] Neary, J. P., and Wijnbergen, S. v. Can an oil discovery lead to a recession? a comment on eastwood and venables. *The Economic Journal 94*, 374 (1984), 390–395.

[33] Pieschacón, A. The value of fiscal discipline for oil-exporting countries. *Journal of Monetary Economics* (2012).

[34] Pigou, A. C. Wage policy and unemployment. *The Economic Journal 37*, 147 (1927), 355–368.

[35] Rogoff, K. The optimal degree of commitment to an intermediate monetary target. *The Quarterly Journal of Economics 100*, 4 (1985), 1169–1189.

[36] Romero, R. E. Monetary policy in oil-producing economies. Working Papers 1053, Princeton University, Department of Economics, Industrial Relations Section., Jan. 2008.

[37] Schmitt-Grohe, S., and Uribe, M. Closing small open economy models. *Journal of International Economics 61*, 1 (2003), 163 – 185.

[38] Taylor, J. B. Discretion versus policy rules in practice. In *Carnegie-Rochester conference series on public policy* (1993), vol. 39, Elsevier, pp. 195–214.

[39] van der Ploeg, F. Bottlenecks in ramping up public investment. *International Tax and Public Finance* (2012), 1–30.

[40] van der Ploeg, F., and Venables, A. Harnessing windfall revenues: Optimal policies for resource-rich developing economies*. *The Economic Journal 121*, 551 (2011), 1–30.

[41] van der Ploeg, F., and Venables, A. J. Natural resource wealth: The challenge of managing a windfall. *Annual Review of Economics 4*, 1 (2012), 315–337.

[42] Van Wijnbergen, S. Inflation, employment, and the dutch disease in oil-exporting countries: A short-run disequilibrium analysis. *The Quarterly Journal of Economics 99*, 2 (1984), 233–250.

[43] Wohltmann, H., and Winkler, R. Rational expectations models with anticipated shocks and optimal policy: a general solution method and a new keynesian example. Tech. rep., Economics working paper/Christian-Albrechts-Universität Kiel, Department of Economics, 2009.

[44] Woodford, M. Inflation stabilization and welfare. *NBER Working Paper 8071* (2001).

[45] Woodford, M. The taylor rule and optimal monetary policy. *The American Economic Review 91*, 2 (2001), pp. 232–237.

[46] Woodford, M. Methods of policy accommodation at the interest-rate lower bound. *presented at the Jackson Hole Symposium, "The Changing Policy Landscape", August 31-September 1* (2012).

7 Appendix: Relative Household Wealth

We start with the definition of the relative wealth ratio, $\Theta_{f,0} = E_0[\sum_{t=0}^{\infty} M_{0,t}(W_t N_t - P_{H,t}T_t)]/E_0[\sum_{t=0}^{\infty} M_{0,t}^f(W_t^f N_t^f - P_{f,t}^f T_t^f)]$ and assume that it is evaluated under flexible prices. If prices are flexible, and if the government provides a small wage subsidy to correct for monopolistic distortions, then $W_t = P_{H,t}\forall t$. Let us also make use of the small open economy assumption and let all foreign variables remain at their steady state, and $r^* = \rho$. Thus, the denominator becomes, $E_0[\sum_0^{\infty} M_{0,t}^f(W_t^f N_t^f - P_{f,t}^f T_t^f)] = P_f^s(N_f^s - T_f^s)(1+\rho)/\rho$. In the steady state we assume that taxes are the same as the rest of the world, so $\gamma_T = \chi$. Therefore, $(N^s - T^s) = (N_f^s - T_f^s) = 1 - \chi$ and $P_H^s/P_f^s = (S_f^s)^{-1} = \Theta_f^s$. Using this, and, $\beta = (1+\rho)^{-1}$, we can log-linearise the definition of $\Theta_{f,0}$ around the steady state assuming prices are flexible,

$$\begin{aligned}
\hat{\vartheta}_{f,0}^n \Theta_f^s &= \frac{P_H^s(N^s - T^s)}{P_f^s(N_f^s - T_f^s)(1+\rho)/\rho} E_0\left[\sum_{t=0}^{\infty} M_{0,t}(\hat{p}_{H,t}^n + \tfrac{1}{1-T^s/N^s}(\hat{n}_t^n - \tfrac{T^s}{N^s}\hat{\mathfrak{t}}_t^n))\right] \\
\hat{\vartheta}_{f,0}^n &= \frac{\rho}{1+\rho} E_0\left[\sum_{t=0}^{\infty} M_{0,t}(\hat{p}_{H,t}^n + \tfrac{1}{1-\chi}(\hat{n}_t^n - \chi\hat{\mathfrak{t}}_t^n))\right]
\end{aligned}$$

Note that taxes are $\hat{\mathfrak{t}}_t = \ln T_t - \ln T^s$, rather than $\hat{t}_t = -\ln(1 - \frac{T_t}{G_t}) + \ln(1 - \frac{T^s}{G^s})$ which we use in the text. We assume that $\hat{t}_t = 0$, so $\hat{\mathfrak{t}}_t = \hat{g}_t$. Under flexible prices it is necessary to fix a numeraire, so we let $\hat{e}_t = 0$ such that $\hat{s}_t^n = -\hat{p}_{H,t}^n$. Also, combining equations 3.3 and 3.6 gives $\varphi\hat{y}_t^n = -\hat{s}_t^n - \hat{\vartheta}_t^n$. So, $\hat{p}_{H,t}^n = \varphi\hat{y}_t^n + \hat{\vartheta}_t^n$ and $\hat{g}_t^n = -\varphi\hat{y}_t^n - \hat{\vartheta}_t^n + \widehat{rb}_t$. Therefore,

