

WELL-POSEDNESS AND TAMED SCHEMES FOR MCKEAN-VLASOV EQUATIONS WITH COMMON NOISE

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In this paper, we first establish well-posedness of McKean–Vlasov stochastic differential equations (McKean–Vlasov SDEs) with common noise, possibly with coefficients of super-linear growth in the state variable. Second, we present stable time-stepping schemes for this class of McKean–Vlasov SDEs. Specifically, we propose an explicit tamed Euler and tamed Milstein scheme for an interacting particle system associated with the McKean–Vlasov equation. We prove stability and strong convergence of order $1/2$ and 1 , respectively. To obtain our main results, we employ techniques from calculus on the Wasserstein space. The proof for the strong convergence of the tamed Milstein scheme only requires the coefficients to be once continuously differentiable in the state and measure component. To demonstrate our theoretical findings, we present several numerical examples, including mean-field versions of the stochastic $3/2$ volatility model and the stochastic double well dynamics with multiplicative noise.

1. Introduction. A McKean–Vlasov equation for an \mathbb{R}^d valued process X (introduced by H. McKean [30]) is an SDE where the drift and diffusion coefficients depend on the state of the process and, additionally, on the marginal laws of X , that is,

$$(1) \quad X_t = X_0 + \int_0^t b_s(X_s, \mathcal{L}_{X_s}) ds + \int_0^t \sigma_s(X_s, \mathcal{L}_{X_s}) dW_s$$

almost surely, for any $t \in [0, T]$, with a given $T > 0$, where W is an m -dimensional Wiener process, X_0 is an \mathbb{R}^d -valued random variable and $(\mathcal{L}_{X_t})_{0 \leq t \leq T}$ denotes the flow of deterministic marginal distributions of X . The theory concerning existence and uniqueness results for strong solutions of McKean–Vlasov SDEs with linearly growing coefficients satisfying Lipschitz-type conditions (in the state and measure component) is well established (see, e.g., [39]). More general results related to the existence and uniqueness for weak and strong solutions of McKean–Vlasov SDEs can be found in [6, 22, 31, 32] and references cited therein. For super-linearly growing drift, and diffusion with linear growth, it is known that such a McKean–Vlasov SDE admits a unique strong solution, provided the drift term satisfies a one-sided Lipschitz condition (see [16]).

The main interest of the present article are McKean–Vlasov equations with common noise W^0 , an m^0 -dimensional Wiener process, that is, SDEs with a solution $(X_t)_{0 \leq t \leq T}$ that satisfies

$$(2) \quad X_t = X_0 + \int_0^t b_s(X_s, \mathcal{L}^1(X_s)) ds + \int_0^t \sigma_s(X_s, \mathcal{L}^1(X_s)) dW_s + \int_0^t \sigma_s^0(X_s, \mathcal{L}^1(X_s)) dW_s^0$$

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almost surely, where $(\mathcal{L}^1(X_t))_{0 \leq t \leq T}$ denotes the flow of marginal conditional distributions of X given the common source of noise (see below for a rigorous probabilistic set-up for this equation). Compared to McKean–Vlasov SDEs without common noise (1), the marginal laws are not deterministic anymore. Intuitively speaking, the classical notion of propagation of chaos refers to the fact that in a large network of N interacting particles, these particles become asymptotically independent as $N \rightarrow \infty$. If the particles have a common source of randomness, we do not expect them to become asymptotically independent in the limit; however, conditioned on the information generated by the common noise, they are asymptotically independent. To put this intuition in other words, the empirical distribution of the particles is expected to converge towards the common conditional distribution of each particle given W^0 . For more details on this topic, we refer to [11, 12].

The existence and uniqueness of strong solutions for McKean–Vlasov SDEs with common noise under Lipschitz-type conditions is well studied (see [12]). Here, we go beyond the classical Lipschitz framework and allow all coefficients of (2) to be only locally Lipschitz continuous, as long as a certain coercivity assumption is satisfied, while we still require Lipschitz dependence on the measure in the Wasserstein metric. For specific classes of McKean–Vlasov SDEs where well-posedness can be shown without Lipschitz measure dependence, but using other stabilising properties, see [1]. An existence and uniqueness result for weak solutions to McKean–Vlasov SDEs with common noise can be found in, for example, [21].

The main difficulty to prove well-posedness of (2) under our set of assumptions is to identify an appropriate space on which one can define a contraction map, which then allows a fixed-point argument. Note that the space considered in [12] seems not to be applicable in our model set-up, as it appears to be difficult to derive a contraction under the assumption of super-linearly growing coefficients.

Our results provide extensions to known well-posedness results for McKean–Vlasov equations even without common noise (see, e.g., [11, 16, 31, 42]) by requiring only a Khasminskii-type monotonicity condition instead of imposing separately a one-sided and two-sided Lipschitz condition on the drift and diffusion coefficient, respectively. Such a condition is classical for the well-posedness of standard SDEs (see, e.g., [29], Chapter 5) and allows us to treat cases with super-linear diffusion coefficients. In [31], the authors prove an existence and uniqueness result for a specific McKean–Vlasov SDE from neuroscience with jump-diffusion noise and coefficients of super-linear growth, allowing additionally for path-dependent delay, but only considering a specific interaction term, that is, the measure dependence of the coefficients is given by the expectation of some regular enough function of the state and a path segment. The precise assumptions for our well-posedness result are given in Section 2.

The contributions of the present paper are the following:

- We study the well-posedness of (2) with common noise and under milder assumptions than those existing in the literature, in particular allowing coefficients which are not globally Lipschitz in the state and have a general (albeit Lipschitz) measure-dependence (see Theorem 2.1).
- We construct a novel tamed Euler and a tamed Milstein scheme to simulate the interacting particle system associated with (2) and prove strong convergence of order 1/2 and 1, respectively (see Theorems 3.5 and 4.9, respectively). We do not rely on an application of Itô's formula to derive our results and only require all coefficients to be once continuously differentiable in the state and measure components.

McKean–Vlasov equations with common noise arise as limiting equations of, for example, N -player games, where each individual player is exposed to an idiosyncratic noise and random shocks common to all the players [12]. Recently, McKean–Vlasov equations with common noise have received significant attention in the optimal control literature; see, for

example, [34], where an optimal control problem for a linear conditional McKean–Vlasov equation with quadratic cost functional was studied. An application of McKean–Vlasov SDEs with common noise in systemic risk was considered in [27]. As pointed out by the authors, their model could serve as a mean-field model for the interplay between common exposures and contagion in large financial systems. Further literature which motivates the consideration of a common noise source is concerned with the mathematical analysis of a large interacting network of neurons, where McKean–Vlasov equations are employed to describe the voltage level of a typical neuron in a network (see, e.g., [40]).

The simulation of McKean–Vlasov SDEs with common noise will involve two steps: At each time t , the unknown measure $\mathcal{L}_{X_t}^1$ is approximated by an empirical measure

$$\mu_t^{X,N}(\cdot) := \frac{1}{N} \sum_{j=1}^N \delta_{X_t^{j,N}}(\cdot),$$

where δ_x is the Dirac measure at point x and $(X^{i,N})_{1 \leq i \leq N}$ (so-called interacting particles) solves the \mathbb{R}^{dN} -dimensional SDE

$$X_t^{i,N} = X_0^i + \int_0^t b_s(X_s^{i,N}, \mu_s^{X,N}) ds + \int_0^t \sigma_s(X_s^{i,N}, \mu_s^{X,N}) dW_s^i + \int_0^t \sigma_s^0(X_s^{i,N}, \mu_s^{X,N}) dW_s^0.$$

Here, W^i , W^0 and X_0^i , $i \in \{1, \dots, N\}$, are independent Wiener processes (also independent of W) and i.i.d. random initial values with $\text{Law}(X_0^i) = \text{Law}(X_0)$, respectively. In a second step, one needs to introduce a stable time-stepping method to approximate the particle system $(X^{i,N})_{1 \leq i \leq N}$ over some finite time horizon $[0, T]$.

One of the earliest works on the numerical approximation of McKean–Vlasov equations is [8], which establishes convergence of an Euler–Maruyama particle scheme with strong order 1/2 in both the time-step and the number of particles, assuming the state and measure dependence of all coefficients is Lipschitz continuous. The study of stable time-stepping schemes for interacting particle systems (without common noise) with a drift that grows super-linearly in the state component and with globally Lipschitz continuous diffusion term was initiated in [15]. In this reference, a tamed Euler scheme in the spirit of [23, 36] was proposed and strong order 1/2 convergence in the time-step is shown. This and the following works all require both coefficients to be globally Lipschitz continuous in the measure component.

In terms of model generalisations, [33] extends the results from the above settings to include additional infinite activity Lévy noise and to allow extra randomness of the coefficients, which encompasses delays and Markovian switching as special cases. In terms of further development of the schemes, an adaptive Euler–Maruyama scheme is proposed and analysed in [35], while an implicit split-step, explicit Euler-type method, which facilitates efficient parallel implementation, is given in [13]. Strong order 1/2 is again shown in both these works.

The tamed Euler–Maruyama schemes were independently extended to higher-order numerical schemes in [5] and [24] for super-linear drift, and including point delays in [4].

This paper further relaxes the assumptions imposed on the diffusion, by also allowing this term to grow super-linearly in the state component as described earlier, and similar to the works for classical SDEs (see [26, 37]). The stability and strong order 1/2 convergence of an adaptive Euler–Maruyama scheme is shown in this setting (but without common noise) in [35]. One motivation for this extension is that it allows us to consider mean-field versions of popular models appearing in mathematical finance, such as the 3/2-model used for pricing VIX options and modelling certain stochastic volatility processes (see [18]). This model will be discussed in more detail in Section 5.

Finally, the results in [3] and [14] (both without common noise) differ crucially from this work in that linear asymptotic growth in the state is imposed, but locally only Hölder conti-

nuity and a certain modulus of continuity, respectively, are required, while [28] allows for a discontinuous drift coefficient under further technical assumptions. This local breakdown of regularity of the coefficients has a substantially different effect on the accuracy and stability of the schemes than super-linear growth.

To our knowledge, this is the first paper on a Milstein-type scheme for McKean–Vlasov SDEs with common noise, even for linearly growing coefficients. To recover the strong convergence of order 1, a term involving the Lions derivative of the diffusion term will appear. These additional terms are of theoretical interest and highlight the inherent difference of McKean–Vlasov SDEs to classical SDEs regarding higher order time-stepping schemes. This notion of differentiability for functions on $\mathcal{P}_2(\mathbb{R}^d)$ was introduced by P.-L. Lions; see [10] and the below subsection for a short introduction to this notion.

The remainder of this article is organised as follows: In Section 1, we prove existence and uniqueness of strong solutions for (2), merely requiring a coercivity and monotonicity condition on the coefficients. To complete this section, we give a quantitative strong conditional propagation of chaos result (see, e.g., [11, 12, 39]). Section 2 and Section 3 are devoted to introducing a tamed Euler and Milstein scheme for the interacting particle system associated with (2), respectively. In both sections, we demonstrate stability of the schemes and prove the expected strong convergence rates of order 1/2 and 1, respectively. In Section 5, we present several numerical examples to support our theoretical findings.

In each section, we give a list all model assumptions that are needed to derive the main results of the respective section.

In the following subsections, we present several notions and auxiliary results, which will be needed throughout this article. Also, we describe in detail the probabilistic framework we work in to analyse the McKean–Vlasov SDE with common noise (2).

Notation. The Euclidean norm of a d -dimensional vector and the Hilbert–Schmidt norm of a $d \times m$ -matrix are both denoted by $|\cdot|$. For the inner product of two vectors $x, y \in \mathbb{R}^d$, we write xy . Also, Ax stands for matrix product of $A \in \mathbb{R}^{d \times m}$ and $x \in \mathbb{R}^m$. The transpose of a matrix A will be denoted by A^* . For a vector $x \in \mathbb{R}^d$, $x^{(l)}$ denotes its l th element. For a matrix $A \in \mathbb{R}^{d \times m}$, $A^{(u,v)}$ and $A^{(l)}$ denote its (u, v) th element and l th column, respectively. The floor function is denoted by $\lfloor \cdot \rfloor$. The gradient of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is denoted by $\partial_x f$. Further, we abbreviate by $\mathcal{P}(\mathbb{R}^d)$ the set of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -field over \mathbb{R}^d , and define the set of probability measures having finite second moment by

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.$$

Also, for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the L^2 -Wasserstein distance is defined by

$$\mathcal{W}_2(\mu, \nu) := \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{1/2},$$

where $\Pi(\mu, \nu)$ is the set of couplings of μ and ν . Clearly, $\mathcal{P}_2(\mathbb{R}^d)$ is a Polish space under the L^2 -Wasserstein metric. Throughout this article, $K > 0$ is a generic constant that might change its value from line to line.

Probabilistic framework. In the sequel, we introduce the precise probabilistic set-up for (2) and follow closely the description presented in [12]. To distinguish between the two underlying sources of randomness, we introduce the complete probability spaces, $(\Omega^0, \mathcal{F}^0, P^0)$ and $(\Omega^1, \mathcal{F}^1, P^1)$, equipped with the filtrations $\mathbb{F}^0 := (\mathcal{F}_t^0)_{t \geq 0}$ and $\mathbb{F}^1 := (\mathcal{F}_t^1)_{t \geq 0}$, satisfying the usual conditions. Here, the Wiener process W^0 will be supported on $(\Omega^0, \mathcal{F}^0, P^0)$ and

W (and the copies thereof used to define the interacting particle system) will be supported on $(\Omega^1, \mathcal{F}^1, P^1)$. Consequently this allows to define a product space (Ω, \mathcal{F}, P) , where $\Omega = \Omega^0 \times \Omega^1$, (\mathcal{F}, P) is the completion of $(\mathcal{F}^0 \otimes \mathcal{F}^1, P^0 \otimes P^1)$ and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the complete and right-continuous augmentation of $(\mathcal{F}_t^0 \otimes \mathcal{F}_t^1)_{t \geq 0}$. It is known from Lemma 2.4 in [12] that for a given random variable $X : \Omega \rightarrow \mathbb{R}^d$, the mapping $\mathcal{L}^1(X) : \Omega^0 \ni \omega^0 \mapsto \mathcal{L}(X(\omega^0, \cdot))$, is P^0 -almost surely well defined and is a random variable from $(\Omega^0, \mathcal{F}^0, P^0)$ into $\mathcal{P}_2(\mathbb{R}^d)$, which can also be seen as a conditional law of X given \mathcal{F}^0 .

As pointed out in [12], it is a priori not guaranteed that $(\mathcal{L}^1(X_t))_{0 \leq t \leq T}$ is adapted to \mathbb{F}^0 . However, if the \mathbb{F} -adapted unique solution $(X_t)_{0 \leq t \leq T}$ of (2) has continuous paths and has uniformly bounded second moment, then one can find a version of $\mathcal{L}^1(X_t)$, for every $t \geq 0$, such that $(\mathcal{L}^1(X_t))_{t \geq 0}$ has continuous paths and is \mathbb{F}^0 -adapted; see Lemma 2.5 in [12].

Further, we remark that the initial value X_0 of (2) is assumed to be defined on $(\Omega^1, \mathcal{F}_0^1, P^1)$, which means that only W^0 plays the role of the common noise. In light of Proposition 2.9 in [12], $\mathcal{L}^1(X_t)$ is a version of the conditional law of X_t given W^0 . For alternative choices of the initial data, we refer to Remark 2.10 in [12].

The coefficients b ., σ . and σ^0 appearing in (2) are measurable functions defined on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ taking values in \mathbb{R}^d , $\mathbb{R}^{d \times m}$ and $\mathbb{R}^{d \times m^0}$, respectively.

