Regularity theory in the calculus of variations and elliptic systems

Christopher Irving

Keble College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2022

Contents

Contents **i**

Abstract **iv**

Acknowledgements **v**

1 Introduction **1**
1.1 Motivation and historical developments . . . 2
1.1.1 The scalar case . . . 2
1.1.2 Semicontinuity and quasiconvexity . . . 5
1.1.3 Regularity in the vectorial case . . . 7
1.2 Outline of the thesis . . . 11

2 Technical groundwork **15**
2.1 Notation and definitions . . . 15
2.1.1 List of notation . . . 15
2.1.2 N-functions . . . 18
2.1.3 Domains in Euclidean space . . . 25
2.1.4 Convexity notions . . . 31
2.2 Harmonic analysis . . . 35
2.2.1 Interpolation and extrapolation . . . 36
2.2.2 Maximal operators . . . 42
2.2.3 The space BMO . . . 45
2.2.4 Miscellaneous estimates . . . 48
2.3 Function spaces . . . 49
2.3.1 Orlicz-Sobolev spaces . . . 49
2.3.2 Poincaré inequalities . . . 51
2.3.3 Traces and extensions . . . 54
2.3.4 Besov spaces . . . 59
2.3.5 Lipschitz truncations . . . 62

3 Linear elliptic estimates **64**
3.1 Ellipticity and Gårding inequalities 64
3.2 Potential theoretic methods 67
3.2.1 Fundamental solutions 68
3.2.2 Poisson kernels 70
3.3 Energy methods 75
3.3.1 Schauder theory 75
3.3.2 The method of Krylov 80
3.3.3 Representation as a divergence 87

4 BMO ε-regularity results for Legendre-Hadamard elliptic systems **90**
4.1 Statement of results 91
4.1.1 Hypotheses and local results 91
4.1.2 Consequences of the main results 93
4.1.3 Controlled growth conditions 94
4.2 The interior case 95
4.2.1 Estimates for the integrand 95
4.2.2 A Caccioppoli-type inequality 96
4.2.3 Harmonic approximation 99
4.2.4 Excess decay 101
4.3 The boundary case 103
4.3.1 Localisation near the boundary 103
4.3.2 Boundary Caccioppoli inequality 105
4.3.3 Boundary harmonic approximation 109
4.3.4 Proof of boundary regularity 110
4.4 The uniformly elliptic case 113

5 Minima for quasiconvex integrands **115**
5.1 Hypotheses and main results 116
5.2 Existence of minima 118
5.2.1 Lower-semicontinuity I 121
5.2.2 Regularisation of minimising sequences 125
5.2.3 Lower-semicontinuity II 130
5.2.4 Coercivity and strict quasiconvexity 133
5.3 Partial regularity 137
5.3.1 Strategy of proof 138
5.3.2 Higher integrability 141
5.3.3 Caccioppoli inequality of the second kind 142
5.3.4 Perturbation via Ekeland 145
5.3.5 Harmonic approximation 149
5.3.6 Excess decay estimates 153

5.3.7 The case of degenerate growth . . . 156

6 Convex problems with (p, q)-growth **161**

6.1 Setup and main results . . . 163

6.1.1 Relaxation . . . 163

6.1.2 Hypotheses and main results . . . 164

6.2 Known interior regularity . . . 165

6.2.1 Improved integrability . . . 166

6.2.2 Characterisation of regular points . . . 166

6.3 Improved differentiability in the interior . . . 167

7 Conclusion and outlook **173**

Bibliography **175**

Abstract

In this thesis we will investigate the regularity of critical points and minimisers of integral functionals of the form

$$\mathcal{F}(w) = \int_\Omega F(x, w, \nabla w)\,\mathrm{d}x, \tag{1}$$

where the mappings w we consider will be vector-valued. This will be done from several different perspectives, with the goal of understanding when solutions are locally regular (of class $C^{1,\alpha}$) near a given point $x_0 \in \Omega$. This is typically done through a so-called "ε-regularity result," which characterises the points x_0 which are regular in the above sense, by a smallness condition for a suitable excess quantity. We will investigate in three different aspects such regularity issues.

In the first part, we will consider the regularity for critical points of functionals satisfying a Legendre-Hadamard ellipticity condition. It is known by the works of Müller & Šverák [MŠ03] that critical points may be highly irregular (in particular nowhere C^1), however we show that regularity does hold if we assume an a-priori smallness condition of the gradient in BMO. Results will be obtained up to the boundary, and global consequences will also be explored.

In the second part, we will consider the existence and partial regularity theory for minimisers of non-autonomous quasiconvex integrands, subject to a general growth condition. By this we mean the growth of the integrand F is governed by an N-function, which we assume satisfies a Δ_2-condition. We will obtain a general existence theorem which will involve selecting a *regularised* minimising sequence along which we establish lower-semicontinuity, along with a partial regularity result. For the regularity theory we will additionally assume a non-degeneracy condition from below in the form of a ∇_2-condition, and we will discuss how this can be partially relaxed.

Finally in the third part, which is joint work with Lukas Koch (MPI Leipzig), we will consider relaxed minimisers for convex integrands satisfying a (p,q)-growth condition of the form

$$|z|^p \lesssim F(x,z) \lesssim 1 + |z|^q. \tag{2}$$

We will establish interior improved differentiability results in the Besov scales $B^{1+s,p}_\infty$, by means of a novel second order difference quotient technique. By using known ε-regularity results for minimisers in this setting, we obtain improved dimension estimates for the singular set.

Acknowledgements

I would like to express my deepest gratitude to my supervisor, Prof. Jan Kristensen, for his immeasurable guidance and support over the course of my studies. Despite his busy schedule he would always be happy to discuss ideas with me, and I was always inspired by his endless enthusiasm for the subject.

I am also grateful to my co-supervisor Prof. Melanie Rupflin for the many pieces of invaluable advice throughout the years, and to Prof. Gui-Qiang Chen for supporting my studies at various stages and the many interesting mathematical discussions. Alongside Prof. Chen, I would like to thank Prof. Andrea Mondino and Prof. Luc Nguyen for the interesting discussions and advice as part of examining my progress for my Transfer and Confirmation of Status. In addition, I would like to give special thanks to Prof. Neshan Wickramasekara for the opportunity to work with him when I was an undergraduate, setting me down the path of elliptic regularity.

I would also like to extend my thanks to Prof. Peter Bella, for inviting me to visit him and speak at TU Dortmund, and for the opportunity to join his research group from October 2022. Many thanks also to Dr. Lukas Koch for suggesting a collaborative project which formed the final chapter of my thesis, and also to J.Prof. Franz Gmeineder for the pleasant correspondences and ongoing collaboration.

I am also thankful for the interesting discussions and correspondences with, Prof. Lars Diening, Prof. Francesco Leonetti, Prof. Giuseppe Mingione, J.Prof. Mathias Schäffner, Prof. Endre Süli which took place in various forms. I would also like to thank my fellow DPhil students in the CDT and OxPDE research group, in particular to Jeremy Wu, Tommaso Seneci and Zhoulin Li for the interesting discussions as part of the reading group we ran.

Finally, I would like to acknowledge to my friends and family who made my time in Oxford an enjoyable experience, particularly those with who joined me on the many walks and adventures. I would also like to extend my gratitude to my parents and my sister, who despite not knowing the details of my studies, have always been supportive of me and my pursuit of mathematics.

Chapter 1

Introduction

The present thesis concerns the existence and regularity of critical points and minimisers for functionals of the form

$$\mathcal{F}(w) = \mathcal{F}(w, \Omega) = \int_\Omega F(x, w(x), \nabla w(x))\,\mathrm{d}x \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $w\colon \Omega \to \mathbb{R}^N$, typically subject to a Dirichlet boundary condition. To ensure minimisers exist in general, we will assume the integrand satisfies a suitable convexity condition. Given this, to establish the existence of minima one uses the so-called *Direct Method* as follows. First one constructs a sequence $\{u_n\}$ such that

$$\mathcal{F}(u_n) \to \inf_{w \in X} \mathcal{F}(w) \tag{1.2}$$

as $n \to \infty$, where we take the infimum with respect to a suitable class of functions X which is to be specified. One then appeals to *compactness* in the space X to extract a convergent subsequence and argues its limit is a minimiser. This poses several challenges; we firstly need to find a space X equipped with a sufficiently weak topology to extract such a subsequence, and then we need to show this convergence is strong enough to show the limit map is actually a minimiser in X. To carry out this procedure one is naturally lead to consider the space of potentially discontinuous functions, where the derivative can only be understood in a weak sense.

This raises the problem of regularity, which will be of central importance in this thesis. In this vector valued setting however, as we will detail in Section 1.1.3 it is known that we cannot expect full regularity for minima in general, and the best we can hope for are *partial* regularity results. Results of this type assert that, away from a possibly non-empty and relatively closed set of singular points, u is regular ($C^{1,\alpha}$ or smooth). We will investigate results of this type from several different aspects, including the case of critical points, and also whether similar results hold under various non-standard growth conditions.

1.1 Motivation and historical developments

We will begin by surveying the development of the existence and regularity theory in the 20$^{\text{th}}$ century, to set the scene and motivate the topics we will consider. Our exposition is far from complete, and we have instead focused on certain aspects of the theory that will connect to the problems we will consider in later sections. In addition to the references below, for a more detailed discussion we refer to the classical texts of MORREY [Mor66], GIAQUINTA [Gia83] and GIUSTI [Giu03], along with the survey articles of BALL [Bal98] and MINGIONE [Min06].

1.1.1 The scalar case

In the single variable case, the systematic treatment of existence of minima to functionals of the form (1.1) was due to TONELLI [Ton15], who popularised the notion of semicontinuity in the variational setting. This was achieved by working with *absolutely continuous* functions introduced by VITALI [Vit04] and using the integration theory of LEBESGUE [Leb02; Leb04]. In this setting (1.1) reduces to

$$\mathcal{F}(y) = \int_a^b F(x,y,y')\,\mathrm{d}y, \tag{1.3}$$

subject to the boundary conditions $y(a) = \alpha$, $y(b) = \beta$. In [Ton34] the existence of minima is established assuming the following.

(1) $F(x,y,y')$ and $F_{y'}(x,y,y')$ exist and are everywhere continuous on $[a,b]\times\mathbb{R}\times\mathbb{R}$.

(2) For each (x,y), the map $y' \mapsto F(x,y,y')$ is a non-constant and convex.

(3) F satisfies the coercivity condition

$$F(x,y,y') \geq \varphi(|y'|), \tag{1.4}$$

where φ is a non-negative function satisfying $\lim_{t\to\infty}\varphi(t)/t = \infty$.

Here the convexity condition (2) is necessary to ensure the existence of minima, and will be a standing assumption in the scalar case. Note the case when $\varphi(t) \gtrsim t^p$ for some $p > 1$ was established much earlier in [Ton15]. The coercivity condition would ensure minimising sequences are uniformly absolutely continuous (in a suitable sense), from which one can extract a uniformly convergent limit map which is itself absolutely continuous. We see that this already raises the problem of regularity; are these minimisers we obtain classically differentiable? This was also considered by Tonelli in [Ton15], who imposed a strict convexity function requiring $\partial_{y'}^2 F$ to be everywhere positive. However he was only able to obtain *partial* regularity results in general, namely that y' exists and is continuous away from an *exceptional set* $E \subset (a, b)$ which is closed and of measure zero. These results were later shown to be sharp by Ball & Mizel in [BM84b; BM85], and can be caused for instance by the *Lavrentiev phenomenon* which was first observed in [Lav27].

While Tonelli also studied the two-dimensional case in [Ton29; Ton33], he encountered significant difficulties in extending the one-dimensional theory. The problem is that minimising sequences need not be equicontinuous, and it was not clear how to generalise the notion of absolute continuity at the time. As such, the only existence results were obtained subject to a superquadratic coercivity condition of the form

$$F(x_1, x_2, y, y_{x_1}, y_{x_2}) \gtrsim (y_{x_1}^2 + y_{x_2}^2)^{\frac{\alpha}{2}} - C \tag{1.5}$$

for $\alpha > 2$ and some constant C. This assumption was used to ensure the minimising sequence converges uniformly to a continuous limit map, and while partial results were also obtained for $\alpha = 2$ it was unclear how to treat the general case. As noted by Morrey [Mor43], Tonelli's approach would only generalise to the n-dimensional case if we assume $\alpha > n$.

To work around this problem, it was necessary to abandon the space of continuous functions and work in a space of *generalised functions* which we recognise today as Sobolev spaces. This notion was independently developed by a number of authors, for instance by Sobolev [Sob63], Evans [Eva33], and Friedrichs [Fri44]. In the context of the Calculus of Variations, Calkin [Cal40] and Morrey [Mor40] introduced one such space involving functions that are "essentially absolutely continuous" along each of the coordinate axes. Using this formulation Morrey [Mor43] was able to extend Tonelli's theory to the general

n-dimensional case, obtaining generalised minima in $W^{1,p}$. These results were later extended by SERRIN [Ser61] who considered a suitable relaxation of $\mathcal{F}$, which was used to establish lower-semicontinuity results in L^1_{loc}.

While the above work provides a comprehensive understanding of the existence of "generalised minima," the theory is incomplete without a corresponding regularity theory. In particular, it is natural to ask whether such minima are classically differentiable, smooth, or even analytic. The corresponding problem of regularity was already raised by HILBERT as his 19th problem in [Hil02], who asked whether solutions to variational problems are always regular (analytic). The first substantial progress in this direction was due to BERNSTEIN [Ber04], who established the analyticity of C^3 solutions in two variables. This was relaxed by LICHTENSTEIN [Lic12] and HOPF [Hop29] to $C^{2,\alpha}$ solutions, where they notably used difference quotients of the form

$$U_h(x,y) = \frac{u(x+h,y)-u(x,y)}{h} \tag{1.6}$$

to linear equations. This was extended independently by CACCIOPPOLI [Cac34] and SCHAUDER [Sch34a; Sch34b] to the case of $C^{1,\alpha}$ solutions and all dimensions.

Here it was understood that one may study the corresponding *Euler-Lagrange equation*, which in the autonomous case when $F = F(z)$ reads as

$$-\operatorname{div} F'(\nabla u) = 0, \tag{1.7}$$

which is a first-order extremality condition for $\mathcal{F}$. If F is convex then the corresponding equation is elliptic, and so if know our solution lies $C^{1,\alpha}$, the system (1.7) can be viewed as a linear elliptic equation with Hölder continuous coefficients. This allowed the aforementioned authors to deduce $C^{2,\alpha}$ regularity of the solution, and hence smoothness via a bootstrap argument. Such results are now known as *Schauder estimates*, and remarkably extend to general elliptic systems; we will survey this in Section 3.3.1 following an approach due to CAMPANATO [Cam65].

Thus we have reduced the problem to showing the Sobolev minimisers in $W^{1,p}$ are of class $C^{1,\alpha}$. Here the growth conditions play an integral role, and one usually considers a

p-growth condition of the form

$$|z|^p \lesssim F(x,u,z) \lesssim 1 + |z|^p, \tag{1.8}$$

with corresponding ellipticity and second derivative bounds on F. This is referred to as the *uniformly elliptic case*, and the case of mixed growth condition was only considered much later (and was initiated by MARCELLINI [Mar89; Mar91]). In the two-dimensional case with $p = 2$, this was established by MORREY [Mor38; Mor40] using difference quotients techniques, however the general case remained open for almost two decades afterwards. The general case was resolved independently by DE GIORGI [De 57] and NASH [Nas58], where they establish Hölder continuity of weak solutions to linear elliptic equations of the form

$$-\operatorname{div}\left(\mathbb{A}(x)\nabla u(x)\right) = 0 \tag{1.9}$$

where the coefficient matrix $\mathbb{A}(x)$ is merely measurable and satisfies the uniform ellipticity condition

$$\lambda|\zeta|^2 \leq \mathbb{A}(x)\zeta\cdot\zeta \leq \Lambda|\zeta|^2 \tag{1.10}$$

for a.e. $x \in \Omega$ and all $\zeta \in \mathbb{R}^n$. Such estimates are sufficient to establish regularity in the nonlinear case; roughly speaking one then linearises the Euler-Lagrange equation to obtain a linear equation for the gradient. Shortly afterwards MOSER [Mos60] simplified the proof, and these results are usually referred to as the *De Giorgi-Nash-Moser theorem*. These results were extended by MORREY [Mor59] and STAMPACCHIA [Sta60] to include lower-order terms, and the case of general quasilinear equations was treated by LADYZHENSKAYA & URAL'TSEVA [LU61]. We will not provide a detailed account of further developments, of which there are many, however this largely settled the regularity problem in the scalar ($N = 1$) case for suitably regular functionals.

1.1.2 Semicontinuity and quasiconvexity

The vector-valued setting remained unresolved however, starting with the question of existence. In the scalar case, it was well understood that to obtain a general existence theory for minima, it was necessary to assume the integrand F is *convex* in the gradient variables. Under this assumption, the results of MORREY [Mor43] and SERRIN [Ser61] can also be extended with little difficultly to the vector-valued case. However it was already observed prior by HADAMARD [Had05] that for elliptic systems, it was often more natural to consider the weaker condition

$$F''(z)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq 0, \tag{1.11}$$

for $z \in \mathbb{R}^{Nn}$, $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$; this is referred to as *Legendre-Hadamard ellipticity* which we will investigate in Chapters 3, 4. In particular, this condition requires F to be convex in the direction of rank-one matrices which take the form $\zeta = \xi \otimes \eta$, which we refer to as *rank-one convexity*. In the context of minimisation problems, this was studied in VAN HOVE [Van47] who observed this condition was necessary for quadratic functionals, however it was unclear if one could establish a general existence theory under this assumption. A systematic study of necessary and sufficient conditions to apply the Direct Method was initiated by MORREY in 1952, where the notion of *quasiconvexity* is introduced in his seminal paper [Mor52]. This condition states that F satisfies

$$\int_\Omega F(z + \nabla \varphi) \,\mathrm{d}x \geq F(z) \mathcal{L}^n(\Omega) \tag{1.12}$$

for all $z \in \mathbb{R}^{Nn}$ and all $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$, for a non-empty bounded domain $\Omega \subset \mathbb{R}^n$. He moreover established that this condition was both necessary and sufficient for the associated functional to be sequentially weakly* lower-semicontinuous in $W^{1,\infty}$. However this is not the end of the story; there appeared to be no easy characterisation of this condition, and while this condition implied rank-one convexity of F it was unknown if they were equivalent. MORREY [Mor52] provided general examples of quasiconvex integrands, which included rank-one convex quadratic functionals which take the form $F(z) = \mathbb{A}z \cdot z$, and functionals involving determinants. The latter was formalised and studied by BALL [Bal76] in the context of elasticity, where he introduced the notion of *polyconvexity*; here one requires $F(z)$ to be expressible as a convex function in its entries and minors. It is relatively easy to verify the sequence of implications

$$\text{convex} \implies \text{polyconvex} \implies \text{quasiconvex} \implies \text{rank-one convex}, \tag{1.13}$$

and that they are equivalent if $n = 1$ or $N = 1$. However if $n, N \geq 2$, an important question is whether these notions coincide, or if they are genuinely different. By considering the determinant, it is evident that polyconvex functions in $\mathbb{R}^{2\times 2}$ need not be convex in general. The construction of Terpstra [Ter39], which was later revisited by Serre [Ser83], gives a quasiconvex quadratic form on $\mathbb{R}^{3\times 3}$ which is not polyconvex. Later Dacorogna & Marcellini [DM88] showed that the quartic polynomial on $\mathbb{R}^{2\times 2}$ defined by

$$F_\gamma(z) = |z|^2 \left(|z|^2 - 2\gamma \det z\right) \tag{1.14}$$

is rank-one convex if and only if $|\gamma| \leq \frac{2}{\sqrt{3}}$, and that it is polyconvex if and only if $|\gamma| \leq 1$. Moreover Alibert & Dacorogna [AD92] showed that there exists $\varepsilon > 1$ such that F_γ is quasiconvex when $|\gamma| \leq 1 + \varepsilon$. We also point out that Šverák [Šve91] gave a general construction of non-convex but quasiconvex functions satisfying a subquadratic growth condition, which in particular cannot be polyconvex, which we will discuss in Section 2.1.4. Whether rank-one convex functions are quasiconvex is known as *Morrey's conjecture*, and this was shown to be negative when $n, N \geq 3$ by Šverák [Šve91]. Recently Grabovsky [Gra18] provided another example of a rank-one convex function which is not quasiconvex when $n = 2$ and $N = 8$.

Returning to the problem of existence, Meyers [Mey65] established a weak lower-semicontinuity result for signed quasiconvex integrals in $W^{1,p}$ subject to fixed boundary conditions, hence establishing existence of minima for integrands satisfying a p-growth condition. The case of non-autonomous integrands was considered by Acerbi & Fusco [AF84] and Marcellini [Mar85], and there have been many generalisations and developments since which we will not discuss here. In Chapter 5 we will extend some of these results to the general growth setting, and present a general existence theorem.

1.1.3 Regularity in the vectorial case

We now turn to the problem of regularity in the vectorial case; in the 1960's there was a significant effort to generalise the results of De Giorgi and Nash to the setting of systems. The only positive results known at the time were in dimension two, where the results of Morrey [Mor38; Mor40] could be straightforwardly extended to the vectorial case. Over a decade after the resolution of scalar case however, it was independently shown by De Giorgi [De 68] and Maz'ya [Maz68] that minima of convex functionals may be singular in general; both for systems and higher-order functionals respectively. The counterexample of De Giorgi involves a quadratic integrand of the form $F(x,z) = \mathbb{A}(x)z \cdot z$, with $\mathbb{A}$ discontinuous, that admits a Lipschitz but non-C^1 minimiser for $n = N \geq 3$. On the other hand Maz'ya exhibited an unbounded solution to a scalar fourth order equation when $n \geq 5$.

Shortly afterwards, an example of a continuous quadratic integrand with $\mathbb{A} = \mathbb{A}(u,z)$ was constructed by Giusti & Miranda [GM68], which held for $n = N$ sufficiently large. Later Nečas [Neč77] constructed an autonomous convex integrand which admits a Lipschitz but non C^1-minimiser for $N = n^2$ sufficiently large, which was adapted by Hao, Leonardi & Nečas [HLN96] to hold for $n \geq 5$. This was improved by Šverák & Yan [ŠY00] who gave a counterexample when $n = 3$ and $N = 5$. Moreover in [SY02] this is extended further in several different directions; they exhibit a non-Lipschitz minimiser when $n = 3$, $N = 5$, a non-C^1 minimiser when $n = 4$, $N = 3$, and an unbounded minimiser when $n = 5$, $N = 14$. More recently Mooney & Savin [MS16] constructed a singular minimiser when $n = 3$ and $N = 2$, which conclusively established irregularity for all dimensions in the convex case (except for when $n \leq 2$ or $N = 1$, where we do have full regularity).

Shortly before De Giorgi and Maz'ya's counterexamples emerged, Morrey [Mor68] established a general *partial regularity result*, for integrands of the form $F = F(x,z)$ satisfying the controlled growth conditions

$$|\partial_z^2 F(x,z)| \leq \Lambda(1+|z|)^{p-2}, \quad \partial_z^2 F(x,z)\zeta \cdot \zeta \geq \lambda(1+|z|)^{p-2}|\zeta|^2 \tag{1.15}$$

where $p \geq 2$, for all $x \in \Omega$, $z \in \mathbb{R}^{Nn}$ and $\zeta \in \mathbb{R}^{Nn}$, with additional bounds on mixed second derivatives. Under these assumptions, it is proven that minimisers u are smooth away from a small singular set $\Sigma \subset \Omega$, with $\mathcal{L}^n(\Sigma) = 0$. We point out that a much more general result for solutions to higher-order elliptic systems was presented in Morrey [Mor68], however we will limit out discussion to second order equations to simplify the exposition. Related results were obtained independently by Giusti & Miranda [GM68], who considered quadratic functionals and showed the dimension estimate $\mathcal{H}^{n-2}(\Sigma) = 0$ for the singular set.

The proof of Morrey [Mor68] involves a blow-up technique, citing the work of Almgren [Alm68] in the context of minimal surfaces and related geometric problems. The strategy is to show a key excess decay estimate holds by means of a contradiction argument;

one assumes we have a sequence of solutions exhibiting the failure of the key estimate, which after rescaling gives the so-called "blow-up" sequence u_k. In [Mor68] difference quotients are used to establish a-priori estimates on the higher derivatives, which can be used to show that, up to a subsequence, u_k converges strongly (in a suitable topology) to a limit map h, which solves a linear elliptic equation. Using the regularity of h, together with the strong convergence of u_k, one can then obtain the desired contradiction. Such ideas also go back to De Giorgi [De 61] in the context of minimal surfaces, and related works.

We remark that there are two key ingredients involved in this argument; a compactness result, and local regularity for the limit harmonic mapping. The former typically appears in the form of a *Caccioppoli inequality*, which provides a reverse Poincaré inequality of the form

$$\int_{B_{R/2}(x_0)} |\nabla u|^2 \,\mathrm{d}x \leq \frac{C}{R^2} \int_{B_R(x_0)} |u - (u)_{B_R(\Omega)}|^2 \,\mathrm{d}x. \tag{1.16}$$

In [Mor68] this is combined with a difference quotient argument, which provides second order estimates of the form

$$\int_{B_{R/2}(x_0)} |\nabla^2 u|^2 \,\mathrm{d}x \leq \frac{C}{R^2} \int_{B_R(x_0)} 1 + |\nabla u|^2 \,\mathrm{d}x, \tag{1.17}$$

which up to a subsequence gives strong $W^{1,2}$ convergence of the blow-up sequence to a harmonic limit h. Here by a harmonic mapping, we mean a solution to the linearised problem

$$-\operatorname{div} \mathbb{A}\nabla h = 0 \qquad \text{where } \mathbb{A} = \partial_z^2 F(x_0, u_0, z_0), \tag{1.18}$$

where u_0 and z_0 are chosen to be respectively locally close to u and ∇u near x_0 in a suitable sense. Bounds for the form (1.15) implies the system (1.18) is elliptic, for which it is well known that h is locally smooth.

While the works of Morrey [Mor68] and Giusti & Miranda [GM68] laid the foundations for the regularity theory in the vectorial setting, many important questions remained. In particular one may ask how to treat the general non-autonomous case where the functional depends on $(x, u, \nabla u)$, and what the optimal bounds are for the singular set Σ. In this direction, it was independently noted by the works of Meyers & Elcrat [ME75] and Giaquinta & Giusti [GG78] that they could apply a result of Gehring [Geh73] to deduce higher integrability of minima. The idea is that (1.16) combined with a Poincaré-Sobolev inequality gives the reverse Hölder inequality

$$\fint_{B_{R/2}(x_0)} |\nabla u|^2 \,\mathrm{d}x \leq C \left(\fint_{B_R(x_0)} |\nabla u|^{\frac{2n}{n+2}} \,\mathrm{d}x \right)^{\frac{n+2}{n}}, \tag{1.19}$$

which implies the existence of $\varepsilon > 0$ such that $\nabla u \in L^{2+\varepsilon}$. While the original result of Gehring cannot be immediately applied since he assumed both sides are integrated over the same ball, it was generalised to this setting by Meyers & Elcrat [ME75] along with Giaquinta & Modica [GM79] and Stredulinsky [Str80]. This not only allowed one to treat u-dependent integrals via a perturbation argument, but it also provided refined singular set estimates of the form $\mathcal{H}^{n-2-\varepsilon}(\Sigma) = 0$ for some $\varepsilon > 0$ (see for instance [GG78]).

These approaches also lead to *direct* proofs of partial regularity, which do not involve a blow-up argument. Here the ingredients remained the same, using a Caccioppoli inequality and also by locally approximating u by the harmonic approximant h solving

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla h = 0 & \text{in } B_R(x_0), \\ h = u & \text{on } \partial B_R(x_0), \end{cases} \tag{1.20}$$

using the notation of (1.18). This strategy was introduced in Giaquinta & Giusti [GG78], and was developed further in [GG82; GG83] to work at the level of the functional. This allows one to assume a uniform Hölder assumption in (x, u), so one need not have a Euler-Lagrange system. We also mention the work of Ivert [Ive79] which involves similar ideas.

The above discussed the case of convex integrands, and a natural question is what happens when F is quasiconvex. This was established by Evans [Eva86], who assumed a *strict quasiconvexity condition* taking the form

$$\int_\Omega F(z_0 + \nabla\varphi) - F(z_0) \,\mathrm{d}x \geq \lambda \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x \tag{1.21}$$

for all $z_0 \in \mathbb{R}^{Nn}$ and $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$. Here the key ingredient in the proof is a *Caccioppoli inequality of the second kind* taking the form

$$\int_{B_{R/2}(x_0)} |\nabla u - \nabla a|^2 \,\mathrm{d}x \leq \frac{C}{R^2} \int_{B_R(x_0)} |u - a|^2 \,\mathrm{d}x, \tag{1.22}$$

for all affine maps $a\colon \mathbb{R}^n \to \mathbb{R}^N$. Now one can apply a standard blow-up argument, noting (1.22) ensures strong $W^{1,2}$-convergence of the blow-up sequence (up to a subsequence).

While EVANS only considered autonomous integrands with $p \geq 2$ growth bounds on F'', this has been relaxed significantly since; we will discuss further developments and strategies in Section 5.3 .

An interesting feature of the quasiconvex case is that we need to consider minimisers; this is in contrast to autonomous convex integrands, where solutions to the Euler-Lagrange system are automatically global minima. In particular the proof the Caccioppoli inequality (1.22) in [Eva86] appears to crucially rely on both the minimising property and the strong quasiconvexity condition. The necessity of minimality was shown in the striking work of MÜLLER & ŠVERÁK [MŠ03], who used and developed convex integration techniques to construct a strictly quasiconvex function F and infinitely many solutions to the associated Euler-Lagrange system which are Lipschitz but nowhere C^1. This was extended to suitably weak local minimisers by KRISTENSEN & TAHERI [KT03] and to polyconvex integrands by SZÉKELYHIDI [Szé05]. On the other hand, KRISTENSEN & TAHERI [KT03] showed that partial regularity results hold if u is a so-called *strong* $W^{1,p}$*-local minimiser*, essentially by establishing a suitable replacement for the Caccioppoli inequality (1.22). In Chapter 4, we will consider a situation where a Caccioppoli-type inequality can be obtained for critical points, and use it to establish regularity for such critical points.

1.2 Outline of the thesis

In this thesis we will focus on three separate problems:

(Q1) Regularity of critical points associated to rank-one convex integrands.

(Q2) Existence and partial regularity of minima of quasiconvex integrands.

(Q3) Improved differentiability results for minima of convex integrands.

The central theme among these topics is that of partial regularity, which involves characterising the set of regular points of solutions or minimisers u in a domain Ω, which is typically defined (in the interior case) as the open set

$$\Omega_0 = \left\{ x_0 \in \Omega : u \text{ is } C^{1,\alpha} \text{ in } B_R(x_0) \text{ for some } R > 0 \right\}. \tag{1.23}$$

This is typically done through an ε*-regularity theorem*, which asserts that x_0 is a regular point for u if a particular excess quantity is small (less than some given $\varepsilon > 0$). If such a criterion holds at Lebesgue points of ∇u, we can infer that u is "almost everywhere regular," and possibly even infer dimension bounds on the complement $\Omega \setminus \Omega_0$.

In Chapter 4 we will address (Q1), and establish a criterion for solutions to Euler-Lagrange systems to be regular near a point. This will involve requiring an a-priori smallness condition of the gradient in BMO, and hence will be referred to as a BMO ε*-regularity theorem.* Compared to other ε-regularity theorems in the literature, in general this smallness condition *need not hold at any point*, which is nevertheless best possible in light of the counterexamples of Müller & Šverák [MŠ03]. The strategy of proof will be very similar to how one usually establishes partial regularity, and we will adapt the method of $\mathbb{A}$-harmonic approximation used by Gmeineder & Kristensen [GK19a] in the context of BV minimisers.

In Chapter 5 we will consider (Q2), where in particular we consider functionals associated to integrands satisfying a quasiconvexity condition and further a *general growth condition* governed by an N-function φ. The overarching goal is to develop an analogous theory as in the case of p-growth, which is by now well-understood, in the case where φ is "almost linear." On the other hand we will not consider integrands that grow too quickly, by imposing a so-called Δ_2-condition (defined in Section 2.1.2). We are able to establish a general existence theorem in this setting, which is new even in the case of p-growth, however the corresponding regularity is a point of ongoing investigation. Nevertheless we will establish a general partial regularity result, subject to the additional assumption that φ satisfies a ∇_2-condition.

Finally we consider (Q3) in Chapter 6, which is joint work with Lukas Koch (MPI Leipzig). Unlike in the quasiconvex case, for convex integrands one can apply the method of difference quotients to deduce higher differentiability of minima, which combined with an ε-regularity result provides dimensional estimates for the singular set $\Sigma = \Omega \setminus \Omega_0$. We will consider relaxed minimisers for convex integrands, subject to a (p,q)-growth condition of the form

$$|z|^p \lesssim F(x,z) \lesssim 1 + |z|^q, \tag{1.24}$$

with $1 < p \leq q < \infty$. We will adapt an approach of Savaré [Sav97], which we modify to use second order difference quotients, to obtain improved fractional differentiability.

Corresponding to the three problems described above, this thesis will consist of results from the following papers, along with some results from ongoing projects.

- C. Irving, "BMO ε-regularity results for solutions to Legendre-Hadamard elliptic systems." *Preprint*, arXiv:`2109.07265` (2021).
- C. Irving, "Partial regularity for minima of higher-order quasiconvex integrands with natural Orlicz growth." *Preprint*, arXiv:`2111.14740` (2021).
- C. Irving, L. Koch, "Boundary regularity results for minimisers of convex functionals with (p, q)-growth." *In preparation*[1] (2022).

We will now give a more detailed breakdown of the contents of each chapter.

Chapter 2 (*Technical groundwork*): We will collect several technical results which we will use throughout the thesis. Much of this involves collecting standard results involving Lebesgue and Sobolev spaces generalised to the Orlicz scales, while keeping careful control of the dependence of the estimates on the domain. To our knowledge the fractional embedding result in Section 2.3.3 is new, though they likely will not surprise experts.

Chapter 3 (*Linear elliptic estimates*): We will survey the theory of linear elliptic problems taking the form

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = F & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

focusing on solvability theorems in Orlicz-Sobolev spaces $W^{1,\varphi}(\Omega, \mathbb{R}^N)$. While the results in this section are known and straightforward extensions of the Calderón-Zygmund theory, we found it difficult to locate references which provide complete proofs of these results. As such we surveyed the two main approaches, one involving potential methods and the other using local energy estimates. The latter approach is a similar flavor of the techniques we use in the nonlinear setting, involving taking local harmonic approximations. We will detail a recent approach of Krylov [Kry07], which provides a unified framework for establishing such estimates.

[1]The title may be subject to modification.

Chapter 4 (BMO ε*-regularity results*): We will establish a regularity theorem for solutions to Legendre-Hadamard elliptic systems satisfying an a-priori local smallness condition in BMO. This section is largely based on the preprint [Irv21a] of the author, with a small extension to the case of controlled $p \geq 2$ growth. In particular, we will establish regularity in both the interior and boundary case, and also discuss global consequences.

Chapter 5 (*Minima of quasiconvex integrands*): We will study the minima of non-autonomous functionals, subject to a general growth condition and a strict quasiconvexity condition. We are primarily interested in growth conditions governed by N-functions satisfying the Δ_2-condition only, however presently many of our results additionally assume a ∇_2-condition. The main result is a general $C^{1,\alpha}$ partial regularity result for minimisers of non-autonomous integrands satisfying a strict quasiconvexity condition, adapting the method of $\mathbb{A}$-harmonic approximation from [GK19a] to this setting. We will also establish a general existence theorem in the case of signed integrands where we do not need the ∇_2-condition.

Chapter 6 (*Regularity of minima for convex non-uniformly elliptic problems*): We will consider minimisers associated to functionals of the form $F = F(x, z)$, which are convex and non-uniformly elliptic in that they satisfy a (p, q)-growth condition. By means of a second order difference quotient technique, we will show improved interior regularity in the Besov space $B^{1+r,p}_{\infty}$ for $r < \min\left\{\frac{1+\alpha}{p}, \frac{1}{p-1}\right\}$, where we assume F satisfies a $C^{1,\alpha}$-dependence in the x-variable and that the exponents satisfy $2 \leq p \leq q < \frac{(n+1)p}{n}$. This is known to be essentially sharp for the p-Laplacian, and provides refined singular set estimates for relaxed minimisers.

Chapter 2

Technical groundwork

We will collect several technical results which we will use in our regularity analysis. While these results are largely known, we found it was difficult to locate a precise reference of many of these results, which in particular keep careful track of the dependence of constants. In particular, we will establish many well-known inequalities for Sobolev functions to hold in the Orlicz setting, with careful control of the dependence of the domain. The modular estimates will be useful in Chapter 5 where we study integrands satisfying a general growth condition, and the careful dependence of constants will be useful in Chapter 4 where we consider boundary regularity. In this thesis we will not need to combine these results, however our presentation allows for a unified approach which we hope to apply in future work. As such, we have referred to this section as *groundwork* instead of *preliminaries*.

2.1 Notation and definitions

2.1.1 List of notation

Constants: Throughout the thesis, C will denote a positive constant that may change form line to line. We may specify the dependence by writing $C_{a,b,\dots}$, or $C = C(a, b, \dots)$. We will also write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$, and $a \simeq b$ if we have $C^{-1}b \leq a \leq Cb$ for some $C > 0$. Also if $\phi(t)$, $\psi(t)$ are functions of a single variable t, we will write $\phi \sim \psi$ if we have $\phi(t) \simeq \psi(t)$ for all t; this will commonly be used with N-functions.

Vectors and matrices: We denote by $\mathbb{R}^N$ the usual Euclidean space, equipped with the norm $|v|$ and the inner product $v \cdot w$. The standard basis will be denote by $e_1, \dots, e_N$. We will also denote by $\mathbb{R}^{Nn}$ the space of $N \times n$ real matrices, which we equip with the inner product $z \cdot w = \operatorname{tr}(z^t w)$ and norm $|z|^2 = z \cdot z$ for $z, w \in \mathbb{R}^{Nn}$. Entries of z will be denoted by z_{ij}.

Measures and integrals: We will denote the Lebesgue on $\mathbb{R}^n$ by $\mathcal{L}^n$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. Integration will be with respect to the Lebesgue measure unless otherwise specified, in which case will use the notation $\int_X \cdots \,\mathrm{d}\mu(x)$ to integrate over the variable $x \in X$ with respect to a measure μ.

If (X, μ) is a finite measure space and f is integrable on X, we define the averaged integral as

$$\fint_X f(x)\,\mathrm{d}\mu = \frac{1}{\mu(X)} \int_X f(x)\,\mathrm{d}\mu(x). \tag{2.1}$$

Derivatives: For a distribution u on some $\Omega \subset \mathbb{R}^N$, its distributional derivatives will be denoted $\partial^\alpha u$ for multi-indices $\alpha \in \mathbb{N}_0^n$. Also if $\alpha = e_i$, then we write $\partial^{\alpha_{e_i}} = \partial_i$. We will denote by $\nabla^k u$ the vector of partial derivatives of order $|\alpha| = \sum_{i=1}^n \alpha_i = k$. If $k = 1$, ∇u will be viewed as $N \times n$ matrix. Also if $G \colon \Omega \to \mathbb{R}^{Kn}$ is a vector field mapping into the space of $K \times n$ matrices, we denote its divergence by $(\operatorname{div} G)_j = \sum_i \partial_i G_{ij}$ which is $\mathbb{R}^K$-valued.

For mappings $F \colon \mathbb{R}^{Nn} \to \mathbb{R}$ we define its derivative $F' \colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ as

$$F'(z)w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(z + tw), \tag{2.2}$$

and for a differentiable map $A \colon \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ its derivative $A'(z)$ will be a linear map $\mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ at each $z \in \mathbb{R}^{Nn}$, defined by

$$A'(z)w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A(z + tw). \tag{2.3}$$

If F is C^2 this allows us to define F'', which satisfies $F''(z)v \cdot w = F''(z)w \cdot v$ for all $z, v, w \in \mathbb{R}^{Nn}$. If $F = F(x, z)$ depends on additional variables, we will write $\partial_x F$, $\partial_z F$ for the derivatives with respect to each variable.

Function spaces: For $\Omega \subset \mathbb{R}^n$ and $M \geq 1$, we will use the notation $X(\Omega, \mathbb{R}^M)$ to denote function spaces consisting of mappings $\Omega \to \mathbb{R}^M$, omitting the dependence of the ambient space if $M = 1$. The associated norm will be denoted $\|\cdot\|_{X(\Omega)}$, where we omit the dependence of the ambient space. In particular we will consider the following function spaces:

- The *Lebesgue spaces* by $L^p(\Omega, \mathbb{R}^M)$. For an arbitrary measure space (X, μ) we may also write $L^p(X, \mu, \mathbb{R}^M)$, and in Definition 2.2.6 we also use $L^p(\Omega, w, \mathbb{R}^M)$ for weighted spaces. We will also consider Orlicz spaces $L^{\varphi}(\Omega, \mathbb{R}^M)$ in Section 2.3.1. In addition $L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^M)$ will denote the measurable functions $\Omega \to \mathbb{R}^M$ whose restriction to each $K \subset \Omega$ compact is L^p-integrable.

- The Hölder spaces $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ for $k \geq 0$ and $\alpha \in [0,1]$ denotes the k-times classical differentiable functions with k-fold derivatives which are α-Hölder continuous up to the boundary. If $\alpha = 0$ we write $C^{k,\alpha} = C^k$, understood as k-times differentiable functions. We denote by $[\,\cdot\,]_{C^{0,\alpha}(\Omega)}$ for the associated seminorm. Additionally $C^{k,\alpha}(\Omega, \mathbb{R}^N)$ denotes the space of functions whose k^{th} order derivatives are locally α-Hölder continuous in Ω, and $C^{k,\alpha}_c(\Omega, \mathbb{R}^N)$ denotes those which are compactly supported in Ω. Finally we write $C^{\infty} = \bigcup_{k \geq 0} C^k$, and similarly for the local versions.

- We write $\mathcal{D}(\Omega, \mathbb{R}^N) = C^{\infty}_c(\Omega, \mathbb{R}^N)$ for the space of test functions, and its dual $\mathcal{D}'(\Omega, \mathbb{R}^N)$ which is the space of distributions.

- The *Sobolev spaces* will be denoted by $W^{k,p}(\Omega, \mathbb{R}^N)$, denoting the space of $u \in L^p(\Omega, \mathbb{R}^N)$ for with the weak derivatives $\partial^{\alpha} u$ exist and lie in L^p for all $|\alpha| \leq k$. We denote by $W^{k,p}_0(\Omega, \mathbb{R}^N)$ the closure of $C^{\infty}_c(\Omega, \mathbb{R}^N)$ with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$, and $W^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for the space of k-times weakly differentiable functions whose derivatives lie in L^p_{loc}. We also will define the Orlicz-Sobolev spaces $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ in Section 2.3.1.

Common subsets of $\mathbb{R}^n$: We will make extensive use of the following notation.

- We denote by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ as the half-space, and by $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n = 0\}$ the associated half-plane. Also S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

- For $x_0 \in \mathbb{R}^n$ and $R > 0$, $B_R(x_0)$ will denote the open ball of radius R centred at x_0. We will write $B = B_1(0)$ for the unit ball, and in general we may drop the x_0-dependence if it is clear from context. The closed ball will be denoted by $\overline{B}_R(x_0)$.

- If $\Omega \subset \mathbb{R}^n$ is open, we write $\Omega_R(x_0) = \Omega \cap B_R(x_0)$. If $\Omega = \mathbb{R}^n_+$ is the half-space, then we write $B_R(x_0) \cap \mathbb{R}^n_+ = B^+_R(x_0)$. Their respective closures will be denoted $\overline{\Omega}_R(x_0)$ and $\overline{B}^+_R(x_0)$, and again the x_0-dependence may be dropped if it is clear from context.

- Cubes will typically be denoted by Q, and a cube with side-length $2R$ centred at x_0 will be denoted by $Q_R(x_0)$.

We also write $\omega \Subset \Omega$ for compact inclusions, that is if $\overline{\omega}$ is compact and contained in Ω. Additionally for any set $A \subset \mathbb{R}^n$ we also write $\operatorname{dist}(x, A) = \inf_{y \in A} |x - y|$, and if $B \subset \mathbb{R}^n$ is another set we write $\operatorname{dist}(A, B) = \inf_{x \in A} \operatorname{dist}(x, A)$.

2.1.2 N-functions

In Chapter 5 we will study variational problems where the integrand satisfies a *general growth condition*. In this section we will make this precise and introduce the scales of interest.

Definition 2.1.1. By an *N-function* we mean a non-decreasing continuous convex function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for all $t > 0$ and

$$\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty. \tag{2.4}$$

Classical examples include the power functions $\varphi(t) = t^p$, which are N-functions for $1 < p < \infty$.

Since an N-function φ is convex, it admits a weak derivative $\varphi'\colon [0,\infty) \to [0,\infty)$ which can be chosen to be left-continuous and non-decreasing. We can then define the generalised inverse of φ' as

$$(\varphi')^{-1}(s) = \inf\{t > 0 : \varphi'(t) > s\}, \tag{2.5}$$

which is also left-continuous and non-decreasing. Using this we can define the *conjugate of* φ as

$$\varphi^*(s) = \int_0^s (\varphi')^{-1}(\sigma)\,\mathrm{d}\sigma, \tag{2.6}$$

and we can verify this is an N-function. This satisfies the following inequalities, which can be found in [RR91, Theorem I.3] and [RR91, Proposition II.1(ii)] respectively.

Lemma 2.1.2 (Young's inequality)**.** Let φ be an N-function, then we have

$$ts \le \varphi(t) + \varphi^*(s) \tag{2.7}$$

for all $s, t \ge 0$. Moreover equality holds if and only if $t = (\varphi^*)'(s)$ or $s = \varphi'(t)$, and in particular we have

$$\varphi^*(s) = \sup_{t > 0} \left(ts - \varphi(t)\right). \tag{2.8}$$

Also if $\delta > 0$, applying this to $\delta\varphi$ gives

$$ts \le \delta\varphi(t) + \delta\varphi^*(s/\delta) \tag{2.9}$$

for all $s, t \ge 0$.

Lemma 2.1.3. For all $t \ge 0$ we have

$$t \le \varphi^{-1}(t)(\varphi^*)^{-1}(t) \le 2t. \tag{2.10}$$

The utility of this estimate is that it implies (2.14) in Lemma 2.1.6 below. Before we state this however, it will be convenient to introduce the notion of doubling N-functions.

Definition 2.1.4. We say an N-function φ satisfies the Δ_2-*condition* if there exists $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t > 0$, and the smallest such C will be denoted by $\Delta_2(\varphi)$. If φ^* satisfies the Δ_2-condition, we say φ satisfies the ∇_2-*condition* and write $\nabla_2(\varphi) = \Delta_2(\varphi^*)$. Note this is equivalent to the existence of $r > 0$ such that $\varphi(rt) \ge 2r\varphi(t)$; indeed one can verify using (2.8) that $r = \frac{1}{2}\nabla_2(\varphi)$ works.

Lemma 2.1.5. Let φ be an N-function. If $\varphi \in \Delta_2$, then for any $s \ge 1$ and $t \ge 0$ we have

$$\varphi(st) \le \Delta_2(\varphi) s^{\log_2 \Delta_2(\varphi)} \varphi(t). \tag{2.11}$$

Similarly if $\varphi \in \nabla_2$, then for any $s \in [0, 1]$ and $t \ge 0$ we have

$$\varphi(st) \le \nabla_2(\varphi)\, s^{1+\frac{1}{\log_2 r}} \varphi(t), \tag{2.12}$$

where $r = \frac{1}{2}\nabla_2(\varphi)$.

Proof. Note that by convexity of φ, φ^*, we have $\Delta_2(\varphi), \nabla_2(\varphi) \ge 2$ which ensures the associated exponents are positive. To establish the first statement, for $k \ge 0$ we can apply the doubling condition k-times to get $\varphi(2^k t) \le \Delta_2(\varphi)^k \varphi(t)$ for all $t \ge 0$. Now given $s \ge 1$, choosing k so that $2^k \le s \le 2^{k+1}$ the estimate follows noting that $\Delta_2(\varphi)^k \le \Delta_2(\varphi)^{1+\log_2 s}$. For the second statement we similarly use the estimate $\varphi(t/r^k) \le (2r)^k \varphi(t)$, where $r = \frac{1}{2}\nabla_2(\varphi)$. Given $0 < s \le 1$ we can choose k so that $r^{-(k+1)} < s \le r^{-k}$, and note that $(2r)^k \le 2r \cdot s^{\frac{\log_2(2r)}{\log_2 r}}$. $\square$

Another observation that will be useful later is that if φ is an N-function, then $\varphi^p \in \Delta_2$ for each $p > 1$. This follows by noting that $\varphi(rt) \ge r\varphi(t)$ whenever $r \ge 1$ by convexity of φ, so taking $r = 2^{\frac{1}{p-1}}$ we get $\varphi(rt)^p \ge r^p\varphi(t)^p = 2r\varphi(t)^p$ as claimed.

Lemma 2.1.6. For an N-function φ, we have the inequalities

$$\frac{t}{2}\varphi'\left(\frac{t}{2}\right) \leq \varphi(t) \leq t\,\varphi'(t), \tag{2.13}$$

$$\varphi^*\left(\frac{\varphi(t)}{t}\right) \leq \varphi(t) \leq \varphi^*\left(\frac{2\varphi(t)}{t}\right), \tag{2.14}$$

$$\frac{1}{2}\,\varphi^*\left(\varphi'\left(\frac{t}{2}\right)\right) \leq \varphi(t) \leq \varphi^*(2\varphi'(t)), \tag{2.15}$$

for all $t \geq 0$. In particular, the following hold.

(a) If $\varphi \in \Delta_2$, we have $\varphi(t) \sim t\varphi'(t)$ and $\varphi^*(\varphi'(t)) \leq C\varphi(t)$.

(b) If $\varphi \in \nabla_2$, we have $\varphi^*(\varphi(t)/t) \sim \varphi(t)$.

(c) If $\varphi \in \Delta_2 \cap \nabla_2$, we have $\varphi^*(\varphi'(t)) \sim \varphi(t))$.

Proof. By monotonicity and non-negativity of φ' we have

$$\varphi'\left(\frac{t}{2}\right)\int_{\frac{t}{2}}^{t} \mathrm{d}s \leq \int_0^t \varphi'(s)\,\mathrm{d}s \leq \varphi'(t)\int_0^t \mathrm{d}s, \tag{2.16}$$

from which (2.13) follows using the fundamental theorem of calculus. Replacing t with $\varphi(t)$ in (2.10) and rearranging gives (2.14), and combining the above estimates gives (2.15). Assertions (a)–(c) follow immediately from estimates (2.13)–(2.15) and the additional assumptions. $\square$

The following consequence of (2.13) asserts that often we can assume our N-functions are C^1, and moreover C^k by a repeated application.

Corollary 2.1.7. If φ is an N-function, we have

$$\widetilde{\varphi}(t) = \int_0^t \frac{\varphi(\tau)}{\tau}\,\mathrm{d}\tau \tag{2.17}$$

is a C^1 N-function satisfying $\frac{1}{2}\widetilde{\varphi}(t/2) \leq \varphi(t) \leq \widetilde{\varphi}(t)$. In particular if $\varphi \in \Delta_2$, we have $\widetilde{\varphi} \sim \varphi$.

Definition 2.1.8. If φ is an N-function and $a > 0$, following [BDW20] we define the *shifted N-functions*

$$\varphi_a(t) = \int_0^t \frac{\tau\varphi'(\max\{a,\tau\})}{\max\{a,\tau\}}\,\mathrm{d}\tau. \tag{2.18}$$

Note that $\varphi_a(t) \sim t^2$ if $t \leq a$ and $\varphi_a(t) \sim \varphi(t)$ if $t \geq a$, and we have $(\varphi_a)^* = (\varphi^*)_{\varphi'(a)}$. Moreover φ_a satisfies the Δ_2 and ∇_2 conditions if φ does, with the associated estimates

$1 \leq \Delta_2(\varphi_a) \leq \Delta_2(\varphi)$ and $1 \leq \nabla_2(\varphi_a) \leq \nabla_2(\varphi)$. We observe that if $\varphi \in \Delta_2$ and $1 \leq a \leq b$, we have $\varphi_a \sim \varphi_b$ uniformly in t, where the constant depends only on $\Delta_2(\varphi)$ and b.

Note one also define shifted N-functions as

$$\widetilde{\varphi}_a(t) = \int_0^t \frac{\tau\varphi'(a+\tau)}{a+\tau}\,\mathrm{d}\tau, \tag{2.19}$$

which is equivalent in the sense that $\widetilde{\varphi}_a(t) \sim \varphi_a(t)$ uniformly in a, t. These appeared earlier in [DE08, Definition 22] in a more general form.

Example 2.1.9 (V-functionals)**.** An important class of examples arises from the V-functionals, which for $p \geq 1$ and $\mu \geq 0$ is defined by

$$V_{p,\mu}(z) = \left(\mu^2 + |z|^2\right)^{\frac{p-2}{4}} z. \tag{2.20}$$

We observe that we have

$$e_{p,\mu}(t) = |V_{p,\mu}(t)|^2 = (\mu + |z|^2)^{\frac{p-2}{2}} |z|^2 \tag{2.21}$$

defines an N-function with the asymptotics

$$e_{p,\mu} \simeq \begin{cases} |z|^2 & \text{if } |z| \leq \mu, \\ |z|^p & \text{if } |z| > \mu. \end{cases} \tag{2.22}$$

These play a similar role to the shifted N-functions in the case of p-growth. Indeed if $\varphi(t) = t^p$ for some $1 \leq p < \infty$, then

$$\varphi_\mu(t) \sim_p e_{p,\mu}(t) \tag{2.23}$$

for all $t \geq 0$. In particular, we have $e^*_{p,\mu} \sim_p e_{p',\mu^{p-1}}$.

The V-functionals often arise very naturally in the context of problems with p-growth; since we often employ a linearisation procedure to study our nonlinear problems of interest, we may encounter quadratic behaviour at small scales. As such, when considering problems with p-growth one often needs estimates valid for N-functions in the case when $\varphi(t) = e_{p,\mu}(t)$. To our knowledge, the proofs of many such estimates do not simplify in the case of V-functionals, including Young's inequality and Poincaré-Sobolev type inequalities (see Section 2.3.2).

We will also briefly discuss the notion of indices associated to N-functions. These will allow us to describe the growth of a given N-function from above and below by power-type growth, which can allow us to understand the associated Orlicz space L^φ (as defined in Section 2.3.1) as an intermediate space between two Lebesgue spaces. There is an extensive literature on this subject which we will not attempt to detail, and instead we will record the results we need. We will closely follow the exposition in Maligranda [Mal89, Chapter 11].

Definition 2.1.10. If φ is an N-function, for $t > 0$ let

$$h_\varphi(s) = \sup_{t>0} \frac{\varphi(st)}{\varphi(t)}. \tag{2.24}$$

Then the *Matuszewska-Orlicz indices* $1 \le p_\varphi \le q_\varphi \le \infty$ are defined as

$$p_\varphi(s) = \lim_{s\to 0} \frac{\log h_\varphi(s)}{\log s} \quad \text{and} \quad q_\varphi(s) = \lim_{s\to\infty} \frac{\log h_\varphi(s)}{\log s}. \tag{2.25}$$

The limits are shown to be well-defined as a consequence of [Mal89, Theorem 11.3].

These indices were introduced in Matuszewska & Orlicz in [MO60]. It is known that they essentially coincide with the *Boyd indices* as shown in [Boy71], which were introduced by Boyd [Boy69]. These indices have also been studied by Gustavsson & Peetre [GP77], where Lemma 2.1.14 is essentially proven in [GP77, Appendix B]. We point out that the latter spaces were introduced precisely to capture intermediate scales where a generalised Marcinkiewicz interpolation can be shown, and we will discuss this in Section 2.2.1.

Remark 2.1.11. By definition of p_φ, assuming it is finite for all $\varepsilon > 0$ we have

$$\varphi(st) \le s^{p_\varphi - \varepsilon}\varphi(t) \tag{2.26}$$

for all $t \ge 0$ and $s > 0$ sufficiently small. Moreover p_φ is the largest constant with this property. Similarly if $q_\varphi < \infty$, then for all $\varepsilon > 0$ we have

$$\varphi(st) \le s^{q_\varphi + \varepsilon}\varphi(t) \tag{2.27}$$

for all $t \ge 0$ and s sufficiently large.

Example 2.1.12. Let $\varphi(t) = e_{p,\mu}(t)$ from Example 2.1.9, with $1 < p < \infty$ and $\mu > 0$. Then since $t^{-2}e_{p,\mu}(t) \to 1$ as $t \to 0$, and $t^{-p}e_{p,\mu}(t) \to 1$ as $t \to \infty$, it follows that $p_\varphi = \min\{2, p\}$ and $q_\varphi = \max\{2, p\}$.

Similarly for an arbitrary N-function φ, for any $a > 0$ the shifted integrand $\varphi(t)$ from Definition 2.1.8 satisfies $\varphi_a(t) = \frac{\varphi'(a)}{2a} t^2$ for $0 \le t \le a$, so it follows that $p_{\varphi_a} \le 2$.

We will record some useful results in this direction.

Lemma 2.1.13. If φ is an N-function, we have $q_\varphi < \infty$ if and only if $\varphi \in \Delta_2$, and $p_\varphi > 1$ if and only if $\varphi \in \nabla_2$.

This follows by applying [Mal89, Theorem 11.7] for the Δ_2 case, from which the ∇_2 case follows from the relation

$$\frac{1}{p_\varphi} + \frac{1}{q_{\varphi^*}} = \frac{1}{p_{\varphi^*}} + \frac{1}{q_\varphi} = 1, \tag{2.28}$$

which is proven in [Mal89, Corollary 11.6].

Lemma 2.1.14. Let φ be an N-function and $1 \le r < \infty$. We have $q_\varphi < r$ if and only if there is $C > 0$ for which

$$\int_t^\infty s^{-r} \varphi(s) \frac{\mathrm{d}s}{s} \le C t^{-r} \varphi(t) \tag{2.29}$$

for all $t \ge 0$. Moreover if $\varphi \in \Delta_2$, then we can choose r and C to depend on $\Delta_2(\varphi)$ only.

Similarly if $p_\varphi < \infty$, then $p_\varphi > r$ if and only if there is $C > 0$ for which

$$\int_0^t s^{-r} \varphi(s) \frac{\mathrm{d}s}{s} \le C t^{-r} \varphi(t). \tag{2.30}$$

Moreover if $\varphi \in \nabla_2$ we can choose r, C to depend on $\nabla_2(\varphi)$ only, and the estimate holds even if $p_\varphi = \infty$.

Sketch of proof. The equivalences are proven in [Mal89, Theorem 11.8], and we only need to show the dependence of constants. If $\varphi \in \Delta_2$, taking $r = \Delta_2(\varphi) + 1$ we have

$$\int_t^\infty \sigma^{-r} \varphi(\sigma) \frac{\mathrm{d}\sigma}{\sigma} = \int_1^\infty (st)^{-r} \varphi(st) \frac{\mathrm{d}s}{s} \le \Delta_2(\varphi) t^{-r} \varphi(t) \int_1^\infty s^{-2} \,\mathrm{d}s, \tag{2.31}$$

where we have used (2.11) in the last inequality. The case when $\varphi \in \nabla_2$ is similar, taking $r = 1 + \frac{1}{\log_2(\nabla_2(\varphi))}$ and using (2.12) instead. $\square$

We now list some examples of N-functions which are not of power-type. In Section 5 we will consider N-functions which satisfy the Δ_2-condition but not necessarily the ∇_2-condition, so we will be interested in examples where this can occur.

Example 2.1.15 (Logarithmic examples)**.** The N-function $\varphi(t) = (1+t)\log(1+t) - t$ satisfies the Δ_2-condition, however $\varphi^*(t) = e^t - t + 1$ which is not doubling. Hence this gives an example of $\varphi \notin \nabla_2$. Other examples can include variations of $\varphi(t) = t\log(1+\log(1+t)) - t$.

Example 2.1.16 (Non-∇_2 with arbitrary polynomial growth)**.** Building on ideas from [KR61, Section I.4], a more pathological example is given by Chlebicka, Gianetti, & Zatorska-Goldstein [CGZ19]. For each $1 < p < q < \infty$ they construct an N-function φ which fails the Δ_2-condition but satisfies $t^p \leq \varphi(t) \leq t^q$ for all $t \geq 0$, which is defined as

$$\varphi(t) = \begin{cases} \text{affine} & \text{if } t \in (a_i, b_i), \\ t^p & \text{otherwise,} \end{cases} \tag{2.32}$$

for suitable (a_i, b_i). While it is not clear whether the resulting N-function satisfies the ∇_2-condition, for each $\varepsilon > 0$ we have $\varphi^{1+\varepsilon}$ satisfies the ∇_2-condition while still failing the Δ_2-condition. Note that if $r' \geq q(1+\varepsilon)$, we have $\psi(t) = (\varphi^{1+\varepsilon})^* \in \Delta_2$ but $\psi \notin \nabla_2$ despite satisfying the growth bound $\varphi(t) \gtrsim t^r$.

Example 2.1.17 (Oscillation between growth rates)**.** Another class of examples was given by Lindberg [Lin73] which take the form

$$\varphi(t) = t^{a+b\sin(\log\log t)}, \tag{2.33}$$

for large t. For suitable choices of a, b this provides examples of N-functions satisfying a (p,q)-growth condition for arbitrary $1 < p < q < \infty$ with $\varphi \in \Delta_2 \cap \nabla_2$. This construction has been recently adapted by De Filippis & Leonetti [DL21] to also satisfy a uniform ellipticity condition.

Finally, the following outlines a procedure described in Kokilashvili & Krbec [KK91], which shows that under suitable conditions, a non-convex function φ is equivalent to an N-function. This will be applied to a particular function, which will allow us to deduce modular estimates that will be used in Chapter 4.

Lemma 2.1.18. Let $1 < p < \infty$ and consider

$$\varphi(t) = \omega(t)t^p, \tag{2.34}$$

where $\omega\colon [0,\infty) \to [0,1]$ is a continuous non-decreasing concave function such that $\omega(0) = 0$. Then there exists an N-function $\widetilde{\varphi} \in \Delta_2 \cap \nabla_2$ and $c \geq 1$ so that

$$c^{-1}\widetilde{\varphi}(t) \leq \varphi(t) \leq c\widetilde{\varphi}(t) \tag{2.35}$$

for all $t > 0$. Moreover $c, \Delta_2(\widetilde{\varphi}), \nabla_2(\varphi)$ can be chosen to depend on p only.

Proof. We will outline the construction described in [KK91]. Since ω is increasing we have $\varphi(t) \leq \frac{1}{2a}\varphi(at)$ with $a = 2^{\frac{1}{p-1}}$, and so by [KK91, Lemmas 1.1.1, 1.2.3] we get

$$\widetilde{\varphi}(t) = \frac{1}{a}\int_0^{\frac{t}{a}} \sup_{0<\tau<s} \left(\omega(\tau)\tau^{p-1}\right) \,\mathrm{d}s \tag{2.36}$$

is convex and increasing on $[0,\infty)$ satisfying

$$\widetilde{\varphi}(t) \leq \varphi(t) \leq \widetilde{\varphi}(2at) \tag{2.37}$$

for all $t \geq 0$. Further since φ satisfies $\varphi(2t) \leq 2^{p+1}\varphi(t)$ and $\varphi(at) \geq 2a\varphi(t)$ we can infer that $\widetilde{\varphi}(2t) \leq 2^{p+1}\widetilde{\varphi}(t)$ and $\widetilde{\varphi}(at) \geq 2a\varphi(t)$ also, which shows that $\widetilde{\varphi} \in \Delta_2 \cap \nabla_2$. $\square$

A similar construction gives the following.

Lemma 2.1.19 (Lemma 1.1.1 in [KK91])**.** Suppose $\varphi\colon [0,\infty) \to [0,\infty)$ is a non-negative function which is positive on $(0,\infty)$, satisfies the superlinearity condition (2.4) and such that there is $a \geq 1$ so that

$$\frac{\varphi(t_1)}{t_1} \leq \frac{a\varphi(at_2)}{t_2} \tag{2.38}$$

for all $0 < t_1 < t_2$. Then there exists an N-function $\widetilde{\varphi}$ and $C > 0$ such that $\widetilde{\varphi}(t) \leq \varphi(t) \leq C\widetilde{\varphi}(Ct)$.

2.1.3 Domains in Euclidean space

To analyse regularity up to the boundary, we will need to precisely define the notions of boundary regularity we need. While we are largely interested in domains that are $C^{1,\alpha}$ regular, we will also consider weaker notions to better understand how the constants in our estimates may vary with the domain. Since the regularity problems we study are local in nature, we will restrict our attention to *bounded domains*.

Definition 2.1.20. For $k \geq 1, \alpha \in [0,1]$ we say a bounded domain $\Omega \subset \mathbb{R}^n$ is a *bounded $C^{k,\alpha}$ domain* if $\partial\Omega$ can locally be written as the graph of $C^{k,\alpha}$ functions in the following sense: there is $R_\Omega > 0$ such that for all $x_0 \in \partial\Omega$ there is a unit vector $\nu_{x_0} \in \mathbb{R}^n$ such that letting $T_{x_0} = \langle \nu_{x_0} \rangle^\perp$ denote the orthogonal complement, there is a map

$$\gamma \colon T_{x_0} \cap B_{R_\Omega} \to \mathbb{R} \tag{2.39}$$

which is of class $C^{k,\alpha}$ such that for all $0 < R \leq R_\Omega$ we have

$$\Omega \cap B_R(x_0) = B_R(x_0) \cap \{x_0 + y + \lambda\nu : y \in T_{x_0} \cap B_R, \lambda < \gamma(y)\}\,, \tag{2.40}$$

$$\partial\Omega \cap B_R(x_0) = B_R(x_0) \cap \{x_0 + y + \gamma(y)\nu : y \in T_{x_0} \cap B_R\}\,. \tag{2.41}$$

We define the associated $C^{k,\alpha}$-constant of Ω as

$$\|\Omega\|_{C^{k,\alpha}} = \sup_{x \in \overline{\Omega}} \|\nabla\gamma_x\|_{C^{k,\alpha}(T_x \cap B_{R_\Omega}(x))}\,. \tag{2.42}$$

The definition also extends to $C^{0,1}$ domains, henceforth referred to as *Lipschitz domains*, and C^k domains.

Definition 2.1.21. We say a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies the *uniform interior (respectively exterior) cone condition* is there exists $r > 0$, $\theta \in (0, \pi/2)$ such that for all $x \in \mathbb{R}^n$, there is a unit vector n such that if

$$h \in C_r(\theta, n) = \{h \in \mathbb{R}^n : |h| \leq r, h \cdot n \geq |h| \cos\theta\}, \tag{2.43}$$

we have

$$(\Omega \setminus B_{3r}(x)) + h \subset \mathbb{R}^n \setminus \Omega \tag{2.44}$$

in the exterior case, and respectively

$$(\Omega \cap B_{3r}(x)) - h \subset \Omega \tag{2.45}$$

in the interior case.

Lemma 2.1.22 (Theorem 1.2.2.2 in [Gri11])**.** A bounded domain is a Lipschitz domain if and only if it satisfies either the uniform interior or exterior cone condition. Moreover if Ω is Lipschitz, we can always take $r \leq R_\Omega$.

Lemma 2.1.23. Let Ω be a bounded $C^{k,\alpha}$ domain with $k \geq 1$ and $\alpha \in [0,1]$. Then there exists a *defining function* $\rho = \rho_\Omega \in C^{k,\alpha}(\mathbb{R}^n)$ which satisfies

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \quad \mathbb{R}^n \setminus \overline{\Omega} = \{x \in \mathbb{R}^n : \rho(x) > 0\}, \tag{2.46}$$

and such that $\nabla \rho \neq 0$ on $\partial\Omega$. Moreover the exterior unit normal $\nu_{\partial\Omega}$ satisfies

$$\nu_{\partial\Omega}(x) = \frac{\nabla \rho(x)}{|\nabla \rho(x)|} \tag{2.47}$$

for all $x \in \partial\Omega$.

Sketch of proof. We will use the graphical representation γ at each $x_0 \in \partial\Omega$, which can be chosen to satisfy $\nabla\gamma(0) = 0$ using the C^1 regularity. Then we locally define $\rho(x) = (x - x_0) \cdot \nu - \gamma(x - x_0)$ in $B_{R_0}(x_0)$, which one can patch using a partition of unity. □

We will also consider a more general class of domains, which will be useful in keeping track of estimates when we localise near the boundary.

