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Evolving communities with individual preferences

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Abstract

The goal of this paper is to provide mathematically rigorous tools for modelling the evolution of a community of interacting individuals. We model the population by a measure space $(\Omega, \mathcal{F}, \nu)$ where ν determines the abundance of individual preferences. The preferences of an individual $\omega \in \Omega$ are described by a measurable choice $X(\omega)$ of a rough path.

We aim to identify, for each individual, a choice for the forward evolution $Y_t(\omega)$ for an individual in the community. These choices $Y_t(\omega)$ must be consistent so that $Y_t(\omega)$ correctly accounts for the individual's preference and correctly models their interaction with the aggregate behaviour of the community.

In general, solutions are continuum of interacting threads analogous to the huge number of individual atomic trajectories that together make up the motion of a fluid. The evolution of the population need not be governed by any over-arching partial differential equation (PDE). Although one can match the standard non-linear parabolic PDEs of McKean–Vlasov type with specific examples of communities in this case. The bulk behaviour of the evolving population provides a solution to the PDE.

We focus on the case of weakly interacting systems, where we are able to exhibit the existence and uniqueness of consistent solutions.

An important technical result is continuity of the behaviour of the system with respect to changes in the measure ν assigning weight to individuals. Replacing the deterministic ν with the empirical distribution of an independent and identically distributed sample from ν leads to many standard models, and applying the continuity result allows easy proofs for propagation of chaos.

The rigorous underpinning presented here leads to uncomplicated models which have wide applicability in both the physical and social sciences. We make no presumption that the macroscopic dynamics are modelled by a PDE.

This work builds on the fine probability literature considering the limit behaviour for systems where a large number of particles are interacting with independent preferences; there is also work on continuum models with preferences described by a semi-martingale measure. We mention some of the key papers.

1. *Introduction*

In the Vlasov [**31**] approach to continuum mechanics, the macroscopic behaviour of a cloud of interacting particles is approximated by a single (non-linear) differential equation. One of the motivations behind this is the simplicity of the resulting equation, which provides a convenient way of summarizing and understanding the influence of inter-particle interactions on the macroscopic properties of the continuum. The intractability of many-body problems makes this a fundamental tool in statistical mechanics, for instance in the Boltzmann kinetic

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theory of gases (see [29] and the references therein). In these and other situations, it is typical to assume that the N-particle community evolves according to a system of differential equation with pairwise mean–field interaction

$$dY_t^i(\omega) = \frac{1}{N}\sum_{j=1}^{N} V^0(Y_t^i(\omega), Y_t^j(\omega))\,dt, \tag{1.1}$$

with starting independent and identically distributed (i.i.d.) starting configurations $\{Y_0^i : i = 1, \ldots, N\}$. As the system evolves the interaction introduces dependence between the trajectories. An important problem in this context is to identify conditions for the system to exhibit *propagation of chaos*, which is motivated by the notion of molecular chaos in kinetic theory. This property, which has been made precise by Kac [18], captures the fact that any fixed finite (1.1) subcollection of states of particles (or individuals) in the system (1.1) will resemble independent trajectories in the limit as $N \to \infty$. A single particle evolving in this large-N population has dynamics determined by a (random) non-linear differential equation of the type

$$dY_t(\omega) = V^0(Y_t(\omega), \nu_* Y_t)\,dt.$$

Here $\nu_* Y_t$ is the pushforward of ν giving the mass distribution of Y at time t, and $Y_t(\omega)$ is a solution to the above equation for almost every ω.

Of course, there are a huge number of similar if less precisely characterized models: in the social sciences, in the modelling of the evolution of cancer, etc. where the evolution of an individual is affected by the evolution of the wider community or individual differences mean that different individuals will respond differently to the same external environment. We can easily make adaptations to the calculus to take such behaviour into account. One is lead to equations of the following kind:

$$dY_t(\omega) = V^0(Y_t(\omega), \nu_* Y_t)\,dt + V(Y_t(\omega), \nu_* Y_t)\,dX_t(\omega),$$

where $X(\omega)$ represents the individual preferences of the individual ω. Now if X is smooth, then there is no additional difficulty. If ν is a probability measure, and X is a semi-martingale under this measure, then (under regularity conditions) Sznitman [29], Kurtz and Protter [20], Méléard [27] and others have proved that the corresponding particle system obtained by taking an i.i.d. sample from X and using this empirical measure in the above equations provides a converging sequence of particle systems. The limit can be identified with the law of a non-linear partial differential equation (PDE) which solves the Vlasov equation. Dawson and Gärtner [7] have important results on the large deviations in the convergence of the weakly interactive system (where $V(Y_t(\omega), \nu_* Y_t) = V(Y_t(\omega))$ and later Dai Pra and den Hollander [6] and Guionnet [2, 8, 9, 15] considered the large deviations for interaction in a random media in problems arising from the dynamics of spin glasses. Kurtz and Xiong promoted more advanced discussion in [21].

In many cases of interest, it is unreasonable to expect the preferences X to be a semi-martingale as evidenced by the success of fractional Brownian motion in the modelling of fluids (see [3, 4, 16, 17] and the references therein). In addition, individuals often have knowledge that makes the previsible assumption equally inappropriate. We now understand that the natural assumption on X that leads to equation with a strong meaning is that X should be rough path. Indeed there are a large number of deterministic (and numerically approximable) systems that evolve without the assistance of a PDE. We will study a mathematical framework which exposes the consequences of persistent differences between individuals in the population dynamics (see [28] for a study of such a phenomenon in the context of red deer populations).

The McKean–Vlasov model leads one, in the limit, to the equation

$$dY_t(\omega) = V^0(Y_t(\omega), \nu_* Y_t)\,dt + V(Y_t(\omega), \nu_* Y_t)\,dW_t(\omega),$$

where the individual preferences are given by a d-dimensional Wiener measure W. However, individuals can have very different volatility and speed of reaction to events. Let σ be a positive real function on the space Ω of individuals. For appropriate V^0, V and paths μ_s in measures on Y-space, one can consider the indexed family of differential equations, one for each ω,

$$\begin{aligned} dY_t(\omega) &= V^0(Y_t(\omega), \mu_t)\,dt + V(Y_t(\omega), \mu_t)\sigma(\omega)\,dW_t(\omega). \\ & Y_0(\omega) \text{ given.} \end{aligned}$$

For almost every ω, the path $t \to \sigma(\omega)W_t(\omega)$ is a geometric rough path of finite p-variation for every $p > 2$, based on rescaling W and its Levy area. If ν_t is a path of finite variation in the space of measures and ϕ, ξ are at least $C^{2+\varepsilon}$, then it will be the case that the rough path solution $Y_t(\omega)$ to this equation will exist and be unique. Considering all ω, we see that $Y_t(\omega)$ is a random variable and we denote its law by the probability measure $\tilde{\mu}_t$. Of course, this new path $t \to \tilde{\mu}_t$ in measures will not in general coincide with the path $t \to \mu_t$. But it makes complete sense to ask whether there is a choice $t \to \mu_t$ so that the resultant measure path $t \to \tilde{\mu}_t$ does coincide with it. In this case, we have a community of individuals evolving according to their individual preferences in a way that is also consistent with the dynamics of the population as a whole.

We note that in general having individuals with different volatility results in a process $t \to \sigma(\omega)W_t(\omega)$ that is far from a semi-martingale against the Wiener measure and using the base filtration. One cannot have $\sigma(\omega)$ measurable in $\mathcal{F}_0$ unless one enlarges the filtration or σ is constant. The lack of previsibility does not impede the rough path perspective, and there is no issue at all in setting up the equations. One theoretically amusing choice for σ is to take

$$\sigma(\omega) = \frac{1}{\sup_{t\in[0,1]} |W_t(\omega)|},$$

which in some sense eliminates enthusiastic outliers in the population.

To move on from posing a meaningful questions to identifying solutions is actually quite challenging. For example, it is not clear, at the level of generality that we introduce, that the path $t \to \tilde{\mu}_t$ will have bounded variation or what Banach space to consider it as a path in even if it does. This raises another issue, in that solving equations such as this we require the pair $(\mu_t, W_t(\omega))$ to be a rough path which normally requires extra data unless one has good control on $t \to \mu_t$ so we have to make some compromises. No doubt there is much that can be refined and taken further.

Let $t \to \mu_t$ be a path in the space of probability measures representing a putative evolution of the population $Y_t(\omega)$. We introduce the 'occupation' measure process $\Gamma_t := \int_0^t \mu_s\,ds$ and note that it is monotone increasing and Lipschitz with norm 1 in the total variation norm on measures. Let $W(y)\mu = \int V^0(y, y')\mu(dy')$, then W can be viewed as a linear map from our space of measures to vector fields on the Y-space.

We make two significant simplifications to make the problem more tractable.

(1) We only allow so-called weak interactions between the individual and the population which take place only in the drift component of the equation, that is, $V(Y_t(\omega), \nu_* Y_t) = V(Y_t(\omega))$.

(2) The interaction between the individual and the population admits a superposition principle.

Together these imply that we can write the interaction between Y, its preferences W and the distribution ν of the community in the following form:

$$dY_t(\omega) = W(Y_t(\omega))\,d\Gamma_t + V(Y_t(\omega))\,dX_t(\omega).$$

We then look for fixed points of the map that takes $t \to \mu_t$ to $t \to \tilde{\mu}_t$. Since Γ has bounded variation this equation poses fewer technical problems than the general case but still allows discussion of the McKean–Vlasov-type problems discussed initially.

The paper is structured as follows. In Section 2, we spend some time setting up our notation for the rough path framework; this is the mathematical technology we use to model the community. Section 3 then explores the special case where the law of the preferences is given by a finitely supported probability measure on the space of (geometric) rough paths. Here we prove an existence and uniqueness theorem for the law of the non-linear McKean–Vlasov rough differential equation (RDE). The methodology here is distinct from that used later to prove the (more general) result for non-discrete measures. But this simple case allows us to see very clearly how the weak interaction assumption, combined with the Lyons-Victoir (LV) Extension Theorem of [**26**] gives rise to the uniqueness of fixed points. In Section 4, we proceed with the roadmap sketched out above. We first present some Gronwall inequalities for rough differential equations, developing the deterministic estimates from [**11**] and focusing particularly on the conditions needed to ensure the integrability of the estimates. We make use of the recent paper [**5**] in showing that these conditions are satisfied for a wide range of preference measures. We then present conditions that ensure the existence and uniqueness of fixed points and discuss their continuity in the measure on preferences. Finally, Section 5 establishes propagation of chaos (à la Sznitman [**29**]) for the convergence of the finite particle system. We note that this paper has already lead to follow-up work (see, for example, [**1**]); we discuss other possible applications of our results.

2. *Preliminaries on rough path theory*

There are now a wealth of resources on rough path theory, for example, [**10**, **11**, **15**, **24**, **25**]. Rather than give an overview, we will focus on the notation we need for the current application and direct the reader to references where appropriate. We first recall the notion of the *truncated signature* of a parametrized path in $C^{1\text{-var}}([0,T],\mathbb{R}^d)$ (the set of continuous paths of bounded variation), this is:

$$S_N(x)_{s,t} := 1 + \sum_{k=1}^{N} \int_{s<t_1<t_2<\cdots<t_k<t} dx_{t_1} \otimes dx_{t_2} \otimes \cdots \otimes dx_{t_k} \in T^N(\mathbb{R}^d).$$

Where $T^N(\mathbb{R}^d) = \bigoplus_{i=0}^{N} (\mathbb{R}^d)^{\otimes i}$ denotes the truncated tensor algebra. We use π_n to denote the canonical projection

$$\pi_n : T^N(\mathbb{R}^d) \longrightarrow (\mathbb{R}^d)^{\otimes n}, \quad n = 0, 1, \ldots, N.$$

For $\mathbf{x}^n$ in $(\mathbb{R}^d)^{\otimes n}$, we define $\mathbf{x}^{n;i_1,\ldots,i_n}$ to be the real number

$$\mathbf{x}^{n;i_1,\ldots,i_n} = (e^*_{i_1} \otimes \cdots \otimes e^*_{i_n})(\mathbf{x}^n) =: \langle e_{i_1,\ldots,i_n}, \mathbf{x}^n \rangle,$$

where $e^*_1, \ldots, e^*_d$ denote the standard dual basis vectors and $e_{i_1,\ldots,i_n} := e_{i_1} \otimes \cdots \otimes e_{i_n}$. If we wish to emphasize the dimension d, then we will write, $e^d_j, (e^*_j)^d$, $e^d_{i_1,\ldots,i_n}$ and so forth. We equip each $(\mathbb{R}^d)^{\otimes n}$ with a compatible tensor norm $|\cdot|_{(\mathbb{R}^d)^{\otimes n}}$, and let

$$d_N(\mathbf{g}, \mathbf{h}) := \max_{i=1,\ldots,N} |\pi_i(\mathbf{g} - \mathbf{h})|_{(\mathbb{R}^d)^{\otimes i}}.$$

