

Multi-row extension to laterally confined actuator disk model using a hybrid inviscid-viscous approach

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Abstract

The effects of blockage and wake mixing are fundamental but still unresolved problems in the aero/hydrodynamics of wind/tidal-stream turbine arrays. This paper aims to extend the linear momentum actuator disc theory (LMADT) to incorporate a large number of aligned or staggered turbine rows with inter-turbine wake mixing. The extension is performed by utilising alternating inviscid and viscous zones to calculate the flow around a turbine and to calculate mixing in the wake, respectively. Using the extended model this study explores the power extraction of a multi-row turbine array for a varying resistance coefficient. This exploration is performed for both aligned and staggered turbine arrangements for a range of blockage ratios and wake mixing rates. Results highlight the importance of the combined effects of blockage and wake mixing, particularly for staggered turbine arrays. The impact of the number of turbine rows and inter-turbine mixing rates on the maximum array performance is also investigated.

Keywords: Actuator Disk Theory, Blockage Effect, Wake Losses, Optimal Turbine Arrangement

1 Introduction

The effect of different turbine arrangements for both wind and tidal farms has been the focus of many studies, and is important in both current and future turbine array design and operation. The optimisation of any one of these layouts will not necessarily be the same for another case. This is due to the range of parameters that impact turbine dynamics, for example, environmental conditions, array size and inter-turbine distances. Thus, we have to consider many designs across the parameter space which can become computationally extensive to perform.

To model all fluid dynamical features of a turbine array would require vast computational resources. Therefore, in order to conduct optimisation of turbine array design across a large parameter space it is useful to make simplifications to the system. One common simplification is to replace the turbine with an actuator disk, which exerts a uniform streamwise resistance to the flow. The actuator disk model has been extended in both tidal and wind turbine contexts. This paper particularly focuses on the additions made by Garret and Cummins [1] (GC07) to model turbines in confined flows.

The foundation of the work in this paper comes from the expansions of GC07 made by Draper, Nishino and Adcock [2] along with the study by Draper and Nishino [3] which compared the efficiency of single row and two aligned or staggered rows of turbines. Their study [3] has shown that staggered arrangements extract more power from the flow when compared to aligned arrangements, although they do not perform as well as a single row with the same number of turbines. This study, however, did not account for streamwise inter-turbine mixing as the upstream conditions for the second row of turbines were determined directly from the downstream conditions of the first row. This paper aims to further develop the aforementioned actuator disk model by introducing an extended analytical quasi-1D model for a multi-row turbine farm with a perfectly staggered and aligned turbine arrangement.

To extend this model a series of alternating inviscid and viscous zones are introduced, with inviscid zones surrounding a turbine row and viscous zones placed in between turbine rows, as seen in figure 1. In practice the formulation for the inviscid region is similar to the formulation by Draper, Nishino and

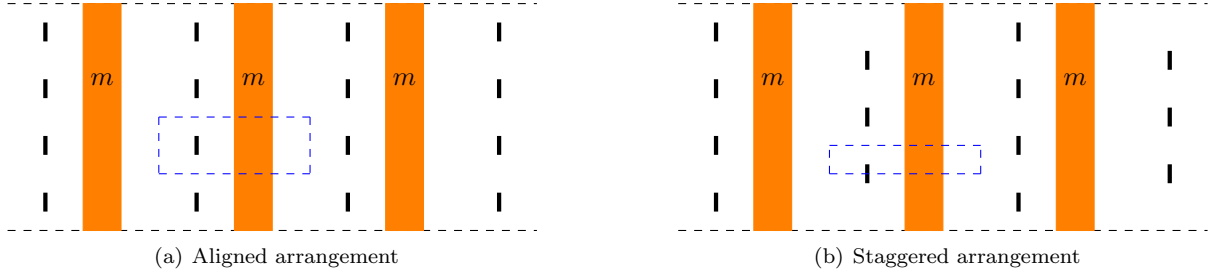


Figure 1: Diagram of four rows of (a) aligned and (b) staggered turbines. The dashed blue square represents the local flow domain depicted in detail in figure 2.

