CURVED SPACES ADMITTING SOLUTIONS

TO TWISTOR EQUATIONS

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Abstract

This thesis comprises three sections. In the first, real space-times admitting a solution to the two-index twistor(Killing spinor) equation are constructed and the separability properties of solutions to zero rest-mass field equations within these space-times are derived. In addition conditions for differential relations between components of opposite helicity are derived.

In the second section a class of half algebraically-special spaces is investigated, a class which may be characterised as those admitting a certain partial two-index twistor (Killing spinor). A formalism based on their structure as a two-dimensional family of flat null two-planes is developed and used to provide an explicit integration of the curvature equations, to discuss Hertz-type potentials for zero rest-mass fields and to examine a class of Einstein-Maxwell spaces and their perturbations.

In the third section we examine the relation between the space of solutions to the twistor equation restricted to a general space--like two-surface of spherical topology (two-surface twistor space) and its dual. This is then used to discuss the nature of the local embedding of the two-surface in a space conformally related to complex Minkowski space.

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Because I know that time is always time
And place is always and only place
And what is actual is actual only for one time
And only for one place
I rejoice that things are as they are and
I renounce the blessèd face
And renounce the voice
Because I cannot hope to turn again
Consequently I rejoice, having to construct something
Upon which to rejoice
And pray to God to have mercy upon us
And pray that I may forget
These matters that with myself I too much discuss
Too much explain
Because I do not hope to turn again
Let these words answer
For what is done, not to be done again
May the judgement not be too heavy upon us

T.S.Eliot 1930

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Introduction

This thesis is divided into three main sections which are intended to be selfcontained and may be read separately. The subjects dealt with are linked by the common thread of dealing with spaces admitting full or partial solutions to various twistor (Killing spinor) equations.

The n-index twistor (Killing spinor) equation is (Walker and Penrose 1970)

$$\nabla_{A'(A} \chi_{B\ldots C)} = 0 \qquad 1$$

for some symmetric n-spinor $\chi_{B\ldots C}$. Conventionally this equation is referred to as a twistor equation if a multi-dimensional space of solutions exists, the Killing spinor equation otherwise. We use two-spinor notation and modified spin coefficient formalism throughout (Penrose and Rindler 1984; Geroch, Held and Penrose 1973).

In the first part we look at real space-times which admit a non-null solution to the two-index Killing spinor equation

$$\nabla_{A'(A} \chi_{BC)} = 0 \qquad 2$$

This work appeared in Jeffryes 1984a, and shows how all such spaces may be constructed and derives properties of the solutions to various other equations within these spaces. This builds on work of Dietz and Rudiger 1980, 1981 and Carter 1968a, on their construction and Carter 1968a,b, Teukolsky 1972, Chandrasekar 1979, Dudley and Finley 1979 regarding the solutions to further equations.

The second section is devoted to complex spaces, whose primed Weyl spinor has a repeated principal null direction (p.n.d.), which is both geodesic and shear-free. Their link with Killing spinors is that they are the largest class of spaces admitting a solutions to the partial two-index Killing spinor equation

$$\nabla_{A(A'} \bar{X}_{B'C')} = -\sigma_A \bar{o}_{A'} \bar{o}_{B'} \bar{o}_{C'} \qquad 3$$

$$\text{or} \quad \bar{o}^{A'} \nabla_{A(A'} \bar{X}_{B'C')} = 0$$

for some σ_A and where $\bar{o}^{A'} \bar{o}^{B'} \nabla_{AA'} \bar{o}_{B'} = 0$

These spaces have been studied chiefly after imposing field equations (Plebanski and Robinson 1977; Garcia, Plebanski and Robinson 1977). In our work we use different methods to those of previous authors and show how a covariant two-spinor approach leads to a complete integrated description of these spaces without any field equations, and enables the discussion of certain field equations to be simplified. Work without the use of field equations has also been carried out recently by Plebanski and Rozga 1984.

Finally we discuss aspects of two-surface twistor space, being the four dimensional space of solutions to the twistor equation, where the only components of the equation required to be

$$(\nabla_{A'(A} \omega_{B)})|_{\mathcal{J}} = 0 \qquad 4$$

satisfied are those tangential to some space-like two-surface of spherical topology $\mathcal{J}$ in a general curved real

space-time, (Penrose 1982). Some of this work appeared in Jeffryes 1984b.

Using this twistor space there is the prospect of a definition of the general relativistic mass and angular momentum surrounded by $\mathcal{J}$, and this section continues the work by a number of authors on the use of such a definition; at spatial and null infinity (Shaw 1983a,b; 1984a,b,c; Dray and Streubel 1984); for certain finite examples (Tod 1983a, 1984b; Tod and Kelly 1984); and other developments (Tod 1983b, 1984a; Jeffryes 1984b)

To conclude this introduction we shall give the algebraic constraints on the Weyl spinor for a space-time to admit a full n-index Killing spinor, and the constraints for a solution to equation 3.

Theorem.

If $\omega^{A\ldots B}$ is an n-index Killing spinor then Ψ_{ABCD} is proportional to $(\omega_{A\ldots B})^{4/n}$

Proof.

By $(\omega_{A\ldots B})^{4/n}$ we mean the following

If $X_{ABCDEFGH} = (\omega_{AB})^4$ then $X_{A\ldots H} = \omega_{(AB}\,\omega_{CD}\,\omega_{EF}\,\omega_{GH)}$

If $X_A = (\omega_{ABCD})^{1/4}$ then $\omega_{ABCD} = X_A\,X_B\,X_C\,X_D$.

Clearly to take the nth root of a symmetric spinor each p.n.d. of that spinor must have a multiplicity divisible by n.

We may write down the second order integrability condition for 1 by applying $\nabla_D^{A'}$ and symmetrising. We find (Penrose 1975)

$$\Psi_{F(ABC\ \omega_{D\ldots E})}{}^{F} = 0 \qquad 5$$

Clearly if $\omega_{DEFG} \alpha \Psi_{DEFG}$ 6

then $\Psi_{F(ABC\ \omega_{D\ldots E})}{}^{F} \alpha \Psi_{F(ABC\ \Psi_{DEG})}{}^{F} = 0$ 7

Although the n = 1 case is also clear from 5 we shall deal with it in conjunction with the other cases by taking components in the notation of Geroch, Held and Penrose 1973 (G.H.P.).

Let $\{o_A, \iota^A\}$ be a spinor dyad, normalised by $o_A \iota^A = 1$. If $\omega^{A\ldots B}$ is an n-index Killing spinor we write

$$\omega_0 = o^A \ldots\ldots\ldots o^D\ \omega_{A\ldots\ldots D} \qquad \text{type } (n,0)$$

$$\omega_1 = \iota^A\ o^B\ \ldots\ldots o^D\ \omega_{A\ldots\ldots D} \qquad \text{type } (n-2,0)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\omega_{n-1} = \iota^A\ \ldots\ldots \iota^C o^D\ \omega_{A\ldots\ldots D} \qquad \text{type } (2-n,0)$$

$$\omega_n = \iota^A\ \ldots\ldots\ldots \iota^D\ \omega_{A\ldots\ldots D} \qquad \text{type } (-n,0)$$

First consider the case of $\omega_{A\ldots B}$ null. If we set o_A as its p.n.d. then all ω_k vanish aside from ω_n, and 1 becomes

$$\kappa\omega_n = \sigma\omega_n = 0 \qquad 8$$

$$(\text{Þ} + n\rho)\omega_n = (\eth + n\tau)\omega_n = 0 \qquad 9$$

$$\eth'\omega_n = \text{Þ}'\omega_n = 0 \qquad 10$$

8 implies $\kappa = \sigma = 0$ and thus (G.H.P. 2.23) Ψ_0 vanishes.

Applying the commutators $[\text{Þ}', \eth]$, and $[\text{Þ}, \eth]$ and $[\text{Þ}', \text{Þ}]$, $[\eth', \eth]$ with G.H.P. 2.26 we find

$$n\,\Psi_3 = 3n\,\Psi_1 = 2n\,\Psi_2 = 0 \qquad 11$$

and thus Ψ_{ABCD} is null, and proportional to $(\omega_{A\ldots B})^{4/n}$.

If $\omega_{A\ldots D}$ is not null it has at least two p.n.d.'s. We choose two of these as our dyad. Thus $\omega_0 = \omega_n = 0$ and 1 becomes

$$\begin{array}{lllll}
\kappa\omega_1 & - & 0 & + & 0 \qquad + \ldots = 0 \\
(\text{Þ} + \rho)\omega_1 & - & n\,\kappa\omega_2 & + & 0 \qquad + \ldots = 0 \\
(\eth' + (n-1)\tau')\omega_1 & - & n(\text{Þ} + 2\rho)\omega_2 & + & \tfrac{1}{2}\,n(n+1)\kappa\omega_3 \qquad + \ldots = 0 \\
\sigma'\omega_1 & - & n(\eth' + (n-2)\tau')\omega_2 & + & \tfrac{1}{2}\,n(n+1)(\text{Þ} + 3\rho)\omega_3 \qquad + \ldots = 0 \\
0 & - & n\sigma'\omega_2 & + & \tfrac{1}{2}\,n(n+1)(\eth' + (n-3)\tau')\omega_3 + \ldots = 0 \\
\vdots & & \vdots & & \vdots
\end{array} \qquad (12)$$

and similarly with σ, $\eth$, Þ', κ' for κ, Þ, $\eth'$, σ' respectively.

Firstly we see $\sigma = \kappa = \sigma' = \kappa' = 0$ and thus Ψ_0 and Ψ_4 vanish. This implies there can be no more than four p.n.d.'s of $\omega_{A\ldots D}$, since all p.n.d.'s of $\omega_{A\ldots D}$ are p.n.d.'s of Ψ_{ABCD}, and this can have only four.

Applying [Þ, $\eth$] to the non-vanishing ω_m with the lowest value of m gives

$$(n - 4m)\Psi_1 = 0 \qquad 13$$

Similarly [Þ', $\eth'$] to the non-vanishing ω_{n-k} with the lowest value of k

$$(n - 4k)\Psi_3 = 0 \qquad 14$$

Firstly consider when n is not divisible by 4.

Ψ_1 and Ψ_3 both vanish and thus Ψ_{ABCD} has only two p.n.d.'s and hence $\omega_{A\ldots D}$ has only two p.n.d.'s and has only one non-vanishing component, say ω_m for which

$$(\text{Þ} + m\rho)\omega_m = (\eth + m\tau)\omega_m = 0 \qquad 15$$

$$(\text{Þ}' + (n-m)\rho')\omega_m = (\eth' + (n-m)\tau')\omega_m = 0 \qquad 16$$

The commutators $[\text{Þ}', \text{Þ}]$ and $[\eth', \eth]$, together with G.H.P. 2.26 give

$$(n - 2m)\Psi_2 = 0 \qquad 17$$

Thus in this case both p.n.d.'s of $\omega_{A\ldots D}$ (and Ψ_{ABCD}) have the same multiplicity $\Psi_{ABCD} \alpha (\omega_{A\ldots D})^{4/n}$.

Finally if n is divisible by four, say n = 4q, we see from 13, 14 that each p.n.d. of $\omega_{A\ldots D}$ must also have a multiplicity divisible by q and thus $\Psi_{ABCD} \alpha (\omega_{A\ldots D})^{1/q}$.

Lemma

If $X_{A'B'}$ is a symmetric two-index spinor and $\bar{o}_{C'}$ a one index spinor, obeying

$$\bar{o}^{C'} \nabla_{A(A'} X_{B'C')} = 0 \qquad 18$$

$$\bar{o}^{D'} \bar{o}^{E'} \nabla_{DD'} \bar{o}_{C'} = 0 \qquad 19$$

then $\bar{o}^{C'}$ is a repeated p.n.d. of $\bar{\Psi}_{A'B'C'D'}$. If $\bar{\Psi}_{A'B'C'D'}$ is not null then it is also a p.n.d. of $X_{A'B'}$.

Proof

Applying $\nabla_{A(A'}\nabla^{A}_{B')}$ to $\bar{o}_{C'}$, we see that

$$\bar{o}^{C'}\ \bar{o}^{D'}\ \bar{o}^{E'}\ \bar{o}^{F'}\ \bar{\Psi}_{C'D'E'F'} = 0 \qquad 20$$

and thus $\bar{o}_{C'}$ is a p.n.d. of $\bar{\Psi}_{A'B'C'D'}$. Similarly applying $\nabla_{A(A'}\nabla^{A}_{D')}$ to $X_{B'C'}$ we find

$$\bar{o}^{B'}\ \bar{o}^{C'}\ \bar{\Psi}_{E'(A'B'C'}\ X_{D')}{}^{E'} = 0 \qquad 21$$

Taking as a spinor dyad $\{\bar{o}_{A'}, \bar{\iota}^{A'}\}$, where $\bar{o}_{A'}\ \bar{\iota}^{A'} = 1$ we may rewrite 20, 21 as

$$\bar{\Psi}_0 = 0 \qquad 22$$

$$X_0\ \bar{\Psi}_1 = 0 \qquad 23$$

$$X_0\ \bar{\Psi}_2 - X_1\ \bar{\Psi}_1 = 0 \qquad 24$$

$$X_0\ \bar{\Psi}_3 - X_2\ \bar{\Psi}_1 = 0 \qquad 25$$

where $X_0 = \bar{o}^{A'}\ \bar{o}^{B'}\ X_{A'B'}$

$X_1 = \bar{o}^{A'}\ \bar{\iota}^{B'}\ X_{A'B'}$

$X_2 = \bar{\iota}^{A'}\ \bar{\iota}^{B'}\ X_{A'B'}$

If we assume that $\bar{o}_{A'}$ is not a repeated p.n.d. of $\bar{\Psi}_{A'B'C'D'}$ then we see that $X_0 = X_1 = X_2 = 0$, and thus $X_{A'B'}$ vanishes, hence $\bar{o}_{A'}$ is a repeated p.n.d. of $\bar{\Psi}_{A'B'C'D'}$. In addition if $o_{A'}$ is not a p.n.d. of $X_{A'B'}$ (i.e. $X_0 \neq 0$) then we see that $\bar{\Psi}_1 = \bar{\Psi}_2 = \bar{\Psi}_3 = 0$ and $\bar{\Psi}_{A'B'C'D'}$ is null.

Space-times with two-index Killing spinors

1. Introduction

2. Preliminaries

3. The existence of two Killing vectors

4. The form of the metric

5. The scalar wave equation

6. The non-scalar wave equation

7. Relations between spin s and spin -s components

8. Remarks on spin 1

9. Conclusion

1. Introduction

This section is devoted to a study of spacetimes admitting a non-null valence-two Killing spinor (valence-two symmetric twistor), namely a symmetric two-spinor X_{AB} satisfying the equation

$$\nabla_{A'(A} X_{BC)} = 0 \tag{1.1}$$

where the n-index Killing spinor equation is

$$\nabla_{A'(A} X_{B\ldots C)} = 0 \tag{1.2}$$

$X_{B\ldots C}$ is an n-index totally symmetric spinor (Walker & Penrose 1970).

If in addition to (1.1) X_{AB} satisfies

$$\nabla_{A'}{}^{B} X_{AB} + \nabla_{A}{}^{B'} \bar{X}_{A'B'} = 0 \tag{1.3}$$

then $Y_{ab} = i(X_{AB}\varepsilon_{A'B'} - \bar{X}_{A'B'}\varepsilon_{AB})$, satisfies the Killing-Yano 2-tensor equation

$$\nabla_{(a} Y_{b)c} = 0 \tag{1.4}$$

(Floyd 1973, Penrose 1973).

Some authors restrict the term Killing spinor for spinors satisfying both (1.1) and (1.3), referring to those for which (1.3) is not satisfied as conformal Killing spinors. (Carter and Mclenaghan 1979). Such terminology will not be used here since it will be seen that not all spinors satisfying (1.1) conformally satisfy (1.3). Spacetimes in which X_{AB} satisfies (1.3) in addition

to (1.1) will be referred to as KY spacetimes. All KY spacetimes have been constructed by Dietz and Rudiger (1980, 1981).

Spacetimes with non-null two-index Killing spinors will be shown to be conformally related to spacetimes in which the Hamilton-Jacobi equation, scalar wave equation and source free non-scalar wave equations are solvable by separation of variables. In addition conditions will be given for the components of opposite helicity of non--scalar waves to be differentially related.

This extends and expands on the work of Carter (1968a,b), Teukolsky (1972), Chandrasekar (1979), Stewart and Walker (1974), Dudley and Finley (1979), and others though in this work no further assumptions other than the existence of the Killing spinor will be made.

Most of the calculation is carried out using the modified spin-coefficient formalism of Geroch, Held and Penrose (1973), (henceforth abbreviated to GHP), a version of the Newman-Penrose formalism modified so that the scaling of the basis spinors is not specified, only their inner product being set.

2. Preliminaries

The existence of a Killing spinor imposes large restrictions on a spacetime. A symmetric n-spinor may be written as the symmetrised outer product of n 1-spinors (known as the principal null directions of the n-spinor). If an n-spinor is to satisfy the n-index Killing spinor equation then each of these 1-spinors must be both geodesic and shear-free. In addition taking a further derivative of (1.2) and symmetrising implies

$$\Psi_A{}^{(BCD}\, X^{E\ldots F)A} = 0 \qquad (2.1)$$

where Ψ_{ABCD} is the Weyl spinor and $X^{E\ldots FA}$ an n-Killing spinor (Penrose (1975)).

We write $X_{AB} = Xo_{(A}\iota_{B)}$ where $o_A\iota^A = 1$. o^A and ι^A will now be used as the basis spinors for the GHP formalism.

Taking components of (1.1) gives

$$\sigma = \sigma' = \kappa = \kappa' = 0 \qquad (2.2)$$

$$(\text{Þ} + \rho)X = 0 \qquad (2.3)$$

$$(\eth + \tau)X = 0 \qquad (2.4)$$

$$(\text{Þ}' + \rho')X = 0 \qquad (2.5)$$

$$(\eth' + \tau')X = 0 \qquad (2.6)$$

GHP equation (2.23) tells us that o_A and ι_A are principal null directions of Ψ_{ABCD} since ψ_0 and ψ_4 vanish. Applying the commmutators [Þ , ð] and [Þ', ð'] (GHP (2.31)

to X tells us that ψ_1 and ψ_3 vanish. Thus ψ_{ABCD} is type D and o_A and ι_A are its two principal null directions. If we apply the commutators [Þ', Þ], [Þ', ð], [Þ, ð'] and [ð , ð'] to X we find further relations among the spin coefficients

$$Þ\rho' - Þ'\rho = \tau\bar{\tau} - \tau'\bar{\tau}' \tag{2.7}$$

$$ð'\rho - Þ\tau' = 0 \tag{2.8}$$

$$ðp' - Þ'\tau = 0 \tag{2.9}$$

$$ð\tau' - ð'\tau = \rho\bar{\rho}' - \bar{\rho}\rho' \tag{2.10}$$

If X_{AB} is a solution to the 2-Killing spinor equation in a spacetime with metric g_{ab} then $\Omega^2 X_{AB}$ will be a solution to the equation in a spacetime with metric $\Omega^2 g_{ab}$ (Penrose (1975)). Thus for all spacetimes with a 2-Killing spinor we can find a conformally related spacetime in which $X\bar{X} = 1$. Using (2.2)-(2.6) we see that in such a spacetime

$$\rho + \bar{\rho} = \rho' + \bar{\rho}' = \tau + \bar{\tau}' = 0 \tag{2.11}$$

Now the tensor $P_{ab} = X_{AB}\ \bar{X}_{A'B'}$ satisfies the trace-free part of the Killing tensor equation (Hughston and Sommers (1973)). We may ask the question of under what conditions is P_{ab} the trace-free part of a Killing tensor.

If K_{ab} is a Killing tensor whose trace-free part is P_{ab}, and letting $K = \frac{3}{4} K^a{}_a$ then K must satisfy

$$\nabla^a P_{ab} + \nabla_b K = 0 \tag{2.12}$$

Taking components gives

$$\text{Þ} K + (\rho + \bar{\rho}\cdot)(X\bar{X}) = 0 \tag{2.13}$$

$$\text{Þ}' K + (\rho' + \bar{\rho}')(X\bar{X}) = 0 \tag{2.14}$$

$$\text{ð} K - (\tau + \bar{\tau}')(X\bar{X}) = 0 \tag{2.15}$$

$$\text{ð}' K - (\bar{\tau} + \tau')(X\bar{X}) = 0 \tag{2.16}$$

If the spacetime is KY then $K = Y_{ab}\, Y^{ab}$, where Y_{ab} satisfies (1.4), will be a solution (Floyd 1973). However we do not know when a spacetime is conformally KY.

If $X\bar{X} = 1$ then (2.13)-(2.16) are satisfied by K = constant, hence in this case P_{ab} is a Killing tensor.

All conformal transformations that preserve the existence of such a Killing tensor are of the form $\Omega^2 = a^2 - b^2$ where

$$\text{ð} a = 0 \qquad (\rho'\text{Þ} - \rho\,\text{Þ}')a = 0 \tag{2.17a,b}$$

$$\text{Þ} b = \text{Þ}' b = 0 \qquad (\tau'\text{ð} - \tau\text{ð}')b = 0 \tag{2.18a,b}$$

where a is real and b imaginary. Such functions exist as it may be shown that $\rho\rho'$ obeys (2.17) and $\tau\tau'$ obeys (2.18). With such a conformal factor

$$X\bar{X} = a^2 - b^2 \tag{2.19}$$

$$K = \text{constant} + a^2 + b^2 \tag{2.20}$$

We may see that conformal transformations that preserve the Killing tensor are of this form by considering a spacetime where $X\bar{X} \neq$ constant and the Killing tensor exists. We see that $a^2 = \frac{1}{2}(X\bar{X} + K)$ and $b^2 = \frac{1}{2}(K - X\bar{X})$ obey (2.17) and (2.18) respectively, so $X\bar{X}$ is of the form

$a^2 - b^2$.

Applying commutators of Þ, $\text{Þ}'$, $ð$ and $ð'$ to K tells us

$$\rho\bar{\rho}' - \bar{\rho}\rho' = 0 \tag{2.21}$$

$$\text{Þ}(\tau + \bar{\tau}') = (2\bar{\rho} + \rho)(\tau + \bar{\tau}') \tag{2.22}$$

$$ð(\rho + \bar{\rho}) = (2\bar{\tau}' + \tau)(\rho + \bar{\rho}) \tag{2.23}$$

$$ð(\rho' + \bar{\rho}') = (2\tau + \bar{\tau}')(\rho' + \bar{\rho}') \tag{2.24}$$

$$\tau\bar{\tau} - \tau'\bar{\tau}' = 0 \tag{2.25}$$

Since all spacetimes under consideration are conformally related to spacetimes in which P_{ab} is the trace-free part of a Killing tensor we shall henceforth consider only these, and thus equations (2.13)-(2.16) and (2.21)-(2.25) will be satisfied.

In order to facilitate discussion we now divide the spacetimes under consideration into six cases, according to their spin coefficients

Case 1 $\rho \neq \bar{\rho}$, $\rho' \neq \bar{\rho}'$, $\tau \neq \bar{\tau}'$

With a conformal factor for which $X\bar{X} = 1$, ρ, ρ', τ, τ' are all non vanishing.

Case 1N $\rho \neq \rho$, $\tau \neq \tau'$, $\rho' = 0$

With a conformal factor as in case 1 only ρ' vanishes. All conformal transformations that preserve the Killing tensor must obey 2.17b, hence in all conformally related spacetimes with a Killing tensor $\rho' = 0$.

Case 2 $\tau \neq \bar{\tau}'$, $\rho = \bar{\rho}$, $\rho' = \bar{\rho}'$

With the above conformal factor $\rho =$ $\rho' = 0$.

Case 3 $\rho \neq \bar{\rho}$, $\rho' \neq \bar{\rho}'$, $\tau = \bar{\tau}'$

With the above conformal factor $\tau = \tau' = 0$.

Case 3N $\rho \neq \bar{\rho}$, $\tau = \bar{\tau}'$, $\rho' = 0$

With the above conformal factor $\tau = \tau' = \rho' = 0$.

Case 4 $\rho = \bar{\rho}$, $\rho' = \bar{\rho}'$, $\tau = \bar{\tau}'$.

With the above conformal factor $\rho = \rho' = \tau = \tau' = 0$.

Differentiating (2.21)-(2.25) gives constraints on the curvature. By using the Bianchi identity components (2.33)-(2.36) of GHP we find

$$0 = (\rho'^2\Phi_{00} - \rho^2\Phi_{22})(\bar{\rho}-\rho) \tag{2.26}$$

$$0 = (\tau'^2\Phi_{02} - \tau^2\Phi_{20})(\bar{\tau}'-\tau) \tag{2.27}$$

$$\rho'\tau'\Phi_{01} = \rho\tau\Phi_{21} \tag{2.28}$$

$$\rho'\Phi_{01} + \rho\Phi_{12} = 3\rho'(\rho\bar{\tau}' + \bar{\rho}\tau) \tag{2.29}$$

$$Þ\Phi_{01} = 3\rho(\rho\bar{\tau}' + \bar{\rho}\tau) + (2\bar{\tau}' + \tau)\Phi_{00} + (3\bar{\rho} - \rho)\Phi_{01} \tag{2.30}$$

$$ð\Phi_{01} = 3\tau(\rho\bar{\tau}' + \bar{\rho}\tau) + (2\bar{\rho} + \rho)\Phi_{02} + (3\bar{\tau}' - \tau)\Phi_{01} \tag{2.31}$$

$$ð\Phi_{00} = 3\rho(\rho\bar{\tau}' + \bar{\rho}\tau) + (3\bar{\tau}' + \tau)\Phi_{00} + (\bar{\rho} - \rho)\Phi_{01} \tag{2.32}$$

$$Þ\Phi_{02} = 3\tau(\rho\bar{\tau}' + \bar{\rho}\tau) + (3\bar{\rho} + \rho)\Phi_{02} + (\bar{\tau}' - \tau)\Phi_{01} \tag{2.33}$$

plus the primed versions of (2.25)-(2.33). From these it can be seen that if Φ_{11} and Λ are the only non-vanishing Ricci components then $\rho\bar{\tau}' + \bar{\rho}\tau = \rho'\bar{\tau}' + \bar{\rho}'\tau = 0$. (2.34)

If the spacetime is KY then in addition to (2.21)-(2.25) we have

$$\rho X = \bar{\rho}\bar{X} \tag{2.35}$$

$$\tau X = -\bar{\tau}'\bar{X} \tag{2.36}$$

and thus (2.34) is true, and ϕ_{11} and Λ are the only Ricci curvature components. It will be seen that the vanishing of all other Ricci curvature implies that the spacetime is KY.

We are now in a position to define two coordinates. We choose a conformal factor so that $\text{Þ}(X\bar{X}) \neq 0$, $\eth(X\bar{X}) \neq 0$ and which preserves the existence of the Killing tensor. Let $R^2 = \frac{1}{2}(X\bar{X} + K)$, $T^2 = \frac{1}{2}(K - X\bar{X})$. R^2 and T^2 are then defined up to the addition of a constant, which we choose so that $R^2 \geqslant 0$ and $T^2 \leqslant 0$. (We may do this since $R^2 - T^2 \geqslant 0$.)

The coordinate derivatives are then

$$\text{Þ}R^2 = (\rho + \bar{\rho})(T^2 - R^2) \quad \eth R^2 = 0 \quad \text{Þ}'R^2 = (\rho' + \bar{\rho}')(T^2 - R^2) \tag{2.37}$$

$$\eth T^2 = (\tau + \bar{\tau}')(R^2 - T^2) \quad \text{Þ}T^2 = 0 \quad \text{Þ}'T^2 = 0 \tag{2.38}$$

Thus on unweighted quantities

$$\frac{\partial}{\partial R} = \frac{-R}{R^2 - T^2}\left(\frac{\text{Þ}}{\rho + \bar{\rho}} + \frac{\text{Þ}'}{\rho' + \bar{\rho}'}\right) \tag{2.39}$$

$$\frac{\partial}{\partial T} = \frac{-T}{T^2 - R^2}\left(\frac{\eth}{\tau + \bar{\tau}'} + \frac{\eth'}{\bar{\tau} + \tau'}\right) \tag{2.40}$$

and in addition in KY spacetimes

$$\frac{\partial}{\partial R} = -\frac{1}{X}\left(\frac{\text{Þ}}{\rho} + \frac{\text{Þ}'}{\rho'}\right) \tag{2.41}$$

$$\frac{\partial}{\partial T} = -\frac{1}{X}\left(\frac{\eth}{\tau} + \frac{\eth'}{\tau'}\right) \tag{2.42}$$

3. The existence of two Killing vectors

A vector k_a may be written

$$k_a = \beta' \ell_a + \beta n_a + \gamma' m_a + \gamma \bar{m}_a \tag{3.1}$$

where $\ell_a = o_A \bar{o}_{A'}$, $n_a = \iota_A \bar{\iota}_{A'}$, $m_a = o_A \bar{\iota}_{A'}$, $\bar{m}_a = \iota_A \bar{o}_{A'}$.