It is a well known that the path $S_N(x)$ in fact takes values in the step-N free nilpotent group with d generators, which we denote $G^N(\mathbb{R}^d)$. Motivated by this, we may consider the set of such group-valued paths

$$\mathbf{x}_t = (1, \mathbf{x}^1_t, \ldots, \mathbf{x}^{\lfloor p \rfloor}_t) \in G^{\lfloor p \rfloor}(\mathbb{R}^d),$$

for $p \geqslant 1$. We can then describe the set of 'norms' on $G^{\lfloor p \rfloor}(\mathbb{R}^d)$ which are *homogeneous* with respect to the natural scaling operation on the tensor algebra (see [11] for definitions and details). The subset of these so-called homogeneous norms which are symmetric and sub-additive [11] give rise to genuine metrics on $G^{\lfloor p \rfloor}(\mathbb{R}^d)$. And these metrics in turn give rise to the notion of a homogeneous p-variation metric $d_{p\text{-var}}$ on the $G^{\lfloor p \rfloor}(\mathbb{R}^d)$-valued paths, a typical example being the Carnot–Caratheodory (CC) metric d_{CC}. The group structure provides a natural notion of increment, namely $\mathbf{x}_{s,t} := \mathbf{x}_s^{-1} \otimes \mathbf{x}_t$ and we may then define

$$d_{p\text{-var};[0,T]}(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_{p\text{-var};[0,T]} := \left(\sup_{D=(t_j)} \sum_{j:t_j \in D} d_{\mathrm{CC}}(\mathbf{x}_{t_j,t_{j+1}}, \mathbf{y}_{t_j,t_{j+1}}) \right)^{1/p}. \tag{2.1}$$

If (2.1) is finite, then, $\omega_{\mathbf{x}}(s,t) := \|\mathbf{x}\|^p_{p\text{-var};[s,t]} (:= \|\mathbf{x} - \mathbf{1}\|^p_{p\text{-var};[s,t]})$ is a control (that is, it is a continuous, non-negative, super-additive function on the simplex $\Delta_{[0,T]} = (s,t) : 0 = s <= t = T$ which vanishes on the diagonal). Also of interest will be the *inhomogeneous rough path metric* defined by

$$\rho_{p\text{-var};[0,T]}(\mathbf{x}, \mathbf{y}) := |\mathbf{x}_0 - \mathbf{y}_0|_{T^{\lfloor p \rfloor}(\mathbb{R}^d)} + \max_{i=1,\dots,\lfloor p \rfloor} \sup_{D=(t_j)} \left(\sum_{j:t_j \in D} |\pi_i(\mathbf{x}_{t_j,t_{j+1}} - \mathbf{y}_{t_j,t_{j+1}})|^{p/i}_{(\mathbb{R}^d)^{\otimes i}} \right)^{i/p}.$$

And the ω-modulus inhomogeneous metric, with respect to a fixed control ω, which is defined by

$$\rho_{p\text{-}\omega;[0,T]}(\mathbf{x}, \mathbf{y}) = |\mathbf{x}_0 - \mathbf{y}_0|_{T^{\lfloor p \rfloor}(\mathbb{R}^d)} + \max_{i=1,\dots,\lfloor p \rfloor} \frac{|\pi_i(\mathbf{x}_{s,t} - \mathbf{y}_{s,t})|}{\omega(s,t)^{1/p}}.$$

The space of *weakly geometric* p-rough paths will be denoted by $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$. This is the set of continuous paths with values in $G^{\lfloor p \rfloor}(\mathbb{R}^d)$ (parametrized over some, usually implicit, time interval) such that (2.1) is finite. A refinement of this notion is the space of geometric p-rough paths, denoted by $G\Omega_p(\mathbb{R}^d)$, which is the closure of

$$\{S_{\lfloor p \rfloor}(x)_{0,\cdot} : x \in C^{1\text{-var}}([0,T], \mathbb{R}^d)\},$$

with respect to the rough path metric $d_{p\text{-var}}$.

We will often end up considering an RDE driven by a path $\mathbf{x}$ in $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$ along a collection of vector fields $V = (V^1, \dots, V^d)$ on $\mathbb{R}^e$. And from the point of view of existence and uniqueness results, the appropriate way to measure the regularity of the vector fields V^i results turns out to be the notion of Lipschitz-γ (or, simply, Lip-γ) in the sense of Stein. This notion provides a norm on the space of such vector fields, which we denote $|\cdot|_{\mathrm{Lip}-\gamma}$. We will often make use of the shorthand

$$|V|_{\mathrm{Lip}-\gamma} = \max_{i=1,\dots,d} |V^i|_{\mathrm{Lip}\text{-}\gamma}.$$

Finally, throughout the article we will consider spaces of probabilities measures on various metric spaces (S, d).

NOTATION 1. Let (S, d) be a metric space. We will use $\mathcal{M}(S)$ to denote the space of probability measures on $(S, \mathcal{B}(S))$. For $p > 0$, $\mathcal{M}_p(S)$ will represent the subset of $\mathcal{M}(S)$ which have finite pth moment in the sense that

$$\int_S d(s_0, s)^p \mu(ds) < \infty,$$

for some (and hence every) $s_0 \in S$.

We will work with the 1-Wasserstein metric on $\mathcal{M}_1(S)$.

DEFINITION 2.1. The 1-Wasserstein metric W on $\mathcal{M}_1(S)$ is defined by

$$W(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{S\times S} d(x,y)\pi(dx,dy),$$

where the infimum is taken over $\Pi(\mu,\nu)$, the set of all couplings of μ and ν.

REMARK 2.2. An important and useful result concerning the 1-Wasserstein metric W is the dual representation obtained from the Kantorovich–Rubinstein theorem (see [**30**, Theorem 1.14]).

3. *Weakly interacting communities*

Let $(\mu_t)_{t\in[0,T]}$ be a family of probability measures on $G^{\lfloor p\rfloor}(\mathbb{R}^d)$ parametrized by time. The main object of study in this paper will be solutions to rough differential equations which incorporate *weak mean–field interactions* with $(\mu_t)_{t\in[0,T]}$. By this, we mean equations of the following type:

$$d\mathbf{y}_t = \int_{G^{\lfloor p\rfloor}(\mathbb{R}^d)} \sigma(y_t,\pi_1\mathbf{y})\mu_t(d\mathbf{y})\,dt + V(y_t)\,d\mathbf{x}, \quad y(0)=y_0\in\mathbb{R}^e. \tag{3.1}$$

The rough path $\mathbf{x}$ flows along the vector fields $V=(V^1,\dots,V^d)$ on $\mathbb{R}^e$, but the resulting trajectory is also influenced by $(\mu_t)_{t\in[0,T]}$ through the *interaction kernel* $\sigma:\mathbb{R}^e\times\mathbb{R}^e\to\mathbb{R}^e$. Assuming enough regularity on the path $t\mapsto\mu$, we may define the integral

$$\gamma_t^{\mu} = \int_0^t \mu_s\,ds,$$

a continuous bounded variation path in an appropriately chosen ambient Banach space. It is convenient to rewrite the main equation (3.1) as

$$d\mathbf{y}_t = V^0(y_t)\,d\gamma_t^{\mu} + V(y_t)\,d\mathbf{x}_t, \quad y(0)=y_0, \tag{3.2}$$

where V^0 and σ are related by

$$V^0(y)(\mu) = \int_{G^{\lfloor p\rfloor}(\mathbb{R}^d)} \sigma(y,\pi_1\mathbf{z})\mu(d\mathbf{z}) \quad \text{with } \mu\in\mathcal{M}(G^{\lfloor p\rfloor}(\mathbb{R}^e)).$$

We will discuss the detail of this construction in Section 4. In the cases we consider, $(\mu_t)_{t\in[0,T]}$ will be derived from the marginal distributions of a probability measure in $\mathcal{M}(G\Omega_p(\mathbb{R}^e))$; that is, by pushing-forward under the evaluation maps $\psi_t(\mathbf{x})=\mathbf{x}_t,\ t\in[0,T]$. We denote a solution to (3.2) by $\Theta_{V^0,V}(\mu,y_0,\mathbf{x})$. A fixed (for the moment) probability measure $u_0\times\nu$ on $\mathbb{R}^e\times \mathrm{WG}\Omega_p(\mathbb{R}^d)$ is such that u_0 describes the initial configuration of the particles, and ν is the law of the preferences (or, more conveniently, the *preference measure*). By taking a realization $(Y_0,\mathbf{X})$ of $u_0\times\nu$ on some probability space $(\Omega,\mathcal{F},P)$, and then using $\mathbf{X}$ to solve (3.2) we will have constructed a well-defined map Ψ_ν from the space $\mathcal{M}(G\Omega_p(\mathbb{R}^e))$ to itself given by the pushforward:

$$\Psi_\nu : \mu \longmapsto [\Theta_{V^0,V}(\mu,\cdot,\cdot)]_*(u_0\times\nu). \tag{3.3}$$

μ will then be fixed point of this map if and only if $\Theta_{V^0,V}(\mu,Y_0,\mathbf{X})$ is a solution the (non-linear) McKean–Vlasov-type RDE

$$\begin{cases} d\mathbf{Y}_t = V(Y_t)\,d\mathbf{X}_t + V^0(Y_t)\,d\gamma_t^{\mu}, \\ \mathrm{Law}(\mathbf{Y}) = \mu,\ \mathrm{Law}(Y_0)=u_0. \end{cases} \tag{3.4}$$

A key objective of this paper is to demonstrate that there exist unique solutions to (3.4) for a class of preference measures which extend far beyond the usual semi-martingale class.

We first spend time developing the important special case where ν is a finitely supported discrete measure of the form

$$\nu = \sum_{i=1}^{N} \lambda_i \delta_{\mathbf{x}_i} \in \mathcal{M}(G\Omega_p(\mathbb{R}^d)).$$

In this setting, we can hope to resolve the fixed-point problem (3.4) by solving the system of RDEs

$$d\mathbf{y}_i(t) = V(y_i(t))d\mathbf{x}_i(t) + \sum_{j=1}^{N} \lambda_j \sigma(y_i(t), y_j(t))\, d\gamma_t^{\mu}, \quad y_i(0) = y_i \in \mathbb{R}^e, \tag{3.5}$$

for $i = 1, \ldots, N$. And then, letting $u_0^{\otimes n}$ denote the n-fold product measure of u_0, we can define the measure

$$\mu = u_0^{\otimes N} * \left(\sum_{i=1}^{N} \lambda_i \delta_{\mathbf{y}_i} \right),$$

by setting

$$\mu(A) = \sum_{i=1}^{N} \lambda_i \int_{\mathbb{R}^e \times \cdots \times \mathbb{R}^e} \delta_{\mathbf{y}_i{}^{y_1,\ldots,y_N}}(A) u_0(dy_1) \cdots u_0(dy_N) \quad \forall A \in \mathcal{B}(G\Omega_p(\mathbb{R}^e)).$$

Here we have written $\mathbf{y}_i^{y_1,\ldots,y_N}$ to emphasize the dependence of $\mathbf{y}_i$ on the starting configuration $y_1, \ldots, y_N$ of the N particles in the system. With μ defined in this way, we would expect that $\Psi_\nu(\mu) = \mu$, and indeed this approach will work for smooth preferences. In the rough case ($p \geqslant 2$), however, things are more complex. Here in order to solve (3.5), we need to define *a priori* the cross-iterated integrals between the (components of) different preferences $\mathbf{x}^i$ and $\mathbf{x}^j$. The LV Extension Theorem [**26**] guarantees that this can always be done, but in general there are many choices for the extension. To ensure uniqueness of the fixed point, we need to check that the resulting solution is not sensitive to this choice; the remainder of this section will present conditions which will ensure that this is the case.

The results of this section will later be subsumed by the general fixed-point theorem of Section 4. Nonetheless, they are important for three reasons. First they expose, in an original and lucid way, the importance of the weakly interacting structure. Secondly, they highlight the main obstacle in extending the analysis to general interactions, in a way that cannot be easily discerned from the general fixed-point result. Thirdly, they crucially underlie our later treatment of the convergent behaviour of the finite particle system.