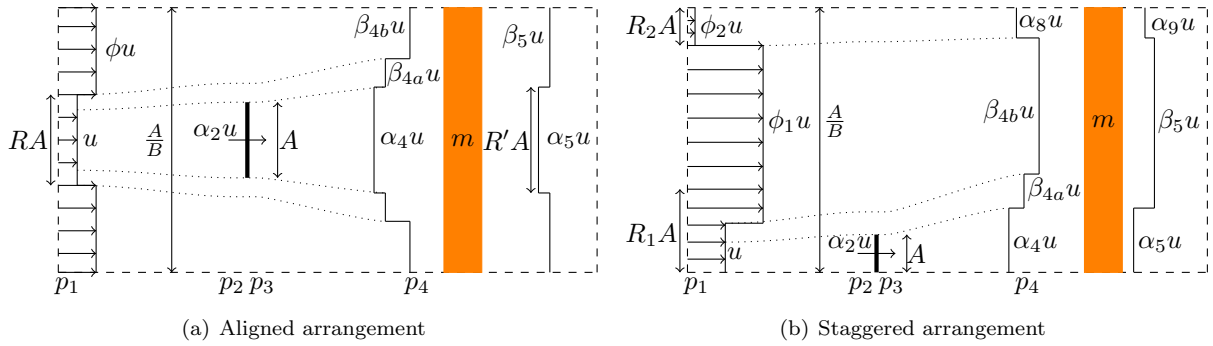


Figure 2: Diagram of the local flow domain for the (a) aligned and (b) staggered geometries.

Adcock [2] but the newly added viscous region acts to mix the wake and the bypass flows downstream of a turbine. A full description of these regions is given in Section 2. This hybrid inviscid-viscous approach was recently proposed by Nishino and Draper [4] for fully periodic arrays. In support of the hybrid inviscid-viscous modelling approach the large-eddy simulations by West and Lele [5] showed the inviscid process is dominant around the disk whereas the viscous process is dominant in the wake.

When examining the power extraction capabilities of a wind farm we often refer back to the seminal studies by Lanchester, Betz and Joukowsky [6] who place the theoretical maximum power coefficient of a wind turbine at 0.593. This limit however can be exceeded within an array of turbines when considering certain array set-ups, like those considered in Nishino and Wilden [7] who placed a higher limiting C_p value at 0.798 for a tidal array and a similar limit can be expected for wind [8]. This increase can be attributed to the effect of local blockage. Even though many of the studies on turbine performance in a channel are primarily targeted at tidal turbines, there is work to suggest that the confined flow model would be applicable to wind as well. This is as, for particularly offshore wind farms, a strong inversion layer in the atmospheric boundary layer (ABL) can act as a rigid lid confining the flow below the inversion layer [9]. Even in cases where the stratification is not large enough to imply a rigid lid we may still calculate an effective blockage ratio [2].

The paper is structured such that in Section 2 the theory is presented along with discussion and validation of the assumptions made in the model. In Section 3 the results of the aligned and staggered cases are discussed. A discussion on the implications of the theoretical predictions made in the model is presented in Section 4, followed by conclusions in Section 5.

2 Theory

We consider many rows of equally spaced turbines, as depicted in figure 1. The rows are considered to be infinitely wide; therefore we consider a local flow domain with symmetric boundary conditions for a single turbine, as shown in figure 2. We consider a uniform inflow for the first row of turbines. The turbines are positioned such that they are perfectly aligned or perfectly staggered. For the aligned case each turbine sits directly behind its upstream turbine, as depicted in figure 1a. For the staggered case

each turbine sits in the bypass flow of the upstream turbine, as depicted in figure 1b.

