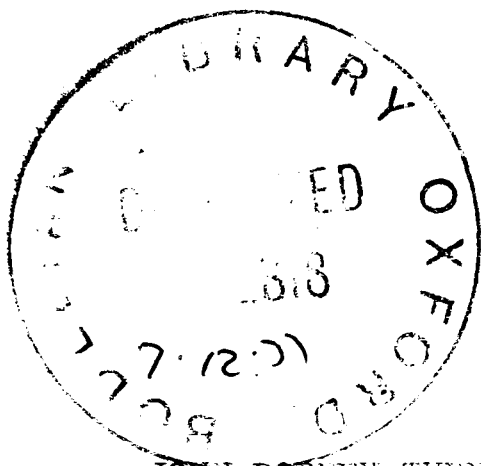


A
VARIATIONAL SOLUTION
OF THE
FRICTIONAL UNLOADING PROBLEM
IN
LINEAR ELASTICITY



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JUNE 1977

.....Brevis esset abor,
Obscurus fio.....

It is when I am struggling to be brief, that I become
unintelligible.

HORACE Ars Poetica 25

Civilised man must feel that he belongs somewhere in
Space and Time, that he consciously looks forward and
back. And for this purpose it is a great convenience
to be able to read and write. .

KENNETH CLARK Civilisation

To my Parents

abstract

A numerical solution is obtained for the frictional unloading problem, of a circular cylindrical rigid punch being removed from a homogeneous, isotropic, incompressible, linear elastic half-space, when conditions of Coulomb friction act during both loading and unloading. Considering a variational formulation of the elasticity equations, it is shown that at the solution the complementary energy achieves a minimum within the subset of admissible stress states; that is those that obey the equilibrium equations and the boundary conditions on stress. By considering the general traction boundary value problem, the complementary energy is reduced to a quadratic functional of the unknown surface stresses alone, the frictional boundary conditions acting as linear constraints on the minimization. It is then shown that this reduces to a fairly simple analytical form for the stated problem. It is required to find a numerical solution of the frictional unloading problem, but first some known solutions to the loading problems in the frictionless, adhesive, and frictional cases are given, all on incompressible half-spaces. A numerical solution, based on a Finite Element approximation to the above energy principles, is then obtained, not only for the unloading problem, but also for the above loading ones. The solution for the three loading problems is known, the numerical solution comparing favourably, and that for the unloading problem confirms the prediction from a physical argument. During the derivation of the above energy principles, a

solution to the general traction boundary value problem is obtained, using the Airy stress function. This is used to calculate the entire stress/displacement state from the numerical solutions. The results obtained for the radial stress are consistent with previously obtained empirical results.

A detailed description of the implementation of the numerical method, and the structure of the FORTRAN programme is given, and convergence of the numerical technique proved. Finally a brief attempt at an analytical solution is made, and some results obtained which further confirm the numerical ones.

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Chapter One

Introduction

§1.1 The Historical Background and Motivation of this Work

The study of the mechanics of contact of deformable bodies is a topic which has been of some interest for now well over a century, such problems arising in Civil, Mechanical, and Biomedical Engineering. The more readily useful results are usually obtained by experimentation, some initial studies being done on rolling friction by REYNOLDS (1875), and now continuing in many fields of engineering. On the other hand, a mathematical solution can be helpful to gain some theoretical insight into the empirical results, and thus used to expand on them. JOHNSON ET AL. (1973) give an example of a way in which one hopes to produce a theoretical explanation of an apparent anomaly in the empirical results. In fact it is towards explaining the observed behaviour on unloading that this thesis is directed. To achieve any progress with a mathematical solution it is necessary to make some fairly broad assumptions about the geometry of the problem, and about the mechanical properties of the bodies. It is normal to assume that the contact is either two-dimensional, or axi-symmetric, and in this thesis the latter assumption is made. Further only problems of normal contact, in which the bodies are pressed together along their common axis, are considered. The assumptions about material properties depend upon the information required, and since this thesis is concerned mainly about the effect of unilateral boundary conditions, it is assumed that the bodies are composed of a linearly elastic, homogeneous, isotropic material. Any of these assumptions could however be relaxed to study its effect on the contact.

The history of the mathematical solution of contact problems, mixed boundary value problems in linear elasticity, began with the famous paper by HERTZ (1881), in which he solved the problem of pressing two elastic spheres together along their common axis, under the assumption that they are both made of the same material, that they are both incompressible, or that the contact is frictionless. In all three cases no shear is transmitted across the interface, and thus it can be shown that after a suitable transformation, the problem is equivalent to pressing a frictionless, rigid, circular cylindrical punch into an elastic half-space, (see for example TURNER (1975)). The solution for the normal stress distribution, or contact pressure, $p(r)$, acting over the surface of the stamp in this case, is a standard result, and after suitable non-dimensionalisation, is

$$p(r) = \frac{(2/\pi)}{\sqrt{1-r^2}} \quad 1.1$$

where r is the distance from the origin, the radius of the punch is unity, and the total normal force, or compliance is $P^* = 2\pi \int_0^1 p(r) r dr = 4$.

Experimental evidence however suggests that the frictionless assumption, in the case of contact between bodies of dissimilar materials, is quite unrealistic. So as a second bound on the solution the case of adhesive contact, in which shear stress acts over the contact surfaces to prevent all relative movement between the bodies at points where they are in contact, was considered. MOSSAKOVSKI (1954) solved the problem of adhesive contact between the half-space and the cylindrical punch, and then in (1963) using an incremental method obtained the solution for the adhesive contact between spheres as a quadrature of this. GOODMAN (1962) found the solution, also as a quadrature, in the incompressible limit, Poisson's ratio $\nu \rightarrow \frac{1}{2}$, using a similar incremental argument, and finally using the above ideas SPENCE (1968) formulated a similarity method to obtain the solution. The non-dimensionalised solution for the axi-symmetric punch gives the following values for the normal pressure, $p(r)$, and shear stress, $q(r)$, (see SPENCE

(1975)),

$$\begin{bmatrix} p(r) \\ q(r) \end{bmatrix} = \left(\frac{2K}{\pi\gamma} \right) \cosh\left(\frac{\pi K}{2}\right) \int_0^1 \begin{bmatrix} \cos K Q_0(\omega) \\ (\omega/r) \sin K Q_0(\omega) \end{bmatrix} \frac{d\omega}{(1-\omega^2)\sqrt{r^2-\omega^2}} \quad 1.2$$

where $Q_0(\omega) = \frac{1}{2} \ln \left(\frac{1+\omega}{1-\omega} \right)$; $K = \frac{1}{\pi} \ln \left(\frac{1+\gamma}{1-\gamma} \right)$; $\gamma = \frac{1-2\nu}{2-2\nu}$

the punch has unit radius, the total normal force $P^* = 4(\pi K/2\gamma)$, and ν is Poissons ratio.

The incompressible limit obtained by GOODMAN (ibid) for the circular punch, and given again by SPENCE (ibid) is

$$p(r) = \frac{2/\pi}{\sqrt{1-r^2}} \quad ; \quad \frac{q(r)}{\gamma} = \frac{p(r)}{\pi r} \ln(1-r^2)^{-1} \quad 1.3$$

Considering the ratio, $\theta(r) = (q(r)/\gamma)/p(r)$ in (1.2), (1.3) above, it is seen to be divergent, and in (1.2) oscillatory, behaving like $\ln(1-r^2)^{-1}$ at the outer edge of the contact region. This is also a fairly unrealistic assumption, but worse, the boundary conditions for both the frictionless and adhesive problems are reversible, and thus the solutions are the same during both quasi-static loading and unloading, P^* increasing and then decreasing monotonically. This is far from the case, as seen by the observations on the fracture of brittle materials, JOHNSON ET AL (ibid), and thus one must allow for the possibility of frictional, or unilateral, boundary conditions.

A physically more realistic assumption is one of Coulomb friction, in which the magnitude of the shear stress, $q(r)$, cannot be greater than some proportion, ρ , called the coefficient of Coulomb friction, of the contact pressure, $p(r)$, (which itself must be positive for contact to be maintained). Since it is found that all the essential physical principles are maintained, and the results inherently the same in character whether contact between spheres, or contact between a punch and a half-space are considered, in the sequel, except in the final chapter, only the latter will be considered.

This greatly simplifies the analysis, and the formulation of the variational method, without losing any of the interesting features of the results. Then during any small change in the stress state, represented by a small change in the normal displacement, $dw(r) = \delta$, a constant, of the punch, the half-space will adhere to the punch if the shear is less than the above value, and slide in a direction opposite to the shear if it is equal. The change in horizontal surface displacement can be split into positive and negative parts, and the above conditions expressed mathematically as

$$\begin{aligned} du_0(r) &= -u_1(r) + u_2(r) \quad ; \quad u_1(r) \geq 0 \quad ; \quad u_2(r) \geq 0 \\ -\rho p(r) &\leq q(r)/\gamma \quad \quad \quad q(r)/\gamma \leq \rho p(r) \quad \quad \quad 1.4 \\ u_2(r)(q(r)/\gamma + \rho p(r)) &= 0 \quad \quad \quad u_1(r)(q(r)/\gamma - \rho p(r)) = 0 \end{aligned}$$

This represents the idea that there will be no displacement, $u_1(r) = u_2(r) = 0$ if $q(r)/\gamma$ lies in the open set $-\rho p(r) < q(r)/\gamma < \rho p(r)$, that sliding will be inwards, $u_1(r)$ non-zero if the upper constraint is satisfied, and outwards, $u_2(r)$ non-zero, if the lower is satisfied. Since for the punch $p(r)$ must always be strictly positive in the contact area, $u_1(r)$ and $u_2(r)$ can never be non-zero together. (The scaling $1/\gamma$ is made because for the incompressible limit to be considered in the sequel $q(r)$ is $O(\gamma)$, and thus $q(r)/\gamma$ remains finite).

The constraints (1.4) are unilateral, irreversible, and thus the loading, and unloading problems will be different in nature.

By inspection of the above adhesive solutions it can be deduced that in the frictional loading problem, the contact region will divide into an inner adhesive region, $u_1(r) = 0$, $q(r)/\gamma < \rho p(r)$, and of unknown radius c , and an outer annulus of outward sliding, $u_1(r) > 0$, $q(r)/\gamma = \rho p(r)$, $c \leq r \leq 1$. Assuming this regime one can then find solutions for $p(r)$, $q(r)$ in terms of the eigenvalue, c , and choose $c = c(\gamma, \rho)$ as that value which gives bounded stress at $r = c$.

SPENCE (1973) found the general analytic solution for the two-dimensional problem, (1975) an analytic solution for the incompressible axi-symmetric problem, and (ibid) a numerical solution for both general problems, based on the Abel equations. The solution he obtained for the incompressible axi-symmetric case was $p(r)$ as in (1.1), and the shear

$$\frac{q(r)/\gamma}{\rho p(r)} = 1 - \psi(r, c) \quad 1.5$$

$$\text{where } \psi(r, c) = \begin{cases} \Lambda_0(\sin^{-1} \frac{\omega}{c}, \sqrt{1-c^2}) - \frac{2\omega K'(c)}{\pi r} + \frac{2}{\pi \rho r} Q_0(\omega) & 0 \leq r \leq c \\ 0 & c \leq r \leq 1 \end{cases}$$

$$\omega = \sqrt{\frac{c^2 - r^2}{1 - r^2}}$$

and c is related to ρ by the implicit relation

$$\rho = \frac{1}{2cK'(c)} \ln \left(\frac{1+c}{1-c} \right) \quad 1.6$$

$K(k)$ is the complete elliptic integral of the first kind, and $\Lambda_0(\beta, k)$ is the Heumann lambda function, (see BYRD AND FRIEDMAN (1954)). Appendix A contains a derivation of (1.5), (1.6), based on the Abel equations, and given by SPENCE in a private communication.

The shear profile in the unloading phase is however not quite so immediately obvious. One might think that the unloading profile would be similar to that obtained by MINDLIN AND DERESIEWICZ (1953) for problems of sideways shear, (referred to as the Mindlin unloading profile in the sequel). In their problem, the above authors demonstrated that the solution could be obtained as an inner region of adhesion, $u_2(r) = 0$, with an ever increasing annulus of outward slip, $u_2(r) > 0$, there being no inward sliding, $u_1(r) = 0$, on the entire contact area. However in a private communication, the author's supervisor has shown that any solution of this form, that gives bounded shear stress in the contact area, except at $r = 1$, cannot be made to satisfy both frictional constraints at the inner edge of the original region of inward slip, $r = c$, and thus it can be deduced that further

inward sliding must occur. SPENCE'S proof is presented in Appendix A.

Moreover there is an extremely fundamental reason why the Mindlin profile cannot be accepted. Introducing now the unloading parameter λ , which is unity at the loading solution and reduces to zero during unloading, then $\lambda = 1$ represents the loading solution, a given horizontal displacement, $u_0(r)$, supported by the stress state (1.5), with $q(r)/\gamma = \rho p(r)$ at the outer edge of the contact region. At some later stage, $\lambda < 1$, the Mindlin profile now requires that $q(r)$ satisfy the lower limit, $q(r)/\gamma = \rho p(r)$, at the edge of the contact region. Working backwards through the stress states required to reach this point, it is seen that this requires a step change in $q(r)/\gamma$ to occur at $\lambda = 1$, within the vicinity of the outer radius $r = 1$. Not only is this physically unacceptable, but since no sliding has occurred by $\lambda = 1$, yet this step change has, the Mindlin profile violates uniqueness, and therefore cannot represent the unloading regime. Not only does this uniqueness argument rule out the Mindlin profile but also any other profile in which regions of outward slip immediately form. The reason MINDLIN AND DERESIEWICZ could obtain such a solution is that in their problem the normal stress is zero at the edge of the contact area, the latter remaining constant in size during loading and unloading, for both the punch and the sphere. Thus a solution must be found in which a region of adhesion forms before any outward sliding occurs.

Following the advice of MINDLIN AND DERESIEWICZ (ibid) the adhesive problem might therefore be solved, starting with the frictional displacements, to deduce the unloading regime, but this proves analytically very cumbersome, and unnecessary since the regime can be deduced from the following heuristic argument.

During the loading phase, points in the surface of the half-space will want to adopt a displacement as in the frictionless Hertzian problem. Points far from the contact region will see only a point load at the origin and thus by St. Venant's principle adopt a full inward Hertzian displacement,

a tensile radial stress developing to cause this, and being balanced by a compressive hoop stress. Frictional forces will however limit the displacement in the vicinity of the contact area, points in the adhesive region will remain where they are, and those in the sliding region will take up only a portion of their full frictionless value. The compressive radial stress set up by the frictional forces will also limit the inward movement of points just outside the contact area, and as a result the maximum tensile radial stress will occur not at the edge of the contact region as predicted from frictionless theory, but a short distance out. This is in fact the behaviour observed in experiments, JOHNSON ET AL (ibid), and supported by the analytic solution (1.5).

Now if the load is reduced slightly, points in the vicinity of the contact area will still be short of their Hertzian displacement, even at the new load, so the last thing one expects is for them to move out against an inward frictional force. In fact points at the inside edge of the contact area will not start moving outwards until points far from the contact region have moved outward enough to create a tensile radial stress at the outside edge of the contact area, sufficient to overcome friction and pull them out. Thus in the initial stages of unloading points at the edge of the contact area will no longer have such a tendency to move in, and thus stay where they are, an outer region of adhesion developing from the outside edge. But the resulting rearrangement of shear will allow points further in to take up more of their Hertzian value, and in fact points that were in the adhesive region will now start to move inwards, the region of inward slip moving towards the origin.

As unloading proceeds, points at the inner edge of the contact area will reach a stage when the tendency is for them to move out, but the very friction that stopped them moving in will stop them moving out, until a further point is reached, when friction will be insufficient, an annulus of outward slip developing at the edge of the contact region. However if

the half-space is made of a brittle material it may fracture at the edge of the contact region, as a result of the tensile radial stress required to support the region of outward slip, predicting the further cracking observed by JOHNSON ET AL (ibid).

As unloading proceeds, the regions of inward and outward slip, and inward slip and the origin, will always be separated by regions of adhesion, but these and the annulus on inward sliding will shrink to the origin as that of outward sliding expands to fill the entire contact area.

It was the author's wish to produce a mathematical basis for this, and indeed to support the above heuristic profile with some form of solution, either analytic, or numerical. In fact a numerical solution, based on a variational formulation, proved relatively straight forward, when in writing the boundary conditions in the form(1.4) it was noticed that the displacement terms $-u_1(r)$, $u_2(r)$ are Lagrange Multipliers on the frictional constraints. In fact an argument based on Lagrange multipliers, using the fact that they cannot be non-zero together, since the constraints never cross, and in which unloading causes a small change in the "cost functional", the complementary energy, leads to a prediction of the same unloading profile. This was earlier reported by the author, TURNER (1975).

The development of this variational formulation, together with the presentation of some numerical results for the loading and unloading problems, obtained from a finite element approximation to the extremum principles for the flat punch, form the major portion of this thesis.

A full analytic solution proved impossible for reasons outlined in Appendix A, but attempts at an asymptotic solution close to the edge of the contact region were made, and are reported in the final chapter.

§1.2 Synopsis.

This thesis concentrates on the solution of mixed boundary value problems in linear elasticity. In the first section of the next chapter, the equations of linear elasticity will be presented, together with some boundary conditions of interest, and the solution derived in the sequel. It was shown in an earlier work by the author, TURNER (1975), how these abstract problems arise from the formulation of physically motivated problems in contact mechanics. The physical model will always be born in mind for the insight it provides, as in the last section, and so a short discussion of contact mechanics will be given in the second section of the next chapter. However a full discussion of the physical model, the formulation of the mathematical problem, and the assumptions involved, and the review of the literature in contact mechanics, will be left to the earlier work. What is of interest here is the solution of a mathematical problem. Any weaknesses in the model do not invalidate the solution as a solution of the mathematics, they merely restrict its validity as a description of the physical situation.

After this introduction, the thesis falls into two parts. The first, Chapters Two to Five, contains the main analytical and numerical results, while the second, Chapters Six and Seven, the details of the numerical method. There is also an eighth conclusive chapter.

In Chapter Two, the equations of a homogeneous, isotropic, linear elastic solid are presented, as well as six types of boundary condition, of both unilateral and bilateral kind, which together describe the five basic mixed

boundary value problems to be solved. Their relation to problems in contact mechanics is also discussed. It is shown how these equations and boundary conditions can be formally represented by self-adjoint operator theory, and after a brief discussion of complementary variational principles, a variational formulation of these abstract problems is derived. It is shown that the solution minimizes the complementary energy, which can be written in terms of the surface stresses alone, with unilateral boundary conditions acting as constraints on the optimization. Existence and uniqueness of the solution are discussed.

Very little can be done with the general problem, so the axi-symmetric limit on a half-space is obtained. First the solution of the traction boundary value problem is calculated, using the Airy stress function, and expressions for the surface stress and displacements, in terms of the contact pressure and shear, derived. Axi-symmetric mixed boundary value problems are then considered, known solutions to some classic problems being presented. These are the Hertz frictionless solution, the adhesive solution, the frictional loading solution, results being given for the incompressible limit. The frictional unloading problem, which cannot be solved by the same methods as the loading problem, is to be solved by a variational approach, so the complementary energy principle is presented for all the above mixed boundary value problems. The principles for the bilateral cases lead to linear integral Euler equations, the equations which are solved by the normal methods, whereas the unilateral problems have linear constraints on the optimization, giving quadratic programming problems. Further the functional at each stage of the unloading phase is obtained from that of the previous stage, by only a small change in the linear term, and thus the unloading regime can be predicted before the numerical results are obtained in Chapter Four. That predicted is the same as by a physical argument in §1.1. In the last section of Chapter Three, singular function expansions of the kernels in the Euler equations are obtained, because

these can be used not only to derive analytic solutions, but also to discuss the convergence and the stability of the numerical method introduced in Chapter Four.

In Chapter Four, the energy principles associated with the four problems to be solved are summarised. These are STEP 1, the frictionless solution, STEP 2, the incompressible adhesive problem, STEP 3, the frictional loading problem, and STEP 4, the frictional unloading problem, the last two also in the incompressible limit. A Rayleigh-Ritz/Finite-Element solution, which reduces the infinite dimensional optimization problems to finite dimensional approximations, is introduced. STEPS 1 and 2 become problems in finite dimensional calculus, and STEPS 3 and 4 similar quadratic programming problems. There exists algorithms to solve these, and hence a numerical solution is obtained not only for the problems for which there is already an analytical solution, STEPS 1 - 3, but also for STEP 4, for which there is no previous solution. Very close agreement was obtained between the results for STEPS 1 and 3, and the known analytical solutions, which have square root singularities, while for STEP 2, which has a stronger singularity, agreement was almost as good, except close to the singularity. Those obtained for STEP 4 not only confirm the predicted unloading profile, but produce acceptable stress values.

In Chapter Five, it is shown that the numerical solution can be used to calculate the stress and displacement fields within the body, through the Airy stress function solution. Expressions are obtained for the radial and hoop stresses in terms of the contact stresses, and it is shown how these reduce to simple matrix expressions in the discrete case. The motivation is to find the radial stress distribution during loading and unloading, which is done, because it is this which controls fracture of the half-space. Results are presented for the numerical profiles of Chapter four, and it is shown that these are consistent with observed behaviour.

The second part deals solely with the numerical method. Chapter Six covers the mechanics of its implementation. Finite dimensional forms are derived for linear transformations of the contact stresses, that is their Hankel transforms, and the compliance, in terms of the surface stress vectors, and these are then used to derive the compliance matrices contained in the quadratic term of the approximate complementary energy. It is also shown how they can be used to derive the approximate expressions for the surface stresses and displacements, through the Airy stress function solution. In the last section the structure of the FORTRAN programme is described.

Convergence and stability of the numerical method are discussed in Chapter Seven. Convergence is proved first for the unconstrained problems, using a standard technique, and this is then extended to include the constrained problems. It is then shown in the last section how two instabilities that can enter the solution can affect the choice of element mesh.

There is a brief final chapter, in which a summary of the major results is given, and an attempt at an analytical solution described, and four appendices.

PART ONE

Chapter Two

ON MIXED BOUNDARY VALUE PROBLEMS
IN LINEAR ELASTICITY AND RELATED
COMPLEMENTARY VARIATIONAL PRINCIPLES

§2.1 The Equations of Linear Elasticity.

The equilibrium equations of linear elasticity relate two second order symmetric tensor fields, and a vector field, namely the stress $\{\sigma_{ij} \mid \sigma_{ij} = \sigma_{ji}\}$, the strain $\{\epsilon_{ij} \mid \epsilon_{ij} = \epsilon_{ji}\}$, and the displacement $\{u_i\}$, defined over a fixed region V of Euclidean point space, which has Cartesian components $(x_1, x_2, x_3) = (x, y, z)$. The boundary conditions define the value or relation between some of the components of the above fields over the boundary, ∂V , of the region. One then asks whether or not the boundary conditions supply necessary and sufficient conditions for the solution to exist, and be uniquely determined, and if so, one then seeks to find that solution.

A derivation of the equations can be found in any of the following sources, LOVE (1922), TIMOSHENKO AND GOODIER (1954), or GREEN AND ZERNA (1953), and are the definition of strain

$$\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \epsilon_{ij} \quad 2.1$$

(the normal summation convention to apply), the equilibrium equations

$$- \frac{\partial \sigma_{ij}}{\partial x_j} = f_i \quad 2.2$$

where $\{f_i\}$ is a defined vector field, the body force, which will be taken as zero in all applications in this work, and the relation between stress and strain derived from the thermodynamics

$$\sigma_{ij} = \frac{1}{2} \left(\frac{\partial \psi}{\partial \epsilon_{ij}} + \frac{\partial \psi}{\partial \epsilon_{ji}} \right) \quad ; \quad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial \phi}{\partial \sigma_{ij}} + \frac{\partial \phi}{\partial \sigma_{ji}} \right) \quad 2.3$$

where ψ is the Helmholtz free energy, (strain energy), and ϕ the Gibbs free energy, (complimentary energy). For a linearly elastic isotropic homogeneous body, with respect to an unstrained configuration

$$\begin{aligned} \psi &= c_{ijkl} \epsilon_{ij} \epsilon_{kl} & ; & & c_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \frac{1}{2} \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\ \phi &= d_{ijkl} \sigma_{ij} \sigma_{kl} & ; & & d_{ijkl} &= -\frac{\nu}{E} \delta_{ij} \delta_{kl} + \frac{1+\nu}{4E} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned} \quad 2.4$$

In the above i, j, k, l take the values $1, 2, 3$; λ, μ are the Lamé constants; $\nu = \lambda / (2\lambda + 2\mu)$, Poissons ratio; $E = 2\mu(1+\nu)$, Young's modulus; and μ the shear modulus G .