3.1. *A two-particle system*

To make clear the structure of the argument, we first deal with the case where $N = 2$ and $p \in [2, 3)$; that is, the preference measure is supported on only two rough paths in $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$. We write $\nu = \lambda\delta_{\mathbf{x}_1} + (1-\lambda)\delta_{\mathbf{x}_2}$. By the LV Extension Theorem, there exists an element $\mathbf{x}$ in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{2d})$ which is an extension consistent with $\mathbf{x}_1$ and $\mathbf{x}_2$ in the sense that for all $(s,t) \in \Delta_{[0,T]}$

$$\mathbf{x}_{s,t}^{1;i} = \begin{cases} \mathbf{x}_1^{1;i}(s,t) & \text{if } i \in \{1, \ldots, d\}, \\ \mathbf{x}_2^{1;i-d}(s,t) & \text{if } i \in \{d+1, \ldots, 2d\} \end{cases}$$

and

$$\mathbf{x}_{s,t}^{2;i,j} = \begin{cases} \mathbf{x}_1^{2;i,j}(s,t) & \text{if } i,j \in \{1, \ldots, d\}, \\ \mathbf{x}_2^{2;i-d,j-d}(s,t) & \text{if } i,j \in \{d+1, \ldots, 2d\}. \end{cases}$$

We can write this more succinctly as

$$\mathbf{x}^1 = (\mathbf{x}_1^1, \mathbf{x}_2^1) \in \mathbb{R}^{2d}, \quad \mathbf{x}^2 = \begin{pmatrix} \mathbf{x}_1^2 & \dagger \\ \dagger & \mathbf{x}_2^2 \end{pmatrix} \in (\mathbb{R}^{2d})^{\otimes 2}, \tag{3.6}$$

by making the obvious identifications. The only constraint on the terms (†) then arises from the need to make $\mathbf{x}$ (weakly) geometric. Given such an extension, we can solve the following RDE uniquely:

$$d\mathbf{y}_t = W^0(y_t)\, dt + W(y_t) d\mathbf{x}_t, \quad y(0) = (y_1(0), y_2(0)) \in \mathbb{R}^{2e}. \tag{3.7}$$

Wherein $W = (W^1, \ldots, W^{2d})$ is the collection of vector fields on $\mathbb{R}^{2e}$ with components relative to the standard basis given by

$$W_j^i(y) := \langle W^i(y), e_j^{2e} \rangle = \begin{cases} \langle V^i(y_1), e_j^e \rangle & \text{if } i \in \{1, \ldots, d\} \text{ and } j \in \{1, \ldots, e\}, \\ \langle V^{i-d}(y_2), e_{j-d}^e \rangle & \text{if } i \in \{d+1, \ldots, 2d\} \text{ and } j \in \{e+1, \ldots, 2e\}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.8}$$

for all $y = (y_1, y_2) \in \mathbb{R}^{2e} \cong \mathbb{R}^e \times \mathbb{R}^e$. The interaction is then expressed through the vector field W^0 whose components are defined by

$$W_j^0(y) := \begin{cases} \lambda\langle \sigma(y_1, y_1), e_j^e \rangle + (1-\lambda)\langle \sigma(y_1, y_2), e_j^e \rangle & \text{if } j \in \{1, \ldots, e\}, \\ \lambda\langle \sigma(y_2, y_1), e_{j-e}^e \rangle + (1-\lambda)\langle \sigma(y_2, y_2), e_{j-e}^e \rangle & \text{if } j \in \{e+1, \ldots, 2e\}. \end{cases} \tag{3.9}$$

By writing the solution $\mathbf{y}$ in terms of its projections

$$\mathbf{y}^1 = (\mathbf{y}_1^1, \mathbf{y}_2^1) \in \mathbb{R}^{2e}, \quad \mathbf{y}^2 = \begin{pmatrix} \mathbf{y}_1^2 & \ddagger \\ \ddagger & \mathbf{y}_2^2 \end{pmatrix} \in (\mathbb{R}^{2e})^{\otimes 2}, \tag{3.10}$$

we can obtain $\mathbf{y}_i = (1, \mathbf{y}_i^1, \mathbf{y}_i^2) \in \mathrm{WG}\,\Omega_p(\mathbb{R}^e)$ for $i = 1, 2$. We will prove that the probability measure

$$\mu = u_0^{\otimes 2} * [\lambda\delta_{\mathbf{y}_1} + (1-\lambda)\delta_{\mathbf{y}_2}] \in \mathcal{M}(G\Omega_p(\mathbb{R}^e)) \tag{3.11}$$

is a fixed point of the map Ψ_ν. We will then show that every fixed point has the form (3.11); that is, it is obtained by taking the convolution of $u_0^{\otimes 2}$ with a measure supported on $\{\mathbf{y}_1, \mathbf{y}_2\}$, where $\mathbf{y}_1, \mathbf{y}_2$ are projections of *the* solution to (3.7) driven by *any* extension $\mathbf{x}$. The uniqueness of the fixed point will follow by proving that the projections $\mathbf{y}_1$ and $\mathbf{y}_2$ do not depend on the extension (and hence neither does the measure (3.11)). This is the essential content of the following proposition.

PROPOSITION 3.1. *Let $2 \leqslant p < 3$ and $y_1(0), y_2(0) \in \mathbb{R}^e$. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are two elements of $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$. Assume that W^0 is given by (3.9) and $W = (W^1, \ldots, W^{2d})$ by (3.8) and suppose that they are, respectively, in $\mathrm{Lip}^\beta(\mathbb{R}^{2e})$ and $\mathrm{Lip}^\gamma(\mathbb{R}^{2e})$ for some $\beta > 1$ and $\gamma > p$. Let $\mathbf{x}$ be any element of $\mathrm{WG}\,\Omega_p(\mathbb{R}^{2d})$ which extends $\mathbf{x}_1$ and $\mathbf{x}_2$ in the sense of (3.6), and let $\mathbf{y}$ be the unique solution in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{2e})$ to the RDE (3.7) driven by $\mathbf{x}$. Then $\mathbf{y}$ has the property that its projections $\mathbf{y}_1, \mathbf{y}_2$ (as given in (3.10)) are elements of $\mathrm{WG}\,\Omega_p(\mathbb{R}^e)$ which depend on $\mathbf{x}_1$ and $\mathbf{x}_2$, but not on the extension $\mathbf{x}$.*

Proof. We prove that $\mathbf{y}_1$ and $\mathbf{y}_2$, the projections of the solution to (3.7), depend only on $\mathbf{x}_1$ and $\mathbf{x}_2$ and not the iterated integral between them. In other words, that $y = \mathbf{y}_1$ and $\mathbf{y}_2$ are uniquely determined independently of the terms † needed to specify the joint lift in (3.6). To see this, we recall [**10**] that

$$\mathbf{y}_{s,t}^1 \simeq W^0(y_s)(t-s) + W(y_s)\mathbf{x}_{s,t}^1 + DW(y_s)W(y_s)\mathbf{x}_{s,t}^2 \tag{3.12}$$

and

$$\mathbf{y}^2_{s,t} \simeq [W(y_s) \otimes W(y_s)]\mathbf{x}^2_{s,t}, \tag{3.13}$$

where $\simeq$ indicates that the difference between the left- and right-hand sides may be controlled, uniformly on $\Delta_{[0,T]}$, by $C\omega(s,t)^\theta$ for some finite $C > 0$ and $\theta > 1$. In the language of [**25**], this say that the right-hand sides of (3.12) and (3.13) constitute a θ-almost p-rough path controlled by ω. The rough paths $\mathbf{y}^1$ and $\mathbf{y}^2$ are determined uniquely by this almost rough path by an appropriate limiting procedure (see the statement and proof of [**25**, Theorem 4.3]). It therefore suffices to prove that the projections of the right-hand sides of (3.12) and (3.13) corresponding to $\mathbf{y}_1$ and $\mathbf{y}_2$ depend only on $\mathbf{x}_1$ and $\mathbf{x}_2$.

To start we observe that the first two terms in (3.12) only depend on y and $\mathbf{x}^1$, in particular they independent of any cross-iterated integrals in $\mathbf{x}$. We then rewrite the last term as

$$\tfrac{1}{2}DW(y_s)W(y_s)[\mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{s,t}] + DW(y_s)W(y_s)\mathbf{x}^{2;a}_{s,t},$$

where $\mathbf{x}^{2;a}_{s,t}$ is the anti-symmetric part of the 2-tensor $\mathbf{x}^2_{s,t}$. The first term only depends on $\mathbf{x}^1_{s,t}$, and the second term can be simplified to

$$\sum_{p,q=1}^{2d} [W^p, W^q](y_s)\mathbf{x}^{2;p,q}_{s,t}.$$

From the definition (3.8) of the vector fields W^i, it is easy to see that the Lie brackets

$$[W^p, W^q] \equiv 0 \quad \forall p \in \{1,\dots,d\} \text{ and } \forall q \in \{d+1,\dots,2d\}$$

(and, therefore, it also vanishes for every $p \in \{d+1,\dots,2d\}$ and $q \in \{1,\dots,d\}$ by antisymmetry). Each summand in (3.12) thus only depends on $\mathbf{x}^1$, $\mathbf{x}^2_1$ and $\mathbf{x}^2_2$, but not on the terms of $\mathbf{x}^2$ corresponding to integrals between $\mathbf{x}^1_1$ and $\mathbf{x}^1_2$. As explained above, the same is hence true for $\mathbf{y}^1$.

Working now on (3.13), we see that, in general, $\mathbf{y}^2$ *will* depend on the extension, however, this dependence will disappear once we take projections. This is can be discerned from the following calculation: for all $i,j \in \{1,\dots,e\}$

$$\begin{aligned}
\langle \mathbf{y}^2_1(s,t), e^e_i \otimes e^e_j\rangle &= \langle \mathbf{y}^2_{s,t}, e^{2e}_i \otimes e^{2e}_j\rangle \\
&\simeq \langle [W(y_s)\otimes W(y_s)]\mathbf{x}^2_{s,t}, e^{2e}_i \otimes e^{2e}_j\rangle \\
&= \sum_{p,q=1}^{2d} \mathbf{x}^{2;p,q}_{s,t} W^p_i(y_s)W^q_j(y_s) \\
&= \sum_{p,q=1}^{d} \mathbf{x}^{2;p,q}_{s,t} W^p_i(y_s)W^q_j(y_s) \\
&= \sum_{p,q=1}^{d} \mathbf{x}^{2;p,q}_1(s,t) W^p_i(y_s)W^q_j(y_s),
\end{aligned}$$

where the fourth line follows from the fact that $W^p_i(y) = W^q_j(y) = 0$ for all $p,q \in \{d+1,\dots,2d\}$ if $i,j \in \{1,\dots,e\}$. A similar calculation shows that $\mathbf{y}^2_2$ depends only on y and $\mathbf{x}^2_2$ but not the cross-terms, whence the proof is complete. $\square$

Corollary 3.2. *Let $\nu = \lambda\delta_{\mathbf{x}_1} + (1-\lambda)\delta_{\mathbf{x}_2} \in \mathcal{M}(G\Omega_p(\mathbb{R}^d))$. There exists a unique fixed point $\mu \in \mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$ of the map Ψ_ν which is given explicitly by*

$$\mu = u_0^{\otimes 2} * [\lambda\delta_{\mathbf{y}_1} + (1-\lambda)\delta_{\mathbf{y}_2}], \tag{3.14}$$

where $\mathbf{y}_i \in \mathrm{WG}\,\Omega_p(\mathbb{R}^{2e})$, $i = 1, 2$ *are the projections of the solution to* (3.7) *driven by any extension* $\mathbf{x}$ *of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$.

Proof. The previous proposition ensures (3.14) is well defined. In other words, for every fixed realization $(y_1(0), y_2(0))$ of $u_0^{\otimes 2}$ the rough paths $\mathbf{y}_1$ and $\mathbf{y}_2$ will not depend on the choice of extension. We must now check that this is the only fixed point. To do so, first note that the assumption on ν implies that any fixed point must have the form

$$\mu = u_0^{\otimes 2} * [\lambda\delta_{\mathbf{z}_1} + (1-\lambda)\delta_{\mathbf{z}_2}],$$

where $\mathbf{z}_i$, $i = 1, 2$ are elements of $\mathrm{WG}\,\Omega_p(\mathbb{R}^e)$. Then, by the definition of the map Ψ_ν, $\mathbf{z}_i$, $i = 1, 2$ must solve the RDEs

$$d\mathbf{z}_i(t) = V^0(z_i(t))d\gamma_t^\mu + V(z_i(t))\,d\mathbf{x}_i(t),$$

with initial conditions $z_i(0)$. Let $\mathbf{x}$ be any path in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{2d})$ whose projections are consistent with $\mathbf{x}_1$ and $\mathbf{x}_2$. Then, since $\mathbf{z}_1$ and $\mathbf{z}_2$ may both be written as solutions to RDEs driven by $\mathbf{x}$, we may define in a canonical way (see [**10**]) a path $\mathbf{z}$ in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{2d})$, which has $\mathbf{z}_1$ and $\mathbf{z}_2$ as its projections. It follows that $\mathbf{z}$ is then the solution of the RDE (3.7) driven along $\mathbf{x}$. □

3.2. *N-particle systems*

We will later want to consider the propagation of chaos phenomenon for rough differential equations, and this requires us to present the treatment of the previous subsection for a population of particles of arbitrary finite size N. We therefore suppose that the preference measure is now given by

$$\nu = \sum_{i=1}^{N} \lambda_i \delta_{\mathbf{x}_i}.$$

Analogously to the two-particle case (recall (3.8)), we introduce vector fields W^0 and $W = (W^1, \ldots, W^{Nd})$ for the extended system on $\mathbb{R}^{Ne} \cong \mathbb{R}^e \times \cdots \times \mathbb{R}^e$. As preparation, for any $l \in \mathbb{N}$ we let $n(l)$ denote the unique integer in $\{1, \ldots, d\}$ such that $n(l) \equiv l (\mathrm{mod}\ d)$, and $m(l)$ the unique integer in $\{1, \ldots, e\}$ such that $m(l) \equiv l (\mathrm{mod}\ e)$. We then define

$$W_j^i(y) := \langle W^i(y), e_j^{Ne}\rangle = \begin{cases} \langle V^{n(i)}(y_{\lceil i/d\rceil}), e^e_{m(j)}\rangle & \text{if } \lceil i/d\rceil = \lceil j/e\rceil, \\ 0 & \text{otherwise,} \end{cases} \tag{3.15}$$

for all $y = (y_1, \ldots, y_N) \in \mathbb{R}^{Ne} \cong \mathbb{R}^e \times \cdots \times \mathbb{R}^e$ and set

$$W_j^0(y) := \langle W^0(y), e_j^{Ne}\rangle = \sum_{i=1}^{N} \lambda_i \langle \sigma(y_{\lceil j/e\rceil}, y_{\lceil i/e\rceil}), e^e_{m(j)}\rangle. \tag{3.16}$$

As before, we will be interested in rough paths in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{Nd})$ whose projections contain each of the rough paths $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ in the support of the measure ν. The following notation is useful in assigning indices to the components of the extension, in terms of the components of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$.