The velocity at a given streamtube cross-section is defined as a scalar times a reference velocity u (e.g. for $(0 \leq \alpha_4 \leq 1)$ we have that $\alpha_4 u$ is the velocity at the end of near wake). Here the near wake is defined to end at the end of the inviscid zone, at which the pressure has equalised across the local flow domain [3], and far wake starts in the viscous mixing zone. Here the reference velocity u is taken as the velocity upstream of the turbine. The value of u does not need to be specified as the outcome of our analysis, such as power coefficient, will depend only on the velocity ratios between streamtubes.

We define the cross sectional areas as a scalar times the disk area A (e.g. $\beta_{4b} u$ has cross sectional area $A_{\beta_{4b}} = R_{\beta_{4b}} A$). The disk is placed in a local flow domain of cross sectional area A/B where B is the local (cross-sectional) blockage ratio. If the inflow is uniform (i.e. for aligned $R = 1/B$ or $\phi = 1$) we get the classical problem from GC07.

The extension made in this model over previous similar models, such as [3][4], is the introduction of a viscous mixing zone between each turbine row to account for wake mixing, as mentioned in Section 1. We define this wake mixing rate by a constant parameter m ($0 \leq m \leq 1$), which determines the 'completeness' of mixing within each viscous zone. We also assume that the narrow bypass flow immediately outside the core flow is fully mixed with the main bypass flow. This is so that the flow will consistently recover to the same number of streamtubes no matter how many turbine rows it passes through. This allows the problem to be solved, even for a large number of rows, without extending the algebra for a high number of streamtubes. The validity of this assumption will be explored in Section 2.2.

2.1 Formulation for the Staggered Case

Only the full formulation for the staggered case will be presented here to conserve space and as the aligned case is simpler.

Application of the Bernoulli equation along the streamlines of the bypass flow immediately outside the core flow and the main bypass flow, respectively, lead to:

$$p_1 - p_4 = \frac{1}{2} \rho u^2 (\beta_{4a}^2 - 1) \quad (1)$$

$$p_1 - p_4 = \frac{1}{2} \rho u^2 (\beta_{4b}^2 - \phi_1^2) \quad (2)$$

Applying the Bernoulli equation to the bypass flow furthest from the core flow we obtain:

$$p_1 - p_4 = \frac{1}{2} \rho u^2 (\alpha_8^2 - \phi_2^2) \quad (3)$$

Equating both (1) and (2) gives us:

$$\beta_{4b}^2 = (\beta_{4a}^2 + \phi_1^2 - 1) \quad (4)$$

Equating both (1) and (3) gives us:

$$\alpha_8^2 = (\beta_{4a}^2 + \phi_2^2 - 1) \quad (5)$$

Conservation of mass across each streamtube gives us the following cross sectional area relationships:

$$R_{\alpha_8} = \frac{R_2 \phi_2}{\alpha_8}; \quad R_{\alpha_4} = \frac{\alpha_2}{\alpha_4}; \quad R_{\beta_{4b}} = \frac{\phi_1 (\frac{1}{B} - (R_1 + R_2))}{\beta_{4b}}; \quad R_{\beta_{4a}} = (\frac{1}{B} - (R_{\alpha_4} + R_{\alpha_8} + R_{\beta_{4b}}))$$

Conservation of mass across the entire flow gives us:

$$\alpha_4 = \frac{-\beta_{4a} \alpha_2}{R_1 - \alpha_2 - \beta_{4a} (\frac{1}{B} - \frac{R_2 \phi_2}{\alpha_8} - \frac{\phi_1}{\beta_{4b}} (\frac{1}{B} - (R_1 + R_2)))} \quad (6)$$

Again using the Bernoulli equation on the streamlines going through the turbine gives us:

$$p_2 - p_3 = p_1 - p_4 + \frac{1}{2} \rho u^2 (1 - \alpha_4^2)$$

which in combination with (1) gives the following:

$$T = \frac{1}{2} \rho A u^2 (\beta_{4a}^2 - \alpha_4^2) \quad (7)$$

where static equilibrium across the disk has been used to write the disk thrust, $T = A(p_2 - p_3)$. Finally, conservation of momentum across the entire flow gives us:

$$T = \frac{A(p_1 - p_4)}{B} - \rho u^2 A \left[(R_1 - \alpha_2)(\beta_{4a} - 1) + \alpha_2(\alpha_4 - 1) + \phi_1 \left(\frac{1}{B} - (R_1 + R_2) \right) (\beta_{4b} - \phi_1) + R_2 \phi_2 (\alpha_8 - \phi_2) \right] \quad (8)$$