Definition 2.1.24. For $\delta \in (0,1)$ we say a bounded domain $\Omega \subset \mathbb{R}^n$ is a *δ-John domain* if there exists a point $x_0 \in \Omega$, called the *John centre*, for which the following holds. For all $x \in D$ there exists a rectifiable curve $\gamma\colon [0,d] \to D$ parametrised by arclength (so $|\dot{\gamma}(t)| = 1$ for almost all t) such that $\gamma(0) = x$, $\gamma(d) = x_0$ and

$$\operatorname{dist}(\gamma(t), \partial D) \geq \delta t \tag{2.48}$$

for all $t \in [0,d]$.

Note that John domains are invariant under dilation; that is if Ω is a δ-John domain, then so is $\lambda\Omega$ for all $\lambda > 0$ with the same constant. This condition can be viewed as a twisted cone condition, by noting the above asserts the so-called *δ-carrot*

$$\operatorname{car}(\gamma,\delta) = \bigcup_{0<t<d} B_{\frac{t}{\delta}}(\gamma(t)) \tag{2.49}$$

is contained in Ω. In particular, this is weaker than the interior cone condition satisfied by Lipschitz domains. Moreover, we can prove the following.

Lemma 2.1.25. Let Ω be a Lipschitz-domain with $\|\Omega\|_{C^{0,1}} < 1$. Then for all $x_0 \in \overline{\Omega}$ and $0 < R < R_\Omega$, we have $\Omega_R(x_0) = \Omega \cap B_R(x_0)$ is a δ-John domain, where δ can be chosen to depend on n and $\|\Omega\|_{C^{0,1}}$ only.

Proof. Put $L := \|\Omega\|_{C^{0,1}} < 1$, and put $\tilde{L} = \frac{L+1}{2} \in (L, 1)$. Consider the graphical representation of γ at each $x_0 \in \overline{\Omega}$ with $R_0 > 0$ so this holds, and by means of a rigid motion assume that $x_0 = 0$ and $T_{x_0} = H = \{x \in \mathbb{R}^n : x_n = 0\}$. Note that $(0, \gamma(0)) = b \in \partial\Omega$ with $b_n = \gamma(0) \leq 0$. Moreover by rescaling, we can assume $R = 1$. By assumption we have $|\nabla\gamma| \leq L$ a.e. in $H \cap B$, which implies that $|\gamma(x') - \gamma(0)| \leq L|x'|$. Therefore noting $x_n = \gamma(x')$ on $\partial\Omega$ we have

$$\partial\Omega \cap B \subset \left\{ x + b \in B : |x'| \geq \frac{|x|}{\sqrt{L^2+1}} \right\} =: S_L. \tag{2.50}$$

Moreover S_L can be seen as the union of all cones $b + C(\theta_L, n)$ intersected with B for all $n \in S^{n-2} \times \{0\}$, where $\cos\theta_L = 1/\sqrt{L^2+1}$. Note that $\theta_L < \frac{\pi}{4}$ if and only if $L < 1$. Also since $\Omega \cap B \supset B^+ \setminus S_L$ where $B^+ = \{x \in B : x_n > 0\}$, we can choose any point y_0 in $B^+ \setminus S_L$ to be our John centre.

We also consider $S_{\tilde{L}}$ which is obtained by replacing L by $\tilde{L}$ in (2.50), and the associated $\theta_{\tilde{L}} \in (\theta_L, 1)$. Now let $x = (x', x_n) \in S_{\tilde{L}}$, then we see that

$$(x + C_1(e_n, \pi/2 - \theta_L)) \cap B \subset \Omega \cap B. \tag{2.51}$$

Now let $y \in \partial B$ is the unique point such that $y \cdot n = |y'| = \cos\theta_{\tilde{L}}$; explicitly this is given by

$$y = (y', y_n) = \left(n\,\cos\theta_{\tilde{L}}, \sin\theta_{\tilde{L}}\right) \tag{2.52}$$

We also let $\omega = (-y', y_n)$, and consider $x_t = x + t\omega$. Then since $\theta_{\tilde{L}} < \frac{\pi}{4} < \frac{\pi}{2} - \theta_L$, we have

$$\operatorname{dist}(x_t, \partial\Omega) \geq \operatorname{dist}(x_t, x + \partial C(e_n, \pi/2 - \theta_L)) \geq \sigma t \tag{2.53}$$

for all $t > 0$, where $\sigma = \tan(\pi/2 - \theta_L - \theta_{\tilde{L}}) > 0$.

Also we claim there is $C > 0$ such that

$$\operatorname{dist}(x_t, \partial B) \geq \delta t \tag{2.54}$$

for all $t > 0$ such that $x_t \in S_{\tilde{L}}$. This follows by noting there is $\rho > 0$ for which

$$x + C(\rho, \omega) \cap \{z \in \partial B : |z'| \leq z_n\} = \varnothing. \tag{2.55}$$

Combining these gives $t_0 > 0$ such that $x_{t_0} \in B^+ \setminus S_L$ and such that

$$\operatorname{dist}(x_t, \partial(\Omega \cap B)) \geq \min\{\delta, \sigma\} t \tag{2.56}$$

for all $0 < t < t_0$. We can then join x_{t_0} to the John centre y_0 via a linear combination to conclude. $\square$

Localising the above result leads to the following corollary. This is the main reason why we have introduced these domains; if we can establish estimates in domains $\Omega_R(x_0)$ where the associated constant only depends on δ, then the constant holds uniformly among these domains.

Corollary 2.1.26. Let Ω be a C^1 domain. Then there is $R_0 > 0$ and $\delta = \delta(n) > 0$ such that for all $x_0 \in \overline{\Omega}$ and $0 < R < R_0$, we have $\Omega_R(x_0)$ is a δ-John domain. Moreover if Ω is a $C^{1,\alpha}$ domain for some $\alpha > 0$, then R_0 can be chosen to depend on n, α, R_Ω and $\|\Omega\|_{C^{0,\alpha}}$ only.

When establishing Sobolev and Poincaré-type inequalities on John domains Ω, we often will obtain constants that depend on the diameter and its Lebesgue measure. The following result tells us that they are essentially equivalent, up to a constant.

Lemma 2.1.27. Let $\Omega \subset \mathbb{R}^n$ be a δ-John domain. Then we have

$$\mathcal{L}^n(\Omega)^{\frac{1}{n}} \sim \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|, \tag{2.57}$$

where the implicit constants depends only on n and δ.

Proof. By definition of the diamter, we can find $x_0 \in \mathbb{R}^n$ such that $\Omega \subset B_{\operatorname{diam}(\Omega)}(x_0)$, so taking measures on both sides one direction follows. Conversely let $x_0 \in \Omega$ be the John centre, then there is $x_1 \in \Omega$ such that $|x_0 - x_1| \geq \frac{1}{3}\operatorname{diam}(\Omega)$. By definition of John domains, there is a rectifiable curve $\gamma \colon [0,d] \to D$ parametrised by arclength such that $\gamma(0) = x_1$, $\gamma(d) = x_0$ and $\operatorname{dist}(\gamma(t), \partial\Omega) \leq \delta t$ for all $0 \leq t \leq d$. In particular, we have $\operatorname{dist}(x_0, \partial\Omega) \geq \delta d \geq \delta|x_0 - x_1|$, noting that γ is an arclength parametrisation. Hence we deduce that

$$\mathcal{L}^n(\Omega) \geq \mathcal{L}^n(B_{\operatorname{dist}(x_0,\partial\Omega}(x_0))) \geq C_n \delta^n \operatorname{diam}(\Omega)^n, \tag{2.58}$$

as required. $\square$

Sometimes it will be useful to replace $\Omega_R(x_0)$ by a more regular domain, for which the following construction of Kristensen & Mingione [KM10] will be useful.

Lemma 2.1.28 (Regularised domains)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\alpha}$ domain, then there is $R_0 = R_0(n, \alpha, R_\Omega, \|\Omega\|_{C^{1,\alpha}}) > 0$ such that for all $x_0 \in \partial\Omega$ and $0 < R < R_0$ there exists a $C^{1,\alpha}$ domain $\widetilde{\Omega}_R(x_0)$ such that

$$\Omega_{R/2}(x_0) \subset \widetilde{\Omega}_R(x_0) \subset \Omega_R(x_0). \tag{2.59}$$

Moreover we have $\left\|\widetilde{\Omega}_R(x_0)\right\|_{C^{1,\alpha}} \leq C\left(1 + \|\Omega\|_{C^{1,\alpha}}\right)$ and $\widetilde{\Omega}_R(x_0)$ is a δ-John domain for some $\delta = \delta(n, \|\Omega\|_{C^{1,\alpha}}) > 0$.

Proof. Fix a $C^{1,\alpha}$ domain $A \subset \mathbb{R}^n$ such that $\overline{B}_{\frac{5}{6}}(0)^+ \subset A \subset B_1(0)^+$. Using the graphical representation from Definition 2.1.20, for $R_0 \leq R_\Omega$ small we can construct a diffeomorphism $\psi \colon B_{R_0}(x_0) \to U \subset B_1(0)$ such that $A \subset U$, $\psi(B_{R_0} \cap \Omega) = U \cap \mathbb{R}^n_+$, and such that $D\psi(x_0)$ is orthogonal. Hence by shrinking R_0 if necessary we can assume that

$$B_{\frac{5R}{6R_0}}(0) \subset \psi(B_R(x_0)) \subset B_{\frac{6R}{5R_0}} \tag{2.60}$$

for all $R \in (0, R_0)$. Thus if we let $\widetilde{\Omega}_R = \psi^{-1}\left(\frac{18R}{25R_0}A\right)$ this satisfies,

$$\Omega_{R/2}(x_0) \subset \psi^{-1}\left(\overline{B}_{\frac{3R}{5R_0}}(0)^+\right) \subset \widetilde{\Omega}_R(x_0) \subset \psi^{-1}\left(B_{\frac{18R}{25R_0}}(0)^+\right) \subset \Omega_R(x_0), \tag{2.61}$$

as claimed. Since the $C^{1,\alpha}$ norm and John constant of $\Omega_R(x_0)$ depends on the associated constants for A and n, ψ, the estimates also follow. $\square$

The utility of John domains is that they provide a way to transfer estimates on balls or cubes to hold on general domains. This is typically done by considering a suitable decomposition of the domain Ω cubes, which often is a Whitney-type decomposition.

Definition 2.1.29. A *Whitney decomposition* of an open set $\Omega \subset \mathbb{R}^n$ is a collection of disjoint open cubes $\{W_i\}$ contained in Ω satisfying

(W1) $\bigcup_i \overline{W}_i = \Omega$,

(W2) $\overline{W}_i \cap \overline{W}_j \neq \varnothing$ implies $\frac{1}{4} \leq \frac{\ell(W_1)}{\ell(W_2)} \leq 4$,

(W3) for each i, we have $\ell(W_i) \leq \operatorname{dist}(W_i, \partial\Omega) \leq 4\sqrt{n}\ell(W_i)$.

Note that such a decomposition always exists, as is shown in [Ste71, Chapter IV].

We will use the following terminology; we say two cubes Q, Q' are connected by a *chain of cubes* $Q_0, \dots, Q_k$ if we have $Q_0 = Q$, $Q_k = Q'$ and $Q_{i-1} \cap Q_i \neq \varnothing$ for all $1 \leq i \leq k$. We say the *length* of the chain is k.

Lemma 2.1.30 (Boman chain condition)**.** A bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if and only if the following *Boman chain condition* is satisfied: there exists $\sigma > 0$, $B > 1$, $M \geq 1$ and a covering $\mathcal{Q}$ of Ω by open cubes for which the following holds.

(B1) We have

$$\sum_{Q \in \mathcal{Q}} \chi_{\sigma Q} \leq M \chi_\Omega. \tag{2.62}$$

(B2) If $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \cap Q_2 \neq \varnothing$, then

$$M^{-1} \mathcal{L}^n(Q_1) \leq \mathcal{L}^n(Q_2) \leq M \mathcal{L}^n(Q_1) \tag{2.63}$$

and also

$$\mathcal{L}^n(Q_1 \cap Q_2) \geq M^{-1} \max\{\mathcal{L}^n(Q_1), \mathcal{L}^n(Q_2)\}. \tag{2.64}$$

(B3) For a distinguished cube $Q_0 \in \mathcal{Q}$, for any other $Q \in \mathcal{Q}$ there is a chain of cubes $Q_1, \dots, Q_k \in \mathcal{Q}$, such that $Q \subset BQ_i$ for each $0 \leq i \leq k-1$.

Moreover if Ω is a δ-John domain, we can choose σ, M to depend on n and B to depend on n, δ only.

The forward implication was proved by Boman [Bom82], to whom this condition is attributed to. Here we take a slightly enlarged version of a Whitney covering from Definition 2.1.29, namely $\mathcal{Q} = \{\sigma W_i\}$ with $\sigma = \frac{9}{8}$ (see also [Hur96]) The converse was proven by Buckley, Koskela & Lu [BKL96], however we will not use it in what follows. A systematic treatment of these domains, including the case of unbounded domains, was considered by Diening, Růžička, & Schumacher [DRS10].

2.1.4 Convexity notions

We will define the relevant notions of convexity for integrands defined on the space of matrices.

Definition 2.1.31. let $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ continuous.

- We say f is *convex* if we have

$$f(tz + (1-t)w) \leq tf(z) + (1-t)f(w) \tag{2.65}$$

  for all $z, w \in \mathbb{R}^{Nn}$ and $t \in (0,1)$.

- We say f is *rank-one convex* if we have (2.65) holds for all $z, w \in \mathbb{R}^{Nn}$ such that $\operatorname{rank}(z-w) \leq 1$; equivalently we can write $z - w = \xi \otimes \eta$ for some $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$.

- We say f is *quasiconvex* (in the sense of MORREY [Mor52]) if for some non-empty bounded open set $\omega \subset \mathbb{R}^n$ we have

$$\fint_\omega f(z + \nabla\xi)\,\mathrm{d}x \geq f(z) \tag{2.66}$$

  for all $z \in \mathbb{R}^{Nn}$ and $\xi \in C_c^\infty(\omega, \mathbb{R}^N)$.

- We say f is *polyconvex* if there is a convex function g such that $f(z) = g(M(z))$, where $M(z)$ denotes the vector of all $k \times k$ minors of z for all $1 \leq k \leq \min\{n, N\}$,

Note we can write $M(z) = (z, \operatorname{adj}_2 z, \dots, \operatorname{adj}_{n-1} z, \det z)$, where $\operatorname{adj}_k z$ denotes the matrix of $k \times k$ minors of z. We will write $\operatorname{adj} z = \operatorname{adj}_{n-1} z$, so in particular we have $M(z) = (z, \det z)$ if $n = 2$ and $M(z) = (z, \operatorname{adj} z, \det z)$ if $n = 3$.

As noted in the introduction (Section 1.1.2), we have the sequence of implications

$$\text{convex} \implies \text{polyconvex} \implies \text{quasiconvex} \implies \text{rank-one convex}, \tag{2.67}$$

with none of the converse statements holding for general n, N. We omit the proof, which can be found for instance in [Giu03, Chapter 5], [Dac07, Chapter 5]. It is also well-known (and shown in the above references) that if F is of class C^2, rank-one convexity is equivalent to the *Legendre-Hadamard ellipticity condition*

$$F''(z_0)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq 0 \tag{2.68}$$

for all $z_0 \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$.

We will later use the general construction of quasiconvex functions introduced by DACOROGNA [Dac82].

Definition 2.1.32. The *quasiconvex envelope* of a continuous function $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is defined as

$$\mathcal{Q}f(z) = \sup\left\{g(z) : g \text{ is quasiconvex and } g \leq f\right\}. \tag{2.69}$$

This approach of defining convex envelopes is standard, and one can also define $\mathcal{C}f, \mathcal{P}f$, and $\mathcal{R}f$ using convex, polyconvex, and rank-one convex functions g respectively. For quasiconvex envelopes we have the following equivalent representation, which was proven by DACOROGNA [Dac82] subject to bounds from below. The general case was established by KINDERLEHRER & PEDREGAL [KP91].

Lemma 2.1.33 (Dual representation of $\mathcal{Q}$, [Dac82; KP91])**.** If $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ is continuous, then the quasiconvex envelope admits the dual representation

$$\mathcal{Q}f(z) = \inf\left\{ \fint_\omega f(z + \nabla\xi(x))\,\mathrm{d}x : \xi \in C_c^\infty(\omega)\right\} \tag{2.70}$$

for any non-empty bounded open subset $\omega \subset \mathbb{R}^n$. Additionally $\mathcal{Q}f \equiv -\infty$ if and only if $\mathcal{Q}f(z_0) = -\infty$ for some $z_0 \in \mathbb{R}^{Nn}$.

We will use this construction later in Section 5.2.4, however we will also show this can be used to construct a quasiconvex function which grows like a given N-function φ. This will be an adaptation of a construction of ŠVERÁK [Šve91], and the idea will be to consider

$$\mathcal{Q}\left(z \mapsto \varphi(\operatorname{dist}(z, K))\right) \tag{2.71}$$

for a suitable subset $K \subset \mathbb{R}^{Nn}$.

Proposition 2.1.34 (Quasiconvex functions with φ-growth, [Šve91])**.** Let φ be an arbitrary N-function satisfying the Δ_2-condition. Then there exists a continuous non-convex, but quasiconvex function $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ such that

$$0 \leq F(z) \leq \varphi(|z|), \tag{2.72}$$

which grows like φ in the sense that

$$\limsup_{|z|\to\infty} \frac{F(z)}{\varphi(|z|)} > 0. \tag{2.73}$$

Proof. Define the subspace

$$L = \left\{z \in \mathbb{R}^{2\times 2} : z_{11} = z_{22},\ z_{12} = -z_{21}\right\} = \left\{\begin{pmatrix} s & -t \\ t & s \end{pmatrix} : t, s \in \mathbb{R}\right\}. \tag{2.74}$$

Note in particular that L does not contain any rank-one directions; that is we have $\operatorname{rank}(z - w) \geq 2$ for all $z, w \in L$ distinct. Then for any non-empty closed and bounded $K \subset L$ we can consider the function

$$f(z) = \varphi\left(\operatorname{dist}(z, K)\right) \tag{2.75}$$

and take the quasiconvex hull $F = \mathcal{Q}f$. This evidently satisfies (2.72). Since L is a convex subspace we have $\varphi(\operatorname{dist}(z, L))$ is convex, and hence by definition (2.69) it follows that

$$\mathcal{Q}f(z) \geq \varphi(\operatorname{dist}(z, L)) \tag{2.76}$$

for all $z \in \mathbb{R}^{Nn}$, so in particular (2.73) is satisfied. Also $\mathcal{Q}f$ vanishes on K using (2.70) and Fatou's lemma.

It remains to show that $\mathcal{Q}f$ is non-convex, for which we claim that $\mathcal{Q}$ is positive everywhere on $L \setminus K$; from here the result will follow by choosing K to be non-convex. Suppose otherwise, so there is $z \in L \setminus K$ such that $\mathcal{Q}f(z) = 0$, then by translation we can assume $z = 0$. Then by Lemma 2.1.33 there exists a bounded non-empty open subset $\omega \subset \mathbb{R}^n$ and a sequence $\xi_j \in C_c^\infty(\omega, \mathbb{R}^2)$ such that

$$\lim_{j\to\infty} \int_\omega f(\nabla \xi_j)\,\mathrm{d}x = 0. \tag{2.77}$$

Now using (2.76) we have

$$f(z) \geq \frac{1}{2}\left(\varphi(|z_{11} - z_{22}|) + \varphi(|z_{21} + z_{12}|)\right), \tag{2.78}$$

so if we set

$$g_j = \begin{pmatrix} \partial_1\xi_{j,1} - \partial_2\xi_{j,2} & \partial_2\xi_{j,1} + \partial_2\xi_{j,1} \\ \partial_2\xi_{j,1} + \partial_2\xi_{j,1} & -(\partial_1\xi_{j,1} - \partial_2\xi_{j,2}) \end{pmatrix}, \tag{2.79}$$

then $\Delta\xi_j = \operatorname{div}\nabla\xi = \operatorname{div} g_j$ and it follows from (2.77), (2.78) that

$$\lim_{j\to\infty} \int_\omega \varphi(|g_j|)\,\mathrm{d}x = 0. \tag{2.80}$$

Since ξ_j is compact, we can write $\xi_j = \nabla I * g_j$ where $I(x) = \frac{1}{2\pi}\log|x - y|$ denotes the fundamental solutions for the Laplacian in two dimensions. Using this we claim that we have the weak-type estimate

$$s\mathcal{L}^n(\{x\in\omega : |\nabla\xi_j|>s\}) \leq C\int_\omega \varphi(|g|)\,\mathrm{d}x \tag{2.81}$$

for all $s>0$, which vanishes as $j\to\infty$. This follows from applying the Calderón-Zygmund estimate established in Theorem 2.2.18 applied to $\nabla^2 I$, in conjunction with the weak-type interpolation result of Lemma 2.2.4 . Hence it follows that $\varphi(|\nabla\xi_j|)$ converges to 0 in measure on ω. Moreover noting that K is bounded, let $R_0>0$ such that $K\subset\{z\in\mathbb{R}^{Nn} : |z|\leq R/2\}$. Then since $f(z)\geq c\varphi(|z|)$ for $|z|\geq R_0$, for $R>R_0$ we have

$$\int_{|\nabla\xi_j|\geq R}\varphi\left(|\nabla\xi_j|\right)\mathrm{d}x \leq c\int_{|\nabla\xi_j|\geq R} f(\nabla\xi_j)\,\mathrm{d}x. \tag{2.82}$$

Since $f(\nabla\xi_j)\to 0$ in L^1, it is uniformly integrable and hence the left-hand side also vanishes uniformly in j as $R\to\infty$. Now as $\varphi(|\nabla\xi|)$ converges to zero in measure and is uniformly integrable, it follows that $\varphi(|\nabla\xi|)\to 0$ in L^1. But this implies that

$$f(0)=\lim_{j\to\infty}\int_\omega f(\nabla\xi)\,\mathrm{d}x = \mathcal{Q}f(0)=0, \tag{2.83}$$

which is absurd as $0\notin K$. $\square$

Remark 2.1.35 (Further examples)**.** We note that there are many more examples of quasiconvex functions satisfying subquadratic, and moreover linear growth conditions. We mention for instance the construction of Müller [Mül92] which is 1-homogeneous, and the general approximation result Gmeineder & Kristensen [GK19a, Section 2.6]. Note that if F is a quasiconvex function satisfying a linear growth condition, we have $F+\varphi(|\cdot|)$ remains quasiconvex and now satisfies a φ-growth condition, thereby giving an abundance of quasiconvex function in our setting.

2.2 Harmonic analysis

Many of our technical estimates will rely on general estimates for various maximal operators and potentials. In this section we will collect the necessary results which will be used in the subsequent chapters, and fix our conventions. The main focus will be in understanding how to extend estimates in L^p to hold in general Orlicz spaces, and we discuss two strategies in Section 2.2.1. From here we will see some well-known consequences and consolidate the particular results we need. We will also discuss the space BMO, and establish some properties which we will use in Chapter 4.

2.2.1 Interpolation and extrapolation

We will often need to establish analogues of estimates in L^p in the Orlicz scales. For instance if we know a linear operator T is bounded on all L^p spaces for $1 < p < \infty$, we would like to infer that it is bounded on the Orlicz spaces L^φ whenever $\varphi \in \Delta_2 \cap \nabla_2$. However a precise reference appears to be difficult to locate, and it is often unclear whether it can be applied more generally. Additionally in some cases we have estimates which cannot be described as a linear operation, for which an alternative approach is needed.

We will establish a Marcinkiewicz-type interpolation result, which was noted by Zygmund [Zyg56] and Riordan [Rio55]. This largely follows by reproducing the classical proof where we replace t^p by a general N-function $\varphi(t)$, and investigating what we obtain. It turns out φ will need to satisfy an integral inequality which is similar to those considered in Lemma 2.1.14, which allows us to impose condition on φ in terms of its Matuszewska-Orlicz indices.

To state the result precisely, we will need the following terminology. Given any set $\Omega \subset \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}^K$ a measurable map, we define the associated *distribution function* as

$$\lambda_f(s) = \mu\left(\{x \in X : |f(x))| > s\}\right). \tag{2.84}$$

Suppose T is an operator between measurable functions $\Omega \to \mathbb{R}^K$ and $\Omega \to \mathbb{R}^L$. We say T is *sublinear* if we have

$$|T(f_1 + f_2)| \leq |Tf_1(x)| + |Tf_2(x)| \text{ and } |Tf(\lambda x)| = |\lambda||Tf(x)|, \tag{2.85}$$

for all measurable maps f, f_1, f_2 and $\lambda \in \mathbb{R}$. We say T is of type *weak*-(p,q) if there is $A > 0$ such that

$$\lambda_{Tf}(s) \leq \left(\frac{A\,\|f\|_{L^p(\Omega)}}{s}\right)^q, \tag{2.86}$$

provided $q < \infty$. If $q = \infty$, we simply require that $\|Tf\|_{L^\infty(\Omega)} \leq A\,\|f\|_{L^\infty(\Omega)}$.

Theorem 2.2.1 (Generalised Marcinkiewicz interpolation)**.** Given $\Omega \subset \mathbb{R}^n$, let T be a sublinear map as above. For $1 \leq p \leq q \leq \infty$, suppose T is of type weak-(p,p) and weak-(q,q). Then for any N-function φ whose indices satisfies $p < p_\varphi < q_\varphi < q$, we have

$$\int_\Omega \varphi(|Tf(y)|)\,\mathrm{d}x \leq C \int_\Omega \varphi(|f(x)|)\,\mathrm{d}x \tag{2.87}$$

for all $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$. Here the implicit constants depend on the weak-type constant for T on L^p, L^q, and the constants in Lemma 2.1.14.

Moreover if $q = \infty$, in the absence of the condition $q_\varphi < \infty$ we still obtain the weaker estimate

$$\int_\Omega \varphi(|Tf(y)|)\,\mathrm{d}x \leq C \int_\Omega \varphi(C|f(x)|)\,\mathrm{d}x \tag{2.88}$$

Note that both sides of the integral may be infinite, in which case the statement holds vacuously. Note that the above formulation is sufficient for our purposes, however the result holds more generally for mappings between arbitrary measure spaces.

Proof. To avoid any issues with divergent integrals, we can work on the set $\Omega \cap \{|f|+|Tf| \leq M\}$ and send $M \to +\infty$ at the end. Given $f \colon X \to \mathbb{R}^K$ be measurable, $s > 0$, and $\mu > 0$ to be determined set

$$f_1 = f 1_{\{|f|>\mu s\}}, \text{ and } f_2 = f - f_1. \tag{2.89}$$

By subadditivity of T we have

$$\lambda_{Tf}(s) \leq \lambda_{Tf_1}\left(\frac{s}{2}\right) + \lambda_{Tf_2}\left(\frac{s}{2}\right). \tag{2.90}$$

Let A_p, A_q be the associated constants in the weak-type estimates for T. If $q < \infty$, then applying these weak-type inequalities we have

$$\lambda_{Tf}(s) \leq \left(\frac{2A_p}{s}\right)^p \int_{\{|f|\leq \mu s\}} |f|^p\,\mathrm{d}x + \left(\frac{2A_p}{s}\right)^q \int_{\{|f|>\mu s\}} |f|^q\,\mathrm{d}x. \tag{2.91}$$

Hence taking $\mu = 2$ we can estimate

$$
\begin{aligned}
&\int_\Omega \varphi(|Tf|)\,\mathrm{d}x \\
&= \int_0^\infty \varphi'(s)\lambda_{Tf}(s)\,\mathrm{d}s \\
&= \int_0^\infty \varphi'(s)\left(\frac{2A_p}{s}\right)^p \int_{\{|f|\leq 2s\}} |f|^p\,\mathrm{d}x\,\mathrm{d}s + \int_0^\infty \varphi'(s)\left(\frac{2A_p}{s}\right)^q \int_{\{|f|>2s\}} |f|^q\,\mathrm{d}x\,\mathrm{d}s \\
&= (2A_p)^p \int_\Omega |f(x)|^p \int_0^{2|f(x)|} \frac{\varphi'(s)}{s^p}\,\mathrm{d}s\,\mathrm{d}x + (2A_q)^q \int_\Omega |f(x)|^q \int_{2|f(x)|}^\infty \frac{\varphi'(s)}{s^q}\,\mathrm{d}s\,\mathrm{d}x \\
&\leq (4A_p)^p \int_\Omega |f(x)|^p \int_0^{2|f(x)|} \frac{\varphi(s/2)}{s^p}\,\mathrm{d}s\,\mathrm{d}x + (4A_q)^q \int_\Omega |f(x)|^q \int_{2|f(x)|}^\infty \frac{\varphi(s/2)}{s^q}\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}
\tag{2.92}
$$

where we have used (2.13) in the last line. Now changing variable $s \mapsto 2s$ and applying Lemma 2.1.14 the result follows.

If $q = \infty$, then taking $\mu = \frac{1}{2A_\infty}$ we have $\lambda_{Tf_1}(s/2) = 0$ for all $s > 0$ and so

$$
\lambda_{Tf}(s) \leq \int_{\{|f|\leq \mu s\}} \left(\frac{2A_p|f|}{s}\right)^p \mathrm{d}x, \tag{2.93}
$$

so we can argue as above. □

We will usually apply the result in the following form.

Corollary 2.2.2. Suppose T is a sublinear operator between spaces of measurable maps $\Omega \to \mathbb{R}^K$ and $\Omega \to \mathbb{R}^L$ which satisfies

$$
\|Tf\|_{L^p(\Omega)} \leq A_p \|f\|_{L^p(\Omega)} \tag{2.94}
$$

for all $1 < p < \infty$ and $f \in L^p(\Omega, \mathbb{R}^K)$. Then for any N-function φ satisfying $\varphi \in \Delta_2 \cap \nabla_2$ we have

$$
\int_\Omega \varphi(|Tf|)\,\mathrm{d}x \leq C \int_\Omega \varphi(|f|)\,\mathrm{d}x \tag{2.95}
$$

for all $f \in L^\varphi(\Omega, \mathbb{R}^K)$. Moreover the constant C depends on n, $\Delta_2(\varphi)$, $\nabla_2(\varphi)$ and A_p for $p = 1 + \log_2 \Delta_2(\varphi)$ and $p = 1 + \frac{1}{\log_2(\frac{1}{2}\nabla_2(\varphi))}$.

It is also possible to interpolate between different scales, and in the Lebesgue scales this is recorded in [Ste71, Appendix B]. Roughly speaking, if T is of type weak(p_0, q_0) and weak(p_1, q_1), then for each $t \in (0, 1)$ it is of type weak(p_t, q_t) where $p_t^{-1} = (1-t)p_0^{-1} + tp_1^{-1}$ and $q_t^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$. In the case of Orlicz growth, we will record one result which will be useful later.

Lemma 2.2.3. For $\Omega \subset \mathbb{R}^n$ any measurable set, let T be a sublinear operator as above. Suppose that T is of type weak(p,q) for some $1 \leq p \leq q < \infty$ and is bounded on L^∞, that is:

$$\mathcal{L}^n\left(\{x \in \Omega : |Tf| > s\}\right) \leq \left(\frac{A_{p,q}\,\|f\|_{L^p(\Omega)}}{s}\right)^q, \tag{2.96}$$

$$\|Tf\|_{L^\infty(\Omega)} \leq A_\infty \|f\|_{L^\infty(\Omega)}, \tag{2.97}$$

for any measurable f. Then for any N-function φ for which $p < p_\varphi < \infty$,

$$\left(\int_\Omega \varphi(|Tf|)^{\frac{q}{p}}\,\mathrm{d}x\right)^{\frac{p}{q}} \leq C \int_\Omega \varphi(|Cf|)\,\mathrm{d}x. \tag{2.98}$$

The same holds if $p = 1$ and $\varphi \in \nabla_2$.

Proof. Similarly as in the proof of Theorem 2.2.1, we set $\mu = \frac{1}{2A_\infty}$ and put $f_1 = f1_{\{|f|>\mu s\}}$ and $f_2 = f - f_s$. Then by subadditivity, definition of μ and the weak-(p,q) inequality we obtain

$$\lambda_{Tf}(s) \leq \lambda_{Tf_1}(s/2) + \lambda_{Tf_2}(s/2) \leq \left(\int_{\{|f|>\mu s\}} \left(\frac{2A_{p,q}|f|}{s}\right)^p \mathrm{d}x\right)^{\frac{q}{p}}. \tag{2.99}$$

Therefore we can estimate

$$\begin{aligned} \int_\Omega \varphi(|Tf|)^{\frac{q}{p}}\,\mathrm{d}x &= \int_0^\infty \lambda_{Tf}(s)\left(\varphi^{\frac{q}{p}}\right)'(s)\,\mathrm{d}s \\ &\leq \int_0^\infty \left(\int_{\{|f|>\mu s\}} \left(\frac{2A_{p,q}|f|}{s}\right)^p \mathrm{d}x\right)^{\frac{q}{p}} \left(\varphi^{\frac{q}{p}}\right)'(s)\,\mathrm{d}s \\ &\leq \left(\int_\Omega \left(\int_0^{|f(x)|/\mu} \left(\frac{2A_{p,q}|f(x)|}{s}\right)^q \left(\varphi^{\frac{q}{p}}\right)'(s)\,\mathrm{d}s\right)^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{q}{p}}, \end{aligned} \tag{2.100}$$

where we have used Minkowski's integral inequality in the last line. It remains to show that there is $C \geq 1$ such that

$$\varphi(t)^{\frac{q}{p}} \leq Ct^p \int_0^{t/\mu} \left(\frac{2A_{p,q}}{s}\right)^q \left(\varphi^{\frac{q}{p}}\right)'(s)\,\mathrm{d}x. \tag{2.101}$$

This follows by applying Lemma 2.1.14 to $\varphi^{\frac{q}{p}}$, noting that $p_{\varphi^{\frac{q}{p}}} = \frac{q}{p}p_\varphi$. $\square$

Finally, we show an endpoint result near $p = 1$. We will only need this in the proof of Lemma 2.1.34, it is a natural consequence of the arguments we have considered. This follows from an adaptation of [KK91, Theorem 1.4.1].

Lemma 2.2.4. For $\Omega \subset \mathbb{R}^n$, let T be a sublinear operator such that T is of type weak$(1,1)$ and weak(q,q) for some $1 < q \leq \infty$. Then for any N-function φ such that $q_\varphi < q < \infty$,

$$\varphi(s)\lambda_{Tf}(s) \leq C \int_\Omega \varphi(|f|)\,\mathrm{d}x. \tag{2.102}$$

If $q = \infty$ we can take φ to be any N-function, except that we replace $\varphi(|f|)$ on the right-hand side by $\varphi(C|f|)$.

Proof. For $s > 0$ write $f_1 = f1_{\{|f|>\mu s\}}$ and $f_2 = f - f_1$. If $q_\varphi < \infty$, then arguing similarly to (2.91) we have

$$\varphi(s)\lambda_{Tf}(s) \leq C\,\frac{\varphi(s)}{s}\int_{\{|f|\leq s\}} |f|\,\mathrm{d}x + C\,\frac{\varphi(s)}{s^q}\int_{\{|f|\geq s\}} |f|^q\,\mathrm{d}x. \tag{2.103}$$

Note that $\varphi(t)/t$ is increasing by convexity of φ, and also since $q < q_\varphi$ by Remark 2.1.11 there is $C > 0$ such that

$$\frac{\varphi(s)}{s^q} \leq C\,\frac{\varphi(t)}{t^q} \tag{2.104}$$

for all $s \geq t$. Hence we have $\frac{\varphi(s)}{s^q}|f|^q \leq \varphi(|f|)$ when $s \geq |f|$ and $\frac{\varphi(s)}{s}|f| \leq \varphi(|f|)$ when $s \leq |f|$. These inequalities show that

$$\varphi(s)\lambda_{Tf}(s) \leq C \int_\Omega \varphi(|f|)\,\mathrm{d}x, \tag{2.105}$$

as required. If T is bounded in L^∞ with constant A_∞, we can take $\mu = \frac{1}{2A_\infty}$ and use convexity of φ to show that

$$\varphi(s)\lambda_{Tf}(s) \leq C\,\frac{\varphi(s)}{s}\int_{\{|f|\leq \mu s\}} |f|\,\mathrm{d}x \leq C\int_\Omega \varphi(2A_\infty|f|)\,\mathrm{d}x, \tag{2.106}$$

as required. □

Remark 2.2.5. Note the constant in Lemma 2.2.4 depends implicitly on φ through the constant in (2.104). However if $\varphi \in \Delta_2$, then by Lemma 2.1.5 we can find q and $C > 0$ depending on $\Delta_2(\varphi)$ only such that (2.104) holds.

This largely settles the problem of interpolation, so now we turn to the problem of extrapolation. Here the idea is to establish *weighted* estimates for some L^p space, from which we can deduce estimates in every L^p space. For this we need to introduce the class of weights we are interested in.

Definition 2.2.6 (Muckenhoupt weights)**.** A *weight* is a non-negative measurable function $w\colon \mathbb{R}^n \to \mathbb{R}$. For $1 < p < \infty$ we say w lies in the *Muckenhoupt A_p class*, denoted $w \in A_p$, if we have

$$[w]_{A_p} = \sup_Q \fint_Q w \,\mathrm{d}x \left(\fint_Q w^{1-p'} \,\mathrm{d}x \right)^{p-1} < \infty, \tag{2.107}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Given $w \in A_p$, we will also define the associated weighted Lebesgue space $L^p_w(\Omega, \mathbb{R}^K)$ as the space of $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f|^p w \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty. \tag{2.108}$$

Muckenhoupt weights were introduced by Muckenhoupt [Muc72], and they are precisely the weights which ensure the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^p_w(\mathbb{R}^n)$. There is a vast literature on this topic which we will not detail, and instead refer the reader to [Ste93, Chapter 5].

Theorem 2.2.7 (Extrapolation in the modular scales, [CMP06])**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and suppose $\mathcal{F}$ is a family of pairs of non-negative measurable functions (f, g) defined in Ω. Assume there is $1 < p_0 < \infty$ such that for all $w \in A_{p_0}$, we have

$$\int_\Omega |g|^{p_0} w \,\mathrm{d}x \leq C(w) \int_\Omega |f|^{p_0} w \,\mathrm{d}x \tag{2.109}$$

for all pairs $(f, g) \in \mathcal{F}$ whenever both sides are finite. Then for any N-function $\varphi \in \Delta_2 \cap \nabla_2$ we have

$$\int_\Omega \varphi(|g|) \,\mathrm{d}x \leq C \int_\Omega \varphi(|f|) \,\mathrm{d}x \tag{2.110}$$

for all $(f, g) \in \mathcal{F}$ whenever both sides are finite.

A detailed proof can be found in [CMP11, Section 4.3]. We point out that the proof heavily relies on the boundedness of the maximal operator on the Orlicz scales, which can be obtained via interpolation as we will show in Section 2.2.2. The benefit of this approach is that we do not need to assume that $g = Tf$ for a suitable operator T. On the other hand, the requirement for the initial estimate to hold with respect to all weights $w \in A_{p_0}$ is a very strong condition, which is difficult to verify in practice.

2.2.2 Maximal operators

We now apply the results established in the previous section to obtain estimates for the maximal operators. For $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$ we define the *Hardy Littlewood maximal operator* as

$$\mathcal{M}_\Omega f(x) = \sup_{x \in B \subset \Omega} \fint_B |f| \,\mathrm{d}x, \tag{2.111}$$

where the supremum is taken over all balls $B \subset \Omega$ containing x. In the full space, we will write $\mathcal{M} = \mathcal{M}_{\mathbb{R}^n}$. We will also define the *Fefferman-Stein sharp maximal function* as

$$\mathcal{M}^\#_\Omega f(x) = \sup_{x \in B \subset \Omega} \fint_B |f - (f)_B| \,\mathrm{d}x. \tag{2.112}$$

There are many different conventions to defining these operators, where one may require the balls B to be centred at x, or one may consider balls B which are not necessarily contained in Ω. For our purposes the above will be most convenient.

The following boundedness properties of the maximal function are well known.

Proposition 2.2.8. For any $\Omega \subset \mathbb{R}^n$ open, the maximal operator $\mathcal{M}_\Omega$ satisfies the weak-$(1,1)$ inequality

$$\lambda_{\mathcal{M}_\Omega}(x) \leq \frac{C(n)}{s} \int_\Omega |f| \,\mathrm{d}x \tag{2.113}$$

and satisfies

$$\int_\Omega \varphi(|\mathcal{M}_\Omega f|) \,\mathrm{d}x \leq C \int_\Omega \varphi(C|f|) \,\mathrm{d}x \tag{2.114}$$

for all N-functions φ satisfying the ∇_2-condition, and all f locally integrable. Moreover the constant C depends on n and $\nabla_2(\varphi)$ only.

Proof. It suffices to prove this in the full space, noting that $\mathcal{M}_\Omega f \leq \mathcal{M} f$ after extending f by zero. The weak-$(1,1)$ inequality is standard, and can be found for instance [Ste71, Section I.3] or [Duo01, Section 2.6]. Also by definition we have $\|\mathcal{M}_\Omega f\|_{L^\infty(\Omega)} \leq \|\mathcal{M}_\Omega f\|_{L^\infty(\Omega)}$, which is immediate from the definition. Now the result follows by interpolation, namely by Theorem 2.2.1. $\square$

Remark 2.2.9. It is known that the ∇_2-condition is also necessary to ensure the maximal operator is bounded. A proof of this can be found in [KK91, Theorem 1.2.1], and follows by choosing $f(x) = t\chi_{B_1(0)}(x)$ in the inequality (2.114) for varying $t > 0$ to deduce constraints on φ. It is also noted in [KK91] that we do however have the weak-type estimate

$$\varphi(s)\lambda_{\mathcal{M}_\Omega}(s) \leq C \int_\Omega \varphi(C|f|)\,\mathrm{d}x. \tag{2.115}$$

for any N-function φ. We will not need this, but this follows by applying Lemma 2.2.4.

For the sharp maximal function, we have the pointwise estimate $\mathcal{M}^\# f \leq 2\mathcal{M}f$ so we also obtain similar boundedness results. An interesting feature is that we have a partial converse, namely that we can control $\mathcal{M}f$ by $\mathcal{M}^\# f$ under suitable integral norms. This is due to Fefferman & Stein [FS72] for the case of the full space, who established an estimate of the form

$$\int_{\mathbb{R}^n} \varphi(|\mathcal{M}f|)\,\mathrm{d}x \leq C \int_{\mathbb{R}^n} \varphi(|\mathcal{M}^\# f|)\,\mathrm{d}x \tag{2.116}$$

for all f for which both sides are finite, and for suitable φ. We will present a generalisation to John domains, which will follow from the work of Iwaniec & Nolder [IN85], and the extrapolation theorem from the previous section. This particular statement was presented by the author in [Irv21a], and will be applied in Chapter 4. We mention that similar results in a more general framework have been obtained by Diening, Růžička, & Schumacher [DRS10].

Theorem 2.2.10 (Fefferman-Stein)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-John domain, and $\varphi \in \Delta_2 \cap \nabla_2$. Then there is $C = C(n, \delta, \Delta_2(\varphi), \nabla_2(\varphi)) > 0$ such that

$$\int_\Omega \varphi(|f - (f)_\Omega|)\,\mathrm{d}x \leq C \int_\Omega \varphi\left(\mathcal{M}_\Omega^\# f\right)\,\mathrm{d}x \tag{2.117}$$

for all $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ such that both sides are finite.

Proof. We first need a weighted L^p estimate in Ω, so let $1 < p < \infty$ and $w \in A_p$. Then for any cube Q we have (see for instance [DRS10, Lemma 7.2])

$$\int_Q |f - (f)_Q|^p w\,\mathrm{d}x \leq C\left(n, p, [w]_{A_p}\right) \int_Q |\mathcal{M}_Q^\# f|^p w\,\mathrm{d}x \tag{2.118}$$

for all $f \in L^p(Q, w, \mathbb{R}^{Nn})$, that is $f\colon Q \to \mathbb{R}^{Nn}$ such that $|f|^p w$ is integrable on Q. To extend this to John domains we can apply [IN85, Theorem 3] together with Lemma 2.1.30 to verify the Boman chain condition they assume. Hence the argument shows that

$$\int_\Omega |\mathcal{M}_\Omega(f - (f)_{Q_0})|^p w\,\mathrm{d}x \leq C\left(n, p, [w]_{A_p}, \delta\right) \int_\Omega |\mathcal{M}_\Omega^\# f|^p w\,\mathrm{d}x \tag{2.119}$$

for all $1 < p < \infty$ and $w \in A_p$, for a distinguished cube $Q_0 \subset \Omega$. Then applying Theorem 2.2.7 to the family of pairs $(|f - f_{Q_0}|1_\Omega, |M_\Omega^\# f|1_\Omega)$ we obtain

$$\int_\Omega \varphi(|f - (f)_{Q_0}|)\,\mathrm{d}x \leq C(n, \delta, \Delta_2(\varphi), \nabla_2(\varphi)) \int_\Omega \varphi\left(\mathcal{M}_\Omega^\# f\right)\,\mathrm{d}x. \tag{2.120}$$

Replacing the average $(f)_{Q_0}$ by $(f)_\Omega$ using the doubling property and convexity of φ, the result follows. $\square$

Remark 2.2.11. If $\Omega = \mathbb{R}^n$ or a cube, one can obtain this estimate by much simpler means. Roughly speaking, it suffices to obtain a *good-λ inequality* of the form

$$\lambda_{\mathcal{M}f}(t) \leq C\lambda_{\mathcal{M}^\# f}(\gamma t) + C\gamma\lambda_{\mathcal{M}f}(Ct) \tag{2.121}$$

for all $\gamma, t > 0$, where $C = C(n) > 0$. We can then integral with respect to $\varphi'(t)\,\mathrm{d}t$ and adjust γ suitably to absorb the second term. This establishes the same inequality assuming $\varphi \in \Delta_2$, as it noted in [KT03]. It is unclear whether the ∇_2-condition is necessary for the general case, however we have not investigated this further.

We point out that this has the following consequence, which was noted earlier by Stampacchia [Sta65] (see also [Cam66]) in the case of a cube.

Corollary 2.2.12 (Stampacchia)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-John domain, and let T be a linear operator acting between measurable functions $\Omega \to \mathbb{R}^K$ to $\Omega \to \mathbb{R}^L$. Suppose that for some $1 \leq p_0 < \infty$ we have

$$\|Tf\|_{L^{p_0}(\Omega)} \leq C\,\|f\|_{L^{p_0}(\Omega)}\,, \tag{2.122}$$

$$\left\|\mathcal{M}^\#(Tf)\right\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^\infty(\Omega)}\,. \tag{2.123}$$

Then T is bounded on L^p for all $p_0 \leq p < \infty$.

This follows by noting $\mathcal{M}^\# \circ T$ is bounded on L^p by Theorem 2.2.1, after which we apply Theorem 2.2.10 to obtain corresponding estimates for T. Here some care is needed since we a-priori do not know whether $Tf \in L^p$; we must first assume $f \in L^{p_0} \cap L^\infty$, after which the general case follows by density.

Finally in Chapter 3 we will need the following variants of the maximal operators

$$\widetilde{\mathcal{M}}_\Omega f(x_0) = \sup_{R>0} \fint_{\Omega_R(x_0)} |f| \,\mathrm{d}x, \tag{2.124}$$

$$\widetilde{\mathcal{M}}_\Omega^{\#} f(x_0) = \sup_{R>0} \fint_{\Omega_R(x_0)} |f - (f)_{\Omega_R(x_0)}| \,\mathrm{d}x. \tag{2.125}$$

Proposition 2.2.13. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and $\varphi \in \Delta_2 \cap \nabla_2$ be an N-function. Then we have

$$\int_\Omega \varphi\left(\widetilde{\mathcal{M}}_\Omega f\right) \mathrm{d}x \leq C(n, \Delta_2(\varphi), \nabla_2(\varphi), \|\Omega\|_{C^{0,1}}, \mathcal{L}^n(\Omega)/R_\Omega^n) \int_\Omega \varphi(|f|) \,\mathrm{d}x \tag{2.126}$$

for all f locally integrable on Ω and

$$\int_\Omega \varphi\left(|f - (f)_\Omega|\right) \mathrm{d}x \leq C(n, \Delta_2(\varphi), \nabla_2(\varphi), \delta) \int_\Omega \varphi\left(\widetilde{\mathcal{M}}_{\Omega,R}^{\#} f\right) \mathrm{d}x, \tag{2.127}$$

for all f such that both sides are finite, where we assume Ω is a δ-John domain.

Proof. By the interior cone condition (see Lemma 2.1.22) there is $\kappa > 0$ such that we have

$$\mathcal{L}^n(\Omega_R(x_0)) \geq \kappa \mathcal{L}^n(B_R(x_0)) \tag{2.128}$$

for all $x_0 \in \overline{\Omega}$ and $0 < R < R_\Omega$. Hence by distinguishing between when $R < \frac{1}{2}R_0$ and $R \geq \frac{1}{2}R_0$ we can estimate

$$\widetilde{\mathcal{M}}_\Omega f(x) \leq C\mathcal{M}(f\chi_{B_{3R}(x_0)})(x) + C\,\frac{\mathcal{L}^n(\Omega)}{R_\Omega^n} \int_\Omega |f| \,\mathrm{d}x \tag{2.129}$$

for all $x \in B_R(x_0)$. Hence integrating over $B_{3R}(x_0)$ and applying Proposition 2.2.8 we have

$$\int_{\Omega_R(x_0)} \varphi\left(\widetilde{\mathcal{M}}_\Omega f(x)\right) \mathrm{d}x \leq C \int_\Omega \varphi(|f|) \,\mathrm{d}x, \tag{2.130}$$

from which the result follows by a covering argument. For (2.125) it simplies suffices to observe that $\mathcal{M}_\Omega^{\#} f \leq \widetilde{\mathcal{M}}_\Omega^{\#} f$ since we are taken the supremum over fewer subests, so the result follows by Theorem 2.2.10. $\square$

2.2.3 The space BMO

Definition 2.2.14. The John-Nirenberg space $\mathrm{BMO}(\Omega, \mathbb{R}^K)$ of functions of *bounded mean oscillation* in Ω will be the space of $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$ for which $\mathcal{M}_\Omega^{\#} f \in L^\infty(\Omega)$. We equip this space with the seminorm $[f]_{\mathrm{BMO}(\Omega)} = \|M_\Omega^{\#} f\|_{L^\infty(\Omega)}$.

As the John-Nirenberg inequality (Proposition 2.2.15) illustrates, BMO is an intermediate scale between the Lebesgue space L^p for $1 < p < \infty$ and L^∞. This plays a fundamental role in the regularity theory for elliptic equations, as it provides a integral scale of smoothness through the *infinitesimal mean oscillation*, which following [MMS10] is defined for $f \in \mathrm{BMO}(\Omega, \mathbb{R}^K)$ as

$$\{f\}_{\mathrm{osc}(\Omega)} = \limsup_{R \to 0} \sup_{\substack{B_r(x) \subset \Omega \\ 0<r<R}} \fint_{B_r(x)} |f - (f)_{B_r(x)}| \,\mathrm{d}x. \tag{2.131}$$

Note that $\{f\}_{\mathrm{osc}(\Omega)} = 0$ if and only if f lies in the Sarason space $\mathrm{VMO}(\Omega, \mathbb{R}^K)$ of *vanishing mean oscillation.*

The following result is essentially proved by Smith & Stegenga [SS91] (see also the paper of Hurri-Syjänen [Hur93]), but we will sketch a proof for completeness and to clarify the dependence of constants. The John domain condition can be relaxed (it suffices to require (2.139) below), but this will suffice for our purposes.

Proposition 2.2.15 (Global John-Nirenberg estimate)**.** Suppose Ω is a bounded δ-John domain, and $f \in \mathrm{BMO}(\Omega, \mathbb{R}^K)$. Then for all $1 \leq p < \infty$, there is $C = C(n, p, \delta) > 0$ such that

$$\left(\fint_\Omega |f - (f)_\Omega| \,\mathrm{d}x \right)^{\frac{1}{p}} \leq C\, [f]_{\mathrm{BMO}(\Omega)} . \tag{2.132}$$

Proof. Let $W = \{Q_j\}$ be a Whitney decomposition of Ω, as given in Definition 2.1.29. We observe that for any $Q \subset Q_j$, there is a concentric ball B with $Q \subset B \subset \Omega$ with $\mathrm{diam}(B) = 2^{\frac{1}{n}} \ell(Q)$, so $\fint_Q |f - (f)_Q| \,\mathrm{d}x \leq C(n)\, [f]_{\mathrm{BMO}(\Omega)}$. Hence up to an equivalent norm we can work with cubes $Q \subset Q_j$ in place of balls. Therefore by the John-Nirenberg inequality in Q_j (see for instance [Giu03, Corollary 2.2]) we obtain

$$\left(\fint_{Q_j} |f - (f)_{Q_j}|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leq C(n,p)\, [f]_{\mathrm{BMO}(\Omega)} . \tag{2.133}$$

To obtain a global estimate let $Q_0 \in W$ to be determined, and suppose $Q \in W$ is connected to Q_0 by a chain $Q_1, \ldots, Q_k = Q$ of length k. If this holds, by [Jon80, Lemma 2.2] we have

$$|(f)_Q - (f)_{Q_0}| \leq \sum_{j=1}^k |(f)_{Q_j} - (f)_{Q_{j-1}}| \leq C(n) k\, [f]_{\mathrm{BMO}(\Omega)} . \tag{2.134}$$

This leads us to consider the minimal length chain connecting Q_0 to Q_j, which we will denote by $k(Q)$. To control this we introduce the *quasi-hyperbolic distance* introduced in [GP76], which for $x_0, x_1 \in \Omega$ is defined as

$$k_\Omega(x_0, x_1) = \inf_\gamma \int_\gamma \frac{1}{\operatorname{dist}(x, \partial\Omega)} \,\mathrm{d}t, \tag{2.135}$$

taking the infimum over all rectifiable curves γ connecting x_0 to x_1. It is observed in [Jon80] (see also [Hur88]) that there exists $C = C(n) > 0$ such that if x_0 is the centre of Q_0 we have

$$C^{-1} k_\Omega(x_0, x) \leq k(Q) \leq C k_\Omega(x_0, x) + 1. \tag{2.136}$$

for all $x \in Q$. Hence combining the above estimates we obtain

$$\begin{aligned} \int_\Omega |f - (f)_\Omega|^p \,\mathrm{d}x &\leq C(p) \sum_{Q \in W} \int_Q |f - (f)_{Q_0}|^p \,\mathrm{d}x \\ &\leq C(p) \sum_{Q \in W} \int_Q |f - (f)_Q|^p \,\mathrm{d}x + C(n,p) \sum_{Q \in W} \mathcal{L}^n(Q) k(Q)^p \, [f]_{\mathrm{BMO}(\Omega)}^p \\ &\leq C(n,p) \left(\mathcal{L}^n(\Omega) + \int_\Omega k_\Omega(x_0, x)^p \,\mathrm{d}x \right) [f]_{\mathrm{BMO}(\Omega)}^p . \end{aligned} \tag{2.137}$$

Finally to show this integral is finite, we use the assumption that Ω is a John domain; letting x_0 be the John centre (and so Q_0 is chosen to contain x_0) it is shown in [GM85] that for all $x \in \Omega$,

$$k_\Omega(x_0, x) \leq \frac{1}{\delta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + \frac{1}{\delta} \left(1 + \log\left(1 + \delta^{-1}\right)\right). \tag{2.138}$$

Using this and keeping track of constants in the proof of [SS90, Theorem 4] we have k_Ω satisfies the integrability condition

$$\int_\Omega k_\Omega(x_0, x)^p \,\mathrm{d}x \leq C(n, p, \delta) \mathcal{L}^n(\Omega), \tag{2.139}$$

from which the result follows. □

Remark 2.2.16. Since the proof only considers cubes Q such that $2Q \subset \Omega$, this shows that we can equivalently define $\mathrm{BMO}(\Omega)$ by taking the supremum over all cubes Q satisfying either $Q \subset \Omega$ or $2Q \subset \Omega$, and similarly with balls; this is a result of Reimann & Rychener [RR75].

We will need the following in Section 4, which follows by applying the Fefferman-Stein inequality (Proposition 2.2.10) to the N-function constructed in Lemma 2.1.18.

Lemma 2.2.17. Suppose $D \subset \mathbb{R}^n$ is a bounded δ-John domain, $1 < p < \infty$, and $\omega \colon [0, \infty) \to [0, 1]$ is non-decreasing, continuous, concave with $\omega(0) = 0$. Then if $f \in \mathrm{BMO}(D, \mathbb{R}^K)$, for each $1 < p < \infty$ there is $C = C(n, p, \delta) > 0$ such that

$$\int_D \omega(|f - (f)_D|)|f - (f)_D|^p \, \mathrm{d}x \leq C\, \omega\left([f]_{\mathrm{BMO}(D)}\right) \int_D |f - (f)_D|^p \, \mathrm{d}x. \tag{2.140}$$

2.2.4 Miscellaneous estimates

We now collect some other results which we will need on occasion later. The first is the *Fourier transform* of a function $u \in L^1(\mathbb{R}^n, \mathbb{R}^K)$ which we define as

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} \, \mathrm{d}x. \tag{2.141}$$

Under these conventions, for $u \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^K)$ we have the differentiation rule

$$\widehat{\nabla u}(\xi) = \hat{u}(\xi) \otimes i\xi, \tag{2.142}$$

and for $u \in L^1 \cap C_0$ we have the inversion formula

$$\mathcal{F}^{-1}(\hat{u})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\xi\cdot x} \, \mathrm{d}\xi = (2\pi)^{-n} \mathcal{F}(\hat{u})(-x). \tag{2.143}$$

We will also need *Plancherel's theorem*, which asserts that the Fourier transform extends uniquely to an isomorphism on L^2, and we have the identity

$$\int_{\mathbb{R}^n} f \cdot \overline{g} \, \mathrm{d}x = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f} \cdot \overline{\hat{g}} \, \mathrm{d}x \tag{2.144}$$

for all $f, g \in L^2(\mathbb{R}^n, \mathbb{R}^K)$, where $\overline{g}$ the componentwise complex conjugation.

We will also record the classical singular integral estimate of Calderón & Zygmund [CZ52], though we will only use it in Section 3.2. There are significantly more general versions of the below result, however the following will be sufficient.

Theorem 2.2.18 (Calderón-Zygmund)**.** Let $K \colon \mathbb{S}^{n-1} \to \mathbb{R}$ be a C^1 map such that

$$\int_{\mathbb{S}^{n-1}} K(\omega) \, \mathrm{d}\mathcal{H}^{n-1}(\omega) = 0. \tag{2.145}$$

Then the mapping

$$Tf = \text{P.V.} \int_{\mathbb{R}^n} \frac{K(y/|y|)}{|y|^n} f(x-y)\,\mathrm{d}y = \lim_{\varepsilon\to 0} \int_{\{y\in\mathbb{R}^n\colon |y|>\varepsilon\}} \frac{K(y/|y|)}{|y|^n} f(x-y)\,\mathrm{d}y, \tag{2.146}$$

which is well defined on $\mathcal{D}(\mathbb{R}^n)$, extends to a bounded linear operator $T\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, which is also of type weak-$(1,1)$. Moreover for any N-function $\varphi \in \Delta_2 \cap \nabla_2$ we have the modular estimate

$$\int_{\mathbb{R}^n} \varphi(|Tf|)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} \varphi(|f|)\,\mathrm{d}x, \tag{2.147}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We omit the well-known proof, and refer the reader to the original paper [CZ52] and the classical text of Stein [Ste71, Chapter 2]. The heart of the proof involves showing T is of type weak-$(1,1)$ by means of the Calderón-Zygmund decomposition. By Fourier analysis one shows T is also bounded on L^2, from which one can conclude using interpolation (Theorem 2.2.1) and duality. Note the above sources only prove boundedness in L^p, but once this is established we can extend this to the modular scales using Corollary 2.2.2. We will not actually need this fact, but this illustrates how most estimates in L^p also extend to these scales.

2.3 Function spaces

2.3.1 Orlicz-Sobolev spaces

In Section 5.3 we will study minima of integral functionals satisfying a general growth condition governed by N-functions. For this we will need to understand the associated spaces our solutions will lie in, and collection some properties which will be used later. We will largely be interested in the case when our N-function satisfies the Δ_2-condition, and we may often assume this to simplify the exposition.

For $\Omega \subset \mathbb{R}^n$ open and $M \ge 1$, we define the Orlicz space $L^\varphi(\Omega, \mathbb{R}^M)$ as the space of $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^M)$ for which

$$\rho^\varphi_\Omega(|f|/\lambda) := \int_\Omega \varphi(|f|/\lambda)\,\mathrm{d}x < \infty \tag{2.148}$$

for some $\lambda > 0$, which we equip with the *Luxemburg norm*

$$\|f\|_{L^\varphi(\Omega)} = \inf\left\{\lambda > 0 : \rho_\Omega^\varphi\left(\frac{|f|}{\lambda}\right) \leq 1\right\}. \tag{2.149}$$

Note that if φ satisfies the Δ_2-condition, then $f \in L^\varphi(\Omega, \mathbb{R}^M)$ if and only if $\rho_\Omega^\varphi(|f|) < \infty$.

For $N \geq 1$ we can define the Orlicz-Sobolev spaces $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ as the space of weakly differentiable $u \in L^\varphi(\Omega, \mathbb{R}^N)$ such that $\nabla u \in L^\varphi(\Omega, \mathbb{R}^{Nn})$. We also define the $W^{1,\varphi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ to be the space of functions for which $u|_{\Omega'} \in W^{1,\varphi}(\Omega', \mathbb{R}^n)$ for all $\Omega' \Subset \Omega$, and $u \in W^{1,\varphi}_0(\Omega, \mathbb{R}^N)$ to be the closure of $C_c^\infty(\Omega, \mathbb{R}^N)$ in $W^{1,\varphi}(\Omega, \mathbb{R}^N)$.

Lemma 2.3.1. Let $\Omega \subset \mathbb{R}^n$, $M \geq 1$ and $\varphi \in \Delta_2$. Then a sequence $\{f_k\} \subset L^\varphi(\Omega, \mathbb{R}^M)$ converges strongly to f if and only if we have

$$\lim_{k\to\infty} \int_\Omega \varphi(|f - f_k|)\,\mathrm{d}x = 0. \tag{2.150}$$

Proof. Strong convergence implies the existence of $\lambda_k \to 0$ such that $\rho_\Omega^\varphi(|f - f_k|/\lambda_k) \leq 1$ for all k, so by convexity of φ we have $\rho_\Omega^\varphi(|f - f_k|) \leq \lambda_k \to 0$. Conversely if (2.150) holds, by the doubling property for all $\lambda \leq 1$ we have

$$\int_\Omega \varphi\left(\frac{|f_k - f|}{\lambda}\right)\mathrm{d}x \leq C\lambda^{-\log \Delta_2(\varphi)} \int_\Omega \varphi(|f_k - f|)\,\mathrm{d}x. \tag{2.151}$$

We can then select a sequence $\lambda_k \to 0$ such that $C\lambda_k^{-\log \Delta_2(\varphi)} \rho_\Omega^\varphi(|f_k - f|) \leq 1$ for all k. $\square$

Discussion 2.3.2 (The predual of L^φ)**.** In order to establish the existence of minimisers in this general growth setting, we will need a suitable weak compactness principle using the Banach-Alaoglu theorem. Given an N-function φ, we define the space $E^\varphi(\Omega, \mathbb{R}^M)$ of measurable functions $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^M)$ such that $\rho_\Omega^\varphi(|f|/\lambda) < \infty$ for all $\lambda > 0$. This is a Banach space under the Luxemburg norm which coincides with $L^\varphi(\Omega, \mathbb{R}^M)$ if $\varphi \in \Delta_2$ (in fact the converse also holds, as shown in [KR61, Section II.10]). Then we have the pairing

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,\mathrm{d}x : E^\varphi(\Omega, \mathbb{R}^M) \times L^{\varphi^*}(\Omega, \mathbb{R}^M) \to \mathbb{R} \tag{2.152}$$

induces an isomorphism $E^\varphi(\Omega, \mathbb{R}^M)^* \cong L^{\varphi^*}(\Omega, \mathbb{R}^M)$, which is proved for instance in [KR61, Section II.14]. Hence viewing $W^{1,\varphi}(\Omega, \mathbb{R}^N)$ as a subspace of $L^\varphi(\Omega, \mathbb{R}^{n+Nn})$ we see it is also a dual space.

Lemma 2.3.3. Let φ be an N-function and $\Omega \subset \mathbb{R}^n$ a bounded domain. If $\varphi \in \nabla_2$, then there is $p \in (1,\infty)$ such that $L^\varphi(\Omega,\mathbb{R}^N) \subset L^p(\Omega,\mathbb{R}^M)$. Also if $\varphi \in \Delta_2$, then there is $q \in (1,\infty)$ such that $L^q(\Omega,\mathbb{R}^M) \subset L^\varphi(\Omega,\mathbb{R}^M)$.

The Δ_2 case follows from simple pointwise estimates for the N-function, while the ∇_2 case uses similar ideas involved in the interpolation results of Theorem 2.2.1.

Proof. Suppose $\varphi \in \nabla_2$, then by Lemma 2.1.14 there is $p > 1$ such that

$$t^p \int_0^1 s^{-p}\varphi(s)\,\frac{\mathrm{d}s}{s} \le C\varphi(p). \tag{2.153}$$

for $t \ge 1$, where we have bounded the integral in (2.30) from below. Now if $f \in L^\varphi(\Omega,\mathbb{R}^N)$, let $\ell > 0$ such that $\varphi(|f|/\ell)$ is integrable in Ω. Then recalling the distribution function from (2.84) we have

$$\begin{aligned}
\ell^{-p}\int_\Omega |f|^p\,\mathrm{d}x &\le \mathcal{L}^n(\Omega) + p\int_1^\infty s^p\lambda_{f/\ell}(s)\frac{\mathrm{d}s}{s} \\
&\le \mathcal{L}^n(\Omega) + C\left(\int_0^1 s^{-p}\varphi(s)\,\mathrm{d}s\right)^{-1}\int_1^\infty \frac{\varphi(s)}{s}\lambda_{f/\ell}(s)\,\mathrm{d}s \\
&\le \mathcal{L}^n(\Omega) + C\int_\Omega \varphi\left(\frac{|f|}{\ell}\right)\mathrm{d}x,
\end{aligned} \tag{2.154}$$

where we have used the fact that $\varphi(t)/t \le \varphi'(t)$ form Lemma 2.1.6.

If $\varphi \in \Delta_2$, then letting $q = \log_2 \Delta_2(\varphi)$ we use Lemma 2.1.5 and (2.13) to estimate

$$t\varphi'(t) \le C\varphi(t) \le Ct^q \tag{2.155}$$

for all $t \ge 1$. Then similarly to above we have

$$\begin{aligned}
\int_\Omega \varphi(|f|)\,\mathrm{d}x &\le \mathcal{L}^n(\Omega) + \int_1^\infty \varphi'(s)\lambda_f(s)\,\mathrm{d}s \\
&\le \mathcal{L}^n(\Omega) + C\int_1^\infty t^{q-1}\lambda_f(s)\,\mathrm{d}s \\
&\le \mathcal{L}^n(\Omega) + C\int_\Omega |f|^q\,\mathrm{d}x,
\end{aligned} \tag{2.156}$$

as required. □

2.3.2 Poincaré inequalities

In our local estimates, we will frequently need to pass between integral estimates of the function and estimates on its derivatives. For these it will be important to keep track of the dependence of constants involved, along with different conditions to ensure such an inequality applies. As such, we will record several Poincaré inequalities in this section, which will be used in later chapters. We note that other variants are also stated in Lemmas 4.3.1, 5.2.10.

We begin by stating the main variant of Poincaré-Sobolev inequality we will use, which we state in the sharp form for L^p-norms.

Lemma 2.3.4 (Poincaré-Sobolev inequality)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-John domain, and $1 \leq p < n$. Then for any $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, we have the estimate

$$\left(\int_\Omega |u - (u)_\Omega|^{\frac{np}{n-p}} \,\mathrm{d}x \right)^{\frac{n-p}{np}} \leq C(n,p,\delta) \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x \right)^{\frac{1}{p}}. \tag{2.157}$$

Moreover if $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$, the $(u)_\Omega$ term can be omitted and the John domain assumption can be dropped.

This is proved in [Boj88, Theorem 5.1]. If u vanishes on $\partial\Omega$, this simply follows by the Gagliardo-Nirenberg inequality in $\mathbb{R}^n$ and so the constant in (2.157) depends on n, p only. In what follows, we will denote this critical exponent by

$$p^* = \frac{np}{n-p}. \tag{2.158}$$

For general exponents we have the following version.

Lemma 2.3.5 (Scale-invariant Poincaré-Sobolev)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-John domain, and $1 \leq p, q < \infty$ such that $q \leq p^*$ if $p < n$, and is otherwise unconstrained. Then if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, we have the estimate

$$\left(\fint_\Omega \frac{|u - (u)_\Omega|^q}{\mathcal{L}^n(\Omega)^{\frac{q}{n}}} \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C(n,p,q,\delta) \left(\fint_\Omega |\nabla u|^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.159}$$

and if $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ we can omit the average term.