The boundary conditions considered in this thesis fall into six types, defined on six disjoint subsets $\partial\mathcal{V}_m$, $m=0, 5$, of the boundary $\partial\mathcal{V}$.

$$\partial\mathcal{V} = \bigcup_{m=0}^5 \partial\mathcal{V}_m \quad \text{such that} \quad \partial\mathcal{V}_m \cap \partial\mathcal{V}_n = \emptyset \quad m \neq n$$

Let n_i be the normal to the surface, $\partial\mathcal{V}$, at the point $x_j \in \partial\mathcal{V}$, then the traction vector, t_i , acting on the surface is defined as $t_i = \sigma_{ij} n_j$. Further any vector v_i , at the surface has normal component, $v_i^N = v_j n_j n_i$, $|v_i^N| = v^N = v_j n_j$, and tangential component, $v_i^T = v_i - v_i^N$, $|v_i^T| = v^T$.

Because quasistatic loading and unloading problems are of interest, the body, $\mathcal{V}$, is assumed to be in an initial known equilibrium stress state, σ_{ij}^0 , obeying (2.1)-(2.4), then undergoes a change in stress state, $d\sigma_{ij}$, also obeying (2.1)-(2.4), to take up a new equilibrium stress state, σ_{ij} , yet again obeying the equations. Thus if

$$\begin{aligned} \sigma_{ij} &= \sigma_{ij}^0 + d\sigma_{ij} & ; & & \epsilon_{ij} &= \epsilon_{ij}^0 + d\epsilon_{ij} & ; & & u_i &= u_i^0 + du_i & & x_k \in \mathcal{V} \\ t_i &= t_i^0 + dt_i & ; & & u_i &= u_i^0 + du_i & & & & & x_k \in \partial\mathcal{V} \end{aligned} \quad 2.5$$

the boundary conditions specifying σ_{ij} , $d\sigma_{ij}$, are four of the bilateral type, and two of the unilateral type. Since the equations are linear, without any loss of generality for the bilateral conditions one may take $\sigma_{ij}^0 \equiv 0$, $d\sigma_{ij} \equiv \sigma_{ij}$, and these conditions then become for

$x_j \in \partial\psi_0$ the surface traction vector $t_i(x_j)$ is given, $t_i(x_j) = p_i(x_j)$, 2.6

$x_j \in \partial\psi_1$ the surface displacement $u_i(x_j)$ is given, $u_i(x_j) = u_{oi}(x_j)$, 2.7

$x_j \in \partial\psi_2$ the tangential traction $t_i^T(x_j)$, and normal displacement $u^N(x_j)$ 2.8
are given, $t_i^T(x_j) = q_i(x_j)$, $u^N(x_j) = -w_o(x_j)$,

$x_j \in \partial\psi_4$ the normal traction $t^N(x_j)$, and tangential displacement $u_i^T(x_j)$ 2.9
are given, $t^N(x_j) = -p(x_j)$, $u_i^T(x_j) = u_{oi}(x_j)$.

The two unilateral cases in contact mechanics correspond to frictional loading and unloading, in which the normal displacement is first defined as being monotonically increasing, and then decreasing. The problems considered also correspond to ones in which, in contact terms, there is no relative tangential rigid body motion of the two bodies. Thus the tangential displacement is zero at the centroid of $\partial\psi_4, \partial\psi_5$. If these regions have local coordinates (θ_1, θ_2) , then the surface components will be assumed to be independent of one of these, θ_2 say, varying only with θ_1 . (In the two-dimensional problem $\theta_1 = x_1$, and in the axi-symmetric one $\theta_1 = r = \sqrt{(x_1^2 + x_2^2)}$). In addition u_i^T, t_i^T will be parallel or anti-parallel to the unit vector in the θ_1 direction, $\{e_i\}$. Thus $u_i^T = u^T e_i$, $t_i^T = t^T e_i$. It is also obvious, again since the equations are linear, that if $w(x_j)$ is monotonic, that is $w(x_j) = \lambda w^o(x_j)$, $\lambda > 1$, (where $-w(x_j)$ is the normal displacement), then there is no loss of generality if $\sigma_{ij} = d\sigma_{ij}$, that is the half space is loaded from a state of zero stress. However this is not true if $w(x_j)$ is not monotonic, and then it is important to realise that the changes in stress state must be made in infinitesimally small incremental steps. Thus for $x_j \in \partial\psi_5$, conditions (2.10), (2.11) below apply only at the instant these changes occur, leading to difficulties later when finite step lengths must be made.

The unilateral conditions correspond to frictional conditions, that is there is some maximum limit to the shear at the surface. Then during

any small change in the stress state, represented by a change in the normal displacement, $dw(x_j)$, if the shear is less than this maximum limit, there will be no change of tangential displacement, or slip, of the surface, and if it is equal there may be slip in the direction opposite to the shear.

Thus writing the coefficient of Coulomb friction as ρ , the two unilateral conditions become,

$x_j \in \partial\mathcal{V}_4$ the normal displacement $u^N(x_j)$ is given, $u^N(x_j) = -w_0(x_j)$, and is monotonic, that is $du^N(x_j) = u^N(x_j)$, with the shear and tangential displacement, $u^T = du^T = -u_1$, obeying the conditions

$$\begin{aligned} t^T(x_j) - \rho g(x_j) &\leq 0 \\ u_1(x_j) &\geq 0 \\ u_1(x_j)(t^T(x_j) - \rho g(x_j)) &= 0 \end{aligned} \quad 2.10$$

$x_j \in \partial\mathcal{V}_5$ the normal displacement is given but not monotonic, so that $u^T = u_0^T + du^T$, $u_0^T = u^{T_0}(x_j)$ is the initial tangential displacement, and du^T can be split into positive and negative parts, $du^T = -u_1 + u_2$, so that together with the shear it obeys conditions (2.10) and

$$\begin{aligned} t^T(x_j) + \rho g(x_j) &\geq 0 \\ u_2(x_j) &\geq 0 \\ u_2(x_j)(t^T(x_j) + \rho g(x_j)) &= 0 \end{aligned} \quad 2.11$$

Two problems can be identified, Problem One in which $g(x_j)$ is a known function of x_j , and Problem Two in which it is the unknown normal pressure, $g(x_j) = t^N(x_j)$. Existence and uniqueness for problem Two can only be discussed as a consequence of Problem One. Further in all problems to be considered the coefficient of Coulomb friction is a constant, independent of position and stress. The former is a reasonable assumption,

though the second is not, but it could easily be overcome by absorbing the variation of ρ into $g(x_j)$. However this thesis is concerned only with the response to frictional loading and unloading, and thus $\rho g(x_j)$ will always be taken as a linear function of the normal stress, $\rho g(x_j) = \rho t^N(x_j)$, ρ a constant, and the latter may or may not be known in advance.

Having introduced the boundary conditions, if one now defines the inner products

$$\begin{aligned} (\sigma | \epsilon) &= \int_V \sigma_{ij} \epsilon_{ij} \, dV \\ [u | f] &= \int_V u_i f_i \, dV \\ \delta(N\sigma | u) &= \int_{\partial V} \sigma_{ij} n_j u_i \, da = \int_{\partial V} (t^N u^N + t^T u^T) \, da \end{aligned} \tag{2.12}$$

then the equations (2.1), (2.2) are related through the following form of Green's theorem

$$\int_V \left\{ \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \sigma_{ij} + u_i \frac{\partial \sigma_{ij}}{\partial x_j} \right\} dV = \int_{\partial V} \sigma_{ij} n_j u_i \, da \tag{2.13}$$

which is the adjoint property to be used in the variational formulation.

Finally let it be stressed as it was in TURNER (1975), that since the equations of linear elasticity are being considered, through all displacements the domain of definition, V , its surface, ∂V , and normal vector, n_i , at a point $x_j \in \partial V$, remain fixed.

§2.2 The Physical Model - Problems in Contact Mechanics.

The boundary conditions (2.6) - (2.11) can be identified with some of the standard problems in contact mechanics discussed in TURNER (1975). A logical development of the model is given in Chapters 1,2,4 of that work, and analytical results derived for two-dimensional problems. The proofs are also valid for axisymmetric problems, thus some of the salient points of contact mechanics with the basic assumptions will be reproduced here, but the final details, including the proof of theorems will be left to the earlier work.

Two linearly elastic, convex bodies, B_1 , B_2 , are pressed together along their common axis of symmetry, from hence forth the z -axis, such that they come into contact over a region Γ of their surfaces. The origin is placed at the point the z -axis crosses the region Γ , (with the linear assumption necessarily the centroid), and the common tangent plane at that point is made the xy -plane.

Assumption 2.1: It was shown in the above work, following SPENCE (1968), that if the dimensions of B_1 , B_2 , are sufficiently large then it is consistent with the linear assumption to apply the equations to the bodies as if they were half-spaces, $B_1 = \{x_j \mid x_3 \leq 0\}$, $B_2 = \{x_j \mid x_3 \geq 0\}$. Then under a suitable transformation of the elasticities of B_1 , the other body, B_2 , may be taken as being rigid. One then need only concentrate on the body B_1 now identified as V , with surface $\partial V = \{x_j \mid x_3 = 0\}$, and B_2 will be referred to as the indenter. The surface ∂V then divides into

two regions, the first ∂V_0 which is stress free, $p_i(x_j) = 0$, and the second, Γ , on which the boundary conditions depend on the type of contact, $\Gamma = \partial V_m$, $m=1,5$.

Assumption 2.2: The rate at which V is loaded is slow compared to the response time of the body, so that dynamic effects may be ignored, and the region V assumed to be in quasi-equilibrium at all times. Thus the equations (2.1) - (2.4) may be applied to all stress states, and changes in stress states.

Proposition 2.3: Using these two assumptions, SPENCE (1973), (1975), showed that for power law indentors, $w_0(x_j) = \delta - \alpha |\theta^r|$, $r = 0, \infty$, and for polynomial indentors, $w_0(x_j) = \delta - \sum_{r=1}^{\infty} \alpha_r |\theta^r|$, and TURNER (ibid) for all convex curved indentors, $w_0(x_j) = \delta - f(\theta_1)$, $f(\theta_1)$ convex, that during loading, ($m = 1,4$) the stress can be found as a quadrature of that for a dye, $w_0(x_j) = \delta$, a constant. Thus in problems to be considered, great simplifications will be achieved by assuming w_0 is a constant in contact area, Γ , (for problems in which u^N is to be prescribed). Further using his similarity argument, SPENCE (1968) showed that if $\Gamma = \partial V_1$, ∂V_4 , the horizontal displacement can be specified for a polynomial indenter, and the problem cast as one of the types introduced in the last section. Having done this, the problem can be scaled such that the contact region remains constant in size, $|\theta_1| \leq 1$, and this removes the need to introduce unilateral conditions on the normal stress, so often done in the literature, SEWELL (1969).

The various problems in contact mechanics can now be identified. Solutions in the cases $m = 1,4$ are given for the two-dimensional problem in TURNER (ibid), but all were known previous to that.

Adhesive Contact, $\Gamma = \partial V_1$: Solved in the two-dimensional case by MUSKHELISHVILI (1953) using complex variables, and in the axi-symmetric case by MOSSAKOVSKI (1954), (1963) and SPENCE (ibid), the latter reducing

the problem to one of Wiener-Hopf.

Frictionless Contact, $\Gamma = \partial\mathcal{V}_2$: The classic problem solved by HERTZ (1881) for an elliptic contact region with both axi-symmetric, and two-dimensional limits.

Adhesive Incompressible Limit, $\Gamma = \partial\mathcal{V}_3$: In the limit $\nu \rightarrow \frac{1}{2}$, it is found that the surface shear is zero, so that the normal pressure can be calculated as in the previous problem. But in such a problem, the limit $\lim_{\gamma \rightarrow 0} t^T(x_j)/\gamma$, $x_j \in \partial\mathcal{V}_3$, $\gamma = (1-2\nu)/(2-2\nu)$, is not zero, and can be found, the normal pressure providing an effective horizontal displacement.

Frictional Loading Problem, $\Gamma = \partial\mathcal{V}_4$: SPENCE (1973), (1975) gives a numerical solution, and an analytical solution in the incompressible limit, $\gamma \rightarrow 0$, for both the axi-symmetric and two-dimensional problems, and a general solution for the latter. He was only able to do this because he could specify the regions of slip and adhesion a priori. A full discussion is given in TURNER (ibid), including a solution by GALIN (1961) based on an approximation in the complex plane.

Frictional Unloading Problem, $\Gamma = \partial\mathcal{V}_5$: A numerical solution is presented in this thesis for the problem in which $u^N(x_j)$ is steadily decreasing from the frictional loading solution, in the incompressible limit. Continued cyclic loading is a trivial extension of this work.

Before proceeding let it be stressed that the model is not perfect, the linear assumption alone is violated in the vicinity of any stress singularity at the edge of the contact region. But what is of interest in this work is the solution of the equations, not the deficiencies of the model. The solution is valid if it obeys the equations and the boundary conditions. Any pathological occurrence is a result of the mathematics, and cannot be explained away by resource to the physical situation or deficiencies of the model. However the physical model can be used to give some insight into the form of the solution.

§2.3 Formal Representation.

The equations (2.1) - (2.4) lie within a broader class of problem, that of self-adjoint operator theory. Such equations have a formal representation presented in this section, and also have a variational formulation, which will be discussed in the later sections of the chapter in terms of the representation below, but with boundary conditions of the form (2.6) - (2.11).

For such a formal representation, the stress is identified as the Hilbert space, $\{\sigma\}$, the strain as its dual, $\{\epsilon\}$, inner product, $(\sigma|\epsilon)$, the displacement as the Hilbert space, $\{u\}$, and the body force as its dual, $\{f\}$, inner product, $[u|f]$. Further functionals $X: \{\sigma u\} \rightarrow \mathcal{R}$, $Y: \{\epsilon u\} \rightarrow \mathcal{R}$, and adjoint operators $T: \{\sigma\} \rightarrow \{f\}$, $T^*: \{u\} \rightarrow \{\epsilon\}$, are introduced. Then equations (2.1), (2.2) have the form

$$T^* u = \frac{\partial X}{\partial \sigma} = \epsilon \quad (\alpha) \quad ; \quad T \sigma = \frac{\partial X}{\partial u} = f \quad (\beta) \quad 2.14$$

with (2.3), (2.4) representing a Legendre transformation connecting σ, ϵ

$$\sigma = \frac{\partial Y}{\partial \epsilon} = E \epsilon \quad ; \quad \epsilon = \frac{\partial X}{\partial \sigma} = E^{-1} \sigma \quad 2.15$$

$$X + Y = (\sigma | \epsilon)$$

The general non-linear form of the problem has been included here, as well as the special case of which linear elasticity is an example, in which X is quadratic in σ , and linear in u , $X = \frac{1}{2}(\sigma|E^{-1}\sigma) + [f|u]$, and likewise Y in ϵ and u respectively.

The adjoint property relating T and T^* through the boundary values of the variables, σ, ϵ, u , is the generalization of Green's theorem, (2.13),

$$(\sigma | T^* u) - [u | T \sigma] = S(u | N \sigma) \quad 2.16$$

Finally gathering (2.14), (2.15) together one gets

$$L u - f(u) = T E T^* u - f = 0 \quad 2.17$$

where $L = T E T^*$ is a self adjoint operator if E is self-adjoint, in linear elasticity a fourth order symmetric tensor.

It is worthwhile to introduce at this point what is called the dual representation of the problem, ODEN AND REDDY (1974). One satisfies (2.14) by introducing further variables χ, η , and adjoint operators R, R^* such that

$$R^* \chi = \sigma \quad (\alpha) \quad ; \quad R \epsilon = -\eta \quad (\beta) \quad 2.18$$

then χ, η obey the equation

$$R E^{-1} R^* \chi + \eta = 0 \quad 2.19$$

and R, R^* are connected by the adjoint property

$$(\epsilon | R^* \chi) - (\chi | R \epsilon) = S(P \chi | N \sigma) \quad 2.20$$

with

$$u = P \chi$$

In linear elasticity with zero body force, χ is the Airy stress function, and R is self adjoint. Thus (2.18) is the formal representation of the equations

$$e_{lms} e_{jer} \frac{\partial^2 \chi_{rs}}{\partial x_l \partial x_m} = \sigma_{ij} \quad ; \quad e_{lms} e_{jer} \frac{\partial^2 \epsilon_{rs}}{\partial x_l \partial x_m} = -\eta_{ij} = 0 \quad 2.21$$

the latter being the compatability equation, and (2.19) is that of the Beltrami-Mitchel equation

$$e_{ikm} e_{jen} e_{pro} e_{qsu} \frac{\partial^4 \chi_{uv}}{\partial x_r \partial x_s \partial x_n \partial x_l} = 0 \quad 2.22$$

It will be shown for the axi-symmetric problem on the half-space, but is too cumbersome to prove in the general case, that on the surface, ∂V , χ is related to the surface traction by $P\chi = KN\sigma$, where K is self-adjoint, $K = K^*$. This will be assumed for now, and proved in the above case in the next chapter. However writing $N\sigma = p$, the second of (2.20) leads to the integral equation relating surface displacements, u , to the surface traction vector, p , namely

$$Kp - u = 0 \tag{2.23}$$

This is a further example of (2.17).

The remainder of this chapter will be devoted to discussing complementary variational principles associated with equations of the form (2.17), (2.23).

§2.4 Complementary Variational Principles.

The recognition of complementary extremum principles associated with linear elasticity dates almost from the time of Cauchy, the greatest contribution being made by CASTIGLIANO (1875), who proved minimization of the strain and complementary energy, showed that in the absence of dislocations these could be written in terms of the surface stresses, and proved the theorem associated with his name, that the surface displacements equal the Fréchet derivative of energy, with respect to the surface stresses. Little has been done beyond this point, other than to show that Castigliano's theory lies within a much broader class of problem, represented by (2.17), and to introduce unilateral boundary conditions into the formulation, DUVAUT AND LIONS (1972).

VON NEUMANN (1932) instigated the study of self-adjoint operator equations, the present state of the art being summarised in a recent paper by ODEN AND REDDY (1974), in which they give a full set of complementary variational principles associated with equations (2.17), (2.19). Independently HILL (1956) showed that the equations of continuum mechanics could be found as the Euler equations of functionals of the physical variables. The application of variational principles to continuum mechanics is a rapidly expanding field, papers by SEWELL (1969), (1973) covering it in some depth, dealing mainly with elasticity and plasticity. The work of the next section is an extension of that in the papers by ODEN AND REDDY (ibid) and SEWELL (1969), with the specific inclusion of unilateral

conditions of the type of interest. Further the dual representation is used to advantage to reduce the full variational formulation to one in terms of the surface variables alone, which has Euler equations of the form (2.23). The casting of problems in surface mechanical form is not new, the idea being developed by CASTIGLIANO (ibid), and being the standard method of converting mixed boundary value problems in potential theory to integral equations. Its usefulness however has not been fully exploited in the variational formulation.

Meanwhile some study has been made of integral equations of the type (2.23). NOBLE (1958) derived variational principles for equations whose kernels are of the form

$$k(x, y) = \int_0^\infty X(x, \rho) X(y, \rho) d\rho \quad 224$$

and recently more general cases have been considered by NOBLE AND HUSSAIN (1969), HUSSAIN AND PU (1973). These latter two papers deal with problems in which the range of definition, or contact area, is not known a priori, and show the way in which the complementary energy varies with the unknown contact radius. They deal mainly with applications in linear elasticity.

Quite unsurprisingly it has become possible to formulate a general theory of complementary variational principles in mathematical physics. ARTHURS (1970), and NOBLE AND SEWELL (1972) cover the theory for general non-linear problems of the form (2.14), (2.15), while DUVAUT AND LIONS (ibid) deal specifically with existence and uniqueness of problems in mechanics with variational inequalities. The motivation behind much of this work seems to be in proving existence and uniqueness of solutions, to obtain simple Rayleigh-Ritz approximations, and to obtain error bounds.

It has become popular in Eastern Europe and Germany, for example the papers by PANAGIOTOPOULOS (1975), (1977), to solve contact problems in

elasticity with unilateral or frictional boundary conditions using the finite element method, by showing that through the variational formulation these naturally give rise to standard problems of constrained optimization. The unilateral boundary conditions act as inequality constraints on the surface variables, rather than equality constraints as bilateral conditions do. This approach is used in this thesis, but rather than working on the entire domain, as has previously been done, the surface mechanical formulation will be used to advantage, the finite element solution being a solution of the integral equation, Green's function, relating the surface variables only.

§2.5 Variational Formulation of Self-Adjoint Operator Equations.

The variational formulation of equations (2.14) - (2.17) subject to the boundary conditions (2.6) - (2.11) will now be introduced. As described in §2.1 all surface vectors will be split into normal and tangential components, the direction of the latter being known through the symmetry of the problem. Thus the surface traction vector has components $t^N(x_j)$, $t^T(x_j)$ and the displacement $u^N(x_j)$, $u^T(x_j)$. The results of this section have wider application, but that is beyond the present need. In this way, the surface energy, which is central to this section, can be written as in (2.12)

$$S(u|N\sigma)_{\partial\mathcal{V}} = \int_{\partial\mathcal{V}} (t^N(x_j)u^N(x_j) + t^T(x_j)u^T(x_j)) da \quad 225$$

Only problems of type One, in which $g(x_j)$ is a predetermined function, will be considered in this section, Problem Two being left to the next.

Uniqueness follows immediately, existence from the minimization theorems.

Uniqueness: To prove there exists at most one solution to problems with boundary conditions of the type (2.6) - (2.11), following SEWELL (1969) the inner product $(\sigma|\epsilon)$ is considered. If σ , ϵ obey (2.15), (2.14) then on the half-space, at least in the axi-symmetric case, this inner product will be positive definite if the elasticities obey

$$\nu \neq 0; \quad -\infty < \nu < 3/4 \quad \text{ie.} \quad -1 < \nu < 1 \quad 226$$

Again this will be assumed for now, but proved in the next chapter. If

the conditions on the elasticities are such that $(\sigma|\epsilon) > 0$, $\forall \sigma, \epsilon \neq 0$ obeying (2.14), (2.15), then uniqueness is proved by the usual method, KNOPS AND PAYNE (1971). Considering two solutions, σ_1, σ_2 , which obey (2.14), (2.15), and (2.6) - (2.11), the difference $\Delta\sigma = \sigma_1 - \sigma_2$ must obey (2.14), (2.15) with $\Delta f \equiv 0$, and hence $(\Delta\sigma|\Delta\epsilon)$ must also be positive definite. But using (2.16), with the fact that T and T^* are linear, and that $\Delta f \equiv 0$, $(\Delta\sigma|\Delta\epsilon)$ can be written in terms of the associated surface energy.

$$\begin{aligned} (\Delta\sigma|\Delta\epsilon) &= (\Delta\sigma|T^*\Delta u) \\ &= [\Delta u|T\Delta\sigma] + S(\Delta u|N\Delta\sigma)_{\partial V} \\ &= \sum_{m=0}^5 S(\Delta u|N\Delta\sigma)_{\partial V_m} \end{aligned}$$

It is obvious from (2.6) - (2.9) that $S(\Delta u|N\Delta\sigma)_{\partial V_m} = 0$, $m=0,3$. Consider therefore ∂V_4 .

$$S(\Delta u|N\Delta\sigma)_{\partial V_4} = \int_{\partial V_4} (\Delta t^N(x_j) \Delta u^N(x_j) + \Delta t^T(x_j) \Delta u^T(x_j)) da$$

Now $\Delta u^N(x_j) = 0$, $\Delta u^T(x_j) = \Delta u_0(x_j) - \Delta u_1(x_j) = -\Delta u_1(x_j)$, and $\Delta g(x_j) = 0$ for Problem One, from which it is deduced that

$$S(\Delta u|N\Delta\sigma)_{\partial V_4} = \int_{\partial V_4} \{\Delta u_1(x_j) \Delta(t^T(x_j) - \rho g(x_j))\} da$$

Further both σ_1 and σ_2 obey equation (2.10), that is $u_1(x_j)(t^T(x_j) - \rho g(x_j))$ is identically zero for both, so

$$\begin{aligned} \Delta\{u_1(x_j)(t^T(x_j) - \rho g(x_j))\} &= 0 = \Delta u_1(x_j) \Delta(t^T(x_j) - \rho g(x_j)) + u_{11}(x_j) \Delta(t^T(x_j) - \rho g(x_j)) \\ &\quad + \Delta u_1(x_j) \Delta(t^T(x_j) - \rho g(x_j)) \end{aligned}$$

$$\begin{aligned} \text{Hence } S(\Delta u|N\Delta\sigma) &= - \int_{\partial V_4} \{\Delta u_1(x_j) \Delta(t^T(x_j) - \rho g(x_j))\} da \\ &= + \int_{\partial V_4} \{\Delta u_1(x_j) (t^T(x_j) - \rho g(x_j)) + u_{11}(x_j) \Delta(t^T(x_j) - \rho g(x_j))\} da \end{aligned}$$

But it can be shown from (2.10) that this latter expression is non-positive, $S(\Delta u|N\Delta\sigma)_{\partial V_4} \leq 0$. Similarly $S(\Delta u|N\Delta\sigma)_{\partial V_5} \leq 0$, and thus if both σ_1, σ_2 , are solutions to the problem, then $(\Delta\sigma|\Delta\epsilon) \leq 0$, contradicting

positive definiteness, unless $\Delta\sigma \equiv 0$. This proves the following theorem

Theorem 2.4: If the elasticities obey (2.26), there exists at most one equilibrium stress state σ , on a half space, obeying the equations (2.1) - (2.4), with boundary conditions (2.6) - (2.11), when $g(x)$ is a prescribed function, (Problem One).