This can be written as:

$$0 = (1 - B)\beta_{4a}^2 - 1 - 2B \left\{ \alpha_2(\alpha_4 - 1) + (\beta_{4a} - 1)(R_1 - \alpha_2) - \frac{\alpha_4^2}{2} + R_2 \phi_2 (\alpha_8 - \phi_2) + \phi_1 \left(\frac{1}{B} - (R_1 + R_2) \right) (\beta_{4b} - \phi_1) \right\} \quad (9)$$

These can then be solved numerically, using inputs $B, R_1, R_2, \phi_1, \phi_2$ and α_2 , to recover the post-turbine velocity profile. The power extracted by the turbine is then

$$P = T \alpha_2 u = \alpha_2 (\beta_{4a}^2 - \alpha_4^2) \frac{1}{2} \rho A u^3$$

We now define the local power coefficient, C_p^* , as the following:

$$C_p^* = \frac{P}{\frac{1}{2} \rho u_{av}^3 A} = \frac{\alpha_2 (\beta_{4a}^2 - \alpha_4^2) u^3}{u_{av}^3} \quad (10)$$

where u_{av} is the average velocity over the cross sectional area. We will adopt the convention of replacing our parameter α_2 with a resistance coefficient for the turbine, $k = (\beta_{4a}^2 - \alpha_4^2)/\alpha_2^2$. We also define the farm averaged C_p^* value, $\langle C_p^* \rangle$, which is the sum of C_p^* at each turbine divided by the number of turbines.

We then calculate $\beta_5, \alpha_5, \alpha_9$ values using mixing between the streamtubes. We define the mixing rate, m , such that at each cross-sectional region $u_{out} = m u_{av} + (1 - m) u_{in}$ where u_{in} is the velocity at the start of the viscous zone, and u_{out} is the velocity at the end of the viscous zone. Due to the assumption that the bypass flow immediately outside the core flow is fully mixed with the main bypass flow we get the following:

$$\beta_5 = m \left(\frac{u_{av}}{u} \right) + (1 - m) \frac{\beta_{4a} R_{4a} + \beta_{4b} R_{4b}}{R_{4a} + R_{4b}}$$

Essentially this is a 3-input-parameter model consisting of B, k and m . It is also worth noting that ϕ_1, ϕ_2, R_1 and R_2 are calculated within the model for each row.

2.2 Instant Mixing Assumption

There is an assumption in this model that the main bypass flow and the bypass flow immediately outside the core flow is fully mixed before it reaches the next downstream turbine. To test this assumption a comparison was made between an aligned case with a single core flow and bypass flow, as depicted in figure 2, and an aligned case in which an additional bypass streamtube is preserved. For clarity, in figure 2, the main bypass flow is the streamtube with velocity $\beta_{4b} u$ and the bypass flow immediately outside the core flow is the streamtube with velocity $\beta_{4a} u$. The formulation of the aligned case with an additional bypass flow (double streamtube case) is similar to the staggered case. To showcase the difference between the single and double bypass cases a comparison of $\langle C_p^* \rangle$ values for fixed k values for 4 turbine rows was constructed. The comparison as plotted in figure 4 shows the relative difference between the two cases, calculated by $(\langle C_p^{*d} \rangle - \langle C_p^{*s} \rangle) / \langle C_p^{*d} \rangle$ at each k value (where the d index indicates double and the s indicates single). The significance of the difference between the double and single streamtube cases varies with both blockage ratio and mixing rate. In figure 4 we observe differences up to 12%, with lesser differences as blockage ratio increases. One consistency across all blockage values is that the