Proof. This follows by noting that for averaged integrals, for $1 \leq r < n$ by Lemma 2.3.4 we have

$$\left(\fint_\Omega |u - (u)_\Omega|^{r^*} \,\mathrm{d}x \right)^{\frac{1}{r^*}} \leq \mathcal{L}^n(\Omega)^{\frac{1}{n}} \left(\fint_\Omega |\nabla u|^r \,\mathrm{d}x \right)^{\frac{1}{r}}, \tag{2.160}$$

so dividing both sides by $\mathcal{L}^n(\Omega)^{\frac{1}{n}}$ establishes the result for $p = r, q = r^*$. In general, if $p < n$ we apply the above with $r = p$ and apply Hölder on the left-hand side, and if $p \geq n$ and $\frac{n}{n-1} \leq q < \infty$ we choose r to satisfy $r^* = q$ and apply Hölder on the right-hand side. In the final case we choose $r = 1$ and apply Hölder on both sides. $\square$

Lemma 2.3.6 (Orlicz Poincaré-Sobolev)**.** Let φ be an N-function satisfying the ∇_2-condition, and $\Omega \subset \mathbb{R}^n$ a δ-John domain. Then for all $1 \leq p \leq \frac{n}{n-1}$, for $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ we have

$$\left(\fint_\Omega \varphi \left(\frac{|u - (u)_\Omega|}{R} \right)^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leq C(n, N, \Delta_2(\varphi), \delta) \fint_\Omega \varphi(|\nabla u|) \,\mathrm{d}x \tag{2.161}$$

where $R = \operatorname{diam}(\Omega)$, and if $u \in W_0^{1,\varphi}(\Omega, \mathbb{R}^n)$ we can omit the $(u)_\Omega$ term.

Proof. Recall from Lemma 2.1.27 that $R \simeq \mathcal{L}^n(\Omega)^{\frac{1}{n}}$. If $\Omega = Q$ is a cube and $p = 1$, this was shown by Bhattacharya & Leonetti [BL91] using the Riesz-potential estimate

$$|u - (u)_Q| \leq C(n) \int_Q \frac{|\nabla u(y)|}{|x - y|^{n-1}} \,\mathrm{d}y. \tag{2.162}$$

This was then extended to John domains by Hurri-Syjänen [Hur96], who applied the above to a covering satisfying the Boman chain condition from Lemma 2.1.30.

For general p, by translation we can assume $(u)_\Omega = 0$. Now noting that $\nabla(\varphi(|u|)) = \varphi'(|u|)\nabla|u|$, we can apply Young's inequality (Lemma 2.1.2) to estimate

$$\begin{aligned} \fint_\Omega |\nabla \varphi(|u|)| \,\mathrm{d}x &\leq \fint_\Omega \delta\, \varphi^* \left(\varphi'(|u|) \right) + \delta\varphi(|\nabla u|/\delta) \,\mathrm{d}x \\ &\leq \fint_\Omega \delta\varphi(2|u|) + \delta\varphi(|\nabla u|/\delta) \,\mathrm{d}x, \end{aligned} \tag{2.163}$$

where $\delta > 0$ is to be determined. This shows that $\varphi(|u|) \in W^{1,1}(\Omega)$, so applying Lemma 2.3.5 we have

$$\begin{aligned} \left(\fint_\Omega \varphi \left(\frac{|u|}{R} \right)^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} &\leq C \left| \left(\varphi \left(\frac{|u|}{R} \right) \right)_\Omega \right| + CR \fint_\Omega \delta\varphi \left(\frac{2|u|}{R} \right) + \delta\varphi \left(\frac{|\nabla u|}{\delta R} \right) \,\mathrm{d}x \\ &\leq \fint_\Omega \varphi \left(\frac{|u|}{R} \right) \,\mathrm{d}x + \fint_\Omega \varphi(|\nabla u|) \,\mathrm{d}x. \end{aligned} \tag{2.164}$$

Here we have applied (2.163) to u/R, used Jensen's inequality for the first term, and set $\delta = \frac{1}{R}$. Now the result follows from the $p = 1$ case.

If u vanishes on the boundary, the result is proven in [CGZ19, Proposition 1.4]; this essentially consists of adapting the argument for the classical Gagliardo-Nirenberg inequality. Note here we use the fact that $R = \operatorname{diam}(\Omega)$. □

We note that the optimal scales for the Orlicz-Sobolev embedding theorem has been identified by CIANCHI [Cia96], however we will find the above version is more convenient for our purposes as it comes with the modular estimate (2.161), even if doesn't capture the sharp improvement in integrability. We will usually apply this for balls $\Omega = B_R(x_0)$, however in light of Lemma 2.1.25 this generality will be useful when establishing estimates up to the boundary. Another important example will be the case of V-functionals, when we take $\varphi = e_{p,\mu}$ from Example 2.1.9.

2.3.3 Traces and extensions

We will also identify the trace space of these Orlicz-Sobolev spaces, and derive a fractional embedding result. This technical step mirrors what as done in [GK19a] when considering regularity of minima in the BV setting, and similar ideas were used by FONSECA & MÁLY [FM97] in the context of lower-semicontinuity.

Assuming $\varphi \in \Delta_2$ for simplicity, we define $Y^\varphi(\partial\Omega, \mathbb{R}^N)$ to be the space of $u \in L^\varphi(\partial\Omega, \mathbb{R}^N)$ for which

$$\varpi^\varphi_{\partial\Omega}(u) := \int_{\partial\Omega}\int_{\partial\Omega} \varphi\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)}{|x-y|^{n-2}} < \infty, \tag{2.165}$$

where $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ denotes the surface measure on $\partial\Omega$. We have the associated seminorm

$$[u]_{Y^\varphi(\partial\Omega)} = \inf\left\{\lambda > 0 : \varpi^\varphi_{\partial\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\} < \infty. \tag{2.166}$$

The characterisation of the trace space of Orlicz-Sobolev spaces was studied by LACROIX [Lac74], where under the ∇_2 condition the trace space was identified as $Y^\varphi(\partial\Omega, \mathbb{R}^N)$. As this reference is not easily accessible, we have sketched the details in the half-space for the convenience of the reader. This largely follows by modifying proof of the $W^{1,p}$ case, and we will loosely follow the exposition in [Neč12]. We will also mention the more recent works [KK13; DK15; DK16] where the trace space is identified under for general N-functions in the weighted setting. We also mention the work of CIANCHI, KERMAN, & PICK [CKP08] and Cianchi & Pick [CP10; CP16] in identifying the embedding of trace embeddings into Orlicz and rearrangement-invariant spaces.

Lemma 2.3.7. Let φ be an N-function such that $\varphi \in \Delta_2 \cap \nabla_2$, and $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain. Then there exists a well-defined trace operator

$$\mathrm{Tr}\colon W^{1,\varphi}(\Omega, \mathbb{R}^N) \to Y^{\varphi}(\partial\Omega, \mathbb{R}^N), \tag{2.167}$$

which satisfies $\mathrm{Tr}(u) = u|_{\partial\Omega}$ for any $u \in C^1(\overline{\Omega}, \mathbb{R}^N)$, and we have the modular estimate

$$\rho^{\varphi}_{\partial\Omega}(|\mathrm{Tr}\, u|) + \varpi^{\varphi}_{\partial\Omega}(\mathrm{Tr}\, u) \leq C \left(\rho^{\varphi}_{\Omega}(|u|) + \rho^{\varphi}_{\Omega}(|\nabla_{\tau} u|) \right). \tag{2.168}$$

Here ∇_{τ} denotes the tangential gradient along $\partial\Omega$. Moreover this operator is surjective, and there exists an extension operator

$$E\colon Y^{\varphi}(\partial\Omega, \mathbb{R}^N) \to W^{1,\varphi}(\Omega, \mathbb{R}^N) \tag{2.169}$$

satisfying $\mathrm{Tr} \circ E = \mathrm{id}_{Y^{\varphi}(\partial\Omega)}$, along with the estimate

$$\rho^{\varphi}_{\Omega}(|Ev|) + \rho^{\varphi}_{\Omega}(|\nabla_{\tau} Ev|) \leq C \left(\rho^{\varphi}_{\partial\Omega}(|v|) + \varpi^{\varphi}_{\partial\Omega}(v) \right). \tag{2.170}$$

Sketch of proof for $\Omega = \mathbb{R}^n_+$. Let $u \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$, and choose a representative which is absolutely continuous on almost all lines. Writing $\overline{x} = (x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, for almost every $x, h \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} &|u(x+h,0) - u(x,0)| \\ &\quad \leq \int_0^{|h|} \left(|\partial_{x_n} u(x,t)| - |\partial_{x_n} u(x+h,t)| + |\partial_v u(x+tv, |h|)| \right) \mathrm{d}x, \end{aligned} \tag{2.171}$$

putting $v = h/|h|$. We now use the pointwise estimate in $\varpi^{\varphi}_{\partial\mathbb{R}^n_+}(u(\cdot,0))$ applied in the radial directions of $\mathbb{R}^{n-1}$, converting to spherical coordinates. From here the desired estimate will follow from an application of the Hardy-type inequality

$$\int_0^{\infty} \varphi\left(\frac{1}{x} \int_0^x f(t)\,\mathrm{d}t \right) \mathrm{d}x \leq C \int_0^{\infty} \varphi(|f|)\,\mathrm{d}x, \tag{2.172}$$

which holds for all $f \in L^{\varphi}((0,\infty))$ since maximal operator is bounded on this space noting $\varphi \in \Delta_2 \cap \nabla_2$ (see Proposition 2.2.8).

Conversely given $v \in Y^{\varphi}(\mathbb{R}^n_+, \mathbb{R}^N)$, we will define an extension using *Gagliardo's extension operator*

$$Ev(x, x_n) = \frac{1}{x_n^{n-1}} \int_{\mathbb{R}^n_+} \omega\left(\frac{x-y}{x_n}\right) v(y)\,\mathrm{d}y. \tag{2.173}$$

where $\omega \in C_c^\infty(\mathbb{R}^{n-1})$ is a radially symmetric mollifier supported on the unit ball. Distinguishing between the cases $1 \leq i \leq n-1$ and $i = n$, we have the pointwise estimate

$$|\partial_{x_i} v(x, x_n)| \leq C(n)\,\|\omega\|_{C^1(\overline{B}(0,1))} \frac{1}{x_n^n} \int_{B(x,x_n)} |v(y) - v(x)|\,\mathrm{d}y, \tag{2.174}$$

so then we can estimate $\rho^{\varphi}_{\mathbb{R}^n_+}(|\nabla(Ev)|)$ using the above and Jensen's inequality. $\square$

We will also need the following fractional embedding result, which is a adaptation of the analogous estimate in $W^{1,1}$ established by Bourgain, Brezis, & Mironescu [BBM04]. We will also record a compactness result for non-critical spaces, which will be a straightforward consequence of the embedding.

Lemma 2.3.8. Let φ be an N-function satisfying the Δ_2-condition, $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, and $1 < q \leq \frac{n}{n-1}$ if $n \geq 3$ and $1 < q < \frac{1}{2}$ if $n = 2$. Then there exists $C > 0$ such that

$$\left(\int_{\partial\Omega}\int_{\partial\Omega} \varphi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)^q \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n-2}}\right)^{\frac{1}{q}} \leq C \int_{\partial\Omega} \varphi(|u|) + \varphi(|\nabla_\tau u|)\,\mathrm{d}x, \tag{2.175}$$

for all $u \in W^{1,\varphi}(\partial\Omega, \mathbb{R}^N)$, where $C = C(n, q, \Delta_2(\varphi)) > 0$. Moreover if $q < \frac{n}{n-1}$, the associated embedding $W^{1,\varphi}(\partial\Omega) \to Y^{\varphi^q}(\partial\Omega)$ is compact.

Proof. We will prove the assertion for $\Omega = \mathbb{R}^n$, and the general case follows by applying this locally. We begin by establishing the following one-dimensional estimate

$$\begin{aligned}
&\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)^q \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{q}} \\
&\leq C\left(\int_{\mathbb{R}} \varphi(|u|) + \varphi(|u'|)\,\mathrm{d}x\right)^{q-1} \left(\int_{\mathbb{R}} \varphi(|u|)^q\,\mathrm{d}x\right)^{\frac{2}{q}-1},
\end{aligned} \tag{2.176}$$

where $C = C(q, \Delta_2(\varphi))$. For this let $0 < \lambda \leq 1$ to be determined and $1 < q < 2$, then we can write

$$\begin{aligned}
\varpi^{\varphi^q}_{\mathbb{R}}(u) &= 2\int_{\mathbb{R}}\int_0^\lambda \varphi\left(\frac{|u(x+h)-u(x)|}{h}\right)^q \mathrm{d}h\,\mathrm{d}x \\
&\quad + 2\int_{\mathbb{R}}\int_\lambda^\infty \varphi\left(\frac{|u(x+h)-u(x)|}{h}\right)^q \mathrm{d}h\,\mathrm{d}x \\
&= 2(I + II),
\end{aligned} \tag{2.177}$$

which we estimate separately. For the first term we use the fundamental theorem of calculus to note that

$$\frac{|u(x+h)-u(x)|}{h} = \frac{1}{h}\int_x^{x+h} u'(t)\,\mathrm{d}t = \int_0^1 u'(x+th)\,\mathrm{d}t, \tag{2.178}$$

so applying Jensen's inequality to φ we have

$$\begin{aligned} I &= \int_{\mathbb{R}}\int_0^\lambda \varphi\left(\frac{1}{h}\int_x^{x+h} u'(t)\,\mathrm{d}t\right)^q \mathrm{d}h\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}}\int_0^\lambda \frac{1}{h^q}\left(\int_x^{x+h} \varphi(|u'(t)|)\,\mathrm{d}t\right)^q \mathrm{d}h\,\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}} \varphi(|u'|)\,\mathrm{d}t\right)^{q-1} \int_{\mathbb{R}}\int_0^\lambda \frac{1}{h^{q-1}}\int_0^1 \varphi(|u'(x+th)|)\,\mathrm{d}t\,\mathrm{d}h\,\mathrm{d}x \\ &= \left(\int_{\mathbb{R}} \varphi(|u'|)\,\mathrm{d}t\right)^{q-1} \int_0^\lambda \frac{1}{h^{q-1}}\int_0^1\int_{\mathbb{R}} \varphi(|u'(x+th)|)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}h \\ &= \beta\,\frac{\lambda^{2-q}}{2-q}\left(\int_{\mathbb{R}} \varphi(|u'|)\,\mathrm{d}t\right)^q, \end{aligned} \tag{2.179}$$

where we have applied Fubini in the penultimate line, and $\beta \geq 1$ is to be chosen. For the second term we can bound

$$\begin{aligned} II &\leq C_1 \int_{\mathbb{R}}\int_0^\lambda \varphi\left(\frac{|u(x+h)|}{h}\right)^q + \varphi\left(\frac{|u(x)|}{h}\right)^q \mathrm{d}h\,\mathrm{d}x \\ &= C_1\int_\lambda^\infty \frac{\lambda^q}{h^q}\,\mathrm{d}h \int_{\mathbb{R}} \varphi\left(\frac{|u(x)|}{\lambda}\right)^q \mathrm{d}x \\ &= \frac{C_1\lambda^{1-q}}{q-1}\int_{\mathbb{R}} \varphi\left(|u(x)|\right)^q \mathrm{d}x, \end{aligned} \tag{2.180}$$

where $C_1 = 2^{q-1}\Delta_2(\varphi)^q$. We wish to extremise in $\lambda \leq 1$, for which we take

$$\lambda = \frac{C_1\int_{\mathbb{R}}\varphi(|u|)^q\,\mathrm{d}x}{\beta\left(\int_{\mathbb{R}}\varphi(|u|)+\varphi(|u'|)\,\mathrm{d}x\right)^q}. \tag{2.181}$$

Since $\varphi(|u|) \in W^{1,1}(\mathbb{R})$, we can choose $\beta = \beta(C_1, q)$ sufficiently large to ensure that $\lambda \leq 1$ independently of u, and so the $n = 1$ case follows.

Now we can argue analogously as in [BBM04, Lemma D.1]; let $n \geq 2$ and fix $q = \frac{n+1}{n} < 2$. Then by taking spherical coordinates in $h = y-x$ and working componentwise we can bound

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi\left(\frac{|u(x)-u(y)|}{|x-y|}\right)^q \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n-1}} \\ &\qquad \leq C\sum_{i=1}^n \int_{\mathbb{R}}\int_0^\infty \varphi\left(\frac{|u(x+he_i)-u(x)|}{h}\right)^q \mathrm{d}h\,\mathrm{d}x, \end{aligned} \tag{2.182}$$

so it suffices to bound each term separately. To achieve this we apply the above one-dimensional estimate in the x_i direction and apply Hölder, using the fact that $\varphi(|u|) \in$

$L^{\frac{n}{n-1}}(\mathbb{R}^n)$ by the Gagliardo-Nirenberg inequality (arguing similarly as in Lemma 2.3.6), and also that $\int_{\mathbb{R}} \varphi(|u|)\,\mathrm{d}x_i$ lies in $L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})$ by integrating over the remaining coordinates.

To show the embedding is compact for $q < \frac{n}{n-1}$ note that if $\{u_n\}$ is bounded in $W^{1,\varphi}(\partial\Omega, \mathbb{R}^N)$ it is precompact in $L^1(\partial\Omega, \mathbb{R}^N)$ by Rellich-Kondrachov, and hence converges in measure. Further by the above embedding we have the argument of $\varpi^{\varphi^q}_{\partial\Omega}(u_n - u)$ is uniformly integrable, so by Vitali it follows that $\varpi^{\varphi^q}_{\partial\Omega}(u_n - u) \to 0$ as $n \to \infty$. $\square$

Combining the above two results gives the following.

Proposition 2.3.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and $\varphi \in \Delta_2$ an N-function. Then there exists an extension operator

$$E : W^{1,\varphi}(\partial\Omega, \mathbb{R}^N) \to W^{1,\varphi^q}(\Omega, \mathbb{R}^N) \tag{2.183}$$

which is bounded for $1 \leq q \leq \frac{n}{n-1}$ if $n \geq 3$, and $1 \leq q < \frac{1}{2}$ if $n = 2$, and moreover compact if $q < \frac{n}{n-1}$. Moreover E satisfies the estimates

$$\left(\int_\Omega \varphi(|Ev|)^q + \varphi(|\nabla(Ev)|)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \int_{\partial\Omega} \varphi(|v|) + \varphi(|\nabla_\tau v|),$$
$$\left(\int_\Omega \varphi(|\nabla(Ev)|)^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq C \int_{\partial\Omega} \varphi(|\nabla_\tau v|).$$

This straightforwardly follows by combining Lemmas 2.3.7 and 2.3.8. Here the second inequality follows by applying the first to $v - (v)_\Omega$ and using the Poincaré-Sobolev inequality (Lemma 2.3.6). In general the constant depends non-trivially on Ω, but we will usually apply this on balls where we can deduce the R-dependence by a scaling argument.

Discussion 2.3.10. Extension operators of this type were used by Fonseca & Malý [FM97] to establish lower-semicontinuity results for quasiconvex integrands satisfying a (p,q)-growth condition with $q < \frac{np}{n-1}$. This range of exponents ensures an extension operator of the above type is defined, and the inequality is strict to ensure said extension is compact. Similar ideas will be used in Section 5.2 by adapting an argument of [CK17]. Moreover extension operators of this type have also been applied to establish regularity; this was done by Schmidt [Sch08a; Sch08b; Sch09] to establish partial regularity for quasiconvex integrands satisfying a (p,q)-growth condition.

2.3.4 Besov spaces

We will briefly introduce and discuss the basic properties of Besov spaces, which provides intermediate scales between the usual Sobolev scale of smoothness. Unlike in the Section 2.3.3, we will restrict our attention to the L^p case, and these spaces will be used in Chapter 6. We will write $\tau_h f(x) = f(x+h)$ for any function f and $x, h \in \mathbb{R}^n$, and for $\Omega \subset \mathbb{R}^n$ open we set

$$\Omega^{[h]} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}. \tag{2.184}$$

Definition 2.3.11. Let $\Omega \subset \mathbb{R}^n$ be either the full space or a bounded Lipschitz domain, $1 \leq p, q \leq \infty$ and $s \in (0,1)$. We define the *Besov space* $B^{s,p}_q(\Omega, \mathbb{R}^N)$ to be the space of functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$[u]_{B^{s,p}_q(\Omega)} = \left(\int_{\mathbb{R}^n} |h|^{-(n+sq)} \left\| \tau_h u - u \right\|^q_{L^p(\Omega^{[h]})} \,\mathrm{d}h \right)^{\frac{1}{q}} < \infty \tag{2.185}$$

if $q < \infty$, and if $q = \infty$ we require

$$[u]_{B^{s,p}_\infty(\Omega)} = \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-s} \left\| \tau_h u - u \right\|_{L^p(\Omega^{[h]})} < \infty \tag{2.186}$$

with the associated norm $\|u\|_{B^{s,p}_q(\Omega)} + \|u\|_{L^p(\Omega)} + [u]_{B^{s,p}_q(\Omega)}$. We also define $B^{1+s,p}_q(\Omega, \mathbb{R}^N)$ to be the space of $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\nabla u \in B^{s,p}_q(\Omega, \mathbb{R}^{Nn})$, with the associated norm $\|u\|_{B^{1+s,p}_q(\Omega)} = \|u\|_{W^{1,p}(\Omega)} + [\nabla u]_{B^{s,p}_q(\Omega)}$. We also can define the corresponding local versions $B^{s,p}_{q,\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

We are largely interested in the case when $q = \infty$, which is also known as *Nikol'skiĭ-spaces.* We will refer to the classical texts of Nikol'skiĭ [Nik12], Triebel [Tri83], Adams & Fournier [AF03] for a detailed discussion and further references. In particular, we remark these spaces can be obtained by means of real interpolation between L^p and $W^{1,p}$, which is often how the below embedding theorems are proven.

We have the following embedding theorem on the full space.

Proposition 2.3.12 (Embeddings in $\mathbb{R}^n$)**.** Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq q \leq \infty$, $0 < s_1 < s < 1$, and assume that

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1} \tag{2.187}$$

Then if $v \in B_q^{s,p}(\mathbb{R}^n)$, we have

$$[v]_{B_q^{s_1,p_1}(\mathbb{R}^n)} \le C\,[v]_{B_q^{s,p}(\mathbb{R}^n)}\,.$$

Moreover if $s_1 = 0$, the above inequalities remain true if we replace $B_q^{s,p}(\mathbb{R}^n,\mathbb{R}^N)$ with the Lorentz space $L^{p,q}(\mathbb{R}^n,\mathbb{R}^N)$.

On bounded domains we have the following slightly different version.

Proposition 2.3.13 (Embedding in Lipschitz domains)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 \le p \le p_1 \le \infty$, $1 \le q_1 \le q \le \infty$, $0 < s_1 < s < 1$. Assume that

$$s - \frac{n}{p} \le s_1 - \frac{n}{p_1} \tag{2.188}$$

and suppose $v \in B_q^{s,p}(\Omega)$. Then,

$$\|v\|_{B_{q_1}^{s_1,p_1}(\Omega)} \le C\,\|v\|_{B_q^{s,p}(\Omega)}\,.$$

Moreover if $s_1 = 0$, the above inequalities remain true if we replace $B_q^{s,p}(\Omega)$ with the Lorentz space $L^{p,q}(\Omega)$. Also if the inequality is strict, the result holds for all q, q_1.

Note that for $\Omega \subset \mathbb{R}^n$, the *Marcinkiewicz space* $L^{p,\infty}(\Omega,\mathbb{R}^N)$ is defined as the space of measurable functions such that

$$\|f\|_{L^{p,\infty}(\Omega)} = \sup_{s\ge 0}\, s\,\lambda_f(s)^{\frac{1}{p}}, \tag{2.189}$$

where $\lambda_f(s)$ is the distribution function defined in (2.84) from Section 2.2.1. We will record the following elementary inequality.

Lemma 2.3.14. If $1 \le q < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set, we have

$$\|f\|_{L^q(\Omega)} \le \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \mathcal{L}^n(\Omega)^{\frac{1}{q}-\frac{1}{p}}\,\|f\|_{L^{p,\infty}(\Omega)}\,. \tag{2.190}$$

Proof. For $s_0 > 0$ to be determined we can estimate

$$\begin{aligned} \|f\|_{L^q(\Omega)}^p &= q\int_0^\infty s^{q-1}\lambda_f(s)\,\mathrm{d}s \\ &= q\mathcal{L}^n(\Omega)\int_0^{s_0} s^{q-1}\,\mathrm{d}s + q\,\|f\|_{L^{p,\infty}(\Omega)}^p \int_{s_0}^\infty s^{q-p-1}\,\mathrm{d}s \\ &= s_0^q\,\mathcal{L}^n(\Omega) + \frac{q}{p-q}\,s_0^{q-p}\,\|f\|_{L^{p,\infty}(\Omega)}^p\,. \end{aligned} \tag{2.191}$$

Extremising s_0 we take $s_0 = \|f\|_{L^{p,\infty}(\Omega)}/\mathcal{L}^n(\Omega)^{\frac{1}{p}}$, from which the result follows. □

For $q = \infty$ and $\Omega \subset \mathbb{R}^n$ we will introduce the localised seminorm

$$[u]_{s,p,d;\Omega} = \sup_{0<|h|<d} |h|^{-s} \|\tau_h u - u\|_{L^p(\Omega[h])}, \tag{2.192}$$

which is defined for $u \in B^{s,p}_\infty(\Omega)$. Using this we obtain the following localised estimate, which is similar to [ELM04, Lemma 3].

Lemma 2.3.15. Let $0 < r < R$, $1 \leq p \leq p_1 \leq \infty$, $0 < s < 1$, and assume that

$$s < \frac{n}{p} - \frac{n}{p_1}. \tag{2.193}$$

then we have

$$[u]_{L^{p_1}(B_r)} \leq C R^{\frac{n}{p}-\frac{n}{p_1}-s} \left([u]_{s,p,\frac{R-r}{4},B_R} + \frac{1}{(R-r)^{\max\{s,1-s\}}} \|u\|_{L^p(B_R)} \right), \tag{2.194}$$

where $C = C(n,p,p_1,s) > 0$.

Proof sketch. Let $\eta \in C^\infty_c(B_R, \mathbb{R}^N)$ be a cutoff such that $\eta = 1$ on B_r, vanishes outsides $B_{(r+R)/2}$ and $|\nabla \eta| \leq \frac{C}{R-r}$. Now we apply the full-space estimate (Proposition 2.3.12) to ηu, which gives

$$\|u\|_{L^{\frac{np}{n-sp},\infty}(B_r)} \leq \|\eta u\|_{L^{\frac{np}{n-sp},\infty}(\mathbb{R}^n)} \leq C\, [\eta u]_{B^{s,p}_\infty(\mathbb{R}^n)}. \tag{2.195}$$

Now we apply Lemma 2.3.14 to the left-hand side, and if $|h| \leq \frac{R-r}{4}$ we see that

$$\begin{aligned} &\left(\frac{1}{|h|^{\alpha p}} \int_{\mathbb{R}^n} |\tau_h(\eta u) - \eta u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &\lesssim \left(\frac{1}{|h|^{sp}} \int_{\mathbb{R}^n} |\eta(\tau_h u - u)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\frac{1}{|h|^{sp}} \int_{\mathbb{R}^n} |u(\tau_h \eta - \eta)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &\lesssim \left(\frac{1}{|h|^{sp}} \int_{B_{r+R}2} |\tau_h u - u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(|h|^{(1-s)p} \int_{B_R} |u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\ &\lesssim [u]_{s,p,\frac{R-r}{4},B_R} + C(R-r)^{-(1-s)} \|u\|_{L^p(B_R)}. \end{aligned} \tag{2.196}$$

On the other hand if $|h| > \frac{R-r}{4}$, then we can estimate $|h|^{-s} \leq (R-r)^{-s}$ to conclude. $\square$

Finally we will need the following property of second order difference quotients; which follows straightforwardly from the identity

$$\frac{1}{2}(\tau_h u + 2u - \tau_{-h} u) = \frac{1}{2}(\tau_h u - \tau_{-h} u) - (u - \tau_{-h} u). \tag{2.197}$$

Lemma 2.3.16. Let $1 \leq p \leq \infty$ and $0 < s < 1$, then we have

$$[[u]]_{B^{s,p}_{\infty}(\mathbb{R}^n)} = \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-s} \, \|\tau_h u - 2u + \tau_{-h} u\|_{L^p(\mathbb{R}^n)} \tag{2.198}$$

defines an equivalent norm on $B^{s,p}_{\infty}(\mathbb{R}^n, \mathbb{R}^N)$.

2.3.5 Lipschitz truncations

We conclude with a Lipschitz truncation result due to Acerbi & Fusco [AF84; AF88]. This has been adapted in [DMS08; DSV12; Die+12] in the context of the $\mathcal{A}$-harmonic approximation results in the modular scales, and while the version we need is essentially contained in the above references, we will provide a precise statement to illustrate that the Δ_2-condition is unnecessary.

We point out that in the context of higher-order problems, it was necessary in [Irv21b] to derive a $W^{k,\infty}$-truncation result in a similar spirit. This was proven in [FJM02] for $k = 2$ which we adapted in [Irv21b, Proposition 2.11], and this is the main additional ingredient needed to consider the higher order case.

Lemma 2.3.17 (Lipschitz truncation)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $q \in W^{1,\varphi}_0(\Omega, \mathbb{R}^N)$. Then for each $\lambda > 0$, there exists $q_\lambda \in W^{1,\infty}_0(\Omega, \mathbb{R}^N)$ such that $\|q_\lambda\|_{L^\infty(\Omega)} \leq C\lambda$ and we have

$$\int_\Omega \varphi(|\nabla q_\lambda|) \, \mathrm{d}x \leq C \int_\Omega \varphi(C|\nabla q|) \, \mathrm{d}x \tag{2.199}$$

$$\varphi(\lambda) \mathcal{L}^n \left(\{x \in \Omega : q(x) \neq q_\lambda(x)\}\right) \leq C \int_\Omega \varphi(C|\nabla q|) \, \mathrm{d}x \tag{2.200}$$

for any N-function φ satisfying the ∇_2-condition.

We will apply this on balls $\Omega = B_R(x_0)$, in which case a scaling argument shows that the constants are independent of x_0 and R.

Proof. Using [DMS08, Theorem 2.3] we can construct $q_\lambda \in W^{\infty}_0(\Omega, \mathbb{R}^N)$ such that $\|\nabla q_\lambda\|_{L^\infty(\Omega)} \leq C\lambda$ and

$$\{x \in \Omega : q(x) \neq q_\lambda(x)\} \subset \{x \in \Omega_L : \mathcal{M}(\nabla q)(x) > \lambda\}, \tag{2.201}$$

where Ω_L is the set of Lebesgue point of ∇q in Ω, and $\mathcal{M}$ is the Hardy-Littlewood maximal function defined for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{Nn})$ by

$$\mathcal{M}f(x) = \sup_{R>0} \fint_{B_R(x)} |f(y)| \,\mathrm{d}y, \tag{2.202}$$

which we apply to the zero extension of ∇q to $\mathbb{R}^n$. Then if $\varphi \in \nabla_2$ is an N-function, note by Proposition 2.2.8 that we have modular estimate

$$\int_{\mathbb{R}^n} \varphi\left(\mathcal{M}(|\nabla q|)\right) \,\mathrm{d}x \leq C \int_{\Omega} \varphi(C|\nabla q|) \,\mathrm{d}x, \tag{2.203}$$

where $C = C(n, \nabla_2(\varphi))$. Hence by Markov's inequality we have

$$\varphi(\lambda)\mathcal{L}^n(\{q \neq q_\lambda\}) \leq \varphi(\lambda)\mathcal{L}^n(\mathcal{M}(|\nabla q|) > \lambda) \leq \int_{\mathbb{R}^n} \varphi(\mathcal{M}(|\nabla q|)) \,\mathrm{d}x, \tag{2.204}$$

so combining with (2.203) the second estimate (2.200) follows. From here (2.199) follows by noting that

$$\int_{\Omega} \varphi(|\nabla q_\lambda|) \leq \int_{\Omega} \varphi(|\nabla q|) \,\mathrm{d}x + \int_{\{q \neq q_\lambda\}} \varphi\left(\|\nabla q_\lambda\|_{L^\infty(\Omega)}\right) \,\mathrm{d}x, \tag{2.205}$$

and that $\|\nabla q_\lambda\|_{L^\infty(\Omega)} \leq C\lambda$, as required. $\square$

Chapter 3

Linear elliptic estimates

Many of our regularity results will involve a linearisation argument, where we compare our solution u with a suitable harmonic approximation. For this we will need to understand the well-posedness of solutions to elliptic systems of the form

$$Lu = -\operatorname{div} \mathbb{A}\nabla u = 0, \tag{3.1}$$

subject to a suitable Legendre-Hadamard ellipticity condition. While results of this type are classical and well-understood, it is difficult to locate a precise reference for many of these results. In particular for elliptic systems, the approach via potentials becomes significantly more involved compared to the scalar case, as we will see in Section 3.2. While there are alternative approaches via local energy estimates, until recently it was not clear whether this approach was equally as powerful. As part of collecting the necessary results we need for the subsequent chapters, we will also survey of these techniques and illustrate the advantages of the various approaches.

3.1 Ellipticity and Gårding inequalities

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$ and $N \geq 1$, and denote by $\mathrm{Sym}(\mathbb{R}^{Nn})$ the space of symmetric bilinear forms on $\mathbb{R}^{Nn}$.

Hypotheses 3.1.1. We say a measurable map $\mathbb{A} : \Omega \to \mathrm{Sym}(\mathbb{R}^{Nn})$ is *uniformly Legendre-Hadamard elliptic* if there is $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2|\eta|^2 \leq \mathbb{A}(x)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \leq \Lambda|\xi|^2|\eta|^2 \tag{3.2}$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^n$.

This can be viewed as a rank-one ellipticity condition, where we only require the bilinear form $\mathbb{A}$ to be positive definite along matrices $\zeta = \xi \otimes \eta$. By considering variants of $u(x) = \sin{(x \cdot \eta)}\, \xi$, for $\mathbb{A}$ constant we see that (3.2) is necessary to establish coercivity estimates. By using the Fourier transform (2.141) we can moreover show that it is sufficient, which is the content of the following lemma.

Proposition 3.1.2 (Gårding inequalities)**.** Let $\mathbb{A}$ be uniformly Legendre Hadamard elliptic on $\Omega \subset \mathbb{R}^N$ open.

(i) If $\mathbb{A}$ is constant, then we have the *strong Gårding inequality*

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla u \,\mathrm{d}x \geq \lambda \int_\Omega |\nabla u|^2 \,\mathrm{d}x \tag{3.3}$$

which holds for all $u \in C_c^\infty(\Omega, \mathbb{R}^N)$ we have

(ii) If $\mathbb{A}$ is uniformly continuous in Ω, then we have the weak Gårding inequality

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla u \,\mathrm{d}x \geq C\lambda \int_\Omega |\nabla u|^2 \,\mathrm{d}x - C\int_\Omega |u|^2 \,\mathrm{d}x. \tag{3.4}$$

Proof. If $\mathbb{A}$ is constant and $u \in C_c^\infty(\Omega, \mathbb{R}^N)$, extending u by zero to $\mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} \mathbb{A}\nabla u \cdot \nabla u \,\mathrm{d}x = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathbb{A}\widehat{\nabla u} \cdot \overline{\widehat{\nabla u}} \,\mathrm{d}x \tag{3.5}$$

by Plancherel. Since $\widehat{\nabla u}(\xi) = \hat{u}(\xi) \otimes (i\xi)$, we can apply the estimate (3.2) pointwise in the Fourier space, and convert back via Plancherel to establish (i).

By a perturbation argument, (3.3) holds for non-constant $\mathbb{A}$ if there is there is $\widetilde{\mathbb{A}}$ constant satisfying (3.2) such that $M = \|\mathbb{A} - \widetilde{\mathbb{A}}\|_{L^\infty(\Omega)} < \frac{\Lambda}{\lambda}$, with constant $\lambda - M\Lambda$. Applying this locally near each $x_0 \in \Omega$ and patching the estimates we deduce (ii). $\square$

Remark 3.1.3. If $\mathbb{A}$ is strongly elliptic in that $\mathbb{A}(x\zeta \cdot \zeta \geq \lambda|\zeta|^2$ for all $\zeta \in \mathbb{R}^{Nn}$, then the strong Gårding inequality immediately follows without further assumptions on the coefficients. However for Legendre-Hadamard elliptic systems this strong form is not gurarenteed, even if $\mathbb{A}$ is assumed to be continuous. This was shown by Zhang [Zha89; Zha96], who considered a map of the form

$$\mathbb{A}(x)z \cdot z = |z|^2 + b(x) \det z, \tag{3.6}$$

for $z \in \mathbb{R}^{2\times 2}$, where $b \in \mathrm{BMO}(\Omega)$. In [Zha96, Theorem 1], he shows the strong Gårding inequality (3.3) holds only if b is sufficiently small with respect to a BMO-type seminorm. Using this we can construct a suitable b continous which violates this smallness condition. In [Zha96, Theorem] this is also adapted to show the weak Gårding inequality (3.4) holds only if a VMO-type condition is satisfied by b. These results were later generalised to three dimension by Lou & McIntosh [LM04, Section 7].

A straightforward application of the Lax-Milgram theorem allows us to obtain solvability results from these energy estimates.

Proposition 3.1.4 (Solvability in L^2)**.** Let $\mathbb{A}$ be uniformly Legendre-Hadamard elliptic, and additionally assume that (3.3) holds. Then if $f \in W^{-1,2}(\Omega, \mathbb{R}^{Nn})$ and $g \in W^{\frac{1}{2},2}(\Omega, \mathbb{R}^N)$, the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = f & \text{in } \Omega \\ \qquad\quad u = g & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

admits a unique solution in $W^{1,2}(\Omega, \mathbb{R}^N)$.

Note that if $\mathbb{A}$ and Ω are sufficiently regular, then we can use difference quotients to obtain the following.

Proposition 3.1.5 (Improved differentiability in L^2)**.** Let $\mathbb{A}$ be of class $C^{0,1}$ satisfying (3.3), and let $\Omega \subset \mathbb{R}^n$ be a $C^{1,1}$ domain. Then if $f \in L^2(\Omega, \mathbb{R}^{Nn})$, the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = f & \text{in } \Omega \\ \qquad\quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

admits a unique solution in $W^{1,2}_0 \cap W^{2,2}(\Omega, \mathbb{R}^N)$.

We omit the standard but lengthy proof, which involves flattening the boundary locally and applying difference quotients; see for instance [Giu03, Theorem 10.3]. We can also iteratively apply the method of difference quotients to deduce interior regularity, which we will state here. We will state a version noted by Carozza, Fusco, & Mingione [CFM98].

Lemma 3.1.6 (Weyl-type lemma, [CFM98, Proposition 2.10])**.** Suppose $\mathbb{A}$ is constant and satisfies (3.2). Then if $B_R(x_0) \subset \mathbb{R}^n$ and $u \in W^{1,1}(B_R(x_0), \mathbb{R}^N)$ solves

$$-\operatorname{div} \mathbb{A}\nabla u = 0 \tag{3.9}$$

in Ω, we have u is smooth on $\overline{B}_{R/2}(x_0)$ and we have the estimate

$$R^k \sup_{B_{R/2}(x_0)} |\nabla^k u| \leq C_k \fint_{B_R(x_0)} |u| \,\mathrm{d}x, \tag{3.10}$$

where $C_k = C_k(n, N, \Lambda/\lambda) > 0$.

Sketch of proof. For $0 < t < s < R$, take a cutoff $\eta \in C_c^\infty(B_R, \mathbb{R}^N)$ such that $1_{B_t} \leq \eta \leq 1_{B_s}$ and $|\nabla \eta| \leq \frac{C}{s-t}$. Testing the equation against $\varphi = \eta^2 u$ and using the Gårding inequality (3.3) we obtain the Caccioppoli inequality

$$\int_{B_t} |\nabla u|^2 \,\mathrm{d}x \leq \frac{C}{(s-t)^2} \fint_{B_s} |u|^2 \,\mathrm{d}x \tag{3.11}$$

Now we use the linearity of the equation to apply this to $\Delta_h u = u(x+h) - u(x)$ and send $h \to 0$ to infer improved Sobolev differentiability. Repeating this and using Sobolev embedding gives

$$\begin{aligned} \sup_{B_t}|u| &\leq \left(\int_{B_t} |\nabla^n u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \frac{C}{(s-t)^n} \left(\int_{B_s} |u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \frac{C}{(s-t)^n} \left(\int_{B_s} |u| \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\sup_{B_s}|u| \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \sup_{B_s}|u| \,\mathrm{d}x + \frac{C}{(s-t)^{2n}} \int_{B_s} |u| \,\mathrm{d}x, \end{aligned} \tag{3.12}$$

using Young's inequality at the end. Now the result follows by an iteration argument, see for instance [Gia83, Lemma V.3.1]. □

3.2 Potential theoretic methods

We will briefly discuss the classical approach to solving linear elliptic equations, via the method of potentials. Given $\mathbb{A}$ constant satisfying (3.2), we consider the operator

$$L = -\operatorname{div} \mathbb{A} \nabla. \tag{3.13}$$

The idea is that we can consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ \ \ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{3.14}$$

and write the solution u as a convolution of f with a suitable potential. We can then appeal to general estimates from Section 2.2 to deduce solvability results in various scales.

3.2.1 Fundamental solutions

When $\Omega = \mathbb{R}^N$, this potential is known as a *fundamental solution* F_L, which solves the distributional equation

$$LF_L(x) = \delta(x)\,\mathrm{Id}_N\,. \tag{3.15}$$

in $\mathbb{R}^n$.

Theorem 3.2.1 (Existence of fundamental solutions, [Joh50])**.** Let L be given by (3.13) satisfying (3.2). Then the distribution

$$F_L(x) = \begin{cases} \frac{1}{4(2\pi i)^{n-1}}\Delta^{\frac{n-1}{2}} \int_{S^{n-1}} |x\cdot\xi| L(\xi)^{-1}\,\mathrm{d}\mathcal{H}^{n-1}(\xi) & \text{if } n \text{ is odd,} \\ \frac{-1}{4(2\pi i)^{n}}\Delta^{\frac{n}{2}} \int_{S^{n-1}} (x\cdot\xi)^2 \log|x\cdot\xi| L(\xi)^{-1}\,\mathrm{d}\mathcal{H}^{n-1}(\xi) & \text{if } n \text{ is even.} \end{cases} \tag{3.16}$$

is a fundamental solution for L. Further F_L is represented by a locally integrable function of the form

$$F_L(x) = \Phi(x) + \Psi(x)\log|x|, \tag{3.17}$$

where Φ, Ψ are positively homogeneous of degree $2-n$ and are smooth away from the origin, and Ψ is a polynomial which is identically zero if $n \geq 3$. Also for any multi-index α with $|\alpha| = s$, there is $C = C(n, N, \Lambda/\lambda, s)$ such that we can estimate

$$|\partial^\alpha F_L(x)| \leq C|x|^{2-n-s}(1 + |\log|x||) \tag{3.18}$$

holds for all $x \in \mathbb{R}^n$, and the logarithm term may be dropped with $n + s \geq 3$.

Sketch of proof. The idea is to use the fundamental solution for the polyharmonic operator

$$\Delta^{\frac{n+q}{2}} = \left(\sum_{i=1}^n \partial_{x_i}^2\right)^{\frac{n+q}{2}}. \tag{3.19}$$

for $q \geq 0$ such that $n + q$ is even; this is given by

$$E_q(x) = \frac{-1}{(2\pi i)^n q!}\int_{S^{n-1}} (x\cdot\xi)^q \log\left(\frac{x\cdot\xi}{i}\right)\,\mathrm{d}\mathcal{H}^{n-1}(\xi), \tag{3.20}$$

From here the strategy is to construct a suitable function $F_{L,q}(x)$ such that

$$LF_{L,q}(x) = E_q(x), \tag{3.21}$$

so then $F_L(x) = \Delta^{\frac{n+q}{2}} F_{L,q}$ will be a fundamental solution for L. Using the regularity of the polyharmonic operator, this allows us to work with classically defined objects in our derivation of F_L. Now for $g \in C^2(\mathbb{R})$ to be determined, for $\xi \neq 0$ we define

$$F_q(x) = \int_{S^{n-1}} g(x \cdot \xi) L(\xi)^{-1} \, \mathrm{d}\mathcal{H}^{n-1}(\xi). \tag{3.22}$$

Here we write

$$L(\xi) = -\xi \cdot \mathbb{A}\xi, \tag{3.23}$$

which is invertible for all $\xi \neq 0$ by (3.2). Then by differentiating under the integral sign we have

$$LF_q(x) = \int_{S^{n-1}} g''(x \cdot \xi) \, \mathrm{d}\mathcal{H}^{n-1}(\xi), \tag{3.24}$$

so we just need to choose g suitably. For this we take

$$g(t) = \frac{-1}{(2\pi i)^n (q+2)!} \, t^{q+2} \log\left(t/i\right), \tag{3.25}$$

which we can verify works. The remaining assertions are shown in detail in [Mit18, Theorem 11.1]. □

Remark 3.2.2. The construction in [Joh50] also applies to higher order operators, but we have omitted this generality for simplicity. It is useful to observe that we use the ellipticity condition to ensure $L(\xi)^{-1}$ exists, and since we only need the existence of a left-inverse this more generally applies to over-determined systems. We also highlight that this approach is significantly more complicated compared to the scalar case, where one can always reduce to the Laplacian by means of a suitable coordinate transformation (see for instance [GT98, Chapter 9]).

The advantage of this approach is that, once we have written the solution u as a convolution with a potential, we can apply known estimates for convolution operators. One such approach is to consider pointwise estimates for $|F(x) - F(y)|$, which leads to estimates in $C^{1,\alpha}$ and $C^{2,\alpha}$. We can also apply the singular integral estimates of Calderón & Zygmund [CZ52] to obtain estimates in L^p.

Proposition 3.2.3 (Solvability in $W^{1,p}$)**.** Let L be given by (3.13) satisfying (3.2) and consider the problem

$$Lu = f = -\operatorname{div} F \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{3.26}$$

where $F \in L^p(\mathbb{R}^n, \mathbb{R}^{Nn})$ with $1 < p < \infty$. Then we have

$$u(x) = \sum_{i=1}^{n} \partial_i F_L * F_i(x) \in \mathcal{D}'(\mathbb{R}^n), \tag{3.27}$$

is well-defined and satisfies $Lu = f$ in $\mathcal{D}'(\mathbb{R}^n)$. Moreover $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and there is $C = C(n, N, \Lambda/\lambda, p) > 0$ such that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C \|F\|_{L^p(\mathbb{R}^n)} . \tag{3.28}$$

Proof sketch. Assume each $F \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^{Nn})$, and we will extend by density in the end. Then it is shown in [Mit18, Theorem 4.69] that u satisfies

$$\nabla u(x) = \sum_{i=1}^{n} \mathrm{P.V.}(\partial_i \nabla F_L) * F_i(x) \tag{3.29}$$

Therefore by applying the Calderón-Zygmund singular integral estimates from [CZ52], we obtain the estimate (3.28). □

By a similar argument, we can obtain estimates in $W^{2,p}$ whenever $L(\nabla)u = f \in L^p(\mathbb{R}^n)$. One can also localise these estimates by taking a cutoff χ and noting that $L(\chi u) = \chi f + [L, \chi]u$, which leads to L^p estimates in the interior.

3.2.2 Poisson kernels

To extend Proposition 3.2.3 from the previous section to hold up to the boundary, one needs to construct suitable potentials on the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. This is where the case of systems is significantly more involved compared to the scalar case; there we can reduce to the Laplacian ($\mathbb{A} = \mathrm{Id}_n$) by a change of coordinates while preserving the boundary. Then one can appeal to the reflection principle to construct the Green's function

$$G(x, y) = F(x - y) - F(\overline{x} - y) \tag{3.30}$$

from the fundamental solution $F(x)$ (see for instance [GT98, Chapter 4]). Here for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we write $\overline{x} = (x', x_n)$. For systems no such general principle is available, and a separate construction is needed to preserve the boundary values. This is done through the construction of *Poisson kernel* P_L, which loosely speaking solve the distributional equation

$$\begin{cases} LP_{L,k}(x) = 0 & \text{in } \mathbb{R}^n_+ \\ P_{L,k}(x',0) = \delta(x')\,\mathrm{Id}_N & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.31}$$

The construction of such kernels is due to AGMON, DOUGLIS, & NIRENBERG [ADN59; ADN64], which applies more generally. In particular for equations of order $2m$, we need m kernels $P_{L,0},\dots,P_{L,m-1}$ to prescribe each $\partial^k_{x_n} u$ on $\mathbb{R}^n_+$, but this simplifies in the second order case. This construction is also discussed in [Mor66, Chapter 6].

Theorem 3.2.4 (Existence of Poisson kernels, [ADN59; ADN64])**.** Let L be given by (3.13) satisfying (3.2). Then there exists a Poisson kernel P_L which is a locally integrable function satisfying

$$LP_L(x) = 0 \tag{3.32}$$

in $\mathcal{D}'(\mathbb{R}^n)$ and additionally

$$\lim_{t\to 0^+} P_L(x',t) = \delta(x')I_N \tag{3.33}$$

in $\mathcal{D}'(\mathbb{R}^{n-1})$. Moreover $P_L(x)$ is smooth away from zero, homogeneous of degree $1-n$, and for any multi-index α with $|\alpha| = s$ there is $C = C(n,m,\ell,\kappa,s) > 0$ such that we can estimate

$$|\partial^\alpha P_L(x)| \leq C|x|^{k-n+1-s} \tag{3.34}$$

holds for all $x \in \mathbb{R}^n_+$.

Sketch of proof. For $q \in \mathbb{Z}$ we introduce

$$G(r,q) = \begin{cases} \frac{-1}{(2\pi i)^n q!} r^q \left(\log(r/i) - \sum_{j=1}^q 1/j\right) & \text{if } q \geq 0, \\ \frac{(-1)^q(-q-1)!}{(2\pi i)^n} r^q & \text{if } q < 0, \end{cases} \tag{3.35}$$

observing that for $q \geq 0$ such that $n+q$ is odd we have

$$\widetilde{E}_q(x) = \int_{S^{n-2}} G(x\cdot\xi, q)\,\mathrm{d}\mathcal{H}^{n-1}(\xi) \tag{3.36}$$

is a fundamental solution for the polyharmonic operator $\Delta_{x'}^{\frac{n+q}{2}}$ in $\mathbb{R}^{n-1}$. Also for any $k > 0$ we have

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k} G(r,q) = G(r,q-k). \tag{3.37}$$

The derivation will involve exploiting the algebraic properties of L, so for each $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ consider the degree $2N$ polynomial $\tau \mapsto \det L(\xi',\tau)$. By the ellipticity and homogeneity assumptions we know there exists N roots $\tau_1^+(\xi'),\ldots,\tau_N^+(\xi')$ with strictly positive imaginary part, and similarly N roots with negative imaginary part. Using this we define

$$M^+(\xi',\tau) = \sum_{k=1}^{N}(\tau - \tau_k^+(\xi')) = \sum_{p=0}^{N} a_p(\xi')\tau^{N-p}, \tag{3.38}$$

and the truncated polynomials $M_k^+(\xi',\tau) = \sum_{p=0}^{k} a_p(\xi')\tau^{k-p}$. Now let γ be a rectifiable Jordan curve in $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ enclosing all the roots $\tau_k^+(\xi')$ for all $\xi' \in S^{n-2}$. Then by [ADN59, Section I.1] we have

$$\frac{1}{2\pi i}\int_\gamma \frac{M_{N-p-1}^+(\xi',\tau)}{M^+(\xi',\tau)}\tau^k \,\mathrm{d}\tau = \delta_{kp}, \tag{3.39}$$

for all $0 \le k,p \le N-1$. Also since $\frac{\det L(\xi',\tau)}{M^+(\xi',\tau)}$ is holomorphic in $\mathbb{C}^+$, we additionally have

$$\frac{1}{2\pi i}\int_\gamma f(\tau)\,\frac{\det L(\xi',\tau)}{M^+(\xi',\tau)}\,\mathrm{d}\tau = 0 \tag{3.40}$$

for any f holomorphic in $\mathbb{C}_+$. Now for a matrix-valued function $A : S^{n-2} \to \mathbb{C}^{N^2}$ define

$$\begin{aligned} &P_{h,q,A}(x) \\ &= \frac{1}{2\pi i}\int_{S^{n-2}}\int_\gamma \operatorname{adj} L(\xi',\tau)\cdot A(\xi')\frac{M_{N-h-1}^+(\xi',\tau)}{M^+(\xi',\tau)}G(x\cdot(\xi',\tau), j+q)\,\mathrm{d}\tau\,\mathrm{d}\mathcal{H}^{n-1}(\xi'), \end{aligned} \tag{3.41}$$

then we observe by differentiating under the integral sign and using (3.40) that $LP_{h,q,A} = 0$ in $\mathbb{R}^n_+$. Now to ensure the boundary conditions are satisfied, we seek $A_0,\ldots,A_{N-1}$ such that

$$\frac{1}{2\pi i}\sum_{h=0}^{N-1}\int_\gamma \operatorname{adj} L(\xi',\tau)\cdot A_h(\xi',\tau)\frac{M_{N-h-1}^+(\xi',\tau)}{M^+(\xi',\tau)}\,\mathrm{d}\tau = \mathrm{Id}_N\,. \tag{3.42}$$

This is achieved by writing

$$\operatorname{adj} L(\xi',\tau) = \sum_{p=0}^{N-1} Q_h(\xi')\tau^p \pmod{M^+(\xi',\tau)}, \tag{3.43}$$

then by (3.39) we seek matrices such that

$$\sum_{p=0}^{N-1} Q_h(\xi')\cdot A_h(\xi') = \mathrm{Id}_N\,. \tag{3.44}$$

Note this is an overdetermined system, and by rank-nullity it suffices to show the dual operator $\mathcal{Q}^T$ sending

$$\mathbb{R}^{N^2} \to \mathbb{R}^{N^2\times N} : C \mapsto \sum_{p=0}^{N-1} C\cdot Q_h(\xi') \tag{3.45}$$

is injective.

To simplify notation, we will supress the ξ' dependence. Consider a Schur decomposition $L(\tau) = Q(\tau)U(\tau)Q^{-1}(\tau)$ of L at each $\tau \in \mathbb{C}$, so Q is unitary and U is upper triangular. Then the diagonal elements $U_{ii}(\tau)$ are eigenvalues of $L(\tau)$, and by means of a Gram-Schmidt construction we can ensure they are polynomials in τ. Since $\operatorname{adj} U$ is also upper-triangular, we have

$$(U \cdot \operatorname{adj} U)_{jj}(\tau) = U_{jj}(\tau) \operatorname{adj} U_{jj}(\tau) = \det U(\tau) = \sum_{k=1}^{\ell} U_{kk}(\tau), \tag{3.46}$$

Since $\det U(\tau)$ is a polynomial of degree $2N$, each $U_{jj}(\tau)$ is of degree 2. Moroever by ellipticity the roots do not lie on the real line, so they appear in complex pairs $\tau_{\pm}$. Hence by noting $M^+(\tau)$ vanishes at precisely the roots of $U_{jj}(\tau)$ in $\mathbb{C}_+ = \{\Im z > 0\}$, it follows that

$$\operatorname{adj} U_{jj}(\tau) = \sum_{j \neq k} U_{kk}(\tau) \not\equiv 0 \pmod{M^+(\tau)}. \tag{3.47}$$

Therefore if $C \in \mathbb{C}^{N \times N}$ satisfies $\mathcal{Q}^T C = 0$, this satisfies

$$C \cdot \operatorname{adj} U = 0 \pmod{M^+(\tau)}, \tag{3.48}$$

so in particular for each $1 \leq i \leq N$ we have

$$C_{i1} \operatorname{adj} U_{11}(\tau) = C_{i1} \prod_{k \neq 1} U_{kk}(\tau) = 0 \pmod{M^+(\tau)}. \tag{3.49}$$

Since C_{i1} is constant while $\operatorname{adj} U_{11}(\tau)$ is a non-constant polynomial, we infer that each $C_{i,1} = 0$. We can proceed inductively to show $C_{i,2} = \cdots = C_{i,N} = 0$, so $C = 0$. This proves $\mathcal{Q}^T$ is injective, and hence we infer the existence of the claimed A_h. Now we can set

$$P_L = \Delta^{\frac{n+q}{2}} \sum_{h=0}^{N-1} P_{h,q,A_h} \tag{3.50}$$

and verify this works. □

Remark 3.2.5. We note the existence of A_h, which involved a lengthy but elementary algebraic argument, encompasses a more general principle regarding the solvability of elliptic systems. In [ADN64], the authors derive what they call a *complementing boundary condition* which provides a necessary and sufficient condition to the boundary condition for the elliptic system to be solvable. This for instance allows us to replace the Dirichlet boundary condition with a more general condition of the form

$$Bu = B_0 u + B_1 \cdot \nabla u = 0 \quad \text{ on } \partial\mathbb{R}^n_+. \tag{3.51}$$

Roughly speaking, the requirement is for the analogue of $\mathcal{Q}^T$ to be injective.

Remark 3.2.6. The above construction shows that there exists $\tilde{P}_L$ locally integrable such that $\Delta_{x'}\tilde{P}_L = P_L$, and such that $\nabla^2 \tilde{P}_L$ is also homogenous of degree $1-n$. This will be used below in obtaining estimates in $W^{1,p}$.

Finally we will briefly discuss how to deduce solvability results in the half space from this. Once again this is an adaptation of the results in [ADN59; ADN64], however the authors only consider $W^{2,p}$ estimates there.

Proposition 3.2.7. Let L be given by (3.13) satisfying (3.2) and $1 < p < \infty$. Then the mapping Then for all $F \in L^p(\mathbb{R}^n_+, \mathbb{R}^{Nn})$, the there exists a solution u solving the Dirichlet problem

$$\begin{cases} Lu = -\operatorname{div} F & \text{in } \mathbb{R}^n_+ \\ \quad u = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{3.52}$$

satisfying the estimate

$$\|\nabla u\|_{L^p(\mathbb{R}^n_+)} \leq C \|F\|_{L^p(\mathbb{R}^n_+)}, \tag{3.53}$$

where $C = C(n, N, p, \Lambda/\lambda) > 0$.

Proof. We will assume $F \in C^\infty_c(\mathbb{R}^n_+, \mathbb{R}^N)$, and extend by density at the end. Using Proposition 3.2.3 we will write $v = \sum_{i=1}^n \partial_i F_L * F_i$, and using the Poisson kernel P_L we consider

$$\begin{aligned} w(x) &= \int_{\mathbb{R}^{n-1}} P_L(x'-y', x_n) v(y', 0) \,\mathrm{d}y' \\ &= -\int_{\mathbb{R}^{n-1}} \int_0^\infty \partial_n \left(P_L(x'-y', x_n + y_n) v(y', y_n) \right) \,\mathrm{d}y' \mathrm{d}y_n. \end{aligned} \tag{3.54}$$

Hence writing $\overline{y} = (y', y_n)$ and using Remark 3.2.6 we have

$$\begin{aligned} \nabla w(x) &= -\int_{\mathbb{R}^n_+} \nabla \Delta_{x'} \partial_{x_n} \tilde{P}_L(x - \overline{y}) v(y) + \nabla P_L(x - \overline{y}) \partial_{x_n} v(y) \\ &= \int_{\mathbb{R}^n_+} \nabla \nabla_{x'} \partial_{x_n} \tilde{P}_L(x - \overline{y}) \cdot \nabla_{x'} v(y) - \nabla P_L(x - \overline{y}) \partial_{x_n} v(y). \end{aligned} \tag{3.55}$$

Note that both $\nabla\nabla_{x'}\partial_{x_n}\tilde{P}_L$ and ∇P_L are homogenous of degree $1-n$, and smooth on S^{n-2}. We wish to apply the Calderón-Zygmund singular integral estimates to these terms, for which we extend $\nabla_{x'}v, \partial_{x_n}v$ by zero to $\mathbb{R}^n \setminus \mathbb{R}^n_+$, and take odd reflection of the kernels. Then applying Theorem 2.2.18 the assertion follows. $\square$

Once the above key estimates have been established, a routine perturbation argument allows one to consider the case when $\mathbb{A}$ is continuous, along with bounded C^1-domains by a flattening argument. These are detailed for instance in [GT98, Chapter 9], where we apply Propositions 3.2.3, 3.2.7 instead of the corresponding statements for the Laplacian. One can also obtain solvability results in other scales, which follows as soon as the relevant potentials have been constructed. This illustrates the power of this potential approach; there is a mountain of literature on singular integral operators which we can now apply to these elliptic systems. In particular this may be useful in treating endpoint cases, however we will not need this here.

3.3 Energy methods

We now describe an alternative approach, which is based on local estimates and perturbation. This approach is due to Campanato [Cam65], and is based on a technique of comparing solutions to elliptic equations to suitable harmonic approximations. We will see similar strategies employed in subsequent chapters of the thesis, and we will illustrate how this can be applied already in the linear setting. This will allow us to obtain estimates for linear elliptic systems in the Hölder and Sobolev scales, which we will use in the subsequent parts. Moreover we will show a modern adaption of the L^p theory by Krylov [Kry07], which allows one to obtain more general statements. These approaches will not only provide a simpler proof of the linear theory compared to the potential methods, but they are more flexible as they do not rely on delicate constructions of kernels.

3.3.1 Schauder theory

We begin by briefly sketching the classical Schauder estimates, which establishes $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity of solutions to linear elliptic problems. The main result of this section is the following.

Proposition 3.3.1 (Local $C^{1,\alpha}$ estimates)**.** Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\alpha}$ domain, and suppose $\mathbb{A}$ is a $C^{1,\alpha}$ map satisfying (3.2). Then there is $R_0 > 0$ taking the form

$$R_0 = c(n, N, \alpha, \Lambda/\lambda, [\mathbb{A}]_{C^{0,\alpha}(\Omega)}, \|\Omega\|_{C^{1,\alpha}(\Omega)}) \min\{1, R_\Omega\} > 0 \tag{3.56}$$

such that the following holds. For any $x_0 \in \overline{\Omega}$ and $R \in (0, R_0)$, suppose $u \in W^{1,2}(\Omega_{2R}(x_0), \mathbb{R}^n)$ solves the localised Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega_{2R}(x_0), \\ \qquad\quad u = 0 & \text{on } \partial\Omega \cap B_{2R}(x_0), \end{cases} \tag{3.57}$$

where $F \in C^{1,\alpha}(\overline{\Omega}_{2R}(x_0), \mathbb{R}^{Nn})$. Then u is $C^{1,\alpha}$ in $\overline{\Omega}_R(x_0)$ and we have the associated estimate

$$[\nabla u]_{C^{0,\alpha}(\Omega_R(x_0))} \leq C \left(\fint_{\Omega_{2R}(x_0)} |\nabla u - (\nabla u)_{\Omega_{2R}(x_0)}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + C\,[F]_{C^{0,\alpha}(\Omega_{2R}(x_0))}. \tag{3.58}$$

where C depends on $n, N, \alpha, \Lambda/\lambda, [\mathbb{A}]_{C^{0,\alpha}(\Omega)}$ and $\|\Omega\|_{C^{1,\alpha}}$ only.

This result is classical, and can be found for instance in [Gia83, Chapter 3] and [Giu03, Chapter 10]. The dependence of constants is usually not stated this way, however it follows by a routine check of the proofs in the aforementioned texts. We will provide an outline of the proof however, to illustrate the key ingredients we use and how this argument can be applied in more general settings.

Sketch of proof. By flattening the boundary, we will consider the case $\Omega = B^+_{R_0}$ is a hall-ball and obtain suitable decay estimates for $R < R_0$ provided R_0 is sufficiently small depending on R_Ω. The interior case is similar, and the case of general $\Omega_R(x_0)$ will then follow by a covering argument. Note that u vanishes on $\Gamma = \overline{B}^+_R \cap \partial\mathbb{R}^n_+$ in the sense of traces, and the weak formulation gives

$$\int_{B^+_R} \mathbb{A}(x)\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_{B^+_R} f \cdot \nabla\varphi \,\mathrm{d}x \tag{3.59}$$

for all $\varphi \in W^{1,2}_0(B^+_R, \mathbb{R}^N)$. For $\theta \in (0,1)$ we consider the linearised problem

$$\begin{cases} -\operatorname{div} \mathbb{A}(0)\nabla v_\theta = 0 & \text{in } B^+_{\theta R} \\ \qquad\qquad v_\theta = u & \text{on } \partial B^+_{\theta R} \end{cases} \tag{3.60}$$

Note that setting $w_\theta = v - v_\theta$ we have

$$\begin{cases} -\operatorname{div} \mathbb{A}(0)\nabla w_\theta = -\operatorname{div} G & \text{in } B^+_{\theta R} \\ \qquad\qquad w_\theta = 0 & \text{on } \partial B^+_{\theta R} \end{cases} \tag{3.61}$$

where

$$G = (\mathbb{A}(0) - \mathbb{A}(x))\,\nabla u + F - F_0. \tag{3.62}$$

for any constant $F_0 \in \mathbb{R}^N$. By Lemma 3.1.4 we see that a unique v_θ exists and satisfies

$$\int_{B^+_{\theta R}} |\nabla v_\theta|^2 \,\mathrm{d}x \le C \int_{B^+_{\theta R}} |\nabla u|^2 \,\mathrm{d}x, \tag{3.63}$$

$$\int_{B^+_{\theta R}} |\nabla w_\theta|^2 \,\mathrm{d}x \le C \int_{B^+_{\theta R}} |\mathbb{A}(0) - \mathbb{A}(x)|^2 |\nabla u|^2 + |F - F_0|^2 \,\mathrm{d}x. \tag{3.64}$$

By the standard method of applying difference quotients in the tangential directions, we deduce that v_θ is smooth in the interior of $B^+_{\theta R}$, and we have

$$\theta^{k-1} \sup_{B^+_{\theta R/2}} |\nabla^k v_\theta| \le C_k \left(\fint_{B^+_{\theta R}} |\nabla v_\theta|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \tag{3.65}$$

for all $k \ge 1$. Using this we deduce that for $\kappa \in (0,1)$,

$$\fint_{B^+_{\kappa\theta R}} |\nabla v_\theta|^2 \,\mathrm{d}x \le C \fint_{B^+_{\theta R}} |\nabla v_\theta|^2 \,\mathrm{d}x, \tag{3.66}$$

$$\fint_{B^+_{\kappa\theta R}} |\nabla v_\theta - (\nabla v_\theta)_{B^+_R}|^2 \,\mathrm{d}x \le C\kappa^2 \fint_{B^+_{\theta R}} |\nabla v_\theta - (\nabla v_\theta)_{B^+_{\theta R}}|^2 \,\mathrm{d}x, \tag{3.67}$$

where we have applied (3.65) to $v - (\nabla v_\theta)_{B^+_{\theta R}} x_n$ to obtain (3.67) when $\kappa < \frac{1}{2}$. By increasing the constant if necessary, this extends to $\kappa \in (0,1)$. Combining (3.64) with (3.66), (3.67) and noting that $u = v_\theta + w_\theta$ we obtain

$$\Phi(\kappa\theta R) \le C \left(\kappa^n + (\theta R)^{2\alpha} \, [\mathbb{A}]^2_{C^{0,\alpha}(\Omega)} \right) \Phi(\theta R) + C(\theta R)^{n+2\alpha} \, [F]^2_{C^{0,\alpha}(\overline{B}^+_R)} \tag{3.68}$$

$$\Psi(\kappa\theta R) \le C\kappa^{n+2}\Psi(R) + C(\theta R)^{2\alpha} \, [\mathbb{A}]^2_{C^{0,\alpha}(\Omega)} \, \Phi(\theta R) + C(\theta R)^{n+2\alpha} \, [F]^2_{C^{0,\alpha}(\overline{B}^+_R)}, \tag{3.69}$$

where for $0 < r \le R$ we define

$$\Phi(r) = \int_{B^+_r} |\nabla u - (\nabla u)_{B^+_R}|^2 \,\mathrm{d}x, \tag{3.70}$$

$$\Psi(r) = \int_{B^+_r} |\nabla u - (\nabla u)_{B^+_r}|^2 \,\mathrm{d}x. \tag{3.71}$$

Now we iterate these in θ and R. Applying [Gia83, Lemma III.2.1] to (3.68), for which we choose R_0 to be sufficiently small, we have

$$\Phi(\kappa\theta R) \leq C\kappa^{n-\alpha}\Phi(\theta R) + C(\kappa\theta)^{n-\alpha}R^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\,. \tag{3.72}$$

for $R < R_0$ sufficiently small. Then applying [Gia83, Proposition III.1.2] we see that

$$\Phi(\theta R) \leq C(\theta R)^{n-\alpha}\sup_{r<R} r^{\alpha-n}\Psi(r) \leq C\theta^{n-\alpha}\left(\Phi(R) + R^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\right). \tag{3.73}$$

Hence noting $\Phi(R) = \Psi(R)$, (3.69) becomes

$$\Psi(\kappa\theta R) \leq C\kappa^{n+2}\Psi(\theta R) + C\theta^{n+\alpha}\left(\Psi(R) + R^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\right), \tag{3.74}$$

so applying [Gia83, Proposition III.1.2] again we obtain

$$\Psi(\theta R) \leq C\theta^{n+\alpha}\left(\Psi(R) + R^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\right). \tag{3.75}$$

Using this we can improve (3.69) to

$$\Psi(\kappa\theta R) \leq C\kappa^{n+2}\Psi(R) + C\theta^{n+2\alpha}\left(\Psi(R) + R^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\right), \tag{3.76}$$

so applying [Gia83, Proposition III.1.2] one last time we get

$$\Psi(\theta R) \leq C\theta^{n+2\alpha}\Psi(R) + C(\theta R)^{n+2\alpha}\,[F]^2_{C^{0,\alpha}(\overline{B}^+_R)}\,. \tag{3.77}$$

By the Campanato-Meyers characterisation of Hölder continuity (see for instance [Gia83, Theorem III.1.2] or [Giu03, Theorem 2.9]) we deduce that u is $C^{0,\alpha}$ in $\overline{B}^+_{R_0}$, with the claimed estimate (3.58). □

Remark 3.3.2. The ingredients in the above proof are the L^2 estimates (3.63), (3.64), and the local uniform estimates (3.65) for the comparison map v_θ. From there we simply combine these estimates and apply it iteratively to show a suitable excess quantity of ∇u exhibits a power-type decay. The idea is that we take a *harmonic approximation* v_θ to the solution u at each scale B^+_θ, which allows us to transfer the regularity of v_θ to u. We will see the same principle be used in the next section to obtain L^p estimates.