$$\begin{aligned}
\hat{p}_{H,t} + \tfrac{1}{1-\chi}(\hat{n}_t - \chi\hat{\mathfrak{t}}_t) &= \varphi\hat{y}_t^n + \hat{\vartheta}_t^n + \tfrac{1}{1-\chi}(\hat{y}_t - \chi(-\varphi\hat{y}_t^n - \hat{\vartheta}_t^n + \widehat{rb}_t)) \\
&= \tfrac{1+\varphi}{1-\chi}\hat{y}_t^n + \tfrac{1}{1-\chi}\hat{\vartheta}_t^n - \tfrac{\chi}{1-\chi}\widehat{rb}_t \\
&= \tfrac{1+\varphi}{1-\chi}\left\{\tfrac{\gamma_G}{(1+\varphi)}\widehat{rb}_t - \tfrac{(\alpha+\gamma_G(1-\alpha))}{(1+\varphi)}\hat{\vartheta}_t^n\right\} + \tfrac{1}{1-\chi}\hat{\vartheta}_t^n - \tfrac{\chi}{1-\chi}\widehat{rb}_t \\
&= \tfrac{\gamma_G - \chi}{1-\chi}\widehat{rb}_t + \tfrac{(1-\gamma_G)(1-\alpha)}{1-\chi}\hat{\vartheta}_t^n
\end{aligned}$$

Substituting this into the above relationship, and using effective (rather than bilateral) relationships gives the following, where we make use of the observation that $E_t[\hat{\vartheta}^n_{t+s}] = \hat{\vartheta}^n_t \forall s$, as relative wealth incorporates all expected future income,

$$\begin{aligned}
\hat{\vartheta}^n_0 &= \tfrac{\rho}{1+\rho} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \left(\tfrac{\gamma_G - \chi}{1-\chi} \widehat{rb}_t + \tfrac{(1-\gamma_G)(1-\alpha)}{1-\chi} \hat{\vartheta}^n_t \right) \right] \\
&= \tfrac{\rho}{1+\rho} E_0 \left[\sum_{j=0}^{\infty} M_{0,t} \tfrac{\gamma_G - \chi}{1-\chi} \widehat{rb}_t \right] + \tfrac{\rho}{1+\rho} \tfrac{(1-\gamma_G)(1-\alpha)}{1-\chi} \tfrac{1+\rho}{\rho} \hat{\vartheta}^n_0 \\
&= \tfrac{(1-\beta)(\gamma_G - \chi)}{(1-\chi)-(1-\gamma_G)(1-\alpha)} E_0 \left[\sum_{j=0}^{\infty} M_{0,t} \widehat{rb}_t \right]
\end{aligned}$$

Setting $\hat{\vartheta}_t = \hat{\vartheta}^n_t$ gives the relationship in equation 2.13. It is important to note that this is an approximation, which is subject to two types of error. The first is that the goods market equilibrium (equation 3.4) is not log-linear, and so may require a higher order approximation. The second is that we approximate $-\alpha\hat{\vartheta}_t \approx \ln\left[(1-\alpha) + \alpha\Theta_t^{-1}\right] - \ln\left[(1-\alpha) + \alpha(\Theta^s)^{-1}\right]$ when approximating the goods market equilibrium. These errors can be reduced by making the jump and the distance from $\Theta^s = 1$ small.

8 Appendix: Steady State

The steady state is found by simultaneously solving four equations,

$$\begin{aligned}
(1-\chi)AN^{-\varphi} &= S^\alpha C && (8.1) \\
AN - T &= S^\alpha C && (8.2) \\
AN &= S^\alpha C(1-\alpha+\alpha\Theta^{-1}) + G && (8.3) \\
C &= S^{1-\alpha}\Theta C^* && (8.4)
\end{aligned}$$

using $T = \gamma_T AN$, $G = \gamma_G AN$, and $C^* = (1-\chi)$. The first equation comes from combining 2.3 and the steady state marginal cost condition $MC^s = W/(P_H A) = 1$. The second equation follows from the household budget constraint, 2.2, if we assume there is no permanent accumulation of foreign assets $D^s = 0$. The third equation is the goods market equilibrium, 3.4. The fourth follows from the international risk sharing condition, 2.12. Simultaneously solving gives the results in Lemma 2.

9 Appendix: Central Bank Loss Function

We begin with the household utility function 2.1,

$$U_t = E_0 \sum_t^{\infty} \beta^t \{(1-\chi)\ln C_t + \chi \ln G_t - \frac{N^{1+\varphi}}{1+\varphi}\}$$

A second order log linear approximation around the steady state is derived for each term in turn. Starting with consumption, using $\gamma_G = G^s/Y^s$, equation 3.4 and $\hat{c}_t = \hat{\vartheta}_t + (1-\alpha)\hat{s}_t$ from 2.12 we get the following,

$$\begin{aligned}
\ln C_t &= \ln(Y_t - G_t) - \alpha s_t - \ln(1-\alpha+\alpha\Theta_t^{-1}) \\
\ln C_t &\approx \ln(Y^s - G^s) + \tfrac{1}{1-\gamma_G}(\hat{y}_t - \gamma_G \hat{g}_t) - \tfrac{1}{2}\tfrac{\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1}{1-\gamma_G}\hat{y}_t^2 - \alpha s_t - \ln(1-\alpha+\alpha\Theta_t^{-1}) \\
\ln C_t &= c^s + \tfrac{1}{1-\gamma_G}(\hat{y}_t - \gamma_G \hat{g}_t) - \tfrac{1}{2}\tfrac{\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1}{1-\gamma_G}\hat{y}_t^2 - \alpha \hat{s}_t + \alpha\hat{\vartheta}_t \\
\hat{c}_t &= \tfrac{1-\alpha}{1-\gamma_G}(\hat{y}_t - \gamma_G \hat{g}_t) - \tfrac{1}{2}\tfrac{\gamma_G(1-\alpha)}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1-\alpha}{1-\gamma_G}\hat{y}_t^2 + \alpha(2-\alpha)\hat{\vartheta}_t
\end{aligned}$$

Alternatively, consumption can be expressed using the terms of trade,

$$\hat{c}_t = (1-\alpha)\hat{s}_t + \hat{\vartheta}_t - \tfrac{1}{2}\tfrac{\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1}{1-\gamma_G}\hat{y}_t^2$$