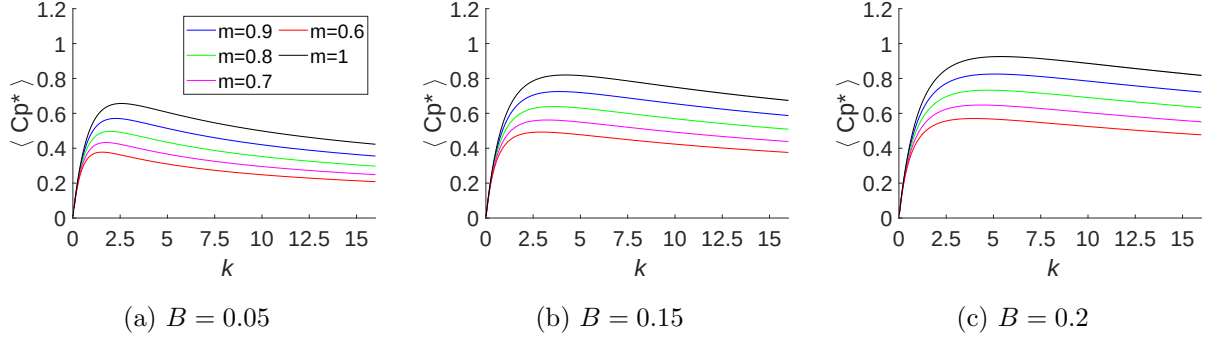


Figure 3: Average local power coefficient $\langle C_p^* \rangle$ for four aligned rows of turbines at three different blockage ratios: (a) $B = 0.05$, (b) $B = 0.15$, and (c) $B = 0.2$. The different colours correspond to different mixing rates.

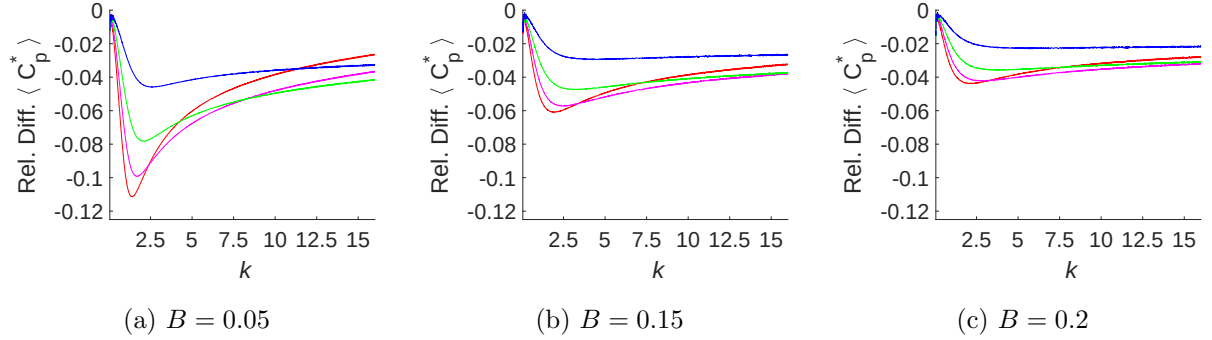


Figure 4: Relative difference in $\langle C_p^* \rangle$, calculated as in section 2.2, between the single bypass case and the double bypass streamtube case for four rows of aligned turbines.