If in addition the strong Gårding inequality (3.3) holds, using Lemma 3.1.4 and applying the above local regularity result about each $x_0 \in \overline{\Omega}$ we deduce the following consequence. While we also have an associated estimate, the dependence of constants on the domain is less explicit due to the global nature of the Hölder seminorm.

Corollary 3.3.3. Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\alpha}$ domain, and $\mathbb{A}$ a $C^{1,\alpha}$ map satisfying (3.2) and (3.3). Then for all $F \in C^{0,\alpha}(\Omega, \mathbb{R}^{Nn})$, there is a unique solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.78}$$

We can also adapt the above using L^p estimates instead of L^2 estimates to weaken the localised estimate. This will rely on the estimates from (3.3.8) in the next section, and will be used to establish Proposition 3.3.9.

Proposition 3.3.4 (Local $C^{1,\alpha}$ estimates in $W^{1,p}$)**.** Suppose the assumptions of Proposition 3.3.1 hold, and let $1 < p < \infty$. Then there is $R_0 > 0$ taking the form (3.56) with additionally a p-dependence such that the following holds. For any $x_0 \in \overline{\Omega}$ and $R \in (0, R_0)$, suppose $u \in W^{1,p}(\Omega_{2R}(x_0), \mathbb{R}^n)$ solves the localised Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega_{2R}(x_0), \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \cap B_{2R}(x_0), \end{cases} \tag{3.79}$$

where $F \in C^{1,\alpha}(\overline{\Omega}_{2R}(x_0), \mathbb{R}^{Nn})$. Then u is $C^{1,\alpha}$ in $\overline{\Omega}_R(x_0)$ and we have the associated estimate

$$R^\alpha \left[\nabla u\right]_{C^{0,\alpha}(\Omega_R(x_0))} \leq C \left(\fint_{\Omega_{2R}(x_0)} |\nabla u - (\nabla u)_{\Omega_{2R}(x_0)}|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + C \left[F\right]_{C^{0,\alpha}(\Omega_{2R}(x_0))}. \tag{3.80}$$

where C depends on $n, N, p, \alpha, \Lambda/\lambda, [\mathbb{A}]_{C^{0,\alpha}(\Omega)}$ and $\|\Omega\|_{C^{1,\alpha}}$ only and R_0 satisfies a similar dependence as in (3.56).

The idea is to repeat the proof from the $p = 2$ case, where we replace the L^2 estimates (Lemma 3.1.4) by the L^p estimates established in Theorem 3.3.8 to v_θ and w_θ. For uniform estimates we can adapt the argument of Carozza, Fusco, & Mingione [CFM98] to hold up to the boundary. A technical difficulty arises in deducing the existence of v_θ however, as $B^+_{\theta R}$ must be a $C^{1,\alpha}$ domain for Theorem 3.3.8 to hold. Here we replace $B^+_{\theta R}$ by regularised domains $\widetilde{B}^+_{\theta R}$ from Lemma 2.1.28, and the rest is largely the same.

One also obtains the following variant for $C^{2,\alpha}$ estimates, with minor modifications to the above argument.

Proposition 3.3.5 (Linear $C^{2,\alpha}$ estimates)**.** Let $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, and suppose $\mathbb{A}$ is a $C^{0,\alpha}$ map satisfying the strong Gårding inequality (3.3). Then there is $R_0 > 0$ such that the following holds. For any $x_0 \in \overline{\Omega}$ and $R \in (0, R_0)$, suppose $u \in W^{2,2}(\Omega_{2R}(x_0), \mathbb{R}^n)$ solves the localised Dirichlet problem Then for all $f \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^{Nn})$, there is a unique solution $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ to the Dirichlet problem

$$\begin{cases} \mathbb{A} : \nabla^2 u = f & \text{in } \Omega_{2R}(x_0), \\ \qquad\quad u = 0 & \text{on } \partial\Omega \cap B_{2R}(x_0). \end{cases} \tag{3.81}$$

where $f \in C^{2,\alpha}(\overline{\Omega}_{2R}(x_0), \mathbb{R}^N)$. Then u is $C^{2,\alpha}$ in $\overline{\Omega}_R(x_0)$ and we have the associated estimate

$$[\nabla^2 u]_{C^{0,\alpha}(\Omega_R(x_0))} \leq C \left(\fint_{\Omega_{2R}(x_0)} |\nabla^2 u - (\nabla^2 u)_{\Omega_{2R}(x_0)}|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + C \, [f]_{C^{0,\alpha}(\Omega_{2R}(x_0))} . \tag{3.82}$$

where C depends on $n, N, \alpha, \Lambda/\lambda$, $[\mathbb{A}]_{C^{0,\alpha}(\Omega)}$ and $\|\Omega\|_{C^{2,\alpha}}$ only.

While we can also obtain a global solvability result, since the equation is not in divergence-form we do not necessarily get the existence of a weak solution from Lemma 3.1.4. One can instead argue using the method of continuity, which is detailed in [GT98, Section 6.3]. If $\mathbb{A}$ is constant however, we can recast the equation as a divergence-form equation, and use difference quotients to show the obtain weak solution in $W^{1,2}$ lies in $W^{2,2}$.

3.3.2 The method of Krylov

We now present an alternative approach to establishing L^p estimates for linear elliptic systems, which is based on interpolating between L^2 and uniform estimates. This method allows one to bypass the complicated construction of potentials from Section 3.2 to obtain the Calderón-Zygmund estimates, and is more easily applicable to other settings as we will see later. This approach is classical, and due to Stampacchia [Sta65] (see also Campanato [Cam66]). It is a natural extension of the proof of Schauder estimates presented in the Section 3.3.1, and is based a on similar approach of L^2 estimates and harmonic approximation. In particular, it simply uses the observation that we can obtain estimates in the finer scale BMO by similar arguments to the case of $C^{1,\alpha}$ estimates, which we can interpolate using Corollary 2.2.12. These ideas were more recently refined by Krylov [Kry07], which provided a more general and streamlined approach to obtaining estimates of this type. We mention that there is also an alternative approach due to Caffarelli & Peral [CP98], which was applied to linear equations by Byun & Wang [BW04]. This approach involves estimating the level sets of the maximal operator $\mathcal{M}(|\nabla u|^2)$, and is a more hands-on approach which is more widely applicable to nonlinear problems.

The key result is the following pointwise estimate involving maximal operators. Here we will use (2.124), (2.125) defined in Section 2.2.2.

Proposition 3.3.6 (Krylov-Stampacchia I)**.** Let $\mathbb{A}$ be a constant-coefficient matrix satisfying (3.2), and $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,\alpha}$ domain. Then for all $F \in L^2(\Omega, \mathbb{R}^{Nn})$, the unique solution $u \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ to the elliptic system

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.83}$$

satisfies the pointwise estimate

$$\widetilde{\mathcal{M}}^{\#}_{\Omega}(\nabla u)(x) \leq C\left(\theta^{\alpha}\widetilde{\mathcal{M}}_{\Omega}\left(|\nabla u|^2\right)(x)^{\frac{1}{2}} + \theta^{-\frac{n}{2}}\widetilde{\mathcal{M}}_{\Omega}\left(|F|^2\right)(x)^{\frac{1}{2}}\right) \tag{3.84}$$

for all $x \in \overline{\Omega}$ and $\theta \in (0,1)$, where $C = C(n, N, \alpha, \Lambda/\lambda, \|\Omega\|_{C^{1,\alpha}}, \mathcal{L}^n(\Omega)/R_0^n) > 0$, with $R_0 > 0$ as in (3.56) from Proposition 3.3.1.

This is reminiscent of a potential-type estimate, except we have maximal-type operators instead. The proof straightforwardly follows by combining the L^2 and Hölder estimates.

Proof. Let $R_0 > 0$ so the Schauder estimate from Proposition 3.3.1 applies. Then for $x_0 \in \overline{\Omega}$ and $0 < R < R_0$ we let $w \in W^{1,2}(\Omega_R(x_0), \mathbb{R}^N)$ be the unique solution to

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla v = 0 & \text{in } \Omega_R(x_0), \\ \qquad\qquad v = u & \text{on } \partial\Omega_R(x_0). \end{cases} \tag{3.85}$$

Then letting $w = u - v \in W_0^{1,2}(\Omega_R(x_0), \mathbb{R}^N)$, by Lemma 3.1.4 we have the energy estimates

$$⨍_{\Omega_R(x_0)} |\nabla v - (\nabla v)_{\Omega_R(x_0)}|^2 \,\mathrm{d}x \leq C ⨍_{\Omega_R(x_0)} |\nabla u|^2 \,\mathrm{d}x, \tag{3.86}$$

$$⨍_{\Omega_R(x_0)} |\nabla w - (\nabla w)_{\Omega_R(x_0)}|^2 \,\mathrm{d}x \leq C ⨍_{\Omega_R(x_0)} |F|^2 \,\mathrm{d}x. \tag{3.87}$$

Now applying Proposition 3.3.1 we have v is $C^{1,\alpha}$ in $\overline{\Omega}_{R/2}(x_0)$, and for $\theta \in (0, 1/2)$ by (3.58) we have

$$⨍_{\Omega_{\theta R}(x_0)} |\nabla v - (\nabla v)_{\Omega_{\theta R}(x_0)}|^2 \,\mathrm{d}x \leq C\theta^{2\alpha} ⨍_{\Omega_R(x_0)} |\nabla v - (\nabla v)_{\Omega_R(x_0)}|^2 \,\mathrm{d}x. \tag{3.88}$$

Moreover by increasing C if necessary, this holds for all $\theta \in (0,1)$. Now we can combine these estimates to show that for all $x_0 \in \overline{\Omega}$, $0 < R < R_0$, $\theta \in (0,1)$, we have

$$\begin{aligned}\left(\fint_{\Omega_{\theta R}(x_0)} |\nabla u - (\nabla u)_{\Omega_{\theta R}(x_0)}| \,\mathrm{d}x\right)^2 &\leq \fint_{\Omega_{\theta R}(x_0)} |\nabla u - (\nabla u)_{\Omega_{\theta R}(x_0)}|^2 \,\mathrm{d}x \\ &\leq C\theta^{2\alpha} \fint_{\Omega_R(x_0)} |\nabla u|^2 \,\mathrm{d}x + C\theta^{-n} \fint_{\Omega_R(x_0)} |F|^2 \,\mathrm{d}x.\end{aligned} \tag{3.89}$$

To extend this to $R > R_0$, we will use the rudimentary estimate

$$\begin{aligned}\fint_{\Omega_{\theta R}(x_0)} |\nabla u - (\nabla u)_{\Omega_R(x_0)}|^2 \,\mathrm{d}x &\leq C \frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\Omega_{\theta R}(x_0))} \fint_{\Omega} |\nabla u|^2 \,\mathrm{d}x \\ &\leq C\theta^{-n} \frac{\mathcal{L}^n(\Omega)}{R_0^n} \fint_{\Omega} |F|^2 \,\mathrm{d}x\end{aligned} \tag{3.90}$$

using the global L^2 estimates from Lemma 3.1.4. Hence combining the two cases we deduce that

$$\widetilde{\mathcal{M}}^{\#}_{\Omega}(\nabla u)(x)^2 \leq C\theta^{2\alpha} \widetilde{\mathcal{M}}_{\Omega}(|\nabla u|^2)(x) + C\theta^{-n} \widetilde{\mathcal{M}}^{\#}_{\Omega}(|F|^2), \tag{3.91}$$

as required. $\square$

Remark 3.3.7. An unfortunate technical point is that our estimates is not scale invariant, as the constant depends on $\mathcal{L}^n(\Omega)/R_0^n$. This appears to arise as a consequence of obtaining a global estimate, where we need to use the strong Gårding inequality (3.3) on the whole space Ω. For our applications we will either apply this estimate on balls, where the correct R-dependence can be deduced by a scaling argument, or on regularised versions $\widetilde{\Omega}_r(x_0)$ of $\Omega_r(x_0)$ constructed in Lemma 2.1.28. In the latter case, we observe that $\mathcal{L}^n(\widetilde{\Omega}_r(x_0)) \simeq r^n$, and $R_0(\widetilde{\Omega}_r(x_0)) \simeq r$ for r sufficiently small. This follows since each $\widetilde{\Omega}_r(x_0)$, up to a $C^{1,\alpha}$-diffeomorphism and rescaling is the same domain A, so

$$R_{\widetilde{\Omega}_r(x_0)} \simeq rR_A \quad \text{and} \quad \left\|\widetilde{\Omega}_r(x_0)\right\|_{C^{1,\alpha}} \lesssim 1 + \|\Omega\|_{C^{1,\alpha}}. \tag{3.92}$$

Here we are using the notation R_A from Definition 2.1.20. Hence this will be sufficient for our purposes, however we believe there should be a better way to state the dependence of constants on the domain.

Theorem 3.3.8 (Solvability in L^{φ})**.** Let $\varphi \in \Delta_2 \cap \nabla_2$ be an N-function and $\mathbb{A}$ be constant satisfying (3.2) as above. Then if $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\alpha}$ domain and δ-John, the problem

$$\begin{cases} -\operatorname{div} \mathbb{A}u = -\operatorname{div} F & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.93}$$

is uniquely solvable in $W_0^{1,\varphi}(\Omega,\mathbb{R}^N)$ for all $F \in L^\varphi(\Omega,\mathbb{R}^{Nn})$, and we have

$$\int_\Omega \varphi\left(|\nabla u|\right)\,\mathrm{d}x \leq C\int_\Omega \varphi(|F|)\,\mathrm{d}x \tag{3.94}$$

where $C = C(n,N,\alpha,\Lambda/\lambda,\Delta_2(\varphi),\nabla_2(\varphi),\delta,\|\Omega\|_{C^{1,\alpha}},\mathcal{L}^n(\Omega)/R_0^n) > 0$.

Proof. We first consider the case when $\varphi(t) = t^p$; note that if $p = 2$ this follows by Lemma 3.1.4. If $p \geq 2$, suppose we have $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ satisfying (3.93) for some $F \in L^p(\Omega,\mathbb{R}^N)$. Now taking L^p-norms of the pointwise estimate (3.84) from Proposition 3.3.6, we can estimate

$$\begin{aligned}\|\nabla u - (\nabla u)_\Omega\|_{L^p(\Omega)} &\leq C\left\|\widetilde{\mathcal{M}}^{\#}_{\Omega,\theta R}(\nabla u)\right\|_{L^p(\Omega)} \\ &\leq C\theta^\alpha\left\|\widetilde{\mathcal{M}}_{\Omega,R}\left(|\nabla u|^2\right)^{\frac12}\right\|^2_{L^{\frac p2}(\Omega)} + C\theta^{-\frac n2}\left\|\widetilde{\mathcal{M}}_{\Omega,R}\left(|F|^2\right)^{\frac12}\right\|^2_{L^{\frac p2}(\Omega)} \\ &\leq C\theta^\alpha\|\nabla u\|_{L^p(\Omega)} + C\theta^{-\frac n2}\|F\|_{L^p(\Omega)},\end{aligned} \tag{3.95}$$

where we have used the estimates (2.126), (2.127) from Proposition 2.2.13 in $L^{\frac p2}$ and L^p respectively. Note also that $(\nabla u)_\Omega = 0$ since u vanishes on the boundary. Hence choosing $\theta \in (0,1)$ so that $C\theta^\alpha \leq \frac12$, we obtain the a-priori estimate (3.94). To deduce solvability we employ an approximation argument; given $F \in L^p(\Omega)$, we choose a sequence $F_j \in C^\infty(\overline{\Omega})$ such that $F_j \to F$ in $L^p(\Omega)$. Then by Corollary 3.3.3 there exists corresponding $u_j \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$ which vanishes on the boundary and satisfies

$$\begin{cases} -\operatorname{div}\mathbb{A}\nabla u_j = -\operatorname{div} F_j & \text{in } \Omega,\\ \qquad\quad u_j = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.96}$$

Using (3.94) applied to $u_j - u_k$, we deduce that u_j convergence strongly to a limit map $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$, which coincides with the unique $W^{1,2}$ solution guaranteed by Lemma 3.1.4.

We now existence this to the case $1 < p < 2$, for which we argue by duality. By a similar reasoning it suffices to prove an a-priori estimate, so suppose $u \in W_0^{1,2}(\Omega,\mathbb{R}^N)$ is a weak solution with $F \in L^2(\Omega,\mathbb{R}^N)$. Then we let φ solve the dual problem

$$\begin{cases} -\operatorname{div}\mathbb{A}\nabla\varphi = |\nabla u|^{p-2}\nabla u & \text{in } \Omega,\\ \qquad\quad \varphi = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.97}$$

then since $p' \geq 2$ we have $\|\nabla\varphi\|_{L^{p'}(\Omega)} \leq C\|\nabla u\|^{p-1}_{L^p(\Omega)}$. Then testing the weak formulation for u with this φ, we obtain

$$\|\nabla u\|^p_{L^p(\Omega)} = \int_\Omega \mathbb{A}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega F\cdot\nabla\varphi\,\mathrm{d}x \leq C\|F\|_{L^p(\Omega)}\|\nabla u\|^{p-1}_{L^p(\Omega)}. \tag{3.98}$$

Hence rearranging we deduce the result when $1 < p < 2$.

We now extend this to the modular scales by interpolation, for which we consider the associated operator

$$T : F \mapsto \nabla u, \tag{3.99}$$

which maps $F \in L^p(\Omega, \mathbb{R}^{Nn})$ to the gradient ∇u of the unique solution in $W_0^{1,p}(\Omega, \mathbb{R}^N)$. Now the desired modular estimate follows by Corollary 2.2.2, and a similar approximation argument gives existence. Uniqueness follows by noting that $F \in L^\varphi(\Omega, \mathbb{R}^{Nn})$ implies $F \in L^p(\Omega, \mathbb{R}^{Nn})$ for some $1 < p < \infty$ by Lemma 2.3.3, so we can appeal to uniqueness in $W_0^{1,p}(\Omega, \mathbb{R}^N)$. □

By using the L^p estimates obtained in Theorem 3.3.8, we can repeat the argument by replacing the L^2 scales with L^p to obtain further estimates. This was noted by Dong & Kim [DK18], and we will briefly outline the procedure.

Proposition 3.3.9 (Krylov-Stampacchia II)**.** Let $\mathbb{A}$ be a matrix satisfying (3.2) and (3.3) with coefficients in BMO, $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,\alpha}$ domain, and $1 < q < \infty$. Then for all $F \in L^q(\Omega, \mathbb{R}^{Nn})$, the unique solution $u \in W_0^{1,q}(\Omega, \mathbb{R}^N)$ to the elliptic system

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.100}$$

satisfies the pointwise estimate

$$\begin{aligned} \widetilde{\mathcal{M}}^{\#}_{\Omega}(\nabla u)(x) \leq C\theta^{\alpha} \widetilde{\mathcal{M}}_{\Omega}\left(|\nabla u|^q\right)(x)^{\frac{1}{q}} + C\theta^{-\frac{n}{q}} \widetilde{\mathcal{M}}_{\Omega}\left(|F|^q\right)(x)^{\frac{1}{q}} \\ + C\theta^{-\frac{n}{q}} \{\mathbb{A}\}_{\mathrm{osc}_R(\Omega)} \widetilde{\mathcal{M}}_{\Omega}\left(|\nabla u|^{\mu q}\right)(x)^{\frac{1}{\mu q}} \end{aligned} \tag{3.101}$$

for all $x \in \overline{\Omega},$, $\mu > 1$, $\theta \in (0,1)$, where we define

$$\{\mathbb{A}\}_{\mathrm{osc}_R(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<r<R}} \fint_{\Omega_r(x)} |\mathbb{A} - (\mathbb{A})_{B_r(x)}| \, \mathrm{d}x \tag{3.102}$$

and $C = C(n, N, q, \mu, \alpha, \Lambda/\lambda, \|\Omega\|_{C^{1,\alpha}}, \mathcal{L}^n(\Omega)/R_0^n) > 0$.

The proof will require a two-step argument where we extend Theorem 3.3.8 to hold for variable-coefficient systems (in particular in L^q), after which we treat the case of general q. However for brevity we have included both steps in the same proof.

Proof. For $x_0 \in \partial\Omega$, let $\widetilde{\Omega}_R(x_0)$ be as in Lemma 2.1.28, shrinking R_0 as necessary. Then we put $\mathbb{A}_0 = (\mathbb{A})_{\widetilde{\Omega}_R(x_0)}$ and let v be the unique solution to the elliptic system

$$\begin{cases} -\operatorname{div} \mathbb{A}_0 \nabla v = 0 & \text{in } \widetilde{\Omega}_R(x_0), \\ v = u & \text{on } \partial\widetilde{\Omega}_R(x_0), \end{cases} \tag{3.103}$$

which exists by Theorem 3.3.8. Moreover putting $w = u - v$ we have have the associated estimates

$$\fint_{\widetilde{\Omega}_R(x_0)} |\nabla v - (\nabla v)_{\widetilde{\Omega}_R(x_0)}|^q \,\mathrm{d}x \leq C \fint_{\widetilde{\Omega}_R(x_0)} |\nabla u|^q \,\mathrm{d}x, \tag{3.104}$$

$$\fint_{\widetilde{\Omega}_R(x_0)} |\nabla w - (\nabla w)_{\widetilde{\Omega}_R(x_0)}|^q \,\mathrm{d}x \leq C \fint_{\widetilde{\Omega}_R(x_0)} |\mathbb{A} - \mathbb{A}_0|^q |\nabla u|^q + |F|^q \,\mathrm{d}x. \tag{3.105}$$

Applying the $C^{1,\alpha}$-estimates from Proposition 3.3.4, we have

$$\fint_{\Omega_{\theta R}(x_0)} |v - (\nabla v)_{\Omega_{\theta R}(x_0)}|^q \,\mathrm{d}x \leq C\theta^{q\alpha} \fint_{\Omega_R(x_0)} |v - (\nabla v)_{\Omega_R(x_0)}|^q \,\mathrm{d}x \tag{3.106}$$

for all $\theta \in (0,1)$. Also note by Hölder with $\mu > 1$ and the John-Nirenberg inequality (Proposition 2.2.15) that we have

$$\fint_{\widetilde{\Omega}_R(x_0)} |\mathbb{A} - \mathbb{A}_0| |\nabla u| \,\mathrm{d}x \leq C \,[\mathbb{A}]^q_{\mathrm{BMO}(\widetilde{\Omega}_R(x_0))} \left(\fint_{\widetilde{\Omega}_R(x_0)} |\nabla u|^{\mu q} \right)^{\frac{1}{\mu}}. \tag{3.107}$$

Hence combining the above estimates we deduce that

$$\begin{aligned} \left(\fint_{\Omega_{\theta R}(x_0)} |\nabla u - (\nabla u)_{\Omega_{\theta R}(x_0)}| \,\mathrm{d}x \right)^q &\leq C\theta^{2\alpha} \fint_{\Omega_R(x_0)} |\nabla u|^q \,\mathrm{d}x + C\theta^{-n} \fint_{\Omega_R(x_0)} |F|^q \,\mathrm{d}x \\ &\quad + C\theta^{-n} \,[\mathbb{A}]^q_{\mathrm{BMO}(\widetilde{\Omega}_R(x_0))} \left(\fint_{\widetilde{\Omega}_R(x_0)} |\nabla u|^{\mu q} \right)^{\frac{1}{\mu}}, \end{aligned} \tag{3.108}$$

Now for $R \geq R_0$, if $q = 2$ then we can use the strong Gårding inequality to conclude. Using this, the argument in Theorem 3.3.8 implies the corresponding L^q estimate

$$\int_\Omega |\nabla u|^q \,\mathrm{d}x \leq C \int_\Omega |F|^q \,\mathrm{d}x, \tag{3.109}$$

from which the case of general q follows. □

While we will not need the following result, we will include it to illustrate how this approach can be used to deduce refined solvability results. Recall the class A_p of Muckenhoupt weights was defined in 2.2.6. Results of this type, and more general results can be found in [DK18] and related references.

Theorem 3.3.10 (Further solvability results)**.** Let $\mathbb{A}$ be a matrix satisfying (3.2) and (3.3) with coefficients in BMO, and $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,\alpha}$ domain. Then the following solvability results hold for the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = -\operatorname{div} F & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.110}$$

(a) If $1 < p < \infty$ and $w \in A_p$, there is $\kappa > 0$ such that if

$$\{\mathbb{A}\}_{\operatorname{osc}_R(\Omega)} \leq \kappa, \tag{3.111}$$

then for all $F \in L^p_w(\Omega, \mathbb{R}^{Nn})$ there is a unique solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ to the Dirichlet problem such that $\nabla u \in L^p_w(\Omega, \mathbb{R}^{Nn})$, and we have the associated estimate

$$\int_\Omega |\nabla u|^p w \,\mathrm{d}x \leq C \int_\Omega |F|^p w \,\mathrm{d}x, \tag{3.112}$$

where $C = C(n, N, p, \Lambda/\lambda, [w]_{A_p}, \delta, \|\Omega\|_{C^{1,\alpha}}, \mathcal{L}^n(\Omega)/R_0^n) > 0$, where we assume Ω is a δ-John domain.

(b) If $\varphi \in \Delta_2 \cap \nabla_2$, there is $\kappa > 0$ for which if (3.111) holds, then for all $F \in L^\varphi(\Omega, \mathbb{R}^{Nn})$ there is a unique solution $u \in W^{1,\varphi}_0(\Omega, \mathbb{R}^N)$ to the Dirichlet problem satisfying the estimate

$$\int_\Omega \varphi(|\nabla u|) \,\mathrm{d}x \leq C \int_\Omega \varphi(|F|) \,\mathrm{d}x, \tag{3.113}$$

where $C = C(n, N, p, \Lambda/\lambda, \Delta_2(\varphi), \nabla_2(\varphi), \delta, \|\Omega\|_{C^{1,\alpha}}, \mathcal{L}^n(\Omega)/R_0^n) > 0$.

The strategy is to argue similarly as in Theorem 3.3.8, where we use the pointwise estimate (3.101) from Proposition 3.3.9 instead. We use several results concerning A_p weights from [Ste93] for the first assertion.

Sketch of proof. In both cases it suffices to establish an a-priori estimate assuming u is sufficiently regular, after which we can argue as in Theorem 3.3.8. If $w \in A_p$ with $1 < p < \infty$, then by [Ste93, Proposition V.3] there is $\varepsilon > 0$ depending on $n, [w]_{A_p}$ such that $w \in A_{p-\varepsilon}$. We then choose $q, \mu > 1$ so that $\mu q = \frac{p}{p-\varepsilon} > 1$, so that $w \in A_{p/(\mu q)} \subset A_{p/q}$ by monotonicity of A_p spaces (see [Ste93, Section V.1]). This ensures by [Ste93, Theorem V.1] that the maximal operator is bounded on $L_w^{\frac{p}{q}}$ and $L_w^{\frac{p}{\mu q}}$ in the full space. We have also seen in the proof of Theorem 2.2.10 that the Fefferman-Stein inequality extends to the weighted scales in the form (2.119). Now a similar argument as in Proposition 2.2.13 allows us to apply these results to the localised operators also (noting that the A_p–seminorm is invariant under the scaling $w \mapsto \lambda w$). Therefore taking L^p_w–norms of (3.101) gives

$$\|\nabla u\|_{L^p_w(\Omega)} \lesssim (\theta^\alpha + \kappa) \|\nabla u\|_{L^p_w(\Omega)} + \|F\|_{L^p_w(\Omega)}, \tag{3.114}$$

so choosing $\theta, \kappa > 0$ to be sufficiently small we can absorb this term to the left-hand side. This establishes (a), and (b) now follows by taking $w \equiv 1$ and interpolating between the L^p estimates (using Theorem 2.2.1). □

Discussion 3.3.11 (Further results for linear elliptic systems)**.** The results of Theorem 3.3.10 are not optimal, and one can say much more. One of the key results in the work of Krylov [Kry07] is that one can further relax the assumptions on the coefficient matrix $\mathbb{A}$, and assume that $\mathbb{A}$ is merely measurable in one direction and satisfy a BMO-smallness condition in the others. In addition one can also relax the $C^{1,\alpha}$ regularity assumption of the boundary, which would follow by a suitable flattening argument.

3.3.3 Representation as a divergence

Often we will consider solvability for the problem

$$-\operatorname{div} \mathbb{A}\nabla u = f, \tag{3.115}$$

where f lines in L^φ. While we do expect solvability in $W^{2,\varphi}$, this requires stringent regularity assumptions on the boundary, and we only wish to infer a solution u exists in $W^{1,\psi}$ for a suitable N-function ψ. To circumvent this, we can seek a suitable vector field F such that $-\operatorname{div} F = f$ in Ω. Abstractly we are looking to establish an embedding of the form $L^\varphi \hookrightarrow W^{-1,\psi}$, where the negative order space is suitably defined. However we have opted to take a more concrete approach in this text, which will allow us to carefully keep track of the constants involved.

We will start with a standard result concerning the Newtonian potentials.

Proposition 3.3.12. Let $\Omega \subset \mathbb{R}^n$ be a bounded δ-John domain, and $\varphi \in \nabla_2$ be an N-function. Then if $f \in L^p(\Omega, \mathbb{R}^N)$ with $1 < p < \infty$, there exists $F \in W^{1,p}(\Omega, \mathbb{R}^{Nn})$ such that

$$- \operatorname{div} F = f \text{ in } \Omega. \tag{3.116}$$

Moreover $F \in L^q(\Omega, \mathbb{R}^{Nn})$ for $1 \leq q \leq \frac{np}{n-p}$ if $p < n$, and is unconstrained otherwise with the associated estimate

$$\left(\fint_\Omega |F|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq CR \left(\fint_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{3.117}$$

where $R = \operatorname{diam}(\Omega)$. Moreover if $p > n$, then for all $\alpha \in \left(0, 1 - \frac{1}{p}\right)$ we have

$$R^{-1} \|F\|_{L^\infty(\Omega)} + R^{1-\alpha} [F]_{C^{0,\alpha}(\Omega)} \leq C \left(\fint_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{3.118}$$

Proof. By rescaling we can assume that $\Omega \subset B_1$, and prove the statement for $R = 1$, noting that $\mathcal{L}^n(\Omega) \leq CR^n$. We consider the Newtonian potential (see for instance [GT98, Section 7.8]) defined as

$$F(x) = \frac{-1}{n\omega_n} \int_\Omega \frac{f(y) \otimes (x-y)}{|x-y|^n} \, \mathrm{d}y, \tag{3.119}$$

which satisfies $- \operatorname{div} F = f\chi_\Omega$ by [GT98, Lemma 4.1, Lemma 7.14]. Therefore we can bound F in terms of the *Riesz potential*

$$|F(x)| \leq CI_1(|f|) = C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \tag{3.120}$$

for a.e. $x \in \Omega$, and we also have

$$\begin{aligned} |F(x_1) - F(x_2)| &\leq C \int_\Omega |f(y)| \left| \frac{1}{|x_1 - y|^{n-1}} - \frac{1}{|x_2 - y|^{n-1}} \right| \mathrm{d}y \\ &\leq C|x_1 - x_2|^\alpha \int_\Omega |f(y)| \left(\frac{1}{|x_1 - y|^{n-1-\alpha}} + \frac{1}{|x_2 - y|^{n-1-\alpha}} \right) \mathrm{d}y. \end{aligned} \tag{3.121}$$

Hence when $p > n$, we can choose $\alpha \in \left(0, 1 - \frac{1}{p}\right)$ and apply Hölder to estimate By translating we can assume $\Omega \subset B_{2R}$, so if $p > n$ we can choose $\alpha \in (0,1)$ so that $p > n + \alpha$ use Hölder to (3.120) to estimate

$$|F(x)| \leq \left(\int_{B_2} |y|^{-(n-1)p'} \, \mathrm{d}y \right)^{\frac{p-1}{p}} \left(\int_\Omega |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \leq C_{n,p} \left(\int_\Omega |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}, \tag{3.122}$$

and similarly for (3.121) to establish the $C^{0,\alpha}$ estimates. On the other hand if $1 \leq p < n$, it is shown in [Ste71, Section V.1.3] that the mapping $f \mapsto F$ is of type weak(p,q) where $q = \frac{np}{n-p}$. Hence using a variant of Marcinkiewicz interpolation proven in [Ste71, Appendix B] the result follows. $\square$

Now we obtain the following non-optimal solvability result.

Proposition 3.3.13. Let $\varphi \in \nabla_2$ be an N-function and $\mathbb{A}$ be constant satisfying (3.2) as above. Then if $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\alpha}$ domain and δ-John, the problem

$$\begin{cases} -\operatorname{div} \mathbb{A}\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.123}$$

is uniquely solvable in $W_0^{1,\varphi^r}(\Omega, \mathbb{R}^N)$ with $r = \frac{n}{n-1}$ for each $f \in L^\varphi(\Omega, \mathbb{R}^N)$, and we have

$$\left(\fint_\Omega \varphi \left(R \left| \nabla u \right| \right)^r \,\mathrm{d}x \right)^{\frac{1}{r}} \leq C \fint_\Omega \varphi(C|f|) \,\mathrm{d}x, \tag{3.124}$$

where $C = C(n, N, \alpha, \Lambda/\lambda, \Delta_2(\varphi), \nabla_2(\varphi), \delta, \|\Omega\|_{C^{1,\alpha}}) > 0$ and $R = \operatorname{diam}(\Omega)$. Moreover if $\varphi(t) = t^p$, then we can take $r = \frac{n}{n-p}$.

Proof. Suppose $\varphi(t) = t^p$ with $1 < p < n$, then by Proposition 3.3.12 where $F \in L^p(\Omega, \mathbb{R}^{Nn})$ such that $-\operatorname{div} F = f$ in Ω, so we can apply Theorem 3.3.8 to obtain a unique solution $u \in W_0^{1,pr}(\Omega, \mathbb{R}^N)$ with $r = \frac{n}{n-p}$, along with the associated estimate

$$\left(\fint_\Omega |R\nabla u|^{pr} \right)^{\frac{1}{r}} \leq C \left(\fint_\Omega |RF|^{pr} \,\mathrm{d}x \right)^{\frac{1}{r}} \leq C \fint_\Omega |f|^p \,\mathrm{d}x. \tag{3.125}$$

If $p > n$, then we know from Proposition 3.3.12 that $F \in C^{0,\alpha}(\Omega, \mathbb{R}^{Nn})$ for some $\alpha > 0$, so by Schauder estimates (Proposition 3.3.1) we have $u \in C^{1,\alpha}(\Omega, \mathbb{R}^N)$ and hence we obtain the estimate

$$\|R\nabla u\|_{L^\infty(\Omega)}^p \leq C \fint_\Omega |f|^p \,\mathrm{d}x. \tag{3.126}$$

Hence by Hölder the mapping $f \mapsto \nabla u$ is bounded from $L^p \to L^{\frac{np}{n-1}}$ for all $p > 1$, and from $L^p \to L^\infty$ for all $p > n$. Now the assertion follows by applying the endpoint interpolation result (Lemma 2.2.3), where the R-dependence is obtained by scaling at the end. $\square$

Chapter 4

BMO ε-regularity results for Legendre-Hadamard elliptic systems

The results in this section are taken from the preprint [Irv21a] of the author. We point out that Theorem 4.1.9 improves the results in the preprint, which follows from mild modification to the arguments in the original paper.

We will study general weak solutions to systems which are elliptic in the sense of Legendre-Hadamard, focusing on the case of the autonomous Euler-Lagrange system

$$-\operatorname{div} F'(\nabla u) = 0 \tag{4.1}$$

in Ω. For arbitrary weak solutions of the above equation however, the work of MÜLLER & ŠVERÁK [MŠ03] shows that we cannot hope for improved regularity results. Developing the theory of convex integration for Lipschitz mappings they constructed highly irregular solutions to (4.1), including Lipschitz solutions that fail to be C^1 in any open subset and compactly supported solutions whose gradient is L^q-integrable if and only if $q \leq 2$. These results have been extended by KRISTENSEN & TAHERI [KT03] for weak local minimisers, and by SZÉKELYHIDI [Szé04] for strongly polyconvex integrands.

We will show that we can infer improved regularity if we additionally assume the gradient is locally small in BMO. This builds upon the work of MOSER [Mos01] who considered the same problem under more stringent assumptions.

4.1 Statement of results

4.1.1 Hypotheses and local results

We will study the following class of integrands.

Hypotheses 4.1.1. For $n \geq 2$ and $N \geq 1$, let $F\colon \mathbb{R}^{Nn} \to \mathbb{R}$ satisfy the following.

(H0) F is of class C^2.

(H1) There is $q \geq 2$ such that F satisfies the natural growth condition

$$|F(z)| \leq \Lambda(1+|z|)^q$$

for all $z \in \mathbb{R}^{Nn}$.

(H2) F'' satisfies a strict Legendre-Hadamard condition, namely for all $z \in \mathbb{R}^{Nn}$ we have

$$F''(z)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq 0$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$, with equality if and only if $\xi \otimes \eta = 0$.

A key feature of our results is that we only need to assume a strict Legendre-Hadamard condition, as opposed to a quasiconvexity condition which is a stronger notion as illustrated by the example of Šverák [Šve92]. We also highlight that we do not require control in the L^q scales from below, so if F satisfies the above hypotheses for some $1 \leq q < 2$ instead, it will hold with $q = 2$.

Theorem 4.1.2 (BMO ε-regularity theorem)**.** Suppose F satisfies Hypotheses 4.1.1. Then for all $M > 0$ and $\alpha \in (0,1)$, there is $\varepsilon > 0$ such that for any ball $B_R(x_0) \subset \mathbb{R}^n$ if u is F-extremal in $B_R(x_0)$ with $|(\nabla u)_{B_R(x_0)}| \leq M$ and

$$[\nabla u]_{\mathrm{BMO}(B_R(x_0))} \leq \varepsilon, \tag{4.2}$$

we have u is $C^{1,\alpha}$ on $\overline{B}_{R/2}(x_0)$.

We will also establish an analogous result up to the boundary, using ideas from Kronz [Kro05] and Campos Cordero [Cam17]. Recall that $\Omega_R(x_0) = \Omega \cap B_R(x_0)$.

Theorem 4.1.3 (Boundary BMO ε-regularity theorem)**.** Suppose F satisfies Hypotheses 4.1.1, $\Omega \subset \mathbb{R}^n$ is a $C^{1,\beta}$ domain for some $\beta \in (0,1)$, $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$, and let $\alpha \in (0,\beta)$, $M > 0$. Then there is $\varepsilon > 0$ and $R_0 > 0$ such that if $u \in W^{1,q}_g(\Omega, \mathbb{R}^N)$ is F-extremal. Then for $x_0 \in \partial\Omega$ and $0 < R < \widetilde{R}_0$ with $|(\nabla u)_{\Omega_R(x_0)}| \leq M$ and

$$[\nabla u]_{\mathrm{BMO}(\Omega_R(x_0))} \leq \varepsilon, \tag{4.3}$$

we have u is $C^{1,\alpha}$ on $\overline{\Omega}_{R/2}(x_0)$.

Discussion 4.1.4. In the context of strictly quasiconvex integrands, there is a close connection between sufficiency results (whether extremals are minimising) and regularity of the extremal. One of the early results in the quasiconvex setting is due to Zhang [Zha92], who showed that a C^2 extremal is absolutely minimising on small balls $B \subset \Omega$. In the opposite direction, it was shown by Kristensen & Taheri in [KT03, Theorem 4.1] that if $u \in W^{1,p} \cap W^{1,q}_{\mathrm{loc}}$ is a strong $W^{1,q}$-local minimiser for some $1 \leq q \leq \infty$, then we can establish a partial regularity theorem (we refer the reader to the aforementioned paper for the precise terminology and results).

It was moreover established in [KT03, Theorems 6.1, 7.1] that if u is a Lipschitz extremal with strictly positive second variation (it is a *weak local minimiser*) then it is minimising among perturbations such that $[\nabla\varphi]_{\mathrm{BMO}(\Omega)}$ is small, but that this is too weak to infer improved regularity though counterexamples. The former statement uses a modular version of the Fefferman-Stein inequality which we also use (see Section 2.2.3), and the latter follows by adapting the construction of Müller & Šverák [MŠ03]. For Lipschitz weak local minimisers however, it was shown by Campos Cordero [Cam17] that we can infer global regularity if we additionally assume that $\nabla u \in \mathrm{VMO}(\Omega)$. Their proof loosely follows the compensated compactness argument used in [KT03, Section 4].

We see that Corollary 4.1.5 below generalises the above result in [Cam17], by removing the condition on the second variation and allowing F to merely satisfy a strict Legendre-Hadamard condition (H2). Here the Legendre-Hadamard condition can be seen to be a natural relaxation in the following sense; it is proved by Kristensen [Kri99] that (H2) implies that F is *locally quasiconvex* in the sense that for each $z_0 \in \mathbb{R}^{Nn}$ there exists a quasiconvex function G such that $F = G$ in a neighbourhood of z_0. Our argument, which builds upon ideas of Moser [Mos01], streamlines this process by establishing regularity directly. In particular we note that the same Fefferman-Stein estimate used for the BMO-sufficiency result in [KT03] is crucially used to obtain a Caccioppoli-type inequality in [Mos01] and Section 4.2.2.

We add that in [Irv21a, Section 5] we extend these results to non-autonomous elliptic systems and higher-order integrands, which essentially follow by the same arguments developed here.

4.1.2 Consequences of the main results

Patching the local regularity results from the previous section, we can infer global consequences. For this we recall the infinitesimal mean oscillation $\{\cdot\}_{\operatorname{osc}(\Omega)}$ from Section 2.2.3.

Corollary 4.1.5 (Regularity of almost VMO Lipschitz solutions). Suppose F satisfies Hypotheses 4.1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0,1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then for each $M > 0$ and $\alpha \in (0,\beta)$, there is $\varepsilon > 0$ such that if $u \in W^{1,\infty}_g(\Omega, \mathbb{R}^N)$ is F-extremal such that $\|\nabla u\|_{L^\infty(\Omega)} \leq M$, and

$$\{\nabla u\}_{\operatorname{osc}(\Omega)} \leq \varepsilon, \tag{4.4}$$

then $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

It is unclear if the Lipschitz assumption can be removed; the infinitesimal mean oscillation assumption requires us to consider balls of arbitrarily small radius, which in turn requires a uniform bound on all averages $|(\nabla u)_{\Omega_R(x_0)}|$ for all $x_0 \in \overline{\Omega}$ and $R > 0$ small. However this is equivalent to assuming ∇u is bounded by the Lebesgue differentiation theorem, and in general we can only infer the following partial regularity result.

Corollary 4.1.6 (Partial regularity of VMO solutions). Suppose F satisfies Hypotheses 4.1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0,1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then if $u \in W^{1,q}_g(\Omega, \mathbb{R}^N)$ is F-extremal such that $\nabla u \in \mathrm{VMO}(\Omega; \mathbb{R}^{Nn})$, letting

$$\mathcal{R}_{\overline{\Omega}} = \left\{ x \in \overline{\Omega} : \limsup_{R \to 0} |(\nabla u)_{\Omega_R(x_0)}| < \infty \right\}, \tag{4.5}$$

we have $\mathcal{R}_{\overline{\Omega}} \subset \overline{\Omega}$ is a relatively open subset of full measure and u is $C^{1,\alpha}$ on $\mathcal{R}_{\overline{\Omega}}$ for all $\alpha \in (0,\beta)$.

We can also obtain a global regularity result if we assume ∇u is suitably small in both L^1 and BMO. The L^1 smallness condition allows us to cover $\overline{\Omega}$ by balls finitely many balls $B_{R_k}(x_k)$ such that each $|(\nabla u)_{\Omega_{R_k}(x_k)}| \leq 1 + [\nabla g]_{L^\infty(\Omega)}$, on which we can apply our ε-regularity result to obtain the following.

Corollary 4.1.7 (Regularity of BMO-small solutions)**.** Suppose F satisfies Hypotheses 4.1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0,1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then for each $\alpha \in (0, \beta)$ there is $\varepsilon > 0$ such that if $u \in W^{1,q}_g(\Omega, \mathbb{R}^N)$ is F-extremal in Ω with $\nabla u \in \mathrm{BMO}(\Omega, \mathbb{R}^{Nn})$ satisfying

$$\|\nabla u - \nabla g\|_{L^1(\Omega)} + [\nabla u]_{\mathrm{BMO}(\Omega)} \leq \varepsilon, \tag{4.6}$$

then $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

4.1.3 Controlled growth conditions

As noted in the previous section, the ε-regularity results we established do not appear to be strong enough to establish regularity if we merely assume $\nabla u \in \mathrm{VMO}(\Omega, \mathbb{R}^{Nn})$. The issue is the dependence of ε on M, and appears to be caused by a lack of uniform ellipticity in our assumption. It turns out this can be avoided if we assume strong controlled conditions on F'', namely the following.

Hypotheses 4.1.8. For $n \geq 2$, $N \geq 1$ and $p \geq 2$, let $F \colon \mathbb{R}^{Nn} \to \mathbb{R}$ satisfy the following.

($\tilde{\mathrm{H}}0$) F is of class C^2.

($\tilde{\mathrm{H}}1$) We have $(1 + |z|^2)^{-(p-2)} F''(z)$ is bounded and uniformly continuous on $\mathbb{R}^{Nn}$.

($\tilde{\mathrm{H}}2$) For all $z \in \mathbb{R}^{Nn}$ we have

$$F''(z)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq \lambda (1 + |z|)^{p-2} |\xi|^2 |\eta|^2$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$.

Note the the growth and continuity hypothesis ($\tilde{\mathrm{H}}1$) is satisfied if F is a polynomial; in particular this includes the example of Šverák [Šve92] (suitably pertubed to satisfy ($\tilde{\mathrm{H}}2$)).

Theorem 4.1.9 (BMO ε-regularity in the uniformly elliptic case)**.** Suppose $F\colon \mathbb{R}^{Nn} \to \mathbb{R}$ satisfies Hypotheses 4.1.8, let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,\beta}$ domain, and let $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$ for some $\beta \in (0,1)$. Then for each $\alpha \in (0,\beta)$ there is $\varepsilon > 0$ such that if $u \in W^{1,p}_g(\Omega;\mathbb{R}^N)$ is F-extremal in Ω and

$$\{\nabla u\}_{\mathrm{osc}(\Omega)} \leq \varepsilon, \tag{4.7}$$

then u is $C^{1,\alpha}$ in $\overline{\Omega}$.

4.2 The interior case

4.2.1 Estimates for the integrand

Given $F\colon \mathbb{R}^{Nn} \to \mathbb{R}$, for $w \in \mathbb{R}^{Nn}$ we introduce the *shifted integrand*

$$F_w(z) = F(z+w) - F(w) - F'(w)z, \tag{4.8}$$

following Acerbi & Fusco [AF87]. This satisfies the following properties.

Lemma 4.2.1. Suppose F satisfies Hypotheses 4.1.1, and fix $M > 0$. Then there is $0 < \lambda_M < \Lambda_M < \infty$ such that for any $w \in \mathbb{R}^{Nn}$ with $|w| \leq M$, we have (4.8) satisfies

$$|F_w(z)| \leq \Lambda_M(|z|^2 + |z|^q), \tag{4.9}$$

$$|F_w'(z)| \leq \Lambda_M(|z| + |z|^{q-1}), \tag{4.10}$$

for all $z \in \mathbb{R}^{Nn}$. We also have

$$F_w''(0)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq \lambda_M |\xi|^2 |\eta|^2 \tag{4.11}$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$. Moreover there exists a non-negative non-decreasing continuous and concave function $\omega_M\colon [0,\infty) \to [0,1]$ such that $\omega_M(0) = 0$ and we have

$$|F_w''(0)z - F_w'(z)| \leq \Lambda_M\, \omega_M(|z|)(|z| + |z|^{q-1}). \tag{4.12}$$

Proof. Since $F''(z)$ is uniformly continuous on compact subsets, there is $K_M > 0$ and a modulus of continuity function $\omega_M\colon [0,\infty) \to [0,1]$ as in the statement of the lemma such that $|F''(z)| \leq K_M$ and

$$|F''(z) - F''(w)| \leq K_M \omega_M(|z-w|) \tag{4.13}$$

for all $z, w \in \mathbb{R}^{Nn}$ with $|z|, |w| \leq M+1$. Since F'' satisfies a Legendre-Hadamard condition, F is rank-one convex and hence we have $|F'(z)| \leq C\Lambda(1+|z|)^{q-1}$. Therefore by distinguishing between the cases when $|z| \leq 1$ and $|z| > 1$, we have (4.9), (4.10), (4.12) follow. Finally (4.11) follows from the fact that the strict Legendre-Hadamard condition holds uniformly on compact subsets of $\mathbb{R}^{Nn}$. □

Under the stronger hypotheses, we obtain better control on the associated constants. While the following estimate is somewhat rudimentary, it will suffice for our purposes.

Lemma 4.2.2. Suppose F satisfies Hypotheses 4.1.8 for some $p \geq 2$. Then there is $0 < \lambda < \Lambda < \infty$ such that for any $w \in \mathbb{R}^{Nn}$ with $|w| \leq M$, we have (4.8) satisfies

$$|F_w(z)| \leq \Lambda(1+|w|)^{p-2}(|z|^2 + |z|^p), \tag{4.14}$$

$$|F_w'(z)| \leq \Lambda(1+|w|)^{p-2}(|z| + |z|^{p-1}), \tag{4.15}$$

and

$$|F_w''(0)z - F_w'(z)| \leq \Lambda(1+|w|)^{p-2}\omega_M(|z|)(|z| + |z|^{p-1}). \tag{4.16}$$

for all $z, w \in \mathbb{R}^{Nn}$, with ω a modulus of continuity as in the statement of Lemma 4.2.1. We also have

$$F_w''(0)(\xi \otimes \eta) \cdot (\xi \otimes \eta) \geq \lambda(1+|w|)^{p-2}|\xi|^2|\eta|^2 \tag{4.17}$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$.

Proof. Note that (4.17) follows immediately from ($\tilde{\text{H}}$2). For the remaining inequalities we observe ($\tilde{\text{H}}$1) implies there is $K > 0$ and ω such that $|F''(z)| \leq K(1+|z|)^{p-2}$ and

$$|F''(z) - F''(w)| \leq K\omega(|z|)(1+|z|+|w|)^{p-2} \tag{4.18}$$

for all $z, w \in \mathbb{R}^{Nn}$. Using these, the result follows by distinguishing between the cases when $|z| \leq |w|$ and $|z| > |w|$. □

4.2.2 A Caccioppoli-type inequality

We now prove the following weakening of the *Caccioppoli inequality of the second kind* introduced by Evans in [Eva86], which is a staple for many partial regularity proofs in the quasiconvex setting. The following estimate was essentially proved by Moser in [Mos01], and involves applying the modular version of the estimate of Fefferman & Stein [FS72] established in Section 2.2.3.

Lemma 4.2.3 (Caccioppoli-type inequality)**.** Suppose F satisfies Hypotheses 4.1.1, and let $M \geq 1$. Then if u is F-extremal in some ball $B_R(x_0) \subset \Omega$ such that $\nabla u \in \mathrm{BMO}(B_R, \mathbb{R}^{Nn})$ with $[\nabla u]_{\mathrm{BMO}(B_R)} \leq 1$ and $|(\nabla u)_{B_R}| \leq M$, then setting

$$a_R(x) = (u)_{B_R} + \left(\frac{n+2}{R^2} \fint_{B_R} u(y) \otimes (y - x_0)\,\mathrm{d}y\right) \cdot (x - x_0), \tag{4.19}$$

there is $\widetilde{M} = C(n)M$ and $C = C(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}) > 0$ such that

$$\fint_{B_{R/2}} |\nabla u - \nabla a_R|^2\,\mathrm{d}x \leq C\,\gamma\left([\nabla u]_{\mathrm{BMO}(B_R)}\right) \fint_{B_R} |\nabla u - \nabla a_R|^2\,\mathrm{d}x + \frac{C}{R^2}\fint_{B_R} |u - a_R|^2\,\mathrm{d}x, \tag{4.20}$$

with $\gamma(t) \colon [0, \infty) \to [0, 1]$ a non-decreasing, continuous function such that $\gamma(0) = 0$, depending on $\omega_{\widetilde{M}}$ and q only.

This choice of a_R is due to Kronz [Kro02], whose significance is illustrated in the lemma below; this is essentially contained in Lemma [Kro02, Lemma 2(ii)], applying the Poincaré inequality in $W^{1,1}$ instead.

Lemma 4.2.4. If $u \in W^{1,2}(B_R, \mathbb{R}^N)$, we have a_R defined as in (4.19) is the unique minimiser to

$$a \mapsto \fint_{B_R} |u - a|^2\,\mathrm{d}x \tag{4.21}$$

amoung affine maps $a \colon \mathbb{R}^n \to \mathbb{R}^N$. Further we have the estimate

$$|\nabla a_R - (\nabla u)_{B_R}| \leq C(n) \fint_{B_R} |\nabla u - (\nabla u)_{B_R(x_0)}|\,\mathrm{d}x. \tag{4.22}$$

In particular if $\nabla u \in \mathrm{BMO}(B_R)$ we have $|\nabla a_R - (\nabla u)_{B_R}| \leq C(n)\,[\nabla u]_{\mathrm{BMO}(B_R)}$.

Proof of Lemma 4.2.3. Set $\widetilde{F}(z) = F_{\nabla a_R}(z)$ as in (4.8), and note by Lemma 4.2.4 that

$$|\nabla a_R| \leq |(\nabla u)_{B_R}| + C(n)\,[\nabla u]_{\mathrm{BMO}(B_R)} \leq \widetilde{M}. \tag{4.23}$$

Also fix a cutoff $\eta \in C_c^\infty(B_R)$ such that $1_{B_{R/2}} \leq \eta \leq 1_{B_R}$ and $|\nabla \eta| \leq \frac{C}{R}$. Putting $\tilde{u} = u - a_R$ we have $\tilde{u}$ is $\widetilde{F}$-extremal since u is F-extremal, and so testing the equation against $\phi = \eta^2 \tilde{u}$

gives

$$0 = \fint_{B_R} \widetilde{F}'(\nabla\tilde{u}) \cdot \nabla(\eta^2\tilde{u})\,\mathrm{d}x. \tag{4.24}$$

Also since $\widetilde{F}''(0) = F''(\nabla a)$ satisfies the strict Legendre-Hadamard condition (4.11) with $|\nabla a| \leq \widetilde{M}$, applying this to $\eta\tilde{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ gives (see for instance [Giu03, Theorem 10.1]),

$$\lambda_{\widetilde{M}} \fint_{B_R} |\nabla(\eta\tilde{u})|^2\,\mathrm{d}x \leq \fint_{B_R} \widetilde{F}''(0)\nabla(\eta\tilde{u}) \cdot \nabla(\eta\tilde{u})\,\mathrm{d}x. \tag{4.25}$$

Taking the difference of (4.24), (4.25) and rearranging we get

$$\begin{aligned}
&\lambda_{\widetilde{M}} \fint_{B_R} |\nabla(\eta\tilde{u})|^2\,\mathrm{d}x \\
&\quad\leq \fint_{B_R} \eta^2 \left(\widetilde{F}''(0)\nabla\tilde{u} - \widetilde{F}'(\nabla\tilde{u})\right) \cdot \nabla\tilde{u}\,\mathrm{d}x \\
&\qquad + \fint_{B_R} 2\widetilde{F}''(0)(\tilde{u}\nabla\eta) \cdot (2\nabla(\eta\tilde{u}) - \tilde{u}\nabla\eta)\,\mathrm{d}x - \fint_{B_R} \widetilde{F}'(\nabla\tilde{u}) \cdot (2\eta\tilde{u}\nabla\eta)\,\mathrm{d}x \\
&\quad\leq \Lambda_{\widetilde{M}} \fint_{B_R} \omega_{\widetilde{M}}(|\nabla\tilde{u}|)\left(|\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^q\right)\,\mathrm{d}x \\
&\qquad + 4\Lambda_{\widetilde{M}} \fint_{B_R} |\tilde{u}\nabla\eta|\left(|\tilde{u}\nabla\eta| + |\nabla(\eta\tilde{u})| + \eta|\nabla\tilde{u}|^{q-1}\right)\,\mathrm{d}x,
\end{aligned} \tag{4.26}$$

where we have used the comparison estimate (4.12) to control the first term along with the fact that $\eta^2 \leq 1$, and the growth estimates (4.13), (4.10) for the additional terms. Now noting by Lemma 4.2.4 that

$$|\nabla\tilde{u}| \leq |\nabla u - (\nabla u)_{B_R}| + C(n)\,[\nabla u]_{\mathrm{BMO}(B_R)}\,, \tag{4.27}$$

we will apply the modular Fefferman-Stein estimate (Lemma 2.2.17) together with subadditivity of $\omega_{\widetilde{M}}$ to estimate

$$\fint_{B_R} \omega_{\widetilde{M}}(|\nabla u|)\left(|\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^q\right)\,\mathrm{d}x \leq C\,\omega_{\widetilde{M}}\left([\nabla u]_{\mathrm{BMO}(B_R)}\right) \fint_{B_R} |\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^q\,\mathrm{d}x. \tag{4.28}$$

Combining these with the earlier estimate and using Young's inequality to absorb the $|\nabla(\eta\tilde{u})|^2$ term we arrive at

$$\begin{aligned}
\fint_{B_{R/2}} |\nabla\tilde{u}|^2\,\mathrm{d}x &\leq C\,\omega_{\widetilde{M}}\left([\nabla u]_{\mathrm{BMO}(B_R)}\right) \fint_{B_R} |\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^q\,\mathrm{d}x \\
&\quad + \frac{C}{R^2} \fint_{B_R} |\tilde{u}|^2\,\mathrm{d}x + C \fint_{B_R} |\nabla\tilde{u}|^{2(q-1)}\,\mathrm{d}x.
\end{aligned} \tag{4.29}$$

Note that if $q = 2$, we do not get the $|\nabla \tilde{u}|^{2(q-1)}$ term. Otherwise by the John-Nirenberg inequality (Proposition 2.2.15) we can bound

$$\fint_{B_R} |\nabla \tilde{u}|^q \,\mathrm{d}x \leq C\,[\nabla u]_{\mathrm{BMO}(B_R)}^{q-2} \fint_{B_R} |\nabla \tilde{u}|^2 \,\mathrm{d}x, \tag{4.30}$$

$$\fint_{B_R} |\nabla \tilde{u}|^{2(q-1)} \,\mathrm{d}x \leq C\,[\nabla u]_{\mathrm{BMO}(B_R)}^{2(q-2)} \fint_{B_R} |\nabla \tilde{u}|^2 \,\mathrm{d}x. \tag{4.31}$$

Therefore if we let $\gamma(t) = \min\{1, \big(\omega_{\widetilde{M}}(t)(1+t^{q-2}) + t^{2(q-2)}\big)\}$ (omitting the t^{q-2} terms if $q = 2$) we deduce that

$$\fint_{B_{R/2}} |\nabla \tilde{u}|^2 \,\mathrm{d}x \leq C\,\gamma\left([\nabla u]_{\mathrm{BMO}(B_R)}\right) \fint_{B_R} |\nabla \tilde{u}|^2 \,\mathrm{d}x + \frac{C}{R^2} \fint_{B_R} |\tilde{u}|^2 \,\mathrm{d}x, \tag{4.32}$$

as required. □

4.2.3 Harmonic approximation

Our second ingredient is a comparison estimate for solutions to the linearised system. The following duality argument is an adaptation of the estimate proved in [GK19a]. The linear theory we need will straightforwardly follow from the strict Legendre-Hadamard condition satisfied by $F''(\nabla a)$, and we will refer to Section 3 and [Giu03, Chapter 10] for details.

Lemma 4.2.5 (Interior harmonic approximation)**.** Suppose F satisfies Hypotheses 4.1.1, let $M > 1$, and suppose u is F-extremal in some ball $B_R(x_0) \subset \Omega$ such that $\nabla u \in \mathrm{BMO}(B_R, \mathbb{R}^{Nn})$ with $[\nabla u]_{\mathrm{BMO}(B_R)} \leq 1$ and $|(\nabla u)_{B_R}| \leq M$. Then letting ∇a_R as in (4.19), we have the unique solution $h \in W^{1,2}(B_R, \mathbb{R}^N)$ to the problem

$$\begin{cases} -\operatorname{div} F''(\nabla a_R)\nabla h = 0 & \text{in } B_R, \\ \qquad\qquad\qquad\quad h = u & \text{on } \partial B_R, \end{cases} \tag{4.33}$$

satisfies the L^2 estimate

$$\fint_{B_R} |\nabla h - \nabla a_R|^2 \,\mathrm{d}x \leq C \fint_{B_R} |\nabla u - \nabla a_R|^2 \,\mathrm{d}x \tag{4.34}$$

with $\widetilde{M} = C(n)M$ and $C = C(n, N, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}) > 0$, and further the comparison estimate

$$\frac{1}{R^2} \fint_{B_R} |u - h|^2 \,\mathrm{d}x \leq C\,\gamma\left([\nabla u]_{\mathrm{BMO}(B_R)}\right) \fint_{B_R} |\nabla u - \nabla a_R|^2 \,\mathrm{d}x, \tag{4.35}$$

with $C = C(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}})$ and $\gamma \colon [0,\infty) \to [0,1]$ is increasing and continuous such that $\gamma(0) = 0$, depending on $\omega_{\widetilde{M}}$ and q only.

Proof. Put $\tilde{u} = u - a_R$ and $\widetilde{F} = \lambda_{\widetilde{M}}^{-1} F_{\nabla a}$. Then the existence of a unique $h \in W_w^{1,2}(B_R, \mathbb{R}^N)$ follows from L^2-coercivity of $\widetilde{F}''(0)$ (see Lemma 3.1.4) which gives (4.34). Then for any $\phi \in W_0^{1,2}(B_R, \mathbb{R}^N)$ we have

$$\begin{aligned}\fint_{B_R} \widetilde{F}''(0)(\nabla u - \nabla h) \cdot \nabla\phi \,\mathrm{d}x &= \fint_{B_R} \left(\widetilde{F}''(0)\nabla\tilde{u} - \widetilde{F}'(\nabla\tilde{u})\right) \cdot \nabla\phi \,\mathrm{d}x \\ &\leq \Lambda_{\widetilde{M}} \fint_{B_R} \omega_{\widetilde{M}}(|\nabla\tilde{u}|) \left(|\nabla\tilde{u}| + |\nabla\tilde{u}|^{q-1}\right) |\nabla\phi| \,\mathrm{d}x,\end{aligned} \tag{4.36}$$

where we have used the fact that $\tilde{u}$ is $\widetilde{F}$-extremal and the comparison estimate (4.12). Now choose ϕ to be the unique solution in $W_0^{1,2} \cap W^{2,2}(B_R, \mathbb{R}^N)$ to the problem

$$-\operatorname{div} \widetilde{F}''(0)\nabla\phi = u - h \tag{4.37}$$

in B_R (see Lemma 3.1.5), so in particular by symmetry of $\widetilde{F}''(0)$ this satisfies

$$\fint_{B_R} \widetilde{F}''(0)(\nabla u - \nabla h) \cdot \nabla\phi \,\mathrm{d}x = \fint_{B_R} |u - h|^2 \,\mathrm{d}x. \tag{4.38}$$

Moreover ϕ satisfies a $W^{2,2}$ estimate which combined with the Poincaré-Sobolev inequality (noting $(\nabla\phi)_{B_R} = 0$) gives

$$\|\nabla\phi\|_{L^{2^*}(B_R)} \leq C(n) \left\|\nabla^2\phi\right\|_{L^2(B_R)} \leq C \|u - h\|_{L^2(B_R)} \tag{4.39}$$

where $2^* = \frac{2n}{n-2}$ provided $n > 2$. For this choice of ϕ, applying Hölder's inequality and rearranging (4.36) using (4.39) we get

$$\frac{1}{R^2} \fint_{B_R} |u - h|^2 \,\mathrm{d}x \leq \frac{C}{R^2} \left\|\omega_{\widetilde{M}}(|\nabla\tilde{u}|)\right\|_{L^n(B_R)}^2 \fint_{B_R} |\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^{2(q-1)} \,\mathrm{d}x. \tag{4.40}$$

If $n = 2$ we use the fact that $\|\nabla\phi\|_{L^4(B_R)} \leq CR^{\frac{1}{2}} \|u - h\|_{L^2(B_R)}$ to get the slightly modified estimate

$$\frac{1}{R^2} \fint_{B_R} |u - h|^2 \, dx \leq \frac{C}{R} \left\|\omega_{\widetilde{M}}(|\nabla\tilde{u}|)\right\|_{L^4(B_R)}^2 \fint_{B_R} |\nabla\tilde{u}|^2 + |\nabla\tilde{u}|^{2(q-1)} \,\mathrm{d}x. \tag{4.41}$$

In both cases since $\omega_{\widetilde{M}} \leq 1$ is concave by Jensen's inequality we have

$$\left\|\omega_{\widetilde{M}}(|\nabla\tilde{u}|)\right\|_{L^p(B_R)} \leq R^{\frac{n}{p}} \left(\fint_{B_R} \omega_{\widetilde{M}}(|\nabla\tilde{u}|) \, dx\right)^{\frac{1}{p}} \leq R^{\frac{n}{p}} \omega_{\widetilde{M}} \left([\nabla\tilde{u}]_{\mathrm{BMO}(B_R)}\right)^{\frac{1}{p}}, \tag{4.42}$$

and by the John-Nirenberg estimate (Proposition 2.2.15) and Lemma 4.2.4 we can also estimate

$$\fint_{B_R} |\nabla\tilde{u}|^{2(q-1)} \,\mathrm{d}x \leq C \,[\nabla u]_{\mathrm{BMO}(B_R)}^{2(q-2)} \fint_{B_R} |\nabla\tilde{u}|^2 \,\mathrm{d}x. \tag{4.43}$$

Putting everything together the result follows by taking $\gamma(t) = \min\{1, \omega_{\widetilde{M}}(t)^{\frac{2}{n}} \left(1 + t^{2(q-2)}\right)\}$, modified suitably if $n = 2$. □

4.2.4 Excess decay

From here Theorem 4.1.2 follows by combining the above estimates to get a suitable decay estimate, which we can then iterate. This approach is standard among many partial regularity proofs, and we follow a similar argument to that found in [GK19a].

Proof of Theorem 4.1.2. We will begin by establishing the following decay estimate for the excess energy

$$E(x,r) = \fint_{B_r(x)} |\nabla u - (\nabla u)_{B_R}|^2 \,\mathrm{d}y. \tag{4.44}$$

Claim: For any $B_r(x) \subset B_R(x_0)$ and $\sigma \in \left(0, \frac{1}{4}\right)$ for which $|(\nabla u)_{B_{2\sigma r}(x)}|, |(\nabla u)_{B_r(x)}| \leq 2^{n+1}M$ and $[\nabla u]_{\mathrm{BMO}_R(x)} \leq 1$ we have

$$E(x,\sigma r) \leq C\left(\sigma^2 + \sigma^{-(n+2)}\gamma\left([\nabla u]_{\mathrm{BMO}(B_r(x))}\right)\right) E(x,r), \tag{4.45}$$

where $C = C(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}) > 0$ and γ satisfies both Lemmas 4.2.3 and 4.2.5 with $2^{n+1}M$ in place of M.