Government spending gives, $\ln G_t = g_s + \hat{g}_t$. Finally, the disutility of labour can be log-linearized to give,

$$\frac{N_t^{1+\varphi}}{1+\varphi} \approx \frac{(N_t^s)^{1+\varphi}}{1+\varphi} + (N_t^s)^{1+\varphi}(\hat{n}_t + \tfrac{1}{2}(1+\varphi)\hat{n}_t^2) + o(\|\hat{g}_t\|^3)$$

To express this in terms of the output gap we use $N_t = \left(\frac{Y_t}{A}\right)\int_0^1 \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} dj$ and $N_t^s = \frac{Y_t^s}{A}$. Thus, we have $\hat{n}_t = \hat{y}_t + z_t$ where $z_t \equiv \ln\int_0^1 \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\varepsilon} dj$. The following Lemma, proved in Gali and Monacelli (2008) shows that z_t is proportional to the cross-sectional variance of relative prices and thus of second order.

Lemma 6. $z_t = \frac{\epsilon}{2} var\{p_{H,t}(j)\} + o(\|\bar{g}_t\|^3)$

Proof. See Gali and Monacelli (2008), Appendix. □

Thus the disutility of labour can be expressed as follows.

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{(N_t^s)^{1+\varphi}}{1+\varphi} + (N_t^s)^{1+\varphi}(\hat{y}_t + z_t + \tfrac{1}{2}(1+\varphi)\hat{y}_t^2) + o(\|\hat{g}_t\|^3)$$

where our definition of the steady state in Section 3.3 gives $N^s = 1$. Collecting terms and simplifying we have the following expression for household utility, making use of the following Lemma for the final equality.

Lemma 7. $\sum_{t=0}^{\infty} \beta^t var\{p_{H,t}(j)\} = \frac{1}{\lambda}\sum_{t=0}^{\infty} \beta^t \pi_{H,t}^2$ *where* $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$

Proof. See Woodford (2001), pp22-23. □

$$
\begin{aligned}
U_t &= E_0 \sum_t^{\infty} \beta^t \Big\{ (1-\chi)\left(c^s + \tfrac{1-\alpha}{1-\gamma_G}(\hat{y}_t - \gamma_G \hat{g}_t) - \tfrac{1}{2}\tfrac{\gamma_G(1-\alpha)}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1-\alpha}{1-\gamma_G}\hat{y}_t^2 + \alpha(2-\alpha)\hat{\vartheta}_t\right) \\
&\quad + \chi\left(g_s + \hat{g}_t\right) \\
&\quad - 1 - (\hat{y}_t + z_t + \tfrac{1}{2}(1+\varphi)\hat{y}_t^2)\Big\} + o(\|\hat{g}_t\|^3) \\
&= E_0 \sum_t^{\infty} \beta^t \Big\{ \left(\tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} - 1\right)\hat{y}_t - \left(\tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} - \tfrac{\chi}{\gamma_G}\right)\gamma_G \hat{g}_t \\
&\quad - \tfrac{1}{2}\left[\tfrac{(1-\chi)(1-\alpha)\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 + \tfrac{\epsilon}{\lambda}\pi_{H,t}^2 + \left((1+\varphi) + \tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G}\right)\hat{y}_t^2\right]\Big\} \\
&\quad + t.i.p + o(\|\hat{g}_t\|^3)
\end{aligned}
$$

where we assume that deviations in the long term ratio of household wealth at home and abroad, $\hat{\vartheta}_t$, cannot be affected by monetary policy. Or, if we express utility using the terms of trade,

$$
\begin{aligned}
U_t &= E_0 \sum_t^{\infty} \beta^t \Big\{ (1-\chi)\left(c^s + (1-\alpha)\hat{s}_t + \hat{\vartheta}_t - \tfrac{1}{2}\tfrac{\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 - \tfrac{1}{2}\tfrac{1}{1-\gamma_G}\hat{y}_t^2\right) \\
&\quad + \chi\left(g_s + \hat{g}_t\right) \\
&\quad - 1 - (\hat{y}_t + z_t + \tfrac{1}{2}(1+\varphi)\hat{y}_t^2)\Big\} + o(\|\hat{g}_t\|^3) \\
&= E_0 \sum_t^{\infty} \beta^t \Big\{ -((1-\gamma_G) - (1-\chi)(1-\alpha))\hat{s}_t - (\gamma_G - \chi)\hat{g}_t \\
&\quad - \tfrac{1}{2}\left[\tfrac{(1-\chi)(1-\alpha)\gamma_G}{(1-\gamma_G)^2}(\hat{g}_t - \hat{y}_t)^2 + \tfrac{\epsilon}{\lambda}\pi_{H,t}^2 + \left((1+\varphi) + \tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G}\right)\hat{y}_t^2\right]\Big\} \\
&\quad + t.i.p + o(\|\hat{g}_t\|^3)
\end{aligned}
$$

This gives the loss function in Proposition 3. If we also assume that the government follows a rule for oil revenues, $\hat{g}_t = \hat{y}_t + (1-\gamma_G)\widehat{rb}_t - (1-\alpha)(1-\gamma_G)\hat{\vartheta}_t$ from 2.123.5 and $\hat{g}_t = \hat{s}_t + \widehat{rb}_t$, the welfare function becomes,

$$\begin{aligned}
U &= E_0 \sum_t^{\infty} \beta^t \Bigg\{ \left(\tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} - 1 \right) \hat{y}_t - \left(\tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} - \tfrac{\chi}{\gamma_G} \right) \gamma_G \hat{y}_t \\
&\quad -\tfrac{1}{2} \left[\tfrac{(1-\chi)(1-\alpha)}{1-\gamma_T} \tfrac{\epsilon}{\lambda} \pi_{H,t}^2 + \left((1+\varphi) + \tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} \right) \hat{y}_t^2 \right] \Bigg\} + t.i.p + o(\|\hat{g}_t\|^3) \\
&= E_0 \sum_t^{\infty} \beta^t \Bigg\{ -\tfrac{1}{2} \left[\tfrac{\epsilon}{\lambda} \pi_{H,t}^2 + \left((1+\varphi) + \tfrac{(1-\chi)(1-\alpha)}{1-\gamma_G} \right) \hat{y}_t^2 \right] \\
&\quad -\alpha(1-\chi)\hat{y}_t \Bigg\} + t.i.p + o(\|\hat{g}_t\|^3)
\end{aligned}$$

This gives the loss function in equation 4.2.