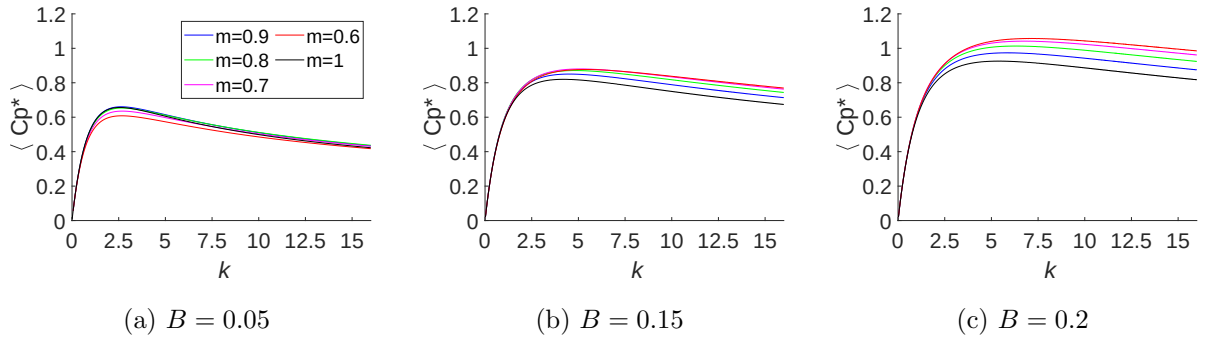


Figure 5: Average local power coefficient $\langle C_p^* \rangle$ for four staggered rows of turbines at three different blockage ratios: (a) $B = 0.05$, (b) $B = 0.15$, and (c) $B = 0.2$.

magnitude of $\langle C_p^* \rangle$ for the double streamtube case is always less than that of the single streamtube case (i.e. $\langle C_p^{*d}(k) \rangle \leq \langle C_p^{*s}(k) \rangle \forall k \in [0, 16]$). This means that the instant mixing assumption tends to overpredict $\langle C_p^* \rangle$.

The k value at which the relative difference is maximised corresponds closely to the point at which the $\langle C_p^* \rangle$ value is maximised. This is more prominent at lower blockage ratios and lower mixing values.

For the aligned case presented in the rest of the paper the double streamtube model is used. It is reasonable to consider that the difference between single and double streamtube is greater than the difference between double streamtube and more complex models with a larger number of streamtubes (i.e. less instant mixing). In support of this a comparison has been made between β_{4a} and β_{4b} for the single

and double streamtube cases. It was found that for the single streamtube case the difference between β_{4a} and β_{4b} was greater than that of the double streamtube cases. If $\frac{\beta_{4b}}{\beta_{4a}} \rightarrow 1$ instant mixing will have a negligible impact on the flow profile. To confirm this it would be useful to construct a model with an additional streamtube (i.e. a triple bypass streamtube case).

The impact of instant mixing for the staggered case is assumed here to be similar to that of the single streamtube for the aligned case. However, due to the staggered configuration the wake of a turbine passes through two mixing zones before it encounters another turbine. For this reason it is likely that, for low blockage ratios, the impact of instant mixing is reduced in the staggered case compared to the aligned.

3 Results

3.1 Aligned case

3.1.1 Combined effects of blockage and mixing on $\langle C_p^* \rangle$

For the aligned case, as expected and can be seen from figure 3, there is an observed increase in $\langle C_p^* \rangle$ values as the inter-turbine mixing increases. This is intuitive as each downstream turbine directly interacts with a faster flow and thus be able to extract more kinetic energy. The effect of mixing rate on $\langle C_p^* \rangle$ appears to have no great qualitative difference between each blockage ratio. For increasing blockage ratio there is also an observed increase in $\langle C_p^* \rangle$ for a given mixing rate. This echoes the work of Garrett and Cummins [1] who gave the analytical formulation for the maximum power coefficient for a given blockage ratio for a single disk ($m = 1$ in this study) as $C_{Pmax} = \frac{16}{27} \left(\frac{1}{1-B} \right)^2$. It is worth noting as well that as $B \rightarrow 0$ the system approaches the unconfined case in which $C_{Pmax} = \frac{16}{27}$, the Lanchester-Betz-Joukowski limit.

Another feature that can be observed as blockage ratio varies is that the shape of the $\langle C_p^* \rangle$ curve, in figure 3, alters. For lower blockage ratios there is a more pronounced peak $\langle C_p^* \rangle$, compared to high blockage ratio cases. It can be concluded from this that the operational range of k for optimal or near-optimal $\langle C_p^* \rangle$ is much narrower for the low blockage ratio case, most notably with low mixing rate, when compared to the high blockage ratio case. The optimal k value is also observed to reduce as mixing rate decreases. In essence the change in optimal k value as mixing rate changes is down to the relationship between u^3/u_{av}^3 and the other terms in (10).