If k_a satisfies the Killing vector equation $\nabla_{(a} k_{b)} = 0$ then by taking components we have

$$\text{Þ}'\beta' = \text{Þ}\beta = 0 \tag{3.2}$$

$$ð'\gamma' = ð\gamma = 0 \tag{3.3}$$

$$\text{Þ}\beta' + \text{Þ}'\beta = \gamma'(\tau + \bar{\tau}') + \gamma(\bar{\tau} + \tau') \tag{3.4}$$

$$ð\gamma' + ð'\gamma = \beta'(\rho + \bar{\rho}) + \beta(\rho' + \bar{\rho}') \tag{3.5}$$

$$(ð + \bar{\tau}')\beta = (\text{Þ} + \bar{\rho})\gamma \tag{3.6}$$

$$(ð' + \bar{\tau})\beta' = (\text{Þ}' + \bar{\rho}')\gamma' \tag{3.7}$$

$$(ð + \tau)\beta' = (\text{Þ}' + \rho')\gamma \tag{3.8}$$

$$(ð' + \tau')\beta = (\text{Þ} + \rho)\gamma' \tag{3.9}$$

We seek a solution independent of the derivatives of R and T, thus we put

$$\beta' = f(\rho' + \bar{\rho}') \tag{3.10}$$

$$\beta = -f(\rho + \bar{\rho}) \tag{3.11}$$

$$\gamma' = -g(\bar{\tau} + \tau') \tag{3.12}$$

$$\gamma = g(\tau + \bar{\tau}') \tag{3.13}$$

(3.4) and (3.5) imply

$$(\rho' + \bar{\rho}')\,\text{Þ}\,f,g = (\rho + \bar{\rho})\,\text{Þ}'f,g \tag{3.14}$$

$$(\bar{\tau} + \tau')\,\eth f,g = (\tau + \bar{\tau}')\,\eth' f,g \tag{3.15}$$

(3.2), (3.3), (3.6)-(3.9) now reduce to

$$\text{Þ}f = -\frac{(\rho^2 + \bar{\rho}^2 + 2\phi_{00})}{(\rho + \bar{\rho})} f \tag{3.16}$$

$$\eth f = (\tau + \bar{\tau}')\ (\{\frac{\bar{\rho} - \rho}{\bar{\rho} + \rho}\}\ g - 2f) \tag{3.17}$$

$$\text{Þ}g = (\rho + \bar{\rho})\ (\{\frac{\bar{\tau}' - \tau}{\bar{\tau}' + \tau}\}\ f - 2g) \tag{3.18}$$

$$\eth g = -\frac{(\tau^2 + \bar{\tau}'^2 + 2\phi_{02})}{\tau + \bar{\tau}'} g \tag{3.19}$$

It may be checked that consistency depends on two conditions

$$\rho'^2\ \phi_{00}\ f = \rho^2\ \phi_{22}\ f \tag{3.20}$$

$$\tau'^2\ \phi_{02}\ g = \tau^2\ \phi_{20}\ g \tag{3.21}$$

For cases 1 and 1N these are equivalent to (2.26) and (2.27) thus two Killing vectors exist. For case 2 we seek a solution with $f = 0$, $g \neq 0$. Under these conditions. (3.20) vanishes and (3.21) is again equivalent to (2.27), hence one Killing vector exists. Similarly for cases 3 and 3N one Killing vector exists. In case 4 we may have no Killing vectors.

4. The form of the metric

We now consider the form of the metric for the different cases.

Cases 1 and 1N

If the pairs of solutions to (3.16)-(3.19) are f,g and εf, νg then equations for ε and ν are given by

$$\text{Þ}\varepsilon = \text{Þ}'\varepsilon = 0 \tag{4.1}$$

$$\eth\nu = 0 \tag{4.2}$$

$$(\rho + \bar{\rho})f\eth\varepsilon = (\tau + \bar{\tau}')(\bar{\rho} - \rho)(\nu - \varepsilon)g \tag{4.3}$$

$$(\tau + \bar{\tau}')g\,\text{Þ}\nu = (\rho + \bar{\rho})(\bar{\tau}' - \tau)(\varepsilon - \nu)f \tag{4.4}$$

Thus ε depends only on T and ν only on R. When the conformal factor is changed in a manner that preserves the Killing tensor f and εf will rescale identically, as will g and νg. Hence ε and ν will remain unchanged, thus by a suitable choice of conformal factor, under which R and T will be rescaled, ν and ε can be set to arbitrary functions of R and T respectively. We choose a factor so that $\varepsilon = T^2$ and $\nu = R^2$. (4.3) and (4.4) become

$$(\rho + \bar{\rho})f = (\bar{\rho} - \rho)g \tag{4.5}$$

$$(\tau + \bar{\tau}')g = (\bar{\tau}' - \tau)f \tag{4.6}$$

$$\text{implying } \rho\bar{\tau}' + \bar{\rho}\tau = 0 \tag{4.7}$$

The condition given by (4.7) is sufficient in general to narrow choice of conformal factor down to two, so long as ρ and τ do not vanish. It is evident that by choosing $\varepsilon = {}^{1}/_{T}\,2$, $\nu = {}^{1}/_{R}\,2$ (4.7) will again result. This corresponds

to a conformal factor of the form $\Omega = {}^{1}/_{\sqrt{-R^2T^2}}$, corresponding to a change in choice of R to ${}^{1}/_{R}$ and T to ${}^{1}/_{T}$. This preserves (4.17) and it will be seen also involves a change over in the roles of the two Killing vectors.

Differentiating (4.7) adds further curvature constraints, adding to (2.36)-(2.32)

$$\tau\Phi_{00} = \rho\Phi_{01} \tag{4.8}$$

$$\rho\Phi_{02} = \tau\Phi_{01} \tag{4.9}$$

$$\text{Þ}\Phi_{00} = 0 \quad \eth\Phi_{02} = 0 \tag{4.10}$$

$$\eth\Phi_{00} = 2\bar{\tau}'\Phi_{00} \tag{4.11}$$

$$\text{Þ}\Phi_{02} = 2\bar{\rho}\ \Phi_{02} \tag{4.12}$$

and their primed equivalents.

If Φ_{11} and Λ are the only non-vanishing Ricci components we set

$$\omega = \rho X \tag{4.13}$$

$$\omega' = \rho' X \tag{4.14}$$

$$\eta = \tau X \tag{4.15}$$

$$\eta' = \tau' X \tag{4.16}$$

If not then

$$\omega = \sqrt{-\Phi_{00}} \tag{4.17}$$

$$\omega' = \sqrt{-\Phi_{22}} \tag{4.18}$$

$$\eta = \sqrt{-\Phi_{02}} \tag{4.19}$$

$$\eta' = \sqrt{-\Phi_{20}} \tag{4.20}$$

(we take $-\Phi_{00}$ and $-\Phi_{02}$ since it will be seen that $\Phi_{00} \leqslant 0$)

Solutions to (3.16)-(3.19) are then

$$f(\rho + \bar{\rho}) = \omega \tag{4.21}$$

$$g(\tau + \tau') = \eta \tag{4.22}$$

We now examine the case where $\Phi_{00} \neq 0$ in more detail and obtain the relationship between the curvature components and spin coefficients.

$$\text{Let } F = \frac{\rho\bar{\rho}X\bar{X}}{\Phi_{00}} = -\frac{\tau\bar{\tau}'X\bar{X}}{\Phi_{02}} \tag{4.23}$$

$$\text{we find } \frac{\partial F}{\partial R^2} = -1 \qquad \frac{\partial F}{\partial T^2} = -1 \tag{4.24}$$

$$\text{hence } \frac{\Phi_{00}}{\rho\bar{\rho}} = -\frac{(R^2-T^2)}{(R^2+T^2+k)} = -\frac{\Phi_{02}}{\tau\bar{\tau}'} \tag{4.25}$$

for some real k.

$$\text{If } G = \frac{(\rho+\bar{\rho})^2(X\bar{X})^2}{\Phi_{00}} \qquad H = \frac{(\tau+\bar{\tau}')^2(X\bar{X})^2}{\Phi_{02}} \tag{4.26}$$

$$\text{then } \frac{\partial G}{\partial T^2} = 0 \qquad \frac{\partial G}{\partial R^2} = -(8R^2 + 4k) \tag{4.27}$$

$$\frac{\partial H}{\partial T^2} = -(8T^2 + 4k) \qquad \frac{\partial H}{\partial R^2} = 0 \tag{4.28}$$

$$\text{Thus } G = -4(R^4 + kR^2 + \ell) \qquad H = -4(T^4 + kT^2 + m) \tag{4.29}$$

for some real k,ℓ, m. The constants k,ℓ,m are restricted since

$$0 \leqslant \frac{(\rho+\bar{\rho})^2}{\rho\bar{\rho}} \leqslant 4 \tag{4.30}$$

$$0 \geqslant \frac{-(\tau+\bar{\tau}')^2}{\tau\bar{\tau}'} = \frac{(\rho-\bar{\rho})^2}{\rho\bar{\rho}} \geqslant -4 \tag{4.31}$$

$$(\rho+\bar{\rho})^2 - (\rho-\bar{\rho})^2 = 4\rho\bar{\rho} \tag{4.32}$$

(4.32) implies $\ell = m$. (4.30) and (4.31) tell us

$$0 \leqslant \frac{(R^4+kR^2+\ell)}{(R^2-T^2)(R^2+T^2+k)} \leqslant 1 \qquad (4.33)$$

$$0 \geqslant \frac{(T^4+kT^2+\ell)}{(R^2-T^2)(R^2+T^2+k)} \geqslant -1 \qquad (4.34)$$

Implying $R^4 + kR^2 + \ell \geqslant 0$, $T^4 + kT^2 + \ell \leqslant 0$

Noting that

$$(R^2+T^2+k) = \frac{(R^4+kR^2+\ell)-(T^4+kT^2+\ell)}{(R^2-T^2)}$$

we find that

$$\frac{\rho}{\bar{\rho}} = \frac{(\hat{R}-\hat{T})}{(\hat{R}+\hat{T})} \qquad (4.35)$$

where $\hat{R} = \sqrt{R^4+kR^2+\ell}$, $\hat{T} = \sqrt{T^4+kT^2+\ell}$

Solving (2.3)-(2.6) we find

$$X^2 = \frac{1}{\sqrt{k^2-4\ell}} ((\hat{R}+\hat{T})^2 - (R^2-T^2)^2) \qquad (4.36)$$

$$\bar{X}^2 = \frac{1}{\sqrt{k^2-4\ell}} ((\hat{R}-\hat{T})^2 - (R^2-T^2)^2) \qquad (4.37)$$

Thus in the limit as k tends to infinity Φ_{00} tends to zero and X becomes R+T. (In this limit the behaviour of ℓ is immaterial as we may change our definition of R^2 and T^2 so that R^2 becomes $R^2 + \ell/k$, and T^2 becomes $T^2 + \ell/k$).

It can be seen therefore that when all Ricci curvature bar Φ_{11} and Λ vanishes (4.35) becomes equivalent to (2.35) and

(2.36) and hence the spacetime is KY.

We are now in a position to write down the metrics. We note first that $\omega\omega'(R^2-T^2)$ is a function only of R and $\eta\eta'(R^2-T^2)$ a function only of T.

Let $P(R) = \omega\omega'(R^2-T^2)$, $Q(T) = \eta\eta'(R^2-T^2)$.

Case 1: $\rho,\rho',\ \tau,\tau' \neq 0$

$$ds^2 = (R^2-T^2)\left(\frac{R^2dR^2}{2P\hat{R}^2} - \frac{T^2dT^2}{2Q\hat{T}^2}\right) - \frac{1}{(R^2-T^2)}\left\{P(du+T^2dv)^2 - Q(du+R^2dv)^2\right\} \quad (4.38)$$

where u and v are coordinates whose gradients are Killing vectors.

We see that this is a subclass of Carter (1968a) metrics with separable Hamilton-Jacobi and scalar wave equation. The Carter (1968a) metrics differ in having $\hat{R}^2$ and $\hat{T}^2$ unrestricted as functions of R and T.

Case 1N: $\rho,\tau,\tau' \neq 0 \quad \rho' = 0$

$$ds^2 = (R^2-T^2)\left(\frac{-T^2dT^2}{2Q\hat{T}^2} + \frac{R}{\hat{R}}\,dR\ (du+T^2dv)\right) + \frac{Q}{(R^2-T^2)}\,(du+R^2dv)^2 \quad (4.39)$$

It is instructive to examine the contravariant metric for case 1.

$$g^{ab} = \frac{1}{(R^2-T^2)}\left\{\frac{2P\hat{R}^2}{R^2}\left(\frac{\partial}{\partial R}\right)^2 - \frac{1}{P}\left(\frac{\partial}{\partial u} - R^2\frac{\partial}{\partial v}\right)^2\right\} \quad (4.40)$$

$$+ \frac{-1}{(R^2-T^2)}\left\{\frac{2Q\hat{T}^2}{T^2}\left(\frac{\partial}{\partial T}\right)^2 - \frac{1}{Q}\left(\frac{\partial}{\partial u} - T^2\frac{\partial}{\partial v}\right)^2\right\}$$

A conformal factor that preserves the Killing tensor changes the factor (R^2-T^2) to $(F(R) + G(T))$ for any functions

F and G. It is clear that any such metric has a separable Hamilton-Jacobi equation, and conversely any transformation that preserves the separability of the Hamilton-Jacobi equation also preserves the Killing tensor.

This is to be expected since from the two Killing vectors and the Killing tensor we may construct three Killing tensors which are sufficiently unrelated to guarantee a separable Hamilton-Jacobi equation by a theorem of Woodhouse (1975).

Cases 2-4

It will be shown that all remaining cases are conformally KY spacetimes. Consider the space-time with a conformal factor for which $X\bar{X} = 1$.

In case 4 it is clear that X = constant, hence (1.3) is satisfied and the spacetime is KY. In Case 2 we have $\rho' = \rho = 0$, hence we may choose as an additional conformal factor. $\frac{\alpha}{X+\bar{X}}$, α real constant. In this conformally related spacetime we have $X + \bar{X}$ = constant, $\rho = \rho' = 0$, $\tau X = -\bar{\tau}'\bar{X}$ and thus (2.35) and (2.36) are satisfied and the spacetime is KY.

Similarly for cases 3 and 3N, with a conformal factor proportional to $X - \bar{X}$ we end with $X - \bar{X}$ = constant, $\tau = \tau' = 0$, $\rho X = \bar{\rho}\bar{X}$ and thus a KY spacetime.

All the above spacetimes have been constructed in Dietz and Rudiger (1980) hence only brief details of cases 3 and 4 will be given. In what follows calculations will not

be given in cases 1N, 2 and 3N since 1N and 3N are very similar to 1 and 3, and case 2 is identical to case 3 with the roles of R and T reversed.

The metric for case 3: $\tau = \bar{\tau}' = 0$, $X = R + \ell$, $\bar{X} = R - \ell$, ℓ constant

$$\begin{pmatrix} \frac{1}{\rho X}\, \ell^a \\ \frac{1}{\rho' X}\, n^a \\ \frac{1}{\alpha}\, m^a \\ \frac{1}{\alpha'}, \bar{m}^a \end{pmatrix} \qquad \nabla_a = \begin{pmatrix} -1 & 0 & \lambda & 0 \\ -1 & 0 & -\lambda & 0 \\ 0 & 1 & \varepsilon & \delta \\ 0 & 1 & -\varepsilon & -\delta \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial T} \\ \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix} \tag{4.41}$$

where T is an imaginary co-ordinate, v is a real co-ordinate and $\frac{\partial}{\partial u}$ is a Killing vector. α is of a type {1,-1} and α' of type {-1,1}

Applying commutators tells us that

$$\rho\rho' X^3 \bar{X} = P(R) \tag{4.42}$$

$$\delta = \delta(T,v), \quad \varepsilon = \varepsilon(T,v) \tag{4.43}$$

$$\alpha\alpha' = Q(T,v)/(R^2-\ell^2) \tag{4.44}$$

$$\lambda = \frac{(R^2-\ell^2)}{P} \tag{4.45}$$

$$\eth\alpha' = \eth'\alpha \tag{4.46}$$

and further

$$\delta\, \eth\alpha' + \alpha\alpha' \frac{\partial \delta}{\partial T} = 0 \tag{4.47}$$

$$\varepsilon\, \eth\alpha' + \alpha\alpha' \frac{\partial \varepsilon}{\partial T} = -2\ell/(R^2-\ell^2) \tag{4.48}$$

if we put $\delta = D(T,v)$ (4.49)

$$\varepsilon = 2\ell C(T,v)\ D(T,v) \tag{4.50}$$

where C is an imaginary function.

Then (4.47) and (4.48) tells us

$$(R^2-\ell^2)\alpha\alpha' = -\frac{1}{DC_T} \tag{4.51}$$

$$(R^2-\ell^2)\text{ð}\alpha' = \frac{D_T}{D^2C_T} \tag{4.52}$$

where $D_T = \frac{\partial D}{\partial T}$, $C_T = \frac{\partial C}{\partial T}$

$$ds^2 = \frac{(R^2-\ell^2)}{2}\ \{\frac{dR^2}{P} + DC_T dT^2\} - \frac{P}{2(R^2-\ell^2)}\ \{(du-2\ell Cdv)^2\} - \frac{C_T}{2D}\ dv^2 \tag{4.53}$$

The metric for case 4: $\tau = \tau' = \rho = \rho' = 0$, $X = 1$

$$\begin{pmatrix} \frac{1}{\beta}\ \ell^a \\ \frac{1}{\beta},\ n^a \\ \frac{1}{\alpha}\ m^a \\ \frac{1}{\alpha},\ \bar{m}^a \end{pmatrix} \qquad \nabla_a = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial T} \\ \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix} \tag{4.54}$$

T is an imaginary co-ordinate, all others are real. β is of type {1,1}, β' of type {-1,-1}, α of type {1,-1} and α' of type {-1,1}. Applying commutators we have

$$\beta\beta' = P(R,u) \tag{4.55}$$

$$\alpha\alpha' = Q(T,v) \tag{4.56}$$

$$\text{Þ}\beta' = \text{Þ}'\beta = 0 \tag{4.57}$$

$$\text{ð}\alpha' = \text{ð}'\alpha = 0 \tag{4.58}$$

$$ds^2 = \frac{1}{2P}\ (dR^2-du^2) - \frac{1}{2Q}\ (dT^2-dv^2) \tag{4.59}$$

For all the metrics we see that a solution to the Hamilton-Jacobi equation of the form $S = A(R,u) + B(T,v)$ will separate into equations for A and B.

5. The scalar wave equation

In the examination of the scalar and non-scalar wave equations the calculations are similar for the three types of metric presented, therefore the details will only be given for the case 1 metric, with the results following for cases 3 and 4.

Case 1

If a function χ is a scalar wave then it obeys the equation

$$(\nabla^2+R/6+m^2)\chi = 0 \qquad (5.1)$$

where R is the scalar curvature, and m the mass of χ .
In GHP this becomes

$$(\text{Þ}'\text{Þ} - \bar{\rho}'\text{Þ} - \rho\,\text{Þ}' - \eth'\eth + \bar{\tau}\eth + \tau\eth' + 2\Lambda + \tfrac{1}{2}m^2)\chi = 0 \qquad (5.2)$$

In order to show separation of variables we must learn something of the dependence of Λ on R and T. By applying the commutator $[\text{Þ},\eth]$ to Φ_{11}, using GHP equations (2.37)-(2.38), and to Λ we are left with an equation for Λ

$$(R^2-T^2)\,\frac{\partial^2\Lambda}{\partial R\partial T} - 2T\,\frac{\partial\Lambda}{\partial R} + 2R\,\frac{\partial\Lambda}{\partial T} = 0 \qquad (5.3)$$

This has solution $\Lambda = \dfrac{F(R)-G(T)}{R^2-T^2}$ for arbitrary differentiable F and G. In what follows it is useful to define two coordinates implicitly in terms of R and T by

$$dr = \frac{R}{\hat{R}}\,dR \qquad d\theta = \frac{T}{\hat{T}}\,dT \qquad (5.4)$$

then $\frac{\partial}{\partial r} = \frac{1}{2}(\frac{\partial}{\omega} + \frac{\partial}{\omega'})$ $\frac{\partial}{\partial \theta} = \frac{1}{2}(\frac{\partial}{\eta} + \frac{\partial}{\eta'})$ (5.5)

Since u and v are ignorable coordinates we seek to expand χ as

$$\chi = e^{\gamma u}\ e^{\beta v}\ A(r)\ B(\theta) \tag{5.6}$$

Taking the most general metric, substituting into (5.2) and multiplying throughout by $R^2 - T^2$ yields

$$\frac{P}{A}\frac{d^2A}{dr^2} + \frac{1}{A}\frac{dP}{dr}\frac{dA}{dr} - \frac{(R^2\gamma+\beta)^2}{P} + 2F + \frac{1}{2}m^2R^2$$

$$- \{\frac{Q}{B}\frac{d^2B}{d\theta^2} + \frac{1}{B}\frac{dQ}{d\theta}\frac{dB}{d\theta} - \frac{(T^2\gamma+\beta)^2}{Q} + 2G + \frac{1}{2}m^2T^2\} = 0 \tag{5.7}$$

implying

$$\frac{d}{dr}\ (P\ \frac{dA}{dr}) - \left\{\frac{(R^2\gamma+\beta)^2}{P} - 2F - \frac{1}{2}m^2R^2\right\} A = \lambda A \tag{5.8}$$

$$\frac{d}{d\theta}\ (Q\ \frac{dB}{d\theta}) - \left\{\frac{(T^2\gamma+\beta)^2}{Q} - 2G - \frac{1}{2}m^2T^2\right\} B = \lambda B \tag{5.9}$$

where as yet we do not know how F and G are related to P and Q. In order to determine F and G we must differentiate P and Q. GHP equation (2.26) relates differentials of P and Q to $\Psi_2 + 2\Lambda$; Φ_{11}, Ψ_2 and Λ are then found using (2.34) and (2.35) of GHP.

Writing $P' = \frac{dP}{dr}$, $P'' = \frac{d^2P}{dr^2}$, $\dot{Q} = \frac{dQ}{d\theta}$ $\ddot{Q} = \frac{d^2Q}{d\theta^2}$

we find

$$\Psi_2 = (P-Q)\ (\frac{1}{(R^2-T^2)} - \frac{2(\hat{R}\ -\hat{T}\ )^2}{(R^2-T^2)^3}) + (P'-\dot{Q})\ (\frac{\hat{R}\ -\hat{T}}{(R^2-T^2)^2}) - \frac{1}{6}\ \frac{(P''-\ddot{Q})}{(R^2-T^2)} \tag{5.10}$$

$$\Lambda = \frac{1}{12}\left(\frac{P''-\ddot{Q}}{R^2-T^2}\right) \tag{5.11}$$

$$\Phi_{11} = \frac{(\hat{R}^2-\hat{T}^2)(Q-P)}{(R^2-T^2)^2} + \frac{\hat{R}\,P'-\hat{T}\,\dot{Q}}{(R^2-T^2)^2} - \frac{1}{4}\frac{(P''+\ddot{Q})}{(R^2-T^2)} \tag{5.12}$$

$$\text{Thus } F = \frac{1}{12}P'', \quad G = \frac{1}{12}\ddot{Q}\ . \tag{5.13}$$

(5.8) and (5.9) may be rewritten

$$(\mathcal{D}^{-}_{+1}\mathcal{D}^{+}_{0} - 2\hat{R}\gamma + \frac{1}{6}P'' + \frac{1}{2}m^2R^2)A = \lambda A \tag{5.14}$$

$$(\mathcal{L}^{-}_{+1}\mathcal{L}^{+}_{0} - 2\hat{T}\gamma + \frac{1}{6}\ddot{Q} + \frac{1}{2}m^2T^2)B = \lambda B \tag{5.15}$$

$$\text{where } \mathcal{D}^{\pm}_{s} = \left(\sqrt{P}\frac{d}{dr} \pm \frac{(R^2\gamma+\beta)}{\sqrt{P}} \mp s\frac{d\sqrt{P}}{dr}\right) \tag{5.16}$$

$$\mathcal{L}^{\pm}_{s} = \left(\sqrt{Q}\frac{d}{d\theta} \pm \frac{(T^2\gamma+\beta)}{\sqrt{Q}} \mp s\frac{d\sqrt{Q}}{d\theta}\right) \tag{5.17}$$

Case 3

$$\Lambda = \frac{1}{12(R^2-\ell^2)}\left(\Gamma + \frac{d^2P}{dR^2}\right) \tag{5.18}$$

$$\Phi_{11} = \frac{\Gamma}{4(R^2-\ell^2)} - \frac{1}{4(R^2-\ell^2)}\frac{d^2P}{dR^2} - \frac{R}{2(R^2-\ell^2)^2}\frac{dP}{dR} - \frac{P}{2(R^2-\ell^2)^2} \tag{5.19}$$

$$\Psi_2 = \frac{1}{(R+\ell)(R^2-\ell^2)}\frac{dP}{dR} - \frac{2P}{(R+\ell)^2(R^2-\ell^2)} - 2\Lambda \tag{5.20}$$

$$\Gamma = \frac{1}{C_T^2}\frac{\partial^2}{\partial T^2}\left(\frac{C_T}{D}\right) - \frac{1}{DC_T}\frac{\partial^2}{\partial v^2}(\ln(DC_T)) + \frac{1}{C_T^3}\frac{\partial^2 C}{\partial T^2}\frac{\partial}{\partial T}\left(\frac{C_T}{D}\right)$$

$$+ \frac{1}{C_T^3}\frac{\partial^2 C}{\partial T\partial v}\frac{\partial}{\partial v}(DC_T) \tag{5.21}$$

$$\chi = e^{\gamma u}\,A(R)B(T,v) \tag{5.22}$$

$$(\mathcal{D}^{-}_{+1}\mathcal{D}^{+}_{0} - 2\gamma R + \frac{1}{6}\frac{\partial^2 P}{\partial R^2} + \frac{1}{2}m^2R^2)A = \lambda A \tag{5.23}$$

$$(\mathcal{L}^{-}_{+1}\mathcal{L}^{+}_{0} - 2\gamma \ell + \frac{1}{6}\Gamma + \frac{1}{2}m^2\ell^2)B = \lambda B \tag{5.24}$$

$$\mathcal{D}^{\pm}_{S} = (\sqrt{P}\frac{\partial}{\partial R} \pm \frac{(R^2-\ell^2)}{\sqrt{P}}\gamma \pm s\frac{d\sqrt{P}}{dR}) \tag{5.25}$$

$$\mathcal{L}^{\pm}_{S} = (\frac{1}{\sqrt{DC_T}}\frac{\partial}{\partial T} \pm \sqrt{(\frac{D}{C_T})}\frac{\partial}{\partial v} \pm 2\ell C\sqrt{(\frac{D}{C_T})}\gamma \pm \frac{S}{C_T}(\frac{\partial}{\partial T}(\frac{D}{C_T})^{\frac{1}{2}} \mp \frac{\partial}{\partial v}(DC_T)^{\frac{1}{2}})) \tag{5.26}$$

Case 4

$$\Lambda = \frac{1}{12}(F-G) \tag{5.27}$$

$$\Phi_{11} = -\frac{1}{4}(F+G) \tag{5.28}$$

$$\Psi_2 = -2\Lambda \tag{5.29}$$

where
$$F = \frac{\partial^2 P}{\partial R^2} - \frac{1}{P}(\frac{\partial P}{\partial R})^2 - \frac{\partial^2 P}{\partial u^2} + \frac{1}{P}(\frac{\partial P}{\partial u})^2 \tag{5.30}$$

$$G = \frac{\partial^2 Q}{\partial T^2} - \frac{1}{Q}(\frac{\partial Q}{\partial T})^2 - \frac{\partial^2 Q}{\partial v^2} + \frac{1}{Q}(\frac{\partial Q}{\partial v})^2 \tag{5.31}$$

$$\chi = A(R,u)B(T,v) \tag{5.32}$$

$$(\mathcal{D}^{-}_{+1}\mathcal{D}^{+}_{0} + \frac{1}{6}F + \frac{1}{2}m^2)A = \lambda A \tag{5.33}$$

$$(\mathcal{L}^{-}_{+1}\mathcal{L}^{+}_{0} + \frac{1}{6}G + \frac{1}{2}m^2)B = \lambda B \tag{5.34}$$

$$\mathcal{D}^{\pm}_{s} = \sqrt{P}(\frac{\partial}{\partial R} \pm \frac{\partial}{\partial u}) \pm S(\frac{\partial\sqrt{P}}{\partial R} \pm \frac{\partial\sqrt{P}}{\partial u}) \tag{5.35}$$

$$\mathcal{L}^{\pm}_{S} = \sqrt{Q}(\frac{\partial}{\partial T} \pm \frac{\partial}{\partial v}) \pm S(\frac{\partial\sqrt{Q}}{\partial T} \pm \frac{\partial\sqrt{Q}}{\partial v}) \tag{5.36}$$

6. The non scalar wave equations

When no background fields are present the scalar wave equation (s=0), the neutrino equation ($s= \pm\frac{1}{2}$), Maxwell's equation ($s= \pm 1$) and the linearised gravitational perturbation equation ($s= \pm 2$) can, in a type D spacetime with vanishing shear and geodesic principal null directions, be reduced to two second order partial differential equations (Stewart and Walker (1974)). If χ_s is the component of spin s when $s > 0$ and $X^{2s}\chi_s$ that for spin $s < 0$ then χ_s obeys

$$[\text{Þ}'\text{Þ} - (2s+1)\rho\,\text{Þ}' - \bar{\rho}'\text{Þ} - \eth'\eth + (2s+1)\tau\eth' + \bar{\tau}\eth + 2(2s+1)\Lambda - s(2s+1)\Psi_2]\chi_s = 0 \qquad (6.1)$$

The solutions to these equations are weighted, so in order to transform (6.1) to a partial differential equation we put

$$\chi_s = (\omega\eta\bar{X})^s\, Y_s$$

(6.1) then becomes

$$[\text{Þ}'\text{Þ} - (2s+1)\rho\,\text{Þ}' + [s(\frac{\text{Þ}'\rho}{\rho} - \bar{\rho}') - \bar{\rho}']\text{Þ} - \eth'\eth + (2s+1)\tau\eth' - [s(\frac{\eth'\tau}{\tau} - \bar{\tau}) - \bar{\tau}]\eth + 2(s+1)(2s+1)\Lambda]Y_s = 0 \qquad (6.2)$$

On writing $Y_s = e^{\gamma u}\, e^{\beta v}\, A_s(r)B_s(\theta)$ (6.2) separates into

$$P^{-s}\frac{d}{dr}(P^{s+1}\,\frac{dA}{dr}) + [\frac{-(R^2\gamma+\beta)^2}{P} + \frac{sP'}{P}(R^2\gamma+\beta) - 4\hat{R}\,\gamma s + \frac{(s+1)(2s+1)P''}{6}]A_s = \lambda A_s \qquad (6.3)$$

$$Q^{-s}\frac{d}{d\theta}(Q^{s+1}\,\frac{dB}{d\theta}) + [\frac{-(T^2\gamma+\beta)^2}{Q} + \frac{s\dot{Q}}{Q}(T^2\gamma+\beta) - 4\hat{T}\,\gamma s + \frac{(s+1)(2s+1)\ddot{Q}}{6}]B_s = \lambda B_s \qquad (6.4)$$

These may be simplified by writing

$$A_s = P^{-s/2}\xi_s \qquad B_s = Q^{-s/2}\zeta_s \tag{6.5}$$

(6.3) and (6.4) become

$$(\mathcal{P}^-_{s+1}\mathcal{P}^+_s - (4s+2)\hat{R}\,\gamma + \frac{(s+1)(2s+1)}{6}P'')\xi_s = \lambda\xi_s \tag{6.6}$$

$$(\mathcal{L}^-_{s+1}\mathcal{L}^+_s - (4s+2)\hat{T}\,\gamma + \frac{(s+1)(2s+1)}{6}\ddot{Q})\zeta_s = \lambda\zeta_s \tag{6.7}$$

If the dependence of Y_s on u and v is periodic then γ and β will be imaginary. In this case the functions ξ_s and ζ_s possess a number of remarkable properties, known as the Teukolsky-Starobinsky identities (Teukolsky (1972), Teukolsky and Press (1972), Starobinsky (1973)).

