

Essays on the Implementation of Monetary Policy



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A thesis submitted for the degree of

Doctor of Philosophy

Hilary 2015

Acknowledgements

This work was completed through support of the University of Oxford Clarendon Fund Scholarship, Merton College Domus A Scholarship, and the Reserve Bank of Australia Post Graduate Study Award.

I would like to thank my M.Phil./D.Phil. supervisor Martin Ellison and M.Phil. supervisor Antonio Mele for all of their guidance and assistance.

I would like to thank the Reserve Bank of Australia for access to their data.

This thesis benefited from the advice of Guido Ascari, Huberto Ennis, Charles Gottlieb, Ricardo Lagos, Pascal Paul, and participants of the Oxford Macroeconomics Working Group.

I am forever indebted to my Australian family and friends, for both their unwavering support, and for not forgetting me, despite my extended absence.

And to my Mertonian family – who ensured my years at Oxford were some of the greatest of my life – thank you.

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Chapter 1 builds a two-bank bargaining model of the overnight interbank market in which, due to the commitment of the central bank to its interest rate target, bargaining between banks impacts loan sizes rather than interest rates (the converse of existing models). As a result, policy changes have a different impact to what is posited by existing models.

The model is applied to a market where the commitment of the central bank is well documented (Australia). With reasonable parameter values, the model replicates five stylised facts of the Australian market. Moreover, the stylised facts are replicated without recourse to any asymmetries.

Chapter 2 extends the two-bank model to incorporate a large number of heterogeneous banks. This model is able to replicate the asymmetric shape of banks' end-of-day central bank deposit distributions (despite symmetric initial distributions); a novel contribution to the literature. Moreover, after inputting recent changes in Australian central bank policy, this model produces percentage changes in interbank trading volumes that closely align with the data.

Central banks typically supply more overnight deposits than banks desire to hold (in aggregate), but this aggregate is typically small relative to interbank lending. With commitment, this is not required for the central bank to achieve its interest rate target (the typical explanation in the literature). So, to explain this phenomenon, Chapter 3 builds a DSGE model that incorporates commitment and the results from the previous chapters.

Due to asymmetric information, there may be stigma associated with borrowing from the central bank's overnight lending facility, which is costly. But while the central bank can reduce use of its lending facility, by increasing aggregate deposits, the resulting fall in interbank lending is also costly; because the interbank market helps banks monitor their counterparties. Therefore, low but positive aggregate deposits can be explained as the welfare-optimising point in the trade-off between stigma and monitoring costs (a novel contribution to the literature).

Official word count: 92,610 (294 pages, including 151 pages of appendices)

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Part I

Central Bank Commitment and the Overnight Interbank Market

Chapter 1

Two-bank Model and Qualitative Results

1.1 Introduction

The overnight interbank market is the market in which banks make overnight loans to each other (henceforth, trades).¹ These trades cause funds to move between the deposit accounts the trading banks have at the central bank (henceforth, deposits). And it is by influencing the interest rates at which these trades occur, that many central banks influence longer-term interbank and non-bank interest rates. In other words, monetary policy is often implemented through this market; so it is crucial to properly understand how the market works.

Many central banks currently implement monetary policy via a ‘corridor system’.² The typical features of corridor systems are:

- the central bank operates a facility (henceforth, lending facility) from which banks

¹The overnight interbank market includes all institutions that hold deposit accounts with the central bank, not just banks. For simplicity, this thesis refers to all of these institutions as banks.

²Central banks operating some form of corridor system include the Federal Reserve, European Central Bank, Bank of England, Bank of Japan, Reserve Bank of Australia, Bank of Canada, and Reserve Bank of New Zealand.

can borrow funds, overnight, at some predefined interest rate (henceforth, lending rate) – the corridor roof;

- the central bank pays interest on overnight deposits that exceed some predetermined level (this deposit rate will be lower than the lending rate, and can be zero) – the corridor floor; and
- the central bank operates such that the average interest rate at which trades occur (henceforth, average trading rate) is equal to a publicly announced point between the lending and deposit rates (inclusive) – the target rate.

In existing models of corridor systems, the central bank can change the average trading rate by shifting the corridor, aggregate deposits need not change. But within the corridor, there is a unique level of aggregate deposits that aligns the average trading rate with the central bank’s target.³

This uniqueness is at odds with the assertions of both academics and practitioners. For example, when commenting on *Bech and Klee* (2011), *Furfine* (2011) explains that:

Interbank market participants trade at an interest rate at or near the central bank’s target rate because they rationally anticipate that the central bank can (and will, if necessary) adjust reserves [deposits] to the level consistent with that rate.

While *Disyatat* (2008) states that:

... a policy rate is simply a commitment by the central bank to use instruments at its disposal to achieve the target and so long as that commitment is credible, the instruments do not actually need to be used for this purpose.

Commenting on the Australian market, the Reserve Bank of Australia (RBA) explains that (*Baker and Jacobs*, 2010):

³In most models of corridor systems, all trades occur at a market clearing rate that banks treat as exogenous; examples include the models of: *Henckel et al.* (1999), *Guthrie and Wright* (2000), and *Woodford* (2001). However, some recent models incorporate the over-the-counter structure of the market by having the terms of trade determined through bilateral bargaining; examples include the models of: *Bech and Klee* (2011), *Afonso and Lagos* (2012), *Ennis and Weinberg* (2013), and *Bech and Monnet* (2013).

In practice, short-term movements in system cash [aggregate deposits] do not tend to affect the cash rate [average trading rate]. In part, this is due to the anchor provided by a publicly announced target and the very strong commitment by the Reserve Bank to ensure that the cash rate stays at the target rate.

To narrow the gap between the modelling literature and the practicalities of the market, I build a bilateral bargaining model in which the commitment of the central bank ensures that all trades occur at the target rate, irrespective of the aggregate volume of deposits.⁴

In existing bilateral bargaining models, changes in the relative bargaining positions of trading banks are reflected in the trading rate, the volume of trade is unaffected. With commitment ensuring that the target rate is the only rate at which both banks are willing to trade, changes in relative bargaining positions leave the trading rate unaffected but impact the trading volume.

As a result, changes to a central bank's operations (i.e. changes in the corridor width or aggregate deposits) have a different impact to what is posited by existing models. This has important implications for both monetary policy implementation (as these changes will impact trading volumes rather than trading rates) and the central bank's balance sheet (these changes will impact the frequency with which banks access the lending facility). Determining how the market will react to a change in operations, and therefore how operations may need to change as a result of an exogenous shock to the market, is the key motivation for this research.⁵

To evaluate the performance of my model, it is applied to a market in which the commitment of the central bank is well documented (Australia). With reasonable parameter values, the model replicates five stylised facts of the Australian market. The

⁴With respect to commitment, my research is positive in nature; determining the optimal strength of central bank commitment is left for future research.

⁵For example, if the volume of interbank market activity has welfare implications (as shown in Chapter 3), an exogenous change to the volume of activity may require the central bank to change its operations. This chapter provides qualitative guidance for markets in which there is strong commitment, while Chapter 2 provides quantitative guidance for Australia.

five stylised facts are: 1) aggregate deposits are greater than reserve requirements; 2) trading occurs well before the end of the day; 3) banks typically finish the day with positive deposits; 4) the average trading rate is almost always equal to the central bank's target; and 5) the central bank's lending facility is rarely used.

In order to replicate these stylised facts, the model incorporates uncertainty, transaction costs, and two trading periods. While these features are part of the existing modelling literature, this is the first time they have been simultaneously included within a bilateral bargaining framework.⁶ The principal mechanics of the model are as follows.

Due to the cost of borrowing from the lending facility, or the opportunity cost of finishing the day with excess deposits (if they could have been lent in the market at the target rate), there are mutual gains from trade if one bank has fewer deposits than it desires and the other bank has more. Exogenous transactions cause deposit levels to differ from banks' desired end-of-day holdings (which will be zero in this model, for simplicity). While banks are able to defer trading until all exogenous transactions have occurred, late-day transaction costs entice banks to trade while the total volume of these flows is uncertain, and typically prevent trades from occurring after this volume is known (i.e. banks trade in the first period but not the second, replicating the second stylised fact).

When banks meet in the first period, the commitment of the central bank ensures that the only interest rate at which they are both willing to trade is the central bank's target rate (replicating the fourth stylised fact). Banks then bargain over the trading volume. When the central bank supplies more deposits than banks desire to hold (i.e. when the first stylised fact holds), someone has to hold the excess deposits at the end of the day. If, after trading, one of the banks had its optimum level of deposits and the other bank held the entire excess, there would be a first-order benefit to the latter

⁶Uncertainty has been included in overnight interbank market models since at least *Poole* (1968). The model in *Ashcraft et al.* (2011) incorporates two trading periods (with uncertainty between periods). The model in *Bindseil et al.* (2006) incorporates transaction costs.

bank from lending more, but only a second-order cost to the former bank (because the expected profit curve is flat at the optimum). So bargaining typically results in banks sharing the excess after trading in the first period (partially replicating the third stylised fact).

To fully replicate the third stylised fact, the exogenous uncertainty must be sufficiently small such that banks' excess holdings are typically not completely eroded by the end of the second period. While it is the parameter calibration that ensures this fact, importantly, one of the calibration requirements is that the central bank supplies excess deposits (i.e. the first stylised fact is required to replicate the third stylised fact). And with banks typically holding excess deposits at the end of the day, they rarely borrow from the central bank's lending facility (the fifth stylised fact).

Importantly, this model replicates these stylised facts without recourse to any of the asymmetries that would be required when applying existing models (such as stigma creating an extra cost of using the lending facility, risk premia associated with interbank lending, or asymmetric bargaining power). This is particularly important for Australia, where there is no evidence of these asymmetries being prevalent (so relying on them to replicate the stylised facts would be inappropriate).

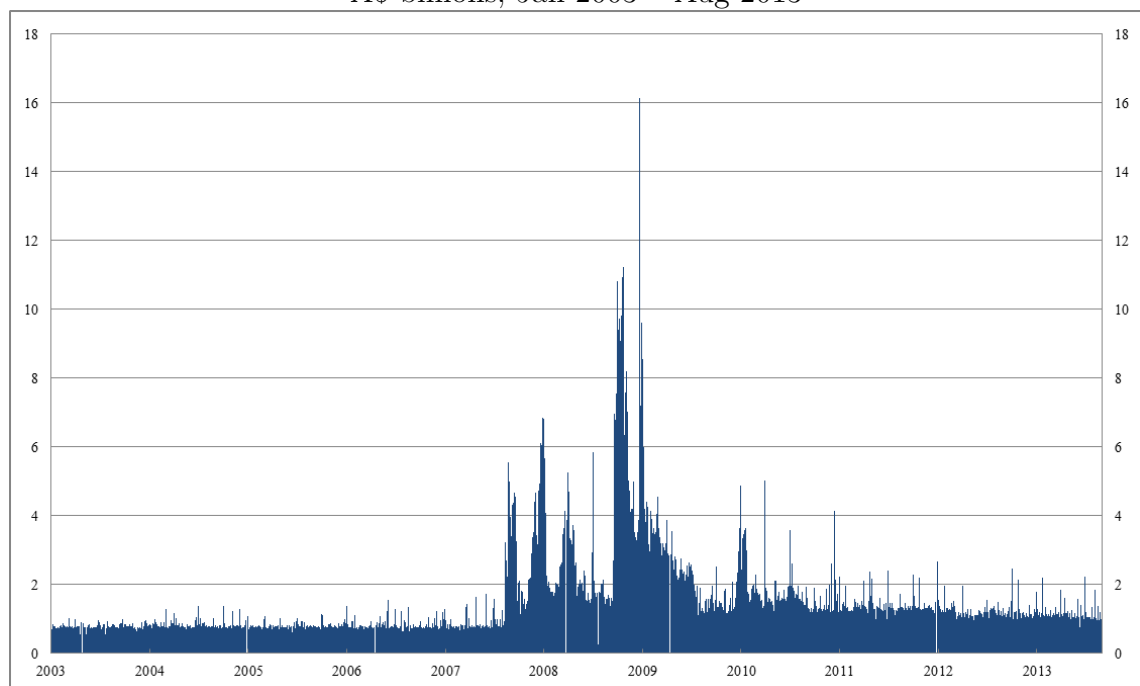
In the remainder of this chapter, Section 1.2 describes five stylised facts of the Australian market. Section 1.3 constructs and solves the model, and gives an intuitive explanation of the model's mechanics. Section 1.4 discusses the calibration/parameterisation used for simulating the Australian market. Section 1.5 discusses the success of the model in replicating the stylised facts, and provides some simulated comparative statics. Section 1.6 concludes. Appendix 1.7.2 discusses the existing literature.

1.2 Stylised facts

1.2.1 Aggregate deposits are greater than reserve requirements

In Australia, central bank deposit accounts are known as Exchange Settlement (ES) accounts. Australian banks are not required to maintain a pre-specified positive ES balance (*Campbell, 1997*). That is, reserve requirements in Australia are zero. However, the aggregate end-of-day ES balance is consistently well above zero (Figure 1.1).⁷ This stylised fact will henceforth be known as the ‘positive aggregate’ fact.

Figure 1.1: Daily Aggregate End-of-day ES Balance*
A\$ billions; Jan 2003 – Aug 2013



* Excludes ES balances borrowed via the lending facility.

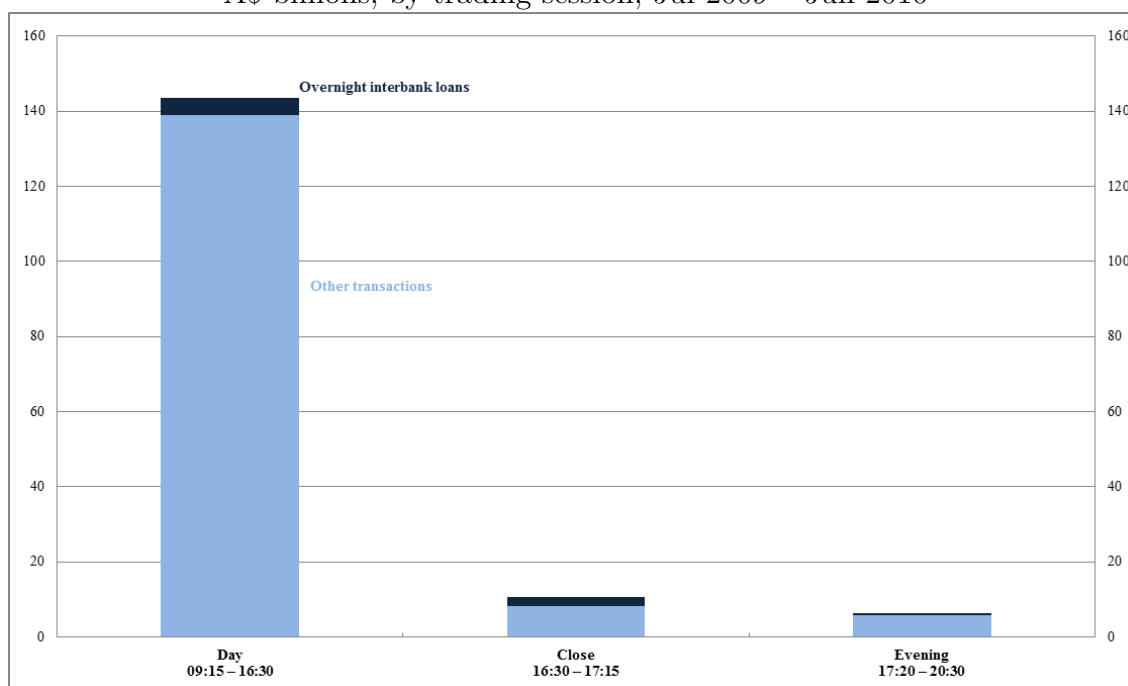
Source: Reserve Bank of Australia Statistical Table A3

⁷In this thesis, references to the aggregate end-of-day ES balance exclude ES balances borrowed via the lending facility.

1.2.2 Trading occurs well before the end of the day

In addition to trading (overnight interbank loans), there are many other transactions that also result in ES balances moving between banks (transactions between customers of different banks, for example). Figure 1.2 shows the average value of daily transactions (including trades) that cause a transfer of funds between banks' ES accounts – split into the three trading sessions of the Australian trading day. 91 per cent of trades are made before 17:15, well before the close of trade. On average (between July 2009 and June 2010), there were less than A\$8 billion of trades per day and more than A\$150 billion of other interbank transactions. This stylised fact will henceforth be known as the 'early trade' fact.

Figure 1.2: Average Value of Transactions
A\$ billions; by trading session; Jul 2009 – Jun 2010



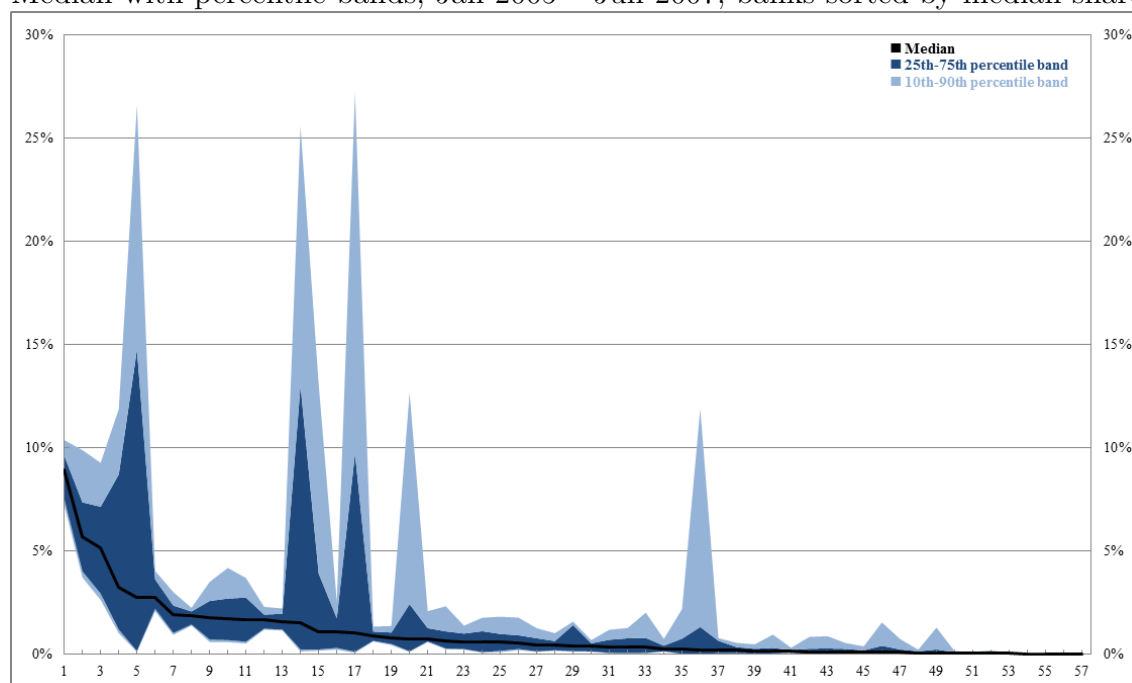
Source: Reserve Bank of Australia (data not publicly available)

1.2.3 Banks typically finish the day with a positive ES balance

Figure 1.3 shows the distributions of each bank's end-of-day share of the aggregate ES balance. With the majority of the distribution above zero for most banks, these data suggest that banks typically finish the day with a positive ES balance. This stylised fact will henceforth be known as the 'positive end-of-day' fact.

Figure 1.3: Individual Bank's End-of-day Shares of the Aggregate ES Balance

Median with percentile bands; Jan 2005 – Jun 2007; banks sorted by median share



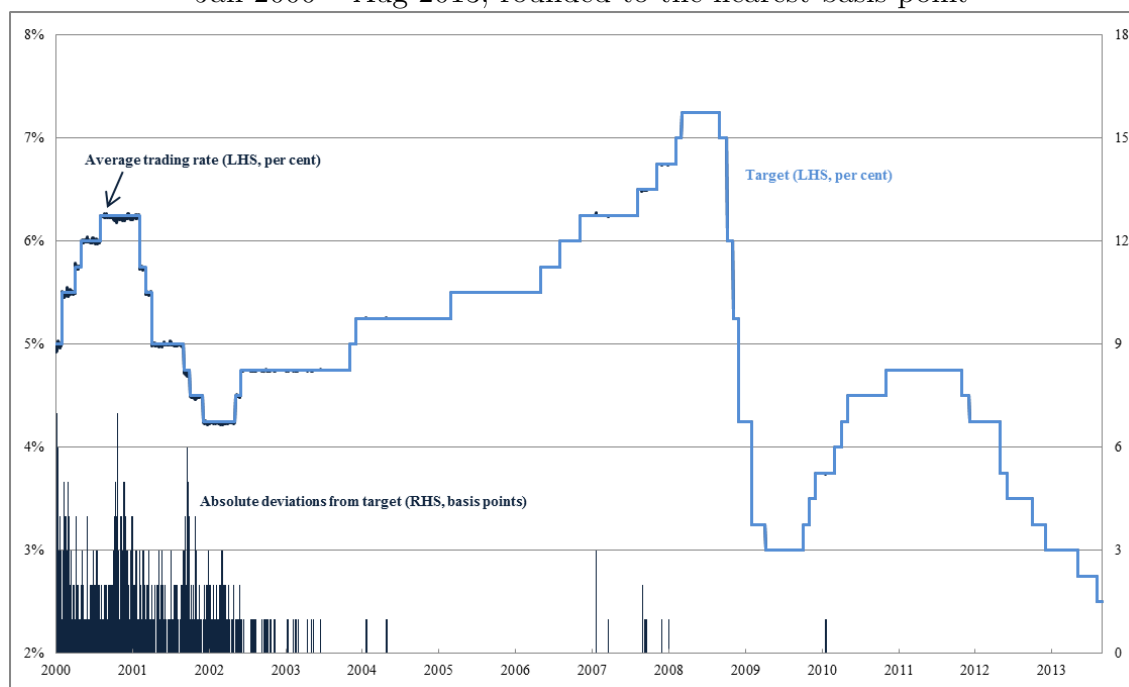
Source: Reserve Bank of Australia (data not publicly available)

1.2.4 The average trading rate is almost always equal to the central bank's target

The RBA administers a symmetric corridor system; i.e. the target rate is halfway between the lending and deposit rates. Volatility of the daily average trading rate in Australia has been almost non-existent in recent years; the average rate has deviated from target only twice since the beginning of 2008, and each time by just 1 basis point

(rounded to the nearest basis point, Figure 1.4). This stylised fact will henceforth be known as the ‘equal to target’ fact.

Figure 1.4: Australian Daily Average Trading Rates
Jan 2000 – Aug 2013; rounded to the nearest basis point



Source: Reserve Bank of Australia Statistical Table F1

1.2.5 The central bank’s lending facility is rarely used

The RBA’s lending facility was accessed an average of 11 times per year between 2005 and 2012 (Table 1.1). This accounts for around 4.5 per cent of trading days. This stylised fact will henceforth be known as the ‘rare use’ fact.

Table 1.1: Use of the RBA’s Lending Facility

	Number of times used	Per cent of trading days
2005/06	10	3.95
2006/07	24	9.56
2007/08	18	7.20
2008/09	15	5.95
2009/10	5	1.99
2010/11	2	0.80
2011/12	5	1.99
Average	11	4.49

Source: Reserve Bank of Australia Annual Reports

1.3 Model

To qualitatively replicate these stylised facts, my model requires just two banks – that are risk neutral, profit maximisers – and two trading periods per day. With two trading periods, banks are able to choose between trading close to the end of the day (the second period) and trading earlier in the day (the first period), thereby facilitating explanation of the ‘early trade’ fact. There are two differences between trading in the first and second periods (i.e. two features that impact the timing-of-trade decision):

- there is a transaction cost associated with second-period trading; and
- first-period trading occurs while there is uncertainty.

Late-day trading costs have previously been modelled by *Bindseil et al.* (2006). They suggest that, since the European interbank market is open from early in the morning, the cost of always staffing the entire trading day would be excessive. Instead, banks face an extra labour cost associated with ‘staying in the office’ that they may choose to pay on a daily basis. Since Australia also has a long trading day, the transaction cost in my model is given the same interpretation. Therefore, the end of the first period can be interpreted as the end of the normal working day (i.e. the majority of the trading

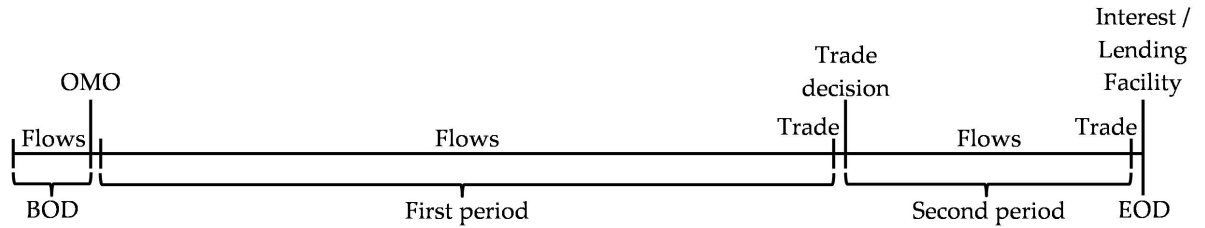
day), while the second period represents a short interval before the close of trade. Both banks must pay the transaction cost before second-period trading can occur.

The ‘other transactions’ discussed in Section 1.2.2 will henceforth be known as ‘payment flows’ and are treated as exogenous in the model (consistent with the existing literature, see Appendix 1.7.2 for examples). When trading occurs in the first period, banks are uncertain about the volume of second-period payment flows; this uncertainty dissipates before any second-period trading.⁸ Uncertainty is necessary for my model to replicate the ‘rare use’ fact. This is because, without uncertainty, the lending facility would never be used.

Due to the uncertainty, and if there was no transaction cost, one of the banks would always prefer to wait until the second period before trading. Therefore, in this model, the transaction cost is necessary for any trading to occur in the first period, and is therefore necessary to replicate the ‘early trade’ fact.

⁸Late-day payment flow uncertainty is common in the literature (e.g. *Poole* (1968), *Henckel et al.* (1999), and *Woodford* (2001)). As suggested by *Furfine* (2000) and *Bindseil et al.* (2006), the uncertainty includes human error, technical problems, unexpected payments, and expected payments that fail to arrive.

The following timeline shows the daily timing of payment flows, interbank trading, open market operations, and use of the central bank's lending and deposit facilities. While the timeline consists of four periods, trading can only occur in two of these periods.⁹



Beginning of day (BOD)

This period sees the occurrence of all flows that change the aggregate ES balance (e.g. flows to/from the Government), and also sees the occurrence of open market operations (OMO). The central bank uses open market operations to move the aggregate ES balance back to its desired level (which is non-negative).¹⁰ The central bank operates such that each bank begins the day with an equal share of the aggregate, and so that banks' profits are unaffected by the operations.¹¹

First period

The first period begins following open market operations. It is this period that sees the occurrence of the majority of payment flows (reflecting the data, see Figure 1.2).¹²

⁹Although the timeline shows gaps between the end of one period and the beginning of the next, this is just to improve readability (i.e. there are no such gaps in the model).

¹⁰In reality, the RBA has the ability to conduct intra-day open market operations, and therefore has complete control over the aggregate ES balance. For simplicity, I assume only one open market operation is needed (which, prior to November 2013, was typically the case in reality, see *Debelle* (2013)), and that the aggregate ES balance is known by both banks. Therefore, in this thesis, the terms 'aggregate end-of-day ES balance' and 'aggregate ES balance' are equivalent.

¹¹Having banks' beginning-of-day ES balances and profits being independent of their previous end-of-day ES balances is a simplifying assumption that allows each trading day to be solved independently (and recursively). The ability to solve each day independently is a common assumption in the literature (the papers discussed in Appendix 1.7.2 are examples).

¹²First-period payment flows include any payment flows that are expected to occur during the second period.

Denote Bank i 's ES balance following first-period payment flows as ES_0^i (where $i \in \{1, 2\}$ to reflect the two banks).

At the end of this period, the banks trade (while second-period payment flows are uncertain). Denote Bank i 's ES balance following any trading as ES_1^i .

Second period

After first-period trading, the second-period payment flows are revealed, and banks decide whether to pay the transaction cost and conduct second-period trading (i.e. there is no uncertainty when banks make this decision).¹³ Once this decision is made, the second-period payment flows occur (i.e. while uncertainty about these flows dissipates at the beginning of the second period, the flows actually occur during the period). Denote Bank i 's ES balance following second-period payment flows as ES_2^i .

At the end of the second period, if both banks decided to conduct second-period trading, the banks trade.¹⁴ Denote Bank i 's ES balance following any second-period trading as ES_T^i .

End of day (EOD)

After any second-period trading, any bank with an ES balance surplus (i.e. $ES_T^i > 0$) receives interest on this surplus (at the deposit rate), all banks have the option to access the lending facility.¹⁵

To determine the decisions of the banks at each decision-making point in the timeline, the remainder of this section solves the model recursively.

¹³Having no uncertainty when deciding whether to conduct second-period trading is a simplifying assumption. A more realistic (but less analytically appealing) version of the model – in which banks make the second-period trading decision while there is still uncertainty – did not produce significantly different results.

¹⁴In the Australian market, there could still be some ES balance uncertainty when second-period trading occurs. While this feature is not captured by this model, another version of the model, that retained some uncertainty in the second period, did not produce significantly different results.

¹⁵In reality, these interest payments will occur the following day. However, as long as the central bank accounts for these interest payments when conducting open market operations, the timing of these interest payments is immaterial.

1.3.1 End-of-day

At the end of the trading day, all exogenous payment flows have occurred and banks are no longer allowed to trade.

Australian banks are never allowed to have a negative ES balance. However, the RBA offers interest-free intra-day loans to banks. Moreover, although these loans are collateralised, banks typically hold more than enough collateral on their balance sheets, so there is no extra cost due to collateralisation.¹⁶ So, for simplicity, the model abstracts from these intra-day loans and allows banks to have negative ES balances during the day (but not overnight).

With banks required to have a non-negative end-of-day ES balance, any bank with an ES balance deficit after the second period must borrow from the central bank's lending facility to make their ES balance non-negative. The interest rate charged on these loans is r^L (the lending rate).¹⁷ Due to the absence of stigma associated with use of the RBA's lending facility (*Debelle, 2008*), and the large value of potential collateral on banks' balance sheets, the true cost of collateralised borrowing from the RBA's lending facility is unlikely to significantly differ from the lending rate (i.e. there are no asymmetries in the Australian market that result from stigma or costly collateral requirements).

With no reserve requirements, banks are not required to maintain a positive ES balance overnight. And with no payment flows occurring overnight (any overnight transactions settle the following day), and banks able to access intra-day repurchase agreements from the opening of trade the next day, overnight ES balances are not needed for liquidity purposes. So banks do not need a positive ES balance overnight.¹⁸

¹⁶In March 2012, banks held around A\$270 billion of eligible collateral (*Heath and Manning, 2012*), while the average daily value of intra-day repurchase agreements (the intra-day loans) was A\$14 billion (RBA Statistical Table A3).

¹⁷In this model, all interest rates are overnight interest rates (as opposed to annualised rates).

¹⁸From November 2013, the RBA's operational framework changed, and some payment flows now occur overnight (*Debelle, 2013*). However, modelling the new system requires only minor changes to the model presented in this chapter, with these changes leaving the mechanics of the model unchanged

In *normal* times, banks' demand for risk-free assets is mostly satiated by assets other than ES balances; this is due to the typically lower returns on ES balances relative to other risk-free assets (*Debelle*, 2013). While this is not always the case, as long as any increased demand for overnight ES balances is satisfied by the central bank, the market should function as *normal*.¹⁹ So, for simplicity, I model the *normal* circumstances and assume the only benefit a bank gets from having a positive end-of-day ES balance is the interest it receives from the central bank (at the deposit rate, r^D).

Since $r^L > r^D$ in a corridor system, a bank with an end of day ES balance deficit will only ever borrow enough to return its ES balance to zero. Therefore, the above discussion can be summarised in the profit Bank i makes from its end-of-day ES balance (where $i \in \{1, 2\}$):

$$\pi_T^i = \begin{cases} r^L ES_T^i & \text{if } ES_T^i < 0 \\ r^D ES_T^i & \text{if } ES_T^i \geq 0 \end{cases} \quad (1.1)$$

Recall that ES_T^i is Bank i 's ES balance after any second-period trading (but before accessing the lending facility).

For use in the next section, Eq. (1.1) can be re-written as:

$$\pi_T^i = r^D [\max(ES_T^i, 0)] + r^L [\min(ES_T^i, 0)] \quad (1.2)$$

1.3.2 Second-period trading

When trading in the second period (i.e. after both banks have paid the transaction cost), all exogenous payment flows have occurred. Recall that ES_2^i is the ES balance (see Appendix 1.7.3 for more detail).

¹⁹An example of abnormal circumstances is late 2007 – mid 2009 in Figure 1.1 (the height of the financial crisis). During this period, increased demand for risk-free assets, combined with the lack of Australian Government debt, increased demand for ES balances (*Kearns* (2009) and *Debelle* (2013)). With the RBA increasing the aggregate ES balance to satisfy the extra demand, the interbank market continued to function as normal (i.e. interbank lending continued, with the average trading rate usually hitting the target).

of Bank i after all payment flows have occurred and after first-period trading. Let b be Bank i 's second-period trading volume ($b < 0$ if Bank i is lending to the other bank, and $b > 0$ if Bank i is borrowing). Therefore:

$$ES_T^i = ES_2^i + b \quad (1.3)$$

Substituting Eq. (1.3) into Eq. (1.2):

$$\pi_T^i = r^D \left[\max(ES_2^i + b, 0) \right] + r^L \left[\min(ES_2^i + b, 0) \right]$$

The interest rates associated with trading will also impact banks' profits. Let the overnight interest rate associated with trade be r . Due to the high credit-quality and small number of Australian banks (*Baker and Jacobs, 2010*), and with the loans being repaid the next day, the typical risk associated with unsecured overnight interbank lending (versus depositing ES balances with the central bank) is likely negligible. So I abstract from risk and assume the return from overnight interbank lending is r .

For trading to occur in the second period, both banks must pay the transaction cost. Let the transaction cost reduce Bank i 's profit by $\gamma > 0$. With a trading rate of r , a bank borrowing b will have its profit reduced by $(r.b)$. Then the profit Bank i derives from second-period trading and end-of-day transactions with the central bank is:

$$\pi_2^i = r^D \left[\max(ES_2^i + b, 0) \right] + r^L \left[\min(ES_2^i + b, 0) \right] - (r.b) - \gamma \quad (1.4)$$

With unconstrained bilateral trading, banks could bargain over both the trading rate (r) and the trading volume (b). However, in my model the central bank is committed to achieving its interest rate target, and I assume that this commitment is strong enough to restrict the set of feasible bargaining outcomes to those in which the interest rate

is equal to the target rate (i.e. $r = (r^L + r^D)/2$).^{20,21} In other words, the central bank's commitment ensures that the target rate is the only rate at which both banks are willing to trade.²² Therefore, the trading volume is the only remaining variable over which banks can bargain, and Eq. (1.4) becomes:

$$\pi_2^i = r^D \left[\max(ES_2^i + b, 0) \right] + r^L \left[\min(ES_2^i + b, 0) \right] - \left[\frac{r^L + r^D}{2} \right] b - \gamma \quad (1.5)$$

1.3.2.1 Trading volume

As is shown in Appendix 1.7.4.1, the profits of both banks can be increased by trading in the second period if, and only if, one of the banks has an ES balance deficit (in which case the other bank has an ES balance surplus, since the aggregate ES balance is non-negative). Intuitively, a bank with an ES balance deficit would prefer to borrow in the market (at the target rate) than from the central bank, while a bank with an ES balance surplus would prefer to lend in the market than deposit its surplus with the central bank. Therefore, there will be a non-zero second-period trading volume only if one of the banks has a pre-trade ES balance deficit, and a Pareto-efficient bargaining solution must have both banks finishing the day with a non-negative ES balance.

To determine the trading volume when trade is mutually profitable, let the deficit bank's ES balance equal $k < 0$, and let the aggregate ES balance equal $M \geq 0$. Therefore, the surplus bank will have an ES balance of $M - k$.

²⁰The commitment assumption is not restrictive in the second period. That is, the NBS with the commitment restriction is in the set of unrestricted NBS (i.e. the set of r and b that produce the unique NBS level of profit for each bank). This will not be the case for first-period trading.

²¹As shown in Figure 1.4, small deviations from target can occur. The commitment to an exact rate (assumed in this model) is a simplification of the commitment to a small range around the target (exhibited in the data).

²²This thesis is focussed on the market outcomes when the central bank can credibly commit to its trading-rate target (as is suggested in the literature, see Section 1.1); the focus is not on how this commitment could be enforced. Therefore, commitment is assumed rather than being explicitly modelled (for simplicity). Some enforcement options discussed in the literature (but not explicitly modelled) include the threat of direct market intervention (e.g. through further open market operations), and moral suasion (see *Breton and Wintrobe* (1978), *Guthrie and Wright* (2000), and *Bowman et al.* (2010), for example).

Assume the outcome of bargaining between the banks equals the Nash Bargaining Solution (NBS). Then, with the NBS satisfying Pareto efficiency, both banks must finish the period with a non-negative ES balance. Using Eq. (1.5), the Nash Product can therefore be written as (see Appendix 1.7.4.2 for a derivation):

$$\left\{ r^D[k + b] - \left[\frac{r^L + r^D}{2} \right] b - r^L k \right\} \left\{ r^D[M - k - b] + \left[\frac{r^L + r^D}{2} \right] b - r^D[M - k] \right\} \quad (1.6)$$

And, as is also shown in Appendix 1.7.4.2, $b = -k$ is the unique solution to the Nash Bargaining Problem (i.e. it maximises Eq. (1.6) subject to the bargaining constraints). That is, the NBS has the deficit bank reach an ES balance of zero, and the surplus bank hold the entire aggregate ES balance.

Substituting the NBS for the trading volume into Eq. (1.5), if both banks decide to partake in second-period trading (i.e. if both banks pay the transaction cost), the profit Bank i derives from second-period trading and end-of-day transactions with the central bank is (if $0 \leq ES_2^i \leq M$, then neither bank has an ES balance deficit and the trading volume is zero):

$$\pi_2^i = \begin{cases} \left[\frac{r^L + r^D}{2} \right] ES_2^i - \gamma & \text{if } ES_2^i < 0 \\ r^D ES_2^i - \gamma & \text{if } 0 \leq ES_2^i \leq M \\ r^D M + \left[\frac{r^L + r^D}{2} \right] [ES_2^i - M] - \gamma & \text{if } M < ES_2^i \end{cases} \quad (1.7)$$

If Bank i pays the transaction cost but the other bank does not:

$$\pi_2^i = \begin{cases} r^L ES_2^i - \gamma & \text{if } ES_2^i < 0 \\ r^D ES_2^i - \gamma & \text{if } ES_2^i \geq 0 \end{cases} \quad (1.8)$$

If Bank i does not pay the transaction cost:

$$\pi_2^i = \begin{cases} r^L ES_2^i & \text{if } ES_2^i < 0 \\ r^D ES_2^i & \text{if } ES_2^i \geq 0 \end{cases} \quad (1.9)$$

With trading in the second period causing at least one bank to finish the day with an ES balance of zero, frequent trading in the second period would violate the ‘positive end-of-day’ fact. Therefore, the parameter values used when simulating the Australian market must ensure that second-period trading is infrequent.

1.3.3 Second-period trading decision

The decision of whether to pay the transaction cost and partake in second-period trading is made at the beginning of the second period. As discussed with the timeline, although the second-period payment flows have not yet occurred, for simplicity, both banks are assumed to know the exact value of these flows before deciding whether to partake in second-period trading (i.e. there is no uncertainty when making this decision).

From Eqs. (1.7) and (1.9), if a bank will have a non-negative ES balance no greater than the aggregate, then it is never profitable to partake in second-period trading. This is because neither bank will have an ES balance deficit, so the NBS trading volume will be zero.

Now suppose that one bank will have an ES balance deficit. Then the payoff from deciding to partake in second-period trading is dependent on whether the surplus bank also decides to partake in second-period trading (a bank’s payoff is equal to its profit). Therefore, the payoffs associated with the decisions of the two banks can be represented in the following payoff matrix, and the “partake / don’t partake” decision can be represented as a game (using Eqs. (1.7) – (1.9), let the deficit bank’s ES balance equal $k < 0$, and let the upper-left part of each cell be the deficit bank’s profit):

		Surplus bank	
		Partake	Don't partake
Deficit bank	Partake	$\left[\frac{r^L+r^D}{2}\right]k - \gamma$ $r^D M - \left[\frac{r^L+r^D}{2}\right]k - \gamma$	$r^L k - \gamma$ $r^D [M - k]$
	Don't partake	$r^L k$ $r^D [M - k] - \gamma$	$r^L k$ $r^D [M - k]$

If either of the banks chooses not to partake in second-period trading, the other bank's best response is also to not partake. So {Don't partake, Don't partake} is always a Nash Equilibrium of this game. If $0 > k > \left[(-2\gamma) / (r^L - r^D)\right]$ then this is the only Nash Equilibrium (see Appendix 1.7.5 for a proof).

If $k \leq \left[(-2\gamma) / (r^L - r^D)\right]$ then another Nash Equilibrium exists at {Partake, Partake}. Moreover, when this condition is met, the profits of both banks are at least as high in the {Partake, Partake} equilibrium than in the {Don't partake, Don't partake} equilibrium (see Appendix 1.7.5 for a proof). Therefore, I assume that both banks choose to partake in second-period trading if, and only if, $k \leq \left[(-2\gamma) / (r^L - r^D)\right]$.²³ Intuitively, second-period trading will only occur if the gains from trade more than offset the transaction cost (for both banks), which only occurs if one bank has a sufficiently large deficit, and the other a sufficiently large surplus.

Therefore, using Eqs. (1.7) – (1.9), the profit Bank i derives from second-period trading and end-of-day transactions with the central bank is (let $A \equiv \left[(2\gamma) / (r^L - r^D)\right]$):

$$\pi_2^i = \begin{cases} \left[\frac{r^L+r^D}{2}\right]ES_2^i - \gamma & \text{if } ES_2^i \leq -A \\ r^L ES_2^i & \text{if } -A < ES_2^i \leq 0 \\ r^D ES_2^i & \text{if } 0 < ES_2^i < A + M \\ r^D M + \left[\frac{r^L+r^D}{2}\right][ES_2^i - M] - \gamma & \text{if } A + M \leq ES_2^i \end{cases} \quad (1.10)$$

²³When faced with two Nash equilibria, one with interbank trade and one without, *Boissay et al.* (2013) also assume that banks coordinate on the Pareto-dominant Nash Equilibrium (the one with trade).

From Eq. (1.10):

Proposition 1.1. *The lending facility is not accessed if banks trade in the second period. Therefore, use of the central bank's lending facility is limited to banks that have a sufficiently small ES balance deficit following second-period payment flows.*

This is consistent with the experience of the Bank of Canada (*Arjani and McVanel, 2006*) (a central bank that also operates a corridor system):

A participant that ends the day with a very small net debit position may choose to take an overnight advance from the Bank of Canada and absorb potential costs related to the interest rate differential rather than search for another participant with these funds.

For use in the next section, Eq. (1.10) can be re-written as the opportunity cost to Bank i of not being able to lend/borrow its entire surplus/deficit in the market (the higher the opportunity cost, the more negative is Eq. (1.11)):

$$\pi_2^i - \left[\frac{r^L + r^D}{2} \right] ES_2^i = \begin{cases} -\gamma & \text{if } ES_2^i \leq -A \\ \left[\frac{r^L - r^D}{2} \right] ES_2^i & \text{if } -A < ES_2^i \leq 0 \\ -\left[\frac{r^L - r^D}{2} \right] ES_2^i & \text{if } 0 < ES_2^i < A + M \\ -\left[\frac{r^L - r^D}{2} \right] M - \gamma & \text{if } A + M \leq ES_2^i \end{cases} \quad (1.11)$$

1.3.4 First-period trading

Interbank payment flows occur between first and second-period trading. Since it is only the impact of payment flows on banks' ES balances that matter for banks' trading decisions, only *net* payment flows need to be modelled. Let w represent the net second-period payment flows to Bank i , and recall that Bank i 's ES balance after first-period trading is ES_1^i . Then Bank i 's ES balance before second-period trading can be written as:

$$ES_2^i = ES_1^i + w \quad (1.12)$$

At the time of first-period trading, these second-period payment flows are unknown. Since w represents *net* interbank second-period payment flows, if w is the shock to Bank i 's ES balance, the shock to the other bank's ES balance is $-w$. Therefore, although there are two banks in the model, there is only one independent second-period shock. The unknown w is modelled as following a Normal probability distribution (with mean zero, since any expected payment flows are modelled as being included with the first-period flows):

$$w \sim \mathcal{N}(0, \sigma^2) \quad (1.13)$$

Using this probability distribution, Eq. (1.10), and Eq. (1.12), Appendix 1.7.6 derives $E[\pi_2^i | ES_1^i]$ (i.e. Bank i 's expected profit from second-period trading and end-of-day transactions with the central bank, given the value of its ES balance after first-period trading). For simplicity, denote this conditional expectation as $\vartheta(ES_1^i)$.²⁴

Recall that Bank i 's ES balance before first-period trading is ES_0^i , and let y be Bank i 's first-period trading volume ($y < 0$ if Bank i is lending to the other bank, and $y > 0$ if Bank i is borrowing). Then, after first-period trading, Bank i 's expected profit from first-period trading, second-period trading, and end-of-day transactions with the central bank is (given the commitment of the central bank to its trading-rate target):

$$E[\pi_1^i | ES_0^i, y] = \vartheta(ES_0^i + y) - \left[\frac{r^L + r^D}{2} \right] y \quad (1.14)$$

By subtracting $\left[\frac{r^L + r^D}{2} \right] ES_0^i$ from both sides of Eq. (1.14):

²⁴Since the unconditional probability distribution of w is symmetric around zero, the unconditional probability distribution of $-w$ is the same as the unconditional probability distribution of w . Therefore, the expected profits of the banks are dependent only on their ES balances after first-period trading (so the $\vartheta(\cdot)$ function can be used for both banks).

$$\mathbb{E}[\pi_1^i | ES_0^i, y] - \left[\frac{r^L + r^D}{2} \right] ES_0^i = \vartheta(ES_1^i) - \left[\frac{r^L + r^D}{2} \right] ES_1^i \quad (1.15)$$

Since ES_0^i is pre-determined when making first-period trading decisions, the value of y that maximises Eq. (1.14) can be found by either maximising Eq. (1.14) with respect to y , or maximising Eq. (1.15) with respect to ES_1^i (where $y = ES_1^i - ES_0^i$). For notational simplicity, define the function $g(z) \equiv \vartheta(z) - \left[\frac{r^L + r^D}{2} \right] z$.

Like the second period, the outcome of bargaining in the first period is assumed to equal the NBS (banks bargain over the value of y). Since ES_0^i is pre-determined, the NBS value of y can be found by maximising the following Nash Product with respect to z :

$$\{g(z) - g(ES_0^i)\} \{g(M - z) - g(M - ES_0^i)\} \quad (1.16)$$

subject to

$$\begin{aligned} g(z) - g(ES_0^i) &\geq 0 \\ g(M - z) - g(M - ES_0^i) &\geq 0 \end{aligned}$$

Where:

- $y = z - ES_0^i$;
- the ES balance of the other bank after trading is $M - z$; and
- the disagreement point occurs where $z = ES_0^i$ (equivalent to $y = 0$).

Let ES_1^* be the value of z that maximises $g(z)$ (i.e. ES_1^* is the optimal end-of-first-period ES balance for both banks); ES_1^* exists and is unique (see Appendix 1.7.8). And let ES_1^N be the end-of-first-period ES balance of Bank i determined by the NBS (so $y^N = ES_1^N - ES_0^i$).

Proposition 1.2. *When $M = 0$, then $ES_1^* = 0$ and $ES_1^N = 0$ (i.e. both banks finish the first period with an ES balance of zero, which is the optimum).*

Proof. Appendix 1.7.7 □

To explain Proposition 1.2 intuitively, note that the only Bank i specific variable on which π_2^i depends is ES_2^i (see Eq. (1.10)); so π_2^i can be expressed as an explicit function of ES_2^i (i.e. $\pi_2(ES_2^i) \equiv \pi_2^i$). Then, from Eq. (1.11), with the aggregate ES balance set to zero:

$$\pi_2(x) - \left[\frac{r^L + r^D}{2} \right] [x] \equiv \pi_2(-x) - \left[\frac{r^L + r^D}{2} \right] [-x] \quad (1.17)$$

So the opportunity cost of having an ES balance deficit in the second period (i.e. following second-period payment flows) is the same as the opportunity cost of having an equivalent surplus. Moreover, the opportunity cost is monotonically increasing the further second-period ES balances get from zero (i.e. both sides of Eq. (1.17) are monotonically decreasing as $|x|$ increases).

From Eq. (1.12) and the definition of $\vartheta(ES_1^i)$, $g(ES_1^i)$ is the expectation of Eq. (1.11) with respect to w (i.e. the expectation of the opportunity cost), given ES_1^i . Therefore, with the probability distribution of w being symmetric and monotonically decreasing away from zero, and with the second-period opportunity cost being symmetric around zero (Eq. (1.17)) and monotonically increasing away from zero, the optimal end-of-first-period ES balance for both banks is zero.

With the aggregate ES balance set to zero, trading is able to get both banks to this optimum; so this point must be the NBS (due to the Pareto efficiency property of the NBS).

With both banks ending the first period with an ES balance of zero (when $M = 0$), at least one of the banks will finish the day with an ES balance of zero (violating the ‘positive end-of-day’ fact). This occurs because second-period payment flows will

cause one of the banks to enter the second period with an ES balance deficit, leading to both/either a high probability of lending facility use (violating the rare use fact), and/or a high probability of second-period trading. Therefore, $M = 0$ would not only violate the ‘positive aggregate’ fact (by definition), it would also cause the violation of other stylised facts.

Proposition 1.3. *When $M > 0$, then $ES_1^* < 0$. And $y^N \neq 0$ iff either $ES_0^i < ES_1^*$ or $M - ES_0^i < ES_1^*$ (i.e. banks trade in the first period iff one of them has a sufficiently large deficit).*

Proof. Appendix 1.7.8 □

To explain Proposition 1.3 intuitively, when $M > 0$, then (from Eq. (1.11)):

$$\pi_2(x) - \left[\frac{r^L + r^D}{2} \right] [x] > \pi_2(-x) - \left[\frac{r^L + r^D}{2} \right] [-x]$$

when $x < -A$

That is, the opportunity cost of having a second-period ES balance deficit that is more negative than $-A$ is less than the opportunity cost of the equivalent surplus (Eq. (1.17) continues to hold when $-A \leq x \leq 0$). Then, with the probability distribution of w being symmetric and monotonically decreasing away from zero, the lower opportunity cost for some deficits means that Bank i ’s expected profit is higher at some negative values of ES_1^i (compared with $ES_1^i \geq 0$); so the optimal end-of-first-period ES balance must be negative.

As is shown in Appendix 1.7.8, each bank’s expected profit is always increased by trading in the first period such that its ES balance moves towards the first-period optimum (and is therefore decreased by trading away from the optimum). Therefore, mutually profitable trade can only occur (i.e. $y^N \neq 0$) if one of the banks has a pre-trade ES balance less than the first-period optimum.

Proposition 1.4. *Suppose $M > 0$ and, without loss of generality, $ES_0^i < ES_1^*$. Then:*

$$M - ES_1^* > ES_1^N > ES_1^*$$

That is, the NBS has both banks finish the first period with an ES balance greater than the optimum.

Proof. Appendix 1.7.9 □

Intuitively, with a smooth and continuous $g(z)$, having either bank finish trading with an ES balance equal to the optimum point cannot be a NBS. This is because, at this point, increasing the trading volume only causes a second-order cost to the bank at the optimum but a first-order benefit to the other bank.

Proposition 1.4 is an important result for replication of the stylised facts. Since if first-period trading resulted in either of the banks having an ES balance equal to the optimum point (which is negative), frequent first-period trading would cause both/either a high probability of lending facility use (violating the ‘positive end-of-day’ and ‘rare use’ facts), and/or a high probability of second-period trading (causing a high probability of one of the banks finishing the day with an ES balance of zero, violating the ‘positive end-of-day’ fact). Therefore, Proposition 1.4 shows that, with $M > 0$, it is possible for this model to replicate the stylised facts.

The exact NBS will depend on banks’ ES balances before first-period trading (the disagreement points) and the model parameters. For the results shown in Section 1.5, the NBS is found by numerically maximising Eq. (1.16) (see Appendix 1.7.10 for more detail).

Except when $M = 0$, the commitment of the central bank causes the total surplus from trade in the first period to be lower than if bargaining were unrestricted. If bargaining were unrestricted (i.e. if there was no commitment), the trading volume would be larger (banks’ ES balances following first-period trade would always be equal) and the trading rate would fall below the central bank’s target. So the commitment

of the central bank is pivotal to simultaneously replicating the ‘positive aggregate’ and ‘equal to target’ facts without any asymmetries.²⁵

1.4 Calibration/identification of the parameters

Let u represent the net payment flows to Bank i in the first period. These flows are modelled as following a Normal probability distribution:

$$u \sim \mathcal{N}(0, \tau^2)$$

Recall that the central bank is assumed to conduct open market operations so that each bank begins the day with an equal share of the aggregate ES balance. Therefore:

$$ES_0^i \sim \mathcal{N}\left(\frac{M}{2}, \tau^2\right) \quad (1.18)$$

The baseline scenario uses the following parameter calibrations:

Parameter	Value
Lending/deposit rate spread $(r^L - r^D)$	0.005/365
Aggregate ES balance (M) (in A\$ millions)	30

As can be seen by substituting Eq. (1.52) (in Appendix 1.7.6) into $g(z)$, the lending and deposit rates do not independently impact the first-period trading volume, only their difference is relevant. Similarly, it is only the difference between these rates that impacts the decision to partake in second-period trading. And the second-period trading volume is independent of the lending and deposit rates. Therefore, to solve the model, only the difference between these rates needs to be calibrated. The difference is calibrated to equal the 50 basis point spread implemented by the RBA (adjusted to a daily spread).

²⁵In existing bilateral bargaining models in which both trading rates and volumes are unrestricted, if all asymmetries are removed, the mechanics of bargaining are the same as the unrestricted bargaining described here (see Appendix 1.7.2 for more detail). This is why these models would require some form of asymmetry to simultaneously replicate the ‘positive aggregate’ and ‘equal to target’ facts.

An aggregate ES balance (M) of A\$30 million is equivalent to the typical pre-crisis aggregate (around A\$750 million, see Figure 1.1) adjusted for a two-bank model (there were around 50 active banks in the Australian market pre-crisis, see Figure 1.3).

The values of the transaction cost (γ), net first-period payment flow variance (τ^2), and net second-period payment flow variance (σ^2) that match the Australian data are identified by matching the ‘average daily trading volume’ and the ‘probability that the central bank’s lending facility is accessed’ (from the model) to the data in Sections 1.2.2 and 1.2.5 (i.e. identification uses the Method of Moments, see Appendix 1.7.13 for more detail); this procedure ‘set identifies’ these parameters.

To further refine the identified set, I discard all subsets in which the transaction cost is either too small or too large (given its interpretation as an extra labour cost).

For parameter values within this ‘refined set’, the mechanics of the model are the same, as is the model’s ability to qualitatively replicate the stylised facts (see Appendix 1.7.13.2 for a detailed explanation). Moreover, all values of σ are small enough to be reasonable representations of second-period payment flow uncertainty (since the second period only represents a short interval before the close of trade, when most payment flows have already occurred). Therefore, within the refined set, the subset of parameter values used for the baseline scenario is not important. The parameter values used for the baseline scenario are:

Parameter	Value (A\$ millions)
Transaction cost (γ)	0.0002
Standard deviation of net first-period payment flows (τ)	21.46
Standard deviation of net second-period payment flows (σ)	0.0043

At A\$200, the transaction cost is similar to the €200 estimated in *Bindseil et al.* (2006). With $\sigma = 0.0043$, the expected absolute value of net second-period payment flows is A\$3,400. With $\tau = 21.46$, the expected absolute value of net first-period payment flows is A\$17 million.

1.5 Results²⁶

1.5.1 Replication of the stylised facts

The ‘positive aggregate’ fact is replicated by calibrating M to replicate the Australian market. As discussed at the end of Section 1.3.4, the commitment of the central bank to its trading-rate target ensures that the ‘equal to target’ fact is simultaneously replicated with the ‘positive aggregate’ fact (as long as some trading occurs).

The parameter identification procedure (see Section 1.4) ensures that the daily average trading volume equals A\$6.15 million and that the daily probability of lending facility use is 0.18 per cent (i.e. the values in the Australian market – A\$7.5 billion and 4.49 per cent, respectively – adjusted for a two-bank model, see Appendix 1.7.13 for more detail). Therefore, interbank trading occurs, and use of the central bank’s lending facility is low (‘rare use’ fact). Moreover, in the ‘refined set’, second-period trading is rare. Therefore, most trading occurs in the first period, well before the end of the day (‘early trade’ fact).

Second-period trading or use of the lending facility occur if, and only if, one of the banks has an ES balance deficit following second-period payment flows. Therefore, with rare second-period trading and lending facility use, it follows that banks’ ES balances following second-period payment flows are rarely in deficit.²⁷ Therefore, banks typically finish the day with a positive ES balance (‘positive end-of-day’ fact).

Hence, with reasonable parameter values, my model qualitatively replicates all five of the stylised facts documented in Section 1.2.

²⁶The results in this section are derived using numerical integration (see Appendix 1.7.11 for details).

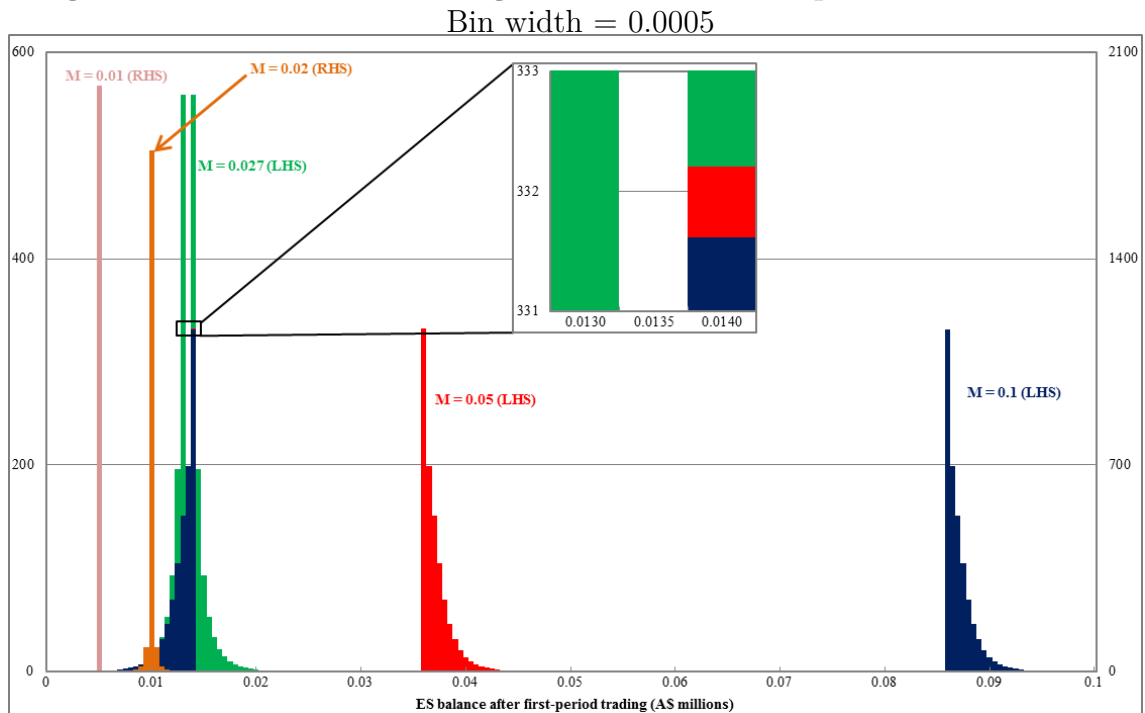
²⁷In more detail, parameter values from the ‘refined set’ ensure that: the majority of payment flows typically occur in the first period (consistent with the data), bargaining usually results in both banks finishing the first period with a positive ES balance, and second-period payment flows are sufficiently small such that banks’ ES balances are rarely pushed into second-period deficit.

1.5.2 The impact of changes in the aggregate ES balance²⁸

1.5.2.1 End-of-first-period ES balances

Figure 1.5 shows an estimate (normalised histogram) of the probability density function of a representative bank's ES balance after first-period trading, for five different settings of the aggregate ES balance.²⁹ Due to the large variance of first-period payment flows (relative to the aggregate ES balances and the optimal end-of-first-period ES balances), the probability of no trade occurring in the first period is low. When trading does occur, bargaining typically leads to both banks having a positive ES balance.

Figure 1.5: Normalised Histogram of End-of-first-period ES Balances



Despite the large variation in pre-trade ES balances, trading causes the majority of the distribution of end-of-first-period ES balances to be tightly spread around two points; one close to zero and the other close to the aggregate (for $M = 0.01$ and

²⁸Except for the aggregate ES balance, results in this section use the baseline parameter values (see Section 1.4).

²⁹Since banks are ex-ante identical (see Eq. (1.18)), both banks will have the same end-of-first-period ES balance distribution.

$M = 0.02$, these two points are too close to be differentiated in a histogram with a bin width of 0.0005). Since banks are ex-ante identical, and the ES balances of the two banks must sum to equal the aggregate, the end-of-first-period ES balance distribution is symmetric around $M/2$.

Increasing the aggregate ES balance moves the half of the distribution that is greater than $M/2$ more than the half that is less than $M/2$. In fact, once the aggregate ES balance is sufficiently high, the lower half is not significantly affected by further increases in the aggregate ES balance (as shown by the left halves of $M = 0.05$ and $M = 0.1$ overlapping in the histogram). This is because, for a given pre-trade deficit ($ES_0^i < ES_1^*$), once the aggregate ES balance is large (relative to the second-period payment flow variance) further increases in the aggregate have little impact on the gains from trade (i.e. at the NBS before the change, any change has little impact on $g(ES_1^N) - g(ES_0^i)$ and $g(M - ES_1^N) - g(M - ES_0^i)$); so ES_1^N is little changed and the impact of the higher aggregate is mainly borne by the surplus bank. That being said, the probability of no-trade increases with the aggregate, which is part of the reason why the height of the $M = 0.1$ peak is slightly lower than the height of the $M = 0.05$ peak.

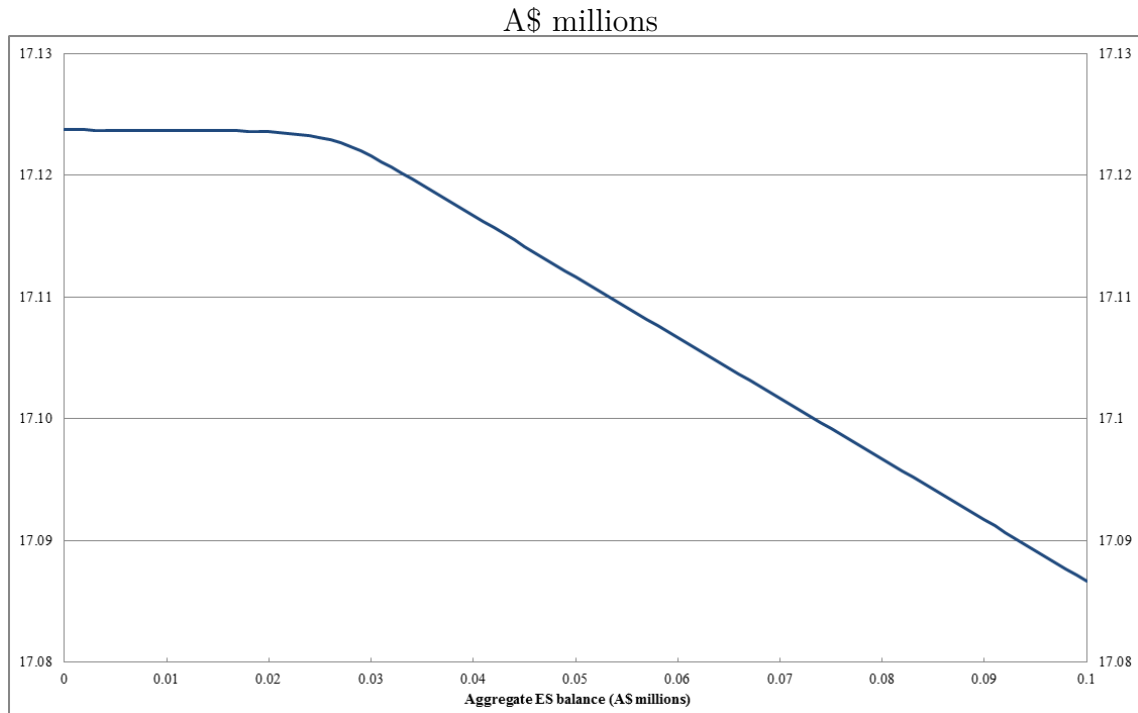
1.5.2.2 First-period trading volume

As discussed in Section 1.3.4, when the aggregate equals zero, all banks finish the first period with an ES balance of zero. Therefore, the expected trading volume is equal to the expected absolute value of net first-period payment flows (A\$17 million, see Section 1.4).

With the baseline parameter values, the expected trading volume is decreasing in the aggregate ES balance (Figure 1.6). This is mostly because increasing the aggregate ES balance increases the region in which no trading occurs (i.e. the region in which both banks have a pre-trade ES balance above the end-of-first-period optimum). For

low levels of the aggregate ES balance, this effect is partly offset by shifts in the NBS, which increase the volume of trade for a given pre-trade deficit.³⁰ For higher levels of the aggregate ES balance, changes in M have little impact on the NBS (as discussed in Section 1.5.2.1).

Figure 1.6: Expected First-period Trading Volume



1.5.2.3 Second-period trading volume

For trade to occur in the second period, one of the banks must have an ES balance deficit lower than $(-2\gamma) / (r^L - r^D)$. With the baseline parameter values, this bound is large (in absolute terms) relative to the standard deviation of second-period payment flows. That is, $|(-2\gamma) / (r^L - r^D)| = 29 \gg \sigma = 0.0043$. Combined with end-of-first-period ES balances that are typically non-negative, the expected volume of second-period trade is very close to zero (for all values of the aggregate ES balance).

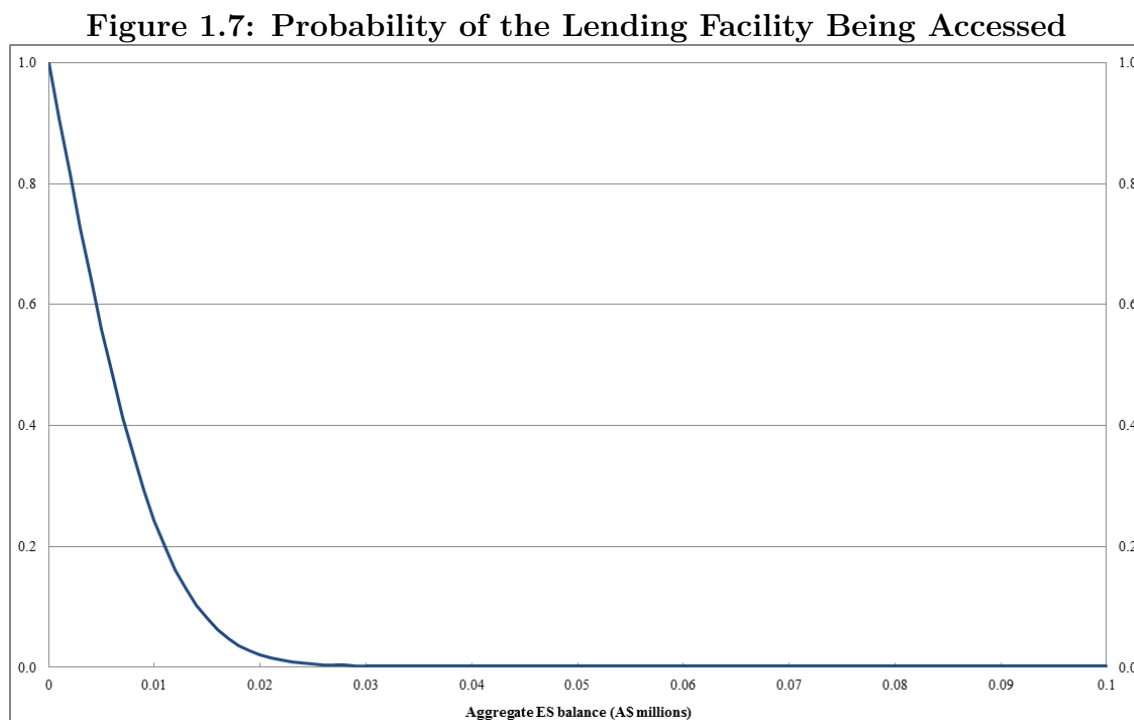
Since there are typically still some loans occurring in the Australian market after

³⁰These changes in the NBS are the main cause of the shifts in the left-half of the probability distributions in Figure 1.5.

17:20 (Figure 1.2), this result suggests that the second period does not begin until after 17:20. Also, having rare second-period trading – and therefore rare payment of the transaction cost – validates my premise that the cost of always staffing the entire trading day would be excessive.

1.5.2.4 Probability that the central bank’s lending facility is accessed

As is shown analytically in Appendix 1.7.12, the probability of the central bank’s lending facility being accessed is maximised when the aggregate ES balance is set to zero. Moreover, Figure 1.7 shows that, with the baseline parameter values, the probability is monotonically decreasing as the aggregate ES balance is increased.



The intuition behind the decreasing probability is that, with a low aggregate ES balance, increasing the aggregate pulls both the left and right halves of the end-of-first-period distribution away from zero (as shown in Figure 1.5); thereby reducing the probability of any bank facing a shock large enough to push its second-period ES balance into

negative territory, and reducing the probability of the lending facility being accessed. Once the aggregate ES balance is sufficiently high, the left half of the distribution is not significantly impacted by further increases, and so the probability only marginally declines (in Figure 1.7, these marginal changes are indistinguishable from a flat line).

The decreasing probability of lending facility use is consistent with the experience of the Bank of Canada (*Bank of Canada*, 2010):

Currently, the Bank will typically target a small positive amount of settlement balances (\$25 million), thus alleviating transactions costs and other frictions from the end-of-day process and reducing the need for participants to take frequent small advances from the Bank.

1.5.3 The impact of changes in the other parameters

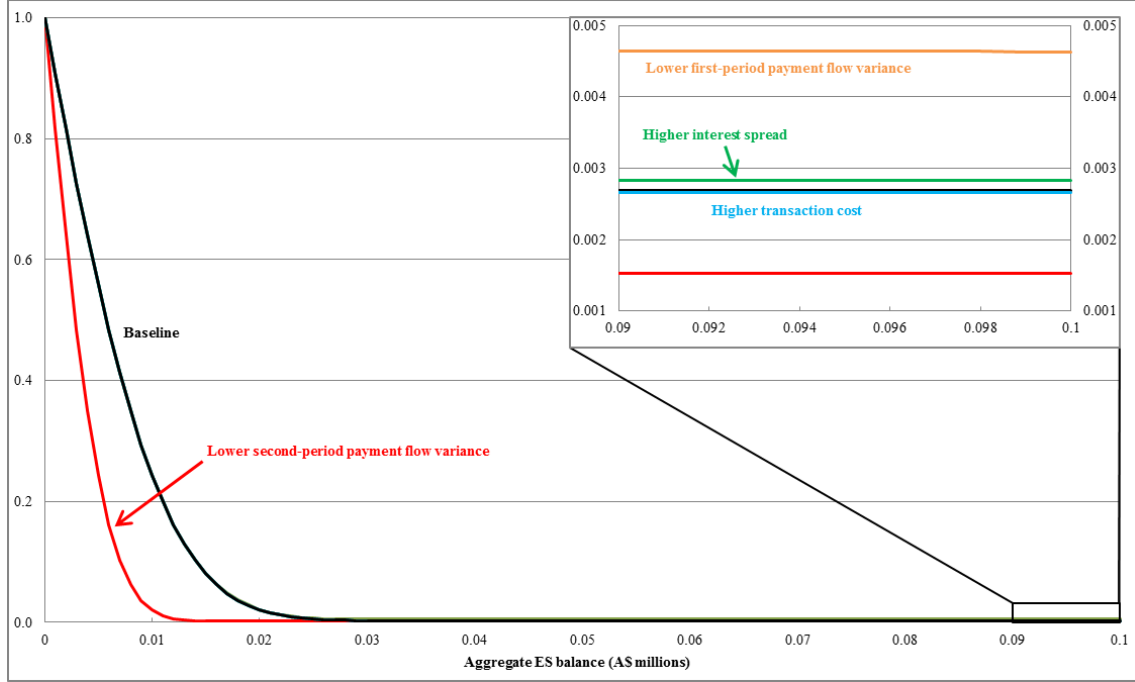
Changes in some of the other parameters can have both direct and indirect effects on use of the central bank's lending facility. The 'direct effect' is the impact of a parameter change on the second-period decision to borrow from the central bank. The 'indirect effect' is the impact that occurs through a change in the end-of-first-period ES balance distribution. These 'effects' can have offsetting impacts on the model, with the dominant 'effect' dependent on the parameter values.

With the impact of a parameter change being dependent on the parameter values, this section provides an example of a small change around the baseline scenario (Figure 1.8). In Figure 1.8:

- the 'higher transaction cost' line represents a doubling of γ , from A\$200 to A\$400;
- the 'higher interest spread' line represents a doubling of $(r^L - r^D)$, from 50 basis points to 100 basis points;
- the 'lower second-period payment flow variance' line represents a halving of σ , from 0.0043 to 0.0021; and

- the ‘lower first-period payment flow variance’ line represents a halving of τ , from 21.46 to 10.73.

Figure 1.8: Probability of the Lending Facility Being Accessed



Note: Due to the scale of the full figure, the ‘Higher transaction cost’, ‘Lower first-period payment flow variance’, and ‘Higher interest spread’ curves are indistinguishable from the ‘Baseline’ curve. In the inset figure, the ‘Baseline’ curve is behind the ‘Higher transaction cost’ curve.