Indeed let

$$a_r(x) = (u)_{B_r} + (\nabla u)_{B_r} \cdot (x - x_0) \tag{4.46}$$

and apply the harmonic approximation result (Lemma 4.2.5) in $B_r(x)$ to get $h \in W^{1,2}_u(B_r(x), \mathbb{R}^N)$ solving

$$-\operatorname{div} F''(\nabla a_r)\nabla h = 0, \tag{4.47}$$

which satisfies

$$\frac{1}{r^2}\fint_{B_r(x)} |u - h|^2 \,\mathrm{d}y \leq \gamma\left([\nabla u]_{\mathrm{BMO}(B_R(x_0))}\right) E(x,r). \tag{4.48}$$

Now put $\tilde{h} = h - a_r$ and let $a_h(y) = \tilde{h}(x) + \nabla\tilde{h}(x)\cdot(y-x)$, since $B_{2\sigma r}(x) \subset B_{r/2}(x)$ we have

$$\begin{aligned}\frac{1}{(2\sigma r)^2}\fint_{B_{2\sigma r}(x)} |\tilde{h} - a_h|^2 \,\mathrm{d}y &\leq \frac{C}{(2\sigma r)^2}\left(\sup_{B_{r/2}(x)} |\nabla^2 \tilde{h}|\right)^2 \fint_{B_{2\sigma r}(x)} |y-x|^4 \,\mathrm{d}y \\ &\leq C\sigma^2 \fint_{B_r(x)} |\nabla(h - a_r)|^2 \,\mathrm{d}y \leq C\sigma^2 E(x,r)\end{aligned} \tag{4.49}$$

using interior regularity for h (namely Lemma 3.1.6). We will use these in conjunction with the Caccioppoli-type inequality (Lemma 4.2.3) applied in $B_{2\sigma r}(x)$, letting $a_{2\sigma r}$ be given by

(4.19) we have

$$\begin{aligned} E(x,\sigma r) &\leq \fint_{B_{\sigma r}(x)} |\nabla u - \nabla a_{2\sigma r}|^2 \,\mathrm{d}y \\ &\leq \frac{C}{(2\sigma r)^2} \fint_{B_{2\sigma r}(x)} |u - a_{2\sigma r}|^2 \,\mathrm{d}y + C\gamma\left([\nabla u]_{\mathrm{BMO}(B_r(x))}\right) E(x, 2\sigma r). \end{aligned} \tag{4.50}$$

Now using the estimates (4.48) and (4.49) and the minimising property (4.21) we can estimate

$$\begin{aligned} &\frac{1}{(2\sigma r)^2} \fint_{B_{2\sigma r}(x)} |u - a_{2\sigma r}|^2 \,\mathrm{d}y \\ &\quad\leq \frac{1}{(2\sigma r)^2} \fint_{B_{2\sigma r}(x)} |u - a_r - a_1|^2 \,\mathrm{d}y \\ &\quad\leq \frac{C}{\sigma^{n+2} r^2} \fint_{B_r(x)} |u - h|^2 \,\mathrm{d}y + \frac{C}{(2\sigma r)^2} \fint_{B_{2\sigma r}(x)} |h - a_h|^2 \,\mathrm{d}y \\ &\quad\leq C\left(\sigma^2 + \sigma^{-(n+2)}\gamma\left([\nabla u]_{\mathrm{BMO}(B_R(x_0))}\right)\right) E(x,r). \end{aligned} \tag{4.51}$$

So the claim follows by combining the above two estimates.

We now iteratively apply the claim for suitably chosen parameters. Since $|(\nabla u)|_{B_R(x_0)} \leq M$, for all $x \in B_{R/2}(x_0)$ we have $|(\nabla u)_{B_{R/2}(x)}| \leq M$ and so

$$|(\nabla u)_{B_{\sigma R/2}(x)}| \leq |(\nabla u)_{B_{R/2}(x)}| + |(\nabla u)_{B_{\sigma R/2}(x)} - (\nabla u)_{B_{R/2}(x)}| \leq 2^n M + \sigma^{-n} E(x, R/2). \tag{4.52}$$

Iteratively applying this therefore gives

$$|(\nabla u)_{B_{\sigma^k R/2}(x)}| \leq 2^n M + \sigma^{-n} \sum_{j=0}^{k-1} E(x, \sigma^j R/2). \tag{4.53}$$

Since $E(x,r) \leq [\nabla u]^2_{\mathrm{BMO}(B_R(x_0))} \leq \varepsilon^2$ for all $r < R/2$, see that if $\sigma^{-n}\varepsilon^2 \leq 2^n M$, we can apply the claimed decay estimate (4.45) to obtain

$$E(x, \sigma R) \leq C\left(\sigma^2 + \sigma^{-n-2}\gamma(\varepsilon)\right) E(x, R/2). \tag{4.54}$$

Fix $\alpha \in (0,1)$, and choose $\sigma \leq \frac{1}{4}$ such that $C\sigma^2 \leq \frac{1}{2}\sigma^{2\alpha}$. Then we can take $\varepsilon > 0$ small enough so $C\sigma^{-(n+2)}\gamma(\varepsilon) \leq \frac{1}{2}\sigma^{2\alpha}$ and $\sigma^{-n}\varepsilon^2 \sum_j \sigma^{\alpha j} \leq 2^n M$. Then we can inductively check that (4.45) gives

$$E(x, \sigma^k R/2) \leq \sigma^{2\alpha k} E(x, R/2), \tag{4.55}$$

and by (4.53) we can ensure

$$|(\nabla u)_{B_{\sigma^k R/2}(x)}| \leq 2^{n+1} M \tag{4.56}$$

for each $k \geq 1$. Hence for each $r \in (0, R/2)$, choosing k such that $\sigma^k R/2 \leq r < \sigma^{k-1} R/2$ we deduce that

$$E(x, r) \leq C \left(\frac{r}{R}\right)^{2\alpha} E(x_0, R). \tag{4.57}$$

This verifies the Campanato-Meyers characterisation of Hölder continuity (see for instance [Giu03, Theorem 2.9]), allowing us to conclude that $u \in C^{1,\alpha}(\overline{B}_{R/2}(x_0), \mathbb{R}^N)$ as required. $\square$

4.3 The boundary case

4.3.1 Localisation near the boundary

Our strategy in the boundary case will be similar to the interior case, but some technical modifications will be needed in our choice of approximant a. This is because the Caccioppoli-type estimate (Lemma 4.2.3) involved testing the equation against $\phi = \eta(u - a)$ with η a suitable cutoff and a an affine approximation to u, but to use this close to the boundary we need $u - a$ to vanish suitably on $\partial\Omega$. This will be done using ideas of Kronz [Kro05] along with the refinements of Campos Cordero [Cam17].

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain, and consider the defining function $\rho = \rho_\Omega \in C^{1,\beta}$ constructed in Lemma 2.1.23. The idea is to use ρ as a replacement for the affine approximation, considering maps of the form

$$a(x) = \xi \, \frac{\rho(x)}{|\nabla \rho(x_0)|}, \tag{4.58}$$

with $\xi \in \mathbb{R}^N$. Since $\nabla a = \xi \otimes \frac{\nabla \rho(x)}{|\nabla \rho(x_0|}$ which is close to $\xi \otimes \nu_{x_0}$ however, taking $\xi = (\nabla v \cdot \nu_{x_0})_{\Omega_R(x_0)}$ only allows us to control the normal component compared to the full derivative $\nabla a = (\nabla u)_{B_R(x_0)}$ from the interior case. It turns out this is sufficient however; this is illustrated by the following result which is an adaptation of an observation of Campos Cordero [Cam17].

Lemma 4.3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain and let $p > \frac{3}{2}$. There is $R_0 > 0$ and $C > 0$ such that for all $x_0 \in \partial\Omega$ and $0 < R < R_0$, for all $v \in W^{1,p}(\Omega_R(x_0), \mathbb{R}^N)$ such that

$v = 0$ on $\partial\Omega \cap B_R(x_0)$ we have

$$\begin{aligned}\left(\fint_{\Omega_R(x_0)} |\nabla v - (\nabla v \cdot \nu_{x_0})_{\Omega_R(x_0)} \otimes \nu_{x_0}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ \leq C\left(\fint_{\Omega_R(x_0)} |\nabla v - (\nabla v)_{\Omega_R(x_0)}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + C|(\nabla v)_{\Omega_R(x_0)}|R^\beta.\end{aligned} \tag{4.59}$$

Proof. Fix $x_0 \in \partial\Omega$, then by translating and rotating we can assume $x_0 = 0$ and $\nu(x_0) = e_n$, and take $R_0 > 0$ small enough so we can write $\Omega_{R_0}(x_0)$ as the graph of some γ. We have

$$\left(\fint_{\Omega_R} |\nabla v - (\nabla_n v)_{\Omega_R} \otimes e_n|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq \left(\fint_{\Omega_R} |\nabla v - (\nabla v)_{\Omega_R}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \sum_{i=1}^{n-1} |(\nabla_i v)_{\Omega_R}|, \tag{4.60}$$

where we write $\nabla_j v = \nabla v \cdot e_j$, so we need to estimate the tangential derivatives. We proceed analogously to [Cam17, Lemma 5.6] with minor modifications to account for the curved boundary, so letting ρ be the defining function for Ω as above we consider

$$\widetilde{v}(x) = v(x) - (\nabla_n v)_{\Omega_R} \frac{\rho(x)}{|\nabla\rho(0)|}. \tag{4.61}$$

Note that $\widetilde{v}$ still vanishes on $\partial\Omega \cap B_R$, so writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, so a similar argument to [Cam17] gives

$$\int_{\Omega_R} \nabla_i \widetilde{v} \,\mathrm{d}x = \int_{\Omega \cap \partial B_R} \widetilde{v}(x) \frac{x_i}{R} \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\Omega_R} \nabla_n \widetilde{v}(x) \frac{x_i}{(R^2 - |x'|^2)^{\frac{1}{2}}} \,\mathrm{d}x, \tag{4.62}$$

where the only difference is that $\widetilde{v}$ vanishes at $(x', \gamma(x'))$ writing $x = (x', x_n)$. This can then be estimated using Hölder's inequality as in [Cam17], and using the fact that ρ is of class $C^{1,\beta}$ we deduce that

$$\begin{aligned}\left|\fint_{\Omega_R} \nabla_i v \,\mathrm{d}x\right| &\leq \left|\fint_{\Omega_R} \nabla_i \widetilde{v} \,\mathrm{d}x\right| + |(\nabla_n v)_{\Omega_R}| \frac{|(\nabla_i \rho)_{\Omega_R}|}{|\nabla\rho(x_0)|} \\ &\leq C(n,p)\left(\fint_{\Omega_R} |\nabla_n v - (\nabla_n v)_{\Omega_R}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + C(n, p\, \|\Omega\|_{C^{1,\beta}})|(\nabla_n v)_{\Omega_R}|R^\beta,\end{aligned} \tag{4.63}$$

from which the result follows. □

We will also record a Poincaré inequality for maps which vanish on part of the boundary. This will follow by flattening the boundary and applying integrating in the x_n-direction.

Lemma 4.3.2 (A Poincaré inequality)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain and let $R_0 > 0$, $\widetilde{\Omega}_R(x_0)$ as in Lemma 4.3.1. Then for all $x_0 \in \partial\Omega$, $0 < R < R_0$, $1 < p < \infty$, for $u \in W^{1,p}(\widetilde{\Omega}_R(x_0), \mathbb{R}^N)$ such that $u = 0$ on $\partial\Omega \cap \overline{\Omega}_R(x_0)$ in the trace sense we have

$$R^{\frac{n}{p} - \frac{n}{q} - 1} \|u\|_{L^q(\widetilde{\Omega}_R)} \leq C \|\nabla u\|_{L^p(\widetilde{\Omega}_R)} \tag{4.64}$$

for all $1 \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$, with $C = C(n, p, q, \beta, \|\Omega\|_{C^{1,\beta}}) > 0$. Also the same conclusion holds for $\Omega_R(x_0)$ in place of $\widetilde{\Omega}_R(x_0)$.

For the remainder of the section, we will fix $R_0 > 0$ and $\delta >$ such that $\Omega_R(x_0)$ is a δ-John domain for all $x_0 \in \partial\Omega$, $0 < R < R_0$ (using Lemma 2.1.25), and given ρ as above we will also assume that we have $\mathcal{L}^n(\Omega_R(x_0)) \geq 4^{-n}\mathcal{L}^n(B_R(x_0))$ and

$$C(n)^{-1}R^2 \leq \fint_{\Omega_R(x_0)} \frac{\rho(x)^2}{|\nabla\rho(x_0)|^2} \,\mathrm{d}x \leq C(n)R^2, \tag{4.65}$$

for all $R < R_0$. Shrinking R_0 further if necessary we will moreover assume the construction of $\widetilde{\Omega}_R(x_0)$ (Lemma 4.3.1) and the above Poincaré inequality (Lemma 4.3.2) holds with this choice of R_0.

4.3.2 Boundary Caccioppoli inequality

Lemma 4.3.3 (Boundary Caccioppoli-type inequality)**.** Suppose F satisfies Hypotheses 4.1.1, let $M \geq 1$, and suppose $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\beta}$ domain for some $\beta \in (0,1)$. Given $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$, there is $R_0 = R_0(n, \Omega) > 0$ such that the following holds. Suppose $x_0 \in \partial\Omega$, $0 < R < R_0$, and $u \in W^{1,q}_g(\Omega, \mathbb{R}^N)$ is F-extremal in $\Omega_R(x_0)$ such that $\nabla u \in \mathrm{BMO}(\Omega_R(x_0), \mathbb{R}^{Nn})$, $[\nabla u]_{\mathrm{BMO}(\Omega_R(x_0))} \leq 1$, and $|(\nabla u)_{\Omega_R}| \leq M$. Then if we define

$$a_R(x) = \xi_R \frac{\rho(x)}{|\nabla\rho(x_0)|} = \frac{((u-g)\rho)_{\Omega_R}}{(\rho^2)_{\Omega_R}} \rho(x), \tag{4.66}$$

with ρ the defining function for Ω as above, we have the estimate

$$\begin{aligned} \fint_{\Omega_{R/2}} |\nabla u - (\nabla u)_{\Omega_{R/2}}|^2 \,\mathrm{d}x \leq C\,\gamma\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \,\mathrm{d}x \\ + \frac{C}{R^2} \fint_{\Omega_R} |u - g - a_R|^2 \,\mathrm{d}x + CM^{2(q-1)}R^{2\beta}, \end{aligned} \tag{4.67}$$

where setting $\widetilde{M} = C(n, \beta, \|\Omega\|_{C^{1,\beta}}, \|\nabla g\|_{C^{0,\beta}(\Omega)})M$, $\gamma\colon [0,\infty) \to [0,1]$ is a non-decreasing continuous function satisfying $\gamma(0) = 0$ depending on $\omega_{\widetilde{M}}$ and q only, and

$$C = C\left(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}, \delta, \|\Omega\|_{C^{1,\beta}}, R_0, [\nabla g]_{C^{0,\beta}(\Omega)}\right) > 0. \tag{4.68}$$

Remark 4.3.4. Similarly as in the interior case, the choice of $a_R(x)$ in (4.66) ensures that

$$\xi \mapsto \int_{\Omega_R} \left| u - g - \xi \frac{\rho}{|\nabla \rho(x_0)|} \right|^2 \mathrm{d}x \tag{4.69}$$

is minimised in $\xi \in \mathbb{R}^N$ when $\xi = \xi_R$, as noted by Kronz [Kro05]. If we set $\xi = (\nabla(u-g) \cdot \nu(x_0))_{\Omega_R}$, these can be compared through estimate

$$\begin{aligned} &|\xi_R - (\nabla(u-g) \cdot \nu(x_0))_{\Omega_R}| \\ &\quad \leq C \left(\fint_{\Omega_R} \left| \nabla(u-g) - (\nabla(u-g) \cdot \nu(x_0))_{\Omega_R} \otimes \frac{\nabla \rho}{|\nabla \rho(x_0)|} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} + CMR^{\beta}, \end{aligned} \tag{4.70}$$

where $C = C(n, \beta, \|\Omega\|_{C^{1,\beta}}) > 0$. This is proved in [Kro05, Lemma 2(ii)], relying on the Poincaré inequality and (4.65).

Proof. Let $R_0 > 0$ as in the beginning of this section, and define

$$\tilde{u}(x) = u(x) - g(x) - a_R(x), \tag{4.71}$$

noting that $\tilde{u} = 0$ on $\partial\Omega \cap B_R(x_0)$. We also fix a cutoff $\eta \in C_c^\infty(B_R(x_0))$ such that $1_{B_{R/2}(x_0)} \leq \eta \leq 1_{B_R(x_0)}$ and $|\nabla \eta| \leq \frac{C}{R}$, and consider the shifted functional $\widetilde{F}(z) = F_{z_R}(z)$ as in (4.8) where

$$z_R = \xi_R \otimes \nu_{x_0} + (\nabla g)_{\Omega_R}, \tag{4.72}$$

with ξ_R as in (4.66). Using the Poincaré inequality, we can choose $\widetilde{M}$ so that

$$|z_R| \leq C(n, \beta, \|\Omega\|_{C^{1,\beta}}) \left(\fint_{\Omega_R} |\nabla u - \nabla g|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + |(\nabla g)_{\Omega_R}| \leq \widetilde{M}. \tag{4.73}$$

Now by the strict Legendre-Hadamard condition applied to $\eta\tilde{u}$ and testing the equation (4.1) against $\eta^2 \tilde{u}$ we have

$$\begin{aligned} \lambda_{\widetilde{M}} \fint_{\Omega_R} |\nabla(\eta\tilde{u})|^2 \, dx &\leq \fint_{\Omega_R} \widetilde{F}''(0) \nabla(\eta\tilde{u}) \cdot \nabla(\eta\tilde{u}) \, \mathrm{d}x - \fint_{\Omega_R} \widetilde{F}'(\nabla u - z_R) \cdot \nabla(\eta^2 \tilde{u}) \, \mathrm{d}x \\ &= \fint_{\Omega_R} \eta \left(\widetilde{F}''(0)(\nabla u - z_R) - \widetilde{F}'(\nabla u - z_R) \right) \cdot \nabla(\eta\tilde{u}) \, \mathrm{d}x \\ &\quad + \fint_{\Omega_R} \eta \widetilde{F}''(0)(z_R - \nabla a_R - \nabla g) \cdot \nabla(\eta\tilde{u}) \, \mathrm{d}x \\ &\quad + \fint_{\Omega_R} \tilde{u} \widetilde{F}''(0) \nabla\eta \cdot \nabla(\eta\tilde{u}) \, \mathrm{d}x - \fint_{\Omega_R} \eta\tilde{u} \widetilde{F}'(\nabla u - z_R) \cdot \nabla\eta \, \mathrm{d}x. \end{aligned} \tag{4.74}$$

We can absorb the $\nabla(\eta\tilde{u})$ terms using Cauchy-Schwarz and Young's inequality; for the last term we can use the growth estimate (4.10) for $\widetilde{F}'$ to estimate

$$\begin{aligned}
&\fint_{\Omega_R} \eta\tilde{u}\widetilde{F}'(\nabla u - z_R)\cdot\nabla\eta \,\mathrm{d}x \\
&\quad\leq \Lambda_{\widetilde{M}} \fint_{\Omega_R} |\tilde{u}\nabla\eta| \left(|\eta(\nabla u - z_R)| + \eta|\nabla u - z_R|^{q-1}\right) \,\mathrm{d}x \\
&\quad\leq \frac{C\Lambda_{\widetilde{M}}^2}{\lambda_{\widetilde{M}}} \fint_{\Omega_R} |\tilde{u}\nabla\eta|^2 \,\mathrm{d}x + \frac{\lambda_{\widetilde{M}}}{8} \fint_{\Omega_R} |\nabla(\eta\tilde{u})|^2 + \eta^2|\nabla\tilde{u}|^{2(q-1)} \,\mathrm{d}x \\
&\qquad + C\lambda_{\widetilde{M}} \fint_{\Omega_R} \eta^2|z_R - a_R - \nabla g|^2 + \eta^2|z_R - a_R - \nabla g|^{2(q-1)} \,\mathrm{d}x.
\end{aligned} \tag{4.75}$$

Hence since $\eta^2 \leq 1$ we deduce that

$$\begin{aligned}
\frac{1}{2}\fint_{\Omega_R} |\nabla(\eta\tilde{u})|^2 \,\mathrm{d}x \leq{}& \frac{4}{\lambda_{\widetilde{M}}} \fint_{\Omega_R} \left|\widetilde{F}''(0)(\nabla u - z_R) - \widetilde{F}'(\nabla u - z_R)\right|^2 \,\mathrm{d}x \\
&+ C\fint_{\Omega_R} \left(|z_R - \nabla a_R - \nabla g|^2 + |z_R - \nabla a_R - \nabla g|^{2(q-1)}\right) \,\mathrm{d}x \\
&+ \frac{C}{R^2}\fint_{\Omega_R} |\tilde{u}|^2 \,\mathrm{d}x + C\fint_{\Omega_R} |\nabla\tilde{u}|^{2(q-1)} \,\mathrm{d}x,
\end{aligned} \tag{4.76}$$

where the final term can be omitted if $q = 2$. For the second term we note that since g, ρ are $C^{1,\beta}$ we have

$$|z_R - \nabla a_R - \nabla g| \leq |\xi_R \otimes \nu_{x_0}| \frac{|\nabla\rho(x) - \nabla\rho(x_0)|}{|\nabla\rho(x_0)|} + |\nabla g - (\nabla g)_{\Omega_R}| \leq CMR^\beta \tag{4.77}$$

in Ω_R, where $C = C(\|\Omega\|_{C^{1,\beta}}, [\nabla g]_{C^{0,\beta}}) > 0$. For the first term we apply the comparison estimate (4.12); writing $\varphi(t) = \omega_{\widetilde{M}}(t)(t^2 + t^{2(q-1)})$ this gives

$$\fint_{\Omega_R} \left|\widetilde{F}''(0)(\nabla u - z_R) - \widetilde{F}'(\nabla u - z_R)\right|^2 \,\mathrm{d}x \leq \frac{\Lambda_{\widetilde{M}}}{\lambda_{\widetilde{M}}} \fint_{\Omega_R} \varphi(|\nabla u - z_R|) \,\mathrm{d}x, \tag{4.78}$$

noting that $\omega_{\widetilde{M}}(t) \leq 1$. Now we estimate

$$\begin{aligned}
|\nabla u - z_R| \leq{}& |\nabla u - (\nabla u)_{\Omega_R}| + |\xi_R - ((\nabla(u-g))\cdot\nu_{x_0})_{\Omega_R}| \\
&+ |(\nabla(u-g))_{\Omega_R} - ((\nabla(u-g))\cdot\nu_{x_0})_{\Omega_R} \otimes \nu_{x_0}|.
\end{aligned} \tag{4.79}$$

By Remark 4.3.4 the second term can be estimated as

$$\begin{aligned}
&|\xi_R - (\nabla(u-g)\cdot\nu_{x_0})_{\Omega_R}| \\
&\quad\leq C\left(\fint_{\Omega_R} |\nabla u - \nabla g - (\nabla(u-g)\cdot\nu_{x_0})_{\Omega_R} \otimes \nu_{x_0}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + CMR^\beta \\
&\quad\leq C|(\nabla(u-g))_{\Omega_R} - ((\nabla(u-g))\cdot\nu_{x_0})_{\Omega_R} \otimes \nu_{x_0}| \\
&\qquad + C\left(\fint_{\Omega_R} |\nabla u - \nabla g - (\nabla(u-g))_{\Omega_R}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + CMR^\beta,
\end{aligned} \tag{4.80}$$

and applying Campos Cordero's trick (Lemma 4.3.1) followed by the John-Nirenberg estimate (Proposition 2.2.15) we have

$$\begin{aligned}
&|(\nabla u - \nabla g)_{\Omega_R} - ((\nabla u - \nabla g)\cdot \nu_{x_0})_{\Omega_R} \otimes \nu_{x_0}| \\
&\quad \le C \left(\fint_{\Omega_R} |\nabla u - \nabla g - (\nabla u - \nabla g)_{\Omega_R}|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + CMR^{\beta} \\
&\quad \le C\, [\nabla u - \nabla g]_{\mathrm{BMO}(\Omega_R)} + CMR^{\beta},
\end{aligned} \tag{4.81}$$

for $p \in \{2, q\}$. Also applying the modular Fefferman-Stein estimate (Lemma 2.2.17) we can bound

$$\begin{aligned}
&\fint_{\Omega_R} \varphi\left(|\nabla u - (\nabla u)_{\Omega_R}|\right) \mathrm{d}x \\
&\qquad \le C\, \omega_{\widetilde{M}}\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 + |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \,\mathrm{d}x.
\end{aligned} \tag{4.82}$$

Now since $[\nabla g]_{\mathrm{BMO}(\Omega_R)} \le CR^{\beta}$ and $\varphi(R^{\beta}) \le \left(1 + R_0^{2(q-2)}\right) R^{2\beta}$, we can combine the above using the doubling property of φ to get

$$\begin{aligned}
&\fint_{\Omega_R} \varphi\left(|\nabla u - z_0|\right) \mathrm{d}x \le CM^{2(q-1)} R^{2\beta} \\
&\qquad + C\, \omega_{\widetilde{M}}\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 + |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \,\mathrm{d}x.
\end{aligned} \tag{4.83}$$

To complete the estimate, note by the John-Nirenberg inequality (Proposition 2.2.15) that

$$\fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \le C\, [\nabla u]_{\mathrm{BMO}(\Omega_R)}^{2(q-2)} \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \,\mathrm{d}x, \tag{4.84}$$

and similarly

$$\fint_{\Omega_R} |\nabla \tilde{u}|^{2(q-1)} \,\mathrm{d}x \le C\, [\nabla u]_{\mathrm{BMO}(\Omega_R)}^{2(q-2)} \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \,\mathrm{d}x + CM^{2(q-1)} R^{2\beta}. \tag{4.85}$$

Hence putting everything together gives

$$\begin{aligned}
\fint_{\Omega_R} |\nabla(\eta \tilde{u})|^2 \,\mathrm{d}x &\le C\, \omega_{\widetilde{M}}\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \left(1 + [\nabla u]_{\mathrm{BMO}(\Omega_R)}^{2(q-2)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \\
&\quad + \frac{C}{R^2} \fint_{\Omega_R} |\tilde{u}|^2 \,\mathrm{d}x + C\, [\nabla u]_{\mathrm{BMO}(\Omega_R)}^{2(q-2)} \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \,\mathrm{d}x \\
&\quad + CM^{2(q-1)} R^{2\beta},
\end{aligned} \tag{4.86}$$

from which the result follows taking $\gamma(t) = \min\{1, \omega_{\widetilde{M}}(t)(1 + t^{2(q-2)}) + t^{2(q-2)}\}$, omitting the $t^{2(q-2)}$ terms if $q = 2$. □

4.3.3 Boundary harmonic approximation

We now wish to establish a harmonic approximation result analogous to the interior case, however a technical difficulty arises when applying the linear existence theory in $\Omega_R(x_0)$, whose boundary is only piecewise $C^{1,\beta}$ in general. Similarly as with the linear L^p estimates (Proposition 3.3.9), we will replace $\Omega_R(x_0)$ by the regularised domain from Lemma 2.1.28. Hence we can apply the linear solvability theory from Section 3 on this modified domain instead, noting that the corresponding estimates will be independent of R.

Lemma 4.3.5 (Boundary harmonic approximation)**.** Suppose F satisfies Hypotheses 4.1.1, let $M \geq 1$, and suppose $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\beta}$ domain and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$, for some $\beta \in (0,1)$. Suppose $x_0 \in \partial\Omega$, $0 < R < R_0$ with $R_0 = R_0(n,\Omega) > 0$ and $u \in W^{1,q}_g(\Omega_R, \mathbb{R}^N)$ is F-extremal in $\Omega_R(x_0)$ with $\nabla u \in \mathrm{BMO}(\Omega_R, \mathbb{R}^{Nn})$, $|(\nabla u)_{\Omega_R}| \leq M$, and $[\nabla u]_{\mathrm{BMO}(\Omega_R)} \leq 1$.

Then letting $\widetilde{\Omega}_R$ as in Lemma 2.1.28, the unique solution $\tilde{h} \in W^{1,2}(\widetilde{\Omega}_R, \mathbb{R}^N)$ to the Dirichlet problem

$$\begin{cases} -\operatorname{div} F''(z_R)\nabla \tilde{h} = 0 & \text{in } \widetilde{\Omega}_R, \\ \tilde{h} = u - g - a_R & \text{on } \partial\widetilde{\Omega}_R, \end{cases} \tag{4.87}$$

with z_R, a_R as in (4.66), (4.72) respectively satisfies

$$\fint_{\widetilde{\Omega}_R} |\nabla \tilde{h}|^2 \,\mathrm{d}x \leq C \fint_{\widetilde{\Omega}_R} |\nabla(u - g - a_R)|^2 \,\mathrm{d}x, \tag{4.88}$$

where $C = C\left(n, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}\right) > 0$ with $\widetilde{M} = C\left(n, \beta, \|\Omega\|_{C^{1,\beta}}, \|\nabla g\|_{C^{0,\beta}(\Omega)}\right) M$. Moreover we have the remainder estimate

$$\frac{1}{R^2}\fint_{\widetilde{\Omega}_R} |u - g - a_R - \tilde{h}|^2 \,\mathrm{d}x \leq C\gamma\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \,\mathrm{d}x + CM^2R^{2\beta}, \tag{4.89}$$

where $C = C\left(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}, \|\Omega\|_{C^{1,\beta}}, [\nabla g]_{C^{0,\beta}(\Omega)}\right) > 0$ and $\gamma\colon [0,\infty) \to [0,1]$ non-decreasing continuous such that $\gamma(0) = 0$, depending on n, q and $\omega_{\widetilde{M}}$ only.

Proof. We will assume $n \geq 3$ so Sobolev embedding applies, taking similar modifications as in the interior case if $n = 2$. Additionally we will use similar arguments used in the proof of Lemma 4.3.3 which we will not reproduce in detail, in particular choosing $R_0, \widetilde{M}$ in the same way. Letting $\widetilde{F} = \lambda_{\widetilde{M}}^{-1} F_{z_R}$ be the shifted functional with z_R as in (4.72) and setting

$\tilde{u} = u - g - a_R$, note for $\phi \in W_0^{1,2}(\widetilde{\Omega}_R, \mathbb{R}^N)$ we have

$$\begin{aligned}
&\fint_{\widetilde{\Omega}_R} \widetilde{F}''(0)(\nabla\tilde{u} - \nabla\tilde{h}) \cdot \nabla\phi \, \mathrm{d}x \\
&\quad = \fint_{\widetilde{\Omega}_R} \left(\widetilde{F}''(0)\nabla\tilde{u} - \widetilde{F}'(\nabla u - z_R)\right) \cdot \nabla\phi \, \mathrm{d}x \\
&\quad \leq \Lambda_{\widetilde{M}} \fint_{\widetilde{\Omega}_R} \omega_M(|\nabla u - z_R|) \left(|\nabla u - z_R| + |\nabla u - z_R|^{q-1}\right) |\nabla\phi| \, \mathrm{d}x \\
&\qquad + \Lambda_{\widetilde{M}} \fint_{\widetilde{\Omega}_R} |\nabla a_R - \nabla g - z_R||\nabla\phi| \, \mathrm{d}x,
\end{aligned} \tag{4.90}$$

where we used the comparison estimate (4.12). We now choose ϕ to be the unique solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} \widetilde{F}''(0)\nabla\phi = \tilde{u} - \tilde{h} & \text{in } \widetilde{\Omega}_R, \\ \phi = 0 & \text{on } \partial\widetilde{\Omega}_R. \end{cases} \tag{4.91}$$

By Theorem 3.3.8 and Proposition 3.3.12 we have a unique solution exists and satisfies the estimate $\|\nabla\phi\|_{L^{2^*}(\widetilde{\Omega}_R)} \leq C \left\|\tilde{u} - \tilde{h}\right\|_{L^2(\widetilde{\Omega}_R)}$. Therefore for this choice of ϕ we get

$$\begin{aligned}
\fint_{\widetilde{\Omega}_R} |\tilde{u} - \tilde{h}|^2 \, \mathrm{d}x \leq C\, \omega_{\widetilde{M}} \left(\fint_{\Omega_R} |\nabla u - z_R| \, \mathrm{d}x\right)^{\frac{2}{n}} \fint_{\Omega_R} |\nabla u - z_R|^2 + |\nabla u - z_R|^{2(q-1)} \, \mathrm{d}x \\
+ C \left(\fint_{\Omega_R} |\nabla a_R - \nabla g - z_R|^{2_*} \, \mathrm{d}x\right)^{\frac{2}{2_*}},
\end{aligned} \tag{4.92}$$

where we have used Hölder and Jensen's inequalities (here $2_* = \frac{2n}{n+2}$), and absorbed the $\fint_{\widetilde{\Omega}_R} |\tilde{u} - \tilde{h}|^2 \, \mathrm{d}x$ term on the right-hand side. Arguing by splitting $|\nabla u - z_R|$ as in (4.79) from the previous section (proof of Lemma 4.3.3) we arrive at the estimate

$$\fint_{\widetilde{\Omega}_R} |\tilde{u} - \tilde{h}|^2 \, \mathrm{d}x \leq C\gamma\left([\nabla u]_{\mathrm{BMO}(\Omega_R)}\right) \fint_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \, \mathrm{d}x + CM^2R^{2\beta} \tag{4.93}$$

with $\gamma(t) = \min\{1, \omega_{\widetilde{M}}(t)^{\frac{2}{n}}(1 + t^{2(q-2)})\}$, as required. □

4.3.4 Proof of boundary regularity

We can now combine the previous two estimates to conclude as in the interior case.

Proof of Theorem 4.1.3. For $B_r(x) \subset B_{R_0}(x_0)$ with $x \in \overline{\Omega}$ we consider the excess energy

$$E(x, r) = \fint_{\Omega_r(x)} |\nabla u - (\nabla u)_{\Omega_r(x)}|^2 \, \mathrm{d}y, \tag{4.94}$$

so by assumption and Proposition 2.2.15 there is $C_1 = C_1(n, \delta) > 0$ such that $E(x, r) \leq C_1\varepsilon^2$, which we can assume is less than 1.

Claim: If $x \in \partial\Omega$ and $r > 0$ so that $\Omega_r(x) \subset \Omega_R(x_0)$ and $\sigma \in \left(0, \frac{1}{4}\right)$ for which

$$|(\nabla u)_{\Omega_{2\sigma r}(x)}|, |(\nabla u)_{\Omega_r(x)}| \leq 2^{3n+1} M, \tag{4.95}$$

we have

$$E(x, \sigma r) \leq C \left(\sigma^{2\beta} + \sigma^{-(n+2)} \gamma \left([\nabla u]_{\mathrm{BMO}(\Omega_r(x))}\right)\right) E(x, r) + C M^{2(q-1)} \sigma^{-(n+2)} r^{2\beta}, \tag{4.96}$$

where γ is as in Lemmas 4.3.3 and 4.3.5 with $2^{3n+1} M$ in place of M, and

$$C = C\left(n, N, q, \Lambda_{\widetilde{M}}/\lambda_{\widetilde{M}}, \delta, \|\Omega\|_{C^{1,\beta}}, R_0, [\nabla g]_{C^{0,\beta}(\Omega)}\right) > 0. \tag{4.97}$$

Proof of claim: Applying the Caccioppoli-type inequality (Lemma 4.3.3) we have

$$\begin{aligned} E(x, \sigma r) &\leq C\, \gamma \left([\nabla u]_{\mathrm{BMO}(\Omega_{2\sigma r}(x))}\right) E(x, 2\sigma r) \\ &\quad + \frac{1}{\sigma^2 r^2} \fint_{\Omega_{2\sigma r}(x)} |u - g - a_{2\sigma r}|^2 \, \mathrm{d}y + C M^{2(q-1)} (\sigma r)^{2\beta}, \end{aligned} \tag{4.98}$$

where $a_{2\sigma r}$ is given by (4.66) in $\Omega_{2\sigma r}(x)$. Also by the boundary harmonic approximation (Lemma 4.3.5) in $\Omega_r(x)$ the unique solution $\tilde{h} \in W^{1,2}(\widetilde{\Omega}_r(x), \mathbb{R}^N)$ solving

$$\begin{cases} -\operatorname{div} F''(z_r) \nabla \tilde{h} = 0 & \text{in } \widetilde{\Omega}_r(x), \\ \qquad\qquad \tilde{h} = u - g - a_r & \text{on } \partial\widetilde{\Omega}_r(x), \end{cases} \tag{4.99}$$

satisfies

$$\frac{1}{\sigma^2 r^2} \fint_{\Omega_{r/2}(x)} |u - g - a_r - \tilde{h}|^2 \, \mathrm{d}y \leq C\, \gamma \left([\nabla u]_{\mathrm{BMO}(\Omega_r(x))}\right) E(x, r) + C M^2 r^{2\beta}, \tag{4.100}$$

noting that $\Omega_{r/2}(x) \subset \widetilde{\Omega}_r(x) \subset \Omega_r(x)$. Now by Remark 4.3.4 we have

$$\frac{1}{\sigma^2 r^2} \fint_{\Omega_{2\sigma r}(x)} |u - g - a_\sigma|^2 \, \mathrm{d}y \leq \frac{1}{\sigma^2 r^2} \fint_{\Omega_{2\sigma r}(x)} \left| u - g - \xi \frac{\rho}{|\nabla \rho(x)|} \right|^2 \mathrm{d}y \tag{4.101}$$

for all $\xi \in \mathbb{R}^N$, so taking $\xi = \xi_r + (\nabla \tilde{h} \cdot \nu_x)_{\Omega_{2\sigma r}(x)}$ we can split

$$\begin{aligned} &\frac{1}{\sigma^2 r^2} \fint_{\Omega_{2\sigma r}(x)} |u - g - a_\sigma|^2 \, \mathrm{d}y \\ &\quad \leq \frac{1}{\sigma^2 r^2} \fint_{\Omega_{2\sigma r}(x)} \left| \tilde{h} - (\nabla \tilde{h} \cdot \nu_x)_{\Omega_{2\sigma r}(x)} \frac{\rho}{|\nabla \rho(x)|} \right|^2 \mathrm{d}y \\ &\qquad + C \sigma^{-(n+2)} \gamma \left([\nabla u]_{\mathrm{BMO}(\Omega_r(x))}\right) E(x, r) + C M^{2(q-1)} \sigma^{-(n+2)} r^{2\beta}. \end{aligned} \tag{4.102}$$

For the second term we use the Poincaré inequality and Lemma 4.3.1 to estimate

$$\begin{aligned}
&\frac{1}{\sigma^2 r^2}\fint_{\Omega_{2\sigma r}(x)} \left|\tilde{h} - (\nabla\tilde{h}\cdot\nu_x)_{\Omega_{2\sigma r}(x)}\frac{\rho}{|\nabla\rho(x)|}\right|^2 \mathrm{d}y \\
&\quad\le C\fint_{\Omega_{2\sigma r}(x)} \left|\nabla\tilde{h} - (\nabla\tilde{h}\cdot\nu_x)_{\Omega_{2\sigma r}(x)}\frac{\nabla\rho}{|\nabla\rho(x)|}\right|^2 \mathrm{d}y \\
&\quad\le C\fint_{\Omega_{2\sigma r}(x)} \left|\nabla\tilde{h} - (\nabla\tilde{h}\cdot\nu_x)_{\Omega_{2\sigma r}(x)}\otimes\nu_x\right|^2 \mathrm{d}y + CM^2(\sigma r)^{2\beta} \\
&\quad\le C\fint_{\Omega_{2\sigma r}(x)} \left|\nabla\tilde{h} - (\nabla\tilde{h})_{\Omega_{2\sigma r}(x)}\right|^2 \mathrm{d}y + CM^2\sigma^{-n}r^{2\beta},
\end{aligned}\tag{4.103}$$

where we have used the bound $|(\nabla\tilde{h})_{\Omega_{2\sigma r}(x)}\cdot\nu_x|^2 \le CM^2\sigma^{-n}$. Now as $\tilde{h}$ vanishes on $\partial\Omega\cap\partial\Omega_r(x)$, using boundary Schauder estimates (Proposition 3.3.1) we have the estimate

$$\left[\nabla\tilde{h}\right]_{C^{0,\beta}(\overline{\Omega}_{r/2}(x))} \le C\fint_{\widetilde{\Omega}_r(x)} |\nabla(u-g-a_R)|\,\mathrm{d}y \le CE(x,r) + CM^2r^{2\beta},\tag{4.104}$$

where the last line is obtained by arguing as in the proof of Lemma 4.3.3. Hence it follows that

$$\frac{1}{\sigma^2 r^2}\fint_{\Omega_{2\sigma r}(x)} \left|\tilde{h} - (\nabla\tilde{h}\cdot\nu_x)_{\Omega_{2\sigma r}(x)}\frac{\rho}{|\nabla\rho(x)|}\right|^2 \mathrm{d}y \le C\sigma^{2\beta}E(x,r) + CM^2\sigma^{-n}r^{2\beta},\tag{4.105}$$

so the claim follows by putting everything together.

We now argue analogously as in the interior case; note for $x\in\partial\Omega\cap B_{R/2}(x_0)$ we have $|(\nabla u)_{\Omega_{R/2}(x)}| \le 2^{3n}M$, and so $|(\nabla u)_{\Omega_{\sigma R/2}(x)}| \le 2^{3n}M + C_1\sigma^{-n}\varepsilon \le 2^{3n+1}M$ for $\varepsilon > 0$ sufficiently small. Hence applying the claim gives

$$E(x,\sigma R/2) \le C\left(\sigma^{2\beta} + \sigma^{-(n+2)}\gamma(\varepsilon)\right)E(x,r/2) + CM^{2(q-1)}\sigma^{-(n+2)}\widetilde{R}_0^{2(\beta-\alpha)}R^{2\alpha}.\tag{4.106}$$

We choose $\sigma\in\left(0,\frac{1}{4}\right)$ such that $C\sigma^{2\beta}\le\frac{1}{4}\sigma^{2\alpha}$, and $\varepsilon > 0$ such that $C\sigma^{-(n+2)}\gamma(\varepsilon)\le\frac{1}{4}\sigma^{2\alpha}$. We then choose $\widetilde{R}_0 > 0$ such that $CM^{2(q-1)}\sigma^{-(n+2)}\widetilde{R}_0^{2(\beta-\alpha)} \le \kappa\sigma^{2\alpha}$ for $0<\kappa<1$ to be chosen to get

$$E(x,\sigma R/2) \le \frac{1}{2}\sigma^{2\alpha}E(x,R/2) + \kappa\left(\sigma R\right)^{2\alpha}.\tag{4.107}$$

Further shrinking $\varepsilon > 0$ if necessary and taking $\kappa > 0$ small enough so

$$\sigma^{-(n+2)}(C_1\varepsilon+\kappa)\sum_j \sigma^{\alpha j} \le 3^n M,\tag{4.108}$$

we can iteratively argue that for all $k\ge 0$,

$$|(\nabla u)_{B_{\sigma^k R/2}(x)}| \le 2^{3n+1}M,\tag{4.109}$$

$$E(x,\sigma^k R/2) \le 2^{-k}\sigma^{2\alpha k}E(x,R/2) + (\sigma^k R)^{2\alpha}.\tag{4.110}$$

Hence it follows that $E(x,r) \leq Cr^{2\alpha}$ for all $r \in (0, R/2)$.

By the interior case we also have $E(x,r) \leq Cr^{2\alpha}$ when $B(x,r) \subset \Omega_R(x_0)$ with $x \in B_{R/2}$. We can extend this to all $x \in \Omega_{R/2}(x_0)$ and $0 < r < R/2$ by a covering argument (adjusting constants as necessary), so by the Campanato-Meyers characterisation we get u is $C^{1,\alpha}$ in $\overline{\Omega}_{R/2}(x_0)$, as required. □

4.4 The uniformly elliptic case

Finally we prove the uniformly elliptic case, which straightforwardly follow from the above using the careful dependence of constants.

Proof of Theorem 4.1.9. By our hypotheses, if $\Lambda_{|z_0|}, \lambda_{|z_0|}$ are the constants associated with the shifted functional F_{z_0} from Lemma 4.2.1, then the quotient $\Lambda_{|z_0|}/\lambda_{|z_0|}$ is bounded uniformly in $z_0 \in \mathbb{R}^{Nn}$, as is $\omega = \omega_{|z_0|}$. Hence arguing as in the proofs of Theorems 4.1.2 and 4.1.3, noting in the intermediate lemmata the constants only dependend on $\Lambda_{|z_0|}/\lambda_{|z_0|}$, we obtain the excess decay estimate

$$\begin{aligned} E(x,\sigma r) \leq C \left(\sigma^{2\beta} + \sigma^{-(n+2)}\gamma\left([\nabla u]_{\mathrm{BMO}(\Omega_r(x))}\right)\right) E(x,r) \\ + C(1 + |(\nabla u)_{\Omega_{2\sigma r}}| + |(\nabla u)_{\Omega_r}|)^2 \sigma^{-(n+2)} r^{2\beta} \end{aligned} \tag{4.111}$$

for all $x \in \overline{\Omega}$, $R > 0$ such that either $B_R(x) \subset \Omega$ or $x \in \overline{\Omega}$ and $0 < R < R_0$ (with $R_0 = R_0(n, \Omega) > 0$). Note in the interior case the second term can be omitted.

Fix $\varepsilon > 0$ to be determined. Then there is $0 < R < \frac{R_0}{2}$ for which there exists a finite covering of Ω by balls $\{B_R(x_j)\}$ where either $B_R(x_j) \subset \Omega$ or $x_j \in \overline{\Omega}$, and $[\nabla u]_{\mathrm{BMO}(\Omega_{2R}(x_j))} \leq 2\varepsilon \leq 1$ for each j. Let $M > 0$ such that $|(\nabla u)_{\Omega_{2R}(x_j)}| \leq M$ for all j, then observe that for all $x \in \overline{\Omega}$ and $0 < r < R$ we have $|(\nabla u)_{\Omega_r(x)}| \leq C(n)M\left(1 + \log(R/r)\right)$. Hence the excess decay estimate becomes

$$E(x,\sigma r) \leq C\left(\sigma^{2\beta} + \sigma^{-(n+2)}\gamma(2\varepsilon)\right) E(x,r) + CM^2\sigma^{-(2n+2)} r^{2\beta}\left(1 + \log(R/r)\right) \tag{4.112}$$

whenever $0 < r < R$, and modifying constants this holds for all $x \in \overline{\Omega}$.

Now choose $\sigma \in (0, \frac{1}{4})$ such that $C\sigma^{2\beta} \leq \frac{1}{4}\sigma^{2\alpha}$, and $\varepsilon > 0$ such that $C\sigma^{-(n+2)}\gamma(2\varepsilon) \leq \frac{1}{4}\sigma^{2\alpha}$. Then choose $0 < r_0 < R$ such that $CM^2\sigma^{-(2n+2)} r_0^{2(\beta-\alpha)}\left(1 + \log(R/r_0)\right) \leq \sigma^{2\alpha}$. This gives

$$E(x,\sigma r) \leq \frac{1}{2}\sigma^{2\alpha} E(x,r) + (\sigma r)^{2\alpha}, \tag{4.113}$$

from which the result follows by iteration as in the proof of Theorem 4.1.3. $\square$

Chapter 5

Minima for quasiconvex integrands

In this chapter we will investigate the existence and regularity theory for minima of functionals of the form

$$\mathcal{F}(w) = \int_\Omega F(x, w(x), \nabla w(x))\,\mathrm{d}x, \tag{5.1}$$

subject to a quasiconvexity and a general growth condition. More precisely the growth of F will be governed by an N-function φ, as defined in Section 2.1.2, and will satisfy the Δ_2-condition.

We will be concerned with integrands $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ which are *strictly quasiconvex*, in the sense that if there is $\lambda > 0$ such that

$$\int_\Omega f(z_0 + \nabla\xi) - f(z_0)\,\mathrm{d}x \geq \lambda \int_\Omega \varphi_{1+|z_0|}(|\nabla\xi|)\,\mathrm{d}x \tag{5.2}$$

for all $z_0 \in \mathbb{R}^{Nn}$ and $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^n$ non-empty and open. Here we recall that φ_a are the shifted N-functions defined in Definition 2.1.8, and that if $\varphi \in \Delta_2$, for each $M > 0$ we have $\varphi_{1+|z_0|} \sim_M \varphi_1$ whenever $|z_0| \leq M$. Also in the case of p-growth, the left-hand side is equivalent (up to modifying λ) to

$$\int_{\mathbb{R}^n} |V_{p,1+|z_0|}(\nabla\xi)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n} (1 + |z_0|^2 + |\nabla\xi|^2)^{\frac{p-2}{2}} |\nabla\xi|^2\,\mathrm{d}x \tag{5.3}$$

where $V_{p,\mu}(t) = (\mu^2 + t^2)^{\frac{p-2}{4}} t$. This was introduced by Evans [Eva86] when $p \geq 2$ in the context of regularity theory. However such a condition also plays a role in the existence theory, as it ensures minimising sequences are bounded in $W^{1,\varphi}(\Omega, \mathbb{R}^N)$; this will be discussed in Section 5.2.4 adapting a result of Chen & Kristensen [CK17].

Remark 5.0.1. In practice it can be easier to verify that $F - \lambda\varphi_1(|\cdot|)$ is quasiconvex for some $\lambda > 0$. In light of Lemma 5.2.17 below, such a condition implies (5.2) assuming φ is C^2, with strictly positive second derivative. This does introduce an implicit $|z_0|$-dependence in the constants, however this does not affect any of our results below.

Using this observation, we can show that strictly quasiconvex integrands with φ-growth exist based on the construction of Šverák [Šve91].

Example 5.0.2. Let $\varphi \in \Delta_2$ be an N-function. Following the argument in Proposition 5.2.16, we can consider the shifted N-function associated to $\varphi(t) + t + (e^{-t} - 1)$ and apply Corollary 2.1.7 to obtain an N-function $\widetilde{\varphi}$ which is C^2 on $[0, \infty)$ such that $\widetilde{\varphi} \sim \varphi_1$ and $\varphi_1''(t) \geq 0$ for all $t \geq 0$.

We then apply Lemma 2.1.34 to obtain a quasiconvex function $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ which is non-convex. Moreover by mollification, which preserves the quasiconvexity of F, we can assume it is smooth. Then by Lemma 5.2.17, we have $F + \widetilde{\varphi}(|\cdot|)$ satisfies the strict quasiconvexity condition (5.2) with φ.

We will start by discussing the existence using the *Direct Method*, which based on the strategies discussed in the introduction (Section 1.1). We will establish a general semicontinuity result for integrands of the form $F = F(x, \nabla u)$ adapting a triangulation argument of Chen & Kristensen [CK17]. These results will also be extended to deduce the existence of minima for integrands with a u-dependence, where we will establish lower-semicontinuity along *regularised* minimising sequences, which are uniformly integrable.

We will then discuss the associated partial regularity theory for the obtained minimisers. Here we will need to additionally assume our N-function satisfies the ∇_2-condition, however we believe this should not be necessary. In this direction we will show that in the autonomous case the ∇_2-condition can be replaced by a growth condition, and this is a point of ongoing investigation.

5.1 Hypotheses and main results

For the corresponding regularity problem, we will consider the following assumptions.

Hypotheses 5.1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ with $n \geq 2$, $N \geq 1$ satisfy the following.

(A1) F is continuous in (x, u, z) and of class C^2 in z, such that F_{zz} is also continuous.

(A2) F satisfies the *natural growth condition*

$$|F(x, u, z)| \leq \Lambda(1 + \varphi(z))$$

for all (x, u, z), where $\Lambda > 0$ and φ is an N-function. By rescaling we can assume that $\varphi(1) = 1$.

(A3) There exists $g\colon \mathbb{R}^{Nn} \to \mathbb{R}$ which is strictly quasiconvex at the origin in the sense of (5.2) with constant $\lambda > 0$ such that

$$g(z) \leq F(x, u, z)$$

for all (x, u, z).

(A4) We have $z \mapsto F(x, u, z)$ is strictly quasiconvex in the sense of 5.2 at each $(x, u) \in \Omega \times \mathbb{R}^N$, with constant $\lambda > 0$.

(A5) F is jointly β-Hölder continuous in (x, u) for some $\beta \in (0, 1)$, in that we have

$$|F(x, u, z) - F(y, v, z)| \leq \Lambda\, \vartheta_{M,\beta}(|x - y| + |u - v|)(1 + \varphi(|z|))$$

for all $x, y \in \Omega$, $u, v \in \mathbb{R}^N$, $z \in \mathbb{R}^{Nn}$ such that $|u| \leq M$, where $\vartheta_{M,\beta}(t) = \min\left\{1, c_M t^\beta\right\}$. for some $c_M > 0$.

Note that for any $\varphi \in \Delta_2$, the autonomous integrand constructed in Example 5.0.2 satisfies the hypotheses of the above theorem.

The main result we will prove in this section is the following. Note that we will establish a much more general existence theorem (see Proposition 5.2.3), however the following captures the main case we are interested in.

Theorem 5.1.2 (Existence and partial regularity)**.** Suppose F satisfies Hypotheses 5.1.1, and the associated N-function satisfies the Δ_2-condition. Then for any $\Omega \subset \mathbb{R}^n$ and $g \in$

$W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N)$, there exists a minimiser $u \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ of $\mathcal{F}$. Moreover if $\varphi \in \nabla_2$ in addition, there is $\Omega_0 \subset \Omega$ such that $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$ and such that u is $C^{1,\alpha}$ in Ω_0 for some $\alpha > 0$.

As is standard for partial regularity results, this will follow from an ε-regularity theorem which is the content of Theorem 5.3.1. We note that in the autonomous ($F = F(z)$) case, the regularity theory in the quasiconvex and general growth setting was first considered by Diening, Lengeler, Stroffolini, & Verde [Die+12], subject to additional quantitative estimates on F'' and stronger conditions on φ. This was relaxed to the natural growth condition by the author in [Irv21b], where the result was also extended to higher-order integrands and strong local minimisers which we will not discuss here. Extensions to the non-autonomous setting have also been considered in [CO20; GSS21; OSS21], however they all assume a strong controlled growth condition building upon the techniques in [Die+12]. We improve improve upon the above results by establishing partial regularity under natural growth conditions in the non-autonomous setting.

5.2 Existence of minima

In this section we will establish the existence of minima for $\mathcal{F}$ using the Direct Method. For this we will require a lower-semicontinuity result, along with a coercivity estimate to obtain boundedness of minimising sequences. Once these results have been established, we can apply the Direct Method since $W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ is equipped with a weak* topology as noted in Discussion 2.3.2.

We will largely consider two approaches to existence; the first will establish a general lower-semicontinuity result for signed integrands, and will follow from an argument due to Chen & Kristensen [CK17] via piecewise affine approximations. The second will involve selecting a regularised minimising sequence using Ekeland's variation principle, building upon ideas of Marcellini & Sbordone [MS83], which was recently considered by Chen [Che16].

There is an extensive literature on lower-semicontinuity results for quasiconvex integrands, starting from the fundamental works of Morrey [Mor52] and Meyers [Mey65].

We will not provide a detailed survey, but we mention the works of Acerbi & Fusco [AF84] and Marcellini [Mar85] for the case of non-autonomous integrands, and the results of Fonseca & Malý [FM97] for mixed growth conditions.

When establishing weak* sequential lower-semicontinuity in the full space $W^{1,\varphi}(\Omega,\mathbb{R}^N)$ however, one requires a growth condition from below of the form

$$-c(1+\psi(|z|)) \leq F(x,u,z) \leq c_1(1+\varphi(|z|)), \tag{5.4}$$

where ψ is an N-function of the form $\lim_{t\to\infty}\frac{\psi(t)}{\varphi(t)}=0$. This imposes a sub-critical growth condition on the negative part of F, and is necessary due to potential concentration near the boundary. This is seen in the following example due to Ball & Murat [BM84a, Section 7].

Example 5.2.1 (Failure of weak continuity of the Jacobian)**.** Consider the case $n=N=2$ and $F(z)=\det z$ on the unit cube $Q=\{x\in\mathbb{R}^2 : |x_1|,|x_2|<1\}$. This is a quasi-convex function satisfying a quadratic growth condition, so we may expect the associated integral operator is sequentially weakly lower-semicontinuous in $W^{1,2}$. However Ball & Murat [BM84a] noted the sequence

$$u_j(x_1,x_2) = j^{-\frac{1}{2}}(1-|y|)^j(\sin jx, \cos jx) \tag{5.5}$$

is weakly bounded in $W^{1,2}(D,\mathbb{R}^2)$ and converges weakly to zero. However letting $Q_a^+ = (-a,a)\times(0,a)\subset Q$ for $0<a<1$ we have

$$\lim_{j\to\infty}\int_{Q_a^+}\det\nabla u_j \,\mathrm{d}x_1\,\mathrm{d}x_2 = -a < 0, \tag{5.6}$$

exhibiting the failure of weak lower-semicontinuity.

The additional condition (5.4) is not needed for the corresponding regularity theory however, and one may wonder if we can still show the existence of minima in the absence of this condition. This was shown already by Meyers [Mey65], who noticed that this can be circumvented if one considers the Dirichlet class $W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$, where the fixed boundary prevents such concentration phenomena. In the autonomous case, this observation has been extended to the setting of mixed growth by Chen & Kristensen [CK17]. In the linear case this can be viewed as the failure of weak* continuity of the trace operator in BV, as observed by Kristensen & Rindler [KR10, Theorem 2].

We will adapt the results in [CK17] to the general growth case, and allow for a continuous x-dependence. While it is not clear whether we can allow a u-dependence in general, we will show that lower-semicontinuity still holds along *minimising sequences* which will be sufficient to obtain the existence of minimisers.

More precisely we will establish the following two results.

Proposition 5.2.2 (Lower semicontinuity in Dirichlet classes)**.** Suppose $\Omega \subset \mathbb{R}^N$ and $g \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$, where φ is an N-function satisfying the Δ_2-condition. Let $F\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a continuous integrand satisfying the natural growth condition (A2), and moreover the joint continuity estimate

$$|F(x,z) - F(y,z)| \leq \Lambda\, \omega(|x-y|)(1+\varphi_1(|z|)) \tag{5.7}$$

for all $x, y \in \Omega$, $z \in \mathbb{R}^{Nn}$ where $\omega : [0,\infty) \to [0,1]$ is a non-decreasing continuous concave function such that $\omega(0) = 0$. Further assume that such that $z \mapsto F(x,z)$ is quasiconvex for all $x \in \Omega$. Then the associated functional $\mathcal{F}$ is weakly* sequentially lower-semicontinuous on $W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$.

Proposition 5.2.3 (Lower semicontinuity along minimising sequences)**.** Suppose $\Omega \subset \mathbb{R}^N$ is a bounded open set and $g \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$, where φ is an N-function satisfying the Δ_2-condition. Let $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a Carathéodory integrand satisfying the natural growth condition (A2), the coercivity condition (A3), and assume that $z \mapsto F(x,u,z)$ is quasiconvex for all $(x,u) \in \Omega \times \mathbb{R}^N$. Assume in addition that there is $\kappa_0 > 0$ for which

$$\varphi(|\nabla g|) \left(\frac{\varphi(|\nabla g|)}{|\nabla g|} \right)^{\kappa_0} \in L^1(\mathbb{R}^n). \tag{5.8}$$

Then if $\{u_j\} \subset W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ is a minimising sequence for $\mathcal{F}$, then for any weak* limit u we have

$$\liminf_{j\to\infty} \mathcal{F}(u_j) \geq \mathcal{F}(u). \tag{5.9}$$

In particular, u is a minimiser of $\mathcal{F}$ in the Dirichlet class $W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$.

In particular, this proves the existence part of Theorem 5.1.2.

5.2.1 Lower-semicontinuity I

In this section we will prove Proposition 5.2.2, for which we will closely follow the argument of Chen & Kristensen [CK17, Theorem 5.1].

Discussion 5.2.4 (Piecewise affine approximation)**.** Our arguments will involve approximation by piecewise affine maps, whose construction we will briefly sketch. We say $\sigma \subset \mathbb{R}^n$ is an *k-simplex* if there exists set of $(k+1)$-points $V = \{v_0, \ldots, v_k\}$ that are affinely independent such that $\sigma = \operatorname{conv}(V)$. Elements of V are called *vertices* of σ, and a *face* of σ is the convex hull of any proper subset of V. A *triangulation* of an open set $\Omega \subset \mathbb{R}^n$ is a collection $\mathcal{T}$ of n-simplicies contained in Ω such that $\overline{\Omega} = \overline{\bigcup \mathcal{T}}$, each $\sigma \in \mathcal{T}$ only intersects finitely many other $\tau \in \mathcal{T}$, and $\sigma \cap \tau$ is non-empty if and only if is a face of both σ and τ. The *mesh* of a triangulation $\mathcal{T}$ refers to the maximal diameter of any simplex $\sigma \in \mathcal{T}$.

For a simplex τ with vertices V, for any function $f \colon V \to \mathbb{R}^N$ there is a unique affine function $\tilde{f} \colon \tau \to \mathbb{R}^N$ which agrees with f on V. This can be constructed by setting

$$\tilde{f}\left(\sum_{v\in V} \lambda_v v\right) = \sum_{v \in V} \lambda_v f(v), \tag{5.10}$$

where each $\lambda_v \geq 0$ such that $\sum_{v\in V} \lambda_v \leq 1$. Now if $\mathcal{T}$ is a triangulation of Ω and $f \colon \Omega \to \mathbb{R}^N$ is any function, we can construct $a \colon \Omega \to \mathbb{R}^N$ piecewise affine such that $a(v) = f(v)$ for any vertex v of any $\tau \in \mathcal{T}$. We say a is obtained from f by *Lagrange interpolation.*

We can triangulate the unit cube $Q = (0,1)^n$ by 2^{n-1} simplicies, where the vertices of each simplex coincides with the vertices of Q. By decomposing $\mathbb{R}^n$ into cubes given by translates of Q we obtain a triangulation $\mathcal{T}_0$ of $\mathbb{R}^n$. Note that each simplex is a translate of one of the 2^{n-1} simplicies in Q, and by scaling these simplicies we can obtain triangulations with arbitrarily small mesh.

The idea will be to approximate the limit map u by a suitable piecewise approximation a with respect to a triangulation $\mathcal{T}$, and apply the quasiconvexity condition on each $\tau \in \mathcal{T}$ to compare $u_j|_\tau$ with $a|_\tau$. However as these need not agree in $\partial\tau$, we instead consider a slightly larger simplex $k\tau$ for some $k > 1$, and extend u_j to $k\tau$ in a way that it agrees with $a|_\tau$ (extending to an affine map on $\mathbb{R}^n$) on $\partial(k\tau)$.

Proof of Proposition 5.2.2. Suppose $u_j, u \in W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$ such that $\nabla u_j \overset{*}{\rightharpoonup} \nabla u$ in $L^\varphi(\Omega,\mathbb{R}^{Nn})$, which extending by g we will view as elements in $W^{1,\varphi}(\mathbb{R}^n,\mathbb{R}^N)$. Given $\delta > 0$ to be determined, let $\mathcal{T}_\delta$ be the uniform triangulation of $\mathbb{R}^n$ constructed in Discussion 5.2.4 which has been rescaled to have mesh δ. Then for any $\xi \in B_1$, we set $\mathcal{T}_{\xi,\delta} = \{\xi + \tau : \tau \in \mathcal{T}_\delta\}$ for the translated triangulation.

Claim 1: For each $\varepsilon > 0$, there is $\delta \in (0,1)$ such that for any $\xi \in B_1$ there exists $a \colon \mathbb{R}^n \to \mathbb{R}^N$ which is piecewise affine with respect to $\mathcal{T}_{\xi,\delta}$ (so $a|_\tau$ is affine for all $\tau \in \mathcal{T}_{\xi,\delta}$) such that

$$\int_{\Omega+B_1} \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x < \varepsilon, \tag{5.11}$$

$$\int_{\Omega+B_1} |F(x,\nabla u) - F(x,\nabla a)|\,\mathrm{d}x < \varepsilon. \tag{5.12}$$

Proof of claim 1: Let $\{\eta_t\}_{t>0}$ be a standard mollifier in $\mathbb{R}^n$, then for $t > 0$ sufficiently small we can ensure that

$$\int_{\Omega+B_1} \varphi_1(|\nabla u - \nabla u * \eta_t|)\,\mathrm{d}x < \frac{\varepsilon}{2\Delta_2(\varphi)}, \tag{5.13}$$

$$\int_{\Omega+B_1} |F(x,\nabla u) - F(x,\nabla u * \eta_t)|\,\mathrm{d}x < \frac{\varepsilon}{2}. \tag{5.14}$$

This follows by noting that both quantities vanish as $t \to 0$, and using the growth condition (A2) and the Δ_2-condition to apply the dominated convergence theorem. Given such a t, we now take $a = a_\delta$ be the map obtained by applying Lagrange interpolation to $u * \eta_t$ with respect to $\mathcal{T}_{\xi,\delta}$. Since $u * \eta_t$ is everywhere smooth, the claim follows by noting that $\nabla a_\delta \to \nabla u * \eta_t$ locally uniformly in $\mathbb{R}^n$ and independently of ξ. Hence the claim holds for $\delta > 0$ sufficiently small.

Claim 2: For almost all $\xi \in B_1$, passing to a subsequence for each $\tau \in \mathcal{T}_{\xi,\delta}$ we have every $u|_{\partial\tau}, u_j|_{\partial\tau} \in W^{1,\varphi}(\partial\tau,\mathbb{R}^N)$ and

$$u_j|_{\partial\tau} \overset{*}{\rightharpoonup} u|_{\partial\tau} \text{ in } W^{1,\varphi}(\partial\tau,\mathbb{R}^N), \tag{5.15}$$

where $u|_{\partial\tau}$ denotes the restriction of u to $\partial\tau$ in the trace sense.

Proof of claim 2: Let $e \in S^{n-1}$ ($e \in \mathbb{R}^n$ such that $|e| = 1$) such that e is non-tangential to all of the faces of $\tau \in \mathcal{T}_\delta$; since it suffices to check the 2^{n-1} simplicies contained in

$Q_\delta = (0,\delta)^n$, this holds for almost all e. Then for each $\tau \in \mathcal{T}_\delta$ an application of Fubini's theorem gives

$$\int_0^\delta \int_{te+\partial\tau} \varphi_1(|\nabla u_j|)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \leq C \int_{\tau+B_\delta} \varphi_1(|\nabla u_j|)\,\mathrm{d}x. \tag{5.16}$$

This is the proved in [CK17, Lemma 5.1]. By Fatou's lemma, (5.16) and the fact that $\{\tau + B_\delta\}_{\tau\in\mathcal{T}_\delta}$ is a locally finite covering of $\mathbb{R}^n$ we can estimate

$$\begin{aligned}\sum_{\tau\in\mathcal{T}_\delta}\int_0^\delta \liminf_{j\to\infty}\int_{te+\partial\tau} \varphi_1(|\nabla u_j|)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t &\leq \liminf_{j\to\infty}\sum_{\tau\in\mathcal{T}_\delta}\int_0^\delta\int_{te+\partial\tau} \varphi_1(|\nabla u_j|)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t \\ &\leq \liminf_{j\to\infty}\sum_{\tau\in\mathcal{T}_\delta} C\int_{\tau+B_\delta}\varphi_1(|\nabla u_j|)\,\mathrm{d}x \\ &\leq \liminf_{j\to\infty} C\int_{\mathbb{R}^n}\varphi(|\nabla u_j|)\,\mathrm{d}x,\end{aligned} \tag{5.17}$$

which is finite by weak* convergence of ∇u_j to ∇u. In particular there is $t \in (0,\delta)$ such that letting $\xi = te$ we have

$$\sum_{\tau\in\mathcal{T}_{\xi,\delta}} \liminf_{j\to\infty}\int_{\partial\tau}\varphi_1(|\nabla u_j|)\,\mathrm{d}\mathcal{H}^{n-1} < \infty. \tag{5.18}$$

A similar argument noting that $u_j \to u$ strongly in $L^\varphi(\mathbb{R}^n,\mathbb{R}^N)$ ensures we can choose t so that in addition,

$$\sum_{\tau\in\mathcal{T}_{\xi,\delta}} \lim_{j\to\infty}\int_{\partial\tau}\varphi_1(|u_j-u|)\,\mathrm{d}\mathcal{H}^{n-1} < \infty. \tag{5.19}$$

Hence passing to a subsequence, for each $\tau \in \mathcal{T}_{\xi,\delta}$ we get $\nabla u_j|_{\partial\tau}$ is uniformly bounded in $L^\varphi(\partial\tau,\mathbb{R}^{Nn})$ and $u_j \to u$ in $L^\varphi(\partial\tau,\mathbb{R}^N)$, so it follows that $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\varphi}(\partial\tau,\mathbb{R}^N)$ establishing the claim.