NOTATION 2. For each k and r in $\mathbb{N}$ and $m = 1, \ldots, N$, define $I_{k,m,r}$, a subset of $\{1, \ldots, Nr\}^k$, by

$$(i_1, \ldots, i_k) \in I_{k,m,r} \text{ if and only if } \{i_1, \ldots, i_k\} \subseteq \{(m-1)r + 1, \ldots, mr\}.$$

We will let $I_{k,r}$ denote the subset $\bigcup_{m=1}^{N} I_{k,m,r}$.

We now formalize the precise sense in which $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ is related to any extension.

DEFINITION 3.3. If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ is a collection of rough paths in $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$, then we say that $\mathbf{x}$ in $\mathrm{WG}\,\Omega_p(\mathbb{R}^{Nd})$ is a lift which is consistent with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$, if for every $k = 1, \ldots, \lfloor p \rfloor$ we have

$$\mathbf{x}^{k;i_1,\ldots,i_k} = \mathbf{x}_m^{k;n(i_1),\ldots,n(i_k)} \quad \forall (i_1, \ldots, i_k) \in I_{k,m,d}, \ \forall m = 1, \ldots, N,$$

where, as above, $n(l)$ denotes the unique integer in $\{1, \ldots, d\}$ such that $n(l) \equiv l (\mathrm{mod}\ d)$.

We now choose any extension $\mathbf{x}$ which is consistent with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$. We want to show that if we solve the RDE

$$d\mathbf{y}_t = W^0(y_t)dt + W(y_t)d\mathbf{x}_t, \quad y(0) = (y_1(0), \ldots, y_N(0)) \in \mathbb{R}^{Ne},$$

along $\mathbf{x}$, then the output $\mathbf{y}$ will have the same projections irrespective of the initial choice of extension. To do so will involve an application of [**26**, Theorem 20], and hence involves identifying a normal subgroup K of $G^n(\mathbb{R}^{Nd})$ so that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ can be identified with a path in the quotient group $G^n(\mathbb{R}^{Nd})/K$.

LEMMA 3.4. *For $n \in \mathbb{N}$, let $\mathfrak{g}^n(\mathbb{R}^{Nd})$ denote the Lie algebra of $G^n(\mathbb{R}^{Nd})$. Suppose that $\mathfrak{k}^n(\mathbb{R}^{Nd})$ is the subset of $\mathfrak{g}^n(\mathbb{R}^{Nd})$ defined by*

$$\mathfrak{k}^n(\mathbb{R}^{Nd}) = \left\{ a \in \mathfrak{g}^n(\mathbb{R}^{Nd}) : \langle e_I, a^{|I|} \rangle = 0, \ \forall I \in \textstyle\bigcup_{k=1}^n I_{k,d} \right\},$$

where, if $I = (i_1, \ldots, i_k)$, then we write $|I| = k$ and $e_I := e_{i_1} \otimes \cdots \otimes e_{i_k}$. Let

$$K^n(\mathbb{R}^{Nd}) := \exp(\mathfrak{k}^n(\mathbb{R}^{Nd})).$$

Then $K^n(\mathbb{R}^{Nd})$ is a connected Lie subgroup of $G^n(\mathbb{R}^{Nd})$, $\mathfrak{k}^n(\mathbb{R}^{Nd})$ is an ideal in $\mathfrak{g}^n(\mathbb{R}^{Nd})$ and hence $K^n(\mathbb{R}^{Nd})$ is a normal subgroup of $G^n(\mathbb{R}^{Nd})$.

Proof. It is immediate that $K^n(\mathbb{R}^{Nd})$ is a connected Lie subgroup. To prove that $\mathfrak{k}^n(\mathbb{R}^{Nd})$ is an ideal $\mathfrak{g}^n(\mathbb{R}^{Nd})$, we need to show that for any a in $\mathfrak{k}^n(\mathbb{R}^{Nd})$ and b in $\mathfrak{g}^n(\mathbb{R}^{Nd})$ we have

$$\langle e_I, [a, b]^{|I|} \rangle = 0 \quad \text{for all } I \in I_{k,d}, \ k = 1, \ldots, n.$$

But this follows by noticing that

$$\langle e_I, (a \otimes b)^k \rangle = \sum_{l=1}^{k-1} \langle e_I, a^l \otimes b^{k-l} \rangle = \sum_{l=1}^{k-1} \langle e_{I(l)}, a^l \rangle \langle e_{I(k-l)}, b^{k-l} \rangle = 0.$$

Where, for every $l = 1, \ldots, k - l$, we have written $I =: (I(l), I(k - l))$ and used the fact that $I \in I_{k,d}$ to deduce $I(l) \in I_{l,d}$ and $I(k - l) \in I_{k-l,d}$. It is easily seen from this that $\langle e_I, [a, b]^{|I|} \rangle = 0$. The assertion that $K^n(\mathbb{R}^{Nd})$ is normal then follows from the well-known correspondence between ideals of Lie algebras and normal subgroups of the Lie group (see, for example, [**22**]). □

REMARK 3.5. In a straightforward way, we may uniquely identify any given collection of rough paths $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ in $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$ with a path, which we denote by $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, in the quotient group:

$$G^{\lfloor p \rfloor}(\mathbb{R}^{Nd})/K^{\lfloor p \rfloor}(\mathbb{R}^{Nd}).$$

Indeed, if we let $G^{\lfloor p\rfloor}(\mathbb{R}^d)^N$ denote the set $G^{\lfloor p\rfloor}(\mathbb{R}^d)\times\cdots\times G^{\lfloor p\rfloor}(\mathbb{R}^d)$ equipped with the direct product group structure, then $\Psi: G(\mathbb{R}^{Nd})\to G(\mathbb{R}^d)^N$ obtained by projecting

$$\Psi(\mathbf{x})=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$$

is a surjective group homomorphism and $\ker\Psi=K^{\lfloor p\rfloor}(\mathbb{R}^{Nd})$. Hence we have the isomorphic relation

$$G(\mathbb{R}^d)^N\cong G^{\lfloor p\rfloor}(\mathbb{R}^{Nd})/K^{\lfloor p\rfloor}(\mathbb{R}^{Nd}),$$

we identify $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}$ with $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$ in the obvious way through this isomorphism, whereupon $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$ will have finite p-variation with respect to the homogenous quotient norm (see [**26**] for details). Any extension $\mathbf{x}\in\mathrm{WG}\Omega_p(\mathbb{R}^{Nd})$ which is consistent with $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$, can then also be said to extend $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$ in the obvious sense that

$$\pi_{G^{\lfloor p\rfloor}(\mathbb{R}^{Nd}),G^{\lfloor p\rfloor}(\mathbb{R}^{Nd})/K^{\lfloor p\rfloor}(\mathbb{R}^{Nd})}(\mathbf{x})=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N),$$

where $\pi_{G^{\lfloor p\rfloor}(\mathbb{R}^{Nd}),G^{\lfloor p\rfloor}(\mathbb{R}^{Nd})/K^{\lfloor p\rfloor}(\mathbb{R}^{Nd})}$ is the canonical homomorphism.

We now prove the generalization of Proposition 3.1 to the N-particle system.

THEOREM 3.6. *Let $p\geqslant 1$, $y_1(0),\ldots,y_N(0)\in\mathbb{R}^e$, and suppose that $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}$ is a collection of rough paths in* $\mathrm{WG}\,\Omega_p(\mathbb{R}^e)$. *Assume that W^0, defined by (3.16), and $W=(W^1,\ldots,W^{Nd})$, defined by (3.15) are, respectively, vector fields in $\mathrm{Lip}^\beta(\mathbb{R}^{Ne})$ and $\mathrm{Lip}^\gamma(\mathbb{R}^{Ne})$ for some $\beta>1$ and $\gamma>p$. For any q in $[p,\gamma)$, let $\mathbf{x}$ be an element of $\mathrm{WG}\,\Omega_q(\mathbb{R}^{Nd})$ which is an extension consistent with $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$. Suppose that $\mathbf{y}$ is the unique solution in $\mathrm{WG}\,\Omega_q(\mathbb{R}^{Ne})$ to the RDE (3.7) driven by $\mathbf{x}$. Then $\mathbf{y}$ has the property that its projections $\mathbf{y}_1,\ldots,\mathbf{y}_N$ to elements of $\mathrm{WG}\,\Omega_q(\mathbb{R}^e)$ depend on $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$, but not on the extension $\mathbf{x}$.*

Proof. From Remark 3.5 and the LV Extension Theorem, there always exists an extension $\mathbf{x}$ of $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$ in $\mathrm{WG}\,\Omega_q(\mathbb{R}^{Nd})$ for any $q>p$ (and any $q\geqslant p$, if p is not an integer). Let us define an algebra homomorphism from the (truncated) tensor algebra $T^{\lfloor p\rfloor}(\mathbb{R}^{Nd})$ into the space of continuous differential operators, by taking the linear extension of

$$F^W(e_{i_1\cdots i_n})=W^{i_1}\circ\cdots\circ W^{i_n}.$$

Restricting F^W to $\mathfrak{g}^{\lfloor p\rfloor}(\mathbb{R}^{Nd})$ gives a Lie algebra homomorphism into the space of vector fields on $\mathbb{R}^{Ne}$. An easy calculation confirms that

$$\ker(F^W|_{\mathfrak{g}^{\lfloor p\rfloor}(\mathbb{R}^{Nd})})\supseteq\mathfrak{k}^{\lfloor p\rfloor}(\mathbb{R}^{Nd}),\tag{3.17}$$

whereupon [**26**, Theorem 20] shows that $\mathbf{y}^1=y$ depends on $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$ but is independent of the other terms in the extension $\mathbf{x}$ of $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$.

In general, $\mathbf{y}^2,\ldots,\mathbf{y}^{\lfloor p\rfloor}$ will still depend on the choice of extension. Nevertheless, the *projections* of $\mathbf{y}$ to the N rough paths $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_N$ will be not do so. This is most easily seen when $p\in[2,3)$ by the same calculation as in the proof of Proposition 3.1. To deal with the general case, we first recall that, since $\mathbf{y}$ is the solution to an RDE driven by $\mathbf{x}$, there exist a collection of *derivative processes*

$$\{y^k: k=1,\ldots,\lfloor p\rfloor-1\},$$

which together with y form an $\mathbf{x}$-controlled rough path in the sense of Gubinelli (see [**10**, **13**, **14**]). Each y^k is an $L((\mathbb{R}^{Nd})^{\otimes k},\mathbb{R}^{Ne})$-valued path which we will detail later in the proof. For the moment, we observe that $\mathbf{y}^n_{s,t}$ can be approximated (in the same sense as in (3.12)) in terms of these derivative processes as follows:

$$\mathbf{y}^n_{s,t} \simeq \sum_{\substack{j_1,\ldots,j_n=1,\\ j_1+\cdots+j_n\leqslant\lfloor p\rfloor}}^{\lfloor p\rfloor-1} y_s^{j_1}\otimes\cdots\otimes y_s^{j_n}\sum_{\pi\in OS(j_1,\ldots,j_n)}\pi^{-1}\mathbf{x}^J_{s,t}, \tag{3.18}$$

where: $J := j_1+\cdots+j_n$, and $OS(j_1,\ldots,j_n)$ is a subset of the symmetric group S_J of permutations of $\{1,\ldots,J\}$ consisting of *ordered shuffles* (see [**25**, p. 72]). As usual, we assume that elements σ of S_J have an action on $(\mathbb{R}^{Nd})^{\otimes J}$ which is determined by $\sigma(x_1\otimes\cdots\otimes x_J) = x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(J)}$.