With the baseline parameter values, the region in which no second-period trading occurs is large (relative to the second-period payment flow variance, see Section 1.5.2.3), and the probability of a bank entering the second period with an ES balance close to the lower bound of this region ($-A \equiv -2\gamma / (r^L - r^D)$) is very small. As a result, a small change in the size of this region (i.e. a change in A) does not significantly impact the results; so the direct effects for some of the parameter changes are negligible. This is the case for the transaction cost and the interest spread, for which the indirect effects are (for given pre-trade balances):

- Increasing the interest spread weakens the relative bargaining position of the pre-trade surplus bank, reducing the first-period trading volume. A reduced

first-period trading volume increases the probability that one of the banks will have an ES balance deficit following second-period payment flows, increasing the probability of lending facility access.

- Increasing the transaction cost weakens the relative bargaining position of the pre-trade deficit bank, increasing the first-period trading volume. An increased first-period trading volume reduces the probability that one of the banks will have an ES balance deficit following second-period payment flows, decreasing the probability of lending facility access.

In contrast to the above, the ‘direct effect’ of lowering the second-period payment flow variance is significant. For any positive end-of-first-period ES balance, lowering σ reduces the probability of lending facility use (since $A \gg \sigma$ in the baseline scenario). On the other hand, lowering σ causes first-period trading volumes to fall, increasing use of the central bank’s lending facility (indirect effect). Figure 1.8 shows that, in this case, the direct effect dominates and the probability of lending facility use falls.

Changing the first-period payment flow variance only changes the probability distribution of pre-trade ES balances, it has no impact on the bargaining outcome for a given pre-trade ES balance, nor does it have any ‘direct effect’. In the first period, increasing a bank’s pre-trade ES balance deficit weakens its bargaining position (i.e. its end-of-first-period ES balance is further from the optimum). Therefore, lowering the first-period payment flow variance reduces the typical end-of-first-period ES balances that result from trade (and reduces the average trading volume), increasing the probability of lending facility use. However, lowering the first-period payment flow variance also increases the probability of no-trade, which may decrease the probability of lending facility use. In Figure 1.8, the former impact dominates, and the probability of lending facility use increases.

When the aggregate ES balance is set to zero, banks always finish the first period with an ES balance of zero. Therefore, there are no indirect effects. For small levels of

the aggregate ES balance, end-of-first-period ES balances will be close to zero. Therefore, indirect effects will be small (for a sufficiently small change in the parameters). A small indirect effect is why the impact of lowering the second-period payment flow variance is much larger for low levels of the aggregate ES balance; it is also why the impact of changing the transaction cost, interest spread, or first-period payment flow variance is negligible (for low levels of the aggregate).

1.6 Conclusion

In this chapter, I have constructed a model of an overnight interbank market in which the central bank operates a symmetric corridor system and is committed to achieving its interest rate target. My model differs from the existing literature by:

- combining multiple trading periods, transaction costs, payment-flow uncertainty, and bilateral bargaining;
- explicitly incorporating the commitment of the central bank; and
- not requiring asymmetries to replicate the stylised facts of the Australian market.

With reasonable parameter values, my model replicates five stylised facts of the Australian market. This chapter also shows, qualitatively, how parameter changes can impact trading volumes and lending facility use; with some impacts differing from the existing literature (which has implications for both monetary policy implementation and the central bank's balance sheet).

That being said, a two-bank model with ex-ante identical banks is too simplistic for quantitative policy analysis; especially since some of the parameters are only 'set identified'. So the next chapter of this thesis expands the simple model of this chapter to a model that can incorporate many heterogeneous banks, and can be directly fit to the data.

In the model of this chapter, the commitment of the central bank ensures that *all* trades occur at the target rate. While this is a reasonably accurate representation of the Australian market, other markets exhibit greater trading-rate volatility (around their targets). Therefore, an avenue for future research could be to explicitly model how this commitment is enforced, as this could facilitate determination of why trading-rate volatility differs between countries.

A model that incorporates the enforcement mechanism could also be used to determine how changes in the central bank's operations might influence its commitment credibility. This is particularly important with some central banks currently implementing unconventional policies, leading to atypically large aggregate ES balances, and potentially impacting their ability to enforce their commitment.

1.7 Appendices

For use throughout the appendices:

- $\phi(\cdot)$ is the standard Normal probability density function
- $\Phi(\cdot)$ is the standard Normal cumulative distribution function

1.7.1 Lemmas for use in the appendices

Lemma 1.1. *When $\theta > 0$ then $\{\Phi(\beta + \theta) - \Phi(\beta - \theta)\} > \{\Phi(\alpha + \theta) - \Phi(\alpha - \theta)\} \forall |\alpha| > |\beta|$*

When $\theta < 0$ then $\{\Phi(\beta + \theta) - \Phi(\beta - \theta)\} < \{\Phi(\alpha + \theta) - \Phi(\alpha - \theta)\} \forall |\alpha| > |\beta|$

Proof.

$$\frac{d\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}}{d\lambda} = \phi(\lambda + \theta) - \phi(\lambda - \theta)$$

Suppose $\theta > 0$ and $\lambda > 0$, then (since $\phi(x)$ is symmetric, monotonically increasing for $x < 0$, and monotonically decreasing for $x > 0$):

$$\frac{d\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}}{d\lambda} < 0$$

Suppose $\theta > 0$ and $\lambda < 0$, then:

$$\frac{d\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}}{d\lambda} > 0$$

Therefore, when $\theta > 0$, increasing $|\lambda|$ decreases $\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}$.

So, $\{\Phi(\beta + \theta) - \Phi(\beta - \theta)\} > \{\Phi(\alpha + \theta) - \Phi(\alpha - \theta)\} \forall |\alpha| > |\beta|$.

Suppose $\theta < 0$ and $\lambda > 0$, then:

$$\frac{d\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}}{d\lambda} > 0$$

Suppose $\theta < 0$ and $\lambda < 0$, then:

$$\frac{d\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}}{d\lambda} < 0$$

Therefore, when $\theta < 0$, increasing $|\lambda|$ increases $\{\Phi(\lambda + \theta) - \Phi(\lambda - \theta)\}$.

So, $\Phi(\beta + \theta) - \Phi(\beta - \theta) < \Phi(\alpha + \theta) - \Phi(\alpha - \theta) \quad \forall \quad |\alpha| > |\beta|$. □

Lemma 1.2. *Let $\beta \geq 0$ and $\mu \geq 0$.*

When $\theta \leq (\mu/2)$ then $\Phi(\beta - \theta) - \Phi(-\theta) \geq \Phi(\beta + \mu - \theta) - \Phi(\mu - \theta)$

When $\theta \geq (\mu/2)$ then $\Phi(\mu - \theta) - \Phi(-\beta + \mu - \theta) \geq \Phi(-\theta) - \Phi(-\beta - \theta)$

Proof. With $\theta \leq (\mu/2)$ then $|x - \theta| \leq |x + \mu - \theta| \quad \forall x \geq 0$.

Therefore, since $\phi(y)$ is symmetric, monotonically increasing for $y < 0$, and monotonically decreasing for $y > 0$, $\phi(x - \theta) \geq \phi(x + \mu - \theta) \quad \forall x \geq 0$.

Then:

$$\begin{aligned} \Phi(\beta - \theta) - \Phi(-\theta) &= \int_0^\beta \phi(x - \theta) dx \\ &\geq \int_0^\beta \phi(x + \mu - \theta) dx \\ &= \Phi(\beta + \mu - \theta) - \Phi(\mu - \theta) \end{aligned}$$

With $\theta \geq (\mu/2)$ then $|x - \theta| \geq |x + \mu - \theta| \quad \forall x \leq 0$.

Therefore, $\phi(x - \theta) \leq \phi(x + \mu - \theta) \quad \forall x \leq 0$.

Then:

$$\begin{aligned} \Phi(\mu - \theta) - \Phi(-\beta + \mu - \theta) &= \int_{-\beta}^0 \phi(x + \mu - \theta) dx \\ &\geq \int_{-\beta}^0 \phi(x - \theta) dx \\ &= \Phi(-\theta) - \Phi(-\beta - \theta) \end{aligned}$$

□

Lemma 1.3. *For any $\varepsilon > 0$ and $\delta > 0$, there is a point ($x = c$) at which $2\Phi(x) - \Phi(x - \varepsilon) - \Phi(x + \varepsilon + \delta) = 0$.*

Moreover:

When $x > c$ then $2\Phi(x) - \Phi(x - \varepsilon) - \Phi(x + \varepsilon + \delta) > 0$.

When $x < c$ then $2\Phi(x) - \Phi(x - \varepsilon) - \Phi(x + \varepsilon + \delta) < 0$.

Proof. Let:

$$y = 2\Phi(x) - \Phi(x - \varepsilon) - \Phi(x + \varepsilon + \delta) \quad (1.19)$$

Therefore:

$$\frac{dy}{dx} = 2\phi(x) - \phi(x - \varepsilon) - \phi(x + \varepsilon + \delta) \quad (1.20)$$

$$\frac{d^2y}{dx^2} = -2x\phi(x) + (x - \varepsilon)\phi(x - \varepsilon) + (x + \varepsilon + \delta)\phi(x + \varepsilon + \delta) \quad (1.21)$$

Substituting Eq. (1.20) into Eq. (1.21):

$$\frac{d^2y}{dx^2} = -\varepsilon\phi(x - \varepsilon) + (\varepsilon + \delta)\phi(x + \varepsilon + \delta) - x\frac{dy}{dx} \quad (1.22)$$

Using Eq. (1.22), at a local maximum point (i.e. where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$):

$$(\varepsilon + \delta)\phi(x + \varepsilon + \delta) < \varepsilon\phi(x - \varepsilon) \quad (1.23)$$

Using the definition of the standard Normal pdf $\{\phi(u) = (1/\sqrt{2\pi})\exp(-u^2/2)\}$ and simplifying, Eq. (1.23) can be re-written as:

$$x > \frac{\ln(\varepsilon + \delta) - \ln(\varepsilon)}{2\varepsilon + \delta} - \frac{\delta}{2} \quad (1.24)$$

Similarly, at a local minimum point (i.e. where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$):

$$(\varepsilon + \delta)\phi(x + \varepsilon + \delta) > \varepsilon\phi(x - \varepsilon)$$

Which can be simplified to:

$$x < \frac{\ln(\varepsilon + \delta) - \ln(\varepsilon)}{2\varepsilon + \delta} - \frac{\delta}{2} \quad (1.25)$$

Therefore, any stationary point in the range given by Eq. (1.24) must be a local maximum point, while any stationary point in the range given by Eq. (1.25) must be a local minimum point.

Since y is continuous, no two stationary points can be local minima without having a local maximum at some point in between, and no two stationary points can be local maxima without having a local minimum at some point in between. Therefore, Eqs. (1.24) and (1.25) imply that there can be *at most* one local minimum, and *at most* one local maximum.

Moreover, from Eqs. (1.24) and (1.25), the point (x) at which any local maximum occurs must be greater than the point at which any local minimum occurs. Therefore, since there can be at most one local maximum, there can be no stationary points when x is greater than the point at which a local maximum occurs. Similarly, there can be no stationary points when x is lower than the point at which a local minimum occurs.

From Eq. (1.20), at $x = 0$:

$$\frac{dy}{dx} > 0 \quad (1.26)$$

At $x = \frac{\varepsilon}{2} + \frac{\ln(2)}{\varepsilon} > 0$:

$$\phi(x - \varepsilon) = 2\phi(x) \quad (1.27)$$

Therefore, using Eqs. (1.20) and (1.27), when $x = \frac{\varepsilon}{2} + \frac{\ln(2)}{\varepsilon}$:

$$\frac{dy}{dx} < 0 \quad (1.28)$$

From the continuity of $\frac{dy}{dx}$ and the intermediate value theorem, since $\frac{dy}{dx} > 0$ when $x = 0$ (Eq. (1.26)) and $\frac{dy}{dx} < 0$ when $x = \frac{\varepsilon}{2} + \frac{\ln(2)}{\varepsilon}$ (Eq. (1.28)), there must be a local

maximum ($x = d$) in the region $d \in \left(0, \frac{\varepsilon}{2} + \frac{\ln(2)}{\varepsilon}\right)$.

Therefore, since no stationary points can occur at points greater than the local maximum:

$$\frac{dy}{dx} < 0 \quad \forall x > d \quad (1.29)$$

From Eq. (1.19), $y \rightarrow 0$ as $x \rightarrow \infty$. Therefore, due to Eq. (1.29), y must decrease towards zero in the limit, implying:

$$y > 0 \quad \forall x \geq d \quad (1.30)$$

From Eq. (1.19), at $x = 0$:

$$\begin{aligned} y &= 1 - \Phi(-\varepsilon) - \Phi(\varepsilon + \delta) \\ &= \Phi(\varepsilon) - \Phi(\varepsilon + \delta) \\ &< 0 \end{aligned} \quad (1.31)$$

Also from Eq. (1.19):

$$y \rightarrow 0 \quad \text{as } x \rightarrow -\infty \quad (1.32)$$

Since the unique local maximum occurs where $x > 0$, Eqs. (1.31) and (1.32) imply that y must increase toward zero as $x \rightarrow -\infty$ (since to decrease towards zero would require a local maximum point at some $x < 0$), and therefore:

$$y < 0 \quad \forall x \leq 0 \quad (1.33)$$

From Eqs. (1.30) and (1.33), $y < 0$ when $x \leq 0$, and $y > 0$ when $x \geq d$. Therefore, due to the continuity of y and the intermediate value theorem, $y = 0$ at some point $x = c$, where $c \in (0, d)$.

Since Eqs. (1.31) and (1.32) imply that y must increase toward zero as $x \rightarrow -\infty$,

combining this with Eq. (1.26) implies that the unique minimum point must exist, and it must be in the region $x < 0$.

Therefore, since the unique local maximum and unique local minimum occur outside the region $[0, d]$, Eq. (1.26) implies $\frac{dy}{dx} \geq 0$ when $0 \leq x \leq d$. Combining this result with Eqs. (1.30) and (1.33), and that $y = 0$ when $x = c$, gives the following:

$$y > 0 \quad \forall x > c$$

$$y < 0 \quad \forall x < c$$

□

1.7.2 The terms of trade in existing models

1.7.2.1 Market clearing models

In the market clearing models of the overnight interbank market (e.g. *Henckel et al.* (1999), *Guthrie and Wright* (2000), and *Woodford* (2001)), Bank i 's demand for excess reserves (i.e. central bank deposits beyond any reserve requirements) is typically of the form:

$$ES^i = -c^i F^{-1}\left(\frac{r - r^D}{r^L - r^D}\right)$$

Where c^i is some (potentially heterogeneous) scaling factor, and $F(\cdot)$ is the scaled payment-flow uncertainty cumulative distribution function.

To clear the market, aggregate demand for excess reserves ($\sum ES^i$) must equal the aggregate supply of excess reserves (M). Therefore, the market clearing rate is:

$$r^* = r^D + [r^L - r^D] F\left(\frac{-M}{\sum c^i}\right)$$

If the central bank operates a symmetric corridor $[r^* = (r^L + r^D)/2]$, then the

aggregate level of excess reserves must be set such that:

$$M = - \left(\sum c^i \right) F^{-1} \left(\frac{1}{2} \right)$$

If the uncertainty distribution is symmetric around zero, then $M = 0$. Therefore, these models would require some form of asymmetry to simultaneously replicate the ‘positive aggregate’ and ‘equal to target’ facts in Australia.

1.7.2.2 Bilateral bargaining models

In the existing bilateral bargaining literature, trading volumes are either fixed or are unrestricted. In *Bech and Klee* (2011), *Ennis and Weinberg* (2013), and *Bianchi and Bigio* (2014), trading volumes are fixed. As a result, any change in the relative bargaining positions of the trading banks (i.e. a change in any of the model parameters) can only be reflected in the trading rates.

In *Ennis and Weinberg* (2013), there is a continuum of illiquid banks that are required to make a payment of one unit but have no funds, and there is a continuum of liquid banks that have one unit of funds but do not need to make a payment. Since each continuum has measure one, aggregate excess reserves in this model are zero. Therefore, this model is not suitable for replicating either the ‘positive aggregate’ or ‘positive end-of-day’ facts in Australia.

The model in *Bech and Klee* (2011) does not include a measure of individual bank’s deposits. Therefore, this model is not suitable for replicating either the ‘positive end-of-day’ or ‘rare use’ facts in Australia.

Bianchi and Bigio (2014) split each bank’s desired lending/borrowing volumes into infinitesimally small packets of equal size; lending packets are then matched with borrowing packets until there are no more possible matches. In each match, the trading volume is fixed to the packet size, and the banks bargain over the trading rate. In each match, the bargaining banks do not account for each bank’s total number of matches

nor the outcome of these matches (even if the other matches are between the same two banks); so banks' ES balances have no impact on bargaining outcomes. Therefore, this model is too simplistic for my purposes.³¹

In bilateral bargaining models of the overnight interbank market, the total surplus from trade is independent of the trading rate (any increase in the profit of one bank from a higher interest rate is exactly offset by a decrease in the other bank's profit). But the total surplus may not be independent of the trading volume; because banks' profits may depend on their post-trade levels of central bank deposits (if there is uncertain late-day payment flows, for example). Therefore, in models with unrestricted trading rates and trading volumes, banks set the trading volume to the level that maximises the total surplus, and then bargain over the trading rate (to split the surplus). This is the case for *Afonso and Lagos* (2012) and *Bech and Monnet* (2013), in which the total-surplus maximising trading volume always results in both banks having the same level of deposits after trade (despite initial deposits varying).³²

While the average trading rate in the general model of *Afonso and Lagos* (2012) does not have an analytical solution, *Afonso and Lagos* (2012) provide a stylised version of their model (see *Afonso and Lagos* (2012) Sections 7 and 8.3.1) in which, at each intra-day point in time, there is a measure of banks with one unit of excess reserves, a measure of banks with a one unit reserve deficiency, and a measure of banks that meet their reserve requirements exactly (and are therefore not willing to trade). Then, if banks have a negligible rate of time preference, and there are negligible intra-day costs/benefits to holding particular levels of central bank deposits (as is the case in Australia due to the availability of interest-free intra-day loans), the trading rate between two banks that trade at intra-day time t is:

³¹*Bianchi and Bigio* (2014) is the only paper in this appendix that does not treat payment flows as exogenous (banks account for the potential cost of overnight borrowing when making investment decisions). However, when modelling the outcome of trade in the interbank market, payments flows are pre-determined.

³²Since the *Afonso and Lagos* (2012) model only allows discrete trading volumes, banks' post-trade deposits are as close to equality as the discrete trading volumes allow.

$$r(t) = r^D + [r^S - r^D][1 - \beta(t)]$$

Where $\beta(t)$ is the borrower's *effective* bargaining power at time t , and r^S is the *effective* corridor ceiling (i.e. it accounts for stigma). One of the determinants of $\beta(t)$ is aggregate reserves. If aggregate reserves equal reserve requirements (i.e. the measure of banks with excess reserves equals the measure with a reserve deficiency), the trading rate simplifies to:

$$r(t) = r^D + [r^S - r^D][1 - \theta]$$

Where θ is the borrower's *fundamental* bargaining power. Then, if banks have symmetric fundamental bargaining power and there is no stigma:

$$r(t) = \frac{r^L + r^D}{2}$$

Afonso and Lagos (2012) also show that if aggregate excess reserves are positive (i.e. the measure of banks with excess reserves is greater than the measure with a reserve deficiency), the trading rate falls below the midpoint of the corridor $((r^L + r^D)/2)$. Therefore, the stylised version of the model in *Afonso and Lagos* (2012) cannot simultaneously replicate the 'positive aggregate' and 'equal to target' facts in Australia without some asymmetries (such as stigma or asymmetric bargaining power).

In *Bech and Monnet* (2013), banks only trade when one bank has excess reserves and the other has a reserve deficiency. Let the surplus bank's deposit level be ES^{Sur} and let the deficit bank's deposit level be ES^{Def} . Then the risk-free trading rate is (setting reserve requirements to zero, without loss of generality):

$$r(ES^{Sur}, ES^{Def}) = r^D + [r^L - r^D] \left[\frac{-ES^{Def}}{ES^{Sur} - ES^{Def}} \right]$$

Each bank's deposit level before trading is equal to its deposit level at the beginning

of the day (banks share aggregate deposits equally at the beginning of the day) plus net payment inflows (each bank's net inflows are drawn independently from a standard Normal distribution). Then, when aggregate excess reserves are set to zero, the conditional expected value of $\frac{-ES^{Def}}{ES^{Sur}-ES^{Def}}$ (conditional on $ES^{Def} < 0$ and $ES^{Sur} > 0$) equals one-half. Therefore, $E[r(ES^{Sur}, ES^{Def})] = ((r^L + r^D)/2)$. Although there is no closed-form solution for the conditional expected value when excess reserves are non-zero, the numerical results in *Bech and Monnet* (2013) show an expected trading rate that is decreasing in aggregate excess reserves. Therefore, the model in *Bech and Monnet* (2013) cannot simultaneously replicate the 'positive aggregate' and 'equal to target' facts in Australia without some asymmetry (the asymmetry included in *Bech and Monnet* (2013) is counterparty credit risk).

1.7.2.3 Cooperative game theory

Tapking (2006) develops a bargaining model of the European overnight interbank in which banks bargain cooperatively (the solution concept applied is the Shapley Value). While the model in *Tapking* (2006) is not useful for my purposes (it incorporates reserve requirements, maintenance periods, trading only once per day, no transaction costs, and no commitment), use of the Shapley Value to determine the bargaining outcome is a novel feature (the bilateral bargaining models discussed above, and my model, all use the Nash Bargaining Solution).

In a model with two banks, constructed in the way I have done in this chapter but without commitment, the Nash Bargaining Solution and Shapley Value Solution are the same (the unrestricted Nash Bargaining Solution result is discussed in Section 1.3.4). As a result, without commitment, using the Shapley Value Solution does not overcome the inability of the model to replicate the 'positive aggregate' and 'equal to target' facts in Australia without some asymmetries.

As discussed in Section 1.3.2, incorporating commitment restricts the set of feasible

bargaining outcomes to those in which the interest rate is equal to the target rate. As a result, the unrestricted solution may not be attainable.

Within this restricted set of feasible bargaining outcomes, the Nash Bargaining Solution and Shapley Value Solution will not, in general, be the same.³³

1.7.3 The new operational framework of the RBA

The changes detailed in this section are based on the information provided in *Debelle* (2013).

In the system prior to November 2013, ‘direct entry’ transactions (which are low-value interbank transactions) were settled in a single batch on the trading day after the transactions were made. From November 2013, these transactions are settled on the same day as the transaction is made (still in batches). Moreover, some of these batches settle after the close of daily trade.

To ensure banks have sufficient liquidity for any overnight settlements, the RBA now offers ‘open repos’ (these are repurchase agreements with no maturity date). The interest rate charged on these repos equals the RBA’s trading-rate target. And at the end of the trading day, banks receive interest on their open repo positions, with the interest rate equal to the RBA’s trading-rate target. Therefore, there is no cost to holding an open repo position overnight, and banks are expected to hold sufficiently large open repo positions to ensure they have sufficient ES balances to make any overnight payments.³⁴

Although holding an open repo position is costless, if a bank’s end-of-day ES balance falls below its open repo position, it will be charged a 25 basis point penalty on the shortfall. And if a bank’s end-of-day ES balance is above its open repo position, an

³³Using the notation from Section 1.3.4, the Shapley Value Solution maximises $g(z) + g(M - z)$ with respect to z , subject to $g(M - z) - g(M - ES_0^i) = g(z) - g(ES_0^i)$.

³⁴As noted in Section 1.3.1, banks previously held more than enough collateral for use in intra-day repurchase agreements. As long as this continues under the new system, collateralisation should not increase the cost of borrowing via open repo.

interest rate 25 basis points below the RBA's trading-rate target will be paid on the surplus.

So, instead of implementing a symmetric corridor around an ES balance of zero (the old system), the new system will incorporate a symmetric corridor around each bank's open repo position.

In my model, 'direct entry' transactions are included in the first-period flows on the trading day after the transactions are made. And there are no overnight settlements of any transactions.

Under the new system, with banks holding sufficiently large open repo positions, any overnight settlements can still be modelled as impacting the first-period flows of the following trading day. This is because the movement in ES balances resulting from overnight payment flows will only impact banks' trading decisions on the following trading day (assuming the RBA does not use its open market operations to offset these overnight flows).³⁵

Moreover, since open repo positions are costless, and the symmetric corridor will operate around each bank's open repo position, the 'aggregate end-of-day ES balance minus the aggregate end-of-day open repo position' in the new system can be modelled in the same way as the 'aggregate end-of-day ES balance' in the old system. This is consistent with the RBA's expectation that the aggregate end-of-day ES balance (excluding open repos) under the new system will be little-changed from the aggregate end-of-day ES balance in the old system, and why the implementation of monetary policy is not expected to change (*Debelle, 2013*).

Therefore, modelling the new system can use the framework I construct in this chapter, with the only required adjustment being that the 'aggregate end-of-day ES balance minus the aggregate end-of-day open repo position' be used instead of the 'aggregate end-of-day ES balance'.

³⁵If the RBA were to use open market operations to offset these flows, overnight flows would have no impact on banks' next-day trading decisions.

That being said, the new system could cause a change in some of the parameter values. With ‘direct entry’ transactions settling in several batches, if any of these batches are settled during the second period, the second-period payment flow variance may increase (causing a corresponding decrease in the first-period payment flow variance). So, even though my model can be used to model the new system, some of the parameter values may change, leading to changes in trading volumes and use of the lending facility (equivalent to payment of the 25 basis point penalty in the new system).

1.7.4 Second-period trading volume

1.7.4.1 Profits increased by trading towards end-of-day ES balance of zero

Suppose $ES_2^i < 0$ and $b < -ES_2^i$. Then (from Eq. (1.5)):

$$\begin{aligned}\pi_2^i &= r^L [ES_2^i + b] - \left[\frac{r^L + r^D}{2} \right] b - \gamma \\ \frac{d\pi_2^i}{db} &= \frac{r^L - r^D}{2} > 0\end{aligned}$$

So the profit of a deficit bank can be increased by more borrowing (as long as its ES balance does not enter the surplus region).

Now suppose $ES_2^i > 0$ and $b > -ES_2^i$. Then:

$$\begin{aligned}\pi_2^i &= r^D [ES_2^i + b] - \left[\frac{r^L + r^D}{2} \right] b - \gamma \\ \frac{d\pi_2^i}{db} &= - \left[\frac{r^L - r^D}{2} \right] < 0\end{aligned}\tag{1.34}$$

So the profit of a surplus bank can be increased by more lending (as long as its ES balance does not enter the deficit region).

Therefore, if one bank has an ES balance deficit, and the other has an ES balance surplus, trading can not be Pareto efficient until either the surplus or the deficit is

extinguished. Since the aggregate ES balance is non-negative, whenever there is a deficit bank, there must also be a surplus bank. Therefore, trading can not be Pareto efficient until both banks have a non-negative ES balance.

If both banks have a non-negative ES balance following second-period payment flows, trading can not take place. This is because any bank that borrows when they already have a non-negative ES balance must be reducing their profit (Eq. (1.34)).

1.7.4.2 Nash Bargaining Solution

Suppose that in the second period, both banks have decided to partake in second-period trading, Bank 1 has a pre-trade ES balance deficit of $k < 0$ (without loss of generality), and Bank 2 a surplus of $M - k$. From Eq. (1.5), the profits of the two banks if no trading occurs are:

$$\pi_2^1 = r^L k - \gamma \quad (1.35)$$

$$\pi_2^2 = r^D [M - k] - \gamma \quad (1.36)$$

Now suppose that Bank 2 lends b to Bank 1. Moreover, since Appendix 1.7.4.1 proves that trading can not be Pareto efficient until both banks have a non-negative ES balance, and since the Nash Bargaining Solution must satisfy Pareto efficiency, the Nash Bargaining Solution must satisfy $-k \leq b \leq M - k$. Then the post-trade profits of the two banks are:

$$\pi_2^1 = r^D [k + b] - \left[\frac{r^L + r^D}{2} \right] b - \gamma \quad (1.37)$$

$$\pi_2^2 = r^D [M - k - b] + \left[\frac{r^L + r^D}{2} \right] b - \gamma \quad (1.38)$$

In the terminology of Nash Bargaining:

- $B = \{b \mid -k \leq b \leq M - k\}$ is the set of possible agreements;

- Eqs. (1.37) and (1.38) are the payoffs for the two banks given agreement; and
- Eqs. (1.35) and (1.36) are the disagreement payoffs for the two banks.

Then the Nash Bargaining Solution of this problem is:

$$b^* = \arg \max_b \left\{ \left(r^D[k + b] - \left[\frac{r^L + r^D}{2} \right] b - r^L k \right) \times \right. \quad (1.39)$$

$$\left. \left(r^D[M - k - b] + \left[\frac{r^L + r^D}{2} \right] b - r^D[M - k] \right) \right\}$$

Subject to:

- $b \in B$
- $r^D[k + b] - \left[\frac{r^L + r^D}{2} \right] b - r^L k \geq 0$
- $r^D[M - k - b] + \left[\frac{r^L + r^D}{2} \right] b - r^D[M - k] \geq 0$

Simplifying Eq. (1.39), and differentiating the equation to be maximised (with respect to b):

$$b^* = \arg \max_b \left\{ - \left[\frac{r^L - r^D}{2} \right]^2 b^2 - 2 \left[\frac{r^L - r^D}{2} \right]^2 bk \right\}$$

$$\frac{d}{db} = -2 \left[\frac{r^L - r^D}{2} \right]^2 b - 2 \left[\frac{r^L - r^D}{2} \right]^2 k$$

$$\frac{d^2}{db^2} = -2 \left[\frac{r^L - r^D}{2} \right]^2 < 0$$

Therefore, the first order condition for an unconstrained global maximum is $b^* = -k$. Moreover, this solution satisfies all of the constraints of Eq. (1.39); so it is the Nash Bargaining Solution.

1.7.5 Second-period trading decision Nash Equilibria

From Section 1.3.3, where the deficit bank's ES balance is equal to $k < 0$, 'Don't partake' will be a strictly dominant strategy for the surplus bank if:

$$r^D[M - k] > r^D M - \left[\frac{r^L + r^D}{2} \right] k - \gamma \quad (1.40)$$

And will be a strictly dominant strategy for the deficit bank if:

$$r^L k > \left[\frac{r^L + r^D}{2} \right] k - \gamma \quad (1.41)$$

Eqs. (1.40) and (1.41) can both be simplified to:

$$k > \frac{-2\gamma}{r^L - r^D}$$

Therefore, when $0 > k > \left[-2\gamma / (r^L - r^D) \right]$, {Don't partake, Don't partake} is the only Nash Equilibrium.

When $k \leq \left[-2\gamma / (r^L - r^D) \right]$, then:

$$r^D[M - k] \leq r^D M - \left[\frac{r^L + r^D}{2} \right] k - \gamma \quad (1.42)$$

$$r^L k \leq \left[\frac{r^L + r^D}{2} \right] k - \gamma \quad (1.43)$$

Therefore, if the deficit bank chooses to 'Partake', the surplus bank's best response is to 'Partake'. And when the surplus bank chooses to 'Partake', the deficit bank's best response is to 'Partake'. So a Nash Equilibrium exists at {Partake, Partake}.

Moreover, the left-hand-side of the inequalities in Eqs. (1.42) and (1.43) are equivalent to the profits of the surplus and deficit banks (respectively) in the {Don't partake, Don't partake} Nash Equilibrium, while the right-hand-side of the inequalities are the profits of the banks in the {Partake, Partake} Nash Equilibrium. Therefore, when $k \leq \left[-2\gamma / (r^L - r^D) \right]$, the profits of both banks are at least as high in the {Partake,

Partake} Nash Equilibrium than in the {Don't partake, Don't partake} Nash Equilibrium.

1.7.6 Derivation of $\mathbf{E}[\pi_2^i | ES_1^i]$

Expressing Eq. (1.10) as an explicit function of ES_2^i (i.e. $\pi_2(ES_2^i) \equiv \pi_2^i$), and using Eq. (1.12):

$$\pi_2(ES_1^i + w) = \begin{cases} \left[\frac{r^L + r^D}{2} \right] [ES_1^i + w] - \gamma & \text{if } ES_1^i + w \leq -A \\ r^L [ES_1^i + w] & \text{if } -A < ES_1^i + w \leq 0 \\ r^D [ES_1^i + w] & \text{if } 0 < ES_1^i + w < A + M \\ r^D M + \left[\frac{r^L + r^D}{2} \right] [ES_1^i + w - M] - \gamma & \text{if } A + M \leq ES_1^i + w \end{cases} \quad (1.44)$$

Given the distribution of w in Eq. (1.13):

$$P(w \leq W) = \Phi\left(\frac{W}{\sigma}\right) \quad (1.45)$$

$$E(w | W_1 \leq w \leq W_2) = \left[\frac{\phi\left(\frac{W_1}{\sigma}\right) - \phi\left(\frac{W_2}{\sigma}\right)}{\Phi\left(\frac{W_2}{\sigma}\right) - \Phi\left(\frac{W_1}{\sigma}\right)} \right] \sigma \quad (1.46)$$

Then, taking expectations of Eq. (1.44) (with respect to w) gives:

$$\begin{aligned} E(\pi_2(ES_1^i + w) | ES_1^i) = & \\ & P(w \leq -A - ES_1^i | ES_1^i) E\left(\left[\frac{r^L + r^D}{2} \right] [ES_1^i + w] - \gamma \middle| w \leq -A - ES_1^i\right) + \\ & P(-A - ES_1^i < w \leq -ES_1^i | ES_1^i) E(r^L [ES_1^i + w] | -A - ES_1^i < w \leq -ES_1^i) + \\ & P(-ES_1^i < w < A + M - ES_1^i | ES_1^i) E(r^D [ES_1^i + w] | -ES_1^i < w < A + M - ES_1^i) + \\ & P(A + M - ES_1^i \leq w | ES_1^i) E\left(r^D M + \left[\frac{r^L + r^D}{2} \right] [ES_1^i + w - M] - \gamma \middle| A + M - ES_1^i \leq w\right) \end{aligned} \quad (1.47)$$

Individually evaluating each of the four components in the sum of Eq. (1.47), and using Eqs. (1.45) and (1.46) to simplify:

$$\begin{aligned} & P\left(w \leq -A - ES_1^i \mid ES_1^i\right) E\left(\left[\frac{r^L + r^D}{2}\right] [ES_1^i + w] - \gamma \mid w \leq -A - ES_1^i\right) = \\ & \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) \left[\frac{r^L + r^D}{2}\right] ES_1^i - \left[\frac{r^L + r^D}{2}\right] \phi\left(\frac{-A - ES_1^i}{\sigma}\right) \sigma - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) \gamma \quad (1.48) \end{aligned}$$

$$\begin{aligned} & P\left(-A - ES_1^i < w \leq -ES_1^i \mid ES_1^i\right) E\left(r^L [ES_1^i + w] \mid -A - ES_1^i < w \leq -ES_1^i\right) = \\ & \left[\Phi\left(\frac{-ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right)\right] r^L ES_1^i + r^L \left[\phi\left(\frac{-A - ES_1^i}{\sigma}\right) - \phi\left(\frac{-ES_1^i}{\sigma}\right)\right] \sigma \quad (1.49) \end{aligned}$$

$$\begin{aligned} & P\left(-ES_1^i < w < A + M - ES_1^i \mid ES_1^i\right) E\left(r^D [ES_1^i + w] \mid -ES_1^i < w < A + M - ES_1^i\right) = \\ & \left[\Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-ES_1^i}{\sigma}\right)\right] r^D ES_1^i + r^D \left[\phi\left(\frac{-ES_1^i}{\sigma}\right) - \phi\left(\frac{A + M - ES_1^i}{\sigma}\right)\right] \sigma \quad (1.50) \end{aligned}$$

$$\begin{aligned} & P\left(A + M - ES_1^i \leq w \mid ES_1^i\right) \times \\ & E\left(r^D M + \left[\frac{r^L + r^D}{2}\right] [ES_1^i + w - M] - \gamma \mid A + M - ES_1^i \leq w\right) = \\ & \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right)\right] r^D M + \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right)\right] \left[\frac{r^L + r^D}{2}\right] [ES_1^i - M] + \\ & \left[\frac{r^L + r^D}{2}\right] \phi\left(\frac{A + M - ES_1^i}{\sigma}\right) \sigma - \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right)\right] \gamma \quad (1.51) \end{aligned}$$

Substituting Eqs. (1.48)-(1.51) into Eq. (1.47), and then simplifying, gives:

$$\begin{aligned}
\vartheta(ES_1^i) \equiv \mathbb{E}[\pi_2^i | ES_1^i] &= [r^L - r^D] \left[\Phi\left(\frac{-ES_1^i}{\sigma}\right) ES_1^i - \phi\left(\frac{-ES_1^i}{\sigma}\right) \sigma \right] - \\
&\quad \left[\frac{r^L - r^D}{2} \right] \left[\Phi\left(\frac{-A - ES_1^i}{\sigma}\right) + \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) \right] ES_1^i + \\
&\quad \left[\frac{r^L - r^D}{2} \right] \left[\phi\left(\frac{-A - ES_1^i}{\sigma}\right) + \phi\left(\frac{A + M - ES_1^i}{\sigma}\right) \right] \sigma - \\
&\quad \left[\frac{r^L - r^D}{2} \right] \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) \right] M - \\
&\quad \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) + \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) \right] \gamma + \left[\frac{r^L + r^D}{2} \right] ES_1^i
\end{aligned} \tag{1.52}$$

1.7.7 Proof of Proposition 1.2

Substituting Eq. (1.52) into $g(z)$, and setting $M = 0$ gives:

$$\begin{aligned}
g(z) &= \left[\frac{r^L - r^D}{2} \right] \left[2\Phi\left(\frac{-z}{\sigma}\right) - \Phi\left(\frac{-A - z}{\sigma}\right) - \Phi\left(\frac{A - z}{\sigma}\right) \right] z - \\
&\quad \left[\frac{r^L - r^D}{2} \right] \left[2\phi\left(\frac{-z}{\sigma}\right) - \phi\left(\frac{-A - z}{\sigma}\right) - \phi\left(\frac{A - z}{\sigma}\right) \right] \sigma - \left[1 - \Phi\left(\frac{A - z}{\sigma}\right) + \Phi\left(\frac{-A - z}{\sigma}\right) \right] \gamma
\end{aligned} \tag{1.53}$$

Differentiating Eq. (1.53) with respect to z gives:

$$\frac{dg(z)}{dz} = \left[\frac{r^L - r^D}{2} \right] \left[2\Phi\left(\frac{-z}{\sigma}\right) - \Phi\left(\frac{-A - z}{\sigma}\right) - \Phi\left(\frac{A - z}{\sigma}\right) \right] \tag{1.54}$$

Which is equivalent to:

$$\frac{dg(z)}{dz} = [r^L - r^D] \Phi\left(\frac{-z}{\sigma}\right) + \left[\frac{r^L - r^D}{2} \right] \left[\Phi\left(\frac{-A + z}{\sigma}\right) - \Phi\left(\frac{-A - z}{\sigma}\right) - 1 \right] \tag{1.55}$$

Using Lemma 1.1 and Eq. (1.55), when $z > 0$:

$$\frac{dg(z)}{dz} < [r^L - r^D] \Phi\left(\frac{-z}{\sigma}\right) + \left[\frac{r^L - r^D}{2} \right] \left[\Phi\left(\frac{z}{\sigma}\right) - \Phi\left(\frac{-z}{\sigma}\right) - 1 \right] = 0 \tag{1.56}$$

Again using Lemma 1.1, when $z < 0$:

$$\frac{dg(z)}{dz} > [r^L - r^D] \Phi\left(\frac{-z}{\sigma}\right) + \left[\frac{r^L - r^D}{2}\right] \left[\Phi\left(\frac{z}{\sigma}\right) - \Phi\left(\frac{-z}{\sigma}\right) - 1\right] = 0 \quad (1.57)$$

Therefore, Eq. (1.53) is maximised at $z = 0$.

Since $z = 0$ is optimal for both banks, and can be achieved by both banks when $M = 0$, $z = 0$ must be the NBS when $M = 0$ (due to the Pareto efficiency property of the NBS). That is, when the aggregate ES balance is set to zero, the NBS has both banks finishing the first period with an ES balance of zero.

1.7.8 Proof of Proposition 1.3

Substituting Eq. (1.52) into $g(z)$ gives:

$$\begin{aligned} g(z) = & \left[\frac{r^L - r^D}{2}\right] \left[2\Phi\left(\frac{-z}{\sigma}\right) - \Phi\left(\frac{-A-z}{\sigma}\right) - \Phi\left(\frac{A+M-z}{\sigma}\right)\right] z - \\ & \left[\frac{r^L - r^D}{2}\right] \left[2\phi\left(\frac{-z}{\sigma}\right) - \phi\left(\frac{-A-z}{\sigma}\right) - \phi\left(\frac{A+M-z}{\sigma}\right)\right] \sigma - \\ & \left[\frac{r^L - r^D}{2}\right] \left[1 - \Phi\left(\frac{A+M-z}{\sigma}\right)\right] M - \left[1 - \Phi\left(\frac{A+M-z}{\sigma}\right) + \Phi\left(\frac{-A-z}{\sigma}\right)\right] \gamma \end{aligned} \quad (1.58)$$

Differentiating Eq. (1.58) with respect to z gives:

$$g'(z) \equiv \frac{dg(z)}{dz} = \left[\frac{r^L - r^D}{2}\right] \left[2\Phi\left(\frac{-z}{\sigma}\right) - \Phi\left(\frac{-A-z}{\sigma}\right) - \Phi\left(\frac{A+M-z}{\sigma}\right)\right] \quad (1.59)$$

Let $\varepsilon = A/\sigma$; $\delta = M/\sigma$; and $x = -z/\sigma$. Then, using Lemma 1.3, there is a unique z at which Eq. (1.59) equals zero (with this point being < 0). Therefore, Eq. (1.58) has a unique stationary point. Moreover, also using Lemma 1.3, Eq. (1.59) is positive when z is lower than the stationary point, and is negative when z is higher than the stationary point (so the stationary point is a local and global maximum). Therefore, profits are always increased by moving ES balances closer to the optimum.

This means that, for a non-zero first-period trading volume, one of the banks must begin the first period with an ES balance less than the optimum (otherwise trade will always move one of the banks away from the optimum). It also means that a Pareto-efficient bargaining solution must have both banks finish the first period with an ES balance greater than or equal to the optimum.

1.7.9 Proof of Proposition 1.4

From Appendix 1.7.8, $g(z)$ is smooth and continuous in $z \in \mathbb{R}$, and has a unique stationary point (which is the maximum point, ES_1^*). Therefore: $\frac{dg(z)}{dz} < 0$ for all $z > ES_1^*$; $\frac{dg(z)}{dz} > 0$ for all $z < ES_1^*$; and $\frac{dg(z)}{dz} = 0$ for $z = ES_1^*$.

Since $ES_1^* < 0$ when $M > 0$ (Proposition 1.3), these derivatives mean that a mutually profitable trade exists if one of the banks has an ES balance below the optimum (since the other bank must then have an ES balance above the optimum). Therefore, a Pareto-efficient bargaining solution must have both banks finish the first period with an ES balance at least as high as the optimum. That is: $ES_1^N \geq ES_1^*$ and $M - ES_1^N \geq ES_1^*$.

Since $ES_0^i < ES_1^*$, then $M - ES_0^i > ES_1^*$. Combining this result with the above values of $\frac{dg(z)}{dz}$ and $M > 0$, ensures that $z = ES_1^*$ satisfies the constraints of Eq. (1.16) as strict inequalities. Therefore, for $z = ES_1^*$ to be the NBS (from differentiating the Nash Product with respect to z):

$$\frac{dg(z)}{dz} \left\{ g(M - z) - g(M - ES_0^i) \right\} + \frac{dg(M - z)}{dz} \left\{ g(z) - g(ES_0^i) \right\} = 0$$

But at $z = ES_1^*$, $\frac{dg(z)}{dz} = 0$, $\frac{dg(M - z)}{dz} > 0$, and $g(z) > g(ES_0^i)$. So the above equality does not hold. Therefore, $z = ES_1^*$ cannot be the NBS.

The above values of $\frac{dg(z)}{dz}$, and $M - ES_0^i > ES_1^*$, ensure that $M - z = ES_1^*$ satisfies $g(M - z) > g(M - ES_0^i)$. But $M - z = ES_1^*$ may not satisfy $g(z) \geq g(ES_0^i)$. If it does not satisfy $g(z) \geq g(ES_0^i)$, then it cannot be the NBS. If it does satisfy this inequality,

then for $M - z = ES_1^*$ to be the NBS:

$$\frac{dg(z)}{dz} \left\{ g(M - z) - g(M - ES_0^i) \right\} + \frac{dg(M - z)}{dz} \left\{ g(z) - g(ES_0^i) \right\} \geq 0$$

With the inequality being an equality unless $g(z) = g(ES_0^i)$.

But at $M - z = ES_1^*$, $\frac{dg(M - z)}{dz} = 0$, $\frac{dg(z)}{dz} < 0$, and $g(M - z) > g(M - ES_0^i)$. So the above inequality does not hold. Therefore, $M - z = ES_1^*$ cannot be the NBS.

Therefore, the NBS must satisfy $M - ES_1^* > ES_1^N > ES_1^*$.

1.7.10 Numerical computation of the first-period NBS

Since the outcome of trading when $M = 0$ is given in Proposition 1.2, for the remainder of this section assume $M > 0$. From Proposition 1.3, when $M > 0$, then $ES_1^* < 0$ (ES_1^* is the value of z that maximises $g(z)$); and banks only trade if either $ES_0^i < ES_1^*$ or $M - ES_0^i < ES_1^*$. So, without loss of generality, for the remainder of this section assume that $ES_0^i < ES_1^*$.

As noted in Section 1.3.4, the first-period NBS trading volume can be found by maximising Eq. (1.16) with respect to z . Re-producing Eq. (1.16):

$$\left\{ g(z) - g(ES_0^i) \right\} \left\{ g(M - z) - g(M - ES_0^i) \right\} \quad (1.60)$$

subject to

$$g(z) - g(ES_0^i) \geq 0$$

$$g(M - z) - g(M - ES_0^i) \geq 0$$

To simplify the analysis, define two functions (both functions take ES_0^i as pre-determined):

$$q(z) \equiv g(z) - g(ES_0^i)$$

$$s(z) \equiv g(M - z) - g(M - ES_0^i)$$

Note: $q(ES_0^i) = s(ES_0^i) = 0$.

Since expected profits are always increased by trading towards the first-period optimum, both $q(z)$ and $s(z)$ are strictly quasi-concave, and the derivatives of both functions are positive at $z = ES_0^i$.

From Section 1.3.4, when $M > 0$, the opportunity cost of having a second-period surplus (i.e. following second-period payment flows) is at least as high as the equivalent deficit (and sometimes higher). Therefore, $q(ES_0^i) > q(-ES_0^i)$. Due to continuity, the intermediate value theorem, the fact that $q(ES_0^i) = 0$, and the positive derivative at ES_0^i : $q(ES_0^i) > q(-ES_0^i)$ implies there exists some point $ES_0^i < z_q < -ES_0^i$ where $q(z_q) = 0$. The strict quasi-concavity also implies ES_0^i and z_q are the only two points at which the $q(\cdot)$ function equals zero, and therefore:

$$q(z) \geq 0 \text{ when } ES_0^i \leq z \leq z_q \text{ and } q(z) < 0 \text{ otherwise}$$

The differences in opportunity costs between second-period deficits and surpluses also imply that $s(ES_0^i) < s(-ES_0^i)$. Therefore, with the strict quasi-concavity of $s(z)$ and the fact that $s(ES_0^i) = 0$:

$$s(z) \geq 0 \text{ when } ES_0^i \leq z \leq z_q$$

Therefore, the inequality conditions of Eq. (1.60) are satisfied as long as:

$$ES_0^i \leq z \leq z_q$$

Therefore, the NBS is:

$$ES_1^N = \arg \max_{ES_0^i \leq z \leq z_q} \{q(z)s(z)\} \quad (1.61)$$

1.7.10.1 Refinement of NBS search region

Differentiating the argument to be maximised in Eq. (1.61) gives:

$$D(z) \equiv \frac{d\{q(z)s(z)\}}{dz} = g'(z)s(z) - g'(M-z)q(z) \quad (1.62)$$

Using Eq. (1.59) (let $z \equiv y + \frac{M}{2}$):

$$\begin{aligned} \frac{g'\left(y + \frac{M}{2}\right) - g'\left(-y + \frac{M}{2}\right)}{r^L - r^D} = \\ \left[\Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) + \Phi\left(\frac{-A - \frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-A - \frac{M}{2} - y}{\sigma}\right) \right] \end{aligned} \quad (1.63)$$

Using Lemma 1.1, when $y > 0$:

$$\Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) > \Phi\left(\frac{-A - \frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-A - \frac{M}{2} - y}{\sigma}\right)$$

Therefore, when $y > 0$:

$$\begin{aligned} \frac{g'\left(y + \frac{M}{2}\right) - g'\left(-y + \frac{M}{2}\right)}{r^L - r^D} < \\ \left[\Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) + \Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) \right] = 0 \end{aligned} \quad (1.64)$$

And therefore, $g'(M-z) \geq g'(z)$ when $z \geq M/2$ (call this Fact A).

Moreover, from Proposition 1.4, recall that $M - ES_1^* > ES_1^N > ES_1^*$ (call this Fact B). And given that $g(z)$ has a unique stationary point (that is a maximum point) at ES_1^* , when $M - ES_1^* > z$, then $M - z > ES_1^*$, which means that $g'(M-z) < 0$ when $M - ES_1^* > z$. Combining this with Fact A gives: $0 > g'(M-z) \geq g'(z)$ when $M - ES_1^* > z \geq M/2$ (call this Fact C).

Also, since $g(ES_0^i) > g(M - ES_0^i)$ (due to the higher opportunity cost of second-period

surpluses when $M > 0$ (so $g(ES_0^i) > g(-ES_0^i)$), and due to the proof in Appendix 1.7.8 (so $g(-ES_0^i) > g(M - ES_0^i)$), and $g(z) \leq g(M - z)$ when $z \geq M/2$ (because $g(z)$ has a unique stationary point, that is a maximum point, at $ES_1^* < 0$), then $s(z) > q(z) > 0$ when $z_q > z \geq M/2$ (call this Fact D).

Therefore, using Fact C and Fact D, if $z_q > M/2$, then $g'(z)s(z) < g'(M - z)q(z) < 0 \forall \min(M - ES_1^*, z_q) > z \geq M/2$. Which means:

$$D(z) < 0 \forall \min(M - ES_1^*, z_q) > z \geq M/2$$

So $z < M/2$ at the NBS.

As a result, the refined set of values that could be the NBS is (this uses Fact B from above):

$$B = \left\{ z \mid ES_1^* \leq z \leq \min\left(z_q, \frac{M}{2}\right) \right\} \quad (1.65)$$

Note that, due to $q(z)$ being strictly increasing when $z < ES_1^*$, this means $z_q > ES_1^*$. Therefore, the set B is non-empty.

Therefore, from Eq. (1.61) and (1.65), the NBS is:

$$ES_1^N = \arg \max_{x \in B} \{q(x)s(x)\} \quad (1.66)$$

Since the set B is compact and convex, and $q(x)s(x)$ is continuous, ES_1^N exists.

1.7.10.2 The NBS must be in the interior of B

Since expected profits are always increased by trading towards the first-period optimum, within the set B , $g'(z) \leq 0$ and $g'(M - z) < 0$.

So, when $z = ES_1^*$:

$$D(ES_1^*) = -g'(M - ES_1^*)q(ES_1^*) > 0$$

When $z = z_q$:

$$D(z_q) = g'(z_q)s(z_q) < 0$$

Therefore, ES_1^N must be an interior solution.

1.7.10.3 The Nash Product has a unique stationary point in the interior of B

At an interior optimum, $D(ES_1^N) = 0$. Therefore:

$$\frac{g'(ES_1^N)}{g'(M - ES_1^N)} = \frac{q(ES_1^N)}{s(ES_1^N)} \quad (1.67)$$

Taking the logarithm of Eq. (1.67) (the numerator and denominator of the left-hand-side have switched signs to ensure the logarithm can be split):

$$\ln(-g'(ES_1^N)) - \ln(-g'(M - ES_1^N)) = \ln(q(ES_1^N)) - \ln(s(ES_1^N)) \quad (1.68)$$

With $q(z)$ strictly decreasing and $s(z)$ strictly increasing (in the interior of B), $[\ln(q(z)) - \ln(s(z))]$ is strictly decreasing in the interior of the set B . Therefore, if $[\ln(-g'(z)) - \ln(-g'(M - z))]$ is non-decreasing in the interior of B , then there is, at most, a single point at which Eq. (1.67) holds. Moreover, since the NBS is an interior solution, this stationary point must exist and it must be a maximum point.

So, to prove that $[\ln(-g'(z)) - \ln(-g'(M - z))]$ is non-decreasing within the interior of B , the derivative of $[\ln(-g'(z)) - \ln(-g'(M - z))]$ must be non-negative:

$$\frac{g''(z)}{g'(z)} + \frac{g''(M - z)}{g'(M - z)} \geq 0 \quad (1.69)$$

$g''(z) < 0$ in the interior of B

Using the proof of Lemma 1.3, $g''(ES_1^*) < 0$ and $g''(z)$ only changes sign once when $z > ES_1^*$.

At $z = M/2$:

$$g''\left(\frac{M}{2}\right) = -\left[\frac{r^L - r^D}{2\sigma}\right]\left[2\phi\left(\frac{-\frac{M}{2}}{\sigma}\right) - \phi\left(\frac{-A - \frac{M}{2}}{\sigma}\right) - \phi\left(\frac{A + \frac{M}{2}}{\sigma}\right)\right] < 0 \quad (1.70)$$

Therefore, since $g''(z)$ is still negative at $z = M/2$, then $g''(z) < 0 \forall z \in B$.

$g'(z) < 0$ and $g'(M - z) < 0$ in the interior of B

Using Lemma 1.1, when $y < 0$:

$$\Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) < \Phi\left(\frac{-A - \frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-A - \frac{M}{2} - y}{\sigma}\right)$$

Therefore, using Eq. (1.63), when $y < 0$:

$$\frac{g'\left(y + \frac{M}{2}\right) - g'\left(-y + \frac{M}{2}\right)}{r^L - r^D} > \left[\Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) + \Phi\left(\frac{-\frac{M}{2} + y}{\sigma}\right) - \Phi\left(\frac{-\frac{M}{2} - y}{\sigma}\right)\right] = 0 \quad (1.71)$$

And therefore, $0 > g'(z) > g'(M - z)$ when z is in the interior of B .

So, within the interior of B : $g'(z) < 0$, $g'(M - z) < 0$, and $g''(z) < 0$.

Therefore, Eq. (1.69) can be re-arranged as:

$$g''(M - z) \leq -g''(z) \left[\frac{g'(M - z)}{g'(z)} \right] \quad (1.72)$$

Moreover, $\frac{g'(M - z)}{g'(z)} > 1$. Therefore, Eq. (1.72) holds if:

$$g''(M - z) + g''(z) \leq 0 \quad (1.73)$$

$g''(M - z) + g''(z) \leq 0$ in the interior of B

Expanding Eq. (1.73):

$$\phi\left(\frac{-M+z}{\sigma}\right) + \phi\left(\frac{-z}{\sigma}\right) - \phi\left(\frac{-A-M+z}{\sigma}\right) - \phi\left(\frac{-A-z}{\sigma}\right) \geq 0 \quad (1.74)$$

The inequality in Eq. (1.74) always holds in the region $M/2 > z > -A/2$ (the inequality also holds for some $z \leq -A/2$, but the values of $z \leq -A/2$ for which Eq. (1.74) holds will depend on the parameter values).

Therefore, to ensure that Eq. (1.74) holds when z is in the interior of B , the lower bound of the set must satisfy $ES_1^* \geq -A/2$.

Totally differentiating Eq. (1.59) (which equals zero at the optimum point) to determine the impact of a change in A on the optimal end-of-first-period ES balance gives (recall that $ES_1^* < 0$):

$$\frac{dES_1^*}{dA} = \frac{\left[\frac{r^L - r^D}{2\sigma}\right] \left[\phi\left(\frac{A+M-ES_1^*}{\sigma}\right) - \phi\left(\frac{-A-ES_1^*}{\sigma}\right)\right]}{g''(ES_1^*)} > 0$$

Therefore, the largest value of A for which $ES_1^* \leq -A/2$ occurs where $ES_1^* = -A/2$.

While it is not possible to analytically determine the value of A for which $ES_1^* = -A/2$, Eq. (1.59) can be used to derive an upper bound (recall $A = (2\gamma) / (r^L - r^D)$):

$$\gamma < \sigma [r^L - r^D] \Phi^{-1}\left(\frac{2}{3}\right) \quad (1.75)$$

With the baseline values of σ and $r^L - r^D$ (see Section 1.4), the right-hand-side of Eq. (1.75) is equal to 2.5×10^{-8} , equivalent to A\$0.03. Since reasonable levels of transaction costs are far larger than this upper bound, $ES_1^* \geq -A/2$ in this chapter. Therefore, Eq. (1.69) holds, and the NBS occurs at the unique point (within the set B) where Eq. (1.67) is satisfied.³⁶

The first-period NBS is found by searching (numerically) within the set B for the unique point where Eq. (1.67) is satisfied.

³⁶While this proof provides a sufficient condition (Eq. (1.75)) for a unique solution to Eq. (1.67) (within the set B), this condition may not be necessary. That is, there may be a unique solution to Eq. (1.67) even when Eq. (1.69) is violated.

1.7.11 Formulas for numerical computation

1.7.11.1 Approximate end-of-first-period ES balance probability density function

The unconditional probability that ES_1^i lies in some region ($\varepsilon > 0$) around x is:

$$P(x - \varepsilon \leq ES_1^i < x + \varepsilon) = \int_{x-\varepsilon}^{x+\varepsilon} f_{ES_1^i}(y) dy$$

where $f_{ES_1^i}(\cdot)$ is the unconditional probability density function of ES_1^i .

Expanding the probability density function of ES_1^i :

$$P(x - \varepsilon \leq ES_1^i < x + \varepsilon) = \int_{x-\varepsilon}^{x+\varepsilon} \left\{ \int_{-\infty}^{\infty} f_{ES_1^i|ES_0^i=z}(y) f_{ES_0^i}(z) dz \right\} dy$$

where $f_{ES_0^i}(\cdot)$ is the unconditional probability density function of ES_0^i .

Switching the order of integration (using Fubini's theorem):

$$P(x - \varepsilon \leq ES_1^i < x + \varepsilon) = \int_{-\infty}^{\infty} \left\{ \int_{x-\varepsilon}^{x+\varepsilon} f_{ES_1^i|ES_0^i=z}(y) dy \right\} f_{ES_0^i}(z) dz \quad (1.76)$$

From Appendix 1.7.10, each observation of ES_0^i corresponds to a single ES_1^i (i.e. there is a unique trading volume). Let $I_{\{x, \varepsilon, ES_0^i\}}$ be an indicator function equal to 1 if the value of ES_1^i for a given ES_0^i lies in the region $[x - \varepsilon, x + \varepsilon)$, and 0 otherwise. Then Eq. (1.76) can be written as (using Eq. (1.18)):

$$P(x - \varepsilon \leq ES_1^i < x + \varepsilon) = \int_{-\infty}^{\infty} I_{\{x, \varepsilon, z\}} \left[\frac{1}{\tau} \phi\left(\frac{z - (M/2)}{\tau}\right) \right] dz \quad (1.77)$$

The end-of-first-period ES balance probability density function is approximated by using Eq. (1.77) in the following estimator:

$$\hat{f}_{ES_1^i}(x) = \frac{1}{2\varepsilon} P(x - \varepsilon \leq ES_1^i < x + \varepsilon) \quad (1.78)$$

From Section 1.3.4, Eq. (1.78) must be zero outside the region $x \in (ES_1^* - \varepsilon,$

$M - ES_1^* + \varepsilon]$. So, to construct Figure 1.5, numerical integration is used to compute Eq. (1.78) for 0.0005 increments of x within the range $ES_1^* \leq x \leq M - ES_1^*$, and using $\varepsilon = 0.00025$ (thereby creating consecutive, non-overlapping intervals).

1.7.11.2 Expected first-period trading volume

From Appendix 1.7.10, each observation of ES_0^i corresponds to a single ES_1^i (i.e. there is a unique trading volume). Therefore, let $ES_1^i = k(ES_0^i)$ such that the first-period trading volume is:

$$V_1(ES_0^i) = |k(ES_0^i) - ES_0^i| \quad (1.79)$$

The unconditional expected first-period trading volume is:

$$E[V_1(ES_0^i)] = \int_{-\infty}^{\infty} |k(z) - z| f_{ES_0^i}(z) dz \quad (1.80)$$

where $f_{ES_0^i}(\cdot)$ is the unconditional probability density function of ES_0^i .

Using Eq. (1.18), Eq. (1.80) can be re-written as:

$$E[V_1(ES_0^i)] = \int_{-\infty}^{\infty} |k(z) - z| \left[\frac{1}{\tau} \phi\left(\frac{z - (M/2)}{\tau}\right) \right] dz \quad (1.81)$$

The expected first-period trading volume is determined by numerically integrating Eq. (1.81).

1.7.11.3 Expected second-period trading volume

From Section 1.3.3, trading in the second period will only occur if one of the banks has an ES balance deficit less than $-A$. From the perspective of Bank i , this will occur if: $ES_2^i \leq -A$ or $M - ES_2^i \leq -A$. In the first case, the trading volume will be $V_2(ES_2^i) = -ES_2^i$. In the second case, the trading volume will be $V_2(ES_2^i) = ES_2^i - M$.

Therefore, after first-period trading, and using Eqs. (1.12) and (1.13), the condi-

tional expected trading volume is:

$$\begin{aligned} & P(w \leq -A - ES_1^i | ES_1^i) E(-ES_1^i - w | w \leq -A - ES_1^i) + \\ & P(w \geq A + M - ES_1^i | ES_1^i) E(ES_1^i + w - M | w \geq A + M - ES_1^i) \end{aligned} \quad (1.82)$$

Using Eqs. (1.45) and (1.46), Eq. (1.82) can be simplified to:

$$\begin{aligned} E[V_2(ES_2^i) | ES_1^i] = & \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) \right] ES_1^i + \\ & \left[\phi\left(\frac{A + M - ES_1^i}{\sigma}\right) + \phi\left(\frac{-A - ES_1^i}{\sigma}\right) \right] \sigma - \\ & \left[1 - \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) \right] M \end{aligned}$$

Then, using the law of iterated expectations and $ES_1^i = k(ES_0^i)$ (see Appendix 1.7.11.2), the unconditional expected second-period trading volume is:

$$E[V_2(ES_2^i)] = E_{ES_0^i} [E[V_2(ES_2^i) | k(ES_0^i)]]$$

Which is equivalent to:

$$E[V_2(ES_2^i)] = \int_{-\infty}^{\infty} \{E[V_2(ES_2^i) | k(z)]\} f_{ES_0^i}(z) dz \quad (1.83)$$

where $f_{ES_0^i}(\cdot)$ is the unconditional probability density function of ES_0^i .

Using Eq. (1.18), Eq. (1.83) can be re-written as:

$$E[V_2(ES_2^i)] = \int_{-\infty}^{\infty} E[V_2(ES_2^i) | k(z)] \left[\frac{1}{\tau} \phi\left(\frac{z - (M/2)}{\tau}\right) \right] dz \quad (1.84)$$

The expected second-period trading volume is determined by numerically integrating Eq. (1.84).

1.7.11.4 Probability of lending facility access

From Section 1.3.3, the central bank's lending facility will be accessed if either of the banks' second-period ES balances falls into the region $(-2\gamma) / (r^L - r^D) < ES_2^i \leq 0$. From the perspective of Bank i , this will occur if (recall that $A = (2\gamma) / (r^L - r^D)$):

$$-A < ES_2^i \leq 0 \quad \text{or} \quad -A < M - ES_2^i \leq 0$$

After first-period trading, and using Eqs. (1.12) and (1.13), the probability of this occurring is:

$$P(-A - ES_1^i < w \leq -ES_1^i | ES_1^i) + P(A + M - ES_1^i > w \geq M - ES_1^i | ES_1^i)$$

Therefore, using Eq. (1.45), the probability of the central bank's lending facility being accessed (given the outcome of first-period trading) is:

$$P_{ES_1^i} = \Phi\left(\frac{-ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) + \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{M - ES_1^i}{\sigma}\right) \quad (1.85)$$

Denote the unconditional probability of lending facility access as P^U , then (with $ES_1^i = k(ES_0^i)$):

$$P^U = \int_{-\infty}^{\infty} P_{k(z)} f_{ES_0^i}(z) dz \quad (1.86)$$

where $f_{ES_0^i}(\cdot)$ is the unconditional probability density function of ES_0^i .

Using Eq. (1.18), Eq. (1.86) can be re-written as:

$$P^U = \int_{-\infty}^{\infty} P_{k(z)} \left[\frac{1}{\tau} \phi\left(\frac{z - (M/2)}{\tau}\right) \right] dz \quad (1.87)$$

The probability of lending facility access is determined by numerically integrating Eq. (1.87).

1.7.12 Maximum probability of lending facility access

Using Lemma 1.2, suppose $ES_1^i \leq (M/2)$, then:

$$\Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{M - ES_1^i}{\sigma}\right) \leq \Phi\left(\frac{A - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-ES_1^i}{\sigma}\right)$$

Using this inequality on Eq. (1.85):

$$P_{ES_1^i} \leq \Phi\left(\frac{A - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right)$$

And, using Lemma 1.1:

$$P_{ES_1^i} \leq \Phi\left(\frac{A - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right) \leq \Phi\left(\frac{A}{\sigma}\right) - \Phi\left(\frac{-A}{\sigma}\right) \quad (1.88)$$

Using Lemma 1.2, suppose $ES_1^i \geq (M/2)$, then:

$$\Phi\left(\frac{M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A + M - ES_1^i}{\sigma}\right) \geq \Phi\left(\frac{-ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A - ES_1^i}{\sigma}\right)$$

Using this inequality on Eq. (1.85):

$$P_{ES_1^i} \leq \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A + M - ES_1^i}{\sigma}\right)$$

And, using Lemma 1.1:

$$P_{ES_1^i} \leq \Phi\left(\frac{A + M - ES_1^i}{\sigma}\right) - \Phi\left(\frac{-A + M - ES_1^i}{\sigma}\right) \leq \Phi\left(\frac{A}{\sigma}\right) - \Phi\left(\frac{-A}{\sigma}\right) \quad (1.89)$$

Therefore, from Eqs. (1.88) and (1.89), and irrespective of the level of ES_1^i or M :

$$P_{ES_1^i} \leq \Phi\left(\frac{A}{\sigma}\right) - \Phi\left(\frac{-A}{\sigma}\right) \quad (1.90)$$

Using Eq. (1.85), when $M = 0$ and $ES_1^i = 0$:

$$P_0 = \Phi\left(\frac{A}{\sigma}\right) - \Phi\left(\frac{-A}{\sigma}\right) \quad (1.91)$$

Since $ES_1^i = 0$ is the NBS when $M = 0$ (see Section 1.7.7), Eq. (1.91) will be the probability of access when the aggregate ES balance is set to zero. Then, comparing Eq. (1.91) to Eq. (1.90), the probability of access is maximised when the aggregate ES balance is set to zero.

1.7.13 Identification of the ‘refined set’

The values of the transaction cost (γ), net first-period payment flow variance (τ^2), and net second-period payment flow variance (σ^2) are identified using the Method of Moments. The population moments are:

- ‘probability that the central bank’s lending facility is accessed’ (from the model, adjusted for a market of 50 ex-ante identical banks)
- ‘average daily trading volume’ (from the model, adjusted for a market of 50 ex-ante identical banks)

The sample moments are:

- average rate of use of the RBA’s lending facility (4.49 per cent, see Table 1.1)
- daily average volume of trading between July 2009 and June 2010 (A\$7.5 billion, see Section 1.2.2)

This procedure ‘set identifies’ these parameters.

1.7.13.1 Adjusting the model-based moments for a market of 50 ex-ante identical banks

With ex-ante identical banks, each bank’s unconditional probability of using the central bank’s lending facility is the same. Moreover, with a non-negative aggregate ES balance and only two banks, only one bank can use the central bank’s lending facility per trading day. Therefore, in the two-bank model, the unconditional probability of a bank using

the lending facility is $P^U/2$ (see Eq. (1.87)). And the unconditional probability of a bank not using the lending facility is $[1 - (P^U/2)]$.

With 50 banks, the probability of the lending facility not being accessed can be deconstructed as (P_i is the probability relating to Bank i):

$$\begin{aligned} P(\text{no use}) &= P_1(\text{no use} | \text{no use by banks 2-50}) \times \\ &P_2(\text{no use} | \text{no use by banks 3-50}) \times \\ &P_3(\text{no use} | \text{no use by banks 4-50}) \times \dots \times P_{50}(\text{no use}) \end{aligned} \quad (1.92)$$

A 50 bank model would be required to deconstruct the probability as in Eq. (1.92). However, given how the two-bank model works (i.e. only one bank can use the lending facility per day), it is reasonable to suspect that the conditional probability of a bank not using the lending facility (conditioned on other banks not using the facility) will be no higher than the bank's unconditional no-use probability. Therefore, with ex-ante identical banks, an upper bound for Eq. (1.92) is:

$$P_{UB}(\text{no use}) = [P_i(\text{no use})]^{50}$$

Equating the sample moment with the population moment, and using $P^U/2$ as the representative bank's unconditional probability of lending facility use, this upper bound implies:

$$[1 - 0.0449] \leq [1 - (P^U/2)]^{50} \quad (1.93)$$

Which can be re-arranged as:

$$P^U \leq 0.0018 \quad (1.94)$$

That is, at the point where the sample moment equals the population moment, $P^U \leq 0.0018$. Moreover, setting P^U to any point within this region produces the

same model mechanics and is consistent with the stylised facts. So, instead of using $P(\text{no use}) = 1 - 0.0449$ as the moment condition (which cannot be solved without a 50 bank model), I use the following equation as the first moment condition:

$$P^U - 0.0018 = 0 \quad (1.95)$$

With ex-ante identical banks, the expected daily trading volume between any two banks is the same. Therefore, with a market of 50 ex-ante identical banks, the total expected daily volume of trade is:

$$E(V) = \frac{50 \times 49}{2} \{E[V_1^{i,j}] + E[V_2^{i,j}]\}$$

Where $V_1^{i,j}$ is the first-period trading volume between banks i and j , and $V_2^{i,j}$ is the second-period trading volume between banks i and j .

Equating the sample moment with the population moment, and using Eqs. (1.81) and (1.84) as the expected trading volumes between two representative banks, the second moment condition is:

$$\text{A\$7.5 billion} - \left\{ \left[\frac{50 \times 49}{2} \right] \{E[V_1] + E[V_2]\} \right\} = 0 \quad (1.96)$$

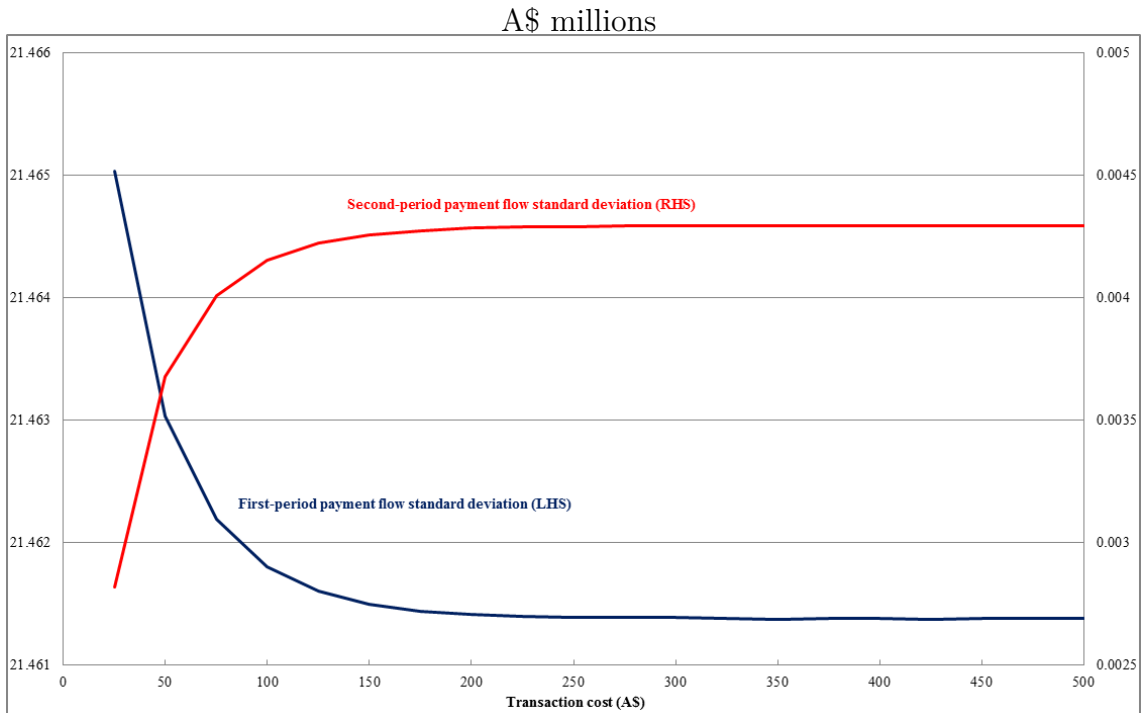
1.7.13.2 Refining the identified set

To further refine the identified set, I discard all subsets in which the transaction cost is either too small or too large (given its interpretation as an extra labour cost). For parameter values within this ‘refined set’, the mechanics of the model are the same, as is the model’s ability to replicate the stylised facts. To see why, first note that, due to the first moment condition (Eq. (1.95)), the model results must satisfy the ‘rare use’ fact. Also, for all of the parameter values within the ‘refined set’, second-period trading is rare. For infrequent second-period trading and lending facility use, both banks must have sufficiently large positive ES balances at the end of the first period (i.e. large

relative to typical second-period payment flows) such that both banks enter the second period, and end the day, with positive ES balances ('positive end-of-day' fact), which requires a positive aggregate ES balance ('positive aggregate' fact). With infrequent second-period trade, trading must typically occur in the first-period (due to the second moment condition, Eq. (1.96)), thereby satisfying the 'early trade' fact. Therefore, for the purposes of this chapter, it does not matter which parameter combination is used as the numerical example.

For various levels of the transaction cost, Figure 1.9 gives examples of first and second-period payment flow standard deviations that are identified using the method described in this section.