10 Appendix: Optimal Monetary Policy

10.1 Optimal Policy under Discretion

Minimising the Lagrangian in equation 4.3 yields the first order conditions,

$$\begin{aligned}
0 &= \phi_\pi \pi_{H,t} + l_{D,t} \\
\lambda(1+\varphi) l_{D,t} &= \phi_{yy} \hat{y}_t + \phi_y
\end{aligned}$$

This can be simplified to,

$$\hat{y}_t = -\frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}} \pi_{H,t} - \frac{\phi_y}{\phi_{yy}} \tag{10.1}$$

Substituting this path for $\hat{y}_t$ into the Phillips curve gives,

$$\begin{aligned}
\pi_{H,t} &= \beta E_t[\pi_{H,t+1}] + \lambda(1+\varphi)(-\frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\pi_{H,t} - \frac{\phi_y}{\phi_{yy}}) - \lambda(1+\varphi)\hat{y}_t^n \\
\pi_{H,t} &= a_D\beta E[\pi_{H,t+1}] - a_D\lambda(1+\varphi)\hat{y}_t^n - a_D\lambda(1+\varphi)\tfrac{\phi_y}{\phi_{yy}}
\end{aligned}$$

Where $a_D = (1 + \frac{\lambda^2(1+\varphi)^2\phi_\pi}{\phi_{yy}})^{-1}$. Iteratively substituting gives,

$$\pi_{H,t} = \left\{-a_D\lambda(1+\varphi)\hat{y}_t^n - a_D\lambda(1+\varphi)\tfrac{\phi_y}{\phi_{yy}}\right\} E\left[\sum_0^\infty (a_D\beta)^s L^{-s}\right] \tag{10.2}$$

This can be rearranged to give,

$$\pi_{H,t} = -a_D\lambda(1+\varphi)\left\{\frac{\phi_y/\phi_{yy}}{1-a_D\beta} + E\left[\sum_0^\infty (a_D\beta)^s \hat{y}_{t+s}^n\right]\right\}$$

The central bank will systematically pursue negative inflation. This increases the value of the windfall in domestic currency. The interest rate rule can be found by combining this expression at time $t+1$ with 10.1 and the IS curve,

$$\begin{aligned}
(i_t - \rho) &= E_t[\hat{y}_{t+1}] - \hat{y}_t + E_t[\pi_{H,t+1}] - (1+\varphi)E_t[\Delta\hat{y}_{t+1}^n] \\
&= \frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\pi_{H,t} + \left(1 - \frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\right) E_t[\pi_{H,t+1}] - (1+\varphi)E_t[\Delta\hat{y}_{t+1}^n] \\
&= d_1\pi_{H,t} - d_2E_t[\Delta\hat{y}_{t+1}^n] + d_3E\left[\sum_0^\infty (a_D\beta)^s \hat{y}_{t+1+s}^n\right] + d_4 \qquad (10.3)
\end{aligned}$$

where $d_1 = \lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}$, $d_2 = 1+\varphi$, $d_3 = \left(\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}} - 1\right) a_D\lambda(1+\varphi)$ and $d_4 = \left(\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}} - 1\right) a_D\lambda(1+\varphi)\frac{\phi_y/\phi_{yy}}{1-a_D\beta}$. This rule gives a unique and determinate solution when $d_1 > 1$ or $\lambda(1+\varphi)\phi_\pi > \phi_{yy}$.

10.2 Optimal Policy under Commitment

Minimising the Lagrangian in 4.7 yields the first order conditions,

$$\begin{aligned}
\phi_\pi \pi_{H,t} + l_{C,t} &= l_{C,t-1} \\
\phi_{yy}\hat{y}_t + \phi_y &= \lambda(1+\varphi) l_{C,t} \\
\beta \pi_{H,t+1} + \lambda(1+\varphi)\hat{y}_t - \lambda\gamma_G(\hat{p}_{o,t} + \hat{o}_t) + \omega\hat{\vartheta}_t &= \pi_{H,t}
\end{aligned}$$

Combining the first two first order conditions yields the following result. The first relationship follows from letting $l_{C,-1} = 0$, because the Phillips curve in period -1 is not a binding constraint on the policymaker choosing the optimal plan in period 0.

$$\begin{aligned}
\hat{y}_0 &= -\frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\pi_{H,0} - \frac{\phi_y}{\phi_{yy}} \\
\hat{y}_t &= \hat{y}_{t-1} - \frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\pi_{H,t}
\end{aligned}$$