Firstly it can be seen that $\mathcal{P}^+_s = \overline{\mathcal{D}^-_{-s}}$ and $\mathcal{L}^+_s(T) = \mathcal{L}^-_{-s}(-T)$, thus we see that $\xi_s = \overline{\xi_{-s}}$ and $\zeta_s(T) = \zeta_{-s}(-T)$, with the same value of λ in each case; implying the spectrum of λ is the same for +s as -s.

In discussing the properties of the operators $\mathcal{P}^\pm$ $\mathcal{L}^\pm$ and the solutions of (6.6) and (6.7) we will initially restrict ourselves to the functions of R, u and v and operators $\mathcal{D}^\pm$. The discussion for T, u, v and $\mathcal{L}^\pm$ is identical with the obvious substitutions.

We observe that the operators $\mathcal{D}^+$ and $\mathcal{D}^-$, considered as operators on functions of R, u and v are 'weighted' operators. If instead of investigating the functions ξ_s we look at $\alpha^s\xi_s$ then if we let

$$\mathcal{D}^+ \leftarrow \alpha\mathcal{D}^+ - s(\mathcal{D}^+\alpha) \tag{6.8}$$

$$\mathcal{D}^- \leftarrow \alpha^{-1}\mathcal{D}^- - s(\mathcal{D}^-\alpha)\alpha^{-2} \tag{6.9}$$

then if f scales as $\alpha^s f$

$$\mathcal{P}^{+}_{f} \leftarrow \alpha^{s+1}\,\mathcal{P}^{+}_{f} \qquad \mathcal{D}^{-}_{f} \leftarrow \alpha^{s-1}\,\mathcal{D}^{-}_{f} \tag{6.10}$$

In addition we have the operator

$$\mathcal{P}^{0} = [\mathcal{P}^{-}, \mathcal{D}^{+}] \tag{6.11}$$

with the original $\mathcal{P}^{+}$ and $\mathcal{D}^{-}$ then

$$\mathcal{D}^{0} = 4R' \frac{\partial}{\partial u} - s\,P'' \tag{6.12}$$

$$\mathcal{D}^{0} \text{ transforms according to } \mathcal{D}^{0} \leftarrow \mathcal{P}^{0} - s(\mathcal{D}^{0}\alpha) \tag{6.13}$$

Thus if λ is restricted to being a function only of R then $\mathcal{P}^{0}$ is invariant. We note that $\mathcal{D}^{\pm n}\, r^{n-1} = 0$.

Case 3

$$\chi_s = e^{\gamma u}\left(\frac{\rho\alpha}{\rho'\alpha'}\right)^{s/2} \xi_s(R)\ \zeta_s(T,V) \tag{6.14}$$

$$(\mathcal{D}^{-}\mathcal{D}^{+} - (4s+2)R\gamma + \frac{(s+1)(2s+1)}{6}\frac{d^2P}{dR^2})\xi_s = \lambda\xi_s \tag{6.15}$$

$$(\mathcal{L}^{-}\mathcal{L}^{+} - (4s+2)\ell\gamma + \frac{(s+1)(2s+1)}{6}\Gamma)\zeta_s = \lambda\zeta_s \tag{6.16}$$

$$\mathcal{D}^{0} = 4R\frac{\partial}{\partial u} - s\frac{d^2P}{dR^2} \tag{6.17}$$

$$\mathcal{L}^{0} = 4\ell\frac{\partial}{\partial u} - s\Gamma \tag{6.18}$$

Case 4

$$\chi_s = \left(\frac{\alpha\beta}{\alpha'\beta'}\right)^{s/2} \xi_s(R,u)\zeta_s(T,v) \tag{6.19}$$

$$(\mathcal{D}^{-}\mathcal{D}^{+} + \frac{(s+1)(2s+1)}{6}F)\xi_s = \lambda\xi_s \tag{6.20}$$

$$(\mathcal{L}^{-}\mathcal{L}^{+} + \frac{(s+1)(2s+1)}{6}G)\zeta_s = \lambda\zeta_s \tag{6.21}$$

$$\mathcal{P}^{0} = -sF \tag{6.22}$$

$$\mathcal{L}^{0} = -sG \tag{6.23}$$

7. Relations between spin s and spin -s components

Spin half

Equation (6.6) may be written

$$\mathcal{P}^- \mathcal{P}^+ \xi_{-\frac{1}{2}} = \lambda \xi_{-\frac{1}{2}} \tag{7.1}$$

$$\mathcal{P} + \mathcal{P}^- \xi_{\frac{1}{2}} = \lambda \xi_{\frac{1}{2}} \tag{7.2}$$

hence $$\mathcal{P}^+ \mathcal{D}^- (\mathcal{P}^+ \xi_{-\frac{1}{2}}) = \lambda (\mathcal{D}^+ \xi_{-\frac{1}{2}}) \tag{7.3}$$

$$\mathcal{P}^- \mathcal{D}^+ (\mathcal{D}^- \xi_{\frac{1}{2}}) = \lambda (\mathcal{D}^- \xi_{\frac{1}{2}}) \tag{7.4}$$

Thus in all cases $\xi_{\frac{1}{2}} \propto \mathcal{D}^+ \xi_{-\frac{1}{2}}$, and $\xi_{-\frac{1}{2}} \propto \mathcal{D}^- \xi_{\frac{1}{2}}$

Spin one

We have

$$(\mathcal{D}^- \mathcal{D}^+ + 2\hat{R}\gamma)\xi_{-1} = \lambda \xi_{-1} \tag{7.5}$$

$$(\mathcal{D}^+ \mathcal{D}^- - 2\hat{R}\gamma)\xi_1 = \lambda \xi_1 \tag{7.6}$$

by analogy we seek when $\mathcal{D}^{+2} \xi_{-1} \alpha \xi_1$ and $\mathcal{D}^{-2} \xi_1 \alpha \xi_{-1}$

$$\mathcal{D}^{+2} (\mathcal{D}^- \mathcal{D}^+ + 2\hat{R}\gamma)\xi_{-1} = \lambda (\mathcal{D}^{+2} \xi_{-1}) \tag{7.7}$$

$$\left[\mathcal{D}^+ (\mathcal{D}^- \mathcal{D}^{+2} - \mathcal{D}^0 \mathcal{D}^+) + 2\hat{R}\gamma \mathcal{D}^{+2} + 4 (\mathcal{D}^+ \hat{R})\gamma \mathcal{D}^+ + 2(\mathcal{D}^{+2}\hat{R})\gamma\right]\xi_{-1}$$

$$= \lambda (\mathcal{D}^{+2} \xi_{-1})$$

$$(\mathcal{D}^+ \mathcal{D}^- - 2\hat{R}\gamma)(\mathcal{D}^{+2} \xi_{-1}) + 2(\mathcal{D}^{+2}\hat{R})\gamma\xi_{-1} = \lambda (\mathcal{D}^{+2} \xi_{-1}) \tag{7.8}$$

Thus $\mathcal{D}^{+2} \xi_{-1} \alpha \xi_1$ only when $\mathcal{D}^{+2}\hat{R} = 0$. This is the same as $\hat{R} = ar + b$ for some a and b. It can only hold in the limit as k tends to infinity (see section 4), when $\hat{R}$, r and R coincide. The same condition suffices for $\mathcal{D}^{-2} \xi_1$ to be proportional to ξ_{-1}.

Spin two

When we seek conditions under which $\mathcal{P}^{+4}\ \xi_{-2}\ \alpha\ \xi_2$ then we find terms in $\mathcal{P}^{+}$ P" as well as $\mathcal{P}^{+}\hat{R}$, however when $\mathcal{P}^{+4}$ is applied to the spin minus-two equations then terms in $\mathcal{P}^{+}$P", $\mathcal{P}^{+2}$P", and $\mathcal{P}^{+}\hat{R}$ vanish. Terms in $\mathcal{P}^{+3}$P" and $\mathcal{P}^{+2}\hat{R}$ remain, so the condition that must be satisfied is that $P^{(v)} = 0$ and $k = \infty$. Similarly we find that $Q^{(v)} = 0$, thus both P and Q must be quartic polynomials. Perhaps not surprisingly this is the minimum condition on P and Q separately to include all conformally vacuum solutions, as for a given quartic polynomial in R we may find a quartic polynomial in T for Q so that the spacetime is conformally vacuum (Plebanski and Demianski (1978)). We are not entitled to use (6.2) except when the spacetime is conformally vacuum, since only then are the perturbations source free. It may also be shown that $P^{(v)} = 0$, $Q^{(v)} = 0$, $k = \infty$ is sufficient for $\mathcal{P}^{\pm 2s}\ \xi_{-s}\ \alpha\ \xi_s$ for any integral or half integral s, though for s other than $\frac{1}{2}$, 1 or 2 this can only be valid in conformally flat spacetime.

Proportionality Constants

Since under certain circumstances $\mathcal{P}^{\pm 2s}\ \xi_s\ \alpha\ \xi_{\pm s}$ and $\mathcal{L}^{\pm 2s}\ \zeta_s\ \alpha\ \zeta_{\pm s}$ we now choose a relative scaling of ξ_s and ξ_{-s}, and ζ_s and ζ_{-s} with which to calculate the constants of proportionality. By using the observation again that with the $\mathcal{P}^{\pm}$, $\mathcal{L}^{\pm}$ of (5.16), (5.17) $\mathcal{P}^{+}_{s} = \overline{\mathcal{P}^{-}_{-s}}$ and $\mathcal{L}^{+}_{s}(T) = \mathcal{L}^{-}_{-s}(-T)$ we see that a scaling can be chosen such that

$$\mathcal{P}^{+2s}\ \xi_{-s} = D_s\ \xi_s \tag{7.9}$$

$$\mathcal{P}^{-2s}\ \xi_s = \overline{D}_s\ \xi_{-s} \tag{7.10}$$

and $\mathcal{L}^{+2s}\ \zeta_{-s} = C_s\ \zeta_s$ (7.11)

$\mathcal{L}^{-2s}\ \zeta_s = C_s\ \zeta_{-s}$ (7.12)

(See Teukolsky and Press (1972), Chandrasekar (1979)).

Thus $\mathcal{D}^{-2s}\ \mathcal{D}^{+2s}\ \xi_{-s} = |D_s|^2\ \xi_{-s}$ (7.13)

$\mathcal{L}^{-2s}\ \mathcal{L}^{2s}\ \zeta_{-s} = C_s^2\ \zeta_{-s}$ (7.14)

By evaluating the left hand side we may find C_s^2 and $|D_s|^2$.

The results are

$$C_{\frac{1}{2}}^2 = |D_{\frac{1}{2}}|^2 = \lambda \tag{7.15}$$

$$C_1^2 = |D_1|^2 = \lambda^2 - 4\gamma\beta \tag{7.16}$$

if we write $P = aR^4 + bR^3 + cR^2 + dR + e$

then

$$\begin{aligned} |D_2|^2 = {} & (\lambda^2 - (12ae-3bd+c^2)^2)^2 + 40\gamma\beta\lambda^2 \\ & - 32\lambda(3\beta^2 a+\gamma\beta c+3\gamma^2 e) + 96(ac\beta^2+3ae\beta\gamma+ec\gamma^2) \\ & - 4(9\beta^2 b^2+4c^2\beta\gamma+9d^2\gamma^2) \end{aligned} \tag{7.17}$$

an identical expression holds for C_2^2 if we write $Q = aT^4 + bT^3 + cT^2 + dT + e$. It is clear that $D_{\frac{1}{2}} = \bar{D}_{\frac{1}{2}}$ but otherwise we need more information to calculate the real and imaginary parts of D_s. It will be shown that if Φ_1 in the background vanishes then $C_1 = D_1 = \bar{D}_1$ but no such evaluation will be attempted for D_2. For the calculation in this case for the Kerr spacetime see Chandrasekar (1977). The sign of C_s may be found by taking the spherical limit and it is found that the positive root should be taken (Teukolsky and Press (1972)).

Case 3

For cases 3 and 4 we consider only spins ½ and 1 since the conditions for spin 2 perturbations to be differentiably related are similar to those of case 1, and may only be satisfied if the spacetime is a limit of the spacetimes considered above.

$$C_{\frac{1}{2}}^2 = D_{\frac{1}{2}}^2 = \lambda \tag{7.18}$$

$$C_1^2 = |D_1|^2 = \lambda^2 - 4\gamma\ell \tag{7.19}$$

Case 4

$$C_{\frac{1}{2}}^2 = D_{\frac{1}{2}}^2 = \lambda \tag{7.20}$$

$$C_1^2 = |D_1|^2 = \lambda^2 \tag{7.21}$$

8. Remarks on spin one

When there are no background spin one fields present we need not make a specific choice of perturbed dyad since ϕ_0 and ϕ_2 are invariant under infinitesimal rotations of o_A and ι_A. When ϕ_1 in the background is non-zero then this is not the case and extra conditions must be imposed on the perturbed o_A and ι_A. The obvious restriction is made by observing the form of the equations obeyed by ϕ_0 and ϕ_2. To first order we have (Stewart and Walker (1974)).

$$\{(\text{Þ}'-\bar{\rho})(\text{Þ}-3\rho) - (\eth'-\bar{\tau})(\eth-3\tau) - 6\Psi_2 - 6\Lambda\}\phi_0$$

$$+ 2(\text{Þ}'\kappa-\eth'\sigma-\kappa(\bar{\rho}'-2\rho') + \sigma(\bar{\tau}-2\tau') + 2\Psi_1)\phi_1 = 0 \qquad (8.1)$$

$$\{(\text{Þ}-\bar{\rho})(\text{Þ}'-3\rho') - (\eth-\bar{\tau}')(\eth'-3\tau') - 6\Psi_2 - 6\Lambda\}\phi_2$$

$$+ 2(\text{Þ}\kappa'-\eth\sigma'-\kappa'(\bar{\rho}-2\rho) + \sigma'(\bar{\tau}'-2\tau) + 2\Psi_3)\phi_1 = 0 \qquad (8.2)$$

Thus if the coefficients of ϕ_1 are set to zero to first order then ϕ_0 and ϕ_2 will obey the same equations as in the case where ϕ_1 vanishes in the background.

If ϕ_1 does not vanish in the background then we cannot simply calculate the perturbed ϕ_1, since terms involving perturbed derivatives appear. When it does vanish however by using (2.39) and (2.40) of GHP we have for ϕ_1

Writing $\phi_1 = \frac{\varphi}{x^2} e^{\gamma u} e^{\beta v}$

$$\mathcal{D}^+\varphi = x^2 \mathcal{L}^- (\frac{\xi_1\zeta_1}{x}) \qquad (8.3)$$

$$\mathcal{D}^-\varphi = x^2 \mathcal{L}^+ (\frac{\xi_{-1}\zeta_{-1}}{x}) \qquad (8.4)$$

$$\mathcal{L}^{+}\varphi = x^2\,\mathcal{D}^{-}\left(\frac{\xi_1\zeta_1}{x}\right) \tag{8.5}$$

$$\mathcal{L}^{-}\varphi = x^2\,\mathcal{D}^{+}\left(\frac{\xi_{-1}\zeta_{-1}}{x}\right) \tag{8.6}$$

First we observe by commuting $\mathcal{D}^{-}$ and $\mathcal{L}^{+}$, and $\mathcal{D}^{+}$ and $\mathcal{L}^{-}$ that

$$x(\mathcal{D}^{-2}\,\xi_1\zeta_1 - \xi_1\,\mathcal{L}^{+2}\,\zeta_{-1}) = 0 \tag{8.7}$$

$$x(\zeta_1\,\mathcal{D}^{+2}\,\xi_{-1} - \xi_1\,\mathcal{L}^{-2}\,\zeta_1) = 0 \tag{8.8}$$

implying $C_1 = D_1 = \bar{D}_1$

φ may be read of immediately since

$\left\{(\mathcal{D}^{-}R)\,\mathcal{D}^{+} - (\mathcal{D}^{+}R)\,\mathcal{D}^{-}\right\}\varphi = 2(R^2\gamma+\beta)\varphi$, however simpler expressions result by calculating $\mathcal{D}^{0}\varphi$ and $\mathcal{L}^{0}\varphi$.

Putting $K = (\mathcal{D}^{-}\,\xi_1)(\mathcal{L}^{-}\,\zeta_1) - (\mathcal{D}^{+}\,\xi_{-1})(\mathcal{L}^{+}\,\zeta_{-1})$ (8.9)

$$L = (\mathcal{D}^{-}R)\,\xi_1\,(\mathcal{L}^{-}\,\zeta_1) - (\mathcal{D}^{+}R)\,\xi_{-1}\,(\mathcal{L}^{+}\,\zeta_{-1}) \tag{8.10}$$

$$M = (\mathcal{L}^{+}T)\,\zeta_{-1}\,(\mathcal{D}^{+}\,\xi_{-1}) - (\mathcal{L}^{-}T)\,\zeta_1\,(\mathcal{D}^{-}\,\xi_1) \tag{8.11}$$

Then $$4R\gamma\varphi = XK + L + M \tag{8.12}$$

$$4T\gamma\varphi = XK - L - M \tag{8.13}$$

implying $$2\gamma\varphi = K \tag{8.14}$$

We may also integrate directly and find

$$C_1\varphi = X(\mathcal{D}^{+}\xi_{-1})(\mathcal{L}^{-}\zeta_1) - (\mathcal{D}^{+}R)\,\xi_{-1}(\mathcal{L}^{-}\zeta_1) - (\mathcal{L}^{-}T)\,\zeta_1(\mathcal{J}^{+}\xi_{-1}) \tag{8.15}$$

$$C_1\varphi = X(\mathcal{D}^{-}\xi_1)(\mathcal{L}^{+}\zeta_{-1}) - (\mathcal{L}^{+}T)\,\zeta_{-1}(\mathcal{D}^{-}\xi_1) - (\mathcal{D}^{-}R)\,\xi_1(\mathcal{L}^{+}\zeta_{-1}) \tag{8.16}$$

Case 3

$$\phi_1 = e^{\gamma u}\varphi \tag{8.17}$$

$$\varphi = \frac{1}{c_1}\,(\mathcal{L}^{+}\,\zeta_{-1})\;\mathcal{D}^{-}(\frac{\xi_1}{X}) \tag{8.18}$$

$$= \frac{1}{c_1}\,(\mathcal{L}^{-}\,\zeta_{+1})\;\mathcal{D}^{+}(\frac{\xi_{-1}}{X}) \tag{8.19}$$

$$2\gamma\varphi = X^2\,(\mathcal{D}^{-}\,\xi_1)(\mathcal{L}^{-}\,\zeta_1) - (\mathcal{D}^{+}\,\xi_{-1})(\mathcal{L}^{+}\,\zeta_{-1}) \tag{8.20}$$

Case 4

$$\phi_1 = \frac{1}{c_1}\,(\mathcal{L}^{+}\,\zeta_{-1})(\mathcal{D}^{-}\,\xi_1) \tag{8.21}$$

$$= \frac{1}{c_1}\,(\mathcal{L}^{-}\,\zeta_1)(\mathcal{D}^{+}\,\xi_{-1}) \tag{8.22}$$

On converting spin two to spin one solutions, and spin one to spin zero solutions.

If $\nabla^{A'(A}\chi^{BC)} = 0$ and $\nabla^{AA'}\psi_{ABCD} = 0$ then $\psi_{ABCD}\chi^{CD}$ will obey $\nabla^{A'A}(\psi_{ABCD}\chi^{CD}) = 0$

(Penrose (1975)). Thus if Ψ_1, Ψ_2 and Ψ_3 are the appropriate components of a Weyl spinor then $\Psi_1 X$, $\Psi_2 X$ and $\Psi_3 X$ are components of a symmetric 2-spinor obeying Maxwell's equations. It can be seen therefore that the problem of evaluating ϕ_0 and ϕ_2 when ϕ_1 is non-zero in the background is formally identical to finding ψ_1 and ψ_3 when ψ_2 does not vanish in the background. This provides an 'explanation' for Chandrasekar's observations (Chandrasekar (1979)) that the choice of dyad made to evaluate ϕ_0 and ϕ_2 in the Kerr-Newman spacetime (with a background ϕ_1) also enables ψ_1 and ψ_3 to be calculated in the Kerr spacetime and that these are simply related to ϕ_0 and ϕ_2 perturbations.

It is straightforward to show that if the spacetime is conformally flat and ϕ_{AB} is a symmetric spin one field then $\phi_{AB}\, X^{AB}$ is a spin zero field. We may show that conformal flatness is also a necessary condition by observing that $\phi_1 X = \frac{\varphi}{X}\, e^{\gamma u}\, e^{\beta v}$ obeys.

$$(\mathcal{D}^{-}\mathcal{D}^{+} - \mathcal{L}^{-}\mathcal{L}^{+} - 2(R-T) + \tfrac{1}{6}(P''-\bar{Q}))\,(\tfrac{\varphi}{X}) = -\Psi_2 (R^2-T^2)\,(\tfrac{\varphi}{X}) \qquad (8.23)$$

$$\text{or } (\nabla^2 + R/6)\,(\tfrac{\varphi}{X}) = -2\Psi_2\,(\tfrac{\varphi}{X}) \qquad (8.24)$$

Conclusion

We have seen how many of the properties of spacetimes with 2-Killing spinors follow directly from the existence of the 2-Killing spinor.† We may also look at the properties of the conformally flat spacetimes included in these spacetimes. In fact most flat spacetime separable coordinate systems, where two of the coordinates have Killing vectors as derivatives may be obtained. In addition to the properties detailed above, in the limit as k tends to infinity Dirac's equation may be separated in the same manner as used by Chandrasekar 1976).

† These being that the Hamilton-Jacobi equation and the various wave equations considered ($s=0,\pm\frac{1}{2},\pm1,\pm2$) separate into two equations, each in only one or two non-ignorable coordinates. The two components of the neutrino equation ($s=\pm\frac{1}{2}$) are always differentialy related. If the spacetime also conformally admits a Killing-Yano two-tensor then the $s=\pm1$ components of Maxwell's equation are also differentially related and in more limited circumstances the $s=\pm2$ components of the gravitational perturbation equation are similarly related.

Half algebraically-special spaces

Introduction

1. Machinery

2. Integration of the curvature equations

3. Increased speciality, zero rest-mass fields and second order field equations

4. A class of Einstein-Maxwell space-times with cosmological constant

5. Perturbations

6. Solutions to Killing spinor equations

Conclusion

Appendix

Introduction

The aim of this section is to develop and utilise a spinorial formalism for the study of a class of complex space-times, which may be loosely referred to as being half algebraically-special. This formalism is intermediate between a fully covariant approach (no choice of basis) and a spin coefficient approach (a choice of normalised null tetrad). By choosing a normalised primed spinor dyad (without choice of scale) we make use of the natural description of these spaces as a two-space of null two-planes. By using this formalism we may integrate without difficulty the curvature quantities of such space-times in terms of an arbitrary symmetric two-index spinor, since the formalism manifests the geometrical structure of the spaces, without complicating matters by introducing specific coordinatisations or choice of scaling of null vectors. This approach and explicit integration considerably simplifies the imposition of further constraints.

We use this integrated description to discuss two areas of study.

Firstly Hertz potentials for zero rest-mass fields (Cohen and Kegeles 1975, Stewart 1979) whose existence is apparent immediately in our formalism, and secondly we show how a large class of field equations may be naturally imposed on these spaces. The equations result in a second-order non-linear differential equation, of which the vacuum equation (Plebanski and Robinson 1976, 1977) is a special case.

We show how the existence of a second order equation may be deduced without calculating its specific form, and how the curvature calculations, described as 'hellish' in Plebanski and Robinson 1977 present no diabolic difficulties. The linearisation properties of such equations are also discussed, displaying immediately how gravitational perturbations result from a Hertz potential, a result previously accessible only after cumbersome and unilluminating algebra (Cohen & Kegeles 1975, Stewart 1979).

We use standard two-index spinor abstract index notation (Penrose and Rindler 1984) and the modified spin-coefficient formalism. (Geroch, Held and Penrose 1973, henceforth abbreviated to G.H.P.). All other notation is described as it is developed. Throughout where algebraic type is mentioned, that of the primed Weyl spinor is being referred to.

The least familiar part of this notation is that relating to the two-index curvature operators

$$\Box_{AB} = \nabla_{A'(A} \nabla_{B)}{}^{A'} \qquad 0.1$$

$$\Box_{A'B'} = \nabla_{A(A'} \nabla_{B')}{}^{A} \qquad 0.2$$

for all α_C, $\bar{\alpha}_{C'}$

$$\Box_{AB}\, \bar{\alpha}_{A'} = \Phi_{ABA'B'}\, \bar{\alpha}^{B'} \qquad \Box_{AB}\, \alpha_C = X_{ABCD}\, \alpha^{D} \qquad 0.3$$

$$\Box_{A'B'}\, \bar{\alpha}_{C'} = \bar{X}_{A'B'C'D'}\, \bar{\alpha}^{D'} \qquad \Box_{A'B'}\, \alpha_A = \Phi_{ABA'B'}\, \alpha^{B}$$

where $X_{ABCD} = \Psi_{ABCD} - 2\, \varepsilon_{A(C}\, \varepsilon_{D)B}\, \Lambda$

$$\bar{X}_{A'B'C'D'} = \bar{\Psi}_{A'B'C'D'} - 2\, \varepsilon_{A'(C'}\, \varepsilon_{D')B'}\, \Lambda \qquad 0.4$$

(bars do not signify complex conjugation, but are used for convenience).

Only one substantial theorem of calculus on manifolds is used, namely Frobenius' theorem.(eg. Flanders 1963)

As applied here it states the following. Let ϑ_i be a collection of differential operators, with commutation relations

$$[\vartheta_i, \vartheta_j] = c_{ij}{}^k \vartheta_k \qquad 0.5$$

then the system of equations

$$\vartheta_i \, a_j = b_{ij} \qquad 0.6$$

will have non-zero solutions a_j if and only if

$$[\vartheta_i, \vartheta_j] \, a_m = c_{ij}{}^k \vartheta_k \, a_m \qquad 0.7$$

or $$2\vartheta_{[i} \, b_{j]m} = c_{ij}{}^k \, b_{km} \; . \qquad 0.8$$

Chapter One - The Machinery

We shall be discussing in this section complex space-times, one of whose Weyl spinors is algebraically special and such that the repeated principal null direction (p.n.d.) is both geodesic and shear-free. The algebraically special Weyl spinor is henceforth taken to be the primed one, and thus in G.H.P. notation for repeated p.n.d. $\bar{o}_{A'}$ we have

$$\bar{\psi}_0 = \bar{\psi}_1 = 0 \qquad 1.1$$

$$\bar{\sigma} = \bar{\kappa} = 0 \qquad 1.2$$

If field equations are imposed - vacuum for instance - then conditions 1.1 and 1.2 may become equivalent but we shall not be considering such field equations until later and so for the moment both conditions must be imposed independently.

Equation 1.2 may be written

$$\bar{o}^{A'} \bar{o}^{B'} \nabla_{AA'} \bar{o}_{B'} = 0 \qquad 1.3$$

This is clearly conformally invariant, as is the Weyl spinor and hence equation 1.1 and thus if a space-time is in the class under consideration so are all conformally related ones.

To facilitate calculation and to demonstrate why particular properties of these spacetimes emerge we choose a conformal factor, such that, in addition to 1.3, for all $\bar{l}^{A'}$

$$\bar{l}^{A'} \bar{o}^{B'} \nabla_{AA'} \bar{o}_{B'} = 0 \qquad 1.4$$

and thus $$\bar{o}^{B'} \nabla_{AA'} \bar{o}_{B'} = 0 \qquad 1.5$$

1.4 rewritten in G.H.P. notation is

$$\bar{\rho} = \bar{\tau} = 0 \qquad 1.6$$

$\bar{\rho}$ and $\bar{\tau}$ transform conformally according to

$$g_{ab} \leftarrow \Omega^2 g_{ab} \qquad 1.7$$

$$\bar{\rho} \leftarrow \Omega^{-1} (\bar{\rho} - \frac{\bar{Þ} \Omega}{\Omega}) \qquad 1.8$$

$$\bar{\tau} \leftarrow \Omega^{-1} (\bar{\tau} - \frac{ð' \Omega}{\Omega}) \qquad 1.9$$

We need to show that there exists a function f such that

$$Þf = \bar{\rho} \qquad ð'f = \bar{\tau} \qquad 1.10$$

We therefore have one integrability condition to check, which may be found by commuting Þ and ð' on f. Doing so we find that there is no obstruction so long as conditions 1.1 and 1.2 hold, and thus by Frobenius' theorem such a conformal factor exists.