Figure 1.9: Estimated Payment-flow Standard Deviations



Chapter 2

Multi-bank Model and Quantitative Results

2.1 Introduction

When a central bank is committed to achieving its interest rate target, the two-bank model of the first chapter provides a simple framework that can be used to qualitatively evaluate the impact of policy changes on the overnight interbank market. However, with interbank markets containing more than two banks, the two-bank model is too simplistic for quantitative analyses. This chapter extends the two-bank model to incorporate a large number of heterogeneous banks. As a result, the model in this chapter can be directly fit to the data, and can be used to quantitatively analyse the impact of changes in central bank policy.

With many banks, each bank can potentially trade (i.e. borrow/lend overnight) many times within each trading period. In fact, with the Reserve Bank of Australia (RBA) stating that banks typically use its lending facility only due to unexpected factors (*Reserve Bank of Australia*, 2012), there must typically be sufficient time during the Australian trading day for the market to satisfy banks' known borrowing require-

ments. With the data suggesting that the RBA typically supplies more ES balances than banks desire to hold overnight (see Section 2.2), and with this surplus typically spread among several banks, the ability to trade until known borrowing requirements are satisfied improves the bargaining positions of the borrowing banks (compared with the two-bank model).

In the multi-bank model of this chapter, these improved bargaining positions allow banks to only borrow their desired amount (banks borrow more than their desired amount in the two-bank model). As a result, in order to replicate the data, the multi-bank model requires a more complex structure for the unexpected second-period payment flows (compared with the two-bank model). But with this complexity comes greater realism, with this new structure replicating more features of the data than the two-bank model (or any other model in the literature), including: the wide range of borrowing volumes when the lending facility is accessed, and that, while being rarely used, the lending facility is sometimes used by multiple banks on the same day.

With respect to each bank's end-of-day ES balance distribution, the multi-bank model can replicate both the wide spread of percentiles above each bank's median, and the negligible spread of percentiles below each bank's median (despite each bank having a symmetric initial distribution). This cannot be achieved by either the two-bank model, nor any of the existing models in the literature. Moreover, by accounting for banks' heterogeneity, the multi-bank model can replicate the different shape of each bank's end-of-day ES balance distribution.

The multi-bank model produces an aggregate ES balance of A\$760 million, close to the A\$751 million in the data.¹ Moreover, increasing the aggregate ES balance from A\$751 million to A\$1.276 billion (corresponding to the pre- and post-crisis levels in Australia) causes a 54 per cent fall in the aggregate daily trading volume, similar to the 47 per cent fall in the data; while decreasing the aggregate ES balance to A\$1.075 billion

¹This is not just a by-product of the data-fitting process, as the aggregate ES balance is not one of the sample statistics used to calibrate the model.

(to match the recent decrease in Australia), produces an aggregate daily trading volume that is 38 per cent lower than pre-crisis, close to the 39 per cent difference in the data. These results are consistent with the qualitative results in the first chapter, and the ability to so closely match the Australian market is novel in the literature.

In the remainder of this chapter, Section 2.2 describes the data that the model is designed to replicate. Section 2.3 constructs and solves the model. Section 2.4 briefly describes the calibration process (details are provided in the appendix). Section 2.5 discusses the success of the model in replicating the Australian market, and gives the predicted impact of policy changes. Section 2.6 concludes. Appendix 2.7.1 relates this model to the existing literature.

2.2 Data

2.2.1 End-of-day ES balance distributions

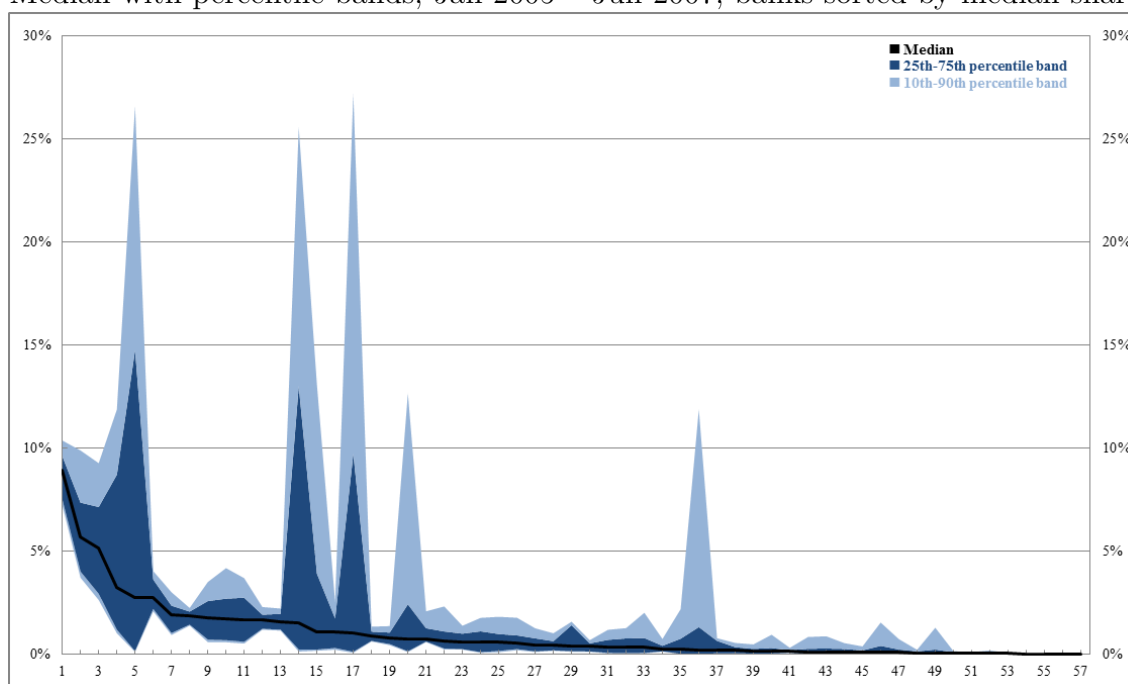
Figure 2.1 shows the distributions of each bank's end-of-day share of the aggregate ES balance. These end-of-day ES balance distributions are typically asymmetric, with a much larger difference between banks' 90th and 75th percentiles than between their 25th and 10th percentiles (with the difference between their 25th and 10th percentiles typically negligible). Moreover, with the 10th percentile above zero for most banks, banks typically finish the day with an ES balance above zero.

If the RBA supplied the exact volume of ES balances that banks desired to hold overnight, the fact that Australian banks typically only use the lending facility due to unexpected factors (*Reserve Bank of Australia*, 2012) would mean that each bank's distribution would be bunched around a single point (allowing for small deviations resulting from uncertainty). Since this does not occur, the RBA must typically be supplying ES balances greater than banks' desired overnight holdings.

If these surplus ES balances were typically stored with a single bank, the spread

between the upper percentiles would be negligible (i.e. banks would either finish the day close to their desired ES balance, or with the majority of the surplus). Since this does not occur, the surplus must typically be spread among several banks.

Figure 2.1: Bank's End-of-day Shares of the Aggregate ES Balance
Median with percentile bands; Jan 2005 – Jun 2007; banks sorted by median share



Source: Reserve Bank of Australia (data not publicly available)

2.2.2 Use of the RBA's lending facility

Table 2.1 details banks' usage of the RBA's lending facility for a pre-crisis period that closely aligns with the data in Figure 2.1. Use of the lending facility is rare, with banks accessing the lending facility only 45 times during the 757 trading days during this period. That said, the lending facility was only accessed on 37 different days, so the lending facility is sometimes used by more than one bank per day.

The volumes borrowed from the lending facility range from as little as A\$1 million per day to as much as A\$1 billion per day – with the expected borrowing volume, conditional on the lending facility being accessed, being A\$120 million. But with the

probability of access being so low, the unconditional expected daily borrowing volume is just A\$7 million (less than A\$6 million per bank per day).

Table 2.1: Use of the RBA's Lending Facility
July 2004 – June 2007

Number of times accessed	45
Number of days accessed	37
Total trading days	757
Expected daily usage (times accessed / trading days)	0.059
Volume borrowed	A\$5.42 billion
Expected borrowing per day (volume borrowed / trading days)	A\$7 million
Conditional expected borrowing per bank (volume borrowed / times accessed)	A\$120 million
Minimum daily borrowing	A\$1 million
Maximum daily borrowing	A\$1.03 billion

Sources: Reserve Bank of Australia Annual Reports;
Reserve Bank of Australia Statistical Table A3

2.2.3 Payment flows, trading volumes, and ES balances

The median daily aggregate ES balance increased from A\$751 million during January 2005 to June 2007 (pre-crisis) to A\$1.276 billion during July 2010 to February 2012 (post-crisis), and then recently fell to A\$1.075 billion during March 2012 to August 2013.² Over the same periods, the average aggregate trading volume decreased from A\$6.11 billion to A\$5.54 billion (a 9 per cent fall), and then to A\$4.72 billion (a 23 per cent fall from pre-crisis).

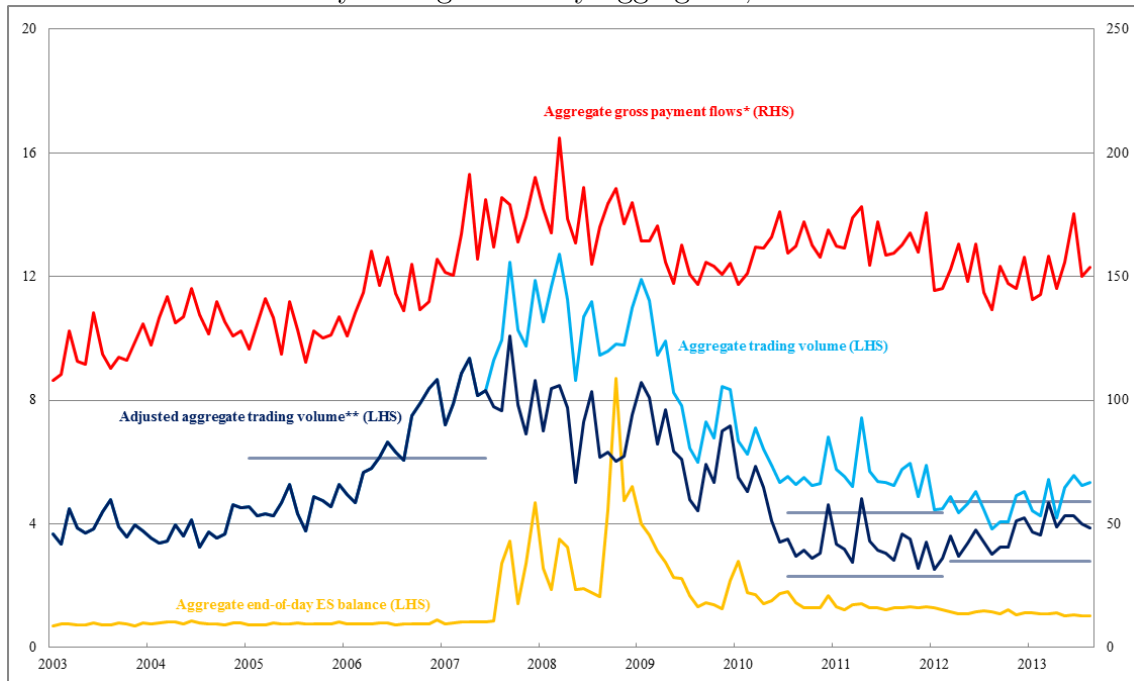
However, as is discussed in Appendix 2.7.2, increasing gross payment flows leads to higher aggregate trading volumes. Therefore, after adjusting the aggregate trading volume for changes in aggregate gross payment flows, the average aggregate trading volume falls to A\$3.27 billion during July 2010 to February 2012 (a 47 per cent fall)

²The aggregate ES balance increased significantly more during the financial crisis, but this was largely due to the abnormal circumstances of the crisis, as explained in Section 1.3.1.

but then increases to A\$3.74 billion during March 2012 to August 2013 (a 39 per cent fall from pre-crisis).³ This is all shown in Figure 2.2.

Figure 2.2: Aggregate Gross Payment Flows, Aggregate Trading Volume, and Aggregate ES Balance

Monthly averages of daily aggregates; A\$ billions



* Consistent with the definition in this thesis, gross payment flows do not include unsecured overnight loans between banks.

** This is the estimated aggregate trading volume if post-June 2007 aggregate gross payment flows returned to their January 2005 to June 2007 average. The translucent dark-blue line corresponding with January 2005 to June 2007 shows the average aggregate trading volume over this period. The sets of these lines corresponding with the ‘post-crisis’ and ‘recent’ periods show the estimated 95 per cent confidence intervals for the period-averages of the adjusted aggregate trading volume. See Appendix 2.7.2 for more detail.

Sources: Author’s calculations; Reserve Bank of Australia (data not publicly available)

2.3 Model

There are N heterogeneous, risk-neutral, profit-maximising banks. Funds move between the deposit accounts banks have at the central bank (i.e. their ES accounts) due to

³The adjustment models the aggregate trading volume and aggregate gross payment flows using a vector error-correction model, and then uses this model to extract the impact of changes in aggregate gross payment flows. See Appendix 2.7.2 for more detail.

either payment flows (treated as exogenous in this model, consistent with the two-bank model) or interbank trades (i.e. overnight borrowing/lending).

The central bank operates a corridor system, allowing banks to borrow from its overnight lending facility (at an interest rate of r^L , known as the lending rate) and paying interest on positive end-of-day ES balances (at an interest rate of r^D , known as the deposit rate) – where $r^L > r^D$. The central bank is committed to achieving its target for the average trading rate, which is $\frac{r^L + r^D}{2}$.

As in Chapter 1, each day is split into four periods. However, unlike the two-bank model, this model does not allow interbank trading following second-period payment flows. While this is a simplifying assumption, it is justified by the results of the first-chapter; that is, with sufficiently small second-period payment flows, late-day labour costs prevent banks from trading in the second period. As is discussed in more detail in Section 2.5.4, expected second-period payment flows in the multi-bank model will satisfy this “sufficiently small” requirement.

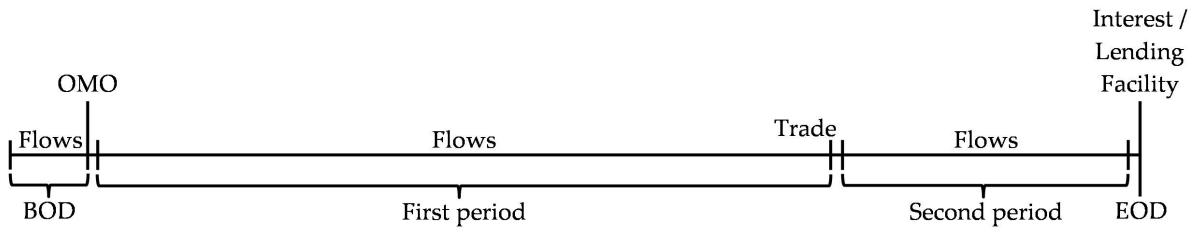
In the two-bank model it is assumed, for simplicity, that the only benefit a bank gets from having a positive end-of-day ES balance is the interest it receives from the central bank. While the two-bank model can qualitatively replicate the stylised facts even with this simplifying assumption, this assumption would prevent the multi-bank model from quantitatively replicating banks’ end-of-day ES balance distributions (this will be explained in Section 2.3.5). So, in the multi-bank model, this simplifying assumption is replaced by the more relaxed assumption that, as long as Bank i ’s end-of-day ES balance is at least as high as some non-negative level (η_i), it will receive an extra benefit of δ (this benefit does not increase the further Bank i ’s ES balance is above η_i).⁴

Importantly, while η_i may be a source of heterogeneity between the banks, it is not a source of asymmetry. As in Chapter 1, banks’ payment-flow distributions in the multi-bank model will be symmetric; a change in η_i will shift the mean of Bank i ’s ES

⁴For example, if banks use ES balances to partially satisfy their regulatory liquid asset requirements, δ could be interpreted as the benefit of satisfying these requirements.

balance distribution following first-period payment flows, but will not change its shape (as will be shown in Section 2.3.2). Therefore, the ability of the model to replicate the asymmetric shape of banks' end-of-day distributions, despite symmetric pre-trade distributions, does not trivially result from the introduction of η_i or δ ; it is commitment-restricted interbank trading that causes the asymmetries (with η_i acting only as a heterogeneous location parameter, this is discussed further in Section 2.3.5).

The following timeline shows the daily timing of payment flows, interbank trading, open market operations, and use of the central bank's lending and deposit facilities (let the aggregate ES balance be denoted by M):⁵



Beginning of day (BOD)

This period sees the occurrence of all flows that change the aggregate ES balance (e.g. flows to/from the Government), and also sees the occurrence of open market operations (OMO). The central bank uses OMO to move the adjusted aggregate ES balance ($\tilde{M} = M - \sum_{i=1}^N \eta_i$) back to its desired level (which is non-negative).⁶ For simplicity, the central bank operates such that each bank's adjusted ES balance ($ES_{OMO}^i - \eta_i$) is a constant share of the adjusted aggregate, and so that banks' profits are unaffected by the operations.⁷ That is, on each day, Bank i 's post-OMO ES balance is

⁵Although the timeline shows gaps between the end of one period and the beginning of the next, this is just to improve readability (i.e. there are no such gaps in the model).

⁶In reality, the RBA has the ability to conduct intra-day OMO, and therefore has complete control over the aggregate ES balance. For simplicity, I assume only one OMO is needed (which, prior to November 2013, was typically the case in reality, see *Debelle (2013)*), and that the adjusted aggregate ES balance is public knowledge. Therefore, in this thesis, the terms 'aggregate end-of-day ES balance' and 'aggregate ES balance' are equivalent.

⁷Having banks' beginning-of-day ES balances and profits being independent of their previous end-of-day ES balances is a simplifying assumption that allows each trading day to be solved inde-

$ES_{OMO}^i = \mu_i \widetilde{M} + \eta_i$ where $i \in \{1, 2, \dots, N\}$, each $\mu_i \in [0, 1]$ is constant over time, and $\sum_{i=1}^N \mu_i = 1$.⁸

First period

The first period begins following open market operations. It is this period that sees the occurrence of the majority of payment flows (reflecting the data, see Section 1.2.2).⁹ Denote Bank i 's ES balance following first-period payment flows as ES_0^i .

At the end of this period, the banks trade (while second-period payment flows are uncertain).¹⁰ The terms of each trade are determined via bilateral bargaining (restricted by the commitment of the central bank to its trading-rate target). For simplicity, each trading opportunity arises via random matching, and matching occurs until all profitable trading opportunities have been exhausted.¹¹ Therefore, banks can trade multiple times during this period. Denote Bank i 's ES balance following any trading as ES_1^i .

Second period

After first-period trading, the second-period payment flows occur. Denote Bank i 's ES balance following second-period payment flows as ES_2^i .

pendently (and recursively). The ability to solve each day independently is a common assumption in the literature (the papers discussed in Appendix 1.7.2 are examples).

⁸Alternatively, it is possible that banks' post-OMO ES balances follow some non-degenerate distribution. However, with only end-of-day data available, such a distribution could not be separately identified from the payment-flow distribution.

⁹First-period payment flows include any payment flows that are expected to occur during the second period.

¹⁰Late-day payment flow uncertainty is common in the literature (e.g. *Poole* (1968), *Henckel et al.* (1999), and *Woodford* (2001)). As suggested by *Furfine* (2000) and *Bindseil et al.* (2006), the uncertainty includes human error, technical problems, unexpected payments, and expected payments that fail to arrive.

¹¹Although there are papers showing that banks form trading relationships (*Afonso et al.* (2013), for example), incorporating such relationships is beyond the scope of this thesis (and not possible with my data).

End of day (EOD)

After any second-period payment flows, banks access the lending facility and interest is paid on ES balances.

For a given pre-trade ES balance, define a bank's "known borrowing requirement" as the amount this bank would borrow from the central bank's lending facility if there were no second-period flows.¹² The RBA states that banks typically use its lending facility only due to unexpected factors (*Reserve Bank of Australia*, 2012). Therefore, there must typically be sufficient time during the Australian trading day for the market to satisfy banks' known borrowing requirements.

As will be shown in Section 2.3.3.1, with $\widetilde{M} \geq 0$ in this model, if there remains a bank with a positive known borrowing requirement in the first period, there must be at least one bank willing to lend. Moreover, once all banks have known borrowing requirements of zero, there will be no banks willing to borrow in the first period (at an interest rate equal to the central bank's trading-rate target). Therefore, in this model, trading until known borrowing requirements are zero is equivalent to trading until all profitable trading opportunities have been exhausted.

So, to replicate the Australian market, random matching occurs in the first period until all profitable trading opportunities have been exhausted.¹³

2.3.1 End of day

At the end of the trading day, all payment flows have occurred and banks are no longer allowed to trade.

As with the two-bank model, the multi-bank model allows banks to have negative

¹²"Known" borrowing requirements must be defined separately from "expected" borrowing requirements since, as long as there is a non-zero probability of a second-period payment flow requiring a bank to borrow from the central bank's lending facility, "expected" borrowing requirements will be positive.

¹³Having random matching finishing *only* after banks' known borrowing requirements are zero is a simplification of reality. As noted above, in reality, known borrowing requirements are *typically* zero following interbank trading.

ES balances during the day (but not overnight), and the cost of borrowing from the central bank's lending facility is equal to the lending rate (i.e. there are no asymmetries that result from stigma or costly collateral requirements).¹⁴ With banks required to have a non-negative end-of-day ES balance, any bank with an ES balance deficit after the second period must borrow from the central bank's lending facility to make their ES balance non-negative.

Since $r^L > r^D$ in a corridor system, if Bank i has an ES balance greater than or equal to η_i , the bank will not access the lending facility, it will receive interest on its end-of-day ES balance, and it will receive the extra benefit of δ . In this case, the profit Bank i makes from its end-of-day dealings with the central bank is:

$$\pi_{EOD}^i = r^D ES_2^i + \delta$$

Recall that ES_2^i is Bank i 's ES balance after any second-period payment flows.

Suppose $ES_2^i < \eta_i$ and the bank decides to access the lending facility only to make its ES balance non-negative (as it must), then the profit Bank i makes from its end-of-day dealings with the central bank is:

$$\pi_{EOD}^i = \begin{cases} r^L ES_2^i & \text{if } ES_2^i < 0 \\ r^D ES_2^i & \text{if } \eta_i > ES_2^i \geq 0 \end{cases} \quad (2.1)$$

Now suppose that $ES_2^i < \eta_i$ and Bank i borrows a sufficient amount from the lending facility such that its ES balance reaches η_i , then the profit Bank i makes from its end-of-day dealings with the central bank is:

$$\pi_{EOD}^i = r^L [ES_2^i - \eta_i] + r^D \eta_i + \delta \quad (2.2)$$

Bank i will never borrow more than is required to reach η_i , since $r^L > r^D$ and δ

¹⁴See Section 1.3.1 for a detailed discussion of these features.

does not increase the further Bank i 's end-of-day ES balance is above η_i .

Using Eqs. (2.1) and (2.2), when $ES_2^i < \eta_i$, a sufficient condition for Bank i to borrow from the lending facility such that its ES balance reaches η_i is $\delta > [r^L - r^D] \eta_i$.

Therefore, assuming $\delta > [r^L - r^D] \max_i (\eta_i)$, the above condition holds for every bank and the profit Bank i makes from its end-of-day dealings with the central bank can be written as:

$$\pi_{EOD}^i = \begin{cases} r^L [ES_2^i - \eta_i] + r^D \eta_i + \delta & \text{if } ES_2^i < \eta_i \\ r^D [ES_2^i - \eta_i] + r^D \eta_i + \delta & \text{if } ES_2^i \geq \eta_i \end{cases} \quad (2.3)$$

Eq. (2.3) can be re-written as:

$$\pi_{EOD}^i = r^D [\max(ES_2^i - \eta_i, 0)] + r^L [\min(ES_2^i - \eta_i, 0)] + r^D \eta_i + \delta \quad (2.4)$$

To simplify the solution of this model, I assume that $\delta > [r^L - r^D] \max_i (\eta_i)$.

When fitting this model to the data, the impact of any stigma or costly collateral requirements would be captured by these η_i terms. This is because the multi-bank model does not allow the impact of stigma/costly-collateral to be captured by an effective lending facility interest rate that is higher than r^L . That being said, this is not a problem when fitting the model to Australia, since there is no stigma associated with use of the RBA's lending facility (*Debelle, 2008*), and Australian banks typically hold more than enough collateral on their balance sheets (see Section 1.3.1).

2.3.2 First-period payment flows

Since it is only the impact of payment flows on banks' ES balances that matter for banks' trading decisions, only *net* payment flows need to be modelled. Let w_i represent the total net payment flows to Bank i in the first period (henceforth, first-period flows). Since these are *net* flows between banks, $\sum_{i=1}^N w_i = 0$. This means that each bank's

first-period flows will be correlated with the first-period flows of every other bank.

To give first-period flows a simple form, while ensuring $\sum_{i=1}^N w_i = 0$, first-period flows are modelled as (there are N x -variables and $\text{COV}(x_i, x_k) = 0 \forall k \neq i$):¹⁵

$$w_i = x_i - \frac{1}{N} \sum_{j=1}^N x_j \quad \text{where} \quad x_i \sim \mathcal{N}(0, \tau_i^2) \quad (2.5)$$

Therefore:

$$V(w_i) = \frac{1}{N^2} \left[N(N-2) \tau_i^2 + \sum_{j=1}^N \tau_j^2 \right] \quad \forall \quad i \in \{1, 2, \dots, N\} \quad (2.6)$$

$$\text{COV}(w_i, w_k) = \frac{1}{N^2} \left[-N \tau_i^2 - N \tau_k^2 + \sum_{j=1}^N \tau_j^2 \right] \quad \forall \quad k \neq i \quad (2.7)$$

And the first-period flows follow a Multivariate Normal distribution:

$$\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \sim \text{MVN} \left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} V(w_1) & \cdots & \text{COV}(w_1, w_N) \\ \vdots & \ddots & \vdots \\ \text{COV}(w_N, w_1) & \cdots & V(w_N) \end{bmatrix} \right) \quad (2.8)$$

Therefore, Bank i 's ES balance after first-period flows is: $ES_0^i = ES_{OMO}^i + w_i = \mu_i \tilde{M} + \eta_i + w_i$.

2.3.3 End of first-period trading

Following first-period flows, bilateral trading occurs between banks that have been randomly matched. To reduce the computational burden of solving this model, the random matching occurs in sets (each match set is indexed by $t \in \{1, 2, \dots\}$). That is, with an even number of banks (as will be the case in this model), all banks are

¹⁵In reality, each bank's net payment flows will equal the sum of the net payment flows between itself and each other bank; amounting to $\frac{1}{2}N(N-1)$ separate net payment flows across the market (one for each possible pairing of banks). Even if the net payment flows between each pair of banks were independent, there would still be too many unknown parameters for successful identification with the available data. Therefore, modelling each bank's total net payment flows requires stringent restrictions (such as the functional form of Eq. (2.5)). As will be noted in the conclusion, these restrictions could be the reason for the model's inability to replicate some features of the data, and could be an avenue of further research.

randomly matched with one other bank (one set of matches). After matching, all banks trade simultaneously (if there is no profitable trade between any matched banks, their trading volume is zero) and are then released from the match. This process continues until there is no possible match that would result in a non-zero trading volume (i.e. when all profitable trading opportunities have been exhausted).¹⁶

Suppose Bank i and Bank j are matched. Let $b_{i,t}^p$ be the sum of all of Bank i 's trades prior to match t (borrowing adds to the sum, lending subtracts from the sum), $b_{i,t}$ be the net amount that Bank i borrows from Bank j in match t , and $f_{i,t}(b_{i,t})$ be the function that determines the sum of all of Bank i 's future trades given its match t trading volume (i.e. the function has complete information about all future matches and all future trading volumes). Then Bank i 's ES balance at the end of the first period is:

$$ES_1^i = \mu_i \widetilde{M} + \eta_i + w_i + b_{i,t}^p + b_{i,t} + f_{i,t}(b_{i,t}) \quad (2.9)$$

Denote Bank i 's second-period net payment flow as y_i (so $ES_2^i = ES_1^i + y_i$). Then, using Eqs. (2.4) and (2.9), Bank i 's daily profit is (recall that, due to commitment, all trades occur at the central bank's target rate):

$$\begin{aligned} \pi^i = & r^D \left[\max \left(\left\{ \mu_i \widetilde{M} + w_i + b_{i,t}^p + b_{i,t} + f_{i,t}(b_{i,t}) + y_i \right\}, 0 \right) \right] \\ & + r^L \left[\min \left(\left\{ \mu_i \widetilde{M} + w_i + b_{i,t}^p + b_{i,t} + f_{i,t}(b_{i,t}) + y_i \right\}, 0 \right) \right] \\ & + r^D \eta_i + \delta - \left[\frac{r^L + r^D}{2} \right] \left[b_{i,t}^p + b_{i,t} + f_{i,t}(b_{i,t}) \right] \end{aligned} \quad (2.10)$$

Eq. (2.10) holds for any t , with $b_{i,1}^p = 0$, and $f_{i,t^*}(b_{i,t^*}) = 0$ if matching finishes after Bank i trades b_{i,t^*} in match t^* .

As in the first chapter, I assume that the trading volume between matched banks is

¹⁶From simulations (using the same bargaining solution), the results of the model when matching in sets are similar to the more general method of generating each new match (from the pool of all banks) after the previous trade has occurred.

determined by the Nash Bargaining Solution (NBS), which can be found by maximising the following Nash Product with respect to $b_{i,t}$ (given the information available at match t); the disagreement point in the Nash Product is equivalent to a trading volume of zero, and since $b_{i,t}$ is the *net* amount that Bank i borrows from Bank j , $b_{j,t} = -b_{i,t}$ is the *net* amount that Bank j borrows from Bank i :

$$\left\{ \mathbb{E}[\pi^i | \Theta_t^i, b_{i,t}, t] - \mathbb{E}[\pi^i | \Theta_t^i, 0, t] \right\} \times \left\{ \mathbb{E}[\pi^j | \Theta_t^j, -b_{i,t}, t] - \mathbb{E}[\pi^j | \Theta_t^j, 0, t] \right\}$$

subject to:

$$\mathbb{E}[\pi^i | \Theta_t^i, b_{i,t}, t] - \mathbb{E}[\pi^i | \Theta_t^i, 0, t] \geq 0 \quad (2.11)$$

$$\mathbb{E}[\pi^j | \Theta_t^j, -b_{i,t}, t] - \mathbb{E}[\pi^j | \Theta_t^j, 0, t] \geq 0 \quad (2.12)$$

where (let Θ_t^i be known as Bank i 's pre-trade *adjusted* ES balance at match t):

$$\Theta_t^i = \mu_i \widetilde{M} + w_i + b_{i,t}^p$$

and (using Eq. (2.10)):

$$\begin{aligned} \mathbb{E}[\pi^i | \Theta_t^i, b_{i,t}, t] - \mathbb{E}[\pi^i | \Theta_t^i, 0, t] = & \quad (2.13) \\ & r^D \mathbb{E} \left[\max \left(\left\{ \Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) + y_i \right\}, 0 \right) \middle| \Theta_t^i, b_{i,t}, t \right] \\ & + r^L \mathbb{E} \left[\min \left(\left\{ \Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) + y_i \right\}, 0 \right) \middle| \Theta_t^i, b_{i,t}, t \right] \\ & - r^D \mathbb{E} \left[\max \left(\left\{ \Theta_t^i + f_{i,t}(0) + y_i \right\}, 0 \right) \middle| \Theta_t^i, t \right] \\ & - r^L \mathbb{E} \left[\min \left(\left\{ \Theta_t^i + f_{i,t}(0) + y_i \right\}, 0 \right) \middle| \Theta_t^i, t \right] \\ & - \left[\frac{r^L + r^D}{2} \right] \left\{ b_{i,t} + \mathbb{E} \left[f_{i,t}(b_{i,t}) \middle| \Theta_t^i, b_{i,t}, t \right] \right. \\ & \left. - \mathbb{E} \left[f_{i,t}(0) \middle| \Theta_t^i, t \right] \right\} \end{aligned}$$

Since η_i , η_j , and δ do not enter Eq. (2.13), and they provide no information about

future matches, they have no impact on the bargaining outcome. Moreover, with banks' payment flows also independent of the η parameters and δ , a change in the value of any of the η parameters leads to an equivalent change in the corresponding bank's end-of-day ES balance.

Therefore, as long as the central bank operates such that any change in the η parameters is offset by an equivalent change in the corresponding bank's post-OMO ES balance (as discussed with the timeline), changes in the η parameters have no impact on the overnight interbank market. This is consistent with the assertion in the first chapter that, as long as any increased demand for overnight ES balances is satisfied by the central bank, the market should function as *normal* (see Section 1.3.1).

When matched, I assume that Bank i and Bank j truthfully reveal Θ_t^i and Θ_t^j (why banks have no incentive to lie is discussed in Appendix 2.7.4) and form common expectations based only on these revealed adjusted ES balances and any common information.¹⁷ Then the unknowns in Eq. (2.13) are the $f_{i,t}(\cdot)$ function and y_i .

Given some restrictions on the probability distribution of y_i and the functional form of $f_{i,t}(\cdot)$, and some simplifying assumptions, only limited information about $f_{i,t}(\cdot)$ and y_i is required to determine the NBS trading volume.

2.3.3.1 Restrictions on the second-period net payment flow distribution

Suppose Bank i were able to choose its adjusted end-of-first-period ES balance, z_i (where $z_i = ES_1^i - \eta_i = \Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) \ \forall t$), then (using Eq. (2.10)) define the adjusted expected profit function as:

¹⁷ r^L and r^D are publicly announced by the central bank and are therefore known to all banks. Since matches occur in sets, the match number is common knowledge. Past matches, past trading volumes, and the other banks' adjusted ES balances are not common knowledge.

$$\begin{aligned}
E[\widetilde{\pi}^i | z_i] &= E[\pi^i | z_i] - \left[\frac{r^L + r^D}{2} \right] [\mu_i \widetilde{M} + w_i] - r^D \eta_i - \delta \\
&= r^D E[\max(\{z_i + y_i\}, 0) | z_i] + r^L E[\min(\{z_i + y_i\}, 0) | z_i] - \left[\frac{r^L + r^D}{2} \right] z_i
\end{aligned} \tag{2.14}$$

Therefore, choosing z_i to maximise Eq. (2.14) also maximises $E[\pi^i | z_i]$.

The only unknown in $E[\widetilde{\pi}^i | z_i]$ is y_i . Since first-period payment flows also include any payment flows that are expected to occur during the second period, $E(y_i) = 0$.

Now suppose that the probability distribution of y_i ensures that $E[\widetilde{\pi}^i | z_i]$ satisfies the following properties (as shown in Appendix 2.7.5, these properties will be satisfied in this model):

- $z_i = 0$ maximises $E[\widetilde{\pi}^i | z_i]$ (so the optimal end-of-first-period ES balance is $ES_1^i = \eta_i$).
- $E[\widetilde{\pi}^i | z_i]$ is strictly quasi-concave in z_i .
- $E[\widetilde{\pi}^i | z_i]$ is symmetric around $z_i = 0$.

With the second-period shock distribution satisfying these properties, a bank's expected profit is increased if, and only if, it trades such that its end-of-first-period ES balance moves closer to its optimum (as in the two-bank model). Therefore, profitable trading opportunities remain as long as banks have adjusted ES balances on either side of zero. This means that random matching cannot finish until either $\Theta_{t+1}^i \geq 0 \forall i$ or $\Theta_{t+1}^i \leq 0 \forall i$ (recall that $\Theta_{t+1}^i = \Theta_t^i + b_{i,t}$).¹⁸

Since $\sum_{i=1}^N w_i = 0$ and $\sum_{i=1}^N b_{i,t}^p = 0 \forall t$, then $\sum_{i=1}^N \Theta_t^i = \widetilde{M} \forall t$. Therefore, for all $\widetilde{M} \geq 0$ (consistent with the operations of the central bank), random matching cannot finish until $\Theta_{t+1}^i \geq 0 \forall i$ (the condition that $\Theta_{t+1}^i \geq 0 \forall i$ is equivalent to all banks having a “known borrowing requirement” of zero).

¹⁸When random matching has finished, $\Theta_{t+1}^i = z_i \forall i$.

Once $\Theta_t^i \geq 0 \forall i$, if trading moves any bank's end-of-first-period ES balance closer to its optimum, another bank's end-of-first-period ES balance must move further from its optimum. Since banks will never trade to reduce their expected profits (and banks are assumed to form common expectations once Θ_t^i and Θ_t^j are revealed), once $\Theta_t^i \geq 0 \forall i$ there is no match that will result in a non-zero trading volume (i.e. all profitable trading opportunities have been exhausted). Therefore, random matching will finish when, and only when, $\Theta_{t+1}^i \geq 0 \forall i$ (i.e. random matching will finish when all banks have zero known borrowing requirements).

Having an adjusted ES balance optimum of zero differs from the two-bank model (where the optimum can be negative). This is because the two-bank model allows second-period trading if banks pay the transaction cost. Not allowing second-period trading is equivalent to letting the transaction cost tend to infinity; in this case, the optimum end-of-first-period ES balance in the two-bank model will tend towards zero. When calibrated, second-period trading was very rare in the two-bank model (i.e. the transaction cost was large relative to second-period payment flows), so the optimum end-of-first-period ES balance was very close to zero.

2.3.3.2 Restrictions on the functional form of $f_{i,t}(\cdot)$

Restriction 1

I restrict the feasible bargaining solutions to only those that are “internally consistent” – where “internally consistent” means that the bargaining solution is derived under the assumption that all future trading volumes are based on the same bargaining solution (this means that the bargaining solution defines the functional form of $f_{i,t}(\cdot)$). This restriction excludes bargaining solutions that are only feasible because they are based on beliefs about future trades that are not consistent with the way banks actually trade.

Restriction 2

I restrict the feasible bargaining solutions to only those that ensure matching will eventually finish (so that the model replicates the Australian market).¹⁹ For matching to finish, $\Theta_{t+1}^i \geq 0 \forall i$ must hold at some t . And at this t , $\Theta_{t+1}^i = z_i \forall i$. Therefore, if an internally-consistent bargaining solution ensures that matching will finish, the following must hold:

$$f_{i,t}(b_{i,t}) \geq -\Theta_t^i - b_{i,t} \quad \forall \quad i, t \quad (2.15)$$

This is because $z_i = \Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) \quad \forall t$.

Restriction 3a

Suppose that Bank i has a pre-trade adjusted ES balance that is non-negative (i.e. $\Theta_t^i \geq 0$). Then $f_{i,t}(\cdot)$ is restricted such that ($b_{i,t}$ and $b'_{i,t}$ are two different values of the *net* amount that Bank i borrows):

$$b_{i,t} + f_{i,t}(b_{i,t}) \leq b'_{i,t} + f_{i,t}(b'_{i,t}) \quad \forall \quad -\Theta_t^i \leq b_{i,t} < b'_{i,t} \quad (2.16)$$

When combined with Restriction 2 and Section 2.3.3.1, Eq. (2.16) implies that lending less (or borrowing more) *in the current match* cannot increase Bank i 's expected profit.

Restriction 3b

In match t , $f_{i,t}(\cdot)$ is unknown (since banks do not know their future matches nor the ES balances of the other banks in those matches), so banks assign probabilities to the different possible values of $f_{i,t}(\cdot)$ in order to form expectations. Therefore, if there is a positive probability that the inequality in Eq. (2.16) is a strict inequality (at each admissible $b_{i,t}$ and $b'_{i,t}$), then Restriction 3a implies that lending more (or borrowing

¹⁹More accurately, admissible bargaining solutions must ensure that $\lim_{t \rightarrow \infty} P(\min_i (\Theta_{t+1}^i) < 0) = 0$. A bargaining solution cannot *always* ensure that matching will finish, since it is possible for random matching to never match every pair of banks.

less) is *expected* to increase Bank i 's profit (when $\Theta_t^i \geq 0$). That is:

$$\mathbb{E}[\pi^i | \Theta_t^i, b'_{i,t}, t] < \mathbb{E}[\pi^i | \Theta_t^i, b_{i,t}, t] \quad \forall \quad -\Theta_t^i \leq b_{i,t} < b'_{i,t} \quad (2.17)$$

Intuitively, since any adjusted aggregate surplus ($\widetilde{M} > 0$) must be held by someone, if there is a positive probability that another bank is currently holding a surplus, then any extra amount Bank i lends (or any reduction in borrowing) potentially reduces Bank i 's end-of-period ES balance (because there is a positive probability that, due to Bank i 's trading in match t , this other bank will hold more of the aggregate surplus (at the end of the period) than otherwise). Since Bank i can be no worse off by lending more (or borrowing less) in match t (Restriction 3a), the fact that Bank i may potentially be better-off means that lending more (or borrowing less) is *expected* to increase Bank i 's profit. This gives Eq. (2.17).

When $\widetilde{M} = 0$, matching does not finish until all banks have an adjusted ES balance of zero (the optimum point). Therefore, Bank i 's trading during match t does not impact its end-of-period ES balance. So Eq. (2.16) is an equality for all $b_{i,t}$ and $b'_{i,t}$, and Eq. (2.17) does not hold.

Since the adjusted ES balances of the other banks are not part of the “common information” that matched banks use to construct expectations (this was discussed earlier in this section), a positive probability of another bank holding a surplus always exists (until matching finishes). Therefore, if $\widetilde{M} > 0$, then Eq. (2.17) holds for any $\Theta_t^i \geq 0$, $-\Theta_t^i \leq b_{i,t}$, and $b_{i,t} < b'_{i,t}$.

Intuitive reasoning for Restriction 3

While banks with positive adjusted ES balances must *eventually* lend in the interbank market to increase their expected profits (from Section 2.3.3.1), this restriction simplifies derivation of the NBS by preventing these banks from being able to borrow (in any match) to improve their bargaining positions in future matches, and by also preventing

these banks from using threats about future trading volumes to improve their current bargaining position.

To give an example of a situation in which these restrictions are realistic, suppose Bank i has $\widetilde{M} \geq \Theta_t^i > 0$. Then, when matched with a bank that has $\Theta_t^j < 0$, any threat Bank i makes about future trading volumes (i.e. threats about the trading volume if these banks are re-matched) would have no impact on the bargaining outcome, as there is a sufficient surplus among other banks for trading to finish without Bank j needing to trade with Bank i . Moreover, if Bank i tried to borrow such that its threats had a greater impact (i.e. borrow such that $\Theta_t^i > \widetilde{M}$, so Bank j may need to trade with Bank i), the cost of borrowing (due to the probability that it would end the period with a higher ES balance) would likely outweigh any improvement to its bargaining position in future trades (recall, from Chapter 1, that even when there are only two banks, the bank with an ES balance deficit does not trade far beyond its optimal end-of-first-period ES balance).

When this model is calibrated to the Australian market, $P(\max_i \{\Theta_1^i\} > \widetilde{M})$ will be very low, so the above example will be the typical scenario in this model. Moreover, if banks with an adjusted ES balance surplus borrowed in an attempt to improve their bargaining positions, the above-median percentiles of banks' end-of-first-period adjusted ES balance distributions would have a tighter spread around the adjusted aggregate (since the entire surplus would be more likely to finish with a single bank) – such borrowing is therefore inconsistent with the data.

2.3.3.3 Nash Bargaining Solution

Suppose the restrictions of sections 2.3.3.1 and 2.3.3.2 hold. And assume, for simplicity, that the bargaining solution in any match can only depend on t indirectly (through the matched banks' adjusted ES balances at match t). Then:

Proposition 2.1. *The NBS will ensure that any Bank i that has $\Theta_t^i \leq 0$ at any t , will*

finish the period with an adjusted ES balance of zero (i.e. $z_i = 0$).

Proof. Appendix 2.7.3 □

Suppose $\Theta_t^i < 0$ and $\widetilde{M} > 0$, then the intuition behind Proposition 2.1 is:

- With the NBS ensuring that matching finishes (Restriction 2), if Bank i does not trade according to the NBS and waits (but all other banks trade according to the NBS), it will eventually be the case that $\Theta_t^k \geq 0 \forall k \neq i$.^{20,21} Moreover, there must exist at least one bank that has $\Theta_t^k > 0$ (since $\widetilde{M} > 0$).
- In the above situation, Bank i is the only bank willing to borrow (due to Restriction 3, a bank with $\Theta_t^k \geq 0$ will never borrow). Therefore, if Bank i is matched with Bank j (where $\Theta_t^j > 0$), any trading volume Bank i could achieve in the current match, it could also achieve by continuing to wait (with no other banks willing to borrow, banks' ES balances remain constant if Bank i continues to wait).²² So Bank i can be no worse-off from further waiting.
- But Bank j expects to be worse-off if Bank i waits (Restriction 3). Therefore, Bank i has all of the bargaining power in this match (and in all future matches in which the other bank in the match has an adjusted ES balance surplus).
- Since there are always sufficient ES balances for Bank i to reach its optimum ES balance, banks know that if Bank i waits sufficiently long, it will have all of the bargaining power, and will therefore finish the period at its optimum ES balance (i.e. $z_i = 0$).

²⁰Or, more accurately, $\lim_{t \rightarrow \infty} P(\Theta_t^k \geq 0 \forall k \neq i) = 1$.

²¹Waiting is equivalent to the disagreement point. Therefore, although the NBS will ensure that there is no incentive to wait (due to Eqs. (2.11) and (2.12)), the NBS must account for what banks could achieve by waiting.

²²The simplifying assumption that the bargaining solution can only depend on t indirectly is used here; since otherwise Bank i may not be able to achieve the same trading volume if it waits.

- Since Bank i always has the option of waiting, the NBS in any match (in which $\Theta_t^i < 0$) must ensure that Bank i has no incentive to wait. Therefore, if $\Theta_t^i < 0$ in any t , then $z_i = 0$.

The intuition for $\widetilde{M} = 0$ and for $\Theta_t^i = 0$ are trivial. When $\widetilde{M} = 0$, all banks finish the period with an adjusted ES balance of zero (since $\sum_{i=1}^N z_i = \widetilde{M}$ and, from Restriction 2, $z_i \geq 0 \forall i$). When $\Theta_t^i = 0$, Bank i already has its optimum ES balance, and will never trade.

Proposition 2.1 only holds because, for Bank i , there is no cost to waiting. If there were some cost to waiting (e.g. if there was a time limit to first-period trading), then the surplus banks would regain some bargaining power, at which point it would be possible for $z_i > 0$. However, as noted earlier, given the RBA's statements about banks typically using its lending facility only due to unexpected factors (*Reserve Bank of Australia*, 2012), the costs to waiting are likely low; so Proposition 2.1 is likely a close approximation of reality.

Proposition 2.1 does not uniquely determine the NBS trading volume in all matches. So I assume the following:

Assumption 1. Suppose Bank i and Bank j are matched, and $\Theta_t^i \leq \Theta_t^j$ (without loss of generality). Then if $\Theta_t^i < 0$ and $\Theta_t^j > 0$, the NBS trading volume $(b_{i,t}^*)$ has Bank i borrow $b_{i,t}^* = -\Theta_t^i$ from Bank j ; otherwise $b_{i,t}^* = 0$.

When $\widetilde{M} = 0$, the NBS is not unique, but all banks must finish the period with an adjusted ES balance of zero (Assumption 1 ensures this). When $\widetilde{M} > 0$:

- If $\Theta_t^i \geq 0$ – Assumption 1 is the unique NBS (because both banks expect to be worse off from borrowing – Restriction 3).
- If $\Theta_t^j \leq 0$ – Then $z_i = 0$ and $z_j = 0$ (Proposition 2.1). In this case, the banks are indifferent with respect to what happens in match t as long they both finish

the period with an ES balance equal to their respective optima. Assumption 1 ensures that $z_i = 0$ and $z_j = 0$.

- If $\Theta_t^j \geq -\Theta_t^i > 0$ – Assumption 1 is the unique NBS. Because $b_{i,t} = -\Theta_t^i$ is the most Bank i is willing to borrow (since if $\Theta_{t+1}^i > 0$, it is possible for $z_i \neq 0$, violating Proposition 2.1), and Bank j always prefers to lend more (Restriction 3).
- If $-\Theta_t^i > \Theta_t^j > 0$ – The NBS must ensure that $\Theta_{t+1}^i \leq 0$ and $\Theta_{t+1}^j \leq 0$ (since this ensures that $z_i = 0$ and $z_j = 0$, and is therefore Pareto efficient); but this requirement does not uniquely determine the NBS. Assumption 1 satisfies this requirement.

From these dot points, when the NBS is not unique, the end-of-period ES balances of the matched banks are unique (they are equal to their respective optima). Therefore, banks' end-of-period ES balance distributions are independent of the chosen NBS (out of the NBS that satisfy Proposition 2.1). However, different Nash Bargaining Solutions lead to different trading volumes. Assumption 1 maximises the aggregate trading volume and gets closest to replicating the daily aggregate trading volume of the Australian market (see Section 2.5.3).

Appendix 2.7.4 shows that an $f_{i,t}(\cdot)$ based on this NBS will satisfy restrictions 1-3.

2.3.4 Second-period payment flows

From Section 2.3.3.3, banks that have an adjusted ES balance deficit at any point during first-period trading will finish the period with an adjusted ES balance of zero. Then, if the probability distributions of each bank's second-period net payment flow are symmetric around zero and have no atom at zero (as with the Normal probability distribution used in the first-chapter model), each of these banks has a (unconditional) 50 per cent chance of facing a second-period net payment flow that forces them to borrow from the central bank's lending facility. As a result, replicating banks' end-

of-day ES balance distributions would lead to lending facility use that is much higher than in the data.

Since a theme of this thesis has been the ability to replicate asymmetries in the data without any asymmetries in the model, in this chapter, the probability distributions of banks' second-period net payment flows will still be symmetric around zero. Therefore, to replicate the data, the probability distributions must allow for an atom at zero; so the Normal probability distribution of the first chapter is no longer sufficient.²³

But with this complexity comes greater realism. For example, a probability distribution with an atom at zero can replicate both the very small unconditional expected borrowing from the lending facility (in the data, less than A\$6 million per bank, see Section 2.2.2) and the much larger expected borrowing conditioned on the lending facility being used (A\$120 million in the data).

In this model, each bank's second-period net payment flow will depend on a single second-period shock:

- The shock occurs with probability φ (so each bank's second-period net payment flow probability distribution has an atom at zero equal to $1 - \varphi$).
- The shock takes the form of either one bank making unexpected payments to all other banks (a negative shock), or one bank failing to make expected payments to all other banks (a positive shock). Given that a shock occurs, the shock takes either of these forms with probability $1/2$.
- Each bank has a $1/N$ probability of being the bank to which the shock occurs.
- The size of the shock varies depending on the bank that is subject to the shock.

Since banks with larger first-period payment flows are also likely subject to larger

²³In the first chapter, the NBS typically had the bank with the ES balance deficit borrowing such that its end-of-first-period ES balance was above zero (the deficit banks had less bargaining power than in the multi-bank model). Therefore, the rare use of the lending facility could be replicated with banks' second-period net payment flows following a Normal distribution (the Normal distribution was assumed for simplicity).

second-period payment flows, in this model, banks with larger net payment flow volatilities in the first period face larger shocks in the second period. A shock to Bank i takes the form of either ($\theta > 0$ is a parameter that determines the average size of the shock, $\sigma_i = \sqrt{V(w_i)}$ is defined in Eq. (2.6), the first of the shocks below is the “negative shock”):

$$-\theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \quad \text{or} \quad \theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right]$$

- Since this shock represents payments to other banks (or expected payments to other banks that fail to occur), this shock is then apportioned among all the banks in the market (including, for simplicity, the bank that is subject to the shock). This apportioning is also dependent on the size of banks’ first-period payment flow volatilities. If Bank i is subject to a negative shock, then Bank q (where $q \in \{1, 2, \dots, N\}$) will receive an unexpected payment of:²⁴

$$\theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left[\frac{\sigma_q}{\sum_{j=1}^N \sigma_j} \right] \quad (2.18)$$

- If Bank i is subject to a positive shock, then it fails to make payments to other banks, so Bank q ’s ES balance will be lower than expected by an amount equal to Eq. (2.18) (i.e. a net payment flow equal to the negative of Eq. (2.18)).

With this shock, each bank is potentially subject to N negative net payment flows and N positive net payment flows, each mutually exclusive, and each occurring with probability $\varphi/2N$. Moreover, for each positive shock there is a negative shock of equal size, so each bank’s second-period net payment flow distribution is symmetric around zero. And since these are payment flows between banks, banks’ net payment flows sum to zero.

With respect to the model’s ability to replicate the features of the data (see Section 2.2.2):

²⁴Since the shock is apportioned among all the banks, if $q = i$, then Bank i ’s total net payment flow that results from the negative shock is $-\theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left\{ 1 - \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \right\}$.

- With each bank's second-period net payment flow probability distribution having an atom at zero, this model can replicate the large difference between the unconditional and conditional expected borrowing from the lending facility.
- By apportioning the impact of a positive shock among all banks, this model is able to replicate the fact that, despite being rarely used, the lending facility is sometimes used by more than one bank per day.
- By varying the shock size based on banks' first-period net payment flow volatilities, this model can better replicate the wide range of volumes borrowed from the lending facility.

Finally, as is shown in Appendix 2.7.5, the probability distributions of banks' second-period net payment flows satisfy the requirements of Section 2.3.3.1.

2.3.5 Banks' ES balance distributions

In this multi-bank model, banks with an adjusted ES balance deficit have all the bargaining power (when $\widetilde{M} > 0$), which is not the case in the two-bank model. This is because, in the multi-bank model, the deficit banks are no worse-off if they wait for future matches, while surplus banks expect to be worse-off if they wait; whereas waiting is not possible in the two-bank model.

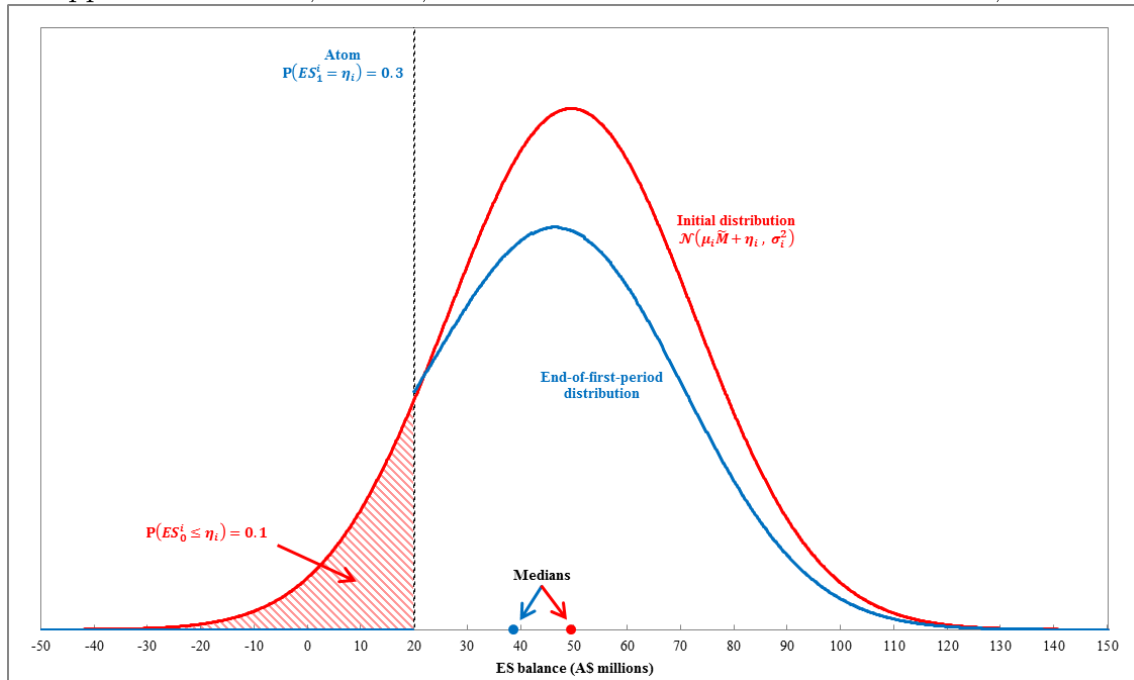
As a result, any bank that has an adjusted ES balance deficit – at any point during first-period trading – will finish the period with an ES balance equal to its optimum level (η_i) (this is Proposition 2.1); thereby creating an atom in each bank's end-of-first-period ES balance probability distribution (the atom is shown by the dashed line in Figure 2.3).

While this means that any bank with an adjusted deficit at the beginning of trade *will* finish at its optimum, banks that begin trading with an adjusted surplus *may* also finish at their optimum. In Figure 2.3, this is shown by the difference between the

probability of the end-of-period atom and the probability of the shaded region (with the shaded region highlighting the probability of Bank i having an adjusted deficit following first-period payment flows).

However, with $\widetilde{M} > 0$, the aggregate adjusted surplus must be held by someone. Since this aggregate can be spread among several banks, banks have a non-atomic distribution for end-of-first-period ES balances above η_i (as shown in Figure 2.3). As a result, banks' end-of-first-period ES balance distributions (from the model) can replicate both the wide spread of percentiles above each bank's median, and the negligible spread of percentiles below each bank's median (in the data, see Section 2.2.1); and this occurs despite each bank having a symmetric initial distribution. This differs from the two-bank model – in which the spread of percentiles above and below each bank's median are the same – and cannot be achieved by any of the existing models in the literature (see Appendix 2.7.1).

Figure 2.3: ES Balance Probability Distribution Example
Approximate PDFs; Bank i ; Based on the calibration in Section 2.4.1; $\widetilde{M} > 0$



Note: The probabilities in this figure are examples. In the model, each bank's probabilities will be different.

However, because the model creates an atom, the negligible spread of below-median percentiles in the data may be replicated in the model by a spread of zero (i.e. percentiles up to the value of the atom's probability will all occur at the atom). This will occur with the calibration used in this chapter (see Section 2.5.1), and is why the ability to set $\eta_i > 0$ is important for quantitative replication of the data (since the negligible spread of percentiles in the data occurs above zero).

A below-median spread of zero is a simplification of reality, resulting from the deficit banks having *all* of the bargaining power. And while this simplification is likely a close approximation of reality (see Section 2.3.3.3), a more complex model – in which surplus banks have some bargaining power – may be able to more closely replicate the non-zero spread in the data.

To replicate the low use of the lending facility in Australia, the model calibration will ensure a low probability of there being any unexpected second-period payment flows (see Section 2.5.4). So banks' end-of-day ES balance distributions will be very similar to their end-of-first-period distributions. Therefore, it is the commitment-restricted interbank trading that is able to replicate banks' asymmetric end-of-day ES balance distributions with symmetric initial distributions.

Since η_i has no impact on first-period trading volumes, any increase in η_i increases the mean of Bank i 's initial distribution and increases the minimum value of Bank i 's end-of-first-period distribution, but has no impact on the shape of either distribution. So η_i acts only as a heterogeneous location parameter (it does not introduce any within-bank asymmetries).

2.4 Solving/calibrating the model

From the model timeline in Section 2.3, Bank i 's post-OMO ES balance is $ES_{OMO}^i = \mu_i \widetilde{M} + \eta_i$. Following open market operations, banks are subject to exogenous payment

flows that move ES balances between the banks, with the probability distribution of the net flows given in Eq. (2.8). Therefore, prior to first-period trading, the probability distribution of banks' ES balances is:

$$\begin{bmatrix} ES_0^1 \\ \vdots \\ ES_0^N \end{bmatrix} \sim \text{MVN} \left(\begin{bmatrix} \mu_1 \widetilde{M} + \eta_1 \\ \vdots \\ \mu_N \widetilde{M} + \eta_N \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \cdots & \text{COV}(w_1, w_N) \\ \vdots & \ddots & \vdots \\ \text{COV}(w_N, w_1) & \cdots & \sigma_N^2 \end{bmatrix} \right) \quad (2.19)$$

The formulas for the variances and covariances in Eq. (2.19) are given in Eqs. (2.6) and (2.7), and are each determined by all of the τ parameters. Therefore, the probability distribution of banks' ES balances prior to first-period trading is determined by the μ , η , and τ parameters (of which there are N of each), and the $\widetilde{M}$ parameter.

If data on banks' pre-trade ES balances were available, most of these parameters could be calibrated by directly fitting Eq. (2.19) to the data.²⁵ Unfortunately, I do not have access to such data; so these parameters can only be calibrated from data on banks' end-of-day ES balance distributions. That is, calibration occurs by determining the parameters necessary for Eq. (2.19), and the model, to produce the end-of-day distributions observed in the data. Therefore, calibration requires the model to be solved; presenting several difficulties.

First, there is no analytical solution for how the model transforms banks' pre-trade ES balance distributions into end-of-day ES balance distributions (when $\widetilde{M} > 0$). Two solutions to this problem that are used in this chapter are discretisation and simulation.

Discretisation involves approximating the infinite continuous set of potential adjusted ES balances ($\mathbb{R}$) by a finite discrete set (denote this set as Ψ).²⁶ Suppose Bank i is matched with Bank j at match t . Then, for every possible $\Theta_t^i \in \Psi$ and $\Theta_t^j \in \Psi$,

²⁵More information would be required to separately identify each μ_i and η_i parameter.

²⁶For a given parameterisation, Eq. (2.19) can be used to determine the proportion of banks' true pre-trade distributions that is captured by the range of this discretised set (increasing the discretised range, increases approximation accuracy). Since banks only trade towards their optimum ES balance (Section 2.3.3.1), this discretised range does not need to be extended to capture the same proportion of banks' end-of-first-period distributions.

the NBS given in Section 2.3.3.3 uniquely determines the post-trade ES balances of the matched banks (Θ_{t+1}^i and Θ_{t+1}^j). Therefore, if the joint probability distribution of adjusted ES balances prior to match t (i.e. $P(\Theta_t^i = X \cap \Theta_t^j = Y) \forall X \in \Psi, Y \in \Psi$) is known for all $i \in N$ and all $\{j \in N | j \neq i\}$, then random matching and the NBS can be used to determine:

$$P(\Theta_{t+1}^i = X \cap \Theta_{t+1}^j = Y) \forall X \in \Psi, Y \in \Psi, i \in N, \{j \in N | j \neq i\}$$

As a result, the model can be solved forwards, starting from the discretisation of Eq. (2.19) (for a given parameterisation), and ending at match T . With a sufficiently large T , $P(\min_i (\Theta_{T+1}^i) < 0) \approx 0$ (i.e. the probability of matching having not finished by match T will be small).

Importantly, these joint distributions can be used to determine each bank's (discretised) marginal end-of-first-period ES balance distribution; from which banks' (discretised) end-of-day distributions can be determined by accounting for the (already discrete) second-period payment flow distribution.

While this is straight-forward in theory, in practice, constructing the joint probability distributions for each potential $t + 1$ match would be computationally burdensome (i.e. for each possible combination of X , Y , i , and j , the probability of each possible match t trade that could have led to this match $t + 1$ combination must be taken into account). Moreover, given the large number of elements in Ψ that is required to accurately approximate the true ES balance distributions (in this chapter, Ψ has 11,657 elements), the memory required to store these joint probability distributions would be large.

Instead, when solving the model using discretisation, I approximate the joint probability distributions with $P(\Theta_t^i = X) \times P(\Theta_t^j = Y)$ (i.e. the product of each matched bank's marginal distribution).²⁷ This approximation saves memory – only N mar-

²⁷This approximation is not conducted for $t = 1$, since there is an analytical solution for the initial joint probability distribution (from Eq. (2.19)).

ginal distributions need to be stored. And is far less computationally intensive (see Appendix 2.7.6 for computation details).

This approximation will be accurate as long as each bank's past trades have little influence on the other banks' ES balance probability distributions (i.e. $P(\Theta_t^j = Y | \Theta_t^i = X) \approx P(\Theta_t^j = Y)$); the accuracy of this approximation is discussed in Section 2.4.2 and Section 2.5.5. Where the approximation is not sufficiently accurate, and simulation is computationally feasible, simulation results are used; simulations are not used exclusively because they are far more computationally burdensome than approximate discretisation.

2.4.1 Calibration

Computational efficiency is particularly important for calibration of the parameters (which is why approximate discretisation is used for calibration, rather than simulation). This is because there are $3N + 3$ parameters to calibrate ($N = 50$ in this chapter).

Since calibration is conducted by matching sample statistics from the Australian data to their model-based equivalents, this large parameter space means that generic algorithms designed to numerically minimise the difference between these statistics (such as quasi-Newton algorithms or direct search algorithms) are computationally burdensome – as they require either numerical derivatives, or a “mesh” of potential update points, to be calculated for each of the parameters and for every iteration (potentially requiring the approximate discretisation solution to be calculated many times per iteration). Moreover, with *each* bank's end-of-day ES balance distribution dependent on *all* of the parameters (potentially non-linearly), the matching loss-function has many (numerical) local minima (some of which are far from the global minimum); with searching among these local minima being computationally infeasible.

To overcome these problems, I use the features of the model to design an algorithm

that is specific to calibrating this model (this algorithm is detailed in Appendix 2.7.7).

Specifically:

- Each bank's value of η_i is calibrated to the location of the 10th percentile of their end-of-day ES balance distribution (from the data) – since this is where the negligible spread of percentiles occurs in the data.
- As shown in Figure 2.3, for ES balances above η_i , the shape of each bank's end-of-first-period distribution is similar to its initial distribution.²⁸ Therefore, the locations of Bank i 's end-of-day ES balance percentiles (that are higher than η_i) are likely heavily dependent on the locations of its initial ES balance percentiles.²⁹ My algorithm exploits this fact by assuming that the ratio of “the difference between two of Bank i 's end-of-day percentiles” (both higher than η_i) and “the difference between the equivalent percentiles from its initial distribution” is the same for each bank. Without any further restrictions, this assumption would mean that $\frac{\widehat{x}_i - \widehat{y}_i}{\sigma_i} = \alpha \ \forall \ i$ (where $\widehat{x}_i - \widehat{y}_i$ is the difference between Bank i 's end-of-day ES balance percentiles (from the data), see Appendix 2.7.7 for further explanation); so that determining all of the τ parameters would reduce to determining α .
- With the restriction that $\tau_i \in \mathbb{R}^+ \ \forall \ i$ (as required by the definition of τ_i , see Section 2.3.2), $\frac{\widehat{x}_i - \widehat{y}_i}{\sigma_i}$ cannot be exactly the same for all i . However, the small adjustments required to satisfy these restrictions do not require computation of the approximate discretisation solution (so they are very fast to compute), and they only need to be computed once. So, to determine the τ parameters, each iteration of the algorithm still only requires a single scale parameter (akin to the α parameter) to be updated (denote this new scale parameter β); the size and direction of each update is determined by comparing the model-based and data-

²⁸This only occurs with a sufficiently large $\widetilde{M}$.

²⁹Recall that, since the probability of a second-period shock occurring will be very low in this model, banks' end-of-day distributions will be very similar to their end-of-first-period distributions.

based percentile differences at each iteration (neither derivatives nor mesh points are required).

- Since each bank's post-OMO ES balance only impacts the other banks via trading (i.e. unlike the τ parameters, the initial (marginal) ES balance distributions of banks other than Bank i are independent of μ_i), changing μ_i should have a much bigger impact on Bank i 's end-of-day ES balance distribution than on the other banks' end-of-day distributions. So, for each update, my algorithm determines the amount to change each bank's $\xi_i \equiv \mu_i \widetilde{M}$ by comparing its end-of-day median in the data, with its model-based end-of-day median at that iteration (i.e. each bank's ξ_i iteration does not account for its impact on the other banks). The advantage of this is that neither numerical derivatives, nor mesh points, are required to determine the update of each bank's ξ_i at each iteration.
- The value of $\widetilde{M}$ is uniquely determined by the summation of banks' calibrated ξ_i parameters (since $\sum_{i=1}^N \mu_i = 1$).
- The updates of the ξ parameters and β parameter, at each iteration, is determined by comparing the relevant data-based *end-of-day* ES balance percentiles to the same percentiles from the model-based *end-of-first-period* distribution. This simplification has no impact on the ξ updates (since, with a symmetric shock, each bank's end-of-day median and end-of-first-period median are equivalent), and likely has little impact on the β parameter updates (given the very small probability of a shock occurring). The advantage of this simplification is that the $3N + 1$ initial-distribution parameters are calibrated independently of the second-period shock distribution parameters.
- The two second-period shock distribution parameters (φ and θ) are calibrated by matching the “expected daily use of the lending facility” and “conditional expected lending facility borrowing per bank” from the data (0.059 and A\$120 million,

respectively), to their model-based counterparts. In this chapter, these moments are matched to within 4 decimal places.

The sample statistics used to calibrate the $3N + 1$ initial-distribution parameters are the 90th percentile, median, and 10th percentile of each bank's end-of-day ES balance distribution (from Section 2.2.1); the differences between the 90th percentiles and the medians are used to determine the τ parameters.^{30,31} The suitability of the algorithm is determined by how close the model-based end-of-first-period percentiles match the data-based end-of-day percentiles.

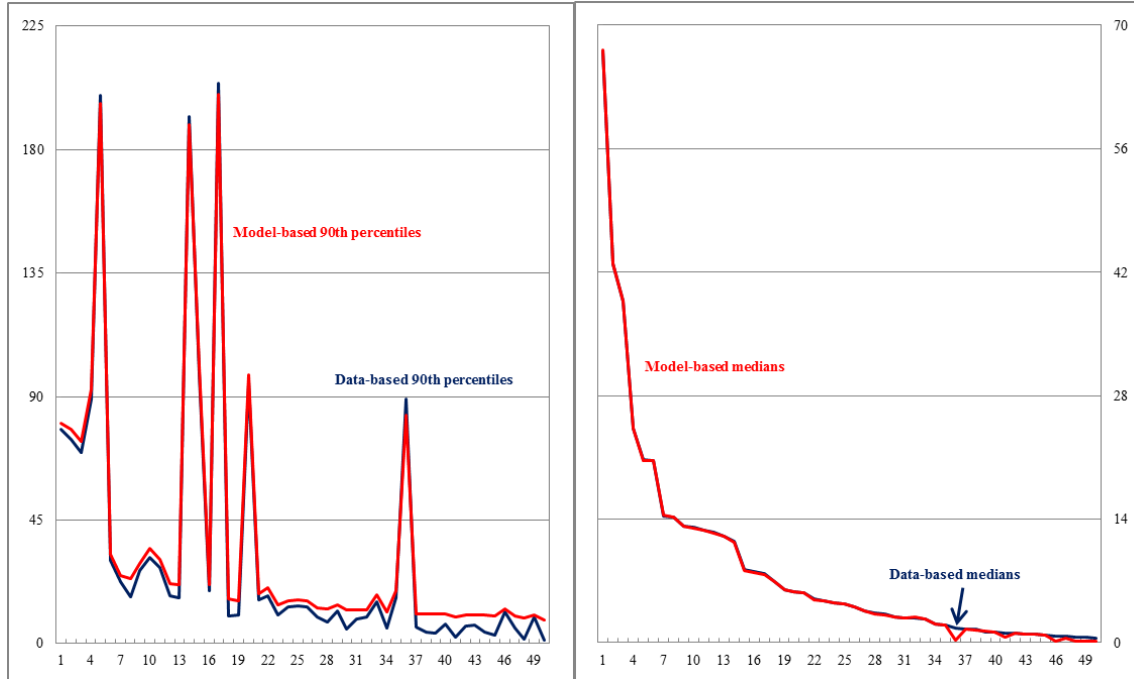
Although I have not proved any generic convergence results, when applied to the Australian data, these percentiles are closely matched. For each bank, $P(ES_1^i = \eta_i) > 0.1$ (i.e. the atom is sufficiently large); so the model-based and data-based 10th percentiles are matched exactly. The close matching of the medians and 90th percentiles is shown in Figure 2.4. The Coefficient of Variation of the Root-mean-square Error (CVRMSE) between the model-based and data-based medians is just 2.95 per cent, while the CVRMSE between the model-based and data-based 90th percentiles is 13.06 per cent.³²

³⁰The data in Section 2.2.1 are percentages of the aggregate ES balance. So, to convert these data into sample statistics that can be used for calibration, these percentages are multiplied by the median aggregate ES balance during the period over which the data is defined (A\$751 million).

³¹Appendix 2.7.8 shows the results from using alternative sample statistics, such as using the 75th percentile instead of the 90th, and using zero instead of the 10th percentile.

³²The CVRMSE between x_i and $\hat{x}_i$ is defined as:
$$\frac{\sqrt{\frac{1}{N} \left[\sum_{i=1}^N (x_i - \hat{x}_i)^2 \right]}}{\frac{1}{N} \left[\sum_{i=1}^N \hat{x}_i \right]}.$$

Figure 2.4: Data-based and Model-based Percentiles for each Bank
50 banks; A\$ millions



Note: Data-based x th percentile = the x th percentile of the bank's end-of-day ES balances between Jan 2005 – Jun 2007. Model-based x th percentile = the x th percentile of the bank's end-of-first-period ES balance distribution that is produced by the model.

Given how closely the medians and 90th percentiles are matched, generic algorithms could not do much better. Moreover, my algorithm achieves these matches after only 5 iterations (with the approximate discretisation solution computed only once per iteration); far quicker than generic methods that would either need to compute many partial derivatives (or a large mesh) at each iteration, or would need many iterations.³³

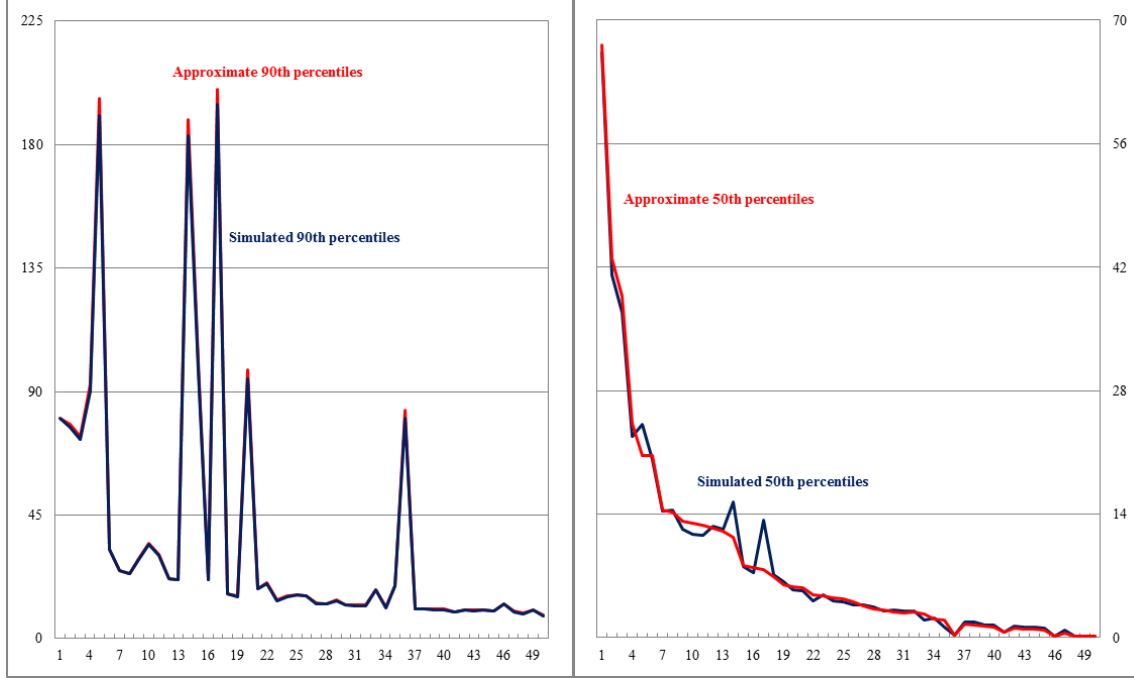
³³A caveat here is that the calibrated parameter values will not necessarily be close to the true parameter values, even if the model-based and data-based percentiles are similar. This could occur either because there is not a unique set of parameter values that causes the percentiles to be matched (due to the non-linear relationship between the parameters and the percentiles), or because the slope of the sum of squared differences between the model-based and data-based percentiles is almost flat (with respect to the parameter values) around the global minimum (so that even parameter values far from the true values could achieve a sum of squared differences close to the global minimum). That being said, this is a problem with calibration via matching rather than the numerical algorithm, so generic algorithms would be just as affected.