Iteratively combining these two relationships allows us to summarise them in a single expression in terms of the deviation of domestic prices from their period -1 level, $\hat{p}_{H,t} = p_{H,t} - p_{H,-1}$,

$$\hat{y}_t = -\frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\hat{p}_{H,t} - \frac{\phi_y}{\phi_{yy}} \tag{10.4}$$

where $\phi_\pi = \frac{\epsilon}{\lambda}$, $\phi_y = \alpha(1-\chi)$ and $\phi_{yy} = \left((1+\varphi) + \frac{1-\chi}{1-\gamma_G}\right)$. To find the associated interest rate rule using the IS curve we must first find an expression for $\pi_{H,t} = \hat{p}_{H,t} - \hat{p}_{H,t-1}$. To do so we substitute the previous equation into the Phillips curve to give,

$$\begin{aligned}
\hat{p}_{H,t} - \hat{p}_{H,t-1} &= \beta E[\hat{p}_{H,t+1}] - \beta\hat{p}_{H,t} + \lambda(1+\varphi)\left(-\frac{\lambda(1+\varphi)\phi_\pi}{\phi_{yy}}\hat{p}_{H,t} - \frac{\phi_y}{\phi_{yy}}\right) \\
&\quad -\lambda(1+\varphi)\hat{y}_t^n \\
E[\hat{p}_{H,t+1}] &= (a\beta)^{-1}\hat{p}_{H,t} - \beta^{-1}\hat{p}_{H,t-1} + \beta^{-1}\lambda(1+\varphi)(\hat{y}_t^n + \frac{\phi_y}{\phi_{yy}}) \qquad (10.5)
\end{aligned}$$

where $a \equiv (1+\beta+\frac{\lambda^2(1+\varphi)^2\phi_\pi}{\phi_{yy}})^{-1}, b \equiv \lambda\gamma_G, \kappa \equiv \lambda(1+\varphi)\frac{\phi_y}{\phi_{yy}}$. To find the stationary solution to this price process we rearrange and make use of the lag operator L, where $\hat{p}_{H,t-1} = L\hat{p}_{H,t}$,

$$\hat{p}_{H,t}E_t[L(\tfrac{1}{\beta} - \tfrac{1}{a\beta}L^{-1} + L^{-2})] = \beta^{-1}\lambda(1+\varphi)(\hat{y}_t^n + \frac{\phi_y}{\phi_{yy}})$$

We now factorise the quadratic expression $(\frac{1}{\beta} - \frac{1}{a\beta}L^{-1} + L^{-2}) = (L^{-1} - \delta_1)(L^{-1} - \delta_2)$. To do so let $z = L^{-1}$, so that $(\frac{1}{\beta} - \frac{1}{a\beta}z + z^2) = (z-\delta_1)(z-\delta_2)$, and $\delta_i = \left(1 \pm \sqrt{1-4a^2\beta}\right)/2a\beta$ for $i = 1, 2$. We assume that $\mid \delta_1 \mid < 1$ and $\mid \delta_2 \mid > 1$. Substituting this factorisation into the above expression yields,

$$\hat{p}_{H,t}(1-\delta_1 L) = E_t\left[\left(\beta^{-1}\lambda(1+\varphi)(\hat{y}_t^n + \frac{\phi_y}{\phi_{yy}})\right)(L^{-1}-\delta_2)^{-1}\right]$$

Now, we can express the term $(L^{-1}-\delta_2)^{-1}$ as an infinite geometric series where the coefficients converge to zero because $\mid \delta_2 \mid > 1$, $(L^{-1}-\delta_2)^{-1} = -\delta_2^{-1}\left(1 + L^{-1}\delta_2^{-1} + L^{-2}\delta_2^{-2} + \ldots\right)$. Substituting this into the above expression yields the following,

$$\hat{p}_{H,t}(1-\delta_1 L) = -(\beta\delta_2)^{-1}\lambda(1+\varphi)\left\{\frac{\phi_y}{\phi_{yy}}\frac{1}{(1-\delta_2^{-1})} + E_t\left[\sum_{s=0}^{\infty}(\delta_2^{-1}\hat{y}_{t+s}^n)\right]\right\}$$

Finally, multiplying the numerator and denominator of each term with $(1-\sqrt{1-4a^2\beta})$, and simplifying notation $\delta_1 = \delta = (1-\sqrt{1-4a^2\beta})/(2a\beta)$ gives the stationary solution in equation 4.8,

$$\hat{p}_{H,t} = \delta\hat{p}_{H,t-1} - \lambda(1+\varphi)\left\{\frac{\phi_y}{\phi_{yy}}\frac{\delta}{(1-\beta\delta)} + \delta E_t\left[\sum_{s=0}^{\infty}(\beta\delta)^s\hat{y}^n_{t+s}\right]\right\} \tag{10.6}$$

An interest rate rule that brings about a unique, determinate equilibrium can be found by combining 10.4, 10.6 and the IS curve,

$$\begin{aligned}
(i_t-\rho) &= E_t[\hat{y}_{t+1}] - \hat{y}_t + E_t[\pi_{H,t+1}] - (1+\varphi)E_t[\Delta\hat{y}^n_{t+1}] \\
&= (-\lambda(1+\varphi)\phi_\pi\phi_{yy}^{-1}+1)E_t[\pi_{H,t+1}] - (1+\varphi)E_t[\Delta\hat{y}^n_{t+1}] \\
&= c_1\hat{p}_{H,t} - c_2E_t[\Delta\hat{y}^n_{t+1}] + c_3E_t\left[\sum_{s=0}^{\infty}(\beta\delta)^s\hat{y}^n_{t+1+s}\right] + c_4
\end{aligned}$$

where $c_1 = (\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}-1)(1-\delta)$, $c_2 = 1+\varphi$, $c_3 = (\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}-1)\lambda(1+\varphi)\delta$ and $c_4 = (\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}-1)\lambda(1+\varphi)\frac{\phi_\pi}{\phi_{yy}}\frac{\delta}{1-\beta\delta}$. This gives a unique and determinate equilibrium when $c_1 > 0$, or $\lambda(1+\varphi)\phi_\pi > \phi_{yy}$, which is the same condition as the discretionary case.

11 Appendix: Calibration

The model is calibrated in line with Gali and Monacelli (2005, 2008), allowing us to compare our results. The calibrated parameter values are summarised in Table 3. This gives a steady state and starting point for our analysis in which government spending accounts for 30% of output, 10% of which is financed by oil income and 20% from lump sum taxes. These taxes are set at the level that maximises welfare in an economy without oil, so we can consider all oil income to be spent on higher government spending. All steady state values are given in Table 4.

We consider scenarios in which oil output increases by 20%, four quarters in the future. We also consider the effect of a 30% temporary shock to the oil price, if output remains constant.