Differentiability of functions of measures. For functions of measures, there are different notions for differentiability; see, for example, [2, 41]. In this article, we use the notion of Lions' derivative. A real-valued function on $\mathcal{P}_2(\mathbb{R}^d)$ is said to be differentiable at $\nu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists an \mathbb{R}^d -valued square integrable random variable Y_0 on some atomless, Polish probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that its law satisfies $\mathcal{L}_{Y_0} := \tilde{P} \circ Y_0^{-1} = \nu_0$ and the lifting of f defined by $F(Z) := f(\mathcal{L}_Z)$ on $L^2(\tilde{\Omega}; \mathbb{R}^d)$ has Fréchet derivative $F'[Y_0]$ at $Y_0 \in L^2(\tilde{\Omega}; \mathbb{R}^d)$. The function f is in class C^1 if its lifting F is in class C^1 . Further on using the Riesz representation theorem, for the bounded linear operator $F'[Y_0] : L^2(\tilde{\Omega}; \mathbb{R}^d) \rightarrow \mathbb{R}$, there exists a unique element $DF(Y_0) \in L^2(\tilde{\Omega}; \mathbb{R}^d)$ satisfying $F'[Y_0](Z) = \tilde{E}\langle DF(Y_0), Z \rangle$ for all $Z \in L^2(\tilde{\Omega}; \mathbb{R}^d)$. Moreover, by Theorem 6.5 (structure of the gradient) in [10], there exists a measurable function $\partial_\mu f(\nu_0) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, independent of the choice of the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and the random variable Y_0 used for the lifting, such that $\int_{\mathbb{R}^d} |\partial_\mu f(\nu_0)(x)|^2 \nu_0(dx) < \infty$ and $DF(Y_0) = \partial_\mu f(\nu_0)(Y_0)$ holds. The function $\partial_\mu f(\nu_0)$ is defined as the *Lions derivative* (abbreviated by *L-derivative*) of f at $\nu_0 = \mathcal{L}_{Y_0}$ and $\partial_\mu f : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is given by $\partial_\mu f(\nu, z) = \partial_\mu f(\nu)(z)$ for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$. Note that $\partial_\mu f(\nu, z)$ is uniquely determined only $\nu(dz)$ -a.e.

Auxiliary lemmas. The following lemma is frequently used in this article.

LEMMA 1.1 (Lemma 3.2 in [19]). *Let $f := (f_t)_{t \geq 0}$ and $g := (g_t)_{t \geq 0}$ be nonnegative continuous \mathcal{F}_t -adapted processes satisfying, for any bounded stopping time $\tau \leq T$,*

$$Ef_\tau I_{g_0 \leq c} \leq Eg_\tau I_{g_0 \leq c},$$

where $c > 0$ and $T \in [0, \infty]$ are any constants. Then, for any bounded stopping time $\tau \leq T$,

$$E \sup_{t \leq \tau} f_t^\gamma I_{g_0 \leq c} \leq \frac{2 - \gamma}{1 - \gamma} E g_\tau^\gamma I_{g_0 \leq c},$$

for any $\gamma \in (0, 1)$.

The above lemma helps us in obtaining a uniform second moment bound of certain SDEs when the diffusion coefficient grows super-linearly (see, e.g., the proof of Theorem 2.1) by

imposing that the initial value of these SDEs is in $L^{2+\epsilon}$ for some $\epsilon > 0$. However, these considerations and additional regularity on the initial value is not required in case the diffusion coefficient grows only linearly. In addition, it allows us to obtain a uniform rate of strong convergence, in Section 3 (see Theorem 3.5) and Section 4 (see Theorem 4.9).

The following lemma is used in the proof of Theorem 2.1.

LEMMA 1.2 (Lemma A.5 in [38]). *Let (X, d) be a complete metric space and let (Ω, \mathcal{F}, P) be a probability space. Let $p \geq 1$. Let $\|\cdot\|_p$ denote the norm in $L^p(\Omega; \mathbb{R})$. Let $\mu_0 : \Omega \rightarrow X$ be a random variable. Let*

$$S := \{\mu : \Omega \rightarrow X \text{ r.v.} : \|d(\mu, \mu_0)\|_p < \infty\}.$$

Let $\rho(\mu, \mu') := \|d(\mu, \mu')\|_p$. Then (S, ρ) is a complete metric space.

We reproduce for the reader's convenience the following lemma from [24], which is used later.

LEMMA 1.3 (Lemma 7 of [24]). *Let $f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be a function such that its derivative $\partial_x f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ and measure derivative $\partial_\mu f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfy a polynomial Lipschitz condition, that is, there exist constants $L > 0$ and $\chi \geq 0$ such that*

$$|\partial_x f(x, \mu) - \partial_x f(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^\chi |x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

$$|\partial_\mu f(x, \mu, y) - \partial_\mu f(\bar{x}, \bar{\mu}, \bar{y})| \leq L\{(1 + |x| + |\bar{x}|)^{\chi+1} |x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}|\},$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. Then,

$$\begin{aligned} & \left| f\left(x^i, \frac{1}{N} \sum_{j=1}^N \delta_{x^j}\right) - f\left(\bar{x}^i, \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}^j}\right) - \partial_x f\left(\bar{x}^i, \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}^j}\right)(x^i - \bar{x}^i) \right. \\ & \quad \left. - \frac{1}{N} \sum_{j=1}^N \partial_\mu f\left(\bar{x}^i, \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}^j}, \bar{x}^j\right)(x^j - \bar{x}^j) \right| \\ & \leq K(1 + |x^i| + |\bar{x}^i|)^\chi |x^i - \bar{x}^i|^2 + K \frac{1}{N} \sum_{j=1}^N |x^j - \bar{x}^j|^2, \end{aligned}$$

for every $i \in \{1, \dots, N\}$, where the constant $K > 0$ does not depend on $N \in \mathbb{N}$.

2. Existence, uniqueness, moment bound and propagation of chaos. In this section, the existence and uniqueness of the solution of (2) is proved under more relaxed assumptions than those existing in the literature (see, e.g. [12]). Indeed, this is the first result on existence and uniqueness of (2) where the coefficients are allowed to grow super-linearly. As a by-product, when $\sigma^0 = 0$ (without common noise), our result is an extension of [16] in the sense that we also allow a super-linear diffusion coefficient for equation (1) by assuming a slightly more regular initial value, that is, $X_0 \in L^{2+\epsilon}$ for any $\epsilon > 0$. Further, moment boundedness of the solution is established.

The following assumptions are made in this section.

ASSUMPTION 1. $E|X_0|^{p_0} < \infty$ for a fixed constant $p_0 > 2$.

ASSUMPTION 2. There exists a constant $L > 0$ such that

$$2xb_t(x, \mu) + (p_0 - 1)|\sigma_t(x, \mu)|^2 + (p_0 - 1)|\sigma_t^0(x, \mu)|^2 \leq L\{(1 + |x|)^2 + \mathcal{W}_2^2(\mu, \delta_0)\},$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 3. There exists a constant $L > 0$ such that

$$\begin{aligned} & 2(x - \bar{x})(b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})) + |\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})|^2 + |\sigma_t^0(x, \mu) - \sigma_t^0(\bar{x}, \bar{\mu})|^2 \\ & \leq L\{|x - \bar{x}|^2 + \mathcal{W}_2^2(\mu, \bar{\mu})\}, \end{aligned}$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 4. For every $t \in [0, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $b_t(x, \mu)$ is a continuous function of $x \in \mathbb{R}^d$ and for every $R > 0$ there exists $N_R \geq 0$ such that $\sup_{|x| \leq R} |b_t(x, \delta_0)| \leq N_R$ for any $t \in [0, T]$.

The main result of this section is given in the following theorem.

THEOREM 2.1 (Existence, uniqueness and moment bound). *Let Assumptions 1, 2, 3 and 4 be satisfied. Then, there exists a unique strong solution of (2) and the following holds:*

$$\sup_{0 \leq t \leq T} E|X_t|^{p_0} \leq K,$$

where $K := K(L, E|X_0|^{p_0}, d, m, m_0) > 0$ is a constant. Moreover,

$$E \sup_{0 \leq t \leq T} |X_t|^q \leq K,$$

for any $q < p_0$.

PROOF. Consider the space

$$\chi := \left\{ \nu : \Omega^0 \rightarrow P_2(\mathbb{R}^d) \text{ r.v.} : E^0 \int_{\mathbb{R}^d} |x|^2 \nu(dx) < \infty \right\}.$$

It follows from Lemma 1.2 that (χ, d) is a complete metric space with metric d is given by

$$d(\nu_1, \nu_2) := (E^0 \mathcal{W}_2^2(\nu_1, \nu_2))^{1/2},$$

and hence the space $C([0, T]; \chi)$ is also a complete metric space. Define an operator

$$\Phi : C([0, T]; \chi) \rightarrow C([0, T]; \chi)$$

by

$$\Phi(\mu) = (\mathcal{L}^1(Y_t^\mu))_{0 \leq t \leq T},$$

where $(Y_t^\mu)_{0 \leq t \leq T}$ is the unique solution of the following stochastic differential equation with random coefficients:

$$Y_t^\mu = X_0 + \int_0^t b_s(Y_s^\mu, \mu_s) ds + \int_0^t \sigma_s(Y_s^\mu, \mu_s) dW_s + \int_0^t \sigma_s^0(Y_s^\mu, \mu_s) dW_s^0,$$

almost surely for any $t \in [0, T]$. From [20], the above SDE has a unique continuous strong solution. Further, it is known that under the assumptions of this article (see, e.g., [25]),

$$(3) \quad \sup_{t \in [0, T]} E|Y_t^\mu|^{p_0} \leq K \quad \text{and} \quad E \sup_{t \in [0, T]} |Y_t^\mu|^p \leq K,$$

for any $p < p_0$ where $K := K(L, E|X_0|^{p_0}, d, m, m_0) > 0$.

Notice that the map Φ is well defined. Indeed, for any $\mu \in C([0, T]; \chi)$,

$$\begin{aligned} |\Phi(\mu)|_{C([0, T]; \chi)} &= \sup_{0 \leq t \leq T} E^0 \int_{\mathbb{R}^d} |x|^2 \mathcal{L}^1(Y_t^\mu)(dx) = \sup_{0 \leq t \leq T} E^0 E^1(|Y_t^\mu|^2 | \mathcal{F}^{W^0}) \\ &= \sup_{0 \leq t \leq T} E|Y_t^\mu|^2 < \infty, \end{aligned}$$

which implies that $\Phi(\mu) \in C([0, T]; \chi)$. In view of Lemma 2.5 in [12] and equation (3), the flow $(\mathcal{L}^1(Y_t^\mu))_{t \geq 0}$ has P_0 -almost surely continuous paths in $\mathcal{P}_2(\mathbb{R}^d)$ and is \mathbb{F}^0 -adapted.

Following a standard stopping time argument (see, e.g., [29], Chapter 2, in particular Lemma 3.2), let $\tau_R := \inf\{t \geq 0 : |Y_t^\mu| > R \text{ or } |Y_t^\nu| > R\}$ for any $R > 0$. By Itô's formula applied to the stopped process,

$$\begin{aligned} |Y_{t \wedge \tau_R}^\mu - Y_{t \wedge \tau_R}^\nu|^2 &= 2 \int_0^{t \wedge \tau_R} (Y_s^\mu - Y_s^\nu)(b_s(Y_s^\mu, \mu_s) - b_s(Y_s^\nu, \nu_s)) ds \\ &\quad + 2 \int_0^{t \wedge \tau_R} (Y_s^\mu - Y_s^\nu)(\sigma_s(Y_s^\mu, \mu_s) - \sigma_s(Y_s^\nu, \nu_s)) dW_s \\ &\quad + 2 \int_0^{t \wedge \tau_R} (Y_s^\mu - Y_s^\nu)(\sigma_s^0(Y_s^\mu, \mu_s) - \sigma_s^0(Y_s^\nu, \nu_s)) dW_s^0 \\ &\quad + \int_0^{t \wedge \tau_R} |\sigma_s(Y_s^\mu, \mu_s) - \sigma_s(Y_s^\nu, \nu_s)|^2 ds \\ &\quad + \int_0^{t \wedge \tau_R} |\sigma_s^0(Y_s^\mu, \mu_s) - \sigma_s^0(Y_s^\nu, \nu_s)|^2 ds, \end{aligned}$$

which on using Assumption 3 gives

$$\begin{aligned} E|Y_{t \wedge \tau_R}^\mu - Y_{t \wedge \tau_R}^\nu|^2 &= E \int_0^{t \wedge \tau_R} \{2(Y_s^\mu - Y_s^\nu)(b_s(Y_s^\mu, \mu_s) - b_s(Y_s^\nu, \nu_s)) \\ &\quad + |\sigma_s(Y_s^\mu, \mu_s) - \sigma_s(Y_s^\nu, \nu_s)|^2 + |\sigma_s^0(Y_s^\mu, \mu_s) - \sigma_s^0(Y_s^\nu, \nu_s)|^2\} ds \\ &\leq K \int_0^t E|Y_{s \wedge \tau_R}^\mu - Y_{s \wedge \tau_R}^\nu|^2 ds + K E \int_0^{t \wedge \tau_R} \mathcal{W}_2^2(\mu_s, \nu_s) ds \end{aligned}$$

for any $t \in [0, T]$. The application of Gronwall's inequality in combination with Fatou's lemma ($R \rightarrow \infty$) yields

$$\sup_{0 \leq t \leq T} E|Y_t^\mu - Y_t^\nu|^2 \leq K \int_0^T \sup_{0 \leq r \leq s} E^0 \mathcal{W}_2^2(\mu_r, \nu_r) ds,$$

for any $t \in [0, T]$, which further implies

$$\begin{aligned} |\Phi(\mu) - \Phi(\nu)|_{C([0, T]; \chi)}^2 &\leq \sup_{0 \leq t \leq T} E|Y_t^\mu - Y_t^\nu|^2 \\ &\leq K \int_0^T \sup_{0 \leq r \leq s} E^0 \mathcal{W}_2^2(\mu_r, \nu_r) ds = K \int_0^T |\mu - \nu|_{C([0, s]; \chi)}^2 ds. \end{aligned}$$

Using the above inequality, one obtains

$$\begin{aligned} |\Phi^2(\mu) - \Phi^2(\nu)|_{C([0, T]; \chi)}^2 &\leq K \int_0^T |\Phi(\mu) - \Phi(\nu)|_{C([0, t_1]; \chi)}^2 dt_1 \\ &\leq K^2 \int_0^T \int_0^{t_1} |\mu - \nu|_{C([0, t_2]; \chi)}^2 dt_2 dt_1, \end{aligned}$$

and iterating further yields

$$\begin{aligned} |\Phi^j(\mu) - \Phi^j(\nu)|_{C([0,T];\chi)}^2 &\leq K^j \int_0^T \int_0^{t_1} \cdots \int_0^{t_{j-1}} |\mu - \nu|_{C([0,t_j];\chi)}^2 dt_j \cdots dt_1 \\ &\leq K^j \int_0^T \frac{(T-t_j)^{j-1}}{(j-1)!} |\mu - \nu|_{C([0,t_j];\chi)}^2 dt_j \\ &\leq \frac{(KT)^j}{j!} |\mu - \nu|_{C([0,T];\chi)}^2. \end{aligned}$$

Since $\sum_{j=1}^{\infty} (KT)^j / j! = e^{KT} < \infty$, the mapping Φ has a unique fixed point which is the solution of (2). \square

Let us now introduce the *interacting particle system* of (2) in order to study the *propagation of chaos* property (see, e.g., [39]). The state of the particle $i \in \{1, \dots, N\}$ in the symmetric system of N SDEs coupled in a mean-field sense is given by

$$\begin{aligned} (4) \quad X_t^{i,N} &= X_0^i + \int_0^t b_s(X_s^{i,N}, \mu_s^{X,N}) ds + \int_0^t \sigma_s(X_s^{i,N}, \mu_s^{X,N}) dW_s^i \\ &\quad + \int_0^t \sigma_s^0(X_s^{i,N}, \mu_s^{X,N}) dW_s^0, \end{aligned}$$

almost surely, where for any $t \in [0, T]$

$$\mu_t^{X,N}(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}(\cdot).$$

Also, consider the following system of conditional noninteracting particles:

$$\begin{aligned} (5) \quad X_t^i &= X_0^i + \int_0^t b_s(X_s^i, \mathcal{L}^1(X_s^i)) ds + \int_0^t \sigma_s(X_s^i, \mathcal{L}^1(X_s^i)) dW_s^i \\ &\quad + \int_0^t \sigma_s^0(X_s^i, \mathcal{L}^1(X_s^i)) dW_s^0, \end{aligned}$$

almost surely for any $t \in [0, T]$ and $i \in \{1, \dots, N\}$. Moreover, by Proposition 2.11 in [12],

$$P^0[\mathcal{L}^1(X_t^i) = \mathcal{L}^1(X_t^1) \text{ for all } t \in [0, T]] = 1.$$

We remark that the applicability of Proposition 2.11 in [12] is justified in our setting. A careful inspection of the proof shows that this result relies on the existence and uniqueness of (2) and the Yamada–Watanabe theorem (Theorem 1.33 in [12]), but does not require Lipschitz continuity or linear growth of the coefficients.