2.4.2 Accuracy of approximate discretisation

With the model calibrated, the accuracy of the approximate discretisation can be determined by comparing the approximate discretisation percentiles to the percentiles derived from simulation (based on 1 million draws). If these two methods do not produce similar results, approximate discretisation may be too inaccurate for use in model calibration (with a sufficiently large number of draws, simulation is the most accurate method).

As shown in Figure 2.5, the results from approximate discretisation closely match the simulation results. For the medians, the CVRMSE is 14.44 per cent, while the CVRMSE for the 90th percentiles is 4.70 per cent. The largest deviations occur for the banks with relatively large first-period payment flow variances, but these deviations are still small.

Figure 2.5: Results from Approximate Discretisation and Simulation
50 banks; A\$ millions; 1 million draws for simulation



Note: Approximate x th percentile = the x th percentile of the bank's end-of-first-period ES balance distribution that is produced by the approximate discretisation solution of the model. Simulated x th percentile = the x th percentile of the bank's end-of-first-period ES balance distribution that is produced by the simulated solution of the model.

From these simulations, the CVRMSE between the end-of-first-period medians and end-of-day medians is just 0.3 per cent (with a symmetric shock, the CVRMSE is theoretically zero); the CVRMSE for the 90th percentiles is 0.32 per cent (similar to the CVRMSE of the medians, which is theoretically zero). Therefore, there is little difference between end-of-first-period percentiles and end-of-day percentiles, as required by the calibration algorithm.

2.5 Results

From the above discussion, the model is calibrated using data from banks' end-of-day ES balance distributions and data on use of the RBA's lending facility. Specifically, the calibration algorithm uses the 90th percentiles, medians, and 10th percentiles from

banks' end-of-day ES balance distributions (Section 2.2.1), and uses the “expected daily use of the lending facility” and “conditional expected lending facility borrowing per bank” from Section 2.2.2. This means that the following data is not used in the calibration process and can therefore be used to evaluate how well the model matches the data:

- The 25th and 75th percentiles from banks' end-of-day ES balance distributions.
- The aggregate ES balance.
- The remaining lending facility use data from Section 2.2.2.

Appendix 2.7.8 shows the results from using alternative sample statistics to calibrate the model. However, the calibration described above produces the closest overall fit to the data; so it is only the results from this calibration that are discussed below.

2.5.1 End-of-day ES balance distributions

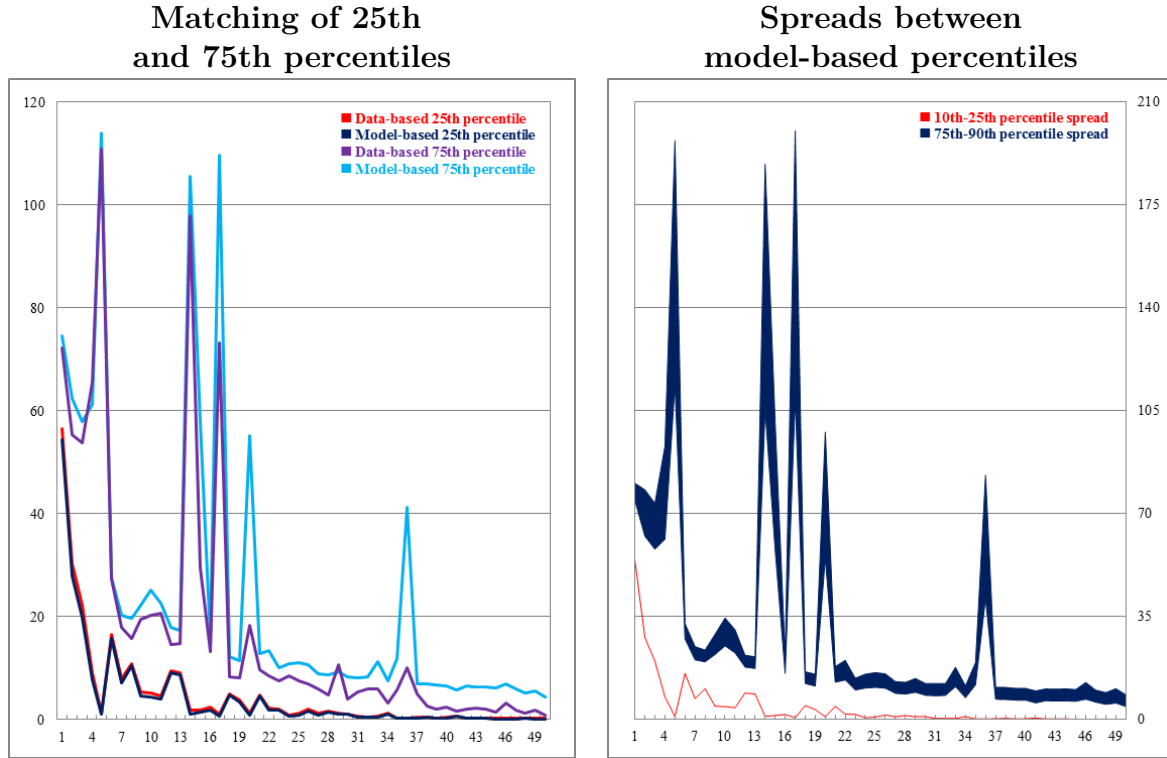
The model closely matches banks' 25th and 75th end-of-day ES balance percentiles (left-hand panel of Figure 2.6) – the CVRMSE between the model-based and data-based 25th percentiles is 15 per cent, while the CVRMSE between the model-based and data-based 75th percentiles is 57 per cent.

The close matching of the 25th percentiles is due to the minimum size of the atom at each bank's optimal end-of-first-period ES balance being 28 per cent (i.e. $\min_i \{P(ES_1^i = \eta_i)\} = 0.28$). Therefore, the model replicates the tight spread between banks' 10th and 25th percentiles with a spread of zero (right-hand panel of Figure 2.6), and matches the fact that these percentiles occur at a point above zero.³⁴ The model also replicates the wide spread between banks' 75th and 90th percentiles (right-hand panel of Figure 2.6). Therefore, the model is able to replicate the positive skew of banks'

³⁴Recall, from Section 2.3.5, that setting $\eta_i > 0$ is required to get the tight spread of percentiles to occur above zero, but is not required to replicate the tight spread.

end-of-day ES balance distributions, despite banks having symmetric pre-trading distributions.

Figure 2.6: Data-based and Model-based Percentiles for each Bank
Model-based percentiles are from the approximate discretisation solution; A\$ millions



Note: Data-based x th percentile = the x th percentile of the bank's end-of-day ES balances between Jan 2005 – Jun 2007. Model-based x th percentile = the x th percentile of the bank's end-of-first-period ES balance distribution that is produced by the model.

2.5.2 Aggregate ES balance

Since banks only trade towards their optimum end-of-first-period ES balance, each bank's end-of-day ES balance median can be no higher than its pre-trade median; and will be smaller as long as $\mu_i \widetilde{M} > 0$ and there is a positive probability of trade occurring (this can be seen in Figure 2.3 of Section 2.3.5). Therefore, since the sum of banks' pre-trading medians equals the aggregate ES balance (in the model) (i.e. $M = \sum_{i=1}^N (\mu_i \widetilde{M} + \eta_i)$), for the model to be able to replicate the data-based ag-

gregate ES balance, banks' end-of-day medians (in the data) must sum to a value less than the data-based aggregate ES balance.

In the data (Section 2.2.1), banks' end-of-day medians sum to A\$430 million, less than the median aggregate ES balance over the period (A\$751 million); the model is therefore qualitatively consistent with the data (with any reasonable calibration).

Moreover, with the calibration used in this chapter, the model produces an aggregate ES balance of A\$760 million, close to the A\$751 million in the data. These figures are remarkably close, especially considering that the aggregate ES balance was not one of the sample statistics used to calibrate the model; further supporting the suitability of this model.³⁵

2.5.3 Trading volumes

The model produces an average aggregate daily trading volume of A\$893 million, significantly less than the A\$6.11 billion average between January 2005 to June 2007 in the data. The reason for this much smaller aggregate trading volume is not clear, and could be the subject of future research (this is discussed further in the conclusion).

2.5.4 Use of the central bank's lending facility

Table 2.2 gives details of banks' use of the central bank's lending facility, in both the model and the Australian data. As noted at the beginning of this section, the "expected daily use of the lending facility" (0.059 banks per day) and "conditional expected lending facility borrowing per bank" (A\$120 million) are used to calibrate the second-period payment flow distribution; with the calibration process ensuring that these sample statistics are matched almost exactly. The small differences, in Table 2.2, between the model-based and data-based values of the statistics used for calibration,

³⁵That said, the ability of the model to get so close to the data seems to be due to the sample statistics chosen for use in calibration; with the model producing an aggregate ES balance of A\$654 million when the 75th percentiles are used for calibration instead of the 90th percentiles (see Appendix 2.7.8).

are because calibration uses the approximate discretisation solution, while this table uses simulation results.

In addition to the data used for calibration:

- This model replicates the large difference between the unconditional and conditional expected borrowing from the lending facility – the maximum unconditional expected borrowing of any bank in the model is just A\$1 million.
- This model replicates the fact that the lending facility is sometimes used by more than one bank per day (despite the lending facility rarely being accessed) – when a second-period shock occurs, there is a 50 per cent chance that the shock causes $N - 1$ banks' ES balances to be lower than expected, potentially causing some of these banks to access the lending facility.
- This model replicates the wide range of volumes borrowed from the lending facility (as can be seen in the last two rows of Table 2.2).

The main discrepancies between the model and the data are due to the simplified payment shock process (i.e. the fact that any shock is apportioned among all of the banks). Because of this apportioning, and in order to match the conditional expected borrowing in the data (the calibration process), the number of banks that typically access the facility will be much larger than in the data – in the model, around 35 banks will access the facility, on average, on days when the facility is accessed by multiple banks (far larger than in the data). This means that, in order to match the expected daily usage, the probability of at least one bank accessing the facility will be much lower than in the data (0.3 per cent compared with 4.9 per cent in the data). This also means that, on days when the facility is accessed, the typical total borrowing volumes will be much larger in the model than in the data (as can be seen in the last two rows of Table 2.2).

Table 2.2: Use of the Lending Facility

	Model (Simulation)	Data (2004/05 – 06/07)
Probability of the lending facility being used	0.3 per cent	4.9 per cent
Expected daily usage (banks per day)	0.059	0.059
Maximum expected borrowing per bank*	A\$1 million	N/A
Conditional expected borrowing per bank**	A\$122 million	A\$120 million
Conditional probability of daily borrowing $\leq$ A\$300 million**	0.8 per cent	89 per cent
Conditional probability of daily borrowing $\geq$ A\$1 billion**	29 per cent	3 per cent

* Maximum across banks – not available in the data.

** Conditional on the lending facility being accessed.

Sources: Author's calculations; Reserve Bank of Australia Annual Reports;
Reserve Bank of Australia Statistical Table A3

These results also support the simplifying assumption that interbank trading only occurs in the first period. Recall, from the first-chapter model, that for second-period trading to occur, one of the banks needed to have an adjusted ES balance deficit lower than $-2\gamma/(r^L - r^D)$.³⁶ The results of this chapter show that the largest unconditional expected second-period borrowing requirement, of any bank, is just A\$1 million. Therefore, a late-day labour cost of just A\$7 would prevent even the bank with the largest expected borrowing requirement from retaining staff for late-day trading.³⁷

2.5.5 Impact of policy changes

This section evaluates the impact of changing $\widetilde{M}$ (i.e. changing the adjusted aggregate ES balance) – keeping all other parameters at their previously calibrated levels. The

³⁶In the first chapter, this was the boundary condition when the deficit bank was sure that there would be a surplus bank that had also paid the transaction cost. In this model, the boundary condition would likely be even further from zero, since a bank with an adjusted ES balance deficit may not be sure that a sufficient number of surplus banks have also paid the transaction cost.

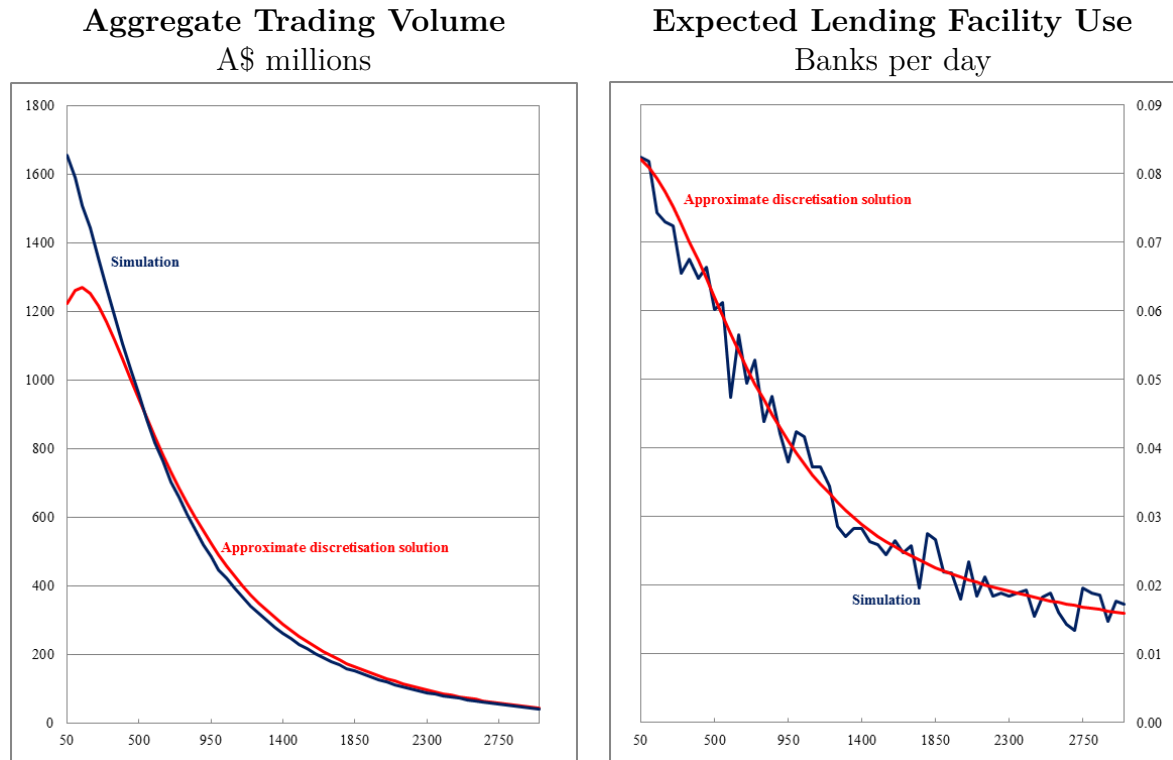
³⁷Recall, from the first chapter, that the timing of the second-period trading decision did not significantly impact the results. This was because, conditional (conditioned on the lending facility being used) and unconditional expected second-period borrowing requirements were both small. Since this no longer holds in the multi-bank model, it is important that unconditional expectations are used to determine the profitability of late-day trading (since this is more realistic).

left-hand panel in Figure 2.7 shows the changes in the aggregate trading volume that result from changing $\widetilde{M}$, the right-hand panel shows the resulting changes in expected lending facility use.

The figures show the impacts of the changes based on both the approximate discretisation solution and simulations. Although Section 2.4.2 shows that the approximate solution is accurate at the calibrated level of $\widetilde{M}$, Figure 2.7 suggests that the accuracy of the approximation falls as $\widetilde{M}$ falls; this is consistent with the fact that the approximation is accurate as long as each bank's past trades have little influence on the other banks' ES balance probability distributions (since the expected number of trades falls as $\widetilde{M}$ increases).

On the other hand, the approximate discretisation solution appears to accurately approximate expected lending facility use at all levels of $\widetilde{M}$. This is because the lending facility is mainly used by banks that finish the first period with low levels of adjusted ES balances, and because the approximate end-of-first-period ES balance distribution (constructed by the approximate discretisation solution) is more accurate for the lower levels of adjusted ES balances. Moreover, since lending facility use is rare, simulation results are much more volatile than the approximate discretisation solution.

So, because of the falling accuracy of the aggregate trading volume approximation, and the low volatility of the simulation results, this section will use the simulation results to evaluate the impact of $\widetilde{M}$ on the aggregate trading volume. Conversely, due to the relative accuracy of the expected lending facility use approximation, this section will use the approximate discretisation solution to evaluate the impact of $\widetilde{M}$ on expected lending facility use.

Figure 2.7: Impact of Adjusted ES Balance Changes100,000 draws for simulation; $\tilde{M}$ on horizontal axes

2.5.5.1 Impact on the aggregate trading volume

From Figure 2.7, increasing the adjusted aggregate ES balance leads to a fall in the aggregate trading volume; consistent with the qualitative results of the first-chapter model. Unfortunately, even with an adjusted aggregate ES balance as low as A\$50 million, the aggregate trading volume is still far lower than in the data (this is discussed further in the conclusion).

As noted in Section 2.2.3, the median daily aggregate ES balance in Australia increased from A\$751 million during January 2005 to June 2007 (pre-crisis) to A\$1.276 billion during July 2010 to February 2012 (post-crisis), and then recently fell to A\$1.075 billion during March 2012 to August 2013. Adjusting these figures for the calibrated η parameters gives a pre-crisis adjusted aggregate ES balance of A\$540 million, in-

creasing to A\$1.065 billion post-crisis, and then falling to A\$864 million (assuming that the η parameters did not change between these periods).³⁸

In the model, equivalent pre-crisis/post-crisis changes cause a A\$483 million (54 per cent) fall in the aggregate daily trading volume, while changing the adjusted aggregate ES balance to A\$864 million (consistent with the recent period), produces an aggregate daily trading volume that is A\$345 million (38 per cent) lower than pre-crisis. Although these *volume* changes are far less than the changes in the adjusted data (i.e. adjusted for changes in aggregate gross payment flows, see Appendix 2.7.2), the point estimates of the *percentage* changes in the adjusted data are similar to the changes in the multi-bank model (the data-based percentage changes are 47 per cent and 39 per cent, respectively, see Section 2.2.3), and the estimated 95 per cent confidence intervals around these point estimates include the multi-bank model results ([29, 62] % for the “post-crisis” period and [23, 54] % for the “recent” period, see Appendix 2.7.2 for more detail).

Although there is insufficient data to determine whether the multi-bank model has overestimated the pre-crisis/post-crisis percentage fall in the adjusted aggregate trading volume (i.e. since the model result is within the 95 per cent confidence interval estimated for the fall in the adjusted data), overestimation of the point estimate would be consistent with the η parameter values remaining above pre-crisis levels (during the July 2010 to February 2012 period), which is consistent with statements made by the RBA (*Reserve Bank of Australia*, 2011):

Over the past year [2010/11], aggregate ES balances averaged a little over \$1.25 billion, with the Reserve Bank facilitating temporarily higher levels around key balancing dates in order to keep the cash market operating smoothly. While this is less than the level of balances held during the fin-

³⁸If the change in the aggregate ES balance between these periods was entirely due to changes in the η parameters, my model suggests that the aggregate ES balance changes would have no impact on the aggregate trading volume. As shown in Appendix 2.7.2, this is not the case in the data; with these aggregate ES balance changes corresponding with statistically significant changes in the long-run equilibrium level of the aggregate trading volume (this can also be seen in the confidence intervals in Figure 2.2). Therefore, although this assumption may be too restrictive (as is discussed further below), it is at least not inconsistent with the data.

ancial crisis, it is higher than the levels recorded before the crisis, reflecting a desire on the part of some ADIs [equivalent to banks in this thesis] to hold higher precautionary balances.

The ability of my model to so closely match both the percentage changes in the data, and RBA statements, is a novel result in the literature.

Figure 2.7 also shows that very high levels of the adjusted aggregate ES balance are predicted to cause very low levels of market activity. This fall in market activity is qualitatively consistent with what has occurred in some markets that have recently implemented large increases in their analogues of the aggregate ES balance (e.g. the US, UK, and Europe, as shown in *Bech and Monnet (2013)*).³⁹

2.5.5.2 Impact on expected lending facility use

From Figure 2.7, increasing the adjusted aggregate ES balance leads to a fall in the expected use of the central bank's lending facility; consistent with the qualitative results of the first-chapter model. Using the aforementioned pre-crisis, post-crisis, and recent values of the adjusted aggregate ES balance, expected lending facility use in the model falls from 0.06 banks per day pre-crisis to 0.037 post crisis, and then increases to 0.044 with the recent fall in the adjusted aggregate ES balance.

This is a much smaller fall than in the data, with expected lending facility use falling to 0.008 banks per day during 2010/11, and then increasing to 0.02 banks per day in 2011/12. However, there are two caveats to these data:

- With lending facility use so rare, the observed use in any given year is likely a poor estimator of the true expected use of the central bank's lending facility.
- The RBA reported that, during 2009/10, "The lower usage of the facility over the past year in part reflects an increase in the willingness of ES counterparties to lend

³⁹Care needs to be taken when using this model to forecast the impact of a large increase in the adjusted aggregate ES balance, since such an increase could impact a central bank's ability to commit to its trading-rate target (especially if the monetary policy focus moves from targeting the overnight interest rate to "unconventional" policies).

late in the settlement session” (*Reserve Bank of Australia*, 2010). With respect to the multi-bank model, this statement could be interpreted as a lengthening of the first period (i.e. a lengthening of the normal working day), which would likely lead to both smaller and less frequent second-period payment flows, thereby reducing use of the central bank’s lending facility beyond that caused by the change in the aggregate ES balance.⁴⁰

2.6 Conclusion

This chapter extends the two-bank framework of the first chapter to a multi-bank model that can be directly fit to the data and that can quantitatively analyse the impact of central bank policy changes.

Due to the complexity of the model, no analytical solution is available. Moreover, simulating the model is computationally intensive. Therefore a discrete approximation is developed, and is shown to be accurate at the calibrated parameter values.

Even with this approximation, the large parameter space means that the standard algorithms used for calibration are computationally burdensome and can get stuck at local minima that are far away from the globally optimal solution. So, to increase computational efficiency, I develop an algorithm that exploits the specific features of the model – this algorithm is shown to leave little room for other techniques to improve on the calibration (i.e. the statistics from the model closely match the data used for calibration).

In addition to closely matching the data used for calibration, the model also closely matches the remaining data (with the ability to so closely match the data in the Australian market being a novel result in the literature):

⁴⁰Using the first-chapter model, this statement could also be interpreted as a reduction in the transaction cost. However, as is noted in the first chapter, quite a large change in the transaction cost would be required to have any noticeable effect on lending facility use. So “a reduction in the transaction cost” (without a corresponding lengthening of the normal working day) is unlikely to be the correct interpretation.

-
- With respect to each bank's end-of-day ES balance distribution, the model replicates both the wide spread of percentiles above each bank's median, and the negligible spread of percentiles below each bank's median (despite each bank having a symmetric initial distribution). This can not be achieved by either the two-bank model, nor any of the existing models in the literature.
 - With respect to use of the central bank's lending facility, the model replicates the wide range of volumes that are borrowed via this facility, and that, while being rarely used, the lending facility is sometimes used by multiple banks on the same day (such features cannot be replicated by any other model in the literature).
 - The model produces an aggregate ES balance of A\$760 million, close to the A\$751 million in the data.
 - Increasing the aggregate ES balance from A\$751 million to A\$1.276 billion (corresponding to the pre- and post-crisis levels in Australia) causes a 54 per cent fall in the aggregate daily trading volume, similar to the 47 per cent fall in the adjusted data (and within the estimated 95 per cent confidence interval); while decreasing the aggregate ES balance to A\$1.075 billion (to match the recent decrease in Australia), produces an aggregate daily trading volume that is 38 per cent lower than pre-crisis, close to the 39 per cent difference in the adjusted data.

Unfortunately, although the model is able to closely replicate the proportional changes in the aggregate trading volume, the levels in the model are far lower than in the data. The reason for this is not clear, and could be the subject of further research; one suspect is the highly restricted functional form of the first-period payment flows (Eq. (2.5)), with more data potentially allowing a less restrictive functional form to be identified.

2.7 Appendices

2.7.1 Literature review

The only other model that allows a large number of banks to trade (bilaterally) multiple times within a single trading period is in *Afonso and Lagos* (2012). However, while the search-theoretic framework of the *Afonso and Lagos* (2012) model would also be applicable to the Australian market (i.e. Australian banks must also search for counterparties), there are several reasons why it is not suitable for the purposes of this chapter:

- As discussed in the first chapter (see Section 1.7.2), by not accounting for the commitment of the central bank to its trading-rate target, the *Afonso and Lagos* (2012) model requires asymmetries (such as stigma associated with using the central bank's lending facility) to simultaneously replicate aggregate excess reserves (equivalent to a positive adjusted aggregate ES balance in my model) and a trading rate equal to the central bank's target. This is problematic when applying the model to the Australian market, as there is no evidence of these asymmetries being prevalent.
- In the *Afonso and Lagos* (2012) model, matched banks always trade such that their post-trade normalised reserve balances (the analogue to my adjusted ES balances) are equal.⁴¹ Therefore, the distribution of normalised reserve balances will converge towards a single point equivalent to the average of these balances. This means that if the *Afonso and Lagos* (2012) model was applied to Australia, matching both the wide spread of percentiles above each bank's end-of-day ES balance median and the negligible spread below each bank's median (while also matching the rare use of the central bank's lending facility and large aggregate

⁴¹Or as close to equal as the discrete reserve balances in *Afonso and Lagos* (2012) allow.

trading volumes) would require very specific functional forms for banks' pre-trading ES balance distributions (if it is even possible), which would not be symmetric. Conversely, my model can match these spreads using symmetric Normal distributions to represent banks' pre-trading ES balance distributions.

- The model of *Afonso and Lagos* (2012) incorporates a search friction (i.e. matching occurs via a Poisson process). While, in reality, banks in Australia must search for trading counterparties, including a search friction in my model would have little impact since it would have to be very small (so there would be little benefit to the increased complexity). This is because, with the RBA stating that banks typically use its lending facility only due to unexpected factors (*Reserve Bank of Australia*, 2012), the search friction would need to be sufficiently small such that banks' "known borrowing requirements" were typically zero at the end of the first period.

As noted in Appendix 1.7.2, an alternative method of determining interbank trading volumes is to assume that banks trade in coalitions, and to use the Shapley Value to determine banks' end-of-first-period ES balances (as in *Tapking* (2006)).

In a model that incorporates commitment, the Shapley Value solution would likely also be able to (at least qualitatively) replicate banks' asymmetric end-of-day ES balance distributions (with symmetric initial distributions). However, as a coalitional solution concept, if applied to the whole market, the Shapley Value solution would not account for the bilateral nature of bargaining; i.e. it would not account for the fact that, even if two banks begin the period with the same surplus, one of the banks may be more successful at finding trading partners than the other bank. As a result, applying the Shapley Value to the entire market would likely result in unrealistic end-of-day ES balance distributions.

2.7.2 Adjustment to the aggregate trading volume

Each bank's net payment flows are equal to the difference between their gross inflows from other banks and their gross outflows to other banks. Therefore, changes to the volatility of banks' gross payment flows will generally be reflected in the volatility of banks' net payment flows. And with net payment flows being the reason that banks need to borrow/lend in the overnight interbank market, changes in the volatilities of these net payment flows will impact trading volumes.

According to *Furfine* (2000), increases in the aggregate level of gross payment flows in the US are associated with increases in the volatility of these flows. If the same is true for Australia, then higher levels of aggregate gross payment flows will likely lead to changes in banks' net payment flow volatilities.

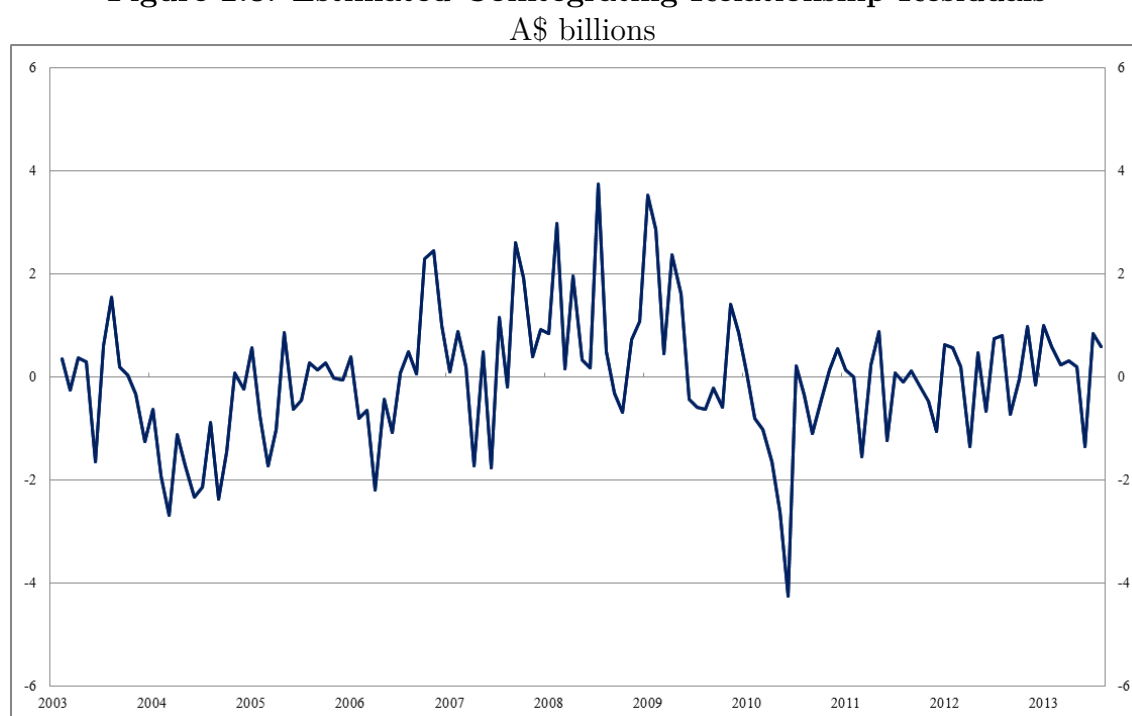
Therefore, to determine the changes in trading volumes in the data that are due to changes in the aggregate ES balance, the impact of changes in aggregate gross payment flows must be extracted.

Extracting the impact of changes in payment flows is complicated by the fact that both the aggregate gross payment flows series and the aggregate trading volume series are likely non-stationary (as determined by non-rejection of the Augmented Dickey-Fuller (ADF) tests). However, testing reveals that these variables are likely cointegrated.

Cointegration testing is conducted via an ADF-type test on the residuals of a potentially cointegrating relationship (proposed by *Engle and Granger* (1987), and implemented in EViews). When the potentially cointegrating relationship is formed by regressing the aggregate trading volume on just a constant and aggregate gross payment flows, this test has insufficient power to reject the null hypothesis of residual non-stationarity. However, as is discussed in *Gregory and Hansen* (1996), these residual non-stationarity tests can have low power in the presence of structural breaks. So, to account for a potential structural break, a dummy variable is included in the cointegrating relation-

ship. In this case, the test produces a p-value of less than 0.025, suggesting that the aggregate trading volume and aggregate gross payment flows are likely cointegrated.⁴² The estimated residuals from this potentially cointegrating relationship (including the dummy variable) are shown in Figure 2.8.

Figure 2.8: Estimated Cointegrating Relationship Residuals



2.7.2.1 Modelling aggregate payment flows and aggregate trading volumes

With the above testing suggesting that a cointegrating relationship exists between the aggregate trading volume and aggregate gross payment flows, I model these variables using a structural vector error-correction model (SVECM).

The SVECM consists of two equations, one with the first-difference of the aggregate trading volume as the dependent variable, and the other with the first-difference of aggregate gross payment flows as the dependent variable. For the initial SVECM, both

⁴²Accounting for a potential structural break in the cointegrating relationship changes the asymptotic distribution of the test statistic under the null hypothesis; therefore, the critical values given in the EViews output are not appropriate. Critical values that account for a potential structural break are given in *Gregory and Hansen (1996)*, and are used to construct the p-value upper bound of 0.025.

equations included the error-correction term (i.e. the cointegrating relationship), 3 lags (in first-differences) of both the aggregate trading volume and aggregate gross payment flows, and orthogonalised seasonal dummy variables (monthly).⁴³

Consistent with the above testing, the error-correction term included the cointegrated variables, a constant, and dummy variables to account for structural breaks. Three dummy variables were initially included to account for three potential changes in the level of the cointegrating relationship (i.e. the constant term) – pre-crisis/crisis (a break at June 2007), crisis/post-crisis (a break at June 2010), and another break to account for the recent fall in the aggregate ES balance (a break at February 2012).

Initial testing also suggested that the structural shocks of the SVECM exhibited greater variance during the crisis period than either before or after (i.e. heteroskedasticity). So the SVECM allows for a different shock covariance matrix during the crisis period.

With SVECMs incorporating the maintained assumption that the structural shocks are contemporaneously orthogonal, moving from the reduced-form VECM (in which the only contemporaneous term in either equation is the residual) to the SVECM requires an identifying restriction. So I assume that there is no contemporaneous causation running from the aggregate trading volume to the aggregate gross payment flows.

This identifying restriction is consistent with what I have assumed throughout this thesis (i.e. interbank trading is simply the actions of banks trying to optimise their end-of-day ES balances, given exogenous payment flows). Moreover, with gross payment flows representing a large number of transactions that result from, for example, transactions between the banks' customers, the A\$ leg of foreign exchange transactions, and the settlement from trading debt securities (*Gallagher et al.*, 2010), it seems reasonable to assume that such diverse transactions do not depend on banks' overnight interbank loans.

⁴³Orthogonality is required so that the seasonal dummy variables do not create a trend in the fitted levels of the data, see *Johansen* (1991).

Following identification of the SVECM, further restrictions are imposed to reduce the estimation burden (in the unrestricted SVECM there are 46 unknown parameters, with just 124 observations in the data). The results from estimating the restricted model are given in Table 2.3 (the coefficients of zero represent these restrictions):

Table 2.3: Restricted SVECM Parameter Estimates

Cointegrating relationship (CR)		SVECM**	d(TV)	d(GPF)	Seasonal dummies	
Constant	9.10	CR (-1)	-0.24	0	Jan	-17.23
TV*	1	d(TV) (-1)	0	0	Feb	-10.78
GPF	-0.11	d(TV) (-2)	0	0	Mar	0
Crisis dummy	0	d(TV) (-3)	0	0	Apr	-3.32
Post-crisis dummy	2.79	d(GPF)	0.04	N/A	May	-18.52
		d(GPF) (-1)	0	-0.49	Jun	0
Recent dummy	2.26	d(GPF) (-2)	0	-0.17	Jul	-19.92
		d(GPF) (-3)	0	0	Aug	-17.18
		Seasonal dummies	0	→	Sep	0
					Oct	-10.82
					Nov	-12.47

TV = Aggregate trading volume; GPF = Aggregate gross payment flows;

(-*t*) = the *t*th lag; d(.) = first difference

* The coefficient of TV in the cointegrating relationship is normalised to 1.

** The coefficients of each equation should be read as if the heading variable (in bold) is on the left-hand side with a coefficient of one, and all other variables (multiplied by their coefficients) are on the right-hand side.

These restrictions reduce the estimation burden to just 20 parameters. A likelihood-ratio test comparing the restricted model to the unrestricted model (with the null hypothesis that the models are the same) returns a p-value of 0.086, so the restrictions are not inconsistent with the data.⁴⁴

The restrictions include the fact that aggregate gross payment flows do not error-correct, and are not dependent on any lags of the aggregate trading volume. With

⁴⁴The only remaining parameter that is not estimated to be significantly different from zero is the April dummy variable coefficient in the GPF equation; it is retained simply because it would be the only seasonal dummy restriction that is not at a quarter-end. Restricting any other parameter values to zero would result in the likelihood-ratio test rejecting the null hypothesis (at the 1 per cent significance level for all but the d(GPF) (-2) coefficient; restricting the d(GPF) (-2) coefficient results in a p-value of 0.05).

no error-correction term in the aggregate gross payment flows equation, there are no cross-equation restrictions in the restricted SVECM. Then, with initial testing on the reduced-form VECM showing that the aggregate trading volume does not “Granger cause” aggregate gross payment flows (*Granger*, 1969), aggregate gross payment flows are “strongly exogenous” with respect to the parameters in the aggregate trading volume equation (*Engle et al.*, 1983).⁴⁵

The restrictions also constrain the aggregate trading volume to only be dependent on any deviations from its long-run equilibrium relationship with the aggregate gross payment flows, and any contemporaneous change in the aggregate gross payment flows – consistent with the modelling assumptions made throughout this thesis.

The dummy variable accounting for a pre-crisis/crisis structural break is estimated to be insignificantly different from zero (both economically and statistically), and so is set to zero in the restricted model. Although there was a large increase in the aggregate ES balance during the crisis period, the fact that this change has an insignificant impact on the long-run equilibrium level of the aggregate trading volume suggests that this increase may have been due to an increase in precautionary demand for ES balances (i.e. increases in the η parameters, see Section 2.3.3), consistent with the statements of *Kearns* (2009) and *Debelle* (2013) (see Section 1.3.1 for more discussion).

The crisis/post-crisis structural break suggests that the long-run equilibrium level of the aggregate trading volume (for a given level of aggregate gross payment flows) is lower post-crisis, consistent with what my model suggests should happen when the aggregate ES balance is higher (and suggesting that the increase in the aggregate ES balance from pre-crisis to post-crisis is not completely due to an increase in the η parameters).

The point estimate for the post-crisis/recent structural break suggests that the long-run equilibrium level of the aggregate trading volume is higher than during the

⁴⁵A likelihood-ratio test with the null hypothesis that, in the reduced-form aggregate gross payment flows equation, the error-correction term and all lagged aggregate trading volume terms have coefficients of zero (i.e. no Granger causality), produced a p-value of 0.98 – so the null hypothesis can not be rejected at any reasonable significance level.

post-crisis period, consistent with what my model suggests should happen when the aggregate ES balance falls. However, the estimated coefficients of the “post-crisis” and “recent” dummy variables are not statistically different (consistent with the overlapping confidence intervals in Figure 2.2) – so it is possible that the “recent” reduction in the aggregate ES balance may be due to a fall in the η parameters (relative to the post-crisis values).

Due to the SVECM allowing for crisis/non-crisis heteroskedasticity (as noted above), the SVECM variables and shocks must be standardised before running model specification tests (as noted by *Mun et al.* (2014), for example, in relation to autocorrelation tests).⁴⁶ Since the true magnitude of the heteroskedasticity is unknown, consistent estimators of the crisis/non-crisis shock variances (derived via maximum likelihood estimation of the SVECM) are used to standardise the variables and shocks.⁴⁷

Following standardisation, model specification testing suggests that the restricted SVECM is well specified.⁴⁸

- White’s heteroskedasticity tests suggest that, once the crisis/non-crisis heteroskedasticity is taken into account, there is insufficient evidence of any heteroskedasticity remaining in the aggregate trading volume equation – with a null hypothesis of homoskedasticity, the p-value is 0.21. Initial testing of the aggregate gross payment flows equation suggested that there may be some remaining heteroskedasticity – with a p-value of 0.004.⁴⁹ However, inspection of the squared

⁴⁶With orthogonal structural shocks, standardisation occurs by dividing each observation of the variables and estimated shocks by that shock’s standard deviation.

⁴⁷Using consistent estimators to standardise the variables is asymptotically equivalent to using the true values. However, in finite samples, using estimators may cause the tests to behave abnormally (i.e. the asymptotic distribution of the test statistics, under the null hypothesis, may no longer provide a good finite-sample approximation). Since the SVECM modelling is not the focus of my thesis, I assume the asymptotic distributions of these tests provide good finite-sample approximations.

⁴⁸With no cross-equation restrictions in the restricted SVECM, and given the identifying restriction, contemporaneous orthogonality of the shocks, and modelling assumptions (i.e. assuming no cross-autocorrelation among the shocks), FIML is equivalent to single-equation estimation for all of the model parameters (from the definition of weak exogeneity in *Engle et al.* (1983)), so some of the model specification tests can be conducted on each equation separately.

⁴⁹Due to the large number of parameters that would result from including cross-terms between the

standardised shocks (the dependent variable in the auxiliary regression) suggests that this finding is driven by a single large outlier. Excluding this outlier from the test increases the p-value to 0.08. So I treat this as insufficient evidence of any un-modelled heteroskedasticity remaining in the aggregate gross payment flows equation.

- From the Jarque–Bera Normality tests, there is insufficient evidence to conclude that the standardised shocks do not follow a Normal distribution – with a null hypothesis of the standardised shocks following a Normal distribution, the p-value from testing the standardised aggregate trading volume shocks is 0.07, while the p-value from testing the standardised aggregate gross payment flow shocks is 0.03 (this p-value increases to 0.96 once the aforementioned outlier is removed).
- Due to the possibility of cross-autocorrelation in multivariate models, standard univariate autocorrelation tests are not appropriate. Instead, I use the multivariate version of the Breusch–Godfrey autocorrelation test described in *Doornik* (1996) (with 12 lags of the standardised multivariate shocks included in the auxiliary regression) – with a null hypothesis of no autocorrelation or cross-autocorrelation, the p-value of this test is 0.19. So there is insufficient evidence to conclude that there exists any autocorrelation or cross-autocorrelation among the standardised shocks.

2.7.2.2 Deriving adjusted aggregate trading volumes

Since the aggregate trading volume does not “Granger cause” aggregate gross payment flows, it is possible to use a given series of aggregate gross payment flows to forecast the aggregate trading volume, without needing to account for any feedback from the forecast aggregate trading volume levels to future aggregate gross payment flows (*Engle*

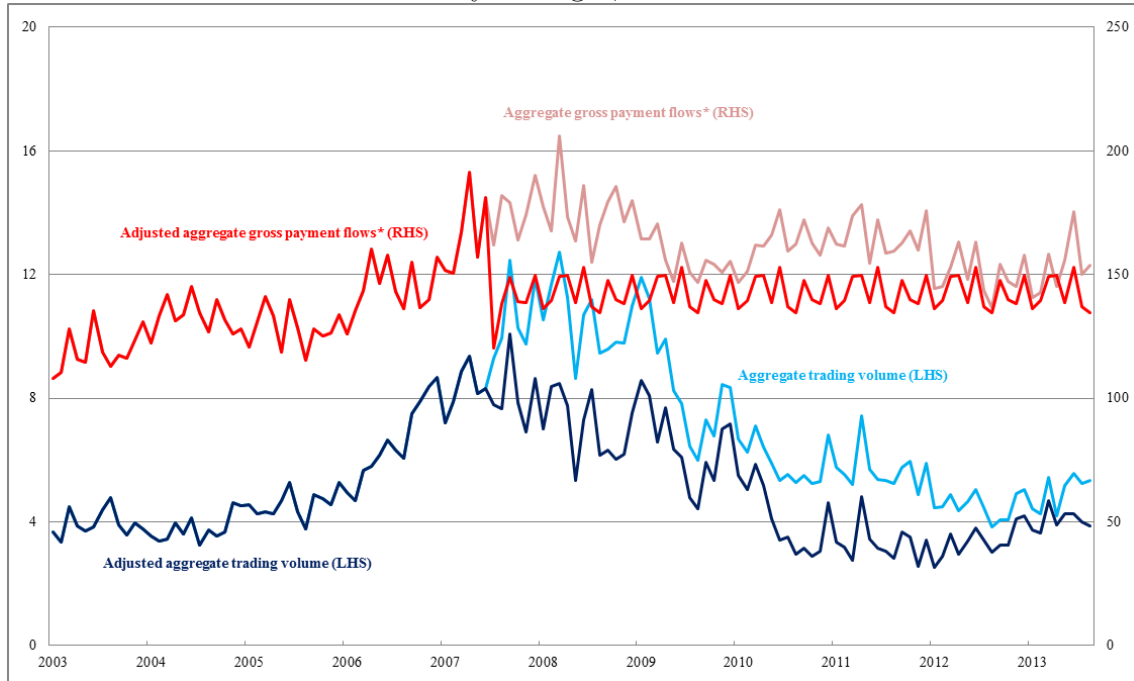
seasonal dummy variables and the other regressors (in the auxiliary regression of this test), I only include cross-terms that do not involve the seasonal dummies.

et al., 1983).

Since my model is parameterised using data from January 2005 - June 2007, the true aggregate gross payment flows are used for this period. Then, to find the levels of the aggregate trading volume that would have occurred if the average aggregate gross payment flows remained at their January 2005 - June 2007 average, I construct a shock in July 2007 (with no shocks thereafter) that moves the average aggregate gross payment flows during July 2010 - August 2013 back to the January 2005 - June 2007 average (with the monthly fluctuations due to the seasonal effects in the SVECM, see Figure 2.9).

The adjusted aggregate trading volume is then constructed by inputting this adjusted aggregate gross payment flow series (instead of the true aggregate gross payment flow series) into the $d(TV)$ equation of the estimated SVECM (see Table 2.3). As a result, instead of the average aggregate trading volume falling from (A\$6.11 billion during January 2005 to June 2007) to (A\$5.54 billion during July 2010 to February 2012) and then to (A\$4.72 billion during March 2012 to August 2013), the average aggregate trading volume (point estimate) falls to (A\$3.27 billion during July 2010 to February 2012) but then increases to (A\$3.74 billion during March 2012 to August 2013).

Figure 2.9: Aggregate Gross Payment Flows and Aggregate Trading Volume
Monthly averages; A\$ billions



* Consistent with the definition in this thesis, gross payment flows do not include unsecured overnight loans between banks.

Sources: Author's calculations; Reserve Bank of Australia (data not publicly available)

I retain the originally estimated $d(TV)$ equation shocks when computing the adjusted aggregate trading volume. This is because I only want to extract the changes in gross payment flows. By constructing the series in this way, I am implicitly assuming that there is no cross-autocorrelation among the shocks (recall that no contemporaneous correlation among shocks is a maintained assumption of the SVECM). This assumption was also used when estimating the SVECM model, with the above model specification tests suggesting that this assumption is reasonable.

2.7.2.3 Estimating confidence intervals

Since the relationship between aggregate trading volumes and aggregate gross payment flows is estimated, the adjusted aggregate trading volumes are subject to estimation uncertainty. To approximate the extent of this uncertainty, I use bootstrapping to

estimate 95 per cent confidence intervals around the point estimate. Since I am only comparing my model's results with the percentage changes between the pre-crisis/post-crisis adjusted aggregate trading volumes and the pre-crisis/recent adjusted aggregate trading volumes, I only construct two confidence intervals (one for each of these percentage changes).

With no contemporaneous correlation, autocorrelation, or cross-autocorrelation in the shocks, each equation's shocks can be re-sampled independently. Moreover, in the restricted SVECM, aggregate gross payment flows are "strongly exogenous" with respect to the parameters in the aggregate trading volume equation. So, for each new set of shocks, each equation can be separately estimated without any loss of information.

Since I am only concerned with finding the aggregate volume of trading that would occur had there been no change in the average aggregate level of gross payment flows, I am not concerned with the estimation uncertainty that results from estimating the aggregate gross payment flows equation. So, to construct the confidence intervals, I do not re-sample from the aggregate gross payment flow shocks. Instead, for each new set of aggregate trading volume shocks, I use the observed levels of aggregate gross payment flows when re-estimating the aggregate trading volume equation.

The bootstrapping method proceeds as follows (references to the SVECM refer to the restricted SVECM):

- Construct 100,000 new sets of shocks (of the same sample size) by re-sampling from the SVECM-estimated shocks of the aggregate trading volume equation (with replacement).
 - Account for the estimated heteroskedasticity by re-sampling the crisis-period shocks from the estimated set of crisis-period shocks, and re-sampling the non-crisis period shocks from the estimated set of non-crisis period shocks.
- Use these new shock samples, the SVECM-estimated parameters (of the aggregate

trading volume equation), and the values of the regressors (the observed aggregate gross payment flows and the dummy variables) to construct 100,000 new series of aggregate trading volumes (treating the initial trading volume as fixed).

- Re-estimate the aggregate trading volume equation of the SVECM (imposing the same restrictions as the original estimation) for each of these new series.
- Construct the adjusted aggregate trading volume for each of these new series:
 - Use the true aggregate trading volume until June 2007 (since the adjusted must equal the true values for this period).
 - From July 2007 onwards, replace the observed aggregate gross payment flows with the adjusted series constructed in Appendix 2.7.2.2. Due to estimation uncertainty, the resulting forecasts of the aggregate trading volume will vary across the 100,000 series.
 - Add the *original* aggregate trading volume shocks to these forecast values; this isolates the estimation uncertainty from the variation in the exogenous shocks (due to re-sampling).
- For each of these adjusted aggregate trading volume series, determine the pre-crisis/post-crisis and pre-crisis/recent percentage changes, and take the 2.5th percentile and 97.5th percentile values (of each) to construct the two estimated 95 per cent confidence intervals.

Confidence intervals are designed to determine the estimation uncertainty around the true values of estimated statistics (in my case, the percentage changes). Therefore, these estimated confidence intervals, that are constructed around the *point estimates*, are only valid if they provide a good approximation of the estimation uncertainty around the *true values*. In support of this approximation, the 100,000 percentages changes constructed above closely approximate Normal distributions (with the shape

of a Normally-distributed estimation-uncertainty distribution being the same for all point estimates, assuming the true variance of the estimation uncertainty is not a function of the mean).

Assuming these estimated confidence intervals are good approximations of the true estimation uncertainty, running a hypothesis test on either of these percentage changes (with the null hypothesis that the percentage change is equal to zero) would be rejected at the 95 per cent significance level. So, based on these bootstrapping results, there are statistically significant percentage changes (in aggregate trading volumes) between pre-crisis/post-crisis and between pre-crisis/recent.

2.7.3 Proof of Proposition 2.1

Proof. Let $b_{i,t}^*$ be the NBS trading volume.

For any non-zero trading volume to be a NBS (i.e. $b_{i,t}^* \neq 0$), neither bank involved in any match can have any incentive to wait until a subsequent match (since the disagreement point of $b_{i,t} = 0$ is equivalent to waiting). But Lemma 2.2 (see below) shows that if $\widetilde{M} > 0$, if Bank i has an ES balance deficit (at any match), and if Bank i waits (instead of trading according to the NBS), then, from the perspective of the banks in the match, Bank i 's end-of-period ES balance will converge (almost surely) to its optimal ES balance (recall that if Bank i reaches its optimum before the end of trading, $z_i = 0$).⁵⁰ Therefore, to ensure Bank i has no incentive to wait, the NBS in any match must ensure that (for $\Theta_t^i < 0$ and $\widetilde{M} > 0$):

$$z_i = \Theta_t^i + f_{i,t}(b_{i,t}^*) + b_{i,t}^* = 0$$

The proof when $\widetilde{M} = 0$ or $\Theta_t^i = 0$ are trivial. When $\widetilde{M} = 0$, all banks finish the period with an adjusted ES balance of zero (since $\sum_{i=1}^N z_i = \widetilde{M}$ and, from Restriction 2, $z_i \geq 0 \forall i$). When $\Theta_t^i = 0$, Bank i already has its optimum ES balance, and will never trade. □

⁵⁰Recall that banks do not know $f_{i,t}(b_{i,t})$ and therefore treat it as a random variable.

Lemma 2.1. *Assume the restrictions in Section 2.3.3.2 hold, and assume that the bargaining solution in any match can only depend on t indirectly. Then, for any given $\Theta_t^i < 0$.⁵¹*

$$\mathbb{E}[\pi^i | \Theta_t^i, t+1] - \mathbb{E}[\pi^i | \Theta_t^i, t] \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty$$

Proof. From Restriction 2 (in Section 2.3.3.2), admissible bargaining solutions must ensure that, if all banks trade according to the bargaining solution, $\lim_{t \rightarrow \infty} \mathbb{P}(\min_i (\Theta_t^i) < 0) = 0$. Now suppose that Bank i has an adjusted ES balance deficit ($\Theta_t^i < 0$) and does not trade according to the bargaining solution but waits (but all other banks trade according to the bargaining solution), then $\lim_{t \rightarrow \infty} \mathbb{P}(\Theta_t^k \geq 0 \forall k \neq i) = 1$ and there must exist at least one bank where $\Theta_t^k > 0$.

Then, from Restriction 3 (when $\widetilde{M} > 0$), $\lim_{t \rightarrow \infty} \mathbb{P}(\Theta_t^k \geq 0 \forall k \neq i) = 1$ implies that, as $t \rightarrow \infty$, the “probability that there will be another bank willing to borrow” approaches zero. This means that, as $t \rightarrow \infty$, if Bank i does not trade in match t (i.e. Bank i continues to wait, equivalent to $b_{i,t} = 0$), the “probability that all banks will have exactly the same ES balances before match $t+1$ as they had before match t ” approaches 1.

Denote Bank i ’s expected profit just before the t th match as $\mathbb{E}[\pi^i | \Theta_t^i, t]$, and Bank i ’s expected profit “given the t th match trading volume” as $\mathbb{E}[\pi^i | \Theta_t^i, b_{i,t}, t]$.

Assume that the bargaining solution in any match can only depend on t indirectly (through the matched banks’ adjusted ES balances at match t). Then if it is known that all banks will have the same adjusted pre-trade ES balances in match t and match $t+1$ (recall, from Restriction 1, that the bargaining solution must be internally consistent):

$$\mathbb{E}[\pi^i | \Theta_t^i, t+1] = \mathbb{E}[\pi^i | \Theta_t^i, t] \tag{2.20}$$

Note that $\mathbb{E}[\pi^i | \Theta_t^i, 0, t] \equiv \mathbb{E}[\pi^i | \Theta_t^i, t+1] \forall t$ (i.e. the expected profit given disagree-

⁵¹By “for any given $\Theta_t^i < 0$ ” I mean that this convergence result holds for any $\Theta_t^i < 0$ that is held constant over time.

ment in match t is equivalent to the expected profit before the next match (with the same pre-trade adjusted ES balance)).

Combining Eq. (2.20) with the fact that, if Bank i continues to wait then the “probability that all banks will have exactly the same ES balances before match $t + 1$ as they had before match t ” approaches 1 as $t \rightarrow \infty$, then (for any given $\Theta_t^i < 0$):

$$\mathbb{E}[\pi^i | \Theta_t^i, t + 1] - \mathbb{E}[\pi^i | \Theta_t^i, t] \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty$$

□

Lemma 2.2. *Assume the restrictions in Section 2.3.3.2 hold, and assume that the bargaining solution in any match can only depend on t indirectly. Then, if $\Theta_t^i < 0$ and $\widetilde{M} > 0$, from the perspective of the banks in the match (as $t \rightarrow \infty$).⁵²*

$$z_i = [\Theta_t^i + b_{i,t}^* + f_{i,t}(b_{i,t}^*)] \xrightarrow{\text{a.s.}} 0$$

Proof. Proof by contradiction. Suppose $\Theta_t^i < 0$ and $\widetilde{M} > 0$.

Now suppose that, in any match t , the banks assign probabilities such that $\mathbb{P}(f_{i,t}(0) > -\Theta_t^i) \neq 0$ (recall from Restriction 2 that $f_{i,t}(0) \geq -\Theta_t^i$ always holds). Then:

$$\mathbb{E}[\pi^i | \Theta_t^i, -\Theta_t^i, t] > \mathbb{E}[\pi^i | \Theta_t^i, 0, t] \quad (2.21)$$

That is, since there is a non-zero probability that Bank i will end the period with an ES balance greater than its optimum value if it does not trade in match t , trading to its optimum ES balance in match t must increase its expected profit (since then Bank i does not need any more trades to reach its optimum).

Now suppose Bank i is matched with Bank j in match t . And Bank j has $\Theta_t^j \geq -\Theta_t^i$. Then, given Restriction 3:

$$\mathbb{E}[\pi^j | \Theta_t^j, \Theta_t^i, t] > \mathbb{E}[\pi^j | \Theta_t^j, 0, t] \quad (2.22)$$

⁵²Recall that, although $f_{i,t}(\cdot)$ is not random, in any match it is unknown to the banks (and is therefore treated by the banks as random). Therefore, for the purposes of this lemma, each $f_{i,t}(b_{i,t})$ is treated as a random variable (and therefore so is z_i).

That is, given Restriction 3, Bank j must expect to increase its profits by lending Θ_t^i to Bank i . Therefore, $b_{i,t} = -\Theta_t^i$ is a Pareto improvement relative to $b_{i,t} = 0$ (from Eqs. (2.21) and (2.22)), so $b_{i,t} = 0$ cannot be a NBS. Moreover, since both banks' expected profits are strictly greater at $b_{i,t} = -\Theta_t^i$ than at the disagreement point, it must be the case that:

$$E[\pi^i | \Theta_t^i, b_{i,t}^*, t] > E[\pi^i | \Theta_t^i, 0, t] \quad (2.23)$$

To see why, note that the Nash Product is strictly greater than zero at $b_{i,t} = -\Theta_t^i$ (from Eqs. (2.21) and (2.22)). But if Eq. (2.23) did not hold, the Nash Product would equal zero at the NBS. Therefore, Eq. (2.23) must hold.

By the law of iterated expectations:

$$E[\pi^i | \Theta_t^i, t] = E_{\Theta_t^j} \{ E[\pi^i | \Theta_t^i, b_{i,t}^*, t] \} \quad (2.24)$$

Since $E[\pi^i | \Theta_t^i, b_{i,t}^*, t] \geq E[\pi^i | \Theta_t^i, 0, t]$ for any Θ_t^j (due to the constraints of the Nash Product), if there is a non-zero probability that $\Theta_t^j \geq -\Theta_t^i$ then (using Eqs. (2.23) and (2.24)):

$$E[\pi^i | \Theta_t^i, t] > E[\pi^i | \Theta_t^i, 0, t] \equiv E[\pi^i | \Theta_t^i, t+1] \quad (2.25)$$

At any point in time, and from the perspective of matched banks, there is always a probability that a Bank i with $\Theta_t^i < 0$ will be matched with a Bank j that has $\Theta_t^j \geq -\Theta_t^i$ (since the adjusted ES balances of the other banks are not “common knowledge”).⁵³ But this means that, from the perspective of matched banks, Eq. (2.25) holds for all $\Theta_t^i < 0$ and all future t . Moreover, for a given $\Theta_t^i < 0$, the probability of Eq. (2.25) holding does not approach zero as $t \rightarrow \infty$. Therefore, for any given $\Theta_t^i < 0$:

$$E[\pi^i | \Theta_t^i, t+1] - E[\pi^i | \Theta_t^i, t] \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty \quad (2.26)$$

But Eq. (2.26) contradicts Lemma 2.1. Therefore, for any given $\Theta_t^i < 0$, Eq. (2.25) cannot hold for all future t (since this causes violation of Lemma 2.1); so there must

⁵³Recall that, if $\Theta_t^i < 0$, then there must exist at least one bank where $\Theta_t^k > 0$. Moreover, this surplus may be concentrated at one bank, in which case $\Theta_t^k \geq -\Theta_t^i$. So, without knowing the ES balances of the other banks, there is always a probability that $\Theta_t^j \geq -\Theta_t^i$; and, for any given $\Theta_t^i < 0$, this probability does not approach zero as $t \rightarrow \infty$.

exist a future t at which $P(f_{i,t}(0) > -\Theta_t^i) = 0$ (let $t = t'$ when this occurs). Since $f_{i,t}(0) \geq -\Theta_t^i$ always holds, due to Restriction 2, $P(f_{i,t'}(0) > -\Theta_{t'}^i) = 0$ implies that $P(f_{i,t'}(0) = -\Theta_{t'}^i) = 1$. Since $f_{i,t}(\cdot)$ is based on an internally consistent NBS (Restriction 1), $P(f_{i,t'}(0) = -\Theta_{t'}^i) = 1$ implies that, for all $t > t'$:

$$P(\Theta_t^i + b_{i,t}^* + f_{i,t}(b_{i,t}^*) = 0) = 1$$

Which therefore implies that, for any given $\Theta_t^i < 0$, as $t \rightarrow \infty$:

$$\Theta_t^i + b_{i,t}^* + f_{i,t}(b_{i,t}^*) \xrightarrow{\text{a.s.}} 0 \quad (2.27)$$

Trivially, if, for any given $\Theta_t^i < 0$, $P(f_{i,t}(0) > -\Theta_t^i) = 0 \forall t > t'$, then Lemma 2.1 is satisfied. Therefore, Lemma 2.1 can be satisfied as long as $P(f_{i,t}(0) > -\Theta_t^i) \neq 0 \forall t$ does not hold. And if $P(f_{i,t}(0) > -\Theta_t^i) \neq 0 \forall t$ does not hold, then Eq. (2.27) does. $\square$

2.7.4 $f_{i,t}(\cdot)$ satisfies restrictions 1-3

Recall, from Restriction 1, that $f_{i,t}(\cdot)$ must be internally consistent. Therefore, for all i and t , $f_{i,t}(\cdot)$ is determined by the NBS given in Assumption 1.

Proof that $f_{i,t}(\cdot)$ satisfies Restriction 2

Suppose Bank i has $\Theta_t^i < 0$. Then, after each match, the NBS ensures that this bank will only retain its adjusted ES balance deficit if it was not matched with a bank that had $\Theta_t^j > 0$. Since $\widetilde{M} \geq 0$, the “probability of Bank i never being matched with a bank that has an adjusted ES balance surplus” falls to zero in the limit. Therefore, as the number of matches increases, the “probability of there being a bank that still has an adjusted ES balance deficit” falls to zero (i.e. $\lim_{t \rightarrow \infty} P(\min_i (\Theta_t^i) < 0) = 0$). And therefore, the NBS satisfies my definition of ensuring that random matching will eventually finish – as required by Restriction 2.

Proof that $f_{i,t}(\cdot)$ satisfies Restriction 3a

Suppose $\Theta_t^i \geq 0$, the current trading volume is $-\Theta_t^i \leq b_{i,t}$, and the NBS ensures that $\Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) = 0$. Then a trading volume of $b'_{i,t} > b_{i,t}$ cannot cause Bank i to end the first period with a lower ES balance (since, matching does not finish until all banks have an adjusted ES balance greater than or equal to zero); therefore, Eq. (2.16) holds.

On the other hand, suppose the current trading volume is $-\Theta_t^i \leq b_{i,t}$, and the NBS ensures that $\Theta_t^i + b_{i,t} + f_{i,t}(b_{i,t}) > 0$. Then, it must be the case that Bank i 's future trading volumes (determined by the NBS) are either zero or are determined by the adjusted ES balances of the other banks in the matches.⁵⁴

Now, by trading $b'_{i,t}$ (where $b'_{i,t} > b_{i,t}$), Θ_{t+1}^i increases by $(b'_{i,t} - b_{i,t})$, while the other bank in match t (Bank j) has its adjusted ES balance fall by the same amount. Having a lower adjusted ES balance impacts Bank j 's future trades but, because of the NBS, the impact cannot be magnified (i.e. this $-(b'_{i,t} - b_{i,t})$ adjusted ES balance difference can be transferred to one other bank or can be spread among many other banks (through trading), but the maximum impact on any single bank remains $-(b'_{i,t} - b_{i,t})$).

Therefore, it is possible that trading $b'_{i,t}$ in match t has no impact on Bank i 's end-of-period ES balance (e.g. if, in future trading, Bank i trades with a bank that has an adjusted ES balance deficit that is larger (i.e. lower ES balance) by $(b'_{i,t} - b_{i,t})$ due to Bank i 's trade in match t), or it is possible that some other banks will end the period with a lower ES balance, thereby leaving Bank i 's end-of-period ES balance higher. In the former case, Eq. (2.16) holds as an equality, while in the latter case Eq. (2.16) holds as a strict inequality.

Therefore, the NBS ensures that Restriction 3a holds when $\Theta_t^i \geq 0$.

⁵⁴If this were not true, then Bank i must, at some point, have an adjusted ES balance deficit, at which point it would end the period with an adjusted ES balance of zero.

There is no incentive for matched banks to lie about Θ_t^i and Θ_t^j

With this NBS, a bank with $\Theta_t^i < 0$ ends the period with an ES balance equal to its optimum. Therefore, in any match, this bank cannot increase its expected profit by lying about Θ_t^i .

For a bank with $\Theta_t^j > 0$, if the bank lies and says that $\Theta_t^j \leq 0$, it either has a trading volume of zero, or it borrows from the other bank (reducing its expected profit, by Restriction 3). Therefore, this bank cannot increase its expected profit by lying about the sign of Θ_t^j . If the bank tells the truth about the sign of its adjusted ES balance (i.e. $\Theta_t^j > 0$), the NBS ensures that, in any match, its trading volume is either zero or is determined by the adjusted ES balance of the other bank. Therefore, lying about the size of Θ_t^j has no impact on its total lending volume; so the bank cannot increase its expected profit by lying about Θ_t^j .

Consequently, with this NBS, no bank has an incentive to lie about their adjusted ES balance.

2.7.5 Proof that the model satisfies Section 2.3.3.1

With the shock outlined in Section 2.3.4, in the second period, each bank is potentially subject to N negative net payment flows and N positive net payment flows, each mutually exclusive, and each occurring with probability $\varphi/2N$.

Suppose the N potential positive net payment flows are ranked according to size, with $s_1 > 0$ the smallest and s_N the largest (each of these positive shocks has a corresponding negative shock of equal size).

Re-writing the adjusted expected profit function (Eq. (2.14)):

$$\begin{aligned}
\mathbb{E}[\widetilde{\pi^i} | z_i] &= r^D \mathbb{E}[\max(\{z_i + y_i\}, 0) | z_i] + r^L \mathbb{E}[\min(\{z_i + y_i\}, 0) | z_i] - \left[\frac{r^L + r^D}{2} \right] z_i \\
&= r^D \mathbb{P}(y_i \geq -z_i | z_i) \mathbb{E}[z_i + y_i | y_i \geq -z_i] \\
&\quad + r^L \mathbb{P}(y_i < -z_i | z_i) \mathbb{E}[z_i + y_i | y_i < -z_i] - \left[\frac{r^L + r^D}{2} \right] z_i
\end{aligned} \tag{2.28}$$

Using the $2N$ potential net payment flows to expand Eq. (2.28):

z_i	$\mathbb{E}[\widetilde{\pi^i} z_i] + \left[\frac{r^L + r^D}{2} \right] z_i =$
$z_i \geq s_N$	$r^D z_i$
$s_N > z_i \geq s_{N-1}$	$r^D \left[1 - \frac{\varphi}{N} \right] z_i + r^D \left[\frac{\varphi}{2N} \right] [z_i + s_N] + r^L \left[\frac{\varphi}{2N} \right] [z_i - s_N]$
$s_{N-1} > z_i \geq s_{N-2}$	$r^D \left[1 - \frac{2\varphi}{N} \right] z_i + \left[r^L + r^D \right] \left[\frac{2\varphi}{2N} \right] z_i$ $+ r^D \left[\frac{\varphi}{2N} \right] [s_N + s_{N-1}] - r^L \left[\frac{\varphi}{2N} \right] [s_N + s_{N-1}]$
$\vdots$	$\vdots$
For $k \in \{1, \dots, (N-1)\}$ $s_{N-k+1} > z_i \geq s_{N-k}$	$r^D z_i + \left[r^L - r^D \right] \left[\frac{k\varphi}{2N} \right] z_i - \left[r^L - r^D \right] \left[\frac{\varphi}{2N} \right] \sum_{j=0}^{k-1} s_{N-j}$
$\vdots$	$\vdots$
$s_1 > z_i \geq 0$	$r^D z_i + \left[r^L - r^D \right] \left[\frac{\varphi}{2} \right] z_i - \left[r^L - r^D \right] \left[\frac{\varphi}{2N} \right] \sum_{j=0}^{N-1} s_{N-j}$
$0 > z_i > -s_1$	$r^L z_i - \left[r^L - r^D \right] \left[\frac{\varphi}{2} \right] z_i - \left[r^L - r^D \right] \left[\frac{\varphi}{2N} \right] \sum_{j=0}^{N-1} s_{N-j}$
$\vdots$	$\vdots$
For $k \in \{1, \dots, (N-1)\}$ $-s_{N-k} \geq z_i > -s_{N-k+1}$	$r^L z_i - \left[r^L - r^D \right] \left[\frac{k\varphi}{2N} \right] z_i - \left[r^L - r^D \right] \left[\frac{\varphi}{2N} \right] \sum_{j=0}^{k-1} s_{N-j}$
$\vdots$	$\vdots$
$-s_{N-2} \geq z_i > -s_{N-1}$	$r^L \left[1 - \frac{2\varphi}{N} \right] z_i + \left[r^L + r^D \right] \left[\frac{2\varphi}{2N} \right] z_i$ $+ r^D \left[\frac{\varphi}{2N} \right] [s_N + s_{N-1}] - r^L \left[\frac{\varphi}{2N} \right] [s_N + s_{N-1}]$
$-s_{N-1} \geq z_i > -s_N$	$r^L \left[1 - \frac{\varphi}{N} \right] z_i + r^D \left[\frac{\varphi}{2N} \right] [z_i + s_N] + r^L \left[\frac{\varphi}{2N} \right] [z_i - s_N]$
$-s_N \geq z_i$	$r^L z_i$

Summarising this information (let $s_0 = 0$):

z_i	$E[\widetilde{\pi}^i z_i] =$
$z_i \geq s_N$	$-\left[\frac{r^L - r^D}{2}\right] z_i$
For $k \in \{1, \dots, N\}$ $s_{N-k+1} > z_i \geq s_{N-k}$ $-s_{N-k} \geq z_i > -s_{N-k+1}$	$-\left[\frac{r^L - r^D}{2}\right] \left\{ \left[1 - \frac{k\varphi}{N}\right] z_i + \left[\frac{\varphi}{N}\right] \sum_{j=0}^{k-1} s_{N-j} \right\}$ $-\left[\frac{r^L - r^D}{2}\right] \left\{ \left[1 - \frac{k\varphi}{N}\right] [-z_i] + \left[\frac{\varphi}{N}\right] \sum_{j=0}^{k-1} s_{N-j} \right\}$
$-s_N \geq z_i$	$-\left[\frac{r^L - r^D}{2}\right] [-z_i]$

From this summary, it can be seen that $E[\widetilde{\pi}^i | z_i]$ is symmetric around $z_i = 0$.

Differentiating $E[\widetilde{\pi}^i | z_i]$ with respect to z_i for $z_i > 0$:

z_i	$\frac{dE[\widetilde{\pi}^i z_i]}{dz_i} =$
$z_i > s_N$	$-\left[\frac{r^L - r^D}{2}\right]$
For $k \in \{1, \dots, N\}$ $s_{N-k+1} > z_i > s_{N-k}$	$-\left[\frac{r^L - r^D}{2}\right] \left[1 - \frac{k\varphi}{N}\right]$

Therefore, when $z_i > 0$, $E[\widetilde{\pi}^i | z_i]$ is decreasing (with respect to z_i) within each segment (and increasing within each segment when $z_i < 0$, due to the symmetry).

At the boundary of each segment (when $z_i \geq 0$), when z_i increases from $s_{N-k} > z_i$ to $s_{N-k+1} > z'_i = z_i + \varepsilon \geq s_{N-k}$, for $k \in \{1, \dots, (N-1)\}$:

$$\Delta E[\widetilde{\pi}^i | z_i] = -\left[\frac{r^L - r^D}{2}\right] \left\{ \left[\frac{\varphi}{N}\right] (z_i + \varepsilon - s_{N-k}) + \left[1 - \frac{(k+1)\varphi}{N}\right] \varepsilon \right\} < 0 \quad (2.29)$$

At the boundary of the N th segment, when z_i increases from $s_N > z_i$ to $z'_i = z_i + \varepsilon \geq s_N$:

$$\Delta E[\widetilde{\pi}^i | z_i] = -\left[\frac{r^L - r^D}{2}\right] \left\{ \left[\frac{\varphi}{N}\right] (z_i + \varepsilon - s_N) + \left[1 - \frac{\varphi}{N}\right] \varepsilon \right\} < 0 \quad (2.30)$$

Therefore, when $z_i > 0$, $E[\widetilde{\pi}^i | z_i]$ is also decreasing (with respect to z_i) when moving between segments.

At $z_i = 0$ and for any $s_1 > \varepsilon > 0$:

$$E[\widetilde{\pi}^i | z_i] = - \left[\frac{r^L - r^D}{2} \right] \left\{ \left[\frac{\varphi}{N} \right] \sum_{j=0}^{N-1} s_{N-j} \right\} > - \left[\frac{r^L - r^D}{2} \right] \left\{ [1 - \varphi]\varepsilon + \left[\frac{\varphi}{N} \right] \sum_{j=0}^{N-1} s_{N-j} \right\}$$

That is, the $E[\widetilde{\pi}^i | z_i]$ at $z_i = 0$ is larger than any other value of z_i within the first positive segment. Combined with the derivatives, Eqs. (2.29) and (2.30), and the symmetry of $E[\widetilde{\pi}^i | z_i]$ around $z_i = 0$, this means that $E[\widetilde{\pi}^i | z_i]$ is maximised at $z_i = 0$ (as required), and also means that $E[\widetilde{\pi}^i | z_i]$ is strictly quasi-concave in z_i .

So the second-period net payment flow distribution in this model satisfies the requirements of Section 2.3.3.1.