KNOPS AND PAYNE (ibid) obtain somewhat weaker conditions on the elasticities for traction, $\partial\psi = \partial\psi_0$, and displacement, $\partial\psi = \partial\psi_1$, boundary problems in the axisymmetric case, and for problems on a half-space in which $\partial\psi = \partial\psi_0$, $\partial\psi = \partial\psi_1$, $\partial\psi = \partial\psi_2$, $\partial\psi = \partial\psi_3$, $\partial\psi = \partial\psi_0 \cup \partial\psi_1$. The conditions at ∞ dealt with at length by the above authors have been somewhat glossed over here, but it is obvious that it is only required that the stress or displacement should vanish sufficiently fast, such that the surface energy at ∞ is zero. From the Hankel transform solution of the next chapter, it can be shown that this will be so, should the displacements and stresses at ∞ be merely required to be bounded. In this case the above theorem proves uniqueness for a somewhat broader class of mixed boundary value problem than KNOPS AND PAYNE (ibid), not only under the stated conditions, but for any region V for which $(\epsilon|\sigma)$ can be shown to be positive definite under a certain range of the elasticities.

Variational Formulation: The variational principles associated with these problems will not be derived yet again, because they can be found in many papers and books, ODEN AND REDDY (1974), ARTHURS (1970), SEWELL (ibid). Only the result for complementary energy will be discussed in depth, because it is central to the remainder of the thesis.

The basic principle is that it is possible to define functionals, involving some or all of the physical variables, which have non-negative variations about the solution to (2.14), (2.15), (2.6) - (2.11), now called (σ, ϵ, u) , or if $\partial\psi_4 = \partial\psi_5 = \emptyset$, have stationary values at that point. It can be further shown that some of these functionals actually

obtain extrema at the solution. Defining $F(u)$ such that $f(u) = \partial F / \partial u$, the four most commonly considered functionals are

The Hu-Washizu Functional: $\delta L \geq 0$ at $(\sigma, \epsilon, u) = (\sigma, \epsilon, u)$

$$L(\sigma, \epsilon, u) = (T^* u | \sigma) + \frac{1}{2} (E \epsilon | \epsilon) + F(u) - (\sigma | \epsilon) - S_L$$

The Hellinger-Reissner Functional: $\delta I \geq 0$ at $(\sigma, u) = (\sigma, u)$

$$I(\sigma, u) = (T^* u | \sigma) - \frac{1}{2} (E^{-1} \sigma | \sigma) + F(u) - S_I$$

The Potential Energy Functional: $\delta H \geq 0$ at $(u) = (u)$

$$H(u) = \frac{1}{2} (E T^* u | T^* u) - S_H$$

The Complementary Energy Functional: $\delta J \geq 0$ at $(\sigma) = (\sigma)$

$$J(\sigma) = \frac{1}{2} (\sigma | E^{-1} \sigma) - S_J$$

2.27

where S_L, S_I, S_H, S_J are boundary integrals defined as follows: letting

$$R_m^N = \int_{\partial \chi_m} t^N(x_j) u^N(x_j) da \quad ; \quad R_m^T = \int_{\partial \chi_m} t^T(x_j) u^T(x_j) da$$

$$S_m^N = \int_{\partial \chi_m} p(x_j) u^N(x_j) da \quad ; \quad S_m^T = \int_{\partial \chi_m} q(x_j) u^T(x_j) da$$

$$T_m^N = \int_{\partial \chi_m} t^N(x_j) w_0(x_j) da \quad ; \quad T_m^T = \int_{\partial \chi_m} t^T(x_j) u_0(x_j) da$$

then

$$S_H = S_0^N + S_0^T + S_2^T + S_3^N + S_4^T + S_5^T$$

$$S_L = S_I = S_H + (R_1^N - T_1^N) + (R_1^T - T_1^T) + (R_2^N - T_2^N) + (R_3^T - T_3^T) \\ + (R_4^N - T_4^N) + (R_5^N - T_5^N)$$

$$S_J = T_1^N + T_1^T + T_2^N + T_3^T + T_4^N + T_4^T + T_5^N + T_5^T$$

where $p(x_j)$ is the boundary value of the normal stress on $\partial \Psi_0, \partial \Psi_3$,

$q(x_j)$ is the boundary value of the shear stress on $\partial \Psi_0, \partial \Psi_2$, and

is $\rho g(x_j)$ on $\partial \Psi_4, \partial \Psi_5$,

$w_0(x_j)$ is the boundary value of the normal displacement on $\partial \Psi_1$,

$\partial \Psi_2, \partial \Psi_4, \partial \Psi_5$,

$u_0(x_j)$ is the boundary value of the horizontal displacement on $\partial\psi_1, \partial\psi_3$,
and is the initial displacement on $\partial\psi_4, \partial\psi_5$.

To prove the last assertion, (2.27), simply consider the first variation, δJ

$$\begin{aligned}\delta J &= (E^{-1}\sigma | \delta\sigma) - \delta S_J \\ &= (T^*u | \delta\sigma) - \delta S_J \\ &= [u | \delta T\sigma] + (\delta(u|N\delta\sigma))_{\partial\psi} - \delta S_J\end{aligned}\quad 2.28$$

but $\delta T\sigma = \delta f = 0$, (in the linear case), and at $\sigma = \sigma_1$, $\delta t^N = 0$ on $\partial\psi_0, \partial\psi_3$,
 $\delta t^T = 0$ on $\partial\psi_0, \partial\psi_2$, $(w - w_0) = 0$ on $\partial\psi_1, \partial\psi_2, \partial\psi_4, \partial\psi_5$, and $(u - u_0) = 0$
on $\partial\psi_1, \partial\psi_3$, so the only non-zero terms in (2.28) are

$$\begin{aligned}\delta J &= \int_{\partial\psi_4 + \partial\psi_5} (u(x_j) - u_0(x_j)) \delta t^T(x_j) da \\ &= \int_{\partial\psi_4 + \partial\psi_5} (-u_1(x_j) \delta t^T(x_j) + u_2(x_j) \delta t^T(x_j)) da \\ &= \int_{\partial\psi_4 + \partial\psi_5} (-u_1(x_j) \delta(t^T(x_j) - \rho g(x_j)) + u_2(x_j) \delta(t^T(x_j) + \rho g(x_j))) da \\ &\geq 0\end{aligned}$$

from (2.10), (2.11). This point of non-negative variation is a global minimum, if the solution is unique, because $(\sigma | E^{-1}\sigma)$ is then a positive definite quadratic term, whereas the boundary term is only linear in stress. The fact that the minimum is obtained in a suitably defined Hilbert space follows from Weierstrass's theorem, LUENBERGER (1969) pg 40, and the Kuhn-Tucker theorem, (ibid) pg 249, further guarantees the existence of non-negative Lagrange multipliers $u_1(x_j), u_2(x_j)$ satisfying (2.10), (2.11), proving the following theorem

Theorem 2.5: If the elasticities obey (2.26), there exists a unique equilibrium stress state, σ , on an axi-symmetric half-space, obeying (2.1) - (2.4), with boundary conditions (2.6) - (2.11); and with $g(x_j)$ a prescribed function, (Problem One).

Further the problem of finding this equilibrium stress state has been reduced to finding that stress field σ , which minimizes J given by (2.27), subject to the constraints on σ , which are $T\sigma = f$ and

$$t^N(x_j) = p(x_j) \text{ on } \partial V_0, \partial V_3; \quad t^T(x_j) = q(x_j) \text{ on } \partial V_0, \partial V_2 \quad 2.29$$

$$t^T(x_j) - \rho g(x_j) \leq 0 \text{ on } \partial V_4, \partial V_5; \quad t^T(x_j) + \rho g(x_j) \geq 0 \text{ on } \partial V_5 \quad 2.30$$

The problem can however be further simplified, because J can be expressed in terms of the unknown surface stresses alone. Then in the first instance the minimization need be performed only on this surface mechanical form to yield the unknown surface stresses, transforming the mixed boundary value problem into an equivalent traction boundary value problem, which can be solved by standard techniques. To transform J in this way, consider the strain energy, $U = \frac{1}{2}(\sigma | \epsilon) = \frac{1}{2}(\sigma | E^{-1}\sigma)$, which can be simplified by introducing the dual variables χ, η , (the latter being identically zero in this case).

$$\begin{aligned} U &= \frac{1}{2}(\sigma | \epsilon) = \frac{1}{2}(R^* \chi | \epsilon) \\ &= \frac{1}{2}(\chi | R\epsilon) + \frac{1}{2} S(P\chi | N\sigma)_{\partial V} \\ &= \frac{1}{2} S(KN\sigma | N\sigma)_{\partial V} \end{aligned} \quad 2.31$$

Thus as required the strain energy can be expressed as a functional of the surface stresses alone, which, for future reference, has Fréchet derivative $\partial U / \partial N\sigma = KN\sigma$.

Considering first the problem in which $\partial V = \partial V_1$, then with (2.31), J becomes

$$J = \frac{1}{2} S(KN\sigma | N\sigma)_{\partial V_1} - S(u_0 | N\sigma)_{\partial V_1}$$

It has been shown that at the solution $\delta J = 0$, so performing this variation yields the Euler equation as Castigliano's theorem

$$u_0 = KN\sigma = \partial U / \partial N\sigma \quad 2.32$$

In the more general case

$$J = \frac{1}{2} S(KN\sigma | N\sigma) - S_J \quad 2.33$$

the solution for the surface stresses being those functions $t^T(x_j)$, $t^N(x_j)$, which minimize (2.33), subject to (2.29), (2.30). Finally considering the frictional unloading problem $\partial\psi = \partial\psi_0 \cup \Gamma$, $\Gamma = \partial\psi_s$, (which includes $\partial\psi_+$ as a special case), introducing now the notation $\underline{p} = (p, q)$, $p(x_j) = -t^N(x_j)$, $q(x_j) = t^T(x_j)$, for the surface stresses in $\partial\psi$, to be used in the remainder of this thesis, and dropping a constant term, J becomes

$$J(p, q; \gamma) = \frac{1}{2} S(K\underline{p} | \underline{p})_{\partial\psi_s} - S(\underline{u}_0 | \underline{p})_{\partial\psi_s} \quad 2.34$$

a positive definite quadratic functional of p, q , if $-1 < \gamma < 1$, for the axi-symmetric half-space), depending on the elasticities only through the parameter γ , since K depends only on γ . The solutions to Problem One are those functions $p(x_j)$, $q(x_j)$, which minimize $J(p, q; \gamma)$ subject to

$$-\rho g(x_j) \leq q(x_j) \leq \rho g(x_j) \quad 2.35$$

Herein lies the important achievement of this section, in that what is inherently a non-linear constraint on the problem (2.10), (2.11), and had therefore caused some difficulty in the past, has been converted to a linear constraint on a quadratic minimization, a quadratic programming problem for which there exist standard methods of solution in the finite dimensional case.

The remainder of this thesis will be primarily devoted to solving (2.34), (2.35) for the axi-symmetric problem on a half-space. It remains in the last section of this chapter to discuss solutions to Problem Two.

§2.6 Solutions to Problem Two.

The results of the last section depend very much on the fact that $g(x_j)$ is a prescribed function, such that $\delta g(x_j) = 0$. In Problem Two defined by the conditions

$$-\rho p(x_j) \leq q(x_j) \leq \rho p(x_j)$$

$\delta p(x_j) \neq 0$, so the proofs of uniqueness, minimization, and the demonstration of existence do not hold in this case. However something can still be said about the existence and uniqueness of solutions to the problem, and though what follows is not at all rigorous, it indicates lines along which research might follow.

Consider the quadratic, (convex) functional, $J(p, q; \chi)$, which is a mapping, $J: H \times H \times \mathcal{R}' \rightarrow \mathcal{R}$, where H is a suitably defined pre-Hilbert, or Hilbert space, the boundary values of $\{\sigma\}$ on $\partial \Psi_5$, and $\mathcal{R}' = (-1, 1)$. From the last section one can define numbers κ_χ , ω , the latter being independent of χ by

$$K_\chi = \min_{\substack{p \in H, q \in H \\ \chi \text{ given}}} J \quad ; \quad \omega = \min_{\substack{p \in H, q \in H \\ \chi \text{ given}}} J$$

Obviously $\omega \geq \kappa_\chi \forall \chi \in \mathcal{R}'$.

One can also define a transformation $T_{\chi\rho}: H \rightarrow H$, where $p(x_j) = T_{\chi\rho}(g(x_j))$ is that function, that together with $q(x_j)$ minimizes $J(p, q; \chi)$ for all $(p, q) \in \{p, q | -\rho g(x_j) \leq q(x_j) \leq \rho g(x_j)\}$, ρ, χ given. Theorem (2.5) guarantees the existence and uniqueness of $p(x_j)$, given $g(x_j)$. Since J is convex and the constraint linear, $T_{\chi\rho}$ must be continuous. In fact, because

by the Kuhn-Tucker theorem the Fréchet derivative of J at the solution must be orthogonal to the constraint, $\partial J/\partial p = 0$, and since J is quadratic, combining this with the previous condition, $T_{\lambda\rho}$ must be uniformly continuous $\forall \lambda \in \mathcal{R}'$, and $\forall \rho \in [0, \infty)$.

Now it can be seen that any fixed point of $T_{\lambda\rho}$ satisfies all the conditions of a solution to Problem Two. From this it can be immediately deduced that if the solution does exist, it does not minimize J subject to (2.35) because, at the solution to that problem, $\partial J/\partial p \neq 0$, unless $\rho = 0$, whereas as shown a fixed point of $T_{\lambda\rho}$ must satisfy $\partial J/\partial p = 0$. Thus one must discuss solutions of Problem Two as fixed points of $T_{\lambda\rho}$.

To prove the existence of such a fixed point, either Schauder's theorem, SNEDDON (1971) pg 413, or more fundamentally, Leray-Schauder degree theory, SATTINGER (1973) could be used. The range $R_{\lambda\rho}$ of $T_{\lambda\rho}$ is obviously contained in the set $\mathcal{K} = \{p \mid J(p, q; \lambda) \leq \omega\} \subset H$, which is closed and bounded. If further conditions can be put on $\mathcal{K}$ arising from the fact that the stress is C^∞ in V , and continuous onto ∂V , to make $\mathcal{K}$ compact, then Schauder's theorem proves existence of a fixed point. Note that compactness of $\mathcal{K}$ is a sufficient condition for the proof of Theorem (2.5). Thus existence of solutions to Problem One, and hence of the operator $T_{\lambda\rho}$, implies existence of solutions to Problem Two. The compactness question has been rather glossed over here, lost in the phrase 'in a suitable space'. It is discussed at length in DUVAUT AND LIONS (1972), but because it is outside the main body of this work will be touched on only in the final chapter. However rather than using Schauder's theorem, it would be more useful to use degree theory to prove existence, because that could be used to prove uniqueness as well.

From the last section it can be deduced that there exists a unique fixed point to the operator $T_{\lambda\rho}$, in the set $\mathcal{K}$, $\forall \rho \in [0, \infty)$, in the special case $\lambda = 0$. Further this fixed point has index +1, and hence $\mathcal{K}$ has degree +1

for this value of λ . Since $T_{\lambda\rho}$ is continuous for all $\lambda \in \mathcal{R}'$, $\mathcal{K}$ is closed, compact(?), and bounded, and must contain all fixed points of $T_{\lambda\rho}$. Leray-Schauder degree theory proves that $\mathcal{K}$ has degree +1 for all $\lambda \in \mathcal{R}'$. Hence $T_{\lambda\rho}$ has at least one fixed point in $\mathcal{K} \ \forall \lambda$, and only one if J being quadratic would imply that all fixed points have index +1. The defence or refutation of the above arguments must be left to the functional analysts.

To find the solution to Problem Two, it is usual to construct the sequence $\{p^n(x_j)\}$, $p^{n+1}(x_j) = T_{\lambda\rho}(p^n(x_j))$, taking as $p^0(x_j)$ the frictionless solution. If $\mathcal{K}$ is compact this sequence must contain a convergent subsequence, the limit of which will be a fixed point if and only if the subsequence is the sequence itself. Any finite dimensional, finite element, approximation to $\mathcal{K}$ will be compact, because all finite dimensional spaces are, and thus such a sequence will converge to the unique fixed point for the finite element solution. But it will be shown in Chapter Seven that convergence of the finite element method also requires $\mathcal{K}$ to be compact.

The following chapters present a finite element solution to Problem One, the solution to Problem Two could be found by constructing the above sequence, but the behaviour of the solution will be the same as for the simpler problem.

PART ONE
Chapter Three

ON AXI-SYMMETRIC MIXED
BOUNDARY VALUE PROBLEMS

§3.1 The Traction Boundary Value Problem.

Having introduced the necessary mathematical tools, they will be applied to axi-symmetric problems in elasticity, specifically to solve the frictional unloading problem. It was shown that all mixed boundary value problems of the type to be considered can be reduced to an equivalent traction boundary value problem, thus it is convenient at this point to derive the solution to such a problem using the Airy stress function, and to further derive results to be used later, relating surface displacements, and in plane surface stresses to the surface stress vector.

The coordinate transformation is made from cartesian to cylindrical polars in the usual way, so that the elastic body occupies the region V , where $V = \{r, \theta, z \mid 0 \leq r < \infty, 0 \leq \theta < 2\pi, -\infty < z < 0\}$, $\partial V = \{r, \theta, z \mid z = 0\}$. The axi-symmetric problem is defined by

$$u_\theta = 0 ; \quad \partial/\partial\theta = 0 ; \quad \text{i.e. } \tau^{r\theta} = \tau^{\theta z} = \frac{\partial \tau^{\theta\theta}}{\partial \theta} = 0$$

To further simplify the equations, the physical components of stress will be denoted by $\sigma_r = \tau^{rr}$, $\sigma_z = \tau^{zz}$, $\sigma_\theta = r^2 \tau^{\theta\theta}$, where τ^{ij} are the contravariant components of the stress tensor, $\tau^{ij} = (\partial\theta^i/\partial x_r)(\partial\theta^j/\partial x_s) \sigma_{rs}$, and the notation $2\sigma = \sigma_r - \sigma_\theta$, $2\pi = \sigma_r + \sigma_\theta$, used. No confusion should arise between the stress π and the number π . The coordinate θ can also be dropped.

Thus with zero body force, the equations (2.1), (2.1) become

$$\epsilon_r = \frac{\partial u_r}{\partial r} ; \epsilon_\theta = \frac{u_r}{r} ; \epsilon_z = \frac{\partial u_z}{\partial z} ; \epsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad 3.1$$

$$\frac{\partial \pi}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \sigma + \frac{\partial^2 \tau}{\partial z^2} = 0 ; \frac{1}{r} \frac{\partial}{\partial r} r \tau + \frac{\partial \sigma_z}{\partial z} = 0 \quad 3.2$$

$(r, z) \in V$

and the Gibb's function

$$\phi = \frac{1}{4\mu(1+\nu)} \left\{ 2(1+\nu)\sigma^2 + 2(1-\nu)\pi^2 + \sigma_z^2 - 4\nu\pi\sigma_z + 2(1+\nu)\tau^2 \right\} \quad 3.3$$

The traction boundary value problem is that in which the surface stress vector is defined over the entire boundary, $\partial V = \partial V_0$, and hence the boundary conditions are

$$t^N(r) = \sigma_z(r, 0) = -p(r) ; t^T(r) = \tau(r, 0) = q(r) \quad 0 \leq r < \infty \quad 3.4$$

with $p(r)$, $q(r)$ prescribed functions. In order that the stress remain bounded at the origin, it is necessary to impose the conditions $p'(0) = 0$, $q(0) = 0$. Further the condition that the stress remain bounded at ∞ is also imposed.

The problem can be further simplified by introducing the Airy stress function, following LOVE (1922), §188. In complete analogy to two-dimensions, the equilibrium equations (3.2) are satisfied with two unknown functions, $U(r, z)$, $V(r, z)$, such that

$$\sigma = r \frac{\partial^2}{\partial r^2} \left(\frac{V}{r} \right) ; \pi = \frac{\partial^2 U}{\partial z^2} - \frac{\partial^2 V}{\partial r^2} ; \tau = -\frac{\partial U}{\partial r \partial z} ; \sigma_z = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial U}{\partial r} \quad 3.5$$

which is the axi-symmetric form of the first of (2.21). When these are substituted into the compatibility equations, which in terms of the stress are, SNEDDON (1950),

$$2(1+\nu) \nabla^2 \pi + \left(\nabla^2 - \frac{\partial^2}{\partial z^2} \right) (2\pi + \sigma_z) = 0 ; 2(1+\nu) \left(\nabla^2 - \frac{4}{r^2} \right) \sigma + \left(r \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \right) (2\pi + \sigma_z) = 0 \quad 3.6$$

$\mu \neq 0$

one obtains the corresponding Beltrami-Mitchel equations

$$U_{zz} - 2V_{rr} = \nu \nabla^2 U ; (1-\nu) \nabla^4 U = 0 \quad 3.7$$

where ∇^2 is the Laplacian operator $\nabla^2 = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)$

Thus if $\nu \neq 1$, $\mu \neq 0$, the traction boundary value problem is solved by

$$\nabla^4 U(r, z) = 0 \quad (r, z) \in V \quad 3.8$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial U}{\partial r} = -p(r) \quad ; \quad \frac{\partial^2 U}{\partial r \partial z} = q(r) \quad (r, z) \in \partial V \quad 3.9$$

with the further condition that U and its derivatives are bounded at ∞ .

Existence and uniqueness follow from the results for the biharmonic equation, yielding weaker conditions on the elasticities than needed in the last chapter.

The solution to (3.8), (3.9) is derived here using Hankel Transforms.

The Hankel transform of order m , $\bar{U}(\rho; z) = \mathcal{H}_m\{U(r, z); \rho\}$ of a function $U(r, z)$ is given by

$$\bar{U}(\rho; z) = \mathcal{H}_m\{U(r, z); \rho\} = \int_0^\infty U(r, z) J_m(\rho r) r dr$$

with inversion theorem, SNEDDON (ibid)

$$U(r, z) = \mathcal{H}_m\{\bar{U}(\rho; z); r\} = \int_0^\infty \bar{U}(\rho; z) J_m(\rho r) \rho d\rho$$

Unless otherwise specified, in the sequel $m = 0$, the most notable exception being $\bar{q}(\rho) = \mathcal{H}_1\{q(r); \rho\}$. With a clash in standard usage the Greek letter ρ has been used for both the independent transform variable, and the coefficient of Coulomb friction. It is hoped that no confusion will occur, especially since the standard alternative is to use μ for the friction coefficient, which could be confused with the shear modulus.