Assuming now that 1.5 holds we see that two curvature conditions follow, namely

$$\bar{o}^{C'} \Box_{A'B'} \bar{o}_{C'} = 0 \qquad 1.11$$

$$\bar{o}^{C'} \Box_{AB} \bar{o}_{C'} = 0 \qquad 1.12$$

1.11 implies

$$\bar{o}^{C'} \bar{o}^{D'} (\bar{\Psi}_{A'B'C'D'} - 2\Lambda \varepsilon_{A'(C'} \varepsilon_{D')B'}) = 0 \qquad 1.13$$

or $\bar{o}^{B'}\ \bar{o}^{C'}\ \bar{o}^{D'}\ \bar{\psi}_{A'B'C'D'} = 0$ 1.14

and $\bar{\psi}_2 + 2\Lambda = 0$ 1.15

1.12 implies

$$\bar{o}^{A'}\ \bar{o}^{B'}\ \Phi_{ABA'B'} = 0 \qquad 1.16$$

1.14 we recognise as condition 1.1 and 1.15 and 1.16 are two more conditions which are not conformally invariant. Henceforth we write all occurrences of $\bar{\psi}_2$ as -2Λ.

We now introduce a primed spinor dyad, comprising $\bar{o}_{A'}$, and another spinor $\bar{\iota}_{A'}$, chosen such that

$$\bar{o}_{A'}\ \bar{\iota}^{A'} = 1 \qquad 1.17$$

The spinor $\bar{o}_{A'}$ is defined only up to scale and thus if

$$\bar{o}_{A'} \to \lambda\ \bar{o}_{A'} \qquad 1.18$$

then $\bar{\iota}_{A'} \to \lambda^{-1}\ \bar{\iota}_{A'}$. 1.19

We introduce a 'weight' q, as in G.H.P. such that if $\bar{o}_{A'}$ is rescaled according to 1.18 and a quantity x has weight q, then x is rescaled as

$$x \to \lambda^q\ x \qquad 1.20$$

We now define a derivative operator, $\Theta_{AA'}$ as in G.H.P. as

$$\Theta_{AA'} = \nabla_{AA'} - q\ \bar{\iota}^{B'}\ \nabla_{AA'}\ \bar{o}_{B'} \qquad 1.21$$

We see that by definition

$$\bar{\iota}^{B'}\ \Theta_{AA'}\ \bar{o}_{B'} = \bar{o}^{B'}\ \Theta_{AA'}\ \bar{\iota}_{B'} = 0 \qquad 1.22$$

and hence from 1.5

$$\Theta_{AA'}\ \bar{o}_{B'} = 0 \qquad 1.23$$

New notation is now introduced. We write

$$Ð_A = \bar{o}_{A'}\ \Theta^{A'}{}_A \qquad Ŧ_A = \bar{\iota}_{A'}\ \Theta^{A'}{}_A \qquad 1.24$$

(Ð is the capital of the letter 'edth' and to distinguish it from the lower case letter it is best pronounced 'ed'.)

We write the two derivatives of $\bar{\iota}_{A'}$ as

$$\rho_A = \bar{\iota}^{A'}\ Ð_A\ \bar{\iota}_{A'} \qquad 1.25$$

$$\sigma_A = \bar{\iota}^{A'}\ Ŧ_A\ \bar{\iota}_{A'} \qquad 1.26$$

$$(q(Ð_A) = 1,\ q(Ŧ_A) = q(\rho_A) = -1,\quad q(\sigma_A) = -3) \qquad 1.27$$

Performing a translation into G.H.P. notation ρ_A is composed of the spin coefficients $\bar{\rho}'$, $\bar{\tau}'$ and σ_A is composed of $\bar{\sigma}'$, $\bar{\kappa}'$.

Having introduced derivative operators the next step is to commute them. First we examine $[Ð_A, Ð_B]$

$$[Ð_A, Ð_B] = \varepsilon_{AB}(\bar{o}_{A'}\ \bar{o}_{B'}\ \Box^{A'B'} - q\ \bar{o}_{A'}\bar{o}_{B'}\bar{\iota}^{C'}\ \Box^{A'B'}\ \bar{o}_{C'}) \qquad 1.28$$

$$= \varepsilon_{AB}(\bar{o}_{A'}\ \bar{o}_{B'}\ \Box^{A'B'}) \quad \text{from 1.14} \qquad 1.29$$

but we see from 1.13 and 1.16 that $\bar{o}_{A'}\ \bar{o}_{B'}\ \Box^{A'B'}$ applied to both primed and unprimed spinors vanishes, hence

$$[Ð_A, Ð_B] = 0 \qquad 1.30$$

The other commutators are

$$[Đ_A, Ŧ_B] = \varepsilon_{AB}(\bar{o}_{A'}\ \bar{\imath}_{B'}\ \Box^{A'B'} + 3q\Lambda)$$

$$+ (\Box_{AB} - q\,\varphi_{AB}) + \rho_A\, Đ_B \qquad 1.31$$

$$[Ŧ_A, Ŧ_B] = \varepsilon_{AB}(\bar{\imath}_{A'}\ \bar{\imath}_{B'}\ \Box^{A'B'} - q\,\bar{\psi}_3 + \sigma_C\, Đ^C) \qquad 1.32$$

where $$\varphi_{AB} = \bar{o}^{A'}\ \bar{\imath}^{B'}\ \Phi_{ABA'B'} \qquad 1.33$$

$$\alpha_{AB} = \bar{\imath}^{A'}\ \bar{\imath}^{B'}\ \Phi_{ABA'B'} \qquad 1.34$$

We see from 1.30 and 1.4 that the null two-surfaces defined by $\bar{o}_{A'}$ are flat, both intrinsically and extrinsically and it is this that will allow our subsequent analysis to proceed without too much difficulty.

Again by Frobenius' theorem there exists a solution $\alpha_{B\ldots D}$ to the equation

$$Đ_A\ \alpha_{B\ldots D} = \beta_{AB\ldots D} \qquad 1.35$$

iff $Đ^A\ \beta_{AB\ldots D} = 0$.

Since $Đ_A\ Đ_B = Đ_B\ Đ_A$ we shall write

$Đ_{AB\ldots E}$ for $Đ_A\ Đ_B \ldots\ Đ_E$

Although all the curvature equations may be both written down and integrated without any assumptions about

$\bar{\iota}_{A'}$ we shall find it convenient to restrict our choice.

If we apply $[\text{Ð}_A, \text{Ð}_B]$ to $\bar{\iota}_{A'}$ we see that

$$\text{Ð}^A \rho_A = 0 \qquad 1.36$$

This implies that $\rho_A = \text{Ð}_A a$ for some a. Thus we make a new choice of $\iota_{A'}$

$$\bar{\iota}_{A'} \leftarrow \bar{\iota}_{A'} - a \bar{o}_{A'} \qquad 1.37$$

and we see now that

$$\bar{\iota}^{A'} \text{Ð}_A \bar{\iota}_{A'} = \rho_A - \rho_A = 0 \qquad 1.38$$

In certain circumstances we may wish to choose a different $\bar{\iota}_{A'}$, in particular if for some $\bar{\iota}_{A'}$, σ_A vanishes, and so the full curvature equations will be given in the appendix, but henceforth we assume that $\rho_A = 0$.

Before writing down the curvature equations it is convenient to note the following.

For all ν_A

$$[\text{Ð}_A, \text{Ŧ}^B]\nu_B = (q+1) \ [\varphi_{AB} \nu^B + 3\Lambda\nu_A] \qquad 1.39$$

$$[\text{Ð}^A, \text{Ŧ}_B]\nu_A = (q-1) \ [\varphi_{AB} \nu^A - 3\Lambda\nu_B] \qquad 1.40$$

Applying commutators to $\bar{\iota}_{A'}$ gives

$$\text{Ð}_A \sigma_B = \alpha_{AB} + \varepsilon_{AB} \bar{\psi}_3 \qquad 1.41$$

$$\text{Ŧ}_A \sigma^A = \bar{\psi}_4 \qquad 1.42$$

The Bianchi identities on the primed Weyl spinor may be found by applying commutators to σ_A

$$\text{Đ}^{A}\ \alpha_{AB} + \text{Đ}_{B}\ \bar{\psi}_{3} = 0 \qquad 1.43$$

$$\text{Đ}_{A}\ \bar{\psi}_{4} + \text{Ŧ}^{B}\ \alpha_{AB} - \text{Ŧ}_{A}\ \bar{\psi}_{3} = 2(\varphi_{AB}\ \sigma^{B} + 3\Lambda\sigma_{A}) \qquad 1.44$$

The Bianchi identities on the unprimed Weyl spinor are

$$\text{Đ}^{A}\ \psi_{ABCD} + \text{Đ}_{B}\ \varphi_{CD} - 2\ \text{Đ}_{(C}\ \Lambda\ \varepsilon_{D)B} = 0 \qquad 1.45$$

$$\text{Ŧ}^{A}\ \psi_{ABCD} - \text{Ŧ}_{B}\ \varphi_{CD} - 2\ \text{Ŧ}_{(C}\ \Lambda\ \varepsilon_{D)B} = -\text{Đ}_{B}\ \alpha_{CD} \qquad 1.46$$

contracting gives $\text{Đ}^{B}\ \varphi_{AB} - 3\ \text{Đ}_{A}\Lambda = 0$ 1.47

$$\text{Ŧ}^{B}\ \varphi_{AB} + 3\ \text{Ŧ}_{A}\Lambda = -\text{Đ}_{A}\ \bar{\psi}_{3} \qquad 1.48$$

Before proceeding it is worthwhile to consider how much choice we have left, both in conformal factor and $\bar{\iota}_{A'}$.

First we note that $\iota_{A'}$ may transform as

$$\bar{\iota}_{A'} \leftarrow \bar{\iota}_{A'} + a\ \bar{o}_{A'} \qquad 1.49$$

so long as $\text{Đ}_{A}a = 0$, without affecting the condition $\rho_{A} = 0$. As regards conformal freedom we have the following. Let g_{ab} transform as

$$g_{ab} \leftarrow \Omega^{2}\ g_{ab}, \text{ and let } \gamma_{AA'} = \nabla_{AA'}\ \ln\ \Omega$$

In order that condition 1.5 be maintained we require

$$\bar{o}^{B'}(\nabla_{AA'}\ \bar{o}_{B'} - \gamma_{AB'}\ \bar{o}_{A'}) = 0 \qquad 1.50$$

Thus $\bar{o}^{B'}\ \gamma_{AB'} = 0$, equivalently $\text{Đ}_{A}\Omega = 0$.

Unfortunately we now find that in order to maintain the condition $\rho_{A} = 0$ we cannot simply rescale our existing

$\bar{\iota}_{A'}$, since we see that for the rescaled $\bar{\iota}_{A'}$

$$\rho_{A} \propto \bar{\iota}^{B'}\ \bar{o}^{A'}(\nabla_{AA'}\ \bar{\iota}_{B'} - \gamma_{AB'}\ \bar{\iota}_{A'}) \qquad 1.51$$

$$\propto \bar{\iota}^{B'}\ \gamma_{AB'} \neq 0$$

and so we see a new choice of $\bar{\iota}_{A'}$ must be made. The main implication of this is that σ_{A} ,although apparently rescaling conformally, does not so long as we maintain $\rho_{A} = 0$.

Chapter Two - Integration of the Curvature Equations

We shall now proceed with the integration of equations 1.41 to 1.48. First examine equations 1.45 and 1.47. These are six linear equations on nine quantities and so we expect three free quantities after integration.

We see that they may be rewritten

$$\text{Đ}^{A}[\psi_{ABCD} + \varepsilon_{AB}\varphi_{CD} - 2\,\varepsilon_{A(C}\,\varepsilon_{D)B}\Lambda] = 0 \qquad 2.1$$

$$\text{Đ}^{B}[\varphi_{AB} + 3\,\varepsilon_{AB}\Lambda] = 0 \qquad 2.2$$

and thus we may write

$$\psi_{ABCD} + \varepsilon_{AB}\varphi_{CD} - 2\,\varepsilon_{A(C}\,\varepsilon_{D)B}\Lambda = \text{Đ}_{A}\,\Gamma_{BCD} \qquad 2.3$$

$$\varphi_{AB} + 3\,\varepsilon_{AB}\Lambda = -\text{Đ}_{B}\,\delta_{A} \qquad 2.4$$

for some δ_{A}, Γ_{ABC} and where $\Gamma_{ABC} = \Gamma_{A(BC)}$

(of course the minus sign in 2.4 is not essential).

Contracting on various indices we see that

$$\varphi_{CD} = \frac{1}{2}\,\text{Đ}_{A}\,\Gamma^{A}{}_{CD} = -\text{Đ}_{(C}\,\delta_{D)} \qquad 2.5$$

$$\text{Đ}_{A}\,\Gamma_{B}{}^{AD} = -\text{Đ}^{D}\,\delta_{B} \qquad 2.6$$

A little work leads us to the conclusion that

$$\Gamma^{A}{}_{CD} = 2\,\text{Đ}_{(C}\,\lambda_{D)}{}^{A} \qquad 2.7$$

$$\delta_{B} = \text{Đ}^{A}\,\lambda_{BA} = \Gamma_{A}{}^{A}{}_{B} \qquad 2.8$$

for a symmetric two-spinor λ_{AB}

$$(q(\delta_A) = q(\Gamma_{ABC}) = -1, \quad q(\lambda_{AB}) = -2)$$

Checking we see that we have indeed three free quantities after integration.

The next equation to integrate will be 1.48. This is rewritten as

$$\not{\partial}^B[\varphi_{AB} + 3\,\varepsilon_{BA}\Lambda] = -\not{D}_A\,\bar{\psi}_3 \qquad 2.9$$

$$\text{or} \quad -\not{\partial}^B\,\not{D}_A\,\delta_B = -\not{D}_A\,\bar{\psi}_3 \qquad 2.10$$

but we note that $[\not{\partial}^B, \not{D}_A]\delta_B = 0$, since $q(\delta_B) = -1$

and thus

$$\not{D}_A(\not{\partial}^B\,\delta_B - \bar{\psi}_3) = 0 \qquad 2.11$$

$$\text{or} \quad \bar{\psi}_3 = \not{\partial}^B\,\delta_B + \bar{\psi} \qquad 2.12$$

$$\text{where} \quad \not{D}_A\,\bar{\psi} = 0 \qquad 2.13$$

In a similar manner we rewrite 1.46 as

$$\not{\partial}^A[\psi_{ABCD} - \varepsilon_{AB}\varphi_{CD} - 2\,\varepsilon_{A(C}\,\varepsilon_{D)B}\Lambda] = -\not{D}_B\,\alpha_{CD} \qquad 2.14$$

$$\not{\partial}^A\,\not{D}_B\,\Gamma_{ACD} = -\not{D}_B\,\alpha_{CD} \qquad 2.15$$

commuting $\not{\partial}^A$ and $\not{D}_B$ on Γ_{ACD} we find

$$\not{D}_B\,\not{\partial}^A\,\Gamma_{ACD} - 2(\not{D}_B\,\Gamma_{AE(C}\Gamma^{AE}{}_{D)}) = -\not{D}_B\,\alpha_{CD} \qquad 2.16$$

$$\text{or} \quad \alpha_{CD} = \not{\partial}_A\,\Gamma^A{}_{CD} - \Gamma_{AE(C}\,\Gamma^{AE}{}_{D)} + A_{CD} \qquad 2.17$$

$$\text{where} \quad \not{D}_A\,A_{BC} = 0 \qquad 2.18$$

We are now in a position to evaluate the right hand side of equation 1.41 and hence integrate for σ_A. This

is a rather more involved operation than earlier integrations and results in

$$\sigma_A = 2(\text{Ŧ}_B \lambda_A{}^B - 2 \lambda_{BC} \text{Đ}^B \lambda^C{}_A) + A_{AB} Z^B + \bar{\psi} Z_A + \Sigma_A \qquad 2.19$$

where $\text{Đ}_A \Sigma_B = 0 \qquad \text{Đ}_A Z_B = \varepsilon_{AB}$ 2.20

$(q(\Sigma_A) = -3, \quad q(Z_A) = -1)$.

Z_A as given is ill-defined, as we may add on any $q = -1$ one-spinor whose Đ_A derivative vanishes.

It is as well now to pause and consider what we are left with after integration. We have an arbitrary symmetric two-spinor, λ_{AB} and a function, a one-spinor and a two-spinor whose Đ_A derivatives vanish $(\bar{\psi}, \Sigma_A, A_{AB})$ plus a one spinor whose Đ_A derivative is a Kronecker delta, Z^B.

λ_{AB} as it stands however is partially undefined, we can add a symmetric two-spinor λ'_{AB} to λ_{AB} where $\text{Đ}_{AB} \lambda'_{CD} = 0$ without affecting ψ_{ABCD}, φ_{AB} or Λ and the purpose of the remainder of this section will be to show how a suitable choice of λ'_{AB} and Z_A can remove $\bar{\psi}$, Σ_A and A_{AB}.

Having integrated curvature information twice (and thus loosely speaking produced 'connection' and 'metric' information) now would seem the moment to find out more about the derivative Ŧ_A, by using commutators 1.31 and 1.32. We find the following after guesswork and trial and error.

Consider the derivative Ɏ_A, where for all α_B

$$\not{V}_A\ \alpha_B = \not{\Gamma}_A\ \alpha_B - (2\ \lambda_{AC}\ \not{D}^C\ \alpha_B + \Gamma_{ABC}\ \alpha^C + q\ \delta_A\ \alpha_B) \qquad 2.21$$

Substituting into 1.31 and 1.32 we find

$$[\not{D}_A,\ \not{V}_B] = 0 \qquad 2.22$$

$$[\not{V}_A,\ \not{V}_B]\alpha_C = \varepsilon_{AB}(A_{CD}\ \alpha^D - q\ \bar{\psi}\ \alpha_C + (A_{ED}\ Z^D + \bar{\psi}\ Z_E + \Sigma_E)\not{D}^E\ \alpha_C) \qquad 2.23$$

There are a number of ways to proceed, by manipulating λ_{AB}, Z_A and the choice of $\bar{\imath}_{A'}$. The one we choose is to fix Z_A, change $\bar{\imath}_{A'}$ once and then to change only λ_{AB}.

First we choose a Z_A and now consider $\not{V}_A\ Z_B$. We see from 2.22 that

$$\not{D}_A\ \not{V}_B\ Z_C = 0 \qquad 2.24$$

and more particularly $\not{D}_A\ \not{V}^B\ Z_B = 0$ 2.25

We now change $\bar{\imath}_{A'}$ to 'remove' $\not{V}^A\ Z_A$.

Consider a new $\bar{\imath}_{A'}$ given by

$$\bar{\imath}_{A'} \leftarrow \bar{\imath}_{A'} + \frac{1}{2}\ (\not{V}^A\ Z_A)\bar{o}_{A'} \qquad 2.26$$

producing a change in $\not{V}_A$

$$\not{V}_A \leftarrow \not{V}_A + \frac{1}{2}\ (\not{V}^B\ Z_B)\not{D}_A \qquad 2.27$$

and thus

$$\not{V}^A\ Z_A \leftarrow \not{V}^A\ Z_A - \not{V}^A\ Z_A = 0 \qquad 2.28$$

and so now $\not{V}_A\ Z_B = \not{V}_{(A\ Z_B)}$.

We now start changing λ_{AB} by eliminating $\not{V}_A\ Z_B$ entirely. Consider the change

$$\lambda_{AB} \leftarrow \lambda_{AB} - \frac{1}{2}\ \not{V}_A\ Z_B \qquad 2.29$$

and

$$\nabla_A \leftarrow \nabla_A + (\nabla_A Z_B)\, \text{Đ}^B \qquad 2.30$$

and thus

$$\nabla_A\, Z_B \leftarrow \nabla_A\, Z_B + \nabla_A\, Z_C\, \varepsilon^C{}_B = 0 \qquad 2.31$$

Applying the commutator $[\nabla_A, \nabla_B]$ to Z_C we find

$$\nabla_A\, \nabla^A\, Z_C = -\Sigma_C = 0 \qquad 2.32$$

Changes in λ_{AB} obeying $\text{Đ}_{AB}\, \lambda'_{CD} = 0$ may be expressed in the following way. They are of the form

$$\lambda_{AB} \leftarrow \lambda_{AB} + \xi_{(ABC)}\, Z^C + 2\eta_{(A}\, Z_{B)} + \zeta_{(AB)} \qquad 2.33$$

where $\text{Đ}_A\, \xi_{BCD} = \text{Đ}_A\, \eta_B = \text{Đ}_A\, \zeta_{BC} = 0$ 2.34

It is straightforward to see, using 2.21 and 2.23, that only changes in ζ_{AB} affect $\nabla_A\, Z_B$, and so we now consider only changes in ξ_{ABC} and η_A. We note that under the transformation 2.33

$$\Gamma_{ABC} \leftarrow \Gamma_{ABC} + 2\, \xi_{ABC} + 2\eta_{(B}\, \varepsilon_{C)A} \qquad 2.35$$

$$\delta_A \leftarrow \delta_A - 3\eta_A \qquad 2.36$$

Consider now ∇_A acting on quantities whose Đ_A derivatives vanish, i.e. functions on the space of $\bar{o}_A$, planes. This space is clearly two dimensional and we have for such quantities

$$[\nabla_A, \nabla_B]\, \alpha_C = \varepsilon_{AB}(A_{CD}\, \alpha^D - q\, \bar{\psi}\, \alpha_C) \qquad 2.37$$

and thus $\mathcal{V}_A$ is a two dimensional 'charged' connection, which therefore has the form

$$\mathcal{V}_A \alpha_B = \mathcal{F}_A \alpha_B + \omega_{ABC} \alpha^C + q \beta_A \alpha_B \qquad 2.38$$

where $\mathcal{F}_A$ is a flat connection, i.e. $[\mathcal{F}_A, \mathcal{F}_B] = 0$, and where

$$\omega_{ABC} = \omega_{A(BC)} .$$

Applying $\mathcal{V}^A \mathcal{V}_A$ to scalars tells us that

$$\beta_A = \omega_{B\ A}^{\ B} \qquad 2.39$$

and thus we see that by setting $\xi_{ABC} = \frac{1}{2} \omega_{(ABC)}$ 2.40

$$\eta_A = \frac{1}{3} \beta_A \qquad 2.41$$

in 2.33 we may effect the transformation

$$\mathcal{V}_A \leftarrow \mathcal{F}_A \qquad 2.42$$

and we have the final result

$$\mathcal{I}_A \alpha_B = \mathcal{V}_A \alpha_B + 2 \lambda_{AC} \mathcal{D}^C \alpha_B + \Gamma_{ABC} \alpha^C + q \delta_A \alpha_B \qquad 2.43$$

where $[\mathcal{D}_A, \mathcal{D}_B] = [\mathcal{D}_A, \mathcal{V}_B] = [\mathcal{V}_A, \mathcal{V}_B] = 0$ 2.44

This construction has been carried out with a particular choice of Z_A and hence a particular choice of $\bar{\iota}_A$, has arisen. Consider the change

$$Z_A \leftarrow Z_A + \theta_A , \quad \mathcal{D}_A \theta_B = 0 \qquad 2.45$$

Then the transformations:

$$\bar{\imath}_{A'} \leftarrow \bar{\imath}_{A'} + \frac{1}{2} (\mathcal{V}^A \theta_A) \bar{o}_{A'} \qquad 2.46$$

$$\lambda_{AB} \leftarrow \lambda_{AB} - \frac{1}{2} \mathcal{V}_{(A} \theta_{B)} \qquad 2.47$$

result in

$$\mathcal{V}_A \leftarrow \mathcal{V}_A + (\mathcal{V}_A \theta_B) Đ^B \qquad 2.48$$

which maintains both $\mathcal{V}_A Z_B = 0$ and $[\mathcal{V}_A, \mathcal{V}_B] = 0$.

Since $\mathcal{V}^A \theta_A$ may be any q =-2 function, constant on $\bar{o}_A$, planes we see that for each choice of $\bar{\imath}_A$, there are choices of Z_A, $\mathcal{V}_B$ such that $\mathcal{V}_A Z_B = 0$.

We still have some remaining freedom, which may be expressed in the following manner.

Let r,S be coordinates on $\bar{o}_A$, planes, i.e. $\mathcal{V}_A$ r,S = 0, $Đ_{AB}$ r,S = 0, chosen such that $r_A S^A = 1$, where $r_A = Đ_A r$, $S_A = Đ_A S$ (note that q(r) + q(S) = -2). Let R = ar + b, where q(a) = 0, q(b) = q(r) and $Đ_A a,b = 0$. Then if λ_{AB} transforms as

$$\lambda_{AB} \leftarrow \lambda_{AB} + \frac{1}{2} \left(\frac{r \mathcal{V}_{AB} W}{a} - (\mathcal{V}_A b) S_C - (S^B \mathcal{V}_B b) S_A r_C\right) \qquad 2.49$$

the transformed $\mathcal{V}_A$ will obey both $[\mathcal{V}_A, \mathcal{V}_B] = 0$ and $\mathcal{V}_A R = 0$. (where $r^A \mathcal{V}_A W = a$).

To sum up the results of this section, the contravariant metric may be written as

$$g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} = \mathcal{V}^C Đ_C - 2 \lambda^{CD} Đ_C Đ_D \qquad 2.50$$

and the curvature components may be written

$$\sigma_A = 2\, \text{Ɪ}_B\, \lambda_A{}^B - 4\, \lambda_{BC}\, Ð^B\, \lambda^C{}_A \qquad 2.51$$

$$= 2\, \text{Ɣ}_B\, \lambda_A{}^B + 4\, \lambda_{BC}\, Ð^B\, \lambda^C{}_A$$

$$\bar{\psi}_3 = \text{Ɪ}^A\, Ð^B\, \lambda_{AB} = \text{Ɪ}^A\, \delta_A \qquad 2.52$$

$$\bar{\psi}_4 = \text{Ɪ}_A\, \sigma^A \qquad 2.53$$

$$\Lambda = -\frac{1}{2}\, \bar{\psi}_2 = -\frac{1}{6}\, Ð^{AB}\, \lambda_{AB} = -\frac{1}{6}\, Ð^A\, \delta_A \qquad 2.54$$

$$\varphi_{AB} = -Ð_{(A}{}^E\, \lambda_{B)E} = -Ð_{(A}\, \delta_{B)} \qquad 2.55$$

$$\Gamma_{ABC} = 2\, Ð_{(B}\, \lambda_{C)A} \qquad 2.56$$

$$\alpha_{AB} = \text{Ɪ}_C\, \Gamma^C{}_{AB} - \Gamma_{CD(A}\, \Gamma^{CD}{}_{B)} \qquad 2.57$$

$$\psi_{ABCD} = 2\, Ð_{(AB}\, \lambda_{CD)}\,. \qquad 2.58$$

Chapter three - Increased speciality, zero-rest mass fields and field equations

For a generic λ_{AB} the primed Weyl spinor will be of type II, however the restriction to type III is quite simple. Unfortunately further restriction is not so straightforward.

Type III implies that $\bar{\psi}_2 = 0$, equivalently $\Lambda = 0$.

Since $$\Lambda = -\frac{1}{6} \text{Ð}^{AB} \lambda_{AB} \qquad 3.1$$

$$\Lambda = 0 \iff \lambda_{AB} = \text{Ð}_{(A} \mu_{B)} \qquad 3.2$$

for some μ_B, $q(\mu_B) = -3$

Should we wish to consider type N we see that

$$\bar{\psi}_3 = \text{ð}^A \delta_A = 0 \qquad 3.3$$

One solution to this is $\delta_A = 0$, or equivalently

$$\lambda_{AB} = \text{Ð}_{AB} \nu \qquad q(\nu) = -4 \qquad 3.4$$

Clearly however, this does not encompass all type Ns, such a λ_{AB} restricts us to those space-times for which $\varphi_{AB} = 0$. Substituting 3.2 into 3.3 we see that μ_B must satisfy

$$\text{ð}^A \text{Ð}_A (\text{Ð}^B \mu_B) = 0 \qquad 3.5$$

or

$$\text{ð}^A \text{Ð}_A (\text{Ð}^B \mu_B) - 2 \text{Ð}_A \mu_C \text{Ð}^{AC} (\text{Ð}^B \mu_B) = 0 \qquad 3.6$$

We shall see that a class of solutions to 3.6 are conformally Einstein Type Ns.

Zero rest-mass fields

We now discuss the source free zero rest-mass field equations, either as background, or as a source term for the space-time. This duplicates and enlarges on the work of a number of authors (Cohen and Kegeles 1974, 1975, Stewart 1979).