2.7.6 Computation of the approximate discretisation solution

For a given parameterisation, the discretised range of adjusted ES balances is:

$$\left[\min_j (\mu_j \widetilde{M} - 4\sigma_j), \max_j (\mu_j \widetilde{M} + 4\sigma_j) \right]$$

This range captures at least 99.99 per cent of each bank's marginal pre-trading ES balance distribution and marginal end-of-first-period distribution.

This range is then discretised into A\$0.1 million increments (denote the number of elements within this discretised range as K).

$T = 1000$ is more than sufficient for this chapter since, even with the lowest values of $\widetilde{M}$ used in this chapter (trading takes longer with lower values of $\widetilde{M}$), $P(\min_i (\Theta_t^i) < 0)$ is less than 0.0001 per cent before $t = 800$.

With these settings, the approximate discretisation solution proceeds as follows:

1. For all K values of Θ^i and all K values of Θ^j within the discretised range (i.e. for each feasible pair of adjusted ES balances at any match), work out the $(K \times K)$ matrix of Bank i 's adjusted post-trade ES balances (i.e. $\Theta^i + b_{i,j}^*$, where $b_{i,j}^*$ is the NBS net amount that a bank with Θ^i borrows from a bank with Θ^j). With

the NBS only dependent on the adjusted ES balances of the banks in the match, this matrix is time independent (hence the lack of time subscript).

2. For each bank (denoted as Bank i):

- (a) Use Eq. (2.19) to determine the marginal Probability Mass Function (marginal PMF) of Bank i 's discretised adjusted pre-trading ES balances, where $P(\Theta_1^i = X) = \Phi\left(\frac{X + \frac{D}{2} - \mu_i \tilde{M}}{\sigma_i}\right) - \Phi\left(\frac{X - \frac{D}{2} - \mu_i \tilde{M}}{\sigma_i}\right)$, $\Phi(\cdot)$ is the standard Normal cumulative distribution function, and D is equal to A\$0.1 million. Then scale the PMF so that the probabilities sum to one – this is a small adjustment given the proportion of the true distribution captured by the discretised range.
- (b) For each value of Θ_1^i , use Eq. (2.19) to determine the conditional PMFs of every other bank's discretised adjusted pre-trading ES balances – conditioned on Bank i 's adjusted ES balance (and scaled so that the probabilities sum to one). With random matching, Bank i has a $1/(N-1)$ probability of being matched with each of the other banks. Therefore, the conditional PMF of adjusted ES balances that Bank i may face at the first match, is equal to the sum of these other banks' conditional PMFs, divided by $(N-1)$; henceforth, the average conditional PMF (denote the elements of this average conditional PMF as Θ_1^j).⁵⁵
- (c) Using the marginal PMF from Step 2a and the average conditional PMF from Step 2b, construct a $(K \times K)$ matrix containing the probabilities of each possible $\{\Theta_1^i, \Theta_1^j\}$ combination in the first match (each element of this

⁵⁵In detail, let $P(\Theta_1^q = X | \Theta_1^i = Y) = \Phi\left(\frac{X + \frac{D}{2} - \mu_q \tilde{M} + \frac{Y}{N-1}}{\sigma_q^c}\right) - \Phi\left(\frac{X - \frac{D}{2} - \mu_q \tilde{M} + \frac{Y}{N-1}}{\sigma_q^c}\right)$ be the probability that Bank q 's discretised adjusted pre-trade ES balance equals X given that Bank i 's adjusted pre-trade ES balance equals Y (where $\sigma_q^c = \frac{1}{N-1} \sqrt{(N-2)^2 \tau_q^2 + \sum_{j \neq i, k}^N \tau_j^2}$). Then the conditional probability of Bank i being matched with a bank that has a discretised adjusted pre-trade ES balance equal to X is: $P(\Theta_1^j = X | \Theta_1^i = Y) = [1/(N-1)] \sum_{q \neq i}^N P(\Theta_1^q = X | \Theta_1^i = Y)$.

matrix is the product of the relevant marginal and conditional probabilities).

- (d) Using the $(K \times K)$ matrix of adjusted post-trade ES balances from Step 1, and the $(K \times K)$ probability matrix from Step 2c, sum the probabilities assigned to each possible adjusted post-trade ES balance of Bank i to derive the marginal PMF of Bank i 's adjusted post-trade ES balances (i.e. a $(K \times 1)$ vector of probabilities that represents the PMF of Θ_2^i).

3. For each successive match, denoted by $t = \{2, 3, \dots, T\}$:

- (a) For each Bank i (denote the set of K feasible Θ^i values as Ψ):
 - i. Sum the marginal pre-trade PMFs of every other bank, and divide by $(N - 1)$ (i.e. for each $X \in \Psi$, derive $[1/(N-1)] \sum_{q \neq i}^N P(\Theta_t^q = X)$). This forms the approximation of the “average conditional PMF that Bank i may face at this match”.
 - ii. Using Bank i 's marginal pre-trade PMF and the approximate PMF from Step 3ai, construct a $(K \times K)$ matrix containing the probabilities of each possible $\{\Theta_t^i, \Theta_t^j\}$ combination in this match.
 - iii. Using the $(K \times K)$ matrix of adjusted post-trade ES balances from Step 1, and the $(K \times K)$ probability matrix from Step 3aii, sum the probabilities assigned to each possible adjusted post-trade ES balance of Bank i to derive the marginal PMF of Bank i 's adjusted post-trade ES balances (i.e. a $(K \times 1)$ vector of probabilities that represents the PMF of Θ_{t+1}^i).

4. The approximation of each bank's end-of-first-period adjusted ES balance distribution is equal to the marginal PMF of its post-trade adjusted ES balances following the final match (i.e. for Bank i , the marginal PMF of Θ_{T+1}^i).

2.7.6.1 Expected trading volume

From above, the set of K feasible Θ^i values is denoted as Ψ . For the remainder of this section, assume that this is an ordered set (smallest to largest).

The total trading volume is equal to the sum of each bank's borrowing during the first period. Therefore, the expected total trading volume is equal to the sum of each bank's expected borrowing during the first period:

$$E(\text{vol}) = \sum_{i=1}^N E(\text{borrowing of Bank } i) = \sum_{i=1}^N \sum_{t=1}^T \sum_{k=1}^K \sum_{m=1}^K \left\{ \hat{P}_{i,t,k} \hat{P}_{j,t,m} \hat{b}_{k,m} \right\}$$

where:

- Ψ_k is the k th element in the ordered set Ψ .
- $\hat{P}_{i,t,k}$ is the probability that Bank i has the k th pre-trade adjusted ES balance in the t th match (i.e. $P(\Theta_t^i = \Psi_k)$).
- $\hat{P}_{j,t,m}$ is the probability that the bank that is matched with Bank i (in the t th match) has the m th pre-trade adjusted ES balance (from the average conditional PMF in Step 2b or the approximate PMF in Step 3ai).
- $b_{k,m}^*$ is the NBS trading volume when Bank i is matched with Bank j , $\Theta^i = \Psi_k$, and $\Theta^j = \Psi_m$ (recall that the NBS is only dependent on the adjusted ES balances of the banks in the match, so there is no time subscript).
- $\hat{b}_{k,m} = b_{k,m}^*$ if $b_{k,m}^* > 0$, $\hat{b}_{k,m} = 0$ otherwise.

2.7.7 Calibration algorithm

$N = 50$ is calibrated to match the 50 active banks in the Australian market (see Figure 2.1). Then there are $3N + 3 = 153$ remaining parameters to calibrate $(\mu_1, \dots, \mu_N, \eta_1, \dots, \eta_N, \tau_1, \dots, \tau_N, \widetilde{M}, \theta, \varphi)$. The μ , η , τ , and $\widetilde{M}$ parameters are calibrated by matching banks' data-based end-of-day percentiles to their model-based end-of-first-period

percentiles. The model-based percentiles used for calibration are derived from banks' approximate end-of-first-period distributions constructed via the approximate discretisation solution (see Appendix 2.7.6).

Denote the $(C \times 100)$ th percentile from Bank i 's approximate end-of-first-period ES balance distribution as c_i (i.e. $P(ES_1^i \leq c_i) = C$). And denote the equivalent percentile from the end-of-day data as $\hat{c}_i$. Then the calibration algorithm is:

Step 1: Calibration of the η parameters

The η parameters are calibrated to match the lowest percentile in the available data. That is, suppose the lowest percentile in the data is W . Then $\eta_i = \hat{w}_i \forall i$.

Step 2: Reducing calibration of the τ parameters to calibration of a single parameter

Let x_i and y_i be two different values of c_i (i.e. $P(ES_1^i \leq x_i) = X$ and $P(ES_1^i \leq y_i) = Y$).

The calibration algorithm assumes that the ratio of “the difference between two of Bank i 's end-of-day percentiles” (both higher than η_i) and “the difference between the equivalent percentiles from its initial distribution” is the same for each bank. From Eq. (2.19), the difference between Bank i 's $(X \times 100)$ th and $(Y \times 100)$ th percentiles in its initial distribution equals $\sigma_i [\Phi^{-1}(X) - \Phi^{-1}(Y)]$ (where $\Phi(\cdot)$ is the standard Normal cumulative distribution function). Therefore, this assumption is equivalent to imposing the following N additional restrictions on the σ parameters:

$$\frac{\hat{x}_i - \hat{y}_i}{\sigma_i} = \alpha \quad \forall \quad i \quad (2.31)$$

So calibrating the σ parameters reduces to calibrating the unknown scalar α .⁵⁶ Moreover, from Eq. (2.6) (the ones are on all of the off-diagonals):

⁵⁶Since the σ 's are standard deviations, if $X > Y$ then $\alpha > 0$ (if $X < Y$ then $\alpha < 0$).

$$\begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_N^2 \end{bmatrix} = \frac{1}{N^2} \begin{bmatrix} (N-1)^2 & & 1 \\ & \ddots & \\ 1 & & (N-1)^2 \end{bmatrix} \begin{bmatrix} \tau_1^2 \\ \vdots \\ \tau_N^2 \end{bmatrix} \quad (2.32)$$

Without any further restrictions, since the square matrix in Eq. (2.32) is invertible, Eq. (2.31) and α would uniquely identify the values of each bank's τ_i^2 . Unfortunately, such unrestricted identification is not sufficient to ensure that each $\tau_i \in \mathbb{R}^+$ (as they must be since the τ parameters are standard deviations). This is because the net payment flows between banks sum to zero, and because of the large variation of $\widehat{x}_i - \widehat{y}_i$ between banks (see Figure 2.1). To see the problem in a simple example, suppose $N = 3$, then (from Eqs. (2.31) and (2.32)):

$$\tau_1^2 = \frac{1}{2\alpha^2} \left[5(\widehat{x}_1 - \widehat{y}_1)^2 - (\widehat{x}_2 - \widehat{y}_2)^2 - (\widehat{x}_3 - \widehat{y}_3)^2 \right]$$

If the difference between Bank 1's data-based percentiles is sufficiently small relative to the other banks (i.e. $5(\widehat{x}_1 - \widehat{y}_1)^2 < (\widehat{x}_2 - \widehat{y}_2)^2 + (\widehat{x}_3 - \widehat{y}_3)^2$), then the identified $\tau_1^2 < 0$. Therefore, to ensure $\tau_i \in \mathbb{R}^+$, the restrictions in Eq. (2.31) must be modified.

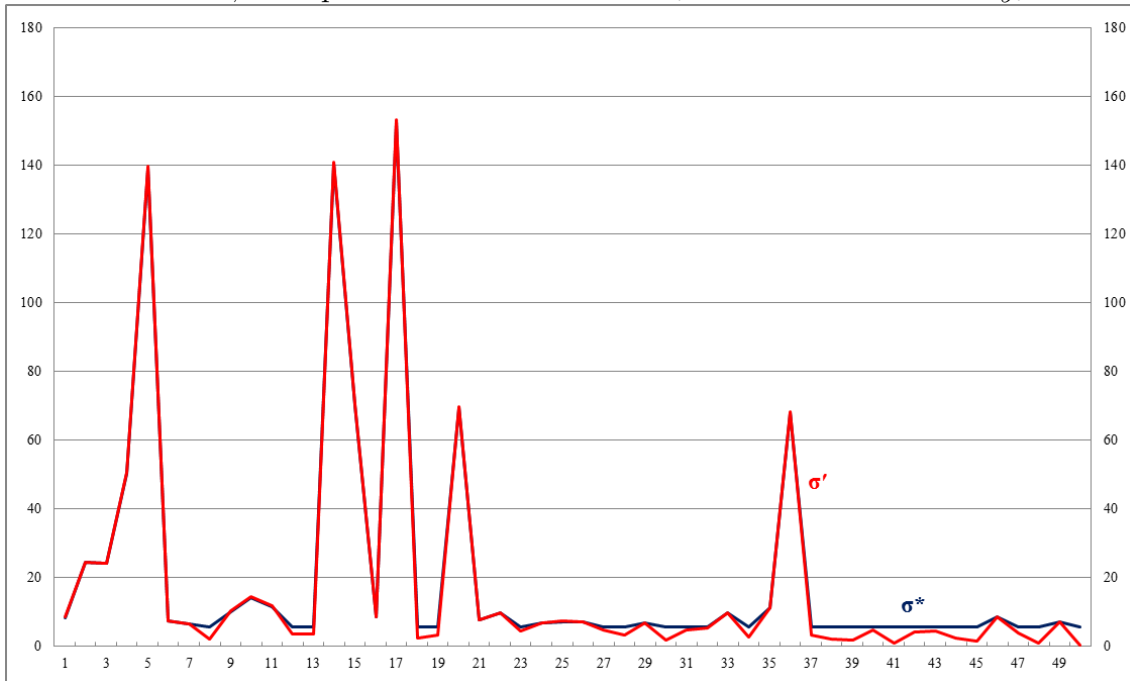
That said, even with additional restrictions to ensure $\tau_i \in \mathbb{R}^+$ for each bank (with Eq. (2.32) still holding), if the $\tau_i \in \mathbb{R}^+$ restrictions hold for any arbitrary set of σ parameters, these restrictions will still hold when this set of σ parameters is multiplied by a positive scalar (due to the Eq. (2.32) relationship). This means that these restrictions can be imposed before the algorithm is iterated (e.g. by adjusting Eq. (2.31), where necessary), so that only a scale parameter (akin to α) needs to be updated at each iteration (i.e. the algorithm does not require extra restrictions to ensure $\tau_i \in \mathbb{R}^+$ holds at each iteration) – greatly increasing computational speed and reducing the number of loss-function local minima.

To impose $\tau_i \in \mathbb{R}^+$, while remaining close to the values of σ_i that would result if these restrictions were not required:

1. Initialise the value of α as $\alpha = \Phi^{-1}(X) - \Phi^{-1}(Y)$.
2. Substitute this α and the data-based percentiles $(\widehat{x}_i, \widehat{y}_i)$ into Eq. (2.31) and call the resulting values σ'_i .
3. Initialise $\tau_i = \sigma'_i \forall i$. This initialisation satisfies the $\tau_i \in \mathbb{R}^+$ restrictions.
4. From Eq. (2.6), let $\sigma_i(\boldsymbol{\tau}) = \frac{1}{N} \sqrt{N(N-2)\tau_i^2 + \sum_{j=1}^N \tau_j^2}$
5. Starting from the initial values of τ_i , run a Pattern Search algorithm to find the values of τ_i that minimise $\sum_{i=1}^N (\sigma_i(\boldsymbol{\tau}) - \sigma'_i)^2$ given the constraint that $\tau_i \in \mathbb{R}^+ \forall i$.
6. Let the resulting values of τ_i and $\sigma_i(\boldsymbol{\tau})$ be known as τ_i^* and σ_i^* , respectively.

When using the 90th percentile values for the $\widehat{x}_i$'s and the medians for the $\widehat{y}_i$'s, the impact of imposing the $\tau_i \in \mathbb{R}^+$ restrictions is not large (Figure 2.10) – the CVRMSE between the σ_i^* values and the σ'_i values is just 10.85 per cent.

Figure 2.10: Impact of the $\tau_i \in \mathbb{R}^+$ Restrictions on σ_i
50 banks; 90th percentile values for the $\widehat{x}_i$'s and medians for the $\widehat{y}_i$'s



Imposing the $\tau_i \in \mathbb{R}^+$ restrictions is equivalent to replacing the N restrictions in Eq. (2.31) with:

$$\sigma_i = \beta \sigma_i^* \quad \forall \quad i \quad (2.33)$$

where $\beta > 0$ is the scale parameter that will be calibrated via iterating the algorithm (see Step 3) – that is, the β parameter plays the role that the α parameter would play if the $\tau_i \in \mathbb{R}^+$ restrictions were not binding.

Step 3: Calibration of the μ parameters and the β parameter

Let $\xi_i \equiv \mu_i \widetilde{M}$. Then the calibration algorithm proceeds as follows:

Step 3-A

Initialise the parameters (where $\widehat{m}_i$ is Bank i 's median end-of-day ES balance in the data):

$$\beta = 1$$

$$\xi_i = \left[\frac{\widehat{m}_i - \eta_i}{\sum_{i=1}^N (\widehat{m}_i - \eta_i)} \right] \left[751 - \sum_{i=1}^N \eta_i \right] \quad \forall \quad i$$

The initialisation of ξ_i apportions the adjusted aggregate ES balance from the data (A\$751 million adjusted for the calibrated values of η_i from Step 1) using each bank's share of the “sum of the adjusted end-of-day medians” in the data.⁵⁷

Step 3-B

Using the initial values of the parameters, approximate banks' end-of-first-period ES balance distributions using the approximate discretisation solution. From these approx-

⁵⁷Although the A\$751 million is used to initialise the ξ parameters, this initialisation has little impact on the results (with initialisations using A\$600 million and A\$800 million producing similar results).

imate distributions, determine each bank's x_i , y_i , and m_i , and compute $\sum_{i=1}^N (m_i - \widehat{m}_i)^2$. To reduce the computational burden, the medians are used for the $(Y \times 100)$ th percentiles.

Step 3-C

Given the current values of the x_i 's, y_i 's, and m_i 's, and the data, the parameter values are updated by setting:

$$\beta^{\text{New}} = \beta \sqrt{\frac{\sum_{i=1}^N (\widehat{x}_i - \widehat{y}_i)^2}{\sum_{i=1}^N (x_i - y_i)^2}}$$

$$\xi_i^{\text{New}} = \xi_i + \widehat{m}_i - m_i \quad \forall \quad i$$

These updates are designed such that if the restrictions in Eq. (2.31) hold exactly (for both model-based and data-based percentiles), and if banks' end-of-first-period medians are only dependent on the medians of their own initial distributions, then only one iteration would be required for banks' data-based and model-based percentiles to be matched exactly (see Appendix 2.7.7.1). Intuitively then, if these conditions are close to holding, iterating the algorithm should cause the model-based percentiles to converge toward the data-based percentiles (as long as the conditions remain close to holding with each iteration).

Step 3-D

Using these new parameter values, approximate banks' end-of-first-period ES balance distributions using the approximate discretisation solution. From these approximate distributions, determine each bank's x_i , y_i , and m_i , and compute $\sum_{i=1}^N (m_i - \widehat{m}_i)^2$. If the $\sum_{i=1}^N (m_i - \widehat{m}_i)^2$ computed at this step is not lower than at the previous iteration, retain the β and ξ values from the previous iteration, and finish the procedure.

Otherwise, go to Step 3-C.⁵⁸

For the results in this chapter, only 5 iterations of the algorithm were required:

Table 2.4: Algorithm Iterations
90th percentile values for the x_i 's and medians for the y_i 's

Iteration	$\sum_i (m_i - \widehat{m}_i)^2$	$\sqrt{\frac{\sum_i (\widehat{x}_i - \widehat{y}_i)^2}{\sum_i (x_i - y_i)^2}}$
Initial	832.46	0.9774
1	120.05	0.9814
2	25.05	0.9979
3	5.74	0.9981
4	5.66	0.9993
5	3.35	0.9990

Step 4

Using the calibrated values of ξ and β from Step 3:

- $\widetilde{M} = \sum_{i=1}^N \xi_i$ – this is because $\sum_{i=1}^N \mu_i = 1$ (from the Timeline in Section 2.3)
- $\mu_i = \frac{1}{\widetilde{M}} \xi_i \forall i$
- $\tau_i = \beta \tau_i^* \forall i$ – where τ_i^* comes from Step 2

Step 5: Calibration of the φ and θ parameters

The two second-period shock distribution parameters (φ and θ) are calibrated by matching the “expected daily use of the lending facility” and “conditional expected lending facility borrowing per bank” from the data (0.059 and A\$120 million, respectively), to the model-based counterparts (a Pattern Search algorithm is used to match these moments). The model-based counterparts are derived below.

From Appendix 2.7.6, the set of K feasible Θ^i values is denoted as Ψ . For the remainder of this section, assume that this is an ordered set (smallest to largest).

⁵⁸ $\sum_{i=1}^N (m_i - \widehat{m}_i)^2$ is used to determine the algorithm stopping point simply because, when applying the algorithm, the β parameter appeared to converge faster.

Given each bank's approximate end-of-first-period ES balance distribution (Appendix 2.7.6), the unconditional expected daily usage of the lending facility is equal to the sum of each bank's unconditional expected daily usage:

$$\mathbb{E}(\text{usage}) = \sum_{i=1}^N \mathbb{E}(\text{usage of Bank } i) = \sum_{i=1}^N \left\{ \frac{\varphi}{2N} \sum_{q=1}^N \sum_{k=1}^{K_{i,q}} \hat{P}_{i,T+1,k} \right\} \quad (2.34)$$

where:

- $K_{i,q}$ is the index of the largest element in the ordered set Ψ such that a negative net payment flow (originating from a shock to Bank q) will require Bank i to access the lending facility (recall that Bank i is potentially subject to N negative net payment flows, each mutually exclusive, and each occurring with probability $\varphi/2N$).⁵⁹
- Ψ_k is the k th element in the ordered set Ψ .
- $\hat{P}_{i,t,k}$ is the probability that Bank i has the k th pre-trade adjusted ES balance in the t th match (i.e. $\mathbb{P}(\Theta_t^i = \Psi_k)$). So $\hat{P}_{i,T+1,k}$ is the k th element of Bank i 's approximate end-of-first-period ES balance distribution.

Using Eq. (2.34), the conditional expected volume of borrowing is equal to:

$$\mathbb{E}(\text{volume} | \text{use}) = \frac{1}{\mathbb{E}(\text{usage})} \left[\sum_{i=1}^N \left\{ \frac{\varphi}{2N} \sum_{q=1}^N \sum_{k=1}^{K_{i,q}} \hat{P}_{i,T+1,k} [-S_{i,q} - \Psi_k] \right\} \right]$$

where:

- $S_{i,q} < 0$ is the negative net payment flow to Bank i (originating from a shock to Bank q).⁶⁰

⁵⁹When $q \neq i$, $K_{i,q}$ indexes the largest element of Θ^i such that $\Theta^i < \theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left[\frac{\sigma_q}{\sum_{j=1}^N \sigma_j} \right]$. When $q = i$, $K_{i,q}$ indexes the largest element of Θ^i such that $\Theta^i < \theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left\{ 1 - \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \right\}$.

⁶⁰When $q \neq i$, $S_{i,q} = -\theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left[\frac{\sigma_q}{\sum_{j=1}^N \sigma_j} \right]$. When $q = i$, $S_{i,q} = -\theta \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \left\{ 1 - \left[\frac{\sigma_i}{\sum_{j=1}^N \sigma_j} \right] \right\}$.

2.7.7.1 Raison d'être for the parameter updates

To show that $\beta^{\text{New}} = \beta$ after only one iteration if the restrictions in Eq. (2.31) hold exactly, if the restrictions in Eq. (2.31) hold exactly then the $\tau_i \in \mathbb{R}^+$ are not binding and (from Step 2):

$$\sigma_i^* = \sigma_i' = \frac{\hat{x}_i - \hat{y}_i}{\Phi^{-1}(X) - \Phi^{-1}(Y)}$$

Substituting this into Eq. (2.33), at each iteration:

$$\sigma_i = \beta \left[\frac{\hat{x}_i - \hat{y}_i}{\Phi^{-1}(X) - \Phi^{-1}(Y)} \right] \quad (2.35)$$

If Eq. (2.31) holds exactly for the model-based end-of-first-period percentiles, then for any σ_i :

$$x_i - y_i = \alpha \sigma_i \quad (2.36)$$

Substituting Eqs. (2.35) and (2.36) into the formula for β^{New} gives:

$$\beta^{\text{New}} = \frac{\Phi^{-1}(X) - \Phi^{-1}(Y)}{\alpha}$$

Substituting this update back into the equation that determines σ_i (Eq. (2.35)) gives:

$$\sigma_i = \frac{\hat{x}_i - \hat{y}_i}{\alpha}$$

Then substituting this equation for σ_i into the equation that determines the model-based end-of-first-period percentile difference (Eq. (2.36)) gives:

$$x_i - y_i = \hat{x}_i - \hat{y}_i$$

Therefore, only one iteration is required for banks' model-based percentile differences to match the data-based percentile differences.

At each iteration, $m_i = \xi_i + \eta_i + g_i(\boldsymbol{\mu}, \boldsymbol{\eta}, \boldsymbol{\tau}, \widetilde{M})$. That is, Bank i 's end-of-first-

period median is dependent on its initial median plus some function (determined by the impact of trading on Bank i 's median) that is dependent on all of the initial distribution parameters. For banks' end-of-first-period medians to only depend on the medians of their own initial distributions, there can be no trading (so that $g_i(\boldsymbol{\mu}, \boldsymbol{\eta}, \boldsymbol{\tau}, \widetilde{M}) = 0$).⁶¹ Then, since η_i is not updated with each iteration of the algorithm, substituting $m_i = \xi_i + \eta_i$ into the formula for ξ_i^{New} gives:

$$\xi_i^{\text{New}} = \widehat{m}_i - \eta_i$$

Then substituting this update into $m_i = \xi_i^{\text{New}} + \eta_i$ gives:

$$m_i = \widehat{m}_i$$

Therefore, only one iteration would be required if banks' end-of-first-period medians only depended on the medians of their own initial distributions.

2.7.8 Alternative calibrations

This appendix shows the results from two alternative calibrations:

- Alternative 1 – the 75th percentile is used instead of the 90th percentile.
- Alternative 2 – the η parameters are calibrated to zero.

Table 2.5 shows how well these alternative calibrations fit the data. For each of the alternatives, the percentiles used in the calibration process are closely matched by the model (i.e. low CVRMSE) – so the calibration algorithm described in Section 2.4.1 remains accurate when using these alternative combinations of sample statistics.

The “Preferred” and “Alternative 1” calibrations exhibit similar CVRMSE values for the sample statistics that are not used in the calibration process. But the “Preferred”

⁶¹As long as $\xi_i > 0$, then any positive probability of trading will cause $g_i(\boldsymbol{\mu}, \boldsymbol{\eta}, \boldsymbol{\tau}, \widetilde{M}) < 0$; since banks only trade towards their optimum end-of-first-period ES balance.

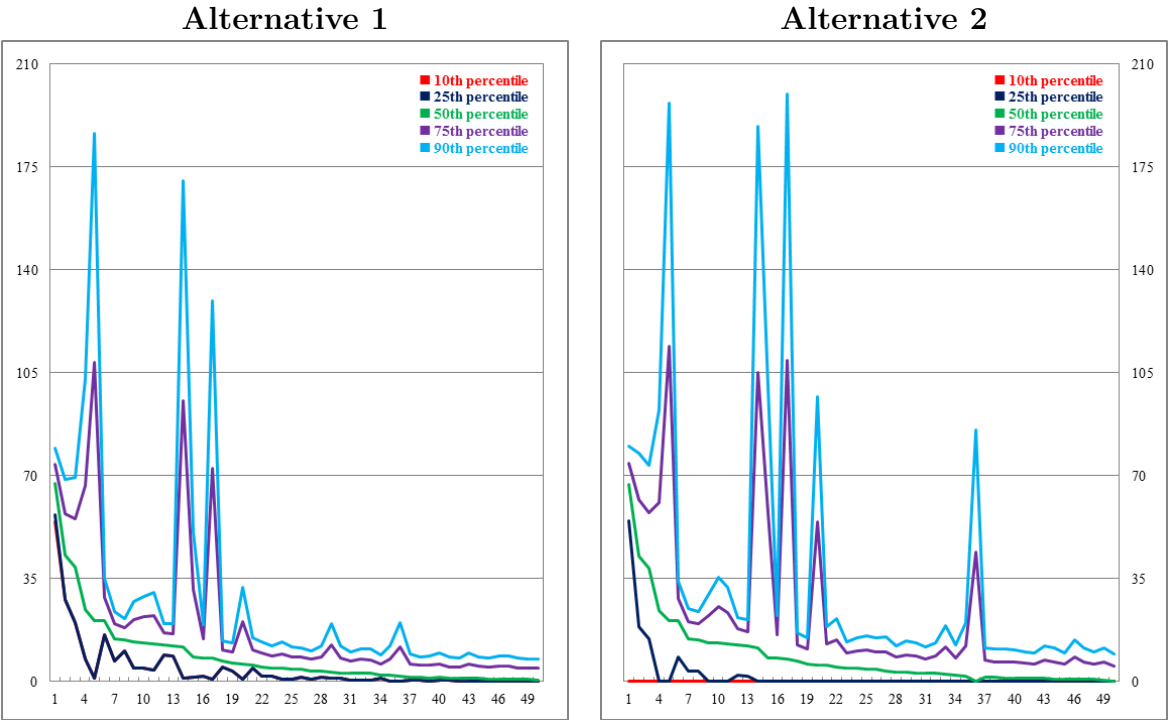
calibration produces an aggregate ES balance much closer to the data.

Table 2.5: Accuracy of the Calibrations

	CVRMSE (%)					Aggregate ES balance A\$ million	Trading Volume A\$ million
	10th	25th	50th	75th	90th		
Data	N/A	N/A	N/A	N/A	N/A	751	6115
Preferred	0.00	14.77	2.95	56.65	13.06	760	893
Alternative 1	0.00	13.03	1.03	12.81	56.91	654	653
Alternative 2	234.00	80.99	2.85	57.29	13.98	713	784

Table 2.5 and Figure 2.11 show that the tight spread of banks' 10th and 25th percentiles (at levels above zero) and wide spread of banks' 75th and 90th percentiles, that is in the data, cannot be simultaneously replicated without the ability to set $\eta_i > 0$ (Alternative 2). This is because, to replicate the data-based 90th percentiles and medians, the model produces a sufficient volume of trading such that the 10th percentiles and, for most banks, the 25th percentiles occur at the atom (which is zero without the ability to set $\eta_i > 0$).

Figure 2.11: Model-based Percentiles
A\$ millions



Note: Model-based x th percentile = the x th percentile of the bank's end-of-first-period ES balance distribution that is produced by the model.

Part II

Welfare-optimising Aggregate ES Balance

Chapter 3

DSGE Model and Qualitative Results

3.1 Introduction

Thus far in this thesis, I have built a model of the overnight interbank market that incorporates a central bank's commitment to its trading-rate target (Chapter 1); and have shown that, when applied to a market where commitment is well documented (Australia), my model fits the data well (Chapter 2). Within these chapters, I also show the implications of exogenously changing the aggregate level of ES balances (as this is a policy tool of the central bank). However, the reasons for central banks typically maintaining an aggregate ES balance in excess of banks' desired end-of-day balances, but small relative to aggregate interbank lending, are not discussed.¹ This chapter provides a capstone to this thesis by providing an argument for why central banks typically set the aggregate ES balance in this way.

¹Recall from Chapter 1 that the aggregate ES balance does not impact the overnight interbank interest rate (due to the commitment of the central bank). Therefore, with a committed central bank, maintaining an aggregate ES balance in excess of banks' desired balances can not be explained by the central bank's interest rate target. This differs from typical overnight interbank models (discussed in Appendix 1.7.2), in which implicit corridor asymmetries can require the central bank to supply excess ES balances in order to achieve its interest rate target.

Recall, from the first two chapters, that reducing the aggregate ES balance leads to increased use of the central bank's lending facility, while increasing the aggregate ES balance reduces trading in the overnight interbank market. If both of these outcomes have costs, then the central bank faces a trade-off when deciding the aggregate ES balance.

There is some concern among banks that use of a central bank's lending facility may be perceived as a negative signal of the borrowing bank's asset quality (i.e. stigma) (see *Committee on the Global Financial System* (2008), *Kohn* (2010), *Armantier et al.* (2011), and *Ennis and Weinberg* (2013), for example) – with *Kohn* (2010) stating that “the problem of discount window stigma is real and serious”.² To the extent that there are costs associated with this negative signal, then the existence of stigma increases the cost of using the central bank's lending facility.

On the other hand, the literature suggests that trade in the interbank market provides valuable information to banks about their counterparties (*Keister et al.*, 2008; *Heath et al.*, 2013). For example, *Keister et al.* (2008) state that:

In addition, some observers argue that the presence of an active overnight market generates valuable information and that some of this information would be lost if market activity declined. For example, market participants must monitor the creditworthiness of borrowers. If the overnight market were substantially less active, such monitoring may not take place on a regular basis; this in turn could make borrowing even harder for a bank that finds itself short of funds. Such monitoring may also play a socially valuable role in exposing banks to market discipline.

Moreover, *Heath et al.* (2013) state that:

One consequence of this [no trade in the interbank market] was a decline in the monitoring of counterparties, which was an inherent feature of trade in the overnight market. ... New Zealand was cited as an example of a country that had adjusted its operations with regard to paying interest on reserves

²Although the Reserve Bank of Australia (RBA) states that there is no stigma associated with use of its lending facility (*Debelle*, 2008), this chapter shows that such an outcome may be due to its policy setting.

because this loss of trade in interbank markets was perceived as being too costly.

Therefore, lowering overnight interbank market activity is costly due to the consequences of the resulting fall in monitoring.

To simply model these costs and determine the welfare-optimising aggregate ES balance, I construct a simple DSGE model with a banking sector. The banks act as intermediaries between the consumers and the firms. However, since the firms need not have accounts with all of the banks, interbank payment flows may be required to channel the savings of consumers to the firms (for use as future capital). As in reality, these interbank payment flows occur via the accounts each bank has with the central bank (ES balances). If these flows cause banks' ES balances to differ from their optimal levels, both banks may benefit from overnight borrowing/lending of ES balances (as in the first two chapters).

As in Chapter 2, total interbank payment flows will not be known until a point late in the day, when interbank lending is no longer possible. Therefore, interbank loans are based on expectations developed earlier in the day. Then, once total payment flows are realised, any bank that has an ES balance deficit must borrow overnight from the central bank (via the central bank's lending facility) to remove this deficit – any difference between the true and the expected payment flows results from a “shock”.

In order to acquire capital goods, the firms issue equity shares to the banks. The banks may then sell these shares to asset managers who expend less effort when managing the shares (i.e. their cost of managing the shares is lower). With symmetric information, this lower effort cost would mean that there is always a mutually profitable sale price. However, I assume asymmetric information between the banks and asset managers; while the banks have full knowledge of any shocks, the asset managers do not observe the shocks, but extrapolate information about the shocks from banks' use of the central bank's lending facility (which is observed by the asset managers).³

³This information asymmetry is analogous to the information asymmetry in *Ennis and Weinberg*

This asymmetric information results in an adverse selection problem. Although both positive and negative shocks could lead to lending facility use, the adverse selection problem may be sufficiently severe that, when the lending facility is used, the shares will only be sold to the asset managers when the shock is negative. In this scenario, use of the lending facility becomes a negative signal about the quality of the assets that can profitably be sold (i.e. stigma).

With the cost of managing the shares being lower for the asset managers than for the banks, an inability to sell the shares results in lower profits for the banks and therefore lower returns on consumers' savings. Therefore, if lending facility stigma leads to fewer asset sales, reducing use of the lending facility – by increasing the aggregate ES balance – increases bank profits, consumer returns, and the consumers' welfare.

On the other hand, by reducing the volume of interbank borrowing/lending, increasing the aggregate ES balance increases the cost of interbank monitoring, reducing banks' profits and reducing consumers' welfare.⁴ So the central bank faces a trade-off when determining the welfare-optimising aggregate ES balance.

While some parameter values lead to a welfare-optimising aggregate ES balance equal to banks' desired end-of-day balances (i.e. parameter values with a sufficiently large monitoring cost), parameter values associated with larger stigma costs can lead to a welfare-optimising aggregate ES balance that is large enough to eliminate lending facility stigma, but still small relative to aggregate interbank lending. There are also parameter values for which the welfare-optimising aggregate ES balance is in between these two extremes (I give numerical examples of all three scenarios).

Therefore, by including stigma and monitoring costs (which, to my knowledge, have never before been simultaneously modelled), this model can explain why central banks typically maintain an aggregate ES balance in excess of banks' desired end-of-day

(2013).

⁴An increase in the cost of monitoring includes any consequences that result from a lower level of monitoring, as suggested by *Keister et al.* (2008) and *Heath et al.* (2013).

balances, but small relative to interbank lending (this is the main contribution of this chapter to the literature).

In the remainder of this chapter: Section 3.2 describes the model's features, derives the model's equations, and solves the model; Section 3.3 derives welfare (using the solutions from Section 3.2), and shows how central bank policy impacts welfare; with the impact of central bank policy depending on the parameter values, Section 3.4 shows the optimal policy in some example parameter settings; Section 3.5 concludes and discusses avenues for further research; and Section 3.6 contains appendices (including a literature review).

3.2 Model

3.2.1 Description of the economic agents

The model economy consists of a $[0, 1]$ continuum of infinitely-lived consumers, a $[0, 1]$ continuum of identical profit-maximising firms, a $[0, 1]$ continuum of identical profit-maximising asset managers, 2 identical profit-maximising banks, and a central bank. There is only one “good” in the economy; this good is infinitely divisible and its market price is normalised to unity. The goods produced each day can be used for either current consumption or future capital. The model does not incorporate any nominal rigidities, and incorporates only a single real shock (a technology shock). All agents form rational expectations.

In Chapter 2, each bank has a desired end-of-day ES balance of η_i . Having $\eta_i > 0$ only shifted each bank's end-of-day ES balance distribution, it did not change any qualitative results. With this chapter seeking only qualitative conclusions, $\eta_i = 0$ for both banks, for simplicity (as in Chapter 1).

3.2.1.1 Consumers

Consumers can save for the future by using some of their goods to purchase equity shares in the banks; this is the only saving mechanism available to consumers (i.e. consumers can not directly invest in either the firms or the asset managers).^{5,6} For simplicity, these shares are completely repaid the following day (i.e. the dividends paid by the banks on the following day equal the total value of the equity on that day).

Although the banks are identical at the beginning of each day (this is discussed further in Section 3.2.1.5), their operations, and therefore their daily dividend payments, will differ. However, when the consumers make their saving decisions, they are not privy to the banks' operational decisions; so the consumers treat the banks as identical.

With respect to consumers' savings, I search for a symmetric equilibrium (i.e. where all consumers invest the same proportion of their savings in each bank). This results in the daily return on savings being the same for all consumers, and simplifies the model by making all consumers identical. As shown in Appendix 3.6.2, a solution where all consumers invest half of their savings in each bank is a symmetric equilibrium. Since Appendix 3.6.2 also shows that this is the only symmetric equilibrium that always exists (irrespective of the parameter values), I assume that consumers invest half of their savings in each bank.

Every day (t), consumers make consumption (C_t), labour (H_t), and savings (D_t) decisions to maximise their expected discounted lifetime utility. For the purposes of this chapter, only a very simple utility function needs to be assumed (this simple utility function has been used in the literature, see *Hansen* (1985), for example):⁷

⁵This assumption can be justified by assuming that consumers cannot cost-effectively make direct investments across the continuum of firms or asset managers (e.g. because they do not have the "monitoring expertise" (*Stein*, 2012)).

⁶With private ownership of the banks, some of the consumers' investments in the banks must be via bank equity. Restricting all saving to occur via bank equity is a simplifying assumption that ensures there is no principal-agent problem of the type in *Bernanke et al.* (1996), for example.

⁷With this utility function, agents are risk-neutral with respect to wealth shocks (*Swanson*, 2012);

$$\mathbb{E} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} [\ln(C_s) - \chi H_s] \right\} \quad (3.1)$$

Subject to their daily budget constraint:

$$C_s + D_s \leq R_s^D [D_{s-1} - \psi M(\bar{S})] + W_s H_s + CB_{s-1} \quad (3.2)$$

Where:

- R_s^D is the stochastic real gross return from shareholdings (half in each bank) purchased the previous day;
- $0 < \beta < 1$ is the discount factor, common to all consumers;
- W_s is the real daily wage, derived from equilibrium in a competitive labour market and common to all consumers;
- CB_{s-1} is a lump-sum transfer from the central bank (or a lump-sum tax if the central bank makes a loss);
- $\chi > 0$ is a scaling parameter representing the marginal disutility of labour, and is common to all consumers;
- $\bar{S}$ is the deterministic steady state level of aggregate interbank lending; and
- $M(\cdot)$ is the function that determines the real cost of interbank monitoring, $\psi > 0$ is a scaling parameter, and $\frac{dM(x)}{dx} < 0$.⁸

As discussed in Section 3.1, trade in the interbank market provides valuable information to the banks about their counterparties, with the amount of information provision falling as market activity declines. Since the exact form of the monitoring cost is

with the technology shock in this model only impacting consumers via wealth.

⁸Although the restriction $D_{s-1} \geq \psi M(\bar{S})$ must hold for every s , with the parameter values used in this chapter, the probability of successive shocks large enough to cause this restriction to bind is infinitesimally small. So, for simplicity, I assume it never binds.

not important for the qualitative conclusions of this chapter, I simplify the model by incorporating an ad hoc monitoring cost function ($M(\cdot)$) that is consistent with the statements of *Keister et al.* (2008) and *Heath et al.* (2013), and leave proper modelling of interbank monitoring for future research.⁹

To further simplify the model: the monitoring cost is modelled as a function of steady state aggregate interbank lending (rather than the daily volume of interbank lending), and is therefore fixed by the model parameters (this simplifies the decision rules that solve the model); and the banks' daily monitoring cost is paid by charging their shareholders a fixed daily fee (this fee is subtracted from their invested equity).¹⁰

Since banks are ex-ante identical, the monitoring cost is the same for both banks. The functional form of the monitoring cost is discussed in Section 3.2.4.

3.2.1.2 Firms

Each firm produces daily output (Y_t) with a constant returns to scale Cobb-Douglas production function:

$$Y_t \equiv A_t H_t^{1-\alpha} K_t^\alpha \quad (3.3)$$

Where:

- A_t represents the stochastic level of technology in the economy;
- H_t represents the daily volume of labour provided by the consumers (in equilibrium, this H_t must equal the H_t chosen by the consumers);
- K_t represents the volume of goods the firm uses as capital in day t ; and

⁹See Appendix 3.6.3 for an example of how monitoring costs, consistent with *Keister et al.* (2008) and *Heath et al.* (2013), could be properly modelled.

¹⁰An alternative modelling option, in which the monitoring cost is subtracted from dividends, causes the monitoring cost to directly impact the consumption Euler equation (in equilibrium, the budget constraint does not change). As a result, there would be no exact analytical solution for the deterministic steady state and the first-order approximation around the steady state would be more complex (increasing the complexity of the welfare result).

- $0 < \alpha < 1$ is the output elasticity of capital.

The level of technology is determined by the following process:

$$\ln(A_t) = \ln(\bar{A}) + \varepsilon_t$$

Where ε_t is a mean-zero daily shock.

3.2.1.3 Banks

There are two identical, risk-neutral, profit-maximising banks in the economy. The banks act as intermediaries between the consumers and the firms. The banks also interact with the central bank, with both banks holding deposit accounts with the central bank (ES balances).

With the banks paying out all profits as dividends on a daily basis, they make decisions to maximise their expected *daily* profits (e.g. they do not account for any impact their day t decisions may have on the future investment decisions of the consumers, which may impact profits beyond day $t + 1$).

Since consumers use goods to purchase equity in the banks, these goods add to the banks' assets (except the goods that are immediately used to pay the monitoring cost). I assume 100 per cent daily depreciation. So, with banks not able to undertake production, goods acquired from consumers during day t are extinguished at the end of day t unless they have been sold to the firms (who can convert the goods into day $t + 1$ capital).¹¹

The firms can purchase these goods by giving the banks equity claims to their day $t + 1$ profits (as in *Kiyotaki and Moore (2012)*, for example). These purchases occur in two steps. First, the firms are given intra-day bank deposits in exchange for

¹¹100 per cent depreciation is a simplifying assumption that ensures banks' return from investment always exceeds their return from simply storing the goods until the following day. The same simplification could be achieved with a smaller depreciation rate, as long as one minus the depreciation rate is smaller than the lowest possible return from sale to the firms. However, since this chapter only seeks to provide qualitative results, a smaller depreciation rate is not necessary.

the equity shares (deposits are intra-day because unused deposits are extinguished at the end of the day). Second, since these deposits are a claim on the bank's assets, the firms use these deposits to acquire goods from the banks.¹²

However, because the bank that receives an equity share from one of the firms need not be the same bank from which this firm must purchase the goods (recall that each bank has half of consumers' savings), purchasing the goods may require asset transfers between the banks. Since interbank asset transfers are a key mechanism in this model, I simplify the model by assuming that, on each day, all firms issue equity shares to only one of the banks, with this bank chosen at random and known as the "lending bank" (each bank has equal probability of being the lending bank).

In reality, payments made between banks are typically settled by transferring ES balances (*Borio, 1997*).¹³ These are the "payment flows" in chapters 1 and 2. To be consistent with reality, when firms purchase goods from the non-lending bank in this model, these payments are settled by ES balances transferring from the lending bank to the non-lending bank.¹⁴

As in chapters 1 and 2, these payment flows may cause the banks to have ES balances that differ from their optimum levels; so both banks may benefit from overnight borrowing/lending of ES balances. Although total payment flows will not be known until a point late in the day when interbank lending is no longer possible (akin to the second period in chapters 1 and 2), expectations are developed earlier in the day, and it is on these expectations that interbank loans are based.¹⁵ I assume that the commitment of the central bank to its interest rate target is sufficient to ensure that all

¹²Intra-day deposits simplify the model by ensuring that firms always use these deposits to acquire goods, and by removing the possibility of firms retaining deposits over time.

¹³Reasons why payments are settled via ES balances include legal requirements and the low settlement risk (*Borio, 1997*). Alternatively, settlement may occur via ES balances if, as is assumed in the models of *Acharya et al. (2012)* and *Ennis and Weinberg (2013)*, a bank's other assets have "some degree of specificity" and are valued less by the other banks.

¹⁴Since the non-lending bank can use these extra ES balances to purchase goods in day $t + 1$, it is through these ES balances that the non-lending bank is able to inter-temporally transfer resources.

¹⁵Interbank loans are uncollateralised, consistent with chapters 1 and 2, and consistent with the Australian market.

interbank loans occur at the target rate (which is halfway between the central bank's deposit rate and lending facility rate) – consistent with the first two chapters.

To be consistent with the first two chapters (and the Australian economy), unexpected late-day payment flows must be small relative to all payment flows. To simplify this model, the technology shock only represents the unexpected exogenous economic activity that causes these late-day payment flows (as opposed to representing *all* exogenous technology shocks, as in a typical DSGE model).¹⁶ Therefore, the technology shocks in this model must be small.

With a small shock, all overnight interbank loans, and overnight loans between the banks and the central bank, are risk-free (consistent with the assumptions in chapters 1 and 2).^{17,18} So banks are always able to find sufficient funding within the interbank market to offset an *expected* ES balance deficit.¹⁹ Once total payment flows are realised (following the shock), any bank that has an ES balance deficit must borrow overnight from the central bank (via the central bank's lending facility) to remove this deficit (recall, from Chapter 1, that end-of-day ES balance deficits are not allowed).

In this model, the lending bank will give the firms a value of deposits sufficient to purchase all of the goods on the banks' balance sheets.²⁰ If interbank payment flows were not costly, achieving this result would only require that firms' equilibrium day $t+1$

¹⁶With this interpretation of the shock, this model is not suitable for explaining the business cycle. However, this model could be incorporated within a full DSGE model. All that would be required is splitting the exogenous shocks into those that are known prior to the late-day period, and those that are unknown until this period.

¹⁷More accurately, small shocks ensure the probability of consecutive shocks large enough to cause loan defaults is infinitesimally small; so I treat the default risk as zero for simplicity.

¹⁸In reality, banks have more interbank assets than just overnight interbank loans. Therefore, even if overnight loans were risk free, there would still be some value in counterparty-monitoring. For simplicity, this model excludes longer-term interbank assets, but retains a simple, ad-hoc monitoring cost function (as discussed in Section 3.2.1.1), and assumes that banks always pay this cost (due to the un-modelled benefits).

¹⁹This is an important difference between my model and *Ennis and Weinberg* (2013) – in which default risk can cause some banks to be unable to find interbank market funding, potentially leading to excessive lending facility use and stigma (see Appendix 3.6.1 for more detail).

²⁰With a constant returns to scale production function, the split of deposits between the firms does not change the lending bank's return. Therefore, with a continuum of profit-maximising firms (each wanting as much capital as possible, see Section 3.2.3.2) and a monopolist lending bank, each firm accepts whatever volume of deposits the lending bank offers them in exchange for their equity.

profits increased with more day $t + 1$ capital (I assume this result holds in this chapter, see Appendix 3.6.4 for supporting evidence), and that the cost of managing the equity claims is sufficiently small. However, since interbank payment flows may be costly to the lending bank (as it may require an overnight loan), some additional simplifying features/assumptions are required to ensure this result (these features/assumptions are detailed in Appendix 3.6.5).²¹

Since the equity (issued by the firms during day t) give their shareholders claims to day $t + 1$ profits only, I assume that firms' day $t + 1$ dividend payments (paid in newly produced goods) equal the value of their day $t + 1$ profits, thereby extinguishing the day t equity claims.

Appendix 3.6.6 provides a detailed explanation of exactly how all of the above transactions impact the banking system over the course of a day, but this appendix is best understood after reading up to the end of Section 3.2.2.

3.2.1.4 Asset managers

With risk-free interbank loans, there must be another avenue for stigma to impact the economy. To facilitate stigma, I follow *Ennis and Weinberg* (2013) and introduce risk-neutral profit-maximising asset managers into the economy. In my model, there is a $[0, 1]$ continuum of identical asset managers willing to buy the equity claims to the firms' profits, from the lending bank, at the end of the day (after the interbank market has closed).

In *Ennis and Weinberg* (2013), banks need to liquidate their assets in order to repay loans from earlier in the day. In my model, banks do not have this liquidity need. So, to generate gains from trade, I assume that to properly manage the equity claims (e.g. to

²¹Having the lending bank issue deposits sufficient to purchase all of the banks' goods, simplifies determination of expected and unexpected payment flows. While removal of these assumptions would increase the model's complexity, as long as the model produced some expected and unexpected payment flows (consistent with reality), the qualitative conclusions of this chapter would be unchanged – so these simplifying assumptions do not drive the results of this chapter.

properly monitor the firms) asset managers need to expend less effort than the banks.²² Therefore, given complete information, there would always be a price at which both the lending bank and the asset managers are willing to trade.

However, as in *Ennis and Weinberg (2013)*, to facilitate stigma I assume asymmetric information between the lending bank and the asset managers. Specifically, I assume the asset managers are unable to observe any shock that occurred earlier in the day (the banks observe the shock), but they are able to observe whether the central bank's lending facility was accessed (but not the volume of borrowing or which bank accessed the facility).²³

The inability to observe any shocks causes asset managers to assign probabilities to all possible shocks, and use the resulting probability distribution to determine the prices at which they are willing to buy the shares. But the highest price at which the asset managers are willing to buy the shares (based on the unconditional probability distribution) may be lower than the lowest price at which the bank is willing to sell (since the bank knows the true value of the shares), so the shares may not always be sold. This possibility negatively impacts asset managers' expectations of asset quality, leading to lower equilibrium prices, fewer sales, and a lower expected quality of the assets sold – i.e. an adverse selection problem.²⁴

Since the asset managers' asset-quality expectation depends on their observations, this expectation can depend on whether the lending facility was used. In this model, lending facility use itself will not signal poor asset quality. However, since asymmetric

²²For example, with a continuum of asset managers and a continuum of firms, each asset manager may be able to specialise and develop expertise in a particular area, thereby lowering costs. This is consistent with the model of *DeMarzo and Duffie (1999)* in which one of the reasons agents may issue securities is because “The issuer could be a broker-dealer whose core business and comparative advantage is intermediation rather than long-term investment.”

²³To ensure the asset managers are unable to extrapolate the day t shock (during day t), they must be unable to observe or extrapolate any of the firms' or consumers' decisions on day t . The features/assumptions of this model ensure this is the case (see Appendix 3.6.5 for details).

²⁴It is an adverse selection problem in the sense that the bank only “selects” to sell the shares if the value of the shares is sufficiently low relative to the price. This can be seen as an extreme example of a more typical adverse selection problem in which there is an array of goods, with differing values, and only those of a sufficiently low value are sold (e.g. *Akerlof (1970)*).

information can cause adverse selection in this model, there may be some prices at which the expected quality of assets that would be offered for sale would be lower following facility use. At these prices, lending facility use is a negative signal of asset quality (stigma).

Asset managers' profits

For simplicity, each asset manager is owned by both banks (with each bank owning an equal share). Therefore, each bank gets half of the asset managers' profits or, if the asset managers' make a loss, each bank pays for half of the loss.²⁵

With consumers making identical saving decisions, having the asset managers owned by the banks simplifies the model by making consumers' returns on their savings independent of the price at which shares are sold to the asset managers.²⁶ This set-up simplifies the welfare analysis of stigma by restricting the welfare impact to be dependent only on whether the equity shares were sold to the asset managers (due to the higher cost of banks managing the equity), as opposed to whether they were sold *and* the sale price. As a result, stigma itself is not welfare decreasing, welfare is only impacted if stigma leads to fewer asset sales.

3.2.1.5 Central bank

The central bank interacts with the banking system via its open market operations (i.e. borrows/lends ES balances), offering unlimited ES balances at a fixed rate (via the lending facility), and paying deposit interest on surplus end-of-day ES balances (i.e. the central bank operates a corridor system).²⁷ To anchor the market for central

²⁵Since each bank only gets half of the asset managers' profits, the lending bank will still seek to sell the shares for the highest possible price (since it gets the total benefit from the higher price, but suffers only half the cost due to the lower profits of the asset managers).

²⁶Independence occurs because any increase (decrease) in the profit of the lending bank due to an increase (decrease) in the sale price is exactly offset by a decrease (increase) in the asset managers' profits (which is transferred to the banks).

²⁷For simplicity, borrowing from the central bank is uncollateralised in this model. While this is inconsistent with the Australian market, collateral is not necessary in this model as overnight

bank deposits to the real economy, all net interest payments between the banks and the central bank must be settled by transferring goods (any principal is repaid in ES balances).²⁸ Since the central bank has no equity, any profits/losses are transferred to the consumers via a lump-sum subsidy/tax (CB_t).

The central bank is able to choose the amount it lends/borrows from each bank via open market operations, the interest rate at which it conducts open market operations (this interest rate can differ for each bank), the interest rate it pays on surplus ES balances, and the interest rate it charges for use of its lending facility. However, its choices are restricted by:

- the open market operation participation constraints of the banks (the banks must be willing to borrow/lend at the given interest rates); and
- to be consistent with chapters 1 and 2, my assumption that the central bank operates so that both banks begin each day with the same expected end-of-day ES balance (excluding lending facility use).

With these restrictions, the central bank is only able to freely choose the aggregate volume of ES balances (excluding any lending facility use), the ES balance deposit rate, and the lending facility rate.

In this model, the central bank's price-stability objective is obsolete (there is only a single good in this model, with a constant market price of unity). Since many central banks (including the RBA), use the level of interest rates to achieve their price-stability objectives, this model is not sufficiently rich to determine how a central bank would realistically set the levels of its deposit and lending facility rates. So I assume the *levels* of these rates are exogenously set outside of this model.

lending is risk-free. Moreover, as discussed in Section 1.3.1, Australian banks hold more than enough collateral for any reasonable use of the central bank's lending facility; so the collateral requirements in the Australian market are not costly and can therefore be excluded without impacting the model's applicability.

²⁸Without this restriction, the only cost to obtaining ES balances, would be paid in ES balances (i.e. there would be no *real* cost).

As will be shown in Section 3.2.3.3, the levels of the deposit and lending facility rates do not independently feature in any of the equations that solve the model in this chapter, only their *spread* matters (this is consistent with the partial equilibrium models of the first two chapters). Moreover, since a central bank committed to its interest rate target can shift the aggregate ES balance without impacting interest rates (as shown in the first two chapters), and since central banks (that operate a corridor system) do not typically move their aggregate ES balances with the business cycle, the welfare-optimising aggregate ES balance can be realistically determined in a model with only late-day payment flow shocks and without a price-stability objective (except in abnormal circumstances, such as the recent financial crisis, which are outside the scope of this thesis). Therefore, the qualitative conclusions in this chapter are the same as what would be derived in a more-complete model with an active price-stability objective and endogenously set interest rates.

Except during the recent financial crisis, there has typically been low daily volatility in Australia's aggregate ES balance (pre-crisis, the coefficient of variation was 16 per cent, see Figure 1.1). Moreover, the RBA seldom changes the spread between its lending and deposit rates. Therefore, I treat these policy variables as fixed parameters in this model (with analysis of changes in these parameters conducted via comparative statics).

3.2.2 Daily timeline

Each day t is split into four periods (Table 3.1):²⁹

²⁹This split of each day is similar to the split in *Berentsen and Monnet* (2008).

Table 3.1: Timeline

	Pre-determined repayments (from day $t - 1$)	Current-day choices / outcomes	Information revealed
Beginning of day		Open market operations	
Trading period		Interbank loans	Lending bank (not revealed to consumers)
		Firms issue shares to the lending bank in return for deposits	
Production period	Interbank loans	Consumers choose labour / firms pay consumers for labour	Shock (not revealed to asset managers)
	Open market operations	Firms pay dividends to lending bank or asset managers	
	Interest on surplus ES balances	Asset managers transfer profits/losses to banks	
	Asset managers pay lending bank for equity shares	Banks pay dividends to consumers	
	Lending facility use	Consumers make consumption decisions	
		Consumers make saving decisions	
		Firms buy goods for day $t + 1$ capital	
End of day		Potential lending facility use	
		Potential sale of firms' shares to asset managers	

3.2.2.1 Beginning of day

The central bank conducts its new open market operations. As in the first two chapters, I assume the central bank conducts open market operations only once per day, at the beginning of the day. When the central bank conducts open market operations, neither the banks nor the central bank know which bank will be the lending bank. As a result, the central bank is unable to determine the direction of the payment flows that will result from the firms purchasing goods, thereby opening an avenue for interbank loans

to occur once this information is revealed.

3.2.2.2 Trading period

The bank that is the “lending bank” is revealed to the banks and the firms at the beginning of the trading period. Although no payment flows occur during the trading period, after learning which bank is the lending bank, banks make interbank loans based on expected payment flows.

As noted in Section 3.2.1.3, simplifying features/assumptions ensure the lending bank gives the firms a value of deposits sufficient to purchase all of the goods on the banks’ balance sheets. Having all deposits issued during the trading period is one of these features (see Appendix 3.6.5 for more detail).

The “lending bank” is not revealed to the consumers since the consumers are not privy to the banks’ day t operational decisions (as discussed in Section 3.2.1.1).

3.2.2.3 Production period

For simplicity, the banks provide no equity to the asset managers initially, but instead underwrite any losses. With no equity, the asset managers have no assets with which to immediately pay the banks for the shares. Instead, they settle payment the following day, when the firms pay their dividends.

With the banks underwriting losses, and with the small shocks, next-day settlement is risk-free. Moreover, there is no opportunity cost associated with next-day settlement, since this is when the bank would have received the dividend payments from the firms if it had not sold its shares.

To prevent any intra-day maturity-mismatch issues, all economic activity that causes goods to move between agents occurs during the production period.³⁰ In fact,

³⁰With all loans being overnight and daily dividend payments equalling the total value of equity, there are also no inter-day maturity-mismatch issues.

all transactions that occur during this period involve some transfer of goods.³¹

With the technology shock impacting the flow of goods (by impacting the value of goods produced), this is also the period in which the shock becomes known (except to the asset managers). Therefore, to be consistent with Chapter 2 (and to ensure the possibility of lending facility use), there can be no new interbank loans made during this period.

3.2.2.4 End of day

Any bank that has an ES balance deficit in this period *must* borrow from the central bank's lending facility – this is consistent with the Australian system. This is also the period when the lending bank may sell its equity shares to the asset managers.

3.2.3 Deriving model equations

3.2.3.1 Consumers

Each day, consumers choose consumption (C_t), labour (H_t), and savings (D_t) to maximise their expected discounted lifetime utility (Eq. (3.1)) subject to their budget constraint (Eq. (3.2)). With a continuum of consumers, each consumer treats as given: returns from shareholdings (dividends), the central bank tax/subsidy, the monitoring cost, and wages. The solution method to such a problem is standard in the literature, so I omit the working and give the first-order conditions for a constrained interior maximum:

$$\mathbb{E} \left[\frac{R_{t+1}^D}{C_{t+1}} \middle| \mathcal{I}_t^C \right] = \frac{1}{\beta C_t} \quad (3.4)$$

$$C_t = \frac{W_t}{\chi} \quad (3.5)$$

³¹Since the net interest on interbank loans impacts banks' dividend payments (which are paid in goods), it is convenient for the net interest on these loans to be paid in goods.

$$C_t + D_t = R_t^D \left[D_{t-1} - \psi M(\bar{S}) \right] + W_t H_t + C B_{t-1} \quad (3.6)$$

$\mathcal{I}_t^C$ is the information available to the consumers during the production period of day t (i.e. it includes the day t shock but not which bank is the lending bank).

3.2.3.2 Firms

When producing during day t , the capital available to the firms (K_t) is fixed (the firms purchased the goods from the banks during the previous day). Then, given the production function (Eq. (3.3)) and the exogenous level of technology (A_t), the only day t choice made by firms that impacts their day t production is the amount of labour they purchase from the consumers (H_t).

With a continuum of firms operating in competitive markets for labour and output, each firm treats the wage (W_t) as given. Then, for a given wage, each firm chooses its desired level of labour to maximise its profit. The resulting maximum occurs at the point where marginal revenue equals marginal cost:

$$W_t H_t = (1 - \alpha) Y_t \quad (3.7)$$

At this maximum point, each firm's profit will equal:

$$\alpha Y_t \quad (3.8)$$

Which is transferred as dividends to the entities that hold the equity claims (either the lending bank or the asset managers).

From the perspective of the firms (i.e. with fixed wages), Eq. (3.8) is increasing in K_t . Therefore, each firm will use all of its deposits to purchase goods from the banks.

3.2.3.3 Banks, asset managers, and the central bank

Let the lending bank in day $t - 1$ be known as Bank i and the non-lending bank be known as Bank j .

During day t , both banks receive interest (from the central bank, denoted $(R^{ES} - 1)$) on their surplus ES balances at the end of the previous day (denoted ES_{t-1}^i and ES_{t-1}^j), they repay (with a gross interest rate of R^F , where $R^F > R^{ES}$) the central bank for any use of their lending facility (denoted F_{t-1}^i and F_{t-1}^j), and they either owe or receive principal and interest (denoted $R_{t-1}^{O,i}$ and $R_{t-1}^{O,j}$) based on the open market operations of the previous day (denoted O_{t-1}^i and O_{t-1}^j).^{32,33}

Both banks also either owe or receive principal and interest (denoted $R^S = \frac{1}{2} [R^{ES} + R^F]$; due to the commitment of the central bank to its interest rate target) based on any interbank loans made during the previous day (S_{t-1}^i denotes the gross amount borrowed by Bank i , while S_{t-1}^j denotes the gross amount borrowed by Bank j).

With the asset managers owned equally by both banks, if the lending bank sold some of its shares (of the firms' equity) to the asset managers at the end of day $t - 1$, then the asset managers transfer half of their net profits to each bank during day t (or each bank pays for half of the losses).

Since it is only the effort *differential* that matters for the qualitative results in this chapter, I simplify the model by assuming the effort expended by the asset managers when managing the shares is zero, and that the cost of the effort expended by the banks when managing any positive volume of shares is fixed at $\vartheta > 0$. I also assume, for simplicity, that this cost is paid out of revenues, rather than by hiring the labour

³² R^{ES} and R^F are treated as fixed parameters in this model (as discussed in Section 3.2.1.5).

³³With net interest payments between the banks and the central bank settled by transferring goods (recall Section 3.2.1.5), exchange rates between ES balances and goods must be agreed between the banks and central bank, with these exchange rates not necessarily the same as the exchange rate agreed between the banks (see Appendix 3.6.5). As it turns out, the central bank is indifferent between exchange rates (due to its control over all interest rates). So, for simplicity, I assume a one-for-one exchange rate (the same as the interbank exchange rate); which means the net interest payments are the same whether expressed in units of goods or units of ES balances.

of consumers.

Therefore, the net profits transferred to each bank (from the asset managers) during day t will equal $\frac{1}{2}(1 - \theta_{t-1})[\alpha Y_t - P_{t-1}]$: where P_{t-1} is the cost of purchasing *all* of the shares from the lending bank at the previously agreed price, αY_t is the gross profit of *all* of the firms, and $(1 - \theta_{t-1})$ is the proportion of equity shares the lending bank sold to the asset managers at the end of day $t - 1$ ($\theta_{t-1} \in [0, 1]$).³⁴

For the non-lending bank, these interbank and central bank transactions, and the net profits of the asset managers, are the only transactions that impact its profit. Since all daily profits must be paid out to the consumers, the net dividend paid by the non-lending bank is (recall that consumers invest half of their savings in each bank):^{35,36}

$$\begin{aligned} (R_t^{D,j} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] &= (R^{ES} - 1) ES_{t-1}^j - (R_{t-1}^{O,j} - 1) O_{t-1}^j \\ &\quad - (R^F - 1) F_{t-1}^j + (R^S - 1) [S_{t-1}^j - S_{t-1}^i] \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] \end{aligned} \quad (3.9)$$

The lending bank derives additional profit from either its equity shares in the firms or from the sale of these shares to the asset managers. The net dividend paid by the lending bank is:

$$\begin{aligned} (R_t^{D,i} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] &= (R^{ES} - 1) ES_{t-1}^i - (R_{t-1}^{O,i} - 1) O_{t-1}^i \\ &\quad - (R^F - 1) F_{t-1}^i + (R^S - 1) [S_{t-1}^j - S_{t-1}^i] \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] - \vartheta I_{t-1} \\ &\quad + \{\theta_{t-1} [\alpha Y_t - K_t] + (1 - \theta_{t-1}) [P_{t-1} - K_t]\} \end{aligned} \quad (3.10)$$

³⁴The *volume* of shares issued by the firms is the same every day (as discussed in Appendix 3.6.5). So the price per share is P_{t-1} divided by the constant number of shares, while the return per share is αY_t divided by the constant number of shares.

³⁵Appendix 3.6.6 derives these dividend equations in greater detail.

³⁶With consumers investing half of their savings in each bank, $\frac{1}{2}\psi M(\bar{S})$ is the monitoring cost that each bank must pay, with $\psi M(\bar{S})$ being the aggregate monitoring cost.

The additional terms in the lending bank's dividend equation include the dividend payments from the firms (net of the originally purchased capital) for the shares that were not sold to the asset managers $\theta_{t-1}(\alpha Y_t - K_t)$, the payment for the shares (net of capital) that were sold to the asset managers $(1 - \theta_{t-1})(P_{t-1} - K_t)$, and the fixed effort cost (ϑ) multiplied by an indicator function equal to zero if all of the shares were sold to the asset managers, and one otherwise (i.e. $I_{t-1} = 0$ iff $\theta_{t-1} = 0$).

With the firms purchasing all of the available goods on the banks' balance sheets for use as future capital: $K_t = D_{t-1} - \psi M(\bar{S})$. Then, with consumers investing half of their savings in each bank, the gross daily return on consumers' bank shares is (from Eqs. (3.9) and (3.10)):

$$\begin{aligned} R_t^D [D_{t-1} - \psi M(\bar{S})] &= \frac{1}{2} [R_t^{D,i} + R_t^{D,j}] [D_{t-1} - \psi M(\bar{S})] \\ &= \alpha Y_t - C B_{t-1} - \vartheta I_{t-1} \end{aligned} \quad (3.11)$$

Where $C B_{t-1}$ is the net profit of the central bank:

$$\begin{aligned} C B_{t-1} &= (R_{t-1}^{O,i} - 1) O_{t-1}^i + (R_{t-1}^{O,j} - 1) O_{t-1}^j \\ &\quad + (R^F - 1) [F_{t-1}^i + F_{t-1}^j] - (R^{ES} - 1) [E S_{t-1}^i + E S_{t-1}^j] \\ &= (R_{t-1}^{O,i} - R^{ES}) O_{t-1}^i + (R_{t-1}^{O,j} - R^{ES}) O_{t-1}^j + \kappa [F_{t-1}^i + F_{t-1}^j] \end{aligned} \quad (3.12)$$

Where $\kappa > 0$ is the spread between the ES balance deposit rate and lending facility rate (i.e. $\kappa = R^F - R^{ES}$). And the bottom line of Eq. (3.12) comes from the fact that, in aggregate, banks can only acquire ES balances $(E S_{t-1}^i + E S_{t-1}^j)$ through either open market operations $(O_{t-1}^i + O_{t-1}^j)$ or through lending facility use $(F_{t-1}^i + F_{t-1}^j)$.

Open market operations (OMO)

Recall from Section 3.2.1.5, that the central bank operates so that both banks begin each day with the same expected end-of-day ES balance (excluding lending facility use). Appendices 3.6.6.6 and 3.6.6.7 show that this is achieved by lending half of the aggregate ES balance (excluding lending facility use) to each bank, each day. Therefore, denoting the fixed aggregate ES balance (excluding any lending facility use) as $\overline{ES}$ (where $\overline{ES} \geq 0$, consistent with the first two chapters), the central bank's daily OMO are:

$$O_t^i = O_t^j = \frac{1}{2}\overline{ES} \quad \forall \quad t \quad (3.13)$$

For the banks to agree to these OMO, the central bank must satisfy the banks' OMO participation constraints. Appendix 3.6.6.7 explains that this can be achieved by setting:

$$R_t^{O,i} = R_t^{O,j} = R^{ES} \quad \forall \quad t \quad (3.14)$$

Substituting Eqs. (3.13) and (3.14) into Eq. (3.12), gives:³⁷

$$CB_{t-1} = \kappa [F_{t-1}^i + F_{t-1}^j] \quad (3.15)$$

Banks' interbank lending/borrowing decisions

Interbank lending/borrowing decisions depend on expectations of production-period payment flows. With the parameter values and shock distribution used in this chapter, if there were no effort costs ($\vartheta = 0$) or monitoring costs ($\psi = 0$), banks' optimal ES balances at the end of the trading period (i.e. after accounting for expected payment flows) would be zero, the same as in Chapter 2 (see Appendix 3.6.6.8 for a proof).

The introduction of monitoring and effort costs would complicate matters; by pre-

³⁷Since $F_{t-1}^i + F_{t-1}^j \geq 0$, the central bank never makes a loss in this model.

venting the model from being solved recursively, and likely preventing the attainment of an analytical solution. So, to simplify the model, I assume each bank is split into two profit-maximising departments.

One department (known as the “Treasury”) makes ES balance lending/borrowing decisions only (i.e. OMO, interbank loans, and lending facility use), treats interbank payment flows as exogenous, and does not account for the impact of its decisions on the other parts of the bank’s operations (i.e. does not account for any effort or monitoring costs) – the Treasury is akin to the banks in chapters 1 and 2. The other department makes all other decisions, fully accounting for the endogeneity between the decisions of the two departments, and with complete knowledge of how the Treasury makes its decisions.³⁸

With this simplifying assumption, banks’ optimal ES balances (from the perspective of the Treasury) are the same as in Chapter 2 (i.e. the proof in Appendix 3.6.6.8 holds). Then, to further simplify the model, I assume banks’ Treasury departments bargain in the same way as Chapter 2 (in Chapter 2, it is these optimal ES balances that determine the bargaining solution).³⁹

During the day $t - 1$ production period, firms are expected to purchase $\frac{1}{2}E\left[K_t|\mathcal{I}_{t-2}^B\right]$ worth of goods from the non-lending bank (recall that each bank holds half of the goods available for the firms to use as next-day capital), which is expected to cause the equivalent value of ES balances to flow from the lending bank to the non-lending bank.^{40,41} If these payment flows are expected to cause the lending bank’s ES balance

³⁸Determining how accurately this assumption approximates the true behaviour of banks is left for future research. Appendix 3.6.6.8 provides some discussion about conditions that would ensure this approximation is sufficiently accurate for the qualitative conclusions of this chapter.

³⁹Appendix 3.6.6.8 provides more detail regarding the choice to use the Chapter 2 results (as opposed to the Chapter 1 results).

⁴⁰ $\mathcal{I}_{t-2}^B$ is the information set of the banks at the end of day $t - 2$. Since the value of payment flows are independent of which bank is the lending bank (due to consumers investing half of their savings in each bank), the expected payment flows derived during the trading period are the same as those that would be derived at the end of the previous day, so the information set $(\mathcal{I}_{t-2}^B)$ is the same.

⁴¹The banks agree on a one-for-one exchange rate between ES balances and goods (see Appendix 3.6.5). So payment flows are the same whether expressed in units of goods or units of ES balances.

to fall into deficit (i.e. if $\frac{1}{2}E[K_t|\mathcal{I}_{t-2}^B] > \frac{1}{2}\overline{ES}$), the lending bank will borrow ES balances, from the non-lending bank, sufficient to return its expected ES balance to zero (this is the Chapter 2 bargaining solution). Therefore (see Appendix 3.6.6.8 for a full derivation):

$$S_{t-1}^i = \begin{cases} \frac{1}{2}E[K_t|\mathcal{I}_{t-2}^B] - \frac{1}{2}\overline{ES} & \text{if } \frac{1}{2}E[K_t|\mathcal{I}_{t-2}^B] > \frac{1}{2}\overline{ES} \\ 0 & \text{otherwise} \end{cases} \quad (3.16)$$

With the corresponding gross interbank borrowing of the non-lending bank being: $S_{t-1}^j = 0$.

Lending facility use

Following the production period, if either of the banks has an ES balance deficit, they must borrow from the central bank's lending facility to remove this deficit. Any borrowing beyond this is discretionary.