Choosing zero order transforms gives a simple relationship between $\bar{U}(\rho; z)$ and the transform of $\nabla^4 U(r, z)$. SNEDDON (ibid) gives the transformed form of (3.8) as

$$\left(\frac{d^2}{dz^2} - \rho^2\right)^2 \bar{U}(\rho; z) = 0 \quad 3.10$$

However no simple relationship exists between $\bar{U}(\rho; z)$ and the transform of the derivative $\partial U / \partial r$, but the following notation was introduced to

overcome this. The transform of the derivative is written as

$$\mathcal{H}_0 \left\{ \frac{\partial U}{\partial r} ; \rho \right\} = -D \bar{U}(\rho; z)$$

where D is an unknown operator in ρ , whence

$$\int_0^\infty D \bar{U}(\rho; z) \mathcal{J}_0(\rho r) \rho d\rho = -\frac{\partial U}{\partial r}(r, z) = -\frac{\partial}{\partial r} \int_0^\infty \bar{U}(\rho; z) \mathcal{J}_0(\rho r) \rho d\rho$$

Further if $V(r, z) = \partial U(r, z)/\partial r$, then $\bar{U} = -D^{-1} \bar{V}$, where

$$D^{-1} \bar{V} = D^{-1} \int_0^\infty V(r, z) \mathcal{J}_0(\rho r) r dr = \frac{1}{\rho} \mathcal{H}_1 \{ V(r, z); \rho \}$$

Transforming the boundary conditions (3.4), using the above notation, with $\bar{U}'(\rho; z) = d\bar{U}(\rho; z)/dz$, gives relations between the boundary value of $\bar{U}$, and the transforms of the surface stresses

$$\bar{\sigma}_z(\rho; 0) = -\rho^2 \bar{U}(\rho, 0) = -\bar{p}(\rho) ; \quad \bar{\tau}(\rho, 0) = D \bar{U}'(\rho, 0) = \mathcal{H}_0 \{ q(r); \rho \} \quad (3.11)$$

Thus the problem becomes to solve (3.10), subject to the initial conditions (3.11), and with the added condition that $\bar{U}(\rho; z)$ and its derivatives are bounded as $z \rightarrow -\infty$, the bounded stress condition. Equation (3.10) has the general solution, bounded at $-\infty$, $\bar{U}(\rho; z) = \{A(\rho) + B(\rho) \rho z\} e^{\rho z}$, ($\rho \geq 0$), which upon substitution from (3.11) gives the solution

$$\bar{U}(\rho; z) = \frac{1}{\rho^2} \{ (1 - \rho^2) \bar{p}(\rho) + \rho^2 \bar{q}(\rho) \} e^{\rho z} \quad (3.12)$$

$U(r, z)$ is found upon inversion. From this solution (3.12), it is possible to find relations for the in-plane surface stresses, and surface displacements, in terms of the surface traction vector, by first finding expressions for them in terms of the Airy stress function.

From the original definition of σ , π , and from equation (3.5), it is easily shown that the radial and hoop stresses, σ_r , σ_θ , respectively are given by

$$\sigma_r(r, z) = U_{zz}(r, z) - \sigma'(r, z) ; \quad \sigma_\theta(r, z) = U \nabla^2 U(r, z) + \sigma'(r, z) \quad (3.13)$$

$$\sigma'(r, z) = \frac{2}{r^2} \int_0^r s s(s, z) s ds$$

The surface value of these stresses is found on letting $z = 0$. In (3.13)

$U_{zz}(r, 0)$, $\nabla^2 U(r, 0)$, are found in terms of the surface stress vector

by differentiating (3.12) twice, and allowing $z \rightarrow 0$. Thus

$$\bar{U}''(\rho; 0) = -\bar{p}(\rho) + 2\bar{q}(\rho) \quad 3.14$$

or
$$U_{zz}(r, 0) = -p(r) + 2 \int_0^\infty \bar{q}(\rho) J_0(\rho r) \rho d\rho$$

and further

$$\overline{\nabla^2 U}(\rho; 0) = \bar{U}''(\rho; 0) - \rho^2 \bar{U}(\rho; 0) = -2\bar{p}(\rho) + 2\bar{q}(\rho) \quad 3.15$$

or
$$\nabla^2 U(r, 0) = -2p(r) + 2 \int_0^\infty \bar{q}(\rho) J_0(\rho r) \rho d\rho$$

Finally by writing $V_{rr}(r, 0)$ as its inverse transform, $\int_0^\infty \overline{V_{rr}}(\rho; 0) J_0(\rho r) \rho d\rho$, in (3.14), and interchanging the order of integration, it follows that

$$\sigma''(r, 0) = \frac{2}{r} \int_0^\infty \overline{V_{rr}}(\rho; 0) J_1(\rho r) d\rho \quad 3.16$$

But $\overline{V_{rr}}(\rho; 0)$ can be expressed in terms of $U_{zz}(r, 0)$ and $\nabla^2 U(r, 0)$ through (3.7), and hence in terms of $\bar{p}(\rho)$, $\bar{q}(\rho)$ as

$$\overline{V_{rr}}(\rho; 0) = \frac{1}{2} (\bar{U}''(\rho; 0) - \rho^2 \bar{U}(\rho; 0)) = -(1-\nu) (\bar{p}(\rho) - \bar{q}(\rho)) \quad 3.17$$

Substituting (3.14) - (3.17) back into (3.13), gives the required form for the in-plane surface stresses. This is dealt with more extensively in Chapter Five.

Expressions for the displacement in terms of the stress function, found in complete analogy to two-dimensions by substituting (3.5) back into (3.1), integrating, and absorbing an arbitrary constant into the stress function, are

$$\begin{aligned} 2\mu u(r, z) &= \frac{\partial W}{\partial z} - \frac{\partial U}{\partial r} \quad ; \quad 2\mu w(r, z) = \frac{1}{r} \frac{\partial}{\partial r} r W - \frac{\partial U}{\partial z} \\ 2\mu \bar{U}(\rho, z) &= \bar{W}' + D \bar{U} \quad ; \quad 2\mu \bar{w}(\rho, z) = \rho^2 D' \bar{W} - \bar{U}' \end{aligned} \quad 3.18$$

where $\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial W}{\partial z}(r, z) = (1-\nu) \nabla^2 U(r, z) \quad ; \quad (r, z) \in \mathcal{V}; \quad \nu \neq \frac{3}{4}$

$$u(r, z) = u_r(r, z); \quad w(r, z) = u_z(r, z)$$

The first of (3.18) is the form $u = P\chi$ discussed in §2.3. Further this can be used to prove existence and uniqueness of solutions to the displacement boundary value problem, $\partial\psi = \partial\psi_1$, if $\nu \neq 1$, $\mu \neq 0$ through the integral equations below. Substituting (3.12) into (3.18), and allowing $z \rightarrow 0$, gives the simultaneous integral equations relating stress and displacement, previously derived by NOBLE AND SPENCE (1971), by following SNEDDON (ibid), and introduced in formal notation as (2.23) They are

$$\underline{u} = K \underline{p} = K N \underline{\sigma}$$

$$\begin{aligned} w(r) &= \int_0^\infty k_{11}(r,s) p(s) s ds - \gamma \int_0^\infty k_{12}(r,s) q(s) s ds \\ u(r) &= -\gamma \int_0^\infty k_{21}(r,s) p(s) s ds + \int_0^\infty k_{22}(r,s) q(s) s ds \end{aligned} \quad 3.19$$

where $k_{ij}(r,s) = K_{ji}(s,r) = \int_0^\infty J_{i,1}(\rho r) J_{j,1}(\rho s) d\rho$

$$w(r) = -(1-\nu) w(r,0); \quad u(r) = (1-\nu) u(r,0)$$

and the stresses have been non-dimensionalised with respect to μ , $p(r) \rightarrow \mu p(r)$, $q(r) \rightarrow \mu q(r)$. The operator K has the form

$$\underline{u} = \begin{pmatrix} K_{11} & -\gamma K_{12} \\ -\gamma K_{21} & K_{22} \end{pmatrix} \underline{p} = T E T^* \underline{p} \quad 3.20$$

where $T \underline{\sigma} = \int_0^\infty \begin{pmatrix} J_0(\rho r) & 0 \\ 0 & J_{11}(\rho r) \end{pmatrix} \begin{pmatrix} \sigma_1(\rho) \\ \sigma_2(\rho) \end{pmatrix} d\rho$

$$E \underline{\epsilon} = \begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix} \underline{\epsilon} = \underline{\sigma}$$

$$T^* \underline{p} = \int_0^\infty \begin{pmatrix} J_0(\rho r) & 0 \\ 0 & J_{11}(\rho r) \end{pmatrix} \begin{pmatrix} p_1(r) \\ p_2(r) \end{pmatrix} r dr$$

and is therefore obviously self-adjoint with respect to the norms

$$\{ \underline{u} | \underline{p} \} = \int_0^\infty \underline{u}(r) \cdot \underline{p}(r) r dr = \int_0^\infty (w(r) p_1(r) + u(r) p_2(r)) r dr$$

$$(\underline{\sigma} | \underline{\epsilon}) = \int_0^\infty \underline{\sigma}(\rho) \cdot \underline{\epsilon}(\rho) d\rho = \int_0^\infty (\bar{\sigma}_1(\rho) \bar{p}_1(\rho) + \bar{\sigma}_2(\rho) \bar{q}_1(\rho)) d\rho$$

As a final note, on the surface $z = 0$ the normal is $\{n_i\} = (0,0,1)$. Therefore the normal displacement is $u^N(r) = u_z(r,0) = -w(r)$, and the tangential displacement is $u^T(r) = u_r(r,0) = u(r)$. In this notation the surface

displacement vector will now be $u = (w, u) = (-u^N, u^T)$, and the surface stress vector $p = (p, q) = (-t^N, t^T)$.

§3.2 Axi-Symmetric Mixed Boundary Value Problems.

Consider now mixed boundary value problems of the type previously discussed, in which the surface, $\partial V = \{r, z | z = 0\}$, splits into two parts, $\partial V = \partial V_0 \cup \Gamma$, where $\partial V_0 = \{r, z | z = 0, r > 1\}$, and $\Gamma = \{r, z | z = 0, r \leq 1\}$ is one of ∂V_m , $m=1,5$. Only punch problems in which ∂V_0 is unloaded, $p(r) = q(r) \equiv 0$, $r \in \partial V_0$, and $w_0(r) = \delta = \text{const.}$, $r \in \Gamma$, are to be considered. The first step in the solution is then to find the unknown surface stresses, $p(r)$, $q(r)$, $r \in \Gamma$, from which the full solution follows from the results of the last section. Existence and uniqueness of the full solution, and of the corresponding traction boundary value problem were proved in the last chapter.

Because they are to be used as a check on the numerical technique to be developed, the results of the four classical problems will be reproduced here with values for the contact pressure, $p(r)$, the horizontal displacement, $u(r)$, contact shear, $q(r)$, and compliance $P^* = 2\pi \int_0^1 p(r) r dr$. The four problems in historical order are:

Hertz's Frictionless Problem, $\Gamma = \partial V_1$: $q(r) = 0$, $w_0(r) = 1$, $r \leq 1$.

$$p(r) = \frac{(2/\pi)}{\sqrt{1-r^2}} \quad ; \quad u(r) = \frac{-\pi(2/\pi)r}{1+\sqrt{1-r^2}} \quad r \leq 1 \quad 3.21$$

$$P^* = 4$$

Adhesive Problem, $\Gamma = \partial V_1$: $w_0(r) = 1$, $u_0(r) = 0$, $r \leq 1$.

$$\begin{bmatrix} p(r) \\ q(r) \end{bmatrix} = \left(\frac{2K}{\pi\delta} \right) \cosh \left(\frac{\pi K}{2} \right) \int_0^1 \begin{bmatrix} \cos K\theta(w) \\ (w/r) \sin K\theta(w) \end{bmatrix} \frac{dw}{(1-w^2)\sqrt{r^2-w^2}} \quad 3.22$$

where $\Theta(\omega) = \frac{1}{2} \ln \left(\frac{1+\omega}{1-\omega} \right)$, $K = \frac{1}{\pi} \ln \left(\frac{1+\gamma}{1-\gamma} \right)$

$$P^* = 4 \left(\frac{\pi K}{2\gamma} \right)$$

Adhesive Problem in the Incompressible Limit, $\gamma \rightarrow 0$, $\Gamma = \partial V_1$: As γ becomes small in equation (3.22), it is found that $p(r)$ tends to the Hertz value (3.21), and $q(r) = 0(\gamma)$. In the limit

$$\lim_{\gamma \rightarrow 0} \frac{q(r)}{\gamma} = -\frac{2}{\pi^2} \frac{\ln(1-r^2)}{r\sqrt{1-r^2}}$$

3.23

Frictional Loading Problem, $\Gamma = \partial V_+$: $w_0(r) = 1$, $u_0(r) = 0$, $r \leq 1$, where here, and in the unloading problem, $u_0(r)$ is the initial horizontal displacement, the final displacement being $u^T(r) = u(r, 0) = u_0(r) - u_1(r)$. For small changes in stress state, the shear, $q(r)$, and differential displacement, $u(r)$, must obey the following conditions

$$u_1(r) \geq 0; \quad q(r) - \rho g(r) \leq 0; \quad u_1(r)(q(r) - \rho g(r)) = 0$$

3.24

Problem One is the case in which $g(r)$ is a known normal pressure distribution, and Problem Two, the unknown normal contact pressure.

SPENCE (1975), (1973) was able to solve the loading problem by arguing that the contact region must divide into an inner adhesive region, $u_1(r) = 0$, $q(r) - \rho p(r) < 0$, $0 \leq r < c$, and an outer region of inward slip, $u_1(r) \geq 0$, $q(r) - \rho p(r) = 0$, $c \leq r \leq 1$. The boundary conditions can therefore be written at the outset as

$$w_0(r) = 0 \quad 0 \leq r \leq 1; \quad u_0(r) = 0 \quad 0 \leq r \leq c; \quad q(r) = \rho p(r) \quad c \leq r \leq 1 \quad 3.25$$

where $c = c(\rho, \gamma)$ is an eigenvalue to be determined from the fact that $p(r)$, $q(r)$ must be bounded at $r = c$. SPENCE (1975) has found a numerical solution to Problem Two based on a finite difference approximation to the Abel equations, in both the axi-symmetric and two-dimensional cases, and in (1973) an analytic solution in the latter case. However it has proved analytically intractable to find a general analytic solution in

the axi-symmetric case.

But again in the incompressible limit, $\gamma \rightarrow 0$, the shear stress vanishes, so $p(r)$ takes its value in (3.21). If in the limit $\gamma \rightarrow 0$, $\rho \rightarrow 0$ as well, in such a way that ρ/γ remains constant, then again it is found that $q(r)/\gamma$ has a finite limit which can be found analytically. Making the transformations $u_1(r) \rightarrow \gamma u_1(r)$, $q(r) \rightarrow \gamma q(r)$, $\rho \rightarrow \gamma \rho$, then $u_1(r)$ and $q(r)$ obey the conditions

$$u_1(r) = 0 \quad 0 \leq r \leq c; \quad q(r) = \rho p(r) \quad c \leq r \leq 1 \quad 3.26$$

where $p(r)$ is known, and thus the problem becomes one of type One. The solution given by SPENCE (1975) is

$$\frac{q(r)}{\rho p(r)} = 1 - \psi(r, c) \quad 3.27$$

where $\psi(r, c) = \Lambda_0(\sin^{-1} \frac{w}{c}, \sqrt{1-c^2}) - \frac{2w K'(c)}{\pi r} + \frac{1}{\pi \rho r} \ln\left(\frac{1+w}{1-w}\right)$

$$w = \sqrt{\frac{c^2 - r^2}{1 - r^2}}$$

and where c and ρ are related by

$$\rho = \frac{1}{2c K'(c)} \ln\left(\frac{1+c}{1-c}\right) \quad 3.28$$

$K(k)$ is the complete elliptic integral of the first kind, and $\Lambda_0(\beta, k)$ the Heumann lambda function, (see BYRD AND FRIEDMAN (1954)).

That as explained in Chapter One is the present limit of knowledge of the problem, but the next logical step is to ask what happens when the half-space is successively loaded and unloaded.

Frictional Unloading Problem, $\Gamma = \partial\mathcal{V}_5$: This problem can be thought of as that in which the half-space has been loaded to some known stress-displacement state, $p^0(r)$, $q^0(r)$, $w^0(r)$, $u^0(r)$, where $w^0(r) = 1 + \delta$, $|\delta| \ll 1$, and is then unloaded to $w_0(r) = 1$. The initial stress state, $p^0(r)$, may either be that obtained by scaling the previous problem up

by $(1 + \delta)$, or else that achieved by loading by an amount $(1 + \delta)$, and then unloading by $(\delta - \delta)$. Further if $\delta < 0$, then this is equivalent to reloading an unloaded problem. Writing the final horizontal displacement as $u^T(r)$, then $u^T(r) = u(r, 0) = u_0(r) - u_1(r) + u_2(r)$, where $u_1(r)$ represents the inward, and $u_2(r)$ the outward slip, and the boundary conditions are as in (2.10), (2.11)

$$\begin{aligned} u_0(r) = 1 \quad ; \quad u_0(r) = u^0(r) \quad 0 \leq r \leq 1 \\ q(r) - \rho g(r) \leq 0 \quad ; \quad u_1(r) \geq 0 \quad ; \quad u_1(r)(q(r) - \rho g(r)) = 0 \\ q(r) + \rho g(r) \geq 0 \quad ; \quad u_2(r) \geq 0 \quad ; \quad u_2(r)(q(r) + \rho g(r)) = 0 \end{aligned} \quad 3.29$$

There are two difficulties here in attempting a classical solution based on the integral equations. First it is impossible to state *a priori* over which regions of the contact area, $0 \leq r \leq 1$, that $u_1(r) = 0$, or $u_2(r) = 0$ during the unloading step, and thus one cannot write the conditions in the form (3.25). But even if one could guess the unloading regime, either from the physical argument given in Chapter One, or from the numerical solution given in the next, the solution as the unloading proceeds contains moving boundaries. Thus after a finite unloading step, $\delta = O(1)$, even if in the assumed final state, $|q(r_i)| < \rho p(r_i)$, at some point $r = r_i$, it is not possible to say that no slip has occurred at that point during unloading, because some may have occurred before that point became adhesive. This latter difficulty does not occur in the solution of the sideways shear problem given by MINDLIN AND DERESIEWICZ (1953), because in their problem the region of outward slip increases monotonically from the outside edge, and thus any point that is adhesive has been so from the moment of unloading. The formulation (3.29) is for δ of differential size, so it was expected that this difficulty would lead to errors even in the numerical solution of the variational problem, where the step lengths δ were finite. Somewhat to the authors surprise this did not occur, the solution obtained being almost the same regardless of the rate of unloading. This may be because the integrals in a

variational solution tend to filter such errors.

It now remains to show how a variational solution of the unloading problem can be obtained, and to this end the axi-symmetric form of the complementary energy principle will be given.

§3.3 Axi-Symmetric Complementary Energy Principles.

It will be recalled from the last chapter that the complementary energy principle is to minimise $2J$

$$2J = S(KN\sigma | N\sigma) - 2S_J$$

subject to the conditions on stress contained in (2.6), (2.8) - (2.11), namely that $p(r)$, $q(r)$ are prescribed on $r \in \partial V_0$, $q(r)$ on $r \in \partial V_2$, $p(r)$ on $r \in \partial V_3$, and the frictional conditions $q(r) \leq \rho g(r)$ on $r \in \partial V_4$, ∂V_5 , and $q(r) \geq -\rho g(r)$ on $r \in \partial V_6$.

For the axi-symmetric case, $da = 2\pi r dr$, so using (3.19), $S(KN\sigma | N\sigma)$ can be written as

$$\begin{aligned} S(KN\sigma | N\sigma) = & 2\pi \int_0^\infty (p(r) \{ \int_0^\infty k_{11}(r,s) p(s) s ds - \gamma \int_0^\infty k_{12}(r,s) q(s) s ds \} \\ & + q(r) \{ -\gamma \int_0^\infty k_{21}(r,s) p(s) s ds + \int_0^\infty k_{22}(r,s) q(s) s ds \}) r dr \end{aligned}$$

or on introducing the expressions for $k_{ij}(r,s)$, and changing the order of integration

$$S(KN\sigma | N\sigma) = 2\pi \int_0^\infty (\bar{p}^2(\rho) - 2\gamma \bar{p}(\rho) \bar{q}(\rho) + \bar{q}^2(\rho)) d\rho \quad 3.30$$

Equation (3.30) is the strain energy, $(\sigma | \epsilon)$, expressed in terms of the surface stresses. It can be written as

$$S(KN\sigma | N\sigma) = 2\pi \int_0^\infty \{ (\bar{p}(\rho) - \gamma \bar{q}(\rho))^2 + (1 - \gamma^2) \bar{q}(\rho)^2 \} d\rho$$

and is thus obviously positive definite if $\gamma^2 < 1$, and since μ has been

absorbed into $p(r)$, $q(r)$, if $\mu \neq 0$. This proves the assertion that the strain energy is positive definite under the conditions on the elasticities, (2.26).

If one further uses the fact that for $r \in \partial\mathcal{V}_0$, $p(r) = q(r) \equiv 0$, then the Hankel transforms become

$$\bar{p}(\rho) = \mathcal{H}_0\{p(r); \rho\} = \int_0^1 p(r) J_0(\rho r) r dr; \quad \bar{q}(\rho) = \mathcal{H}_1\{q(r); \rho\} = \int_0^1 q(r) J_1(\rho r) r dr \quad 3.31$$

The integral S_J contains terms over $r \in [0, 1]$ only, and so can be written

$$S_J = 2\pi \int_0^1 (w_0(r) p(r) + u_0(r) q(r)) r dr \quad 3.32$$

where $u_0(r) \equiv 0$, if $\Gamma = \partial\mathcal{V}_2$, and $w_0(r) \equiv 0$, if $\Gamma = \partial\mathcal{V}_3$. This leads to the variational principles for the five cases $\Gamma = \partial\mathcal{V}_m$, $m = 1, 5$. In each of these a $(1/\pi)$ will be absorbed into J .

The Adhesive Problem: $w_0(r)$, $u_0(r)$ prescribed. Combining (3.30), (3.32) the energy principle is to

$$\min_{\substack{p(r) \in H \\ q(r) \in H}} J = \int_0^\infty (\bar{p}^2(\rho) - 2\gamma \bar{p}(\rho) \bar{q}(\rho) + \bar{q}^2(\rho)) d\rho - 2 \int_0^1 (w_0(r) p(r) + u_0(r) q(r)) r dr \quad 3.33$$

Equation (3.33) is a standard unconstrained minimization of a quadratic functional, on an infinite dimensional Hilbert space, the Euler equations being, NOBLE (1958)

$$\begin{aligned} w_0(r) &= \int_0^1 k_{11}(r, s) p(s) s ds - \gamma \int_0^1 k_{12}(r, s) q(s) s ds \\ u_0(r) &= -\gamma \int_0^1 k_{21}(r, s) p(s) s ds + \int_0^1 k_{22}(r, s) q(s) s ds \end{aligned} \quad 3.34$$

which are the classical simultaneous integral equations of contact, obtained by putting $p(r) = q(r) \equiv 0$, for $r > 1$, in (3.19).

Consider now the incompressible limit, $\gamma \rightarrow 0$. With the scalings $q(r) \rightarrow \gamma q(r)$, $u(r) \rightarrow \gamma u(r)$, equation (3.33) splits into two parts, one $O(1)$, and the other $O(\gamma^2)$,

$$J = \int_0^\infty \bar{p}^2(\rho) d\rho - 2 \int_0^1 \bar{p}(r) w_0(r) r dr \\ + \gamma^2 \left\{ \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \left(\int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho + \int_0^1 q(r) w_0(r) r dr \right) \right.$$

Thus, as $\gamma \rightarrow 0$, the problem for finding $p(r)$ becomes to minimize

$$J_1 = \int_0^\infty \bar{p}^2(\rho) d\rho - 2 \int_0^1 \bar{p}(r) w_0(r) r dr \quad 3.35$$

and then that for $q(r)$ to minimize

$$J_2 = \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \left(\int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho + \int_0^1 q(r) w_0(r) r dr \right) \quad 3.36$$

where $\bar{p}(\rho)$ is a function known from (3.35). The Euler equations for (3.35), (3.36) are respectively, NOBLE (ibid),

$$\int_0^1 K_1(r,s) p(s) s ds = w_0(r) \quad 3.37$$

$$\int_0^1 K_2(r,s) q(s) s ds = \hat{w}_0(r) = \int_0^1 K_1(r,s) p(s) s ds + w_0(r) \quad 3.38$$

There are many standard methods for inverting (3.34), (3.37), (3.38), NOBLE AND SPENCE (1971), which lead to the solutions of the last section, but will not be discussed here. What is of interest is the fact that (3.37), (3.38) are Fredholm equations of the first kind whose kernels are L_2 on $[0,1] \times [0,1]$, and thus there exists singular expansions of the form, $\sum_{n=0}^\infty \kappa_n u^n(r) v^n(r)$, leading to a method of solution in which the variables are expanded as Fourier series of $u^n(r)$, $v^n(r)$. This will be discussed in the next section.