Calibration

Parameter	Value
α	0.40
β	0.96
ρ	0.04
χ	0.20
φ	3.00
ϵ	6.00
ϱ	0.75
A, A^*	1.00
ρ_o	0.70

Table 3: Parameter values are calibrated following Gali and Monacelli (2005, 2008).

Steady State Values

Variable	Steady State	Symmetric State
N	1.00	1.00
G	0.30	0.20
T	0.20	0.20
O	0.15	0.00
P_o	1.00	1.00
Θ	1.45	1.00
S	0.69	1.00
C	0.93	0.80

Table 4: The steady state in the cases with and without oil

12 Appendix: Exchange Rate Peg

Here we derive the dynamics of the economy under a nominal exchange rate peg, $\Delta e_t = 0$. We begin by finding the implications of the nominal exchange rate peg for our key variables, $\pi_{H,t}$ and $\hat{y}_t$. To do this we follow a similar approach to Appendix 4.2.

First, taking first differences of the effective nominal exchange rate yields $\Delta s_t = \Delta e_t + \Delta p_t^* - \Delta p_{H,t} = -\pi_{H,t}$. Also, taking first differences of 2.12 and 3.5 gives $\Delta \hat{s}_t = \frac{1}{1-\gamma_G}\Delta \hat{y}_t - \frac{\gamma_G}{1-\gamma_G}\Delta \hat{g}_t - (1-\alpha)\Delta \hat{\vartheta}_t$. Combining these two expressions gives,

$$\pi_{H,t} = \tfrac{\gamma_G}{1-\gamma_G}\Delta\hat{g}_t - \tfrac{1}{1-\gamma_G}\Delta\hat{y}_t + (1-\alpha)\Delta\hat{\vartheta}_t$$

Taking a similar approach to that used in the commitment case we evaluate this expression at time zero, assuming that the economy was in the steady state at all times prior to this,

$$p_{H,0} - p_{H,0-1} = \tfrac{\gamma_G}{1-\gamma_G}\hat{g}_0 - \tfrac{1}{1-\gamma_G}\hat{y}_0 + (1-\alpha)\hat{\vartheta}_0$$

Iteratively combining these two relationships allows us to express this relationship in terms of the deviation of the domestic price level, $\hat{p}_{H,t} = p_{H,t} - p_{H,-1}$,

$$\hat{p}_{H,t} = \tfrac{\gamma_G}{1-\gamma_G}\hat{g}_t - \tfrac{1}{1-\gamma_G}\hat{y}_t + (1-\alpha)\hat{\vartheta}_t$$

If we express government spending in terms of the resource balance then, $\hat{p}_{H,t} = \gamma_G \widehat{rb}_t - \hat{y}_t + (1-\gamma_G)(1-\alpha)\hat{\vartheta}_t$ or $\hat{p}_{H,t} = -\hat{y}_t + (1+\varphi)\hat{y}_t^n + \hat{\vartheta}_t$. Using this to substitute out $\hat{y}_t$ in the Phillips curve gives,

$$\hat{p}_{H,t} - c\hat{p}_{H,t-1} - c\beta E_t[\hat{p}_{H,t+1}] \quad = \quad c\lambda(1+\varphi)(\varphi\hat{y}_t^n + \hat{\vartheta}_t)$$

where $c = [1 + \beta + \lambda(1+\varphi)]^{-1}$. We can find the closed-form stationary solution to this linear difference equation following the method described in Appendix 10,

$$\hat{p}_{H,t} \quad = \delta_c \quad \hat{p}_{H,t-1} + \lambda(1+\varphi)\left\{\varphi\delta_c E_t\left[\sum_{s=0}^{\infty}(\beta\delta_c)^s \hat{y}_{t+s}^n\right] + \frac{\delta_c}{1-\beta\delta_c}\hat{\vartheta}_t\right\}$$

where $\delta_c = \frac{1-\sqrt{1-4\beta c^2}}{2\beta c}$. We can use this to derive the output gap using the relationship defined above, and for the nominal interest rate by substituting our results into the IS curve. By uncovered interest parity the nominal interest rate will be constant. Uncovered interest parity states that, $(i_t - i_t^*) - E_t[\pi_{H,t+1}] = E_t[s_{t+1} - s_t]$. The definition of the terms of trade yields, $s_t = e_t + p_t^* - p_{H,t}$. So, $(i_t - i_t^*) = E_t[e_{t+1} - e_t] = 0$. While the nominal interest rate stays constant there will be relatively large fluctuations in the real interest rate. Note that while a currency peg is consistent with a constant nominal interest rate, it is not uniquely associated with it. A commitment to a currency peg remains crucial.

13 Appendix: Taylor Rules

Here we derive the dynamics of the economy under a variety of Taylor rules. To do this we extend the Blanchard and Kahn (1980) method to allow for anticipated changes in the government's resource balance. As in Blanchard and Kahn this uses the widely implementable eigen-decomposition of a matrix. The method is similar to that used by Wohltmann and Winkler (2009), who use a generalized Schur matrix decomposition. As we will be interested in interest rate rules that respond to the output gap, we arrange the Phillips curve and IS curve into a system of two equations in domestic inflation and the output gap,

$$\begin{aligned} E_t[\pi_{H,t+1}] &= \beta^{-1}\left(\pi_{H,t} - \lambda(1+\varphi)\tilde{y}_t\right) \\ E_t[\tilde{y}_{t+1}] &= \tilde{y}_t + (i_t - \rho) - \beta^{-1}\left(\pi_{H,t} - \lambda(1+\varphi)\tilde{y}_t\right) + \varphi E_t[\Delta\hat{y}^n_{t+1}] \end{aligned}$$

where $(i_t - \rho) = \phi_\pi \pi_{H,t} + \phi_y \tilde{y}_t + \phi_{yn} E_t[\Delta\hat{y}^n_{t+1}]$. This can be arranged in matrix notation, splitting the variables into control variables, $x_t = [\pi_{H,t}, \tilde{y}_t]'$, and state variables $w_t = \emptyset$ and $v_t = [\hat{y}^n_t, \hat{y}^n_{t+1}]'$,

$$E_t \begin{bmatrix} x_{t+1} \\ w_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ w_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v_t$$