Now suppose $(i_1,\ldots,i_n)\in I_{n,m,e}$ for some $m\in\{1,\ldots,N\}$. We need to calculate

$$\mathbf{y}^{n;i_1,\ldots,i_n}_{s,t} = \langle e^{Ne}_{i_1,\ldots,i_n}, \mathbf{y}^n_{s,t}\rangle, \tag{3.19}$$

and prove that this depends only on $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N$ and y. We first compute the corresponding coefficient of the summands in the approximation (3.18). To this end, we fix $j_1,\ldots,j_n\in\{1,\ldots,\lfloor p\rfloor-1\}$ satisfying $j_1+\cdots+j_n=J\leqslant\lfloor p\rfloor$ and let $\pi\in OS(j_1,\ldots,j_n)$. A simple calculation then shows that we can rewrite

$$\langle e^{Ne}_{i_1,\ldots,i_n}, (y_s^{j_1}\otimes\cdots\otimes y_s^{j_n})\pi^{-1}\mathbf{x}^J_{s,t}\rangle,$$

as

$$\sum_{p_1,\ldots,p_J=1}^{Nd}\langle e^{Nd}_{p_1,\ldots,p_J},\pi^{-1}\mathbf{x}^J_{s,t}\rangle\prod_{l=1}^{n}\langle e^{Ne}_{i_l}, y_s^{j_l;p_{j_1+\cdots+j_{l-1}+1},\ldots,p_{j_1+\cdots+j_l}}\rangle, \tag{3.20}$$

where $y_s^{k;i_1,\ldots,i_k}\in\mathbb{R}^{Ne}$ is determined from y_s^k by

$$y_s^{k;i_1,\ldots,i_k} := y_s^k(e^{Nd}_{i_1,\ldots,i_k}).$$

We will prove that for every $i=1,\ldots,Ne$, and $k=1,\ldots,\lfloor p\rfloor-1$ the coefficient

$$\langle e^{Ne}_i, y_s^{k;l_1,\ldots,l_k}\rangle = 0, \tag{3.21}$$

for any collection of indices $(l_1,\ldots,l_k)\in\{1,\ldots,Nd\}^k$, unless $\lceil l_r/d\rceil = \lceil i/e\rceil$ for every $r = 1,\ldots,k$. Once this has been established, we may use it together with the fact that $(i_1,\ldots,i_n)\in I_{n,m,e}$, which implies

$$\lceil i_1/e\rceil = \lceil i_2/e\rceil = \cdots = \lceil i_n/e\rceil = m,$$

to reduce the summands in (3.20) to a product of terms involving only the coefficients of y_s^k and

$$\langle e^{Nd}_{(p_1,\ldots,p_J)},\pi^{-1}\mathbf{x}^J_{s,t}\rangle = \langle e^d_{(p_1-m-1,\ldots,p_J-m-1)},\pi^{-1}\mathbf{x}^J_m(s,t)\rangle \quad \text{for } (p_1,\ldots,p_J)\in I_{J,m,d}. \tag{3.22}$$

The required result will then follow from two observations. The first being that the term on the right-hand side of (3.22) depends only on $\mathbf{x}_m$, but not the other terms in the extension of $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)$. The second observation is that the coefficients of y_s^k are functions of y_s. Indeed, because $\mathbf{y}$ is the solution to the RDE (3.7) there exist functions

$$F_k:\mathbb{R}^{Ne}\to L((\mathbb{R}^{Nd})^{\otimes k},\mathbb{R}^{Ne}) \quad \text{for } k=1,\ldots,\lfloor p\rfloor-1,$$

such that $y_s^k = F_k(y)$ for $k\geqslant 1$, and these functions are determined explicitly (see [**19**]) by the recurrence relation

$$\begin{aligned} F_1(y)(v) &= W(y)(v),\\ F_k(y)(v_1\otimes\cdots\otimes v_k) &= DF_{k-1}(y)(W(y)(v_1))(v_2\otimes\cdots\otimes v_k)\\ &= \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} F_{k-1}(y+\epsilon W(y)(v_1))(v_2\otimes\cdots\otimes v_k). \end{aligned} \tag{3.23}$$

What remains to be proved is the assertion (3.21). To do so, we first introduce

$$F_k^{j_1\dots,j_k}(y) := F_k(y)(e^{Nd}_{j_1,\dots,j_k}) \in \mathbb{R}^{Ne},$$

and then prove the following statement $\chi(k)$ for every $k = 1,\dots,\lfloor p\rfloor - 1$:

$\chi(k)$: For every $y = (y_1,\dots,y_N) \in \mathbb{R}^{Ne}$, $i \in \{1,\dots,Ne\}$ and $j_1,\dots,j_k \in \{1,\dots,Nd\}$

$$\langle e_i^{Ne}, F_k^{j_1,\dots,j_k}(y)\rangle = 0,$$

unless there exists $n^* \in \{1,\dots,N\}$ such that

$$\left\lceil\frac{j_1}{d}\right\rceil = \left\lceil\frac{j_2}{d}\right\rceil = \cdots = \left\lceil\frac{j_k}{d}\right\rceil = \left\lceil\frac{i}{e}\right\rceil = n^*, \tag{3.24}$$

and if (3.24) does hold, then $\langle e_i^{Ne}, F_k^{j_1,\dots,j_k}(y)\rangle$ depends only on y_{n^*}.

The proof is by induction on k. In the case $k = 1$, we note that $F_1^j(y) = W^j(y)$ and the conclusion is immediate from the definition of the vector fields W^j in formula (3.15). Assuming $\chi(k)$ to hold for each $k = 1,\dots,r-1$ we use (3.23) to see that

$$\langle e_i^{Ne}, F_r^{j_1,\dots,j_r}(y)\rangle = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} \langle e_i^{Ne}, F_{r-1}^{j_2,\dots,j_r}(y+\epsilon F_1^{j_1}(y))\rangle.$$

Using $\chi(r-1)$, we deduce $F_{r-1}^{j_2,\dots,j_r} \equiv 0$, unless for some n^* we have

$$\left\lceil\frac{j_2}{d}\right\rceil = \cdots = \left\lceil\frac{j_k}{d}\right\rceil = \left\lceil\frac{i}{e}\right\rceil = n^* \in \{1,\dots,N\}, \tag{3.25}$$

and in the case described by (3.25) $F_{r-1}^{j_2,\dots,j_r}(y)$ depends only on y_{n^*}. By combining this with $\chi(1)$, we learn that

$$\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} F_{r-1}^{j_2,\dots,j_r}(y+\epsilon F_1^{j_1}(y)) \tag{3.26}$$

will vanish unless we also have $\lceil j_1/d\rceil = n^*$ (in which case (3.26) will again depend only on y_{n^*}). This completes the induction and the proof. □

Remark 3.7. Each $\mathbf{y}_i = \Theta_{V^0,V}(\mu, y_i(0), \mathbf{x}_i)$ solves an RDE driven by $\mathbf{x}_i \in \mathrm{WG}\Omega_p(\mathbb{R}^d)$ and classical RDE theory therefore guarantees that $\mathbf{y}_i$ is in $\mathrm{WG}\,\Omega_p(\mathbb{R}^e)$ and not just $\mathrm{WG}\,\Omega_q(\mathbb{R}^e)$. If $\mathbf{x}_i \in G\Omega_p(\mathbb{R}^d)$, then the universal limit theorem ensures that $\mathbf{y}_i$ is an element of $G\Omega_p(\mathbb{R}^e)$; this observation will be useful later on. It follows from Theorem 3.6, together with a suitable elaboration of the arguments of Corollary 3.2, that

$$\mu = u_0^{\otimes N} * \sum_{i=1}^N \lambda_i\delta_{\mathbf{y}_i}$$

is the unique fixed point of Ψ_ν.

4. *A fixed point and continuity theorem*

We now want to consider the case where the preference measure ν is a non-discrete measure on rough path space. The main problem we address is to find a condition on ν to force the existence of a unique fixed point to the map Ψ_ν. A key feature will be the use of estimates controlling:

$$\rho_{p\text{-var};[0,T]}(\mathbf{y}^1, \mathbf{y}^2),$$

the $\rho_{p\text{-var};[0,T]}$-distance between two RDE solutions $\mathbf{y}^1$ and $\mathbf{y}^2$ driven by $\mathbf{x}$. These estimates need have two properties: they need to be Lipschitz in the defining data (starting point, vector fields, etc.) *and* the Lipschitz constant must have integrable dependence on $\mathbf{x}$, when $\mathbf{x}$ is realized according to a wide class of measures. Classical RDE estimates typically satisfy the first of these criteria, the latter needs more work. For example, in [**11**], the authors have proved estimates of the form

$$\rho_{p\text{-var};[0,T]}(\mathbf{y}^1,\mathbf{y}^2) \leqslant (*) \exp(\|\mathbf{x}\|^p_{p\text{-var};[0,T]}), \tag{4.1}$$

where the terms $(*)$ are Lipschitz in the data. The drawback of this estimate is that the right-hand side fails to be integrable, for example, when $\mathbf{x}$ is the lift of a wide class of common process including Brownian motion and fractional Brownian motion with $H < \frac{1}{2}$; this has implications for our fixed-point theorem. Fortunately, it is possible to replace $\|\mathbf{x}\|^p_{p\text{-var};[0,T]}$ in (4.1) by a quantity called the *accumulated α-local p-variation* (see below). By then making use of the recent tail estimates involving this quantity [**5**], we are able to cover these interesting examples.

For the sake of orientation, we briefly sketch the steps that we need to work through in this section.

Step 1: We enhance the Lipschitz estimate (4.1) and exhibit a Lipschitz constant of the form $\exp(M_{\mathbf{x}})$. The functional $M_\cdot$ will be defined on the space of weakly geometric rough paths, and will satisfy

$$\forall q > 0 : \int \exp(qM_{\mathbf{x}})\nu(d\mathbf{x}) < \infty,$$

for a wide class of measures $\nu \in \mathcal{M}(\mathrm{WG}\Omega_p(\mathbb{R}^d))$.

Step 2: Fixing a measure $u_0 \times \nu$ in a subset of $\mathcal{M}(\mathbb{R}^e \times \mathrm{WG}\,\Omega_p(\mathbb{R}^d))$, we use Step 1 to prove that the map $\Psi_\nu : \mathcal{M}_1(G\Omega_p(\mathbb{R}^e)) \to \mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$ introduced in (3.3) has a unique fixed point. Beforehand we spend some time setting up the problem. In particular, we identify an appropriate choice of Banach space in which to accommodate the occupation measure path:

$$\gamma^\mu_t = \int_0^t \mu_s\, ds,$$

for any path $(\mu_t)_{t\geqslant 0}$ in $\mathcal{M}(G^{\lfloor p\rfloor}(\mathbb{R}^e))$.

Step 3: With u_0 still fixed, we leverage the rough path topology to establish the continuity of the map which takes ν in $\mathcal{M}(\mathrm{WG}\Omega_p(\mathbb{R}^d))$ to the unique fixed point of Ψ_ν.

4.1. *Lipschitz-continuity for RDEs with drift*

We recall the definition of the following function from [**5**].

DEFINITION 4.1. Let $\alpha > 0$ and $I \subseteq \mathbb{R}$ be a compact interval. Suppose that $\omega : I \times I \to \mathbb{R}^+$ is a control. We define the accumulated α-local ω-variation by

$$M_{\alpha,I}(\omega) = \sup_{\substack{D(I)=(t_i)\\ \omega(t_i,t_{i+1})\leqslant\alpha}} \sum_{i:t_i\in D(I)} \omega(t_i,t_{i+1}).$$

REMARK 4.2. Note that for two controls ω_1 and ω_2, we have $M_{\alpha,I}(\omega_1+\omega_2) \leqslant M_{\alpha,I}(\omega_1) + M_{\alpha,I}(\omega_2)$.

The following lemma is a Lipschitz estimate on the RDE solution (with drift), when we vary the defining data of the differential equation.