The Frictionless Problem: $w_0(r) = 1$, $q(r) = 0$, prescribed on $r \leq 1$. If $q(r) = 0$ in (3.33), then J becomes J_1 confirming that the pressure distribution is the same in both the frictionless and incompressible limits. If as in the problem to be considered $w_0(r) = 1$, $r \leq 1$, then J_1 becomes

$$J_1 = \int_0^\infty \bar{p}^2(\rho) d\rho - 2 \int_0^1 p(r) r dr \quad 3.39$$

This unconstrained minimization is STEP 1, for which a numerical solution has been obtained, the results comparing favourably with (3.21).

The Incompressible Adhesive Problem: $u_0(r) = 0$, $p(r) = (2/\pi) (1-r^2)^{-1/2}$ prescribed on $r \leq 1$. Having found $p(r)$ from STEP 1, $q(r)$ is then calculated by minimizing $J = J_1$ in (3.36), with $u_0(r) = 0$, that is

$$J_1 = \int_0^\infty (\bar{q}^2(\rho) - 2\bar{p}(\rho) \bar{q}(\rho)) d\rho \quad 3.40$$

This is STEP 2, a further unconstrained minimization, for which the numerical results compare favourably with (3.23), though, with the stronger singularity not quite as well as in STEP 1.

The Frictional Loading Problem: In the incompressible limit, $p(r)$ is again prescribed from (3.39), so if $u_0(r) = 0$, $q(r)$ is calculated by minimizing $J = J_3$ as in (3.40), but now subject to the constraint $q(r) \leq \rho p(r)$. The slip, $-u_1(r)$, is given by Castigliano's theorem as

$$-u_1(r) = \frac{\partial J_3}{\partial q} = \int_0^1 k_{22}(r,s) q(s) s ds - \int_0^1 k_{21}(r,s) p(s) s ds \quad 3.41$$

which by the Kuhn-Tucker theorem satisfies the conditions of (2.10). This is STEP 3, a familiar problem in Operations Research or Control Theory, that of minimizing a quadratic functional on a Hilbert space, subject to linear constraints, LUENBERGER (1969). The numerical approximation leads to a more standard quadratic programming problem, that on a finite dimensional space, which can be solved by known methods, WAGNER (1969). The numerical results so obtained, almost exactly correspond with (3.26), the singularity being square root as in STEP 1.

The Frictional Unloading Problem: As before in the incompressible limit, $p(r)$ is prescribed at each stage from STEP 1, and $u_0(r)$ from the stage before. Therefore $q(r)$ is found by minimizing $J = J_4$ as in (3.36)

$$J_4 = \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \left(\int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho + \int_0^1 u_0(r) q(r) r dr \right) \quad 3.42$$

subject to $-\rho p(r) \leq q(r) \leq \rho p(r)$. The slip is again given by Castigliano's Theorem

$$-u_1(r) + u_2(r) = \frac{\partial J_4}{\partial q} = \int_0^1 k_{22}(r,s) q(s) s ds - \int_0^1 k_{21}(r,s) p(s) s ds + u_0(r) \quad 3.43$$

which by the Kuhn-Tucker theorem satisfies (2.10), (2.11). At the first stage, the initial displacement is given from STEP 3 by $u_0^1(r) = -(1+\delta) u_1(r)$, and for each successive stage it is given by the initial displacement and slip of the previous stage through the relation

$$u_0^k(r) = (1+\delta) (u_0^{k-1}(r) - u_1^{k-1}(r) + u_2^{k-1}(r))$$

where δ represents the amount of unloading to occur. If λ is the unloading parameter introduced in the first chapter, then $\lambda = (1+\delta)$.

It will be seen that the cost functional at one stage is obtained from that of the previous stage by only a small change in the linear term. The effect of small changes in the cost functional is well known to workers in Operations Research, the resulting unloading regime being that predicted in Chapter One. This is STEP 4, the numerical results being entirely new, and confirming the predictions.

Existence and uniqueness of both the solution and the numerical approximation of all the above problems was discussed in the last chapter. Convergence of the numerical technique will be proved by a similar argument in a later chapter. Secondly it has already been mentioned that STEP 1 and STEP 2 provide the solution of a Fredholm equation of the first kind. Therefore even though the solution of the numerical method may converge, it might still contain an unstable component arising from a vanishing eigenvalue. To study the stability of the numerical approximation it is necessary to be able again to say something about the singular functions of the kernels in (3.37), (3.38). This is the topic of the last section of this chapter.

§3.4 Singular Function Expansions of the Kernels $k_{11}(r,s)$, $k_{12}(r,s)$.

Consider integral equations of the form

$$\int_0^1 k_{ii}(r,s) p_i(s) s ds = u_i(r) \quad ; \quad K p = \underline{u} \quad 3.44$$

where $k_{ii}(r,s) = \int_0^\infty J_\mu(\rho r) J_\mu(\rho s) d\rho \quad i = \mu+1$

It can be shown that these kernels are L_2 on $[0,1] \times [0,1]$, and hence there exist singular expansions, in this case of the form

$$k(r,s) = \sum_{m=0}^{\infty} (\kappa^m / \lambda^m) u^m(r) u^m(s) \quad ; \quad (i \text{ understood}) \quad 3.45$$

where if $\mu = 0$, $\kappa_m^0 \sim (1/m)$ and $\lambda_m^0 \sim (1/m)$ as $m \rightarrow \infty$.

Thus if one can expand $u(r)$ as a Fourier expansion of the $u^m(r)$, and if there exists an orthogonal set of functions, $p^m(r)$, such that

$$\{u^m | p^n\} = \int_0^1 u^m(r) p^n(r) r dr = \lambda^m \delta^{mn} \quad 3.46$$

the solution $p(r)$ can then be found as a Fourier expansion of the $p^m(r)$, that is

$$u(r) = \sum_{m=0}^{\infty} b^m u^m(r) \quad ; \quad p(r) = \sum_{m=0}^{\infty} a^m p^m(r) \quad 3.47$$

where $a^m = (b^m / \kappa^m)$, and the surface energy becomes

$$\{u | p\} = \sum_{m=0}^{\infty} \lambda^m a^m b^m = \sum_{m=0}^{\infty} (\lambda^m / \kappa^m) (b^m)^2 \quad 3.48$$

Hence if a solution introduces even a small component of $u^m(r)$ into the numerical approximation of $w_0(r)$, it can lead to quite large errors

in $p(r)$, but making negligible difference to the surface energy, $\{u|p\}$, which is used as a test on convergence.

It has been shown in the Russian literature, POPOV (1963), (1964), (1966), that it is possible to express the singular functions of Weber-Scharfheitlin kernels, $W_{\nu\lambda}^0(r,s) = \int_0^\infty \rho^\nu J_\nu(\rho r) J_\lambda(\rho s) d\rho$, in terms of Jacobi polynomials. In the special case under consideration, $k_{ii}(r,s) = W_{\nu\nu}^0(r,s)$, the singular functions can be expressed in terms of Gegenbauer's ultraspherical polynomials, $C_n^{(\lambda)}(x)$, ABRAMOWITZ AND STEGUN (1966), which then leads directly to the solution of the frictionless problem, namely

$$\int_0^1 k_{ii}(r,s) p(s) s ds = 1 \quad ; \quad u_0(r) = -\gamma \int_0^1 k_{ii}(r,s) p(s) s ds$$

Consider the kernel (3.44)

$$k_{ii}(r,s) = \int_0^\infty J_\nu(\rho r) J_\nu(\rho s) d\rho \quad 3.49$$

Then from the following result proved in the appendix for general μ

$$J_\nu(2\sin\theta) = \left(\frac{2\pi}{2}\right)^{1/2} \sin^\nu\theta \sum_{m=0}^\infty (2m+\nu+\frac{1}{2}) \frac{(2m-1)!!}{2^m(m+\nu)!} J_{2m+\nu+\frac{1}{2}}(2) \\ \times \frac{(2\nu-1)!!}{2^\nu} C_{2m}^{(\nu+\frac{1}{2})}(\cos\theta) \quad 3.50$$

and the orthogonality relation for half-integer Bessel functions, (μ an integer),

$$\int_0^\infty \frac{1}{\rho} J_{2m+\nu+\frac{1}{2}}(a\rho) J_{2n+\nu+\frac{1}{2}}(b\rho) d\rho = \begin{cases} 0 & m \neq n \\ (4m+2\nu+1)^{-1} & m=n \end{cases} \quad 3.51$$

writing

$$u_i^m = \sin^\nu\theta C_{2m}^{(\nu+\frac{1}{2})}(\cos\theta) \quad 3.52$$

and substituting into (3.49), it is found that

$$k_{ii}(r,s) = \sum_{m=0}^\infty (K_i^m / \lambda_i^m) u_i^m(r) u_i^m(s) \quad 3.53$$

where $K_i^m = \frac{\pi}{2} \frac{(2m+2\nu-1)!!}{(2m+2\nu)!!} \frac{2m!}{4^m(m!)^2} ; \quad \lambda_i^m = \frac{(2m+2\nu)!}{2m!(4m+2\nu+1)(2\nu-1)!!}$

Further the orthogonality relation for the ultraspherical polynomials

$$\int_0^\pi \{ \sin^\nu \theta C_{2n}^{(\nu+1/2)}(\cos \theta) \} \{ \sin^\nu \theta C_{2m}^{(\nu+1/2)}(\cos \theta) \cos^\nu \theta \} \sin \theta \cos \theta d\theta$$

$$= \begin{cases} 0 & m \neq n \\ \lambda_{\nu+1}^m & m = n \end{cases} \quad 3.54$$

shows that $p_i^m(r)$ may be chosen as

$$p_i^m(\sin \theta) = \frac{\sin^\nu \theta C_{2m}^{(\nu+1/2)}(\cos \theta)}{\cos \theta} \quad 3.55$$

and then

$$\bar{p}_i^m(\rho) = \mathcal{H}_\nu \{ p_i^m(r); \rho \} = \left(\frac{\pi}{2\rho} \right)^{1/2} \left(\frac{(2m+2\nu-1)!!}{2^m m!} \right) J_{2m+\nu+1/2}(\rho) \quad 3.56$$

The solution for the frictionless problem follows from the special case $\mu = 0$. Using the fact that $C_n^{(1/2)}(x) = P_n(x)$ the Legendre polynomial, it can be deduced from the above equations that

$$K_1^m P_{2m}(\sqrt{1-r^2}) = \int_0^1 K_{11}(r,s) \left(\frac{P_{2m}(\sqrt{1-s^2})}{\sqrt{1-s^2}} \right) s ds \quad 3.57$$

where $K_1^m = \frac{\pi}{2} \left(\frac{(2m-1)!!}{2^m m!} \right)^2$

Thus if $w_0(r) = 1 = P_0(\sqrt{1-r^2})$, it follows that

$$p(r) = \left(\frac{1}{K_1^0} \right) \frac{P_0(\sqrt{1-r^2})}{\sqrt{1-r^2}}$$

$$= \frac{(2/\pi)}{\sqrt{1-r^2}}$$

Equation (3.56) gives the Hankel transform of $p(r)$ as

$$\bar{p}(\rho) = \bar{p}_i^0(\rho) = \left(\frac{\pi}{2\rho} \right)^{1/2} J_{1/2}(\rho) = \frac{\sin \rho}{\rho} \quad 3.58$$

and thus $u(r) = -\gamma \int_0^1 \bar{p}(\rho) J_1(\rho r) d\rho$

$$= \begin{cases} -\gamma r / (1 + \sqrt{1-r^2}) & r < 1 \\ -\gamma / r & r > 1 \end{cases} \quad 3.59$$

the result quoted in (3.21).

Equations (3.49), (3.51), (3.59) above are special cases of the Weber-

Scharfheitlin integral, to be found in MAGNUS ET AL (1966). The eigenvalues, and eigenvectors of $k_{ij}(r,s)$ could further be found using the methods of TRICOMI (1957), §2.4, but this proves analytically intractable, and there is sufficient information in the above to discuss convergence and stability of the numerical technique. POPOV (ibid) uses these singular functions himself to find series solutions to problems in which $\Gamma = \partial\psi$, as discussed earlier.

PART ONE

Chapter Four

ON THE NUMERICAL SOLUTION

AND THE RESULTS OBTAINED

§4.1 The Problem and Method of Solution.

This chapter contains what the author considers to be the major contribution of this thesis, a numerical solution for the frictional unloading problem, of a flat circular cylinder being removed from an elastic half-space with Coulomb friction acting at the interface. This in the notation of §3.4 is STEP 4. Numerical results were also obtained for the frictionless loading problem, STEP 1, the adhesive loading problem, STEP 2, and the frictional loading problem, STEP 3, not only to provide data for later steps, but also to justify the method by reproducing results obtained previously by both analytic and numerical techniques.

It will be recalled from the last chapter that the four problems are to find those functions $p(r) \in H_1$, and $q(r) \in H_2$, where $r = \sin \theta \in [0,1]$, and H_1 is the space of functions spanned by $\{P_{2n}(\cos \theta)/\cos \theta\}$ and H_2 by $\{\sin \theta C_{2n}^{(3/2)}(\cos \theta)/\cos \theta\}$, satisfying the following conditions:

STEP 1 Solution of the frictionless problem.

$$\min_{p \in H_1} J_1 = \int_0^\infty \bar{p}^2(\rho) d\rho - 2 \int_0^1 p(r) r dr \quad 4.1$$

STEP 2 Solution of the incompressible adhesive problem.

$$\min_{q \in H_2} J_2 = \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho \quad 4.2$$

where $p(r)$ is known from (4.1).

STEP 3 Solution of the frictional loading problem.

$$\min_{q \in H_2^1 \subset H_2} J_3 = \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho \quad 4.3$$

$$\text{where } H_2^1 = \{ q(r) \in H_2 \mid q(r) \leq \rho p(r) \} \quad 4.4$$

ρ , the coefficient of Coulomb friction is given, and $p(r)$ is known from (4.1).

In this step the initial horizontal displacement is zero, $u_0^0(r) \equiv 0$, as is the outward slip, $u_1^0(r) \equiv 0$, but the inward slip is the Fréchet derivative of J_3 at the solution

$$-u_1^0(r) = \partial J_3 / \partial q = \int_0^1 (k_{22}(r,s) q(s) - k_{21}(r,s) p(s)) s ds \quad 4.5$$

The superscript '0' has been introduced to represent the zeroth stage of unloading. It should also be noted at this point that though it is not true that H_2 is a subset of H_1 for $r \in [0,1]$, it is for $r \in (0,1]$. Thus the inequality in (4.4) can be satisfied as long as for the solution to (4.1) $p(0) \geq 0$. This is in fact the condition that contact be maintained, and will be satisfied if $w_0(r)$ is positive, and weakly convex, which it is in this case.

STEP 4. Solution of the frictional unloading problem.

$$\min_{q \in H_2^1 \subset H_2} J_4 = \int_0^\infty \bar{q}^2(\rho) d\rho - 2 \left\{ \int_0^\infty \bar{p}(\rho) \bar{q}(\rho) d\rho + \int_0^1 u_0^k(r) q(r) r dr \right\} \quad 4.6$$

$$\text{where } H_2^1 = \{ q(r) \in H_2 \mid -\rho p(r) \leq q(r) \leq \rho p(r) \} \quad 4.7$$

Again the coefficient of Coulomb friction is given, $p(r)$ is known from (4.1). The minimization (4.6) is performed in successive stages to find the complete unloading profile. At the k th unloading stage, $u_0^k(r)$ is the initial horizontal displacement, found from the sum of that at the $(k-1)$ th stage, and the slip occurring in that stage. The idea is that unloading from $t^N(r) = p(r)$ to $t^N(r) = \lambda p(r)$, is equivalent to unloading from $(1/\lambda)p(r)$ to $p(r)$, where λ is the unloading parameter, and $\delta = 1 - \lambda$ is the 'time' step involved. Thus

$$u_0^k(r) = (1/\lambda) [u_0^{k-1}(r) - u_1^{k-1}(r) + u_2^{k-1}(r)] \quad 4.8$$

and the slip occurring in the k th stage, $-u_1^k(r) + u_2^k(r)$, is the Fréchet derivative of J_4 at the solution

$$-u_1^k(r) + u_2^k(r) = \partial J_4 / \partial q = \int_0^1 (R_{22}(r,s)q(s) - R_{21}(r,s)p(s)) s ds - u_0^k(r) \quad 4.9$$

$$\text{or } u^k(r) = u_0^k(r) - u_1^k(r) + u_2^k(r) = \int_0^1 (R_{22}(r,s)q(s) - R_{21}(r,s)p(s)) s ds \quad 4.10$$

which is the second of (3.19).

Note that the reason the restriction in STEP 3 need only be $q \in H_2^1$, which is also the reason ∂V_4 was considered as separate from ∂V_5 , is that it so happens that the solution, $q(r)$, to STEP 3 is such that $q(r) \in H_2''$. In fact $q(r) \in H_2''' \subset H_2''$ where

$$H_2''' = \{ q(r) \in H_2 \mid 0 \leq q(r) \leq \rho p(r) \}$$

As in Chapter Three, the Hankel transforms above are

$$\bar{p}(\rho) = \int_0^1 p(r) J_0(\rho r) r dr \quad ; \quad \bar{q}(\rho) = \int_0^1 q(r) J_1(\rho r) r dr \quad 4.11$$

with $p(r) = q(r) \equiv 0$, for $r > 1$. Further, requiring that $p \in H_1$, $q \in H_2$, automatically satisfies the conditions that the stress be bounded at the origin, namely $p'(0) = q(0) = 0$.

The philosophy of the Rayleigh-Ritz method is to construct finite dimensional subspaces, $H_1^m \subset H_1$, $H_2^m \subset H_2$, of dimension m , approximating elements $p(r)$, $q(r)$ of H_1 and H_2 . One finds the elements, $\hat{p}(r) \in H_1^m$, $\hat{q}(r) \in H_2^m$, which minimize the functionals in those subspaces, and then expects to be able to show that the error between the minimizing elements in H_1^m , H_2^m , and those in H_1 , H_2 (the true solutions), vanishes as m becomes infinite.

For the purpose of the numerical work conducted for this thesis, the finite element method was used to construct the finite dimensional subspaces, H_1^m , H_2^m , that is they were spanned by a basis of splines. The

line $[0,1]$ is divided into n elements, separated by $m = n + 1$ nodes at radii r_j , $j = 1, m$, with $r_1 = 0$, and $r_m = 1$. Initially a series of spline functions $f_j(r)$, $j = 1, m$, were chosen such that $f_j(r_i) = \delta_{ji}$, which were zero except in the two elements containing the node at $r = r_j$, and which had parabolic variation over those elements.

$$\begin{aligned} f_j(r) &= 0 & r \leq r_{j-1} \quad r \geq r_{j+1} \\ &= \left(\frac{r^2 - r_{j-1}^2}{r_j^2 - r_{j-1}^2} \right) & r_{j-1} \leq r \leq r_j \\ &= \left(\frac{r^2 - r_{j+1}^2}{r_j^2 - r_{j+1}^2} \right) & r_j \leq r \leq r_{j+1} \end{aligned} \quad 4.12$$

The functions $\hat{p}(r) \in H_1^m$, and $\hat{q}(r) \in H_2^m$ were then determined by

$$\hat{p}(r) = \sum_{j=1}^m f_j(r) p_j \quad ; \quad \hat{q}(r) = r \sum_{j=1}^m f_j(r) (q_j / r_j) \quad 4.13$$

where the vectors $\underline{p} = \{p_j\}$, $\underline{q} = \{q_j\}$, are the nodal values of compressive and shear stress to be calculated. These are similar to the spline functions used by BENTALL AND JOHNSON (1967), to solve the two-dimensional frictional rolling problem, but with two noticeable differences. It is well known that in the latter problem, $k_{11}(x,y) = k_{22}(x,y) = \ln |x-y|$, and so the above authors used linear splines, with both $\hat{p}(r)$, and $\hat{q}(r)$ determined as for $\hat{p}(r)$ above. Parabolic splines were chosen here for two reasons, the first being that the definitions (4.13) give an even function for $\hat{p}(r)$, and an odd for $\hat{q}(r)$, as expected from the respective bases of H_1 , and H_2 , and the second that in finding the transforms of the approximate functions, it was necessary to find indefinite integrals of the form

$$\int r^\mu J_0(\rho r) dr \quad ; \quad \int r^\nu J_1(\rho r) dr \quad 4.14$$

where μ and ν are integers. The integrals (4.14) can be found exactly in terms of Bessel functions of the first kind of integer order, if μ is odd, and ν even, which is the case with definitions of the form (4.12), (4.13).

However it was still found that the above formulation was insufficient for the present need. First because (4.13) takes no account of the expected singularity at $r = 1$. In solving the rolling problem BENTALL AND JOHNSON (ibid) were dealing with a problem with vanishing stress at the edge of the contact region, but in this case the analytical solutions have square root singularities at $r = 1$, and in (3.23), the adhesive problem, one of the form $((1-r)^{-1/2} \ln(1-r))$. This can only be accommodated in the above formulation if p_m and q_m are both large, which will certainly give inaccurate results. Secondly, (4.13) automatically gives $q(0) = 0$, independently of (q_1/r_1) , and so this formulation can be expected to lead to a singular matrix.

To overcome these problems, the following modified formulation was used. In the j th element, $\hat{p}(r)$, and $\hat{q}(r)$ were determined by

$$\begin{aligned} \hat{p}(r) &= \alpha_1^j + \alpha_2^j r^2 & \hat{q}(r) &= \beta_1^j r + \beta_2^j r^3 & r_1 \leq r \leq r_2 & j=1, n-1 \\ \hat{p}(r) &= \frac{\alpha_1^n + \alpha_2^n r^2}{\sqrt{1-r^2}} & \hat{q}(r) &= \frac{\beta_1^n r + \beta_2^n r^3}{\sqrt{1-r^2}} & r_n \leq r \leq r_m & j=n \end{aligned} \quad 4.15$$

and required to be continuous at each node. The vectors $p = \{p_j\}$, $q = \{q_j\}$ were chosen such that p_j , $j = 1, n$ and q_j , $j = 2, n$ are the nodal values of compressive and shear stress, $q_1 = r_1 q'(0)$ and p_m , q_m , are given by

$$p_m = \lim_{r \rightarrow 1} \left(\frac{\sqrt{1-r^2} \hat{p}(r)}{\sqrt{1-a^2}} \right) \quad ; \quad q_m = \lim_{r \rightarrow 1} \left(\frac{\sqrt{1-r^2} \hat{q}(r)}{\sqrt{1-a^2}} \right) \quad 4.16$$

where $a = r_n$. This still gives an unbounded value for q_m at the expected solution to the adhesive problem, but with only a logarithmic singularity, but p_m and q_m are bounded for the frictionless, and frictional problems.

Letting $\underline{\alpha}^j = \begin{pmatrix} \alpha_1^j \\ \alpha_2^j \end{pmatrix}$; $\underline{p}^j = \begin{pmatrix} p_1^j \\ p_2^j \end{pmatrix} = \begin{pmatrix} p_j \\ p_{j+1} \end{pmatrix}$; $\underline{r}^j = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_j \\ r_{j+1} \end{pmatrix}$ 4.17

then $\underline{\alpha}^j$ can be determined in terms of $\underline{p}^j$, for $j = 1, n-1$ by

$$\underline{p}^j = \begin{pmatrix} 1 & r_1^2 \\ 1 & r_2^2 \end{pmatrix} \underline{\alpha}^j = (T^j)^{-1} \underline{\alpha}^j \quad \text{or} \quad \underline{\alpha}^j = \frac{1}{(r_2^2 - r_1^2)} \begin{pmatrix} r_2^2 - r_1^2 \\ -1 & 1 \end{pmatrix} \underline{p}^j = T^j \underline{p}^j \quad 4.18$$

and for $j = n$ by

$$\begin{pmatrix} \sqrt{1-a^2} p_n \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & a^2 \\ 1 & 1 \end{pmatrix} \underline{\alpha}^n \quad \text{or} \quad \underline{\alpha}^n = \frac{1}{(1-a^2)} \begin{pmatrix} 1 & -a^2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-a^2} p_n \\ p_{n+1} \end{pmatrix} = T^n \underline{p}^n \quad 4.19$$

Similarly defining $\underline{\beta}^j$, $\underline{q}^j$ as in (4.17) it is found that for

$$j = 2, n-1 \quad \underline{\beta}^j = S^j \underline{q}^j \quad S^j = \frac{1}{r_2 r_1 (r_2^2 - r_1^2)} \begin{pmatrix} r_2^3 - r_1^3 \\ -r_2 & r_1 \end{pmatrix} \quad 4.20$$

$$j = n \quad \underline{\beta}^n = T^n \begin{pmatrix} \sqrt{1-a^2} q_n / a \\ q_{n+1} \end{pmatrix} \quad 4.21$$

but for $j = 1$, with the different definition of q_1

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} r_2 & 0 \\ r_2 & r_2^3 \end{pmatrix} \underline{\beta}^1 \quad \text{or} \quad \underline{\beta}^1 = \frac{1}{r_2^3} \begin{pmatrix} r_2^2 & 0 \\ -1 & 1 \end{pmatrix} \underline{q}^1 = S^1 \underline{q}^1 \quad 4.22$$

Equations (4.15) - (4.22) define an isomorphism between H_1^{∞} , H_2^{∞} and $\mathcal{R}^{\infty}$, reducing the problems to ones in finite dimensional algebra.