The following proposition gives the propagation of chaos under the assumptions of this paper. Note that the interacting particle system introduced in (4) can be readily rewritten as an \mathbb{R}^{dN} -dimensional classical SDE (with drift and diffusion coefficient taking values in \mathbb{R}^{dN} and $\mathbb{R}^{(m+m_0)N}$, respectively). These coefficients inherit the properties in Assumptions 2, 3 and 4. Well-posedness and moment stability up to order p_0 is then guaranteed by, for example, [20].

PROPOSITION 1 (Propagation of chaos). *Let Assumptions 1, 2, 3 and 4 be satisfied with $p_0 > 4$. Then,*

$$\sup_{i \in \{1, \dots, N\}} \sup_{t \in [0, T]} E |X_t^i - X_t^{i,N}|^2 \leq K \begin{cases} N^{-1/2} & \text{if } d < 4, \\ N^{-1/2} \ln(N) & \text{if } d = 4, \\ N^{-2/d} & \text{if } d > 4, \end{cases}$$

where the constant $K > 0$ does not depend on N .

PROOF. From equations (4), (5) and Itô’s formula,

$$\begin{aligned} |X_t^i - X_t^{i,N}|^2 &= 2 \int_0^t (X_s^i - X_s^{i,N})(b_s(X_s^i, \mathcal{L}^1(X_s^i)) - b_s(X_s^{i,N}, \mu_s^{X,N})) ds \\ &\quad + 2 \int_0^t (X_s^i - X_s^{i,N})(\sigma_s(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s(X_s^{i,N}, \mu_s^{X,N})) dW_s \\ &\quad + 2 \int_0^t (X_s^i - X_s^{i,N})(\sigma_s^0(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s^0(X_s^{i,N}, \mu_s^{X,N})) dW_s^0 \\ &\quad + \int_0^t |\sigma_s(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s(X_s^{i,N}, \mu_s^{X,N})|^2 ds \\ &\quad + \int_0^t |\sigma_s^0(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s^0(X_s^{i,N}, \mu_s^{X,N})|^2 ds, \end{aligned}$$

which due to Assumption 3 yields

$$\begin{aligned} E|X_t^i - X_t^{i,N}|^2 &= E \int_0^t \{2(X_s^i - X_s^{i,N})(b_s(X_s^i, \mathcal{L}^1(X_s^i)) - b_s(X_s^{i,N}, \mu_s^{X,N})) \\ &\quad + |\sigma_s(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s(X_s^{i,N}, \mu_s^{X,N})|^2 \\ &\quad + |\sigma_s^0(X_s^i, \mathcal{L}^1(X_s^i)) - \sigma_s^0(X_s^{i,N}, \mu_s^{X,N})|^2\} ds \\ &\leq KE \int_0^t \mathcal{W}_2^2(\mathcal{L}^1(X_s^i), \mu_s^{X,N}) ds + K \int_0^t E|X_s^i - X_s^{i,N}|^2 ds \end{aligned} \tag{6}$$

for any $t \in [0, T]$. For simplicity, above and in the following sections, we do not explicitly formulate a stopping time argument to control the martingale terms. Notice that

$$\mathcal{W}_2^2(\mathcal{L}^1(X_s^i), \mu_s^{X,N}) \leq \frac{2}{N} \sum_{i=1}^N |X_s^i - X_s^{i,N}|^2 + 2\mathcal{W}_2^2\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_s^i}, \mathcal{L}^1(X_s^1)\right),$$

and then applying Theorem 5.8 and Remark 5.9 of [11], one obtains

$$E^1 \left[\mathcal{W}_2^2\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_s^i}, \mathcal{L}^1(X_s^1)\right) \right] \leq K [E^1 |X_0^1|^{p_0}]^{2/p_0} \begin{cases} N^{-1/2} & \text{if } d < 4, \\ N^{-1/2} \ln(N) & \text{if } d = 4, \\ N^{-2/d} & \text{if } d > 4, \end{cases} \tag{7}$$

P_0 -almost surely, where $K := K(d, p_0) > 0$. Taking expectation with respect to P_0 in equation (7), substituting this bound in equation (6) and applying Gronwall’s lemma completes the proof. \square

3. Tamed Euler scheme. In this section, we construct a tamed Euler scheme for the interacting particle system (4) associated with (2) when the coefficients b , σ and σ^0 are allowed to grow super-linearly. This is the first numerical scheme for interacting particles associated with equation (2), with linearly or super-linearly growing coefficients. If $\sigma^0 = 0$, that is, in the absence of common noise, a tamed Euler scheme has been studied in [15], for equations where only the drift coefficient is allowed to grow super-linearly. In contrast, our results allow the diffusion coefficient to grow super-linearly also, subject to a coercivity condition.

The proposed tamed Euler scheme is given below in equation (9). We investigate the moment bounds and the rate of convergence of our tamed Euler scheme in Lemma 3.2 and Theorem 3.5, respectively. Indeed, the rate of convergence of the scheme is shown to be equal to $1/2$, which is consistent with the classical Euler scheme for SDEs. For this purpose, we replace Assumption 3 by a slightly stronger Assumption 5 and add more assumptions on the regularity of the coefficients as stated below.

ASSUMPTION 5. For some $p_1 > 2$, there exists a constant $L > 0$ such that

$$2(x - \bar{x})(b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})) + (p_1 - 1)|\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})|^2 \\ + (p_1 - 1)|\sigma_t^0(x, \mu) - \sigma_t^0(\bar{x}, \bar{\mu})|^2 \leq L\{|x - \bar{x}|^2 + \mathcal{W}_2^2(\mu, \bar{\mu})\},$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 6. There exist constants $L > 0$ and $\rho > 0$ such that

$$|b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/2}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 7. There exists a constant $L > 0$ such that

$$|b_t(x, \mu) - b_s(x, \mu)| + |\sigma_t(x, \mu) - \sigma_s(x, \mu)| + |\sigma_t^0(x, \mu) - \sigma_s^0(x, \mu)| \leq L|t - s|^{1/2},$$

for all $t, s \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

REMARK 1. Due to Assumption 5 and 6, there exists a constant $K := K(L) > 0$ such that

$$|\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})| + |\sigma_t^0(x, \mu) - \sigma_t^0(\bar{x}, \bar{\mu})| \leq K\{(1 + |x| + |\bar{x}|)^{\rho/4}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

REMARK 2. Due to Assumptions 2 and 6, there exists a constant $K := K(L, T) > 0$ such that

$$|b_t(x, \mu)| \leq K\{(1 + |x|)^{\rho/2+1} + \mathcal{W}_2(\mu, \delta_0)\},$$

$$|\sigma_t(x, \mu)| + |\sigma_t^0(x, \mu)| \leq K\{(1 + |x|)^{\rho/4+1} + \mathcal{W}_2(\mu, \delta_0)\},$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

REMARK 3. It is easy to see that Assumption 4 follows from Assumption 6.

For introducing the tamed Euler scheme for the interacting particle system (4) associated with (2), we partition $[0, T]$ into n sub-intervals of size $h := T/n$ and define $\kappa_n(t) := \lfloor nt \rfloor / n$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. Further, for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we define

$$(8) \quad b_t^n(x, \mu) := \frac{b_t(x, \mu)}{1 + n^{-1/2}|x|^{\rho/2}}, \quad \sigma_t^n(x, \mu) := \frac{\sigma_t(x, \mu)}{1 + n^{-1/2}|x|^{\rho/2}}, \\ \sigma_t^{0,n}(x, \mu) := \frac{\sigma_t^0(x, \mu)}{1 + n^{-1/2}|x|^{\rho/2}},$$

and propose the tamed Euler scheme given by

$$(9) \quad X_t^{i,N,n} = X_0^i + \int_0^t b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds + \int_0^t \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^i \\ + \int_0^t \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^0,$$

for each $i \in \{1, \dots, N\}$, where

$$\mu_t^{X,N,n}(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N,n}}(\cdot),$$

almost surely for any $t \in [0, T]$ and $n \in \mathbb{N}$.

REMARK 4. Using equation (8) and Remark 2, one obtains

$$\begin{aligned} |b_t^n(x, \mu)| &\leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |b_t(x, \mu)|\}, \\ |\sigma_t^n(x, \mu)| &\leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\sigma_t(x, \mu)|\}, \\ |\sigma_t^{0,n}(x, \mu)| &\leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\sigma_t^0(x, \mu)|\}, \end{aligned}$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and for some constant $K > 0$ independent of n .

REMARK 5. When $\rho = 0$, then Assumption 6 and Remark 1 lead to Lipschitz continuity of b and σ in the state variable in which case no taming is needed in equation (8).

Before proving the moment bound of the tamed Euler scheme (9), we require the following lemma. The proof can be found in Appendix A.1.

LEMMA 3.1. *Let Assumptions 1, 2, 5 and 6 hold. Then,*

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \leq K n^{-p_0/4} E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0},$$

for any $i \in \{1, \dots, N\}$, $s \in [0, T]$ and $n, N \in \mathbb{N}$.

In the following lemma, we prove moment boundedness of the tamed Euler scheme (9).

LEMMA 3.2 (Moment bound). *Let Assumptions 1, 2, 5 and 6 hold. Then,*

$$\sup_{i \in \{1, \dots, N\}} \sup_{t \in [0, T]} E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \leq K,$$

where $K > 0$ does not depend on n , $N \in \mathbb{N}$. Moreover,

$$\sup_{i \in \{1, \dots, N\}} E \sup_{t \in [0, T]} (1 + |X_t^{i,N,n}|^2)^{q/2} \leq K,$$

for any $q < p_0$.

PROOF. Itô's formula gives

$$\begin{aligned} &(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\ &= (1 + |X_0^{i,N,n}|^2)^{p_0/2} \\ &\quad + p_0 \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} X_s^{i,N,n} b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\ &\quad + p_0 \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} X_s^{i,N,n} \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^i \\ &\quad + p_0 \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} X_s^{i,N,n} \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^0 \\ &\quad + \frac{p_0(p_0-2)}{2} \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-2} |\sigma_{\kappa_n(s)}^{n,*}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) X_s^{i,N,n}|^2 ds \\ &\quad + \frac{p_0(p_0-2)}{2} \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-2} |\sigma_{\kappa_n(s)}^{0,n,*}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) X_s^{i,N,n}|^2 ds \\ &\quad + \frac{p_0}{2} \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} |\sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds \\ &\quad + \frac{p_0}{2} \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} |\sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds \end{aligned}$$

almost surely for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Thus, on taking expectation and using the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned}
 & E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\
 & \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} \\
 (10) \quad & + \frac{p_0}{2} E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \{2X_{\kappa_n(s)}^{i,N,n} b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
 & + (p_0 - 1) |\sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 + (p_0 - 1) |\sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2\} ds \\
 & + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds
 \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. From equation (8), notice that the denominators of b^n , σ^n and $\sigma^{0,n}$ are the same. Hence, on using equation (9), Assumption 2 and Young's inequality, one obtains

$$\begin{aligned}
 & E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\
 & \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} \\
 & + K E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \{(1 + |X_{\kappa_n(s)}^{i,N,n}|^2) + \mathcal{W}_2^2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\} ds \\
 & + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
 & \times \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
 & + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
 & \times \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
 & + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
 & \times \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds
 \end{aligned}$$

which, on using Remark 4 and Young's inequality, gives

$$\begin{aligned}
 (11) \quad & E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
 & + K E \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds + F_1 + F_2
 \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$, where F_1, F_2 are defined below. Notice that

$$\begin{aligned}
 F_1 & := p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
 & \times \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds
 \end{aligned}$$

$$\begin{aligned}
&\leq p_0 E \int_0^t (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2-1} \\
&\quad \times \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
&\quad + p_0 E \int_0^t |(1 + |X_s^{i,N,n}|^2)^{p_0/2-1} - (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2-1}| \\
&\quad \times \left| \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \right| \\
&\leq K E \int_0^t (1 + |X_s^{i,N,n}|^2 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{(p_0-3)/2} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}| \\
&\quad \times \left| \int_{\kappa_n(s)}^s b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right| ds
\end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$, where the last inequality follows due to the following expansion, for some $\theta \in (0, 1)$:

$$\begin{aligned}
&|(1 + |x|^2)^{p_0/2-1} - (1 + |y|^2)^{p_0/2-1}| \\
(12) \quad &= |(p_0 - 2)(1 + |\theta x + (1 - \theta)y|^2)^{p_0/2-2}(\theta x + (1 - \theta)y)(x - y)| \\
&\leq (p_0 - 2)\{1 + |x|^2 + |y|^2\}^{(p_0-3)/2}|x - y|
\end{aligned}$$

for any $x, y \in \mathbb{R}^d$. Therefore, the application of Young's inequality, Burkholder–Davis–Gundy inequality and Lemma 3.1 yields

$$\begin{aligned}
F_1 &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K E \int_0^t n^{p_0/12} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0/3} \\
&\quad \times n^{-p_0/12} \left| \int_{\kappa_n(s)}^s b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^{p_0/3} ds \\
(13) \quad &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t n^{p_0/4} E |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} ds \\
&\quad + K n^{-p_0/8} \int_0^t E \left| \int_{\kappa_n(s)}^s b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^{p_0/2} ds \\
&\leq K + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K n^{-p_0/8} n^{-p_0/4+1} \\
&\quad \times \int_0^t E \int_{\kappa_n(s)}^s |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0/2} |\sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0/2} dr ds
\end{aligned}$$

and then one uses Remark 4 to obtain the following:

$$(14) \quad F_1 \leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K E \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. By adapting similar arguments as used in the estimation of F_1 , one can obtain

$$\begin{aligned}
 F_2 &:= p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
 &\quad \times \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n} (X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 b_{\kappa_n(s)}^n (X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
 &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K E \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds,
 \end{aligned}
 \tag{15}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

On substituting estimates of equations (14) and (15) in equation (11), one gets

$$\begin{aligned}
 E(1 + |X_t^{i,N,n}|^2)^{p_0/2} &\leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
 &\quad + K E \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds,
 \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. By a simple calculation (see, e.g., Lemma 2.3 in [16]), one can observe that

$$\mathcal{W}_2^2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) = \frac{1}{N} \sum_{j=1}^N |X_{\kappa_n(s)}^{i,N,n}|^2,
 \tag{16}$$

which yields

$$\begin{aligned}
 &\sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq t} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} \\
 &\leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds
 \end{aligned}$$

and the application of Gronwall's inequality completes the proof of the first inequality. The second inequality follows due to Lemma 1.1. \square

Before proceeding with the proof of rate of convergence of the tamed Euler scheme (9), we establish some lemmas. The following lemma can be proved by replicating the proof of Lemma 3.1 and then using Remark 2 and Lemma 3.2.