Now let $k > 1$, and if $\sigma = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ is the standard simplex in $\mathbb{R}^n$, we can define the extension operator

$$E_\sigma\colon W^{1,\varphi}(\partial\sigma,\mathbb{R}^N) \to W^{1,\varphi}(k\sigma\setminus\sigma,\mathbb{R}^N), \tag{5.20}$$

which follows from Proposition 2.3.9 with $q = 1$. This operator is compact, and can moreover been chosen to preserve affine maps. Then for each $\tau \in \mathcal{T}_{\xi,\delta}$, by an affine transformation we can apply this on each τ to obtain $w_j, w \in W^{1,\varphi}(k\tau\setminus\tau,\mathbb{R}^N)$ such that $w_j|_{\partial\tau} = u_j|_{\partial\tau}$, $w|_{\partial\tau} = u|_{\partial\tau}$ and $w_j \to w$ in $W^{1,\varphi}(k\tau\setminus\tau,\mathbb{R}^N)$.

Now let $\rho \in C_c^\infty(k\tau, \mathbb{R}^N)$ such that $1_\tau \leq \rho \leq 1_{k\tau}$ and $|\nabla \rho| \leq \frac{C}{k-1}$ and define

$$v_j = \begin{cases} u_j + b & \text{in } \tau, \\ a_\tau + \rho(w_j - a_\tau + b_j) & \text{in } k\tau \setminus \tau, \end{cases} \tag{5.21}$$

where a_τ is the affine map which agrees with a on τ, and $b_j = \fint_{k\tau\setminus\tau} a_\tau - w_j \,\mathrm{d}x$. Then setting $x_\tau \in \tau$ to be the barycentre, noting $v_j|_{\partial(k\tau)} = a_\tau|_{\partial(k\tau)}$ for each j we can use the quasiconvexity of F at ∇a_τ to estimate

$$\begin{aligned} F(x_\tau, \nabla a_\tau)\mathcal{L}^n(k\tau) &\leq \int_{k\tau} F(x_\tau, \nabla v_j)\,\mathrm{d}x \\ &= \int_\tau F(x_\tau, \nabla u_j)\,\mathrm{d}x \\ &\quad + \int_{k\tau\setminus\tau} F\left(x_\tau, \nabla a_\tau + \rho\nabla(w_j - a_\tau) + (w_j - a_\tau + b_j)\nabla\rho\right)\mathrm{d}x \\ &= I_j + II_j. \end{aligned} \tag{5.22}$$

For the second term we can use the growth condition (A2), the doubling property of φ and the Poincaré inequality (noting our choice of b_j and using a scaling argument to deduce the k-dependence) to estimate

$$\begin{aligned} |II_j| &\leq C \int_{k\tau\setminus\tau} 1 + \varphi_1(|\nabla a_\tau|) + \varphi_1(|\nabla w_j - a_\tau|) + \varphi_1\left(\frac{1}{k-1}(|w_j - a_\tau + b_j|)\right)\mathrm{d}x \\ &\leq C(k^n - 1)(1 + \varphi_1(|\nabla a_\tau|))\mathcal{L}^n(\tau) + C\int_{k\tau\setminus\tau} \varphi_1(|\nabla w_j - \nabla a_\tau|)\,\mathrm{d}x. \end{aligned} \tag{5.23}$$

Now by compactness of the extension operator (noting it is linear and $E_\tau a_\tau = a_\tau$) we have

$$\lim_{j\to\infty} \int_{k\tau\setminus\tau} \varphi_1(|\nabla w_j - \nabla a_\tau|)\,\mathrm{d}x = \int_{k\tau\setminus\tau} \varphi_1(|\nabla w - \nabla a|)\,\mathrm{d}x \leq C\int_\tau \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x. \tag{5.24}$$

Hence summing over all simplicies in $\mathcal{T}_{\xi,\delta}^\Omega := \{\tau \in \mathcal{T}_{\xi,\delta} : \tau \cap \Omega \neq \varnothing\}$ and using the continuity estimate (5.7) to observe that

$$|F(x_\tau, z) - F(x, \tau)| \leq C\,\omega(\delta)(1 + \varphi_1(|z|)) \tag{5.25}$$

for all $z \in \mathbb{R}^{Nn}$ and $x \in k\tau$ we have

$$\begin{aligned} k^n \int_{\bigcup \mathcal{T}_{\xi,\delta}^\Omega} F(x, \nabla a)\,\mathrm{d}x &\leq \liminf_{j\to\infty} \int_{\bigcup \mathcal{T}_{\xi,\delta}^\Omega} F(x, \nabla u_j)\,\mathrm{d}x \\ &\quad + c\int_{\bigcup \mathcal{T}_{\xi,\delta}^\Omega} \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x + c(k^n - 1)\int_{\bigcup \mathcal{T}_{\xi,\delta}^\Omega} 1 + \varphi_1(|\nabla a|)\,\mathrm{d}x \\ &\quad + C\,\omega(\delta)\limsup_{j\to\infty} \int_{\bigcup \mathcal{T}_{\xi,\delta}^\Omega} 1 + \varphi_1(|\nabla u|) + \varphi_1(|\nabla u_j|)\,\mathrm{d}x, \end{aligned} \tag{5.26}$$

where the last term is finite by weak* convergence of ∇u_j. Now using the approximation bounds (5.13), (5.14) and that $u = u_j = g$ on $\mathbb{R}^n \setminus \Omega$ we have

$$\begin{aligned} k^n \int_\Omega F(x, \nabla u)\,\mathrm{d}x \leq \liminf_{j\to\infty} &\int_\Omega F(x, \nabla u_j)\,\mathrm{d}x \\ &+ C\varepsilon + C(k^n - 1)\int_{\Omega + B_1} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x \\ &+ C\,\omega(\delta)\limsup_{j\to\infty}\int_{\bigcup \mathcal{T}^\Omega_{\xi,\delta}} 1 + \varphi_1(|\nabla u|) + \varphi_1(|\nabla u_j|)\,\mathrm{d}x. \end{aligned} \tag{5.27}$$

Hence the result follows by sending $k \to 1$, $\varepsilon \to 0$, and $\delta \to 0$. $\square$

5.2.2 Regularisation of minimising sequences

We now turn to the Carathéodory case, for which we will need the notion of *quasi-minima*, which are u satisfying

$$\mathcal{F}(u, \mathcal{O}) \leq Q\mathcal{F}(u + \xi, \mathcal{O}) \tag{5.28}$$

for all $\mathcal{O} \subset \Omega$ open and $\xi \in C^\infty_c(\mathcal{O}, \mathbb{R}^N)$, for constant some $Q \geq 1$. Note that u is an ordinary minimiser if $Q = 1$.

Remark 5.2.5. We point out that for ordinary minimisers it suffices to verify the case when $A = \Omega$, as we can simply extend ξ by zero to a function on Ω. However this is a genuinely weaker notion for quasi-minima, as the corresponding integrals on $\Omega \setminus A$ no longer cancel out.

Lemma 5.2.6. Let $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a Carathéodory integrand satisfying (A2) and (A3), and suppose $u \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ almost minimises $\mathcal{F}$ in the sense that

$$\mathcal{F}(u) \leq \mathcal{F}(u + \xi) + 1 \tag{5.29}$$

for all $\xi \in C^\infty_c(\Omega, \mathbb{R}^N)$. Then there exists $v \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ such that

$$\mathcal{F}(v) \leq \mathcal{F}(u), \tag{5.30}$$

and v is a quasi-minima of

$$\mathcal{E}_{\varphi,g}(u, \Omega) = \mathcal{E}_{\varphi,g}(u) = \int_\Omega 1 + \varphi_1(|\nabla u|) + \varphi_1(|\nabla g|)\,\mathrm{d}x. \tag{5.31}$$

for some $Q = Q(n, N, \Lambda, \lambda, \Delta_2(\varphi))$.

Remark 5.2.7. As the proof will illustrate, if we moreover have $\mathcal{F}(u) \leq \mathcal{F}(u+\xi) + \varepsilon$ for some $\varepsilon \in (0,1)$ and all $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$, then the corresponding v can be chosen to satisfy

$$\int_\Omega |\nabla(u-v)| \,\mathrm{d}x < \varepsilon. \tag{5.32}$$

Hence if $\{u_j\}$ is a minimising sequence for $\mathcal{F}$, using this observation we can construct a regularised sequence $\{v_j\}$ each satisfying the claimed quasi-minima property, which moreover satisfies $u_j - v_j \overset{*}{\rightharpoonup} 0$ as $j \to \infty$ in $W_g^{1,\varphi}(\Omega, \mathbb{R}^N$.)

We will construct the quasi-minima v using the variational principle of Ekeland [Eke74] which we will recall here. We refer the reader to [Giu03, Theorem 5.6] for a proof.

Proposition 5.2.8 (Ekeland's variational principle, [Eke74])**.** Let (X, d) be a complete metric space and $\mathcal{F} \colon X \to \mathbb{R} \cup \{\infty\}$ a lower-semicontinuous function, which is bounded from below and is not identically $+\infty$. Suppose there is $\varepsilon > 0$ and $u \in X$ such that $\mathcal{F}(u) \leq \mathcal{F}(w) + \varepsilon$ for all $w \in X$. Then there is $v \in X$ such that $\mathrm{d}(u,v) \leq 1$, $\mathcal{F}(v) \leq \mathcal{F}(u)$ and

$$\mathcal{F}(v) \leq \mathcal{F}(w) + \varepsilon\, \mathrm{d}(v,w) \tag{5.33}$$

for all $w \in X$.

Proof of Lemma 5.2.6. Applying Ekeland's variational principle (Proposition 5.2.8), there exists $v \in W_g^{1,1}(\Omega, \mathbb{R}^N)$ satisfying (5.30),

$$\int_\Omega |\nabla(u-v)| \,\mathrm{d}x \leq 1, \tag{5.34}$$

and

$$\mathcal{F}(v) \leq \mathcal{F}(v+\xi) + \int_\Omega |\nabla \xi| \,\mathrm{d}x \tag{5.35}$$

for all $\xi \in W_0^{1,1}(\Omega, \mathbb{R}^N)$. Then for any such ξ, letting $A = \operatorname{supp} \xi$ note by Young's inequality and the Δ_2-condition that

$$\begin{aligned} \int_A |\nabla \xi| \,\mathrm{d}x &\leq \frac{\delta}{2\Delta_2(\varphi)} \int_A \varphi_1(|\nabla \xi|) \,\mathrm{d}x + C_\delta \mathcal{L}^n(A) \\ &\leq \delta \int_A \varphi_1(|\nabla v|) \,\mathrm{d}x + C_\delta \int_A 1 + \varphi_1(|\nabla(v+\xi)|) \,\mathrm{d}x \end{aligned} \tag{5.36}$$

for each $\delta > 0$. Also applying (A2), (A3) we get

$$\begin{aligned}\lambda \int_A \varphi_1(|\nabla(v-g)|)\,\mathrm{d}x &\leq \int_A f(\nabla(v-g)) - f(0)\,\mathrm{d}x \\ &\leq \mathcal{F}(v,A) + \Lambda\mathcal{L}^n(A) \\ &\quad + \int_A F(x,v,\nabla(v-g)) - F(x,v,\nabla v)\,\mathrm{d}x.\end{aligned} \tag{5.37}$$

For the third term we use (A2) to estimate

$$\begin{aligned}&\int_A F(x,v,\nabla(v-g)) - F(x,v,\nabla v)\,\mathrm{d}x \\ &= -\int_A\int_0^1 \partial_z F(x,v,\nabla v - t\nabla g)\nabla g\,\mathrm{d}t\,\mathrm{d}x \\ &\leq C\Lambda \int_A \left(1 + \varphi_1'(|\nabla v|) + \varphi_1'(|\nabla(v+g)|)\right)|\nabla g|\,\mathrm{d}x \\ &\leq \Lambda\delta \int_A \varphi_1(|\nabla v|)\,\mathrm{d}x + C_\delta\Lambda \int_{\Omega_R} 1 + \varphi_1(|\nabla g|)\,\mathrm{d}x,\end{aligned} \tag{5.38}$$

using Young's inequality and the Δ_2-condition in the last line (using Lemma 2.1.6(a)). Combining the above estimates and using (5.35) we have

$$\begin{aligned}\lambda \int_A \varphi_1(|\nabla v|)\,\mathrm{d}x \leq C\int_A \varphi_1(|\nabla(v+\xi)|)\,\mathrm{d}x + C\delta\int_A \varphi_1(|\nabla v|)\,\mathrm{d}x \\ + C_\delta \int_A 1 + \varphi_1(|\nabla g|)\,\mathrm{d}x.\end{aligned} \tag{5.39}$$

From here choosing $\delta > 0$ to be sufficiently small the result follows. $\square$

Lemma 5.2.9 (Caccioppoli inequalities for quasi-minima)**.** Let $v \in W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$ be a quasi-minima of $\mathcal{E}_{\varphi,g}$ as in (5.31) with constant Q, where $\varphi \in \Delta_2$. Then if $x_0 \in \overline{\Omega}$ and $R > 0$ we have

$$⨍_{B_{R/2}(x_0)} \varphi_1(|\nabla v|)\,\mathrm{d}x \leq C ⨍_{B_R(x_0)} \varphi_1\left(\frac{|v-(v)_{\Omega_R}|}{R}\right)\mathrm{d}x + C ⨍_{B_R(x_0)} 1 + \varphi_1(|\nabla g|)\,\mathrm{d}x, \tag{5.40}$$

where $C = C(n,N,\Delta_2(\varphi),Q) > 0$ provided we have $\varphi(1) = 1$, where we have extended v to $\mathbb{R}^n$ by g.

Note that, as the proof will illustrate, the ∇g term can be omitted if $B_R(x_0) \Subset \Omega$.

Proof. To simplify notation we will suppress the x_0-dependence, and by translation we will assume $(v-g)_{\Omega_R} = 0$. Then for $\frac{R}{2} \leq t < s \leq R$ we fix $\eta \in W^{1,\infty}_0(B_R(x_0))$ such that $1_{B_t(x_0)} \leq \eta \leq 1_{B_s(x_0)}$ and $|\nabla\eta| \leq \frac{1}{s-t}$. By the quasi-minima property in $B_t(x_0)$ we have

$$\mathcal{E}_\varphi(v,\Omega_t) \leq \mathcal{E}_\varphi(v,\Omega_s) \leq Q\mathcal{E}_\varphi((1-\eta)v,\Omega_s), \tag{5.41}$$

so then using the Δ_2-condition we have

$$\mathcal{E}_\varphi((1-\eta)v,\Omega_s) \leq C\int_{\Omega_s\setminus\Omega_t} \varphi_1(|\nabla v|) + C\int_{\Omega_R} 1+\varphi_1\left(\frac{|v|}{s-t}\right) + \varphi_1(|\nabla g|)\,\mathrm{d}x. \tag{5.42}$$

Extending to the full ball using g and filling the hole we get $\theta\in(0,1)$ for which

$$\int_{B_t} \varphi_1(|\nabla v|)\,\mathrm{d}x \leq \theta\int_{B_s} \varphi_1(|\nabla v|)\,\mathrm{d}x + \int_{B_R} 1+\varphi_1\left(\frac{|v|}{s-t}\right) + \varphi_1(|\nabla g|)\,\mathrm{d}x, \tag{5.43}$$

from which an iteration argument using [Die+12, Lemma 3.1] the result follows. □

We will now need the following results due to Cianchi & Fusco [CF99]. We note a similar higher integrability result was also proven prior by Greco, Iwaniec & Sbordone [GIS97]. For an N-function φ and $\kappa>0$ we will set

$$\varphi^{[\kappa]}(t) = \varphi(t)\left(\frac{\varphi(t)}{t}\right)^{\kappa}. \tag{5.44}$$

Lemma 5.2.10 (Theorem 1.2 in [CF99])**.** Suppose $\varphi\in\Delta_2$ is an N-function, then for any ball $B_R(x_0)\subset\mathbb{R}^n$ and $u\in W^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$ we have

$$\begin{aligned}\varphi^{-1}&\left(\fint_{B_R(x_0)} \varphi\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\,\mathrm{d}x\right)\\ &\leq C\left(\varphi^{\left[\frac{-1}{n}\right]}\right)^{-1}\left(\fint_{B_R(x_0)} \varphi^{\left[\frac{-1}{n}\right]}(|\nabla u|)\,\mathrm{d}x\right).\end{aligned} \tag{5.45}$$

Lemma 5.2.11 (Theorem 1.3 in [CF99])**.** Suppose $\varphi\in\Delta_2$ and for a ball $B_R(x_0)$ we have $f,g\in L^1(B_R(x_0))$ are non-negative functions satisfying

$$\fint_{B_{r/2}(x)} \varphi(f)\,\mathrm{d}x \leq C\varphi\left(\fint_{B_r(x)} f\,\mathrm{d}x\right) + C\fint_{B_r(x)} \varphi(g)\,\mathrm{d}x \tag{5.46}$$

for all balls $B_r(x)\subset B_R(x_0)$. Moreover assume that $g\in L^{\varphi^{[\kappa_0]}}(B_R(x_0))$ for some $\kappa_0>0$. Then there is $0<\kappa<\kappa_0$ such that

$$\fint_{B_{R/2}(x_0)} \varphi^{[\kappa]}(f)\,\mathrm{d}x < \infty. \tag{5.47}$$

From here the strategy is clear; if $\{u_j\}\subset W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$ is a minimising sequence, we can use Lemma 5.2.6 to construct a *regularised* minimising sequence v_j which satisfies a Caccioppoli inequality uniformly in j. We would then like to infer uniform higher integrability estimates by Gehring's lemma, using Lemmas 5.2.10, 5.2.11 in tandem. A technical difficulty arises in that we need to higher integrability of g to apply Lemma 5.2.11 directly, which gives an undesirable additional assumption.

Lemma 5.2.12 (de la Vallée-Poisson [De 15])**.** Given $\Omega \subset \mathbb{R}^n$ open, we have $\{f_j\} \subset L^1(\Omega)$ is uniformly integrable on $\mathbb{R}^n$ if and only if there exists an N-function θ such that

$$\sup_j \int_\Omega \theta(|f_j|)\,\mathrm{d}x < \infty. \tag{5.48}$$

Further θ can be chosen so that $\theta \in \Delta_2$.

We refer the reader for instance to [KR61, Section 11.1] for a proof. The fact that we can choose θ to satisfy the Δ_2-condition was observed in [del+19, Proposition 3.4], where the authors showed that if this is not the case, one can instead consider $\widetilde{\theta}$ which is defined to satisfy $\widetilde{\theta}^{-1}(t) = t^{\frac{1}{2}}\theta^{-1}(t)^{\frac{1}{2}}$.

Proposition 5.2.13. Let $g \in W^{1,\varphi}(\mathbb{R}^n,\mathbb{R}^N)$ and $\{u_j\} \subset W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$ be a minimising sequence for $\mathcal{F}$, that is

$$\lim_{j\to\infty} \mathcal{F}(u_j) = \inf_{v \in W^{1,\varphi}_g(\Omega,\mathbb{R}^N)} \mathcal{F}(v). \tag{5.49}$$

Assume moreover that $\nabla g \in L^{\varphi^{[\chi_0]}}(\mathbb{R}^n,\mathbb{R}^{Nn})$ for some $\chi_0 > 0$. Then there exists a minimising sequence $\{v_j\} \subset W^{1,\varphi}_g(\Omega,\mathbb{R}^N)$ such that $u_j - v_j \overset{*}{\rightharpoonup} 0$ in $W^{1,\varphi}(\Omega,\mathbb{R}^N)$, and such that $\{\varphi_1(|\nabla v_j|)\}_j$ is uniformly integrable in Ω.

Proof. By Corollary 2.1.7 we can replace φ_1 with a C^2 N-function $\widetilde{\varphi}$; also since $\varphi_1(t) = \varphi'(1)\frac{t^2}{2}$ for $t \leq 1$, we also have $\lim_{t\to 0} \widetilde{\varphi}''(t) = \varphi'(1)$. By applying Lemmas 5.2.6 and 5.2.9 to each u_j, we obtain a minimising sequence $\{v_j\}$ which extending by g to $\mathbb{R}^n$ satisfies the Caccioppoli inequality

$$\fint_{B_{R/2}(x_0)} \widetilde{\varphi}(|\nabla v_j|)\,\mathrm{d}x \leq C \fint_{B_R(x_0)} \widetilde{\varphi}\left(\frac{|v_j - (v_j)_{B_R}|}{R}\right)\mathrm{d}x + C\fint_{B_R(x_0)} 1 + \widetilde{\varphi}(|\nabla g|)\,\mathrm{d}x, \tag{5.50}$$

for any $x_0 \in \overline{\Omega}$ and $R > 0$, holding uniformly in j. By applying Lemma 5.2.10 and setting $\rho = \varphi_1(t)(\varphi_1(t)/t)^{-\frac{1}{n}}$ we have

$$\varphi^{-1}\left(\fint_{B_R(x_0)} \varphi_1\left(\frac{|v_j - (v_j)_{\Omega_R(x_0)}|}{R}\right)\mathrm{d}x\right) \leq \rho^{-1}\left(C\fint_{B_R(x_0)} \rho(|\nabla v_j|)\,\mathrm{d}x\right). \tag{5.51}$$

Hence putting $\psi = \varphi \circ \rho^{-1}$ we deduce that

$$\begin{aligned}\psi^{-1}\left(\fint_{B_{R/2}(x_0)} \psi\circ\rho(|\nabla v_j|)\,\mathrm{d}x\right) &\leq C\fint_{B_R(x_0)} \rho(|\nabla v_j|)\,\mathrm{d}x \\ &\quad + C\psi^{-1}\left(\fint_{B_R(x_0)} 1 + \psi\circ\rho(|\nabla g|)\,\mathrm{d}x\right).\end{aligned} \tag{5.52}$$

By hypothesis there is $\chi_0 > 0$ such that $\nabla g \in L^{\varphi^{[\chi_0]}}(\mathbb{R}^n, \mathbb{R}^{Nn})$, and so we infer that $\rho(|\nabla g|) \in L^{\psi^{[\chi_0/n]}}(\mathbb{R}^n)$.

Now applying the Gehring lemma (Lemma 5.2.11) with ψ we deduce there is $0 < \chi < \chi_0/n$ such that

$$\sup_j \int_\Omega \psi^{[\chi]} \circ \rho(|\nabla v_j|)\,\mathrm{d}x < \infty, \tag{5.53}$$

and so the assertion follows by Lemma 5.2.12. $\square$

Remark 5.2.14 (Extra integrability of g)**.** The additional integrability we assume on ∇g is rather unnatural, and one may wonder if this could be removed. We point out that g is always slightly more integrable; by Lemma 5.2.12 applied to the singleton family $\{\nabla g\}$, there exists an N-function $\theta \in \Delta_2$ such that $\theta \circ \varphi(|\nabla g|) \in L^1(\mathbb{R}^n)$. We would like to use this additional integrability to infer improved integrability of $\{\nabla v_j\}$, however we were not able to achieve this. We note that a similar result was established prior by D'Onofrio & Iwaniec [DI04] in the context of p-growth, and in future work we seek to understand if their argument generalises to our setting.

5.2.3 Lower-semicontinuity II

We will use the regularised sequence constructed in the previous section to establish lower-semicontinuity along minimising sequences. We will argue similarly as in the proof of Proposition 5.2.2, except that we use the uniform integrability estimate from the previous section to handle the extra terms we obtain from perturbation. Note that we will not use any improved integrability properties of ∇v_j, and only that $\varphi_1(|\nabla v_j|)$ is uniformly integrable.

Proof of Proposition 5.2.3. By Proposition 5.2.13 we obtain a minimising sequence $\{v_j\} \subset W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ satisfying $\mathcal{F}(v_j) \le \mathcal{F}(u_j)$ for all j, and moreover $\varphi_1(|\nabla v_j|)$ is uniformly integrable in Ω. By the Poincaré-Sobolev inequality applied in a sufficiently large ball containing Ω (extending v_j by g outside Ω as usual) boundedness of ∇v_j in L^φ also implies uniform integrability of $\varphi_1(|v_j|)$. Fix $\varepsilon > 0$, then there is $L \ge 0$ for which setting

$$V_L = \{x \in \Omega + B_1 : |v_j|, |\nabla v_j| \ge L\} \tag{5.54}$$

we have

$$\int_{V_L} 1 + \varphi_1(|v_j|) + \varphi_1(|\nabla v_j|)\,\mathrm{d}x < \varepsilon \tag{5.55}$$

for all j, where we have one again extended v_j to $\mathbb{R}^n$ via g. Passing to a subsequence, we will also assume v_j converges to a weak* limit map u, and we see that (5.55) also holds with u. Then by the Scorza-Dragoni theorem (see for instance [Giu03, Lemma 4.6]), for each $\kappa > 0$ to be determined there is $A_\kappa \subset \Omega + B_1$ with $\mathcal{L}^n(\Omega + B_1 \setminus A) < \kappa$ such that F is uniformly continuous on $A_\kappa \times \{(u,z) \in \mathbb{R}^N \times \mathbb{R}^{Nn} : |u|, |z| \leq L\}$, with joint modulus of continuity ω.

Now let $\mathcal{T}_{\xi,\delta}$ be as in the proof of Proposition 5.2.2, with $\xi \in B_1$ and $\delta > 0$ to be determined. We will then define the piecewise constant map

$$T^L_{\xi,\delta} v_j(x) = v^\delta_{j,\tau} = (v_j 1_{\{|v_j| \leq L\}})_\tau \quad \text{if } x \in \tau \in \mathcal{T}^\Omega_{\xi,\delta}. \tag{5.56}$$

and similarly for u. By Lebesgue differentiation we have $T^L_{\xi,\delta} v_j \to v_j 1_{\{|v_j| \leq L\}}$ a.e. as $\delta \to 0$, and is bounded a.e. by L. Choosing $\kappa < 2^{-2n}\delta^n = \mathcal{L}^n(\tau)$ for each $\tau \in \mathcal{T}^\Omega_{\xi,\delta}$ ensures that $\tau \setminus A_\kappa \neq \varnothing$ for each $\tau \in \mathcal{T}^\Omega_{v,\delta}$, so there is some $\tilde{x}_\tau$ in the intersection which we fix. Now the two claims in the proof of Proposition 5.2.2 ensure that for δ sufficiently small there exists a piecewise affine map with respect to $\mathcal{T}^\Omega_{\xi,\delta}$ such that

$$\int_{\Omega + B_1} \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x < \varepsilon, \tag{5.57}$$

$$\int_{\Omega + B_1} |F(\tilde{x}_\tau, T^L_{\xi,\delta} u, \nabla u) - F(\tilde{x}_\tau, T^L_{\xi,\delta} v, \nabla a)|\,\mathrm{d}x < \varepsilon. \tag{5.58}$$

Further for almost all $\xi \in B_\delta$, for each $\tau \in \mathcal{T}^\Omega_{\xi,\delta}$ we have $v_j|_{\partial\tau}, u|_{\partial\tau} \in W^{1,\varphi}(\partial\tau, \mathbb{R}^N)$ and $v_j|_{\partial\tau} \overset{*}{\rightharpoonup} u|_{\partial\tau}$ in $W^{1,\varphi}(\partial\tau, \mathbb{R}^N)$.

Now arguing as in Proposition 5.2.2, for each $\tau \in \mathcal{T}^\Omega_{\xi,\delta}$ we have the estimate

$$\begin{aligned} F(\tilde{x}_\tau, v^L_{j,\tau}, \nabla a_\tau)\mathcal{L}^n(\lambda\tau) \leq \liminf_{j\to\infty} \int_\tau F(\tilde{x}_\tau, v^L_{j,\tau}, \nabla v_j)\,\mathrm{d}x + C\int_\tau \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x \\ + C(k^n - 1)(1 + \varphi_1(|\nabla a_\tau|))\mathcal{L}^n(\tau). \end{aligned} \tag{5.59}$$

Here we have crucially used the regularised extension of v_j to $k\tau \setminus \tau$ and the quasiconvexity condition. Now summing over $\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}$ and noting we can estimate

$$
\begin{aligned}
&\left| \sum_{\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}} \int_\tau F(\tilde{x}_\tau, u^L_\tau, \nabla a_\tau) - F(x, u, \nabla u)\,\mathrm{d}x \right| \\
&\quad \leq \varepsilon + \sum_{\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}} \int_\tau F(\tilde{x}_\tau, u^L_\tau, \nabla u) - F(x, u, \nabla u)\,\mathrm{d}x \\
&\quad \leq \varepsilon + \int_{V_L} 1 + \varphi(|\nabla u|) \\
&\quad + C \sum_{\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}} \int_{\tau \setminus V_L} \omega(\delta + |u^L_\tau - u|)\,\mathrm{d}x \\
&\quad \leq (1+C)\varepsilon + C\,\omega(\delta)\mathcal{L}^n(\Omega + B_1) + C \int_{\bigcup \mathcal{T}^{\Omega}_{\xi,\delta} \setminus V_L} \omega\left(|T^L_{\xi,\delta} u - u|\right)\,\mathrm{d}x.
\end{aligned}
\tag{5.60}
$$

We note that the last term vanishes as $\delta \to 0$ by dominated convergence. By a similar argument we can estimate

$$
\begin{aligned}
&\liminf_{j \to \infty} \int_\tau F(\tilde{x}_\tau, T^L_{\xi,\delta} v_j, \nabla v_j)\,\mathrm{d}x \\
&\quad \leq \liminf_{j \to \infty} F(x, v, \nabla v_j)\,\mathrm{d}x + (1+C)\varepsilon + C\,\omega(\delta)\mathcal{L}^n(\Omega + B_1) \\
&\qquad + C\mathcal{L}^n(\Omega + B_1) \sup_j \omega\left(\fint_{\bigcup \mathcal{T}^{\Omega}_{\xi,\delta} \setminus V_L} |T^L_{\xi,\delta} v_j - v_j|\,\mathrm{d}x \right),
\end{aligned}
\tag{5.61}
$$

where we have used Jensen's inequality for the last term. To show the last term can be made arbitrarily small, we use the Poincaré inequality on each $\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}$ to estimate

$$
\begin{aligned}
&\int_{\bigcup \mathcal{T}^{\Omega}_{\xi,\delta}} |T^L_{\xi,\delta} v_j - v_j|\,\mathrm{d}x \\
&\quad \leq \sum_{\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}} \int_{\tau \cap V_L} |v_j|\,\mathrm{d}x + \sum_{\tau \in \mathcal{T}^{\Omega}_{\xi,\delta}} \int_\tau |v_j - (v_j)_\tau|\,\mathrm{d}x \\
&\quad \leq \mathcal{L}^n(\Omega + B_1)\varphi_1^{-1}\left(\frac{1}{\mathcal{L}^n(\Omega + B_1)} \int_{V_L} \varphi_1(|v_j|)\,\mathrm{d}x \right) \\
&\qquad + \delta^2 \mathcal{L}^n(\Omega + B_1)\varphi_1^{-1}\left(\frac{1}{\mathcal{L}^n(\Omega + B_1)} \int_{\Omega + B_1} \varphi_1(|\nabla v_j|)\,\mathrm{d}x \right),
\end{aligned}
\tag{5.62}
$$

using Jensen to obtain estimates in terms of φ_1. Using (5.55) and the boundless of ∇v_j in L^φ, this can be made arbitrarily small by sending $\varepsilon, \delta \to 0$. Hence combining the estimates the result follows. $\square$

Remark 5.2.15. One can relax the Carathéodory assumption and assume F is *normal integrand*, which is a Borel measurable function $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R} \cup \{\infty\}$ that is lower-semicontinuous in the (u, z) variables. This can be shown via Young measures as noted in

Kristensen [Kri99, Theorem 2.4] using the general results of Balder [Bal84]. We have opted for a proof using the triangulations to illustrate the flexibility of the argument, and also to avoid introducing the language of Young measures.

5.2.4 Coercivity and strict quasiconvexity

We will conclude our discussion of the existence theory by establishing a characterisation of coercivity in the general growth setting. This follows by a straightforward adaptation of the results in Chen & Kristensen [CK17, Theorem 1.1], and is restricted to autonomous integrands.

Proposition 5.2.16 (Characterisation of coercivity)**.** Let $F \in C(\mathbb{R}^{Nn})$ satisfy the growth condition (A2) with $\varphi \in \Delta_2$, then the following are equivalent.

(i) For any non-empty bounded open subset $\Omega \subset \mathbb{R}^n$ and $g \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$, we have F is L^φ-*coercive* in the sense that for any sequence $\{u_j\} \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ we have

$$\lim_{j\to\infty} \int_\Omega F(\nabla u_j)\,\mathrm{d}x = +\infty \quad\text{implies}\quad \lim_{j\to\infty} \int_\Omega \varphi(|\nabla u_j|)\,\mathrm{d}x = +\infty. \tag{5.63}$$

(ii) There exists $\delta > 0$ and $z_0 \in \mathbb{R}^{Nn}$ such that $F - \delta\,\varphi_1(|\cdot|)$ is strictly quasiconvex at z_0.

(iii) There exists $\delta > 0$ and $z_0 \in \mathbb{R}^{Nn}$ such that F satisfies the strict quasiconvexity condition (5.2) at z_0; recall this asserts that

$$\int_{\mathbb{R}^n} F(z_0 + \nabla\xi) - F(z_0)\,\mathrm{d}x \geq \delta \int_{\mathbb{R}^n} \varphi_{1+|z_0|}(|\nabla \xi|)\,\mathrm{d}x \tag{5.64}$$

for all $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$.

Moreover it suffices to verify (i) for a single pair Ω, g, and if either of the equivalent conditions hold we can infer that F is *mean* L^φ-*coercive* in the sense that there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \geq c_1 \int_\Omega \varphi_1(|\nabla u|)\,\mathrm{d}x + c_2 \tag{5.65}$$

for all $u \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$.

The strict quasiconvexity condition (5.2) was introduced by Evans [Eva86] in the context of regularity theory, however we see that it naturally arises as part of the existence theory.

A key tool in proving this assertion will be the quasiconvex envelope introduced in Definition 2.1.32, along with the duality representation from Lemma 2.1.33.

Proof of Proposition 5.2.16. We note that it suffices to verify (i) for $\Omega = B$ the unit ball and $g \equiv 0$. This is proven in [CK17, Proposition 3.1], and the idea is to find balls $B_r(x_0) \Subset \Omega \Subset B_R(x_0)$ and use a cut-off argument to extend elements of $W_0^{1,\varphi}(B_r(x_0), \mathbb{R}^N)$ to $W_g^{1,\varphi}(\Omega, \mathbb{R}^N)$ and elements of $W_g^{1,\varphi}(\Omega, \mathbb{R}^N)$ to $W_0^{1,\varphi}(B_R(x_0), \mathbb{R}^N)$. By rescaling and using the doubling property of φ_1 the reduction follows.

We then introduce the auxiliary function $\Theta = \Theta_F \colon [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\Theta(t) = \Theta_F(t) = \inf\left\{ \fint_\Omega F(\nabla\xi)\,\mathrm{d}x : \xi \in W_0^{1,\varphi}(\Omega, \mathbb{R}^N), \fint_\Omega \varphi_1(|\nabla \xi|) \geq t \right\}. \tag{5.66}$$

An application of Vitali's covering theorem (see [CK17, Lemma 2.1]) shows that Θ is independent of Ω; indeed if $\omega \subset \mathbb{R}^n$ is any other open set and $\xi \in C_c^\infty(\omega, \mathbb{R}^N)$, take $\widetilde{\omega} \subset \omega$ such that ξ is still compactly supported in ω and $\mathcal{L}^n(\widetilde{\omega}) = 0$. Then we can apply Vitali's theorem (see for instance [EG15, Theorem 1.26]) to find $x_j \in \Omega, r_j > 0$ such that each $\widetilde{\omega}_j := x_j + r_j\widetilde{\omega} \subset \Omega$ are pairwise disjoint and $\mathcal{L}^n(\Omega \setminus \bigcup_j \widetilde{\omega}_j) = 0$. We can then consider

$$\tilde{\xi}(x) = \sum_j r_j\, \xi\left(\frac{x - x_j}{r_j}\right), \tag{5.67}$$

which lies in $W_0^{1,\infty}(\Omega, \mathbb{R}^N)$ and satisfies

$$\fint_\omega \varphi_1(|\nabla \xi|)\,\mathrm{d}x = \fint_\Omega \varphi_1(|\nabla \tilde{\xi}|)\,\mathrm{d}x, \tag{5.68}$$

$$\fint_\omega F(\nabla \xi)\,\mathrm{d}x = \fint_\Omega F(\nabla \tilde{\xi})\,\mathrm{d}x. \tag{5.69}$$

From this construction the assertion follows, extending by density to $W_0^{1,\varphi}(\omega, \mathbb{R}^N)$. We can further use this to show Θ is convex in $t \geq 0$; indeed if $\lambda \in (0,1)$, setting $Q_0 = (0,1)^{n-1} \times (0,\lambda)$ and $Q_1 = (0,1)^{n-1} \times (\lambda, 1)$ for $t_0, t_1 \geq 1$ and each $\varepsilon > 0$ we can find $\xi_i \in W_0^{1,\varphi}(Q_i, \mathbb{R}^N)$ such that

$$\fint_{Q_i} \varphi_1(|\nabla \xi_i|)\,\mathrm{d}x \geq t_i, \quad \text{and} \quad \fint_{Q_i} F(\nabla \xi_i)\,\mathrm{d}x < \Theta(t_i) + \varepsilon \tag{5.70}$$

for each $i = 0, 1$. Then putting $Q = (0,1)^n$ and defining $\xi \in W_0^{1,\varphi}(Q, \mathbb{R}^N)$ by

$$\xi = \begin{cases} \xi_0 & \text{on } Q_0, \\ \xi_1 & \text{on } Q_1, \end{cases} \tag{5.71}$$

we see that $\fint_Q \varphi_1(|\nabla\xi|)\,\mathrm{d}x \geq (1-\lambda)t_0 + \lambda t_1$ and $\fint_Q F(\nabla\xi)\,\mathrm{d}x < (1-\lambda)\Theta(t_0) + \lambda\Theta(t_1) + \varepsilon$, from which convexity follows.

Now observe that F is L^φ coercive with $g = 0$ if and only if $\Theta(t) \to \infty$ as $t \to \infty$; where we notice that $\int_\Omega \varphi(|\nabla\xi_j|)\,\mathrm{d}x \to \infty$ if and only if $\int_\Omega \varphi_1(|\nabla\xi_j|)\,\mathrm{d}x \to \infty$. If this holds, by convexity there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $\Theta(t) \geq c_1 t + c_2$. This implies mean coercivity (5.65). Now taking $\delta < c_1$, setting $G(z) = F(z) - \delta\varphi_1(|z|)$ we have

$$\fint_\Omega G(\nabla\xi)\,\mathrm{d}x \geq (c_1 - \delta)\fint_\Omega \varphi_1(|\nabla\xi|)\,\mathrm{d}x + c_2 \tag{5.72}$$

for any $\xi \in W_0^{1,\varphi}(\Omega, \mathbb{R}^N)$. Taking a minimising sequence ξ_j which attains $\mathcal{Q}G(0)$ (using Lemma 2.1.33) we have

$$(c_1 - \delta)\limsup_{j\to\infty}\fint_\Omega \varphi_1(|\nabla\xi_j|)\,\mathrm{d}x + c_2 \leq \lim_{j\to\infty}\fint_\Omega G(\nabla\xi_j)\,\mathrm{d}x = \mathcal{Q}G(0) < \infty. \tag{5.73}$$

Note also that $\mathcal{Q}G(0) \leq \fint_\Omega \mathcal{Q}G(\nabla\xi_j)\,\mathrm{d}x \leq \fint_\Omega G(\nabla\xi_j)\,\mathrm{d}x \to \mathcal{Q}G(0)$ at $j \to \infty$. Hence by (5.73) the sequence of probability measures $\mu_j = (\nabla\xi_j)_\#\left(\frac{\mathcal{L}^n \llcorner \Omega}{\mathcal{L}^n(\Omega)}\right)$ satisfies

$$\sup_j \int_{\mathbb{R}^{Nn}} \varphi_1(|z|)\,\mathrm{d}\mu_j(z) < \infty, \tag{5.74}$$

so passing to a subsequence we get $\mu_j \overset{*}{\rightharpoonup} \mu$ in $C_0(\mathbb{R}^{Nn})^*$. By using Markov's inequality and (5.74) we can infer that the limit μ is also a probability measure, and in particular that μ is non-trivial. Moreover by weak* convergence, for all $\rho \in C_c^\infty(\mathbb{R}^{Nn})$ non-negative we have

$$\begin{aligned}\int_{\mathbb{R}^{Nn}} \rho(z)\,(G(z) - \mathcal{Q}G(z))\,\mathrm{d}\mu(z) &= \lim_{j\to\infty}\int_{\mathbb{R}^{Nn}} \rho(z)\,(G(z) - \mathcal{Q}G(z))\,\mathrm{d}\mu_j(z) \\ &\leq \|\rho\|_{L^\infty(\mathbb{R}^{Nn})}\liminf_{j\to\infty}\fint_\Omega (G(\nabla\xi_j) - \mathcal{Q}G(\nabla\xi_j))\,\mathrm{d}x = 0\end{aligned} \tag{5.75}$$

noting that $\mathcal{Q}G \leq G$, so letting $\rho \uparrow 1_{\mathbb{R}^{Nn}}$ gives

$$\int_{\mathbb{R}^{Nn}} G(z)\,\mathrm{d}\mu(z) = \int_{\mathbb{R}^{Nn}} \mathcal{Q}G(z)\,\mathrm{d}\mu(z). \tag{5.76}$$

Since μ is non-trivial, taking any point in $\operatorname{spt}\mu$ gives $z_0 \in \mathbb{R}^{Nn}$ such that $G(z_0) = \mathcal{Q}G(z_0)$, that is $F - \delta\varphi_1(|\cdot|)$ is quasiconvex at z_0. Conversely it follows from the definition that quasiconvexity of $F - \delta\varphi_1(|\cdot|)$ implies (i) with $g = 0$, and this immediately implies the mean coercivity estimate (5.65).

To show that (i) implies (iii), observe the above argument also works if we replace φ_1 by an N-function $\widetilde{\varphi}$ satisfying $\varphi_1(t) \sim \widetilde{\varphi}(t)$ for t sufficiently large. We claim that we can find such $\widetilde{\varphi}$ which is C^2 and satisfies $\widetilde{\varphi}''(t) > 0$ for all $t \in [0, \infty)$. This can be constructed by taking the shifted N-function associated to $\varphi + t + (e^{-t} - 1)$ and applying Corollary 2.1.7. Note that $t + (e^{-t} - 1)$ is a convex function which is superlinear at the origin, with strictly positive second derivative. Then by Lemma 5.2.17 below applied to $\widetilde{\varphi}$ we obtain (iii), which in turn evidently implies (i). $\square$

Lemma 5.2.17. Let $\varphi \in \Delta_2$ be an N-function which is C^2 with strictly positive second derivative on $[0, \infty)$. Then for any $z_0 \in \mathbb{R}^{Nn}$, there is $C > 0$ such that

$$\int_\Omega \varphi(|z_0 + \nabla\xi|) - \varphi(|z_0|)\,\mathrm{d}x \geq C \int_\Omega \varphi_{1+|z_0|}(|\nabla\xi|)\,\mathrm{d}x \tag{5.77}$$

for all $\Omega \subset \mathbb{R}^n$ bounded open and $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$.

Proof. For any $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$ we have

$$\begin{aligned}
&\fint_\Omega \varphi\left(|z_0 + \nabla\xi|\right) - \varphi\left(|z_0|\right) - 2\varphi'\left(|z_0|\right)\nabla\xi \cdot \frac{z_0}{|z_0|}\,\mathrm{d}x \\
&\quad = \fint_\Omega \int_0^1 (1-t)\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\varphi(|z_0 + t\nabla\xi|)\right)\mathrm{d}s\,\mathrm{d}x,
\end{aligned} \tag{5.78}$$

noting this equals the left-hand side of (5.77) since $\int_\Omega \nabla\xi\,\mathrm{d}x = 0$. Now for $w \in \mathbb{R}^{Nn}$ such that $z_0 + tw \neq 0$ we can estimate

$$\begin{aligned}
I(w) &:= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\varphi(|z_0 + tw|^2)\right) \\
&= \varphi''(|z_0 + tw|)\left(\frac{w \cdot (z_0 + tw)}{|z_0 + tw|}\right)^2 \\
&\quad + \frac{\varphi'(|z_0 + tw|)}{|z_0 + tw|)}\left(|w|^2 - \left(\frac{w \cdot (z_0 + tw)}{|z_0 + tw|}\right)^2\right).
\end{aligned} \tag{5.79}$$

Since $\varphi''(t) > 0$ for $t \geq 0$, there is $c > 0$ (depending on z_0) such that $\varphi''(|z_0 + tw|) \geq c$ and $\varphi'(|z_0 + tw|) \geq c|z_0 + tw|$ for all $|w| \leq 1 + 2|z_0|$ and $t \in (0, 1)$. Hence we can estimate

$$I(w) \geq c\left(\frac{w \cdot (z_0 + tw)}{|z_0 + tw|}\right)^2 + c\left(|w|^2 - \left(\frac{w \cdot (z_0 + tw)}{|z_0 + tw|}\right)^2\right) = c|w|^2 \tag{5.80}$$

for $|w| \leq 1 + |z_0|$. On the other hand if $|w| > 1 + 2|z_0|$, we have $|z_0 + w| \geq \frac{1}{2}|w|$ and so noting $\varphi \in \Delta_2$,

$$\varphi(|z_0 + w|) - \varphi(|z_0|) \geq \frac{1}{\Delta_2(\varphi)}\varphi(|w|) - \varphi\left(\frac{1}{2}|w|\right) \geq \left(\frac{1}{\Delta_2(\varphi)} - 1\right)\varphi(|w|). \tag{5.81}$$

Hence by combining the two cases, it follows that

$$\varphi\left(|z_0 + \nabla\xi|\right) - \varphi\left(|z_0|\right) - 2\varphi'\left(|z_0|\right)\nabla\xi \cdot \frac{z_0}{|z_0|} \simeq \varphi_{1+|z_0|}(|\nabla w|) \tag{5.82}$$

where the implicit constant depends on z_0. □

Remark 5.2.18. It is clear that the associated constants depend non-trivially on $|z_0|$, through the non-negativity of φ''; this is unavoidable as we need a linearisation argument to obtain the quadratic behaviour for small $\nabla\xi$. This is not an issue however, since we always fix $M > 0$ such that $|z_0| \leq M$.

5.3 Partial regularity

We now consider the regularity of the minimisers we have obtained, and establish a partial $C^{1,\alpha}$ regularity result. This follows from the following characterisation of $C^{1,\alpha}$-regular points for minimisers.

Theorem 5.3.1 (ε-regularity theorem)**.** Suppose F satisfies Hypotheses 5.1.1 with $\varphi \in \Delta_2 \cap \nabla_2$ and $M > 0$. Then there is $\varepsilon > 0$ and $\alpha > 0$ such that the following holds. Suppose $\Omega \subset \mathbb{R}^n$ is open and $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ minimises the associated functional $\mathcal{F}$ in its Dirichlet class. Then if $B_R(x_0) \subset \Omega$ for which we have

$$|(\nabla u)_{B_R(x_0)}| \leq M \text{ and } E(x_0, R) \tag{5.83}$$

and

$$E(x_0, R) = \fint_{B_R(x_0)} \varphi_1(|\nabla u - (\nabla u)_{B_R(x_0)}|)\,\mathrm{d}x < \varepsilon, \tag{5.84}$$

then u is $C^{1,\alpha}$ in $B_{R/2}(x_0)$.

We have made no attempt to optimise the exponent α in our proof, which has been considered by Hamburger [Ham07]. Instead we note that if the integrand F is smooth, the above is sufficient to show that u is smooth by iteratively applying Schauder estimates; we refer to Giusti [Giu03, Section 10.5] for details.

5.3.1 Strategy of proof

Before turning to the proof, we will briefly discuss the development leading up to this result, and the various techniques we will use in our argument.

The first partial regularity result in the quasiconvex setting was established by Evans [Eva86], who considered autonomous integrands. This was achieved through a blow-up argument to obtain decay estimates on the *excess energy* associated to the minimiser u, defined in (5.84). By means of a blow-up argument, he showed this excess energy satisfies suitable decay estimates from which one can infer Hölder continuity of the gradient, provided the excess energy is sufficient small to begin with. The key technical ingredient was the use of a *Caccioppoli inequality of the second kind*, where one estimates u minus a suitable affine approximant; this allows one deduce that the blow-up sequence converges *strongly* in $W^{1,p}$ to a limit map h which is harmonic. While Evans only considered the case when $p \geq 2$ under controlled growth conditions (growth bounds on F''), in the autonomous case this was relaxed to natural $p \geq 2$ growth by Acerbi & Fusco [AF87]. The case $1 < p < 2$ remained open however, and was brought to the spotlight when Šverák [Šve91] constructed examples of quasiconvex integrands satisfying such a condition (see Lemma 2.1.34). The range was improved by Carozza & Passarelli di Napoli [CP96], who extended the range to include $\frac{2n}{n+2} < p < 2$. They identified the key difficulty in extending this range relied on the validity of a Poincaré-Sobolev inequality for V-functions; that is an inequality of the form

$$\left(\fint_{B_R(x_0)} \left| V_{p,\mu}\left(\frac{u-(u)_{B_R(x_0)}}{R} \right) \right|^{2\theta} \mathrm{d}x \right)^{\frac{1}{\theta}} \leq C \fint_{B_R(x_0)} |V_{p,\mu}(\nabla u)|^2 \,\mathrm{d}x \tag{5.85}$$

for some $\theta > 1$. This was eventually settled by Carozza, Fusco & Mingione [CFM98], who established a suitable variant of the above through a maximal estimate for $|V_{p,\mu}(\cdot)|^{2\alpha}$ with $\alpha < 1$. This illustrates that, to a large degree, the main challenge in the subquadratic case is that one needs to work with the scale $(1+t^2)^{\frac{p-2}{2}} t^2$ instead of t^p. The use of the V-functions suggests that the homogeneity of the p-norms does not play an essential role in the regularity analysis, and indeed it turns out that one can just treat the case of general N-functions with only minor modifications. We will exploit this viewpoint in the present work, which provides a unified account of the regularity theory without distinguishing between different growth ranges.

The above results only consider the autonomous case $F = F(z)$, however the general non-autonomous case was also being developed around the same time. By this point the non-autonomous case had been extensively studied when the functional was strictly convex, as discussed in the introduction (Section 1.1.3). In particular, it was well-understood that while the x-dependence could be treated by means of a perturbation argument, the u-dependence required a more delicate argument using Gehring's lemma. In the quasiconvex case the same strategy could be applied, which was done independently by Fusco & Hutchinson [FH85] and Giaquinta & Modica [GM86]. In the former [FH85] the authors employ a blow-up argument, where at each step they apply the variational principle of Ekeland [Eke74] to construct perturbations of the original minima which *almost minimises* a suitably frozen autonomous functional; this strategy was later extended by Acerbi & Fusco [AF89]. On the other hand in [GM86] the authors employ a direct argument where they consider a *harmonic replacement* in suitable balls $B_R(x_0) \subset \Omega$. We note that since both approaches rely on a perturbation argument to treat the lower-order terms, it suffices to assume a Hölder continuity condition in the u-variable; this means minima need not satisfy an Euler-Lagrange system. Additionally both approaches assumed a controlled growth condition with $p \geq 2$, but this could be relaxed once the corresponding results were established in the autonomous case.

Shortly afterwards, a major technical improvement came from the so-called *method of* $\mathbb{A}$*-harmonic approximation*. This approach originated from results in geometric problems, and was formulated precisely by Simon [Sim83; Sim96] in the case of quadratic growth. Loosely speaking, the lemma states that if a function u is *almost harmonic* in a suitable sense, we can find a genuinely harmonic function h for which the difference $u-h$ is small. By a linearisation argument we can show this is locally satisfied by our minima u, to which this lemma can be applied to construct an $\mathbb{A}$-harmonic function which closely approximates u. This was applied to variational problems by Duzaar & Grotowski [DG00], and has been used by many author since; for instance in the quasiconvex setting Kronz [Kro02; Kro05] applied similar results to consider higher-order integrands and boundary regularity. Results of this type were further generalised by Duzaar & Mingione [DM04] to the p-Laplace equation, which was later applied in [DM09, Section 3.1] to treat degenerate quasiconvex problems where the integrand exhibits a p-Laplace type behaviour near the origin. We also mention the recent work in the setting of nonlinear potentials [KM16; De 22; DS22].

This technique was further developed by DIENING, LENGELER, STROFFOLINI, & VERDE [Die+12] who provided a direct proof of the linear $\mathbb{A}$-harmonic approximation result in the Orlicz scales, which was applied to establish partial regularity for quasiconvex integrands satisfying an Orlicz growth condition. Their strategy involved solving the associated dual problem, and using a Lipschitz truncation technique due to ACERBI & FUSCO [AF84; AF87; AF88]. We note that in the case of p-growth, refined quantitative versions of this technique has been obtained in [CD19]. We point out that in [Die+12] the authors assumed a controlled growth condition on the integrand, and while their results have been extended to the non-autonomous setting in [CO20; GSS21; OSS21] this condition is also assumed there.

Recently this harmonic approximation technique was further refined by GMEINEDER & KRISTENSEN [GK19a] to allow for integrands satisfying a linear growth condition, where the corresponding minimisers lie in BV . By means of a suitable duality argument, similar to the one featured in the direct proof of the method of $\mathbb{A}$-harmonic approximation, the authors obtained quantitative control on the difference $u-h$, where h solves the linearised problem

$$\begin{cases} -\operatorname{div} F''(\nabla a)\nabla h = 0 & \text{in } B_R(x_0) \\ \hfill h = u & \text{on } \partial B_R(x_0) \end{cases}. \tag{5.86}$$

In the BV setting this technique has been extended by a number of authors, including [Gme21; Fra19; Li22], and we also applied this in the case of quadratic growth in Chapter 4. We will adapt this technique in the general growth setting to prove Theorem 5.3.1. This extends the work of the author in [Irv21a] where the autonomous case was considered.

Putting together the above, we see there are largely three ingredients that are necessary to establish partial regularity in this setting; the first is a Caccioppoli inequality, the second is a perturbation result to freeze the (x, u)-dependence, and finally the method of $\mathbb{A}$-harmonic approximation. This will be established in Section 5.3.2, 5.3.4, 5.3.5 respectively, which will combined with a suitable iteration argument in Section 5.3.6 to conclude.

5.3.2 Higher integrability

We begin by recording the following preliminary Caccioppoli inequality, from which we will infer improved integrability results. Note this essentially follows from the results in Section 5.2.2.

Lemma 5.3.2 (Caccioppoli inequality of the first kind)**.** Suppose F satisfies Hypotheses 5.1.1, $\varphi \in \Delta_2$ is an N-function, and let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be a minimiser of $\mathcal{F}$ in Ω. Then if $B_R(x_0) \subset \Omega$ we have

$$\fint_{B_{R/2}(x_0)} \varphi_1(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B_R(x_0)} 1 + \varphi_1\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\mathrm{d}x \tag{5.87}$$

where $C = C(n, N, K/\nu, \Delta_2(\varphi))$.

This follows by noting that (A2) and (A3) implies u is a quasi-minima of $\mathcal{E}_{\varphi,0}$ in the interior of Ω, so we can argue as in the proof of Lemma 5.2.9 for balls $B_R(x_0) \subset \Omega$. We will omit the details as they are largely identical.

We will apply this in conjunction with the following higher integrability result. This is where we crucially use the assumption that $\varphi \in \nabla_2$ to obtain a power-type improvement in integrability, in contrast to the improvement given by Lemma 5.2.11.

Lemma 5.3.3 (Higher integrability)**.** Suppose $\varphi \in \Delta_2 \cap \nabla_2$, $\Omega \subset \mathbb{R}^n$ is open and let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ which we assume satisfies the Caccioppoli inequality

$$\fint_{B_{R/2}(x_0)} \varphi_1(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B_R(x_0)} 1 + |G| + \varphi_1\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\mathrm{d}x \tag{5.88}$$

for all $B_{R/2}(x_0) \subset \Omega$, and assume that $G \in L^{\chi}$ for some $\tilde{\chi} > 0$. Then there is $\chi \in (0, \tilde{\chi})$ such that $\nabla u \in L^{\varphi^{\chi}}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and for all $B_R(x_0) \subset \Omega$ we have

$$\left(\fint_{B_{R/2}(x_0)} \varphi_1(|\nabla u|)^{\chi}\,\mathrm{d}x\right)^{\frac{1}{\chi}} \leq C \fint_{B_R(x_0)} \varphi_1(|\nabla u|)\,\mathrm{d}x + \left(\fint_{B_R(x_0)} 1 + |G|^{\chi}\,\mathrm{d}x\right)^{\frac{1}{\chi}}. \tag{5.89}$$

Proof. Since $\varphi \in \Delta_2 \cap \nabla_2$, by [KK91, Theorem 1.2.1] there exists an N-function $\rho \in \Delta_2$ and $\alpha \in (0,1)$ such that $\rho \sim \varphi^{\alpha}$. We can increase α if necessary to ensure $\alpha \geq \frac{n-1}{n}$ so by applying the Poincaré-Sobolev inequality (Lemma 2.3.6) to ρ together with Hölder we have

$$\fint_{B_R(x_0)} \varphi_1\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\mathrm{d}x \leq C\left(\fint_{B_R(x_0)} \varphi_1(|\nabla u|)^{\alpha}\,\mathrm{d}x\right)^{\frac{1}{\alpha}}. \tag{5.90}$$

Hence by our hypothesis we have

$$\fint_{B_{R/2}(x_0)} \varphi_1(|\nabla u|)\,\mathrm{d}x \leq C\left(\fint_{B_R(x_0)} \varphi_1(|\nabla u|)^\alpha\,\mathrm{d}x\right)^{\frac{1}{\alpha}} + C\fint_{B_R(x_0)} 1+|G|\,\mathrm{d}x, \tag{5.91}$$

so the result follows by applying Gehring's lemma, in the form proven in [Gia83, Theorem V.1.2] and [Giu03, Theorem 6.6]. □

5.3.3 Caccioppoli inequality of the second kind

To obtain decay estimates for the excess energy we will need the so-called *Caccioppoli inequality of the second kind*, which was established by Evans [Eva86]. Rather than using the coercivity condition (A3), we linearise the integrand near some $z_0 \in \mathbb{R}^{Nn}$ and use the strict quasiconvexity condition (A4). In the autonomous case this gives the estimate

$$\fint_{B_{R/2}(x_0)} \varphi_1(|\nabla u - \nabla a|)\,\mathrm{d}x \leq C\fint_{B_R(x_0)} \varphi_1\left(\frac{|u-a|}{R}\right)\mathrm{d}x, \tag{5.92}$$

for $a : \mathbb{R}^n \to \mathbb{R}^N$ affine, where the constant C depends on ∇a. In the non-autonomous case we obtain something similar, however we obtain an additional term as part of a perturbation argument. The ∇_2 condition is used to obtain improved integrability of power-type, however we would be interested in understanding if this is really necessary.

By applying the first Caccioppoli inequality (Lemma 5.3.2) and the higher integrability result (Lemma 5.3.3) with $G \equiv 0$, we obtain

$$\chi_0 = \chi_0(n, N, \Delta_2(\varphi), \nabla_2(\varphi), \Lambda/\lambda) > 0 \tag{5.93}$$

such that $\nabla u \in L^{\varphi^\chi}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn})$ and the corresponding estimate (5.89) holds.

We will also use the shifted integrands (4.8) introduced in Section 4.2.1; recall for $f\colon \mathbb{R}^{Nn} \to \mathbb{R}$ this is defined as

$$f_w(z) = f(z+w) - f(w) - f'(w)z. \tag{5.94}$$

Following Lemma 4.2.1, if f is C^2, quasiconvex and satisfies (A2), by distinguishing between the cases $|z| \leq 1$ and $|z| > 1$ we obtain the estimates

$$|f_w(z)| \leq C\varphi_1(|z|), \tag{5.95}$$

$$|f'_w(z)| \leq C\varphi_1'(|z|). \tag{5.96}$$

Lemma 5.3.4 (Caccioppoli inequality of the second kind)**.** Let F satisfy Hypotheses 5.1.1 with $\varphi \in \Delta_2 \cap \nabla_2$, $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be a minimiser of $\mathcal{F}$, and let $\chi_0 > 0$ as in (5.93). Then for each $M > 0$, there is $C > 0$ such that for all $1 < \chi \leq \chi_0$ which the following holds. If any $a\colon \mathbb{R}^n \to \mathbb{R}^N$ affine satisfying $|a(x_0)| + |\nabla a(x_0)| \leq M$ and $B_R(x_0) \subset \Omega$, we have

$$\begin{aligned} \fint_{B_R(x_0)} \varphi_1(|\nabla u - \nabla a|) \,\mathrm{d}x \leq C \fint_{B_R(x_0)} \varphi_1\left(\frac{|u-a|}{R}\right) \mathrm{d}x \\ + C\vartheta_{M,\beta}^{\frac{1}{\chi}} \left(R + \fint_{B_R(x_0)} |u - u_0| \,\mathrm{d}x\right) \fint_{B_R(x_0)} 1 + \varphi_1(|\nabla u|) \,\mathrm{d}x, \end{aligned} \tag{5.97}$$

where $u_0 = a(x_0)$.

We will use the shifted integrands we recall in our setting in (5.98). In particular, we will use the estimates from Lemma 4.2.1.

Proof. To simplify notation we will suppress the x_0-dependence. Put $\tilde{u} = u - a$ and set

$$\widetilde{F}(z) = F(x_0, u_0, z_0 + z) - F(x_0, u_0, z_0) - \partial_z F(x_0, u_0, z_0) \cdot z, \tag{5.98}$$

where $a(x) = u_0 + z_0 \cdot (x - x_0)$.

For $0 < t < s < R$, let $\eta \in W_0^{1,\infty}(B_R(x_0))$ be a cutoff such that $1_{B_t} \leq \eta \leq 1_{B_s}$ and $|\nabla \eta| \leq \frac{1}{s-t}$. Then since $\xi := \eta \tilde{u} \in W_0^{1,\varphi}(B_R(x_0), \mathbb{R}^N)$, by strict quasiconvexity of $\tilde{F}$ we have

$$\nu \int_{B_R} \varphi_1(|\nabla \xi|) \,\mathrm{d}x \leq \int_{B_R} \widetilde{F}(\nabla \xi) \,\mathrm{d}x. \tag{5.99}$$

We will also use the minimality of u which gives

$$\int_{B_R(x_0)} F(x, u, \nabla u) \,\mathrm{d}x \leq \int_{B_R(x_0)} F(x, u - \xi, \nabla(u - \xi)) \,\mathrm{d}x. \tag{5.100}$$

To combine these estimates, we observe that

$$\begin{aligned} &\int_{B_R} \widetilde{F}(\nabla \xi) \,\mathrm{d}x - \int_{B_R} F(x, u, \nabla u) \,\mathrm{d}x \\ &\quad = \int_{B_R} \widetilde{F}(\nabla \xi) - \widetilde{F}(\nabla \tilde{u}) \,\mathrm{d}x + \int_{B_R} \widetilde{F}(\nabla \tilde{u}) - F(x_0, u_0, \nabla u) \,\mathrm{d}x \\ &\qquad + \int_{B_R} F(x_0, u_0, \nabla u) - F(x, u, \nabla u) \,\mathrm{d}x \\ &\quad \leq C \int_{B_R} \left(\varphi_1'(|\nabla \xi|) + \varphi_1'(|\nabla \tilde{u}|)\right) |\nabla(u - \xi)| \,\mathrm{d}x + \int_{B_R} \widetilde{F}(\nabla \tilde{u}) - F(x_0, u_0, \nabla u) \,\mathrm{d}x \\ &\qquad + C \int_{B_R} \vartheta_{M,\beta}(|x - x_0| + |u - u_0|)(1 + \varphi_1(|\nabla u|)) \,\mathrm{d}x, \end{aligned} \tag{5.101}$$

where we have used the growth bound (5.96) and (A5) in the last line. Similarly we have

$$\begin{aligned}
&\int_{B_R} F(x, u-\xi, \nabla(u-\xi))\,\mathrm{d}x - \int_{B_R} \widetilde{F}(\nabla(\tilde{u}-\xi))\,\mathrm{d}x \\
&\quad \leq C\int_{B_R} F(x_0, u_0, \nabla(u-\xi)) - \widetilde{F}(\nabla(\tilde{u}-\xi))\,\mathrm{d}x \\
&\quad\quad + C\int_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)(1+\varphi_1(|\nabla(u-\xi)|))\,\mathrm{d}x.
\end{aligned} \tag{5.102}$$

and we can deduce similar estimates for $\tilde{u}$ replaced by $(1-\eta)\tilde{u}$. Further observe that since $(1-\eta)\tilde{u}$ and $\tilde{u}$ agree on ∂B_R, a calculation shows that

$$\int_{B_R} \widetilde{F}(\nabla\tilde{u}) - F(x_0, u_0, \nabla u)\,\mathrm{d}x = \int_{B_R} \widetilde{F}(\nabla((1-\eta)\tilde{u})) - F(x_0, u_0, \nabla((1-\eta)\tilde{u}) + \nabla a). \tag{5.103}$$

Hence combining (5.99), (5.100), (5.101), (5.103) and using growth bound (5.95) for $\widetilde{F}$ we see that

$$\begin{aligned}
&\int_{B_s} \varphi_1(|\nabla\xi|)\,\mathrm{d}x \leq C\int_{B_R} \varphi_1(|\nabla(\tilde{u}-\xi)|)\,\mathrm{d}x \\
&\quad + C\int_{B_R} (\varphi_1'(|\nabla\xi|) + \varphi_1'(|\nabla\tilde{u}|))|\nabla(\tilde{u}-\xi)|\,\mathrm{d}x \\
&\quad + C\int_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)(1+\varphi_1(|\nabla u|))\,\mathrm{d}x \\
&\quad + C\int_{B_R} \vartheta_{M,\beta}(|x-x_0|+|(1-\eta)\tilde{u}+a-u_0|)(1+\varphi_1(|\nabla((1-\eta)\tilde{u})|))\,\mathrm{d}x \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.104}$$

Noting that $u-\xi$ vanishes in B_t, so by Young's inequality we have

$$I_1 + I_2 \leq C\int_{B_s\setminus B_t} \varphi_1(|\nabla\tilde{u}|)\,\mathrm{d}x + \int_{B_s} \varphi_1\left(\frac{|\tilde{u}|}{s-t}\right)\,\mathrm{d}x. \tag{5.105}$$

Note that

$$\begin{aligned}
|(1-\eta)\tilde{u}+a-u_0| &\leq |\eta|\,(|u-u_0|+|u_0-a|) + |a-u_0| \\
&\leq |u-u_0| + 2M|x-x_0|,
\end{aligned} \tag{5.106}$$

and since $\vartheta_{M,\beta} \leq 1$ and it is doubling we can estimate

$$\begin{aligned}
I_4 + I_5 &\leq C\int_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)(1+\varphi_1(|\nabla(u-\xi)|)+\varphi_1(|\nabla u|))\,\mathrm{d}x \\
&\leq C\int_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)(1+\varphi_1(|\nabla u|))\,\mathrm{d}x + \int_{B_R} \varphi_1\left(\frac{|\tilde{u}|}{s-t}\right)\,\mathrm{d}x.
\end{aligned} \tag{5.107}$$

We now use the improved integrability from Lemmas 5.3.2, 5.3.3, so for $0 < \chi \leq \chi_0$ we can estimate

$$\begin{aligned}
&\int_{B_R} \vartheta_{M,\beta}(|x-x_0| + |u-u_0|)(1+\varphi_1(|\nabla u|))\,\mathrm{d}x \\
&\quad\leq CR^n \left(\fint_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)\,\mathrm{d}x \right)^{\frac{1}{\chi'}} \left(\fint_{B_R} 1+\varphi_1(|\nabla u|)^{\chi}\,\mathrm{d}x \right)^{\frac{1}{\chi}} \\
&\quad\leq CR^n \left(\fint_{B_R} \vartheta_{M,\beta}(|x-x_0|+|u-u_0|)\,\mathrm{d}x \right)^{\frac{1}{\chi'}} \fint_{B_{2R}} 1+\varphi_1(|\nabla u|)\,\mathrm{d}x \\
&\quad\leq C\vartheta_{M,\beta}^{\frac{1}{\chi}} \left(R + \fint_{B_R} |u-u_0|\,\mathrm{d}x \right) \int_{B_{2R}} 1+\varphi_1(|\nabla u|)\,\mathrm{d}x
\end{aligned} \tag{5.108}$$

using Jensen's inequality in the last line, noting that $\vartheta_{M,\beta} \leq 1$ and that it is concave. Putting everything together we arrive at the estimate

$$\begin{aligned}
\int_{B_s} \widetilde{\varphi}(|\nabla\tilde{u}|)\,\mathrm{d}x \leq C_* \int_{B_t\setminus B_s} \widetilde{\varphi}(|\nabla\tilde{u}|)\,\mathrm{d}x + C\int_{B_t} \varphi\left(\frac{|\tilde{u}|}{s-t}\right)\,\mathrm{d}x \\
+ C\vartheta_{M,\beta}^{\frac{1}{\chi}} \left(R + \fint_{B_R} |u-u_0|\,\mathrm{d}x \right) \int_{B_{2R}} 1+\varphi_1(|\nabla\tilde{u}|)\,\mathrm{d}x.
\end{aligned} \tag{5.109}$$

We now fill the hole by adding $C_* \int_{B_s} \varphi_1(|\nabla\tilde{u}|)\,\mathrm{d}x$ to both sides and dividing by $(1+C_*)$. Then by an iteration argument (see [Die+12, Lemma 3.1]) we can absorb the gradient term on the right-hand side to get

$$\begin{aligned}
\int_{B_{R/2}} \varphi_1(|\nabla\tilde{u}|)\,\mathrm{d}x \leq C\int_{B_R} \varphi_1\left(\frac{|\tilde{u}|}{R}\right)\,\mathrm{d}x \\
+ C\vartheta_{M,\beta}^{\frac{1}{\chi}} \left(R + \fint_{B_R} |u-u_0|\,\mathrm{d}x \right) \int_{B_{2R}} 1+\varphi(|\nabla\tilde{u}|)\,\mathrm{d}x.
\end{aligned} \tag{5.110}$$

By a covering argument we can replace the B_{2R} by B_R to conclude. □

5.3.4 Perturbation via Ekeland

We wish to combine our refined Caccioppoli inequality with a suitable harmonic approximation argument, which involves linearising the Euler-Lagrange system satisfied by our minimiser u. However in the non-autonomous case, under our assumptions u need not satisfy an Euler-Lagrange system in general. Instead we will construct a almost-minimiser v of an autonomous functional obtained by freezing the lower-order terms, and show this satisfies an Euler-Lagrange *inequality*. The construction of v will make use of Ekeland's variational principle (Proposition 5.2.8). We will closely follow the argument in Acerbi & Fusco [AF89], adapted to the setting of general Orlicz growth.