First spin one half

Spin one half

If ν_A is a source free field then it obeys the equation

$$\nabla^{AA'} \nu_A = 0 \qquad 3.7$$

or $$Ð^A \nu_A = 0 \qquad 3.8$$

$$Ŧ^A \nu_A = 0 \qquad 3.9$$

3.8 tells us $\nu_A = Ð_A\chi$ for some χ and thus

$$Ŧ^A Ð_A\chi = 0 \qquad q(\chi) = -1 \qquad 3.10$$

If $\bar{\nu}_{A'}$ is a primed source free field then, writing

$$\bar{\nu}_{A'} = \bar{\nu}_0 \bar{\iota}_{A'} - \bar{\nu}_1 \bar{o}_{A'} \qquad 3.11$$

we have $$\nabla^{A'}{}_A (\bar{\nu}_0 \bar{\iota}_{A'} - \bar{\nu}_1 \bar{o}_{A'}) = 0 \qquad 3.12$$

or $$Ŧ_A \bar{\nu}_0 - Ð_A \bar{\nu}_1 = 0 \qquad 3.13$$

and thus $\bar{\nu}_0$ satisfies

$$Ð^A Ŧ_A \bar{\nu}_0 = 0 \qquad 3.14$$

or $$(Ŧ^A Ð_A + 6\Lambda)\bar{\nu}_0 = 0 \qquad 3.15$$

since $q(\bar{\nu}_0) = 1$

The operator $\text{ɪ}^A \text{Đ}_A$, acting on scalars, may be rewritten

$$(\text{ɯ}^A \text{Đ}_A - 2\,\lambda_{AB}\,\text{Đ}^{AB} + (\text{q}+2)\,\delta^A\,\text{Đ}_A) \qquad 3.16$$

Spin one. (Maxwell fields)

If ϕ_{AB} is a Maxwell field it satisfies the equation

$$\nabla^B_{A'}\,\phi_{AB} = -J_{AA'} \qquad 3.17$$

where $J_{AA'}$ is the current. Applying $\nabla^{A'A}$ we note

$$\Box^{AB}\,\phi_{AB} = 0 = \nabla^{AA'}\,J_{AA'} \qquad 3.18$$

If the field is source free then $J_{AA'} = 0$, and 3.17 may be rewritten

$$\text{Đ}^A\,\phi_{AB} = 0 \qquad 3.19$$

$$\text{ɪ}^A\,\phi_{AB} = 0 \qquad 3.20$$

3.19 implies $\phi_{AB} = \text{Đ}_{AB}\chi$, where $q(\chi) = -2$

3.20 becomes $$\text{ɪ}^A\,\text{Đ}_{BA}\chi = 0 \qquad 3.21$$

We remember that the commutator $[\text{ɪ}^B, \text{Đ}_A]\nu_B = 0$ if $q(\nu_B) = -1$. (This is a reflection of the universal identity $[\nabla^{AA'}, \nabla^{BB'}]\lambda_{AB'} = 0 \;\; \forall\, \lambda_{AB'}$)

and hence $$\text{Đ}_B\,\text{ɪ}^A\,\text{Đ}_A\chi = 0 \qquad 3.22$$

or $$\text{ɪ}^A\,\text{Đ}_A\chi = -\bar{\phi} \qquad \text{Đ}_B\bar{\phi} = 0 \qquad 3.23$$

equivalently

$$(\text{ɯ}^A\,\text{Đ}_A - 2\,\lambda_{AC}\,\text{Đ}^{AC})\chi = -\bar{\phi} \qquad 3.24$$

The potential χ is 'really' a primed two-spinor

$$\chi_{A'B'} = \chi\ \bar{o}_{A'}\ \bar{o}_{B'} \qquad 3.25$$

and we produce the Maxwell field by applying $\nabla^{A'}_{(A}\ \nabla^{B'}_{B)}$

(Here the field is automatically symmetric) and thus we have the heirarchy

$$\chi_{A'B'} = \chi\ \bar{o}_{A'}\ \bar{o}_{B'} \qquad 3.26$$

$$\lambda_{AA'} = \nabla^{B'}_{A}\ \chi_{A'B'} \qquad 3.27$$

$$\phi_{AB} = \nabla^{A'}_{(B}\ \lambda_{A)A'} \qquad 3.28$$

involving also a vector potential $\lambda_{AA'}$. With such a potential we may just as well attempt to produce a primed field.

$$\bar{\phi}_{A'B'} = \nabla^{B}_{(B'}\ \lambda_{A')B} \qquad 3.29$$

$$\text{Now } \nabla^{A'}_{A}\ \bar{\phi}_{A'B'} = \nabla^{A'}_{\ A}\ \nabla^{B}_{(A'}\ \lambda_{B')B} \qquad 3.30$$

$$= \nabla^{B}_{\ B'}\ \nabla^{A'}_{(A}\ \lambda_{B)A'} \qquad 3.31$$

$$= \nabla^{B}_{\ B'}\ \phi_{AB} \qquad 3.32$$

Thus the two fields will have the same current. In this case

$$\bar{\phi}_{A'B'} = \nabla^{A}_{\ B'}\ \text{Đ}_{A}\chi\ \bar{o}_{A'} = -\bar{o}_{B'}\ \text{Ŧ}^{A}\ \text{Đ}_{A}\chi\ \bar{o}_{A'} = \bar{\phi}\ \bar{o}_{A'}\ \bar{o}_{B'} \qquad 3.33$$

which is indeed a source free Maxwell field, since

$$\nabla^{A'}_{A}\ \bar{\phi}_{A'B'} = -\bar{o}_{B'}\ \text{Ð}_{A}\bar{\phi} = 0 \qquad 3.34$$

Examining those potentials, x, for which ϕ_{AB} vanishes they must satisfy both

$$\text{Ð}_{AB}x = 0 \qquad 3.35$$

$$\text{Ŧ}^{A}\ \text{Ð}_{A}x = -\bar{\phi} \qquad 3.36$$

$$\text{or} \quad \text{Ѵ}^{A}\ \text{Ð}_{A}x = -\bar{\phi} \qquad 3.37$$

$$\text{putting } x = x_0 + \xi_A\ Z^A \quad \text{Ð}_A\ \xi_B = \text{Ð}_A\ x_0 = 0 \qquad 3.38$$

we find $\bar{\phi} = \text{Ѵ}_A\ \xi^A$.

and so we may write a general solution to 3.23 as

$$\chi = \chi_0 + \xi_A\ Z^A \qquad 3.39$$

$$\text{Ŧ}^{A}\ \text{Ð}_{A}\ \chi_0 = 0 \qquad \text{Ѵ}_{A}\ \xi^{A} = \bar{\phi} \qquad 3.40$$

In special circumstances we may produce primed fields which are not aligned radiation, in particular if $[\text{Ŧ}_A, \text{Ŧ}_B] = 0$, i.e.

$$\sigma_A = \bar{\psi}_3 = \bar{\psi}_4 = \alpha_{AB} = 0 \qquad 3.41$$

Consider $\chi_{A'B'} = 2\chi\ \bar{o}_{(A'}\ \bar{\iota}_{B')} \quad q(\chi) = 0 \qquad 3.42$

then $\lambda_{AA'} = \bar{\iota}_{A'}\ \text{Ð}_{A}\chi + \bar{o}_{A'}\ \text{Ŧ}_{A}\chi \qquad 3.43$

and $\phi_{AB} = \nabla^{A'}_{B}\ \lambda_{AA'} = (\text{Ŧ}_{B}\ \text{Ð}_{A} + \text{Ð}_{B}\ \text{Ŧ}_{A})\chi \qquad 3.44$

(ϕ_{AB} is automatically tracefree since $\phi^{A}_{A} = (\text{Ŧ}^{A}\ \text{Ð}_{A} + \text{Ð}^{A}\ \text{Ŧ}_{A})\chi = 0$).

Applying $\nabla^{A}_{A'}$ we find

$$\text{Đ}_B(\text{Đ}^A\ \text{Ŧ}_A\chi) = \text{Ŧ}_B(\text{Ŧ}^A\ \text{Đ}_A\chi) = 0 \qquad 3.45$$

or $$\text{Ŧ}^A\ \text{Đ}_A\chi = Q \qquad Q \text{ constant} \qquad 3.46$$

and we find that the primed field $\phi_{A'B'} = \nabla_{A'}^{\ A}\ \lambda_{AB'}$ is given by

$$\bar{\phi}_{A'B'} = 2Q\ \bar{o}_{(B'}\ \bar{\iota}_{A')} \qquad 3.47$$

Those potentials, x, for which ϕ_{AB} vanish must satisfy

$$(\text{Ŧ}_A\ \text{Đ}_B + \text{Đ}_A\ \text{Ŧ}_B)x = 0 \qquad 3.48$$

which appears only to have the solution $\text{Đ}_A x = 0$, and such x do not produce a primed field either - and so it is more difficult to isolate those potentials that produce primed fields.

Turning now to general primed Maxwell fields, we write

$$\bar{\phi}_{A'B'} = \bar{\phi}_0\ \bar{\iota}_{A'}\ \bar{\iota}_{B'} - 2\ \bar{\phi}_1\ \bar{o}_{(A'}\ \bar{\iota}_{B')} + \bar{\phi}_2\ \bar{o}_{A'}\ \bar{o}_{B'} \qquad 3.49$$

and produce

$$\text{Đ}_A\ \bar{\phi}_1 - \text{I}_A\ \bar{\phi}_0 = 0 \qquad 3.50$$

$$\text{Đ}_A\ \bar{\phi}_2 - \text{Ŧ}_A\ \bar{\phi}_1 + \sigma_A\ \bar{\phi}_0 = 0 \qquad 3.51$$

$\bar{\phi}_0$ then obeys

$$\text{Đ}^A\ \text{Ŧ}_A\ \bar{\phi}_0 = 0 \qquad 3.52$$

or $$(\text{Ŧ}^A\ \text{Đ}_A + 12\Lambda)\bar{\phi}_0 = 0 \qquad 3.53$$

Spin greater than one.

Spins greater than one are difficult since if

$$\nabla^{A'A}\ \phi_{ABC\ldots E} = 0 \qquad 3.54$$

then $$\Box^{AB}\ \phi_{ABC\ldots E} = 0 = \psi^{ABC}{}_{(D}\ \phi_{\ldots E)ABC} \qquad 3.55$$

(Buchdahl 1958). Of course for primed fields since the Weyl spinor is algebraically special there will be some solutions, but these themselves will in general be algebraically special and will not be discussed further.

This would appear to be the point to discuss Weyl spinor perturbations, this however must wait on a discussion of field equations in general, and the conformal vacuum equations in particular.

Second order field equations

We wish to examine the circumstances in which we can reduce the field equations for a spacetime to the form

$$\text{ɪ}^{A}\ Ð_{A}\chi = E(\chi,\ Ð_{A}\chi,\ Ð_{AB}\chi,\ P) \qquad 3.56$$

for some expression E, and source terms P. The source terms may also be differentially coupled with the potential χ. Consider a spacetime where, in some conformally related spacetime

$$\bar{o}_{A'}\ \hat{\nabla}^{AA'}\ \psi_{ABCD} = 0 \qquad 3.57$$

where $\hat{\nabla}^{AA'}$ is the conformally related connection and the conformally related metric $\hat{g}_{ab}$ is

$$\hat{g}_{ab} = \Omega^{2}\ g_{ab} \qquad 3.58$$

3.56 may be written

$$(\text{Đ}^A + (\text{Đ}^A \ln \Omega))\psi_{ABCD} = 0 \qquad 3.59$$

or $$\psi_{ABCD} = \frac{1}{\Omega} \text{Đ}_{ABCD}\chi \quad \text{for some } \chi \qquad 3.60$$

Since $\text{Đ}^A \text{Đ}_{ABCD}\chi = 0$ it follows that

$$\text{Ŧ}^B \text{Đ}^A \text{Đ}_{ABCD}\chi = 0 \qquad 3.61$$

and $$\text{Đ}^A(\text{Ŧ}^B \text{Đ}_{BCDA}\chi) = [\text{Đ}^A, \text{Ŧ}^B]\text{Đ}_{BCDA}\chi \qquad 3.62$$

$$= \square^{AB} \text{Đ}_{ABCD}\chi \qquad 3.63$$

$$= \psi^{AB}{}_{E(C} \text{Đ}_{D)}{}^{E}{}_{AB}\chi = 0 \quad \text{by 3.60} \qquad 3.64$$

and thus

$$\text{Ŧ}^B \text{Đ}_{ACDB}\chi = \text{Đ}_{ACD} \text{Ŧ}^B \text{Đ}_B\chi - \text{Đ}_{ACD} E(\chi, \text{Đ}_F\chi, \text{Đ}_{FG}\chi, P) \qquad 3.65$$

To maintain consistency with earlier material we put

$$\frac{1}{2} \psi_{ABCD} = \frac{1}{\Omega} \text{Đ}_{ABCD}\chi = q \text{Đ}_{ABCD}\chi \qquad 3.66$$

† q is of a restricted form since

$$\frac{1}{2} \psi_{ABCD} = q \text{Đ}_{ABCD}\chi = \text{Đ}_{(AB} \lambda_{CD)} \qquad 3.67$$

writing $q_A = \text{Đ}_A q$, $q_{AB} = \text{Đ}_{AB} q$ etc. we apply Đ^{AB} to 3.67

$$\frac{1}{6} \text{Đ}_{CD}(\text{Đ}^{AB} \lambda_{AB}) = q^{AB} \text{Đ}_{ABCD}\chi \qquad 3.68$$

and therefore $$0 = q^{ABC} \text{Đ}_{ABCD}\chi \qquad 3.69$$

for ψ_{ABCD} of general algebraic type this implies

$$q_{ABC} = 0 \qquad 3.70$$

† Henceforth q will be used as meaning $1/\sqrt{2}$ and should not be confused with its use as a 'weight'

or q is quadratic in $\bar{o}_A$, plane coordinates.

Integration by parts yields

$$\lambda_{AB} = q\, Ð_{AB}\chi - 2\, q_{(A}\, Ð_{B)}\chi + 3\, q_{AB}\chi + \lambda'_{AB} \qquad 3.71$$

where $Ð_{(AB}\, \lambda'_{CD)} = 0$ 3.72

and hence $\delta_A = 2(q^B\, Ð_{AB}\chi + 2\, q_A{}^B\, Ð_B\chi) + Ð^B\lambda'_{AB}$

$$\Lambda = -q^{AB}\, Ð_{AB}\chi + Ð^{AB}\, \lambda'_{AB} \qquad 3.73$$

(It follows from 3.71 that $Ð_{CD}(Ð^{AB}\, \lambda'_{AB}) = 0$, as required by 3.68).

The conformal Bianchi identities consist of 3.59 and

$$(Ŧ^A + (Ŧ^A \ln \Omega))\psi_{ABCD} = \text{source terms} \qquad 3.74$$

which we write as

$$Ŧ^A\, Ð_{BCDA}\chi = Ð_{BCD}\, P \qquad 3.75$$

since we know from 3.64 that it can always be so written.

A little work reveals that

$$Ð_{BCD}(Ŧ^A\, Ð_A\chi + 6\Lambda\chi)$$

$$= Ð_{BCD}(-q\, Ð^{EF}\chi\, Ð_{EF}\chi + 4\, q^E\, Ð_{EF}\chi\, Ð^F\chi$$

$$- 4\, q^{EF}\, Ð_E\chi\, Ð_F\chi - 6\, q^{EF}\chi\, Ð_{EF}\chi + P). \qquad 3.76$$

Equation 3.76 possesses a remarkable property. First remember that not all the terms non-linear in χ are on the right hand side of 3.76, since $Ŧ^A$ involves λ^{AB} and $Ð^B\, \lambda^A{}_B$ and Λ involves $Ð^{AB}\, \lambda_{AB}$. If we wish to linearise 3.76 about a fixed χ, P i.e. put

$$\chi \leftarrow \chi + \varepsilon x \qquad P \leftarrow P + \varepsilon p \tag{3.77}$$

then 3.76 becomes

$$Ð_{BCD}(Ŧ^{A}\, Ð_{A}x + 6\Lambda x - p) = 0 \tag{3.78}$$

where $Ŧ^{A}$, Λ are the background quantities.

Chapter Four - A Class of Einstein-Maxwell Space-times.

The simplest example of a space-time obeying

$$\bar{o}_{A'}\, \hat{\nabla}^{AA'}\, \psi_{ABCD} = 0 \qquad 4.1$$

is a conformally vacuum space-time i.e.

$$\hat{\nabla}^{AA'}\, \psi_{ABCD} = 0 \qquad 4.2$$

and $\hat{R}_{ab} = 0$

(Carets will denote quantities in the space-time with a conformally related metric).

This case was studied by Plebanski and Robinson 1977. We shall generalize to a degree and examine a class of electro-vac space-times with cosmological constant. (This was done along with more general Maxwell fields in Garcia, Plebanski and Robinson 1977).

The Einstein-Maxwell source free field equations are

$$\nabla^{AA'}\, \psi_{ABCD} = \bar{\phi}^{A'}_{B'}\, \nabla^{B'}_{(B}\, \phi_{CD)} \qquad 4.3$$

$$\nabla^{A'A}\, \bar{\psi}_{A'B'C'D'} = \phi^{A}_{B}\, \nabla^{B}_{(B'}\, \bar{\phi}_{C'D')} \qquad 4.4$$

We need to transform these equations into ones involving the connection, Weyl spinors and Maxwell spinors with our chosen conformal factor. Writing $\gamma_{AA'} = \nabla_{AA'} \ln \Omega$

$$(\nabla^{AA'} + \gamma^{AA'})\psi_{ABCD} = \Omega^{-2}\, \bar{\phi}^{A'}_{B'}(\nabla^{B'}_{(B}\, \phi_{CD)} - 3\gamma^{B'}_{(B}\, \phi_{CD)}) \qquad 4.5$$

$$(\nabla^{AA'} + \gamma^{AA'})\bar{\psi}_{A'B'C'D'} = \Omega^{-2}\phi^{A}_{B}(\nabla^{B}_{(B'}\, \bar{\phi}_{C'D')} - 3\gamma^{B}_{(B'}\, \bar{\phi}_{C'D')}) \qquad 4.6$$

remembering condition 4.1 we require

$$\bar{o}_{A'}\ \bar{\phi}^{A'}_{B'}(\nabla^{B'}_{(B}\ \phi_{CD)} - 3\gamma^{B'}_{(B}\ \phi_{CD)}) = 0 \qquad 4.7$$

and so we consider $\bar{\phi}_{A'B'}$ of the form

$$\bar{\phi}_{A'B'} = \bar{\phi}\ \bar{o}_{A'}\ \bar{o}_{B'}$$

We know from chapter three that such fields may be described by a scalar potential, which we call here α, obeying

$$\text{Ŧ}^{A}\ \text{Đ}_{A}\alpha = -\bar{\phi} \quad , \quad \text{where } \text{Đ}_{AB}\alpha = \phi_{AB} \qquad 4.8$$

Before translating 4.5 and 4.6 we put

$$\nu^{A} = \bar{o}_{A'}\ \gamma^{A'A} \qquad \mu^{A} = \bar{\iota}_{A'}\ \gamma^{A'A} \qquad 4.9$$

and find

$$(\text{Đ}^{A} + \nu^{A})\psi_{ABCD} = 0 \qquad 4.10$$

$$(\text{Ŧ}^{A} + \mu^{A})\psi_{ABCD} = -\Omega^{-2}\ \bar{\phi}(\text{Đ}_{(B}\ \phi_{CD)} - 3\nu_{(B}\ \phi_{CD)}) \qquad 4.11$$

$$(\text{Đ}^{A} + \nu^{A})\Lambda = 0 \qquad 4.12$$

$$(\text{Đ}^{A} + \nu^{A})\bar{\psi}_{3} - 2(\text{Ŧ}^{A} + \mu^{A})\Lambda = \Omega^{-2}\ \bar{\phi}\ \phi^{A}_{\ B}\ \nu^{B} \qquad 4.13$$

$$(\text{Đ}^{A} + \nu^{A})\bar{\psi}_{4} - (\text{Ŧ}^{A} + \mu^{A})\bar{\psi}_{3} - 6\sigma^{A}\Lambda = -\Omega^{-2}\ \phi^{A}_{\ B}(\text{Ŧ}^{B} - 3\mu^{B})\bar{\phi} \qquad 4.14$$

In addition there are the conformal curvature equations.

$$0 = \text{Đ}_{A}\ \nu_{B} - \nu_{A}\ \nu_{B} \qquad 4.15$$

$$0 = \wp_{AB} - \text{Đ}_{(A}\ \mu_{B)} + \mu_{(A}\ \nu_{B)} \qquad 4.16$$

$$0 = \wp_{AB} - \text{Ŧ}_{(A}\ \nu_{B)} + \mu_{(A}\ \nu_{B)} \qquad 4.17$$

$$\bar{\phi}\ \phi_{AB}\ \Omega^{-2} = \alpha_{AB} - \text{Ŧ}_{(A}\ \mu_{B)} + \mu_{A}\ \mu_{B} + \sigma_{(A}\ \nu_{B)} \qquad 4.18$$

$$2\Lambda_0\,\Omega^2 = 2\Lambda - \text{Đ}^A\,\mu_A - \nu^A\,\mu_A \qquad 4.19$$

$$= 2\Lambda + \text{Ŧ}^A\,\nu_A + \mu^A\,\nu_A \qquad 4.20$$

where Λ_0 is the cosmological constant in the conformally related spacetime.

We see immediately from 4.15 that

$$0 = \text{Đ}_A\left(\frac{\text{Đ}_B\Omega}{\Omega}\right) - \frac{\text{Đ}_A\Omega\ \text{Đ}_B\Omega}{\Omega^2} \qquad 4.21$$

or $$\text{Đ}_{AB}\,q = 0 \qquad 4.22$$

and so the conformal factor is less general than that considered in the last chapter, as q is linear, not quadratic, in $\bar{o}_A$, plane coordinates and we write

$$q = ar + b \qquad 4.23$$

where $$\text{Đ}_{AB}r = \text{Ŧ}_A r = \text{Đ}_A a = \text{Đ}_A b = 0 \qquad 4.24$$

If $a \neq 0$ then we may perform a transformation such as that given by 2.49 to produce q = r.

If a = 0 then we may conformally transform to give q = 1.

Consider the case q = r. Equation 4.12 tells us that

$$\Lambda = \varepsilon r \qquad \text{Đ}_A\varepsilon = 0 \qquad 4.25$$

however, we have not exploited our conformal freedom. Since under allowable conformal transformations

$$\Lambda \leftarrow \Lambda\,\Omega^{-2} \qquad 4.26$$

we may rescale so that either

$$\Lambda = q = r \qquad 4.27$$

or $$\Lambda = 0 \ . \qquad 4.28$$

Of course in the Type III case such a rescaling is not necessary since $\Lambda = 0$ already, and so we put

$$\Lambda = \varepsilon r \qquad \varepsilon = 0,1 \qquad 4.29$$

In the light of the above comments we rewrite equations 4.11 to 4.20 putting $r_A = Ð_A r$, $r\bar{\varphi}_4 = \bar{\Psi}_4$, $r\bar{\varphi}_3 = \bar{\Psi}_3$

$$2\, Ŧ^A\, Ð_{BCDA}\chi = -\bar{\phi}\, Ð_{BCD} r\alpha \qquad 4.30$$

$$Ð_A \bar{\varphi}_3 = -\bar{\phi}\, r^B\, Ð_{AB}\alpha \qquad 4.31$$

$$Ð_A \bar{\varphi}_4 - Ŧ_A \bar{\varphi}_3 - \frac{6\sigma_A \Lambda}{r} = -\ (r Ŧ^B \bar{\phi} + 3\,\tilde{\phi}\, Ŧ^B r) Ð_{AB}\alpha \qquad 4.32$$

$$r\hat{\varphi}_{AB} = 0 = r\varphi_{AB} + Ŧ_{(B}\, r_{A)} \qquad 4.33$$

$$r\hat{\alpha}_{AB} = \bar{\phi}\, r^3\, Ð_{AB}\alpha = r\alpha_{AB} + Ŧ_{(A}\, Ŧ_{B)} r - \sigma_{(A}\, r_{B)} \qquad 4.34$$

$$2\hat{\Lambda} = 2\Lambda_0 = 2\Lambda r^2 - r\, Ŧ^A\, r_A - 2 r^A\, Ŧ_A r \qquad 4.35$$

and we know that 4.30 is equivalent to 3.76 with $q = r$, $P = -\frac{1}{2}\,\tilde{\phi}\, r\alpha$

$$(Ŧ^A\, Ð_A + 6\Lambda)\chi - P_2$$

$$= -\frac{1}{2}\, r\bar{\phi}\alpha - r\, Ð^{EF}\chi\, Ð_{EF}\chi + 4\, r^E\, Ð_{EF}\chi\, Ð^F\chi \qquad 4.36$$

where $Ð_{ABC}\, P_2 = 0$ 4.37

(with a slight change from previous notation we put

$$P_2 = P + P_A \, Z^A + \frac{1}{2} P_{AB} \, Z^A \, Z^B \qquad 4.38$$

where $Ð_A P = Ð_A P_B = Ð_A P_{BC} = 0$)

To reduce to the case q = 1, substitute r = 1, $r^A = 0$ in 4.30 to 4.36.

It is interesting to note exactly what 4.36 is the solution to. Recall that λ_{AB} is given by 3.70 as

$$\lambda_{AB} = r \, Ð_{AB}\chi - 2r_{(A} \, Ð_{B)}\chi + \lambda'_{CD} \qquad 4.39$$

where $Ð_{(AB} \, \lambda'_{CD)} = 0$ 4.40

λ'_{CD} may be expanded as

$$\lambda'_{CD} = k \, Z_C \, Z_D + \pi_{A(C} \, Z_{D)} \, Z^A + \xi_{CDE} \, Z^E + 2\eta_{(C} \, Z_{D)} + \zeta_{CD} \qquad 4.41$$

where $k = -\Lambda_0$ if q = 1 4.42

$$k = -\frac{\varepsilon r}{2} \quad \text{if } q = r \qquad 4.43$$

and all other new symbols are symmetric and have zero $Ð_A$ derivatives. So far we know nothing of

$$P, \; P_A, \; P_{AB}, \; \pi_{AB}, \; \xi_{ABC}, \; \eta_A \text{ or } \zeta_{AB}$$

However, what we have solved in the conformally related spacetime is only the equation

$$\hat{\nabla}^{AA'} \, \Psi_{ABCD} = \hat{\bar{\phi}}^{A'}_{B'} \, (\hat{\nabla}^{B'}_{(B} \, \hat{\phi}_{CD)}) \qquad 4.44$$

plus imposing

$$\bar{o}^{A'} \, \bar{o}^{B'} \, \bar{\iota}^{C'} \, (\hat{\nabla}^{D'}_{A} \, \bar{\Psi}_{A'B'C'D'} - \hat{\phi}^{A}_{B} \, \hat{\nabla}^{B}_{(B'} \, \hat{\bar{\phi}}_{C'D')}) = 0 \qquad 4.45$$

$$\bar{o}^{A'} \, \bar{o}^{B'} \, \hat{\Phi}_{ABA'B'} = 0 \qquad 4.46$$

4.45 and 4.46 being 4.12 and 4.15 translated. There could be extra Ricci curvature satisfying

$$\hat{\nabla}^{A'}_{A}\hat{\Phi}_{BCB'A'} - 2\,\hat{\varepsilon}_{A(B}\,\hat{\nabla}_{C)B'}\hat{\Lambda} = 0 \qquad 4.47$$

$$\bar{o}^{(A'}\,\bar{o}^{B'}\,\bar{\iota}^{C')}\,(\hat{\nabla}^{A}_{A'}\,\hat{\Phi}_{B'C'AB} - 2\,\hat{\varepsilon}_{A'(B'}\,\hat{\nabla}_{C')B}\hat{\Lambda}) = 0 \qquad 4.48$$

Writing $\hat{\Phi}_{ABA'B'} = -2\,\hat{\varphi}_{AB}\,\bar{o}_{(A'}\,\bar{\iota}_{B')} + \hat{\alpha}_{AB}\,\bar{o}_{A'}\,\bar{o}_{B'}$ 4.49

and considering the case q = r we find

$$Þ_{(A}\,r\,\hat{\varphi}_{BC)} = 0 \qquad 4.50$$

$$Ŧ_{(A}\,r\,\hat{\varphi}_{BC)} = Þ_{(A}\,r\,\hat{\alpha}_{BC)} \qquad 4.51$$

$$rÞ^{A}\left(\frac{\hat{\varphi}_{AB}}{r^2}\right) - 3Þ_{B}\hat{\Lambda} = 0 \qquad 4.52$$

$$rŦ^{A}\left(\frac{\hat{\varphi}_{AB}}{r^2}\right) + 3Ŧ_{B}\hat{\Lambda} = r\,Þ^{A}\left(\frac{\hat{\alpha}_{AB}}{r^2}\right) \qquad 4.53$$

$$Þ^{A}(r\,\hat{\varphi}_{AB}) = 0 \qquad 4.54$$

Combining 4.50 and 4.54 we see

$$Þ_{A}\,r\,\hat{\varphi}_{BC} = 0 \qquad 4.55$$

and thus by checking the order 1/r component of $\hat{\varphi}_{AB}$ as given in 4.33 we may set $\hat{\varphi}_{BC}$ to zero.

If $\hat{\varphi}_{BC} = 0$ then 4.52 implies

$$Þ_{A}\hat{\Lambda} = 0 \qquad 4.56$$

and so only the order r^0 part of Λ, as given in 4.35 need be checked.

Assuming $\hat{\Lambda} = \Lambda_0$ we have finally

$$Đ_{(A}\, r\, \hat{\alpha}_{BC)} = Đ^{A}\, (\frac{1}{r^2}\, \hat{\alpha}_{AB}) = 0 \qquad 4.57$$

whose combined solution is

$$\hat{\alpha}_{BC} = \frac{1}{r}\, (\alpha_0\, r_B\, r_C + \alpha_1\, r_{(B}\, Z_{C)} + \alpha_2\, Z_B\, Z_C) \qquad 4.58$$

If we check in addition that the equation for $\bar{\Psi}_3$, 4.31 holds, then it will imply

$$Đ^{A}(r\, \hat{\alpha}_{AB}) = 0 \qquad 4.59$$

and we need then only set to zero the 1/r component of $\hat{\alpha}_{AB}$ as given in 4.34.

To recap; we set to zero the order 1/r component of φ_{BC} and put the r^0 component of Λ equal to Λ_0.

We then calculate $\bar{\Psi}_3$ and $Đ_A\, \bar{\Psi}_3$ and substitute into 4.31, using this to set to zero α_1 and α_2 in 4.58. Finally we set to zero the order 1/r component of $\hat{\alpha}_{AB}(\alpha_0)$.

