10/Aug/04:

The dynamics of city formation: finance and governance*

J. Vernon Henderson
Brown University

Anthony J. Venables
LSE and CEPR

Abstract:
This paper examines city formation in a country whose urban population is growing steadily over time, with new cities required to accommodate this growth. In contrast to most of the literature there is immobility of housing and urban infrastructure, and investment in these assets is taken on the basis of forward-looking behavior. In the presence of these fixed assets cities form sequentially, without the population swings in existing cities that arise in current models. Equilibrium city size, absent government, may be larger or smaller than is efficient, depending on how urban externalities vary with population. Efficient formation of cities involves local government borrowing to finance development. The institutions governing land markets, leases, local taxation, and local borrowing and debt affect the efficiency of outcomes. The paper explores the effects of different fiscal constraints, and shows that borrowing constraints lead cities to be larger than is efficient.

JEL classification: R1, R5, O18, H7
Keywords: Urbanization, city size, urban developers, city governance.

* Henderson thanks the Leverhulme foundation for support as a visiting professor at the LSE. Venables' work is supported by the UK ESRC funded Centre for Economic Performance at LSE. The authors thank G. Duranton, E. Rossi-Hansberg, J. Rappaport, W. Strange, and seminar participants at the CEPR conference on "European Integration and Technological Change" for helpful comments.

J. V. Henderson
Dept. Of Economics
Brown University
Providence RI
USA 02912

J_Henderson@brown.edu
http://www.econ.brown.edu/faculty/henderson/

A. J. Venables
Dept of Economics
London School of Economics
Houghton Street, London
UK WC2A 2AE

a.j.venables@lse.ac.uk
http://econ.lse.ac.uk/staff/ajv/

I: Introduction

Understanding city formation and the financing requirements of cities is critical to effective policy formulation in developing countries that face rapid urbanization. The rapid growth of urban populations in developing countries is well known, but what is less well known is the growth in the number of cities. Between 1960 and 2000 the number of metro areas over 100,000 in the developed world grew by 40%, while the number in the rest of the world grew by 185% – i.e. almost tripled (Henderson and Wang, 2004). Moreover the UN's projected 2 billion person increase in the world urban population over the next 45 years ensures this growth in city numbers will continue. How do we start to think about whether the proliferation of cities in developing countries is following an efficient development path, and how policies may assist or constrain achievement of better outcomes?

In thinking about the development of cities we start with two fundamental premises. The first is that city formation requires investment in non-malleable, immobile capital, in the form of public infrastructure, housing, and business capital. Owner-occupied housing capital is immobile and long lived, depreciating at a gross rate well under 1% a year and a net rate after maintenance of almost zero. Urbanization also involves heavy investment in roads, water mains, sewers and the like that are immobile and depreciate slowly. The second premise is that, in developing countries, a key local public finance need is for cities to tax and to borrow, and/or to use central government funds to finance infrastructure investments and subsidize the development of industrial parks so as to attract businesses (World Bank, 2000).

Why does immobility of capital matter to the analysis of city formation? We consider a context in which the urban population of a country is growing steadily, with ongoing rural-urban migration as resources shift out of agriculture into urban industrial and service production. In models with perfect mobility of resources (Henderson, 1974 and Anas 1992), initial urbanization is characterized by huge swings in population of initial cities. In an economy with just one final output good and hence one type of city, urbanization proceeds by the first city growing until at some point a second city forms, with the timing depending on the details of the city formation mechanism and institutions. Regardless of that timing, when a second city forms the first city loses half its population who migrate to that second city. Then the first city resumes growth and the second city grows in parallel until a third city forms, at which point both existing cities lose 1/3 of their population who migrate to this new third city. And so on. That process is implausible. It is not just that it requires huge, costly population movements; with non-malleable and immobile urban infrastructure it also requires cities to go through periods of intensive investment followed by under-occupation. Moreover, the data do not support the idea. From 1900-1990 when the USA moved from being 40% urban to 75% urban, Black and Henderson (2003) show that almost no cities experienced population losses between decades. A similar statement holds for world cities from 1960-2000.

This paper will start at the opposite extreme and assume that the sunk capital costs of urban infrastructure are sufficiently high that urbanization occurs without population swings. In smaller developing countries such as Bolivia, Cameroon, Dominican Republic, Ecuador, South Korea, Portugal, or Yemen, the world wide city data set used in Henderson and Wang (2004) covering the last forty years suggests a pattern where one metro area absorbs most initial rural-urban migration then slows its growth as a second city starts to absorb a disproportionate share of migrants, before it too slows and a third city becomes the target of migrants. Interpretation of the data must be done carefully because ongoing technological change tends to increase all city sizes and because in larger economies, with many types of cities, there is parallel growth of a diverse set of cities which then slows as a second set of cities becomes the target of migrants.

We model an economy with a single final good, in which each new city starts off small and grows through rural to urban migration until the next new city becomes the target of migrants. Given immobility of capital, when a new city forms the prices of fixed assets in old cities adjust in order to maintain occupancy in both new and old cities. Forward-looking agents anticipate income streams that will be earned in new and in old cities and make investment decisions accordingly. This analysis of how asset prices vary within existing cities as a new city grows will be a fundamental insight of the paper.

We start the paper by analyzing the benchmark case of socially efficient city formation in this dynamic context, and show that efficient city size is larger than in models in which resources are perfectly mobile. We then look at city formation in which there are no ‘large agents’ – all individuals are price-takers in all markets. The simple coordination failures that arise in static models do not occur, because agents are forward looking, correctly anticipating population flows and housing market conditions in new and old cities. However, small agents do not internalize city externalities, so equilibrium city size may be larger or smaller than socially optimal depending, in an intuitive manner, on the way in which externalities vary with city size. We then turn to city formation with large agents: private developers or public city governments. Such cities will borrow in order to attract migrants during the period in which urban scale economies are not fully developed. If cities face borrowing and/or tax constraints then formation of new cities is inhibited and cities will be larger than is socially optimal. Timely formation also requires development of institutions governing land markets, leases, and taxation. We articulate the key policy and institutional elements needed for efficient outcomes, and the effect on city sizes and real incomes if such elements are not in place.

In terms of relevant literature there are growth models with city formation (e.g. Black and Henderson 1999 and Rossi-Hansberg and Wright 2004), but they assume that cities form with perfect mobility of all resources and without a local public sector that must borrow to finance development. Incorporating immobility and financing considerations requires a different approach. The effect of having durable, immobile capital on individual city growth has been tackled in Brueckner (2000). Glaeser and Gyourko (2003) analyse urban growth and decline in a stochastic model with durable housing. As in this paper, the role of forward-looking agents, in particular competitive housing builders, and the role of variation in asset prices across cities, are central. However, these papers do not examine the subject studied in this paper – new city formation as the urban sector grows. This paper develops a model of city formation under immobility of capital, building on Venables (2004) who illustrates that population immobility will affect the city formation process. There is a complementary paper on city formation with durable capital by Helsley and Strange (1994) in which large land developers form cities simultaneously in a static context, using durable capital as a strategic commitment device. We have a dynamic context and for much of the paper there are no large land developers; but the Helsley and Strange paper points to interesting extensions.

2. The model

In order to isolate the key elements in the urbanization process, we make four simplifying assumptions. First we assume a small open economy where agents can borrow and lend at a fixed interest rate δ in world capital markets. We do not embed the process in a closed economy model with capital accumulation and an endogenous interest rate, although in section 7 we will examine the effects of capital market imperfections. Second, we assume that the urban sector grows in population by a constant amount, ν, each instant, as if there were a steady stream of population out of agriculture and into the urban sector. Constancy of this rate is not critical to the concepts developed in the paper. For example, having the migration rate to cities respond to rural-urban income differences would affect the rate of population flow into cities and affect our precise calculations of the rate of cities' population growth. But it would not affect the process of how new cities form or the analysis of the effect of borrowing constraints and other policies and institutions. Third, we abstract from ongoing technological change which would tend to increase equilibrium and efficient city sizes over time. This is an easy extension, but does not affect the basic principles developed in the paper, although it is critical to any interpretation of data on city growth and formation. Finally we assume that there is just one final output good, ruling out a situation with multiple types of cities where each specializes in production of a different final good(s). An interesting extension would be to have multiple types of cities, where there could be simultaneously formation and growth of cities, each of different types. Again, the principles developed here would apply in that situation.

We start with a description of a city in the economy, setting out both the urban agglomeration benefits and the urban diseconomies associated with city population growth. Cities form on a "flat featureless plain" with an unexhausted supply of identical city sites, and land is available for urban development at zero opportunity cost. There are $n(t)$ workers in a particular city at date t and we define a worker's real income, $y(t)$, as

$$y(t) = x(n(t)) - cn(t)^{\gamma - 1} + s(t). \tag{1}$$

As we will see, the first term is the worker's output, the second is land rent plus commuting costs, and the final term is any subsidy (or, if negative, a tax) that the worker receives. This real income expression contains all the components of earnings, subsidies, and expenditures that vary directly with city size. This sum is then available to be spent on final consumption and on housing. We discuss each of the components of (1) in turn, as well as housing.

Production: Firms in a city produce a single homogenous good with internal constant returns to scale but subject to citywide scale externalities. With constant returns we simply assume that each worker is also a firm. The city work force is n and per worker output is $x(n)$, with $x'(n) > 0$ and bounded away from infinity at $n = 0$, and $x''(n) < 0$. This represents urban scale economies where per worker output rises at a decreasing rate with city population, as workers benefit from interaction with each other.[1] Output per worker may continue to rise indefinitely with n, or may pass a turning point as congestion sets in.

Commuting and land rent: The second term on the right-hand side of equation (1) is land rent plus commuting costs in a city of size $n(t)$. It generalizes the standard approach in the urban systems literature (Duranton and Puga, 2004). All production in a city takes place in the city's central business district (CBD), to which all workers commute from residential lots of fixed size. Free mobility of workers requires all workers in the city to have the same disposable income after rent and commuting costs are paid. Thus, there is a land rent gradient such that, at all points within the city, land rent plus commuting costs per person equal the commuting costs of the edge worker whose rent is zero. Edge commuting costs take the form $cn(t)^{\gamma-1}$ (the term in (1)) which is derived, along with expressions for rent and commuting costs, in Appendix 1. The parameter c measures the level of commuting costs and γ combines relevant information on the shape and commuting technology of the city. If commuting costs per unit distance are constant then, in a linear city $\gamma = 2$, and in a circular or pie shaped city $\gamma = 3/2$. Our generalization encompasses these cases, and also allows commuting costs to be an iso-elastic function of distance, as shown in Appendix 1. We require merely that $\gamma > 1$, so average commuting costs, as well as average land rent, rise with city population. Integrating over the commuting costs paid by people at each distance from the centre and over their rents gives the functions $TC(n)$ and $TR(n)$ reported in Table 1.

Housing: A plot of city land can be occupied by a worker only after a capital expenditure of H has been incurred. This represents the construction of a house, although it could also include other aspects of infrastructure such as roads and water supply. The housing construction, sale, and rental markets are all assumed to be perfectly competitive, and the spot market rent of a house at time t is denoted $h(t)$, this paid in addition to the rent on land. Throughout the paper we assume that the two rent components are separable; housing rent, $h(t)$, is paid separately from land rent which is determined by the city land rent gradient. We also assume that the two sources of income can be taxed separately. Thus, house builders may rent land from land owners with an infinite lease and pay land rents according to the perfectly foreseen city land rent gradient. Alternatively builders could initially buy the land from the land owners, capitalizing the land rents. And a model with owner-occupancy where residents buy land and housing would yield equivalent results. Land owners are people outside the urban sector, although the same results on city formation hold if they are nationwide Arrow-Debreu share holders in the land of all cities.[2]

Subsidies and taxes: The final term in equation (1) denotes a per worker subsidy at rate $s(t)$ (tax if negative) to workers in the city at date t. We will investigate use of this under different city governance structures. We note that since workers are also firms, the subsidy could be viewed as going to firms, a common element of city finance.

Table 1 summarizes some key relationships in a city with population n. The left-hand block of the table reports the basic relationships between commuting costs and land rent derived in Appendix 1. The right-hand block defines relationships which we will use repeatedly through the paper. Total surplus, $TS(n)$, is the output minus commuting costs of a city of size n; notice that this is defined without including housing costs. Average surplus $AS(n)$ and marginal surplus $MS(n)$ follow from this in the obvious way. $LS(n)$ is the surplus per worker net of average land rent paid to landowners, $LS(n) \equiv AS(n) - AR(n)$; it follows that $y(t) = LS(n(t)) + s(t)$ (from equation 1). Finally, $EX(n) = MS(n) - LS(n)$ is the production externality associated with adding a worker-firm to the city: it is the increase in output of all other workers in the city when a further worker is employed.