LEMMA 3.3. *Let Assumptions 1, 2, 5 and 6 hold. Then,*

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p \leq K n^{-p/2},$$

for any $i \in \{1, \dots, N\}$, $s \in [0, T]$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

The proof of the following lemma can be found in Appendix A.2.

LEMMA 3.4. *Let Assumptions 1, 2, 5, 6 and 7 hold. Then,*

$$\begin{aligned}
 E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p &\leq K n^{-p/2}, \\
 E|\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p &\leq K n^{-p/2}, \\
 E|\sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{n,0}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p &\leq K n^{-p/2},
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$, where the constant $K > 0$ does not depend on n and N .

The following is the main result of this section.

THEOREM 3.5 (Rate of convergence). *Let Assumptions 1, 2, 5, 6 and 7 be satisfied. Then, the tamed Euler scheme (9) converges to the true solution of the interacting particle system (4) associated with (2) in a strong sense with L^p rate of convergence $1/2$, that is,*

$$\sup_{i \in \{1, \dots, N\}} E \sup_{t \in [0, T]} |X_t^{i, N} - X_t^{i, N, n}|^p \leq K n^{-p/2},$$

for any $p < \min\{p_1, p_0/(\rho + 1)\}$, where the constant $K > 0$ does not depend on $n, N \in \mathbb{N}$.

PROOF. Let us first assume that $p < \min\{p_1, p_0/(\rho + 1)\}$. From equations (4) and (9),

$$\begin{aligned} X_t^{i, N} - X_t^{i, N, n} &= \int_0^t (b_s(X_s^{i, N}, \mu_s^{X, N}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) ds \\ &\quad + \int_0^t (\sigma_s(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) dW_s^i \\ &\quad + \int_0^t (\sigma_s^0(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^{0, n}(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) dW_s^0 \end{aligned} \quad (17)$$

almost surely for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. By Itô's formula,

$$\begin{aligned} &|X_t^{i, N} - X_t^{i, N, n}|^p \\ &= p \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} (X_s^{i, N} - X_s^{i, N, n}) \\ &\quad \times (b_s(X_s^{i, N}, \mu_s^{X, N}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) ds \\ &\quad + p \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} (X_s^{i, N} - X_s^{i, N, n}) \\ &\quad \times (\sigma_s(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) dW_s^i \\ &\quad + p \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} (X_s^{i, N} - X_s^{i, N, n}) \\ &\quad \times (\sigma_s^0(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^{0, n}(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) dW_s^0 \\ &\quad + \frac{p(p-2)}{2} \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-4} \\ &\quad \times |(\sigma_s(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n}))^* (X_s^{i, N} - X_s^{i, N, n})|^2 ds \\ &\quad + \frac{p(p-2)}{2} \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-4} \\ &\quad \times |(\sigma_s(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n}))^* (X_s^{i, N} - X_s^{i, N, n})|^2 ds \\ &\quad + \frac{p}{2} \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} |\sigma_s^0(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^{0, n}(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})|^2 ds \\ &\quad + \frac{p}{2} \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} |\sigma_s(X_s^{i, N}, \mu_s^{X, N}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})|^2 ds \end{aligned}$$

which on taking expectation along with the Cauchy–Schwarz inequality yields

$$\begin{aligned} E|X_t^{i, N} - X_t^{i, N, n}|^p &\leq \frac{p}{2} E \int_0^t |X_s^{i, N} - X_s^{i, N, n}|^{p-2} \{2(X_s^{i, N} - X_s^{i, N, n}) \\ &\quad \times (b_s(X_s^{i, N}, \mu_s^{X, N}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i, N, n}, \mu_{\kappa_n(s)}^{X, N, n})) + b_s(X_s^{i, N, n}, \mu_s^{X, N, n}) \} \end{aligned}$$

$$\begin{aligned}
& -b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
& + (p-1)|\sigma_s(X_s^{i,N}, \mu_s^{X,N}) - \sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) \\
& + \sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 \\
& + (p-1)|\sigma_s^0(X_s^{i,N}, \mu_s^{X,N}) - \sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) \\
& + \sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2\} ds
\end{aligned}$$

and then using Young's inequality gives

$$\begin{aligned}
& E|X_t^{i,N} - X_t^{i,N,n}|^p \\
& \leq \frac{p}{2} E \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \{2(X_s^{i,N} - X_s^{i,N,n}) \\
& \quad \times (b_s(X_s^{i,N}, \mu_s^{X,N}) - b_s(X_s^{i,N,n}, \mu_s^{X,N,n})) \\
& \quad + (p_1 - 1)|\sigma_s(X_s^{i,N}, \mu_s^{X,N}) - \sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n})|^2 \\
& \quad + (p_1 - 1)|\sigma_s^0(X_s^{i,N}, \mu_s^{X,N}) - \sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n})|^2\} ds \\
& \quad + KE \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) \\
& \quad \times (b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) ds \\
& \quad + KE \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^{p-2} |\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds \\
& \quad + KE \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^{p-2} |\sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds
\end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. By using the Cauchy-Schwarz inequality, Young's inequality and Assumption 5, one obtains

$$\begin{aligned}
(18) \quad E|X_t^{i,N} - X_t^{i,N,n}|^p & \leq E \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^p ds + E \int_0^t \mathcal{W}_2^p(\mu_s^{X,N}, \mu_s^{X,N,n}) ds \\
& \quad + KE \int_0^t |b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p ds \\
& \quad + KE \int_0^t |\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p ds \\
& \quad + KE \int_0^t |\sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p ds,
\end{aligned}$$

which on the application of Lemma 3.4 along with the following elementary estimate:

$$(19) \quad \mathcal{W}_2^2(\mu_s^{X,N}, \mu_s^{X,N,n}) \leq \frac{1}{N} \sum_{j=1}^N |X_s^{i,N} - X_s^{i,N,n}|^2,$$

yields

$$\sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq t} E|X_r^{i,N} - X_r^{i,N,n}|^p \leq \int_0^t \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E|X_r^{i,N} - X_r^{i,N,n}|^p ds + Kn^{-p/2},$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Application of Gronwall's inequality gives

$$\sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq T} E|X_t^{i,N} - X_t^{i,N,n}|^p \leq Kn^{-p/2},$$

for any $p < \min\{p_1, p_0/(\rho + 1)\}$, $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Then, the application of Lemma 1.1 completes the proof. \square

4. Tamed Milstein scheme. In this section, we propose a tamed Milstein scheme for the interacting particle system (4) associated with McKean–Vlasov SDE (2) when the coefficients b , σ and σ^0 are allowed to grow super-linearly. The proposed Milstein scheme is given below in equation (21). Note that we use the same notation $X^{i,N,n}$ for the tamed Milstein scheme (21) and the tamed Euler scheme (9) (discussed in Section 3), which should not cause any confusion in the reader's mind. We study the moment bounds and the rate of convergence of our scheme in Lemma 4.3 and Theorem 4.9, respectively. Indeed, the rate of convergence of our tamed Milstein scheme is shown to be equal to 1, which is consistent with the classical Milstein scheme for SDEs. For this purpose, we replace Assumption 7 by Assumption 8 and add more assumptions on the regularity of the coefficients as stated below.

ASSUMPTION 8. There exists a constant $L > 0$ such that

$$|b_t(x, \mu) - b_s(x, \mu)| + |\sigma_t(x, \mu) - \sigma_s(x, \mu)| + |\sigma_t^0(x, \mu) - \sigma_s^0(x, \mu)| \leq L|t - s|,$$

for all $t, s \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 9. There exists a constant $L > 0$ such that, for every $j \in \{1, \dots, m\}$ and $j' \in \{1, \dots, m^0\}$

$$|\partial_x b_t(x, \mu) - \partial_x b_t(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/2-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

$$|\partial_x \sigma_t^{(j)}(x, \mu) - \partial_x \sigma_t^{(j)}(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

$$|\partial_x \sigma_t^{0,(j')}(x, \mu) - \partial_x \sigma_t^{0,(j')}(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

ASSUMPTION 10. There exists a constant $L > 0$ such that, for every $k \in \{1, \dots, d\}$, $j \in \{1, \dots, m\}$ and $j' \in \{1, \dots, m^0\}$ we have versions of $b_t^{(k)}(x, \mu, \cdot)$, $\partial_\mu \sigma_t^{(k,j)}(x, \mu, \cdot)$ and $\partial_\mu \sigma_t^{0,(k,j')}(x, \mu, \cdot)$ satisfying

$$|\partial_\mu b_t^{(k)}(x, \mu, y) - \partial_\mu b_t^{(k)}(\bar{x}, \bar{\mu}, \bar{y})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/2}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}|\},$$

$$|\partial_\mu \sigma_t^{(k,j)}(x, \mu, y) - \partial_\mu \sigma_t^{(k,j)}(\bar{x}, \bar{\mu}, \bar{y})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}|\},$$

$$|\partial_\mu \sigma_t^{0,(k,j')}(x, \mu, y) - \partial_\mu \sigma_t^{0,(k,j')}(\bar{x}, \bar{\mu}, \bar{y})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4}|x - \bar{x}|^2 + \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}|\},$$

for all $t \in [0, T]$, $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

REMARK 6. As a consequence of Assumptions 6 and Remark 1, there is a constant $K := K(L) > 0$ such that, for every $k \in \{1, \dots, d\}$, $j \in \{1, \dots, m\}$ and $j' \in \{1, \dots, m^0\}$

$$|\partial_x b_t(x, \mu)| \leq K(1 + |x|)^{\rho/2},$$

$$|\partial_x \sigma_t^{(j)}(x, \mu)| + |\partial_x \sigma_t^{0,(j')}(x, \mu)| \leq K(1 + |x|)^{\rho/4},$$

$$|\partial_\mu b_t^{(k)}(x, \mu, y)| + |\partial_\mu \sigma_t^{(k,j)}(x, \mu, y)| + |\partial_\mu \sigma_t^{0,(k,j')}(x, \mu, y)| \leq K,$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

As in Section 2, for introducing the tamed Milstein scheme for the interacting particle system (4) associated with McKean–Vlasov SDE (2), we partition $[0, T]$ into n sub-intervals of size $h := T/n$ and define $\kappa_n(t) := \lfloor nt \rfloor / n$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. Further, for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we define

$$(20) \quad \begin{aligned} b_t^n(x, \mu) &:= \frac{b_t(x, \mu)}{1 + n^{-1}|x|^\rho}, & \sigma_t^n(x, \mu) &:= \frac{\sigma_t(x, \mu)}{1 + n^{-1}|x|^\rho}, \\ \sigma_t^{0,n}(x, \mu) &:= \frac{\sigma_t^0(x, \mu)}{1 + n^{-1}|x|^\rho}, \end{aligned}$$

and propose the tamed Milstein scheme given by

$$(21) \quad \begin{aligned} X_t^{i,N,n} &= X_0^i + \int_0^t b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds + \int_0^t \tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^i \\ &\quad + \int_0^t \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) dW_s^0 \end{aligned}$$

almost surely for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. The coefficients $\tilde{\sigma}^n$ and $\tilde{\sigma}^{0,n}$ are defined below. For any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$,

$$(22) \quad \tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) := \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}),$$

where $\Gamma^{n,\sigma}$ is further expressed as a sum of four matrices, that is,

$$\begin{aligned} \Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \Lambda_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}), \end{aligned}$$

where $\Lambda^{n,\sigma\sigma}$, $\Lambda^{n,\sigma\sigma^0}$, $\bar{\Lambda}^{n,\sigma\sigma}$ and $\bar{\Lambda}^{n,\sigma\sigma^0}$ are $d \times m$ -matrices defined as follows:

$$\begin{aligned} \Lambda_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i, \\ \Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0, \\ \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j, \\ \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0, \end{aligned}$$

for every $u \in \{1, \dots, d\}$ and $v \in \{1, \dots, m\}$.

Again, for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$,

$$(23) \quad \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) := \sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}),$$

where $\Gamma_{\kappa_n(s)}^{n,\sigma^0}$ is further expressed as a sum of four matrices, that is,

$$\begin{aligned} \Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) &:= \Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}), \end{aligned}$$

where $\Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma}$, $\Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma^0}$, $\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma}$ and $\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma^0}$ are $d \times m^0$ -matrices whose (u, v) th elements are given in this order by

$$\begin{aligned} &\Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad := \partial_x \sigma_{\kappa_n(s)}^{0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i, \\ &\Lambda_{\kappa_n(s)}^{n,\sigma^0\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad := \partial_x \sigma_{\kappa_n(s)}^{0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0, \\ &\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad := \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j, \\ &\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma^0\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad := \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0, \end{aligned}$$

for every $u \in \{1, \dots, d\}$ and $v \in \{1, \dots, m^0\}$.

REMARK 7. Using equation (20), Remarks 2 and 6, one obtains

$$\begin{aligned} |b_t^n(x, \mu)| &\leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |b_t(x, \mu)|\}, \\ |\sigma_t^n(x, \mu)| &\leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\sigma_t(x, \mu)|\}, \\ |\sigma_t^{0,n}(x, \mu)| &\leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\sigma_t^0(x, \mu)|\}, \\ |\partial_x \sigma_t^{(u,v)}(x, \mu)| |\sigma_t^n(x, \mu)| \\ &\leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{(u,v)}(x, \mu)| |\sigma_t(x, \mu)|\}, \\ |\partial_x \sigma_t^{(u,v)}(x, \mu)| |\sigma_t^{0,n}(x, \mu)| \\ &\leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{(u,v)}(x, \mu)| |\sigma_t^0(x, \mu)|\}, \\ |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)| |\sigma_t^n(x, \mu)| \\ &\leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)| |\sigma_t(x, \mu)|\}, \end{aligned}$$

$$\begin{aligned}
& |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)| |\sigma_t^{0,n}(x, \mu)| \\
& \leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)| |\sigma_t^0(x, \mu)|\}, \\
& |\partial_x \sigma_t^{0,(u,v)}(x, \mu)| |\sigma_t^n(x, \mu)| \\
& \leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{0,(u,v)}(x, \mu)| |\sigma_t(x, \mu)|\}, \\
& |\partial_x \sigma_t^{0,(u,v)}(x, \mu)| |\sigma_t^{0,n}(x, \mu)| \\
& \leq K \min\{n^{1/2}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{0,(u,v)}(x, \mu)| |\sigma_t^0(x, \mu)|\}, \\
& |\partial_\mu \sigma_t^{0,(u,v)}(x, \mu, y)| |\sigma_t^n(x, \mu)| \\
& \leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{0,(u,v)}(x, \mu, y)| |\sigma_t(x, \mu)|\}, \\
& |\partial_\mu \sigma_t^{0,(u,v)}(x, \mu, y)| |\sigma_t^{0,n}(x, \mu)| \\
& \leq K \min\{n^{1/4}(1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{0,(u,v)}(x, \mu, y)| |\sigma_t^0(x, \mu)|\},
\end{aligned}$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and for some constant $K > 0$ independent of n .

REMARK 8. In view of Remark 5, no taming is needed in equation (20) when $\rho = 0$.

Before presenting the result on moment boundedness of the Milstein scheme (21), we establish some lemmas and corollaries as given below. The proof of the following lemma can be found in Appendix A.3.

LEMMA 4.1. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then, for each $i \in \{1, \dots, N\}$,*

$$\begin{aligned}
E |\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} & \leq K E (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0), \\
E |\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} & \leq K E (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0),
\end{aligned}$$

for all $s \in [0, T]$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

As a consequence of the above lemma and Remark 7, one obtains the following corollary.

COROLLARY 1. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then, for each $i \in \{1, \dots, N\}$,*

$$\begin{aligned}
E |\tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} & \leq K n^{p_0/4} \{E (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\}, \\
E |\tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} & \leq K n^{p_0/4} \{E (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\},
\end{aligned}$$

for all $s \in [0, T]$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

The proof of the following lemma can be found in Appendix A.4.