3.1.2 Effect of the number of rows on $\langle C_p^* \rangle$

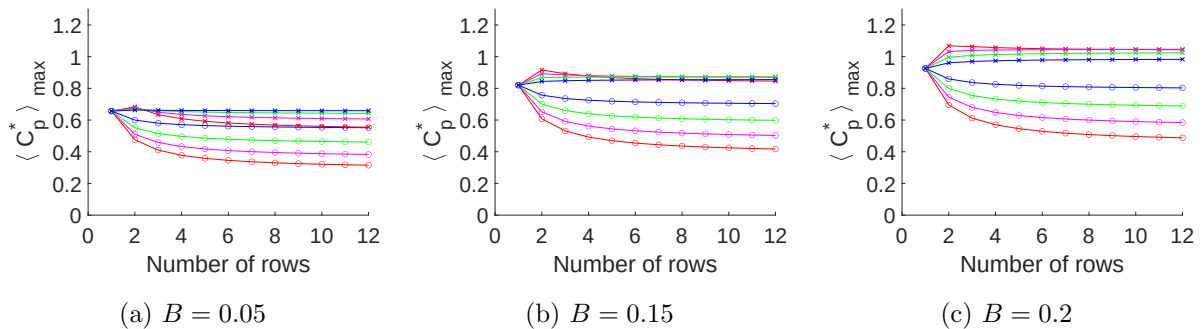


Figure 6: The maximum $\langle C_p^* \rangle$ value achieved by a given number of rows with a fixed k value for the whole array. The curves with crosses (x) and circles (o) are for the staggered and aligned cases, respectively.

Figure 6 shows how the maximum of the farm-average power coefficient, $\langle C_p^* \rangle_{max}$, approaches an equilibrium value (for a given set of B and m) as we increase the number of rows from 1 to 12. It is worth noting that systems with higher mixing rates are observed to converge earlier than those with low mixing rates. There is, however, no significant impact of the blockage ratio on this convergence. This is not to say there is no difference between the pre-converged $\langle C_p^* \rangle$ values at different blockage ratios. Note, the plot for $m = 1$ has been omitted here as for both staggered and aligned arrangements it is in the converged state immediately as they return to the GC07 case of a turbine in a channel of uniform inflow. The maximum $\langle C_p^* \rangle$ value for the $m = 1$ case is exactly the first point in each of figures 6a, 6b and 6c.

3.2 Staggered case

3.2.1 Combined effects of blockage and mixing on $\langle C_p^* \rangle$

In contrast to the aligned case, the staggered case does not, for all three blockage ratios, attain a maximum C_p^* value when $m = 1$, as seen in figure 5. It is worth noting, however, that in figure 5a the difference in maximum $\langle C_p^* \rangle$ between $m = 1$ and $m = 0.9$ is negligible. The optimal m can be seen as dependent on the blockage ratio B . Higher blockage ratio cases tend to show lower mixing values as optimal. In contrast, low blockage ratios tend towards the ordering observed for the aligned case in figure 3, with $\langle C_p^*(m_1) \rangle < \langle C_p^*(m_2) \rangle$ for $m_1 < m_2$. This can be understood intuitively as the upstream turbine diverts flow into the bypass flow and, unless the blockage ratio is negligibly small, the bypass flow speeds up, allowing the downstream turbine to extract more energy. Therefore, in cases with lower mixing values the higher velocity bypass flow approaching a downstream turbine is preserved, whereas a mixing value of $m = 1$ leads us back to the GC07 case. Another contrast to the aligned case is that the optimal k value required to maximise $\langle C_p^* \rangle$ increases as mixing rate decreases.