Since the only benefit from holding positive ES balances overnight is the central bank's deposit rate, and since the deposit rate is less than the lending facility interest rate (i.e. $\kappa > 0$), there will be no discretionary borrowing from the lending facility by either bank's Treasury (consistent with Chapter 1).⁴²

With the parameter values and shock distribution used in this chapter, there will always be some interbank loans agreed during the trading period (i.e. expected payment flows will always be greater than banks' post-OMO ES balances).⁴³ As a result, the lending bank will finish the trading period with an expected ES balance of zero; so any unexpected payment flows to the non-lending bank (i.e. if consumers save more than expected; $K_t > E[K_t|\mathcal{I}_{t-2}^B]$) will require the lending bank to borrow from the central

⁴²With both banks' daily borrowing via OMO, and the rate at which this borrowing occurs, being independent of their ES balance at the end of the previous day, banks' expected end-of-day ES balances (evaluated at the beginning of each day) are independent of their ES balances on any previous days – so the only benefit from holding positive ES balances overnight is the central bank's deposit rate.

⁴³More accurately, there is an infinitesimally small probability of consecutive shocks large enough to cause expected payment flows to fall below post-OMO ES balances.

bank's lending facility. Since consumers invest half of their savings in each bank, unexpected payment flows will equal half of any unexpected savings (see Appendix 3.6.6.9 for a mathematical derivation):

$$F_{t-1}^i = \begin{cases} \frac{1}{2}K_t - \frac{1}{2}\mathbb{E}[K_t|\mathcal{I}_{t-2}^B] & \text{if } K_t > \mathbb{E}[K_t|\mathcal{I}_{t-2}^B] \\ 0 & \text{otherwise} \end{cases} \quad (3.17)$$

The non-lending bank finishes the trading period with an expected ES balance equal to the aggregate. So the difference between actual payment flows and expected payment flows would need to be sufficiently negative (larger than the aggregate ES balance, in absolute value) for the non-lending bank to access the lending facility:

$$F_{t-1}^j = \begin{cases} -\overline{ES} - \frac{1}{2}[K_t - \mathbb{E}[K_t|\mathcal{I}_{t-2}^B]] & \text{if } \overline{ES} < \frac{1}{2}\mathbb{E}[K_t|\mathcal{I}_{t-2}^B] - \frac{1}{2}K_t \\ 0 & \text{otherwise} \end{cases} \quad (3.18)$$

Importantly, both positive and negative shocks can lead to borrowing from the lending facility. But increasing $\overline{ES}$ only reduces the proportion of negative shocks that lead to lending facility use (for a given $K_t - \mathbb{E}[K_t|\mathcal{I}_{t-2}^B]$).

3.2.3.4 Simplified set of equations for day t

From the consumers' daily optimisation problem, there are three equations (including the budget constraint):

$$\mathbb{E}\left[\frac{R_{t+1}^D}{C_{t+1}} \middle| \mathcal{I}_t^C\right] = \frac{1}{\beta C_t} \quad (3.19)$$

$$C_t = \frac{W_t}{\chi} \quad (3.20)$$

The budget constraint (Eq. (3.6)) can be simplified, in equilibrium, by substituting in Eqs. (3.11) and (3.7), and using the condition $(K_{t+1} = D_t - \psi M(\overline{S}))$:

$$Y_t = C_t + K_{t+1} + \psi M(\bar{S}) + \vartheta I_{t-1} \quad (3.21)$$

From the firms' profit-maximisation problem, labour demand satisfies:

$$W_t H_t = (1 - \alpha) Y_t \quad (3.22)$$

Where the production function and stochastic technology process are:

$$Y_t = A_t H_t^{1-\alpha} K_t^\alpha \quad (3.23)$$

$$\ln(A_t) = \ln(\bar{A}) + \varepsilon_t \quad (3.24)$$

From the banking system, dividend payments to the consumers during day t are (from Eqs. (3.11) and (3.15)):

$$R_t^D K_t = \alpha Y_t - \kappa [F_{t-1}^i + F_{t-1}^j] - \vartheta I_{t-1} \quad (3.25)$$

From Section 3.2.3.3, banks' aggregate gross interbank lending is (let $S_t \equiv S_t^i + S_t^j$):

$$S_t = -\min \left\{ \frac{1}{2} [\bar{ES} - E[K_{t+1} | \mathcal{I}_{t-1}^B]], 0 \right\} \quad (3.26)$$

From Section 3.2.3.3, banks' aggregate use of the lending facility is (for ease of notation in the next equation let $EK_{t+1} \equiv E[K_{t+1} | \mathcal{I}_{t-1}^B]$ and $F_t \equiv F_t^i + F_t^j$):

$$F_t = \begin{cases} \frac{1}{2} [K_{t+1} - EK_{t+1}] & \text{if } K_{t+1} > EK_{t+1} \\ -\bar{ES} - \frac{1}{2} [K_{t+1} - EK_{t+1}] & \text{if } \bar{ES} < \frac{1}{2} EK_{t+1} - \frac{1}{2} K_t \\ 0 & \text{otherwise} \end{cases} \quad (3.27)$$

From all of these equations, when solving the model in day t :

- The pre-determined variables are: K_t , F_{t-1} , and I_{t-1} .
- The parameters $(\alpha, \beta, \chi, \psi, \vartheta, \kappa, \bar{ES}, \bar{A})$ and the technology shock (ε_t) are

exogenous, while steady state aggregate interbank lending $(\bar{S})$ is uniquely determined by the parameters (see Section 3.2.4).

- The choice variables are: C_t , H_t , K_{t+1} , S_t , F_t , and I_t .
- With endogenous A_t , R_t^D , and Y_t , and with W_t determined from equilibrium in the competitive labour market.

At present, the model is under-determined (10 choice/endogenous variables and only 9 equations). However, as will be shown in Section 3.2.5, none of Eqs. (3.19)-(3.27) depend on I_t (recall that $I_t = 0$ iff the lending bank sells all of its shares in the firms to the asset managers, $I_t = 1$ otherwise). Therefore, I_t can be determined given all the other choice and endogenous variables – Section 3.2.5.1 describes how this is done. As a result, the model will be exactly-determined.

With none of Eqs. (3.19)-(3.27) dependent on either R^{ES} or R^F , only the spread between these rates (κ) impacts the model. Therefore, $\bar{ES}$ and κ are the only policy parameters available to the central bank.

Due to the monitoring cost, effort cost, and the cost of lending facility use, there is no exact analytical solution to this model. So I derive an approximate solution by: deriving the unique deterministic steady state of the model (Section 3.2.4); deriving first-order approximations of the model equations (where necessary) around the deterministic steady state (Section 3.2.5); and finally, with the Blanchard-Kahn conditions satisfied, deriving the unique stable decision rules that satisfy these first-order approximations (Section 3.2.5).

3.2.4 Deterministic steady state

Let a bar over the variable and the removal of the time subscript denote a deterministic steady state variable. In a deterministic steady state, there is no technology shock. This means that $\bar{K} = E[\bar{K}]$ and that there is no use of the central bank's lending facility.

With no technology shock, there is no asymmetric information between the banks and the asset managers. Therefore, the equity shares are always sold to the asset managers (i.e. $\bar{I} = 0$).

With no lending facility use and $\bar{I} = 0$, then $\bar{R}^D$, $\bar{W}$, $\bar{C}$, $\bar{H}$, and $\bar{Y}$, can be written as functions of the parameters and $\bar{K}$ (using the equations in Section 3.2.3.4, see Appendix 3.6.7). Due to the monitoring cost, $\bar{K}$ and $\bar{S}$ are interdependent, and dependent on the functional form of the monitoring cost function ($M(\cdot)$). The monitoring cost function used in this chapter is given in Appendix 3.6.7; the important features of this functional form are (for all $\bar{ES} \geq 0$):

- For each setting of $\bar{ES}$, there is a unique $\bar{K}$ and $\bar{S}$.
- Increasing $\bar{ES}$ causes $\bar{K}$ to increase, but at a slower rate than $\bar{ES}$ ($0 < \frac{d\bar{K}}{d\bar{ES}} < 1$). This means that increasing $\bar{ES}$ causes steady state interbank lending to fall ($\frac{d\bar{S}}{d\bar{ES}} < 0$) – consistent with the first two chapters. However, there is always some interbank lending in steady state ($\bar{S} > 0$).
- Decreasing $\bar{S}$ causes the monitoring cost to increase ($\frac{dM(\bar{S})}{d\bar{S}} < 0$) – consistent with the statements of *Keister et al.* (2008) and *Heath et al.* (2013). So the monitoring cost is minimised at $\bar{ES} = 0$; at which point $M(\bar{S}) = 0$.

3.2.5 First-order approximate solution

As is the case in standard DSGE models, I assume that consumers are sufficiently sophisticated to understand the workings of the real economy (i.e. they know and understand Eqs. (3.19) - (3.25)). Therefore, consumers can use this information when constructing expectations of the future.⁴⁴ However, the consumers are not privy to the banks' day t operational decisions (such as lending facility use or sales to the asset managers), and I assume that they are not sufficiently sophisticated to understand

⁴⁴This assumption is necessary to avoid indeterminacy.

how the banks make these decisions. Instead, when constructing expectations to make day t choices, consumers form unbiased and stationary unconditional-expectations of the necessary banking system variables.^{45,46}

With these assumptions, and using the simplified set of equations in Section 3.2.3.4, equations can be derived for C_t , Y_t , R_t^D , W_t , A_t , S_t , and F_t , that are each only dependent on: the parameters, exogenous variables, pre-determined variables, H_t , and K_{t+1} . Then, using the same equations, equations can be derived for H_t and K_{t+1} that are only dependent on each other, the parameters, exogenous variables, and pre-determined variables. Finally, since none of the aforementioned day t choice/endogenous variables depend on I_t , an algorithm can be derived to determine I_t (with this algorithm treating all other day t variables as pre-determined).

Therefore, the model can be solved recursively in three stages. In the first stage, the system of equations that determine H_t and K_{t+1} is solved to a first-order approximation (see Appendix 3.6.9). In the second stage, the solution for H_t and K_{t+1} is used to determine the values of the remaining choice/endogenous variables (except I_t). Finally, the value of I_t is determined (see Section 3.2.5.1).

From Appendix 3.6.9, with the first-order approximate system of equations that determine H_t and K_{t+1} satisfying the Blanchard-Kahn conditions, the unique stable decision rules that satisfy this system are:⁴⁷

$$h_t = -N_1 [\alpha k_t + \varepsilon_t] + N_2 I_{t-1} - N_3 E[F | \mathcal{I}^C] - N_4 E[I | \mathcal{I}^C] \quad (3.28)$$

$$k_{t+1} = \alpha k_t + \varepsilon_t - N_5 I_{t-1} - N_6 E[F | \mathcal{I}^C] - N_7 E[I | \mathcal{I}^C] \quad (3.29)$$

⁴⁵The consumers cannot just ignore the unknown variables, as this would lead to systematic over-estimation of the returns on their bank shares.

⁴⁶While it may be unrealistic to assume such sophisticated understanding of the real economy but not the banking system, this assumption greatly simplifies the model (by making the day t decisions of consumers independent of the day t banking system decisions).

⁴⁷While I do not determine the accuracy of these first-order approximations, with the shock only representing small unexpected late-day payment flows, this model will remain in a small region around the steady state.

Where:

- lower case variables denote percentage deviations from their deterministic steady state (e.g. $h_t = \left(\frac{H_t - \bar{H}}{\bar{H}}\right)$ and $\left(\frac{A_t - \bar{A}}{\bar{A}}\right) \approx \ln(A_t) - \ln(\bar{A}) = \varepsilon_t$;
- $N_1 - N_7$ are non-negative reduced-form parameters (see Appendix 3.6.9 for their relationship with the model's structural parameters); and
- $E[I|\mathcal{I}^C]$ and $E[F|\mathcal{I}^C]$ are the unbiased and stationary unconditional-expectations of I_t and F_t , respectively, formed by the consumers.⁴⁸

Assuming the banks know how the consumers form their expectations, at the end of day $t - 1$, the only term in Eq. (3.29) unknown to the banks is ε_t . Therefore:

$$k_{t+1} - E[k_{t+1}|\mathcal{I}_{t-1}^B] = \varepsilon_t \quad (3.30)$$

Substituting Eq. (3.30) into Eq. (3.27) gives:

$$F_t = \begin{cases} \left[\frac{\bar{K}}{2}\right] \varepsilon_t & \text{if } \varepsilon_t > 0 \\ -\left[\frac{\bar{K}}{2}\right] \varepsilon_t - \bar{ES} & \text{if } \varepsilon_t < \frac{-2\bar{ES}}{\bar{K}} \\ 0 & \text{if } \frac{-2\bar{ES}}{\bar{K}} \leq \varepsilon_t \leq 0 \end{cases} \quad (3.31)$$

Exactly how unexpected payment flows lead to lending facility use is discussed in Section 3.2.3.3, and is not repeated here. But there are several important takeaways from expressing aggregate lending facility use in the Eq. (3.31) format:

- Eq. (3.31) is independent of $\mathcal{I}_{t-1}^B$. So consumers' unbiased and stationary unconditional expectation of F_t (i.e. $E[F|\mathcal{I}^C]$), can be derived by using the probability distribution of ε_t and Eq. (3.31) (see Appendix 3.6.10).
- Positive shocks always cause lending facility use, while only some negative shocks cause facility use. With a symmetric shock (as is the case in this chapter), and

⁴⁸The consumers are able to derive I_{t-1} from their daily dividends (see Eq. (3.25)) and the lump-sum tax/subsidy levied by the central bank (CB_{t-1}).

since a positive shock increases the equilibrium value of the firms' expected day $t+1$ profits, Eq. (3.31) means that lending facility use ($F_t > 0$) is a positive signal of asset quality (relative to when $F_t = 0$); so it can only be when lending facility use is combined with adverse selection that the signal may become negative.

- Increasing $\overline{ES}$ reduces the proportion of negative shocks that cause lending facility use. So it may be possible for the central bank to alleviate the stigma problem. In fact, if the shock is bounded (as it is in this chapter), a sufficiently high $\overline{ES}$ will eliminate lending facility stigma (since there will be a point at which only positive shocks lead to facility use).

Exogenous shock probability distribution

To be consistent with Chapter 2, the shock in this model has an atom at zero. That is: $P(\varepsilon_t = 0) = \varphi$, where $0 < \varphi < 1$. Then, with probability $1 - \varphi$, the shock follows a continuous uniform probability distribution with bounds $[-B, B]$ (where $B > 0$).⁴⁹

For the qualitative results in this chapter, the exact value of B is not important. However, B must be set sufficiently small such that interbank and central bank lending is essentially risk-free (i.e. so that the probability of consecutive shocks sufficiently large to cause default is very close to zero), and to ensure there is always some interbank lending when $\overline{ES}$ is set to a realistic level (consistent with reality). Moreover, with this shock representing unknown late-day payment flows (consistent with the first two chapters), B must be small relative to the expectations of payment flows constructed during the trading period. The setting of B is detailed in Section 3.2.7.

3.2.5.1 Determining I_t

Recall, from Section 3.2.3.3 that:

⁴⁹For the model in this chapter, a continuous uniform distribution for the non-zero shocks is easier to work with than the discrete shock distribution in Chapter 2.

- $(1 - \theta_t)$ is the proportion of equity shares the lending bank sells to the asset managers at the end of day t ($\theta_t \in [0, 1]$);
- $I_t = 0$ iff $\theta_t = 0$ (with $I_t = 1$ otherwise);
- P_t is the cost of purchasing *all* of the shares from the lending bank at the end of day t ;
- αY_{t+1} is the gross profit of the firms during day $t + 1$ (i.e. the value of *all* the shares); and
- each share is identical (see Appendix 3.6.5).

Assume asset sales occur in a public market; so the terms of all trades are public knowledge. Then with identical shares, and a continuum of identical asset managers (whose only marginal cost is the purchase price), any shares that are sold to the asset managers must all sell for the same price (in equilibrium) – at this price, the cost of purchasing all of the shares would be P_t .

Moreover, if $(1 - \theta_t)$ of shares are sold to the asset managers in equilibrium, then this $\{\theta_t, P_t\}$ combination must be the most profitable deal the lending bank can feasibly obtain. This is because, in equilibrium, there can be no asset manager that can profitably offer the lending bank a more profitable $\{\theta_t, P_t\}$ combination, and the lending bank must not be able to increase its profit by exiting the market and not selling any shares ($\theta_t = 1$).

Since the lending bank observes the day t shock (but the asset managers do not), the ability of the lending bank to exit the market and not sell any shares allows the asset managers to extrapolate information about the shock from the equilibrium θ_t and P_t that occur in the market. For example, for $\theta_t < 1$, the asset managers are able to determine the shocks for which it would be more profitable for the lending bank to have exited the market (so they know these shocks must not have occurred). Therefore,

θ_t and P_t impact the asset managers expectations of firms' day $t + 1$ profits, thereby impacting their equilibrium expected return on any purchased shares, and subsequently the equilibrium market outcome.

Given their knowledge of Eq. (3.31), the asset managers also know which shocks lead to use of the lending facility. So the asset managers condition their expectations of firms' day $t + 1$ profits on their observation of whether the lending facility was used (recall that the asset managers only observe whether $F_t > 0$ or $F_t = 0$, and do not observe which bank accessed the facility). This allows stigma to develop.

Appendix 3.6.11 derives the equilibrium values of θ_t and P_t . Since these variables only impact the model via their impact on I_t , it is only necessary to show how I_t is determined:

Theorem 3.1. *Let $\mathbb{F}_t$ denote the subset of potential day t shocks that are feasible given the observed facility use (i.e. $\mathbb{F}_t \subset [-B, B]$). Then let $\varepsilon_t^{\dagger\dagger}$ be the largest feasible value of $\varepsilon_t^{\dagger}$ such that the following holds:*

$$E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\dagger}] \geq \varepsilon_t^{\dagger} - \mathbb{C}\vartheta \quad (3.32)$$

Where feasible $\varepsilon_t^{\dagger}$ values are all points within $\mathbb{F}_t$ and the supremum of any segment of $\mathbb{F}_t$ ($\mathbb{F}_t$ may be non-convex and have segments with open upper bounds).⁵⁰ And where $\mathbb{C} > 0$ is a reduced-form parameter dependent only on the structural parameters of the model (see Appendix 3.6.11 for details).

Then, given that $\varepsilon_t \in \mathbb{F}_t$, the subset of $\mathbb{F}_t$ that leads to $I_t = 0$ can be approximated by: $I_t = 0$ iff the day t value of the shock satisfies $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$. $I_t = 1$ otherwise.

Proof. See Appendix 3.6.11. □

The subset of $\mathbb{F}_t$ that leads to $I_t = 0$, in Theorem 3.1, is only an approximation of the true subset because, in the proof, the firms' production function is replaced with a first-order Taylor approximation around the deterministic steady state. As long as

⁵⁰Appendix 3.6.11 shows that there is always at least one feasible $\varepsilon_t^{\dagger}$ that satisfies Eq. (3.32).

the approximate subset and true subset lead to the same qualitative conclusions, the approximate subset is sufficient for this chapter. I use the approximate subset because it ensures a simple analytical solution for $\varepsilon_t^{\dagger\dagger}$ can always be found (see Appendix 3.6.11), and leads to simple analytical expressions for $P(I_t = 1)$ that would not be available without the approximation. For the remainder of this chapter, I treat the approximate subset as if it were the true subset.

To explain Theorem 3.1 in more detail: if $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$, then there exists a P_t at which the lending bank is willing to sell all of its shares and at which the asset managers are willing to buy them (so $\theta_t = 0$ and $I_t = 0$). If $\varepsilon_t > \varepsilon_t^{\dagger\dagger}$, then there is no P_t at which both the lending bank is willing to sell all of its shares and at which the asset managers are willing to buy them (due to asymmetric information); so $I_t = 1$.

$E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\dagger}]$ is derived from the unconditional shock distribution (detailed in Section 3.2.5), updated for the conditioning information using Bayes' rule (this process is detailed in Appendix 3.6.11). Therefore, $\varepsilon_t^{\dagger\dagger}$ depends only on the parameter values and whether the central bank's lending facility was used – with the dependence on lending facility use opening an avenue for stigma (an algorithm for determining $\varepsilon_t^{\dagger\dagger}$ is given in Appendix 3.6.11).

3.2.6 Adverse selection, stigma, and the aggregate ES balance

Adverse selection

From Eq. (3.32), asymmetric information between the banks and asset managers may cause an adverse selection problem (for a sufficiently small effort cost (ϑ)).⁵¹

To see why, suppose $\vartheta = 0$ and suppose $\varepsilon_t^{\dagger} > \min(\mathbb{F}_t)$, then $E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\dagger}] < \varepsilon_t^{\dagger}$. Therefore, this $\varepsilon_t^{\dagger}$ would not satisfy Eq. (3.32). The only $\varepsilon_t^{\dagger}$ that would satisfy Eq. (3.32) is $\varepsilon_t^{\dagger} = \min(\mathbb{F}_t)$; resulting in the shares being sold to the asset managers *iff*

⁵¹For the purposes of this thesis, adverse selection does not only refer to banks only offering assets of a sufficiently low quality (for a given price), but also refers to the resulting lower price and lower average quality of assets sold in equilibrium.

the day t shock is equal to its lowest possible value (for the given facility use). That is, the market for equity shares suffers from Akerlof's lemons problem (*Akerlof*, 1970).

Now suppose $\vartheta > 0$. Then some $\varepsilon_t^\dagger > \min(\mathbb{F}_t)$ will satisfy Eq. (3.32); larger values of ϑ support larger values of $\varepsilon_t^\dagger$ (with a sufficiently large ϑ , there is no adverse selection problem).

Adverse selection causing lending facility stigma

Although the average shock value that leads to lending facility use is at least as high as the average shock value that doesn't (see Eq. (3.31)), since asymmetric information can cause adverse selection in this model, there may be some prices at which the expected quality of assets that would be offered for sale would be lower following facility use. At these prices, lending facility use is a negative signal of asset quality (stigma). If the market outcome following facility use is one of these prices, then this stigma may lead to fewer asset sales, negatively impacting welfare.

In relation to the model, lending facility stigma occurs in equilibrium when ϑ is sufficiently small such that lending facility use ($F_t > 0$) causes $\varepsilon_t^{\dagger\dagger}$ to be lower than if the lending facility had not been used ($F_t = 0$). If this stigma leads to fewer asset sales, welfare will be negatively impacted.

An example of a sequence of events in which lending facility use causes the lending bank to keep the shares, is as follows:

- A shock causes one of the banks to have fewer ES balances than it expected. If this bank's level of ES balances following the shock is negative, this bank will access the central bank's lending facility (Eq. (3.31)).
- Since the asset managers know which shocks could have caused the lending facility to have been accessed ($\mathbb{F}_t$), the fact that the facility was accessed changes the asset managers' expectation of the shock value (i.e. $E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t] \neq E[\varepsilon_t]$), and therefore their valuation of the shares the lending bank may try to sell.

- Since the lending bank does not need to sell the shares, the action of offering the shares for sale (and the price at which they are offered) provides even more information to the asset managers about the value of the shock.
- In equilibrium, any sale of the shares must occur at a price at which the bank is willing to sell and at which the asset managers are willing to buy (with the equilibrium price impacting the asset managers conditional expectation of the shock). From Theorem 3.1, if the shock is sufficiently large (i.e. $\varepsilon_t > \varepsilon_t^{\dagger\dagger}$), then no such price exists, and the lending bank does not sell the shares to the asset managers.
- Therefore, lending facility use prevents the shares from being sold if the value of $\varepsilon_t^{\dagger\dagger}$ following lending facility use is lower than if the lending facility had not been used (i.e. stigma), and if the shock that caused the facility to be used is too large for the shares to be sold (i.e. $\varepsilon_t > \varepsilon_t^{\dagger\dagger}$).⁵²

Relationship between the aggregate ES balance and lending facility stigma

As can be seen from Eq. (3.31), increasing $\overline{ES}$ reduces the proportion of negative shocks that can lead to lending facility use; thereby reducing the proportion of shocks that lead to stigma. In fact, when $\overline{ES} \geq \frac{B\overline{K}}{2}$, only positive shocks lead to lending facility use (recall that B is the upper bound of the shock distribution), and increases in $\overline{ES}$ beyond $\frac{B\overline{K}}{2}$ have no impact on the lending facility signal.

Conversely, increases in $\overline{ES}$ beyond $\frac{B\overline{K}}{2}$ would still increase the monitoring cost, and would still increase the interest cost of using the lending facility (by increasing $\overline{K}$); both of which have a negative impact on welfare (as will be explained in more detail in Section 3.3). Therefore, I can restrict my search for the welfare-optimising $\overline{ES}$ to $\overline{ES} \in \left[0, \frac{B\overline{K}}{2}\right]$.

⁵²With the calibrations used in this chapter, the $\varepsilon_t^{\dagger\dagger}$ that occurs when the lending facility is not used will be sufficiently high such that the shares are always sold.

While increasing $\overline{ES}$ to $\frac{BK}{2}$ eliminates any lending facility stigma, when $\overline{ES} < \frac{BK}{2}$ the existence of lending facility stigma depends on the other parameters. For the parameter values used in this chapter, $\varepsilon_t^{\dagger\dagger} < 0$ (when $F_t > 0$) for the majority of the $0 \leq \overline{ES} < \frac{BK}{2}$ region, leading to lending facility stigma.⁵³ Within this $\varepsilon_t^{\dagger\dagger} < 0$ region, and with the parameter values used in this chapter, the model produces the following results (see Appendix 3.6.11 for a detailed derivation):

- Increasing $\overline{ES}$, decreases the value of $\varepsilon_t^{\dagger\dagger}$ that occurs when a shock causes the lending facility to be accessed (i.e. $F_t > 0$). If the set of shocks that lead to lending facility use did not change with the change in $\overline{ES}$, the lower value of $\varepsilon_t^{\dagger\dagger}$ would mean that there are some values of ε_t for which I_t would have equalled zero under the old value of $\overline{ES}$, but will equal one (i.e. shares not sold to the asset managers) under the new value of $\overline{ES}$.
- However, the range of shocks that lead to $F_t = 0$ increases with an increase in $\overline{ES}$ (see Eq. (3.31)), with the shares always sold to the asset managers when there is no use of the lending facility.
- The later impact dominates, so increasing $\overline{ES}$ has a non-positive impact on the probability (before realisation of the shock) of shares not being sold to the asset managers (i.e. $\frac{dP(I_t=1)}{d\overline{ES}} \leq 0$) – with this impact being strictly negative for the majority of the $0 \leq \overline{ES} < \frac{BK}{2}$ region (i.e. by reducing the prevalence of stigma, increasing $\overline{ES}$ increases asset sales).

This is an important result, as it means stigma need not be completely eliminated for there to be a welfare benefit to increasing $\overline{ES}$; so welfare maxima in the interior of $\left[0, \frac{BK}{2}\right]$ are possible.

⁵³If $\varepsilon_t^{\dagger\dagger} < 0$ when $F_t > 0$, this $\varepsilon_t^{\dagger\dagger}$ must be less than the $\varepsilon_t^{\dagger\dagger}$ when $F_t = 0$ (see Eq. (3.31)); so there is stigma.

3.2.7 Calibration and parameterisation

The parameters in this model are: α , β , χ , ψ , ϑ , κ , $\overline{ES}$, $\overline{A}$, φ , and B ; most of these parameters are calibrated using Australian data. While a detailed description of the calibration is given in Appendix 3.6.12, there are two points worth noting here:

- From Chapter 1, the typical pre-crisis aggregate ES balance (adjusted for a two bank model) was A\$30 million. Moreover, according to the Reserve Bank of Australia (RBA), there is no stigma associated with lending facility use in Australia (*Debelle*, 2008). Since there is no lending facility stigma in this model when $\overline{ES} = \frac{BK}{2}$, I calibrate B (the upper bound of the shock) so that $\overline{ES} = 30$ when $\overline{ES} = \frac{BK}{2}$.
- When $\overline{ES} = 30$, only positive shocks require borrowing from the lending facility. Since the probability of a positive shock is $\frac{1-\varphi}{2}$, I calibrate φ so that the probability of lending facility use in the model (when $\overline{ES} = 30$) is the same as the probability of lending facility use in the Australian market (adjusted for a two-bank model, see Appendix 1.7.13). This results in $\varphi = 0.9964$.

The parameters that are not calibrated using Australian data are ψ (the monitoring cost scaling parameter) and ϑ (the effort cost). Since I have no data to calibrate these parameters, Section 3.4 shows how varying these parameters impacts the qualitative results of this chapter.

Since lending facility stigma is a key feature of this model, I set ϑ sufficiently small such that there is lending facility stigma for the majority of $\overline{ES} \in [0, 30]$. On the other hand, if ϑ is too small, there may be an adverse selection problem when the lending facility is not used. With rare lending facility use (in Australia), adverse selection following no facility use would be more costly than lending facility stigma, resulting in the central bank never eliminating lending facility stigma (in this model). Since this is inconsistent with Australia, I restrict ϑ to be sufficiently large such that there is no

adverse selection problem when the lending facility is not used.

The parameters calibrated to Australian data, and a parameterisation of ψ and ϑ that ensures the welfare-optimising aggregate ES balance is close to $\overline{ES} = 30$, will collectively be known as the “baseline” parameter values.

With the baseline parameter values, unexpected flows produced by the shock are small relative to the level of payment flows (the largest possible unexpected flows are around 3.12 per cent of expected flows); as is required given the interpretation of the shock as unknown late-day payment flows. Also, with the probability of a negative shock on each day being just 0.18 per cent (due to $\varphi = 0.9964$), and with the short half-life of the impact of shocks (2 days), there is an infinitesimally small risk of banks defaulting on their loans, and a similarly small probability of the interbank market drying-up. This justifies the simplifying assumptions that bank loans are risk-free and that there is always some interbank lending.

3.3 Welfare

With the model solved and calibrated, this section: derives the welfare function; details how the monitoring cost, adverse selection, and lending facility use can impact welfare; and shows how the central bank can use its policy parameters to change these costs and maximise welfare.

After explaining the impacts several of the non-policy parameters have on the welfare-maximising aggregate ES balance, I show (in Section 3.4) that, even with the majority of parameters calibrated to Australian data, the freedom of ψ and ϑ can lead to different qualitative conclusions with respect to the welfare-maximising aggregate ES balance. However, these qualitative conclusions include welfare-maximising aggregate ES balances that are greater than banks’ desired end-of-day balances but small relative to aggregate interbank lending, consistent with, and providing an explanation

for, central bank practice.

From Eq. (3.1), the expected discounted lifetime utility of the representative consumer at the end of day t (henceforth, welfare) is:

$$\mathbb{W}_t \equiv \mathbb{E} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} [\ln(C_s) - \chi H_s] \right\}$$

Using the set of equations that defines the model (Section 3.2.3.4), welfare can be expressed as a function of the parameters, H_s , K_s , and ε_s . Then, using the decision rules from Section 3.2.5, welfare can be approximated by a function of the parameters, day t shock (ε_t), day t state variables (k_t and I_{t-1}), and I_t (see Appendix 3.6.13 for a detailed derivation).

Suppose the central bank sets the value of $\overline{ES}$ to maximise the welfare of the representative consumer. Since the welfare-optimising $\overline{ES}$ at day t will differ depending on the values of k_t , ε_t , I_{t-1} , and I_t , I assume the central bank sets $\overline{ES}$ at day zero to maximise welfare evaluated at some point in the future (day t) when the economy is at its stochastic steady state.⁵⁴ When the economy is at its stochastic steady state at day t , welfare can be approximated by (see Appendix 3.6.13 for a detailed derivation):

$$\begin{aligned} \mathbb{W}_t \approx & \left[\frac{1}{1-\beta} \right] \left\{ G_1 - G_2 \vartheta \mathbb{E}[I|\mathcal{I}^C] \right\} \\ & - G_3 \left\{ \vartheta \mathbb{E}[I|\mathcal{I}^C] \right\} - G_4 \left\{ \kappa \mathbb{E}[F|\mathcal{I}^C] \right\} \end{aligned} \quad (3.33)$$

Where $G_1, G_2 > 0$, $G_3 > 0$, and $G_4 > 0$ are reduced-form parameters that depend on $\overline{ES}$ through $\overline{K}$ (Appendix 3.6.13 details how these parameters relate to the structural parameters of the model). Importantly, none of these reduced-form parameters include a $\frac{1}{1-\beta}$ term. Since $1 - \beta = 0.00008$, expressing welfare as in Eq. (3.33) highlights that terms multiplied by $\frac{1}{1-\beta}$ have a much larger impact on welfare than the other terms.

Since $\mathbb{E}[F|\mathcal{I}^C] > 0$ (see Appendix 3.6.10), and $\mathbb{E}[I|\mathcal{I}^C] > 0$ (see Appendix 3.6.11), k_t and h_t will not be zero at the stochastic steady state of the economy.⁵⁵ That being

⁵⁴Since $\overline{ES}$ is modelled as a fixed parameter, this model is not capable of evaluating the impact of a day t shift in $\overline{ES}$.

⁵⁵Since I_t is an indicator function, $\mathbb{E}[I|\mathcal{I}^C] = \mathbb{P}(I = 1)$ (see Appendix 3.6.11 for derivation of

said, as long as $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$ are both close to zero, the deterministic steady state of K and H will remain a close approximation of the stochastic steady state.⁵⁶ With the baseline parameter values, the largest difference between the deterministic and stochastic steady states of K and H is just 0.000009 per cent of the deterministic steady state.

Differentiating welfare with respect to $\overline{ES}$ gives:

$$\begin{aligned} \frac{d\mathbb{W}_t}{d\overline{ES}} \approx & -\left[\frac{1}{1-\beta}\right] \left\{ G_5 \frac{d\overline{K}}{d\overline{ES}} + G_2 \vartheta \frac{dE[I|\mathcal{I}^C]}{d\overline{ES}} \right\} \\ & -G_3 \left\{ \vartheta \frac{dE[I|\mathcal{I}^C]}{d\overline{ES}} \right\} - G_4 \left\{ \kappa \frac{dE[F|\mathcal{I}^C]}{d\overline{ES}} \right\} \\ & +G_6 \left\{ \vartheta E[I|\mathcal{I}^C] + \kappa E[F|\mathcal{I}^C] \right\} \frac{d\overline{K}}{d\overline{ES}} \end{aligned} \quad (3.34)$$

Where $G_5 > 0$ and $G_6 > 0$ are reduced-form parameters (see Appendix 3.6.13 for details); neither of which include a $\frac{1}{1-\beta}$ term.

From Eq. (3.34), changes in the aggregate ES balance have three different impacts on steady state welfare:

- $\frac{d\overline{K}}{d\overline{ES}}$ represents the impact a change in $\overline{ES}$ has on the model via its impact on the deterministic steady state (which only occurs via the impact $\overline{ES}$ has on the monitoring cost);
- $\frac{dE[I|\mathcal{I}^C]}{d\overline{ES}}$ represents the impact a change in $\overline{ES}$ has on the probability that banks will be unable to sell their equity claims (mainly via the impact $\overline{ES}$ has on the probability that banks will use the lending facility); and
- $\frac{dE[F|\mathcal{I}^C]}{d\overline{ES}}$ represents the impact a change in $\overline{ES}$ has on the expected amount that the banks borrow from the central bank's lending facility.

$P(I = 1)$.

⁵⁶ Having the deterministic steady state closely approximate the stochastic steady state is particularly important for the monitoring cost, which, for simplicity, is based on the volume of interbank lending in the deterministic steady state.

Within the region of $\left[0, \frac{B\bar{K}}{2}\right]$ in which $\varepsilon_t^{\dagger\dagger} < 0$ when $F_t > 0$ (i.e. a region in which there is lending facility stigma), $\bar{K}$, $E[I|\mathcal{I}^C]$, and $E[F|\mathcal{I}^C]$ are all continuous with respect to $\bar{ES}$. Moreover, within this region, the derivatives of $\bar{K}$, $E[I|\mathcal{I}^C]$, and $E[F|\mathcal{I}^C]$ exist (almost everywhere), and have the following signs: $\frac{d\bar{K}}{d\bar{ES}} > 0$, $\frac{dE[I|\mathcal{I}^C]}{d\bar{ES}} \leq 0$, while $\frac{dE[F|\mathcal{I}^C]}{d\bar{ES}}$ can be either positive or negative (see Appendices 3.6.8, 3.6.11 and 3.6.10 for further details).⁵⁷ Therefore, whether a maximum exists (since this $\varepsilon_t^{\dagger\dagger} < 0$ region is a half-open set) and, if it exists, whether it is an interior or boundary maximum, will depend on the parameter values. In Section 3.4, I give examples of parameter values for which the maximum is at the lower bound (zero), close to the upper bound, and far from either bound.

To give some intuition about how the various costs impact welfare, and why there is a trade-off between stigma and monitoring costs, the welfare-maximising $\bar{ES}$ in some limit scenarios are:

- As $\psi \rightarrow 0$ (i.e. when the monitoring cost becomes very small), $\frac{d\bar{K}}{d\bar{ES}} \rightarrow 0$ and Eq. (3.34) becomes positive. Intuitively, when the monitoring cost becomes sufficiently small, the impact $\bar{ES}$ changes have on welfare is dominated by the ability of an increase in $\bar{ES}$ to reduce the probability that a bank will need to use the lending facility; thereby reducing both the expected interest cost and the probability that the bank will be unable to sell its shares.⁵⁸ In this case, the maximum point is not within the $\varepsilon_t^{\dagger\dagger} < 0$ region, but at the upper bound $\left(\bar{ES} = \frac{B\bar{K}}{2}\right)$ – at which point there is no stigma.
- As $\varphi \rightarrow 1$ (i.e. when the probability of a shock occurring becomes very small), $\frac{dE[I|\mathcal{I}^C]}{d\bar{ES}} \rightarrow 0$, $\frac{dE[F|\mathcal{I}^C]}{d\bar{ES}} \rightarrow 0$, $E[I|\mathcal{I}^C] \rightarrow 0$ and, $E[F|\mathcal{I}^C] \rightarrow 0$ (see Appendices 3.6.11 and 3.6.10), so Eq. (3.34) becomes negative, and the welfare max-

⁵⁷There is one point within this region in which the derivative of $E[I|\mathcal{I}^C]$ does not exist. See Appendix 3.6.11 for further details.

⁵⁸When $\frac{d\bar{K}}{d\bar{ES}} \rightarrow 0$, the upper bound of $\frac{dE[F|\mathcal{I}^C]}{d\bar{ES}}$ also approaches zero (see Appendix 3.6.10).

imum is $\overline{ES} = 0$. Intuitively, when the probability of a shock becomes very small, the impact changes in $\overline{ES}$ have on welfare is dominated by the impact on the deterministic steady state, which occurs via the impact on the monitoring cost. Since the monitoring cost is minimised at $\overline{ES} = 0$, this is the welfare maximum.

- As $\vartheta \rightarrow 0$ (i.e. when the effort cost differential becomes very small), Eq. (3.34) becomes dominated by the $\frac{d\overline{K}}{d\overline{ES}}$ and $\frac{dE[F|\mathcal{I}^C]}{d\overline{ES}}$ terms, at which point Eq. (3.34) could be either positive or negative. However, with κ close to zero and the first line in Eq. (3.34) divided by $(1 - \beta)$, Eq. (3.34) will be negative as $\vartheta \rightarrow 0$ unless $\frac{d\overline{K}}{d\overline{ES}}$ is also very small. Intuitively, when the cost of not selling the equity claims gets very small, the impact changes in $\overline{ES}$ have on welfare is dominated by its impact on the monitoring cost (unless the monitoring cost impact is also very small).

From Eq. (3.33), there is only one term that is impacted by κ (none of the reduced-form parameters depend on κ), and this term is not divided by $(1 - \beta)$. When combined with the fact that $E[F|\mathcal{I}^C]$ is close to zero with any realistic calibration (calibrated to Australia), changes in $\overline{ES}$ have a much larger impact on welfare than changes in κ (with the parameter values used in this chapter).⁵⁹ Moreover, the welfare impact of an increase in κ is always negative. So, to maximise welfare in this model, central banks should set $\kappa > 0$ as small as possible, and use $\overline{ES}$ to offset the monitoring and effort costs – this is consistent with the operations of the Reserve Bank of Australia, that sets κ at an annual rate of 0.5 per cent.⁶⁰

⁵⁹Both $\frac{d\overline{K}}{d\overline{ES}}$ and $\frac{dE[F|\mathcal{I}^C]}{d\overline{ES}}$ would need to be very close to zero for this not to be the case.

⁶⁰Although it is not modelled in this thesis, in reality there is some risk differential between central bank deposits and interbank loans (it is ignored in this thesis since it is typically very small, see Section 1.3.2). As a result, if κ is set too low, there would be no interbank lending, impacting the monitoring cost. Therefore, central banks will typically not set κ too low.

3.4 Numerical results

In the above sections, the impact of changing $\overline{ES}$ was only evaluated for the subset of $\left[0, \frac{B\overline{K}}{2}\right]$ in which $\varepsilon_t^{\dagger\dagger} < 0$ when $F_t > 0$ (henceforth known as the “modelled” subset). This is because the equations used to determine $\varepsilon_t^{\dagger\dagger}$ differ over the $\left[0, \frac{B\overline{K}}{2}\right]$ range (see Appendix 3.6.11 for more detail) – creating continuity issues and potentially impacting the existence of welfare maxima.

With the parameter values used in this chapter, the smallest size of the modelled subset is $[0, 29.99995)$, with the corresponding non-modelled subset being just $[29.99995, 30]$. So the focus on the modelled subset is sufficient for the qualitative conclusions of this chapter.

The interbank monitoring cost scaling parameter (ψ) impacts welfare only via its impact on the deterministic steady state ($\overline{K}$) – with $\frac{d\overline{K}}{d\psi} > 0$ for all $\overline{ES} > 0$.⁶¹ However, while $\overline{K}$ impacts welfare directly (via the reduced-form parameters), it also impacts $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$.

With the majority of parameters calibrated to the Australian market, and with small monitoring and effort costs, increases in $\overline{K}$, that result from an increase in ψ , have a negative impact on welfare – both directly and indirectly via $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$ (in the modelled subset). Therefore, lowering ψ increases welfare (within the modelled subset, except when $\overline{ES} = 0$).

Lowering ψ also impacts the welfare-maximising $\overline{ES}$ of the modelled subset. Figure 3.1 shows some examples of the impact of reducing ψ (all other parameters are at the baseline setting). With the values of ψ I have chosen, the welfare-maximising aggregate ES balance (of the modelled subset) increases from zero, to A\$16.44 million, and then to A\$29.89 million (close to the upper bound of A\$30 million).

For the qualitative conclusions of this chapter, finding the welfare maximum of the non-modelled subset (if it exists) would be an unnecessary complication. Instead, I

⁶¹ $M(\overline{S}) = 0$ when $\overline{ES} = 0$ (see Section 3.2.4), so ψ has no impact.

construct a welfare upper bound for the non-modelled subset and compare this to the welfare maximum in the modelled subset.⁶² As can be seen from Figure 3.1, with these parameter settings, the welfare upper-bound of the non-modelled subset is not far from the surrounding modelled levels of welfare. In fact, only when $\psi = 3.5$ is the welfare upper-bound of the non-modelled subset greater than the maximum level of welfare in the modelled subset – but even then, the difference is small (see the inset in Figure 3.1).

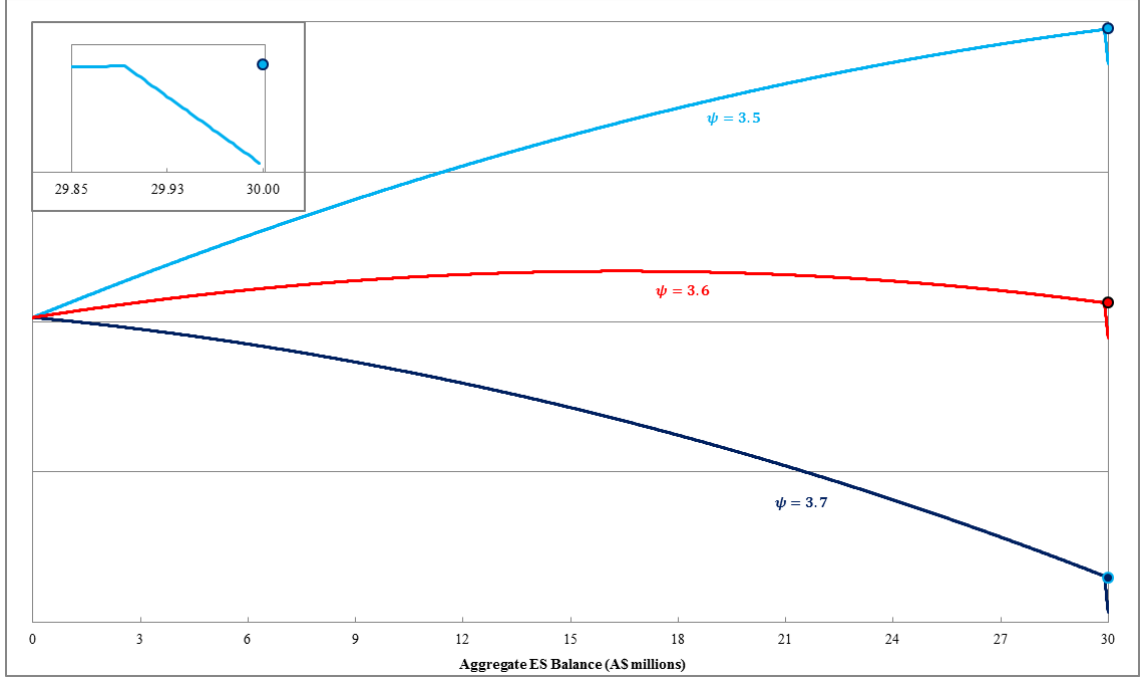
Therefore, by incorporating the market features discussed in the literature (i.e. stigma and interbank monitoring), this model can produce a welfare-maximising aggregate ES balance that is greater than banks' desired end-of-day balances (zero) but still small relative to aggregate interbank lending; consistent with, and providing an explanation for, central bank practice.⁶³ Moreover, the model can produce a welfare-maximising aggregate ES balance very close to (and potentially equal to) the point where there is no lending facility stigma – consistent with the RBA's statements that there is no lending facility stigma in Australia (*Debelle*, 2008).

⁶²Appendix 3.6.14 derives the welfare upper bound for the non-modelled subset.

⁶³At the baseline setting, steady state aggregate interbank lending is around A\$950 million when the aggregate ES balance equals A\$30 million.

Figure 3.1: Welfare for Different Values of ψ

All parameters except ψ (the monitoring cost scaling parameter) are at the baseline setting; circles are the upper bounds of the non-modelled subset



Note: The welfare scale is arbitrary and is omitted from the figure.

With $\vartheta > 0$, there is a region of $\overline{ES}$ (within the modelled subset) in which lending facility stigma remains, but the set of $\varepsilon_t < 0$ that leads to the lending facility being accessed is sufficiently small that there is no adverse selection within the set. With no adverse selection within the $\varepsilon_t < 0$ set, changing $\overline{ES}$ does not impact the banks' ability to sell their shares, so $\frac{dE[I|\mathcal{I}^C]}{d\overline{ES}} = 0$. This is the cause of the downward spike of the welfare curves in Figure 3.1 (close to the upper bound).⁶⁴ With larger values of ϑ , this spike will occur further from the upper bound of the modelled subset, potentially impacting the welfare-maximising aggregate ES balance.

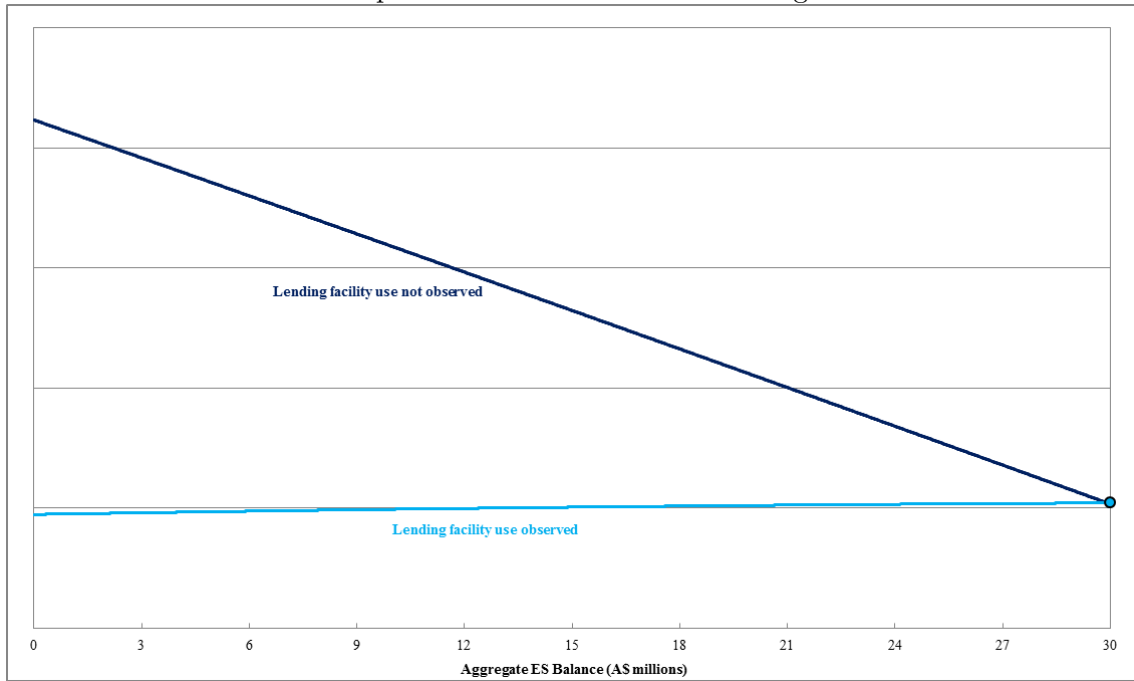
With full information, there can be no adverse selection problem and therefore no stigma. With partial information (i.e. knowledge that the lending facility has been

⁶⁴It creates a spike (rather than a smooth transition) because $\frac{dE[I|\mathcal{I}^C]}{d\overline{ES}}$ does not exist at the lower bound of this no adverse selection subset (although $E[I|\mathcal{I}^C]$ remains continuous) – see Appendix 3.6.11 for more detail.

used), and a sufficiently small ϑ , lending facility stigma may develop, potentially leading to fewer asset sales and welfare losses. But with no information, lending facility use is not observed, so there can be no stigma. With a sufficiently large ϑ , this lack of stigma may increase asset sales (relative to the partial-information scenario), leading to higher welfare for consumers. Figure 3.2 shows that this “Curse of Partial Transparency” exists for the parameter values used in this chapter.

Figure 3.2: Welfare with Different Information

All parameters at the baseline setting



Note: The welfare scale is arbitrary and is omitted from the figure.

With no observance of lending facility use, the only impact $\overline{ES}$ can have on $E[I|\mathcal{I}^C]$ is through the monitoring cost. As a result, $\frac{dE[I|\mathcal{I}^C]}{d\overline{ES}} > 0$ with the parameter values used in this chapter (the opposite of the non-positive derivative in the partial-information case). This means that the first line in Eq. (3.34) is always negative; and since the first line dominates the welfare derivative (with the parameter values used in this chapter, due to being multiplied by $\frac{1}{1-\beta}$), the welfare derivative becomes negative – causing the downward sloping welfare curve in Figure 3.2 and a welfare maximum of $\overline{ES} = 0$.

This “Curse of Partial Transparency” may explain why central banks typically do not publish timely information about lending facility usage. For example, prior to the November 2013 operational changes (see Appendix 1.7.3), the RBA only published aggregate lending facility usage information with a one-month lag, while the US Federal Reserve publishes weekly aggregates, and detailed data with a 2-year lag. However, if there is some probability of this information being discovered anyway (which must be possible for stigma to be a reasonable fear), this model does not explain why central banks do not release all of the available information (since full information maximises welfare in this model). While this question is left for future research, an obvious candidate is that the central banks may not have access to *all* the information necessary to provide “full information”; e.g. in a fuller model with more than one shock type, the central bank may not know which type of shock has occurred.

Figure 3.3 shows the impact of changing some of the parameters. The effort cost (ϑ) has both a direct impact on welfare and an indirect impact through its impact on $E[I|\mathcal{I}^C]$ (see Eq. (3.33)). The direct impact is negative (increasing the effort cost reduces welfare for a given $E[I|\mathcal{I}^C]$), while the indirect impact is non-negative (increasing the effort cost alleviates the adverse selection problem, reducing $E[I|\mathcal{I}^C]$). For the parameter changes in Figure 3.3, the direct effect dominates, with an increase in ϑ causing a decrease in the level of welfare.

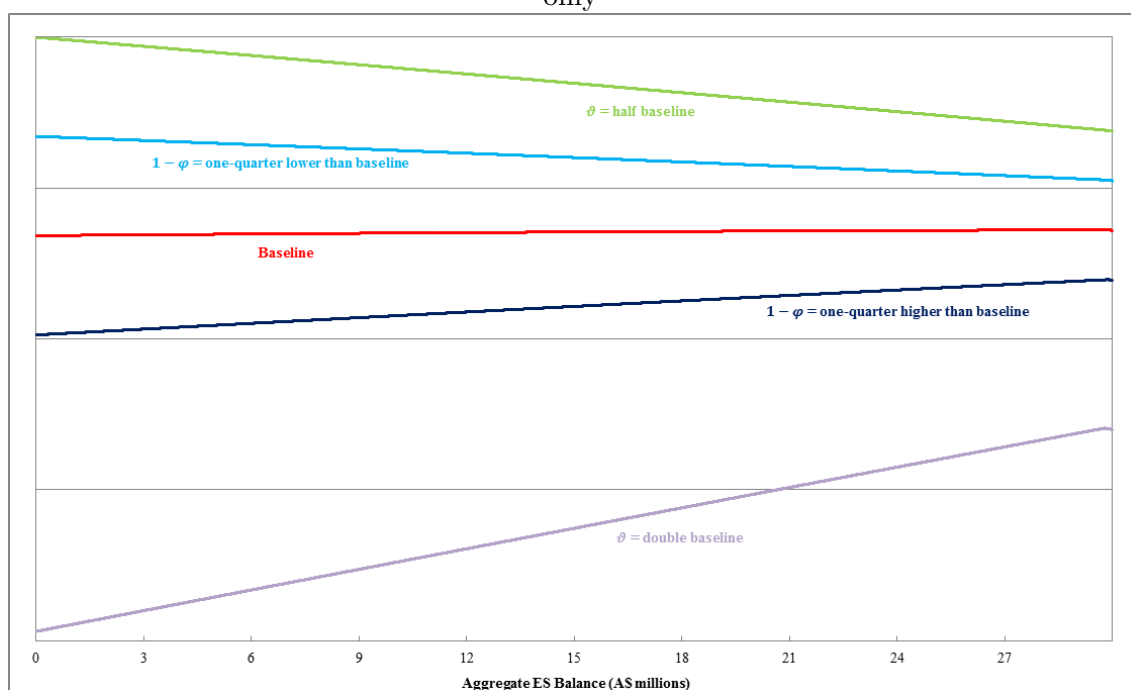
For values of $\overline{ES}$ at which $\frac{dE[I|\mathcal{I}^C]}{dES} < 0$, increasing ϑ reduces the absolute value of the derivative, reducing the slope of the welfare curve. However, the direct impact of ϑ on the welfare curve slope is positive. For the parameter changes in Figure 3.3, the direct effect dominates, with an increase in ϑ causing an increase in the slope of the welfare curve, potentially leading to a higher welfare-maximising aggregate ES balance.

Changing the probability of a non-zero shock ($1 - \varphi$) only has an indirect impact on welfare, via its impact on the $E[I|\mathcal{I}^C]$ and $E[F|\mathcal{I}^C]$. It’s impact on these components is positive, and it’s impact on welfare is therefore negative. Similarly, changing $1 - \varphi$

only has an indirect impact on the slope of the welfare curve. However, its impact on the slope is ambiguous. For the parameter changes in Figure 3.3, increasing $1 - \varphi$ increases the slope of the welfare curve, potentially leading to a higher welfare-maximising aggregate ES balance.

Figure 3.3: Impact of Parameter Changes on Welfare

Other than the changes specified, parameters are at the baseline setting; modelled subset only



Note: The welfare scale is arbitrary and is omitted from the figure. The spikes that are visible in Figure 3.1 are not visible here because of the enlarged scale of the vertical axis.

As noted in Section 3.3, the central bank interest rate spread (κ) has only a small impact on welfare (relative to $\overline{ES}$). In fact, its impact is so small, that it would not be visible in Figure 3.3. For example, across all values of $\overline{ES}$ (within the modelled subset), a doubling of κ from the baseline scenario causes, at most, a 1.1×10^{-14} per cent fall in welfare. This compares with welfare increasing 1.1×10^{-10} per cent between the baseline scenario welfare-minimising $\overline{ES}$ and welfare-maximising $\overline{ES}$. Similarly, a halving of κ from the baseline scenario causes, at most, a 6.1×10^{-15} per cent increase in welfare.

With the probability of lending facility use calibrated to Australian data, $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$ are both close to zero. Then, with an effort cost sufficiently low to generate stigma, generating a welfare-maximising $\overline{ES}$ that is above zero requires the monitoring cost to also be small. As a result, the aggregate ES balance only has a small impact on the real economy and consumer welfare – with the baseline parameterisation, increasing $\overline{ES}$ from 0 to A\$30 million increases steady state daily output by just 0.00000005 per cent. But this is to be expected since, if the aggregate ES balance had a major impact on the real economy, it would be used by central banks to guide economies through the business cycle – the fact that this does not occur in Australia can be seen by comparing the aggregate ES balances in Figure 1.1 to the nominal interest rates in Figure 1.4.

3.5 Conclusion

In this chapter, I construct a simple DSGE model with a banking sector, incorporating the commitment of the central bank to its trading-rate target, interbank monitoring, and asset managers who are better at managing the equity claims to the firms' profits.

The model shows that asymmetric information between the lending bank and asset managers can lead to an adverse selection problem when the banks try to sell their assets. Moreover, if the asset managers are able to observe use of the central bank's lending facility, this adverse selection can cause lending facility use to become a negative signal about the quality of the assets that can profitably be sold (i.e. lending facility stigma).

The model shows how the central bank can alleviate the welfare costs of lending facility stigma – by increasing the aggregate ES balance to reduce the proportion of negative shocks that lead to lending facility use. In fact, a sufficiently large increase in the aggregate ES balance can eliminate lending facility stigma.

However, as suggested in the literature, there are costs associated with the corres-

ponding fall in interbank trade (e.g. the consequences of a lower level of monitoring). Therefore, the central bank faces a trade-off when determining the welfare-optimising aggregate ES balance.

While some parameter values lead to a welfare-optimising aggregate ES balance equal to banks' desired end-of-day balances (i.e. parameter values with a sufficiently large monitoring cost), parameter values associated with larger stigma costs can lead to a welfare-optimising aggregate ES balance that is large enough to eliminate lending facility stigma, but still small relative to aggregate interbank lending. Therefore, this model can explain why central banks typically maintain an aggregate ES balance in excess of banks' desired end-of-day balances but small relative to interbank lending (this is the main contribution of this chapter to the literature).

That being said, although this model can produce a result that is consistent with the Australian outcome, this is only a very simple model, with only a single source of shocks and a single source of potential stigma. So the mechanisms of the model may not be the same mechanisms that cause the outcome in reality.

Since the aim of this chapter is only to provide a simple framework to show how some features of the market (that are based on statements from the literature) could lead to such an outcome (since such a framework does not exist in the literature), the fact that my model does not incorporate all possible sources of stigma is not problematic. However, if future research desires to quantitatively determine the welfare-maximising aggregate ES balance, my simple model will not be sufficient.

Other issues that will need to be overcome before undertaking quantitative modelling include:

- Ensuring that the relative sizes of interbank payment flows and economic output are realistic (to ensure that reasonable late-day payment flows do not require unrealistic real economy movements).⁶⁵ Potential ways to fix this problem could

⁶⁵In Australia, daily payment flows were around A\$150 billion on average during 2012-13 (see

be to include shocks that originate within the banking system, or to have the banking system magnify any real economy shocks.

- Replacing the ad hoc monitoring and effort costs with proper modelling of the features that lead to such costs (such as the example provided in Appendix 3.6.3).
- In reality, dividends are not paid every day, and banks may not sell their assets on the day they are created (i.e. my model is not sufficiently smooth for reality). Incorporating smoothing into the model could reduce the impact of an individual shock, and lead to scenarios in which it is only persistent use of the lending facility that leads to stigma, or where stigma forms only if lending facility use is combined with other negative information (such as a financial crisis).⁶⁶
- Removing the simplifying assumption that banks are split into two departments. Removing this assumption would be complicated, as the model would no longer have a recursive structure.

As a final note, this model is only suitable for explaining central bank policy in “normal times”. This model is not suitable for situations in which the central bank moves the aggregate ES balance in order to achieve its price stability objective (such as, for example, when the nominal interest rate hits the zero lower bound and the central bank implements unconventional monetary policies).

Figure 2.2), while average daily output was A\$4.17 billion. This is because, in reality, it is not just real production that causes interbank payment flows. In this model however, since payment flows are equal to half of consumers’ daily savings, payment flows are lower than daily output.

⁶⁶This would be consistent with *Armantier et al.* (2011), who provide some evidence in favour of the hypothesis that the fear of discount window stigma in the US increased during the recent financial crisis.

3.6 Appendices

3.6.1 Literature review

By including stigma and monitoring costs (which, to my knowledge, have never before been simultaneously modelled), the model in this chapter can explain why central banks typically maintain an aggregate ES balance in excess of banks' desired end-of-day balances, but small relative to interbank lending (this is the main contribution of this chapter to the literature). This contribution is particularly important in light of the models that exclude monitoring costs, and therefore espouse application of the "Friedman Rule" (e.g. *Ennis and Weinberg* (2007), *Cúrdia and Woodford* (2011), and *Berentsen et al.* (2014)) – the Friedman Rule requires supplying unlimited ES balances at no cost (*Keister et al.*, 2008; *Kashyap and Stein*, 2012), thereby eliminating trade in the overnight interbank market.⁶⁷

3.6.1.1 Modelling interbank markets

The impact of the interbank market on the broader economy has been extensively modelled. However, as far as I'm aware, there is no existing model that includes stigma, monitoring costs, and commitment of the central bank to its trading-rate target.

In order to incorporate these features, and to examine the impact on welfare in as simple a framework as possible, the model in this chapter significantly departs from the existing literature. The main departures from the standard literature are: incorporating only two banks within a DSGE framework (instead of the typical continuum of banks), the inclusion of a nominal resource used as a medium of exchange (rather than borrowing/lending real resources), and the simplifying assumptions required to ensure the lending bank always provides sufficient deposits to the firms (i.e. so that the interbank market has no impact on real economic activity, see Appendix 3.6.5 for

⁶⁷Both *Ennis and Weinberg* (2007) and *Cúrdia and Woodford* (2011) suggest that a model more complex than theirs is required to properly determine optimal central bank policy.

details of these assumptions). While these deviations are costly (in the sense that the model becomes increasingly non-standard with every extra deviation), I believe that the ability to develop a sufficiently simple framework – such that the welfare impact of central bank policy can be analytically determined (e.g. in Section 3.3) – is worth the cost.

Many models in the literature incorporate an endowment economy and three time periods. These models typically have two different types of shocks:

- The models of *Bhattacharya and Gale* (1986) and *Allen et al.* (2009), for example, have liquidity shocks.⁶⁸ In the first period, consumers deposit their endowments with the banks, but the banks do not know when these deposits will be withdrawn. Since banks undertake maturity transformation, a bank that is subject to greater than expected withdrawals (in the second period) requires additional liquidity (i.e. a liquidity shock). However, with only liquidity shocks, there can be no stigma. Therefore, these models are unsuitable for my analysis.
- The model of *Kashyap and Stein* (2012) (based on the model of *Stein* (2012)), for example, has asset quality shocks. In the first period, banks are endowed with equity, but also fund some of their long-term assets with short-term debt. In the second period, banks' long-term asset quality may be called into question, resulting in an inability to roll-over their short-term debt. In their model, higher market levels of short-term debt increase each bank's cost of not being able to roll-over their short-term debt (e.g. depressed asset prices from fire-sales impact all banks, as in *Stein* (2012)) – i.e. there is an externality associated with short-term debt issuance. Binding reserve requirements are designed as a Pigouvian tax to prevent socially sub-optimal issuance of short-term debt. However, although the tax is implemented via an interbank market for reserves, there is no lending

⁶⁸ *Bhattacharya and Gale* (1986) expands the seminal work of *Diamond and Dybvig* (1983) to include an interbank market.

facility in their model, and no excess reserves.⁶⁹

- *Heider et al.* (2009) combine both liquidity shocks and asset value shocks. In their model it is the liquidity shocks that lead to interbank trading. However, since the value of borrowers' long-term assets impact their ability to repay these loans, interbank lending is subject to idiosyncratic credit risk. Banks also have the ability to liquidate some of their long-term assets, with a liquidation cost that is inversely related to asset quality (this could be interpreted as a collateralised central bank lending facility with haircuts related to asset quality). With a centralised interbank market, in some scenarios, asymmetric information causes banks with high quality long-term assets to liquidate some of their assets instead of obtaining interbank funding at the market rate (i.e. the interbank market suffers from a lemons problem).⁷⁰ Therefore, there is no stigma attached to liquidation. Moreover, with banks choosing their initial portfolio allocation between short and long-term assets, the central bank plays no role in determining the volume of liquidity in the system (i.e. it does not set the aggregate ES balance).⁷¹
- *Acharya et al.* (2012) is similar to *Heider et al.* (2009) in that the illiquid bank's ability to repay loans is dependent on the returns from its risky investments.⁷² However, instead of asymmetric information about asset quality, the illiquid bank is able to choose (privately) whether it monitors its investments or shirks and receives a private benefit (as in *Holmstrom and Tirole* (1997), for example). The availability of a private benefit creates a moral hazard if the illiquid bank borrows

⁶⁹Although *Kashyap and Stein* (2012) suggest an extension in which excess reserves are paid a higher interest rate than required reserves, having banks desire excess reserves (in their model) would require incorporating: interbank payments that are settled with reserves; and some form of either payment uncertainty or interbank trading friction.

⁷⁰*Vollmer and Wiese* (2014) also model an endowment economy with interbank credit risk and liquidity shocks. But their focus is on the bilateral trade within the interbank market, so they abstract from the central bank's deposit and lending facilities, and any asymmetric information.

⁷¹Although *Heider et al.* (2009) discuss the possibility of central bank liquidity provision, in their model, such provision would crowd-out the interbank market.

⁷²Unlike *Heider et al.* (2009), banks in *Acharya et al.* (2012) are not subject to liquidity shocks; instead there are only two banks, one of which has risky assets and requires liquidity with certainty.

too much (due to limited liability), limiting its ability to borrow. If the interest rate or liquidity need is high, borrowing limits may cause the illiquid bank to sell its assets instead of borrowing; asset returns are higher with the originating bank, so asset sales are inefficient. By offering a lending facility, the central bank can limit the interest rate, thereby encouraging interbank lending and increasing welfare. However, the lending facility in their model plays the role of the “outside option” during interbank bargaining, it is never actually used. And the aggregate supply of liquidity (i.e. the aggregate ES balance) has no impact on their model.

There are also many papers that introduce a banking sector into either flexible-price DSGE models (e.g. *Brunnermeier and Sannikov* (2013)) or New Keynesian DSGE models (e.g. *Christiano et al.* (2004) and *Cúrdia and Woodford* (2011)). However, these models typically do not allow for interbank lending, and therefore do not require the central bank to offer a lending facility (so there can be no stigma).

That being said, there are exceptions. *Dib* (2010) and *Hilberg and Hollmayr* (2011), for example, include a banking sector, within a New Keynesian DSGE model, that allows for interbank trading. However, there is no payment flow uncertainty (and therefore no lending facility), and interbank trading only occurs because either a subset of banks are unable to partake in open market operations (*Hilberg and Hollmayr*, 2011), or because each bank is either a deposit-taking bank or an investment bank (*Dib*, 2010).

Gertler and Kiyotaki (2010) is similar to *Dib* (2010) in that, at each point in time, only some banks receive an investment opportunity; these banks then borrow real resources from the other banks. However, unlike *Dib* (2010), the banks that receive the investment opportunity are chosen randomly (and can differ at each point in time); akin to the randomly chosen “lending bank” in my framework.

That said, banks in *Gertler and Kiyotaki* (2010) lend/borrow real resources in the interbank market (rather than lending/borrowing central bank deposits to offset payment flows); so there is no central bank supplied liquidity (i.e. no ES balances). While

the central bank's lending facility is accessed concurrently with the interbank market, and there is no stigma associated with its use. Moreover, the *Gertler and Kiyotaki* (2010) model incorporates a continuum of banks that trade in a perfectly competitive market (so there is no bilateral trading), and the focus of *Gertler and Kiyotaki* (2010) is on the impact the interbank market can have on real economic activity (my framework abstracts from this impact in order to focus on the impact of stigma and monitoring costs).⁷³

Berentsen and Monnet (2008) and *Martin and Monnet* (2011) construct general equilibrium frameworks that include a lending facility. However, interbank trading only occurs after the agents receive an imperfect signal regarding their liquidity needs, with the lending facility only used because this signal is imperfect. Therefore, there is no stigma attached to its use. Moreover, all borrowing in their models is collateralised.

Berentsen et al. (2014) construct a general equilibrium framework similar to that of *Berentsen and Monnet* (2008), but in which banks use the lending facility only because they are exogenously excluded from the interbank market. So there can be no stigma attached to facility use.

The closest framework to what I develop in this chapter is the framework of *Bianchi and Bigio* (2014). In fact, despite very different modelling assumptions, the outcome of interbank bargaining in their model is the same as in my model.⁷⁴ However, with no post-interbank market shock and no matching frictions, the lending facility is never used in their baseline model (when the aggregate ES balance is set at least as high as reserve requirements); so there can be no stigma.

⁷³Incorporating central bank supplied liquidity and bilateral trading into the *Gertler and Kiyotaki* (2010) framework, and abstracting from the impact of the interbank market on real activity, would require simplifying assumptions similar to those in my framework.

⁷⁴They assume that banks split their desired lending/borrowing volume into infinitesimally small parcels, with each parcel matched and traded independently. Therefore, the aggregate ES balance has no impact on the trading outcome, only on the matching probability. With assumed equal bargaining power, an aggregate ES balance greater than or equal to reserve requirements, and no matching frictions, the interbank interest rate is the midpoint of the corridor, and an ES balance deficit bank will always borrow the exact amount required to return its ES balance to its reserve requirement.

Even when they introduce a matching friction (they shock the matching technology so that there is no interbank market), lending facility use still only occurs following a liquidity shock, resulting in facility use that is unrelated to asset values and therefore provides no signal about asset quality. So the *Bianchi and Bigio* (2014) model is not suitable for my purposes.

3.6.1.2 Lending facility stigma

This chapter is the first time stigma has been modelled as an adverse selection problem within a DSGE framework.

Although the literature highlights stigma as a potential problem for overnight interbank markets in some countries (e.g. *Committee on the Global Financial System* (2008), *Kohn* (2010) and *Armantier et al.* (2011)), there is little existing research that formally models either how lending facility stigma could develop, or its implications for the overnight interbank market. *Ennis and Weinberg* (2013) and *La'O* (2014) are exceptions.⁷⁵ While the *Ennis and Weinberg* (2013) framework is too simple for my purposes (e.g. they build an endowment economy with fixed interbank lending volumes), the way stigma develops in my model (i.e. asymmetric information leading to adverse selection) is based on the approach used in *Ennis and Weinberg* (2013). That being said, there are differences in the way stigma develops:

- In both my model and the model of *Ennis and Weinberg* (2013), the quality of the assets for sale is unknown to the potential buyers (i.e. the asset managers in my model).
- The banks in *Ennis and Weinberg* (2013) must sell their assets to repay any

⁷⁵*Philippon and Skreta* (2012) also model a situation in which there is stigma associated with participation in a Government program. However, their focus is on optimal mechanism design in this scenario, rather than looking at how a signal may develop in response to an existing program (the lending facility), and the impact this signal has on the market. Moreover, their model is an endowment economy in which adverse selection occurs between lenders and borrowers (akin to adverse selection within the interbank market), so it is not appropriate for my purposes.

loans. In my model, banks may want to sell their assets since the asset managers can manage the assets for a lower cost, and may therefore be willing to pay a price higher than the level at which the bank values the assets (i.e. trade can be mutually profitable).

- In *Ennis and Weinberg* (2013), banks that require funds (henceforth, illiquid banks) may not find a trading partner in the interbank market, and may therefore borrow from the central bank in order to meet their liquidity needs; this can happen to banks holding long-term assets of any quality. Even when a bank finds a counterparty, if the illiquid bank has a low-quality long-term asset, its counterparty may refuse to lend (there is no asymmetric information between banks), at which point the illiquid bank may obtain funds from the central bank.⁷⁶ Therefore, if the asset managers observe lending facility use, the expected quality of the assets for sale (from the point-of-view of the asset managers) may be lower if the selling bank borrowed from the central bank (i.e. stigma). **The adverse selection in their model arises because banks with low-quality assets have a higher probability of borrowing from the central bank than banks with high-quality assets.**

- In my model, the true value of the assets (shares in the firms) is only observed by the banks after the interbank market has closed; with the shock that causes an unexpected change in asset value also causing unexpected interbank payment flows, and potentially requiring one of the banks to borrow from the central bank. If the central bank's lending facility is accessed, the expected value of the shares is greater than or equal to the expected value if the facility had not been used.

However, due to asymmetric information, there may be positive shocks sufficiently

⁷⁶In the model of *Ennis and Weinberg* (2013), lending facility stigma requires some positive probability that banks with low-quality assets are able to access the central bank's lending facility (they assume a probability of one for simplicity). While they admit that this is at odds with standard central bank practice, they justify their assumption by assuming the central bank cannot perfectly determine banks' asset quality. With no default risk in my model, such an assumption is not required.

large that the bank would not be willing to sell the shares at a price the asset managers are willing to pay (i.e. adverse selection). If the adverse selection problem is sufficiently severe, lending facility use can become a negative signal of expected asset quality (i.e. stigma). **The adverse selection in my model arises because sale of the bank's assets is a choice, and the bank will only sell if it is profitable (i.e. a classic lemons problem à la *Akerlof* (1970)).**

- Although this is not included in their model (they also leave the modelling of central bank policy for future research), in a reasonable extension in which the central bank could increase the aggregate ES balance, the probability of illiquid banks finding a counterparty should increase (since there is more liquidity in the system); this would reduce the share of lending facility use accounted for by banks with high-quality assets, thereby making the stigma problem worse.⁷⁷ This is the opposite of my model, in which increasing the aggregate ES balance reduces use of the lending facility following a negative shock, eventually leading to the elimination of stigma.