Table 1: Commuting costs, land rents, and surplus.

edge commuting cost = land rent+commuting cost	$cn^{\gamma-1}$	Total surplus: $TS(n)$	$nx(n) - n^{\gamma}(c/\gamma)$
Total commuting costs: $TC(n)$	$n^{\gamma}(c/\gamma)$	Average surplus: $AS(n)$	$x(n) - n^{\gamma-1}(c/\gamma)$
Total land rent: $TR(n)$	$n^{\gamma}c(\gamma-1)/\gamma$	Marginal surplus: $MS(n)$	$x(n) + nx'(n) - n^{\gamma-1}c$
Average land rent: $AR(n)$	$n^{\gamma-1}c(\gamma-1)/\gamma$	Labour surplus $LS(n) \equiv AS(n) - AR(n)$	$x(n) - n^{\gamma-1}c$
		Externality: $EX(n) = MS(n) - LS(n)$	$nx'(n)$

The shapes of these functions are critical in the following analysis, and we now state assumptions that are sufficient for the propositions that follow.

A1: $LS(n)$ is strictly concave in n with unique interior maximum at n_L, $x'(n_L) = n_L^{\gamma-2}c(\gamma-1)$, and such that as $n \to \infty$, $LS(n) < LS(0)$.

It follows that $AS(n)$ is strictly concave (since $\gamma > 1$), but we also assume:

A2: $AS(n)$ has interior maximum at n_A, $x'(n_A) = n_A^{\gamma-2}c(\gamma-1)/\gamma$.

A1 and A2, together with $\gamma > 1$ and our assumptions on $x(n)$ imply that: **(i)** $n_A > n_L$. **(ii)** $MS(0) = AS(0)$. **(iii)** $MS(n)$ intersects $AS(n)$ from above at n_A. $MS(n)$ is initially increasing and is decreasing for all $n > n_A$, since after n_A, $MS'(n) = 2AS'(n) + nAS''(n) < 0$; however, it is also convenient to assume explicitly that:

A3: Starting from $n = 0$, $MS(n)$ is strictly increasing in n until it peaks, after which it is strictly decreasing.

These relationships are illustrated in Figure 1. The average surplus curve is strictly concave with maximum at point n_A. Marginal surplus and average surplus start at the same point, then marginal lies above average until they intersect at n_A. Surplus net of land rents, $LS(n)$, lies below $AS(n)$, with maximum at $n_L < n_A$. (This and other figures use functional forms that are described in section 7.3; until then the figures simply illustrate general properties of the model).

Our analysis also requires an assumption that the magnitude of housing construction costs, H, be large enough to ensure that housing is never left empty; this prevents jumps in city size. The issue arises in different contexts and here we state a condition sufficient to apply in all cases:

A4: $\delta H \geq AS(n_A) - AS(0)$.

The interest charge on housing per worker is therefore at least as large as the difference between the maximum level of surplus per worker and its level in a new city with zero initial population. This implies housing costs are high relative to net agglomeration benefits of cities, and the aptness of the assumption could be debated empirically. However we note that assumption A4 is an all-purpose sufficient condition; in many of the situations we examine much lower relative magnitudes of housing costs are necessary.

With these ingredients in place we look next at the social welfare maximum, then at the competitive equilibrium without city government (section 4) before turning to analysis of city government (sections 5 - 7).

3. Socially optimal city formation

At date 0 new urban population (arriving at ν per unit time) starts to flow into one or more new cities. How should this population be allocated across cities, and how large should cities consequently be?

City formation can be sequenced in a number of ways. One possibility is that cities grow sequentially, so at any point in time there is only one growing city which, t periods into its growth, has population $n(t) = \nu \cdot t$. A second possibility is that population goes temporarily into existing cities and then, at some date, a new city forms with a jump to some discrete size. This has the advantage of delivering returns to scale instantaneously, as the new city can jump to efficient size n_A giving the maximum value of *AS*; old cities also have population approximately equal to n_A. But the cost of jumps is that, after residents have left to form the new city, some housing is left empty in old cities. This cost depends on the magnitude of the sunk housing costs, *H*, and in Appendix 2 we prove that assumption A4 is sufficient to ensure that jumps of any type, large or small, are inefficient. A third possibility is that cities develop in parallel, with several cities growing simultaneously. Such an outcome is inefficient, as gains from increasing returns are slowed, and this too is demonstrated in Appendix 2. We therefore focus on the first case in which population enters each city in turn, without jumps, until each city's growth period is complete.

The new city is born at a date $t = 0$, grows at rate ν for $t \in [0, T]$ with population $n(t) = \nu t$, and is then stationary at final size νT. Population then flows into another new city for *T* periods, and so on. The problem is to choose *T* to maximize the present value of the total surplus earned in all future cities, net of house construction costs. In general we are going to write expressions in a form that combines production, commuting, and rent into the functions *TS*, *AS*, and similar given in Table 1. For this first problem we also write out the elements of the problem in full, but then immediately combine terms into *TS*, *AS*, and *MS* expressions.

The present value of the total surplus in all future cities from time 0 can be expressed as

$$\begin{aligned}\Omega &= \frac{1}{1-e^{-\delta T}}\left[\int_0^T (\nu t x(\nu t) - (\nu t)^\gamma c/\gamma) e^{-\delta t} dt + \frac{(\nu T x(\nu T) - (\nu T)^\gamma c/\gamma) e^{-\delta T}}{\delta}\right] - \int_0^\infty \nu t H e^{-\delta t} dt \\ &= \frac{1}{1-e^{-\delta T}}\left[\int_0^T TS(\nu t) e^{-\delta t} dt + \frac{TS(\nu T) e^{-\delta T}}{\delta}\right] - \int_0^\infty \nu t H e^{-\delta t} dt.\end{aligned} \qquad (2)$$

The term in square brackets is the present value of the output minus commuting costs (total surplus, see Table 1) in a city founded at date zero. While a city is growing the surplus at any instant is $TS(\nu t)$. At date *T* the city stops growing and has a surplus at every instant from then on of $TS(\nu T)$. At that point another city is founded and the process repeats itself. The term multiplying the square brackets is the sum of the geometric series $1 + e^{-\delta T} + e^{-\delta 2T}$ ...; it therefore sums the present value of all such future cities. The final term is total housing costs. These do not depend on *T* because, as long as all houses remain occupied at all dates, the total number of houses built is simply equal to the total inflow of new workers, regardless of where they live.

Optimization with respect to T gives first order condition,

$$\frac{[1-e^{-\delta T}]^2}{\delta e^{-\delta T}}\,\frac{d\Omega}{dT} = -\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right] + MS(\nu T)\,\frac{[1-e^{-\delta T}]\nu}{\delta^2} = 0. \qquad (3)$$

Integrating by parts, this first order condition can be expressed as

$$\int_0^T [MS(\nu t) - MS(\nu T)]\,e^{-\delta t}dt = 0 \qquad (4)$$

(see Appendix 3, equation A9).

We denote the welfare maximizing population as n_{opt} and the corresponding solution of the first order condition as T_{opt} ($\equiv n_{opt}/\nu$). First order condition (4) is readily interpreted. It says that city size must be chosen so that the present value of adding a worker to a new city, $MS(\nu t)$, equals the present value over the same time frame of the marginal surplus from adding the worker to an existing city, $MS(\nu T)$. We can now state our first proposition:

Proposition 1. There exists a unique optimal city size. This city size is larger than that which maximizes surplus per worker, i.e. $T_{opt} > T_A$ ($\equiv n_A/\nu$).

Proof: In the static model welfare is maximized at the peak of the AS schedule, T_A ($\equiv n_A/\nu$) where $AS(\nu T_A) = MS(\nu T_A)$. Writing $TS(\nu t) = \nu t AS(\nu t)$ and using an integral given in Appendix 3 (equation A10) first order condition (4) can be written as

$$\int_0^T [MS(\nu T) - AS(\nu t)]\nu t e^{-\delta t}dt + [MS(\nu T) - AS(\nu T)]\nu T e^{-\delta T}/\delta = 0. \qquad (4')$$

To prove that $T_{opt} > T_A$, suppose not. If $T_{opt} < T_A$, then from assumption A2 and its implications, both terms in square brackets in (4') are greater than zero (see Figure 1). At $T = T_A$ the first term is strictly positive and the second zero. At $T > T_A$ the second term is negative and strictly decreasing. The first term is strictly decreasing and eventually becomes negative given that $MS(\nu T_A)$ declines continuously for $T > T_A$. Thus equation (4') has a unique solution at $T_{opt} > T_A$. Note that at the solution to the first order condition, the second derivative of the objective is $d^2\Omega/dT^2 = MS'(\nu T)\nu^2/(\delta(e^{\delta T}-1))$. This is negative for $T > T_A$, ensuring a maximum. ■

The intuition underlying the result that $T_{opt} > T_A$ comes from the fact that cities in this economy do not jump to their optimal size, but instead grow to it. Since a new city undergoes a period where average surplus is low, it is optimal to expand the growing city beyond size n_A before switching to a new city. Thus, in an efficient solution, average surplus follows a rising path as the city grows and then falls somewhat before it is optimal to start the development of a new city. Furthermore, it is possible that $x'(\nu T_{opt}) < 0$; i.e. it could be efficient to expand to a size at which negative externalities dominate positive ones. The magnitude of the gap between T_{opt} and T_A is smaller the lower is the interest rate, as can be seen from the second term in equation (4'). As the discount rate goes to zero, the optimal city size approaches the size where $MS(\nu T)$ and $AS(\nu T)$ are equal and hence intersect at the maximum of AS, so $T_{opt} \rightarrow T_A$.

4. Competitive equilibrium without city governments

With the efficient outcome as a benchmark we now turn to equilibria with different forms of governance. We look first at the equilibrium in which there are no large agents – neither governments nor large property developers. We seek to find the equilibrium steady state city size, i.e. the length of time T for which a new city grows before it becomes stationary and growth commences in the next new city.[3] In the steady state all cities will have the same value of T, but in setting out the analysis we will initially need notation for individual cities. The first is city 1 and it attracts population for T_1 periods, the second city 2 for T_2 periods, and so on. Technology is identical in all cities, although subsidy rates may differ, so the subsidy function for the ith city is $s_i(t)$. In our base equilibrium in which there is no government, $s_i(t) = 0$, but we carry the terms in the analysis for future reference.

There are three types of economic agents. (i) Landowners, who are completely passive. They are price takers, simply receiving rent according to the city land rent gradient, as discussed in Section 2. (ii) Workers, who are perfectly mobile between cities and must occupy a house in the city in which they work. This mobility implies that their real income net of housing rent is the same in all cities, so at any date $LS(\nu t) + s_i(t) - h_i(t)$ is the same for all occupied cities, i. (iii) Perfectly competitive 'builders' who provide housing. From Section 2, housing is available on a spot rental market, and house construction incurs sunk cost H. The private decision to build is based on a comparison of H with the future rents that a house will earn. Given this, builders' forward-looking decisions of whether to build in new versus old cities, based upon anticipated future rents, is central to the analysis of the competitive equilibrium city size.

The equilibrium condition for supply of housing in a growing city is that the construction cost equals the present value of rents earned. Thus, in city 1, at any date $\tau \in [0, T_1]$ at which construction is taking place

$$H \equiv \int_{\tau}^{T_1} h_1(t) e^{-\delta(t-\tau)} dt + \hat{H}_1 e^{-\delta(T_1 - \tau)}. \tag{5}$$

The first term on the right-hand side is the present value of rents earned while the city is growing where housing rent during this period is denoted as $h_1(t)$. $\hat{H}_1$ is the present value (discounted to date T_1) of rents earned from date T_1 onwards. Construction takes place at all dates in the interval $[0, T_1]$, implying two things. First, for $t \in [0, T_1]$, $h_1(t) = \delta H$, which comes from differentiating equation (5) with respect to τ. Essentially the zero profit condition on construction, (5), means that housing rent in a growing city must be constant, equal to the interest charge on the capital cost. Second, $\hat{H}_1 = H$, necessary for construction to break even at the last date at which it occurs, T_1.