Lemma 5.3.5 (Euler-Lagrange inequality)**.** Let F satisfy Hypotheses 5.1.1 with $\varphi \in \Delta_2 \cap \nabla_2$, and let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be a minimiser of $\mathcal{F}$. Then for $M > 0$, there is $C > 0$, $R_0 > 0$ and $0 < \beta_1 < \beta$ such that if $B_{2R}(x_0) \subset \Omega$ for which $|(u)_{B_{2R}(x_0)}| + |(\nabla u)_{B_{2R}(x_0)}| \leq 2^n M$ and $E(x_0, 2R) \leq 1$, there is $v \in W^{1,\varphi}_u(B_R(x_0), \mathbb{R}^N)$ satisfying

$$\fint_{B_R(x_0)} \varphi(|\nabla u - \nabla v|)\,\mathrm{d}x \leq CR^{\beta_1}. \tag{5.111}$$

Moreover v satisfies the Euler-Lagrange inequality

$$\left|\fint_{B_R} \partial_z F(x_0, u_0, \nabla v) \cdot \nabla\xi \,\mathrm{d}x\right| \leq CR^{\beta_1} \fint_{B_R} |\nabla \xi|\,\mathrm{d}x \tag{5.112}$$

for all $\xi \in W^{1,\varphi}_0(B_R(x_0), \mathbb{R}^N)$, where $u_0 = (u)_{B_R(x_0)}$.

Proof. Introduce the frozen functional

$$\mathcal{F}_0(v, \mathcal{O}) = \int_{\mathcal{O}} F(x_0, u_0, \nabla v)\,\mathrm{d}x \tag{5.113}$$

for all $\mathcal{O} \subset \Omega$ and $v \in W^{1,\varphi}(\mathcal{O}, \mathbb{R}^N)$. We can extend $\mathcal{F}_0(\cdot, \mathcal{O})$ by $+\infty$ to a (strongly) lower-semicontinuous functional on $W^{1,1}(\mathcal{O}, \mathbb{R}^N)$. We wish to apply Ekeland's variational principle to $\mathcal{F}_0$, for which we first claim that

$$\mathcal{F}_0(u, B_R) \leq \mathcal{F}_0(w, B_R) + C_* R^{n + \frac{\beta}{\chi'}} \tag{5.114}$$

for all $w \in W^{1,1}_u(B_R, \mathbb{R}^N)$, suppressing the x_0-dependence to simplify notation. To see this, first observe that our assumptions gives $C_M > 0$ for which

$$\fint_{B_{2R}} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x \leq C_M. \tag{5.115}$$

Also since the integrand $z \mapsto F(x_0, u_0, z)$ is strictly quasiconvex with φ-growth, by Proposition 5.2.2 we have $\mathcal{F}_0(\cdot, B_R)$ is weakly sequentially lower-semicontinuous in $W^{1,\varphi}_u(B_R, \mathbb{R}^N)$, and hence we infer the existence of a minimiser $\overline{u}$ using the Direct Method. Further by the quasiconvexity and growth conditions we have

$$\begin{aligned} \nu \int_{B_R} \varphi_1(|\nabla \overline{u}|)\,\mathrm{d}x &\leq \mathcal{F}_0(\nabla \overline{u}, B_R) + CR^n \\ &\leq \mathcal{F}_0(\nabla u, B_R) + CR^n \\ &\leq C \int_{B_R} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x. \end{aligned} \tag{5.116}$$

In addition, by Lemma 5.3.3 we know there is $\chi_0 > 0$ such that $\varphi_1(|\nabla u|) \in L^{\chi_0}(B_R)$, and by arguing as in (5.2.6) (noting we have (5.35) is satisfied) we have $\overline{u}$ is a quasi-minima of the φ-Dirichlet energy $\mathcal{E}_{\varphi,u}$ in B_R. Hence we have a corresponding Caccioppoli inequality by Lemma 5.2.9, so applying the Gehring lemma (Lemma 5.3.3) we there is $\chi_1 \in (0, \chi_0)$ such that

$$\left(\fint_{B_R} \varphi_1(|\nabla \overline{u}|)^{\chi_1}\,\mathrm{d}x\right)^{\frac{1}{\chi_1}} \leq C\left(\fint_{B_R} 1 + \varphi_1(|\nabla u|)^{\chi_1}\,\mathrm{d}x\right)^{\frac{1}{\chi_1}} \leq C\fint_{B_{2R}} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x. \tag{5.117}$$

We now wish to use the minimality of u to derive estimates in terms of $\mathcal{F}_0$, for which we will perform a perturbation argument. Indeed using (A5) we have

$$\begin{aligned}
\mathcal{F}_0(u, B_R) &\leq \mathcal{F}(\overline{u}, B_R) + (\mathcal{F}_0(u, B_R) - \mathcal{F}(u, B_R)) \\
&= \mathcal{F}_0(\overline{u}, B_R) + (\mathcal{F}_0(u, B_R) - \mathcal{F}(u, B_R)) + (\mathcal{F}(\overline{u}, B_R) - \mathcal{F}_0(\overline{u}, B_R)) \\
&\leq \mathcal{F}_0(\overline{u}, B_R) + C\int_{B_R} \vartheta_{M,\beta}(|x-x_0| + |u-u_0|)\,(1+\varphi_1(|\nabla u|))\,\mathrm{d}x \\
&\quad + C\int_{B_R} \vartheta_{M,\beta}(|x-x_0| + |\overline{u}-u_0|)\,(1+\varphi_1(|\nabla \overline{u}|))\,\mathrm{d}x \\
&= \mathcal{F}_0(\overline{u}, B_R) + I_1 + I_2.
\end{aligned} \tag{5.118}$$

For the latter two terms we will use Hölder, the fact that $\vartheta_{M,\beta} \leq 1$, Jensen's inequality, and the improved integrability properties of u and $\overline{u}$ to bound

$$\begin{aligned}
I &\leq CR^n\left(\fint_{B_R} \vartheta_{M,\beta}(|x-x_0| + |u-u_0|)^{\chi_1'}\,\mathrm{d}x\right)^{\frac{1}{\chi_1'}}\left(\fint_{B_R} 1 + \varphi_1(|\nabla u|)^{\chi_1}\,\mathrm{d}x\right)^{\frac{1}{\chi_1}} \\
&\leq CR^n\vartheta_{M,\beta}\left(R + \fint_{B_R} |u-u_0|\,\mathrm{d}x\right)^{\frac{1}{\chi_1'}}\fint_{B_R} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x \\
&\leq CR^n\vartheta_{M,\beta}\left(R + R\fint_{B_R} |\nabla u|\,\mathrm{d}x\right)\fint_{B_R} 1 + \varphi_1(|\nabla u|)\,\mathrm{d}x.
\end{aligned} \tag{5.119}$$

where we have used the Poincaré inequality noting $u_0 = (u)_{B_R}$. By Jensen and (5.115) we have

$$\fint_{B_R} |\nabla u|\,\mathrm{d}x \leq \varphi_1^{-1}\left(\fint_{B_R} \varphi_1(|\nabla u|)\,\mathrm{d}x\right) \leq C, \tag{5.120}$$

so there is $R_0 > 0$ sufficiently small so that $I_1 \leq CR^{n+\frac{\beta}{\chi_1'}}$ for all $0 < R < R_0$. We estimate

I_2 similarly, however we will need to estimate

$$\begin{aligned}\fint_{B_R} |\overline{u} - u_0| \,\mathrm{d}x &\le \fint_{B_R} |\overline{u} - u| + |u - u_0| \,\mathrm{d}x \\ &\le \fint_{B_R} |\nabla u - \nabla \overline{u}| + |\nabla u| \,\mathrm{d}x \\ &\le 3\varphi_1^{-1}\left(\fint_{B_R} \varphi_1(|\nabla u|)\,\mathrm{d}x\right) \le C,\end{aligned} \tag{5.121}$$

noting that $\overline{u} = u$ on ∂B_R which allows us to use the Poincaré inequality. Hence combining with (5.117), by shrinking R_0 if necessary we can ensure that $I_2 \le CR^{n+\frac{\beta}{\chi_1'}}$ for all $0 < R < R_0$. Since $\overline{u}$ minimises $\mathcal{F}_0$ in $W^{1,\varphi}_u(B_R,\mathbb{R}^N)$, this establishes the claimed estimate (5.114)

Now we apply Ekeland's variational principle (Proposition 5.2.8) to $\mathcal{F}_0$ in $W^{1,1}_u(B_R,\mathbb{R}^N)$ with respect to the metric

$$\mathrm{d}(v,w) = C_*^{-1} R^{-n-\frac{\beta}{2\chi_1'}} \int_{B_R} |\nabla v - \nabla w| \,\mathrm{d}x \tag{5.122}$$

This gives $v \in W^{1,1}_u(B_R,\mathbb{R}^N)$ such that

$$\mathcal{F}_0(v,B_R) \le \mathcal{F}_0(u,B_R) \tag{5.123}$$

$$\int_{B_R} |\nabla u - \nabla v| \,\mathrm{d}x \le C_* R^{n+\frac{\beta}{2\chi_1'}}, \tag{5.124}$$

and for all $w \in W^{1,1}_u(B_R,\mathbb{R}^N)$ we have the quasi-minimising property

$$\mathcal{F}_0(v,B_R) \le \mathcal{F}_0(w,B_R) + R^{\frac{\beta}{2\chi_1'}} \int_{B_R} |\nabla v - \nabla w| \,\mathrm{d}x. \tag{5.125}$$

Taking $w = u \pm t\xi$ for $\xi \in W^{1,\varphi}_0(B_R,\mathbb{R}^N)$ in the quasi-minimising property and sending $t \to 0$, the Euler-Lagrange inequality (5.112) follows.

Now by the quasi-minimising property of v, we can use Lemmas 5.2.6, 5.2.9, 5.3.3 similarly as in how we estimated $\overline{u}$ to infer the existence of $\chi_2 \in (0,\chi_1)$ such that

$$\left(\fint_{B_R} \varphi_1(|\nabla v|)^{\chi_2} \,\mathrm{d}x\right)^{\frac{1}{\chi_2}} \le \left(\fint_{B_R} 1 + \varphi_1(|\nabla u|)^{\chi_2} \,\mathrm{d}x\right)^{\frac{1}{\chi_2}} \le \fint_{B_{2R}} 1 + \varphi_1(|\nabla v|) \,\mathrm{d}x. \tag{5.126}$$

Now for $\theta \in (0,1)$ to be determined we use Hölder to estimate

$$\fint_{B_R} \varphi_1(|\nabla u - \nabla v|) \,\mathrm{d}x \le \left(\fint_{B_R} |\nabla u - \nabla v| \,\mathrm{d}x\right)^{\theta} \left(\fint_{B_R} \frac{\varphi_1(|\nabla u - \nabla v|)^{\frac{1}{1-\theta}}}{|\nabla u - \nabla v|^{\frac{\theta}{1-\theta}}} \,\mathrm{d}x\right)^{1-\theta}. \tag{5.127}$$

Choosing θ sufficiently small so that $\frac{1}{1-\theta} \le \chi_2$ and $\theta \le \frac{1}{2}$, and noting that $\varphi_1(t) \simeq t^2$ for $t \le 1$ we have

$$\frac{\varphi_1(t)^{\frac{1}{1-\theta}}}{t^{\frac{\theta}{1-\theta}}} \lesssim \begin{cases} t^{\frac{1}{1-\theta}-\frac{2\theta}{1-\theta}} & \text{if } t \le 1, \\ \varphi_1(t)^{\chi_2} & \text{if } t > 1. \end{cases} \tag{5.128}$$

Hence using the above together with (5.126) we have

$$\fint_{B_R} \varphi_1(|\nabla u - \nabla v|) \,\mathrm{d}x \leq \left(\fint_{B_R} |\nabla u - \nabla v| \,\mathrm{d}x \right)^{\theta} \left(\fint_{B_R} 1 + \varphi_1(|\nabla u|) \,\mathrm{d}x \right)^{(1-\theta)\chi_2} . \tag{5.129}$$

Combing with (5.115) and (5.124) we deduce that

$$\fint_{B_R} \varphi_1(|\nabla u - \nabla v|) \,\mathrm{d}x \leq C R^{\frac{\theta\beta}{2\chi_1'}}, \tag{5.130}$$

so the result follows by taking $\beta_1 = \frac{\theta\beta}{2\chi_1'}$. □

Remark 5.3.6. In the proof we considered a genuine minimiser of the frozen functional $\mathcal{F}_0$ to obtain estimates for u, however we are not able to work with this directly as this may not be close to u in general; in particular we cannot guarantee an inequality of the form (5.111) to hold.

5.3.5 Harmonic approximation

We now use the Euler-Lagrange inequality derived in the previous section to show it is suitably close to a linearised system. This involves the $\mathbb{A}$-harmonic approximation method, adapting the argument of Gmeineder & Kristensen [GK19a] which we already used in Chapter 4. We will apply this to the function v constructed in Lemma 5.3.5, noting it is sufficient to have an Euler-Lagrange inequality to perform the argument.

Similarly as in Chapter 4, we will need a continuity estimate for the shifted integrand f_w from (5.94), taking the form

$$|f_w'(z) - f_w''(0)z| \leq C\omega_M(|z|) \left(|z| + \varphi_1'(|z|) \right), \tag{5.131}$$

for $|w| \leq M$, where $\omega_M \colon [0,\infty) \to [0,1]$ is the modulus of continuity for f'' in $\{z \in \mathbb{R}^{Nn} : |z| \leq M+1\}$; this can be chosen to be non-decreasing continuous and concave such that $\omega_M(0) = 0$.

Lemma 5.3.7 (Harmonic approximation)**.** Let $F_0 = F_0(z)$ be an autonomous integrand satisfying (A1), (A2), (A4) from Hypotheses 5.1.1 with $\varphi \in \Delta_2 \cap \nabla_2$, and $\beta_1 \in (0,1)$. Then for $M > 0$ and $\delta \in (0,1)$, there is $R_0 > 0$ and $\varepsilon_0 > 0$ such that the following holds. Suppose $B_R(x_0) \subset \Omega$ with $R < R_0$ and $v \in W^{1,\varphi}(B_R(x_0), \mathbb{R}^N)$ satisfies the Euler-Lagrange inequality

(5.112) with parameter $\beta_1 \in (0,1)$, and let $a\colon \mathbb{R}^n \to \mathbb{R}^N$ with $|\nabla a| \le M$. Assume further that

$$\fint_{B_R} \varphi_1\left(|\nabla v - \nabla a|\right) \mathrm{d}x \le \varepsilon_0. \tag{5.132}$$

Then the unique solution to the linearised problem

$$\begin{cases} -\operatorname{div} F_0''(\nabla)h = 0 & \text{in } B_R(x_0), \\ h = v & \text{on } \partial B_R, \end{cases} \tag{5.133}$$

satisfies the estimates

$$\fint_{B_R} \varphi_1(|\nabla(h-a)|)\,\mathrm{d}x \le \fint_{B_R} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x, \tag{5.134}$$

and

$$\begin{aligned} \fint_{B_R} \varphi_1\left(\frac{|v-h|}{R}\right) \mathrm{d}x &\le C\delta \fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x + CR^{2\beta_1} \\ &\quad + C_\delta \gamma_M \left(\fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x\right) \fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x. \end{aligned} \tag{5.135}$$

Here $\gamma_M\colon [0,\infty) \to [0,1]$ is a non-decreasing continuous function such that $\gamma_M(0) = 0$.

Proof. By the Theorem 3.3.8 we have the existence of a unique such $h \in W^{1,\varphi}(B_R(x_0), \mathbb{R}^N)$ satisfying (5.134), so it remains to estimate the difference $u - h$. For this we suppress the x_0-dependence and let $\widetilde{F}_0 = F_{0,\nabla a}$ be the shifted integrand and put $\tilde{v} = v - a$, $\tilde{h} = h - a$. Observe by the Poincaré inequality (Lemma 2.161) and (5.134)

$$\fint_{B_R} \varphi_1\left(\frac{|\tilde{v} - \tilde{h}|}{R}\right) \mathrm{d}x \le C \fint_{B_R} \varphi_1(|\nabla(\tilde{v} - \tilde{h})|)\,\mathrm{d}x \le C \fint_{B_R} \varphi_1(|\nabla \tilde{v}|)\,\mathrm{d}x \le C\varepsilon_0. \tag{5.136}$$

We will chose $\varepsilon_0 > 0$ sufficiently small so this is less than 1. For $q \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$, since h is harmonic and v satisfies the Euler-Lagrange inequality (5.112) we have

$$\begin{aligned} &\fint_{B_R} \widetilde{F}_0''(0)\nabla q \cdot \nabla(\tilde{v} - \tilde{h})\,\mathrm{d}x \\ &\quad \le \fint_{B_R} \left(\widetilde{F}_0''(0)\nabla\tilde{v} - \widetilde{F}_0'(\nabla\tilde{v})\right) \cdot \nabla q\,\mathrm{d}x + CR^{2\beta_1} \fint_{B_R} |\nabla q|\,\mathrm{d}x. \end{aligned} \tag{5.137}$$

We wish to choose q to satisfy

$$\begin{cases} -\operatorname{div} \widetilde{F}_0''(0)\nabla q = g & \text{in } B_R, \\ q = 0 & \text{on } \partial B_R, \end{cases} \tag{5.138}$$

where

$$g = \varphi_1\left(\frac{|\tilde{u}-\tilde{h}|}{R}\right)\frac{\tilde{u}-\tilde{h}}{|\tilde{u}-\tilde{h}|^2}. \tag{5.139}$$

Since $\varphi_1^*(\varphi_1(t)/t) \leq \varphi_1(t)$ by Lemma 2.1.6, we have $g \in L^{\varphi^*}(B_R)$ with the associated estimate

$$\fint_{B_R} \varphi_1^*(R|g|)\,\mathrm{d}x \leq \fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right)\mathrm{d}x. \tag{5.140}$$

Hence by Proposition 3.3.12 and Theorem 3.3.8 we infer the existence of a unique $q \in W_0^{1,(\varphi^*)^{\frac{n}{n-1}}}(B_R,\mathbb{R}^N)$ satisfying the estimate

$$\left(\fint_{B_R} \varphi_1^*(|\nabla q|)^{\frac{n}{n-1}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} \leq C\fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right)\mathrm{d}x. \tag{5.141}$$

We cannot test the Euler-Lagrange inequality against this q in general however, as it need not be Lipschitz. Instead we will take a Lipschitz truncation using Lemma 2.3.17. For each $\lambda > 0$, this gives $q_\lambda \in W_0^{1,\infty}(B_R,\mathbb{R}^N)$ for which $\|\nabla q_\lambda\|_{L^\infty(B_R)} \leq C\lambda$. For this choice of test function, using the continuity estimate (5.131) for $\widetilde{F}_0$ we obtain

$$\begin{aligned}
\fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right)\mathrm{d}x &= \fint_{B_R} \left(\widetilde{F}_0''(0)\nabla\tilde{v} - \widetilde{F}_0'(\nabla\tilde{v})\right)\cdot\nabla q_\lambda\,\mathrm{d}x + CR^{2\beta_1}\fint_{B_R}|\nabla q_\lambda|\,\mathrm{d}x \\
&\quad + \fint_{B_R} \widetilde{F}_0''(0)\nabla\tilde{v}\cdot\nabla(q-q_\lambda)\,\mathrm{d}x \\
&\leq C\fint_{B_R} \omega_M(|\nabla\tilde{v}|)\left(|\nabla\tilde{v}| + \varphi_1'(|\nabla\tilde{v}|)\right)|\nabla q_\lambda|\,\mathrm{d}x \\
&\quad + C\fint_{B_R} |\nabla\tilde{v}||\nabla(q-q_\lambda)|\,\mathrm{d}x + CR^{2\beta_1}\lambda \\
&\leq \delta\fint_{B_R} \varphi_1(|\nabla\tilde{v}|)\,\mathrm{d}x \\
&\quad + C_\delta\fint_{B_R} \varphi_1\left(\lambda\omega_M(|\nabla\tilde{v}|)\right) + \varphi_1^*\left(\lambda\omega_M(|\nabla\tilde{v}|)\right)\mathrm{d}x \\
&\quad + C_\delta\fint_{B_R} \varphi_1^*(|\nabla(q-q_\lambda)|)\,\mathrm{d}x + CR^{2\beta_1}\lambda \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{5.142}$$

where we have used Young's inequality (Lemma 2.1.2) to split the products, and the fact that $\varphi \in \Delta_2 \cap \nabla_2$. Estimating the remaining terms separately, note that by Hölder, (5.141) and by the properties (2.199), (2.200) of the Lipschitz truncation we have

$$
\begin{aligned}
I_3 &\leq C \left(\frac{\mathcal{L}^n(B_R \cap \{q \neq q_\lambda\})}{\mathcal{L}^n(B_R)} \right)^{\frac{1}{n}} \left(\fint_{B_R} \varphi_1^*(|\nabla(q - q_\lambda)|)^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} \\
&\leq C \left(\frac{\fint_{B_R} \varphi_1(|\nabla q|)^{\frac{n}{n-1}} \,\mathrm{d}x}{\varphi_1^*(\lambda)^{\frac{n}{n-1}}} \right)^{\frac{1}{n}} \left(\fint_{B_R} \varphi_1^*(|\nabla q|)^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} \\
&\leq C \left(\frac{\fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right) \mathrm{d}x}{\varphi_1^*(\lambda)} \right)^{\frac{1}{n-1}} \fint_{B_R} \varphi_1 \left(\frac{|\tilde{v} - \tilde{h}|}{R} \right) \mathrm{d}x.
\end{aligned}
\tag{5.143}
$$

We then choose $\lambda > 0$ so that

$$
C \left(\frac{\fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right) \mathrm{d}x}{\varphi_1^*(\lambda)} \right)^{\frac{1}{n-1}} = \frac{1}{2},
\tag{5.144}
$$

so we can absorb I_3 to the left-hand side. By (5.136) we have $\varphi_1^*(\lambda) \leq C\varepsilon_0$, so shrinking $\varepsilon_0 > 0$ if necessary we can assume that $\lambda \leq 1$. Hence it follows that $\lambda^2 \sim \fint_{B_R} \varphi_1\left(\frac{|\tilde{v}-\tilde{h}|}{R}\right) \mathrm{d}x$. This also allows us to estimate I_4 using Young's inequality, absorbing the λ term to the left-hand side.

For the second term, noting $\omega_M \leq 1$, using the convexity of φ_1, φ_1^*, and by Jensen's inequality applied to $\omega_M \circ \varphi_1^{-1}$ we can estimate

$$
\begin{aligned}
I_2 &\leq C\left(\varphi_1(\lambda) + \varphi_1^*(\lambda)\right) \fint_{B_R} \omega_M(|\nabla \tilde{v}|) \,\mathrm{d}x \\
&\leq C \omega_M \circ \varphi_1^{-1} \left(\fint_{B_R} \varphi_1(|\nabla \tilde{v}|) \,\mathrm{d}x \right) \fint_{B_R} \varphi_1 \left(\frac{|\tilde{v} - \tilde{h}|}{R} \right) \mathrm{d}x
\end{aligned}
\tag{5.145}
$$

by choice of λ, noting also that $\varphi(\lambda) \sim \varphi^*(\lambda) \sim \lambda^2$. Using (5.136) and combining the estimates, the result follows with $\gamma = \omega_M \circ \varphi_1^{-1}$. □

Remark 5.3.8. We note that the ∇_2-condition is used at several points in the proof, namely to show the existence of h, q with the associated estimates, and in the estimate (5.140) for ∇q which uses (5.140). In addition, implicit in the proof is the estimate

$$
\varphi_1^*(C\varphi(t)/t) \lesssim \varphi_1(t),
\tag{5.146}
$$

which we know by Lemma 2.1.6 holds for any N-function if $C = 1$, however in general we can only pull out the constant if $\varphi_1^* \in \Delta$. As we will see in Section 5.3.7 we can still establish the existence of a suitable h in the absence of the ∇_2-condition, and we can also infer the existence of a suitable q using the results in Section 3.3.3, however the inequality (5.146) does not seem to be easy to work around. We will state a partial result in Section 5.3.7, where we additionally assume φ satisfies a certain growth condition.

5.3.6 Excess decay estimates

Finally we combine the above estimates to obtain decay estimates for the excess energy, which we can iterate to establish Theorem 5.3.1. Recall that for a minimiser u we consider the excess energy

$$E(x_0, R) = \fint_{B_R(x_0)} \varphi_1(|\nabla u - (\nabla u)_{B_R(x_0)}|)\,\mathrm{d}x. \tag{5.147}$$

Lemma 5.3.9 (Excess decay estimate)**.** Let F satisfy Hypotheses 5.1.1 with $\varphi \in \Delta_2 \cap \nabla_2$, and $M > 0$. Then there is $\varepsilon_1 > 0$ and $R_0 > 0$ such that the following holds. Suppose $\Omega \subset \mathbb{R}^n$ is open and $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ minimises the associated functional in its Dirichlet class, and $B_{2R}(x_0) \subset \Omega$ with $R \leq R_0$ such that

$$|(u)_{B_R(x_0)}| + |(\nabla u)_{B_R(x_0)}| \leq M \quad \text{and} \quad E(x_0, 2R) < \varepsilon_1. \tag{5.148}$$

Then for all $\sigma \in (0,1)$ we have

$$E(x_0, \sigma R) \leq C\left(\sigma^2 + C_\sigma \delta + C_{\sigma,\delta}\gamma(E(x_0,R))\right) E(x_0, R) + C_\sigma R^{2\beta_1}, \tag{5.149}$$

where $\beta_1 \in (0, \beta)$ is as in Lemma 5.3.5, and $\gamma \colon [0,\infty) \to [0,1]$ is a non-decreasing continuous function such that $\gamma(0) = 0$.

Proof. Suppressing the x_0 dependence, we will take $\varepsilon_1 \leq 1$ applying Lemma 5.3.5 to get $v \in W^{1,\varphi}(B_R, \mathbb{R}^N)$ and $\beta_1 \in (0, \beta_1)$ such that the Euler-Lagrange inequality (5.112) along with the comparison estimate

$$\fint_{B_R} \varphi_1(|\nabla u - \nabla v|)\,\mathrm{d}x \leq CR^{2\beta_1} \tag{5.150}$$

holds. We will let $h \in W^{1,\varphi}_u(B_R, \mathbb{R}^N)$ be the unique solution to (5.133) from Lemma 5.3.7, and set $\widetilde{h} = h - a$ where

$$a(x) = (u)_{B_R(x_0)} + (\nabla u)_{B_R(x_0)} \cdot (x - x_0). \tag{5.151}$$

Now let $\sigma \in (0, \frac{1}{4})$, and set

$$a_1(x) = a(x) + (u)_{B_{2\sigma R}} + \left(\nabla \tilde{h}(x_0) - (\nabla u)_{B_R(x_0)}\right) \cdot (x - x_0). \tag{5.152}$$

Applying the Caccioppoli inequality of the second kind (Lemma 5.3.4) on $B_{2\sigma R}$ we have

$$\begin{aligned} E(x_0, \sigma R) &\le C \fint_{B_{\sigma R}} \varphi_1 \left(\nabla u - \nabla a_1\right) \mathrm{d}x \\ &\le C \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|u - a_1|}{2\sigma R}\right) \mathrm{d}x \\ &\quad + C \vartheta_{M,\beta}^{\beta_1/\beta} \left(2\sigma R + \fint_{B_{2\sigma R}} |u - (u)_{B_{2\sigma R}}| \, \mathrm{d}x\right) \fint_{B_R} 1 + \varphi_1(|\nabla u|) \\ &\le C_\delta \fint_{B_R} \varphi_1 \left(\frac{|u - h|}{R}\right) \mathrm{d}x + C \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|h - a|}{2\sigma R}\right) \mathrm{d}x \\ &\quad + C \vartheta_{M,\beta}^{\beta_1/\beta} \left(2\sigma R + C_\sigma R \varphi_1^{-1} \left(\fint_{B_R} \varphi_1(|\nabla u|) \, \mathrm{d}x\right)\right) \fint_{B_R} 1 + \varphi_1(|\nabla u|) \, \mathrm{d}x, \end{aligned} \tag{5.153}$$

where we have used the Poincaré inequality and Jensen in the argument of $\vartheta_{M,\beta}$. Now since $|(\nabla u)_{B_R(x_0)}| \le M$ and $E(x_0, R) \le 1$, we have $\fint_{B_R} \varphi_1(|\nabla u|) \, \mathrm{d}x \le C$. Hence taking R_0 sufficiently small, we can ensure that

$$\vartheta_{M,\beta}^{\beta_1/\beta} \left(2\sigma R + C_\sigma R \varphi_1^{-1} \left(\fint_{B_R} \varphi_1(|\nabla u|) \, \mathrm{d}x\right)\right) \le \vartheta_{M,\beta}^{\beta_1/\beta}(CR) \le C R^{2\beta_1}. \tag{5.154}$$

Also letting $a_h(x) = \tilde{h}(x_0) + \nabla \tilde{h}(x_0) \cdot (x - x_0)$, by Lemma 3.1.6 together with the Poincaré and Jensen inequalities we have

$$\begin{aligned} \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|\tilde{h} - a_h|}{2\sigma R}\right) \mathrm{d}x &\le \varphi_1 \left(C \sigma R \fint_{B_R} |u - a| \, \mathrm{d}x\right) \\ &\le C \varphi_1 \left(\sigma \varphi_1^{-1}(E(x_0, R))\right) \\ &\le C \sigma^2 E(x_0, R), \end{aligned} \tag{5.155}$$

where we used the fact that $\varphi_1(E(x_0, R)) \le 1$ in the last line. Hence using the above and Jensen's inequality,

$$\begin{aligned} \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|h - a_1|}{2\sigma R}\right) \mathrm{d}x &\le C \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|h - a_h|}{2\sigma R}\right) \mathrm{d}x \\ &\quad + C \varphi_1 \left(\frac{|(u)_{B_R} - (h)_{B_R}|}{2\sigma R}\right) \mathrm{d}x \\ &\le C \sigma^2 E(x_0, R) + C \fint_{B_{2\sigma R}} \varphi_1 \left(\frac{|u - h|}{2\sigma R}\right) \mathrm{d}x. \end{aligned} \tag{5.156}$$

We now apply the harmonic approximation lemma 5.3.7 with the above choice of a, taking $\varepsilon_1 \leq \varepsilon_0$ so it applies. Observe by (5.150) we can replace v by u in the estimate (5.135) for $v-h$ to deduce that for $\delta \in (0,1)$ we have

$$\fint_{B_R} \varphi_1\left(\frac{|v-h|}{R}\right)\,\mathrm{d}x \leq \delta E(x_0,R) + C_\delta \gamma\left(E(x_0,R)\right)E(x_0,R) + CR^{2\beta_1}. \tag{5.157}$$

Hence by combining (5.153), (5.156) and (5.157), the claimed estimate (5.149) follows for $\sigma \in (0,\frac{1}{4})$. Increasing the constants if necessary, this also holds for $\frac{1}{4} \leq \sigma \leq 1$. □

Proof of Theorem 5.3.1. Let $x \in B_{R/2}(x_0)$, so we have $|(u)_{B_{R/2}(x)}| + |(\nabla u)_{B_{R/2}(x)}| \leq 2^n M$ and $E(x,R/2) \leq 4^n\varepsilon$. We apply Lemma 5.3.9 with $4^{n+1}M$, taking $\varepsilon > 0$ sufficiently small so it applies to get for all $\sigma \in (0,1)$ and $\delta \in (0,1)$ the estimate

$$E(x_0,\sigma R) \leq (C\sigma^2 + C_\sigma\delta + C_{\sigma,\delta}\gamma(\varepsilon))E(x,\sigma R) + C_\sigma R^{2\beta_1}. \tag{5.158}$$

Now fixing $0 < \alpha < \beta_1$, we will choose σ so that $C\sigma^2 \leq \frac{1}{6}\sigma^{2\alpha}$, δ so that $C_\sigma\delta \leq \frac{1}{6}\sigma^{2\alpha}$, ε so that $C_{\sigma,\delta}\gamma(\varepsilon) \leq \frac{1}{6}\sigma^{2\alpha}$, and for $\kappa \in (0,1)$ to be determined we can shrink R_0 further so that $C_\sigma R^{2\beta_1 - 2\alpha} \leq \kappa\sigma^{2\alpha}$. This gives

$$E(x,\sigma R) \leq \frac{1}{2}\sigma^{2\alpha}E(x,R) + \kappa(\sigma R)^{2\alpha}. \tag{5.159}$$

Now we iterate the above and argue by induction to show that if $\varepsilon,\kappa > 0$ are sufficiently small, then

$$|(\nabla u)_{B_{\sigma^{j-1}R}(x)}| \leq 4^{n+1}M, \tag{5.160}$$

$$E(x,\sigma^j R) \leq 2^{-j}\sigma^{2j\alpha}E(x,R) + (\sigma^j R)^{2\alpha} \tag{5.161}$$

for all $j \geq 1$. If this holds for for all $1 \leq i \leq j$, then note first that for each i we can use Jensen's inequality to bound

$$\begin{aligned}
&|(\nabla u)_{B_{\sigma^i R}(x)} - (\nabla u)_{B_{\sigma^{i-1}R}(x)}| \\
&\quad\leq \sigma^{-n}\fint_{B_{\sigma^{i-1}R}(x)} |\nabla u - (\nabla u)_{B_{\sigma^{i-1}R}(x)}|\,\mathrm{d}x \\
&\quad\leq \sigma^{-n}\varphi_1^{-1}\left(E(x,\sigma^{i-1}R)\right) \\
&\quad\leq \sigma^{-n}\varphi_1^{-1}\left(2^{-(i-1)}\sigma^{2(i-1)\alpha}\varepsilon + \sigma^{2(j-1)\alpha}R^{2\alpha}\right).
\end{aligned} \tag{5.162}$$

Hence by shrinking $\varepsilon, R_0 > 0$ further if necessary to ensure the argument is less than 1 and noting $\varphi_1^{-1}(s) \leq \sqrt{s}$ for small $s \leq 1$, we have

$$\begin{aligned}
|(\nabla u)_{B_{\sigma^j R}(x)}| &\leq 4^n M + \sum_{i=1}^{j} |(\nabla u)_{B_{\sigma^i R}(x)} - (\nabla u)_{B_{\sigma^{i-1} R}(x)}| \\
&\leq 4^n M + \sigma^{-n} \sum_{i=1}^{j-1} \left(2^{-(i-1)} \sigma^{2(i-1)\alpha} \varepsilon + \sigma^{2(j-1)\alpha} R^{2\alpha} \right) \\
&\leq 4^n M + \sigma^{-n} \sqrt{\varepsilon} \sum_{i=0}^{\infty} \left(\frac{\sigma^\alpha}{\sqrt{2}} \right)^i + \sigma^{-n} R_0^\alpha \sum_{i=0}^{\infty} \sigma^{i\alpha},
\end{aligned} \tag{5.163}$$

which is less than $4^{n+1}M$ for $\varepsilon, R_0 > 0$ sufficiently small. Also applying (5.159) iteratively we deduce that

$$E(x, \sigma^j R) \leq 2^{-j} \sigma^{2j\alpha} E(x, R) + \kappa (\sigma^j R)^{2\alpha} \sum_{j=0}^{\infty} 2^{-(j-1)} \sigma^{2(j-1)\alpha}, \tag{5.164}$$

so choosing $\kappa > 0$ we can also ensure (5.161) holds. Hence for any $r \in (0, R/2)$ we have $E(x, r) \leq \varphi_1(2^{n+1}\varepsilon) \leq 1$ and so

$$\fint_{B_r(x)} |\nabla u - (\nabla u)_{B_r(x)}| \, \mathrm{d}y \leq C \sqrt{E(x, r)} \leq C r^\alpha. \tag{5.165}$$

Since this holds for all $x \in B_{R/2}(x)$ and $0 < r < R/4$, by the Campanato-Meyers characterisation of Hölder continuity (see for instance [Giu03, Theorem 2.9]) we conclude that ∇u is $C^{0,\alpha}$ in $\overline{B}_{R/2}(x_0)$, as required. $\square$

5.3.7 The case of degenerate growth

We would like to prove Theorem 5.3.1 without assuming the ∇_2-condition, which would capture the case of almost-linear growth, however we were not able to achieve this. In this section we will establish a partial result, where we assume $F = F(x, z)$ is independent of u, and additionally impose a suitable growth bound on φ.

Assuming F still satisfies Hypotheses 5.1.1, only without the u-dependence, we can still carry out the main steps of the proof. In this case the Caccioppoli inequality simplifies to

$$\begin{aligned}
\fint_{B_{R/2}} \varphi_1(|\nabla(u - a)|) \, \mathrm{d}x &\leq C \fint_{B_R} \varphi_1 \left(\frac{|u - a|}{R} \right) \mathrm{d}x \\
&\quad + C R^\alpha \int_{B_R} 1 + \varphi_1(|\nabla u|) \, \mathrm{d}x,
\end{aligned} \tag{5.166}$$

for all $B_R \subset \Omega$ and $a\colon \mathbb{R}^n \to \mathbb{R}^N$ such that $|\nabla a| \le M$. Here we only needed the higher integrability results to manage the u-dependence, so the argument goes through in the absence of the ∇_2-condition. Similarly we can carry out the procedure in Lemma 5.3.5 to find $v \in W^{1,p}_u(B_R, \mathbb{R}^N)$ satisfying the Euler-Lagrange inequality (5.112). The main difference lies in the proof of the harmonic approximation theorem, which we modify as follows.

Lemma 5.3.10 (Harmonic approximation without ∇_2)**.** Let $F_0 = F_0(z)$ be an autonomous integrand satisfying (A1), (A2), (A4) from Hypotheses 5.1.1, where $\varphi \in \Delta_2$ and additionally satisfies

$$\varphi_1(t) \le C t^{\frac{r}{r-1}}. \tag{5.167}$$

for some $r > n$. Then given $\beta > 0$, $M > 0$ and $\delta \in (0,1)$, there is $R_0 > 0$ and $\varepsilon_0 > 0$ such that the following holds. Suppose $B_R(x_0) \subset \Omega$ with $R < R_0$ and $v \in W^{1,\varphi}(B_R(x_0), \mathbb{R}^N)$ satisfies the Euler-Lagrange inequality (5.112) with parameter $\beta \in (0,1)$, and let $a\colon \mathbb{R}^n \to \mathbb{R}^N$ with $|\nabla a| \le M$. Assume further that

$$\fint_{B_R} \varphi_1\left(|\nabla v - \nabla a|\right)\,\mathrm{d}x \le \varepsilon_0. \tag{5.168}$$

Then for almost all $\kappa \in (\frac{9}{10}, 1)$, the unique solution to the linearised problem

$$\begin{cases} -\operatorname{div} F_0''(\nabla)h = 0 & \text{in } B_{\kappa R}(x_0) \\ h = v & \text{on } \partial B_{\kappa R} \end{cases} \tag{5.169}$$

satisfies the estimates

$$\fint_{B_{\kappa R}} \varphi_1(|\nabla(h-a)|)\,\mathrm{d}x \le \fint_{B_R} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x. \tag{5.170}$$

and

$$\begin{aligned} \fint_{B_{\kappa R}} \varphi_1\left(\frac{|v-h|}{R}\right)\mathrm{d}x &\le C\delta \fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x \\ &+ CR^{\beta}\left(\fint_{B_R(x_0)} \varphi_1\left(|\nabla(v-a)|\right)\,\mathrm{d}x\right)^{\frac{1}{r}} \\ &+ C_\delta \gamma_{M,\delta}\left(\fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x\right) \fint_{B_R(x_0)} \varphi_1(|\nabla(v-a)|)\,\mathrm{d}x \end{aligned} \tag{5.171}$$

Here $\gamma_{M,\delta}\colon [0,\infty) \to [0,1]$ is a non-decreasing continuous function such that $\gamma_{M,\delta}(0) = 0$.

Discussion 5.3.11 (Necessity of the growth condition)**.** We mention that in [GK19b] a partial regularity result is announced for BV minimisers of autonomous integrands satisfying a $(1,q)$-growth conditions with $q < \frac{n}{n-1}$, which is consistent with our results. This also encompasses many examples of interest including $\varphi(t) = (1+t)\log(1+t) - t$, where the N-function is asymptotically bounded from above by t^p for all $p > 1$. However we do not believe this assumption is necessary, as it is only used to show the solution q to the dual problem is Lipschitz. This is in contract in the ∇_2-case where we only use the fact that $\nabla q \in L^{(\varphi^*)^{\frac{n}{n-1}}}$, which is a significantly weaker embedding result. We also mention that SCHMIDT [Sch08a] observed that for problems with (p,q)-growth with $p > 1$, the harmonic approximation step only needs $q \leq p+1$, though for such problems further restrictions are assumed to obtain a Caccioppoli-type inequality.

Proof. We suppress the x_0-dependence, let $\widetilde{F}_0 = F_{0,\nabla a}$ be the shifted integrand and put $\tilde{v} = v - a$. By the coarea formula we know that for $\mathcal{L}^1$-almost all $\kappa \in \left(\frac{9}{10}, 1\right)$ we have $v|_{\partial B_{\kappa R}}$ lies in $W^{1,\varphi}(\partial B_{\kappa R}, \mathbb{R}^N)$ with the associated estimate

$$\int_{\partial B_{\kappa R}} \widetilde{\varphi}(|\nabla_\tau \tilde{v}|)\,\mathrm{d}x \leq \frac{10}{R} \int_{B_R} \widetilde{\varphi}(|\nabla \tilde{v}|)\,\mathrm{d}x, \tag{5.172}$$

where ∇_τ denotes the tangential gradient of $\partial B_{\kappa R}$. Hence applying Proposition 2.3.9 we see $v|_{\partial B_{\kappa R}}$ admits an extension $\overline{v}_\kappa$ to $W^{1,\varphi^{q_0}}(B_{\kappa R}, \mathbb{R}^N)$ satisfying the estimate

$$\left(\fint_{B_{\kappa R}} \varphi_1(|\nabla \overline{v}_\kappa|)^{q_0}\,\mathrm{d}x \right)^{\frac{1}{q_0}} \leq \frac{C}{R} \int_{\partial B_{\kappa R}} \varphi_1(|\nabla_\tau \tilde{v}|)\,\mathrm{d}x, \tag{5.173}$$

where the R-depends follows from a scaling argument. Hence noting $\widetilde{\varphi}^{q_0} \in \Delta_2 \cap \nabla_2$ (as observed in Section 2.1.2) we can use Theorem 3.3.8 to deduce the exist of a unique solution h, satisfying the claimed estimate (5.170). We will put $\tilde{h} = h - a$.

We proceed similarly as in the proof of Lemma 5.3.7, and we will only indicate the necessary changes. Our choice of q will be similar to before and solve

$$\begin{cases} -\operatorname{div} F_0''(\nabla a)\nabla q = g & \text{in } B_{\kappa R} \\ \qquad\qquad q = 0 & \text{on } \partial B_{\kappa R} \end{cases}, \tag{5.174}$$

where

$$g = \varphi_1\left(\frac{|v-h|}{\kappa R}\right) \frac{v-h}{|v-h|^2}. \tag{5.175}$$

We will show below that $g \in L^r(B_{\kappa R})$, from which by Proposition 3.3.12 we can write g as the divergence of a function in $C^{0,1-\frac{n}{r}}$, and so by Schauder estimates (Corollary 3.3.3) we get a solution $q \in W_0^{1,\infty}(B_{\kappa R}, \mathbb{R}^N)$. Indeed by the growth condition (5.167) we can estimate

$$
\begin{aligned}
\|\nabla q\|_{L^\infty(B_{\kappa R})} &\leq C \left(\fint_{B_{\kappa R}} |Rg|^r \,\mathrm{d}x \right)^{\frac{1}{r}} \\
&\leq C \left(\fint_{B_{\kappa R}} \varphi_1 \left(\frac{|v-h|}{R} \right)^r \left(\frac{|v-h|}{R} \right)^{-r} \mathrm{d}x \right)^{\frac{1}{r}} \\
&\leq C \left(\fint_{B_{\kappa R}} \varphi_1 \left(\frac{|v-h|}{R} \right) \mathrm{d}x \right)^{\frac{1}{r}} \\
&\leq C \left(\fint_{B_{\kappa R}} \varphi_1(|\nabla v - \nabla h|) \,\mathrm{d}x \right)^{\frac{1}{r}} \\
&\leq C \left(\fint_{B_{\kappa R}} \varphi_1 \left(|\nabla \tilde{v}| \right) \mathrm{d}x \right)^{\frac{1}{r}},
\end{aligned}
\tag{5.176}
$$

where we have used the Poincaré inequality (Lemma 2.3.6) and (5.170) in the last line. Now using q in the Euler-Lagrange inequality (5.112), the linearisation estimate (5.131) noting $\varphi'(t) \leq Ct$ by (5.167), and Young's inequality with $\delta > 0$ (Lemma 2.1.2) we obtain

$$
\begin{aligned}
\int_{B_{\kappa R}} \varphi_1 \left(\frac{|v-h|}{R} \right) \mathrm{d}x &= \int_{B_{\kappa R}} (\widetilde{F}_0''(0)\nabla\tilde{v} - \widetilde{F}_0'(\nabla\tilde{v})) \cdot \nabla q \,\mathrm{d}x + CR^\beta \|\nabla q\|_{L^\infty(B_{\kappa R})} \\
&\leq C \int_{B_{\kappa R}} \omega(|\nabla\tilde{v}|)|\nabla u||\nabla q| \,\mathrm{d}x + CR^\beta \|\nabla q\|_{L^\infty(B_{\kappa R})} \\
&\leq \delta \int_{B_{\kappa R}} \varphi_1(|\nabla\tilde{v}|) \,\mathrm{d}x + CR^\beta \|\nabla q\|_{L^\infty(B_{\kappa R})} \\
&\quad + \delta\varphi_1^* \left(C\delta^{-1} \|\nabla q\|_{L^\infty(B_{\kappa R})} \right) \fint_{B_{\kappa R}} \omega(|\nabla\tilde{v}|) \,\mathrm{d}x,
\end{aligned}
\tag{5.177}
$$

where we have used convexity of φ^* and the fact that $\omega \leq 1$ in the last line. As before, we can use Jensen's inequality to estimate

$$
\fint_{B_{\kappa R}} \omega(|\nabla\tilde{v}|) \,\mathrm{d}x \leq C\, \omega \circ \varphi_1^{-1} \left(\fint_{B_R} \varphi_1(|\nabla\tilde{v}|) \,\mathrm{d}x \right) \tag{5.178}
$$

Now by dualising (5.167) we have $\varphi^*\left(t^{\frac{1}{r}}\right) \geq t$, so if we define

$$
\gamma_{M,\delta}(t) = \omega \circ \varphi_1(t) \frac{\varphi_1^*\left(C\delta^{-1}t^{\frac{1}{r}}\right)}{t} \tag{5.179}
$$

then we have

$$
\begin{aligned}
\int_{B_{\kappa R}} \varphi_1 \left(\frac{|v-h|}{R} \right) \mathrm{d}x &\leq \delta \int_{B_{\kappa R}} \varphi_1(|\nabla\tilde{v}|) \,\mathrm{d}x + CR^\beta \|\nabla q\|_{L^\infty(B_{\kappa R})} \\
&\quad + C\gamma_{M,\delta} \left(\int_{B_{\kappa R}} \varphi_1(|\nabla\tilde{v}|) \,\mathrm{d}x \right) \int_{B_{\kappa R}} \varphi_1(|\nabla\tilde{v}|) \,\mathrm{d}x,
\end{aligned}
\tag{5.180}
$$

from which the result follows using (5.176) to estimate the R^β term. $\square$

Using this we can obtain the following partial regularity theorem by a straightforward adaptation of the excess decay argument.

Theorem 5.3.12. Suppose $F = F(x, z)$ is independent of u and satisfies Hypotheses 5.1.1, with $\varphi \in \Delta_2$ an N-function satisfying

$$\varphi_1(t) \leq Ct^{\frac{r}{r-1}} \tag{5.181}$$

for some $r > n$. Then if $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ is a minimiser of $\mathcal{F}$, there is $\Omega_0 \subset \Omega$ and $\alpha \in (0, \beta)$ such that $u \in C^{1,\alpha}(\Omega_0)$ and $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$.

We stress however that in light of Example 2.1.16, for any $r > 1$ there are examples of N-functions failing the ∇_2-condition while satisfying the growth bound $\varphi \gtrsim t^r$. Hence the polynomial bound we impose in the above example is genuinely restrictive.

Chapter 6

Convex problems with (p, q)-growth

The results in this chapter were obtained in collaboration with Lukas Koch *(MPI Leipzig), and is currently being prepared for publication.*

In this chapter we will consider regularity of minimisers for functionals of the form

$$\mathcal{F}(u) = \int_\Omega F(x, \nabla u) - f \cdot u \,\mathrm{d}x, \tag{6.1}$$

where the integrand $F = F(x, z)$ is *convex* in the gradient variable z and satisfies a natural (p, q)-growth condition of the form

$$|z|^p - 1 \lesssim F(x, z) \lesssim 1 + |z|^q. \tag{6.2}$$

We will detail the precise assumptions we impose in Hypotheses 6.1.2 below. Such problems are said to be *non-uniformly elliptic* when $p < q$, and this gap in exponents presents significant difficulties in establishing the regularity of minima.

Note that unlike in the quasiconvex case considered in Chapter 5, the convexity assumption implies that the functional $\mathcal{F}$ is convex on $W^{1,p}(\Omega, \mathbb{R}^N)$, so we are always guaranteed sequential weak lower-semicontinuity by an application of Mazur's lemma (see for instance [Bre11, Corollary 3.9]). Hence using (6.2), noting we have coercivity in $W^{1,p}$, we can use the Direct method to obtain minimisers in any Dirichlet class $W^{1,p}_g(\Omega, \mathbb{R}^N)$. However this already presents significant difficulties due to possibility that

$$\inf_{u \in W^{1,p}_g(\Omega, \mathbb{R}^N)} \mathcal{F}(u) < \inf_{u \in W^{1,q}_g(\Omega, \mathbb{R}^N)} \mathcal{F}(u). \tag{6.3}$$

This is referred to as the *Lavrentiev phenomenon* named after Lavrentiev [Lav27] who demonstrated, by means of an example, this can occur in the one-dimensional case (in the setting of Tonelli's existence results detailed in Section 1.1.1). Note that if this occurs, the minimiser u over $W^{1,p}_g$ cannot be attained by a smooth function, thereby acting as an obstruction to regularity. There is a vast literature on this topic, and we will refer to the works of Buttazzo & Belloni [BB95] and Foss [Fos01] for a detailed discussion, along with the work of Foss, Hrusa, & Mizel [FHM03] for connections to elasticity theory. We also highlight, as surveyed in [BB95], the examples in [Zhi87; De 89; CS94; De 95] which illustrates this phenomenon occurs in our setting of convex (x,z)-dependent integrands.

Additionally, even in the scalar case, we can encounter irregularity of minima if the gap between p and q is too large. This was noted by Giaquinta [Gia87] and Marcellini [Mar87; Mar89], and there have been many other examples since. We also mention the constructions of Zhikov [Zhi87], Esposito, Leonetti, Mingione [ELM04], Fonseca, Mály, Mingione [FMM04], and the recent work of Balci, Diening, Weimar [BDW20] which illustrates examples of singularity and Lavrentiev in the non-autonomous setting we consider.

In this chapter we will focus on the corresponding regularity problem in the general vector-valued case, and establish improved differentiability for *relaxed minima* which we will define precisely in Section 6.1.1 below. The key result is an improved fractional differentiability result which has the following consequence for partial regularity, where we assume F satisfies a $C^{1,\alpha}$-dependence in x.

Theorem 6.0.1. Let $\Omega \subset \mathbb{R}^n$ be bounded open, assume F satisfies Hypotheses 6.1.2 including (B4), and that

$$2 \leq p \leq q < \frac{(n+1)p}{n}. \tag{6.4}$$

Also let $f \in L^{p'}(\Omega, \mathbb{R}^N)$. Then if u is a relaxed minimiser for $\mathcal{F}$, there is $\Omega_0 \subset \Omega$ open for which u is locally $C^{1,\beta}$ in Ω_0 for all $\beta \in (0,1)$, and we have

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_0) \leq n - \min\left\{1+\alpha, p'\right\}. \tag{6.5}$$

Here we recall the Hausdorff dimension is defined as $\dim_{\mathcal{H}} A = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}$. In particular this shows that the singular set has dimension strictly less than $n-1$ under our assumptions. This provides hope towards establishing the existence of regularity boundary points, however we are not able to achieve this at the time of writing.

This follows from Theorem 6.1.3, where we show the fractional differentiability result

$$\nabla u \in B^{s,p}_{\infty,\mathrm{loc}}(\Omega, \mathbb{R}^N), \text{ where } s < \min\left\{\frac{1+\alpha}{p}, \frac{1}{p-1}\right\}. \tag{6.6}$$

Note this is essentially sharp for the inhomogenous p-Laplace equation, as Simon [Sim76a; Sim76b; Sim77; Sim78; Sim81] established interior $B_\infty^{1+\frac{1}{p-1},p}$ regularity for $p \geq 2$, and moreover proved this is sharp in the degree of differentiability for data $f \in L^{p'}$.

6.1 Setup and main results

To state our result more precisely, we will first need to define the notion of minimisers we consider.

6.1.1 Relaxation

Instead of considering minimisers in $W^{1,p}$, which we refer to as *pointwise minimisers*, we will study minimisers obtained through *relaxation*. We will adopt the systematic approach introduced by Buttazzo & Mizel [BM92], however similar ideas were explored prior, such as by Serrin [Ser61].

Definition 6.1.1. Let X be a topological space, and $Y \subset X$ a dense subspace. Given a functional $F\colon X \to [0,\infty]$, we define

$$\overline{\mathcal{F}}_X(u) = \sup\{\mathcal{G}\colon X \to [0,\infty] : \mathcal{G} \text{ is slsc}, \mathcal{G} \leq \mathcal{F} \text{ on } X\}, \tag{6.7}$$

$$\overline{\mathcal{F}}_Y(u) = \sup\{\mathcal{G}\colon X \to [0,\infty] : \mathcal{G} \text{ is slsc}, \mathcal{G} \leq \mathcal{F} \text{ on } Y\}, \tag{6.8}$$

where "slsc" is short for sequentially lower-semicontinuous. The associated *Lavrentiev gap functional* is defined as the difference

$$\mathcal{L}(u) = \overline{\mathcal{F}}_Y(u) - \overline{\mathcal{F}}_X(u), \tag{6.9}$$

taking $\mathcal{L}(u) = 0$ if $\overline{\mathcal{F}}_X(u) = +\infty$.

Moreover it is not difficult to see that $\overline{\mathcal{F}}_Y$ admits the dual representation

$$\overline{\mathcal{F}}_X(u) = \inf\left\{\liminf_{k\to\infty} \mathcal{F}(u_k) : \{u_k\} \subset X, u_k \to u \text{ in } X\right\}, \tag{6.10}$$

$$\overline{\mathcal{F}}_Y(u) = \inf\left\{\liminf_{k\to\infty} \mathcal{F}(u_k) : \{u_k\} \subset Y, u_k \to u \text{ in } X\right\}, \tag{6.11}$$

where the convergence is taken in the topology of X. Note the "$\leq$" inequality follows by using the semicontinuity of $\overline{\mathcal{F}}_X$ and $\overline{\mathcal{F}}_Y$ respectively. Conversely, by means of a diagonal argument, we can show the functional defined by the respective the right-hand sides are themselves sequentially lower-semicontinuous and are majorised by $\mathcal{F}$ on X, Y respectively.

In the context of Dirichlet boundary conditions, following [Koc21] we take $X = W_g^{1,p}(\Omega, \mathbb{R}^N)$ and $Y = W_g^{1,q}(\Omega, \mathbb{R}^N)$ equipped with the weak topology of $W^{1,p}$, and denote $\overline{\mathcal{F}}_Y = \overline{\mathcal{F}}_D$. Now we can study minimisers of the relaxed functional $\overline{\mathcal{F}}_D$, which we refer to as *relaxed minimisers*. Note also by the lower-semicontinuity results described at the start of the chapter, we have $\overline{\mathcal{F}}_X = \mathcal{F}$ if F is convex in z. In the context of interior regularity however, for $\omega \Subset \Omega$ we instead take $Y = X \cap W^{1,p}_{\mathrm{loc}}(\omega, \mathbb{R}^N)$ and write $\overline{\mathcal{F}}(\cdot, \omega) = \overline{\mathcal{F}}_Y$.

6.1.2 Hypotheses and main results

Hypotheses 6.1.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider $F\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}$ with $n \geq 2$, $N \geq 1$, and $1 < p < \infty$ satisfying the following.

(B0) F is continuous in (x, z) and continuously differentiable in z such that $\partial_z F$ is continuous.

(B1) F satisfies the natural q-growth condition

$$|F(x, z)| \leq \Lambda(1 + |z|^2)^{\frac{q}{2}}$$

for all $x \in \Omega$ and $z \in \mathbb{R}^{Nn}$.

(B2) F satisfies the strict convexity condition

$$F(x, z) - F(x, w) - \partial_z F(x, w) \cdot (z - w) \geq \lambda(\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} |z - w|^2,$$

for all $x \in \Omega$ and $z, w \in \mathbb{R}^{Nn}$, where $\lambda > 0$.

(B3) F satisfies the α-Hölder continuity assumption

$$|F(x, z) - F(y, z)| \leq \Lambda |x - y|^{\alpha} \left(1 + |z|^2\right)^{\frac{q}{2}}$$

for all $x, y \in \Omega$ and $z \in \mathbb{R}^{Nn}$, where $\alpha \in (0, 1]$.

Additionally for some of our results, we will strengthen (B3) to the following $C^{1,\alpha}$ condition.

(B4) F is continuous differentiable in x, and satisfies the estimate

$$|\partial_x F(x,z) - \partial_x F(y,z)| \leq \Lambda |x-y|^\alpha \left(1+|z|^2\right)^{\frac{q}{2}} \tag{6.12}$$

for all $x, y \in \Omega$ and $z \in \mathbb{R}^{Nn}$, where $\alpha \in (0,1]$.

Note that (B2) ensures $F(x,z)$ is convex in z, and combined with (B1) we see the growth bound (6.2) holds. In addition (B4) implies (B3) with $\alpha = 1$. We note that a similar condition was considered in [CKP11, Remark 5] in the context of bounded minimisers.

Using this, the main result of this chapter is the following. Recall the Besov spaces $B^{s,p}_\infty$ were introduced in Section 2.3.4.

Theorem 6.1.3. Let $\Omega \subset \mathbb{R}^n$ be an open set, $f \in L^{p'}(\Omega, \mathbb{R}^N)$, and suppose that F satisfies (B0)–(B4) with $\alpha \in (0,1]$ and $2 \leq p \leq q < \frac{(n+1)p}{n}$. Then if u is a relaxed minimiser of $\mathcal{F}$, we have $u \in B^{1+s,p}_{\infty,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all

$$s < s_{\alpha,p} = \min\left\{\frac{1+\alpha}{p}, \frac{1}{p-1}\right\}. \tag{6.13}$$

Moreover for $B_R(x_0) \Subset \Omega$ and $0 < r < R \leq 1$ we have the associated estimate

$$\|\nabla u\|_{B^{s,p}_\infty(B_r)} \leq \frac{C}{(R-r)^\gamma}\left(1 + \overline{\mathcal{F}}(u, B_R(x_0)) + \|f\|_{L^{p'}(B_R(x_0))}\right)^\gamma, \tag{6.14}$$

where $C, \gamma > 0$ depend on $n, N, p, q, \alpha, s, \Lambda, \lambda$ only.

Remark 6.1.4. As the proof will illustrate, once we have $W^{1,q}_{\mathrm{loc}}$-regularity of u we only need to assume that $q < \frac{np}{n-1}$. This holds if $F = F(z)$ is autonomous, which was proved by Carozza, Kristensen, & Passarelli di Napoli [CKN14] in the interior case. We mention this was recently extended to the case of Dirichlet boundary by Koch [Koc22].

6.2 Known interior regularity

We will record some known results which we will need in proving Theorem 6.0.1.

6.2.1 Improved integrability

The following result due to Esposito, Leonetti, & Mingione [ELM04] establishes $W^{1,q}$ regularity, and an improved fractional differentiability result provided the gap between p and q is sufficiently small. They moreover show their results are sharp under the assumption (B3). A corresponding global version was established by Koch [Koc21] subject to a Dirichlet boundary condition.

Theorem 6.2.1 (Interior $W^{1,q}$ integrability, [ELM04, Theorem 4])**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $f \in L^{q'}(\Omega, \mathbb{R}^N)$, and suppose F satisfies (B0)–(B3) with $\alpha \in (0,1]$ and $2 \leq p \leq q < \frac{(n+\alpha)p}{n}$. Then if u be a relaxed minimiser of $\mathcal{F}$, we have

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap B^{1+\frac{\alpha}{p},p}_{\infty,\mathrm{loc}}(\Omega, \mathbb{R}^N), \tag{6.15}$$

and for all $B_R(x_0) \Subset \Omega$, $0 < r < R \leq 1$ we have

$$\|\nabla u\|_{L^q(B_r)} \leq \frac{C}{(R-r)^\gamma}\left(1 + \overline{\mathcal{F}}(u, B_R(x_0)) + \|f\|_{L^{q'}(B_R(x_0))}\right)^\gamma, \tag{6.16}$$

$$\|\nabla u\|_{B^{\frac{\alpha}{p},p}_{\infty}(B_r)} \leq \frac{C}{(R-r)^\gamma}\left(1 + \overline{\mathcal{F}}(u, B_R(x_0)) + \|f\|_{L^{q'}(B_R(x_0))}\right)^\gamma, \tag{6.17}$$

where $C, \gamma > 0$ depend on $n, N, p, q, \alpha, \Lambda, \lambda$ only.

While they do not explicitly state the differentiability in $B^{1+\frac{\alpha}{p},p}_{\infty}$ along with the dependence on the radii, they follow from the proof. We will see similar arguments in the proof of Theorem 6.1.3.

6.2.2 Characterisation of regular points

We will also recall the following characterisation of regular singular points, which follows by combining the results in Schmidt [Sch08a; Sch09] and De Maria & Passarelli di Napoli [DP11].

Theorem 6.2.2 (ε-regularity, [Sch09, Theorem 6.4],[DP11, Theorem 1.2])**.** Let $F = F(x,z)$ be C^2 in z and satisfy (B0)–(B3), and let $1 < p \leq q < \infty$ satisfy

$$q < \min\left\{\frac{np}{n-1}, p+1\right\}. \tag{6.18}$$

Then if u is a relaxed minimiser for $\mathcal{F}$, then for all $M > 0$ there is $\varepsilon > 0$ such that if for $B_R(x_0) \subset \Omega$ we have

$$|(\nabla u)_{B_R(x_0)}| \leq M, \text{ and } \fint_{B_R(x_0)} |V_p(\nabla u - (\nabla u)_{B_R(x_0)})|^2 \,\mathrm{d}x < \varepsilon, \tag{6.19}$$

then u is $C^{1,\beta}$ in $\overline{B}_{R/2}(x_0)$ for all $\beta < \alpha$.

Note that if try to apply our arguments from Section 5.3, we will fail in establishing the Caccioppoli inequality of the second kind (Lemma 5.3.4). This is because the q-growth from (B1) is not matched by the p-growth from (B2), and so we cannot employ the hole-filling trick. Moreover in the case of relaxed minima, it is not clear how to even perform a energy comparison at the level of the relaxed functional since u merely lies in $W^{1,p}$. To work around this, Schmidt [Sch08a] introduced a variant of the Caccioppoli inequality using the smoothing operator of Fonseca & Málý [FM97]. In addition, to carry out the energy comparison at the level of the relaxed functional, it is necessary to derive a suitable additivity property established in [Sch09, Lemma 7.10]. Once this is established, the rest of the proof goes through with minimal modifications, as it detailed in [DP11] (through the authors only consider pointwise minimisers). We also point out that for the harmonic approximation step one needs to derive the Euler-Lagrange system for relaxed minima; recently for convex integrands this was shown to hold in general by Koch & Kristensen [KK22].

We record the following consequence on the singular set, which follows from the ε-regularity results and for instance [Min03, Section 6].

Corollary 6.2.3. Suppose the assumptions of Theorem 6.2.2 hold, and that $\nabla u \in B^{s,p}_{\infty,\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for some $s > 0$. Then if we define the *singular set* of u as $\Sigma = \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \left\{ x \in \Omega : \limsup_{r \to 0} |(\nabla v)_{B_r(x)}| = \infty \right\} \tag{6.20}$$

$$\Sigma_2 = \left\{ x \in \Omega : \liminf_{r \to 0} \fint_{B_r(x)} \left|V_p(\nabla u - (\nabla u)_{\Omega_r(x)})\right|^2 \,\mathrm{d}x > 0 \right\}, \tag{6.21}$$

we have u is $C^{1,\alpha}$ in $\Omega \setminus \Sigma$ and $\dim_{\mathcal{H}}(\Sigma) = n - sp$.

6.3 Improved differentiability in the interior

We now prove the main differentiability result, which involves the use of a second order difference quotient.

Proof of Theorem 6.1.3. Note by Theorem 6.2.1 applied with $\alpha = 1$, we have $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. In particular we may use ηu as a test function for a suitable cutoff η. We will let $B_R(x_0) \subset \Omega$, and suppress the x_0-dependence in what follows.