La'O (2014) models a situation in which both interbank borrowing and lending facility borrowing may signal financial weakness. In her model, distressed banks suffer a negative liquidity shock. For some of these banks, their current levels of liquidity would be insufficient to manage the shock, requiring liquidation of their entire portfolio (which is costly). Banks with sufficient liquidity to manage such a shock will never obtain a loan, but banks in danger of needing to liquidate may borrow to increase their liquidity. However, since each bank's liquidity level is private information, borrowing signals that a bank is of a lower-liquidity type (i.e. stigma).⁷⁸ This signal may then entice a

⁷⁷The assumption that banks with low-quality assets are able to access the central bank's lending facility would become increasingly dubious as the frequency of access by banks with high quality assets falls.

⁷⁸Crucially, a bank seeking a loan cannot borrow enough to fully offset any liquidity shock. While

predatory trader to begin a “predation war” in an attempt to force the liquidation of the distressed bank (so it can gain from the fire-sale) – this is the potential cost of the stigma.

The model of *La’O* (2014) is not suitable for my purposes: it does not distinguish between interbank borrowing and central bank borrowing; borrowing volumes are exogenous and stochastic; and it only models the actions of the banks (in an endowment economy). Moreover, since the interbank market is typically very active (in reality), it is presumably only in extreme scenarios that any stigma would be attached to interbank borrowing.

Stigma in the *La’O* (2014) model only exists because distressed banks are restricted from borrowing sufficient liquidity from the central bank. This seems unrealistic given the quality of these banks’ underlying assets.⁷⁹ That is, with good assets, a realistic central bank would be willing to lend as much liquidity as is necessary to prevent costly liquidation and potential contagion (this is the lender-of-last-resort function); and since this would alleviate the liquidity problem, there would be no stigma attached to such borrowing (this, and other issues with *La’O* (2014), are discussed in *Lowery* (2014)). So I am not convinced of the ability of liquidity problems to generate stigma without some underlying asset quality issues (like those in this chapter and in *Ennis and Weinberg* (2013)).

3.6.2 Investing in both banks equally is a symmetric equilibrium

Note: This appendix is best understood after reading Section 3.2 and Appendix 3.6.6.

From Eqs. (3.1) and (3.2), suppose the consumers choose C_t , D_t , H_t , and δ_t to maximise the following expected discounted lifetime utility:

the loan reduces the probability that it will need to liquidate, its liquidity level remains lower than that of the banks that are in no danger of liquidation.

⁷⁹This criticism only refers to financial institutions that are able to tap central bank liquidity.

$$\mathbb{E} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} [\ln(C_s) - \chi H_s] \right\}$$

Subject to their daily budget constraint:

$$C_s + D_s \leq \left[\delta_{s-1} R_s^{D,1} + (1 - \delta_{s-1}) R_s^{D,2} \right] \left[D_{s-1} - \psi M(\bar{S}) \right] + W_s H_s + C B_{s-1}$$

The variables in these equations are the same as in Section 3.2.1.1, except that the total per unit return is split into the per unit return of each bank ($R_s^{D,1}$ for Bank 1 and $R_s^{D,2}$ for Bank 2), and the consumer invests $0 < \delta_s < 1$ of its savings in Bank 1 and $(1 - \delta_s)$ of its savings in Bank 2.

With a continuum of consumers, each consumer treats as exogenous: $R_s^{D,1}$, $R_s^{D,2}$, $\psi M(\bar{S})$, W_s , and $C B_{s-1}$ (for all s). But the consumers have sufficient knowledge of the real economy such that they know the true probability distributions of these variables (for $s > t$).

For the purposes of this proof, let C_s^* , D_s^* , and H_s^* be the global optimum values of C_s , D_s , and H_s (respectively) given the parameters, variables treated as exogenous by the consumers, and δ_t . Since these global optima will be interior optima, the Envelope Theorem applies, and the first-order condition for an interior optimal δ_t is:

$$\frac{d}{d\delta_t} = \beta \mathbb{E} \left\{ \frac{1}{C_{t+1}^*} \left[R_{t+1}^{D,1} - R_{t+1}^{D,2} \right] \right\} \left[D_t^* - \psi M(\bar{S}) \right] = 0$$

From the labour supply first-order condition, the interior optimum must satisfy:

$$C_{t+1}^* = \frac{W_{t+1}}{\chi}$$

Since W_{t+1} is not dependent on which bank is the “lending bank” (since the decisions of the consumers and firms are not dependent on which bank is the “lending bank”) then, using the law of iterated expectations:

$$\frac{d}{d\delta_t} = \beta E \left\{ \frac{\chi}{W_{t+1}} E \left[R_{t+1}^{D,1} - R_{t+1}^{D,2} \middle| \mathcal{I}_{t+1} \right] \right\} \left[D_t^* - \psi M(\bar{S}) \right] = 0 \quad (3.35)$$

Where $\mathcal{I}_{t+1}$ is an information set containing all the information available during day $t+1$ except for which bank was the day t lending bank.

Since banks are identical at the beginning of each day, and have equal probability of being the “lending bank”, when all consumers invest half of their savings in each bank, the lending bank’s dividend (per unit of savings) is the same irrespective of which bank is the lending bank (similarly for the non-lending bank, see Appendix 3.6.6 for more detail). Therefore: $E[R_{t+1}^{D,1} | \mathcal{I}_{t+1}] = E[R_{t+1}^{D,2} | \mathcal{I}_{t+1}]$ and $\frac{d}{d\delta_t} = 0$. This means that, when all consumers invest half of their savings in each bank, each individual consumer is indifferent between values of δ_t , and so is happy to remain at $\delta_t = \frac{1}{2}$; so $\delta_t = \frac{1}{2}$ is a symmetric equilibrium.

Using the same process as in Appendix 3.6.6, suppose that all consumers invest δ_t of savings in Bank 1 and $(1 - \delta_t)$ of savings in Bank 2, then with each bank having an equal probability of being the “lending bank” (recall that, in a symmetric equilibrium, $K_{t+1} = D_t^* - \psi M(\bar{S})$):

$$\begin{aligned} E \left[R_{t+1}^{D,1} - R^{ES} \middle| \mathcal{I}_{t+1} \right] 2\delta_t K_{t+1} &= -R^{ES} K_{t+1} - 2 \left(R_t^{O,1} - R^{ES} \right) O_t^1 \\ &\quad - \kappa F_t^{1,L} - \kappa F_t^{1,NL} + \alpha Y_{t+1} - \vartheta I_t \\ &\quad + \frac{\kappa}{2} \left[S_t^{2,NL} - S_t^{1,L} \right] + \frac{\kappa}{2} \left[S_t^{2,L} - S_t^{1,NL} \right] \end{aligned} \quad (3.36)$$

$$\begin{aligned} E \left[R_{t+1}^{D,2} - R^{ES} \middle| \mathcal{I}_{t+1} \right] 2(1 - \delta_t) K_{t+1} &= -R^{ES} K_{t+1} - 2 \left(R_t^{O,2} - R^{ES} \right) O_t^2 \\ &\quad - \kappa F_t^{2,L} - \kappa F_t^{2,NL} + \alpha Y_{t+1} - \vartheta I_t \\ &\quad + \frac{\kappa}{2} \left[S_t^{1,NL} - S_t^{2,L} \right] + \frac{\kappa}{2} \left[S_t^{1,L} - S_t^{2,NL} \right] \end{aligned} \quad (3.37)$$

Where a variable with a superscript “ i, L ” denotes that this is the variable for Bank i

when Bank i is the “lending bank”, while a variable with a superscript “ i, NL ” denotes the variable when Bank i is the “non-lending bank”.

With the central bank operating as described in Appendix 3.6.6.6, O_t^1 and O_t^2 are set so that both banks begin each day with the same expected end-of-day ES balance (excluding lending facility use). This gives:

$$O_t^1 = \frac{1}{2}\overline{ES} + E[S_t^2 - S_t^1 | \mathcal{I}_{t-1}^B] + \frac{1}{2}(1 - 2\delta_t) E[K_{t+1} | \mathcal{I}_{t-1}^B]$$

$$O_t^2 = \frac{1}{2}\overline{ES} + E[S_t^1 - S_t^2 | \mathcal{I}_{t-1}^B] - \frac{1}{2}(1 - 2\delta_t) E[K_{t+1} | \mathcal{I}_{t-1}^B]$$

If the central bank also operates to ensure $E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B] = 0$ (as in Appendix 3.6.6.7), then it can be shown (using the same process as in Appendix 3.6.6), that the interbank loan volumes would be the same as those given in Appendix 3.6.6.8. Therefore, $\frac{\kappa}{2} [S_t^{1,NL} - S_t^{2,L}] + \frac{\kappa}{2} [S_t^{1,L} - S_t^{2,NL}] = 0$.

Therefore, Eqs. (3.36) and (3.37) can be simplified to:

$$\begin{aligned} E[R_{t+1}^{D,1} - R^{ES} | \mathcal{I}_{t+1}] 2\delta_t K_{t+1} &= -R^{ES} K_{t+1} + \alpha Y_{t+1} - \vartheta I_t \\ &\quad - \kappa F_t^{1,L} - \kappa F_t^{1,NL} \\ &\quad - (R_t^{O,1} - R^{ES}) \left\{ \overline{ES} + (1 - 2\delta_t) E[K_{t+1} | \mathcal{I}_{t-1}^B] \right\} \end{aligned} \quad (3.38)$$

$$\begin{aligned} E[R_{t+1}^{D,2} - R^{ES} | \mathcal{I}_{t+1}] 2(1 - \delta_t) K_{t+1} &= -R^{ES} K_{t+1} + \alpha Y_{t+1} - \vartheta I_t \\ &\quad - \kappa F_t^{2,L} - \kappa F_t^{2,NL} \\ &\quad - (R_t^{O,2} - R^{ES}) \left\{ \overline{ES} - (1 - 2\delta_t) E[K_{t+1} | \mathcal{I}_{t-1}^B] \right\} \end{aligned} \quad (3.39)$$

As in Appendix 3.6.6.7, the banks would be willing to borrow any amount via open market operations if the interest rate is set at R^{ES} . Similarly, both banks would be willing to lend any amount if the interest rate is set at $R^{ES} + \kappa$ (the lending facil-

ity interest rate). Deriving lending facility use in the same way as Appendix 3.6.6.9 gives lending facility use that is independent of κ . Therefore, as $\kappa \rightarrow 0$, substituting Eqs. (3.38) and (3.39) into Eq. (3.35) gives:

$$\frac{d}{d\delta_t} \rightarrow \beta E \left\{ \frac{\chi [\alpha Y_{t+1} - \vartheta I_t - R^{ES} K_{t+1}]}{W_{t+1}} \right\} \frac{1}{2} \left[\frac{1}{\delta_t} - \frac{1}{1 - \delta_t} \right] \quad (3.40)$$

Since the δ_t in Eq. (3.40) is the δ_t of the continuum of consumers (since it is part of the $E[R_{t+1}^{D,1} - R_{t+1}^{D,2} | \mathcal{I}_{t+1}]$ term in Eq. (3.35)), each individual consumer cannot impact the value of Eq. (3.35) by changing its individual δ_t . Therefore, a symmetric equilibrium only occurs where $\frac{d}{d\delta_t} = 0$, such that each consumer is indifferent between all possible values of its individual δ_t , and is therefore willing to remain at the point where $\frac{d}{d\delta_t} = 0$.

From Eq. (3.40), if $\frac{d}{d\delta_t} = 0$ at a value other than $\delta_t = \frac{1}{2}$, this value of δ_t will depend on the value of $E \left\{ \frac{\chi [\alpha Y_{t+1} - \vartheta I_t - R^{ES} K_{t+1}]}{W_{t+1}} \right\}$, the value of κ , and the value of the terms multiplied by κ ; all of which depend on the parameter values. Therefore, $\delta_t = \frac{1}{2}$ is the only symmetric equilibrium that is independent of the other parameter values.

3.6.3 Outline of a Monitoring Cost Model

As a simple illustrative example (of a market with many banks), suppose each bank invests in a monitoring technology that gives it some probability of discovering if another bank is shirking (shirking = acting to increase private benefits at the expense of higher risk, as in *Rochet and Tirole* (1996) and *Holmstrom and Tirole* (1997)). Then suppose that, if a bank discovers that another bank is shirking, it is optimal for the former bank to limit lending to the shirking bank. If the overnight interbank market is vibrant, the banks that did not receive this shirking signal may see that the shirking bank is having unusual difficulty in obtaining funding, and may subsequently infer that it is shirking.⁸⁰ Therefore, the overnight interbank market may provide valuable

⁸⁰If shirking banks are also more likely to access the central bank's lending facility, this process could also lead to lending facility stigma (of a different type to the stigma considered in this chapter, and more like the stigma in *Ennis and Weinberg* (2013)).

information to the banks.

In this scenario, for a given investment in the monitoring technology, a reduction in overnight interbank market activity reduces the probability that each bank would discover whether another bank is shirking. This could make interbank borrowing (at all maturities) more difficult for all banks (through either higher risk premia or lower lending limits, for example), leading to lower profits (this impact is akin to the statement of *Keister et al.* (2008) quoted in Section 3.1). Alternatively, each bank could offset this impact by increasing its investment in the monitoring technology. In either case, the fall in overnight interbank market activity leads to higher “monitoring costs”.⁸¹

While the above example is useful as an intuitive illustration, formally modelling this mechanism would drastically increase the complexity of the model. For example, it would require determining: the optimal shirk/non-shirk decision of each bank; the optimal investment in the monitoring technology; the optimal exposures to each bank after both discovery of shirking or no discovery; and how to optimally interpret information from the overnight interbank market. It would also require a market of more than two banks.

3.6.4 Increasing firms' capital increases aggregate profits

Since each individual firm treats the market wage as constant, the profit-maximising actions of individual firms may lead to a sub-optimal result in equilibrium. So it is possible, in general, for an increase in day $t + 1$ capital to decrease the equilibrium day $t + 1$ profits of the firms (i.e. $\frac{dY_{t+1}}{dK_{t+1}} < 0$). I assume that $\frac{dY_{t+1}}{dK_{t+1}} < 0$ does not occur in this chapter; while I am unable to prove this result, I give some supporting evidence below.

From the equations in Section 3.2.3.4, one of the equations that determines equilib-

⁸¹This contrasts with some of the literature that include a monitoring cost that is increasing with the volume of overnight interbank trade (*Martin et al.* (2013), for example). Such models do not account for the information inherent in the overnight interbank market.

rium H_{t+1} can be expressed as:

$$\mathbb{E} \left[\alpha \beta H_{t+2} - \frac{\beta \kappa [F_{t+1}^i + F_{t+1}^j] + \beta \vartheta I_{t+1}}{A_{t+2} H_{t+2}^{-\alpha} K_{t+2}^{\alpha}} \middle| \mathcal{I}_{t+1}^C \right] = H_{t+1} - \frac{(1-\alpha)}{\chi} - \frac{\vartheta I_t + \psi M(\bar{S})}{A_{t+1} H_{t+1}^{-\alpha} K_{t+1}^{\alpha}} \quad (3.41)$$

While the impact of day $t + 1$ capital on the equilibrium day $t + 1$ profits of the firms is (from Section 3.2.3.2):

$$\frac{d(\alpha Y_{t+1})}{dK_{t+1}} = \alpha W_{t+1} \frac{dH_{t+1}}{dK_{t+1}} + \alpha^2 \frac{Y_{t+1}}{K_{t+1}}$$

From Eq. (3.41), with costless interbank payment flows ($\kappa = 0$), no monitoring cost ($\psi = 0$), and no cost of managing the shares ($\vartheta = 0$), the unique stable solution to the model has $H_{t+1} = \frac{1-\alpha}{(1-\alpha\beta)\chi} \forall t$ (in equilibrium). Therefore, $\frac{dY_{t+1}}{dK_{t+1}} = \frac{\alpha Y_{t+1}}{K_{t+1}} > 0 \forall K_{t+1}$. So increasing the day $t + 1$ capital available to the firms, by increasing their day t deposits, increases the day $t + 1$ equilibrium profits of the firms. In fact, as long as the model remains in a small region around the steady state, $\min \left[\frac{\alpha Y_{t+1}}{K_{t+1}} \right] \approx \beta^{-1} \forall K_{t+1} \leq D_t$ (since the derivative is decreasing in K_{t+1} , and β^{-1} is the value of the derivative when $K_{t+1} = D_t$ and the economy is in steady state).

Once overnight borrowing costs, monitoring costs, and effort costs are added, Eq. (3.41) can no longer be solved analytically. However, for small κ , ϑ , and ψ , K_{t+1} only has a small impact on Eq. (3.41). Therefore, with sufficiently small κ , ϑ , and ψ , and with $\min \left[\frac{\alpha Y_{t+1}}{K_{t+1}} \right] \approx \beta^{-1} \forall K_{t+1} \leq D_t$ when these costs are zero, $\frac{dH_{t+1}}{dK_{t+1}}$ will not be sufficiently negative to cause $\frac{dY_{t+1}}{dK_{t+1}} < 0$. I assume this sufficiently small condition holds with the parameter values used in this chapter, so that $\frac{dY_{t+1}}{dK_{t+1}} > 0 \forall K_{t+1} \leq [D_t - \psi M(\bar{S})]$.

As further evidence in favour of $\frac{dY_{t+1}}{dK_{t+1}} > 0 \forall K_{t+1} \leq [D_t - \psi M(\bar{S})]$ holding in this chapter, I next show that it holds when $K_{t+1} = D_t - \psi M(\bar{S})$ and when the economy is at the deterministic steady state. At the deterministic steady state:

$$\frac{dY_t}{dK_t} = \bar{W} \frac{dH_t}{dK_t} + \beta^{-1} \quad (3.42)$$

From the unique stable decision rule (Eq. (3.28)), and using the deterministic steady states in Appendix 3.6.7:

$$\begin{aligned}\frac{dH_t}{dK_t} &= - \left[\frac{\alpha \left[(1 - \alpha\beta) \chi \bar{H} - (1 - \alpha) \right] \bar{H}}{(\chi \bar{H} + \alpha) (1 - \alpha) \bar{K}} \right] \\ &= - \left[\frac{\alpha \left[\alpha\beta \bar{A} \right]^{\frac{-1}{1-\alpha}} \left[(1 - \alpha\beta) \chi \left[\alpha\beta \bar{A} \right]^{\frac{-1}{1-\alpha}} \alpha\beta \psi M(\bar{S}) \right]}{\left(1 - \alpha^2\beta + \chi \left[\alpha\beta \bar{A} \right]^{\frac{-1}{1-\alpha}} \alpha\beta \psi M(\bar{S}) \right) (1 - \alpha)} \right]\end{aligned}$$

Substituting this into (3.42) gives (and substituting in the value for $\bar{W}$ in terms of the parameters):

$$\frac{dY_t}{dK_t} = \beta^{-1} \left\{ 1 - \left[\frac{(1 - \alpha\beta) \chi \left[\alpha\beta \bar{A} \right]^{\frac{-1}{1-\alpha}} \alpha\beta \psi M(\bar{S})}{1 - \alpha^2\beta + \chi \left[\alpha\beta \bar{A} \right]^{\frac{-1}{1-\alpha}} \alpha\beta \psi M(\bar{S})} \right] \right\} > 0$$

With the parameter values used in this chapter, the smallest this steady state derivative gets is 1.000080192 (as opposed to 1.000080202 when $\psi = 0$).

3.6.5 Simplifying features/assumptions

Features/assumptions to ensure the lending bank provides sufficient deposits to the firms

Ensuring the lending bank always issues sufficient deposits requires: determining the price the non-lending bank sets for its goods (in terms of ES balances), and ensuring the profit from issuing deposits always outweighs the cost. These requirements are met through the following features/assumptions:

- The banks sign a binding contract at the beginning of the day (before they know which bank will be the lending bank) specifying an exchange rate between ES balances and goods. Then, since the probability that each bank sends ES balances is the same as the probability it will receive ES balances (each bank has equal

probability of being the lending bank), and the expected volume of transfers is independent of which bank is the lending bank, the banks are indifferent with respect to the contracted rate.^{82,83} So I assume that, between the banks, one unit of ES balances can always be exchanged for one unit of the good.

- If (and only if) the lending bank refuses to issue sufficient deposits for the firms to purchase all of the goods from the non-lending bank, then the non-lending bank may sell their remaining goods to new firms in exchange for equity.⁸⁴ As long as it is optimal for each bank to issue sufficient deposits for firms to purchase their own goods (which I assume holds in this chapter, see Appendix 3.6.4 for supporting evidence), this feature means that the *total* profits of the firms is independent of the value of deposits issued by the lending bank, it's only the lending bank's *share* of the profits that changes.⁸⁵ With the non-lending bank's equity going to new firms (i.e. firms that have not issued any equity to the lending bank), the lending bank's share of total profits is equal to the value of deposits it issues divided by the total value of next-day capital. The advantage of this feature is that the *marginal* return on equity (from an increase in deposits issued by the lending bank) is equal to the *average* return on equity, which is independent of the value of deposits already issued by the lending bank.⁸⁶
- The banks always accept the same volume of equity shares in aggregate (i.e. the aggregate volume of shares is constant), and always issue the same *volume* of

⁸²For day t transfers that are based on day $t - 1$ decisions, indifference requires that the day $t - 1$ exchange rate be used.

⁸³Although the volume of transfers can not yet be determined (since the shock is not yet known), both banks are privy to the same information, so they agree on how the size of these transfers will be determined. So this contract is like a Forward Contract, except with an agreed formula to determine the volumes rather than agreeing on the volumes.

⁸⁴There are no barriers to entry for new firms.

⁸⁵If both banks issue deposits, but they are not able to sell their assets to the asset managers, I assume the cost of managing the equity claims is shared among the banks. This prevents banks from doubling-up on management costs, thereby decreasing consumers' returns and impacting firms' profits.

⁸⁶Without this feature, the marginal return would be dependent on the value of issued deposits (since the monopolist lending bank would account for any change in equilibrium wages).

deposits for each share. As a result, the proportion of shares the lending bank offers for sale, and the proportion of shares purchased by the asset managers, is common knowledge.

- Equity shares are exchanged for bank deposits during the trading-period only. And the *value* of each deposit is conditional on the shock (such that the total value of deposits always equals the total value of the goods on the banks' balance sheets). When combined with the assumption above, this feature ensures that each share is identical, irrespective of which firm issued the share (since the firms face a constant returns to scale production function). These features also mean the banks never evaluate the cost of issuing extra deposits following a shock.^{87,88} Instead, the cost of issuing the original deposits incorporates the expected cost of a shock (incorporating the cost of any lending-facility use, sale-price change, or non-sale of the equity that results).
- The central bank sets its lending facility and deposit interest rates sufficiently low such that the expected average return on equity is always greater than the expected marginal cost of issuing deposits (this expected cost includes both the cost of interbank loans and the expected marginal cost of a shock), and such that the expected total return is greater than the expected total cost (including any fixed costs).
 - With the shock in this model only representing unknown late-day payment flows (which will be rare and small in this chapter), the expected cost of

⁸⁷Evaluating the cost of issuing extra deposits following a shock is problematic because issuing these deposits may lead to lending facility use, and subsequently either a lower sales price for, or an inability to sell, *all* the equity to the asset managers (when there is stigma). As a result, for a sufficiently small shock, the cost of issuing extra deposits could be large relative to the benefit.

⁸⁸An alternative modelling assumption that could achieve the same result is to assume that, after a positive shock, the extra capital is not only required to increase production but also to ensure the original projects succeed (this is similar to the “liquidity shock” in *Rochet and Tirole* (1996) and *Freixas and Jorje* (2008), for example). Then, although the cost of issuing new deposits could be large, the benefit would also be large.

- the shock will be small (including both marginal and fixed costs). Then, assuming the equity shares are always sold to the asset managers if there is no shock (as will be the case with the parameter values used in this chapter), the expected cost of issuing deposits will be dominated by the cost of interbank loans.⁸⁹ Since the cost of these interbank loans are determined by the central bank's interest rates, sufficiently low central bank interest rates will always exist.
- Assuming the level of interest rates is always “sufficiently low” is possible in this chapter, because the shock only represents small late-day payment flows, and the qualitative conclusions of the model are the same irrespective of the economy's position in the business cycle (this is explained in Section 3.2.1.5).

Additional features/assumptions to ensure the asset managers are unable to extrapolate the day t shock (during day t)

- Since the asset managers that purchased the day $t - 1$ shares will receive the firms' dividend payments during day t , these asset managers could extrapolate the day t shock. Therefore, these asset managers cannot purchase the day t shares. To ensure this, the continuum of asset managers only exist for one day, and are replaced by a new continuum every day (with no transfer of information).
- I assume there is no transfer of information between the non-lending bank and the asset managers. This assumption is required since there may be situations in which it is profitable for the non-lending bank to remove the asymmetric information.

⁸⁹Even if the shares were never sold to the asset managers, the cost of issuing deposits would still be dominated by the cost of interbank loans as long as the cost of managing the shares was small (as it will be with the parameter values used in this chapter).

3.6.6 Banking system equations

To derive the banking system equations (e.g. dividend payments, interbank loans, open market operations, etc.) I start with the balance sheets of the two banks at the end of day $t - 1$. I then evaluate the impact, on the balance sheets, of all the day t transactions; this gives formulae for the balance sheets at the end of day t .⁹⁰ The banking system equations are then derived by imposing restrictions (such as the restriction that all bank profits are paid out as dividends on a daily basis, see Section 3.2.1.1) and through assumptions about how banks make their decisions (e.g. how banks choose their interbank lending volumes).

With all bank profits paid out as dividends on a daily basis, and with the firms purchasing all of the goods on the banks' balance sheets (see Section 3.2.1.3), the only items remaining on the banks' balance sheets at the end of the day are: ES balances, interbank loans, open market operation loans, loans from the central bank's lending facility, the banks' equity claims on the firms (or the goods owed to them by asset managers), and consumers shareholdings in the banks (recall that consumers invest half of their savings in each bank, see Section 3.2.1.1).

Denote the banks as Bank i and Bank j , and assume, without loss of generality, that Bank i was the lending bank during day $t - 1$. Then the balance sheets at the end of day $t - 1$ were (these are book-value balance sheets – i.e. entries are valued at their purchase price rather than their future values):

$$ES_{t-1}^i + K_t + S_{t-1}^j \equiv \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right] + O_{t-1}^i + F_{t-1}^i + S_{t-1}^i \quad (3.43)$$

$$ES_{t-1}^j + S_{t-1}^i \equiv \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right] + O_{t-1}^j + F_{t-1}^j + S_{t-1}^j \quad (3.44)$$

Where:

⁹⁰*Fernandez-Corugedo et al. (2011)* construct an account of their banking system transactions in the same way.

- The right-hand-side of the balance sheets are the liabilities and equity. The left-hand-side are the assets.
- ES_{t-1}^i denotes Bank i 's surplus ES balance at the end of day $t - 1$.
- F_{t-1}^i denotes any borrowing by Bank i from the central bank's lending facility.
- O_{t-1}^i denotes the net amount borrowed from the central bank during open market operations (this variable will be negative if the central bank borrowed from the bank to extract ES balances).
- S_{t-1}^i denotes the gross amount borrowed overnight by Bank i from Bank j .
- K_t is the book-value of the equity claims on the firms profits (since the firms use the purchased goods as future capital). Any change in the value of this asset (due to future returns or sales to the asset managers) are not realised until the following day, when the firms repay dividends or the asset managers pay for the shares. So the book-value of this asset at the end of period $t - 1$ does not depend on whether K_t represents the original shares or the claims to payment from the asset managers.
- $[D_{t-1} - \psi M(\bar{S})]$ is the value of consumers' equity claims on the banks' profits.

With the firms purchasing all of the goods on the banks' balance sheets for use as future capital, $K_t = D_{t-1} - \psi M(\bar{S})$. This condition is reflected in the above balance sheets since adding the two balance sheets and imposing this condition gives:

$$ES_{t-1}^i + ES_{t-1}^j = O_{t-1}^i + O_{t-1}^j + F_{t-1}^i + F_{t-1}^j$$

Which must hold as an identity since, in aggregate, the banks can only acquire ES balances $(ES_{t-1}^i + ES_{t-1}^j)$ through open market operations $(O_{t-1}^i + O_{t-1}^j)$, or through lending facility use $(F_{t-1}^i + F_{t-1}^j)$.

3.6.6.1 Beginning of day

During this period, the central bank conducts new open market operations (denoted O_t^i and O_t^j). After these transactions, the balance sheets of the banks are:

$$ES_t^i + K_t + S_{t-1}^j \equiv \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right] + O_{t-1}^i + O_t^i + F_{t-1}^i + S_{t-1}^i$$

$$ES_t^j + S_{t-1}^i \equiv \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right] + O_{t-1}^j + O_t^j + F_{t-1}^j + S_{t-1}^j$$

While the ES balances of the banks are:

$$ES_t^i = ES_{t-1}^i + O_t^i$$

$$ES_t^j = ES_{t-1}^j + O_t^j$$

ES_t^i and ES_t^j denote the ES balances at the end of day t . However, at each point during this appendix, they will only include the transactions that have occurred up to that point.

3.6.6.2 Trading period

During the trading period of day t , the firms issue equity shares to the lending bank in exchange for deposits. Without loss of generality, assume Bank i is the lending bank during day t (the balance sheets when Bank j is the lending bank are little changed and are shown later).

Since the shock has not yet been revealed, the banks do not know the day t value of consumers' savings and therefore don't know K_{t+1} . So, during the trading period, K_{t+1} is valued at its expected value: $E[K_{t+1} | \mathcal{I}_{t-1}^B]$. Since the value of firms' deposits are always equal to the value of goods on the banks' balance sheets, during the trading period, the expected value of the deposits is $E[K_{t+1} | \mathcal{I}_{t-1}^B]$. So $E[K_{t+1} | \mathcal{I}_{t-1}^B]$ enters

on both sides of Bank i 's balance sheet, on the left-hand-side to denote the expected value of its equity claims, and on the right-hand-side to denote the expected value of its deposit liabilities:

$$ES_t^i + K_t + S_{t-1}^j + E[K_{t+1} | \mathcal{I}_{t-1}^B] \equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_{t-1}^i + O_t^i + F_{t-1}^i + S_{t-1}^i + E[K_{t+1} | \mathcal{I}_{t-1}^B]$$

$$ES_t^j + S_{t-1}^i \equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_{t-1}^j + O_t^j + F_{t-1}^j + S_{t-1}^j$$

$\mathcal{I}_{t-1}^B$ is the information set available to the banks and the central bank at the end of day $t-1$. Since the value of consumers' savings is independent of which bank is the lending bank, this is also the relevant information set for determining expectations of K_{t+1} during the trading period.

This is also the period when new interbank loans are made (denoted S_t^i and S_t^j). As a result, the balance sheets update to:

$$ES_t^i + K_t + S_{t-1}^j + E[K_{t+1} | \mathcal{I}_{t-1}^B] + S_t^j \equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_{t-1}^i + O_t^i + F_{t-1}^i + S_{t-1}^i + E[K_{t+1} | \mathcal{I}_{t-1}^B] + S_t^i$$

$$ES_t^j + S_{t-1}^i + S_t^i \equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_{t-1}^j + O_t^j + F_{t-1}^j + S_{t-1}^j + S_t^j$$

While the ES balances of the banks are:

$$ES_t^i = ES_{t-1}^i + O_t^i - S_t^j + S_t^i$$

$$ES_t^j = ES_{t-1}^j + O_t^j - S_t^i + S_t^j$$

For example, S_t^i causes ES balances to flow from Bank j to Bank i . In exchange for

this transfer of ES balances, Bank j gains a loan asset (with book-value equal to the value of transferred ES balances), while Bank i gains a loan liability.

3.6.6.3 Production period

During the production period of day t , the banks repay their day $t - 1$ use of the lending facility (with gross interest of R^F), repay any day $t - 1$ borrowing from the central bank via open market operations (or get repaid if they lent to the central bank) – this occurs with gross interest rates of $R_{t-1}^{O,i}$ and $R_{t-1}^{O,j}$ – and receive net interest based on their surplus ES balances at the end of day $t - 1$ (denoted $(R^{ES} - 1)$). The banks also repay any interbank lending (with gross interest of R^S , fixed by the commitment of the central bank to its interest rate target). All of these loans are risk-free in this model, so they are repaid in full. When loans are repaid, the balance sheet items are removed from the balance sheets, while the net interest is booked as a net profit:

$$\begin{aligned}
 ES_t^i + K_t + \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B] + S_t^j + \pi_t^i &\equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_t^i \\
 &\quad + \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B] + S_t^i + \pi_t^i \\
 ES_t^j + S_t^i + \pi_t^j &\equiv \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] + O_t^j + S_t^j + \pi_t^j
 \end{aligned} \tag{3.45}$$

Where:

$$ES_t^i = ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i$$

$$ES_t^j = ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j$$

$$\begin{aligned}\pi_t^i &= (R^{ES} - 1) ES_{t-1}^i - (R_{t-1}^{O,i} - 1) O_{t-1}^i \\ &\quad - (R^F - 1) F_{t-1}^i + (R^S - 1) [S_{t-1}^j - S_{t-1}^i]\end{aligned}$$

$$\begin{aligned}\pi_t^j &= (R^{ES} - 1) ES_{t-1}^j - (R_{t-1}^{O,j} - 1) O_{t-1}^j \\ &\quad - (R^F - 1) F_{t-1}^j + (R^S - 1) [S_{t-1}^i - S_{t-1}^j]\end{aligned}$$

π_t^i and π_t^j are the day t net profits of the banks from the transactions that have occurred up to this point during the day. The net profit terms on the right-hand-side of the balance sheet represent the increase (or decrease) in the equity of the banks due to the net profits. The left-hand-side profit terms represent the goods that are either transferred to the banks or that the banks must transfer to settle the relevant transactions (recall from sections 3.2.1.5 and 3.2.2.3 that all of the transactions included in π_t^i and π_t^j are settled using goods).

During this period: the firms pay dividends to either Bank i or the asset managers (these dividends equal αY_t – with $(1 - \theta_{t-1})$ being the proportion of dividends going to the asset managers and θ_{t-1} the proportion going directly to Bank i); the asset managers pay Bank i for any shares purchased at the end of day $t - 1$ (this payment is denoted $(1 - \theta_{t-1}) P_{t-1}$); the asset managers distribute any profits equally among the banks or are equally subsidised by the banks to offset any losses (the net profits distributed to each bank are $\frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}]$); Bank i 's profit is reduced by the effort cost (ϑ) if it did not sell *all* of its shares to the asset managers at the end of day $t - 1$; the banks pay dividends to their shareholders (the gross dividend payments are denoted $R_t^{D,i}$ and $R_t^{D,j}$); and consumers make their new savings decisions – the new equity the consumers purchase in the banks is denoted D_t , while the book-value of the goods the consumers use to pay for these shares is equal to $K_{t+1} + \psi M(\bar{S})$ (i.e. the goods that

are sold to the firms for use as future capital plus the goods that are used to pay the monitoring cost). Recall that each consumer invests half of their savings in each bank.

The monitoring cost is paid immediately (by levying a fixed fee on the shareholders), reducing the value of the goods on banks' balance sheets by $\psi M(\bar{S})$ and reducing the value of the consumers' equity in the banks by the same amount. Since this occurs immediately, this payment impacts the day t balance sheet (rather than impacting day $t + 1$ profits).

Only part of the αY_t or P_{t-1} is classified as net profit, since K_t of these goods were from the consumers' original equity investment.

With the consumers making their new savings decisions, the banks now know the true value of K_{t+1} . Since both the equity claims and the value of the deposits issued to the firms are both equally impacted by this new knowledge, $E[K_{t+1} | \mathcal{I}_{t-1}^B]$ is replaced on Bank i 's balance sheet by K_{t+1} .

Following these transactions (recall that payment of dividends to the consumers extinguishes their equity claims to the banks' assets):

$$ES_t^i + K_{t+1} + \frac{K_{t+1}}{2} + S_t^j + \pi_t^i \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^i + K_{t+1} + S_t^i + \pi_t^i$$

$$ES_t^j + \frac{K_{t+1}}{2} + S_t^i + \pi_t^j \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^j + S_t^j + \pi_t^j$$

Where:

$$ES_t^i = ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i + PF_t^2$$

$$ES_t^j = ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j - PF_t^2$$

$$PF_t^2 = \frac{K_t}{2} \tag{3.46}$$

$$\begin{aligned}
\pi_t^i = & \left(R^{ES} - 1 \right) ES_{t-1}^i - \left(R_{t-1}^{O,i} - 1 \right) O_{t-1}^i \\
& - \left(R^F - 1 \right) F_{t-1}^i + \left(R^S - 1 \right) \left[S_{t-1}^j - S_{t-1}^i \right] \\
& + \{ \theta_{t-1} [\alpha Y_t - K_t] + (1 - \theta_{t-1}) [P_{t-1} - K_t] \} - \vartheta I_{t-1} \\
& + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] - \left(R_t^{D,i} - 1 \right) \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right]
\end{aligned}$$

$$\begin{aligned}
\pi_t^j = & \left(R^{ES} - 1 \right) ES_{t-1}^j - \left(R_{t-1}^{O,j} - 1 \right) O_{t-1}^j \\
& - \left(R^F - 1 \right) F_{t-1}^j + \left(R^S - 1 \right) \left[S_{t-1}^i - S_{t-1}^j \right] \\
& + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] - \left(R_t^{D,j} - 1 \right) \frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right]
\end{aligned}$$

I_{t-1} is an indicator function equal to zero if Bank i sold *all* of the equity claims to the asset managers at the end of day $t - 1$ (i.e. *iff* $\theta_{t-1} = 0$), and one if Bank i retained some of the equity claims.

With all profits paid out as dividends on a daily basis, and all of the transactions in the profit equations being settled in goods, if Bank j 's net dividend payment to its shareholders is positive, then the value of extra goods it requires to complete its *gross* dividend payment equals $\frac{1}{2} \left[D_{t-1} - \psi M(\bar{S}) \right] = \frac{K_t}{2}$ (i.e. the value of the original equity investment).

Now suppose Bank j 's net dividend payment is negative. Then Bank j will require $R_t^{D,j} \frac{K_t}{2} < \frac{K_t}{2}$ of goods to pay the gross dividend to its shareholders and $(1 - R_t^{D,j}) \frac{K_t}{2}$ of goods to settle the transactions in its profit equation (I assume $R_t^{D,j} > 0$, since the bankruptcy risk is infinitesimally small with the parameter values used in this chapter). The sum of these two requirements is $\frac{K_t}{2}$. Therefore, in either case, Bank j needs $\frac{K_t}{2}$ of extra goods to settle all required payments.

Since Bank j has no goods on its balance sheet (aside from any positive net profit

transfers, see Eq. (3.45)), Bank j will need to purchase these $\frac{K_t}{2}$ of goods from Bank i .⁹¹ I assume that payment for these goods occur via ES balance transfers, with the resulting payment flows denoted PF_t^2 .⁹² For Bank i , $\frac{K_t}{2}$ is the exact value of goods remaining on its balance sheet after it has made all required payments.⁹³

This is also the period when the firms use their deposits to purchase the goods from the banks. This process results in the following balance sheet entries:

- The firms use half of their deposits to purchase the goods on Bank i 's balance sheet. This reduces the goods on Bank i 's balance sheet by $\frac{K_{t+1}}{2}$ and reduces their deposit liabilities by the same amount.
- The firms use the other half of their deposits to purchase the goods on Bank j 's balance sheet. This reduces the goods on Bank j 's balance sheet by $\frac{K_{t+1}}{2}$ and reduces Bank i 's deposit liabilities by the same amount.
- In order for this purchase to occur, Bank i must transfer $\frac{K_{t+1}}{2}$ of ES balances to Bank j , this transaction is denoted PF_t^1 .⁹⁴

Following these transactions, the banks' balance sheets are:

$$ES_t^i + K_{t+1} + S_t^j + \pi_t^i \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^i + S_t^i + \pi_t^i$$

$$ES_t^j + S_t^i + \pi_t^j \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^j + S_t^j + \pi_t^j$$

Where:

⁹¹Although Bank j 's assets during the day t production period will include consumers' new savings, in reality, consumers could not make these new savings without first receiving their day $t-1$ dividends. So these goods cannot be used to settle Bank j 's payments.

⁹²To ensure exchange-rate indifference, I assume that the exchange-rate for these PF_t^2 purchases is determined based on the day $t-1$ contract (determined during the beginning-of-day period of day $t-1$, see Appendix 3.6.5). This is because, before determining the day t exchange rate, the banks would know which bank will need to make these purchases, so they would not be indifferent.

⁹³Since the day $t-1$ contract is assumed to specify the formula for determining the value of PF_t^2 , Bank i must sell the goods at the pre-specified exchange rate.

⁹⁴The exchange rate is fixed based on the day t exchange-rate contract (see Appendix 3.6.5).

$$ES_t^i = ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i + PF_t^2 - PF_t^1$$

$$ES_t^j = ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j - PF_t^2 + PF_t^1$$

$$PF_t^1 = \frac{K_{t+1}}{2} \quad (3.47)$$

3.6.6.4 End of day

During the end-of-day period of day t , banks may borrow from the central bank's lending facility (F_t^i and F_t^j). This is also the period when banks may sell their shares of the firm's equity to the asset managers. Since sale of the shares does not impact the book-value of the equity, share sales have no impact on the banks' balance sheets. So the only end-of-day impact on the banks' balance sheets are the addition of any lending facility loans:

$$ES_t^i + K_{t+1} + S_t^j + \pi_t^i \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^i + S_t^i + \pi_t^i + F_t^i$$

$$ES_t^j + S_t^i + \pi_t^j \equiv \frac{1}{2} [D_t - \psi M(\bar{S})] + O_t^j + S_t^j + \pi_t^j + F_t^j$$

Where:

$$ES_t^i = ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i + PF_t^2 - PF_t^1 + F_t^i \quad (3.48)$$

$$ES_t^j = ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j - PF_t^2 + PF_t^1 + F_t^j \quad (3.49)$$

Since all profits must be paid out as dividends every day, $\pi_t^i = \pi_t^j = 0$ at the end of day t , so the banks' balance sheets are:

$$ES_t^i + K_{t+1} + S_t^j \equiv \frac{1}{2} \left[D_t - \psi M(\bar{S}) \right] + O_t^i + S_t^i + F_t^i$$

$$ES_t^j + S_t^i \equiv \frac{1}{2} \left[D_t - \psi M(\bar{S}) \right] + O_t^j + S_t^j + F_t^j$$

Which are the same as Eqs. (3.43) and (3.44) but moved forward one day.

Bank j is the day t lending bank

When Bank j is the lending bank in day t , all that changes are the balance sheets and the ES balances of the banks (the new payment flows, PF_t^1 , change direction); profits and the volume of the payment flows remain unchanged. Therefore, when Bank j is the lending bank in day t , the balance sheets of the banks at the end of day t are:

$$ES_t^i + S_t^j \equiv \frac{1}{2} \left[D_t - \psi M(\bar{S}) \right] + O_t^i + S_t^i + F_t^i$$

$$ES_t^j + K_{t+1} + S_t^i \equiv \frac{1}{2} \left[D_t - \psi M(\bar{S}) \right] + O_t^j + S_t^j + F_t^j$$

Where:

$$ES_t^i = ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i + PF_t^2 + PF_t^1 + F_t^i \quad (3.50)$$

$$ES_t^j = ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j - PF_t^2 - PF_t^1 + F_t^j \quad (3.51)$$

3.6.6.5 Bank dividends

From Section 3.6.6.3, banks' day t profits do not depend on which bank is the lending bank in day t , they only depend on which bank was the lending bank during day $t - 1$. Therefore, the day t profit equations when Bank i is the lending bank during day $t - 1$

(without loss of generality) are:

$$\begin{aligned}\pi_t^i &= (R^{ES} - 1) ES_{t-1}^i - (R_{t-1}^{O,i} - 1) O_{t-1}^i \\ &\quad - (R^F - 1) F_{t-1}^i + (R^S - 1) [S_{t-1}^j - S_{t-1}^i] \\ &\quad + \{\theta_{t-1} [\alpha Y_t - K_t] + (1 - \theta_{t-1}) [P_{t-1} - K_t]\} - \vartheta I_{t-1} \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] - (R_t^{D,i} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})]\end{aligned}$$

$$\begin{aligned}\pi_t^j &= (R^{ES} - 1) ES_{t-1}^j - (R_{t-1}^{O,j} - 1) O_{t-1}^j \\ &\quad - (R^F - 1) F_{t-1}^j + (R^S - 1) [S_{t-1}^i - S_{t-1}^j] \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] - (R_t^{D,j} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})]\end{aligned}$$

With the restriction that all profits must be paid out as dividends to the consumers on a daily basis, the dividend equations can be derived by setting the above profit equations to zero and re-arranging:

$$\begin{aligned}(R_t^{D,i} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] &= (R^{ES} - 1) ES_{t-1}^i - (R_{t-1}^{O,i} - 1) O_{t-1}^i \\ &\quad - (R^F - 1) F_{t-1}^i + (R^S - 1) [S_{t-1}^j - S_{t-1}^i] \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}] \\ &\quad + \{\theta_{t-1} [\alpha Y_t - K_t] + (1 - \theta_{t-1}) [P_{t-1} - K_t]\} - \vartheta I_{t-1}\end{aligned}$$

$$\begin{aligned}(R_t^{D,j} - 1) \frac{1}{2} [D_{t-1} - \psi M(\bar{S})] &= (R^{ES} - 1) ES_{t-1}^j - (R_{t-1}^{O,j} - 1) O_{t-1}^j \\ &\quad - (R^F - 1) F_{t-1}^j + (R^S - 1) [S_{t-1}^i - S_{t-1}^j] \\ &\quad + \frac{1}{2} (1 - \theta_{t-1}) [\alpha Y_t - P_{t-1}]\end{aligned}$$

3.6.6.6 Open market operations

Recall from Section 3.2.1.5, that the central bank operates so that both banks begin each day with the same expected end-of-day ES balance (excluding lending facility use). When open market operations are conducted, the day t lending bank has not yet been determined.

Therefore, the expected end-of-day ES balance (pre lending facility use) for Bank i is (using Eqs. (3.48) and (3.50)):

$$\begin{aligned}
 \mathbb{E}[ES_t^i - F_t^i | \mathcal{I}_{t-1}^B] & \quad (3.52) \\
 &= \frac{1}{2} \left\{ \mathbb{E}[ES_t^i - F_t^i | \text{Lend}, \mathcal{I}_{t-1}^B] + \mathbb{E}[ES_t^i - F_t^i | \text{Non-lend}, \mathcal{I}_{t-1}^B] \right\} \\
 &= \frac{1}{2} \mathbb{E}[ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j | \text{Lend}, \mathcal{I}_{t-1}^B] \\
 &\quad + \frac{1}{2} \mathbb{E}[O_t^i - S_t^j + S_t^i - PF_t^1 + PF_t^2 | \text{Lend}, \mathcal{I}_{t-1}^B] \\
 &\quad + \frac{1}{2} \mathbb{E}[ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j | \text{Non-lend}, \mathcal{I}_{t-1}^B] \\
 &\quad + \frac{1}{2} \mathbb{E}[O_t^i - S_t^j + S_t^i + PF_t^1 + PF_t^2 | \text{Non-lend}, \mathcal{I}_{t-1}^B] \\
 &= \mathbb{E}[ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i + PF_t^2 | \mathcal{I}_{t-1}^B]
 \end{aligned}$$

Where $\mathcal{I}_{t-1}^B$ is the information set of the banks (and the central bank) at the end of day $t - 1$.

From Section 3.2.3.3, the fixed aggregate ES balance (excluding any lending facility use) is $\overline{ES}$. Therefore, the central bank is assumed to operate such that $\mathbb{E}[ES_t^i - F_t^i | \mathcal{I}_{t-1}^B] = \mathbb{E}[ES_t^j - F_t^j | \mathcal{I}_{t-1}^B] = \frac{1}{2} \overline{ES}$ (where $\overline{ES} \geq 0$, consistent with the first two chapters).

Then, substituting Eqs. (3.43) and (3.46) into Eq. (3.52), gives the formula for open market operations with Bank i :

$$O_t^i = \frac{1}{2} \overline{ES} + \mathbb{E}[S_t^j - S_t^i | \mathcal{I}_{t-1}^B] \quad (3.53)$$

Similarly, using Eqs. (3.49) and (3.51) to get the expected end-of-day ES balance for Bank j , and then using Eqs. (3.44) and (3.46), gives the formula for open market operations with Bank j :

$$O_t^j = \frac{1}{2}\overline{ES} + E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B] \quad (3.54)$$

Eqs. (3.53) and (3.54) determine the net amount of ES balances the central bank wants to lend to the banks via open market operations (it is net lending because a negative value signifies that the central bank is willing to borrow ES balances). For the banks to agree to these open market operations, the central bank must satisfy the banks' open market operations participation constraints (i.e. the banks must at least be indifferent between accepting these open market operations and not participating). This is done through the appropriate setting of the open market operations interest rates ($R_t^{O,i}$ and $R_t^{O,j}$).

3.6.6.7 Open market operations interest rates

Suppose $R_t^{O,i} = R_t^{O,j} = R^{ES}$. At this interest rate, both banks would be willing to borrow any amount via open market operations. This is because, at this open market operation interest rate, extra ES balances are costless (since the bank could always choose to simply retain the ES balances and get the deposit interest of R^{ES}), but may provide some benefit by reducing borrowing requirements (since borrowing either in the market or via the lending facility costs more than R^{ES}). Therefore, as long as the central bank never borrows from the banks via open market operations (i.e. as long as $O_t^i \geq 0$ and $O_t^j \geq 0$), the banks' open market operations participation constraints will be satisfied with $R_t^{O,i} = R_t^{O,j} = R^{ES}$.⁹⁵

⁹⁵With $R_t^{O,i} = R_t^{O,j} = R^{ES}$, the banks strictly prefer to borrow via open market operations. As a result, the central bank could still achieve its desired O_t^i and O_t^j by setting the $R_t^{O,i}$ and $R_t^{O,j}$ slightly higher (and a profit-maximising central bank would do so). Setting $R_t^{O,i} = R_t^{O,j} = R^{ES}$, or setting the interest rates at the level a profit-maximising central bank would set, makes no difference to the qualitative results of this chapter.

Since it is possible for the central bank to set $\overline{ES} = 0$, the only value of $E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B]$ that will *always* satisfy the banks' open market operations participation constraints (with $R_t^{O,i} = R_t^{O,j} = R^{ES}$) is $E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B] = 0$. So, in my model, I assume the central bank conducts open market operations such that $R_t^{O,i} = R_t^{O,j} = R^{ES}$ and:

$$O_t^i = O_t^j = \frac{1}{2} \overline{ES} \quad (3.55)$$

To prove that these are valid open market operations (i.e. to show that Eq. (3.55) satisfies Eqs. (3.53) and (3.54)), Appendix 3.6.6.8 shows that with these open market operations, the interbank borrowing/lending decisions of the banks will satisfy $E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B] = 0$.⁹⁶

3.6.6.8 Interbank loans

Optimal interbank lending/borrowing decisions with no monitoring or effort costs

With $\vartheta = 0$, the shares would never be sold to the asset managers, but this would not be costly for the banks. In this scenario, and using the information in Appendix 3.6.6, the expected day $t + 1$ net profits of the banks are (without loss of generality, Bank i is the day t lending bank):

$$E[\pi_{t+1}^i] = - (R^{ES} - 1) \frac{1}{2} E[K_{t+1}] - \kappa E[F_t^i] - \frac{\kappa}{2} S_t + E[\alpha Y_{t+1} - K_{t+1}] \quad (3.56)$$

$$E[\pi_{t+1}^j] = (R^{ES} - 1) \frac{1}{2} E[K_{t+1}] - \kappa E[F_t^j] + \frac{\kappa}{2} S_t \quad (3.57)$$

⁹⁶For some values of $\overline{ES}$, there may be multiple values of O_t^i and O_t^j that satisfy both Eqs. (3.53) and (3.54), and the banks' open market operations participation constraints (this can occur because $E[S_t^i - S_t^j | \mathcal{I}_{t-1}^B]$ is dependent on O_t^i and O_t^j). For a given $\overline{ES}$, the choice of O_t^i and O_t^j within this set will not impact the qualitative results of the model (as long as there is some steady state interbank lending). However, since choices of O_t^i and O_t^j other than Eq. (3.55) will not be valid for all values of $\overline{ES}$, choosing O_t^i and O_t^j different to Eq. (3.55) would complicate the model.

The expectations in the above equations are evaluated given the information available to the banks during the day t trading period (i.e. $\mathcal{I}_{t-1}^B$ and which bank is the lending bank). Since it is only net interbank borrowing that impacts these equations, I have set $S_t = S_t^i - S_t^j$.

Banks' day t choice of S_t does not impact consumers' choices of D_t (since the consumers are not privy to the operational decisions of the banks). Moreover, with sufficiently low central bank interest rates (recall Appendix 3.6.5), changing the volume of interbank lending/borrowing would not impact the lending bank's issuance of equity to the firms. Therefore, with no monitoring costs, S_t has no impact on K_{t+1} or on interbank payment flows.

Since borrowing from the central bank's lending facility is costly, and there is no signalling when $\vartheta = 0$, banks will only borrow the minimum possible amount from the lending facility. Therefore, from Appendix 3.6.6.9, lending facility use will equal:

$$F_t^i = -\min \left\{ \left[\frac{\overline{ES}}{2} + S_t - \frac{K_{t+1}}{2} \right], 0 \right\}$$

$$F_t^j = -\min \left\{ \left[\frac{\overline{ES}}{2} - S_t + \frac{K_{t+1}}{2} \right], 0 \right\}$$

As a result, the expected profit equations simplify to:

$$\begin{aligned} \mathbb{E}[\pi_{t+1}^i] &= -\left(R^{ES} - 1\right) \frac{1}{2} \mathbb{E}[K_{t+1}] - \kappa \mathbb{P}(F_t^i > 0) \mathbb{E}[F_t^i | F_t^i > 0] - \frac{\kappa}{2} S_t \\ &\quad + \mathbb{E}[\alpha Y_{t+1} - K_{t+1}] \end{aligned}$$

$$\mathbb{E}[\pi_{t+1}^j] = \left(R^{ES} - 1\right) \frac{1}{2} \mathbb{E}[K_{t+1}] - \kappa \mathbb{P}(F_t^j > 0) \mathbb{E}[F_t^j | F_t^j > 0] + \frac{\kappa}{2} S_t$$

From the unique stable decision rule for K_{t+1} (see Section 3.2.5): $K_{t+1} = \overline{K} \varepsilon_t + \mathbb{E}[K_{t+1}]$.

Therefore:

$$P(F_t^i > 0) = P\left(\varepsilon_t > \frac{1}{\bar{K}} [\bar{ES} + 2S_t - E[K_{t+1}]]\right)$$

$$P(F_t^j > 0) = P\left(\varepsilon_t < \frac{1}{\bar{K}} [-\bar{ES} + 2S_t - E[K_{t+1}]]\right)$$

Suppose $\bar{ES} \geq 2S_t - E[K_{t+1}] \geq -\bar{ES}$ then, using details of the shock distribution given in Section 3.2.5 (recall that B is the upper bound of the shock):

$$\frac{dE[\pi_{t+1}^i]}{dS_t} = \begin{cases} -\frac{\kappa}{2} < 0 & \text{if } \bar{ES} + 2S_t - E[K_{t+1}] \geq \bar{K}B \\ -\frac{\kappa}{2\bar{K}B} [\bar{ES} + 2S_t - E[K_{t+1}]] \leq 0 & \text{if } \bar{ES} + 2S_t - E[K_{t+1}] < \bar{K}B \end{cases}$$

$$\frac{dE[\pi_{t+1}^j]}{dS_t} = \begin{cases} \frac{\kappa}{2} > 0 & \text{if } -\bar{ES} + 2S_t - E[K_{t+1}] \leq -\bar{K}B \\ \frac{-\kappa}{2\bar{K}B} [-\bar{ES} + 2S_t - E[K_{t+1}]] \geq 0 & \text{if } -\bar{ES} + 2S_t - E[K_{t+1}] > -\bar{K}B \end{cases}$$

Therefore, within this region, the optimal S_t for Bank i is $S_t = \frac{1}{2} [E[K_{t+1}] - \bar{ES}]$, while the optimal S_t for Bank j is $S_t = \frac{1}{2} [E[K_{t+1}] + \bar{ES}]$.

Although I omit the working here, using the shock distribution to derive the derivatives in the same way as above, it can be shown that:

- In the region $2S_t - E[K_{t+1}] > \bar{ES}$, the profit derivatives with respect to S_t are negative for both banks.
- In the region $-\bar{ES} > 2S_t - E[K_{t+1}]$, the profit derivatives with respect to S_t are positive for both banks.

Therefore, $S_t = \frac{1}{2} [E[K_{t+1}] - \bar{ES}]$ is the global optimum for Bank i , $S_t = \frac{1}{2} [E[K_{t+1}] + \bar{ES}]$ is the global optimum for Bank j , and the outcome of bargaining between the banks must be an S_t between these points (inclusive). This is consistent with Chapter 2 (when comparing these results to Chapter 2, recall that $\eta_i = 0$ in this chapter).

Discussion of the two-department assumption

Since the interbank lending/borrowing decisions “net-out” when determining the consumers’ returns (see Eq. (3.11)), it is only the impact of these decisions on lending facility use that impacts either the ability of the lending bank to sell the shares, or the sale price. So, as long as interbank lending/borrowing decisions (without the two-department assumption) ensure that increases in the aggregate ES balance reduce the probability of lending facility use, then the qualitative results of this model (with respect to the effort cost) will be the same as the results with the two-department assumption.

If the two-department assumption were removed, as long as increases in the aggregate ES balance cause a reduction in the steady-state volume of interbank lending (as they do with two-department assumption, see Section 3.2.4), the qualitative conclusions of this model (with respect to the monitoring cost) will not be impacted by the two-department assumption.

The two-department assumption further simplifies the model by preventing the banks from trying to use the lending facility (as a signalling device) to manipulate transactions between the lending bank and the asset managers. The possibility of manipulation would increase the complexity of the model; because the asset managers would account for this possibility when interpreting the signal, which would feed back into the effectiveness of the signal and the benefit of manipulation. So it is not clear whether the possibility of manipulation (without the two-department assumption) would impact the qualitative results of the model.

Interbank lending/borrowing decisions based on the Chapter 2 bargaining solution

Interbank lending/borrowing decisions are based on expectations of production-period payment flows and the impact these payment flows are expected to have on the banks’ ES balances. As noted in Section 3.2.3.3, to determine banks’ interbank lending/borrowing

decisions in this chapter, I assume banks make these decisions in the same way as in Chapter 2 (with $\eta_i = 0$ for both banks, for simplicity). That is, if expected payment flows (akin to first-period payment flows in the Chapter 2 model) are expected to cause one of the bank's ES balances (prior to any lending facility use) to fall into deficit, this bank will seek to borrow sufficient ES balances to return its expected (pre-lending facility) ES balance to zero, with the other bank always willing to lend this amount (since $\overline{ES} \geq 0$).

Although there are only two banks in this model, which is consistent with the Chapter 1 model (rather than the greater than two banks in the Chapter 2 model), there are several reasons why it is better to use the Chapter 2 interbank lending/borrowing decisions:

- The Chapter 2 model is a better reflection of a realistic banking system (i.e. banking systems typically don't just have two banks).
- The technology shock in this model is similar to the second-period shock in the Chapter 2 model (i.e. the shock probability distribution has an atom at zero, see Section 3.2.5).
- The Chapter 2 decision is simpler – the Chapter 1 interbank lending/borrowing decision is non-linear and not available analytically.
- The Chapter 1 and Chapter 2 decisions are similar – the borrowing bank in Chapter 1 does not end the first-period with an ES balance far from zero.
- The Chapter 2 decision is a feasible bargaining outcome in the Chapter 1 framework (i.e. it satisfies the Nash Bargaining Problem disagreement points).

When Bank i is the lending bank, the banks' end-of-day ES balances (excluding lending facility use) are (using Eqs. (3.43), (3.44), (3.48), (3.49), (3.47), and (3.46)):

$$ES_t^i - F_t^i = O_t^i - S_t^j + S_t^i - \frac{1}{2}K_{t+1}$$

$$ES_t^j - F_t^j = O_t^j - S_t^i + S_t^j + \frac{1}{2}K_{t+1}$$

Then, substituting in Eq. (3.55) for the open market operations gives:

$$ES_t^i - F_t^i = \frac{1}{2}\overline{ES} - S_t^j + S_t^i - \frac{1}{2}K_{t+1}$$

$$ES_t^j - F_t^j = \frac{1}{2}\overline{ES} - S_t^i + S_t^j + \frac{1}{2}K_{t+1}$$

Since interbank lending/borrowing decisions are made before knowledge of the shock, the banks make these decisions using expected payment flows. Since the payment flows are independent of which bank is the lending bank (with consumers investing half of their savings in each bank), the expected payment flows derived during the trading period are the same as those that would be derived at the end of the previous day, so I can use the same information set. Replacing the actual payment flows with the expected payment flows gives:

$$ES_t^i - F_t^i = \frac{1}{2}\overline{ES} - S_t^j + S_t^i - \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]$$

$$ES_t^j - F_t^j = \frac{1}{2}\overline{ES} - S_t^i + S_t^j + \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]$$

From these equations, it is only net borrowing (i.e. $S_t^i - S_t^j$) that impacts pre-lending facility ES balances (recall that S_t^i and S_t^j are gross borrowing and are therefore non-negative). So, for simplicity, I assume that only one bank borrows per day (e.g. if $S_t^i > 0$ then $S_t^j = 0$). Then, using the Chapter 2 interbank lending/borrowing decision rule (i.e. borrowing only to return their expected pre-lending facility ES balance to zero), and re-arranging, gives :

$$S_t^i = -\min\left\{\left[\frac{1}{2}\overline{ES} - S_t^j - \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.58)$$

$$S_t^j = -\min\left\{\left[\frac{1}{2}\overline{ES} - S_t^i + \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.59)$$

Using the same information as above (and Eqs. (3.50) and (3.51)), when Bank j is the lending bank:

$$S_t^i = -\min\left\{\left[\frac{1}{2}\overline{ES} - S_t^j + \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.60)$$

$$S_t^j = -\min\left\{\left[\frac{1}{2}\overline{ES} - S_t^i - \frac{1}{2}\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.61)$$

From Eq. (3.59), if $S_t^i = 0$ then $S_t^j = 0$ (since $\overline{ES} \geq 0$ and $\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B] \geq 0$).

Therefore, when Bank i is the lending bank (using Eqs. (3.58) and (3.59)):

$$S_t^i = -\min\left\{\frac{1}{2}\left[\overline{ES} - \mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.62)$$

$$S_t^j = 0 \quad (3.63)$$

From Eq. (3.60), if $S_t^j = 0$ then $S_t^i = 0$. Therefore, when Bank j is the lending bank (using Eqs. (3.60) and (3.61)):

$$S_t^i = 0 \quad (3.64)$$

$$S_t^j = -\min\left\{\frac{1}{2}\left[\overline{ES} - \mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} \quad (3.65)$$

Finally, since each bank has an equal probability of being the lending bank, and $\mathbb{E}[K_{t+1}|\mathcal{I}_{t-1}^B]$ is not dependent on which bank is the lending bank (using Eqs. (3.62), (3.63), (3.64), and (3.65)):

$$\begin{aligned}
\mathbb{E}[S_t^i - S_t^j | \mathcal{I}_{t-1}^B] &= \frac{1}{2} \mathbb{E} \left[-\min \left\{ \frac{1}{2} [\overline{ES} - \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B]], 0 \right\} \middle| \text{Bank } i \text{ lends, } \mathcal{I}_{t-1}^B \right] \\
&\quad + \frac{1}{2} \mathbb{E} \left[\min \left\{ \frac{1}{2} [\overline{ES} - \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B]], 0 \right\} \middle| \text{Bank } j \text{ lends, } \mathcal{I}_{t-1}^B \right] \\
&= 0
\end{aligned}$$

3.6.6.9 Lending facility use

From Section 3.2.3.3, each bank will only borrow from the central bank's lending facility if it has an ES balance deficit at the end of the production period, and will only borrow an amount sufficient to return its ES balance to zero.

So, if Bank i is the lending bank during day t , lending facility use is determined by (using Eqs. (3.48) and (3.49)):

$$F_t^i = -\min \left\{ [ES_{t-1}^i - O_{t-1}^i - F_{t-1}^i - S_{t-1}^i + S_{t-1}^j + O_t^i - S_t^j + S_t^i - PF_t^1 + PF_t^2], 0 \right\}$$

$$F_t^j = -\min \left\{ [ES_{t-1}^j - O_{t-1}^j - F_{t-1}^j - S_{t-1}^j + S_{t-1}^i + O_t^j - S_t^i + S_t^j + PF_t^1 - PF_t^2], 0 \right\}$$

Then, substituting in Eqs. (3.43), (3.44), (3.47), (3.46), (3.55), (3.62), and (3.63), lending facility use is:

$$F_t^i = -\min \left\{ \left[\frac{1}{2} \overline{ES} - \min \left\{ \left[\frac{1}{2} \overline{ES} - \frac{1}{2} \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B] \right], 0 \right\} - \frac{1}{2} K_{t+1} \right], 0 \right\} \quad (3.66)$$

$$F_t^j = -\min \left\{ \left[\frac{1}{2} \overline{ES} + \min \left\{ \left[\frac{1}{2} \overline{ES} - \frac{1}{2} \mathbb{E}[K_{t+1} | \mathcal{I}_{t-1}^B] \right], 0 \right\} + \frac{1}{2} K_{t+1} \right], 0 \right\} \quad (3.67)$$

Similarly, when Bank j is the lending bank during day t , lending facility use is determined by (using Eqs. (3.43), (3.44), (3.47), (3.46), (3.50), (3.51), (3.55), (3.64),

and (3.65)):

$$F_t^i = -\min\left\{\left[\frac{1}{2}\overline{ES} + \min\left\{\left[\frac{1}{2}\overline{ES} - \frac{1}{2}E[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} + \frac{1}{2}K_{t+1}\right], 0\right\} \quad (3.68)$$

$$F_t^j = -\min\left\{\left[\frac{1}{2}\overline{ES} - \min\left\{\left[\frac{1}{2}\overline{ES} - \frac{1}{2}E[K_{t+1}|\mathcal{I}_{t-1}^B]\right], 0\right\} - \frac{1}{2}K_{t+1}\right], 0\right\} \quad (3.69)$$

From Eqs. (3.66), (3.67), (3.68), and (3.69), lending facility use for the lending bank is the same irrespective of which bank is the lending bank (similarly for the non-lending bank). So the remainder of the analysis in this section assumes, without loss of generality, that Bank i is the lending bank.

These lending facility use equations can be split into the case where there is interbank lending and the case where there is no interbank lending. When there is interbank lending during the day t trading period (i.e. when $E[K_{t+1}|\mathcal{I}_{t-1}^B] > \overline{ES}$):

$$F_t^i = \begin{cases} \frac{1}{2}[K_{t+1} - E[K_{t+1}|\mathcal{I}_{t-1}^B]] & \text{if } K_{t+1} > E[K_{t+1}|\mathcal{I}_{t-1}^B] \\ 0 & \text{otherwise} \end{cases}$$

$$F_t^j = \begin{cases} -\overline{ES} - \frac{1}{2}[K_{t+1} - E[K_{t+1}|\mathcal{I}_{t-1}^B]] & \text{if } K_{t+1} < E[K_{t+1}|\mathcal{I}_{t-1}^B] - 2\overline{ES} \\ 0 & \text{otherwise} \end{cases}$$

When there is no interbank lending during the day t trading period:

$$F_t^i = \begin{cases} \frac{1}{2}[K_{t+1} - \overline{ES}] & \text{if } K_{t+1} > \overline{ES} \\ 0 & \text{otherwise} \end{cases}$$

$$F_t^j = 0$$

3.6.7 Deterministic steady state equations

From Eqs. (3.19), (3.20), (3.22), (3.23), and (3.25), and for a given $\bar{K}$, the steady state values of the other real economy variables are (after cross-substituting and rearranging):

$$\bar{R}^D = \beta^{-1} \quad (3.70)$$

$$\bar{W} = (1 - \alpha) [\alpha\beta]^{\frac{\alpha}{1-\alpha}} \bar{A}^{\frac{1}{1-\alpha}} \quad (3.71)$$

$$\bar{C} = \frac{1}{\chi} (1 - \alpha) [\alpha\beta]^{\frac{\alpha}{1-\alpha}} \bar{A}^{\frac{1}{1-\alpha}} \quad (3.72)$$

$$\bar{H} = [\alpha\beta\bar{A}]^{\frac{-1}{1-\alpha}} \bar{K} \quad (3.73)$$

$$\bar{Y} = [\alpha\beta]^{-1} \bar{K} \quad (3.74)$$

Recall that the steady state value of $S_t^i + S_t^j$ is $\bar{S}$. Then, from Eqs. (3.21) and (3.26), and substituting in the above steady state values, $\bar{K}$ and $\bar{S}$ are jointly determined by the following equations:

$$\bar{K} = \frac{1 - \alpha}{\chi(1 - \alpha\beta)} [\alpha\beta\bar{A}]^{\frac{1}{1-\alpha}} + \frac{\alpha\beta\psi M(\bar{S})}{1 - \alpha\beta} \quad (3.75)$$

$$\bar{S} = -\min\left\{\frac{1}{2} [\bar{E}\bar{S} - \bar{K}], 0\right\} \quad (3.76)$$

Since this chapter only seeks to derive qualitative conclusions, the exact form of $M(\cdot)$ is not important; all that is required is that the monitoring cost increases as interbank market activity declines $\left(\frac{dM(\bar{S})}{d\bar{S}} < 0\right)$ (to be consistent with the statements of *Keister et al.* (2008) and *Heath et al.* (2013) in Section 3.1). I use the following functional form:

$$M(\bar{S}) = (\bar{S})^{-1} - (\tilde{S})^{-1} \quad (3.77)$$

Where:

$$\tilde{S} = \frac{1 - \alpha}{2\chi(1 - \alpha\beta)} [\alpha\beta\bar{A}]^{\frac{1}{1-\alpha}} \quad (3.78)$$

$\tilde{S}$ is the volume of steady state interbank lending if $\bar{E}\bar{S} = 0$ and $\psi = 0$.

Using this functional form, Appendix 3.6.8 solves for $\bar{K}$ and $\bar{S}$ as a function of the parameters, and derives the important properties of the result that are outlined in Section 3.2.4.

3.6.8 Deriving steady state capital and interbank lending

First, note that if $\bar{K} \leq \bar{E}\bar{S}$, then $\bar{S} = 0$, at which point the monitoring cost is undefined. Therefore, a feasible solution must have $\bar{E}\bar{S} < \bar{K}$. This appendix shows that a feasible solution always exists.

Suppose $\bar{E}\bar{S} < \bar{K}$, then substituting Eqs. (3.76), and (3.77) into Eq. (3.75), and re-arranging, gives a quadratic function for $\bar{K}$:

$$[\tilde{S}] \bar{K}^2 + \left[\left(\frac{\alpha\beta\psi}{1 - \alpha\beta} \right) - 2\tilde{S}^2 - \tilde{S}\bar{E}\bar{S} \right] \bar{K} - \left\{ \left[\left(\frac{\alpha\beta\psi}{1 - \alpha\beta} \right) - 2\tilde{S}^2 \right] \bar{E}\bar{S} + 2 \left(\frac{\alpha\beta\psi}{1 - \alpha\beta} \right) \tilde{S} \right\} = 0 \quad (3.79)$$

Where:

$$\tilde{S} = \frac{1 - \alpha}{2\chi(1 - \alpha\beta)} [\alpha\beta\bar{A}]^{\frac{1}{1-\alpha}} > 0$$

The two roots of this quadratic function are (after some simplification):

$$\bar{K} = \frac{- \left[\left(\frac{\alpha\beta\psi}{1 - \alpha\beta} \right) - 2\tilde{S}^2 - \tilde{S}\bar{E}\bar{S} \right] \pm \sqrt{\left[\left(\frac{\alpha\beta\psi}{1 - \alpha\beta} \right) - 2\tilde{S}^2 - \tilde{S}\bar{E}\bar{S} \right]^2 - 8\tilde{S}^3\bar{E}\bar{S}}}{2\tilde{S}}$$

When $\psi = 0$, the discriminant is positive. As ψ is increased, the discriminant

increases. Therefore, the discriminant is positive for all $\psi \geq 0$. So $\bar{K}$ has two real roots.

Also, as ψ increases, the first term in the numerator decreases. Therefore, the smaller root is decreasing in ψ .

3.6.8.1 Proof that the smaller root is not feasible

When $\psi = 0$, the smaller root is:

$$\bar{K} = \frac{2\tilde{S} + \overline{ES} - |2\tilde{S} - \overline{ES}|}{2}$$

Suppose $2\tilde{S} \geq \overline{ES}$, then the smaller root is (when $\psi = 0$):

$$\bar{K} = \overline{ES}$$

Since the smaller root is decreasing in ψ , as ψ is increased above zero, $\bar{K} < \overline{ES}$. But this is not a feasible result since the quadratic function was formed based on the assumption that $\bar{K} > \overline{ES}$. So the smaller root is not a feasible root when $2\tilde{S} \geq \overline{ES}$.

Suppose $2\tilde{S} < \overline{ES}$, then the smaller root is (when $\psi = 0$):

$$\bar{K} = 2\tilde{S}$$

But then $\bar{K} < \overline{ES}$, and remains less than $\overline{ES}$ as ψ is increased. So the smaller root is not a feasible root when $2\tilde{S} < \overline{ES}$.

Therefore, the smaller root is not a feasible root. □

3.6.8.2 Proof that the larger root is always feasible

Re-arranging the larger root of Eq. (3.79) gives:

$$\begin{aligned}
\bar{K} &= \frac{-\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 - \tilde{S}\overline{ES}\right] + \sqrt{\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) + 2\tilde{S}^2 + \tilde{S}\overline{ES}\right]^2 - 8\tilde{S}^3\overline{ES}}}{2\tilde{S}} \\
&= \frac{-\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 - \tilde{S}\overline{ES}\right] + \sqrt{\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 + \tilde{S}\overline{ES}\right]^2 + 8\tilde{S}^2\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right)}}{2\tilde{S}} \\
&> \frac{-\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 - \tilde{S}\overline{ES}\right] + \sqrt{\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 + \tilde{S}\overline{ES}\right]^2}}{2\tilde{S}} \\
&\geq \overline{ES}
\end{aligned}$$

Therefore, the larger root is always feasible. $\square$

The following conclusions can be derived from this result:

- There is always some interbank lending in steady state ($\bar{S} > 0$).
- For each setting of the parameters, there is a unique $\bar{K}$ and $\bar{S}$ that solve Eqs. (3.75) and (3.76).