While doing this we may also check which of the terms π_{AB}, ξ_{ABC}, η_A, ζ_{AB} may be removed by incorporating them into χ appropriately. This depends on whether q = 1 or q = r.

The Case q = 1.

Using 4.41 we see that

$$\delta_A = 3\, \Lambda_0\, Z_A - 2\, Z^B\, \pi_{AB} - 3\eta_A \quad . \qquad 4.60$$

and $$\varphi_{AB} = -Đ_{(A}\, \delta_{B)} = 2\, \pi_{AB} \qquad 4.61$$

and thus we set $$\pi_{AB} = 0 \qquad 4.62$$

If we transform χ according to

$$\chi \leftarrow \chi + \frac{1}{6}\, \xi_{ABC}\, Z^A\, Z^B\, Z^C + \frac{1}{2}\, \zeta_{AB}\, Z^A\, Z^B \qquad 4.63$$

we incorporate all but the Λ_0 and η_A terms into χ. The terms $-\Lambda_0 Z_A Z_B$ and $2\eta_{(A} Z_{B)}$ may not be incorporated into χ, since

$$\text{Đ}^A(\eta_{(A} Z_{B)}) \neq 0 \neq \text{Đ}^A(\Lambda_0 Z_A Z_B) \qquad 4.64$$

and thus neither term can equal $\text{Đ}_{AB}x$ for any x.

$$\text{Now } \bar{\Psi}_3 = \text{Ŧ}^A \delta_A \qquad 4.65$$

$$= \text{Ⱬ}^A \delta_A - 2 \lambda^{AC} \text{Đ}_{(A} \delta_{C)} \qquad 4.66$$

$$= \text{Ⱬ}^A \delta_A \qquad 4.67$$

$$\bar{\Psi}_3 = 3 \text{Ⱬ}_A \eta^A \qquad 4.68$$

Next we must calculate σ_A and then $\alpha_{AB} = \text{Đ}_{(A}\sigma_{B)}$.

$$\sigma_A = 2(\text{Ⱬ}_B \lambda_A{}^B + \lambda_{BC} \Gamma_A{}^{BC}) \qquad 4.69$$

a little work reveals

$$\sigma_A = 2(Z_A \text{Ⱬ}_B \eta^B + Z^B \text{Ⱬ}_B \eta_A - P_A - P_{AB} Z^B + 2 \eta_A \eta_C Z^C)$$
$$+ \bar{\phi} \text{Đ}_A\alpha \qquad 4.70$$

and thus

$$\alpha_{AB} = \bar{\phi} \text{Đ}_{AB}\alpha + 2(\text{Ⱬ}_{(A} \eta_{B)} + 2 \eta_A \eta_B - P_{AB}) \qquad 4.71$$

and so we require

$$P_{AB} = \text{Ⱬ}_{(A} \eta_{B)} + 2 \eta_A \eta_B \qquad 4.72$$

Substituting back into 4.70 reveals

$$\sigma_A = \bar{\phi} \text{Đ}_A\alpha - 2P_A + Z_A \bar{\Psi}_3 \qquad 4.73$$

Finally we calculate $\bar{\Psi}_4 = \text{Ð}_A \sigma^A$

$$\bar{\Psi}_4 = (\text{Ø}_A - 2\delta_A)\bar{\phi}\,\text{Ð}^A\alpha + \bar{\phi}\,\text{Ð}_A\text{Ð}^A\alpha$$

$$+ (\text{Ø}_A - 2\delta_A)(Z^A\bar{\Psi}_3 - 2P^A) \qquad 4.74$$

$$= (2\,\text{Ø}^A + 12\,\eta^A - 12\,\Lambda_0 Z^A)P_A + Z^A(\text{Ø}_A + 6\,\eta_A)\bar{\Psi}_3$$

$$+(\text{Ø}_A\bar{\phi})\ (\text{Ð}^A\alpha) - 6\bar{\phi}(\Lambda_0 Z_A - \eta_A)\text{Ð}^A\alpha + \frac{1}{2}\bar{\phi}^2 \qquad 4.75$$

where we recall χ, α obey

$$\text{Ø}^A\,\text{Ð}_A\chi = \text{Ð}^{AB}\chi\,\text{Ð}_{AB}\chi + 2\,\eta^A(2\,Z^B\,\text{Ð}_{AB}\chi - 3\,\text{Ð}_A\chi)$$

$$- 2\,\Lambda_0(Z^A\,Z^B\,\text{Ð}_{AB}\chi - 3\,Z^A\,\text{Ð}_A\chi + 3\chi) \qquad 4.76$$

$$+ P + P_A\,Z^A + \frac{1}{2}(\text{Ø}_{(A}\,\eta_{B)} + 2\,\eta_A\,\eta_B)Z^A\,Z^B - \frac{1}{2}\bar{\phi}\alpha$$

$$\text{Ø}^A\,\text{Ð}_A\alpha = 2(\text{Ð}^{AB}\chi + 2\,\eta^A\,Z^B - \Lambda_0\,Z^A\,Z^B)\text{Ð}_{AB}\alpha - \bar{\phi} \qquad 4.77$$

Which of the terms P_1 P_A and η_A may be removed by appropriate transformations is complicated. Firstly P may always be set to zero, either by adding $\gamma_A\,Z^A$ to χ for some suitable γ_A, or if $\Lambda_0 \neq 0$ by adding a term $-P/6\Lambda_0$ to χ.

If $\text{V}^A\,\eta_A = 0$ (i.e. $\bar{\Psi}_3 = 0$) then we may remove η_A by a transformation that alters V_A.

If $\bar{\Psi}_3 = \Lambda_0 = 0$ and $\text{Ø}^A\,P_A = 0$ then a change in $\bar{\iota}_A$, can be carried out that will remove P_A.

If $\Lambda_0 \neq 0$ then a change in Z_A such that

$$Z_A \leftarrow Z_A - \frac{\eta_A}{\Lambda_0} \qquad 4.78$$

removes the η_A terms. This must be followed by a change in $\bar{\iota}_A$, (in order to remove $\text{Ø}^A\,Z_A$) and a change in λ_{AB} (to remove $\text{Ø}_{(A}\,Z_{B)}$)).

To sum up.

If the space-time is type II ($\Lambda_0 \neq 0$) then $\eta_A = 0$,

If the space-time is type III ($\Lambda_0 = 0$) then we may have no further simplifications

If the space-time is type N ($\Lambda_0 = \bar{\Psi}_3 = 0$) then $\eta_A = 0$

If the space-time is vacuum and half-flat ($\Lambda_0 = \bar{\Psi}_3 = \bar{\Psi}_4 = 0$) then $\eta_A = P_A = 0$.

The case q = r

Firstly we note that we may add a cubic term to χ in order to remove π_{AB}

$$\chi \leftarrow \chi + \frac{1}{6} P_{ABC} z^A z^B z^C , \quad r_C P_{AB}{}^C = \pi_{AB} \qquad 4.79$$

thence

$$\lambda_{AB} = r \text{Þ}_{AB}\chi - 2r_{(A} \text{Þ}_{B)}\chi - \frac{1}{2} \varepsilon r z_A z_B + \xi_{ABC} z^C + 2\eta_{(A} z_{B)} + \zeta_{AB} \qquad 4.80$$

$$\delta_A = 2r^B \text{Þ}_{AB}\chi + 2 \varepsilon r z_A - 3\eta_A \qquad 4.81$$

$$\text{Þ}_{(A} \delta_{B)} = 2r^C \text{Þ}_{ABC}\chi + 2 \varepsilon r_{(A} z_{B)} \qquad 4.82$$

$$\Gamma_{ABC} = 2r \text{Þ}_{ABC}\chi - 2r_A \text{Þ}_{BC}\chi + \varepsilon(r_A z_B z_C - 2 z_A r_{(B} z_{C)}) \qquad 4.83$$

$$+ 2 (\xi_{ABC} - \varepsilon_{A(B} \eta_{C)})$$

We now impose the condition that

$$r\hat{\varphi}_{AB} = r\varphi_{AB} + \text{Þ}_{(A} \text{Ŧ}_{B)} r = 0 \qquad 4.84$$

we know that it can at worst be order r^0, so we need only check this part of the right hand side of 4.84

$$Đ_{(A}\, Ŧ_{B)}\, r = 2\, Đ_{(A}\, \lambda_{B)C}\, r^C = r^C\, \Gamma_{CAB}$$

$$= r^C\, (\xi_{ABC} - \varepsilon_{C(A}\, \eta_{B)}) = 0 \qquad 4.85$$

we can see from 4.83 that this is the condition required to enable us to add a quadratic term to χ and remove ξ_{ABC} and η_A entirely, and so now

$$\xi_{ABC} = \eta_A = 0 \qquad 4.86$$

Note that Γ_{ABC} does not contain explicitly any terms of order r^0. Next we examine the order r^0 part of $\hat{\Lambda}$

$$2\, \Lambda_0 = -2\, r^A \times 2\, \lambda_{AB}\, r^B = -4\, \xi_{AB}\, r^A\, r^B \qquad 4.87$$

If we add a linear term to χ we can only remove a term from ζ_{AB} of the form $r_{(A}\, \beta_{B)}$ for some β_B, hence we may remove all of ζ_{AB} except for a term of the form

$$\zeta_{AB} = -\frac{1}{2}\, \Lambda_0\, S_A\, S_B \qquad 4.88$$

where we recall $r_A\, S^A = 1$, $Ɐ_A\, S_B = 0$ $\quad (q(S^A) = -1)$

note that if $S = S_A\, Z^A$ then $Z_A = r\, S_A - S\, r_A$ 4.89

We are now ready to discuss α_{AB}, σ_A, $\bar{\Psi}_3$ and thence $\bar{\Psi}_4$. Consider $\hat{\alpha}_{AB}$; we have seen that, apart from $\bar{\phi}\, r^2\, Đ_{AB}\alpha$, there can be three terms in $\hat{\alpha}_{AB}$ that must be set to zero; of the form $\frac{1}{r}\, \alpha_0\, r_A\, r_B$, $\frac{1}{r}\, \alpha_1\, r_{(A}\, Z_{B)}$ and $\frac{1}{r}\, \alpha_2\, Z_A\, Z_B$.

First consider the $\frac{1}{r}\, \alpha_0\, r_A\, r_B$ term.

Examining 4.34 this must come from

$$\frac{\alpha_0 r_A r_B}{r} = \frac{1}{r}\, [(Ŧ_{(A}\, Ŧ_{B)}\, r - \sigma_{(A}\, r_{B)}]_{\text{order } r^0} \qquad 4.90$$

$$\alpha_0\, r_A\, r_B = 2\, [Ɐ_{(A}\, \lambda_{B)C} r^C - 2\, Ɐ_C\, \lambda^C{}_{(A}\, r_{B)}]_{\text{order } r^0} \qquad 4.91$$

(this is because order r^0 terms are not present in Γ_{ABC})

$$\alpha_0 \, r_A \, r_B = 4 \, r_A \, r_B \, P \qquad 4.92$$

and hence $P = 0$. 4.93

Now consider $\bar{\Psi}_3$, 4.31 tells us

$$\bar{\Psi}_3 = -r \, \bar{\phi} \, r^B \, Ð_B \alpha + \bar{\Psi} r \, , Ð_A \, \bar{\Psi} = 0 \qquad 4.94$$

From the form of $\hat{\alpha}_{AB}$ we know that $\bar{\Psi}_3$, as calculated by $\bar{\Psi}_3 = Ŧ^A \, \delta_A$, will contain at worst quadratic terms to be set to zero.

$$\bar{\Psi}_3 = Ŧ^A \, \delta_A = \not{V}^A \, \delta_A - 2 \, \lambda^{AC} \, Ð_{(A} \, \delta_{C)} \, . \qquad 4.95$$

$$= 2 \, r^B \, Ð_B (V^A \, Ð_A \chi) - (\chi \text{ terms}) - 4 \, \zeta^{AC} \, \varepsilon r_A \, Z_C \qquad 4.96$$

$$= 2 \, r^B (P_B + P_{BA} \, Z^A) + 2 \, \varepsilon r^A \, Z^C \, \Lambda_0 \, S_A \, S_C - r^B r \, Ð_B \alpha \, \bar{\phi} \qquad 4.97$$

$$= -r \, \bar{\phi} \, r^B \, Ð_B \alpha + 2 \, r^B \, P_B + 2 \, r^B \, Z^A (P_{AB} + \varepsilon \Lambda_0 \, S_A \, S_B) \qquad 4.98$$

and thus $r^B \, P_B = 0$ 4.99

$$r^A \, r^B \, P_{AB} = -\varepsilon \, \Lambda_0 \qquad 4.100$$

$$2 \, r^B \, S^A \, P_{AB} = \bar{\Psi} \qquad 4.101$$

$$\bar{\Psi}_3 = r(-\bar{\phi} \, r^B \, Ð_B \alpha + \bar{\Psi}) \text{ as required} \qquad 4.102$$

and we have set the conformal Ricci curvature to the form required.. Calculating σ_A and $\bar{\Psi}_4$ is now straightforward and not unduly time-consuming, since σ_A is found to be a brief expression

$$\text{Putting } \beta = S^A \, S^B \, P_{AB}, \quad \mu = S^A \, P_B \qquad 4.103$$

$$\sigma_A = \bar{\Psi}\, r^2\, S_A - \beta\, r^2\, r_A - 2\, r^B\, \eth_A\, \nabla_B \chi + r^2\, \bar{\phi}\, \eth_A \alpha \qquad 4.104$$

$$\frac{1}{r}\,\bar{\Psi}_4 = Z^A\, \nabla_A\, \bar{\Psi} + 2\,\varepsilon\, r^2\, S\, \bar{\Psi} + 2\,\beta\,\Lambda_0 - 2\,\varepsilon\beta\, r^3 + 2\, r^B\, \nabla_B \mu$$

$$- 12\,\varepsilon r^B\, \nabla_B \chi + r(\nabla_A\, \bar{\phi})\eth^A \alpha - \bar{\phi}\, r^B\, \nabla_B \alpha + r\,\bar{\phi}^2$$

$$+ 2\,\varepsilon r^2\, \bar{\phi}\, Z^A\, \eth_A \alpha + 4\,\bar{\phi}(r^C\, \eth_C \chi) r^A\, \eth_A \alpha \qquad 4.105$$

where χ, α obey

$$\nabla^A\, \eth_A \chi = (r\, \eth^{AB} \chi\, \eth_{AB} \chi - 4\, r^A\, \eth_{AB} \chi\, \eth^B \chi)$$

$$- \varepsilon r(Z^A\, Z^B\, \eth_{AB} \chi - 4\, Z^A\, \eth_A \chi + 6\chi) - \tfrac{1}{2}\,\varepsilon\,\Lambda_0\, S^2$$

$$- \tfrac{1}{2}\, r\, \bar{\phi}\, \alpha - \Lambda_0\, S^A\, S^B\, \eth_{AB} \chi + \mu r + \tfrac{1}{2}\,\beta\, r^2 - \tfrac{1}{2}\,\bar{\Psi}\, rS \qquad 4.106$$

$$\nabla^A\, \eth_A \alpha = 2(r\, \eth^{AB} \chi - 2 r^A\, \eth^B \chi)\eth_{AB} \alpha - \varepsilon r\, Z^A\, Z^B\, \eth_{AB} \alpha$$

$$- \Lambda_0\, S^A\, S^B\, \eth_{AB} \alpha - \bar{\phi} \qquad 4.107$$

Again various of the terms in 4.10 may be removed by appropriate transformations. First consider the change in χ

$$\chi \leftarrow \chi + \tfrac{1}{3}\, f\, r^3 \qquad \eth_A f = 0 \qquad 4.108$$

then $\lambda_{AB} \leftarrow \lambda_{AB} + 2\, r^2\, f\, r_A\, r_B - 2\, r^2\, f\, r_A\, r_B$

$$\leftarrow \lambda_{AB}$$

and the only terms affected in 4.10 are

$$\beta \leftarrow \beta + 2\, r_A\, \nabla^A\, f \qquad 4.109$$

$$\mu \leftarrow \mu - 2\,\Lambda_0\, f \qquad 4.110$$

and thus by suitable choice of f we may remove β entirely.

If $\varepsilon = 1$ then the transformation

$$\chi \leftarrow \chi - \frac{1}{6}\mu \qquad 4.111$$

will remove μ.

It is clear from 4.93 that by changing ι_A, we may make $\bar{\Psi}$ vanish, but it is not immediately obvious from 4.10 how to do this.

If we change $\bar{\iota}_A$, and ζ_{AB} then we have

$$\mathcal{V}_A \leftarrow \mathcal{V}_A + (\mathcal{V}_A \gamma_B)\eth^B \qquad 4.112$$

for some γ_B. We also require $\mathcal{V}_A r$ to be maintained, hence $\gamma_B r^B = 0$ or $\gamma_\beta = \gamma r_\beta$

and rewriting $\mathcal{V}_A \leftarrow \mathcal{V}_A + (\mathcal{V}_A \gamma) r_B \eth^B$ 4.113

and $$\lambda_{AB} \leftarrow \lambda_{AB} - \frac{1}{2} r_{(B} \mathcal{V}_{A)} \gamma \qquad 4.114$$

hence $$\chi \leftarrow \chi + \frac{1}{4} z^A \mathcal{V}_A \gamma \qquad 4.115$$

The right hand side of 4.10 then changes by

$$\beta \leftarrow \beta - s^A \mathcal{V}_A \gamma \qquad 4.116$$

$$\bar{\Psi} \leftarrow \bar{\Psi} - r^A \mathcal{V}_A \gamma \qquad 4.117$$

and thus by an appropriate choice of γ, $\bar{\Psi}$ may be set to zero (β may be ignored since a further r^3 term in χ will maintain $\beta = 0$).

In conclusion we may always put $\beta = 0$.

If $\varepsilon = 1$ then we may set $\mu = \bar{\Psi} = 0$.

It is immediately clear why there are many less Type D vacuum solutions than Type III, Type N or half-flat. We see that for a general Type II vacuum we may put $\mu = \bar{\Psi} = 0$, and find

$$\bar{\Psi}_3 = 0 \qquad \bar{\Psi}_4 = -12\, r^B \nabla_B \chi \qquad 4.118$$

To produce a Type D solution, in addition to solving the differential equation 4.10 we must impose a constraint $r^B \nabla_B \chi = 0$. In contrast for Type III or , Type N or half-flat there are no constraints in addition to the differential equation.

Curiously though there are many more solutions which are Type D with cosmological constant, where $\bar{\Psi}_2 + 2\,\Lambda_0 = 0$. In the class of spaces where $q = 1$, if we impose $P_A = \eta_A = 0$ then $\bar{\Psi}_3 = \bar{\Psi}_4 = 0$ and we have one differential equation , 4.76 with no additional constraints.

Chapter Five - Perturbations

We shall only consider conformally Einstein spacetimes, the extension to the more general class considered in Chapter Four being straightforward. It is now meaningful to consider perturbing the metric, or at a differential level, perturbing the Weyl curvature. We have however, two approaches. We are seeking to perturb the Weyl curvature in the conformally transformed space-time, i.e. the transformation after which the space-time is Einstein. The Weyl curvature in the untransformed space-time will be

$$\frac{1}{2}\,\Psi_{ABCD} = r\,Ð_{ABCD}\chi + \delta\,\varphi_{ABCD} \qquad 5.1$$

where δ is the perturbation parameter.

We may either make the perturbation conformally invariant

i) $$\frac{1}{2}\,\hat{\Psi}_{ABCD} = r\,Ð_{ABCD}\chi + \delta\,\varphi_{ABCD} \qquad 5.2$$

or transform as Ω^{-1}, like a massless field.

ii) $$\frac{1}{2}\,\hat{\Psi}_{ABCD} = r\,Ð_{ABCD}\chi + \delta\,r\,\varphi_{ABCD} \qquad 5.3$$

Consider case i) first. The $\bar{o}_A$, Bianchi identity gives

$$(Ð^A + \nu^A)(r\,Ð_{ABCD}\chi + \delta\,\varphi_{ABCD}) = 0 \qquad 5.4$$

thus $$\varphi_{ABCD} = r\,Ð_{ABCD}x \qquad 5.5$$

and $$\hat{\Psi}_{ABCD} = 2\,r\,Ð_{ABCD}(\chi + \delta x) \qquad 5.6$$

Consider case ii). Here we have

$$(Ð^A + \nu^A)(r\,Ð_{ABCD}\chi + \delta\,r\,\varphi_{ABCD}) = 0 \qquad 5.7$$

or $$\varphi_{ABCD} = \text{Ð}_{ABCD}x \qquad 5.8$$

and $$\hat{\Psi}_{ABCD} = 2\ r\ \text{Ð}_{ABCD}(\chi + \delta\ x) \qquad 5.9$$

We know from considering field equations in Chapter Three that x will then satisfy

$$(\text{Ð}^{A}\ \text{Ð}_{A} + 6\Lambda)x = \mu' r - \frac{1}{2}\ \bar{\Psi}'\ rS - \frac{1}{2}\ \epsilon\ \Lambda_0'\ S^2 \qquad 5.10$$

where Ð^{A}, Λ are the unperturbed quantities and μ', $\bar{\Psi}'$ and Λ_0' the perturbations in μ, $\bar{\Psi}$ and Λ_0

The two cases are clearly the same, the explanation of which is as follows. If we perturb a metric g_{ab} by

$$g_{ab} \leftarrow g_{ab} + \delta\ h_{ab} \qquad 5.11$$

where $h^{a}_{\ a} = 0$, then

$$\Psi_{ABCD} \leftarrow \Psi_{ABCD} + \delta\ \nabla^{A'}_{(C}\ \nabla^{B'}_{\ D}\ h_{AB)A'B'} \qquad 5.12$$

Now consider the conformal transformation

$$\hat{g}_{ab} \leftarrow \Omega^2(g_{ab} + \delta\ h_{ab}) \qquad 5.13$$

$$\hat{\Psi}_{ABCD} \leftarrow \delta(\nabla^{A'}_{(C} - \gamma^{A'}_{(C})(\nabla^{A'}_{D} + \gamma^{B'}_{D})h_{AB)A'B'} + \Psi_{ABCD} \qquad 5.14$$

If $h_{ABA'B'} = l_{AB}\ \bar{o}_{A'}\ \bar{o}_{B'}$, and $\Omega = \frac{1}{r}$ then we find that Ψ_{ABCD} is conformally invariant and hence in this case

$$\hat{h}_{ab} = \frac{2}{r^2}\ (r\ \text{Ð}_{AB}x - 2\ r_{(A}\ \text{Ð}_{B)}x)\ \bar{o}_{A'}\ \bar{o}_{B'} \qquad 5.15$$

Now we consider the second case.

Let us assume that the perturbation h_{ab} is of the form

$$h_{ABA'B'} = \nabla^{C'}_{(A}\ \theta_{B)A'B'C'} \quad \text{for some } \theta_{BA'B'C'} \qquad 5.16$$

and therefore

$$\Psi_{ABCD} \leftarrow \delta \nabla^{A'}_{(A} \nabla^{B'}_{D} \nabla^{C'}_{C} \theta_{B)A'B'C'} + \Psi_{ABCD} \qquad 5.17$$

We let $\theta_{AA'B'C'}$ conformally transform as

$$\hat{\theta}_{AA'B'C'} = \Omega^2 \theta_{AA'B'C'} \qquad 5.18$$

and we find

$$\hat{\Psi}_{ABCD} \leftarrow \Omega^{-1} \delta(\nabla^{A'}_{(A} - 2 \gamma^{A'}_{(A}) \nabla^{B'}_{B} (\nabla^{C'}_{C} + 2 \gamma^{C'}_{C}) \theta_{D)A'B'C'} + \Psi_{ABCD} \qquad 5.19$$

so long as $\theta_{DA'B'C'} = \theta_D \bar{o}_{A'} \bar{o}_{B'} \bar{o}_{C'}$ and $\Omega = \frac{1}{r}$ the perturbation in Ψ_{ABCD} rescales as Ω^{-1} as required. These conditions are satisfied since we have $\theta_D = Đ_D x$. The metric perturbation however does not rescale but instead

$$\hat{h}_{ab} = \Omega(\nabla^{C'}_{(B} + 2 \gamma^{C'}_{(B}) \theta_{A)A'B'C'} \qquad 5.20$$

$$= \frac{2}{r^2} (r Đ_{AB} x - 2 r_{(A} Đ_{B)} x) \bar{o}_{A'} \bar{o}_{B'} \qquad 5.21$$

This is the same perturbation as in case i).

6. Solutions to Killing spinor equations

In this chapter we demonstrate a solution to the partial Killing spinor equation

$$\overset{-}{o}^{A'} \nabla_{A(A'} X_{B'C')} = 0 \qquad 6.1$$

where $X_{B'C'}$ is symmetric, and investigate the conditions for solutions to the full Killing spinor equations

$$\nabla_{A(A'} \omega_{B')} = 0 \qquad 6.2$$

$$\nabla_{A(A'} X_{B'C')} = 0 \qquad X_{B'C'} \neq \omega_{B'} \omega_{C'} \qquad 6.3$$

It may be seen that

$$X_{B'C'} = \overset{-}{o}_{(B'} \overset{-}{\iota}_{C')} \qquad 6.4$$

is a solution to 6.1 for any $\overset{-}{\iota}_{C'}$ satisfying

$$Đ_A \overset{-}{\iota}_{C'} = 0 \qquad 6.5$$

However 6.1 is conformally invariant although 6.5 is not and thus we see that

$$X_{B'C'} = \overset{-}{o}_{(B'} \overset{-}{\alpha}_{C')} \qquad 6.6$$

is a solution for any $\overset{-}{\alpha}_{C'}$ where after a conformal rescaling obeying

$$Đ_A \Omega = 0 \qquad \hat{Đ}_A \hat{\overset{=}{\alpha}}_{C'} = 0 \qquad 6.7$$

In order that 6.2 have solution we know the Weyl spinor must be type N or half-flat with the repeated p.n.d. in the same direction as $\omega_{B'}$, and hence we may put

$$\omega_{B'} = \omega\bar{o}_{B'} \qquad \text{weight}(\omega) = -1 \qquad 6.8$$

6.2 becomes

$$\bar{o}_{(B'} \, \Theta_{A')A}\omega = 0 \qquad 6.9$$

or $$Ð_A\omega = Ɨ_A\omega = 0 \qquad 6.10$$

Commuting $Ð_A$ and $Ɨ_B$ on ω we see that

$$\varphi_{AB} = \Lambda = \bar{\Psi}_3 = 0 \qquad 6.11$$

and thus $$Ð_A \, \delta_B = 0 = \text{ð}^A \, \delta_A \qquad 6.12$$

We may make a change in choice of λ_{AB} that eliminates δ_A obeying 6.12 and thus (see 3.4)

$$\delta_A = 0 \qquad 6.13$$

or $$\lambda_{AB} = Ð_{AB}\chi \quad \text{for some} \quad \chi \qquad 6.14$$

Equation 6.3 is rather more difficult. Firstly we note that $X_{B'C'}$ has two p.n.d.'s, both of which are geodesic shear-free and repeated p.n.d.'s of the Weyl spinor, thus we take them for our $\bar{o}_{A'}$, $\bar{\iota}_{B'}$ where $\sigma_A = 0$, but ρ_A does not necessarily vanish.

Writing $$X_{B'C'} = X\bar{o}_{(B'} \, \bar{\iota}_{C')} \qquad 6.15$$

we convert 6.3 to

$$Ð_A X = 0 \qquad 6.16$$

$$Ɨ_A X - \rho_A X = 0 \qquad 6.17$$

Since $Ð_A X = 0$ and X conformally rescales as

$$X \leftarrow \Omega X \qquad 6.18$$

we may conformally rescale so that $X = 1$ and thus $\rho_A = 0$. Characterising further these spaces is difficult, we have $\sigma_A = \rho_A = 0$ and thus

$$\frac{1}{2}\sigma_A = \text{Ŧ}_B \lambda_A{}^B - \lambda_{BC} \Gamma_A{}^{BC} = 0 \qquad 6.19$$

$$= \text{Ѵ}_B \lambda_A{}^B + 2 \lambda_{BC} \text{Đ}^B \lambda^C{}_A$$

and thus $[\text{Ŧ}_A, \text{Ŧ}_B] = 0$. 6.20

Since

$$\bar{\Psi}_3 = \text{Ŧ}^A \delta_A = 0 \qquad 6.21$$

we must have

$$\delta_A = \text{Ŧ}_A F = \text{Đ}^B \lambda_{AB} \text{ for some } F. \qquad 6.22$$

6.19 is two constraints on three variables (λ_{AB}), 6.22 is only one constraint and thus if we may write down a general solution to 6.22 substituting into 6.19 will leave one constraint on two variables.

6.22 may be written

$$\text{Ѵ}_A F + 2 \lambda_{AB} \text{Đ}^B F = \text{Đ}^B \lambda_{AB} \qquad 6.23$$

We find as a solution

$$\lambda_{AB} = -\frac{1}{2\Box f} (\text{Ѵ}_{AB} f + \text{Đ}_{AB} g) \qquad f \neq 0$$

$$= \text{Đ}_{AB} g \qquad f = 0 \qquad 6.24$$

where $\Box f = \text{Ѵ}^A \text{Đ}_A f$, $F = \frac{1}{2} \ln(\Box f)$.

Substituting into 6.19 we find

$$\text{Þ}_A \left(\frac{(\text{Ý}_{BC} f + \text{Þ}_{BC} g)(\text{Ý}^{BC} f + \text{Þ}^{BC} g) - 2\square f \square g}{\square f} \right) = 0 \qquad 6.25$$

$$\text{if } f \neq 0$$

$$\text{Þ}_A ((\text{Þ}_{BC} g)(\text{Þ}^{BC} g) - \square g) = 0 \quad \text{if } f = 0 \qquad 6.26$$

If $f = 0$ then $\delta_A = 0$ and the space is half-flat and 6.26 is the vacuum condition. If $f \neq 0$ then the space will be type D.