Although house rent is constant at $h_1 = \delta H$ in a growing city, rent will vary with time in each stationary city to give a path of rent that clears the housing market in that city. Consider the rents earned on housing in city 1 in the period in which city 2 is growing $t \in [T_1, T_1 + T_2]$. Workers are fully mobile between cities, and rents will adjust to clear the housing market, i.e. to hold mobile workers indifferent between living in stationary city 1 or in growing city 2. Thus, city 1 housing rent during the period in which city 2 is growing, denoted $h_{12}(t)$, must equate real incomes net of housing rent across cities which, using equation (1), means that they satisfy

$$LS(\nu(t-T_1)) + s_2(t) - \delta H = LS(\nu T_1) + s_1(T_1) - h_{12}(t), \qquad t \in [T_1, T_1+T_2]. \tag{6}$$

As terms on the left-hand side of this vary through the growth cycle of city 2, $t \in [T_1, T_1 + T_2]$, so rent in city 1 must adjust to hold workers indifferent, so that city 1 housing stock continues to be occupied. Of course, the condition holds only for $h_{12}(t) \geq 0$; if the income gap between cities is too great then rents in stationary cities go to zero and housing in these cities is left empty as workers migrate to the growing city. Our assumption A4 is sufficient to secure $h_{12}(t) \geq 0$ (see below).

Equation (6) defines stationary city rent $h_{12}(t)$ for the period $t \in [T_1, T_1 + T_2]$ in which city 2 is growing. During time interval $t \in [T_1 + T_2, T_1 + T_2 + T_3]$ city 3 is growing and housing rents in both the stationary cities, cities 1 and 2, are set by the path of returns in city 3, so $h_{13}(t) = h_{23}(t)$, determined by an equation analogous to (6), and so on. Extending this analysis through infinitely many time periods, the present value (discounted to date T_1) of these rents, $\hat{H}_1$, is given by

$$\hat{H}_1 = \sum_{i=2}^{\infty} \int_{\Gamma_{i-1}}^{\Gamma_i} h_{1i}(t) e^{-\delta(t - T_1)} dt. \tag{7}$$

where Γ_i is the date at which city i stops growing, $\Gamma_i \equiv \sum_{j=1}^{i} T_j$. The key equilibrium condition is that the date T_1 at which city 1 becomes stationary, is the value of T_1 at which this present value equals construction costs, $\hat{H}_1 = H$. This date is a function of all future T_i, $i > 1$, and these dates are in turn determined by equations analogous to (7) and $\hat{H}_i = H$.

To solve, we invoke a symmetric steady state (with function $s(\cdot)$ the same in all cities), where symmetry follows from the sequential nature of the process: each new city forms under exactly the same circumstances as the previous one. We rewrite equation (6) to give house rents in all old cities in a symmetric steady state. Thus, if all old cities had growth period T then house rent in each such city at date t in the growth cycle of a city born at date 0 is given by $\hat{h}(t)$, defined by

$$\hat{h}(t) = LS(\nu T) + s(T) + \delta H - [LS(\nu t) + s(t)], \qquad t \in [0, T]. \tag{8}$$

These growth cycles repeat indefinitely, so summing their present value over all future cycles gives (discounting to date 0, the date at which the last old city stopped growing):[4]

$$\hat{H} = \frac{1}{1 - e^{-\delta T}} \int_0^T \hat{h}(t) e^{-\delta t} dt = \frac{1}{1 - e^{-\delta T}} \int_0^T [LS(\nu T) + s(T) + \delta H - [LS(\nu t) + s(t)]] e^{-\delta t} dt. \tag{9}$$

Setting $\hat{H} = H$ and integrating, the H terms cancel out so that housing market equilibrium requires[5]

$$\int_0^T [LS(\nu t) + s(t) - LS(\nu T) - s(T)]e^{-\delta t}dt = 0. \tag{10}$$

The value of T solving equation (10) is the last date, T, at which it is profitable to build a house in a growing city. Prior to T, real income in the growing city is large enough to make building still profitable; a moment after T, real income in that city would have fallen sufficiently to make building in a new city relatively more attractive, so builders switch to the next new city.

Equation (10) is sufficient to define the equilibrium, focusing on the last date at which it is profitable to build in a city. However, for future reference it is helpful to also write down an inequality condition which ensures that, for all $t \in [0, T]$, builders do not want to switch construction to an alternative existing city. A necessary condition for builders not to switch is that, at each date τ in which building is occurring in a new city,

$$\int_\tau^T [\delta H - \hat{h}(t)]e^{-\delta(t-\tau)}dt = \int_\tau^T [LS(\nu t) + s(t) - LS(\nu T) - s(T)]e^{-\delta(t-\tau)}dt \geq 0, \quad \tau \in [0, T]. \tag{11}$$

This inequality we call this the 'no-switch' condition. It says that builders cannot earn more rent from building in an old city, with rent path $\hat{h}(t)$, than from building in the one that is currently growing and in which rents are δH. Notice that in this case the differential housing rent expression is defined just to run up to T; beyond T, in the equilibrium, new and old cities would both give the same present value rent $\hat{H}$. The expression holds with equality at $\tau = 0$ and $\tau = T$, the dates at which builders switch cities, as in (10).

We now summarise results for the competitive equilibrium without city government $s(t) = s(T) = 0$. We label the value of T solving equation (10) T_{eq}, with corresponding population size $n_{eq} = \nu T_{eq}$. This gives the following proposition:

Proposition 2: Without city government there exists a unique steady-state equilibrium city size n_{eq}. Workers' real income increases then decreases during the growth of a city, with this variation in real income being transmitted to all existing cities via variation in housing rent.

Proof: The left-hand side of equation (10) takes value zero at $T = 0$. Its gradient is given by $\partial(\int_0^T [LS(\nu t) - LS(\nu T)]e^{-\delta t}dt)/\partial T = -LS'(\nu T)[1 - e^{-\delta T}]\nu/\delta$ and is therefore decreasing until T_L and strictly increasing thereafter (by strict concavity of the function *LS*, assumption A1). The value of the integral is therefore strictly increasing through zero at $T = T_{eq}$, ensuring existence of a unique solution. Housing rents satisfy equation (8), so that income net of housing rent in both new and old cities is $LS(\nu t) - \delta H$. The no-switch condition (11) is satisfied for all $\tau \in [0, T]$, since *LS* is initially strictly increasing and then decreasing. Finally we note existence of the equilibrium requires $h(t) > 0$ for all *t*. This condition will be satisfied if $\delta H > LS(\nu T_L) - LS(\nu T_{eq})$. Given the right-hand side is less than $AS(\nu T_A) - AS(0)$ in assumption A4, (the peak value of *LS* is less than that for *AS*), this is a weaker condition on *H* than is A4.■

The time paths of income and rent are illustrated in Figure 2. The top line gives the output minus land rent and commuting costs of a worker in a city founded at date 0, $LS(\nu t)$. During the life of the city this rises to a peak at T_L, and then starts to decline until date T_{eq} is reached, after which it is stationary. The worker also pays housing rent which, during the growth of the city is simply δH. The worker's real income net of housing costs is the difference between these, given by the middle line $LS(\nu t) - \delta H$, which varies over the life of the city.

In the time interval $[T_{eq}, 2T_{eq}]$ another city is growing and offering its inhabitants the income schedule $LS(\nu t) - \delta H$. Workers in the stationary city are mobile, and remain in the stationary city only if rents follow the path $\hat{h}(t)$ (equation (8)). Thus, there are housing rent cycles in old cities as the housing market adjusts to conditions in the current growing city. As illustrated in Figure 2 house rents in old cities jump up when a new city is born as this city is initially unattractive; they are then U-shaped, reaching δH at the point where the new city is the same size as old ones. The process repeats indefinitely with periodicity T_{eq}, so stationary cities have a rent cycle in response to the possibility of migration to the growing city.

Viewing Figure 2, one might ask why, once a new city starts, builders do not continue to build in old cities in which rents are higher. Once building starts in a new city (at dates T_{eq}, $2T_{eq}$, etc), the no-switch condition is satisfied along the equilibrium path of house rents so it is profitable to continue building there. Initial builders in the new city know that they will be followed by further builders in that city. The key is that housing investment is irreversible; any further housing built in old cities cannot be moved to a new city when rents in old cities start to fall.

Intuition on the actual value of T_{eq} can also be gained from Figure 2. Suppose that the first city stops growing just before T_{eq}. Then its $LS(\nu T_{eq})$ would be somewhat greater, which shifts up its $\hat{h}(t)$ curve at all future dates, given path $LS(\nu t) - \delta H$ of new growing cities. This means, looking to the future, that house rents in this city would be somewhat higher, making it profitable to continue building, rather than stopping and switching to a new one. And if we looked at a potential equilibrium where all cities operated with a lower T_{eq}, not only are the $\hat{h}(t)$ curves shifted up, their later parts where rents are less than opportunity costs are cut off, furthering the incentive to continue building in cities until T_{eq} is reached. Similarly, if a builder supplies housing beyond T_{eq}, that lowers $LS(\nu T_{eq})$ and shifts down the $\hat{h}(t)$ path the builder will receive once the city is stationary, lowering rents so that their present value will no longer cover housing cost.

With this discussion of the equilibrium in place, we now move on to draw out some of its properties. The first is the comparison of the equilibrium with the social optimum.

Proposition 3: The competitive equilibrium without city government gives larger cities than optimum, $T_{eq} > T_{opt}$, if

$$\int_0^{T_{opt}} [EX(\nu t) - EX(\nu T_{opt})] e^{-\delta t} dt > 0. \tag{12}$$

and conversely.

Proof: Subtracting equation (10) from equation (4), equation (12) may be rewritten as

$$\int_0^{T_{opt}} [EX(\nu t) - EX(\nu T_{opt})] e^{-\delta t} dt = \int_{T_{opt}}^{T_{eq}} [LS(\nu t) - LS(\nu T_{eq})] e^{-\delta t} dt.$$

From Proposition 2, the term in the right-hand side integral is positive iff $T_{eq} > T_{opt}$. Thus $T_{eq} > T_{opt}$ iff the term on the left-hand side is positive. ■

The interpretation of (12) is direct. Cities are too large [small] if the *present value* of externalities created by a marginal migrant in a new city is greater [less] than the present value of externalities created by that migrant in a stationary city, over the new city's growth interval. The condition depends on technology. For example, cities are too large if the value of the externality declines monotonically with city size. The present value of externalities in a new city then exceeds that in an old city, and these external benefits are ignored when agents choose to start a new city. Old cities are too big because the ignored benefits of diverting migrants to a new city are less than the ignored benefits of adding people to an old city, so a new city starts too late. Conversely if the externality is increasing in city size, as with the commonly used case in which $x(n)$ is isoelastic, then new cities form too early and old cities are too small, given that the relatively high externalities in an old city compared to those in a new city are not internalized.

The fact that this equilibrium without city governments can result in smaller city sizes than the social optimum contrasts with the static analysis of this problem. There, under perfect mobility, a new city only forms when the real income of a worker in a growing city falls to the level of $LS(0)$ (i.e. $LS(0) = LS(\nu t)$), where it pays people to leave the city, regardless of whether others follow. The problem in static models is coordination failure, where efficient new city formation requires en mass co-ordinated movement of workers before the date at which $LS(0) = LS(\nu t)$. Here, the coordination problem is solved because our agents, in particular builders, commit to new city development through initial fixed H investments, and are sequentially rational. The comparison of equilibrium with optimum size just turns on the present value of marginal externalities in new versus old cities, as one would expect from applied welfare economics in a dynamic context.

This notion is apparent also in another property of the equilibrium, which concerns its efficiency in maximizing the incomes of residents, given that externalities are not internalized. The property is that T_{eq} gives a size which maximizes the present values at date of their entry of the incomes of all entrants. For entrants at date τ, the present value of income net of housing costs is

$$\int_{\tau}^{T} LS(\nu t) e^{-\delta(t-\tau)} dt \;+\; \frac{e^{-\delta(T-\tau)}}{1-e^{-\delta T}} \int_{0}^{T} LS(\nu z) e^{-\delta z} dz \;-\; H \qquad (13)$$

The first term is the present value to entrants at time τ of their income during the remaining growth time of the city. In the second term, the integral expression gives the present value of income net of housing costs for any resident of the city in steady state during the growth cycle of each successive new city. This cycle repeats indefinitely but only starts after a time length (T - τ)

(hence the term $e^{-\delta(T-\tau)}/(1-e^{-\delta T})$ before the second integral). Maximizing this expression with respect to T gives eq. (10), for any τ. The intuition is that changing T only changes final income and the future income net of rent cycles after the city stops growing; these changes apply to everyone regardless of date of entry.