3.2.2 Effect of the number of rows on $\langle C_p^* \rangle$

The effects of this enhanced bypass flow can further be seen in figure 6. There is a peak in maximum $\langle C_p^* \rangle$ at two rows of turbines, and this peak is clearly more prominent in certain cases. As can be expected intuitively, a greater peak is observed for lower mixing values as it preserves the higher velocity bypass flow. As for the effect of blockage ratio it is expected that a higher blockage ratio will cause a greater peak. This is as, by conservation of mass and momentum, the flow diverted into the bypass will have a greater effect on the acceleration of the bypass flow.

4 Discussion

4.1 Implications of the Study

The theoretical analysis presented above is to model the 'internal' sub-problem of a wider turbine array modelling problem. Focusing on wind energy research, due to the complex multi-scale interaction between wind farms and the atmospheric boundary layer (ABL) it is often necessary to split the problem into two coupled sub-problems, the 'external' problem and the 'internal' problem [10]. The external problem is primarily composed of meso-macroscale geophysical flows and wind-farm scale flow patterns. The internal problem is primarily composed of turbine design, operating conditions and turbine arrangement. The value of u_{av} in this study, which is used to define the 'local' (or internal) power coefficient C_p^* , depends on the external problem. The full effects of the external problem on the model presented here are outside the scope of this paper. However, it is worth noting that the optimal k values to maximise the 'global' power coefficient is expected to be lower than the values obtained in this paper as some decrease in u_{av} due to the global blockage effect is inevitable [10].

This study has primarily concerned itself with the differences in performance for an array consisting of four turbine rows. Although the maximum $\langle C_p^* \rangle$, and the corresponding optimal fixed k for the entire farm, have been explored for varying number of rows, there is yet to be an assessment of varying k operating conditions across the farm.

As mentioned in Section 2.2, it is assumed that the staggered case will be impacted by instant mixing to a similar extent as the aligned single bypass case. For the staggered case, this assumption is expected to hold best for low blockage ratios where the system behaves similar to that of the aligned case with additional inter-turbine mixing. It is still unclear how the impact of instant mixing will be for the staggered case at higher blockage ratios.

The use of actuator disc models for quantitative insight should be considered as, most likely, an upper bound for the performance. This is as actuator discs are an approximation of ideal turbines and do not include losses associated with, for example, tip and hub vortices generated by real turbine rotors.

4.2 Limitations and Future Work

One assumption made in this model that can be improved upon in future studies is that of the constant mixing value m . Currently the value of m is not linked to the flow physics and the whole flow is mixed at

the same rate. In reality different streamtubes would mix at different rates and this depends on factors such as turbulent intensity and the velocity ratio between the flows.

5 Conclusions

In conclusion a study into the performance of an array of turbines with varying blockage ratios, local resistance coefficients and inter-turbine mixing rates was performed using an extended inviscid-viscous actuator disk model. The study is relevant to the design of both wind and tidal-stream turbine arrays.

It was shown that the performance of an aligned and staggered array is highly dependent on inter-turbine mixing rate, m , and blockage ratio, B . The greatest array performance was achieved at the highest blockage ratio. The performance of an aligned array was also maximised for higher mixing rates, with the GC07 case as the theoretical maximum. Although the low blockage ratio staggered case (C_p^*) is maximised for high mixing rates, when blockage ratio is increased the lower mixing rates start to outperform higher mixing rates. This is due to the effect of an accelerated bypass flow which is best preserved for low mixing values. The second row of staggered turbines outperform all other turbine rows as they benefit from the local inflow with highest velocity.

The validity of the instant mixing assumption utilised in the new model was also investigated for the aligned case. It was found that the impact of the instant mixing assumption increases for low blockage ratios and lower mixing rates. The impact decreases as more bypass streamtubes are explicitly calculated in the model. The effect of instant mixing on the staggered arrangement is likely similar at low blockage ratios to the effect in the aligned case. However, effect at high blockage ratios, at which lower mixing rates are expected to be beneficial, is still unclear and will require further study.

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