Conclusion

We have shown how the choice of a conformal factor of a particular kind greatly facilitates the discussion of half algebraically-special space-times; however there are disadvantages. In particular the formalism as developed is not idealy suited to the study of real algebraically-special space-times since it both distinguishes between primed and un-primed spinors and in general necessitates a complex conformal transformation on the real space-time before examination.

Despite this there is considerable scope for further work; for instance electro-vac space-times whose primed Maxwell spinor is not null may be studied and the significance of the governing field equation 3.76 when q is quadratic, not linear, is not clear. If we wish to study space-times where there is an $\bar{\iota}_{A'}$ satisfying not only $\rho_A = 0$ but also $\sigma_A = 0$ then we have symmetry between $\bar{o}_{A'}$ and $\bar{\iota}_{A'}$, but the construction as given is asymmetric. It may be possible to write the curvature in terms of the derivatives of one free function in a symmetric manner, but it is not yet clear how.

Appendix; The curvature equations and their integration for a general choice of $\bar{\iota}_{A'}$.

If $\bar{\iota}_{A'}$ is chosen only so that $\bar{o}_{A}$, $\bar{\iota}^{A'} = 1$ then we have the following

$$\rho_A = \bar{\iota}^{A'} \text{Đ}_A \bar{\iota}_{A'}$$

$$\sigma_A = \bar{\iota}^{A'} \text{Ŧ}_A \bar{\iota}_{A'}$$

$$\text{Đ}_A \rho^A = 0$$

$$\text{Đ}_A \sigma_B - \text{Ŧ}_B \rho_A = \alpha_{AB} + \varepsilon_{AB} \bar{\Psi}_3 + \rho_A \rho_B$$

$$\text{Ŧ}_A \sigma^A = \bar{\Psi}_4 + \sigma_A \rho^A$$

The primed Bianchi identities are

$$\text{Đ}^A \alpha_{AB} + \text{Đ}_B \bar{\Psi}_3 = 2(\varphi_{AB} \rho^A - 3 \Lambda \rho_B)$$

$$\text{Ŧ}^B \alpha_{AB} - \text{Ŧ}_A \bar{\Psi}_3 + \text{Đ}_A \bar{\Psi}_4 = 2(\varphi_{AB} \sigma^B + 3 \Lambda \sigma_A) + 4 \bar{\psi}_3 \rho_A$$

and the unprimed are

$$\text{Đ}^A \Psi_{ABCD} + \text{Đ}_B \varphi_{CD} - 2 \text{Đ}_{(D} \Lambda \varepsilon_{C)B} = 0$$

$$\text{Ŧ}^A \Psi_{ABCD} - \text{Ŧ}_B \varphi_{CD} - 2 \text{Ŧ}_{(D} \Lambda \varepsilon_{C)B} = -\text{Đ}_B \alpha_{CD} + 2 \rho_B \varphi_{CD}$$

$$\text{Đ}^B \varphi_{AB} - 3 \text{Đ}_A \Lambda = 0$$

$$\text{Ŧ}^B \varphi_{AB} + 3 \text{Ŧ}_A \Lambda = -\text{Đ}_A \bar{\Psi}_3 - 6 \Lambda \rho_A$$

These may be integrated in terms of a symmetric two-spinor λ_{AB} and a scalar $\beta (q(\lambda_{AB}) = -2 = q(\beta))$

Putting $\Gamma_{ABC} = 2 \text{Đ}_{(B} \lambda_{C)A}$ we find

$$\Psi_{ABCD} = 2\, Đ_{(AB}\, \lambda_{CD)} = Đ_{(A}\, \Gamma_{BCD)}$$

$$\varphi_{AB} = -Đ_{(A}{}^{E}\, \lambda_{B)E} = -Đ_{(A}\, \delta_{B)}$$

$$\Lambda = -\frac{1}{6}\, Đ^{AB}\, \lambda_{AB} = -\frac{1}{6}\, Đ^{A}\, \delta_{A}$$

(where $\delta_A = Đ^B\, \lambda_{AB} = \Gamma_B{}^B{}_A$),

and

$$\alpha_{AB} = Ɨ_C\, \Gamma^C{}_{AB} - \Gamma_{CD(A}\, \Gamma^{CD}{}_{B)}$$

$$\bar{\Psi}_3 = Ɨ^A\, \delta_A$$

$$\sigma_A = 2(Ɨ_B\, \lambda_A{}^B - 2\, \lambda_{BC}\, \Gamma_A{}^{BC}) + Ɨ_A\beta + 2\, \delta_A\, \beta$$

$$\rho_A = Đ_A\, \beta$$

and we may write I_A in the form

$$Ɨ_A\, \alpha_B = Ⱶ_A\, \alpha_B + 2\, \lambda_{AC}\, Đ^C\, \alpha_B + \Gamma_{ABC}\, \alpha^C + q\, \delta_A\, \alpha_B + \beta\, Đ_A\, \alpha_B$$

where $[Đ_A, Đ_B] = [Đ_A, Ⱶ_B] = [Ⱶ_A, Ⱶ_B]$.

Two-surface twistors and conformal embedding

Introduction

1. Two-surface twistors $\mathbb{T}(\mathcal{Y})$

2. Local embeddings in spaces conformal to real Minkowski space

3. Local embeddings in spaces conformal to complex Minkowski space

4. Dual angular-momentum

Concluding remarks

Introduction

Penrose 1982 introduces the concept of a twistor space associated with a space-like two-surface of spherical topology and shows from this how the concepts of energy and angular momentum within linearised general relativity may be generalised to cover curved space-times. We refer the reader to this reference and to Penrose and MacCallum 1973, Penrose and Rindler 1985 for discussion of this and twistor theory in general.

The construction involves a symmetric two-twistor $A_{\alpha\beta}$ whose inner product with two one-twistors $\underset{1}{Z}^{\alpha}$, $\underset{2}{Z}^{\beta}$ is given by

$$A_{\alpha\beta}\ \underset{1}{Z}^{\alpha}\ \underset{2}{Z}^{\beta} = \frac{i}{8\pi G}\int_{\mathcal{J}} R_{abcd}\ \underset{1}{\omega}^{A}\ \underset{2}{\omega}^{B}\ \varepsilon^{A'B'}\ d\mathcal{J}^{cd} \qquad 0.0$$

where in the linearised theory $\underset{1}{\omega}^{A}$, $\underset{2}{\omega}^{B}$ are the primary parts of $\underset{1}{Z}^{\alpha}$, $\underset{2}{Z}^{\beta}$ respectively, R_{abcd} is the linearised Riemann tensor and $d\mathcal{J}^{cd}$ is the two-surface area two-form. In the new construction $\underset{1}{Z}^{\alpha}$, $\underset{2}{Z}^{\beta}$ are two-surface twistors with primary parts $\underset{1}{\omega}^{A}$, $\underset{2}{\omega}^{B}$, and R_{abcd} is the full Riemann tensor.

In order to interpret $A_{\alpha\beta}$ in flat space we make use of two structures on the two-surface twistor space ($\mathbb{T}$). Firstly there is a natural isomorphism between the dual of $\mathbb{T}$ ($\mathbb{T}^*$) and the complex conjugate of $\mathbb{T}$ ($\overline{\mathbb{T}}$), which provides a norm on $\mathbb{T}$ and enables the mass surrounded by the two-surface $\mathcal{J}$ to be calculated. Secondly there is the anti-symmetric infinity twistor $I^{\alpha\beta}$ whose relationship to $A_{\alpha\beta}$

$$A_{\alpha\gamma}\ I^{\beta\gamma} = \overline{A_{\beta\gamma}\ I^{\alpha\gamma}} \qquad 0.1$$

shows how the ten complex components of $A_{\alpha\beta}$ reduce to ten real components, and enables the angular momentum components of $A_{\alpha\beta}$ to be isolated.

In this section we do not consider the infinity twistor for a curved space $\mathcal{PT}$, but instead examine the dual space $\mathcal{PT}^*$ and show how the existence of a natural isomorphism between $\mathcal{PT}^*$ and $\overline{\mathcal{PT}}$ is equivalent to the two-surface being locally embeddable in a space conformally related to real Minkowski space, and how more generally $\mathcal{PT}^*$ provides information on the local embedding of the two-surface in a space conformal to complex Minkowski space.

It is worthwhile considering twistors in a space conformal to complex Minkowski space before discussing two-surface twistors in general. In flat space we have the twistor equation (Penrose and MacCallum 1973)

$$\nabla_{A'\,A}\,\omega_B = -i\varepsilon_{AB}\,\pi_{A'} \qquad 0.2$$

Applying the curvature operators $\Box_{A'B'}$ and $\Box_{AB}$ to ω_C we find

$$\nabla_{A'A}\,\pi_B = 0 \qquad 0.3$$

Let us introduce a conformal rescaling

$$\varepsilon_{AB} \leftarrow \Omega\varepsilon_{AB} \qquad \varepsilon_{A'B'} \leftarrow \tilde{\Omega}\varepsilon_{A'B'} \qquad 0.4,\ 0.5$$

where we introduce torsion by (Penrose 1983)

$$\nabla_{AA'}\,\xi_B \leftarrow \nabla_{AA'}\,\xi_B - \gamma_{A'B}\,\xi_A \qquad 0.6$$

$$\nabla_{AA'}\,\bar{\xi}_{B'} \leftarrow \nabla_{AA'}\,\bar{\xi}_{B'} - \tilde{\gamma}_{AB'}\,\bar{\xi}_{A'} \qquad 0.7$$

where $\gamma_{AA'} = \nabla_{AA'}\ln\Omega \qquad \tilde{\gamma}_{AA'} = \nabla_{AA'}\ln\tilde{\Omega}$ 0.8, 0.9

and thus $\nabla_{[a}\nabla_{b]}\phi = i\varepsilon_{abcd}(\gamma^d - \tilde{\gamma}^d)\nabla^c\phi$ 0.10

We see that $\pi_{A'}$ transforms according to

$$\pi_{A'} \leftarrow \pi_{A'} + i\,\gamma_{AA'}\,\omega^A \qquad 0.11$$

and †

$$\nabla_{AA'}\,\pi_{B'} \leftarrow (\gamma_{AB'} - \tilde{\gamma}_{AB'})\pi_{A'} + i(\nabla_{AA'}\,\gamma_{BB'} - \tilde{\gamma}_{AB'}\,\gamma_{BA'})\omega^B \qquad 0.12$$

Similarly we have a twistor with primed and unprimed indices interchanged.

† thus $\nabla_{AA'}\pi_{B'} = (\gamma_{AB'} - \tilde{\gamma}_{AB'})\pi_{A'} - i\,(\tilde{\Phi}_{ABA'B'} - \Lambda\varepsilon_{AB}\varepsilon_{A'B'})\omega^B$

for definitions see p 104.

$$\nabla_{AA'}\ \bar{\omega}_{B'} = i\ \varepsilon_{A'B'}\ \bar{\pi}_A \qquad 0.13$$

which transforms under the conformal transformation given above as

$$\bar{\pi}_A \leftarrow \bar{\pi}_A - i\ \tilde{\gamma}_{AA'}\ \bar{\omega}^{A'} \qquad 0.14$$

and

$$\nabla_{A'A}\ \bar{\pi}_B \leftarrow (\tilde{\gamma}_{A'B} - \gamma_{A'B})\bar{\pi}_A - i(\nabla_{AA'}\ \tilde{\gamma}_{BB'} - \gamma_{BA'}\ \tilde{\gamma}_{AB'})\bar{\omega}^{B'} \qquad 0.15$$

We see that in flat space the quantity

$$\Sigma = \omega^A\ \bar{\pi}_A + \bar{\omega}^{A'}\ \pi_{A'}, \text{ is constant} \qquad 0.16$$

becoming in general

$$\Sigma = \omega^A\ \bar{\pi}_A + \bar{\omega}^{A'}\ \pi_{A'} - i(\gamma_{AA'} - \tilde{\gamma}_{AA'})\omega^A\ \bar{\omega}^{A'} \qquad 0.17$$

which we may rewrite

$$\Sigma = \omega^A\ \tilde{\pi}_A + \tilde{\omega}^{A'}\ \pi_{A'} \qquad 0.18$$

where $\tilde{\omega}^{A'} = \bar{\omega}^{A'}$

$$\tilde{\pi}_A = \bar{\pi}_A + i(\tilde{\gamma}_{AA'} - \gamma_{AA'})\bar{\omega}^{A'} \qquad 0.19$$

and thus $(\tilde{\omega}^{A'}, \tilde{\pi}_A) = (\bar{\omega}^{A'}, \bar{\pi}_A)$ iff $\Omega = \tilde{\Omega}$ 0.20

We choose a null tetrad and hence spinor dyad (Newman and Penrose 1962) so that l_a and n_a are the outgoing and ingoing null normals of a spacelike two-surface $\mathcal{J}$, normalised so that $l^a n_a = 1$. m_a and $\bar{m}_a$ are thus tangential to $\mathcal{J}$ and we recall

$$l_a = o_A\ \bar{o}_{A'} \qquad n_a = \iota_A\ \bar{\iota}_{A'} \qquad 0.21$$

$$m_a = o_A \bar{\iota}_{A'}, \qquad \bar{m}_a = \iota_A \bar{o}_{A'}, \qquad 0.22$$

We write the tangential components of 0.2, 0.12 with respect to this tetrad. Putting

$$\omega^A = \omega^0 \, o^A + \omega^1 \, \iota^A \qquad 0.23$$

$$\pi_{A'} = \pi_{0'} \, \bar{\iota}_{A'} - \pi_{1'} \, \bar{o}_{A'} \qquad 0.24$$

and $$\nu = \ln(\Omega/\tilde{\Omega}) \qquad 0.25$$

then we have

$$\eth' \omega^0 = \sigma' \omega^1 \qquad \eth \omega^1 = \sigma \omega^0 \qquad 0.26,\ 0.27$$

$$-i\,\pi_{1'} = \eth \omega^0 - \rho' \omega^1 \qquad -i\,\pi_{0'} = \eth' \omega^1 - \rho \omega^0 \qquad 0.28,\ 0.29$$

$$i(\eth \pi_{0'} + \rho \pi_{1'}) = (\Phi_{11} + \Lambda)\omega^1 + \Phi_{01}\omega^0 \qquad 0.30$$

$$i(\eth' \pi_{1'} + \rho' \pi_{0'}) = (\Phi_{11} + \Lambda)\omega^0 + \Phi_{21}\omega^1 \qquad 0.31$$

$$i(\eth \pi_{1'} + \bar{\sigma}' \pi_{0'} + (\eth \nu)\pi_{1'}) = \Phi_{12}\omega^1 + \Phi_{02}\omega^0 \qquad 0.32$$

$$i(\eth' \pi_{0'} + \bar{\sigma} \pi_{1'} + (\eth' \nu)\pi_{0'}) = \Phi_{10}\omega^0 + \Phi_{20}\omega^1 \qquad 0.33$$

Note that $\text{Þ}\nu$ and $\text{Þ}'\nu$ do not appear, nor does $\bar{\rho}$ $(\neq \rho)$ or $\bar{\rho}'$, as the 'ρ' term in 0.30 is in fact made up of

$$\bar{\rho} - \text{Þ}\nu = \rho \qquad 0.34$$

where $\tilde{R}_{ab} = R_{ab} + i * R_{acb}{}^{c}$, $\tilde{\Phi}_{ab} = \tilde{R}_{ab} - \frac{1}{4}\tilde{R}_{c}{}^{c} g_{ab}$

$\Lambda = \frac{1}{24}\tilde{R}_{c}{}^{c}$, $\phi_{11} = o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} \tilde{\Phi}_{ABA'B'}$, $\phi_{01} = o^A D^B \bar{o}^{A'} \bar{\iota}^{B'} \tilde{\Phi}_{ABA'B'}$

$\phi_{02} = o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} \tilde{\Phi}_{ABA'B'}$, $\phi_{20} = \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'} \tilde{\Phi}_{ABA'B'}$, $\phi_{12} = D^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} \tilde{\Phi}_{ABA'B'}$

$\phi_{21} = \iota^A \ell^B \bar{o}^{A'} \bar{\iota}^{B'} \tilde{\Phi}_{ABA'B'}$, $\phi_{10} = o^A \iota^B \bar{o}^{A'} \bar{o}^{B'} \tilde{\Phi}_{ABA'B'}$

1. Two-surface Twistors $\mathbb{PT}(\mathcal{Y})$

In a general space-time we know from Penrose 1982 that we still have at least four linearly independent solutions to 0.26, 0.27; in what follows we assume that there are exactly four linearly independant solutions. By commuting $\eth$ and $\eth'$ on ω^0 and ω^1 we obtain in addition the analogues of 0.30, 0.31, i.e.

$$\eth'\omega^0 = \sigma'\omega^1 \qquad \eth\omega^1 = \sigma\omega^0 \qquad 1.1,\ 1.2$$

$$-i\pi_{1'} = \eth\omega^0 - \rho'\omega^1 \,, \quad -i\pi_{0'} = \eth'\omega^1 - \rho\omega^0 \qquad 1.3,\ 1.4$$

$$-i(\eth\pi_{0'} + \rho\pi_{1'}) = X_2\omega^1 + X_1\omega^0 \qquad 1.5$$

$$-i(\eth'\pi_{1'} + \rho'\pi_{0'}) = X_2\omega^0 + X_3\omega^1 \qquad 1.6$$

where $X_1 = \Psi_1 - \phi_{01}$ 1.7

$$X_2 = \Psi_2 - \phi_{11} - \Lambda \qquad 1.8$$

$$X_3 = \Psi_3 - \phi_{21} \,. \qquad 1.9$$

Here we have no information on the analogues of 0.32, 0.33, however we have assumed four solutions. The argument that follows is due to K.P. Tod.

Consider the representation of $\mathbb{PT}^\alpha$ as a four-dimensional space of twistors $Z^\alpha = (\omega^A, \pi_{A'})$, which we may write as

$$\mathbb{PT}^\alpha = \begin{pmatrix} \omega^0_1 & \pi_{1'1'} & \omega^1_1 & \pi_{0'1'} \\ \omega^0_2 & \pi_{1'2'} & \omega^1_2 & \pi_{0'2'} \\ \omega^0_3 & \pi_{1'3'} & \omega^1_3 & \pi_{0'3'} \\ \omega^0_4 & \pi_{1'4'} & \omega^1_4 & \pi_{0'4'} \end{pmatrix} \qquad 1.10$$

consider G = Det(${}_i\Pi^{\alpha}$) and put B' = $\eth$ ln G. We see that

$$\begin{vmatrix} \omega^0 & (\eth-B')\pi_{1'} & \omega^1 & \pi_{0'} \\ 1 & 1 & 1 & 1 \\ \omega^0 & (\eth-B')\pi_{1'} & \omega^1 & \pi_{0'} \\ 2 & 2 & 2 & 2 \\ \omega^0 & (\eth-B')\pi_{1'} & \omega^1 & \pi_{0'} \\ 3 & 3 & 3 & 3 \\ \omega^0 & (\eth-B')\pi_{1'} & \omega^1 & \pi_{0'} \\ 4 & 4 & 4 & 4 \end{vmatrix} = 0 \qquad 1.11$$

and hence column two is linearly dependent on columns 1, 3, 4

and similarly for $\eth'\pi_{0'}$. Thus for some $\bar{s}$, B, C, D & $\bar{s}'$, B', C', D'

$$-i(\eth\pi_{1'} + \bar{s}'\pi_{0'} - B'\pi_{1'}) = C'\omega^1 + D'\omega^0 \qquad 1.12$$

$$-i(\eth'\pi_{0'} + \bar{s}\pi_{1'} - B\pi_{0'}) = C\omega^0 + D\omega^1 \qquad 1.13$$

By applying the commutator [$\eth$,$\eth'$] to $\pi_{0'}$ and $\pi_{1'}$, we obtain the equations satisfied by $\bar{s}$, B, C, D etc.

$$\eth B + (\bar{s}\bar{s}' - \bar{\sigma}\bar{\sigma}') = X_2 - \bar{X}_2 \qquad 1.14$$

$$C + (\eth' - B)\rho - (\eth + B')\bar{s} = 0 \qquad 1.15$$

$$\eth C - (\eth' - B)X_1 + \sigma D - \bar{s}D' = 0 \qquad 1.16$$

$$\eth D + \rho X_3 + \rho' C - \sigma' X_1 - \bar{s}C' - (\eth'-B)X_2 = 0 \qquad 1.17$$

and their primed versions. We shall find it convenient to rewrite 1.14, 1.16 and 1.17 in the following form.

Putting $d = D + \rho\sigma' + \rho'\bar{s}$ 1.18

$$d' = D' + \rho\bar{s}' + \rho'\sigma \qquad 1.19$$

we see that $\eth B + (\bar{s}\bar{s}' - \bar{\sigma}\bar{\sigma}') = X_2 - \bar{X}_2$ 1.20

$$(\eth(\eth+B') - d')\bar{s} = ((\eth'-B)\eth' - d)\sigma \qquad 1.21$$

$$\text{ð}d = (\text{ð}'-B)(X_2 + \rho\rho') + \sigma'\text{ð}'\sigma + \bar{s}(\text{ð}'+B)\bar{s}' \qquad 1.22$$

note that $B = \text{ð}' \ln(\text{Det}\,\Pi)$ $\qquad B' = \text{ð} \ln(\text{Det}\,\Pi)$ $\qquad 1.23,\ 1.24$

and the quantities in 1.20-1.22 are all unaffected by conformal transformations which change only $\text{Þ}\,\Omega$, $\text{Þ}'\Omega$ but not $\Omega|_\gamma$

To obtain the space dual to ${}_i\Pi^\alpha$, namely ${}^i\Pi^*_\alpha$ we simply invert ${}_i\Pi^\alpha$

$$ {}^i\Pi^*_\alpha = \begin{pmatrix} 1\tilde{\pi}_0 & 1\tilde{\omega}1' & 1\tilde{\pi}_1 & 1\tilde{\omega}0' \\ 2\tilde{\pi}_0 & 2\tilde{\omega}1' & 2\tilde{\pi}_1 & 2\tilde{\omega}0' \\ 3\tilde{\pi}_0 & 3\tilde{\omega}1' & 3\tilde{\pi}_1 & 3\tilde{\omega}0' \\ 4\tilde{\pi}_0 & 4\tilde{\omega}1' & 4\tilde{\pi}_1 & 4\tilde{\omega}0' \end{pmatrix} \qquad 1.25$$

and we see that

$$i\tilde{\pi}_0 = \frac{1}{\text{Det}\,\Pi}\,(\varepsilon^{ijkl}\,{}_j\omega^1\,{}_k\pi_{0'}\,{}_l\pi_{1'}) \qquad 1.26$$

$$i\tilde{\pi}_1 = -\frac{1}{\text{Det}\,\Pi}\,(\varepsilon^{ijkl}\,{}_j\omega^0\,{}_k\pi_{0'}\,{}_l\pi_{1'}) \qquad 1.27$$

$$i\tilde{\omega}0' = \frac{1}{\text{Det}\,\Pi}\,(\varepsilon^{ijkl}\,{}_j\omega^0\,{}_k\omega^1\,{}_l\pi_{1'}) \qquad 1.28$$

$$i\tilde{\omega}1' = -\frac{1}{\text{Det}\,\Pi}\,(\varepsilon^{ijkl}\,{}_j\omega^0\,{}_k\omega^1\,{}_l\pi_{0'}) \qquad 1.29$$

Differentiating we see that

$$\text{ð}\tilde{\omega}^{0'} = \bar{s}'\tilde{\omega}^{1'} \qquad \text{ð}'\tilde{\omega}^{1'} = \bar{s}\tilde{\omega}^{0'} \qquad 1.30,\ 1.31$$

$$i\tilde{\pi}_0 = \text{ð}\tilde{\omega}^{1'} - \rho\tilde{\omega}^{0'} + B'\,\tilde{\omega}^{1'} \qquad 1.32$$

$$i\tilde{\pi}_1 = \text{ð}'\tilde{\omega}^{0'} - \rho'\tilde{\omega}^{1'} + B\,\tilde{\omega}^{0'} \qquad 1.33$$

$$i(\text{ð}'\tilde{\pi}_0 + \rho\tilde{\pi}_1) = X_2\tilde{\omega}^{1'} + C\,\tilde{\omega}^{0'} \qquad 1.34$$

$$i(\eth\tilde{\pi}_1 + \rho'\tilde{\pi}_0) = X_2\tilde{\omega}^{0'} + C'\tilde{\omega}^{1'} \qquad 1.35$$

$$i(\eth'\tilde{\pi}_1 + \sigma'\tilde{\pi}_0) = X_3\tilde{\omega}^{1'} + D\,\tilde{\omega}^{0'} \qquad 1.36$$

$$i(\eth\tilde{\pi}_0 + \sigma\tilde{\pi}_1) = X_1\tilde{\omega}^{0'} + D'\tilde{\omega}^{1'} \qquad 1.37$$

In general we can say little about the functions $\bar{s}$, $\bar{s}'$, however we do know the following. The imaginary part of the curvature of $\mathcal{J}$ is given by

$$2\ \mathrm{im}\ K = (X_2 - \bar{X}_2) - (\sigma\sigma' - \bar{\sigma}\bar{\sigma}') \qquad 1.38$$

and we know (Penrose and Rindler 1984) that

$$\int_{\mathcal{J}} \mathrm{im}\ K\ d\mathcal{J} = 0 \qquad \text{and thus im } K = \eth\eth'f \text{ for some } f.$$

Since $\eth\eth'\ \ln\ (\mathrm{Det}\,\Pi) = (X_2 - \bar{X}_2) - (\bar{s}\bar{s}' - \bar{\sigma}\bar{\sigma}')$ (1.20)

we see that $(\bar{s}\bar{s}' - \sigma\sigma') = \eth\eth'f$ for some f . 1.39

In addition if ${}_i\Pi^{\alpha}$ has any simplifying properties then these are reflected in ${}^{i}\Pi^{*}_{\alpha}$ and hence in $\bar{s}$ and $\bar{s}'$. Consider the case $\sigma' = 0$. We have two solutions to 0.26, 0.27 for which $\omega^0 = 0$ $\eth'\omega^1 = 0$, and hence for such solutions $-i\pi_{1'} = -\rho'\omega^1$. If we conformally transform so that

$$\Omega\Big|_{\mathcal{J}} = 1\ , \qquad \text{Þ}\Omega = \rho\Omega \qquad\qquad \text{Þ}'\Omega = \rho'\Omega$$

then σ,σ', $\bar{s}$, $\bar{s}'$ are unaffected and we see that the upper left hand corner of ${}_i\Pi^{\alpha}$ vanishes, and hence the bottom right hand corner of ${}^{i}\Pi^{*}_{\alpha}$ vanishes, implying there are two solutions to the dual equations 1.30, 1.31 for which $\tilde{\omega}^{0'}$ vanishes, and hence $\bar{s}' = 0$. In a similar manner we have

$$\sigma(\text{resp } \sigma') = 0 \quad \text{iff} \quad \bar{s}(\text{resp } \bar{s}') = 0$$

2. Local embeddings in spaces conformal to real Minkowski space

This section is based on the following. If the curvature restricted to the two-surface satisfies the Bianchi identities for a conformally flat space, then there exists a conformal transformation that will remove all the curvature and hence the two-surface will then be embeddable in locally Minkowski space, although not necessarily with the usual Minkowski global topology. Conversely the original two-surface will be embeddable in a space locally conformal to Minkowski space. The appropriate components of the Bianchi identities are

$$\text{ð}'\phi_{01} - \text{ð}\phi_{10} = \sigma\phi_{20} - \bar{\sigma}\phi_{02} \qquad 2.1$$

$$\text{ð}\phi_{21} - \text{ð}'\phi_{12} = \sigma'\phi_{02} - \bar{\sigma}'\phi_{20} \qquad 2.2$$

$$\text{ð}\phi_{20} - \text{ð}'(\phi_{11}+\Lambda) = \bar{\sigma}\phi_{12} + \sigma'\phi_{01} - \rho'\phi_{10} - \rho\phi_{21} \qquad 2.3$$

$$\text{ð}'\phi_{02} - \text{ð}(\phi_{11}+\Lambda) = \sigma\phi_{21} + \bar{\sigma}'\phi_{10} - \rho'\phi_{01} - \rho\phi_{12} \qquad 2.4$$

Lemma 1. (This has also been shown by K.P. Tod).

1) $\Psi_2 = \bar{\Psi}_2$, $\bar{s} = \bar{\sigma}$, $\bar{s}' = \bar{\sigma}'$

2) $\mathcal{J}$ is locally embeddable in real conformally flat space

3) $\Sigma_{\mathbb{R}} = \omega^{A\dot{B}}\pi_A + \bar{\omega}^{\dot{A}'}\pi_{A'}$ is constant

For a two-surface $\mathcal{J}$ the above three conditions are equivalent, such two-surfaces will be termed 'uncontorted' all others are 'contorted'.

Proof

1) implies 2): Substituting condition 1) into equations 1.14-1.17 we see firstly that $\text{ðð}'(\text{Det}\,\Pi) = 0$. (2.5)

Since $\eth\eth'$ is conformally the spherical Laplacian, Det$\mathcal{T}$ must be constant and B = B' = 0. We then see that

$$C = \eth\bar{\sigma} - \eth'\rho = \bar{X}_1 \qquad 2.6$$

$$C' = \eth'\bar{\sigma}' - \eth\rho' = \bar{X}_3 \qquad 2.7$$

and equations 1.16, 1.17 become

$$\eth\bar{X}_1 - \eth'X_1 = \bar{\sigma}D' - \sigma D \qquad 2.8$$

$$\eth D - \eth'X_2 = \bar{\sigma}\bar{X}_1 + \sigma'X_1 - \rho X_3 - \rho'\bar{X}_3 \qquad 2.9$$

and their primed counterparts. We see these are exactly the Bianchi identities given above on replacing X_1 by $-\phi_{01}$, D by $-\phi_{20}$ and X_2 by $-(\phi_{11}+\Lambda)$ and hence $\mathcal{J}$ is locally embeddable in real conformally flat space.