Note that although T_{eq} maximizes the present values of income of successive entrants, these entrants have different present value incomes according to their date of entry. The first and last entrants have the same present value of income that they would have if they started a new city or entered an old one. However, entrants at intermediate dates receive a higher present value than if they started a new city or entered an old one. Conditional on their date of entry, intermediate entrants get a "surplus." Specifically, the difference between (13) evaluated at some date $\tau \in (0, T)$ and at 0 or T is $\int_0^{\tau}[LS(\nu T)-LS(\nu t)]e^{-\delta(t-\tau)}dt$. This expression is positive, following the proof in proposition 2. It represents the fact that intermediate entrants to a city avoid the low incomes of a start-up city.[6]

Finally, we note some comparative statics of city size. It is possible to show that a faster rate of population inflow, ν, reduces T_{eq}, although it has an ambiguous effect on city size νT_{eq}. The discount rate, δ has an unambiguous effect, with a higher discount rate giving larger city size. This can be seen by totally differentiating (10) to give

$$\frac{\partial(\int_0^T[LS(\nu t)\ -\ LS(\nu T)]e^{-\delta t}dt)}{\partial T}dT\ =\ \left(\int_0^T t[LS(\nu t)\ -\ LS(\nu T)]e^{-\delta t}dt\right)d\delta. \tag{14}$$

The partial derivative on the left-hand side is positive in the neighborhood of T_{eq}, as noted in the proof of proposition 2. To show that the right-hand side is positive, we observe that, compared to (10) with $s(t) = s(T) = 0$, the term in square brackets which switches negative to positive is now weighted by t. Since (10) holds, with weighting the right-hand side of (14) must be positive, so $dT/d\delta > 0$. The result that an increase in the discount rate leads to larger cities is intuitive, since a higher discount rate puts more weight on the low income levels that are initially earned in a new city, discouraging city formation. This suggests that in a model with capital market imperfections, where private agents discount the future more heavily than is socially optimal, equilibrium cities will tend to be larger relative to the optimum.

5. Competitive equilibrium with national government

The remainder of the paper deals with large agents who can intervene to offer subsidy schedules $s(t)$, and thereby potentially internalize externalities. We first look at the case of a benevolent national government, and then turn to local governments, private and public.

Suppose that a national government announces a subsidy schedule in which subsidies are a function of city size (or equivalently here, time from date of city birth). Builders thinking of starting construction in a new city know migrants to the city are guaranteed a schedule of subsidies as the city grows, and then when it is stationary. The subsidies are financed out of lump sum national taxes which could be on the entire population, on all urban residents, or on land rents.

Proposition 4. If the national government enacts a Pigouvian subsidy schedule for residents of all cities, $s(t) = EX(\nu t)$, then the competitive equilibrium without city governments will be socially optimal.

Proof: In equation (10) if $s(t) = EX(\nu t)$, then equation (4) for an optimum will be satisfied, given that $LS(\nu t) + EX(\nu t) = MS(\nu t)$. ■

The proposition is intuitive, since the only distortion present in the competitive equilibrium is workers' failure to internalize the externalities they create for other workers. Notice that this solution, like the competitive one without city governments, has fluctuating housing rents according to equation (8) where now

$$\hat{h}(t) = LS(\nu T_{opt}) + s(T_{opt}) + \delta H - [LS(\nu t) + s(t)] \;=\; MS(\nu T_{opt}) - MS(\nu t) + \delta H$$

In fact, the swings in housing prices and hence also real income will be greater than without national government intervention, since MS has larger swings than LS.[7]

It is not essential that the subsidy path employed by the national government follow the Pigouvian one. By comparing (4) and (10) it is only necessary that $s(t)$ be constructed to satisfy

$$\int_0^T [s(t) - s(T)] e^{-\delta t} dt = \int_0^T [EX(\nu t) - EX(\nu T)] e^{-\delta t} dt. \qquad (15)$$

Thus, the present value of subsidies in a new city compared to an old city must be set equal to the difference between the present value of externalities in a growing and a stationary city. (Although subsidy paths are also constrained such that rents in old cities never fall below zero). As an example of a potential alternative subsidy path, the national government could set $s(T) = 0$. Then (15) requires that the present value of subsidies offered over the growth of a new city must equal the difference between the present value of the externality in that city, and the present value of the externality in the old city. This present value of subsidies could then be positive or negative according to whether competitive equilibrium cities, absent policy, are too large or too small, as when the externality declines versus increases with city size (see Proposition 3).

6 Competitive equilibrium with private government

We now assume that national government is inactive, and turn to the case where each city has its own government which has the abilities to tax land rents, borrow in capital markets, and subsidize worker-firms. We look first at private local governments, or the large developer case, before turning in the next section to public governments.

Following Henderson (1974) we assume that, at any instant, there is an unexhausted supply of potential large developers who each own all the land that will ultimately be used in their individual city and who collect all land rents in their city. However, they face competition from existing and other potential new cities and are induced to offer migrants subsidies to enter their city. These subsidies are guaranteed for all time. As in the preceding section, we assume that housing is constructed by perfectly competitive builders and rented on a spot rental market. We continue to separate out housing rents from land rents. Land rents paid to the developer at each instant equal the rent from the urban land rent gradient, while rents on housing cannot be taxed by the developer.

To find the equilibrium we proceed in three steps. First, we consider the behavior of a single developer establishing a new city; the developer's city can attract population only if it offers migrants a sufficiently high income that they enter this city rather than an old one, *and* migrants pay housing rents sufficient to induce building in the new city. Second, subject to this constraint, the developer announces a subsidy schedule and size of the city to maximize the present value of profits, defined as rents net of subsidies. The size chosen must be consistent with building and migration decisions so, for example, builders would not choose to continue building beyond the announced size. Finally, we move from the decision of a single developer to the full equilibrium with free entry of developers. Competition between potential new developers bids the present value of profits down to zero. This zero profit condition ensures that no more than one developer actually enters at any date, validating our focus on a single developer at step one.

We have already developed the apparatus for the first of these steps. Suppose that all old cities have population size νT and building in the new city starts at date 0. Then rents in the old cities are $\hat{h}(t)$, defined by equation (8). The developer in choosing a subsidy schedule $s(t)$ is constrained by the no-switch condition, equation (11) of section 4; the subsidy schedule must produce an income path in the developer's city and a corresponding housing rent path in old cities such that, at all dates $\tau \in [0, T]$, builders do not want to resume building in old cities. That is, from (11)

$$\begin{aligned} \int_\tau^T [\delta H - \hat{h}(t)] e^{-\delta(t-\tau)} dt &= \int_\tau^T [LS(\nu t) + s(t) - LS(\nu T) - s(T)] e^{-\delta(t-\tau)} dt \ \geq \ 0, \\ &= \int_\tau^T [LS(\nu t) + s(t) - \bar{y}] e^{-\delta(t-\tau)} dt \ \geq \ 0. \end{aligned} \tag{16}$$

From the point of view of a single developer $LS(\nu T) + s(T)$ is exogenous. To emphasize this we denote it $\bar{y}$ in the second equation, although it is endogenous to the full equilibrium.

The objective of the developer is to maximize rent net of subsidy payments, subject to the constraint above. The instruments are the subsidy schedule $s(t)$, $t \in [0, T]$ together with terminal date T at which the city stops growing and after which the subsidy $s(T)$ is constant. Thus, we solve the program,

$$\max_{s(t),\,T} \quad R \equiv \int_0^T \big[TR(\nu t) - \nu t s(t)\big] e^{-\delta t} dt + \big[TR(\nu T) - \nu T s(T)\big] \frac{e^{-\delta T}}{\delta}$$

$$s.t. \quad \int_\tau^T \big[[LS(\nu t) + s(t)] - \bar{y}\big] e^{-\delta(t-\tau)} dt \; \geq \; 0, \qquad \tau \in [0, T]. \tag{17}$$

We show in Appendix 4, by setting up the Lagrangean, that the constraints hold with equality for all $\tau \in [0, T]$. It follows that, differentiating with respect to τ across the constraints, $s(t) + LS(\nu t) = \bar{y}$. While the level of the subsidy is yet to be determined (depending on $\bar{y}$), its shape is set to deliver a flat income path. Intuition about the constraint can be understood by thinking about the equilibrium of Section 4. In that case the no-switch condition (11) holds with equality at the dates on which the city commenced and ceased growing, and holds with inequality at intermediate dates. At these intermediate dates workers in the growing city have higher present values of incomes than if they were in an old city or were the initial entrants to a new city. In the present case, optimization by the developer extracts this surplus, so that incomes of all entrants are the same and equal to those in old cities. This also implies that house rents in old cities are constant and equal to those in new cities at δH.

To solve the optimization problem we therefore use the constraint, $s(t) + LS(\nu t) = \bar{y}$, in the objective, together with the fact that $TR(\nu t) + \nu t LS(\nu t) = TS(\nu T)$ from Table 1, to give,

$$\max_T \quad R = \int_0^T \big[TS(\nu t) - \nu t \bar{y}\big] e^{-\delta t} dt + \big[TS(\nu T) - \nu T \bar{y}\big] \frac{e^{-\delta T}}{\delta}. \tag{17'}$$

Choice of T gives first order condition

$$MS(\nu T) = \bar{y}. \tag{18}$$

This condition gives the optimal value of T for a single developer, depending on $\bar{y}$.

The third and final step of the analysis comes from the assumption that there is free entry of large developers. Their profits, R, must therefore be zero. Consequently $\bar{y}$ and subsidy levels must be bid up to the point at which this condition holds. Equilibrium is therefore characterized by substituting (18) into (17') and setting the consequent level of R equal to zero to give,

$$R\delta/\nu = \int_0^T [MS(\nu t) - MS(\nu T)] e^{-\delta t} dt = 0. \quad (19)$$

where the expression is derived from integrating by parts (see equations A9 and A10 in Appendix 3). The value of T solving equation (19) characterizes city size in the large developer case. This gives the following proposition:

Proposition 5. A unique steady state equilibrium with competitive private city governments supports the social optimum. Workers' real income is constant through time in all cities at level $\bar{y} = MS(\nu T_{opt})$. This income consists of wages net of land rent and commuting costs, $LS(\nu t)$, plus subsidy payments $s(t) = MS(\nu T_{opt}) - LS(\nu t)$ from a guaranteed schedule.

Proof: By a comparison of (19) with (4) we know that developers set $T = T_{opt}$, which we showed earlier has a unique value. Paying $s(t) = \bar{y} - LS(\nu t)$ supports the constant real income path, $\bar{y} = MS(\nu T_{opt})$. There remain two issues. First, we wrote the optimization problem with workers flowing into the city at rate ν, giving population νt. Could a developer profitably engineer a jump in population? The best possible jump is to instantaneously create a city of size n_A and maximal real income, $AS(n_A)$. However, this is not profitable. Creating this new city would reduce house rents according to equation (8), inducing residents to stay in the old city; to induce inter-city migration the developer would have to offer migrants enough income to drive rents in old cities to zero. But doing so is not profitable; assumption A4 is sufficient to ensure that $MS(\nu T_{opt}) > AS(n_A) - \delta H$, where $AS(n_A) - \delta H$ is the maximum income net of housing rent demanded by builders which a new city jumping to n_A can pay migrants. Second there is the issue of why the subsidy path needs to be guaranteed. Consider dates $t > T_{opt}$. At such dates all housing construction in the city is sunk, so any reduction in $s(t)$ would be exactly matched by a reduction in house rents. The developer can therefore expropriate whoever owns the housing stock. In order for housing construction to take place, the developer has to commit to not do this, so the full time path of subsidy payments to workers must be guaranteed. ■

6.1. Financing city development

Critical to being able to offer the constant real income $\bar{y} = MS(\nu T_{opt})$ at each instant is the ability of the developer to borrow and accumulate debt. The path of debt incurred by the developer is implicit in the equilibrium outlined above, and can be drawn out explicitly. City debt at date $\tau \in [0, T]$ is given by the value of cumulated subsidy payments less land rents collected, or

$$D(\tau) = \int_0^\tau (\nu t s(t) - TR(\nu t)) e^{\delta(\tau - t)} dt \tag{20}$$

$$= e^{\delta\tau} \int_0^\tau (\nu t[\bar{y} - AS(\nu t)]) e^{-\delta t} dt \tag{20'}$$

where integration and the discount factor cumulate past expenditures and the interest on them. Equation (20') comes from noting that $s(t) = \bar{y} - LS(\nu t) = \bar{y} - AS(\nu t) + AR(\nu t)$. The debt path is described by the following corollary.