The exact details of the numerical implementation are left to Chapter Six, but the following finite dimensional forms of the equations were obtained.

Substituting (4.15) - (4.22) into (4.11) gives expressions for the Hankel transforms of the approximate functions in the form

$$\bar{p}(\rho) = \underline{k}^T(\rho) \underline{p} \quad ; \quad \bar{q}(\rho) = \underline{n}^T(\rho) \underline{q} \quad 4.23$$

where $\underline{k}(\rho)$, $\underline{n}(\rho)$ are vector functions of ρ . It must be stressed that these expressions, and those below for the functionals $J_1 - J_4$ below are exact for functions $\hat{p}(r) \in H_1^{\infty}$, $\hat{q}(r) \in H_2^{\infty}$, the only approximation being in limiting $p(r)$ and $q(r)$ to the finite dimensional subspaces. If (4.23) are substituted into the quadratic terms of the functionals $J_1 - J_4$ they become

$$\begin{aligned} \int_0^{\infty} \bar{p}^2(\rho) d\rho &= \underline{p}^T \int_0^{\infty} \underline{k}(\rho) \underline{k}^T(\rho) d\rho \underline{p} = \underline{p}^T A_{11} \underline{p} & A_{11} &= \int_0^{\infty} \underline{k}(\rho) \underline{k}^T(\rho) d\rho \\ \int_0^{\infty} \bar{q}^2(\rho) d\rho &= \underline{q}^T \int_0^{\infty} \underline{n}(\rho) \underline{n}^T(\rho) d\rho \underline{q} = \underline{q}^T A_{22} \underline{q} & A_{22} &= \int_0^{\infty} \underline{n}(\rho) \underline{n}^T(\rho) d\rho \\ \int_0^{\infty} \bar{p}(\rho) \bar{q}(\rho) d\rho &= \underline{p}^T \int_0^{\infty} \underline{k}(\rho) \underline{n}^T(\rho) d\rho \underline{q} = \underline{p}^T A_{12} \underline{q} & A_{12} &= \int_0^{\infty} \underline{k}(\rho) \underline{n}^T(\rho) d\rho \end{aligned} \quad 4.24$$

The matrices A_{11} , A_{22} should be positive definite, though in fact almost singular, but for some meshes, $\{r_j\}$, round off error actually caused A_{22} to have one small negative eigenvalue. They are however necessarily symmetric.

Expressions can also be found for the linear terms in $J_1 - J_4$. Substituting (4.15) - (4.22) into a linear functional the following is obtained

$$\int_0^1 f(r) p(r) r dr = \underline{f}^T \underline{p} \quad 4.25$$

In the special case $f(r) \equiv 1$, the linear term in J_1 , this becomes

$$p^* = \int_0^1 p(r) r dr = \underline{b}^T \underline{p} \quad 4.26$$

Thus for functions $\hat{p}(r)$, $\hat{q}(r)$, STEPS 1 - 4 simplify considerably to become:

$$\text{STEP 1} \quad \min_{\underline{p} \in \mathbb{R}^m} \hat{J}_1 = \underline{p}^T A_{11} \underline{p} - 2 \underline{b}^T \underline{p} \quad 4.27$$

Since A_{11} is positive definite, the minimum exists, and for a symmetric matrix is given by $\underline{p} = A_{11}^{-1} \underline{b}$.

$$\text{STEP 2} \quad \min_{\underline{q} \in \mathbb{R}^m} \hat{J}_2 = \underline{q}^T A_{22} \underline{q} - 2 \underline{p}^T A_{12} \underline{q} \quad 4.28$$

with $\underline{p} = A_{11}^{-1} \underline{b}$, known from (4.27). Again since A_{22} is positive definite, and symmetric, the solution exists and is given by $\underline{q} = A_{22}^{-1} A_{12}^T \underline{p}$.

$$\text{STEP 3} \quad \min_{\underline{q} \in \mathbb{R}^m} \hat{J}_3 = \underline{q}^T A_{22} \underline{q} - 2 \underline{p}^T A_{12} \underline{q} \quad 4.29$$

subject to $\underline{q} \leq \underline{p} \underline{p}_j \quad \text{ie} \quad I \underline{q} \leq \underline{p} \underline{p}$

with $\underline{p} = A_{11}^{-1} \underline{b}$, known from (4.27). This step cannot be solved by simple matrix inversion as were the previous two, but (4.29) is a standard problem, quadratic programming with linear constraints, for which there are many methods of solution. The one used for the purposes of the numerical method was the Quadratic Simplex algorithm, WAGNER (1969) page 540, for which there is a NAG (1976) library subroutine.

At the solution, the vector $\underline{e}^0$ is defined as

$$\underline{e}^0 = \partial \hat{J}_3 / \partial \underline{q} = A_{12} \underline{q} - A_{11}^T \underline{p} \quad 4.30$$

It is not true that $\underline{e}_j^0$ equals the nodal value of $-u_1^0(r_j)$. However considering $\delta J_3 = \partial J_3 / \partial \underline{q} \delta \underline{q}$, then if one lets $\delta \underline{q} = \hat{\underline{q}}(r)$, the approximate solution, it can be shown that

$$\begin{aligned} \delta J_3 &= \int_0^1 (\partial J_3 / \partial \underline{q}) \hat{\underline{q}}(r) r dr = \int_0^1 (-u_1^0(r) \hat{\underline{q}}(r)) r dr \\ &= \partial \hat{J}_3 / \partial \underline{q} \underline{q} = \underline{e}^{0T} \underline{q} \end{aligned} \quad 4.31$$

as in (4.25). Thus $\underline{e}^0$ provides the initial condition for the first stage of STEP 4, the vector $\underline{u}^0$ associated with the initial horizontal displacement $u_0^0(r)$ being obviously zero. The vector $\underline{e}^0$ acts as a Lagrange multiplier on the minimization (4.29), and is such that $\underline{e}_j^0 (q_j - \rho p_j) = 0$.

STEP 4 At each stage k of the unloading, the solution is given by

$$\min_{\underline{q} \in \mathbb{R}^m} \hat{J}_4 = \underline{q}^T A_{12} \underline{q} - 2 \underline{b}^{kT} \underline{q} \quad 4.32$$

subject to $-\rho p_j \leq q_j \leq \rho p_j$ i.e. $-\rho \underline{p} \leq \underline{I} \underline{q} \leq \rho \underline{p}$

also a quadratic programming problem with linear constraints, with $\underline{p} = A_{11}^{-1} \underline{b}$, known from (4.27), and

$$\begin{aligned} \underline{b}^k &= A_{12}^T \underline{p} + \underline{u}^k ; \quad \underline{u}^{k+1} = (1/\lambda) (\underline{u}^k + \underline{e}^k) \\ \underline{e}^k &= \partial \hat{J}_4 / \partial \underline{q} = A_{12} \underline{q} - \underline{b}^k \end{aligned} \quad 4.33$$

The Fréchet derivative can be further divided into two vectors, $\underline{e}^k = \underline{e}^+ + \underline{e}^-$, where

$$\underline{e}_j^+ = \begin{cases} \underline{e}_j^k & \underline{e}_j^k \geq 0 \\ 0 & \underline{e}_j^k \leq 0 \end{cases} \quad \underline{e}_j^- = \begin{cases} 0 & \underline{e}_j^k \geq 0 \\ \underline{e}_j^k & \underline{e}_j^k \leq 0 \end{cases}$$

Again $\underline{e}^+$, and $\underline{e}^-$ are related to $u_1(r)$, $-u_1(r)$ through the equations of the form (4.31), and they act as Lagrange multipliers on the minimization (4.23), that is

$$\underline{e}_j^+ (q_j + \rho p_j) = 0 ; \quad \underline{e}_j^- (q_j - \rho p_j) = 0$$

A FORTRAN programme was written to solve STEPS 1 - 4 above, and is described in further detail in Chapter Six. Results were obtained using the ICL 1900 machine at the University of Oxford, and are presented in the second section of this Chapter. Finally stability and convergence of this method are discussed in Chapter Seven.

§4.2 Presentation of the Numerical Results.

Numerical solutions were obtained for the problems of the last section, for a number of different meshes, of different number and distribution of nodes. The final choice of mesh was controlled by the stability of the results obtained, and this is discussed at some length in Chapter Seven. But in brief, instabilities arose from two sources. First the equations are ill-conditioned, and thus the numerical approximation tends to introduce eigenvectors, with small eigenvalues and hence heavy oscillations, into the solution in disproportionate amounts. To prevent this happening it was necessary to take r_1 no smaller than 0.1 for m about 20, and 0.2 for m about 40. Secondly, the calculation of the matrices A_{11} , A_{22} , A_{12} , involves the subtraction of large numbers, typically $O(10^3)$ to $O(10^4)$, to obtain numbers of order 1. Thus it was necessary to choose the spacing of the mesh points to give consistent errors in the large numbers, to cancel in obtaining the small. This was not the case if linear spacing of nodes was used, nor if sudden changes in spacing were made. In fact a parabolic spacing of nodes was found to give very accurate results for STEPS 1 & 3, and even allowed some concentration of nodes towards the singularity, without unduly introducing oscillatory errors.

The actual mesh used in the calculation of the results presented here was one of 37 elements and 38 nodes, in which the nodal radii were determined to three or four figures in the following way:

$$\begin{aligned}
 R(I) &= \text{SQRT}(0.04*(I-1)) & I &= 1,11 \\
 &= \text{SQRT}(0.4 + 0.03*(I-11)) & I &= 12,19 \\
 &= \text{SQRT}(0.64 + 0.02*(I-19)) & I &= 20,36 \\
 &= 0.9950 & I &= 37 \\
 &= 1.0000 & I &= 38
 \end{aligned}$$

4.34

Even introducing one extra node, or making small changes in the position of a few, other than the 37th node, could destroy the accuracy of the results, introducing oscillatory errors in the vicinity of the change.

The results obtained for STEPS 1 and 2 represent the finite element solution of two standard integral equations. The finite element method is of course more commonly applied to differential equations, but even though the present application to integral equations did not lead to such common simplifications as banded matrices, (as a result of the double integral), the author believes that the accuracy of the results obtained justifies the application to other integral equations, and especially to STEPS 3 and 4. This is not the first time the finite element method has been used to solve an integral equation, the work done by BENTALL AND JOHNSON (1967) being effectively an earlier example. But it would appear that the full potential of the finite element method for solving integral equations has not been realised.

The obtained nodal values for $p(r)$ from STEP 1, and $q(r)$ from STEP 2, are plotted against the theoretical values (3.21), (3.23), in FIG 4.1, and in TABLE 4.1 the ratio of the numerical value to the theoretical value, (which obviously should be unity), are listed under STEP 1 and STEP 2 respectively. The good fit obtained in FIG 4.1 is slightly misleading in that, as can be seen from TABLE 4.1, the least accurate of the points are close to the singularity, and hence off the graph. However for the mesh chosen, the least accurate of the results for STEP 1 has an error of about 7%, and all the rest less than 3%, the greatest errors occurring

close to the singularity as expected, and at the change in mesh size at $r = 0.8+$. The results for STEP 2 are not quite so good, the values towards the singularity, where it will be recalled one still expects an infinite value, being in error by up to 40%, but apart from the last three points, and the slope at the first, the results are correct to about 5%.

Thus the results obtained produce a reasonably accurate description of the surface stress values, and inspecting the value of compliance

$$P^*/4 = \pi/2 \int_0^1 p(r) r dr = \pi/2 \mathbf{b}^T \mathbf{p} \quad 435$$

which, it is shown in Chapter Seven, gives an indication of convergence, a value of $P^* = 0.9996896$, as compared to unity, was obtained. This is accurate to about 3 in 10^4 , as predicted from the convergence proofs of Chapter Seven, but even meshes that gave quite gross errors in $p(r)$, gave values of compliance this accurate. Thus even though the actual surface stress values may have appreciable errors, these tend to cancel in any integration process, such as those involved in finding not only the compliance, but also the linear term in STEPS 2, 3 and 4, namely $\int_0^1 k_{21}(r,s) p(s) ds$, and the surface and displacement field within the body. In fact the in-plane surface stresses were calculated, the results being presented in the next chapter.

Solutions were obtained for STEPS 3 and 4 for three values of Coulomb friction, which were chosen to give, through equation (3.28), expected adhesive radii for the loading problem of $c = 0.6, 0.7, 0.8$, respectively. It is impossible to say what the resulting numerical value for the adhesive radius is, the problem being compounded by the fact that the e_j do not represent the nodal values of displacement, though they are zero if the corresponding nodal value q_j is less than ρp_j . If the adhesive radii were required, the best possibility would be to interpolate through the obtained shear values, perhaps trying to fit an inverse sine curve to those values of $\theta_j = q_j/\rho p_j$ that are less than unity. This would both

take account of the slope singularity at the expected adhesive radius, and pay lip service to the expected distribution. However the shear distribution, and the resulting full stress and displacement field, are all that of interest here, and it must be accepted that the finite element method can never produce a true picture of behaviour at a mesh point, but only of overall behaviour. It is precisely the overall behaviour on unloading that is required, and obtained in this thesis. The adhesive radius in STEP 3, and the radii of slip and adhesion in STEP 4, can be found reasonably accurately by inspection, and to this end, in TABLE 4.2 numerical results obtained in STEP 3 for the values of $\theta_j = q_j/\rho p_j$, Q_S , and the corresponding e_j^0 , EJ , as well as the theoretical values for θ_j , QE , are listed for the three values of Coulomb friction at the nodes immediately around the expected adhesive radius. Knowing that typical values of e_j^0 are of order unity, it can be seen that if the adhesive radius is not exactly the expected value in each case, it is that value to within about 10^{-3} .

Numerical values for θ_j obtained in STEP 3 with an expected adhesive radius of 0.7 are listed for the entire mesh in TABLE 4.1 under STEP 3A, and the theoretical values under STEP 3B. The same values are plotted against the theoretical values in FIG 4.2. The results obtained for this value of ρ , $\rho = 0.6652$, are all accurate to about 1/10 of 1%, except those close to the slope singularity, the worst being 5%. STEP 3 gave better results than STEP 2, because the constraint gives a sharper minimum, and the singularity at $r = 1$ is weaker. Overall the accuracy of STEP 3 gives some encouragement for the results obtained in STEP 4.

Unloading curves were obtained for each value of ρ , with time steps of $\lambda_1 = 0.99$, $\lambda_2 = 0.99^{10}$, $\lambda_3 = 0.99^{100}$, and $\lambda_4 = 0.99^{1000}$. However it was consistently found that taking say λ_1 ten steps, gave exactly the same shear values as taking λ_2 one, and thus one can have some faith in the results for quite large time steps. The reason for this is that the outer

adhesive region does not expand very rapidly, until the region of outward slip forms, and then both move together to the origin, the annulus of adhesion always remaining fairly thin. Thus the errors arising from a region of adhesion moving into a region of slip are always small, and are smoothed by the integration.

Unloading regimes, numerical values for θ_j are plotted in FIGS 4.3 - 4.6, for $\rho = 0.6652$ and at time steps of λ_3 , λ_3^3 , λ_3^4 , and λ_3^7 respectively. The distribution of $\theta(r)$ in the inner adhesive region maintains its general inverse-sine shape, with the slope singularity at the inner adhesive radius, the latter moving towards the origin. In the outer adhesive region, it again seems to have an inverse-sine shape, with a finite slope at the outer edge of the region of inward slip, and a slope singularity at $r = 1$, or at the inner edge of the region of outward slip once it forms. This suggests an alternative Rayleigh-Ritz approach to the problem, assuming a distribution of the form

$$\begin{aligned}\theta(r) &= (2/\pi) \sin^{-1}(r/c_1) & 0 \leq r \leq c_1 \\ &= 1 & c_1 \leq r \leq c_2 \\ &= 1 - a \sin^{-1}\left(\frac{r-c_2}{1-c_2}\right) & c_2 \leq r \leq 1\end{aligned}$$

if $c_3 \geq 1$, or

$$\begin{aligned}\theta(r) &= (2/\pi) \sin^{-1}(r/c_1) & 0 \leq r \leq c_1 \\ &= 1 & c_1 \leq r \leq c_2 \\ &= 1 - (4/\pi) \sin^{-1}\left(\frac{r-c_2}{c_3-c_2}\right) & c_2 \leq r \leq c_3 \\ &= -1 & c_3 \leq r \leq 1\end{aligned}$$

if $c_3 \leq 1$, and then minimizing J_4 to find c_1 , c_2 , c_3 and a . However this would have to be entirely numerical, based on the singular functions, because the integrations involved in finding the Hankel transforms, $\bar{q}(\rho)$, are impossible analytically.

Two further curves are plotted. In FIG 4.7, a plot is made of the value of

$\theta(r) = q(r)/\rho p(r)$ at the edge of the contact area, $r = 1$, against the unloading parameter, λ . This is done to determine the value of λ at which the region of outward slip forms. The value obtained by extrapolation is $\lambda = 0.5151$. Finally in FIG 4.8 a plot is made of the unloading regime. The boundaries between the inner region of adhesion, and the annulus of inward slip, the latter and the annulus of adhesion, and this and the region of outward slip, are drawn against λ decreasing from 1 to 0. Also drawn, as a dashed line, is the ratio $\theta(r)$ for the loading solution. It can be seen that the inner radius of adhesion follows this fairly closely until the region of outward slip forms. Since λ represents the ratio of the present normal stress to the initial normal stress, this means that inward slip starts at a point when the initial shear at that point no longer satisfies the present constraint. It also tends to indicate that during the initial stages of unloading the shear remains approximately constant in the original region of adhesion, until the region of outward slip forms. This is to be expected, because the same shear would be required to maintain adhesion at a point, inward slip starting when the constraint falls below this shear. It is not until the region of outward slip forms, and the tendency becomes for points to move out, that the radius of adhesion falls noticeably below the $\theta(r)$ curve.

Similar unloading regimes were obtained using different meshes, the radii of slip and adhesion always lying in a consistent element as unloading proceeds, but oscillations in stress values producing little confidence in the actual values for $\theta(r)$, until the region of outward slip develops. Results were tabulated for this mesh, because it produced the most consistently accurate results for STEPS 1 - 3, and also results for STEP 4 in which one could have the most confidence. Even still, in the early stages the numbers for $\theta(r)$, in the outer annulus, individually have little meaning, but collectively indicate a pattern consistent with later stages.

As a check on the unloading regime, the in-plane surface stresses are presented in the next chapter. The values taken by the maximum radial stress during unloading confirm the empirical results of JOHNSON ET AL (1973). Beyond this the author can only present the above unloading regime, hope that the physical argument at least supports it, and ask for reasoned criticism.

TABLE 4.1 Results for STEPS 1, 2 and 3

I	R(I)	STEP 1	STEP 2	STEP 3A	STEP 3B
1	0.0000	0.9999	0.8331	0.1295	*****
2	0.2000	1.0006	1.0613	0.1325	0.1329
3	0.2828	0.9998	0.9713	0.1944	0.1939
4	0.3464	1.0007	1.0126	0.2452	0.2455
5	0.4000	0.9999	0.9949	0.2941	0.2938
6	0.4472	1.0005	1.0000	0.3408	0.3412
7	0.4899	1.0000	1.0017	0.3897	0.3897
8	0.5292	1.0005	0.9974	0.4403	0.4406
9	0.5657	0.9999	1.0021	0.4953	0.4956
10	0.6000	1.0003	0.9983	0.5570	0.5573
11	0.6326	1.0005	0.9995	0.6253	0.6298
12	0.6557	1.0001	1.0003	0.7055	0.6960
13	0.6782	1.0008	1.0012	0.7460	0.7837
14	0.7000	0.9998	0.9991	1.0000	1.0000
15	0.7211	1.0006	0.9979	1.0000	1.0000
16	0.7416	1.0013	1.0058	1.0000	1.0000
17	0.7616	0.9982	0.9891	1.0000	1.0000
18	0.7810	1.0047	1.0169	1.0000	1.0000
19	0.8000	0.9903	0.9705	1.0000	1.0000
20	0.8124	1.0254	1.0390	1.0000	1.0000
21	0.8246	0.9670	0.9782	1.0000	1.0000
22	0.8367	1.0278	1.0004	1.0000	1.0000
23	0.8485	0.9897	1.0050	1.0000	1.0000
24	0.8600	1.0015	1.0026	1.0000	1.0000
25	0.8718	1.0010	0.9883	1.0000	1.0000
26	0.8832	1.0046	1.0151	1.0000	1.0000
27	0.8944	0.9914	0.9963	1.0000	1.0000
28	0.9055	1.0090	0.9814	1.0000	1.0000
29	0.9165	1.0025	1.0207	1.0000	1.0000
30	0.9274	0.9940	0.9857	1.0000	1.0000
31	0.9381	1.0065	1.0104	1.0000	1.0000
32	0.9487	0.9907	0.9723	1.0000	1.0000
33	0.9592	1.0203	1.0254	1.0000	1.0000
34	0.9695	0.9705	0.9412	1.0000	1.0000
35	0.9798	1.0146	1.0450	1.0000	1.0000
36	0.9899	0.9380	0.6986	1.0000	1.0000
37	0.9950	0.9859	1.3790	1.0000	1.0000
38	1.0000	0.9905	1.4384	1.0000	1.0000

TABLE 4.2 Adhesive Radii for STEP 3

ρ	J	R(J)	QS	QE	EJ
0.5790	9	0.5657	0.6858	0.7328	-0.0000
	10	0.6000	1.0000	1.0000	-0.0043
	11	0.6325	1.0000	1.0000	-0.0901
0.6652	13	0.6782	0.7460	0.7837	-0.0000
	14	0.7000	1.0000	1.0000	-0.0025
	15	0.7211	1.0000	1.0000	-0.0619
0.7844	18	0.7810	0.7603	0.7863	0.0000
	19	0.8000	0.9868	1.0000	0.0000
	20	0.8124	1.0000	1.0000	-0.0288

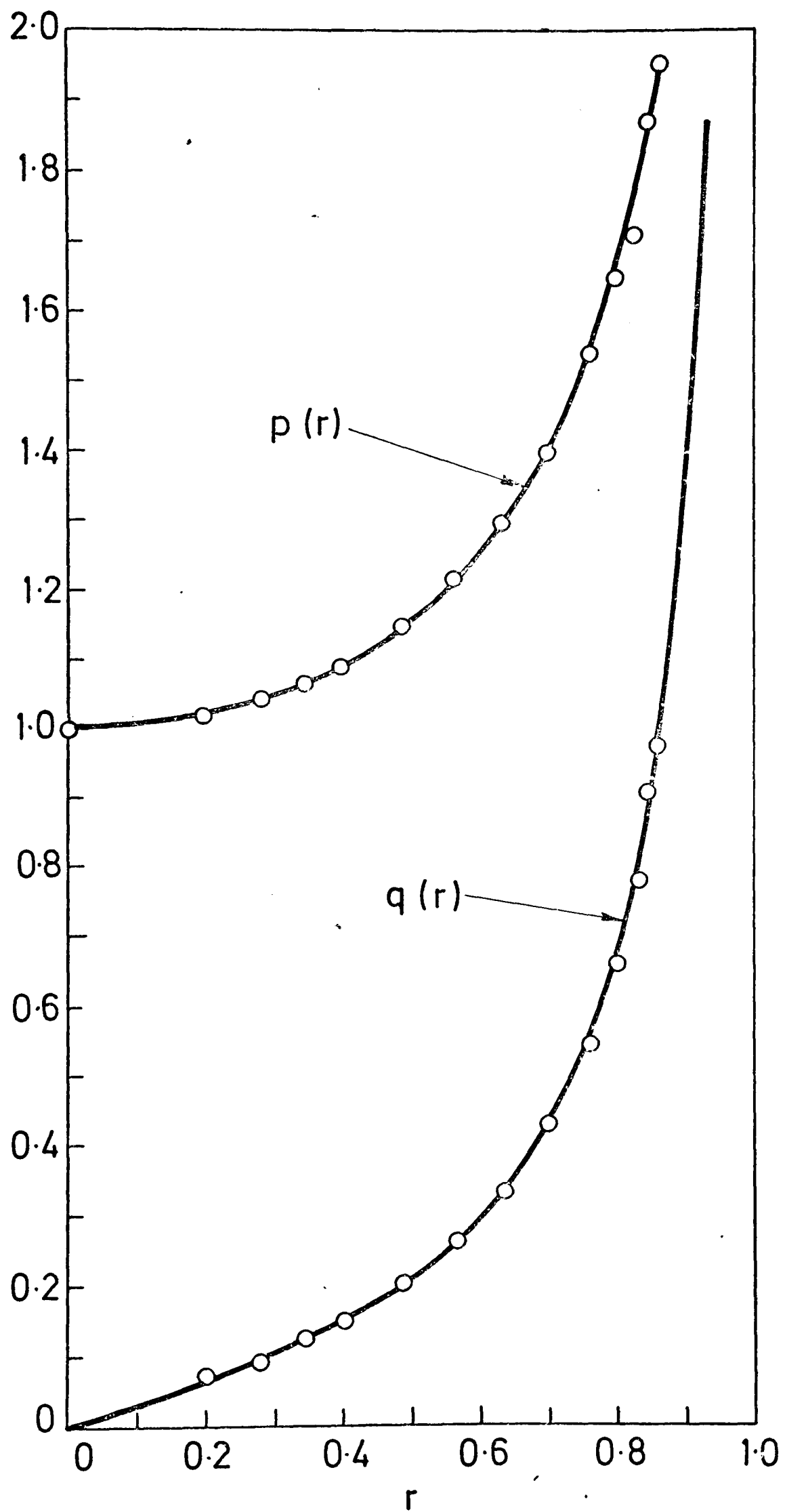


Fig. 4.1 Theoretical curves for normal stress $p(r)$ and shear stress $q(r)$ with computed numerical values.