3.6.8.3 Derivative of the larger root

Taking the derivative of the larger root with respect to $\overline{ES}$ gives:

$$\frac{d\bar{K}}{d\overline{ES}} = \frac{1}{2} + \frac{\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 + \tilde{S}\overline{ES}}{2\sqrt{\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) + 2\tilde{S}^2 + \tilde{S}\overline{ES}\right]^2 - 8\tilde{S}^3\overline{ES}}}$$

Re-arranging the denominator of this derivative gives:

$$\frac{d\bar{K}}{d\overline{ES}} = \frac{1}{2} + \frac{\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 + \tilde{S}\overline{ES}}{2\sqrt{\left[\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right) - 2\tilde{S}^2 + \tilde{S}\overline{ES}\right]^2 + 8\tilde{S}^2\left(\frac{\alpha\beta\psi}{1-\alpha\beta}\right)}} \quad (3.80)$$

From this version of the derivative, it can be seen that the square-root in the denominator is larger than the numerator. Therefore: $0 < \frac{d\bar{K}}{d\overline{ES}} < 1$ for all $\overline{ES}$. That is, $\bar{K}$ is always increasing, but at a slower rate than $\overline{ES}$.

The following conclusions can be derived from this result:

- Since $\bar{S} = -\frac{1}{2} [\bar{ES} - \bar{K}]$, steady state interbank lending decreases as $\bar{ES}$ increases ($\frac{d\bar{S}}{d\bar{ES}} < 0$); consistent with the first two chapters.
- Steady state interbank market activity is maximised when $\bar{ES} = 0$, at which point $\bar{K} = 2\tilde{S}$ (from substituting $\bar{ES} = 0$ into the larger root), $\bar{S} = \tilde{S}$, and therefore $M(\bar{S}) = 0$.
- When combined with the result that $\bar{K} > \bar{ES}$, $\bar{K}$ always increasing at a slower rate than $\bar{ES}$ implies that, as the aggregate ES balance increases without bound ($\bar{ES} \rightarrow \infty$), interbank market activity approaches zero ($\bar{S} \rightarrow 0$), and the monitoring cost increases without bound ($M(\bar{S}) \rightarrow \infty$).

3.6.9 Deriving the unique stable decision rules

Using the simplified set of equations in Section 3.2.3.4 (to substitute out the other choice/endogenous variables), the equations that determine H_t and K_{t+1} are:

$$\mathbb{E} \left[\alpha H_{t+1} - \frac{\{\kappa F_t + \vartheta I_t\} H_{t+1}^\alpha}{A_{t+1} K_{t+1}^\alpha} \middle| \mathcal{I}_t^C \right] = \frac{K_{t+1} H_t^\alpha}{\beta A_t K_t^\alpha} \quad (3.81)$$

$$A_t H_t^{1-\alpha} K_t^\alpha = \frac{(1-\alpha)}{\chi} [A_t H_t^{-\alpha} K_t^\alpha] + K_{t+1} + \psi M(\bar{S}) + \vartheta I_{t-1} \quad (3.82)$$

Since there is no exact analytical solution to these equations, I construct first-order approximations of Eqs. (3.81) and (3.82) around the deterministic steady state (these approximations are simplified using the steady state relationship ($\bar{K} = \alpha\beta\bar{Y}$)):

$$k_{t+1} - \mathbb{E} [h_{t+1} | \mathcal{I}_t^C] \approx -\alpha h_t + \alpha k_t + \varepsilon_t - \left[\frac{\vartheta}{\alpha \bar{Y}} \right] \mathbb{E} [I_t | \mathcal{I}_t^C] - \left[\frac{\kappa}{\alpha \bar{Y}} \right] \mathbb{E} [F_t | \mathcal{I}_t^C] \quad (3.83)$$

$$\alpha\beta k_{t+1} \approx \left[\frac{(\chi\bar{H} + \alpha)(1 - \alpha)}{\chi\bar{H}} \right] h_t + \left[\frac{\alpha[\chi\bar{H} - (1 - \alpha)]}{\chi\bar{H}} \right] k_t - \left[\frac{\vartheta}{\bar{Y}} \right] I_{t-1} + \left[\frac{\chi\bar{H} - (1 - \alpha)}{\chi\bar{H}} \right] \varepsilon_t \quad (3.84)$$

Where lower case variables denote percentage deviations from their deterministic steady state (e.g. $h_t = \left(\frac{H_t - \bar{H}}{\bar{H}}\right)$) and $\left(\frac{A_t - \bar{A}}{\bar{A}}\right) \approx \ln(A_t) - \ln(\bar{A}) = \varepsilon_t$.

Since I_t is a binary variable, a partial derivative of Eqs. (3.81) and (3.82) with respect to I_t does not exist (similarly for I_{t-1}). So I derive first-order approximations by computing the partial derivatives of each function with respect to the continuous variables (as in a first-order Taylor approximation), and by computing the finite difference of each function with respect to the binary variables (to use in place of the partial derivatives).⁹⁷

As noted in Section 3.2.5, consumers form stationary unconditional-expectations of the necessary banking system variables. So $E[I_t | \mathcal{I}_t^C]$ and $E[F_t | \mathcal{I}_t^C]$ are replaced by the constants $E[I | \mathcal{I}^C]$ and $E[F | \mathcal{I}^C]$, respectively.

To use the Blanchard-Kahn approach, define (re-defining the variables in this way removes all constants from the state-space form):

$$\begin{aligned} \tilde{h}_t = & h_t + \left[\frac{(1 - \alpha) - (1 - \beta)\chi\bar{H} - (1 - \alpha)\chi\bar{H}}{(1 - \alpha\beta)\chi\bar{H}} \right] \left[\frac{\vartheta}{(1 - \alpha)\bar{Y}} \right] E[I | \mathcal{I}^C] \\ & + \left[\frac{(1 - \alpha) - (1 - \beta)\chi\bar{H}}{(1 - \alpha\beta)\chi\bar{H}} \right] \left[\frac{\kappa}{(1 - \alpha)\bar{Y}} \right] E[F | \mathcal{I}^C] \end{aligned} \quad (3.85)$$

⁹⁷In Eq. (3.82), the finite difference with respect to I_{t-1} is equivalent to the true impact of I_{t-1} changing from zero to one; so the finite difference is not an approximation in this equation. For Eq. (3.81), I_t only enters the equation in the expectation. Then, since $P(I_t = 1 | \mathcal{I}_t^C) = E[I_t | \mathcal{I}_t^C]$, the expectation in Eq. (3.81) can be re-written as:

$$E\left[\alpha H_{t+1} - \frac{\kappa F_t H_{t+1}^\alpha}{A_{t+1} K_{t+1}^\alpha} \middle| \mathcal{I}_t^C\right] - E[I_t | \mathcal{I}_t^C] E\left[\frac{\vartheta H_{t+1}}{Y_{t+1}} \middle| \mathcal{I}_t^C, I_t = 1\right]$$

Therefore, the finite-difference approximation for I_t in Eq. (3.81) will be accurate as long as $E\left[\frac{H_{t+1}}{Y_{t+1}} \middle| \mathcal{I}_t^C, I_t = 1\right] \approx \frac{\bar{H}}{\bar{Y}}$.

$$\tilde{k}_t = k_t + \left[\frac{\chi\bar{H} + \alpha - \alpha\chi\bar{H}}{(1 - \alpha\beta)\chi\bar{H}} \right] \left[\frac{\vartheta}{\alpha\bar{Y}} \right] E[I|\mathcal{I}^C] + \left[\frac{\chi\bar{H} + \alpha}{(1 - \alpha\beta)\chi\bar{H}} \right] \left[\frac{\kappa}{\alpha\bar{Y}} \right] E[F|\mathcal{I}^C] \quad (3.86)$$

$$\tilde{I}_t = I_t - E[I|\mathcal{I}^C] \quad (3.87)$$

Then define three mean-zero prediction errors (all are unknown to consumers during day t):

$$u_{h,t+1} = \tilde{h}_{t+1} - E[\tilde{h}_{t+1}|\mathcal{I}_t^C]$$

$$u_{\varepsilon,t+1} = \varepsilon_{t+1}$$

$$u_{I,t} = \tilde{I}_t$$

Then Eqs. (3.83) and (3.84) can be re-written in state-space form as:

$$\begin{aligned} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & \alpha\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{h}_{t+1} \\ \tilde{k}_{t+1} \\ \varepsilon_{t+1} \\ \tilde{I}_t \end{bmatrix} \approx \\ & \begin{bmatrix} \begin{bmatrix} -\alpha \\ (\chi\bar{H} + \alpha)(1 - \alpha) \\ \chi\bar{H} \end{bmatrix} & \begin{bmatrix} \alpha \\ \alpha[\chi\bar{H} - (1 - \alpha)] \\ \chi\bar{H} \end{bmatrix} & \begin{bmatrix} 1 \\ \chi\bar{H} - (1 - \alpha) \\ \chi\bar{H} \end{bmatrix} & \begin{bmatrix} 0 \\ -\vartheta \\ \bar{Y} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \tilde{h}_t \\ \tilde{k}_t \\ \varepsilon_t \\ \tilde{I}_{t-1} \end{bmatrix} + \\ & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{h,t+1} \\ u_{\varepsilon,t+1} \\ u_{I,t} \end{bmatrix} \end{aligned}$$

Re-arranging and simplifying gives:

$$\begin{bmatrix} \frac{(\chi\bar{H}+\alpha)(1-\alpha)+\alpha^2\beta\chi\bar{H}}{\alpha\beta\chi\bar{H}} \\ \frac{(\chi\bar{H}+\alpha)(1-\alpha)}{\alpha\beta\chi\bar{H}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\alpha[\chi\bar{H}-(1-\alpha)]-\alpha^2\beta\chi\bar{H}}{\alpha\beta\chi\bar{H}} \\ \frac{\alpha[\chi\bar{H}-(1-\alpha)]}{\alpha\beta\chi\bar{H}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\chi\bar{H}-(1-\alpha)-\alpha\beta\chi\bar{H}}{\alpha\beta\chi\bar{H}} \\ \frac{\chi\bar{H}-(1-\alpha)}{\alpha\beta\chi\bar{H}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{-\vartheta}{\alpha\beta\bar{Y}} \\ \frac{-\vartheta}{\alpha\beta\bar{Y}} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{h}_{t+1} \\ \tilde{k}_{t+1} \\ \varepsilon_{t+1} \\ \tilde{I}_t \end{bmatrix} \approx \\
+ \begin{bmatrix} \tilde{h}_t \\ \tilde{k}_t \\ \varepsilon_t \\ \tilde{I}_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{h,t+1} \\ u_{\varepsilon,t+1} \\ u_{I,t} \end{bmatrix}$$

The eigenvalues of the reduced-form state-variable coefficient matrix are: $\lambda_1 = \frac{1}{\alpha\beta} > 1$, $\lambda_2 = \alpha < 1$, and $\lambda_3 = \lambda_4 = 0$. Therefore, there are three eigenvalues inside the unit circle and one outside the unit circle. Since there is only one control variable (h_t), and three pre-determined/exogenous variables, the Blanchard-Kahn conditions are satisfied and there is a unique stable solution.

To derive the unique stable solution (via the Blanchard-Kahn approach), the reduced-form coefficient matrix is decomposed using an Eigen-decomposition (i.e. $D\Lambda D^{-1}$ where Λ is a diagonal matrix of eigenvalues and D is a matrix of corresponding eigenvectors). Then the state-space form can be re-written as:

$$\begin{bmatrix} \hat{h}_{t+1} \\ \hat{k}_{t+1} \\ \hat{\varepsilon}_{t+1} \\ \hat{I}_t \end{bmatrix} \approx \begin{bmatrix} \frac{1}{\alpha\beta} & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{h}_t \\ \hat{k}_t \\ \hat{\varepsilon}_t \\ \hat{I}_{t-1} \end{bmatrix} + D^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{h,t+1} \\ u_{\varepsilon,t+1} \\ u_{I,t} \end{bmatrix}$$

Where:

$$\begin{bmatrix} \hat{h}_t \\ \hat{k}_t \\ \hat{\varepsilon}_t \\ \hat{I}_{t-1} \end{bmatrix} = D^{-1} \begin{bmatrix} \tilde{h}_t \\ \tilde{k}_t \\ \varepsilon_t \\ \tilde{I}_{t-1} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & [\alpha(1-\alpha) - \alpha(1-\alpha\beta)\chi\bar{H}] & 1 & 0 \\ 1 & (\chi\bar{H} + \alpha)(1-\alpha) & 0 & 1 \\ 0 & 0 & \alpha & -\alpha \\ 0 & 0 & [\frac{\bar{Y}}{\vartheta}] & 0 \end{bmatrix}$$

Since $\frac{1}{\alpha\beta} > 1$, avoiding explosive behaviour (i.e. satisfying the transversality condition) requires $\hat{h}_t = 0 \forall t$. This requirement restricts consumers' choice of h_t to be a linear function of the parameters and day t state variables. That is, $\hat{h}_t = 0$ defines the unique stable decision rule for h_t (the following equation also uses Eqs. (3.85), (3.86), and (3.87)):

$$\begin{aligned} h_t = & - \left[\frac{\alpha [(1-\alpha\beta)\chi\bar{H} - (1-\alpha)]}{(\chi\bar{H} + \alpha)(1-\alpha)} \right] k_t - \left[\frac{(1-\alpha\beta)\chi\bar{H} - (1-\alpha)}{(\chi\bar{H} + \alpha)(1-\alpha)} \right] \varepsilon_t \\ & + \left[\frac{(1-\alpha^2\beta)\chi\bar{H}\vartheta}{(\chi\bar{H} + \alpha)(1-\alpha)\bar{Y}} \right] I_{t-1} - \left[\frac{\beta\kappa}{(1-\alpha\beta)\bar{Y}} \right] \mathbb{E}[F|\mathcal{I}^C] \\ & - \left[\frac{[(\chi\bar{H} + \alpha)(1-\alpha) - \alpha(1-\alpha^2\beta)\chi\bar{H}]\beta\vartheta}{(\chi\bar{H} + \alpha)(1-\alpha)(1-\alpha\beta)\bar{Y}} \right] \mathbb{E}[I|\mathcal{I}^C] \end{aligned}$$

Substituting this decision rule into Eq. (3.84) gives the unique stable day t decision rule for k_{t+1} :

$$\begin{aligned} k_{t+1} = & \alpha k_t + \varepsilon_t - \left[\frac{\alpha\vartheta}{\bar{Y}} \right] I_{t-1} - \left[\frac{(\chi\bar{H} + \alpha)(1-\alpha)\kappa}{\alpha(1-\alpha\beta)\chi\bar{H}\bar{Y}} \right] \mathbb{E}[F|\mathcal{I}^C] \\ & - \left[\frac{[(\chi\bar{H} + \alpha)(1-\alpha) - \alpha(1-\alpha^2\beta)\chi\bar{H}]\vartheta}{\alpha(1-\alpha\beta)\chi\bar{H}\bar{Y}} \right] \mathbb{E}[I|\mathcal{I}^C] \end{aligned}$$

The above two equations are the same as Eqs. (3.28) and (3.29), except using the model's structural parameters rather than the $N_1 - N_7$ reduced-form parameters.

3.6.10 Expected lending facility use

Using Eq. (3.31) and the shock distribution outlined in Section 3.2.5, and where $\frac{2\overline{ES}}{\overline{K}} < B$, $E[F|\mathcal{I}^C]$ is equal to:

$$E[F|\mathcal{I}^C] = \left[\frac{(1-\varphi)\overline{K}B}{8} \right] + \left[\frac{(1-\varphi)\overline{K}}{8B} \right] \left[\frac{2\overline{ES}}{\overline{K}} - B \right]^2 > 0 \quad (3.88)$$

If $\overline{ES}$ is increased to the point where $\frac{2\overline{ES}}{\overline{K}} \geq B$, then:

$$E[F|\mathcal{I}^C] = \left[\frac{(1-\varphi)\overline{K}B}{8} \right] > 0 \quad (3.89)$$

Derivative of $E[F|\mathcal{I}^C]$

From Eq. (3.88) (let $L = \frac{2\overline{ES}}{\overline{K}} \leq B$):

$$E[F|\mathcal{I}^C] = \left[\frac{(1-\varphi)\overline{K}B}{8} \right] + \left[\frac{(1-\varphi)\overline{K}}{8B} \right] [B - L]^2$$

This expectation is continuous with respect to $\overline{ES}$. Differentiating with respect to $\overline{ES}$:

$$\frac{dE[F|\mathcal{I}^C]}{d\overline{ES}} = \left[\frac{(1-\varphi)}{8B} \right] [2B^2 - L^2] \frac{d\overline{K}}{d\overline{ES}} - \left[\frac{(1-\varphi)}{2B} \right] [B - L]$$

Therefore:

$$-\frac{(1-\varphi)}{2} < \frac{dE[F|\mathcal{I}^C]}{d\overline{ES}} < \left[\frac{(1-\varphi)B}{4} \right] \frac{d\overline{K}}{d\overline{ES}}$$

3.6.11 Deriving I_t and $P(I_t = 1)$

3.6.11.1 Assumptions, propositions and corollaries for the proof of Theorem 3.1

To derive the optimal choices of the lending bank and asset managers, I make the following simplifying assumptions (these assumptions simplify the proofs but do not impact the outcome of I_t):

Assumption 2. *At the end of day t , the lending bank and asset managers make their choices sequentially. The lending bank goes first and offers a proportion $(1 - x_t)$ of its shares for sale, at a price such that P_t would be the cost of purchasing all of the shares ($x_t \in [0, 1]$). Then the asset managers make their share purchase choices. The asset managers cannot purchase more shares than the lending bank offers (so $1 \geq \theta_t \geq x_t$), and they can only buy at the offered price.*

Assumption 3. *The lending bank always sets x_t and P_t such that an equilibrium θ_t exists.⁹⁸*

Assumption 4. *The lending bank will set $0 < x_t < 1$ iff its expected profit (given optimal setting of P_t) is strictly greater than when $x_t = 1$ or $x_t = 0$.*

The lending bank will set $x_t = 0$ iff its expected profit (given optimal setting of P_t) is greater than or equal to its expected profit when $x_t > 0$ (i.e. $x_t = 0$ if indifferent).

$x_t = 1$ otherwise.

As discussed in Section 3.2.5.1, in equilibrium, the lending bank always gets the best deal possible. Therefore, it makes no difference whether the lending bank proposes this deal to the asset managers (which it will since it is the best possible deal) and the asset managers accept, or whether it is achieved through some other equilibrium-attaining

⁹⁸An equilibrium will not exist iff $x_t = 0$ and $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t = 1, P_t] \geq P_t > \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t = 0, P_t]$.

process. Having the lending bank and asset managers make their choices sequentially simplifies derivation of the equilibrium solution.

Given the information available to the asset managers at the end of day t , their expectation of firms' day $t + 1$ profits is: $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t, P_t]$. Where:

- $\mathcal{F}_t$ denotes knowledge of whether the lending facility was accessed during day t (but not the volume of use or which bank accessed the facility);
- For simplicity, I assume the asset managers have access to the information that was available to the banks at the end of the previous day ($\mathcal{I}_{t-1}^B$);
- I_t impacts the expectation through its impact on consumers' day $t + 1$ dividends (see Section 3.2.3.4); and
- the asset managers can extrapolate information about ε_t from the lending bank's choice of x_t and P_t (since the lending bank observes the day t shock).

Although the day $t + 1$ profits of the firms depend on the day t actions of the asset managers (through the impact of I_t on consumers' day $t + 1$ bank dividends), with a continuum of identical asset manager (whose only cost is the purchase price), the equilibrium value of I_t is independent of each individual asset manager's actions. Therefore, each asset manager treats $E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t, P_t]$ as independent of its own actions, and asset managers will be willing to buy shares as long as the equilibrium expected return from these shares is greater than or equal to the purchase price.^{99,100}

Since each share is identical, and the purchase price is the same for each share, then in equilibrium the asset managers will either be willing to buy as many shares as possible, if $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t, P_t] \geq P_t$. Whereas, if $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t, P_t] < P_t$, the asset managers will not be willing to buy any shares.

⁹⁹As discussed in Appendix 3.6.5, the continuum of asset managers only exist for one day, and are replaced by a new continuum every day. As a result, the day t asset managers are only concerned with their day $t + 1$ profits.

¹⁰⁰For simplicity, I assume the asset managers are willing to buy if indifferent.

From Eq. (3.10), the part of the lending bank's expected day $t + 1$ profit that is impacted by asset sales is:¹⁰¹

$$(1 - \theta_t) \left[\frac{\alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t] + P_t}{2} \right] + \theta_t \alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t] - \vartheta I_t \quad (3.90)$$

Importantly, $\mathcal{I}_t^B$ includes knowledge of the day t shock.

With this set-up:

Proposition 3.1. *When $x_t \in (0, 1]$, the asset managers will not buy any shares (i.e. $\theta_t = 1$). Therefore, the lending bank will never set $x_t \in (0, 1)$.*

Proof. From Eq. (3.90) and Assumption 4, for the lending bank to set $0 < x_t < 1$, the following must hold:

$$\begin{aligned} \frac{1}{2} (1 - \theta_t) [\alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] + P_t] + \theta_t \alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] - \vartheta \\ > \alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] - \vartheta \end{aligned} \quad (3.91)$$

This is because when $x_t = 1$, $\theta_t = 1$ (from the restriction that the asset managers cannot buy more shares than the lending bank offers for sale – i.e. $1 \geq \theta_t \geq x_t$). So the θ_t that results from the $0 < x_t < 1$ (and the setting of P_t) must give the bank an expected profit greater than when $\theta_t = 1$.

But re-arranging Eq. (3.91) (when $\theta_t < 1$) and simplifying gives:

$$P_t > \alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] \quad (3.92)$$

Therefore, it can only be optimal for the lending bank to set $0 < x_t < 1$ if the lending bank also sets $P_t > \alpha \mathbb{E}[Y_{t+1} | \mathcal{I}_t^B, I_t = 1]$.

Since the banks have more information than the asset managers (i.e. they know the value of the shock), the asset managers would expect to make a loss if they buy the shares when $0 < x_t < 1$ (due to Eq. (3.92)). Therefore, the asset managers will not buy any of the shares. $\square$

¹⁰¹Recall, from Section 3.2.1.3, that banks are only concerned with maximising their daily profits, they are not concerned with any impact their actions may have beyond day $t + 1$.

Proposition 3.2. $1 > \theta_t > x_t$ cannot be an equilibrium.

Proof. Recall that θ_t only impacts $E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t, P_t]$ through I_t . And since $I_t = 0$ iff $\theta_t = 0$, if $\theta_t > x_t$ then $I_t = 1$. Then consider two possible scenarios:

If $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t = 1, P_t] \geq P_t$, the asset managers will want to buy all of the shares the lending bank offers. Therefore $\theta_t > x_t$ cannot be an equilibrium.

If $\alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t, I_t = 1, P_t] < P_t$, the asset managers will not want to buy any shares. Therefore $\theta_t < 1$ cannot be an equilibrium.

Since these two cases encompass all possible P_t , there is no P_t for which $1 > \theta_t > x_t$ is an equilibrium. $\square$

Proposition 3.3. *If there exists a $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then the lending bank will set $x_t = 0$ and P_t equal to the largest value of P_t such that $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$. The resulting equilibrium value of θ_t will be $\theta_t = 0$.*

$$\frac{1}{2} [\alpha E[Y_{t+1} | \mathcal{I}_t^B, I_t = 0] + P_t] \geq \alpha E[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] - \vartheta \quad (3.93)$$

Proof. Part 1: $x_t = 0$ is required for $\theta_t = 0$ (from the restriction that the asset managers cannot buy more shares than the lending bank offers for sale).

$P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ is required for the asset managers to be willing to purchase all of the shares (i.e. for $\theta_t = 0$ to be an equilibrium).

The left-hand-side of Eq. (3.93) is equal to Eq. (3.90) when $\theta_t = 0$, while the right-hand-side is equal to Eq. (3.90) when $\theta_t = 1$.

Eq. (3.93) is a necessary condition for x_t and P_t to be set such that $\theta_t = 0$ (in equilibrium), because the lending bank could always achieve the right-hand-side profit by setting $x_t = 1$.

Eq. (3.93) is a sufficient condition because, from Proposition 3.1, the lending bank will never set $x_t \in (0, 1)$. And, from Proposition 3.2, $\theta_t = 1$ is the only other potential equilibrium when $x_t = 0$. Eq. (3.93) is a weak inequality due to Assumption 4.

This part proves that Eq. (3.93) is a necessary and sufficient condition for the lending bank to optimally set $x_t = 0$ and $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$.

Part 2: From Eq. (3.90), as long as a particular increase in P_t does not impact θ_t , this increase in P_t will increase the lending bank's profit. Therefore, the lending bank will always set P_t to the largest value of P_t such that $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ (since any $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ will lead to $\theta_t = 0$ when $x_t = 0$).

Suppose the largest value of $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ is $P_t^* < \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*]$.

And suppose P_t is increased to $P_t^{**} < \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*]$ (i.e. the price remains lower than α times the conditional expectation from before the change).

Using Eqs. (3.23), (3.28), and (3.29), ε_t has a greater positive impact on $E[Y_{t+1} | \mathcal{I}_t^B, I_t = 1]$ than on $E[Y_{t+1} | \mathcal{I}_t^B, I_t = 0]$. Therefore, from the asset managers' perspective, following a change from P_t^* to P_t^{**} , the upper bound of the range of shocks (ε_t) that would cause Eq. (3.93) to hold would be at least as high as before the change.¹⁰² This means that:

$$E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*] \leq E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^{**}]$$

That is, since an increase in P_t cannot shrink the set of $E[Y_{t+1} | \mathcal{I}_t^B, I_t]$ that causes Eq. (3.93) to hold, but can increase the set by adding some values of $E[Y_{t+1} | \mathcal{I}_t^B, I_t]$ that would have been too high with P_t^* , the asset managers' conditional expectation of Y_{t+1} cannot fall with the increase in P_t .¹⁰³

But this means that:

$$P_t^{**} < \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*] \leq E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^{**}]$$

¹⁰²The asset managers are assumed to know how the banks form their expectations, so they know which day t shocks lead to which expectations.

¹⁰³If $E[Y_{t+1} | \mathcal{I}_t^B, I_t]$ was too high for Eq. (3.93) to hold given P_t^* , the asset managers know that if such a shock had occurred, the bank would not set $x_t = 0$ and $P_t = P_t^*$, since this would cause its profit to be lower than if $x_t = 1$.

Which contradicts the original proposition that P_t^* was the largest value of P_t such that $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$. Therefore, since P_t is a continuous choice variable, for P_t^* to be the largest value of P_t such that $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$, it must be the case that $P_t^* = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*]$. $\square$

Corollary 3.1. *If there is no $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then the lending bank will set $x_t = 0$ and P_t sufficiently large such that $\theta_t = 1$.*

Proof. If there is no $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then any equilibrium in which $\theta_t = 0$ would be less profitable for the lending bank than an equilibrium in which $\theta_t = 1$ (recall that $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ is required for $\theta_t = 0$ to be an equilibrium). Since $\theta_t = 0$ and $\theta_t = 1$ are the only two possible equilibria (from propositions 3.1 and 3.2), the lending bank in this scenario will set x_t and P_t such that $\theta_t = 1$.

From Assumption 4, if the lending bank is indifferent between $x_t = 0$ and another x_t , the lending bank will set $x_t = 0$. If the lending bank prefers $\theta_t = 1$, it can achieve this by setting $x_t = 0$ and a P_t sufficiently high that it knows the asset managers will never purchase the shares (such a P_t exists because the shock is bounded, and therefore so is Y_{t+1}). Therefore, $x_t = 0$ in this scenario.

From Proposition 3.3, if there exists a $P_t < \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then there must also exist a $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds. Therefore, if there is no $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then there is no $P_t \leq \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds. $\square$

Definition 3.1. From the perspective of the asset managers (who do not know the true value of the day t shock (ε_t)), let $\mathbb{F}_t$ denote the subset of potential day t shocks that are feasible given $\mathcal{F}_t$ (i.e. $\mathbb{F}_t \subset [-B, B]$). Then let $\varepsilon_t^{\sup}$ be the supremum of a

subset of $\mathbb{F}_t$ that contains all values of $\varepsilon_t \in \mathbb{F}_t$ for which the following inequality holds (changing ε_t only impacts the right-hand-side of the inequality):

$$\begin{aligned} & \mathbb{E}\left[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{\sup}\right] \\ & \geq 2\mathbb{E}\left[Y_{t+1} | \mathcal{I}_t^B, I_t = 1\right] - \mathbb{E}\left[Y_{t+1} | \mathcal{I}_t^B, I_t = 0\right] - \frac{2\vartheta}{\alpha} \end{aligned} \quad (3.94)$$

There may be many values of $\varepsilon_t^{\sup}$ that satisfy this definition.

Proposition 3.4. *There exists a $P_t = \alpha \mathbb{E}\left[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t\right]$ such that Eq. (3.93) holds iff there exists an $\varepsilon_t^{\sup}$ (satisfying Definition 3.1) such that the true day t shock satisfies $\varepsilon_t \leq \varepsilon_t^{\sup}$. Therefore, given the true day t shock, the requirement for $I_t = 0$ (from Proposition 3.3 and Corollary 3.1) reduces to finding a sufficiently large $\varepsilon_t^{\sup}$.*

Proof. Lemmas 3.1 and 3.2. □

Lemma 3.1. *If there exists an $\varepsilon_t^{\sup}$ (satisfying Definition 3.1) such that the true day t shock satisfies $\varepsilon_t \leq \varepsilon_t^{\sup}$, then there exists a $P_t = \alpha \mathbb{E}\left[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t\right]$ such that Eq. (3.93) holds.*

Proof. Since ε_t has a greater positive impact on $\mathbb{E}\left[Y_{t+1} | \mathcal{I}_t^B, I_t = 1\right]$ than on $\mathbb{E}\left[Y_{t+1} | \mathcal{I}_t^B, I_t = 0\right]$, for any $\varepsilon_t^{\sup}$ that satisfies Definition 3.1, Eq. (3.94) holds for all $\varepsilon_t \leq \varepsilon_t^{\sup}$, but does not hold for any $\varepsilon_t > \varepsilon_t^{\sup}$ that are also within the $\mathbb{F}_t$ subset (by the definition of $\varepsilon_t^{\sup}$).¹⁰⁴

After finding an $\varepsilon_t^{\sup}$ that satisfies definition 3.1, suppose the lending bank sets $P_t = \alpha \mathbb{E}\left[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{\sup}\right]$.¹⁰⁵ Substituting this P_t into Eq. (3.94) gives Eq. (3.93). Therefore, at this price, Eq. (3.93) holds for all $\varepsilon_t \leq \varepsilon_t^{\sup}$ but does not hold for any $\varepsilon_t > \varepsilon_t^{\sup}$ that are also within the $\mathbb{F}_t$ subset.

¹⁰⁴Although it is possible that $\varepsilon_t^{\sup} \notin \mathbb{F}_t$ (e.g. if $\mathbb{F}_t$ is non-convex or has an open upper bound), since Eq. (3.94) is a weak inequality and the ε_t distribution is continuous, $\varepsilon_t = \varepsilon_t^{\sup}$ will always satisfy Eq. (3.94).

¹⁰⁵The lending bank is assumed to know how the asset managers form expectations and therefore has sufficient information to compute this P_t .

Since P_t does not directly impact Y_{t+1} (only indirectly via I_t), the only role P_t plays in determining $E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ is by determining which values of ε_t would satisfy Eq. (3.93) for the given P_t (and given $\mathcal{I}_{t-1}^B$ and $\mathcal{F}_t$) – recall that, for $I_t = 0$, Eq. (3.93) must be satisfied. That is, P_t only tells the asset managers which shocks could have occurred for the lending bank to be willing to sell the shares at this price; the asset managers then use this knowledge to form their expectation of Y_{t+1} and then to determine whether the offered price is sufficiently low for them to profit (in expectation) from buying the shares.

From the above information, at $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{\sup}]$, Eq. (3.93) holds for all $\varepsilon_t \leq \varepsilon_t^{\sup}$ but does not hold for any $\varepsilon_t > \varepsilon_t^{\sup}$ that are also within the $\mathbb{F}_t$ subset. Therefore, $\varepsilon_t^{\sup}$ and $\mathcal{F}_t$ uniquely determine the values of ε_t (that are feasible given $\mathcal{F}_t$) that would satisfy Eq. (3.93) for the given P_t (i.e. $\varepsilon_t^{\sup}$ and P_t provide the same information to the asset managers when forming expectations). So:

$$\begin{aligned} P_t &= \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{\sup}] \\ &= \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t] \end{aligned} \quad (3.95)$$

□

Lemma 3.2. *If there exists a $P_t = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t]$ such that Eq. (3.93) holds, then there exists an $\varepsilon_t^{\sup}$ (satisfying Definition 3.1) such that the true day t shock satisfies $\varepsilon_t \leq \varepsilon_t^{\sup}$.*

Proof. Denote the true day t shock as ε_t^* . Now suppose there is a $P_t = P_t^* = \alpha E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*]$ such that Eq. (3.93) holds at $\varepsilon_t = \varepsilon_t^*$. Then:

$$\begin{aligned} &E[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*] \\ &\geq 2E[Y_{t+1} | \mathcal{I}_t^B, I_t = 1] - E[Y_{t+1} | \mathcal{I}_t^B, I_t = 0] - \frac{2\vartheta}{\alpha} \end{aligned} \quad (3.96)$$

For a given P_t^* , there must exist a value (call it ε_t^{**}) such that Eq. (3.93) holds for

all $\varepsilon_t \leq \varepsilon_t^{**}$, does not hold for any $\varepsilon_t > \varepsilon_t^{**}$ that are also within the $\mathbb{F}_t$ subset, and is the supremum of the subset of $\mathbb{F}_t$ for which Eq. (3.93) holds. Since P_t only tells the asset managers which shocks could have occurred for the lending bank to be willing to sell the shares at this price (see the proof of Lemma 3.1):

$$\mathbb{E}[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, P_t^*] = \mathbb{E}[Y_{t+1} | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{**}]$$

But since $\varepsilon_t^* \leq \varepsilon_t^{**}$, and ε_t^{**} satisfies all of the requirements of an $\varepsilon_t^{\sup}$ in Definition 3.1 (using the fact that Eq. (3.96) holds only for the subset of $\mathbb{F}_t$ for which $\varepsilon_t \leq \varepsilon_t^{**}$), then there exists an $\varepsilon_t^{\sup}$ (satisfying Definition 3.1) such that the true day t shock satisfies $\varepsilon_t \leq \varepsilon_t^{\sup}$. $\square$

3.6.11.2 Proof of Theorem 3.1

Re-stating Theorem 3.1: *Finding the largest $\varepsilon_t^{\sup}$ that satisfies Definition 3.1 approximates to finding the largest $\varepsilon_t^{\dagger}$ such that:*

$$\mathbb{E}[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\dagger}] \geq \varepsilon_t^{\dagger} - \mathbb{C}\vartheta \quad (3.97)$$

Where:

$$\mathbb{C} = \frac{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right]}{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}}$$

And where $\varepsilon_t^{\dagger}$ is restricted to be either $\varepsilon_t^{\dagger} \in \mathbb{F}_t$ or the supremum of any segment of $\mathbb{F}_t$ ($\mathbb{F}_t$ may be non-convex and have segments with open upper bounds). Call the solution $\varepsilon_t^{\dagger\dagger}$.

Then, from Proposition 3.4 and given that $\varepsilon_t \in \mathbb{F}_t$, the subset of $\mathbb{F}_t$ that leads to $I_t = 0$ can be approximated by: $I_t = 0$ iff the day t value of the shock satisfies $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$. $I_t = 1$ otherwise.

Proof. Part 1: Eq. (3.97) is derived by first substituting a first-order Taylor approximation of Y_{t+1} around the deterministic steady state (using Eq. (3.23)) into Eq. (3.94)

and simplifying using Eqs. (3.28) and (3.29). After substituting in $\varepsilon_t^{\sup} = \varepsilon_t^\dagger$, the result of this approximation is:

$$\mathbb{E}[\varepsilon_t | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^\dagger] \geq \varepsilon_t - \left[\frac{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right] \vartheta}{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}} \right] \quad (3.98)$$

For $\varepsilon_t^\dagger$ to be an $\varepsilon_t^{\sup}$ according to Definition 3.1, Eq. (3.94) must hold for all $\varepsilon_t \leq \varepsilon_t^\dagger$ that are within $\mathbb{F}_t$ but not hold for any $\varepsilon_t > \varepsilon_t^\dagger$ that are within $\mathbb{F}_t$ (see the proof of Lemma 3.1). Now suppose $\varepsilon_t^\dagger$ is substituted in the right-hand-side of Eq. (3.98) in place of ε_t :

$$\mathbb{E}[\varepsilon_t | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^\dagger] \geq \varepsilon_t^\dagger - \left[\frac{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right] \vartheta}{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}} \right] \quad (3.99)$$

Then suppose each feasible value of $\varepsilon_t^\dagger$ is searched (from highest to lowest) until a value is found that satisfies Eq. (3.99) (feasibility is based on the restrictions in Theorem 3.1). Any higher feasible value of $\varepsilon_t^\dagger$ does not satisfy Eq. (3.98) for all $\varepsilon_t \leq \varepsilon_t^\dagger$ that are within $\mathbb{F}_t$ (since this requires Eq. (3.99) to hold); so, when $\varepsilon_t^\dagger \in \mathbb{F}_t$, no higher value can be an $\varepsilon_t^{\sup}$ that satisfies Definition 3.1 (based on this first-order approximation). Also, at the highest feasible value of $\varepsilon_t^\dagger$, no $\varepsilon_t > \varepsilon_t^\dagger$ that is within $\mathbb{F}_t$ can satisfy Eq. (3.98) (since then a higher feasible value of $\varepsilon_t^\dagger$ that satisfies Eq. (3.99) would exist). Therefore, when the highest feasible value of $\varepsilon_t^\dagger$ that satisfies Eq. (3.99) also satisfies $\varepsilon_t^\dagger \in \mathbb{F}_t$, then it is the largest $\varepsilon_t^{\sup}$ that satisfies Definition 3.1 (based on this first-order approximation).

If $\varepsilon_t^{\dagger\dagger} \notin \mathbb{F}_t$ (i.e. the supremum is not within the subset), then Eq. (3.94) need not be satisfied at $\varepsilon_t = \varepsilon_t^{\dagger\dagger}$ (since this point is not within $\mathbb{F}_t$). However, due to the continuity of ε_t , Eq. (3.94) must be satisfied as $\varepsilon_t \rightarrow \varepsilon_t^{\dagger\dagger}$ (by definition of a supremum). Therefore, to a first-order approximation, finding $\varepsilon_t^{\dagger\dagger} \notin \mathbb{F}_t$ to be the highest feasible value of $\varepsilon_t^\dagger$ that satisfies Eq. (3.99) remains a necessary and sufficient condition for $\varepsilon_t^{\dagger\dagger}$ to be the largest $\varepsilon_t^{\sup}$ that satisfies Definition 3.1.

Part 2: Since the shocks are exogenous, the only relevant information for determining the expected value of the shock is any information that can reduce the set of possible

day t shocks (this is the expectation from the perspective of the asset managers, who do not observe the day t shock until day $t + 1$).

From Proposition 3.3 and Corollary 3.1, the lending bank always sets $x_t = 0$. Therefore, observing $x_t = 0$ does not change the probability distribution of the day t shocks and can be removed from the conditioning information.

Day t shocks are independent of past information, so $\mathcal{I}_{t-1}^B$ can be removed from the conditioning information.

From Definition 3.1, $\mathcal{F}_t$ can be replaced by $\varepsilon_t \in \mathbb{F}_t$ (since they provide the same information).

From Proposition 3.4, if $\varepsilon_t^{\dagger\dagger}$ is the highest $\varepsilon_t^{\sup}$ that satisfies Definition 3.1, then $I_t = 0$ iff $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$. Therefore, $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$ and $I_t = 0$ provide the same information. So, when searching for the largest $\varepsilon_t^{\dagger}$ that satisfies Eq. (3.99), $I_t = 0$ can be removed from the conditioning information.

The result of all of the above is:

$$\mathbb{E}[\varepsilon_t | \mathcal{I}_{t-1}^B, \mathcal{F}_t, x_t = 0, I_t = 0, \varepsilon_t \leq \varepsilon_t^{\dagger}] \equiv \mathbb{E}[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\dagger}]$$

□

3.6.11.3 Deriving $\mathbb{E}[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\sup}]$

From the unconditional shock distribution (see Section 3.2.5): $P(\varepsilon_t = 0) = \varphi$, and with probability $1 - \varphi$, the shock follows a continuous uniform probability distribution with bounds $[-B, B]$ (where $B > 0$).

And from Eq. (3.31):

$$F_t = \begin{cases} \left\lceil \frac{K}{2} \right\rceil \varepsilon_t & \text{if } \varepsilon_t > 0 \\ -\left\lceil \frac{K}{2} \right\rceil \varepsilon_t - \overline{ES} & \text{if } \varepsilon_t < \frac{-2\overline{ES}}{K} \\ 0 & \text{if } \frac{-2\overline{ES}}{K} \leq \varepsilon_t \leq 0 \end{cases}$$

For ease of notation, let $L = \frac{2\overline{ES}}{K}$ and recall from Section 3.2.6 that $\overline{ES}$ will never be set such that $L > B$.

Then, using Bayes' rule and the above information, if $F_t = 0$ and $\varepsilon_t^{\text{sup}} = 0$ (with $F_t = 0$, the maximum possible value of $\varepsilon_t^{\text{sup}}$ is $\varepsilon_t^{\text{sup}} = 0$):

$$\begin{aligned} E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\text{sup}}] &= \frac{(1 - \varphi) P(-L \leq \varepsilon_t < 0) E[\varepsilon_t | -L \leq \varepsilon_t < 0]}{\varphi + (1 - \varphi) P(-L \leq \varepsilon_t < 0)} \\ &= \frac{-(1 - \varphi) L^2}{4B\varphi + 2(1 - \varphi)L} \end{aligned} \quad (3.100)$$

If $F_t = 0$ and $-L \leq \varepsilon_t^{\text{sup}} < 0$:

$$\begin{aligned} E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\text{sup}}] &= E[\varepsilon_t | -L \leq \varepsilon_t \leq \varepsilon_t^{\text{sup}}] \\ &= \frac{\varepsilon_t^{\text{sup}} - L}{2} \end{aligned} \quad (3.101)$$

If $F_t > 0$ and $B \geq \varepsilon_t^{\text{sup}} > 0$:

$$\begin{aligned} E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\text{sup}}] &= \frac{P(0 < \varepsilon_t \leq \varepsilon_t^{\text{sup}}) E[\varepsilon_t | 0 < \varepsilon_t \leq \varepsilon_t^{\text{sup}}]}{P(0 < \varepsilon_t \leq \varepsilon_t^{\text{sup}}) + P(-B \leq \varepsilon_t < -L)} \\ &\quad + \frac{P(-B \leq \varepsilon_t < -L) E[\varepsilon_t | -B \leq \varepsilon_t < -L]}{P(0 < \varepsilon_t \leq \varepsilon_t^{\text{sup}}) + P(-B \leq \varepsilon_t < -L)} \\ &= \frac{[\varepsilon_t^{\text{sup}}]^2 + L^2 - B^2}{2[\varepsilon_t^{\text{sup}} + B - L]} \end{aligned} \quad (3.102)$$

If $F_t > 0$ and $-L \geq \varepsilon_t^{\text{sup}} \geq -B$:

$$\begin{aligned} E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\text{sup}}] &= E[\varepsilon_t | -B \leq \varepsilon_t \leq \varepsilon_t^{\text{sup}}, \varepsilon_t \neq 0] \\ &= \frac{\varepsilon_t^{\text{sup}} - B}{2} \end{aligned} \quad (3.103)$$

In this last case, if $\varepsilon_t^{\text{sup}} = -L$ then $\varepsilon_t^{\text{sup}} \notin \mathbb{F}_t$. However, since the shock distribution is continuous and there are no probability atoms when $\varepsilon_t \neq 0$, the conditioning region can still be written $-B \leq \varepsilon_t \leq \varepsilon_t^{\text{sup}}$ when $L \neq 0$. The $\varepsilon_t \neq 0$ condition is added since

there is a probability atom at $\varepsilon_t = 0$, so the conditioning region must explicitly exclude $\varepsilon_t = 0$ when $L = 0$.

3.6.11.4 Existence of $\varepsilon_t^\dagger$ that satisfies the requirements

Re-producing Eq. (3.97):

$$\mathbb{E}[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^\dagger] \geq \varepsilon_t^\dagger - \left[\frac{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right] \vartheta}{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}} \right]$$

For ease of notation, let:

$$\frac{Z}{2} = \frac{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right] \vartheta}{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}} > 0$$

Suppose $F_t = 0$ and $\varepsilon_t^\dagger = -L$, then from Eqs. (3.97) and (3.101): $\frac{Z}{2} - L \geq -L$. Therefore, $\varepsilon_t^\dagger = -L$ is always a potential solution when $F_t = 0$, so a $\varepsilon_t^\dagger$ that satisfies Theorem 3.1 exists when $F_t = 0$.

Suppose $F_t > 0$ and $\varepsilon_t^\dagger = -B$, then from Eqs. (3.97) and (3.103): $\frac{Z}{2} - B \geq -B$. Therefore, $\varepsilon_t^\dagger = -B$ is always a potential solution when $F_t > 0$, so a $\varepsilon_t^\dagger$ that satisfies Theorem 3.1 exists when $F_t > 0$.

3.6.11.5 Algorithm to determine $\varepsilon_t^{\dagger\dagger}$

When $F_t = 0$

From Section 3.2.7, ϑ is set sufficiently large such that the shares are always sold to the asset managers when the lending facility is not used. Using Eqs. (3.97) and (3.100), this means ϑ must be set such that (for all $\overline{ES} \in [0, \frac{B\overline{K}}{2}]$):

$$\vartheta \geq \left[\frac{\alpha^2 (1 + \alpha \beta \chi \overline{H}) \overline{Y}}{2 \left[(1 - \alpha) \chi \overline{H} + \alpha (1 + \alpha^2 \beta \chi \overline{H}) \right]} \right] \left[\frac{(1 - \varphi) L^2}{4B\varphi + 2(1 - \varphi) L} \right]$$

With this condition met, when $F_t = 0$, no algorithm is required to find $\varepsilon_t^{\dagger\dagger}$, since $\varepsilon_t^\dagger = 0$ is the largest possible value when $F_t = 0$ and it satisfies the requirements.

When $F_t > 0$

When searching for the largest $\varepsilon_t^\dagger$ in the region $(0, B]$, the condition that must be satisfied is (from Eqs. (3.97) and (3.102)):

$$\frac{[\varepsilon_t^\dagger]^2 + L^2 - B^2}{\varepsilon_t^\dagger + B - L} \geq 2\varepsilon_t^\dagger - Z \quad (3.104)$$

In fact, if there are some $\varepsilon_t^\dagger \in (0, B]$ that satisfy Eq. (3.104), then the largest value must be either $\varepsilon_t^{\dagger\dagger} = B$ or it must satisfy Eq. (3.104) as an equality (since if Eq. (3.104) was satisfied as a strict inequality (with $\varepsilon_t^\dagger < B$), a larger $\varepsilon_t^\dagger$ that satisfies Eq. (3.104) must also exist).

Re-arranging Eq. (3.104) and setting it as an equality gives:

$$\varepsilon_t^\dagger = \frac{-[2B - 2L - Z] \pm \sqrt{[2B - 2L - Z]^2 - 4[B - L][B + L - Z]}}{2} \quad (3.105)$$

Since the right-hand-side of Eq. (3.105) is only dependent on the parameters, there are, at most, two distinct real points where Eq. (3.105) holds.

So, to find the largest $\varepsilon_t^\dagger$ that satisfies the requirements:

1. Substitute $\varepsilon_t^\dagger = B$ into Eq. (3.104). If Eq. (3.104) holds, then $\varepsilon_t^{\dagger\dagger} = B$ is the solution.
2. If Eq. (3.104) does not hold at $\varepsilon_t^\dagger = B$, compute the (at most) two real points where Eq. (3.105) holds. If either of these points satisfy $\varepsilon_t^\dagger \in (0, B)$, then the largest that satisfies $\varepsilon_t^\dagger \in (0, B)$ is the solution.
3. Otherwise, the solution must be in the $[-B, -L]$ region (recall that a solution in this region always exists).

When searching in the $[-B, -L]$ region, the condition that must be satisfied is (from Eqs. (3.97) and (3.103)): $\varepsilon_t^\dagger \leq Z - B$. Therefore, the largest value of $\varepsilon_t^\dagger$ within $[-B, -L]$ that satisfies Theorem 3.1 is $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$.

3.6.11.6 Values of $\overline{ES}$ for which $\varepsilon_t^{\dagger\dagger} < 0$ when $F_t > 0$

$\varepsilon_t^{\dagger\dagger} = B$ is never a solution

From Eq. (3.104), for $\varepsilon_t^{\dagger\dagger} = B$ to be the solution:

$$L^2 - [2B - Z][2B - L] \geq 0$$

Assume $B > Z$ (as will be the case with the parameter values in this chapter), then, since $B > L$:

$$L^2 - [2B - Z][2B - L] < -B[B - Z] < 0$$

Since this contradicts the requirement for $\varepsilon_t^{\dagger\dagger} = B$ to be the solution, $\varepsilon_t^{\dagger\dagger} = B$ is never the solution.

Necessary conditions for the solution to satisfy $\varepsilon_t^{\dagger\dagger} \in (0, B)$

For a solution to satisfy $\varepsilon_t^{\dagger\dagger} \in (0, B)$, then the roots of Eq. (3.105) must be real, this requires:

$$[2B - 2L - Z]^2 - 4[B - L][B + L - Z] \geq 0$$

In the case where the roots are real, only when $2B - 2L - Z < 0$ will it be possible for the roots of Eq. (3.105) to lie within the $(0, B)$ region (when $2B - 2L - Z \geq 0$ any real roots satisfy $\varepsilon_t^{\dagger} < 0$).

Both of these conditions are necessary, but not sufficient.

Simplifying the first necessary condition gives:

$$[2B - 2L - Z]^2 - 4[B - L][B + L - Z] = 8L^2 - 8BL + Z^2 \geq 0 \quad (3.106)$$

Sufficient condition for solution to be $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$

From Eq. (3.106), if $8L^2 - 8BL + Z^2 < 0$ then the solution must be $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$.

Using the deterministic steady state equations (Appendix 3.6.7), and the fact that $0 < \frac{d\bar{K}}{d\bar{E}\bar{S}} < 1$ (from Appendix 3.6.8):

$$\frac{dZ}{d\bar{E}\bar{S}} = \left\{ -\frac{4\vartheta}{\alpha(1 + \alpha\beta\chi\bar{H})\bar{Y}\bar{K}} - \frac{Z}{\bar{K}} \left[\frac{\alpha\beta\chi\bar{H}}{1 + \alpha\beta\chi\bar{H}} \right] \right\} \frac{d\bar{K}}{d\bar{E}\bar{S}} < 0$$

Therefore, the largest value of Z occurs at $\bar{E}\bar{S} = 0$. Denote this value as Z_0 . Then:

$$8L^2 - 8BL + Z^2 \leq 8L^2 - 8BL + Z_0^2$$

With respect to L , $8L^2 - 8BL + Z_0^2$ has a global minimum point at $L = \frac{B}{2}$ (at which point $8L^2 - 8BL + Z_0^2 < 0$). Therefore, $8L^2 - 8BL + Z_0^2 < 0$ for all:

$$\frac{B}{2} < L < \frac{2B + \sqrt{4B^2 - 2Z_0^2}}{4}$$

Since $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$ is the solution when $L \leq \frac{B}{2}$ (substitute any $L \leq \frac{B}{2}$ into the $2B - 2L - Z \geq 0$ condition), a sufficient condition for $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$ to be the solution is:

$$L < \frac{2B + \sqrt{4B^2 - 2Z_0^2}}{4}$$

$\left[0, \frac{2B + \sqrt{4B^2 - 2Z_0^2}}{4}\right)$ will be the region (of L) of the “modelled” subset (see Section 3.4).

From Appendix 3.6.8, $0 < \frac{d\bar{K}}{d\bar{E}\bar{S}} < 1$ and $0 \leq \frac{\bar{E}\bar{S}}{\bar{K}} < 1$. Therefore:

$$\frac{dL}{d\bar{E}\bar{S}} = \frac{2}{\bar{K}} \left[1 - \frac{\bar{E}\bar{S}}{\bar{K}} \frac{d\bar{K}}{d\bar{E}\bar{S}} \right] > 0$$

Since the function that maps $\bar{E}\bar{S}$ to L is strictly-increasing and monotonic, with the baseline parameter settings used in this chapter (see Section 3.2.7), the smallest size of the modelled subset (with respect to $\bar{E}\bar{S}$) is $[0, 29.99995)$, with the corresponding

non-modelled subset being just $[29.99995, 30]$.

Importantly, $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$ is continuous (with respect to $\overline{ES}$) within the modelled subset. Whereas there would be discontinuities when switching between $\varepsilon_t^{\dagger\dagger} = \min\{[Z - B], -L\}$ and $\varepsilon_t^{\dagger\dagger} \in (0, B)$.

3.6.11.7 Deriving $P(I_t = 1)$ for the “modelled” subset

Since consumers form unbiased and stationary expectations of I_t , and since I_t is an indicator function, then $E[I|\mathcal{I}^C] = P(I_t = 1)$.

$I_t = 0$ iff $\varepsilon_t \leq \varepsilon_t^{\dagger\dagger}$. Therefore:

$$\begin{aligned} P(I_t = 1) &= P(\varepsilon_t > \varepsilon_t^{\dagger\dagger}) \\ &= P(\varepsilon_t > \varepsilon_t^{\dagger\dagger} | F_t = 0) P(F_t = 0) + P(\varepsilon_t > \varepsilon_t^{\dagger\dagger} | F_t > 0) P(F_t > 0) \end{aligned}$$

Note that the $\varepsilon_t^{\dagger\dagger}$ when $F_t > 0$ will be different from the $\varepsilon_t^{\dagger\dagger}$ when $F_t = 0$.

With the parameters used in this chapter, the shares are always sold to the asset managers when $F_t = 0$. Therefore, $P(\varepsilon_t > \varepsilon_t^{\dagger\dagger} | F_t = 0) = 0$.

From Eq. (3.31) and given the earlier results from this appendix, in the modelled subset of L :

$$\begin{aligned} P(F_t > 0) &= \frac{(1-\varphi)}{2} + \frac{(1-\varphi)}{2} \left[\frac{B-L}{B} \right] \\ P(\varepsilon_t > \varepsilon_t^{\dagger\dagger} | F_t > 0) &= \frac{\frac{(1-\varphi)}{2} + \frac{(1-\varphi)}{2} \left[\frac{-L - \min\{[Z-B], -L\}}{B} \right]}{\frac{(1-\varphi)}{2} + \frac{(1-\varphi)}{2} \left[\frac{B-L}{B} \right]} \end{aligned}$$

Therefore:

$$P(I_t = 1) = \frac{(1-\varphi)}{2} \left[1 - \frac{L + \min\{[Z-B], -L\}}{B} \right] \geq \frac{(1-\varphi)}{2} \quad (3.107)$$

Importantly, since $\varepsilon_t^{\dagger\dagger}$ and L are both continuous (with respect to $\overline{ES}$) within the modelled subset, $P(I_t = 1)$ is continuous (with respect to $\overline{ES}$) within the modelled

subset.

Given that $\frac{dZ}{d\overline{ES}} < 0$ and $\frac{dL}{d\overline{ES}} > 0$, increasing $\overline{ES}$ (within the modelled subset) decreases $\varepsilon_t^{\dagger\dagger}$. However, increasing $\overline{ES}$ decreases $P(F_t > 0)$. When $L > B - Z$, the resulting impact on $P(I_t = 1)$ is:

$$\frac{dP(I_t = 1)}{d\overline{ES}} = 0$$

That is, the decrease in $\varepsilon_t^{\dagger\dagger}$ is exactly offset by the decrease in $P(F_t > 0)$, so the proportion of shocks that lead to asset sales does not change.

When $L < B - Z$:

$$\begin{aligned} \frac{dP(I_t = 1)}{d\overline{ES}} &= \frac{-(1 - \varphi)}{2B} \left[\frac{dL}{d\overline{ES}} + \frac{dZ}{d\overline{ES}} \right] \\ &= \frac{(1 - \varphi)}{2B} \left[\frac{2}{\overline{K}} \left[\frac{\overline{ES}}{\overline{K}} \frac{d\overline{K}}{d\overline{ES}} - 1 \right] - \frac{dZ}{d\overline{K}} \frac{d\overline{K}}{d\overline{ES}} \right] \end{aligned}$$

To determine the upper bound of this derivative:

$$\begin{aligned} \frac{dZ}{d\overline{K}} &= -\frac{4\vartheta}{\alpha(1 + \alpha\beta\chi\overline{H})\overline{Y}\overline{K}} - \frac{Z}{\overline{K}} \left[\frac{\alpha\beta\chi\overline{H}}{1 + \alpha\beta\chi\overline{H}} \right] < 0 \\ \frac{d^2Z}{d\overline{K}^2} &= \left\{ \frac{4\vartheta \left[\alpha(1 + \alpha\beta\chi\overline{H})\overline{Y} + \alpha(1 + \alpha\beta\chi\overline{H})\overline{Y} + \alpha(\alpha\beta\chi\overline{H})\overline{Y} \right]}{\left[\alpha(1 + \alpha\beta\chi\overline{H})\overline{Y}\overline{K} \right]^2} - \frac{1}{\overline{K}} \left[\frac{\alpha\beta\chi\overline{H}}{1 + \alpha\beta\chi\overline{H}} \right] \frac{dZ}{d\overline{K}} + \frac{Z}{\overline{K}^2} \left[\frac{\alpha\beta\chi\overline{H}}{1 + \alpha\beta\chi\overline{H}} \right]^2 \right\} > 0 \end{aligned}$$

Therefore, the most negative value of $\frac{dZ}{d\overline{K}}$ occurs when $\overline{ES} = 0$. So (using the equations in Appendix 3.6.7 and the fact that $M(\overline{S}) = 0$ when $\overline{ES} = 0$):

$$\begin{aligned} -\frac{dZ}{d\overline{K}} &\leq \left[\frac{4\beta(1 - \alpha\beta)\vartheta}{(1 - \alpha^2\beta)\overline{K}^2} \right] \left\{ 1 + \left[\frac{\alpha(1 - \alpha)\overline{K}Z}{4(1 - \alpha\beta)\vartheta} \right] \right\} \\ &< \left[\frac{12\vartheta}{\overline{K}^2} \right] \end{aligned} \tag{3.108}$$

The second step uses the fact that $\alpha < \frac{1}{2}$ and $\beta < 1$ (with the parameter values used in this chapter, see Appendix 3.6.12), and that (this upper bound uses the deterministic steady state equations when $M(\bar{S}) = 0$):

$$Z < \frac{8\vartheta}{\alpha\bar{K}}$$

Then, using Eq. (3.108), and with $\bar{ES} < \frac{\bar{K}B}{2}$, when $L < B - Z$:

$$\frac{dP(I_t = 1)}{d\bar{ES}} < \frac{(1 - \varphi)}{\bar{K}B} \left[\frac{B}{2} + \frac{6\vartheta}{\bar{K}} - 1 \right]$$

Which, for the parameter values used in this chapter, is always less than zero.

Note that the derivative of $P(I_t = 1)$ with respect to $\bar{ES}$ does not exist at the point $L = B - Z$.

3.6.12 Calibration and parameterisation

The parameters in this model are: α , β , χ , ψ , ϑ , κ , $\bar{ES}$, $\bar{A}$, φ , and B ; most of these parameters are calibrated using Australian data:

- α (the output elasticity of capital) is calibrated so that labour's share of output in the deterministic steady state $(1 - \alpha)$ is equal to the average compensation of employees (as a share of total factor income) in Australia since 2000. Since 2000, this share has ranged from 0.52 to 0.55 (with an average 0.54). So $\alpha = 0.46$.
- β (the discount factor) is calibrated so that the deterministic steady state daily gross return on equity $(\bar{R}^D)$ is equal to the average daily gross real earnings yield in Australia between 1987 and 2013 (based on the S&P/ASX 200 and CPI). The average annual net real earnings yield over this period was 2.97 per cent and $\beta = 0.99992$.
- κ (the central bank daily interest rate spread) is calibrated to equal the daily interest spread between the RBA's deposit and lending rates (consistent with

chapters 1 and 2). The annualised interest spread is 50 basis points, so $\kappa = 0.005/365$.

- From Chapter 1, the typical pre-crisis aggregate ES balance (adjusted for a two bank model) was A\$30 million. Moreover, according to the Reserve Bank of Australia (RBA), there is no stigma associated with lending facility use in Australia (*Debelle*, 2008). Since there is no lending facility stigma when $\overline{ES} = \frac{B\overline{K}}{2}$, I calibrate B (the upper bound of the shock) such that, with the baseline parameter values, when $\overline{ES} = 30$ then $\overline{ES} = \frac{B\overline{K}}{2}$.¹⁰⁶ The resulting value of B is 0.0312. That is, the largest technology shock is 3.12 per cent of the deterministic steady state level of technology.
- When $\overline{ES} = 30$, only positive shocks require borrowing from the lending facility. Since the probability of a positive shock is $\frac{1-\varphi}{2}$, I calibrate φ so that the probability of lending facility use in the model is the same as the probability of lending facility use in the Australian market (adjusted for a two-bank model, see Appendix 1.7.13). This results in $\varphi = 0.9964$.
- χ (the marginal disutility of labour) is calibrated so that the deterministic steady state level of labour ($\overline{H}$) is equal to the average proportion of each day that people spend working in Australia. Since 2000, trend average daily working hours of employed persons (including weekends) has been between 4.5 and 4.9 hours, with an average of 4.7 hours (around one-fifth of each day). Since $\overline{H}$ is dependent on $\overline{ES}$, I calibrate χ so that $\overline{H} = 0.196$ when $\overline{ES} = 30$; the result is $\chi = 5.11$.
- $\overline{A}$ (the steady state level of technology) is calibrated so that the deterministic steady state level of output ($\overline{Y}$) is equal to average daily GDP in Australia

¹⁰⁶Since $\overline{K}$ is not dependent on B (see Appendix 3.6.8), there will be a unique value of B that ensures $\overline{ES} = 30 = \frac{B\overline{K}}{2}$.

during 2012-13 (A\$4.17 billion).¹⁰⁷ Since $\bar{Y}$ is dependent on $\bar{ES}$, I calibrate $\bar{A}$ so that $\bar{Y} = 4174$ when $\bar{ES} = 30$; the result is $\bar{A} = 311.12$.

With $\bar{ES}$ varied within the range $[0, 30]$, the only remaining parameters without values are ψ (the monitoring cost scaling parameter) and ϑ (the effort cost).

For the baseline level of ψ , I set the parameter such that the welfare-optimising aggregate ES balance is close to $\bar{ES} = 30$. The baseline ψ is $\psi = 3.5$.

As discussed in Section 3.2.7, ϑ must be set sufficiently small to ensure there is a stigma problem to solve, but sufficiently large such that there is no adverse selection problem when the lending facility is not used. The baseline ϑ is $\vartheta = 0.033$.

3.6.13 Deriving Welfare

From Eq. (3.1) (and using Eqs. (3.20)-(3.24) to substitute for C_s), the expected discounted lifetime utility of the representative consumer at the end of day t (henceforth, welfare) is approximately equal to:

$$\begin{aligned} \mathbb{W}_t \approx & \mathbb{E} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\ln(1 - \alpha) - \ln(\chi) + \ln(\bar{A}) + \varepsilon_s + \alpha k_s \right] \right\} \\ & + \mathbb{E} \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[\alpha \ln(\bar{K}) - (\chi \bar{H} + \alpha) h_s - \alpha \ln(\bar{H}) - \chi \bar{H} \right] \right\} \end{aligned} \quad (3.109)$$

Where the approximation is that: $\ln(K_s) - \ln(\bar{K}) \approx k_s$ and $\ln(H_s) - \ln(\bar{H}) \approx h_s$.

From Theorem 3.1, each I_s is only dependent on the parameter values and the day s shock. Therefore, expectations of future I_s are independent of the day t state variables. Then, since consumers' expectations of I_s are unbiased and stationary: $\mathbb{E}[I_s | s > t] \equiv \mathbb{E}[I | \mathcal{I}^C]$.

Then, using Eqs. (3.28) and (3.29) for the k_s ($s > t$) and h_s ($s \geq t$) terms, Eq. (3.109) can be simplified to:

¹⁰⁷I calibrate to a single year since my model does not incorporate output growth. I also calibrate to average GDP per day (rather than per day per person) because the number of consumers is normalised to a $[0, 1]$ continuum.

$$\begin{aligned}
\mathbb{W}_t \approx & \left[\frac{1}{1-\beta} \right] \left[\ln(1-\alpha) - \ln(\chi) + \ln(\bar{A}) \right] \\
& + \left[\frac{1}{1-\beta} \right] \left[\alpha \ln(\bar{K}) - \alpha \ln(\bar{H}) - \chi \bar{H} \right] \\
& + \left[\frac{\chi \bar{H}}{(1-\alpha)} \right] \varepsilon_t + \left[\frac{\alpha \chi \bar{H}}{(1-\alpha)} \right] k_t \\
& - \left[\frac{\chi \bar{H} \vartheta}{(1-\alpha) \bar{Y}} \right] \left\{ I_{t-1} + \beta I_t + \left[\frac{\beta^2}{(1-\beta)} \right] \mathbb{E}[I | \mathcal{I}^C] \right\}
\end{aligned} \tag{3.110}$$

From Eqs. (3.28) and (3.29), the stochastic steady state values of h_t and k_t are:

$$\begin{aligned}
k = & - \left[\frac{\left[(\chi \bar{H} + \alpha) (1-\alpha) - \alpha (1-\alpha^2 \beta) \chi \bar{H} \right] \vartheta}{\alpha (1-\alpha) (1-\alpha \beta) \chi \bar{H} \bar{Y}} \right] \mathbb{E}[I | \mathcal{I}^C] \\
& - \left[\frac{(\chi \bar{H} + \alpha) \kappa}{\alpha (1-\alpha \beta) \chi \bar{H} \bar{Y}} \right] \mathbb{E}[F | \mathcal{I}^C]
\end{aligned} \tag{3.111}$$

$$\begin{aligned}
h = & - \left[\frac{\left[(1-\alpha) - (1-\beta) \chi \bar{H} \right] \kappa}{(1-\alpha) (1-\alpha \beta) \chi \bar{H} \bar{Y}} \right] \mathbb{E}[F | \mathcal{I}^C] \\
& - \left[\frac{\left[(1-\alpha) - (1-\beta) \chi \bar{H} \right] \left[(\chi \bar{H} + \alpha) (1-\alpha) - \alpha (1-\alpha^2 \beta) \chi \bar{H} \right] \vartheta}{(\chi \bar{H} + \alpha) (1-\alpha)^2 (1-\alpha \beta) \chi \bar{H} \bar{Y}} \right] \mathbb{E}[I | \mathcal{I}^C]
\end{aligned} \tag{3.112}$$

When the economy is in its stochastic steady state at day t , Eq. (3.110) becomes (using Eq. (3.111)):

$$\begin{aligned}
\mathbb{W}_t \approx & \left[\frac{1}{1-\beta} \right] \left[\ln(1-\alpha) - \ln(\chi) + \ln(\bar{A}) \right] \\
& + \left[\frac{1}{1-\beta} \right] \left[\alpha \ln(\bar{K}) - \alpha \ln(\bar{H}) - \chi \bar{H} \right] \\
& - \left[\frac{\left[(\chi \bar{H} + \alpha) (1-\alpha) - \alpha (1-\alpha^2 \beta) \chi \bar{H} \right] \vartheta}{(1-\alpha)^2 (1-\alpha \beta) \bar{Y}} \right] \mathbb{E}[I | \mathcal{I}^C] \\
& - \left[\frac{(\chi \bar{H} + \alpha) \kappa}{(1-\alpha) (1-\alpha \beta) \bar{Y}} \right] \mathbb{E}[F | \mathcal{I}^C] \\
& - \left[\frac{\beta^2 \chi \bar{H} \vartheta}{(1-\alpha) (1-\beta) \bar{Y}} \right] \mathbb{E}[I | \mathcal{I}^C]
\end{aligned} \tag{3.113}$$

Therefore:

$$\begin{aligned}
 G_1 &= \ln(1 - \alpha) - \ln(\chi) + \ln(\bar{A}) \\
 &\quad + \alpha \ln(\bar{K}) - \alpha \ln(\bar{H}) - \chi \bar{H} \\
 G_2 &= \frac{\beta^2 \chi \bar{H}}{(1 - \alpha) \bar{Y}} \\
 G_3 &= \frac{(\chi \bar{H} + \alpha) (1 - \alpha) - \alpha (1 - \alpha^2 \beta) \chi \bar{H}}{(1 - \alpha)^2 (1 - \alpha \beta) \bar{Y}} \\
 G_4 &= \frac{(\chi \bar{H} + \alpha)}{(1 - \alpha) (1 - \alpha \beta) \bar{Y}}
 \end{aligned}$$

Note that $\bar{H}$ and $\bar{Y}$ can both be expressed as a function of the structural parameters and $\bar{K}$. Therefore, the impact of $\bar{E}\bar{S}$ on these reduced-form parameters occurs only through its impact on $\bar{K}$.

Using the deterministic steady state values given in Appendix 3.6.7, differentiating Eq. (3.113) with respect to $\bar{E}\bar{S}$ gives:

$$\begin{aligned}
 \frac{d\mathbb{W}_t}{d\bar{E}\bar{S}} \approx & - \left[\frac{\chi \bar{H}}{(1 - \beta) \bar{K}} \right] \left\{ \frac{d\bar{K}}{d\bar{E}\bar{S}} + \left[\frac{\alpha \beta^3 \vartheta}{(1 - \alpha)} \right] \frac{d\mathbb{E}[I|\mathcal{I}^C]}{d\bar{E}\bar{S}} \right\} \\
 & - \left[\frac{[(\chi \bar{H} + \alpha) (1 - \alpha) - \alpha (1 - \alpha^2 \beta) \chi \bar{H}] \vartheta}{(1 - \alpha)^2 (1 - \alpha \beta) \bar{Y}} \right] \frac{d\mathbb{E}[I|\mathcal{I}^C]}{d\bar{E}\bar{S}} \\
 & - \left[\frac{(\chi \bar{H} + \alpha) \kappa}{(1 - \alpha) (1 - \alpha \beta) \bar{Y}} \right] \frac{d\mathbb{E}[F|\mathcal{I}^C]}{d\bar{E}\bar{S}} \\
 & + \left[\frac{\alpha \vartheta \mathbb{E}[I|\mathcal{I}^C] + \alpha \kappa \mathbb{E}[F|\mathcal{I}^C]}{(1 - \alpha) (1 - \alpha \beta) \bar{Y} \bar{K}} \right] \frac{d\bar{K}}{d\bar{E}\bar{S}}
 \end{aligned} \tag{3.114}$$

So:

$$\begin{aligned}
 G_5 &= \frac{\chi \bar{H}}{\bar{K}} \\
 G_6 &= \frac{\alpha}{(1 - \alpha) (1 - \alpha \beta) \bar{Y} \bar{K}}
 \end{aligned}$$

3.6.14 Upper bound of welfare in the non-modelled subset

Both $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$ have a negative impact on welfare. Suppose the value of $\bar{K}$ is given. Instead of using the algorithm (in Appendix 3.6.11) to determine the exact value of welfare at $\bar{K}$, and then finding the value of $\bar{K}$ that maximises welfare (within the non-modelled subset), it is sufficient (for the qualitative results of this chapter) to simply determine the welfare upper bound for a given $\bar{K}$ and then to find the $\bar{K}$ that maximises this welfare upper bound (this removes any potential continuity and maxima existence issues).

From Eq. (3.31), positive shocks will always lead to lending facility use. But the central bank can set the aggregate ES balance such that negative shocks do not cause lending facility use. Let $E[F^{\min}|\bar{K}]$ be the value of $E[F|\mathcal{I}^C]$ assuming that only positive shocks lead to lending facility use. That is:

$$E[F^{\min}|\bar{K}] = \frac{(1 - \varphi)\bar{K}B}{8}$$

For a given $\bar{K}$, the expected use of the lending facility cannot be lower than this value.

Similarly, with the parameter values in this chapter ensuring the shares are always sold to the asset managers whenever the lending facility is not used, the largest proportion of shocks that lead to asset sales occurs when only positive shocks lead to lending facility use.¹⁰⁸ In this case:

$$\begin{aligned} E[\varepsilon_t | \varepsilon_t \in \mathbb{F}_t, \varepsilon_t \leq \varepsilon_t^{\sup}] &= E[\varepsilon_t | 0 < \varepsilon_t \leq \varepsilon_t^{\sup}] \\ &= \frac{\varepsilon_t^{\sup}}{2} \end{aligned}$$

Then, using the same process as Appendix 3.6.11, the largest value of $\varepsilon_t^{\dagger}$ (within

¹⁰⁸If any other shocks lead to lending facility use, they must be less than or equal to zero, thereby reducing the share sale price and potentially leading to fewer sales.

the $(0, B]$ range) that satisfies Theorem 3.1 is $\varepsilon_t^{\dagger\dagger} = Z$ (recall that $Z < B$ with the parameter values used in this chapter). Therefore, the lowest possible value of $E[I|\mathcal{I}^C]$ (for a given $\overline{K}$) is:

$$\begin{aligned} E[I^{\min}|\overline{K}] &= P(\varepsilon_t > Z | F_t > 0) P(F_t > 0) \\ &= \left[\frac{B - Z}{B} \right] \frac{(1 - \varphi)}{2} \end{aligned}$$

Replacing $E[F|\mathcal{I}^C]$ and $E[I|\mathcal{I}^C]$ in Eq. (3.33) (the welfare function) with the functions for $E[F^{\min}|\overline{K}]$ and $E[I^{\min}|\overline{K}]$, respectively, produces a function that is continuous in $\overline{ES}$ (henceforth, the welfare upper bound function). Therefore, with the non-modelled subset being closed, bounded, and convex, there exists a value of $\overline{ES}$ that maximises the welfare upper bound function within the non-modelled subset.

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