Corollary 1. Total city debt rises monotonically with city growth up to the last instant of development where the increase is zero. Per person debt is declining towards the end of a city's growth path. Post-growth, land rents collected exactly equal subsidies paid plus interest payments on the debt.

Proof: Differentiating (20), the change in debt with respect to time is

$$D'(\tau) = \delta D(\tau) + \nu\tau[\bar{y} - AS(\nu\tau)] = \nu e^{\delta\tau} \int_0^\tau [\bar{y} - MS(\nu t)] e^{-\delta t} dt \tag{21}$$

where the second equation is derived in Appendix 4, equation (A16). Given that $\bar{y} = MS(\nu T)$, $D'(\tau) \to 0$ as $\tau \to T_{opt}$, from eq.(4). The term in the integral is positive for small τ, and then eventually declines monotonically, given assumption A3 (see Figure 1). Thus the integral starts positive, increases, and then decreases monotonically until it is zero at T_{opt}, implying that total debt is always increasing up to T_{opt}. Since $D'(\tau) \to 0$ as $\tau \to T_{opt}$, it must be the case that, with strictly positive population growth until T_{opt}, debt *per worker* peaks at some point and then declines. Finally, we show in Appendix 4 that from time T_{opt} on, debt payments plus subsidies must equal rents, or $\delta D(T) + \nu T s(T) = TR(\nu T)$.■

The underlying paths describing city finance are illustrated in Figure 3 which shows an example of a subsidy schedule, per worker interest charges on the debt, and the level of subsidy minus land rent. Note that subsidies per worker decline and then rise again, mimicking the inverse of the *LS* path so as to maintain constant income. Debt accumulates according to the gap between rents collected versus subsidies paid plus accumulated interest on the debt. Using this information there is a second corollary to Proposition 5.

Corollary 2. Given a balanced budget constraint for the developer from time T on, which requires that rents equal subsidies plus interest on the public debt, the developer by setting $T = T_{opt}$ maximizes the real income payable to residents once the city is stationary.

Proof. Income from time T on in a city is $y = LS(\nu T) + s(T)$ and the balanced budget requires $s(T) = AR(\nu T) - \delta D(T)/(\nu T)$. Combining, we get $y = AS(\nu T) - \delta D(T)/(\nu T)$. In Appendix 4 (equation A17) we show that this can be written as

$$y = AS(\nu T) - \delta D(T)/(\nu T) = \frac{\nu}{1 - e^{-\delta T}}\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right]. \qquad (22)$$

This expression is a linear transformation of social welfare, Ω, the maximand in equation (2). It is therefore maximized at T_{opt}.■

We argued above that developer profit maximization constrained by the no-switch condition (11) meant that the developer extracted the surplus earnings of later entrants compared to initial ones, so a constant income is paid to all entrants. The corollary confirms what then must result from competition: profits are bid away so that the developer pays the highest constant income possible subject to a zero profit constraint and the requirement that debt be paid off. A complimentary perspective on the developer is that competition to form the current new city requires the developer to pay the highest income possible to the initial residents (so they do not go to other potential new cities), subject to the no-switch constraint that building does not later resume in old cities. That constraint requires that later residents are paid no less than initial ones; i.e. everyone is paid a constant income. If later residents were paid less, then rents would rise in old cities and induce builders to construct houses there because the no switch condition is violated. Equivalently, the developers' solution mimics the outcome of a situation where the city's objective is simply to maximise per worker income once the city is stationary, given the debt repayment constraint. It is the debt repayment constraint that yields a city size and borrowing interval defined by T_{opt}.

This discussion leads to another corollary, which is the Henry George theorem amended to a dynamic context.

Corollary 3: In the large developer equilibrium, (I) the present value of land rents collected in a city equal the present value of subsidies paid out, so land rents cover all needed "public" revenues." (II) The present value of externalities created by the marginal entrant from a city's initial occupation onward equals the present value of subsidies paid to that entrant.

Proof: The first result follows trivially from imposing the zero profit equilibrium condition on the objective function in eq. (17). For the second, given $\bar{y} = MS(\nu T) = LS(\nu T) + s(T)$ and given the definition of the externality in Table 1, from T onwards the subsidy equals the Pigouvian tax on externalities or $s(T) = EX(\nu T)$. Thus for (II) the remaining requirement is to show that the present value of externalities from 0 to T of a marginal entrant,

$$\int_0^T EX(\nu t)e^{-\delta t}dt = \int_0^T [MS(\nu t) - LS(\nu t)]e^{-\delta t}dt,$$

equals the present value of subsidies paid to that entrant:

$$\int_0^T s(t)e^{-\delta t}dt = \int_0^T [MS(\nu T) - LS(\nu t)]e^{-\delta t}dt.$$

Comparing the two, the LS terms cancel out and the two remaining terms are equal from (19).■

Note that, in contrast to the static version of the Henry George theorem, there is no equality between subsidies paid at any instant and externalities at that instant. Nor is there any equality between the present value of total subsidies and total externalities. In fact, it is possible to show that total externalities, $\int_0^T \nu t EX(\nu t)e^{-\delta t}dt$, exceed total subsidies, $\int_0^T \nu t s(t)e^{-\delta t}dt$, given

$\int_0^T \nu t[MS(\nu t) - MS(\nu T)]e^{-\delta t}dt > 0$ (see the second section of Appendix 2). What matters is the present value of the externalities created by the marginal entrant.

6.2. Other aspects of the developer equilibrium

We have several further comments on the developer equilibrium. First, since developers earn zero profits, the order in which they develop is arbitrary. Second, we could construct an identical equilibrium in which the developer owns the housing and subsidy payments are not guaranteed. We need to adjust our equations to subtract the present value of housing costs from the developer's profits and reduce $s(T)$ by δH, so that in the zero profit condition all housing terms net out to zero. Once the developer owns housing, in order to retain residents and cover all costs including the debt payments (which now increase by housing costs), the developer would choose to offer the $s(t)$ schedule, net of housing costs, that we constructed above.

The third comment concerns the starting point for the steady-state, the first city. As long as potential developers are always out there, the first city must follow the specified income and $s(t)$ paths in order to survive competition from a city that would replace it otherwise. But what about a historical, pre-existing (pre-developer) city founded before a country starts the urbanization process? Once new cities form, competition among developers forces the efficient solution with constant real income, $\bar{y} = MS(\nu T_{opt})$. Corollary 2 tells us that it is the best a developer can offer new migrants. In the historical city, whatever its size, housing and possibly land rents (depending on its institutions) will adjust so residents also now receive $\bar{y} = MS(\nu T_{opt})$. And population adjustments may occur also. If for example, the historical city is sufficiently oversized, keeping all residents might require negative housing rents. In that case the city would lose population (and some builders at the city fringe will go out of business), until non-negative rents are restored.

7. Public city governments

The large developer or private government case is widely used in the urban literature as a convenient paradigm to model city formation. While such institutions with varying degrees of power are observed in countries like Mexico, Indonesia, USA and Canada, even within these countries either they are not widespread, or their scope of operation is sharply limited and overseen by local public governments. To what extent do results from Section 6 apply to cities that are controlled by public governments?

To address this question we assume that local public governments set policies at each instant to reflect the choices of voters, under a competitive perfect information electoral process in which only current city residents vote. Thus at each instant the city government chooses policy to maximize the real income of current residents, and these payments are constrained by the city's debt obligations. In particular, we assume that a city can neither renounce its debt, nor expropriate house-owners. Clearly, there is an incentive for a stationary city – one which has finished borrowing and in which house construction is complete – to renounce debt. The only collateral for this debt are the city assets, in our model just the housing stock. However builders will not provide housing if it is likely to be seized by debt holders, and lenders will not lend unless there is assurance of repayment. Once house construction is complete there is also an incentive for residents to expropriate nonresident house-owners by voting themselves a higher current subsidy payment in return for higher future debt. The final incidence of this increased debt obligation falls on housing rent which must decline as debt payments in the future rise, to induce mobile workers to stay in the city. Once again, builders will not provide housing if they are vulnerable to such expropriation.

These are fundamental problems in city finance, especially in developing countries, where localities have been able to renounce debt, with national governments sometimes then taking on the debt. Home ownership may mitigate the problems, as might constitutional requirements imposed by higher level governments. For current purposes, we simply assume that these default possibilities and moral hazard issues do not arise. City residents vote recognising the implications of their decisions for city debt, for the future growth of the city, and for the incentives of forward-looking house-builders to undertake construction.

7.1. Local governments with full taxation and borrowing powers

Suppose first that city governments can fully tax land rents and borrow without legislated debt limits, while residents bear future debt obligations in line with the preceding discussion. Voters choose $s(t)$ for the city at each instant, where the relevant voting population grows as the city grows. Voters also choose, at some date, to stop growth, this setting the value of T after which $s(T)$ is constant. A dynamic voting game is a difficult problem and our treatment and findings are limited to showing that, under our assumptions, local public governments can duplicate the outcome of the large developer case.

Proposition 6. A steady state solution with city governments exists and supports the social optimum. Workers' real income is at a constant maximal level in all cities, $\bar{y} = MS(\nu T_{opt})$.

Proof: The income path $\bar{y} = MS(\nu T_{opt})$, as constructed in proposition 1, is feasible. Two sorts of deviation from this path are possible. First, voters may choose to halt growth at some date other than T_{opt}. However, from corollary 2, given cumulated debt, T_{opt} is the date that maximises the income flow at all future dates. A referendum on the date at which to stop growth therefore chooses $T = T_{opt}$. The second possible deviation is that at some date prior to T_{opt} residents choose a subsidy rate that pays themselves an income level greater than $\bar{y} = MS(\nu T_{opt})$. Such a payment increases debt and debt service obligations so what the city can pay in the future is reduced, or $y(T) < \bar{y}$. Any change in the stopping date, $T \neq T_{opt}$, further reduces the city's capacity to both pay $\bar{y}$ and meet debt service (by Corollary 2). This means that once the city is stationary, in order to stop out-migration to other stationary cities (equation (9)) builders would have to offer lower rents and thus would be unable to collect enough rents to cover housing costs. Knowing this, forward-looking builders will cease building (at a non-optimal time for residents) if residents deviate to pay themselves a subsidy so income at any instant exceeds $\bar{y}$. ■

As in the discussion of equilibrium without city governments, sequentially rational builders play a critical role. If citizens try to borrow excessively then it is builders who foresee that the city is not sustainable and will not supply housing. The equivalence of the private and public city outcomes is analogous to that obtained in the static literature.

7.2 Financing constraints; partial land taxation and no access to capital markets.

Now we turn to situations in which city governments face financing constraints. Such constraints include limited taxation powers and caps on the amount of debt that may be accumulated. We have covered one extreme case already; the Section 4 analysis of competitive equilibrium without government. In this section we consider two cases. The first, perhaps the most realistic in many countries, is that local governments have no ability to borrow but may tax land rents to either a full or limited extent. In many Latin American and Asian countries permission to borrow is either nonexistent or limited to a few special cities (Bahl and Linn, 1992). Taxation powers of local governments are often restricted – for example tax rates may be set by the national government with localities receiving a set fraction of taxes raised in their locality (e.g. Indonesia). In the second case, we explore what happens when local governments may borrow but face explicit caps on debt accumulation, a common occurrence in situations where local government borrowing is permitted (Bahl and Linn, 1992).

Let us assume then that city governments cannot access capital markets but can tax away an exogenously fixed fraction of land rents, α, and redistribute the money as $s(t)$ payments to workers. Thus, the subsidy payment becomes

$$s(t) = \alpha AR(\nu t). \tag{23}$$

Given this, with rearrangement, the equilibrium condition (10) becomes

$$\int_0^T \big([LS(\nu t) - LS(\nu T)] + \alpha[AR(\nu t) - AR(\nu T)]\big)e^{-\delta t}dt = 0. \tag{24}$$

and city size is given by the value of T solving this equation. The path of subsidy payments given by (23) gives workers an income profile $LS(\nu t) + s(t)$ that is not constant through time; housing rents are therefore not constant, but go through a path derived from using (23) in equation (8). Our main concern is with city size, and we have the following proposition.