We decompose A into its eigenvectors, V, and eigenvalues, Λ, such that $AV = V\Lambda$. The matrix Λ is diagonal with the eigenvalues arranged in descending order along the diagonal. Replacing $A = V\Lambda V^{-1}$ and pre-multiplying by V^{-1}we get,

$$V^{-1}E_t \begin{bmatrix} x_{t+1} \\ w_{t+1} \end{bmatrix} = \Lambda V^{-1} \begin{bmatrix} x_t \\ w_t \end{bmatrix} + V^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} v_t$$

Let $V^{-1} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$. For this work, where the state variables evolve independently of the controls, $Y_{21} = 0$. The matrix Λ_1 contains unstable eigenvalues (> 1) which are equal in number to the control variables, and Λ_2 contains stable eigenvalues (< 1) which are equal in number to the state variables, as imposed by Blanchard and Kahn (1980). Using $\begin{bmatrix} \tilde{x}_t \\ \tilde{w}_t \end{bmatrix} = V^{-1} \begin{bmatrix} x_t \\ w_t \end{bmatrix}$ we can describe the system in independent equations,

$$E_t \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{w}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{w}_t \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} v_t$$

where $\tilde{B}_1 = Y_{11}B$ and $\tilde{B}_2 = Y_{21}B$. First taking $E_t[\tilde{x}_{t+1}] = \Lambda_1\tilde{x}_t + \tilde{B}_1 v_t$. This can be expressed as, $(E_t[L^{-1}] - \Lambda_1)\tilde{x}_t = \tilde{B}_1 v_t$ where L^{-1} is a scalar inverse lag operator. Rearranging gives $\tilde{x}_t = (E_t[L^{-1}] - \Lambda_1)^{-1}\tilde{B}_1 v_t = -\Lambda_1^{-1}(I - E_t[L^{-1}]\Lambda_1^{-1})^{-1}\tilde{B}_1 v_t$. We can only accept stable paths for the control variables. As all the elements of $\Lambda_1$are greater than one, the eigenvalues of Λ_1^{-1} will be less than one and the matrix geometric series, $(I - E_t[L^{-1}]\Lambda_1^{-1})^{-1} = \sum_{s=0}^{\infty}(E_t[L^{-1}]\Lambda_1^{-1})^s$ will converge. Thus,

$$\begin{aligned} \tilde{x}_t &= -\Lambda_1^{-1}\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+s}] \\ x_t &= -(Y_{11})^{-1}Y_{12}w_t - (Y_{11})^{-1}\Lambda_1^{-1}\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+s}] \qquad (13.1) \end{aligned}$$

Turning now to the path of the state variables, $E_t[\tilde{w}_{t+1}] = \Lambda_2\tilde{w}_t + \tilde{B}_2 v_t$. We have $\tilde{w}_t = Y_{21}x_t + Y_{22}w_t = Kw_t - J\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+s}]$, where $K = -Y_{21}(Y_{11})^{-1}Y_{12} + Y_{22}$ and $J = Y_{21}(Y_{11})^{-1}\Lambda_1^{-1}$. Therefore, the path of the state variables can be described as,

$$
\begin{aligned}
E_t[\tilde{w}_{t+1}] &= \Lambda_2\tilde{w}_t + \tilde{B}_2 v_t \\
E_t[Kw_{t+1} - J\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+1+s}]] &= \Lambda_2(Kw_t - J\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+s}]) + \tilde{B}_2 v_t \\
E_t[w_{t+1}] &= K^{-1}\Lambda_2 K w_t - K^{-1}\Lambda_2 J\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+s}] \\
&\quad + K^{-1}J\sum_{s=0}^{\infty}(\Lambda_1^{-1})^s\tilde{B}_1 E_t[v_{t+1+s}] \\
&\quad + K^{-1}\tilde{B}_2 v_t \qquad (13.2)
\end{aligned}
$$

The paths for the control and state variables described by 13.1 and 13.2 respectively are illustrated in the plots in Section 5.

14 Appendix: Robustness

14.1 Approximating relative household wealth

In log-linearising the goods market equilibrium (equation 3.4) we approximate relative household wealth. Specifically, we let $-\alpha\hat{\vartheta}_t \approx \ln(1-\alpha+\alpha\Theta_t^{-1}) - \ln(1-\alpha+\alpha\Theta_0^{-1})$. This approximation makes the analysis far more tractable and permits a closed form solution, without materially affecting our results.

To test how accurate this is we compare three different approximations, each implemented in the numerical package Dynare. The first uses a Newton-Raphson algorithm applied directly to the model in levels ("*NR*"). This is our benchmark. The second takes a first-order log-linearisation of the model around the initial steady state, but keeps track of both $\hat{\theta}_t = \ln(1-\alpha+\alpha\Theta_t^{-1}) - \ln(1-\alpha+\alpha\Theta_0^{-1})$ and $\hat{\vartheta}_t = \ln(\Theta_t) - \ln(\Theta_0)$ ("$\theta+\vartheta$"). The third