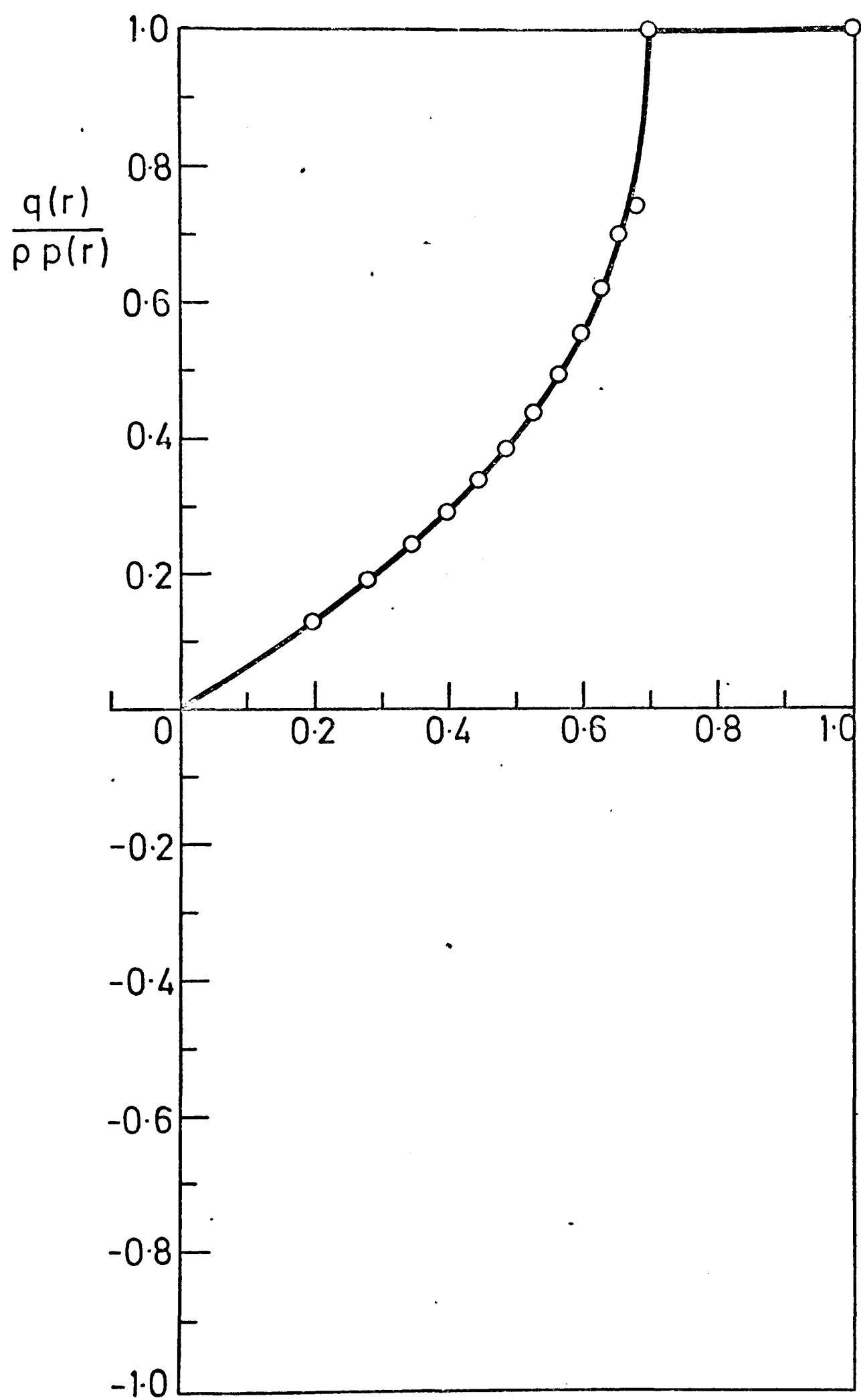


Fig.4.2 Theoretical loading regime, $\theta(r)$, with computed numerical values, $\rho = 0.6652$

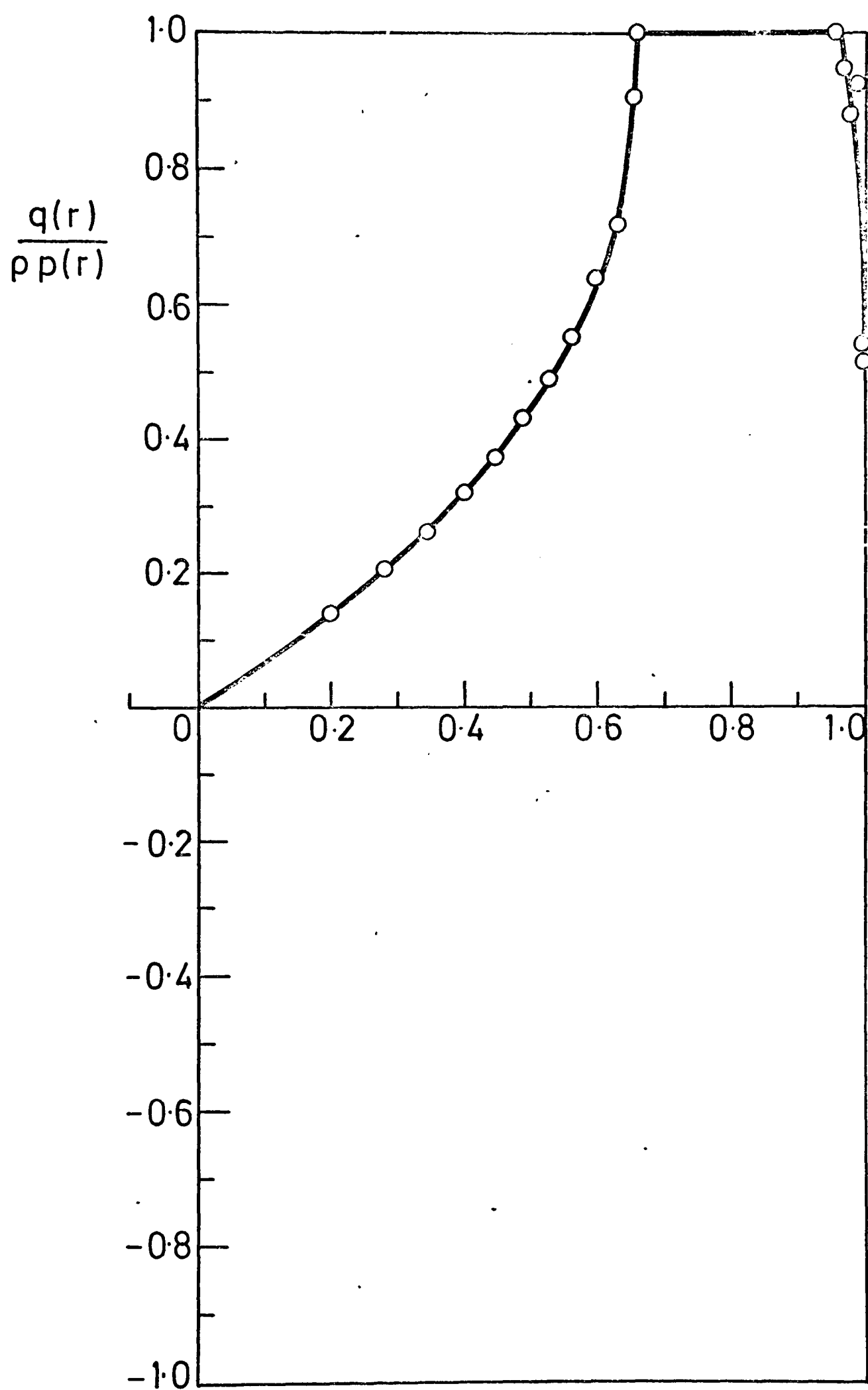


Fig.4.3 Computed unloading regime, $\theta(r)$, at $\lambda=0.8179$, for $p=0.6652$

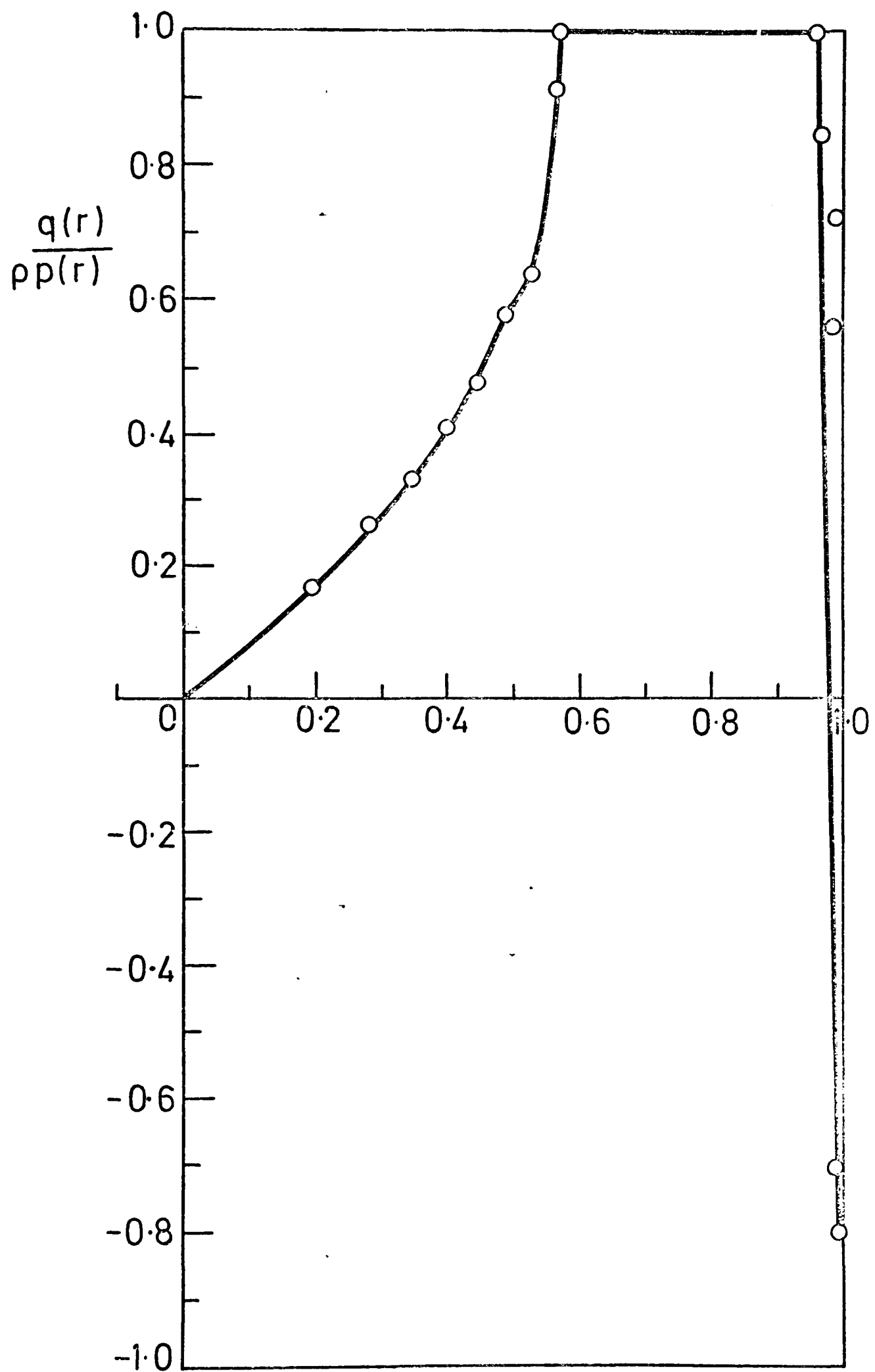


Fig. 4.4 Computed unloading régime, $\Theta(r)$, at $\lambda = 0.5472$, for $\rho = 0.6652$

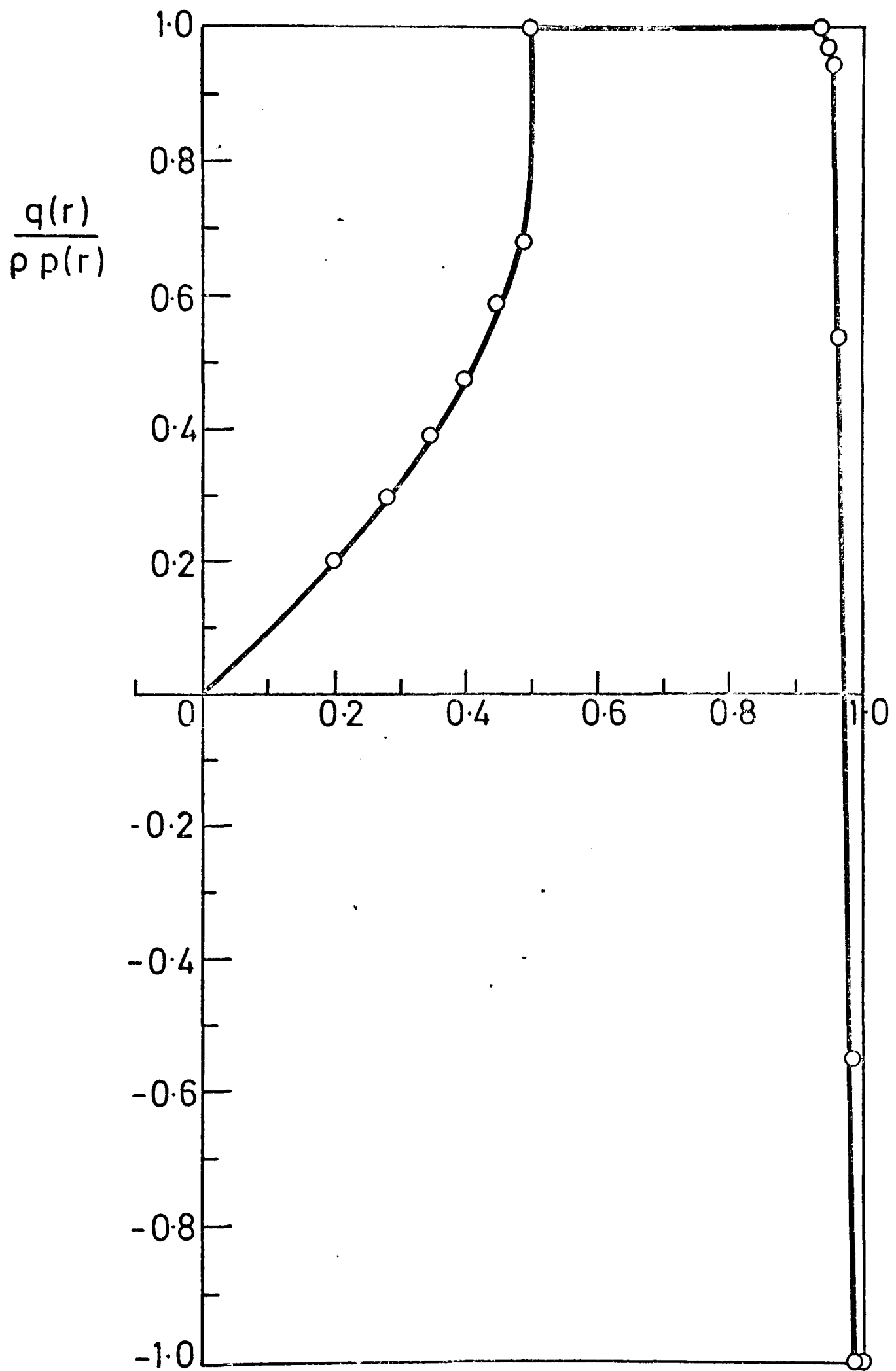


Fig.4.5 Computed unloading regime, $\theta(r)$, at $\lambda = 0.4475$, for $\rho = 0.6652$

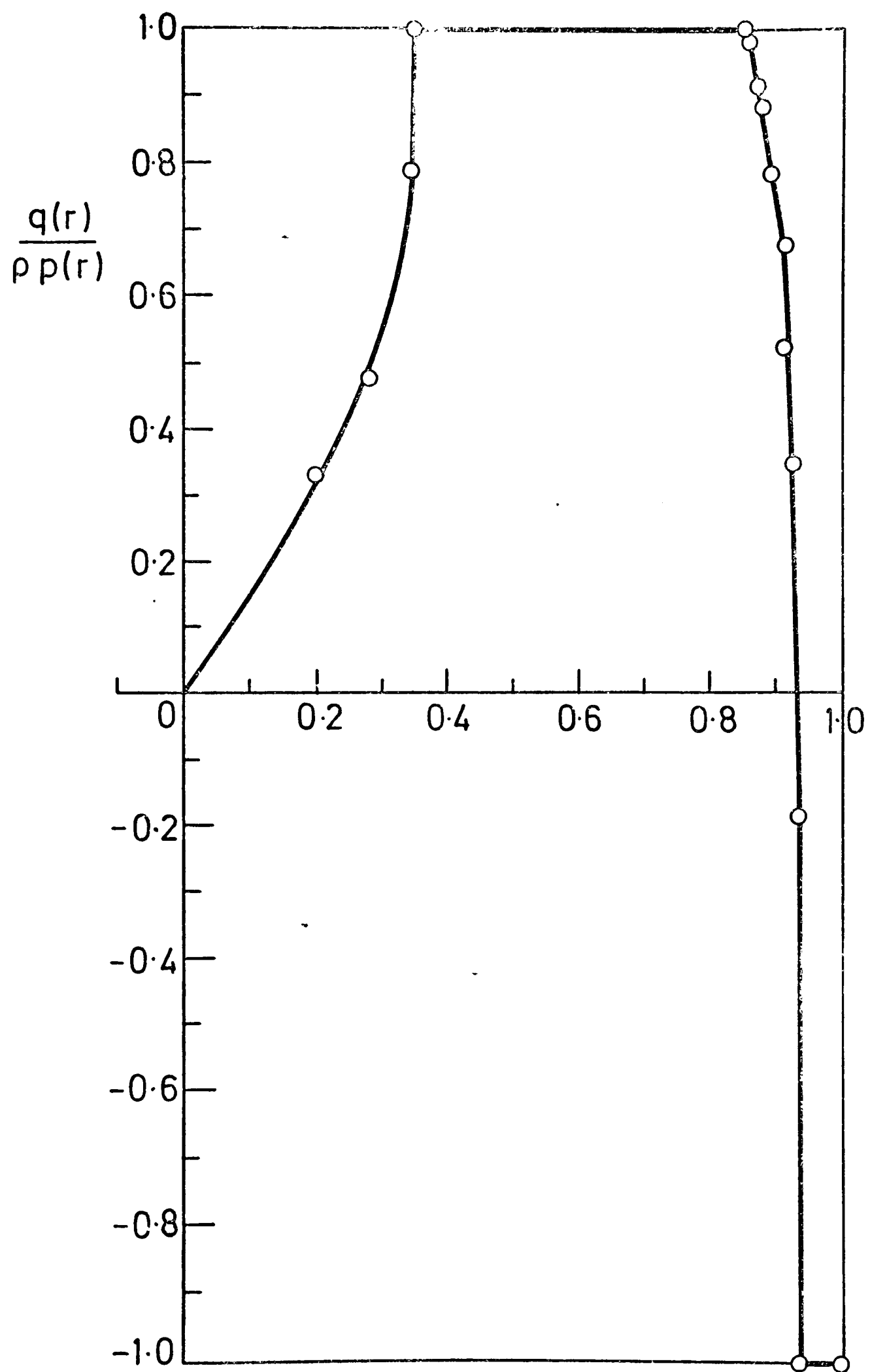


Fig. 4.6 Computed unloading regime, $\Theta(r)$, at $\lambda=0.2449$, for $\rho=0.6652$

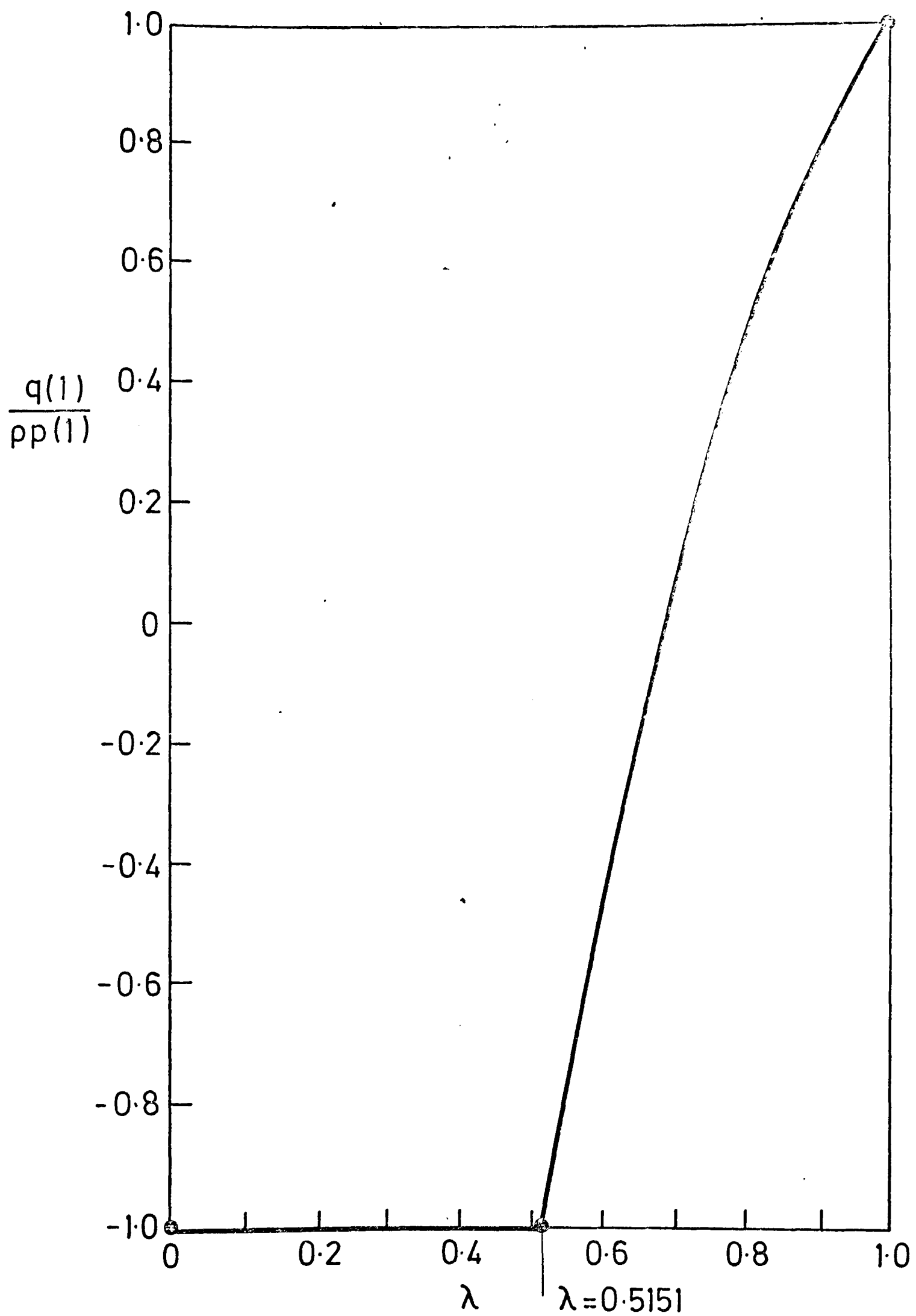


Fig.4.7 $q(1)/p(1)$ against the unloading parameter λ , to determine the point of formation of the region of outward slip, $p = 0.6652$