LEMMA 4.2. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then,*

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \leq K n^{-p_0/4} \{E(1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\},$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ is independent of N and n .

LEMMA 4.3 (Moment bounds). *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then,*

$$\sup_{i \in \{1, \dots, N\}} \sup_{t \in [0, T]} E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \leq K,$$

for any $n, N \in \mathbb{N}$ where $K > 0$ is a constant independent of n and N . Moreover,

$$\sup_{i \in \{1, \dots, N\}} E \sup_{t \in [0, T]} (1 + |X_t^{i,N,n}|^2)^{q/2} \leq K,$$

for any $q < p_0$.

PROOF. On using the arguments used in the proof of Lemma 3.2 with equation (21), we obtain the following equation analogous to equation (10):

$$\begin{aligned} & E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\ & \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} \\ & \quad + \frac{p_0}{2} E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \{2X_{\kappa_n(s)}^{i,N,n} b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ & \quad + (p_0 - 1)|\tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 + (p_0 - 1)|\tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2\} ds \\ & \quad + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Observe that $\tilde{\sigma}^n$ and $\tilde{\sigma}^{0,n}$ are sum of two matrices, see equations (22) and (23). Thus, on using $|A + B|^2 = |A|^2 + |B|^2 + 2 \sum_{u=1}^d \sum_{v=1}^m A^{(u,v)} B^{(u,v)}$ for matrices A and B , one obtains

$$\begin{aligned} & E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\ & \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} \\ & \quad + \frac{p_0}{2} E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \{X_{\kappa_n(s)}^{i,N,n} b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ & \quad + (p_0 - 1)|\sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 + (p_0 - 1)|\sigma_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2\} ds \\ & \quad + p_0 E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\ & \quad + \frac{p_0(p_0 - 1)}{2} E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} |\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds \\ & \quad + \frac{p_0(p_0 - 1)}{2} E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} |\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^2 ds \\ & \quad + p_0(p_0 - 1) E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ & \quad \times \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \end{aligned}$$

$$\begin{aligned}
& + p_0(p_0 - 1)E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad \times \Gamma_{\kappa_n(s)}^{n,\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds
\end{aligned}$$

which on using Assumption 2, Young's inequality, equation (21) and Lemma 4.1 yields

$$\begin{aligned}
& E(1 + |X_t^{i,N,n}|^2)^{p_0/2} \\
& \leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
& \quad + KE \int_0^t |\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} ds \\
& \quad + KE \int_0^t |\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} ds \\
& \quad + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
& \quad \times \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& \quad + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
& \quad \times \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& \quad + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
& \quad \times \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& \quad + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad \times \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& \quad + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad \times \Gamma_{\kappa_n(s)}^{n,\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& =: E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + \sum_{i=1}^7 \Pi_i,
\end{aligned}
\tag{24}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

By Lemma 4.1, Remark 7 and Young's inequality, one obtains

$$\begin{aligned}
& \Pi_1 + \Pi_2 + \Pi_3 \\
& := KE \int_0^t |\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} ds \\
& \quad + KE \int_0^t |\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} ds + KE \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1}
\end{aligned}
\tag{25}$$

$$\begin{aligned}
& \times \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
& \leq K E (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2} + K E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)
\end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

Notice that Π_4 is similar to F_1 in the proof of Lemma 3.2 and hence by adapting the same technique, one can obtain an analogue of inequality (13) with σ^n replaced by $\tilde{\sigma}^n$, that is,

$$\begin{aligned}
\Pi_4 &:= K E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
&\quad \times \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E (1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
&\quad + K n^{-p_0/8} n^{-p_0/4+1} \int_0^t E \int_{\kappa_n(s)}^s |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0/2} \\
&\quad \times |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0/2} dr ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E (1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
&\quad + K n \int_0^t E \int_{\kappa_n(s)}^s n^{-p_0/4} |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0/2} n^{-p_0/8} \\
&\quad \times |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0/2} dr ds,
\end{aligned}$$

which on the application of Young's inequality, Corollary 1 and Remark 7 yields

$$\begin{aligned}
\Pi_4 &\leq K \int_0^t \sup_{0 \leq r \leq s} E (1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
&\quad + K n \int_0^t \int_{\kappa_n(s)}^s \{n^{-p_0/2} E |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\
(26) \quad &\quad + n^{-p_0/4} E |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0}\} dr ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E (1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds
\end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

By following the steps of estimating Π_4 , one easily obtains

$$\begin{aligned}
\Pi_5 &:= K E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \\
(27) \quad &\quad \times \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
&\leq K \int_0^t \sup_{0 \leq r \leq s} E (1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds
\end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

It remains to analyse Π_6 and Π_7 now. For Π_6 , it is easy to see that

$$\begin{aligned} \Pi_6 &:= K E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad \times \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\ &\leq K E \int_0^t (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad \times \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\ &\quad + K E \int_0^t |(1 + |X_s^{i,N,n}|^2)^{p_0/2-1} - (1 + |X_{\kappa_n(s)}^{i,N,n}|^2)^{p_0/2-1}| \\ &\quad \times \left| \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \right| \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Notice that $\Gamma^{n,\sigma}$ is a martingale and thus the first term in the right-hand side vanishes. For the second term, one uses inequality (12) and Young's inequality to obtain

$$\begin{aligned} \Pi_6 &\leq K E \int_0^t \{1 + |X_s^{i,N,n}|^2 + |X_{\kappa_n(s)}^{i,N,n}|^2\}^{(p_0-3)/2} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}| \\ &\quad \times \left| \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \right| \\ &\leq K \int_0^t \{1 + |X_s^{i,N,n}|^2 + |X_{\kappa_n(s)}^{i,N,n}|^2\}^{p_0/2} ds + K E \int_0^t n^{p_0/12} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0/3} \\ &\quad \times n^{-p_0/12} \left| \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \right|^{p_0/3} ds \\ &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t n^{p_0/4} E |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} ds \\ &\quad + K n^{-p_0/8} \\ &\quad \times E \int_0^t \left| \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \right|^{p_0/2} ds, \end{aligned}$$

which on the application of Hölder's inequality, Remark 7 and Lemma 4.1 yields

$$\begin{aligned} \Pi_6 &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t n^{p_0/4} E |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} ds \\ &\quad + K n^{-p_0/8} \int_0^t \sum_{u=1}^d \sum_{v=1}^m \{E |\sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0}\}^{1/2} \\ (28) \quad &\quad \times \{E |\Gamma_{\kappa_n(s)}^{n,\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0}\}^{1/2} ds \\ &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Notice that Π_6 and Π_7 are similar terms and hence one also obtains

$$\begin{aligned}
 \Pi_7 &:= K E \int_0^t (1 + |X_s^{i,N,n}|^2)^{p_0/2-1} \sum_{u=1}^d \sum_{v=1}^m \sigma_{\kappa_n(s)}^{n,0,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
 &\quad \times \Gamma_{\kappa_n(s)}^{n,\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) ds \\
 &\leq K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds + K \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds
 \end{aligned}
 \tag{29}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

On substituting the estimates obtained in (25) to (29) in equation (24) gives

$$\begin{aligned}
 E(1 + |X_t^{i,N,n}|^2)^{p_0/2} &\leq E(1 + |X_0^{i,N,n}|^2)^{p_0/2} + K \int_0^t \sup_{0 \leq r \leq s} E(1 + |X_r^{i,N,n}|^2)^{p_0/2} ds \\
 &\quad + K \int_0^t \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0) ds
 \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Thus, the proof of the first inequality can be completed by using equation (16) and Gronwall's inequality. The second inequality follows by Lemma 1.1. \square

We now proceed to the rate of convergence of the tamed Milstein scheme (21). For this, we prove some lemmas and corollaries as given below. The proof of the following lemma can be found in Appendix A.5.

LEMMA 4.4. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then,*

$$E|\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq K n^{-p/2},$$

$$E|\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq K n^{-p/2},$$

for all $p \leq p_0/(\rho/2 + 1)$, $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

As a consequence of Remark 7, Lemmas 4.3 and 4.4, one obtains the following corollary.

COROLLARY 2. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then,*

$$E|\tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq K,$$

$$E|\tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq K,$$

for any $p \leq p_0/(\rho/2 + 1)$, $s \in [0, T]$, $i \in \{1, \dots, N\}$, $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

The following lemma can be proved by adapting arguments similar to the proof of Lemma 4.2 along with Remark 2, Lemma 4.3 and Corollary 2.

LEMMA 4.5. *Let Assumptions 1, 2, 6, 9 and 10 be satisfied. Then,*

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p \leq K n^{-p/2},$$

for any $p \leq p_0/(\rho/2 + 1)$, $s \in [0, T]$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

The following three lemmas play an important role in the proof of the main result of this section, that is, Theorem 4.9. The proofs of these lemmas are given in Appendices A.6, A.7 and A.8, respectively.

LEMMA 4.6. *Let Assumptions 1, 2, 5, 6, 8, 9 and 10 be satisfied. Then,*

$$\begin{aligned} E|\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p &\leq Kn^{-p}, \\ E|\sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p &\leq Kn^{-p}, \end{aligned}$$

for any $p \leq p_0/(2\rho + 4)$, $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on n and N .

LEMMA 4.7. *Let Assumptions 1, 2, 5, 6, 8, 9 and 10 be satisfied. Then,*

$$E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq Kn^{-p/2},$$

for any $p \leq p_0/(2\rho + 4)$, $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$ where constant $K > 0$ does not depend on n and N .

LEMMA 4.8. *Let Assumptions 1, 2, 5, 6, 8, 9 and 10 be satisfied. Then,*

$$\begin{aligned} E|X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) (b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\ \leq Kn^{-p} + K \sup_{i \in \{1, \dots, N\}} \sup_{r \in [0, s]} E|X_r^{i,N} - X_r^{i,N,n}|^p, \end{aligned}$$

for any $p \leq p_0/(2\rho + 4)$, $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$ where constant $K > 0$ does not depend on n and N .

The following is the main result of this section.

THEOREM 4.9 (Rate of convergence). *Let Assumptions 1, 2, 5, 6, 8, 9 and 10 be satisfied. Then, the explicit Milstein-type scheme (21) converges to the true solution of the interacting particle system (4) associated with McKean–Vlasov SDE (2) in a strong sense with L^p rate of convergence 1, that is,*

$$\sup_{i \in \{1, \dots, N\}} E \sup_{t \in [0, T]} |X_t^{i,N} - X_t^{i,N,n}|^p \leq Kn^{-p},$$

for any $p < \min\{p_1, p_0/(2\rho + 4)\}$, where the constant $K > 0$ does not depend on $n, N \in \mathbb{N}$.

PROOF. The proof follows by applying arguments used in the proof of Theorem 3.5. Indeed, one replaces equation (17) by the following equation:

$$\begin{aligned} X_t^{i,N} - X_t^{i,N,n} &= \int_0^t (b_s(X_s^{i,N}, \mu_s^{X,N}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) ds \\ &\quad + \int_0^t (\sigma_s(X_s^{i,N}, \mu_s^{X,N}) - \tilde{\sigma}_{\kappa_n(s)}^n(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) dW_s^i \\ &\quad + \int_0^t (\sigma_s^0(X_s^{i,N}, \mu_s^{X,N}) - \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) dW_s^0 \end{aligned}$$

and thus equation (18) is replaced by

$$\begin{aligned} E|X_t^{i,N} - X_t^{i,N,n}|^p &\leq K E \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^p ds \\ &\quad + K E \int_0^t |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) \\ &\quad \times (b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) ds \\ &\quad + K E \int_0^t |\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p ds \\ &\quad + K E \int_0^t |\sigma_s^0(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{0,n}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p ds \end{aligned}$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Application of Lemmas 4.6 and 4.8 yields

$$\sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq t} E|X_r^{i,N} - X_r^{i,N,n}|^p \leq K \int_0^t \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E|X_r^{i,N} - X_r^{i,N,n}|^p ds + Kn^{-p},$$

for any $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. The use of Gronwall’s inequality yields,

$$\sup_{i \in \{1, \dots, N\}} \sup_{0 \leq t \leq T} E|X_t^{i,N} - X_t^{i,N,n}|^p \leq Kn^{-p},$$

for any $p < \min\{p_1, p_0/(2\rho + 4)\}$, $t \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Then, the application of Lemma 1.1 completes the proof. \square

5. Numerical results. The aim of this section is to demonstrate the practical performance of the schemes proposed in this article. To approximate the law $\mathcal{L}_{X_{t_n}}$ (or the conditional law $\mathcal{L}_{X_{t_n}}^1$) at each time-step t_n contained in a uniform time-grid on $[0, T]$, we use a standard particle method with N particles for each realisation of W^0 . For our numerical experiments, we used $N = 10^3$ to approximate the conditional law (e.g., a conditional expectation) and perform 100 independent outer simulations over Ω^0 , in order to estimate the L^p -error on the product space. This resembles in our last example the estimation of nested expectations as, for example, in [9] or the survey paper [17] and references given therein, and suggests further research concerning efficient multi-level Monte Carlo methods for estimation with conditional laws.

Since the exact solution of the examples considered below is not known, we determined the strong convergence rates, in terms of time-steps, by comparing two numerical solutions (at time $T = 1$) obtained from simulations based on a fine and coarse time-grid, respectively. To obtain a coupling between these two levels, the same Brownian motions are used for both. In order to test the strong convergence in h , we thus compute the L^p -error, for $p = 2, 4, 6$, denoted by RMSE,

$$\text{RMSE} := \left(\frac{1}{N} \sum_{i=1}^N |X_T^{i,N,l} - X_T^{i,N,l-1}|^p \right)^{1/p},$$

where $X_T^{i,N,l}$ denotes the numerical solution of X at time T computed with N particles and 2^l time-steps, where $l \geq 1$.

This section numerically illustrates the convergence of the tamed Euler and Milstein schemes for interacting particle systems, with and without common noise.

To demonstrate numerically the performance of our proposed tamed Euler scheme, we present the following McKean–Vlasov equation, which is a mean-field version of the well-known 3/2-model used for modelling certain stochastic volatility processes.

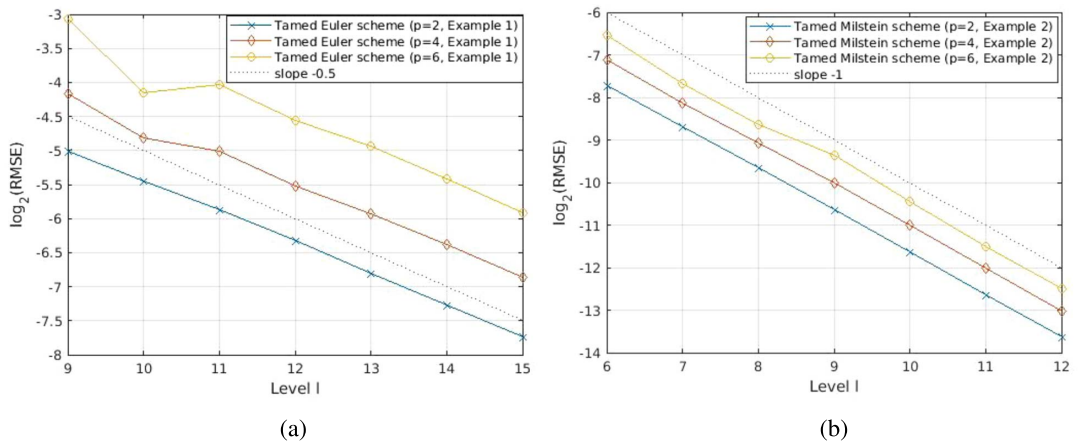


FIG. 1. Left: Strong convergence of the tamed Euler scheme for Example 1. Right: Strong convergence of the tamed Milstein scheme for Example 2.