LEMMA 4.3. *Let $\gamma > p \geqslant 1$ and $\beta > 1$. Suppose that $\mathbf{x}$ is a weakly geometric p-rough path in $\mathrm{WG}\,\Omega_p(\mathbb{R}^d)$, and assume that γ^1 and γ^2 are two paths which take values in some Banach* *space E, and belong to $C^{1\text{-var}}([0,T],E)$. Then $\omega : \Delta_{[0,T]} \to \mathbb{R}^+$ defined by*

$$\omega(s,t) := \sum_{i=1}^{2} \|\gamma^i\|_{1-\text{var};[s,t]} + \sum_{i=1}^{2} \|\mathbf{x}^i\|^p_{p\text{-var};[s,t]}$$

is a control. Furthermore, if $V = (V^1, \ldots, V^d)$ is a collection of vector fields in $\mathrm{Lip}^\gamma(\mathbb{R}^e)$, and V^0 is in $\mathrm{Lip}^\beta(\mathbb{R}^e, L(E, \mathbb{R}^e))$, then for $i = 1, 2$ the RDEs

$$d\mathbf{y}^i_t = V(y^i_t)\, d\mathbf{x}^i_t + V^0(y^i_t)\, d\gamma^i_t,$$
$$\pi_1 \mathbf{y}^i_t = y^i_0$$

have unique solutions. And for every $\alpha > 0$ and some $C = C(v,\alpha) > 0$, we also have the following Lipschitz-continuity of the solutions:

$$\rho_{p,\omega}(\mathbf{y}^1, \mathbf{y}^2) \leqslant C[|y^1_0 - y^2_0| + \rho_{1,\omega}(\gamma^1,\gamma^2) + \rho_{p,\omega}(\mathbf{x}^1,\mathbf{x}^2)] \exp(CM_{\alpha,[0,T]}(\omega)).$$

Proof. The proof is obtained by following the arguments of [**11**, Theorem 12.10] on RDEs with drift; two enhancements are necessary. The first is allow the drift term to take values in an arbitrary (infinite-dimensional) Banach space. This is elementary, because in the current lemma γ^1 and γ^2 have bounded variation, and hence classical ordinary differential equation estimates can be used everywhere. The second, more subtle, enhancement is to end up with the accumulated α-local ω-variation featuring in the exponential on the right-hand side (as opposed to the usual $\omega(0,T)$). For this, we refer the reader to [**5**; **11**, Remark 10.64]. □

By exploiting the relationship between $\rho_{p,\omega}$ and $\rho_{p\text{-var}}$, we can obtain a Lipschitz estimate in $\rho_{p\text{-var}}$-distance.

Corollary 4.4. *With the notation of, and under the same conditions as, Lemma 4.3, we have*

$$\rho_{p\text{-var};[0,T]}(\mathbf{y}^1, \mathbf{y}^2) \leqslant C\omega(0,T)^N [|y^1_0 - y^2_0| + \|\gamma^1 - \gamma^2\|_{1\text{-var};[0,T]} + \|\mathbf{x}^1 - \mathbf{x}^2\|_{p\text{-var};[s,t]}] \exp(CM_{\alpha,[0,T]}(\omega)),$$

for some $N > 0$.

4.2. *Measure-valued paths*

For the current application, the main interest in these Lipschitz estimates will occur when the space of probability measures $\mathcal{M}(G^{\lfloor p \rfloor}(\mathbb{R}^e))$ is embedded in a Banach space E. In the typically case, γ will then be constructed from $\mu \in \mathcal{M}(G\Omega_p(\mathbb{R}^e))$ by setting $\gamma_t := \int_0^t \mu_s ds$. For the moment, we develop this more abstractly by letting $\mathrm{Lip}^1(S)^*$denote the dual of Lip-1 functions (that is, the bounded Lipschitz functions) on a metric space (S,d). There is a canonical injection $\mu \mapsto T_\mu$ from $\mathcal{M}(S)$ into $\mathrm{Lip}^1(S)^*$ defined by the integration of functions in $\mathrm{Lip}^1(S)$ against μ:

$$T_\mu(\phi) = \langle T_\mu, \phi \rangle := \int_S \phi(s)\mu(ds). \tag{4.2}$$

In this setting, two metric spaces will be of special interest as already mentioned in Section 2. The first is the step-N free nilpotent group with e generators, $G^N(\mathbb{R}^e)$, with the (inhomogeneous) metric it inherits from the tensor algebra:

$$d_N(\mathbf{g}, \mathbf{h}) := \max_{i=1,\ldots,N} |\pi_i(\mathbf{g} - \mathbf{h})|.$$

The second is the space of geometric p-rough paths $G\Omega_p(\mathbb{R}^d)$ equipped with $\rho_{p\text{-var};[0,T]}$.

The next lemma examines the regularity of the paths which result from (4.3) the pushforward of μ under the evaluation maps, that is,

$$\mu_t \equiv (\psi_t)_* \mu \in \mathcal{M}(G^{\lfloor p \rfloor}(\mathbb{R}^e)). \tag{4.3}$$

LEMMA 4.5. *Suppose $p \geqslant 1$ and let μ be an element of $\mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$. Let $(\mu_t)_{t\in[0,T]}$ be the path in $\mathcal{M}(G^{\lfloor p \rfloor}(\mathbb{R}^e))$ defined by (4.3), and $(T_{\mu_t})_{t\in[0,T]}$ be the path in $\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*$ obtained by the injection of $\mathcal{M}(G^{\lfloor p \rfloor}(\mathbb{R}^e))$ into $\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*$. Then for every $0 \leqslant s \leqslant t \leqslant T$, we have*

$$\|T_{\mu_t} - T_{\mu_s}\|_{\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*} \leqslant \int_{G\Omega_p(\mathbb{R}^e)} d_{\lfloor p \rfloor}(\mathbf{y}_s, \mathbf{y}_t)\mu(d\mathbf{y}) \leqslant \int_{G\Omega_p(\mathbb{R}^e)} \rho_{p\text{-var};[0,t]}(1, \mathbf{y})\mu(d\mathbf{y}),$$

where $\mathbf{y}_u = \psi_u(\mathbf{y})$. In particular, $t \mapsto T_{\mu_t}$ is a continuous path in $\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*$.

Proof. Take $\phi \in \mathrm{Lip}^1(S_p)$ with $\|\phi\|_{\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))} = 1$. The result then follows from the proceeding calculation:

$$\begin{aligned}
|\langle T_{\mu_t} - T_{\mu_s}, \phi \rangle| &= \left| \int_{G^{\lfloor p \rfloor}(\mathbb{R}^e)} \phi(\mathbf{g})[(\psi_t)_*\mu](d\mathbf{g}) - \int_{G^{\lfloor p \rfloor}(\mathbb{R}^e)} \phi(\mathbf{g})[(\psi_s)_*\mu](d\mathbf{g}) \right| \\
&= \left| \int_{G\Omega_p(\mathbb{R}^e)} (\phi \circ \psi_t - \phi \circ \psi_s)(\mathbf{y})\mu(d\mathbf{y}) \right| \\
&\leqslant \int_{G\Omega_p(\mathbb{R}^e)} d_{\lfloor p \rfloor}(\psi_s(\mathbf{y}), \psi_t(\mathbf{y}))\mu(d\mathbf{y}) \\
&\leqslant \int_{G\Omega_p(\mathbb{R}^e)} \rho_{p\text{-var};[s,t]}(1, \mathbf{y})\mu(d\mathbf{y}).
\end{aligned}$$

The right-hand side is finite since μ is in $\mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$. Using the fact that $d_{\lfloor p \rfloor}(\psi_s(\mathbf{y}), \psi_t(\mathbf{y})) \to 0$, μ-almost surely (a.s.) as $|t-s| \to 0$ and the dominated convergence theorem concludes the proof. □

It follows from this lemma that

$$\gamma_{s,t} := \int_s^t T_{\mu_r}\, dr := \int_0^T 1_{[s,t]}(r) T_{\mu_r}\, dr \in \mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^* \tag{4.4}$$

is well defined for every $(s,t) \subseteq [0,T]$, where the integral is understood in the sense of Bochner integration. For any $\phi \in \mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))$, standard properties of the integral yield that

$$\langle \gamma_{s,t}, \phi \rangle = \int_s^t \langle T_{\mu_r}, \phi \rangle\, dr = \int_s^t \int_{G^{\lfloor p \rfloor}(\mathbb{R}^e)} \phi(\mathbf{g})\mu_r(d\mathbf{g})\, dr. \tag{4.5}$$

The space $G\Omega_p(\mathbb{R}^e)$ carries with it an implicit time interval $[0,T]$ which we suppress in the notation. Occasionally, we might want to make this explicit by writing $G\Omega_p([0,T];\mathbb{R}^e)$. For example, if we start with a probability measure μ in $\mathcal{M}(G\Omega_p([0,T];\mathbb{R}^e))$, then we will need to consider its restriction, $\mu|_{[0,t]}$ to a probability measure in $\mathcal{M}(G\Omega_p([0,t];\mathbb{R}^e))$. We then let W_t denote the Wasserstein metric on $\mathcal{M}(G\Omega_p([0,t];\mathbb{R}^e))$, and write $W_t(\mu^1,\mu^2)$ to mean $W_t(\mu^1|_{[0,t]}, \mu^2|_{[0,t]})$.

COROLLARY 4.6. *Suppose $p \geqslant 1$ and let $T > 0$. Assume that μ^1 and μ^2 are two elements of $\mathcal{M}_1(G\Omega_p([0,T];\mathbb{R}^e))$, and for $i = 1,2$ let $\gamma^i : \Delta_{[0,T]} \to \mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*$ be the function* *defined by*

$$\gamma^i_{s,t} = \int_s^t T_{\mu^i_r}\, dr.$$

Then for every $0 \leqslant s < t \leqslant T$, we have that

$$|\gamma^1_{s,t} - \gamma^2_{s,t}|_{\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*} \leqslant C(t-s) W_t(\mu^1, \mu^2). \tag{4.6}$$

In particular,

$$\|\gamma^1 - \gamma^2\|_{1\text{-var};[0,T]} \leqslant C \int_0^T W_t(\mu^1, \mu^2)\, dt. \tag{4.7}$$

Proof. Let $\phi \in \mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))$ with $\|\phi\|_{Lip^1(S_p)^*} = 1$, then from (4.5) we can deduce that

$$\begin{aligned} |\langle \gamma^1_{s,t} - \gamma^2_{s,t}, \phi\rangle| &= \left| \int_s^t \left[\int_{G^{\lfloor p \rfloor}(\mathbb{R}^e)} \phi(\mathbf{g}) \mu^1_r(d\mathbf{g}) - \int_{S_p} \phi(\mathbf{g}) \mu^2_r(d\mathbf{g}) \right] dr \right| \\ &\leqslant (t-s) \sup_{r \in [s,t]} \int_{G^{\lfloor p \rfloor}(\mathbb{R}^e) \times G^{\lfloor p \rfloor}(\mathbb{R}^e)} d_{\lfloor p \rfloor}(\mathbf{g}^1, \mathbf{g}^2) \pi_r(d\mathbf{g}^1, d\mathbf{g}^2), \end{aligned}$$

where π_r is an element of $\mathcal{M}_1(G^{\lfloor p \rfloor}(\mathbb{R}^e) \times G^{\lfloor p \rfloor}(\mathbb{R}^e))$ with marginal distributions μ^1_r and μ^2_r. For any such π_r and every $r \in [s,t]$, we have

$$\begin{aligned} &\int_{G^{\lfloor p \rfloor}(\mathbb{R}^e) \times G^{\lfloor p \rfloor}(\mathbb{R}^e)} d_{\lfloor p \rfloor}(\mathbf{g}^1, \mathbf{g}^2) \pi_r(d\mathbf{g}^1, d\mathbf{g}^2) \\ &\quad \leqslant C \int_{G\Omega_p([0,t];\mathbb{R}^e) \times G\Omega_p([0,t];\mathbb{R}^e)} \rho_{p\text{-var};[0,t]}(\mathbf{y}^1, \mathbf{y}^2) \pi(d\mathbf{y}^1, d\mathbf{y}^2), \end{aligned}$$

where π in $\mathcal{M}_1(G\Omega_p([0,t];\mathbb{R}^e) \times G\Omega_p([0,t];\mathbb{R}^e))$ is any coupling of $\mu^1|_{[0,t]}$ and $\mu^2|_{[0,t]}$. This yields

$$\sup_{r \in [s,t]} \int_{G^{\lfloor p \rfloor}(\mathbb{R}^e) \times G^{\lfloor p \rfloor}(\mathbb{R}^e)} d_{\lfloor p \rfloor}(\mathbf{g}^1, \mathbf{g}^2) \pi_r(d\mathbf{g}^1, d\mathbf{g}^2) \leqslant W_t(\mu^1, \mu^2),$$

which implies (4.6) at once. Deducing (4.7) from (4.6) is then elementary. □

4.3. *A fixed-point theorem*

Suppose that $p \geqslant 1$ and $\mathbf{x}$ is an element of $G\Omega_p(\mathbb{R}^d)$, then we write $\omega_{\mathbf{x}}$ for the control induced by $\mathbf{x}$ via

$$\omega_{\mathbf{x}}(s,t) \equiv \|\mathbf{x}\|^p_{p\text{-var};[s,t]}.$$

The following lemma gives a useful way of controlling $\omega_{\mathbf{x}}(0,T)$ in terms of the α-local p-variation.