Step 1 (The difference quotient estimate): Let $0 < t < s < R$ and $h \in \mathbb{R}^n$ such that $|h| \leq \frac{s-t}{4}$. Then fix a smooth cutoff $0 \leq \eta = \eta_{t,s} \leq 1$ supported in $B_s(x_0)$ with $\eta(x) = 1$ in $B_{(t+s)/2}(x_0)$ and $|\nabla^k \eta(x)| \leq C_k(s-t)^{-k}$ for some $C_k > 0$ and $k \in \mathbb{N}$. Now given any function w, we define

$$T'_h w = \eta\left(\frac{1}{2}w_h + \frac{1}{2}w_{-h}\right) + (1-\eta)w \tag{6.22}$$

and denote $\Delta^2_h w = \frac{1}{2}w_h - w + \frac{1}{2}w_{-h}$, so $T'_h w - w = \eta\Delta^2_h w$. Using (B2) and noting $p \geq 2$ we obtain

$$\mathcal{F}(T'_h w) - \mathcal{F}(w) \gtrsim \left\|\nabla(\eta\Delta^2_h w)\right\|^p_{L^p(B_s(x_0))}. \tag{6.23}$$

By noting the identity

$$\Delta^2_h(\eta w) = \eta\Delta^2_h w + \frac{1}{2}(\eta_h - \eta)w_h + \frac{1}{2}(\eta - \eta_{-h})w_{-h}, \tag{6.24}$$

we have

$$\int_{B_s} |\nabla(\eta\Delta^2_h w)|^p \,\mathrm{d}x \leq \int_{\mathbb{R}^n} |\Delta^2_h(\eta w)|^p \,\mathrm{d}x - C|h|^p \int_{B_s} \left|\frac{\nabla w}{s-t}\right|^p + \left|\frac{w}{(s-t)^2}\right|^p \mathrm{d}x. \tag{6.25}$$

Additionally by Lemma 2.3.16 we can estimate

$$\begin{aligned}\sup_{|h|\neq 0} \frac{1}{|h|^{\alpha p}} \int_{B_s} |\Delta_h(\eta w)|^p \,\mathrm{d}x &\leq C \sup_{|h|\neq 0} \frac{1}{|h|^{\alpha p}} \int_{B_s} |\Delta'_h(\eta w)|^p \,\mathrm{d}x \\ &\leq C \sup_{0<|h|<\frac{s-t}{4}} \frac{1}{|h|^{\alpha p}} \int_{B_s} |\Delta'_h(\eta w)|^p \,\mathrm{d}x \\ &\quad + C|h|^p \int_{B_s} \left|\frac{\nabla w}{s-t}\right|^p + \left|\frac{w}{(s-t)^2}\right|^p \mathrm{d}x,\end{aligned} \tag{6.26}$$

so combining (6.25) and (6.26) it follows that

$$\begin{aligned}&c \sup_{|h|\neq 0} \frac{1}{|h|^{\alpha p}} \int_{B_s} |\Delta_h(\eta w)|^p \,\mathrm{d}x \\ &\quad \leq \sup_{0<|h|<\frac{s-r}{4}} \frac{\mathcal{F}(T'_h w) - \mathcal{F}(w)}{|h|^{\alpha p}} + C(s-r)^{-\alpha p} \int_{B_s} |\nabla w|^p + \left|\frac{w}{s-r}\right|^p \mathrm{d}x.\end{aligned} \tag{6.27}$$

Now noting that $T'_h u \in W^{1,p}(\Omega)$, we may estimate

$$
\begin{aligned}
&\mathcal{F}(T'_h u) - \mathcal{F}(u) \\
&= \int_\Omega F\left(x, T'_h \nabla u + \nabla\eta \otimes \left(\frac{1}{2}u_h - u + \frac{1}{2}u_{-h}\right)\right) - F(x, T'_h\nabla u)\mathrm{d}x \\
&\quad + \int_\Omega F(x, T'_h\nabla u) - F(x, \nabla u)\mathrm{d}x - \int_\Omega f\cdot (T'_h u - u)\mathrm{d}x \\
&= A_1 + A_2 + A_3.
\end{aligned}
\tag{6.28}
$$

Using (B2) and Hölder's inequality, we find

$$
|A_1| \leq C\int_\Omega |\nabla\eta\otimes\Delta_h^2 u|\big(1 + |T'_h\nabla u|^2 + |\nabla\eta\otimes\Delta_h^2 u|^2\big)^{\frac{q-1}{2}}\mathrm{d}x \tag{6.29}
$$

We now turn to A_2. By convexity of $F(x,\cdot)$, we have the pointwise estimate

$$
\begin{aligned}
F(x, T'_h\nabla u) - F(x,\nabla u) &\leq \eta F\left(x, \frac{1}{2}\nabla(u_h + u_{-h})\right) + (1-\eta)F(x,\nabla u_h) - F(x,\nabla u) \\
&\leq \frac{1}{2}\eta(F(x,\nabla u_h) - 2F(x,\nabla u) + F(x,\nabla u_{-h})).
\end{aligned}
\tag{6.30}
$$

In particular, by using a change of coordinates, noting η vanishes outside of $B_{(t+s)/2}$, and using (B4) we have

$$
\begin{aligned}
A_2 &\leq \frac{1}{2}\int_{B_{(t+s)/2+h}} \eta(x-h)F(x-h,\nabla u)\mathrm{d}x - \int_{B_{(t+s)/2}}\eta(x)F(x,\nabla u)\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{B_{(t+s)/2-h}}\eta(x+h)F(x+h,\nabla u)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{B_s}\eta(x-h)F(x-h,\nabla u) - 2\eta(x)F(x,\nabla u) + \eta(x+h)F(x+h,\nabla u)\,\mathrm{d}x \\
&= \int_{B_s}\int_0^1 \frac{\partial}{\partial t}\big(\eta(x+th)F(x+th,\nabla u) + \eta(x-th)F(x-th,\nabla u)\big)\mathrm{d}t\mathrm{d}x \\
&= h\cdot\int_{B_s}\int_0^1 \nabla\eta(x+th)F(x+th,\nabla u) - \nabla\eta(x-th)F(x-th,\nabla u)\mathrm{d}t\mathrm{d}x \\
&\quad + h\cdot\int_{B_s}\int_0^1 \eta(x+th)\partial_x F(x+th,\nabla u) - \eta(x-th)\partial_x F(x-th,\nabla u)\mathrm{d}x \\
&\leq |h|\int_{B_s}\int_0^1 |\nabla\eta(x+th) - \nabla\eta(x-th)||F(x+th,\nabla u)|\,\mathrm{d}x \\
&\quad + |h|\int_{B_s}\int_0^1 |\nabla\eta(x-th)||F(x+th,\nabla u) - F(x-th,\nabla u)|\,\mathrm{d}x \\
&\quad + |h|\int_{B_s}\int_0^1 |\eta(x+th) - \eta(x-th)||\partial_x F(x+th,\nabla u)|\,\mathrm{d}x \\
&\quad + |h|\int_{B_s}\int_0^1 |\eta(x-th)||\partial_x F(x+th,\nabla u) - \partial_x F(x-th,\nabla u)|\,\mathrm{d}x \\
&\lesssim |h|^{1+\alpha}(s-t)^{-2}\int_{B_s}(1+|\nabla u|^q).
\end{aligned}
\tag{6.31}
$$

Also using Hölder we have

$$|A_3| \le \int_{B_s} |f \cdot (\eta \Delta_h^2 u)| \le \|f\|_{L^{p'}(B_s)} \left\|\eta \Delta_h^2 u\right\|_{L^p(B_s)}. \tag{6.32}$$

Thus we have estimates for A_1, A_2, A_3, which we will use in the next step.

Step 2 (Improved differentiability): To proceed, we will establish an inductive improvement in differentiability which we iteratively apply for a carefully chosen set of parameters. More precisely, assume there is $\delta > 0$ such that

$$\|u\|_{B^{1+\delta,p}_\infty(B_t)} \le \frac{c}{(s-t)^\gamma} A^\gamma = \frac{c}{(s-t)^\gamma}\left(1 + \|u\|^p_{W^{1,p}(B_s)} + \|f\|^{p'}_{L^{p'}}\right), \tag{6.33}$$

for all $0 < t < s < R$, and suppose there is $\tau_1, \tau_2 \ge 1$ and $\varepsilon, \sigma > 0$ such that we have

$$\frac{1}{\tau_2} + \frac{q-1}{\tau_1 - \varepsilon} = 1, \quad \frac{1}{\tau_1} \ge \frac{1}{p} - \frac{\delta_k}{n}, \quad \frac{1}{\tau_2} \ge \frac{1}{p} - \frac{\sigma - \delta}{n}. \tag{6.34}$$

Then we claim that $u \in B^{1+\delta',p}_{\infty,\mathrm{loc}}(B_R, \mathbb{R}^N)$ where

$$\delta' = \min\left\{\frac{1+\alpha}{p}, \frac{1+\delta}{p}, \frac{1+\sigma}{p}\right\}, \tag{6.35}$$

and we have the associated estimate

$$\|u\|_{B^{1+\delta',p}_\infty(B_t)} \le \frac{c'}{(s-t)^{\gamma'}} A^{\gamma'}, \tag{6.36}$$

for all $0 < t < s < R$, where $c', \gamma' > 0$ depend on the structural constants and additionally on $\tau_1, \tau_2, \sigma, \varepsilon$.

Note the latter two inequalities in (6.34) ensures we have $B^{1+\delta,p}_\infty$ embeds into B^{1,τ_1}_∞ and $B^{1+\sigma,\tau_2}_\infty$, which we will use to estimate A_1. Indeed by Lemma 2.3.15 and Proposition 2.3.13 there is $\gamma_1 > 0$ such that

$$\begin{aligned} |A_1| &\le \left\|\nabla\eta \otimes \Delta_h^2 u\right\|_{L^{\tau_2}(B_s)} \left(1 + \|\nabla u\|_{L^{\tau_1-\varepsilon}(B_s)} + (s-t)^{-1}\|u\|_{L^{\tau_1-\varepsilon}(B_s)}\right)^{q-1} \\ &\le \frac{C|h|^{1+\sigma}}{(s-r)^{q+\gamma_1}}\left(1 + \|u\|_{B^{1+\delta,p}_\infty(B_s)}\right)^q \\ &\le \frac{C|h|^{1+\sigma}}{(s-r)^{(1+\gamma)q+\gamma_1}} A^{\gamma q}, \end{aligned} \tag{6.37}$$

where we have used (6.33) in the last line. For the second term we use Lemma 6.2.1, adjusting γ if necessary, and for the third we use (6.33) to estimate

$$A_2 \le \frac{C|h|^{1+\alpha}}{(s-t)^\gamma} A^\gamma, \tag{6.38}$$

$$A_3 \le \frac{C|h|^{1+\delta}}{(s-t)^\gamma} A^\gamma. \tag{6.39}$$

Hence the claim follows by taking the minimum of these exponents.

Step 3 (Iteration): We now repeatedly use the previous step for a carefully chosen set of parameters $\delta, \sigma, \tau_1, \tau_2, \varepsilon$. To do this, we will set $\delta_0 = \frac{1}{p}$ and show by induction that $u \in B^{1+\delta_k,p}_{\infty,\mathrm{loc}}(B_R, \mathbb{R}^N)$, where

$$\delta_k = \min\left\{\frac{1+\alpha}{p}, \sum_{i=1}^{k} \frac{1}{p^i}\right\}. \tag{6.40}$$

Note that if we take $\tau = \frac{np}{n-\delta_k p}$, then since $q < \frac{np}{n-1}$ and $\delta_k \geq p^{-1}$, in the equality case by shrinking $\varepsilon > 0$ we can assume that $\tau_1 - \varepsilon > q$, and hence that $\tau_2 > 1$. We will consider two cases depending on the possible values of τ_2; note that if $\tau_2 \leq p$ then we may take $\sigma = \delta_k$, and otherwise we will need to choose our parameters more carefully.

Case (a): Suppose that for some δ_k we have $\tau_2 > p$ for all $\varepsilon > 0$ sufficiently small, which is equivalent to the condition

$$\delta_k < \frac{n}{q-1}\left(\frac{q}{p} - 1\right). \tag{6.41}$$

Given $\varepsilon > 0$ to be determined, defining $\tau_1 = \frac{np}{n-\delta_k p}$ and τ_2 by (6.34), we will choose σ to satisfy

$$\frac{\sigma}{n} = \frac{\delta_k}{n} + \frac{1}{\tau_2} - \frac{1}{p}. \tag{6.42}$$

Note that

$$\lim_{\varepsilon \to 0} \sigma = q\left(\delta_k - \frac{1}{p}\right) + n - (n-1)\frac{q}{p} = q\left(\delta_k - \frac{1}{p}\right) + \kappa_0, \tag{6.43}$$

where $\kappa_0 > 0$. Hence we can choose $\varepsilon > 0$ so that

$$\sigma \geq q\left(\delta_k - \frac{1}{p}\right) + \frac{\kappa_0}{2}. \tag{6.44}$$

Then by step 2, we deduce that $u \in B^{1+\delta_{k,1},p}_{\infty,\mathrm{loc}}(B_R, \mathbb{R}^N)$ where

$$\delta_{k,1} = \min\left\{\delta_{k+1}, \frac{2+\kappa_0}{2p} + \frac{q}{p}\left(\delta_k - \frac{1}{p}\right)\right\} \geq \min\left\{\delta_{k+1}, \frac{1}{p} + \frac{\kappa_0}{2p}\right\}. \tag{6.45}$$

We can now replace $\delta_k = \delta_{k,0}$ by $\delta_{k,1}$ and apply this iteratively, which gives $\delta_{k,j}$ such that

$$\delta_{k,j} \geq \min\left\{\delta_{k+1}, \frac{1}{p} + \frac{\kappa_0}{2q}\sum_{i=1}^{j}\left(\frac{q}{p}\right)^i\right\}. \tag{6.46}$$

Since the second term is divergent, there is some $j_0 \in \mathbb{N}$ for which either $\delta_{k,j_0} = \delta_{k+1}$ or $\delta_{k,j_0} \geq \frac{n}{q-1}\left(\frac{q}{p} - 1\right)$. In the former case we are done, otherwise we can turn to case (b).

Case (b): Suppose for some $\delta' \geq \delta_k$, we have $u \in B^{1+\delta',p}_{\infty,\mathrm{loc}}(B_R, \mathbb{R}^N)$ and $\delta' \geq \frac{n}{q-1}\left(\frac{q}{p} - 1\right)$. Then we will take $\tau_2 = p$, and for $\varepsilon > 0$ sufficiently small we can choose $\tau_1 \leq \frac{np}{n-\delta' p}$ using (6.34). Hence applying step 2 with $\sigma = \delta_k$ we get $u \in B^{1+\delta_{k+1},p}_{\infty,\mathrm{loc}}(B, \mathbb{R}^N)$. This completes our induction argument, thereby establishing the result. $\square$

Remark 6.3.1. We were unable to extend this to the case when $p \leq 2$, where we expect a similar result to hold at the level of V-functionals. The main difficulty arises due to the presence of the second order difference quotient, where we need a suitable analogue of Lemma 2.3.16 which we were unable to establish. We hope to revisit this in future work.

From here Theorem 6.0.1 follows by combining Theorem 6.1.3 with Corollary 6.2.3.

Chapter 7

Conclusion and outlook

In this thesis we have considered several different aspects of the regularity theory for vectorial problems in the calculus of variations, contributing to and extending the existing literature of regularity results. While our results largely considered general and non-standard growth conditions, some of our results were also of interest in the case of standard p-growth.

We conclude with a brief discussion of some unresolved questions which naturally lead from our work, that we hope to address in future work.

(1) A corresponding existence theorem for extremals in $W^{1,\mathrm{BMO}}$ and $W^{1,\mathrm{VMO}}$, for which the results in Chapter 4 apply. More specifically, we may seek to use the implicit function theorem to deduce the existence of solutions in these spaces, provided the gradient of the boundary datum is sufficiently small in BMO. It is well-known (see for instance [Val88; Zha91]) that such results can be obtained if we work in the Hölder scales, and impose a $C^{1,\alpha}$-smallness condition on the boundary data g. This is also motivated by recent work of Campos Cordero & Kristensen (in preparation) who show such results do hold for strictly quasiconvex integrands, by means of a variational approach.

(2) An analogue of the partial regularity theorem in Chapter 5 for N-functions satisfying only the Δ_2-condition. In this case there are two primary difficulties. Firstly, the improved integrability results guaranteed by Lemma 5.2.11 for general N-functions is significantly weaker in the absence of the ∇_2-condition, so it is not clear if a similar perturbation argument can be carried out. Additionally, in the $\mathbb{A}$-harmonic approximation

method (Lemmas 5.3.7, 5.3.10) we are presently unable to remove the ∇_2-condition, and this will need further consideration.

(3) A more interesting, but seemingly harder problem is what happens in the absence of the Δ_2-condition. In this situation very little is known, and the key obstruction in the vectorial setting is the seeming lack of a Caccioppoli inequality. One of the few results in this area was given by Marcellini [Mar96], who established Lipschitz (and hence $C^{1,\alpha}$) regularity for integrands of type $F(z) = \varphi(|z|)$, where φ is an N-function function satisfying a non-oscillatory condition at infinity. We will not give a complete summary of results in this direction, but we will mention the recent paper of Beck & Mingione [BM20] and the references therein. Nevertheless, the present author is not aware of any results in the context of partial regularity, and it would be interesting to understand if any results carry over. In particular, it was already noted by Marcellini [Mar93] that fast growth problems exhibit behaviour that is analogous to non-uniform ellipticity, which is crucial to obtain a Caccioppoli inequality. However the gap in ellipticity shrinks if we assume "slow exponential growth," for instance $\varphi(t) = \exp(t^\alpha)$ for $\alpha > 0$ small, in which case one may hope to obtain partial regularity.

(4) Extend the improved differentiability results in Chapter 6 to hold up to the boundary. The initial motivation for the results in this section was to prove the existence of regular boundary points, similar to the results of Kristensen & Mingione [KM10] in the uniformly elliptic case. The optimal boundary differentiability appears to be open even for the p-Laplacian, where the results of Simon [Sim81] only gives $B_\infty^{1+\frac{1}{(p-1)^2},p}$ regularity. We note the method of Savaré [Sav97]) gives regularity in $B_\infty^{1+\frac{1}{p},p}$ which for large p improves the results of Simon, which was adapted by Koch [Koc21] to our setting, however this is insufficient weak to infer partial boundary regularity.

Bibliography

[AF89] E. Acerbi and N. Fusco. "Local regularity for minimizers of non convex integrals". In: *Ann. Della Scuola Norm. Super. Pisa - Cl. Sci.* 16.4 (1989), pp. 603–636.

[AF84] E. Acerbi and N. Fusco. "Semicontinuity Problems in the Calculus of Variations". In: *Arch. Ration. Mech. Anal.* 86.2 (1984), pp. 125–145. ISSN: 0003-9527.

[AF87] E. Acerbi and N. Fusco. "A Regularity Theorem for Minimizers of Quasiconvex Integrals". In: *Arch. Ration. Mech. Anal.* 99.3 (Sept. 1987), pp. 261–281. ISSN: 0003-9527.

[AF88] E. Acerbi and N. Fusco. "An Approximation Lemma for $W^{1,p}$ Functions". In: *Proc. Symp. Year Mater. Instab. Contin. Mech.* Edinburgh: Oxford Sci. Publ, 1988, pp. 1–5.

[AF03] R. A. Adams and J. Fournier. *Sobolev Spaces.* Second. Vol. 140. Pure and Applied Mathematics. Elsevier, 2003, p. 305. ISBN: 978-0-08-054129-7.

[ADN59] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions. I". In: *Commun. Pure Appl. Math.* 12.4 (Nov. 1959), pp. 623–727. ISSN: 00103640.

[ADN64] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions II". In: *Commun. Pure Appl. Math.* 17.1 (Feb. 1964), pp. 35–92. ISSN: 00103640.

[AD92] J.-J. Alibert and B. Dacorogna. "An Example of a Quasiconvex Function That Is Not Polyconvex in Two Dimensions". In: *Arch. Rational Mech. Anal.* 117.2 (June 1992), pp. 155–166. ISSN: 1432-0673.

[Alm68] F. J. Almgren. "Existence and Regularity Almost Everywhere of Solutions to Elliptic Variational Problems among Surfaces of Varying Topological Type and Singularity Structure". In: *Ann. Math.* 87.2 (1968), pp. 321–391. ISSN: 0003-486X.

[BDW20] A. K. Balci, L. Diening, and M. Weimar. "Higher Order Calderón-Zygmund Estimates for the p-Laplace Equation". In: *Journal of Differential Equations* 268.2 (Jan. 2020), pp. 590–635. ISSN: 0022-0396.

[Bal84] E. J. Balder. "A General Approach to Lower Semicontinuity and Lower Closure in Optimal Control Theory". In: *SIAM J. Control Optim.* 22.4 (July 1984), pp. 570–598. ISSN: 0363-0129.

[BM84a] J. M. Ball and F. Murat. "$W^{1,p}$-Quasiconvexity and Variational Problems for Multiple Integrals". In: *Journal of Functional Analysis* 58.3 (Oct. 1984), pp. 225–253. ISSN: 0022-1236.

[Bal98] J. M. Ball. "The Calculus of Variations and Materials Science". In: *Quart. Appl. Math.* 56.4 (1998), pp. 719–740. ISSN: 0033-569X, 1552-4485.

[BM85] J. M. Ball and V. J. Mizel. "One-Dimensional Variational Problems Whose Minimizers Do Not Satisfy the Euler-Lagrange Equation". In: *Arch. Rational Mech. Anal.* 90.4 (Dec. 1985), pp. 325–388. ISSN: 1432-0673.

[Bal76] J. M. Ball. "Convexity Conditions and Existence Theorems in Nonlinear Elasticity". In: *Arch. Rational Mech. Anal.* 63.4 (Dec. 1976), pp. 337–403. ISSN: 1432-0673.

[BM84b] J. M. Ball and V. J. Mizel. "Singular Minimizers for Regular One-Dimensional Problems in the Calculus of Variations". In: *Bull. Amer. Math. Soc.* 11.1 (1984), pp. 143–146. ISSN: 0273-0979, 1088-9485.

[BM20] L. Beck and G. Mingione. "Lipschitz Bounds and Nonuniform Ellipticity". In: *Commun. Pure Appl. Math.* 73.5 (2020), pp. 944–1034. ISSN: 1097-0312.

[Ber04] S. Bernstein. "Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre". In: *Math. Ann.* 59.1 (Mar. 1904), pp. 20–76. ISSN: 1432-1807.

[BL91] T. Bhattacharya and F. Leonetti. "A New Poincaré Inequality and Its Application to the Regularity of Minimizers of Integral Functionals with Nonstandard Growth". In: *Nonlinear Analysis: Theory, Methods & Applications* 17.9 (1991), pp. 833–839. ISSN: 0362546X.

[Boj88] B. Bojarski. "Remarks on Sobolev Imbedding Inequalities". In: *Complex Anal. Joensuu 1987*. Ed. by I. Laine, T. Sorvali, and S. Rickman. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1988, pp. 52–68. ISBN: 978-3-540-45992-7.

[Bom82] J. Boman. *L^p-Estimates for Very Strongly Elliptic Systems.* Tech. rep. Department of Mathematics, University of Stockholm, Sweden, 1982.

[BBM04] J. Bourgain, H. Brezis, and P. Mironescu. "$H^{1/2}$ maps with values into the circle : minimal connections, lifting, and the Ginzburg-Landau equation". In: *Publ. Mathématiques IHÉS* 99 (2004), pp. 1–115.

[Boy71] D. Boyd. "Indices for the Orlicz Spaces". In: *Pac. J. Math.* 38.2 (Aug. 1971), pp. 315–323. ISSN: 0030-8730.

[Boy69] D. W. Boyd. "Indices of Function Spaces and Their Relationship to Interpolation". In: *Can. J. Math.* 21 (1969), pp. 1245–1254. ISSN: 0008-414X, 1496-4279.

[Bre11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* New York, NY: Springer New York, 2011. ISBN: 978-0-387-70913-0 978-0-387-70914-7.

[BKL96] S. M. Buckley, P. Koskela, and G. Lu. "Boman Equals John". In: *Proc. XVI Rolf Nevanlinna Colloq.* Berlin: W. de Gruyter, 1996, pp. 91–99.

[BB95] G. Buttazzo and M. Belloni. "A Survey on Old and Recent Results about the Gap Phenomenon in the Calculus of Variations". In: *Recent Developments in Well-Posed Variational Problems.* Ed. by R. Lucchetti and J. Revalski. Mathematics and Its Applications. Dordrecht: Springer Netherlands, 1995, pp. 1–27. ISBN: 978-94-015-8472-2.

[BM92] G. Buttazzo and V. J. Mizel. "Interpretation of the Lavrentiev Phenomenon by Relaxation". In: *Journal of Functional Analysis* 110.2 (Dec. 1992), pp. 434–460. ISSN: 00221236.

[BW04] S.-S. Byun and L. H. Wang. "Elliptic Equations with BMO Coefficients in Reifenberg Domains". In: *Commun. Pure Appl. Math.* 57.10 (Oct. 2004), pp. 1283–1310. ISSN: 0010-3640.

[Cac34] R. Caccioppoli. "Sulle Equazioni Ellittiche a Derivate Parziali Con n Variabili Indipendenti". In: *Red. Accad. Naz. Lincei* 19 (1934), pp. 83–89.

[CP98] L. Caffarelli and I. Peral. "On $W^{1,p}$ Estimates for Elliptic Equations in Divergence Form". In: *Commun. Pure Appl. Math.* 51.1 (Jan. 1998), pp. 1–21. ISSN: 0010-3640.

[CZ52] A. P. Calderón and A. Zygmund. "On the Existence of Certain Singular Integrals". In: *Acta Math.* 88.1 (Dec. 1952), pp. 85–139. ISSN: 00015962.

[Cal40] J. W. Calkin. "Functions of Several Variables and Absolute Continuity, I". In: *dmj* 6.1 (Jan. 1940), pp. 170–186. ISSN: 0012-7094, 1547-7398.

[Cam65] S. Campanato. "Equazioni ellittiche del IIo ordine e spazi $\mathfrak{L}^{(2,\lambda)}$." In: *Annali di Matematica* 69.1 (Dec. 1965), pp. 321–381. ISSN: 1618-1891.

[Cam66] S. Campanato. "Su un teorema di interpolazione di G. Stampacchia". In: *Ann. Della Scuola Norm. Super. Pisa - Cl. Sci.* 20.3 (1966), pp. 649–652. ISSN: 0391-173X.

[Cam17] J. Campos Cordero. "Boundary Regularity and Sufficient Conditions for Strong Local Minimizers". In: *J. Funct. Anal.* 272.11 (June 2017), pp. 4513–4587. ISSN: 00221236.

[CFM98] M. Carozza, N. Fusco, and G. Mingione. "Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth". In: *Ann Mat Pura Ed Appl* 175.1 (Dec. 1998), pp. 141–164. ISSN: 0373-3114.

[CKN14] M. Carozza, J. Kristensen, and A. P. di Napoli. "Regularity of Minimizers of Autonomous Convex Variational Integrals". In: *Ann. Della Scuola Norm. Super. Pisa- Cl. Sci.* 13.4 (2014), pp. 1065–1089.

[CKP11] M. Carozza, J. Kristensen, and A. Passarelli di Napoli. "Higher Differentiability of Minimizers of Convex Variational Integrals". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28.3 (June 2011), pp. 395–411. ISSN: 0294-1449, 1873-1430.

[CP96] M. Carozza and A. P. di Passarelli di Napoli. "A Regularity Theorem for Minimisers of Quasiconvex Integrals: The Case $1 < p < 2^*$". In: *Proc. R. Soc. Edinb. Sect. Math.* 126.6 (1996), pp. 1181–1199. ISSN: 1473-7124, 0308-2105.

[CO20] P. Celada and J. Ok. "Partial Regularity for Non-Autonomous Degenerate Quasi-Convex Functionals with General Growth". In: *Nonlinear Analysis.* Nonlinear Potential Theory 194 (May 2020), p. 111473. ISSN: 0362-546X.

[Che16] C. Y. Chen. "Existence and Global Higher Integrability of Quasiminimizers among Minimizing Sequences of Variational Integrals". In: *Malays. J. Math. Sci.* 10 (2016), pp. 85–100.

[CK17] C. Y. Chen and J. Kristensen. "On Coercive Variational Integrals". In: *Nonlinear Anal Theory Methods Appl* 153 (2017), pp. 213–229.

[CS94] V. Chiadò Piat and F. Serra Cassano. “Some Remarks about the Density of Smooth Functions in Weighted Sobolev Spaces.” In: *J. Convex Anal.* 1 (1994), pp. 135–142.

[CGZ19] I. Chlebicka, F. Giannetti, and A. Zatorska-Goldstein. “Elliptic Problems with Growth in Nonreflexive Orlicz Spaces and with Measure or L^1 Data”. In: *Journal of Mathematical Analysis and Applications* 479.1 (Nov. 2019), pp. 185–213. ISSN: 0022-247X.

[Cia96] A. Cianchi. “A Sharp Embedding Theorem for Orlicz-Sobolev Spaces”. In: *Indiana Univ. Math. J.* 45.1 (1996), pp. 39–65. ISSN: 0022-2518.

[CF99] A. Cianchi and N. Fusco. “Gradient Regularity for Minimizers under General Growth Conditions”. In: 1999.507 (Feb. 1999), pp. 15–36. ISSN: 1435-5345.

[CKP08] A. Cianchi, R. Kerman, and L. Pick. “Boundary Trace Inequalities and Rearrangements”. In: *J Anal Math* 105.1 (Jan. 2008), pp. 241–265. ISSN: 1565-8538.

[CP10] A. Cianchi and L. Pick. “An optimal endpoint trace embedding”. In: *Ann. Inst. Fourier* 60.3 (2010), pp. 939–951.

[CP16] A. Cianchi and L. Pick. “Optimal Sobolev Trace Embeddings”. In: *Trans. Amer. Math. Soc.* 368.12 (Dec. 2016), pp. 8349–8382. ISSN: 0002-9947, 1088-6850.

[CD19] D. Cruz-Uribe and L. Diening. “Sharp $\mathscr{A}$-Harmonic Approximation”. In: *Appl. Anal.* 98.1-2 (Jan. 2019), pp. 374–380. ISSN: 0003-6811.

[CMP06] D. Cruz-Uribe, J. M. Martell, and C. Pérez. “Extensions of Rubio de Francia’s Extrapolation Theorem”. In: *Collect. Math.* Extra Volu (2006), pp. 195–231.

[CMP11] D. Cruz-Uribe, J. M. Martell, and C. Pérez. “Weights, Extrapolation and the Theory of Rubio de Francia”. In: *Oper. Theory Adv. Appl.* Basel: Springer Basel, 2011. ISBN: 978-3-0348-0071-6.

[DI04] L. D’Onofrio and T. Iwaniec. “The P-Harmonic Transform beyond Its Natural Domain of Definition”. In: *Indiana Univ. Math. J.* 53.3 (2004), pp. 683–718. ISSN: 0022-2518.

[Dac82] B. Dacorogna. “Quasiconvexity and Relaxation of Nonconvex Problems in the Calculus of Variations”. In: *Journal of Functional Analysis* 46.1 (Mar. 1982), pp. 102–118. ISSN: 0022-1236.

[Dac07] B. Dacorogna. *Direct Methods in the Calculus of Variations.* Vol. 78. Applied Mathematical Sciences. New York, NY: Springer New York, 2007. ISBN: 978-0-387-35779-9.

[DM88] B. Dacorogna and P. Marcellini. “A Counterexample in the Vectorial Calculus of Variations”. In: *Mater. Instab. Contin. Mech.* (Dec. 1988), pp. 77–83.

[De 89] R. De Arcangelis. “Some Remarks on the Identity between a Variational Integral and Its Relaxed Functional”. In: *Ann. Univ. Ferrara* 35.1 (Jan. 1989), pp. 135–145. ISSN: 1827-1510.

[De 95] R. De Arcangelis. “The Lavrentieff Phenomenon for Quadratic Functionals”. In: *Proc. R. Soc. Edinb. Sect. Math.* 125.2 (1995), pp. 329–339. ISSN: 1473-7124, 0308-2105.

[De 22] C. De Filippis. "Quasiconvexity and Partial Regularity via Nonlinear Potentials". In: *Journal de Mathématiques Pures et Appliquées* 163 (July 2022), pp. 11–82. ISSN: 0021-7824.

[DL21] C. De Filippis and F. Leonetti. "Uniform Ellipticity and (p, q) Growth". In: *Journal of Mathematical Analysis and Applications*. New Developments in Non-Uniformly Elliptic and Nonstandard Growth Problems 501.1 (Sept. 2021), p. 124451. ISSN: 0022-247X.

[DS22] C. De Filippis and B. Stroffolini. *Singular Multiple Integrals and Nonlinear Potentials*. Mar. 2022. arXiv: `2203.05519 [math]`.

[De 57] E. De Giorgi. "Sulla Differenziabilità e l'analiticità Delle Estremali Degli Integrali Multipli Regolari". In: *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.* 3.3 (1957), pp. 25–43.

[De 61] E. De Giorgi. *Frontiere orientate di misura minima*. Pisa: Editr. tecnico scientifica, 1961.

[De 68] E. De Giorgi. "Un Esempio Di Estremali Discontinue per Un Problema Variazionale Di Tipo Ellittico". In: *Boll DellUnione Mat Ital* 4 (1968), pp. 135–137.

[De 15] C. De La Vallée Poussin. "Sur L'Integrale de Lebesgue". In: *Trans. Am. Math. Soc.* 16.4 (1915), pp. 435–501. ISSN: 0002-9947.

[DP11] B. De Maria and A. Passarelli di Napoli. "A New Partial Regularity Result for Non-Autonomous Convex Integrals with Non-Standard Growth Conditions". In: *Journal of Differential Equations* 250.3 (Feb. 2011), pp. 1363–1385. ISSN: 0022-0396.

[del+19] R. del Campo, A. Fernández, F. Mayoral, and F. Naranjo. "The de La Vallée-Poussin Theorem and Orlicz Spaces Associated to a Vector Measure". In: *Journal of Mathematical Analysis and Applications* 470.1 (Feb. 2019), pp. 279–291. ISSN: 0022247X.

[DK15] R. N. Dhara and A. Kałamajska. "On One Extension Theorem Dealing with Weighted Orlicz-Slobodetskii Space. Analysis on Cube". In: *Math. Inequalities Appl.* 1 (2015), pp. 61–89. ISSN: 1331-4343.

[DK16] R. N. Dhara and A. Kałamajska. "On One Extension Theorem Dealing with Weighted Orlicz-Slobodetskii Space. Analysis on Lipschitz Subgraph and Lipschitz Domain". In: *Math. Inequalities Appl.* 2 (2016), pp. 451–488. ISSN: 1331-4343.

[DE08] L. Diening and F. Ettwein. "Fractional Estimates for Non-Differentiable Elliptic Systems with General Growth". In: 20.3 (May 2008), pp. 523–556. ISSN: 1435-5337.

[Die+12] L. Diening, D. Lengeler, B. Stroffolini, and A. Verde. "Partial Regularity for Minimizers of Quasi-Convex Functionals with General Growth". In: *SIAM Jounral Math. Anal.* 44 (2012), pp. 3594–3616.

[DMS08] L. Diening, J. Málek, and M. Steinhauer. "On Lipschitz Truncations of Sobolev Functions (with Variable Exponent) and Their Selected Applications". In: *ESAIM Control Optim. Calc. Var.* 14.2 (Apr. 2008), pp. 211–232. ISSN: 1292-8119, 1262-3377.

[DRS10] L. Diening, M. Ružička, and K. Schumacher. "A Decomposition Technique for John Domains". In: *Ann. Acad. Sci. Fenn. Math.* 35 (Mar. 2010), pp. 87–114. ISSN: 1239629X.

[DSV12] L. Diening, B. Stroffolini, and A. Verde. "The φ-Harmonic Approximation and the Regularity of φ-Harmonic Maps". In: *Journal of Differential Equations* 253.7 (Oct. 2012), pp. 1943–1958. ISSN: 0022-0396.

[DK18] H. Dong and D. Kim. "On L_p-Estimates for Elliptic and Parabolic Equations with A_p Weights". In: *Trans. Am. Math. Soc.* 370.7 (Feb. 2018), pp. 5081–5130. ISSN: 0002-9947.

[Duo01] J. Duoandikoetxea. *Fourier Analysis.* Graduate Studies in Mathematics, Vol 29. American Mathematical Society, 2001. ISBN: 978-0-8218-2172-5.

[DG00] F. Duzaar and J. F. Grotowski. "Optimal Interior Partial Regularity for Nonlinear Elliptic Systems: The Method of A-harmonic Approximation". In: *manuscripta math.* 103.3 (Nov. 2000), pp. 267–298. ISSN: 1432-1785.

[DM04] F. Duzaar and G. Mingione. "The p-Harmonic Approximation and the Regularity of p-Harmonic Maps". In: *Cal Var* 20.3 (July 2004), pp. 235–256. ISSN: 1432-0835.

[DM09] F. Duzaar and G. Mingione. "Harmonic Type Approximation Lemmas". In: *Journal of Mathematical Analysis and Applications.* Degenerate and Singular PDEs and Phenomena in Analysis and Mathematical Physics 352.1 (Apr. 2009), pp. 301–335. ISSN: 0022-247X.

[Eke74] I. Ekeland. "On the Variational Principle". In: *Journal of Mathematical Analysis and Applications* 47.2 (Aug. 1974), pp. 324–353. ISSN: 0022-247X.

[ELM04] L. Esposito, F. Leonetti, and G. Mingione. "Sharp Regularity for Functionals with (p,q) Growth". In: *Journal of Differential Equations* 204.1 (Sept. 2004), pp. 5–55. ISSN: 00220396.

[Eva33] G. C. Evans. "Complements of Potential Theory. Part Ii". In: *Am. J. Math.* 55.1 (1933), pp. 29–49. ISSN: 0002-9327.

[Eva86] L. C. Evans. "Quasiconvexity and Partial Regularity in the Calculus of Variations". In: *Arch. Ration. Mech. Anal.* 95.3 (Sept. 1986), pp. 227–252. ISSN: 0003-9527.

[EG15] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* 2015. ISBN: 978-1-4822-4238-6.

[FS72] C. Fefferman and E. M. Stein. "H^p Spaces of Several Variables". In: *Acta Math.* 129.1 (1972), pp. 137–193. ISSN: 18712509.

[FM97] I. Fonseca and J. Malý. "Relaxation of Multiple Integrals below the Growth Exponent". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* 14.3 (Jan. 1997), pp. 309–338. ISSN: 0294-1449.

[FMM04] I. Fonseca, J. Malý, and G. Mingione. "Scalar Minimizers with Fractal Singular Sets". In: *Arch. Rational Mech. Anal.* 172.2 (May 2004), pp. 295–307. ISSN: 1432-0673.

[FHM03] M. Foss, W. J. Hrusa, and V. J. Mizel. "The Lavrentiev Gap Phenomenon in Nonlinear Elasticity". In: *Arch. Rational Mech. Anal.* 167.4 (May 2003), pp. 337–365. ISSN: 1432-0673.

[Fos01] M. D. Foss. "On Lavrentiev's Phenomenon". PhD thesis. Carnegie Mellon University, 2001.

[Fra19] F. Franceschini. "Partial Regularity for $\mathrm{BV}^{\mathscr{B}}$ Local Minimizers". MA thesis. Universit'a degli Studi di Pisa, 2019.

[Fri44] K. O. Friedrichs. "The Identity of Weak and Strong Extensions of Differential Operators". In: *Trans. Am. Math. Soc.* 55.1 (1944), pp. 132–151. ISSN: 0002-9947.

[FJM02] G. Friesecke, R. D. James, and S. Müller. "A Theorem on Geometric Rigidity and the Derivation of Nonlinear Plate Theory from Three-Dimensional Elasticity". In: *Commun. Pure Appl. Math.* 55.11 (2002), pp. 1461–1506. ISSN: 1097-0312.

[FH85] N. Fusco and J. Hutchinson. "$C^{1,\alpha}$ Partial Regularity of Functions Minimising Quasiconvex Integrals". In: *Manuscripta Math* 54.1 (Mar. 1985), pp. 121–143. ISSN: 1432-1785.

[Geh73] F. W. Gehring. "The L^p-Integrability of the Partial Derivatives of a Quasiconformal Mapping". In: *Acta Math.* 130.1 (July 1973), pp. 265–277. ISSN: 1871-2509.

[GM85] F. W. Gehring and O. Martio. "Lipschitz Classes and Quasiconformla Mappings". In: *Ann Acad Sci Fenn Ser Math* 10 (1985), pp. 203–219. ISSN: 00661953.

[GP76] F. W. Gehring and B. P. Palka. "Quasiconformally Homogeneous Domains". In: *J Anal. Mathématique* 30.1 (Dec. 1976), pp. 172–199. ISSN: 0021-7670.

[Gia83] M. Giaquinta. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*. Annals of Mathematical Studies 105. Princeton University Press, 1983.

[Gia87] M. Giaquinta. "Growth Conditions and Regularity, a Counterexample". In: *Manuscripta Math* 59.2 (June 1987), pp. 245–248. ISSN: 1432-1785.

[GG78] M. Giaquinta and E. Giusti. "Nonlinear Elliptic Systems with Quadratic Growth". In: *Manuscripta Math* 24.3 (Sept. 1978), pp. 323–349. ISSN: 1432-1785.

[GG82] M. Giaquinta and E. Giusti. "On the Regularity of the Minima of Variational Integrals". In: *Acta Math* 148.1 (July 1982), pp. 31–46. ISSN: 1871-2509.

[GG83] M. Giaquinta and E. Giusti. "Differentiability of Minima of Non-Differentiable Functionals". In: *Invent Math* 72.2 (June 1983), pp. 285–298. ISSN: 1432-1297.

[GM79] M. Giaquinta and G. Modica. "Regularity Results for Some Classes of Higher Order Non Linear Elliptic Systems". In: *J Für Reine Angew Math Crelles J.* 1979.311-312 (Nov. 1979), pp. 145–169. ISSN: 0075-4102.

[GM86] M. Giaquinta and G. Modica. "Partial Regularity of Minimizers of Quasiconvex Integrals". In: *Ann IHP Anal Non Linéaire* 3.3 (1986), pp. 185–208.

[GT98] D. Gilbert and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag Berlin Heidelberg, 1998.

[Giu03] E. Giusti. "Direct Methods in the Calculus of Variations". In: *Direct Methods Calc. Var.* WORLD SCIENTIFIC, Jan. 2003. ISBN: 978-981-238-043-2.

[GM68] E. Giusti and M. Miranda. "Sulla Regolarità Delle Soluzioni Deboli Di Una Classe Di Sistemi Ellittici Quasi-Lineari". In: *Arch. Ration. Mech. Anal.* 31.3 (Jan. 1968), pp. 173–184. ISSN: 0003-9527.

[Gme21] F. Gmeineder. "Partial Regularity for Symmetric Quasiconvex Functionals on BD". In: *Journal de Mathématiques Pures et Appliquées* 145 (Jan. 2021), pp. 83–129. ISSN: 0021-7824.

[GK19a] F. Gmeineder and J. Kristensen. "Partial Regularity for BV Minimizers". In: *Arch. Ration. Mech. Anal.* 232.3 (June 2019), pp. 1429–1473. ISSN: 0003-9527.

[GK19b] F. Gmeineder and J. Kristensen. "Regularity for Higher Order Quasiconvex Problems with Linear Growth from Below". In: *arXiv* (Mar. 2019). arXiv: `1903.08124 [math]`.

[GSS21] C. Goodrich, G. Scilla, and B. Stroffolini. "Partial Regularity for Minimizers of Discontinuous Quasiconvex Integrals with General Growth". In: *Proc. R. Soc. Edinb. Sect. Math.* (Dec. 2021), pp. 1–42. ISSN: 0308-2105, 1473-7124.

[Gra18] Y. Grabovsky. "From Microstructure-Independent Formulas for Composite Materials to Rank-One Convex, Non-Quasiconvex Functions". In: *Arch Rational Mech Anal* 227.2 (Feb. 2018), pp. 607–636. ISSN: 1432-0673.

[GIS97] L. Greco, T. Iwaniec, and C. Sbordone. "Variational Integrals of Nearly Linear Growth". In: *Differ. Integral Equ.* 10.4 (Jan. 1997), pp. 687–716. ISSN: 0893-4983.

[Gri11] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, Jan. 2011. ISBN: 978-1-61197-202-3.

[GP77] J. Gustavsson and J. Peetre. "Interpolation of Orlicz Spaces". In: *Stud. Math.* 60.1 (1977), pp. 33–59. ISSN: 0039-3223.

[Had05] J. Hadamard. "Sur quelques questions de calcul des variations". In: *Bull. Société Mathématique Fr.* 33 (1905), pp. 73–80.

[Ham07] C. Hamburger. "Optimal Partial Regularity of Minimizers of Quasiconvex Variational Integrals". In: *ESAIM Control Optim. Calc. Var.* 13.4 (Oct. 2007), pp. 639–656. ISSN: 1292-8119, 1262-3377.

[HLN96] W. Hao, S. Leonardi, and J. Necas. "An Example of Irregular Solution to a Nonlinear Euler-Lagrange Elliptic System with Real Analytic Coefficients". In: *Ann. Della Scuola Norm. Super. Pisa Cl. Sci.* 23.1 (1996), pp. 57–67. ISSN: 0391-173X.

[Hil02] D. Hilbert. "Mathematical Problems". In: *Bull. Amer. Math. Soc.* 8.10 (1902), pp. 437–479. ISSN: 0002-9904, 1936-881X.

[Hop29] E. Hopf. "Zum analytischen Charakter der Lösungen regulärer zweidimensionaler Variationsprobleme". In: *Math Z* 30.1 (Dec. 1929), pp. 404–413. ISSN: 1432-1823.

[Hur88] R. Hurri. "Poincare Domains in $\mathbb{R}^n$". In: *Ann. Acad. Sci. Fenn. Math. Diss.* 71 (1988), pp. 1–42.

[Hur93] R. Hurri-Syrjänen. "The John-Nirenberg Inequality and a Sobolev Inequality in General Domains". In: *J. Math. Anal. Appl.* 175.2 (May 1993), pp. 579–587. ISSN: 0022247X.

[Hur96] R. Hurri-Syrjänen. "A Generalization of an Inequality of Bhattacharya and Leonetti". In: *Can. Math. Bull.* 39.4 (Dec. 1996), pp. 438–447. ISSN: 0008-4395, 1496-4287.

[Irv21a] C. Irving. "BMO ε-Regularity Results for Solutions to Legendre-Hadamard Elliptic Systems". In: *arXiv* (Sept. 2021). arXiv: `2109.07265 [math]`.

[Irv21b] C. Irving. "Partial Regularity for Minima of Higher-Order Quasiconvex Integrands with Natural Orlicz Growth". In: *arXiv* (Nov. 2021). arXiv: `2111.14740 [math]`.

[Ive79] P.-A. Ivert. "Regularitätsuntersuchungen von Lösungen Elliptischer Systeme von Quasilinearen Differentialgleichungen Zweiter Ordnung". In: *Manuscripta Math* 30.1 (Mar. 1979), pp. 53–88. ISSN: 1432-1785.

[IN85] T. Iwaniec and C. Nolder. "Hardy-Littlewood Inequality for Quasiregular Mappings in Certain Domains in $\mathbb{R}^n$". In: *Ann Acad Sci Fenn Ser Math* 10 (1985), pp. 267–282. ISSN: 00661953.

[Joh50] F. John. "The Fundamental Solution of Linear Elliptic Differential Equations with Analytic Coefficients". In: *Commun. Pure Appl. Math.* 3.3 (1950), pp. 273–304. ISSN: 1097-0312.

[Jon80] P. Jones. "Extension Theorems for BMO". In: *Indiana Univ. Math. J.* 29.1 (1980), pp. 41–66.

[KK13] A. Kałamajska and M. Krbec. "Traces of Orlicz-Sobolev Functions under General Growth Restrictions". In: *Math. Nachrichten* 286.7 (2013), pp. 730–742. ISSN: 1522-2616.

[KP91] D. Kinderlehrer and P. Pedregal. "Characterizations of Young Measures Generated by Gradients". In: *Arch. Rational Mech. Anal.* 115.4 (Dec. 1991), pp. 329–365. ISSN: 1432-0673.

[Koc21] L. Koch. "Global Higher Integrability for Minimisers of Convex Functionals with (p, q)-Growth". In: *Calc. Var.* 60.2 (Apr. 2021), p. 63. ISSN: 1432-0835.

[Koc22] L. Koch. "Global Higher Integrability for Minimisers of Convex Obstacle Problems with (p,q)-Growth". In: *Calc. Var.* 61.3 (Mar. 2022), p. 88. ISSN: 1432-0835.

[KK22] L. Koch and J. Kristensen. *On the Validity of the Euler-Lagrange System without Growth Assumptions.* Mar. 2022. arXiv: `2203.00333 [math]`.

[KK91] V. Kokilashvili and M. Krbec. *Weighted Inequalities in Lorentz and Orlicz Spaces.* WORLD SCIENTIFIC, Dec. 1991. ISBN: 978-981-02-0612-3.

[KR61] M. A. Krasnosel'skii and Y. B. Rutickii. *Convex Functions and Orlicz Spaces.* P. Noordhoff, 1961.

[Kri99] J. Kristensen. "On the Non-Locality of Quasiconvexity". In: *Ann Inst. Henri Poincaré C Anal Non Linéaire* 16.1 (Jan. 1999), pp. 1–13. ISSN: 02941449.

[KM10] J. Kristensen and G. Mingione. "Boundary Regularity in Variational Problems". In: *Arch Rational Mech Anal* 198.2 (Nov. 2010), pp. 369–455. ISSN: 1432-0673.

[KR10] J. Kristensen and F. Rindler. "Characterization of Generalized Gradient Young Measures Generated by Sequences in $W^{1,1}$ and BV". In: *Arch Rational Mech Anal* 197.2 (Aug. 2010), pp. 539–598. ISSN: 1432-0673.

[KT03] J. Kristensen and A. Taheri. "Partial Regularity of Strong Local Minimizers in the Multi-Dimensional Calculus of Variations". In: *Arch. Ration. Mech. Anal.* 170.1 (Nov. 2003), pp. 63–89. ISSN: 0003-9527.

[Kro02] M. Kronz. "Partial Regularity Results for Minimizers of Quasiconvex Functionals of Higher Order". In: *Ann Inst. Henri Poincaré C Anal Non Linéaire* 19.1 (2002), pp. 81–112. ISSN: 02941449.

[Kro05] M. Kronz. “Boundary Regularity for Almost Minimizers of Quasiconvex Variational Problems”. In: *Nonlinear Differ Equ. Appl NoDEA* 12.3 (Dec. 2005), pp. 351–382. ISSN: 1021-9722.

[Kry07] N. V. Krylov. “Parabolic and Elliptic Equations with VMO Coefficients”. In: *Commun. Partial Differ. Equ.* 32.3 (Mar. 2007), pp. 453–475. ISSN: 0360-5302.

[KM16] T. Kuusi and G. Mingione. “Partial regularity and potentials”. In: *J. L'École Polytech. — Mathématiques* 3 (2016), pp. 309–363.

[Lac74] M.-T. Lacroix. “Espaces de Traces Des Espaces d'Orlicz-Sobolev et Applications”. In: *J. Math. Pures. Appl.* 53.9 (1974), pp. 439–458.

[LU61] O. A. Ladyzhenskaya and N. N. Ural'tseva. “Quasilinear Elliptic Equations and Variational Problems with Several Independent Variables”. In: *Russ. Math. Surv.* 16.1 (1961), pp. 17–91. ISSN: 0036-0279, 1468-4829.

[Lav27] M. Lavrentieff. “Sur quelques problèmes du calcul des variations”. In: *Annali di Matematica* 4.1 (Dec. 1927), pp. 7–28. ISSN: 1618-1891.

[Leb02] H. Lebesgue. “Intégrale, Longueur, Aire”. In: *Annali di Matematica, Serie III* 7.1 (Dec. 1902), pp. 231–359. ISSN: 0373-3114.

[Leb04] H. Lebesgue. *Leçons sur l'intégration et la recherche des fonctions primitives, professées au Collège de France*. Paris: Gauthier-Villars, 1904.

[Li22] Z. Li. “Partial Regularity for ω-Minimizers of Quasiconvex Functionals”. In: *Calc. Var.* 61.5 (July 2022), p. 178. ISSN: 1432-0835.

[Lic12] L. Lichtenstein. “Über Den Analytischen Charakter Der Lösungen Regulärer Zweidimensionaler Variationsprobleme”. In: *Bull Acad Sci Cracoive Cl Sci Math Nat A* (1912), pp. 915–941.

[Lin73] K. Lindberg. “On Subspaces of Orlicz Sequences Spaces”. In: *Stud. Math.* 45 (1973), pp. 119–146. ISSN: 0039-3223, 1730-6337.

[LM04] Z. Lou and A. McIntosh. “Hardy Spaces of Exact Forms on Lipschitz Domains in $\mathbb{R}^n$”. In: *Indiana Univ. Math. J.* 53.2 (2004), pp. 583–611. ISSN: 0022-2518.

[Mal89] L. Maligranda. *Orlicz Spaces and Interpolation*. Seminars in Mathematics 5. Departamento de Matemática, Universidade Estadual de Campinas, 1989.

[Mar85] P. Marcellini. “Approximation of Quasiconvex Functions, and Lower Semicontinuity of Multiple Integrals”. In: *Manuscripta Math.* 51.1-3 (1985), pp. 1–28. ISSN: 0025-2611.

[Mar87] P. Marcellini. *Un Example de Solution Discontinue d'un Problème Variationnel Dans Le Cas Scalaire*. Mat. Univ. Firenze, 1987.

[Mar89] P. Marcellini. “Regularity of Minimizers of Integrals of the Calculus of Variations with Non Standard Growth Conditions”. In: *Arch. Rational Mech. Anal.* 105.3 (Sept. 1989), pp. 267–284. ISSN: 1432-0673.

[Mar91] P. Marcellini. “Regularity and Existence of Solutions of Elliptic Equations with p,q-Growth Conditions”. In: *Journal of Differential Equations* 90.1 (Mar. 1991), pp. 1–30. ISSN: 0022-0396.

[Mar93] P. Marcellini. “Regularity for Elliptic Equations with General Growth Conditions”. In: *Journal of Differential Equations* 105.2 (Oct. 1993), pp. 296–333. ISSN: 0022-0396.

[Mar96] P. Marcellini. “Everywhere Regularity for a Class of Elliptic Systems without Growth Conditions”. In: *Ann. Della Scuola Norm. Super. Pisa - Cl. Sci.* 23.1 (1996), pp. 1–25. ISSN: 0391-173X.

[MS83] P. Marcellini and C. Sbordone. “On the Existence of Minima of Multiple Integrals of the Calculus of Variations”. In: *Journal de Mathématiques Pures et Appliqués* 62 (Jan. 1983), pp. 1–9.

[MO60] W. Matuszewska and W. Orlicz. “On Certain Properties of φ-Functions”. In: *Bull. Acad. Polon. Sci.* 8 (1960), pp. 439–443.

[MMS10] V. Maz’ya, M. Mitrea, and T. Shaposhnikova. “The Dirichlet Problem in Lipschitz Domains for Higher Order Elliptic Systems with Rough Coefficients”. In: *J Anal. Mathématique* 110.1 (Jan. 2010), pp. 167–239. ISSN: 0021-7670.

[Maz68] V. G. Maz’ya. “Examples of Nonregular Solutions of Quasilinear Elliptic Equations with Analytic Coefficients”. In: *Funct. Anal. Its Appl.* 2.3 (1968), pp. 230–234. ISSN: 0016-2663.

[Mey65] N. G. Meyers. “Quasi-Convexity and Lower Semi-Continuity of Multiple Variational Integrals of Any Order”. In: *Trans. Amer. Math. Soc.* 119 (1965), pp. 125–149. ISSN: 0002-9947.

[ME75] N. G. Meyers and A. Elcrat. “Some Results on Regularity for Solutions of Non-Linear Elliptic Systems and Quasi-Regular Functions”. In: *Duke Math. J.* 42.1 (Mar. 1975), pp. 121–136. ISSN: 0012-7094, 1547-7398.

[Min03] G. Mingione. “Bounds for the Singular Set of Solutions to Non Linear Elliptic Systems”. In: *Cal Var* 18.4 (Dec. 2003), pp. 373–400. ISSN: 1432-0835.

[Min06] G. Mingione. “Regularity of Minima: An Invitation to the Dark Side of the Calculus of Variations”. In: *Appl Math* 51.4 (Aug. 2006), p. 355. ISSN: 1572-9109.

[Mit18] D. Mitrea. *Distributions, Partial Differential Equations, and Harmonic Analysis.* Universitext. Cham: Springer International Publishing, 2018. ISBN: 978-3-030-03295-1.

[MS16] C. Mooney and O. Savin. “Some Singular Minimizers in Low Dimensions in the Calculus of Variations”. In: *Arch. Ration. Mech. Anal.* 221.1 (July 2016), pp. 1–22. ISSN: 0003-9527.

[Mor38] C. B. Morrey. “On the Solutions of Quasi-Linear Elliptic Partial Differential Equations”. In: *Trans. Am. Math. Soc.* 43.1 (1938), pp. 126–166. ISSN: 0002-9947.

[Mor59] C. B. Morrey. “Second Order Elliptic Equations in Several Variables and Hölder Continuity”. In: *Math Z* 72.1 (Dec. 1959), pp. 146–164. ISSN: 1432-1823.

[Mor68] C. B. Morrey. “Partial Regularity Results for Non-Linear Elliptic Systems”. In: *J. Math. Mech.* 17.7 (1968), pp. 649–670. ISSN: 0095-9057.

[Mor40] C. B. Morrey. “Functions of Several Variables and Absolute Continuity, II”. In: *dmj* 6.1 (Jan. 1940), pp. 187–215. ISSN: 0012-7094, 1547-7398.

[Mor43] C. B. Morrey. “Multiple Integral Problems in the Calculus of Variations and Related Topics”. In: *Univ Calif. Publ Math NS* 1 (1943), pp. 1–130.

[Mor52] C. B. Morrey. “Quasi-Convexity and the Lower Semicontinuity of Multiple Integrals.” In: *Pacific J. Math.* 2.1 (1952), pp. 25–53. ISSN: 0030-8730.

[Mor66] C. B. Morrey. *Multiple Integrals in the Calculus of Variations*. Springer, 1966, p. 506. ISBN: 3-540-69952-X.

[Mos60] J. Moser. "A New Proof of de Giorgi's Theorem Concerning the Regularity Problem for Elliptic Differential Equations". In: *Commun. Pure Appl. Math.* 13.3 (1960), pp. 457–468. ISSN: 1097-0312.

[Mos01] R. Moser. *Vanishing Mean Oscillation and Regularity in the Calculus of Variations*. Preprint. 2001.

[Muc72] B. Muckenhoupt. "Weighted Norm Inequalities for the Hardy Maximal Function". In: *Trans. Amer. Math. Soc.* 165 (1972), pp. 207–226. ISSN: 0002-9947, 1088-6850.

[Mül92] S. Müller. "On Quasiconvex Functions Which Are Homogeneous of Degree 1". In: *Indiana Univ. Math. J.* 41.1 (1992), pp. 295–301. ISSN: 0022-2518.

[MŠ03] S. Müller and V. Šverák. "Convex Integration for Lipschitz Mappings and Counterexamples to Regularity". In: *Ann. Math.* 157.3 (May 2003), pp. 715–742. ISSN: 0003-486X.

[Nas58] J. Nash. "Continuity of Solutions of Parabolic and Elliptic Equations". In: *Am. J. Math.* 80.4 (1958), pp. 931–954. ISSN: 0002-9327.

[Neč77] J. Nečas. "Example of an Irregular Solution to a Nonliear Elliptic System with Analytic Coefficients and Conditions for Regularity". In: *Theory Nonlinear Oper. Constr. Asp. (Proceedings Fourth Int. Summer Sch.* Ed. by R. Kluge and W. Müller. Berlin: Akademie-Verlag, 1977, pp. 197–206.

[Neč12] J. Nečas. *Direct Methods in the Theory of Elliptic Equations*. Springer Monographs in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012. ISBN: 978-3-642-10454-1.

[Nik12] S. M. Nikol'skii. *Approximation of Functions of Several Variables and Imbedding Theorems*. Springer Science & Business Media, Dec. 2012. ISBN: 978-3-642-65711-5.

[OSS21] J. Ok, G. Scilla, and B. Stroffolini. "Boundary Partial Regularity for Minimizers of Discontinuous Quasiconvex Integrals with General Growth". In: *arXiv* (Aug. 2021). arXiv: `2108.11796 [math]`.

[RR91] M. Rao and Z. Ren. "Theory of Orlicz Spaces". In: *undefined* (1991).

[RR75] H. M. Reimann and T. Rychener. *Funktionen Beschränkter Mittlerer Oszillation*. Vol. 487. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1975. ISBN: 978-3-540-07404-5.

[Rio55] W. J. Riordan. "On the Interpolation of Operations". PhD thesis. University of Chicago, 1955.

[Sav97] G. Savaré. "Regularity and Perturbation Results for Mixed Second Order Elliptic Problems". In: *Commun. Partial Differ. Equ.* 22.5-6 (Jan. 1997), pp. 869–899. ISSN: 0360-5302.

[Sch34a] J. Schauder. "Numerische Abschätzungen in elliptischen linearen Differentialgleichungen". In: *Stud. Math.* 5.1 (1934), pp. 34–42. ISSN: 0039-3223.

[Sch34b] J. Schauder. "Über lineare elliptische Differentialgleichungen zweiter Ordnung." In: *Math Z* 38.1 (Dec. 1934), pp. 257–282. ISSN: 1432-1823.

[Sch08a] T. Schmidt. “Regularity of Minimizers of $W^{1,p}$-Quasiconvex Variational Integrals with (p, q)-Growth”. In: *Calc. Var.* 32.1 (May 2008), pp. 1–24. ISSN: 1432-0835.

[Sch08b] T. Schmidt. “Regularity Theorems for Degenerate Quasiconvex Energies with (p, q)-Growth”. In: 1.3 (Oct. 2008), pp. 241–270. ISSN: 1864-8266.

[Sch09] T. Schmidt. “Regularity of Relaxed Minimizers of Quasiconvex Variational Integrals with (p, q)-Growth”. In: *Arch Rational Mech Anal* 193.2 (Aug. 2009), pp. 311–337. ISSN: 1432-0673.

[Ser83] D. Serre. “Formes Quadratiques et Calcul Des Variations”. In: *J. Math. pures appl.* 62 (1983), pp. 177–196.

[Ser61] J. Serrin. “On the Definition and Properties of Certain Variational Integrals”. In: *Trans. Amer. Math. Soc.* 101.1 (1961), pp. 139–167. ISSN: 0002-9947, 1088-6850.

[Sim76a] J. Simon. “Régularité de Solutions de Problèmes Non Linéaires”. In: *C R Acad Sci Paris Sér A* 282 (1976), pp. 1351–1354.

[Sim76b] J. Simon. *Régularité Locale Des Solutions d’une Équation Non Linéaire*. Université de Paris 6, 1976.

[Sim77] J. Simon. “Sur Des Équations Aux Dérivées Partielles Non Linéaires”. Introduction Du Mémoire de Thèse d’État. Université de Paris, 1977.

[Sim78] J. Simon. “Regularite de La Solution d’une Equation Non Lineaire Dans $\mathbb{R}^n$”. In: *Journées d’Analyse Non Linéaire*. Ed. by P. Bénilan and J. Robert. Vol. 665. Berlin, Heidelberg: Springer Berlin Heidelberg, 1978, pp. 205–227. ISBN: 978-3-540-08922-3 978-3-540-35756-8.

[Sim81] J. Simon. “Régularité de la solution d’un problème aux limites non linéaires”. In: *Ann. Fac. Sci. Toulouse Mathématiques* 3.3-4 (1981), pp. 247–274. ISSN: 0240-2963.

[Sim83] L. Simon. *Lectures on Geometric Measure Theory*. Centre for Mathematical Analysis, Australian National University, 1983, p. 272. ISBN: 0-86784-429-9.

[Sim96] L. Simon. *Theorems on Regularity and Singularity of Energy Minimizing Maps*. Basel: Birkhäuser Basel, 1996. ISBN: 978-3-7643-5397-1 978-3-0348-9193-6.

[SS90] W. Smith and D. Stegenga. “Hölder Domains and Poincaré Domains”. In: *Trans. Am. Math. Soc.* 319.1 (May 1990), p. 67. ISSN: 00029947.

[SS91] W. Smith and D. Stegenga. “Exponential Integrability of the Quasi-Hyperbolic Metric on Hölder Domains”. In: *Ann Acad Sci Fenn Ser Math* 16.2 (1991), pp. 345–360.

[Sob63] S. L. Sobolev. “On a Theorem of Functional Analysis”. In: *American Mathematical Society Translations: Series 2*. Vol. 34. Providence, Rhode Island: American Mathematical Society, 1963, pp. 39–68. ISBN: 978-0-8218-1734-6 978-1-4704-3245-4.

[Sta65] G. Stampacchia. “The Spaces $L^{p,\lambda}$, $N^{p,\lambda}$ and Interpolation”. In: *Ann Della Sc Norm Super Pisa Cl Sci* 19.3 (1965), pp. 443–462.

[Sta60] G. Stampacchia. “Problemi al contorno ellittici, con dati discontinui, dotati di soluzioni hölderiane”. In: *Annali di Matematica* 51.1 (Dec. 1960), pp. 1–37. ISSN: 1618-1891.

[Ste93] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Aug. 1993. ISBN: 978-0-691-03216-0.

[Ste71] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton: Princeton University Press, Dec. 1971. ISBN: 978-1-4008-8388-2.

[Str80] E. W. Stredulinsky. "Higher Integrability from Rerverse Hölder Inequalities". In: *Indiana Univ. Math. J.* 29.3 (1980), pp. 407–413.

[SY02] V. Sverák and X. Yan. "Non-Lipschitz Minimizers of Smooth Uniformly Convex Functionals". In: *Proc Natl Acad Sci U S A* 99.24 (Nov. 2002), pp. 15269–15276. ISSN: 0027-8424.

[Šve92] V. Šverák. "Rank-One Convexity Does Not Imply Quasiconvexity". In: *Proc R Soc Edinb. Sect Math* 120.1-2 (Nov. 1992), pp. 185–189. ISSN: 0308-2105.

[Šve91] V. Šverák. "Quasiconvex Functions with Subquadratic Growth". In: *Proc. Math. Phys. Sci.* 433.1889 (1991), pp. 723–725. ISSN: 0962-8444.

[ŠY00] V. Šverák and X. Yan. "A Singular Minimizer of a Smooth Strongly Convex Functional in Three Dimensions". In: *Calc Var* 10.3 (Apr. 2000), pp. 213–221. ISSN: 1432-0835.

[Szé04] L. Székelyhidi. "The Regularity of Critical Points of Polyconvex Functionals". In: *Arch. Ration. Mech. Anal.* 172.1 (Apr. 2004), pp. 133–152. ISSN: 0003-9527.

[Szé05] L. Székelyhidi. "Rank-One Convex Hulls in $\mathbb{R}^{2\times 2}$". In: *Calc. Var.* 22.3 (Mar. 2005), pp. 253–281. ISSN: 1432-0835.

[Ter39] F. J. Terpstra. "Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung". In: *Math. Ann.* 116.1 (Dec. 1939), pp. 166–180. ISSN: 1432-1807.

[Ton15] L. Tonelli. "Sur une méthode directe du calcul des variations". In: *Rend. Circ. Mat. Palermo* 39.1 (Dec. 1915), pp. 233–264. ISSN: 0009-725X.

[Ton29] L. Tonelli. "Sur La Semi-Continuité Des Intégrales Doubles Du Calcul Des Variations". In: *Acta Math.* 53.none (Jan. 1929), pp. 325–346. ISSN: 0001-5962, 1871-2509.

[Ton33] L. Tonelli. "L'estremo assoluto degli integrali doppi". In: *Ann. Della Scuola Norm. Super. Pisa - Cl. Sci.* 2.1 (1933), pp. 89–130.

[Ton34] L. Tonelli. "Su Gli Integrali Del Calcolo Delle Variazioni in Forma Ordinaria". In: *Ann. Della Scuola Norm. Super. Pisa-Cl. Sci.* 3.3-4 (1934), pp. 401–450.

[Tri83] H. Triebel. *Theory of Function Spaces.* Basel: Springer Basel, 1983. ISBN: 978-3-0346-0415-4.

[Val88] T. Valent. *Boundary Value Problems of Finite Elasticity.* Vol. 31. Springer Tracts in Natural Philosophy. New York, NY: Springer New York, 1988. ISBN: 978-1-4612-8326-3.

[Van47] L. Van Hove. "Sur l'extension de la condition de Legendre du calcul de variations aux intégrales multiples à plusieurs fonctions inconnues". In: *Proc. K. Ned. Akad. Van Wet.* 50.1 (1947), pp. 18–23. ISSN: 0924-8323.

[Vit04] G. Vitali. "Sulle Funzioni Integrali". In: *Atti Accad Sci Torino Cl Sci Fis Mat Nat.* 1905 (1904), pp. 1021–1034.

[Zha89] K. Zhang. “A Counterexample in the Theory of Coerciveness for Elliptic Systems”. In: *J Partial Differ. Equ.* 2.3 (1989), pp. 79–82.

[Zha91] K. Zhang. “Energy Minimizers in Nonlinear Elastostatics and the Implicit Function Theorem”. In: *Arch. Rational Mech. Anal.* 114.2 (June 1991), pp. 95–117. ISSN: 1432-0673.

[Zha92] K. Zhang. “Remarks on Quasiconvexity and Stability of Equilibria for Variational Integrals”. In: *Proc. Am. Math. Soc.* 114.4 (Apr. 1992), pp. 927–927. ISSN: 0002-9939.

[Zha96] K. Zhang. “On the Coercivity of Elliptic Systems in Two Dimensional Spaces”. In: *Bull. Aust. Math. Soc.* 54.3 (1996), pp. 423–430. ISSN: 00049727.

[Zhi87] V. V. Zhikov. “Averaging of Functionals of the Calculus of Variations and Elasticity Theory”. In: *Math. USSR Izv.* 29.1 (Feb. 1987), p. 33. ISSN: 0025-5726.

[Zyg56] A. Zygmund. “On a Theorem of Marcinkiewicz Concerning Interpolation of Operators”. In: *J. Math. Pures Appl.* 35 (1956), pp. 223–248.