EXAMPLE 1 (Mean-field 3/2 stochastic volatility model). Consider the two-dimensional McKean–Vlasov SDE

$$X_t = X_0 + \int_0^t (\lambda X_s(\mu - |X_s|) + E X_s) ds + \int_0^t \xi |X_s|^{3/2} dW_s,$$

where we choose $X_0 = [1, 1]^T$, $\lambda = 2.5$, $\mu = 1$ and

$$\xi = \begin{bmatrix} 2/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 2/\sqrt{10} \end{bmatrix}.$$

The above model without the mean-field has been studied numerically in [37]. Figure 1(a) depicts the strong convergence rate for this example and we observe a rate of $1/2$.

To illustrate the convergence behaviour of our proposed tamed Milstein scheme, we consider the following one-dimensional example, which has been studied numerically in [7, 26].

EXAMPLE 2 (Mean-field stochastic double well dynamics). Consider the one-dimensional McKean–Vlasov SDE

$$X_t = X_0 + \int_0^t (X_s(1 - X_s^2) + E X_s) ds + \int_0^t \sigma(1 - X_s^2) dW_s,$$

with $X_0 = 1$ and $\sigma = 0.3$. Note that the diffusion term does not depend on the marginal distributions of X . Hence, the L -derivative terms appearing in the Milstein scheme (21) are zero. In the case that the diffusion depends on the marginal distributions of X , we remark that the L -derivative terms (involving N different Brownian motions) are expected to be close to zero, in an L_2 -sense, for a large number of particles. However, for a small number of particles (e.g., 5 or 10), they have to be considered to obtain a convergence rate of 1; see [5] for a more rigorous discussion on the numerical role of the L -derivative terms (in the case of zero common noise). Figure 1(b) and Figure 2 reveal the expected strong convergence rate of 1.

EXAMPLE 3 (Mean-field stochastic double well dynamics with common noise). Consider the one-dimensional McKean–Vlasov SDE

$$X_t = X_0 + \int_0^t (X_s(1 - X_s^2) + E^1 X_s) ds + \int_0^t \sigma(1 - X_s^2) dW_s + \int_0^t \sigma(1 - X_s^2) dW_s^0,$$

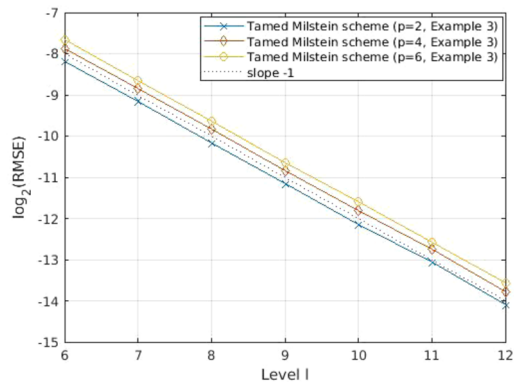


FIG. 2. Strong convergence of the tamed Milstein scheme for Example 3.

with $X_0 = 1$ and $\sigma = 0.1$. This example is a slight modification of Example 2 and involves additionally a common noise term. We remark that in this case iterated stochastic integrals (the Brownian motion W integrated against W^0 and the other way around) appear, but due to the antisymmetry property of the Lévy area these terms will cancel.

APPENDIX: PROOFS OF SUPPLEMENTARY RESULTS

A.1. Proof of Lemma 3.1. Due to equation (9), one obtains

$$\begin{aligned} &|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \\ &\leq K \left| \int_{\kappa_n(s)}^s b_{\kappa_n(u)}^n(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n}) du \right|^{p_0} \\ &\quad + K \left| \int_{\kappa_n(s)}^s \sigma_{\kappa_n(u)}^n(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n}) dW_u^i \right|^{p_0} + K \left| \int_{\kappa_n(s)}^s \sigma_{\kappa_n(u)}^{0,n}(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n}) dW_u^0 \right|^{p_0}, \end{aligned}$$

which on applying Hölder’s inequality and Burkholder–Davis–Gundy inequality yields,

$$\begin{aligned} &E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \\ &\leq K n^{-p_0+1} E \int_{\kappa_n(s)}^s |b_{\kappa_n(u)}^n(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n})|^{p_0} du \\ &\quad + K n^{-p_0/2+1} E \int_{\kappa_n(s)}^s \{ |\sigma_{\kappa_n(u)}^n(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n})|^{p_0} + |\sigma_{\kappa_n(u)}^{0,n}(X_{\kappa_n(u)}^{i,N,n}, \mu_{\kappa_n(u)}^{X,N,n})|^{p_0} \} du \end{aligned}$$

and then the result follows by using Remark 4.

A.2. Proof of Lemma 3.4. On using Remark 1 and Assumption 7, one obtains

$$\begin{aligned} &E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ &\leq K E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_s(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ &\quad + K E|b_s(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ &\quad + K E|b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ &\leq K E(1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho p/2} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p + K E \mathcal{W}_2^p(\mu_s^{X,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\ &\quad + K |s - \kappa_n(s)|^{p/2} + K n^{-p}, \end{aligned}$$

which on the application of Hölder's inequality, Lemma 3.2, Lemma 3.3 and the following elementary estimate:

$$(30) \quad \mathcal{W}_2^2(\mu_s^{X,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \leq \frac{1}{N} \sum_{j=1}^n |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^2,$$

proves the estimate for b . The proof is complete by performing similar calculations for σ and σ^0 .

A.3. Proof of Lemma 4.1. Using Remark 7, one obtains

$$\begin{aligned} & E|\Lambda_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &= E\left|\partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i\right|^{p_0} \\ &\leq K n^{-p_0/2} E|\partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &\leq K E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0} \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. Similarly, one obtains

$$E|\Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \leq K E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$.

Again, the application of Remark 7 yields

$$\begin{aligned} & E|\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &= E\left|\frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j\right|^{p_0} \\ &\leq K \frac{1}{N} \sum_{j=1}^N E\left|\partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j\right|^{p_0} \\ &\leq K n^{-p_0/2} \frac{1}{N} \sum_{j=1}^N E\{|\partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})| |\sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{j,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|\}^{p_0} \\ &\leq K n^{-p_0/4} K E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0} \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. Similarly,

$$E|\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \leq K n^{-p_0/4} K E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. Adding the above inequalities leads to

$$\begin{aligned} & E|\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &\leq K E|\Lambda_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &\quad + K E|\Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} + K E|\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \\ &\quad + K E|\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{p_0} \leq K E(1 + |X_{\kappa_n(s)}^{i,N,n}| + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0))^{p_0} \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. This completes the proof of the first inequality. The second inequality can be proved similarly.

A.4. Proof of Lemma 4.2. From (21), one obtains

$$\begin{aligned} & |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \\ & \leq K \left| \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \right|^{p_0} \\ & \quad + K \left| \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^{p_0} \\ & \quad + K \left| \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right|^{p_0}, \end{aligned}$$

which on applying Hölder's inequality and Burkholder–Davis–Gundy inequality yields

$$\begin{aligned} & E |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \\ & \leq K n^{-p_0+1} E \int_{\kappa_n(s)}^s |b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0} dr \\ & \quad + K n^{-p_0/2+1} E \int_{\kappa_n(s)}^s \{ |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0} + |\tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p_0} \} dr \end{aligned}$$

and then the result follows on using Corollary 1 and Remark 7.

A.5. Proof of Lemma 4.4. Using Remarks 6, 7, Lemma 4.3 and equation (16),

$$\begin{aligned} & E |\Lambda_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ & = E \left| \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^p \\ & \leq K n^{-p/2} E |\partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \sigma_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq K n^{-p/2} \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. Similarly, one obtains

$$\begin{aligned} & E |\Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ & = E \left| \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right|^p \leq K n^{-p/2}, \\ & E |\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ & = E \left| \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \right|^p \\ & \leq K n^{-p/2}, \\ & E |\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ & = E \left| \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \sigma_{\kappa_n(r)}^{0,n}(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right|^p \\ & \leq K n^{-p/2}, \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. By using the above estimates, one obtains

$$\begin{aligned} & E |\Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\ & \leq K E |\Lambda_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \end{aligned}$$

$$\begin{aligned}
& + KE |\Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p + KE |\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& + KE |\bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \leq Kn^{-p/2}
\end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{R}^d$. This completes the proof for the first inequality. The second inequality follows similarly.

A.6. Proof of Lemma 4.6. From equation (21),

$$\begin{aligned}
(31) \quad & \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})(X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
& = \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
& \quad + \Lambda_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \\
& \quad + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0
\end{aligned}$$

and also,

$$\begin{aligned}
(32) \quad & \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})(X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \\
& = \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
& \quad + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \\
& \quad + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0
\end{aligned}$$

almost surely for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Furthermore,

$$\begin{aligned}
& \sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{n,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& = \sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \Lambda_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad - \Lambda_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad - \bar{\Lambda}_{\kappa_n(s)}^{n,\sigma\sigma^0,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})(X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
& \quad - \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})(X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n})
\end{aligned}$$

$$\begin{aligned}
& + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})(X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
& + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})(X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}),
\end{aligned}$$

which on using equations (31) and (32) yields

$$\begin{aligned}
& \sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{n,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& = \sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad - \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})(X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
& \quad - \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})(X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \\
& \quad + \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& \quad + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
& \quad + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \\
& \quad + \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \\
& \quad + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
& \quad + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \\
& \quad + \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0
\end{aligned}$$

almost surely for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. On using Lemma 1.3 with $f = \sigma_{\kappa_n(s)}$ and Remark 1, one obtains

$$\begin{aligned}
& E |\sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{n,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \leq K E (1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho p/4} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{2p} \\
& \quad + K \frac{1}{N} \sum_{j=1}^N E |X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}|^{2p} \\
& \quad + K E |\sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \sigma_{\kappa_n(s)}^{n,(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \quad + K E \left| \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \right|^p \\
& \quad + K E \left| \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^p
\end{aligned}$$

$$\begin{aligned}
& + KE \left| \partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right|^p \\
& + KE \left| \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \right|^p \\
& + KE \left| \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \right|^p \\
& + KE \left| \frac{1}{N} \sum_{j=1}^N \partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right|^p
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Notice that due to equation (20), Remark 2, equation (16) and Lemma 4.3,

$$\begin{aligned}
(33) \quad & E \left| \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \frac{\sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})}{1 + n^{-1} |X_{\kappa_n(s)}^{i,N,n}|^\rho} \right|^p \\
& \leq Kn^{-p} E |\sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p |X_{\kappa_n(s)}^{i,N,n}|^{\rho p} \leq Kn^{-p}
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Furthermore, one uses Hölder's inequality, equation (33), Lemmas 4.5, 4.3 and Remarks 2, 6 to obtain

$$\begin{aligned}
& E |\sigma_{\kappa_n(s)}^{(u,v)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^{n,(u,v)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \leq K \{E(1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho p/2}\}^{1/2} \{E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{4p}\}^{1/2} + Kn^{-p} \\
& \quad + Kn^{-p} E |\partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \quad + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s |\partial_x \sigma_{\kappa_n(r)}^{(u,v)}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p |\Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p dr \\
& \quad + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s |\partial_x \sigma_{\kappa_n(r)}^{(u,v)}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p |\Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p dr \\
& \quad + Kn^{-p} \frac{1}{N} \sum_{j=1}^N E |\partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})|^p |b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{j,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \quad + Kn^{-p/2+1} \frac{1}{N} \sum_{j=1}^N E \int_{\kappa_n(s)}^s |\partial_\mu \sigma_{\kappa_n(r)}^{(u,v)}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}, X_{\kappa_n(r)}^{j,N,n})|^p \\
& \quad \times |\Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p dr \\
& \quad + Kn^{-p/2+1} \frac{1}{N} \sum_{j=1}^N E \int_{\kappa_n(s)}^s |\partial_\mu \sigma_{\kappa_n(r)}^{(u,v)}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}, X_{\kappa_n(r)}^{j,N,n})|^p \\
& \quad \times |\Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p dr \\
& \leq Kn^{-p} + Kn^{-p/2+1} \int_{\kappa_n(s)}^s \{E |\partial_x \sigma_{\kappa_n(r)}^{(u,v)}(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2p}\}^{1/2} \\
& \quad \times \{E |\Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2p}\}^{1/2} dr
\end{aligned}$$

$$\begin{aligned}
& + Kn^{-p/2+1} \int_{\kappa_n(s)}^s \{E|\partial_x \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^{2p}\}^{1/2} \\
& \times \{E|\Gamma_{\kappa_n(s)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2p}\}^{1/2} dr \\
& + Kn^{-p/2+1} \frac{1}{N} \sum_{j=1}^N \int_{\kappa_n(s)}^s \{E|\partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})|^{2p}\}^{1/2} \\
& \times \{E|\Gamma_{\kappa_n(r)}^{n,\sigma}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2p}\}^{\frac{1}{2}} dr \\
& + Kn^{-p/2+1} \frac{1}{N} \sum_{j=1}^N \int_{\kappa_n(s)}^s \{E|\partial_\mu \sigma_{\kappa_n(s)}^{(u,v)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})|^{2p}\}^{\frac{1}{2}} \\
& \times \{E|\Gamma_{\kappa_n(r)}^{n,\sigma^0}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2p}\}^{\frac{1}{2}} dr
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. On using Lemma 4.4, Remarks 6 and equation (16), one obtains

$$E|\sigma_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^n(s, X_s^{i,N,n}, \mu_s^{X,N,n})|^p \leq Kn^{-p},$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Furthermore, the application of Assumption 8 yields

$$KE|\sigma_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - \sigma_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n})|^p \leq Kn^{-p},$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Also, similar calculations give the second inequality.

A.7. Poof of Lemma 4.7. On using Assumptions 6, 8, equation (20), Hölder's inequality, equation (30) and Remark 2, one obtains

$$\begin{aligned}
& E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_s^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \leq KE|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n})|^p \\
& \quad + KE|b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \quad + KE|b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \leq Kn^{-p} + KE(1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho p/2} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p \\
& \quad + KE\mathcal{W}_2^p(\mu_s^{X,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) + KE\left|b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \frac{b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})}{1 + n^{-1}|X_{\kappa_n(s)}^{i,N,n}|^\rho}\right|^p \\
& \leq Kn^{-p} + K\{E(1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho p} E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{2p}\}^{1/2} \\
& \quad + \frac{1}{N} \sum_{j=1}^n E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p + n^{-p} E(1 + |X_{\kappa_n(s)}^{i,N,n}|^{3\rho p/2})
\end{aligned}$$

and then the application of Lemmas 4.3 and 4.5 completes the proof.