Proposition 7. (I) If all land rent can be taxed ($\alpha = 1$) then city size is larger than socially optimal. (II) City size is monotonically increasing in α.

Proof: See Appendix 5.■

Intuition on part I of the proposition can be derived by comparing the subsidy schedule based on full distribution of land rents ($\alpha = 1$) with the Pigouvian subsidy schedule of Proposition 4. The Pigouvian subsidy equals the marginal externality at any instant, and average land rent is large relative to that externality when cities get larger. This can be checked from the fact that $EX - AR = MS - AS$ (Table 1) is negative beyond n_A (Figure 1). Thus, with $\alpha = 1$, the high subsidy rate from land rent redistribution in a city as it gets larger has the effect of postponing the development of a new city where land rents are minuscule, causing city sizes to be larger than socially optimal.

Intuition for part II can be gained by noting that if we start with $\alpha = 0$, then (24) is satisfied at T_{eq} from Proposition 2. Having $\alpha > 0$ introduces a negative term in the second square brackets in (24), given that average rents rise continuously. The term in the first square brackets must therefore become positive for (24) to hold and, from the proof of Proposition 2, that term only increases as T increases. As α rises in the debt constrained city, so must T, until the limit where $\alpha = 1$. The idea that city size under developers or city governments is monotonically increasing in α is consistent with the literature on determination of city sizes in a static context (see Abdel-Rahman and Anas, 2004).

7.3 Financing constraints; debt ceilings.

As we saw in Section 6, it is in the interest of the early population of a growing city to borrow, financing current consumption and shifting the burden of debt onto future – and larger – city populations. This, together with the no-switch condition, flattens out the per worker real income stream payable in the city. However, the presence of a borrowing constraint means that a debt ceiling will be hit at an endogenously determined date $\hat{T}$, limiting the ability of the city to do this smoothing. What is the impact of such a constraint on equilibrium income and city size?

Suppose that all land rent is taxed ($\alpha = 1$ in terms of the last subsection), workers receive $LS(\nu t) + s(t)$, and the debt ceiling is denoted $\bar{D}$. The city goes through three phases. The first is from date 0 to $\hat{T}$, in which subsidies $s(t)$ can be freely chosen. However, debt accumulates according to the difference between subsidies paid and land rents collected, so $\hat{T}$ is reached when this accumulated debt reaches $\bar{D}$,

$$\bar{D} = \int_0^{\hat{T}} \nu t[s(t) - AR(\nu t)] e^{\delta(\hat{T}-t)} dt. \tag{25}$$

In the second phase, between $\hat{T}$ and T, the city is still growing, but subsidy payments are constrained to be land rent receipts minus debt service per worker, so $s(t) = AR(\nu t) - \delta\bar{D}/\nu t$, $t \in [\hat{T}, T]$. In the third phase the city is stationary and the subsidy remains average rent minus debt service,

$s(T) = AR(\nu T) - \delta\bar{D}/\nu T.$

As usual, labor mobility and housing market behavior constrain the time path of subsidies offered. This is captured by the no-switch condition, equation (11), which can be written, for each date $\tau \in [0, \hat{T}]$, as

$$\int_{\tau}^{\hat{T}}[LS(\nu t) + s(t)]e^{-\delta(t-\tau)}dt + \int_{\hat{T}}^{T}[LS(\nu t) + AR(\nu t) - \delta\bar{D}/\nu t]e^{-\delta(t-\tau)}dt$$
$$\geq \int_{\tau}^{T}[LS(\nu T) + s(T)]e^{-\delta(t-\tau)}dt. \tag{26}$$

Early inhabitants of the city pay themselves the highest values of $s(t)$ that they can, subject to this condition not being violated at any date. In the first phase, $\tau \in [0, \hat{T}]$, the condition holds with equality; any slack would be bid away by increased subsidy, just as in Section 6. In the second phase, $t \in (\hat{T}, T]$, it holds with inequality, as $s(t)$ cannot be freely varied, just as in Section 4.

The no-switch condition holds on present values, and its implications for instantaneous subsidy payments are given in equations (27),

$$\begin{aligned} LS(\nu t) + s(t) &= LS(\nu T) + s(T) = LS(\nu T) + AR(\nu T) - \delta\bar{D}/\nu T \qquad t \in [0, \hat{T}], \\ LS(\nu t) + s(t) &= LS(\nu t) + AR(\nu t) - \delta\bar{D}/\nu t, \qquad t \in [\hat{T}, T]. \end{aligned} \tag{27}$$

For $t \in [0, \hat{T}]$ the subsidy is set such that income takes its stationary value (as can be checked by differentiating (26), holding with equality, with respect to τ), while beyond this it is determined by debt repayments. The outcome is therefore given by $s(t)$ from (27), and dates $\hat{T}$ and T given by equation (25) and by (26) holding with equality at $\tau = 0$.[8]

This is illustrated in Figure 4, on which the path of income is given by solid line ABCD. This equals the stationary value $LS(\nu T) + AR(\nu T) - \delta\bar{D}/\nu T$, until date $\hat{T}$ at which the borrowing constraint binds. It then drops discontinuously to point B, and between points B and C follows the curve given by the second equation in (27).

While this characterizes the equilibrium, analytical results on its properties are not transparent and we do comparative statics through numerical simulation. In particular, we investigate the effects of lowering the debt ceiling, and find that a lower ceiling reduces $\hat{T}$, the length of time subsidies can be paid, and raises T. Intuitively, when cities are more financially constrained both the subsidy level and $\hat{T}$ decline, implying lower steady state incomes and hence bigger cities (with lower $LS(\nu T)$ + $AR(\nu T)$ values in (27)). Consistent with earlier results, in the limit as the debt ceiling goes to zero so $\hat{T}$ goes to zero and the outcome is as in Proposition 7 with $\alpha = 1$. Conversely, as the debt constraint ceases to bind points $\hat{T}$ and T both converge to T_{opt}, as achieved by an unconstrained city government (Section 6).

These and other points made in the paper are summarised in the numerical simulation results presented in Table 2. To obtain these results we set $c = 0.2$, $\gamma = 2$ and production function $x(n) = 0.9 + n - 0.4n^2$. The population increase, ν, and interest rate δ are set such that $\delta = 1/T_A$; this scales the time dimension such that if it took 20 years for the city to grow to size νT_A then the interest rate would be set at 5% pa; equivalently, if ν were twice as large then the interest rate would be 10% pa. These parameter values are used in all figures in the paper.

Table 2: City size, income and welfare: quadratic production

	Debt	T	$AS(\nu T) - \delta D/\nu T$	Ω
Social optimum / large developer	1	1	1	1
Competitive equilibrium: no government	0	1.352	0.971	0.973
City government: no debt: $\alpha = 0.2$	0	1.387	0.959	0.967
$\alpha = 0.4$	0	1.423	0.946	0.961
$\alpha = 0.6$	0	1.459	0.933	0.956
$\alpha = 0.8$	0	1.495	0.918	0.949
$\alpha = 1.0$	0	1.531	0.902	0.942
City government: $\alpha = 1$: Debt constraint factor:	0.2	1.384 (0.253)	0.956	0.968
	0.4	1.314 (0.384)	0.973	0.978
	0.6	1.249 (0.510)	0.984	0.986
	0.8	1.102 (0.659)	0.993	0.993

The first row of the table is the social optimum, constructed as a benchmark in which all variables of interest take value unity. We first illustrate results from earlier sections of the paper.

Thus, the equilibrium without a city government gives city size 35% larger than optimal (Proposition 3), and delivers a present value of social welfare, Ω, around 3% below that of the optimum. The next group of rows give outcomes when city government is able to tax and redistribute fraction α of land rent, but is unable to borrow. The case with no government is $\alpha = 0$, and increasing α from this point increases city size monotonically (Proposition 7), eventually giving city size more than 50% larger than optimal, while delivering a 6% lower level of welfare.

The final block reports results on debt ceilings. The first of these rows is when the borrowing constraint is set at 20% of what would be borrowed by large developers. City size is 38% larger than socially optimal; the term in brackets is $\hat{T}$, indicating that borrowing occurs for a length of time equal to 25% of the socially optimal city life. Relaxing the borrowing constraint (increasing the debt ceiling) shortens T, or reduces city size, and increases $\hat{T}$, as discussed earlier. Welfare rises towards the socially optimal level. Notice that welfare increases at a sharply diminishing rate; this is as would be expected from relaxation of a constraint, but makes the point that permitting even small amounts of borrowing may be quite socially valuable.

8. Concluding comments.

In this paper we have developed a dynamic model in which to analyze the problem of city formation and city size in an economy in which total urban population is increasing. We think that this environment is relevant for many developing countries experiencing rapid urbanization. The dynamic context has two further advantages. It enables the competitive equilibrium to be analyzed free of simple coordination failures. And it enables us to address important questions of sunk infrastructure investments and city borrowing and debt constraints.

We find that socially optimal city size is larger than in a static model; cities should grow beyond the point at which surplus per worker is maximized. The competitive equilibrium with no large agents may support cities that are larger or smaller than socially efficient, depending on how externalities vary with city size. Large developers or public city government can internalize these externalities and support the social optimum. However, capital market imperfections – either debt ceilings or interest premia – increase city size, tending to lead to outcomes in which cities are inefficiently large.

There are many possible extensions that we have considered. Having ongoing technological change which shifts *AS* curves out and up would be a simple one, needed for data interpretation of ongoing growth of city sizes. One could introduce a stochastic element affecting individual city success, which might work to delay dates of new city formation. One really intriguing extension would be to further analyze the effect of financing constraints and limited applicability of institutions guaranteeing non-expropriation of rents, especially looking to the issue of squatter settlements. Certain types of capital infrastructure could be variable and investment in them postponed due to borrowing constraints or issues of rent expropriation. These remain subjects for future research.

Appendix 1: Commuting costs and rent gradients

Population at distance z from the CBD is kz^{θ} and commuting costs from this distance are cz^{η}, where $\theta = 0$ or 1, in respectively a linear or circular city and $\eta \geq 1$. Total population in a city of radius $\bar{z}$ is:

$$n = \int_0^{\bar{z}} kz^{\theta} dz = \bar{z}^{1+\theta} k/(1+\theta), \text{ so } \bar{z} = \left[n(1+\theta)/k\right]^{1/(1+\theta)}. \tag{A1}$$

Edge commuting costs are: $$c\bar{z}^{\eta} = c\left[n(1+\theta)/k\right]^{\eta/(1+\theta)}. \tag{A2}$$

Total commuting costs are

$$TC = \int_0^{\bar{z}} kcz^{\eta} z^{\theta} dz = \frac{ck\bar{z}^{1+\theta+\eta}}{1+\theta+\eta} = \frac{ck}{1+\theta+\eta}\left[\frac{n(1+\theta)}{k}\right]^{\frac{1+\theta+\eta}{1+\theta}}. \tag{A3}$$

We define the parameter $\gamma \equiv (1+\theta+\eta)/(1+\theta) > 1$ and choose units such that $k = 1+\theta$. Edge commuting costs and total commuting costs in Table 1 follow directly. Total land rent is population, n, times edge commuting cost times minus total commuting costs.

Appendix 2: The social optimum

In this appendix we develop three properties that define the form we give to equation (2).

1) Inefficiency of jump solution

If workers can be moved between cities at zero cost then a new city size can jump to some size. Suppose that instead of growing continuously through interval $[0, T]$ a new city jumps at date ζ to population $\nu\zeta$. For $t \in [0, \zeta]$ new migrants accumulate in old cities. The cost of this jump is that new migrants have to be accommodated in existing cities until they jump to a new city where housing is then built for them. The present value of the extra housing costs incurred over $[0, T]$ is:

$$C = \delta H\nu\zeta \int_0^T e^{-\delta t} dt + H\nu\zeta e^{-\delta\zeta} - H\nu \int_0^{\zeta} e^{-\delta t} dt.. \tag{A4}$$

The first term is the cost of holding $\nu\zeta$ houses in old cities. The remaining terms give the cost saving in the new city from the fact that $\nu\zeta$ units of housing are constructed at date ζ (second term) rather than being constructed continuously through $t \in [0, \zeta]$. Using integral (A10) below, this expression integrates to:

$$C = \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - \nu t)e^{-\delta t}dt\right] \tag{A5}$$

which can be interpreted directly as the present value of the rental income foregone on having empty houses. The term outside the square brackets is the rental income per house. Inside, the first term is the number of houses that are empty for $t \in [\zeta, T]$, and the second the number that are empty at date $t \in [0, \zeta]$, each discounted back to date zero.