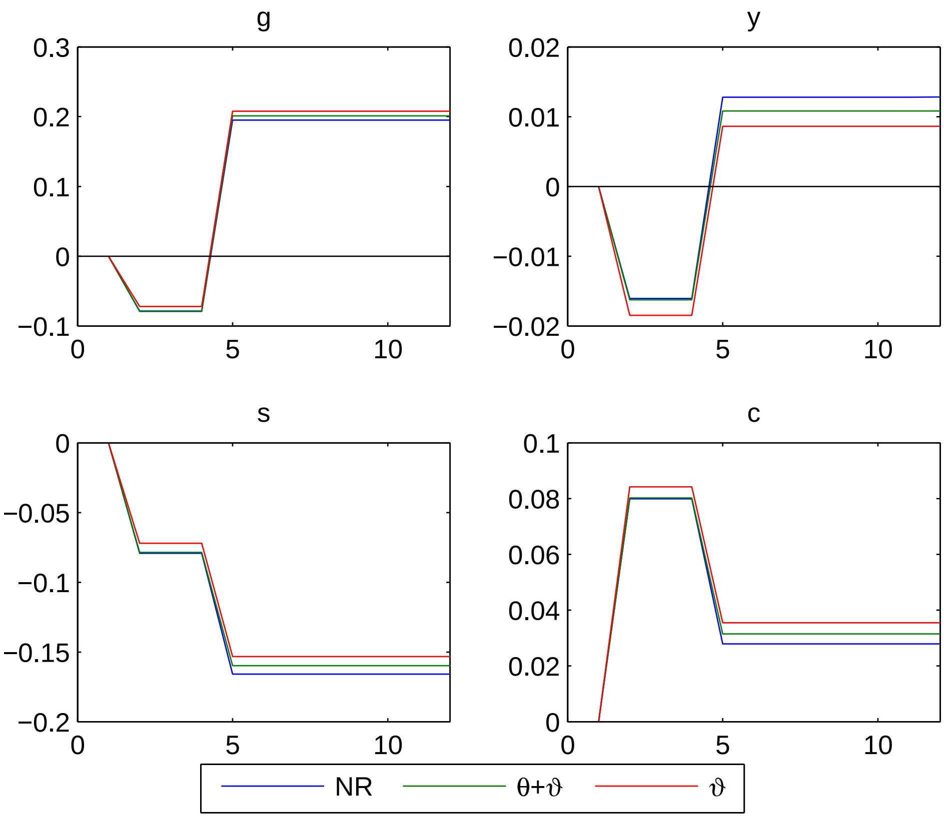

Figure 14.1: Robustness check on simplifying assumptions. The spend all rule is calculated under flexible prices in three cases: numerically using the Newton Raphson method (NR), log-linearised making a distinction between θ and ϑ ($\theta + \vartheta$), and log-linearised using the approximation, $\theta \approx -\alpha\vartheta$ (ϑ). The ϑ (red) case is reported in the main body of the paper.

again takes a first-order log-linearisation of the model, but approximates $-\alpha\hat{\vartheta}_t \approx \hat{\theta}_t$ as reported in the main text ("ϑ"). Prices are assumed to be flexible in each. As illustrated in Figure 14.1, these approximations make little material difference to our analysis.

To check the accuracy of our solution method we can also compare the results from our closed-form solution to those calculated in Dynare. We find that the third case ("ϑ") perfectly replicates the results calculated by hand and reported in the main text.

14.2 Lump sum taxes

In log-linearising the government spending rule, $\hat{g}_t = \hat{s}_t - \hat{p}_t^* + \widehat{rb}_t - \hat{t}_t$, we make a simplifying assumption regarding lump sum taxes. We define $t_t = -\ln(1 - \frac{T_t}{G_t})$, so that the log-linearized government spending rule is exact. We also assume that $\hat{t}_t = 0$, so the ratio of lump sum taxes to government spending remains constant. Alternatively, we could have set the level of taxes to be constant. As illustrated in the simulations in Figure 14.2, this assumption changes the magnitude of the government spending shock but the qualitative results remain the same.

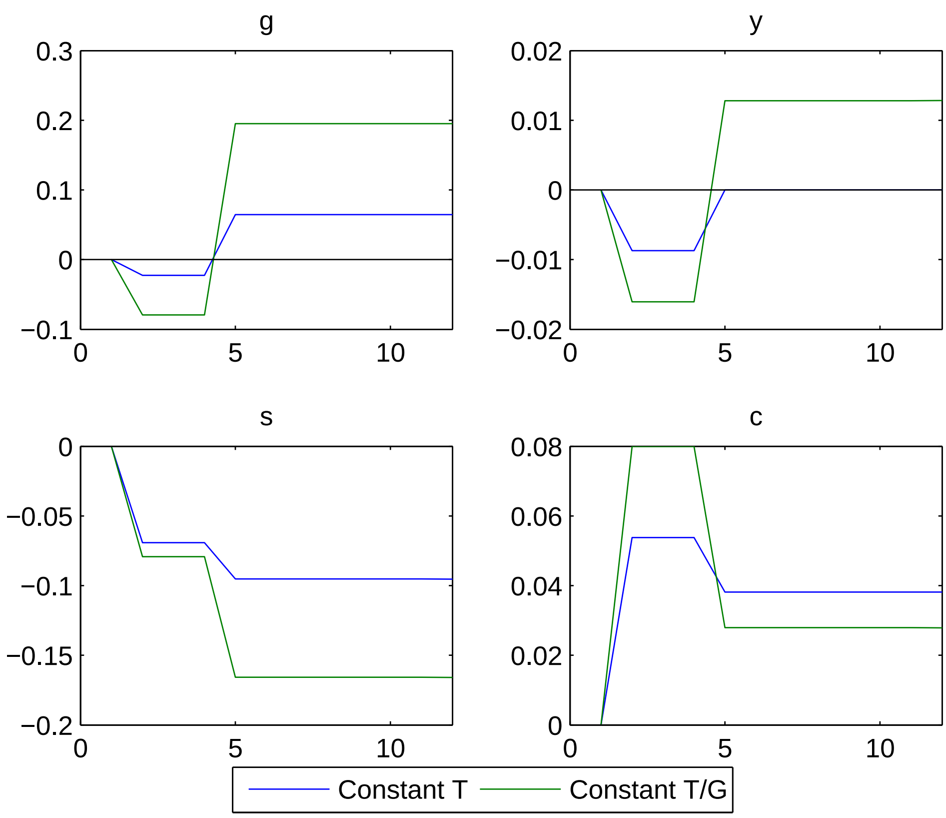

Figure 14.2: Comparing different tax assumptions: constant T and constant T/G.