A.8. Poof of Lemma 4.8. We first prove

$$\begin{aligned}
& E|X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n})(b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
(34) \quad & \leq Kn^{-p} + K \sup_{i \in \{1, \dots, N\}} \sup_{r \in [0, s]} E|X_r^{i,N} - X_r^{i,N,n}|^p
\end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$. For this, notice that

$$\begin{aligned}
 & E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) (b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) \\
 & \quad - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
 & = E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \left\{ b_{\kappa_n(s)}^{(k)}(X_s^{i,N,n}, \mu_s^{X,N,n}) \right. \\
 & \quad - b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
 & \quad \left. - \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) (X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \right\} \\
 & \quad + E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
 & \quad \times (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) + E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 & \quad \times \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) (X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \\
 & =: T_1 + T_2 + T_3
 \end{aligned} \tag{35}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. By using Lemma 1.3 and Young's inequality,

$$\begin{aligned}
 T_1 & := E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \left\{ b_{\kappa_n(s)}^{(k)}(X_s^{i,N,n}, \mu_s^{X,N,n}) \right. \\
 & \quad - b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
 & \quad \left. - \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) (X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \right\} \\
 & \leq K E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d |X_s^{(k),i,N} - X_s^{(k),i,N,n}| \\
 & \quad \times \left\{ (1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho/2-1} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^2 + \frac{1}{N} \sum_{j=1}^N |X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}|^2 \right\} \\
 & \leq K E |X_s^{i,N} - X_s^{i,N,n}|^p + K E \left\{ (1 + |X_s^{i,N,n}| + |X_{\kappa_n(s)}^{i,N,n}|)^{\rho/2-1} |X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^2 \right. \\
 & \quad \left. + \frac{1}{N} \sum_{j=1}^N |X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}|^2 \right\}^p,
 \end{aligned}$$

which on the application of Hölder's inequality, Lemmas 4.3 and 4.5 yields

$$T_1 \leq K E |X_s^{i,N} - X_s^{i,N,n}|^p + K n^{-p}, \tag{36}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

Notice that T_2 can be written as

$$\begin{aligned}
 T_2 &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})(X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}) \\
 (37) \quad &= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
 &\quad \times \left\{ \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr + \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right. \\
 &\quad \left. + \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right\} =: T_{21} + T_{22} + T_{23}
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. For T_{21} , one uses the Cauchy–Schwarz inequality, Young’s inequality, equation (16) and Remarks 2, 6 and Lemma 4.3 to obtain

$$\begin{aligned}
 T_{21} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
 (38) \quad &\leq K E |X_s^{i,N} - X_s^{i,N,n}|^{p-1} n^{-1} \{1 + |X_{\kappa_n(s)}^{i,N,n}|^{\rho+1} + \mathcal{W}_2(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0)\} \\
 &\leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_s^{i,N} - X_s^{i,N,n}|^p + K n^{-p}
 \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

For T_{22} , let us define, for $k = 1, \dots, d$,

$$(39) \quad \mathcal{M}^{(k)}(\kappa_n(s), s) := \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i,$$

and notice that, due to Burkholder–Davis–Gundy inequality, Hölder’s inequality and Remark 6,

$$\begin{aligned}
 &E |\mathcal{M}^{(k)}(\kappa_n(s), s)|^q \\
 &= E \left| \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \right|^q \\
 &\leq K n^{-q/2+1} E \int_{\kappa_n(s)}^s |\partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^q |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^q dr \\
 &\leq K n^{-q/2+1} E \int_{\kappa_n(s)}^s (1 + |X_{\kappa_n(s)}^{i,N,n}|)^{(q\rho)/2} |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^q dr \\
 &\leq K n^{-q/2+1} \int_{\kappa_n(s)}^s \{E(1 + |X_{\kappa_n(s)}^{i,N,n}|)^{q\rho p_0} \{E |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{2q}\}^{1/2} dr,
 \end{aligned}$$

which on using Lemma 4.3 and Corollary 2 yields

$$(40) \quad E |\mathcal{M}(\kappa_n(s), s)|^q \leq K n^{-q/2}$$

for any $q \leq p_0/(\rho + 2)$ and $s \in [0, T]$. Using the notation in equation (39) along with Lemma 4.3 and the Cauchy–Schwarz inequality,

$$\begin{aligned}
 T_{22} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \partial_X b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^i \\
 &= E |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \sum_{k=1}^d (X_{\kappa_n(s)}^{(k),i,N} - X_{\kappa_n(s)}^{(k),i,N,n}) \mathcal{M}^{(k)}(\kappa_n(s), s) \\
 &\quad + E \left\{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \right. \\
 &\quad \left. - |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \sum_{k=1}^d (X_{\kappa_n(s)}^{(k),i,N} - X_{\kappa_n(s)}^{(k),i,N,n}) \right\} \mathcal{M}^{(k)}(\kappa_n(s), s) \\
 &= E \sum_{k=1}^d \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad - |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} (X_{\kappa_n(s)}^{(k),i,N} - X_{\kappa_n(s)}^{(k),i,N,n}) \} \mathcal{M}^{(k)}(\kappa_n(s), s) \\
 &\leq E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) \\
 &\quad - |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} (X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}) ||\mathcal{M}(\kappa_n(s), s)|
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Also, for an \mathbb{R}^d -valued function $f(z) = |z|^{p-2}z$ for any $z \in \mathbb{R}^d$ and a $\theta \in (0, 1)$,

$$\begin{aligned}
 |x|^{p-2}x - |y|^{p-2}y &= |f(x) - f(y)| \\
 &\leq K(p-1)|\theta x + (1-\theta)y|^{p-2}|x-y| \leq K\{|x|^{p-2} + |y|^{p-2}\}|x-y|
 \end{aligned}$$

for any $x, y \in \mathbb{R}^d$, which further implies on taking $x = X_s^{i,N} - X_s^{i,N,n}$ and $y = X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}$,

$$\begin{aligned}
 &||X_s^{i,N} - X_s^{i,N,n}|^{p-2}(X_s^{i,N} - X_s^{i,N,n}) - |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}(X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n})| \\
 (41) \quad &\leq K\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
 &\quad \times |X_s^{i,N} - X_s^{i,N,n} - (X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n})|
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Thus,

$$\begin{aligned}
 T_{22} &\leq KE\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
 &\quad \times |X_s^{i,N} - X_{\kappa_n(s)}^{i,N} - (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n})| |\mathcal{M}(\kappa_n(s), s)| \\
 &\leq KE\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
 (42) \quad &\times \left\{ \left| \int_{\kappa_n(s)}^s \{b_r(X_r^{i,N}, \mu_r^{X,N}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dr \right| \right. \\
 &\quad \left. + \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dW_r^i \right| \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \left| \int_{\kappa_n(s)}^s \{ \sigma_r^0(X_r^{i,N}, \mu_r^{X,N}) - \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) \} dW_r^0 \right| |\mathcal{M}(\kappa_n(s), s)| \\
& =: T_{22A} + T_{22B} + T_{22C}
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Also, by Assumption 6,

$$\begin{aligned}
T_{22A} & := KE \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \} \\
& \quad \times \left| \int_{\kappa_n(s)}^s \{ b_r(X_r^{i,N}, \mu_r^{X,N}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) \} dr \right| |\mathcal{M}(\kappa_n(s), s)| \\
& \leq KE \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \} \\
& \quad \times \int_{\kappa_n(s)}^s \{ |b_r(X_r^{i,N}, \mu_r^{X,N}) - b_r(X_r^{i,N,n}, \mu_r^{X,N,n})| \\
& \quad + |b_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})| \} dr |\mathcal{M}(\kappa_n(s), s)| \\
& \leq KE \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \} \\
& \quad \times \int_{\kappa_n(s)}^s \{ (1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{\rho/2} |X_r^{i,N} - X_r^{i,N,n}| + \mathcal{W}_2(\mu_r^{X,N}, \mu_r^{X,N,n}) \\
& \quad + |b_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})| \} dr |\mathcal{M}(\kappa_n(s), s)|
\end{aligned}$$

and the application of Young's inequality and Hölder's inequality yields

$$\begin{aligned}
T_{22A} & \leq KE \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \}^{p/(p-2)} \\
& \quad + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s \{ (1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{\rho/2} |X_r^{i,N} - X_r^{i,N,n}| \\
& \quad + \mathcal{W}_2(\mu_r^{X,N}, \mu_r^{X,N,n}) + |b_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})| \}^{p/2} dr \\
& \quad \times |\mathcal{M}(\kappa_n(s), s)|^{p/2} \\
& \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p \\
& \quad + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s (1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{(p\rho)/4} |X_r^{i,N} - X_r^{i,N,n}|^{p/2} dr \\
& \quad \times |\mathcal{M}(\kappa_n(s), s)|^{p/2} + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s \mathcal{W}_2^{p/2}(\mu_r^{X,N}, \mu_r^{X,N,n}) dr |\mathcal{M}(\kappa_n(s), s)|^{p/2} \\
& \quad + Kn^{-p/2+1} E \int_{\kappa_n(s)}^s |b_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^{p/2} dr \\
& \quad \times |\mathcal{M}(\kappa_n(s), s)|^{p/2} \\
& \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p \\
& \quad + Kn^{-p/2+1} \int_{\kappa_n(s)}^s \{ E |X_r^{i,N} - X_r^{i,N,n}|^p \}^{1/2} \{ E (1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{(p\rho)/2} \\
& \quad \times |\mathcal{M}(\kappa_n(s), s)|^p \}^{1/2} dr \\
& \quad + Kn^{-p/2+1} \int_{\kappa_n(s)}^s \{ E \mathcal{W}_2^p(\mu_r^{X,N}, \mu_r^{X,N,n}) \}^{1/2} \{ E |\mathcal{M}(\kappa_n(s), s)|^p \}^{1/2} dr
\end{aligned}$$

$$\begin{aligned}
& + K n^{-p/2+1} \int_{\kappa_n(s)}^s \{E|b_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p\}^{1/2} \\
& \times \{E|\mathcal{M}(\kappa_n(s), s)|^p\}^{1/2} dr
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. On using estimates from equation (40), Lemmas 4.3 and 4.7,

$$\begin{aligned}
T_{22A} & \leq K \sup_{0 \leq r \leq s} E|X_r^{i,N} - X_r^{i,N,n}|^p \\
& + K n^{-p/2+1} \int_{\kappa_n(s)}^s \{E|X_r^{i,N} - X_r^{i,N,n}|^p\}^{1/2} \\
& \times \{E(1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{p\rho} E|\mathcal{M}(\kappa_n(s), s)|^{2p}\}^{1/4} dr \\
& + K n^{-3p/4+1} \int_{\kappa_n(s)}^s \{E\mathcal{W}_2^p(\mu_r^{X,N}, \mu_r^{X,N,n})\}^{1/2} dr + K n^{-p}
\end{aligned}$$

and thus the application of estimates in equation (19) and Young's inequality yields

$$(43) \quad T_{22A} \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E|X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p},$$

for any $s \in [0, T]$, and $n, N \in \mathbb{N}$.

For estimating T_{22B} , one notices that

$$\begin{aligned}
T_{22B} & := K E\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
& \times \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dW_r^i \right| |\mathcal{M}(\kappa_n(s), s)| \\
& \leq K E\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
& \times \left\{ \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) \right. \right. \\
& \quad \left. \left. + \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dW_r^i \right| \right\} |\mathcal{M}(\kappa_n(s), s)| \\
& \leq K E\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
& \times \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n})\} dW_r^i \right| |\mathcal{M}(\kappa_n(s), s)| \\
& + K E\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\} \\
& \times \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dW_r^i \right| |\mathcal{M}(\kappa_n(s), s)|,
\end{aligned}$$

which on using Hölder's inequality and Young's inequality yields

$$\begin{aligned}
T_{22B} & \leq K E\{|X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2}\}^{p/(p-2)} \\
& + K E \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n})\} dW_r^i \right|^{p/2} |\mathcal{M}(\kappa_n(s), s)|^{p/2} \\
& + K E \left| \int_{\kappa_n(s)}^s \{\sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})\} dW_r^i \right|^{p/2} \\
& \times |\mathcal{M}(\kappa_n(s), s)|^{p/2} \leq K \sup_{0 \leq r \leq s} E|X_r^{i,N} - X_r^{i,N,n}|^p
\end{aligned}$$

$$\begin{aligned}
& + K \left\{ E \left| \int_{\kappa_n(s)}^s \{ \sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) \} dW_r^i \right|^{3p/4} \right\}^{2/3} \\
& \times \{ E |\mathcal{M}(\kappa_n(s), s)|^{3p/2} \}^{1/3} \\
& + K \left\{ E \left| \int_{\kappa_n(s)}^s \{ \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) \} dW_r^i \right|^p \right\}^{1/2} \\
& \times \{ E |\mathcal{M}(\kappa_n(s), s)|^p \}^{1/2}
\end{aligned}$$

and then one applies the estimates in equation (40), Remark 1 and Lemma 4.6 to obtain

$$\begin{aligned}
T_{22B} & \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p \\
& + Kn^{-p/4} \left\{ n^{-3p/8+1} E \int_{\kappa_n(s)}^s |\sigma_r(X_r^{i,N}, \mu_r^{X,N}) - \sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n})|^{3p/4} dr \right\}^{2/3} \\
& + Kn^{-p/4} \left\{ n^{-p/2+1} E \int_{\kappa_n(s)}^s |\sigma_r(X_r^{i,N,n}, \mu_r^{X,N,n}) - \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^p dr \right\}^{1/2} \\
& \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + Kn^{-p} \\
& + Kn^{-p/4} \left\{ n^{-3p/8+1} \int_{\kappa_n(s)}^s \{ E(1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{3\rho p/16} |X_r^{i,N} - X_r^{i,N,n}|^{3p/4} \right. \\
& \quad \left. + E \mathcal{W}_2^{3p/4}(\mu_r^{X,N}, \mu_r^{X,N,n}) \} dr \right\}^{2/3} \\
& \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + Kn^{-p} \\
& + Kn^{-p/4} \left\{ n^{-3p/8+1} \int_{\kappa_n(s)}^s \{ (E(1 + |X_r^{i,N}| + |X_r^{i,N,n}|)^{3\rho p/4})^{1/4} \right. \\
& \quad \left. \times (E |X_r^{i,N} - X_r^{i,N,n}|^p)^{3/4} + (E \mathcal{W}_2^p(\mu_r^{X,N}, \mu_r^{X,N,n}))^{3/4} \} dr \right\}^{2/3}
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Further, one uses Lemma 4.3, equation (19) and Young's inequality to obtain

$$\begin{aligned}
(44) \quad T_{22B} & \leq K \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + Kn^{-p} \\
& + Kn^{-p/2} \left\{ \left(\sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p \right)^{3/4} \right. \\
& \quad \left. + \left(\sup_{0 \leq r \leq s} \frac{1}{N} \sum_{j=1}^N E |X_r^{j,N} - X_r^{j,N,n}|^p \right)^{3/4} \right\}^{2/3} \\
& \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + Kn^{-p}
\end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

By adapting arguments similar to the one used in the estimation of T_{22B} , one obtains

$$\begin{aligned}
 T_{22C} &:= K E \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \} \\
 (45) \quad &\times \left| \int_{\kappa_n(s)}^s \{ \sigma_r^0(X_r^{i,N}, \mu_r^{X,N}) - \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) \} dW_r^0 \right| |\mathcal{M}(\kappa_n(s), s)| \\
 &\leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}
 \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

Hence, on combining estimates obtained in equations (43), (44) and (45) in equation (42), one obtains

$$(46) \quad T_{22} \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

By using methods similar to the one used in estimating T_{22} , one also obtains

$$\begin{aligned}
 T_{23} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 (47) \quad &\times \partial_x b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{i,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \\
 &\leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}
 \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

Thus, merging the estimates in equations (38), (46) and (47) yields

$$(48) \quad T_2 \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

For estimating T_3 , use equation (21) to get

$$\begin{aligned}
 T_3 &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\times \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) (X_s^{j,N,n} - X_{\kappa_n(s)}^{j,N,n}) \\
 &= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\times \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \\
 &\times \left\{ \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr + \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \right. \\
 &\left. + \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \right\} =: T_{31} + T_{32} + T_{33}
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$.