LEMMA 4.7. *For any $\mathbf{x}$ in $G\Omega_p(\mathbb{R}^d)$ and any $\alpha > 0$, we have that*

$$\|\mathbf{x}\|^p_{p\text{-var};[0,T]} = \omega_{\mathbf{x}}(0,T) \leqslant 2^{p-1} \alpha \max\{1, \alpha^{-p} M_{\alpha,[0,T]}(\omega_{\mathbf{x}})^p\} \leqslant 2^{p-1} \alpha \exp[c_{p,\alpha} M_{\alpha,[0,T]}(\omega_{\mathbf{x}})], \tag{4.8}$$

where $c_{p,\alpha} = \alpha^{-1} e^{-1} p$.

Proof. Fix $\alpha > 0$, and let $D = (t_i : i = 0, 1, \ldots, n)$ be an arbitrary partition of $[0,T]$. We aim to estimate

$$\sum_{i=1}^n \|\mathbf{x}_{t_{i-1},t_i}\|^p := \sum_{i=1}^n d_{\mathrm{CC}}(\mathbf{x}_{t_{i-1}}, \mathbf{x}_{t_i})^p.$$

Let t_{j-1} and t_j be any two consecutive points in D, and define $\sigma_0 = t_{j-1}$ and

$$\sigma_{i+1} = \inf\{t > \sigma_i : \omega(\sigma_i, t) = \alpha\} \wedge t_j,$$

for $i \in \mathbb{N}$. Define

$$N_{\alpha,[t_{j-1},t_j]}(\omega_{\mathbf{x}}) = \sup\{n \in \mathbb{N} \cup \{0\} : \sigma_n < t_j\}.$$

A simple calculation shows that $t_N = t_j$ if $N = N_{\alpha,[t_{j-1},t_j]}(\omega_{\mathbf{x}}) + 1$, and therefore

$$\begin{aligned}\|\mathbf{x}_{t_{j-1},t_j}\|^p &\leqslant \left(\sum_{i=1}^{N+1} \|\mathbf{x}_{\sigma_{i-1},\sigma_j}\|\right)^p \\ &\leqslant (N+1)^{p-1} \sum_{i=1}^{N+1} \|\mathbf{x}_{\sigma_{i-1},\sigma_j}\|^p \leqslant (N_{\alpha,[0,T]}(\omega_{\mathbf{x}}) + 1)^{p-1} \sum_{i=1}^{N+1} \|\mathbf{x}_{\sigma_{i-1},\sigma_j}\|^p.\end{aligned}$$

Using this observation, it is easy to deduce that

$$\sum_{i=1}^{n} \|\mathbf{x}_{t_{i-1},t_i}\|^p \leqslant (N_{\alpha,[0,T]}(\omega_{\mathbf{x}}) + 1)^{p-1} M_{\alpha,[0,T]}(\omega_{\mathbf{x}}). \tag{4.9}$$

The claimed bound follows by first noticing that $N_{\alpha,[0,T]}(\omega_{\mathbf{x}}) \leqslant \alpha^{-1} M_{\alpha,[0,T]}(\omega_{\mathbf{x}})$, and then taking the supremum over all partitions D in (4.9). The last estimate in (4.8) follows from the classical estimate $x^p \leqslant \exp[pe^{-1}x]$ for $x > 0$. □

In order to prove the fixed-point theorem, we require some integrability conditions on the preference measure. The subset of $\mathcal{M}(G\Omega_p(\mathbb{R}^d))$ for which the fixed-point theorem will hold is described by the following condition.

Condition 1. *Let $p \geqslant 1$. Suppose that ν is a probability measure in $\mathcal{M}(G\Omega_p(\mathbb{R}^d))$, and let ϕ_ν be the pushforward measure in $\mathcal{M}([0,\infty))$ defined by*

$$\phi_\nu := [M_{1,[0,T]}(\omega_\cdot)]_*(\nu).$$

We will assume that ϕ_ν has well-defined moment-generating function; that is, for every θ in $\mathbb{R}$ we have

$$\int_{[0,\infty)} \exp[\theta y]\phi_\nu(dy) = \int_{G\Omega_p(\mathbb{R}^d)} \exp[\theta M_{1,[0,T]}(\omega_{\mathbf{x}})]\nu(d\mathbf{x}) < \infty.$$

Remark 4.8. If Condition 1 is in force, then $\phi_\nu^\alpha := [M_{\alpha,[0,T]}(\omega_\cdot)]_*(\nu)$ will also have a well-defined moment-generating function for any α in $(0,1)$.

For the reader's convenience, we recall some notation from Section 3. The map $\Psi : \mathcal{M}_1(G\Omega_p(\mathbb{R}^e)) \to \mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$ is defined by

$$\Psi = \Psi_\nu : \mu \longmapsto [\Theta_{V^0,V}(\mu,\cdot,\cdot)]_*(u_0 \times \nu) \in \mathcal{M}_1(G\Omega_p(\mathbb{R}^e)),$$

and fixed points of Ψ_ν correspond to solutions of the non-linear McKean–Vlasov RDE

$$\begin{cases} d\mathbf{Y}_t = V(Y_t^\mu)d\mathbf{X}_t + V^0(Y_t^\mu)\, d\gamma_t^\mu, \\ \text{Law}(\mathbf{Y}) = \mu,\ \text{Law}(Y_0) = u_0. \end{cases} \tag{4.10}$$

We now formulate and prove our main existence and uniqueness theorem for solutions to (4.10).

Theorem 4.9. *Let $\gamma > p \geqslant 1$ and $\beta > 1$. Suppose that ν is an element of $\mathcal{M}_1(G\Omega_p(\mathbb{R}^d))$ which satisfies Condition 1, and let u_0 be an element of $\mathcal{M}_1(\mathbb{R}^e)$ (when $\mathbb{R}^e$ is equipped with* *the standard Euclidean metric). Let $(Y_0, \mathbf{X})$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, taking values in $\mathbb{R}^e \times G\Omega_p(\mathbb{R}^d)$ and having law $u_0 \times \nu$. Then for any collection of vector fields $V = (V^1, \ldots, V^d)$ in $\mathrm{Lip}^\gamma(\mathbb{R}^e)$ and V^0 in $\mathrm{Lip}^\beta(\mathbb{R}^e, L(\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*, \mathbb{R}^e))$, there exists a unique solution to the non-linear McKean–Vlasov RDE (4.10).*

Proof. The space $(G\Omega_p(\mathbb{R}^e), \rho_p)$ is complete, and hence (see, for example, [**30**]) so is $(\mathcal{M}_1(G\Omega_p(\mathbb{R}^e)), W)$. Suppose that μ_1 and μ_2 are in $\mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$, and let

$$\mathbf{Y}^i = \Theta_{V^0,V}(\mu_i, Y_0, \mathbf{X}).$$

The conditions on $\mathbf{X}$ are sufficient to guarantee that the laws of $\mathbf{Y}^i$, $i = 1, 2$, are in $\mathcal{M}_1(G\Omega_p(\mathbb{R}^e))$. Using Corollary 4.4 together with Lemma 4.7, we obtain for any α in $(0, 1]$ the bound

$$\rho_{p\text{-var};[0,T]}(\mathbf{Y}^1, \mathbf{Y}^2) \leqslant C \|\gamma^{\mu_1} - \gamma^{\mu_2}\|_{1\text{-var};[0,T]} \exp(C M_{\alpha,[0,T]}(\omega_{\mathbf{X}})),$$

for some $C = C_1(\alpha) > 0$. It follows from Corollary 4.6 that

$$\|\gamma^{\mu_1} - \gamma^{\mu_2}\|_{1\text{-var};[0,T]} \lesssim \int_0^T W_t(\mu_1, \mu_2)\, dt.$$

And therefore by taking expectations in the previous inequality, we obtain

$$\begin{aligned} W_T(\Psi_\nu(\mu_1), \Psi_\nu(\mu_2)) &\leqslant E[\rho_{p\text{-var};[0,T]}(\mathbf{Y}^1, \mathbf{Y}^2)] \\ &\leqslant C E[\exp(C M_{\alpha,[0,T]}(\omega_{\mathbf{X}}))] \int_0^T W_t(\mu_1, \mu_2)\, dt \\ &=: \tilde{C} \int_0^T W_t(\mu_1, \mu_2)\, dt, \end{aligned} \tag{4.11}$$

where Condition 1 ensures that $\tilde{C} < \infty$. Let $\Psi_\nu^k = \Psi_\nu \circ \Psi_\nu \circ \cdots \circ \Psi_\nu$ denote the k-fold composition of Ψ_ν with itself, then iterating (4.11) k times yields

$$\begin{aligned} W_T(\Psi_\nu^k(\mu_1), \Psi_\nu^k(\mu_2)) &\leqslant \tilde{C}^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} W_{t_1}(\mu_1, \mu_2)\, dt_1\, dt_2 \cdots dt_k \\ &\leqslant \frac{\tilde{C}^k}{k!} W_T(\mu_1, \mu_2) \\ &=: \alpha_k W_T(\mu_1, \mu_2). \end{aligned} \tag{4.12}$$

Setting $k = K := \inf\{n \in \mathbb{N} : \tilde{C}^n < n!\}$, it follows that $\alpha_k < 1$, that is, Ψ_ν^k is a contraction and thus has a unique fixed point, μ. But applying (4.12) to $\mu_1 = \Psi_\nu(\mu)$ and $\mu_2 = \mu$, then implies

$$W_T(\Psi_\nu(\mu), \mu) \leqslant \alpha_k W_T(\Psi_\nu(\mu), \mu),$$

so that $\Psi_\nu(\mu) = \mu$, and μ must also be the unique fixed point of Ψ_ν. □

In light of the conditions of this theorem, it is useful to make some observation about the type of processes which satisfy the key integrability condition (Condition 1). In the recent paper [**5**], we consider a continuous Gaussian process $X = (X^1, \ldots, X^d)$ with i.i.d. components such that:

(1) X has a natural lift to a geometric p-rough path $\mathbf{X}$;
(2) the Cameron–Martin space associated to X has the embedding property

$$\mathcal{H} \hookrightarrow C^{q\text{-var}}([0, T], \mathbb{R}^d),$$

for some $1/p + 1/q > 1$.

We then prove that for some $\eta > 0$, we have

$$E[\exp[\eta M_{\alpha,[0,T]}(\omega_{\mathbf{X}})^{2/q}]] < \infty.$$

This class of examples is rich enough to include fractional Brownian motion $H > \frac{1}{4}$ (for which q can be chosen to ensure $2/q > 1$), and other examples of Gaussian processes which are genuinely rougher than Brownian motion (see [**12**]). The importance of the Lipschitz estimate in Corollary 4.4 can now be grasped more clearly. Since, as an immediate corollary, we see that Condition 1 holds for the class of measures described.

Remark 4.10. In some valuable recent work [**1**], a flow-based approach is used to derive continuity estimates for RDEs. An existence and uniqueness theorem is proved, under the following condition on $\mathbf{X}$: for some family of random variables $\{C_s : s \in [0,T]\}$, which is bounded in L^1,

$$|E[\mathbf{X}^k_{s,t}|\mathcal{F}_s]| \leqslant C_s(t-s) \quad \forall [s,t] \subseteq [0,T],\ k = 1,\ldots,\lfloor p \rfloor. \tag{4.13}$$

This requirement forces some structure on X (for example: smoothness, or independence of increments). It does not hold in general for the examples illustrated above, where the sample paths are less regular than Brownian motion. Indeed if X is fBm with $H < \frac{1}{2}$, then we have

$$t^{-1}E[X^2_{0,t}] = t^{2H-1} \uparrow \infty \quad \text{as } t \downarrow 0,$$

which violates (4.13) when $s = 0$. By contrast, the exponential integrability required in Condition 1 holds both in this example, and for the much wider class of Gaussian processes highlighted above. We note, however, that the result in [**1**] will also apply in some cases not considered here. For instance, an existence result for the fixed point is established in the strongly interacting case, that is, where an interaction term also features in the coefficients of the noise.

4.4. *Continuity in ν*

Suppose that we have a set of preference measures and for each measure in the set the conditions of Theorem 4.9 hold, so that $\Psi_\nu(\cdot)$ has a unique fixed point. A very natural question is to ask about the stability properties of this map. The rough path setup is well-suited to tackle this sort of problem. To this end, let $K : (0,\infty) \to (0,\infty)$ be a monotone increasing real-valued function. Define a subset of $\mathcal{M}_1(G\Omega_p(\mathbb{R}^d))$ by

$$\mathcal{E}(K) = \{\nu \in \mathcal{M}_1(G\Omega_p(\mathbb{R}^d)) : \forall\theta \in (0,\infty), \psi_\nu(\theta) \leqslant K(\theta)\},$$

where, as above,

$$\psi_\nu(\theta) = \int_{[0,\infty)} \exp(\theta y)\phi_\nu(dy).$$

It is easy to see that $\mathcal{E}(K)$ is a closed subset of $\mathcal{M}_1(G\Omega_p(\mathbb{R}^d))$ in the topology of weak convergence of measures.