2) implies 3): From the introduction we see that in flat space $\Sigma_{\mathbb{R}}$ is constant and is invariant under conformal transformations so long as $\Omega = \tilde{\Omega}$.

3) implies 1): Since $\Sigma_{\mathbb{R}}$ is constant $(\bar{\omega}^{A'}, \bar{\pi}_A)$ must be an element of ${}^{i}\mathcal{T}^{*}_{\alpha}$. Comparing the equations satisfied by $(\bar{\omega}^{A'}, \bar{\pi}_A)$ to those of $(\tilde{\omega}^{A'}, \tilde{\pi}_A)$ we see $\bar{\sigma} = \bar{s}$, $\bar{\sigma}' = \bar{s}'$, B = B' = 0 and hence $\Psi_2 - \bar{\Psi}_2 = 0$. □

Example 1. The examples given in Tod 1983 are all uncontorted, which enables the magnitude of $A_{\alpha\beta}$ to be calculated. These include spherically symmetric two-surfaces, a special case of the following.

If $\mathcal{J}$ obeys $\sigma = \sigma' = \Psi_2 - \bar{\Psi}_2 = 0$ then it is conformal to a metric sphere in flat space.

By the properties given in chapter one we see that

$\bar{s} = \bar{\sigma} = 0$ and $\bar{s}' = \bar{\sigma}' = 0$, hence the conditions of Lemma 1 are satisfied. To see that it is conformal to a metric sphere we note that when conformally transformed we have

$$\eth\rho = \eth'\rho = 0 \qquad 2.10$$

$$\eth\rho' = \eth'\rho' = 0 \qquad 2.11$$

and hence the intrinsic curvature $(-\rho\rho')$ is constant, and since $\mathcal{J}$ is compact it must be a sphere.

Such $\mathcal{J}$ include static black-hole Killing horizons (Tod 1984).

Example 2. We shall see how $\Pi(\mathcal{J})$ for $\mathcal{J}$ with an orthogonal timelike Killing vector has special properties. If in addition $\mathcal{J}$ possesses a tangential (rotational) Killing vector then it may be shown that $\mathcal{J}$ is uncontorted so long as both Killing vectors are hyper-surface orthogonal.

Lemma 2

1) $\bar{s} = \bar{\sigma}$, $\bar{s}' = \bar{\sigma}'$, $\Psi_2 \neq \bar{\Psi}_2$

2) $\mathcal{J}$ is locally embeddable in real conformally flat space, with torsion generated by a complex conformal factor $(\tilde{\Omega} = \bar{\Omega} \neq \Omega)$.

3) $\Sigma_T = \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'} + iB\omega^1\bar{\omega}^{0'} + iB\omega^0\bar{\omega}^{1'}$ is constant.

For a two-surface $\mathcal{J}$ the above three conditions are equivalent

Such $\mathcal{J}$ will be termed 'minimally contorted'

Proof

1) implies 2): We see from 1.14 that $\eth B = \Psi_2 - \bar{\Psi}_2 \neq 0$ and hence (Det Π) $\neq$ constant. We observe that under the (torsion inducing) conformal transformation

$$\varepsilon_{AB} \leftarrow \Omega\, \varepsilon_{AB} \qquad \varepsilon_{A'B'} \leftarrow \tilde{\Omega}\, \varepsilon_{A'B'} \qquad 2.12,\ 2.13$$

$\det\Pi$ transforms as

$$(\det\Pi') \leftarrow (\frac{\tilde{\Omega}}{\Omega})\ (\det\Pi) \qquad 2.14$$

Therefore if we put

$$\Omega = (\det\Pi)^{\frac{1}{2}}\ ,\quad \tilde{\Omega} = (\det\Pi)^{-\frac{1}{2}} \qquad 2.15,\ 2.16$$

$$\text{Þ}\Omega = \text{Þ}'\Omega = \text{Þ}\tilde{\Omega} = \text{Þ}'\tilde{\Omega} = 0 \qquad 2.17$$

then the transformed $\det\Pi$ will be constant. We note that originally

$$\eth\eth'\ \ln\ (\det\Pi') = \Psi_2 - \bar{\Psi}_2 \quad \in i\,\mathbb{R} \qquad 2.18$$

and thus $\Omega = e^{i\theta}$ for some $\theta \in \mathbb{R}$, $\tilde{\Omega} = \bar{\Omega}$.

We now consider $\mathcal{J}$ after the conformal transformation 2.15, 2.16. Since $\det\Pi$ is now constant $B = B' = X_2 - X_2 = 0$. (Φ_{11} and Λ are no longer real), and the conditions of Lemma 1 are satisfied and $\mathcal{J}$ with conformal torsion is locally embeddable in conformally flat space. Equivalently the original $\mathcal{J}$ is locally embeddable in conformally flat space with conformal torsion.

2) implies 3): From the introduction we see that after the conformal transformation $\varepsilon_{AB} \leftarrow \Omega\varepsilon_{AB}$, $\varepsilon_{A'B'} \leftarrow \bar{\Omega}\varepsilon_{A'B'}$, on flat space $\Sigma = \omega^{A-}\pi_A + \bar{\omega}^{-A'}\pi_{A'} - i(\tilde{\gamma}_{AA'}{}^{-}\gamma_{AA'})\omega^{A-A'}$ is constant, which is the expression given with $\Omega/\bar{\Omega} = (\det\Pi)$,

$$\text{Þ}\,(\Omega/\bar{\Omega}) = \text{Þ}'(\Omega/\bar{\Omega}) = 0.$$

3) implies 1): Since Σ_T is constant it follows that $(\omega^0, \omega^1, \pi_0, + iB'\bar{\omega}^{-1'}, \pi_{1'} + iB\bar{\omega}^{-0'})$ is a member of Π^*_α. Comparing equations again we see that $\bar{\sigma} = \bar{s}$, $\bar{\sigma}' = \bar{s}'$, and $B \neq 0$ implies $\Psi_2 \neq \bar{\Psi}_2$. □

Example 3. Consider a two-surface with $\sigma = \sigma' = 0$, $\Psi_2 \neq \bar{\Psi}_2$ (e.g. the Killing horizon of a stationary black-hole), then as before. $\bar{s} = \bar{s}' = 0$ and the conditions of Lemma 2 are satisfied and, as in the preceeding example, $\mathcal{J}$ must be conformal to a flat space metric sphere.

In general we have no reason to expect $\sigma\sigma' = ss'$, since in that case $\eth\eth'$ ln (Det $\bar{\Pi}$) = $\Psi_2 - \bar{\Psi}_2$, and this may not have any solutions.

3. Local embeddings in spaces conformal to complex Minkowski space

By embeddings in this section we mean only a partial connection embedding, not a metric embedding. We may construct the embedding geometrically (Tod 1983) by considering at each point of $\mathcal{J}$ the linear subspace of $\mathbb{T}^{\alpha}$ for which $\omega^{A} = 0$ (i.e. $Z^{\alpha} = (0,\pi_{A'})$). This subspace is two-dimensional and hence defines a point in complex Minkowski space (Penrose and MacCallum 1973), and thus we have an embedding of the original $\mathcal{J}$ in complex Minkowski space. It is not clear from this however how much of the connection and curvature is preserved by the embedding .

Lemma 3. If $\det \mathbb{T}$ is constant then $\mathcal{J}$ is embeddable in complex conformally flat space with no torsion.

Proof.

$\mathrm{Det}\,\mathbb{T}$ constant is equivalent to $B = B' = 0$. Consider equations 1.16 and 1.17 with $B = B' = 0$. Comparing with 2.1-2.4 we see that they are exactly the Bianchi identities for a conformally flat space in a similar manner to the real case, except that while for the connection and curvature on unprimed spinors $(\rho,\rho',\sigma,\sigma', X_1,X_2,X_3)$ we are equating quantities in the original space with those in the conformally flat space, for the connection and curvature on primed spinors we use instead $(\rho,\rho', \bar{s},\bar{s}', C, X_2, C')$, and it is in this sense that $\mathcal{J}$ is embeddable in complex conformally flat space.

Example 4. Consider $\mathcal{J}$ with $\sigma' = \Psi_2 - \bar{\Psi}_2 = 0$. From chapter one we know that $\bar{s}' = 0$ and hence $\eth B = 0$ and $\mathrm{Det}\,\mathbb{T}$ is constant.

The two-surface twistor equations and their duals, and the equations governing $\bar{s}$ are then

$$\eth' \omega^0 = 0 \qquad \eth \omega^1 = \sigma \omega^0 \qquad 3.1,\ 3.2$$

$$\eth \tilde{\omega}^{0'} = 0 \qquad \eth' \tilde{\omega}^{1'} = \bar{s} \tilde{\omega}^{0'} \qquad 3.3,\ 3.4$$

$$(\eth^2 - d')\bar{s} = (\eth'^2 - d)\sigma \qquad 3.5$$

$$\eth d = \eth'(X_2 + \rho\rho') \qquad \eth' d' = \eth(X_2 + \rho\rho') \qquad 3.6,\ 3.7$$

3.5 is soluble by noting that the commutator

$$[(\eth^2 - d'),\quad (\eth'^2 - d)]\alpha = 0 \qquad 3.8$$

For all α such that $p(\alpha) = q(\alpha)$.

Hence we see that

$$\bar{s} = (\eth'^2 - d)\alpha \qquad 3.9$$

$$\sigma = (\eth^2 - d')\alpha \qquad \text{for some } \alpha \qquad 3.10$$

Note that $\bar{s} = \bar{\sigma}$ iff $\alpha = \bar{\alpha}$.

It is instructive to solve equations 3.1, 3.2 and 3.3, 3.4.

We see that the four solutions to 3.1, 3.2 divide into two types

Type I $\qquad \omega^0 = 0 \quad , \quad \eth \omega^1 = 0$

Type II $\qquad \omega^0 \neq 0 \qquad \eth \omega^1 = \sigma \omega^0$

We see also that 1.12, 1.13 may be rewritten

$$(\eth^2 - B'\eth - d')\omega^0 = \omega^1(\eth' + B)\bar{s}' - \bar{s}'\,\eth'\omega^1 \qquad 3.11$$

$$(\eth'^2 - B\eth' - d)\omega^1 = \omega^0(\eth + B')\bar{s} - \bar{s}\,\eth\omega^0 \qquad 3.12$$

In these circumstances we have

$$(\eth^2 - d')\omega^0 = 0 \qquad 3.13$$

$$(\eth'^2 - d)\omega^1 = \omega^0\ \eth\bar{s} - \bar{s}\ \eth\omega^0 \qquad 3.14$$

and thus for Type I solutions $(\eth'^2 - d)\omega^1 = 0$ 3.15

and for Type II solutions, using 3.13

$$\omega^1 = (\eth\alpha)\omega^0 - \alpha\ \eth\omega^0 \qquad 3.16$$

in a manner identical to that found on $\mathscr{I}$ (Penrose 1982), where $d = d' = 0$.

Similarly we find

$$\tilde{\omega}^{0'} = \bar{\omega}^{0'} \qquad 3.17$$

$$\tilde{\omega}^{1'} = (\eth'\alpha)\bar{\omega}^{0'} - \alpha\ \eth'\bar{\omega}^{0'} \qquad 3.18$$

Note that there is a natural isomorphism between Type I solutions and the dual Type I solutions, but not for Type II. This is discussed further for the case of $\mathscr{I}$ in Shaw (1984a,b).

Lemma 4. Generically $\mathcal{J}$ is locally embeddable in complex conformally flat space with torsion resulting from a conformal transformation.

Proof.

In an identical manner to the real case we see that by making the torsion inducing conformal transformation

$$\varepsilon_{AB} \leftarrow \Omega\varepsilon_{AB} \qquad \varepsilon_{A'B'} \leftarrow \Omega^{-1}\varepsilon_{A'B'} \qquad 3.19$$

where $\Omega = (\mathrm{Det}\ \Pi)^{\frac{1}{2}}$ (a complex function), $(\mathbb{P}\Omega = \mathbb{P}'\Omega = 0)$ we may induce the preceeding case where $\mathrm{Det}\,\Pi$ is constant, and thus the resulting $\mathcal{J}$ is locally embeddable in complex conformally flat space (no torsion). Hence the original $\mathcal{J}$ is

locally embeddable in complex conformally flat space with torsion resulting from a conformal transformation.

Example 5. As an example of an $\mathcal{J}$ for which $\ln(\mathrm{Det}\,\Pi)$ is neither constant, real or imaginary we consider the leading order terms in an expression for two-surfaces near $\mathscr{I}$ (null infinity). We use the spin coefficient expressions as given in Shaw 1984b.

In what follows r is an affine parameter on the null ray approaching $\mathscr{I}$, and $\dot{}$ (dot) denotes a time derivative up $\mathscr{I}$. To the order we require

$$\rho = -\frac{1}{r} \qquad \sigma = \frac{\sigma^0}{r^2} \qquad 3.20,\ 3.21$$

$$\rho' = \frac{1}{r} \qquad \sigma' = -\frac{\dot{\bar{\sigma}}^0}{r} \qquad 3.22,\ 3.23$$

$$\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\sigma}^0\dot{\sigma}^0 - \sigma^0\dot{\bar{\sigma}}^0 + {}^{o}\eth'^2\sigma^0 - {}^{o}\eth^2\bar{\sigma}^0 \qquad 3.24$$

where $\Psi_2 = \frac{\Psi_2^0}{r^3}$ 3.25

and $\eth = \frac{1}{r}\,{}^{o}\eth - \frac{\sigma^0}{r^2}\,{}^{o}\eth' - q\frac{({}^{o}\eth'\sigma^0)}{r^2}$ 3.26

where ${}^{o}\eth$ is the $\eth$ for a metric sphere ($[{}^{o}\eth, {}^{o}\eth'] = -(p-q)$).

Substituting the above into 1.20, 1.27 we see (writing $\eth$ for ${}^{o}\eth$)

$$\eth B = \frac{1}{r^3}\left(\eth'^2\sigma^0 - \eth^2\bar{\sigma}^0 - \sigma^0\dot{\bar{\sigma}}^0\right) - \bar{s}\bar{s}' \qquad 3.27$$

$$\left(\frac{1}{r^2}\eth^2 - d'\right)\bar{s} = \left(\frac{\eth'^2}{r^2} - d\right)\frac{\sigma^0}{r^2} \qquad 3.28$$

$$\left(\frac{1}{r^2}\eth'^2 - d\right)\bar{s}' = \left(\frac{\eth^2}{r^2} - d'\right)\left(-\frac{\dot{\bar{\sigma}}^0}{r^2}\right) \qquad 3.29$$

$$\eth d = 0(r^{-4}) \qquad \eth' d' = 0(r^{-4}) \qquad 3.30$$

3.30 tells us that d,d' are $0(r^{-3})$, and hence we may put

$\bar{s} = \frac{\bar{s}^0}{r^2}$, $\bar{s}' = \frac{\bar{s}'^0}{r}$ and obtain (to leading order),

$$r^3 \eth B = \eth'^2 \sigma^0 - \eth^2 \bar{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0 - \bar{s}^0 \bar{s}'^0 \qquad 3.31$$

$$\eth^2 \bar{s}^0 = \eth'^2 \sigma^0 \qquad 3.32$$

$$\eth'^2 \bar{s}'^0 = \eth^2 (-\dot{\bar{\sigma}}^0) \qquad 3.33$$

and thus putting $\sigma^0 = \eth^2 \alpha$ we have

$$\bar{s}^0 = \eth'^2 \alpha \qquad \bar{s}'^0 = -\eth^2 \dot{\bar{\alpha}} \qquad 3.34,\ 3.35$$

and

$$r^3 \eth B = \eth^2 \eth'^2 (\alpha - \bar{\alpha}) - (\eth^2 \alpha)(\eth'^2 \dot{\bar{\alpha}}) + (\eth'^2 \alpha)(\eth^2 \dot{\bar{\alpha}}) \qquad 3.36$$

The right hand side of 3.36 is in general complex and we may confirm a solution since clearly

$$\eth^2 \eth'^2 (\bar{\alpha} - \alpha) = \eth \eth' (\eth \eth' + 2)(\bar{\alpha} - \alpha) . \qquad 3.37$$

and

$$\int_J ((\eth^2 \alpha)(\eth'^2 \dot{\bar{\alpha}}) - (\eth'^2 \alpha)(\eth^2 \dot{\bar{\alpha}}))\, dJ = \int_J \alpha (\eth^2 \eth'^2 - \eth'^2 \eth^2) \dot{\bar{\alpha}}\, dJ = 0 . \qquad 3.38$$

Since $\eth B = \eth \eth' \ln(\mathrm{Det}\, \Pi)$, $\mathrm{Det}\, \Pi$ will be complex, and it will have constant modulus if either the shear is 'electric' ($\alpha = \bar{\alpha}$), in which case the conditions of Lemma 2 are satisfied, or $\dot{\alpha} = 0$ (stationary case).

Although $\mathrm{Det}\, \Pi$ generically has no special form we shall give two examples of cases where it does have a special form.

Example 6. $\sigma' = 0$, $\Psi_2 \neq \bar{\Psi}_2$, $|\mathrm{Det}\, \Pi|$ = constant.

This is clearly similar to Example 3, and again $\bar{s}' = 0$ but here

$$\eth\eth' \ln(\mathrm{Det}\,\Pi) = \Psi_2 - \bar{\Psi}_2 \qquad 3.39$$

hence $\ln(\mathrm{Det}\,\Pi)$ is imaginary and $|\mathrm{Det}\,\Pi|$ is constant. Again we write down the equations governing the dual curvature terms.

$$(\eth(\eth+B') - d')\bar{s} = ((\eth'-B)\eth' - d)\sigma \qquad 3.40$$

$$\eth d = (\eth'-B)(X_2+\rho\rho') \qquad 3.41$$

$$\eth' d' = (\eth-B')(X_2+\rho\rho') \qquad 3.42$$

and one finds similarly to before that the commutator

$$[\eth(\eth+B') - d',\ (\eth'-B)\eth' - d]\alpha = 0 \qquad 3.43$$

if $p(\alpha) = q(\alpha)$. This may be seen either by direct computation, or by using 3.8 and making a torsion inducing conformal transformation.

Thus
$$\bar{s} = [(\eth'-B)\eth' - d]\alpha \qquad 3.44$$

$$\sigma = [\eth(\eth+B') - d']\alpha \qquad 3.45$$

for some α, and

$$(\eth^2 - B\eth - d')\omega^0 = 0 \qquad 3.46$$

$$(\eth'^2 - B\eth' - d)\omega^1 = \omega^0(\eth+B')\bar{s} - \bar{s}\,\eth\,\omega^0 \qquad 3.47$$

Using 3.46 we may solve the twistor equations and their duals and find

$$\omega^1 = \omega^0(\eth+B')\alpha - \alpha\,\eth\omega^0 + \xi^1 \qquad 3.48$$

$$\tilde{\omega}^{1'} = \bar{\omega}^{0'}(\eth'+B)\alpha - \alpha\,\eth'\bar{\omega}^0 + \bar{\xi}^{1'} \qquad 3.49$$

where $\eth'\xi^{1'} = \eth'\omega^0 = 0$, $\qquad \tilde{\omega}^{0'} = \bar{\omega}^{0'}$ $\qquad$ 3.50, 3.51

Example 7. Static $\mathcal{J}$ in a static space-time. Det $\pi \in \mathbb{R}$

In this example we assume there exists a timelike, normal hypersurface-orthogonal Killing vector t_a.

Applying commutators we have $R_{ab[cd}t_{e]}t^b = 0$ 3.52

If we choose a boost such that

$$t_a = \frac{|t|}{\sqrt{2}}(\ell_a + n_a) \qquad 3.53$$

then we find

$$\rho = -\rho' \qquad \sigma = -\bar{\sigma}' \qquad 3.54,\ 3.55$$

$$\Psi_2 = \bar{\Psi}_2 \qquad X_1 = -\bar{X}_3 \qquad 3.56,\ 3.57$$

We see that $\sigma\sigma' - \bar{\sigma}\bar{\sigma}' = 0$ 3.58

thus $[\eth, \eth']\alpha = (p-q)(\rho\rho' - \sigma\sigma' + X_2)\alpha$ 3.59

We recall that

$$\eth = \delta + \frac{1}{2}(p+q)(\beta - \bar{\beta}') + \frac{1}{2}(p-q)(\beta + \bar{\beta}') \qquad 3.60$$

$$= \eth_s + \frac{1}{2}(p+q)(\beta - \bar{\beta}') \qquad 3.61$$

Substituting 3.61 into the curvature equations

$$\eth'\sigma - \eth\rho = X_1 \qquad \eth\bar{\sigma}' - \eth'\rho' = X_3 \qquad 3.62,\ 3.63$$

we see that $\beta - \bar{\beta}' = 0$ 3.64

Now the two-surface twistor equations may be written

$$\eth'\omega^0 = -\bar{\sigma}\omega^1 \qquad \eth\omega^1 = \sigma\omega^0 \qquad 3.65,\ 3.66$$

and thus using 3.64 we find that if

$$(\omega^0, \omega^1, \pi_{0'}, \pi_{1'}) \in \mathbb{T}^\alpha \qquad 3.67$$

then $$(-\bar{\omega}^{1'}, \bar{\omega}^{0'}, \bar{\pi}_1, -\bar{\pi}_0) \in \overline{\mathbb{T}}^\alpha \qquad 3.68$$

Let $Z^\alpha \in \mathbb{T}^\alpha$ be a basis element of $\mathbb{T}^\alpha$ and V_α the corresponding dual element of $\mathbb{T}^*_\alpha$, and thus if $\overline{Z^\alpha} \in \overline{\mathbb{T}}^\alpha$ as we have seen then we must also have $\overline{V}_\alpha \in \overline{\mathbb{T}}^*_\alpha$, and thus we find

$$s = -\bar{s}' \qquad 3.69$$

Substituting into 1.14 we see that

$$\eth B = \eth\eth' \ln(\mathrm{Det}\, \mathbb{T}) = s\bar{s} - \sigma\bar{\sigma} \qquad 3.70$$

and hence $\mathrm{Det}\, \mathbb{T} \in \mathbb{R}$

We recall the definition of $A_{\alpha\beta}$ (the angular-momentum twistor) for uncontorted two-surfaces and the suggested modification for contorted two-surfaces.

$$A_{\alpha\beta} Z^\alpha_1 Z^\beta_2 = \frac{-i}{4\pi G} \int_J (\mathrm{Det}\, \mathbb{T})^n (X_1\, \omega^0_1 \omega^0_2 + X_2 (\omega^0_1 \omega^1_2 + \omega^1_1 \omega^0_2) + X_3\, \omega^1_1 \omega^1_2)\, d\mathcal{J} \qquad 3.71$$

where for the original definition $n = 0$, the modified version requires $n = 1$ (Penrose 1982, Penrose and Rindler 1985).

Since $X_1 = -\bar{X}_3$ and $X_2 = \bar{X}_2$ we see that if

$$A_{\alpha\beta} Z^\alpha_1 Z^\beta_2 = x \qquad 3.72$$

then $$A_{\alpha\beta} W^\alpha_3 W^\beta_4 = \bar{x} \qquad 3.73$$

where $W^\alpha_3 = \overline{Z^\alpha_1}$, $W^\alpha_4 = \overline{Z^\alpha_2}$. We see that $A_{\alpha\beta}$ has four complex components and two real if we take as a basis Z^α_1, Z^α_2, W^α_3 and W^α_4.

If there is in addition a tangential Killing vector (say $\frac{\partial}{\partial\phi}$) then the twistors Z^α_1, Z^α_2 will have ϕ dependence of the

form $e^{im\phi}$ for some m; W_3^{α}, W_4^{α} dependence of the form $e^{-im\phi}$,
and then only four real numbers in $A_{\alpha\beta}$ survive (a four-momentum).

4. Dual Angular Momentum

So far we have hardly touched on the subject of the angular-momentum twistor $A_{\alpha\beta}$ (actually an element of $\mathbb{T}^* \otimes \mathbb{T}^*$ The question of what definition to use for $A_{\alpha\beta}$ for contorted two-surfaces is a matter of current debate (Penrose 1984, Penrose and Rindler 1985). It seems reasonable to try and define a suitable dual angular-momentum twistor $\tilde{A}^{\alpha\beta}$ (an element of $\mathbb{T} \otimes \mathbb{T}$), and one then has a quantity in units of mass

$$\mu^2 = -\frac{1}{2} A_{\alpha\beta}\tilde{A}^{\alpha\beta} \tag{4.1}$$

The uncontorted definition of $A_{\alpha\beta}$ is

$$A_{\alpha\beta}\, Z_1^{\alpha} Z_2^{\beta} = \frac{-i}{4\pi G} \int_{\mathcal{I}} (X_1\, \omega_1^0 \omega_2^0 + X_2(\omega_1^0 \omega_2^1 + \omega_1^1 \omega_2^0) + X_3\, \omega_1^1 \omega_2^1)\, d\mathcal{I} \tag{4.2}$$

$$= \frac{-i}{4\pi G} \int_{\mathcal{I}} (\pi_1^0 \pi_2^{1'} + \pi_1^1 \pi_2^{0'})\, d\mathcal{I}$$

It seems reasonable to define (4.3)

$$\tilde{A}^{\alpha\beta}\tilde{Z}_1^{\alpha}\tilde{Z}_2^{\beta} = \frac{i}{4\pi G} \int_{\mathcal{I}} (\det\mathbb{T})(C\, \tilde{\omega}_1^{0'} \tilde{\omega}_2^{0'} + X_2(\tilde{\omega}_1^{0'} \tilde{\omega}_2^{1'} + \tilde{\omega}_1^{1'} \tilde{\omega}_2^{0'}) + C'\tilde{\omega}_1^{1'} \tilde{\omega}_2^{1'})\, d\mathcal{I}$$

$$= \frac{i}{4\pi G} \int_{\mathcal{I}} (\det\mathbb{T})(\tilde{\pi}_1^0 \tilde{\pi}_2^1 + \tilde{\pi}_1^1 \tilde{\pi}_2^0)\, d\mathcal{I} \tag{4.4}$$

For the two-surfaces considered in Example 6 $\tilde{A}^{\alpha\beta}$ has ten real components, just as $A_{\alpha\beta}$.

The obvious two-surface to test any definition on is one at $\mathscr{I}$, however problems arise.

At $\mathscr{I}$ one has $\sigma' = 0$ (to leading order), and the angular momentum twistor may be split into three. (see Example 4).

i) both Z^α type I 'anti-angular-momentum' $\frac{1}{r} G_{A'B'}$

ii) one Z^α type I, one type II 'Bondi momentum' P_a

iii) both Z^α type II 'angular-momentum' M_{AB}

The anti-angular-momentum vanishes as $\frac{1}{r}$ and one finds that with any reasonable choice of norm (Penrose 1982, Shaw 1984a,b)

$$-\tfrac{1}{2}|A_{\alpha\beta}|^2 = P_a P^a$$

P_a and M_{AB} are finite on $\mathscr{I}$ because the curvature components Ψ_1, Ψ_2, Ψ_3 scale respectively as r^{-4}, r^{-3}, r^{-2}, however the dual curvature components C, Ψ_2, C' scale as $\underline{r^{-3}}$, r^{-3}, r^{-2} and when we define $\tilde{G}^{AB}$, $\tilde{P}^a$, $\tilde{M}^{A'B'}$ similarly we find $\tilde{M}^{AB}$ diverges as r on $\mathscr{I}$. The inner product with $A_{\alpha\beta}$ will however be finite and one finds

$$-\tfrac{1}{2}A_{\alpha\beta}\tilde{A}^{\alpha\beta} = P_a \tilde{P}^a - \frac{1}{2r} G_{A'B'} \tilde{M}^{A'B'}$$

This has so far not been calculated, and depends not only on the 'correct' definition of $\tilde{A}^{\alpha\beta}$, but also on which definition of $A_{\alpha\beta}$ to use, as this affects G_{AB}.

Concluding remarks

Although in general there are many obstacles to understanding (or even defining) quasi-local mass and angular momentum the two-surface twistor space and its dual are both well defined and reflect aspects of the surrounding space. Little is known however about how symmetries neither tangential or orthogonal to the two-surface are reflected in the twistor space structure. When the symmetries are reflected in the two-surface in particular stationary axi-symmetry or spherical symmetry (Tod 1983), then the two surfaces appear to be at worst minimally contorted, and hence possess angular-momentum twistors amenable to calculation and interpretation.

In addition the question of embeddings in other spaces, and their usefulness - such as conformally vacuum half-flat spaces - has not yet been investigated.

Addendum

I have discovered some new results since writing the preceeding material, in particular concerning two-surfaces with a tangential (rotational) Killing vector. One finds in these cases , after aligning the imaginary part of m_a along the direction of the Killing vector, that

$$\bar{s} = \sigma \qquad \bar{s}' = \sigma'$$

and thus for any choice of phase

$$\gamma\gamma' \ln \mathrm{Det}\,\Pi = \Psi_2 - \bar{\Psi}_2 + \bar{\sigma}\bar{\sigma}' - \sigma\sigma' = 2i\,\mathrm{Im}\,K$$

implying that $|\mathrm{Det}\,\Pi|$ is constant. If the Killing vector is hypersurface orthogonal one has in addition that $\Psi_2 = \bar{\Psi}_2$ and for the above choice of phase $\sigma = \bar{\sigma}$, $\sigma' = \bar{\sigma}'$. In this case $\mathcal{J}$ is uncontorted and also only four real components remain in $A_{\alpha\beta}$.

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