For estimating T_{31} , one uses the Cauchy–Schwarz inequality, Remarks 2, 6 and Hölder’s inequality to obtain

$$\begin{aligned}
 T_{31} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dr \\
 &\leq E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d |X_s^{(k),i,N} - X_s^{(k),i,N,n}| \\
 &\quad \times \frac{1}{N} \sum_{j=1}^N |\partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})| \int_{\kappa_n(s)}^s |b_{\kappa_n(r)}^n(X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})| dr \\
 &\leq K n^{-1} E |X_s^{i,N} - X_s^{i,N,n}|^{p-1} \{(1 + |X_{\kappa_n(s)}^{j,N,n}|)^{\rho/2+1} + \mathcal{W}_2(\mu_{\kappa_n(r)}^{X,N,n}, \delta_0)\},
 \end{aligned}$$

which on using Young’s inequality, equation (16) and Lemma 4.3 yields

$$(49) \quad T_{31} \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_s^{i,N} - X_s^{i,N,n}|^p + K n^{-p}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

For estimating T_{32} , let us define, for $k = 1, \dots, d$,

$$\begin{aligned}
 &\mathcal{M}^{N,(k)}(\kappa_n(s), s) \\
 (50) \quad &:= \frac{1}{N} \sum_{j=1}^N \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j
 \end{aligned}$$

and observe that due to Burkholder–Davis–Gundy inequality, Hölder’s inequality and Remark 6,

$$\begin{aligned}
 &E |\mathcal{M}^{N,(k)}(\kappa_n(s), s)|^q \\
 &\leq K E \frac{1}{N} \sum_{j=1}^N \left| \int_{\kappa_n(s)}^s \partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \right|^q \\
 &\leq K n^{-q/2+1} E \frac{1}{N} \sum_{j=1}^N \int_{\kappa_n(s)}^s |\partial_\mu b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n})|^q \\
 &\quad \times |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^q dr \\
 &\leq K n^{-q/2+1} \frac{1}{N} \sum_{j=1}^N \int_{\kappa_n(s)}^s E |\tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n})|^q dr,
 \end{aligned}$$

which on using Corollary 2 yields

$$(51) \quad E |\mathcal{M}^N(\kappa_n(s), s)|^q \leq K n^{-q/2}$$

for any $q \leq p_0/(\rho/2 + 1)$. Thus, using the notation defined in equation (50),

$$\begin{aligned}
 T_{32} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \frac{1}{N} \sum_{j=1}^N \partial_{\mu} b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^n(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^j \\
 &= E \left\{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{i,N} - X_s^{i,N,n}) \right. \\
 &\quad \left. - |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \sum_{k=1}^d (X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}) \right\} \mathcal{M}^N(\kappa_n(s), s) \\
 &\quad + E |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} (X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}) \mathcal{M}^N(\kappa_n(s), s)
 \end{aligned}$$

and then notice that the third term in the above equation vanishes because in view of Lemma 4.3, $\mathcal{M}^{N,(k)}(\kappa_n(s), s)$ is a martingale. Thus, equation (41) yields

$$\begin{aligned}
 T_{32} &\leq K E \{ |X_s^{i,N} - X_s^{i,N,n}|^{p-2} + |X_{\kappa_n(s)}^{i,N} - X_{\kappa_n(s)}^{i,N,n}|^{p-2} \} \\
 &\quad \times |X_s^{i,N} - X_{\kappa_n(s)}^{i,N} - (X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n})| \mathcal{M}^N(\kappa_n(s), s)
 \end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. Now, replacing $\mathcal{M}(\kappa_n(s), s)$ by $\mathcal{M}^N(\kappa_n(s), s)$ in equation (42) and in what follows along with the estimates in equation (51), one can obtain the following estimates:

$$(52) \quad T_{32} \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

By using methods similar to the one used in estimating T_{32} , one also obtains

$$\begin{aligned}
 (53) \quad T_{33} &:= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} \sum_{k=1}^d (X_s^{(k),i,N} - X_s^{(k),i,N,n}) \\
 &\quad \times \frac{1}{N} \sum_{j=1}^N \partial_{\mu} b_{\kappa_n(s)}^{(k)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}, X_{\kappa_n(s)}^{j,N,n}) \int_{\kappa_n(s)}^s \tilde{\sigma}_{\kappa_n(r)}^{0,n}(r, X_{\kappa_n(r)}^{j,N,n}, \mu_{\kappa_n(r)}^{X,N,n}) dW_r^0 \\
 &\leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}
 \end{aligned}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$.

Hence, on combining the estimates from equations (49), (52) and (53), one obtains

$$(54) \quad T_3 \leq K \sup_{i \in \{1, \dots, N\}} \sup_{0 \leq r \leq s} E |X_r^{i,N} - X_r^{i,N,n}|^p + K n^{-p}$$

for any $s \in [0, T]$ and $n, N \in \mathbb{N}$. The proof of equation (34) is complete by substituting estimates from equations (36), (48) and (54) in equation (35).

In order to complete the proof, we consider

$$\begin{aligned}
 &E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) (b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
 &= E |X_s^{i,N} - X_s^{i,N,n}|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) (b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}))
 \end{aligned}$$

$$\begin{aligned}
& + b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \\
& + b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}),
\end{aligned}$$

which on using Young's inequality, equation (20), Assumption 8 and equation (34) yields

$$\begin{aligned}
& E|X_s^{i,N} - X_s^{i,N,n}|^{p-2}(X_s^{i,N} - X_s^{i,N,n})(b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
& \leq KE|X_s^{i,N} - X_s^{i,N,n}|^p + E|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n})|^p \\
& \quad + E|X_s^{i,N} - X_s^{i,N,n}|^{p-2}(X_s^{i,N} - X_s^{i,N,n}) \\
& \quad \times (b_{\kappa_n(s)}(X_s^{i,N,n}, \mu_s^{X,N,n}) - b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})) \\
& \quad + E|b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - b_{\kappa_n(s)}^n(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})|^p \\
& \leq Kn^{-p} + E\left|b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) - \frac{b_{\kappa_n(s)}(X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n})}{1 + n^{-1}|X_{\kappa_n(s)}^{i,N,n}|^\rho}\right|^p
\end{aligned}$$

for any $s \in [0, T]$, $i \in \{1, \dots, N\}$ and $n, N \in \mathbb{N}$. The proof is complete by using Remark 2, equation (16) and Lemma 4.3.

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REFERENCES

- [1] ADAMS, D., DOS REIS, G., RAVAILLE, R., SALKELD, W. and TUGAUT, J. (2020). Large deviations and exit-times for reflected McKean–Vlasov equations with self-stabilizing terms and superlinear drifts. Preprint. Available at [arXiv:2005.10057](https://arxiv.org/abs/2005.10057).
- [2] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2008). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. [MR2401600](https://doi.org/10.1007/978-3-03910-611-1)
- [3] BAO, J. and HUANG, H. (2021). Approximations of McKean–Vlasov SDEs with irregular coefficients. *J. Theor. Probab.* **159**. <https://doi.org/10.1007/s10959-021-01082-9>
- [4] BAO, J., REISINGER, C., REN, P. and STOCKINGER, W. (2020). Milstein schemes for delay McKean–Vlasov equations and interacting particle systems. Preprint. Available at [arXiv:2005.01165](https://arxiv.org/abs/2005.01165).
- [5] BAO, J., REISINGER, C., REN, P. and STOCKINGER, W. (2021). First-order convergence of Milstein schemes for McKean–Vlasov equations and interacting particle systems. *Proc. R. Soc. A* **477** Paper No. 20200258, 27 pp. [MR4212406](https://doi.org/10.1098/rspa.2020.0258)
- [6] BAUER, M., MEYER-BRANDIS, T. and PROSKE, F. (2018). Strong solutions of mean-field stochastic differential equations with irregular drift. *Electron. J. Probab.* **23** Paper No. 132, 35 pp. [MR3896869](https://doi.org/10.1214/18-EJP259)
- [7] BEYN, W.-J., ISAAK, E. and KRUSE, R. (2017). Stochastic C-stability and B-consistency of explicit and implicit Milstein-type schemes. *J. Sci. Comput.* **70** 1042–1077. [MR3608332](https://doi.org/10.1007/s10915-016-0290-x)
- [8] BOSSY, M. and TALAY, D. (1997). A stochastic particle method for the McKean–Vlasov and the Burgers equation. *Math. Comp.* **66** 157–192. [MR1370849](https://doi.org/10.1090/S0025-5718-97-00776-X)
- [9] BUJOK, K., HAMBLY, B. M. and REISINGER, C. (2015). Multilevel simulation of functionals of Bernoulli random variables with application to basket credit derivatives. *Methodol. Comput. Appl. Probab.* **17** 579–604. [MR3377850](https://doi.org/10.1007/s11009-013-9380-5)

- [10] CARDALIAGUET, P. (2013). Notes on mean-field games, notes from P. L. Lions lectures at Collège de France. Available at <https://www.ceremade.dauphine.fr/cardalia/MFG100629.pdf>.
- [11] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications I: Mean Field FBSDEs, Control, and Games. Probability Theory and Stochastic Modelling* **83**. Springer, Cham. MR3752669
- [12] CARMONA, R. and DELARUE, F. (2018). *Probabilistic Theory of Mean Field Games with Applications II: Mean Field Games with Common Noise and Master Equations. Probability Theory and Stochastic Modelling* **84**. Springer, Cham. MR3753660
- [13] CHEN, X. and DOS REIS, G. (2021). A flexible split-step scheme for MV-SDEs. Preprint. Available at [arXiv:2105.09688](https://arxiv.org/abs/2105.09688).
- [14] DING, X. and QIAO, H. (2021). Euler–Maruyama approximations for stochastic McKean–Vlasov equations with non-Lipschitz coefficients. *J. Theoret. Probab.* **34** 1408–1425. MR4289889 <https://doi.org/10.1007/s10959-020-01041-w>
- [15] DOS REIS, G., ENGELHARDT, S. and SMITH, G. (2022). Simulation of McKean–Vlasov SDEs with super-linear growth. *IMA J. Numer. Anal.* **42** 874–922. MR4367675 <https://doi.org/10.1093/imanum/draa099>
- [16] DOS REIS, G., SALKELD, W. and TUGAUT, J. (2019). Freidlin–Wentzell LDP in path space for McKean–Vlasov equations and the functional iterated logarithm law. *Ann. Appl. Probab.* **29** 1487–1540. MR3914550 <https://doi.org/10.1214/18-AAP1416>
- [17] GILES, M. B. (2015). Multilevel Monte Carlo methods. *Acta Numer.* **24** 259–328. MR3349310 <https://doi.org/10.1017/S096249291500001X>
- [18] GOARD, J. and MAZUR, M. (2013). Stochastic volatility models and the pricing of VIX options. *Math. Finance* **23** 439–458. MR3070371 <https://doi.org/10.1111/j.1467-9965.2011.00506.x>
- [19] GYÖNGY, I. and KRYLOV, N. (2003). On the rate of convergence of splitting-up approximations for SPDEs. In *Stochastic Inequalities and Applications. Progress in Probability* **56** 301–321. Birkhäuser, Basel. MR2073438
- [20] GYÖNGY, I. and KRYLOV, N. V. (1980/81). On stochastic equations with respect to semimartingales. I. *Stochastics* **4** 1–21. MR0587426 <https://doi.org/10.1080/03610918008833154>
- [21] HAMMERSLEY, W. R. P., ŠIŠKA, D. and SZPRUCH, Ł. (2021). Weak existence and uniqueness for McKean–Vlasov SDEs with common noise. *Ann. Probab.* **49** 527–555. MR4255126 <https://doi.org/10.1214/20-aop1454>
- [22] HAMMERSLEY, W. R. P., ŠIŠKA, D. and SZPRUCH, Ł. (2021). McKean–Vlasov SDEs under measure dependent Lyapunov conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1032–1057. MR4260494 <https://doi.org/10.1214/20-aihp1106>
- [23] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2012). Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.* **22** 1611–1641. MR2985171 <https://doi.org/10.1214/11-AAP803>
- [24] KUMAR, C. and NEELIMA (2021). On explicit Milstein-type scheme for McKean–Vlasov stochastic differential equations with super-linear drift coefficient. *Electron. J. Probab.* **26** Paper No. 111, 32 pp. MR4302574 <https://doi.org/10.1214/21-ejp676>
- [25] KUMAR, C. and SABANIS, S. (2017). On explicit approximations for Lévy driven SDEs with super-linear diffusion coefficients. *Electron. J. Probab.* **22** Paper No. 73, 19 pp. MR3698742 <https://doi.org/10.1214/17-EJP89>
- [26] KUMAR, C. and SABANIS, S. (2019). On Milstein approximations with varying coefficients: The case of super-linear diffusion coefficients. *BIT* **59** 929–968. MR4032893 <https://doi.org/10.1007/s10543-019-00756-5>
- [27] LEDGER, S. and SØJMARK, A. (2021). At the mercy of the common noise: Blow-ups in a conditional McKean–Vlasov problem. *Electron. J. Probab.* **26** Paper No. 35, 39 pp. MR4235486 <https://doi.org/10.1214/21-EJP597>
- [28] LEOBACHER, G., REISINGER, C. and STOCKINGER, W. (2020). Well-posedness and numerical schemes for one-dimensional McKean–Vlasov equations and interacting particle systems with discontinuous drift. Preprint. Available at [arXiv:2006.14892](https://arxiv.org/abs/2006.14892).
- [29] MAO, X. (1997). *Stochastic Differential Equations and Their Applications. Horwood Publishing Series in Mathematics & Applications*. Horwood Publishing Limited, Chichester. MR1475218
- [30] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595 <https://doi.org/10.1073/pnas.56.6.1907>
- [31] MEHRI, S., SCHEUTZOW, M., STANNAT, W. and ZANGENEH, B. Z. (2020). Propagation of chaos for stochastic spatially structured neuronal networks with delay driven by jump diffusions. *Ann. Appl. Probab.* **30** 175–207. MR4068309 <https://doi.org/10.1214/19-AAP1499>
- [32] MISHURA, Y. S. and VERETENNIKOV, A. Y. (2020). Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. *Theory Probab. Math. Statist.* **103** 59–101.

- [33] NEELIMA, BISWAS, S., KUMAR, C., DOS REIS, G. and REISINGER, C. (2020). Well-posedness and tamed Euler schemes for McKean–Vlasov equations driven by Lévy noise. Preprint. Available at [arXiv:2010.08585](https://arxiv.org/abs/2010.08585).
- [34] PHAM, H. and PHAM, H. (2016). Linear quadratic optimal control of conditional McKean–Vlasov equation with random coefficients and applications. *Probab. Uncertain. Quant. Risk* **1** Paper No. 7, 26 pp. [MR3583182 https://doi.org/10.1186/s41546-016-0008-x](https://doi.org/10.1186/s41546-016-0008-x)
- [35] REISINGER, C. and STOCKINGER, W. (2022). An adaptive Euler–Maruyama scheme for McKean–Vlasov SDEs with super-linear growth and application to the mean-field FitzHugh–Nagumo model. *J. Comput. Appl. Math.* **400** Paper No. 113725, 23 pp. [MR4293705 https://doi.org/10.1016/j.cam.2021.113725](https://doi.org/10.1016/j.cam.2021.113725)
- [36] SABANIS, S. (2013). A note on tamed Euler approximations. *Electron. Commun. Probab.* **18** Paper No. 47, 10 pp. [MR3070913 https://doi.org/10.1214/ECP.v18-2824](https://doi.org/10.1214/ECP.v18-2824)
- [37] SABANIS, S. (2016). Euler approximations with varying coefficients: The case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab.* **26** 2083–2105. [MR3543890 https://doi.org/10.1214/15-AAP1140](https://doi.org/10.1214/15-AAP1140)
- [38] ŠIŠKA, D. and SZPRUCH, Ł. (2020). Gradient flows for regularized stochastic control problems. Preprint. Available at [arXiv:2006.05956v3](https://arxiv.org/abs/2006.05956).
- [39] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D’Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. [MR1108185 https://doi.org/10.1007/BFb0085169](https://doi.org/10.1007/BFb0085169)
- [40] ULLNER, E., POLITI, A. and TORCINI, A. (2018). Ubiquity of collective irregular dynamics in balanced networks of spiking neurons. *Chaos* **28** 081106, 5 pp. [MR3848177 https://doi.org/10.1063/1.5049902](https://doi.org/10.1063/1.5049902)
- [41] VILLANI, C. (2011). Optimal transport: Monge meets Riemann and Fourier. *J. Egyptian Math. Soc.* **19** 95–96. [MR2914136 https://doi.org/10.1016/j.joems.2011.09.007](https://doi.org/10.1016/j.joems.2011.09.007)
- [42] WANG, F.-Y. (2018). Distribution dependent SDEs for Landau type equations. *Stochastic Process. Appl.* **128** 595–621. [MR3739509 https://doi.org/10.1016/j.spa.2017.05.006](https://doi.org/10.1016/j.spa.2017.05.006)