The benefit of jumping is the value of putting new workers arriving during $t \in [0, \zeta]$ in an established city with average surplus $AS(\nu T)$ rather than in a growing city with surplus $AS(\nu t)$,

$$B = \int_{0}^{\zeta}\nu t AS(\nu T)e^{-\delta t}dt - \int_{0}^{\zeta}\nu t AS(\nu t)e^{-\delta t}dt. \tag{A6}$$

Hence, the net benefit is

$$\begin{aligned} C - B &= \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - \nu t)e^{-\delta t}dt\right] - \int_{0}^{\zeta}\nu t[AS(\nu T) - AS(\nu t)]e^{-\delta t}dt \\ &= \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - 2\nu t)e^{-\delta t}dt\right] + \int_{0}^{\zeta}\nu t\left[\delta H - [AS(\nu T) - AS(\nu t)]\right]e^{-\delta t}dt. \end{aligned} \tag{A7}$$

Setting (A7) equal to zero defines the value of H above which it is not profitable to jump. In the second line, the first square bracketed expression is positive. [Note the second integral in that first square brackets is positive. In that integral; the term in parentheses under that integral is declining in t; the integral is zero without discounting; and thus it is positive with discounting.]. A sufficient condition for $C > B$, then is that the second square bracketed term is also positive. Assumption A4 ensures this.

2) Inefficiency of simultaneous development of multiple new cities

Redefine the problem in equation (2) to have β cities form at time 0 and each grow at a rate ν/β for a length of time T. Optimising gives the same first order condition for T as in the text (although different optimal values, as functions depend on $\nu t/\beta$). The first order condition for β is

$$[1-e^{-\delta T}]\frac{\beta}{\nu}\frac{d\Omega}{d\beta} = \delta^{-1}\left[\int_0^T (MS(\frac{\nu}{\beta}t)-MS(\frac{\nu}{\beta}T))e^{-\delta t}dt\right] - \left[\int_0^T (MS(\frac{\nu}{\beta}t)-MS(\frac{\nu}{\beta}T))te^{-\delta t}dt\right] \qquad \text{(A8)}$$

On the RHS, given eq (4), the first term in square brackets is zero (implied by the first order condition defining the optimal T). Following the analysis of equation (4) and assumption A3, given the first term is zero, then the second term in square brackets must be positive (negative items in the expression get low t weights and positive ones high t weights). Thus the whole condition is negative, indicating that increases in β reduce welfare. β is bounded below by one, the case we solve for in the text. Having multiple cities growing at different rates (i.e. a non-symmetric outcome) doesn't change the principle. Whatever the growth rates, we still want to minimize the number of cities.

3) Inefficiency of stopping and restarting growth

Why not halt a city's growth before an optimal *T*, start a new city and then later resume growth of the first city? Consider any sequence of this type where at any instant only one city is growing. Pick a date at which growth cycles of all existing cities are complete. At that date, the undiscounted total surplus is the same regardless of the sequencing. However starting with the first city, given discounting, prematurely stopping growth advances [delays] the date of low [high] *MS* values, lowering the present value of total surplus.

Appendix 3: Derivations, Section 3

Derivation of equation (4): integration by parts gives the expression:

$$\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right] = \left(\frac{\nu}{\delta}\right)\int_0^T MS(\nu t)e^{-\delta t}dt. \qquad \text{(A9)}$$

Using this in equation (3) gives equation (4).

Derivation of equation (4'): We replace the term $\left[1-e^{-\delta T}\right]/\delta^2$ in (3) using the following expression, derived by integration by parts.

$$\left[\frac{1-e^{-\delta T}}{\delta^2}\right] = \int_0^T te^{-\delta t}dt + \left[\frac{Te^{-\delta T}}{\delta}\right]. \qquad \text{(A10)}$$

Appendix 4: Derivations, Section 6

1) Optimization problem (17)

The Lagrangean corresponding to (17) is:

$$L \equiv \int_0^T \left[TR(\nu t) - \nu t s(t)\right] e^{-\delta t} dt + \left[TR(\nu T) - \nu T s(T)\right] \frac{e^{-\delta T}}{\delta}$$
$$+ \int_0^T \lambda(\tau) \left[\int_\tau^T \left[LS(\nu t) + s(t) - \bar{y}\right] e^{-\delta(t-\tau)} dt\right] d\tau \qquad \text{(A11)}$$

where the function is written with the constraint as an integral over τ from 0 to T with multipliers $\lambda(\tau)$. The first order condition at date z is

$$\nu z e^{-\delta z} = \int_0^z \lambda(\tau) e^{-\delta(z-\tau)} d\tau \qquad \text{(A12)}$$

from which $\nu e^{-\delta z} = \lambda(z)$. This is strictly positive at all dates, so the constraint binds.

2) Corollary 1

For the debt expressions, from equation (20),

$$D(\tau) = e^{\delta\tau} \int_0^\tau (\nu t \bar{y} - TS(\nu t)]) e^{-\delta t} dt. \qquad \text{(A13)}$$

Using (A9) and (A10) this can be integrated to give

$$\delta D(\tau) = \nu \bar{y} \left[\frac{e^{\delta\tau} - 1}{\delta} - \tau\right] + TS(\nu\tau) - \nu e^{\delta\tau} \int_0^\tau MS(\nu t) e^{-\delta t} dt. \qquad \text{(A14)}$$

At date $\tau = T_{opt}$ this expression reduces to

$$\delta D(T_{opt}) = TS(\nu T_{opt}) - MS(\nu T_{opt}) \nu T_{opt}. \qquad \text{(A15)}$$

(derived using equation (4) and noting $\bar{y} = MS(\nu T_{opt})$). This says that debt service is equal to total surplus minus real income payment to workers, equal in turn to rents minus subsidies.
Differentiating (A14) with respect to time, τ,

$$D'(\tau)\frac{e^{-\delta\tau}}{\nu} = \bar{y}\left[\frac{1-e^{-\delta\tau}}{\delta}\right] - \int_0^\tau MS(\nu t)e^{-\delta t}dt = \int_0^\tau[\bar{y} - MS(\nu t)]e^{-\delta t}dt \qquad \text{(A16)}$$

which gives equation (21) of the text.

3) Corollary 2

Steady state income is $y = AS(\nu T) - \delta D(T)/\nu T$. Using A11 and rearranging gives

$$y = \frac{\delta}{1-e^{-\delta T}}\int_0^T MS(\nu t)e^{-\delta t}dt = \frac{\delta^2/\nu}{1-e^{-\delta T}}\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right] \qquad \text{(A17)}$$

where the second equation uses (A9) again. This is a positive linear transformation of the maximand (2), so is maximized at T_{opt}.

Appendix 5: Proof of Proposition 7, Section 7.2

(I) If $\alpha = 1$ then, using the fact that *AS* - *LS* = *AR*, (24) can be written as $\int_0^T[AS(\nu t) - AS(\nu T)]e^{-\delta t} = 0$. This integral is negative at $T = T_A$ (condition A2), and strictly increasing thereafter ($-AS'(\nu T)e^{-\delta T} > 0$ for $T > T_A$), so it is only zero at $T > T_{opt}$, if at $T = T_{opt}$ it is negative. Define $\phi(t) \equiv MS(\nu t) - AS(\nu t)$, where $\phi(t)$ is positive for $T < T_A$, zero at T_A, and negative and declining for $T > T_A$. (the last follows given concavity of AS and given AS′ < 0 beyond T_A). Given eq (4) holds at T_{opt} we need to prove that $\int_0^T[\phi(t) - \phi(T)]e^{-\delta t}dt$ is positive at $T = T_{opt}$. This is the case since $\phi(t) - \phi(T_{opt}) > 0$ for $t < T_A$ and, from T_A to T_{opt}, it declines in value to zero at T_{opt}. Therefore all terms in $\int_0^T[\phi(t) - \phi(T)]e^{-\delta t}dt$ are positive for $T = T_{opt}$.

(II) Differentiating equation (24) and using the fact that *AS* - *LS* = *AR*,

$$d\alpha\int_0^T[AR(\nu t) - AR(\nu T)]e^{-\delta t}dt = dT[(1-\alpha)LS'(\nu T) + \alpha AS'(\nu T)](1-e^{-\delta T})\nu/\delta. \qquad \text{(A18)}$$

The integral on the left hand side is negative, since *AR* is increasing. The square brackets on the right hand side is negative for $T > t_A$. It is therefore the case that $dT/d\alpha > 0$.

Endnotes:

1. These urban scale economies can be given a variety of micro foundations, see Duranton and Puga (2004).

2. In this case, we would add an income term to (1), the worker's share in national urban land rents, which is perceived as fixed by any worker.

3. New development does not occur simultaneously in more than one new city for 'stability' reasons. Having migrants go to more than one new city is not robust to population perturbations because urban agglomeration effects cause real incomes to rise with city scale

4. If we continue with the notation in (5) - (7), substituting (6) into (7), the equilibrium condition for housing construction at date T_1 is

$$H = \hat{H}_1 = \sum_{i=2}^{\infty} \int_{\Gamma_{i-1}}^{\Gamma_i} \Big([LS(\nu T_1) + s_1(T_1)] + \delta H - [LS(\nu(t - \Gamma_{i-1}) + s_i(t)] \Big) e^{-\delta(t-T_1)} dt.$$

Imposing symmetry, so all T's and $s(\cdot)$ schedules are the same, evaluating gives equation (9). To see this, in evaluating the expression pull $e^{\delta T_1}$ outside the summation sign and recognize that the summation infinitely repeats a fixed expression so that expression is multiplied by $e^{-\delta T} + e^{-\delta 2T}$ +....which equals $1/(1- e^{-\delta T}) - 1$.

5. We can rearrange (10) to read

$$\int_0^T LS(\nu t) e^{-\delta t} dt + \frac{e^{-\delta T}}{\delta} LS(\nu T) = \frac{1}{\delta} LS(\nu T).$$

This is the equation that arises in Venables (2004), where workers are assumed to be perfectly immobile. The left-hand side is the PV of being the first person in a new city, and the right-hand side the PV of being the last worker to enter an old city. The equilibrium date at which a new city is founded is given by this indifference condition between migrating permanently to a new versus an old city.

6. Note from (10) that $\int_0^{\tau} [LS(\nu T) - LS(\nu t)] e^{-\delta(t-\tau)} dt = \int_{\tau}^{T} [LS(\nu t) - LS(\nu T)] e^{-\delta(t-\tau)} dt.$

7. These larger swings raise the possibility that the condition for house rents to be non-negative could be violated. Assumption A3 is not sufficient to ensure that $\delta H > MS(\nu T_{opt}) - MS(\nu t)$ for all t.

8.Using (27) for $s(t)$, (25) and (26) can be rearranged as functions of $\hat{T}$ and T:

$$\int_{\hat{T}}^{T} [AS(\nu t) - AS(\nu T)] e^{-\delta t} = \frac{\delta \bar{D}}{\nu} \int_{\hat{T}}^{T} \left(\frac{1}{t} - \frac{1}{T} \right) e^{-\delta t} dt$$

$$\int_{0}^{\hat{T}} [AS(\nu T) - AS(\nu t)] \nu t e^{-\delta t} = \left[1 - e^{-\delta \hat{T}} \left(1 - \delta (T - \hat{T}) \right) \right] \frac{\bar{D}}{T \delta}.$$

References:

Abdel-Rahman, H, and A. Anas (2004), “Theories of Systems of Cities”,” in J.V. Henderson and J-F Thisse (eds), *Handbook of Urban and Regional Economics, Volume 4, Cities and Geography,* North Holland, in press.

Anas, A (1992), “On the Birth and Growth of Cities,” *Regional Science and Urban Economics*, 22, 243-258

Bahl R. and J. Linn (1992), *Urban Public Policy in Developing Countries*, New York: Oxford University Press.

Black D. and J.V. Henderson (1999), "A Theory of Urban Growth," *Journal of Political Economy*, 107, 252-284.