Lemma 4.11. *Let $K : (0,\infty) \to (0,\infty)$ be a monotone increasing, then the map*

$$\Xi : \mathcal{E}(K) \longrightarrow \mathcal{M}_1(P_{e,d}),$$
$$\Xi : \nu \longmapsto \text{ fixed point of } \Psi_\nu(\cdot)$$

is well defined and continuous in the topology of weak convergence of measures on $\mathcal{E}(K)$.

Proof. Suppose that $(\nu_n)_{n=1}^\infty$ is a sequence of measures in $\mathcal{E}(K)$ such that $\nu_n \Rightarrow \nu \in \mathcal{E}(K)$ as $n \to \infty$; we will show that $W_T(\Xi(\nu_n), \Xi(\nu)) \to 0$ as $n \to \infty$. By Skorohod's lemma, there exists a probability space carrying: (i) an $\mathbb{R}^e$-valued random variable Y_0 with law u_0 and (ii) a sequence of ($G\Omega_p(\mathbb{R}^d)$-valued) random variables $(\mathbf{X}^{\nu_n})_{n=1}^{\infty}$ and $\mathbf{X}^{\nu}$, such that $\mathbf{X}^{\nu}$ has law ν, $\mathbf{X}^{\nu_n}$ has law ν_n for every n, and

$$\mathbf{X}^{\nu_n} \longrightarrow \mathbf{X}^{\nu} \text{ a.s. in } \rho_{p\text{-var}}.$$

Let $\Xi(\nu_n) = \mu_n$, $\Xi(\nu) = \mu$ and $\mathbf{Y}^{\nu_n} = \Theta_{V^0,V}(\mu_n, Y_0, \mathbf{X}^{\nu_n})$, $\mathbf{Y}^{\nu} = \Theta_{V^0,V}(\mu, Y_0, \mathbf{X}^{\nu})$. We then have that

$$\mathrm{Law}(\mathbf{Y}^{\nu_n}) = \mu_n = [\Theta_{V_0,V}(\mu_n, \cdot, \cdot)]_*(u_0, \nu_n),$$

and similarly for $\mathrm{Law}(\mathbf{Y}^{\nu})$. The estimates of Corollary 4.4 and Lemma 4.7 show that for some non-random $C_1 > 0$:

$$\rho(\mathbf{Y}^{\nu_n}, \mathbf{Y}^{\nu}) \leqslant C_1[\|\gamma^{\nu_n} - \gamma^{\nu}\|_{1\text{-var};[0,T]} + \|\mathbf{X}^{\nu_n} - \mathbf{X}^{\nu}\|_{p\text{-var};[0,T]}] \exp(C_1 M_{\alpha,[0,T]}(\omega^{\nu_n,\nu})),$$

where

$$\omega^{\nu_n,\nu}(s,t) \equiv \|\mathbf{X}^{\nu_n}\|_{p\text{-var};[s,t]}^p + \|\mathbf{X}^{\nu}\|_{p\text{-var};[s,t]}^p.$$

Taking expectations and then making use of Corollary 4.6 and Condition 1, it is easy to derive that

$$W_T(\mu_n, \mu) \leqslant C_2 \int_0^T W_t(\mu_n, \mu)\, dt + C_1 a_n, \tag{4.14}$$

where

$$a_n := E[A_n] := E[\|\mathbf{X}^{\nu_n} - \mathbf{X}^{\nu}\|_{p\text{-var};[0,T]} \exp[C_1 M_{\alpha,[0,T]}(\omega^{\nu_n,\nu})]].$$

It is easily seen by using the estimate (4.8) that

$$\begin{aligned}\|\mathbf{X}^{\nu_n} - \mathbf{X}^{\nu}\|_{p\text{-var};[s,t]} &\leqslant \|\mathbf{X}^{\nu_n}\|_{p\text{-var};[s,t]} + \|\mathbf{X}^{\nu}\|_{p\text{-var};[s,t]} \\ &\leqslant 2\omega^{\nu_n,\nu}(s,t)^{1/p} \\ &\leqslant C_2 \exp[C_2 M_{\alpha,[0,T]}(\omega^{\nu_n,\nu})],\end{aligned}$$

for some non-random constant C_2 depending only on p and α. An easy calculation making use of Remark 4.2 then shows that A_n may be bounded by

$$B_n := C_3 \exp[C_3 M_{\alpha,[0,T]}(\omega^{\nu_n})] \exp[C_3 M_{\alpha,[0,T]}(\omega^{\nu})],$$

where again C_3 is non-random and depends only on p and α. An application of Hölder's inequality and the assumption $\{\nu_n : n \in \mathbb{N}\} \subset \mathcal{E}(K)$ shows that $\{B_n : n \in \mathbb{N}\}$ is bounded in L^q for any $q > 0$, and hence $\{A_n : n \in \mathbb{N}\}$ is uniformly integrable. Since $A_n \to 0$ a.s. as $n \to \infty$, we deduce also that $a_n = E[A_n] \to 0$ as $n \to \infty$. Finally, we use Gronwall's inequality in (4.14) to give

$$W_T(\mu_n, \mu) \leqslant C_3 a_n \exp(C_3 T) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

and the proof is thereby completed. □

5. *Applications*

As an application of our uniqueness theorem, we prove the classical *propagation of chaos* phenomenon (see Sznitman [**29**]) for the finite interacting particle system. This is the observation that, granted sufficient symmetry to the interaction and initial configuration, then in the large-population limit any finite subcollection of particles resembles the evolution independent particles, each having the law of the non-linear McKean–Vlasov RDE. To make progress, let ν be a fixed preference measure in $\mathcal{M}_1(G\Omega_p(\mathbb{R}^d))$ which satisfies Condition 1. Assume that $\{(\mathbf{X}^i, Y_0^i) : i \in \mathbb{N}\}$ and $(\mathbf{X}, Y_0)$ are i.i.d. $\mathbb{R}^e \times G\Omega_p(\mathbb{R}^d)$-valued random variables each with law

$u_0 \times \nu$ and defined on the same probability space $(\Omega, \mathcal{F}, P)$. Suppose that μ is the unique fixed point of $\Psi_\nu(\cdot)$, and let

$$\mathbf{Y} = \Theta_{V^0,V}(\mu, Y_0, \mathbf{X}).$$

From Lemma 3.2, we can interpret the trajectories of individual particles in the community

$$d\mathbf{Y}_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} \sigma(Y_t^{i,N}, Y_t^{j,N})\,dt + V(Y_t^i)\,d\mathbf{X}_t^i, \quad Y_0^{i,N} = Y_0^i,$$

as projections of the solution to a system of rough differential equation driven by any rough path in $P_{p,Nd}$ which consistently lifts $\mathbf{X}^1, \ldots, \mathbf{X}^N$. Indeed, we showed that $\mathbf{Y}^i$ is then well defined as

$$\mathbf{Y}^{i,N} = \Theta_{V^0,V}(\mu^N, Y_0^i, \mathbf{X}^i), \quad i = 1, \ldots, N,$$

where V^0 in $\mathrm{Lip}^\beta(\mathbb{R}^e, L(\mathrm{Lip}^1(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*, \mathbb{R}^e))$ is defined in terms of the interaction kernel σ by

$$V^0(y)(\mu) = \langle \mu, \sigma(y, \cdot)\rangle, \tag{5.1}$$

and $\mu^N = \mu^N(\omega)$ is the empirical measure

$$\mu^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\mathbf{Y}^{j,N}}.$$

For every N, $\{\mathbf{Y}^{1,N}, \mathbf{Y}^{2,N}, \ldots, \mathbf{Y}^{N,N}\}$ is an exchangeable system of random variables, and a classical result of [**29**] shows that propagation of chaos is equivalent to proving that

$$\mu^N \Longrightarrow \mu.$$

REMARK 5.1. Explicitly, this assertion says that the law of the random variable

$$\mu^N(\omega) \in \mathcal{M}_1(P_p),$$

which is a probability measure in $\mathcal{M}_1(\mathcal{M}_1(P_p))$, converges weakly as $N \to \infty$ to the probability measure δ_μ, which is the law of the constant random variable μ.

THEOREM 5.2. *Let $\gamma > p \geqslant 1$, $\beta > 1$ and $y_0 \in \mathbb{R}^e$. Suppose that ν is a given preference measure in $\mathcal{M}_1(G\Omega_p(\mathbb{R}^d))$ which satisfies Condition* 1, *and let u_0 be an element of $\mathcal{M}_1(\mathbb{R}^e)$ (when $\mathbb{R}^e$ is equipped with the standard Euclidean metric). Assume that the vector fields $V = (V^1, \ldots, V^d)$ belong to $\mathrm{Lip}^\gamma(\mathbb{R}^e)$, and V^0 defined by* (5.1) *is in $\mathrm{Lip}^\beta(\mathbb{R}^e, L(\mathrm{Lip}(G^{\lfloor p \rfloor}(\mathbb{R}^e))^*, \mathbb{R}^e))$. Let μ denote the unique fixed point of map $\Psi_\nu(\cdot)$ which results from Theorem* 4.9. *Assume further that $\{(Y_0^i, \mathbf{X}^i) : i \in \mathbb{N}\}$ is a collection of i.i.d. $\mathbb{R}^e \times G\Omega_p(\mathbb{R}^d)$-valued random variables, with law $u_0 \times \nu$, defined on the same probability space. For each $N \in \mathbb{N}$, let $\{\mathbf{Y}^{i,N} : 1 = 1, \ldots, N\}$ be the solution to the particle system*

$$d\mathbf{Y}_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} \sigma(Y_t^{i,N}, Y_t^{j,N})\,dt + V(Y_t^i)\,d\mathbf{X}_t^i, \mathbf{Y}_0^{i,N} = Y_0^i \in \mathbb{R}^e.$$

Then as $N \to \infty$

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\mathbf{Y}^{j,N}} =: \mu^N \Rightarrow \mu, \tag{5.2}$$

and the particle system exhibits propagation of chaos.

Proof. Fix $N \in \mathbb{N}$ and suppose $i \in \{1, \dots, N\}$. Let $\mathbf{Y}^i = \Theta_{V^0,V}(\mu, Y_0^i, \mathbf{X}^i)$ so that the law $\mathbf{Y}^i$ is μ, and $\{\mathbf{Y}^1, \dots, \mathbf{Y}^N\}$ are N independent copies of the solution of the rough McKean–Vlasov equation. Then using Corollary 4.4 and Lemma 4.7, we have (for α in $(0, 1]$) that

$$\rho_{p\text{-var};[0,T]}(\mathbf{Y}^i, \mathbf{Y}^{i,N}) \leqslant C_1 \|\gamma^\mu - \gamma^{\mu^N}\|_{1\text{-var};[0,T]} \exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}^i})). \tag{5.3}$$

Using Corollary 4.6, we observe

$$\|\gamma^\mu - \gamma^{\mu^N}\|_{1\text{-var};[0,T]} \leqslant C_2 \int_0^T W_t(\mu, \mu^N)\, dt,$$

and hence by summing (5.3) over $i = 1, \dots, N$ we obtain

$$\frac{1}{N} \sum_{i=1}^N \rho_{p\text{-var};[0,T]}(\mathbf{Y}^i, \mathbf{Y}^{i,N}) \leqslant C_2 \int_0^T W_t(\mu, \mu^N) dt \frac{1}{N} \sum_{i=1}^N \exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}^i})). \tag{5.4}$$

Let

$$\bar{\mu}^N = \frac{1}{N} \sum_{j=1}^N \delta_{\mathbf{Y}^j}.$$

Then we also have

$$W_T(\mu, \mu^N) \leqslant W_T(\mu, \bar{\mu}^N) + W_T(\bar{\mu}^N, \mu^N). \tag{5.5}$$

And, on the other hand using (5.4) we have the bound

$$W_T(\bar{\mu}^N, \mu^N) \leqslant C_2 \int_0^T W_t(\mu, \mu^N) dt \frac{1}{N} \sum_{i=1}^N \exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}^i})). \tag{5.6}$$

Putting (5.6) into (5.5) and using Gronwall's lemma, we deduce that

$$W_T(\mu, \mu^N) \leqslant C_2 W_T(\mu, \bar{\mu}^N) \exp\left[\frac{C_1}{N} \sum_{i=1}^N \exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}^i}))\right].$$

It is a simple matter to conclude from the strong law of large numbers that both $W_T(\mu, \bar{\mu}^N) \to 0$ a.s., and

$$\frac{1}{N} \sum_{i=1}^N \exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}^i})) \to E[\exp(C_1 M_{\alpha,[0,T]}(\omega_{\mathbf{X}}))] < \infty,$$

a.s. as $N \to \infty$. It is then easy to deduce (5.2). Propagation of chaos is then a consequence of the classical result of [**29**] we cited earlier. □

There are a number of follow-up results that seem worth pursuing. In particular, Sanov-type theorems à la Dawson–Gartner [**7**] will be possible for (5.2) in the weakly interacting case. The presence of the rough path topology we believe should allow for a simplified treatment of these results. We intend to return to this discussion in future work.

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