Black D. and J.V. Henderson (2003), “Urban Evolution in the USA”, *Journal of Economic Geography*, 3, 343-372

Brueckner J. (2000) “Urban Growth Models with Durable Housing” in *Economics of Cities* (J-M Huriot and J-F Thisse (eds), Cambridge University Press.

Duranton G, and Puga D (2004) “Micro-Foundations of Urban Agglomeration Economies” in J.V. Henderson and J-F Thisse (eds), *Handbook of Urban and Regional Economics, Volume 4, Cities and Geography*, North Holland, in press.

Glaeser E.L. and J. Gyourko (2003) “Urban Decline and Durable Housing,” Harvard University mimeo

Helsley R. and W. Strange(1994), “City Formation with Commitment,” *Regional Science and Urban Economics*, 24, 373-390.

Henderson, J.V. (1974), “The Sizes and Types of Cities”, *American Economic Review*, 61, 640-56.

Henderson J.V. and H-G Wang (2004) “Urbanization and City Growth,” Brown University mimeo

Rossi-Hansberg E. and M.L.J. Wright (2004) “Urban Structure and Growth” Stanford University mimeo.

Venables, A.J. (2004), "Spatial Disparities in Developing Countries: Cities, Regions, and International Trade", *Journal of Economic Geography*, forthcoming

World Bank (2000), *Entering the 21st Century: World Development Report 1999/2000*, Oxford University Press.

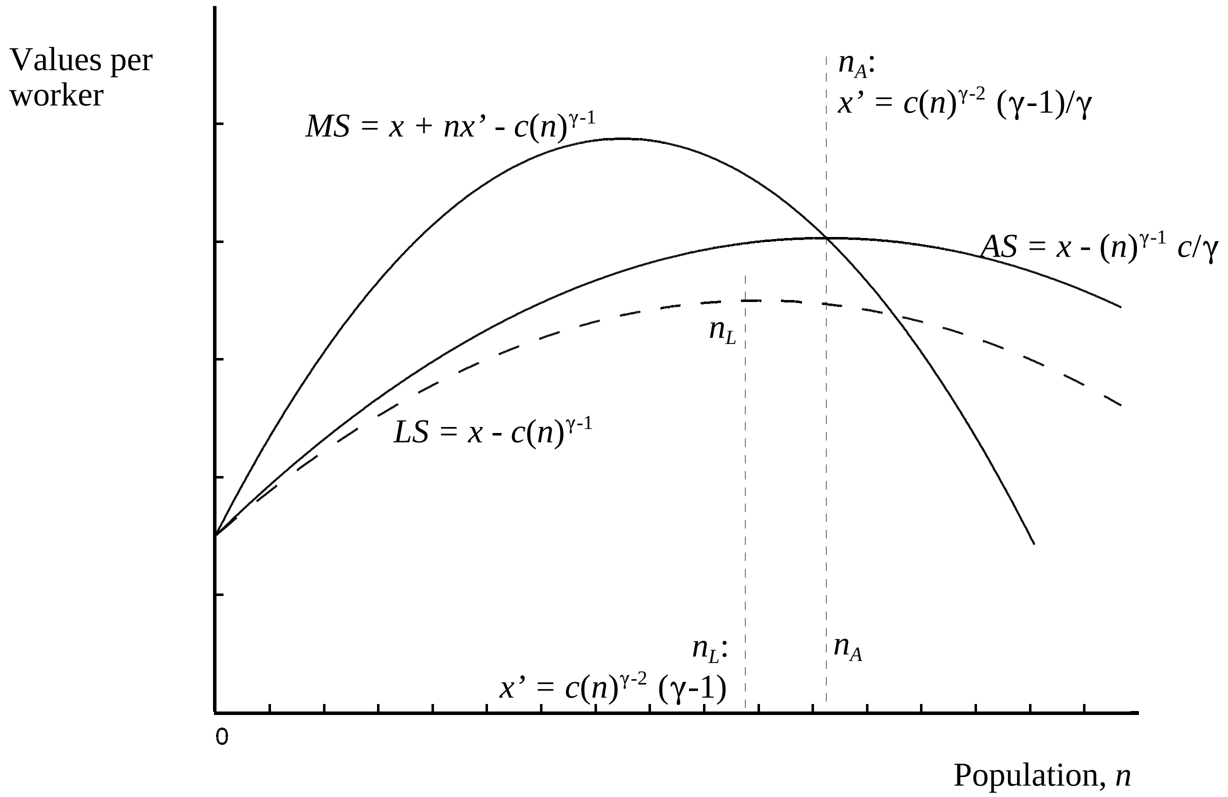

Figure 1: Surplus per worker

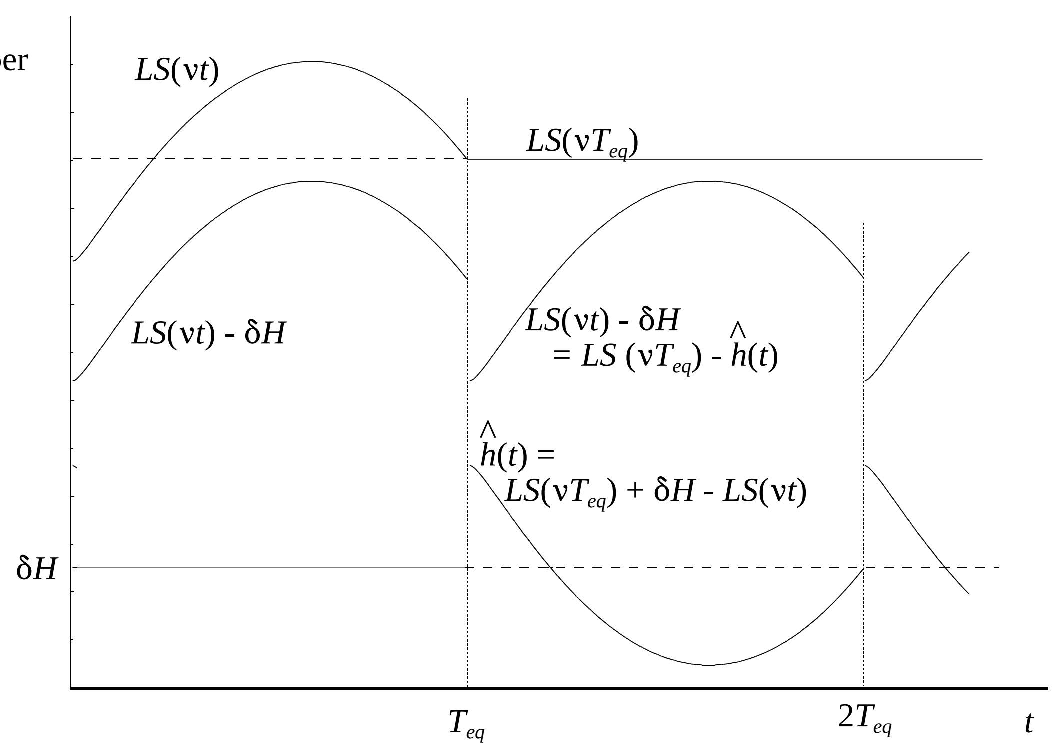

Figure 2: Income and housing rent in the competitive equilibrium:

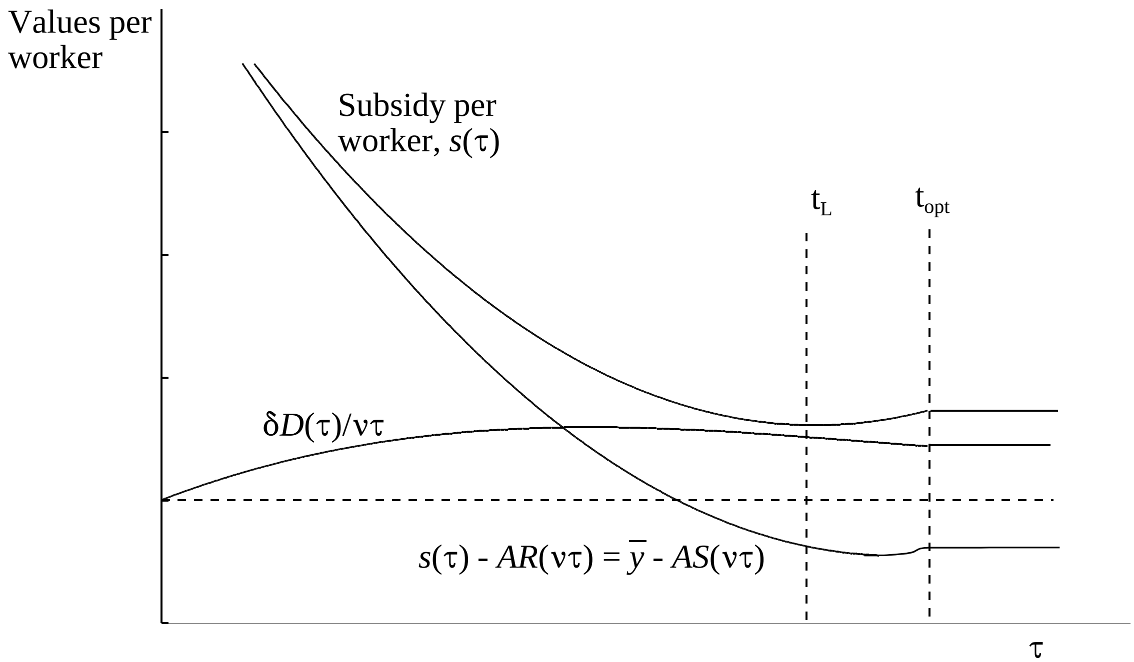

Figure 3: Large developer's subsidy and debt service (per worker)

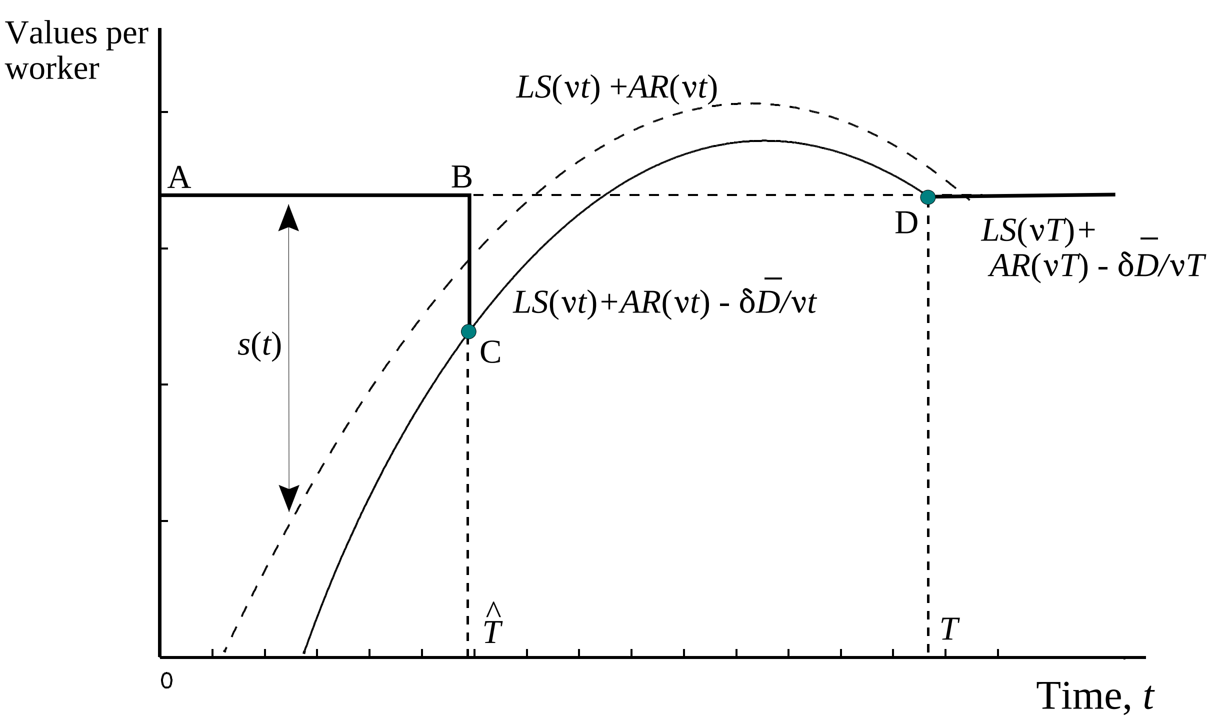

Figure 4: Borrowing constraints (= 0.8)