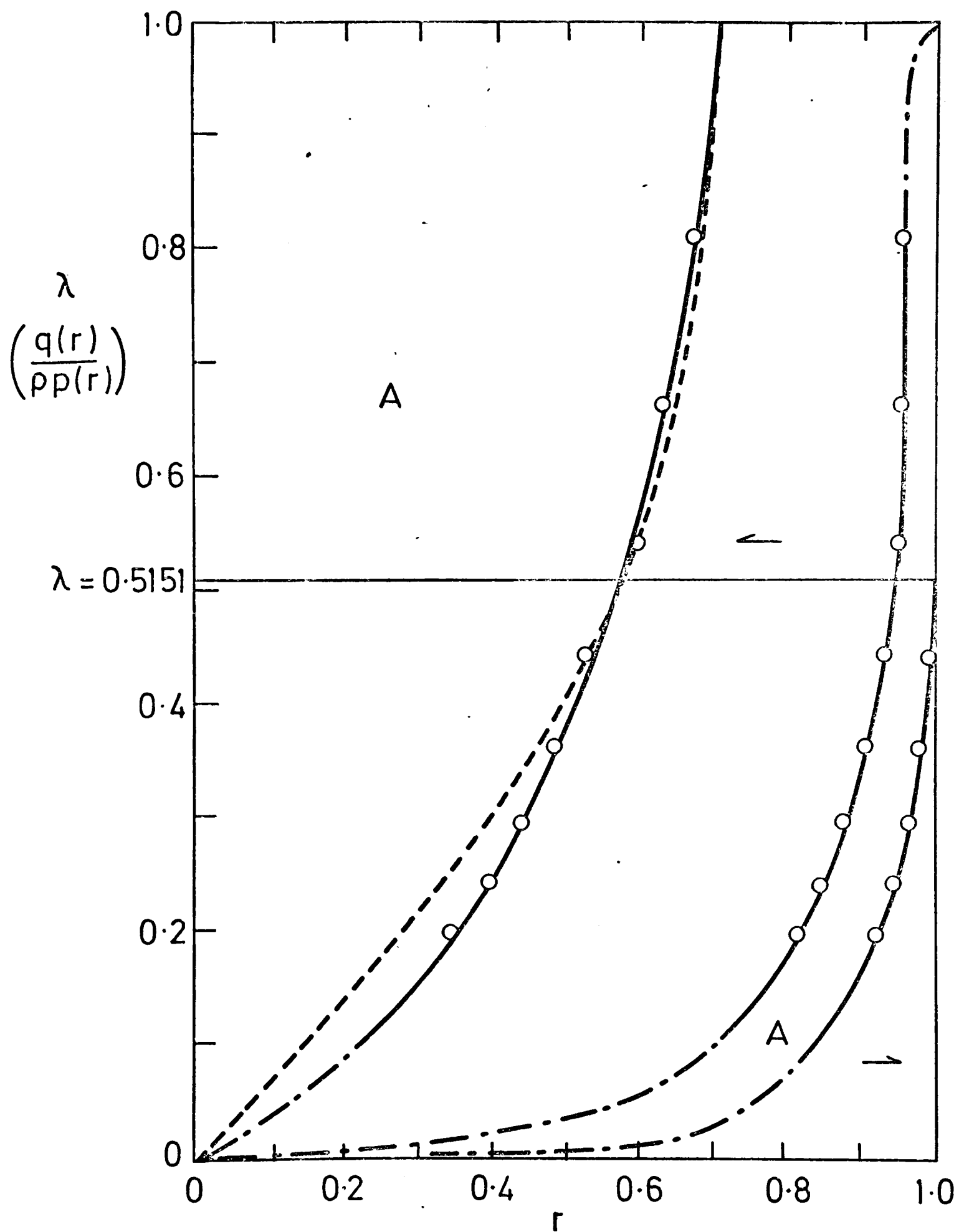


Fig. 4.8 The unloading regime showing regions of adhesion A, inward slip $\leftarrow$ and outward slip $\rightarrow$, against the loading parameter λ
 $\rho = 0.6652$

PART ONE
Chapter Five

ON THE IN-PLANE
SURFACE STRESSES

§5.1 Expressions for the Radial and Hoop Stress.

From a practical viewpoint, quantities of special interest are the surface displacements, both in- and outside the contact area, and the surface radial and hoop stress, the hoop stress controlling the fracture of the surface of brittle materials, JOHNSON ET AL (1973). Once the contact stresses, $p(r)$, $q(r)$, are known, the surface displacements, $w(r)$, $u(r)$, follow from equation (3.19), and the stresses, $\sigma_r(r)$, $\sigma_\theta(r)$, from (3.13) - (3.17). In the incompressible limit, $\nu \rightarrow 0$, the normal displacement, $w(r)$, is as for the frictionless problem, and $u(r)$ is related to the radial and hoop stress as shown below. In this chapter, expressions which relate the latter two to the numerical solution of the last chapter will be derived, and the resulting numerical distribution of $\sigma_r(r)$ presented for STEPS 1 - 4.

It was shown in Chapter Three, equation (3.13), that the radial and hoop stress are in general given by

$$\sigma_r(r, z) = U_{22}(r, z) - \sigma'(r, z) ; \quad \sigma_\theta(r, z) = \nu \nabla^2 U(r, z) + \sigma'(r, z) \quad 5.1$$

where

$$\begin{aligned} \sigma'(r, z) &= \frac{2}{r^2} \int_0^r V_{ss}(s, z) s \, ds \\ &= \frac{2}{r^2} \int_0^\infty \overline{V_{rr}}(\rho, z) J_1(\rho r) \, d\rho \end{aligned}$$

The Airy stress function is given in terms of the transforms of the surface stresses, $\bar{p}(\rho)$, $\bar{q}(\rho)$, through the inverse of (3.12)

$$\bar{U}(\rho, z) = \frac{1}{\rho^2} \{ (1 - \rho^2) \bar{p}(\rho) + \rho^2 \bar{q}(\rho) \} e^{\rho^2} \quad 5.2$$

By substituting (5.2) back into (5.1) and taking the limit $z \rightarrow 0$, expressions can be found for the radial and hoop stress in the surface, in terms of $\bar{p}(\rho)$, $\bar{q}(\rho)$, and hence in terms of $p(r)$, $q(r)$. As shown in Chapter Three these are

$$\begin{aligned} \sigma_r(r, 0) &= -p(r) + 2\sigma_3(r) + 2(1-\nu)(\gamma\sigma_1(r) - \sigma_2(r)) \\ \sigma_\theta(r, 0) &= -2\nu p(r) + 2\nu\sigma_3(r) - 2(1-\nu)(\gamma\sigma_1(r) - \sigma_2(r)) \end{aligned} \quad 5.3$$

where

$$\begin{aligned} \sigma_1(r) &= \frac{1}{r} \int_0^\infty \bar{p}(\rho) J_1(\rho r) d\rho \\ \sigma_2(r) &= \frac{1}{r} \int_0^\infty \bar{q}(\rho) J_1(\rho r) d\rho \\ \sigma_3(r) &= \int_0^\infty \bar{q}(\rho) J_0(\rho r) \rho d\rho \end{aligned} \quad 5.4$$

Note that by substituting (5.4), into (3.18), an expression can also be derived for the radial surface displacements

$$u(r) = r(\sigma_2(r) - \gamma\sigma_1(r)) \quad 5.5$$

which is the combination appearing in (5.3).

If the scaling, $q(r) \rightarrow \gamma q(r)$, is introduced as before, then in the limit $\nu \rightarrow \frac{1}{2}$, the above equation (5.3) separates into terms of $O(1)$, $O(\gamma)$, and higher order. The $O(1)$ terms are

$$\sigma_r(r, 0) = -p(r) ; \quad \sigma_\theta(r, 0) = -p(r) \quad 5.6$$

Thus in the limit $\gamma \rightarrow 0$, the surface of the half-space is entirely stress free outside the contact area, and within the contact area, both the radial and hoop stress are compressive, and equal to the normal pressure. An incompressible brittle material would therefore not be expected to fracture, so the explanation of this must come from the $O(\gamma)$ terms. The $O(\gamma)$ terms are

$$\sigma_r(r,0) = \sigma_1(r) - \sigma_2(r) + 2\sigma_3(r)$$

5.7

$$\sigma_\theta(r,0) = -\sigma_1(r) + \sigma_2(r) + \sigma_3(r) + p(r)$$

Before discussing the numerical calculation of these surface stresses, consider those developed by the frictionless solution. For the frictionless solution, $q(r) \equiv 0$, and thus too are $\sigma_r(r)$, $\sigma_\theta(r)$. Further the normal stress is given by (3.21), $p(r) = (2/\pi)(1-r^2)^{1/2}$, or recalling (3.58), (3.59)

$$\bar{p}(\rho) = \frac{\sin \rho}{\rho} ; \quad \sigma_1(r) = -\frac{U(r)}{r} = \begin{cases} \frac{1}{1+\sqrt{1-r^2}} & r < 1 \\ \frac{1}{r^2} & r > 1 \end{cases}$$

5.8

This term of the radial stress, which is tensile, and balanced by a compressive hoop stress, is the only contribution outside the contact area, causing the inward horizontal displacement. It further has a maximum value of unity at $r = 1$. Because this tensile term is overshadowed by the compressive $\sigma_1(r)$ term within the contact area, $r < 1$, it is expected that the half space will fracture outside the contact area, at the point where the $\sigma_1(r)$ term is greatest, that is at $r = 1^+$. The empirical evidence is, however, that most real materials do not exhibit this simple behaviour, JOHNSON ET AL (ibid), actually fracturing some distance out from the indenter, typically at $r = 1.1$ to $r = 1.3$ for experiments with glass plates and steel balls. This sort of behaviour can be explained by introducing almost any positive shear distribution, because then $(-\sigma_2 + 2\sigma_3)$ would be negative, (compressive), and decay like $(1/r^3)$, whereas σ_1 decays like $(1/r^2)$. Thus the terms arising from the shear may balance σ_1 close to the contact area, but on moving away, very quickly have little effect, giving the maximum stress a short distance out. The numerical evidence is that for the flat punch the maximum radial stress will occur at $r = 2$ in the adhesive case, and at smaller radii for finite values of ρ . The compressive effect of the shear terms will also reduce the value of the

maximum tensile stress, in the particular case covered in the next section to about 15% of the frictionless value for the punch, meaning that the half-space can sustain about 6 times the pressure predicted by frictionless theory, without fracturing during loading.

But as explained in the first chapter, the radial stress can be expected to increase further during unloading, since, as the shear becomes negative, the shear term above will become tensile. Thus if this term increases faster than the pressure term decreases, (which it does so linearly), then the maximum radial stress will, at some point of the unloading, increase above the loading value. Again for the case presented in the next section, the numerical indication is that it reaches a maximum value of about half the frictionless value, when the normal stress has fallen to about one quarter of its loading value. This maximum occurs at the outside edge of the contact area. The above stress can be found for a given numerical solution through equation (5.4) in the following manner.

For a stress distribution $p(r)$, $q(r)$, the in-plane surface stresses can be calculated from the numerical values, $\hat{p}(r)$, $\hat{q}(r)$, by substituting (4.23) into (5.4). On making this substitution σ_1 , σ_2 , σ_3 become

$$\begin{aligned} \sigma_1(r) &= \underline{K}^T(r) \underline{p} & ; & \quad \underline{K}(r) = \frac{1}{r} \int_0^\infty \underline{k}(\rho) \mathcal{J}_1(\rho r) d\rho \\ \sigma_2(r) &= \underline{N}^T(r) \underline{q} & ; & \quad \underline{N}(r) = \frac{1}{r} \int_0^\infty \underline{n}(\rho) \mathcal{J}_1(\rho r) d\rho \\ \sigma_3(r) &= \underline{M}^T(r) \underline{q} & ; & \quad \underline{M}(r) = \int_0^\infty \underline{m}(\rho) \mathcal{J}_0(\rho r) \rho d\rho \end{aligned} \quad 5.9$$

The radial and hoop stresses are found by further substituting these back into (5.7). The vectors, $\underline{K}(r)$, $\underline{N}(r)$, $\underline{M}(r)$, are functions of r and the mesh only, and so, for a given mesh, can be calculated in advance, at a number of different values of r .

In the incompressible limit, $\nu \rightarrow 0$, the value of $\sigma_1(r)$ obtained from (5.9) is the numerical approximation to (5.8). It was calculated with $\underline{K}(r)$, $\underline{N}(r)$,

$\underline{M}(r)$, at the end of STEP 1, and the results found to compare favourably. Then at the end of each stage of loading and unloading, $\sigma_2(r)$ and $\sigma_3(r)$ were calculated from (5.9), using the stored values of $\underline{N}(r)$, $\underline{M}(r)$. The numerical values could be checked, $\sigma_2(r)$ by using the fact that from (5.5), in any adhesive region, $\sigma_2(r) = \sigma_1(r)$. For the numerical values this was found to be so to less than 5×10^{-4} . It can also be shown that $\sigma_3(r) \sim -\sigma_2(r)$ as $r \rightarrow \infty$. This can be done in two ways. For both, the first step is to find the asymptotic behaviour of $\sigma_2(r)$ for large r . Letting $\rho r = \eta$ in (5.4)

$$\sigma_2(r) = \frac{1}{r} \int_0^\infty \bar{q}(\rho) J_1(\rho r) d\rho = \frac{1}{r^2} \int_0^\infty \bar{q}\left(\frac{\eta}{r}\right) J_1(\eta) d\eta \quad 5.10$$

The asymptotic behaviour of $\bar{q}(\eta/r)$ can be obtained from the definition of $\bar{q}(\rho)$

$$\begin{aligned} \bar{q}\left(\frac{\eta}{r}\right) &= \int_0^1 q(s) J_1\left(\frac{\eta s}{r}\right) s ds \\ &\sim \frac{\alpha_1 \eta}{r^2} \int_0^1 q(s) s^2 ds \\ &\sim \frac{\alpha_2 \eta}{r^2} \end{aligned} \quad 5.11$$

$$\begin{aligned} \text{Thus } \sigma_2(r) &\sim \frac{\alpha_2}{r^3} \int_0^\infty J_1(\eta) \eta d\eta \\ &\sim \frac{\alpha}{r^3} \end{aligned} \quad 5.12$$

Similarly the asymptotic behaviour of $\sigma_3(r)$ can be found after an integration by parts

$$\begin{aligned} \sigma_3(r) &= \frac{1}{r^2} \int_0^\infty \bar{q}\left(\frac{\eta}{r}\right) J_0(\eta) \eta d\eta \\ &\sim \frac{\alpha_2}{r^3} \int_0^\infty J_0(\eta) \eta^2 d\eta \\ &\sim -\frac{\alpha_2}{r^3} \int_0^\infty J_1(\eta) \eta d\eta + \frac{\alpha_2}{r^3} \eta^2 J_1(\eta) \Big|_0^\infty \\ &\sim -\frac{\alpha}{r^3} \sim -\sigma_2(r) \end{aligned} \quad 5.13$$

The same result can be proved by noticing from the definition of $\sigma_2(r)$ and $\sigma_3(r)$, that

$$\sigma_3(r) = \frac{1}{r} \frac{d}{dr} (r^2 \sigma_2(r)) \quad 5.14$$

and then substituting (5.12) into (5.14) gives (5.13). The numerical values exhibited this behaviour for $r < 10$, for all distributions of $q(r)$, but for larger values of r , inaccuracies of the order of the actual size of the solution entered. Thus no faith could be put in the numerical stress values for $r > 8$, say, but this was of little matter, since all the important effects occurred in $r < 3$.

The numerical implementation of (5.19) is covered in Chapter Six, and some results obtained for the numerical stress distributions of the last chapter outlined in the next section.

§5.2 Results of the Stress Calculations.

Numerical values for both the radial and hoop stresses, σ_r and σ_θ respectively, were obtained for the frictionless problem, the adhesive problem, and the frictional loading problem, for which the adhesive radius is $c = 0.7$, that is $\rho = 0.6652$, and finally for the unloading problem for this value of Coulomb friction, but only the results for the radial stress proved to be of any significance.

The numerical values obtained for $\sigma_r(r) = p_1(r)$ in the frictionless case, provide the pressure term for later steps, and were found to agree with the expected distribution, (5.8), to within about 2×10^{-4} . The obtained values are plotted as part of FIG 5.1.

The numerical values obtained for $\sigma_r(r)$ in the adhesive and frictional loading solutions, showed the expected pattern, though the results in the adhesive case were rather obviously inaccurate for $r > 8.0$. This was probably caused by the poor accuracy of the values for $q(r)$ itself, close to the singularity, the stress becoming more and more dependent on these alone, far from the contact area. The radial stress obtained a maximum value at $r = 2$ in the adhesive case, and at $r = 1.75$ for the above frictional problem. The results are plotted for $r > 1$ for the frictional loading problem, for this particular value of Coulomb friction, in FIG 5.1. The contributions from the normal pressure, $\sigma_1(r)$, and the shear, $(-\sigma_2(r) + 2\sigma_3(r))$, are plotted along with the actual stress distribution, $\sigma_r(r)$, which is the sum of the above two. The radial stress obtains a maximum value of

$\sigma_r(1.75) = 0.1509$, as compared to unity for the frictionless case.

The unloading solution also demonstrates the predicted behaviour. A full set of curves is not given, because they carry very little information, but in FIG 5.2 the position and magnitude of the maximum radial stress is plotted as a function of the loading and unloading parameter, λ . (It will be recalled that if the maximum normal load achieved during loading is $P^* = 1$, then the loading and unloading parameter is the value of the load at some other time, $P^* = \lambda$. Thus the normal stress at that point is $p(r) = (2/\pi) \lambda (1-r^2)^{-1/2}$). As mentioned above, the position of the maximum stress maintains a constant position at $r = 1.75$ during the loading phase, and its magnitude increases linearly to $\sigma_r(1.75) = 0.1509$. As unloading commences, the position of the maximum stress starts to move immediately toward the contact area, reaching the outside edge slightly after the region of outward slip forms, at $\lambda = 0.485$ as compared to $\lambda = 0.515$. The reason for this must be that even during the early stages of outward slip, the region of inward slip still has a noticeable compressive effect. Meanwhile the value of the maximum stress starts to decrease almost linearly with λ , (the radial stress at a point actually decreasing linearly), though at a slower rate than the increase during the loading phase, until the region of outward slip forms. At this point it takes a sudden sharp increase, going through an almost parabolic loop, reaching a maximum value of 0.55, (for this particular value of ρ), at about $\lambda = 0.25$. Though this maximum stress is still short of the frictionless value, the half-space could be loaded to just over one quarter of the frictional value, and it would still fracture during the unloading phase. Further, unless it has been loaded to just short of failure, it will fracture at the edge of the contact region during unloading.

Though it is expected that spherical indentors will exhibit similar behaviour, it cannot be truly predicted from the above results what the

ratio of maximum stress in loading and unloading will be in a particular case, nor at what value of the unloading parameter the absolute maximum will occur. The above results therefore do not confirm the experiments reported by JOHNSON ET AL (1973), nor the experiments confirm these results. What can be said is that the numerical results are at least consistent with the empirical ones, and further they predict a delay before fracture occurs on unloading, even if the half-space has been loaded to the point of frictional failure, whereas any regime that postulates immediate outward slip would probably predict immediate cracking. In fact, in an attempt to confirm the above results, that portion of the maximum stress curve marked AB in FIG 5.2, could perhaps be found experimentally.

This completes the major results of this thesis.

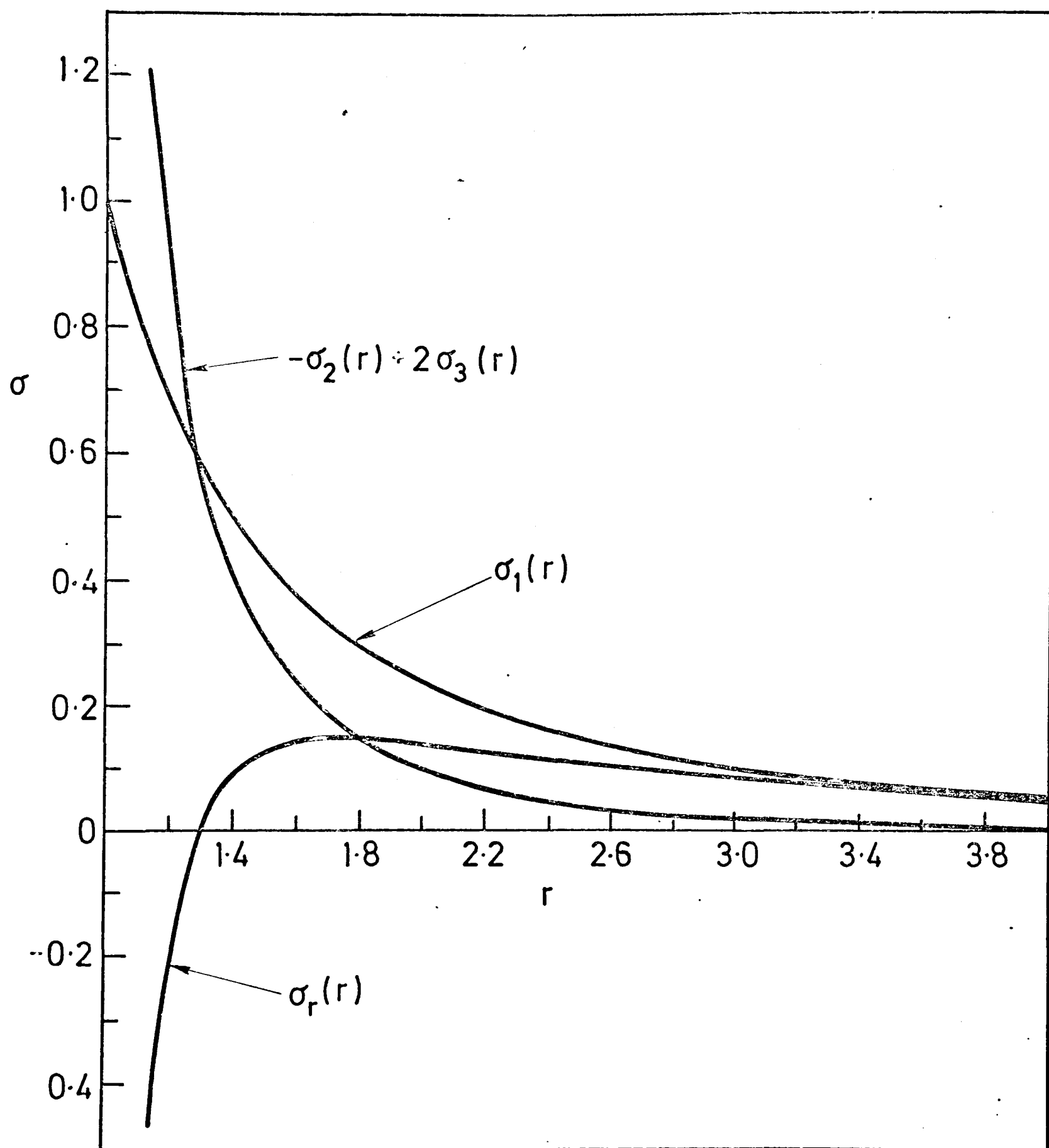


Fig.5.1 Surface radial stress $\sigma_r(r)$, outside the contact area $r \geq 1$, for the frictional loading case $p=0.6652$, with its pressure, $\sigma_1(r)$, and shear $-\sigma_2(r) + 2\sigma_3(r)$, components.

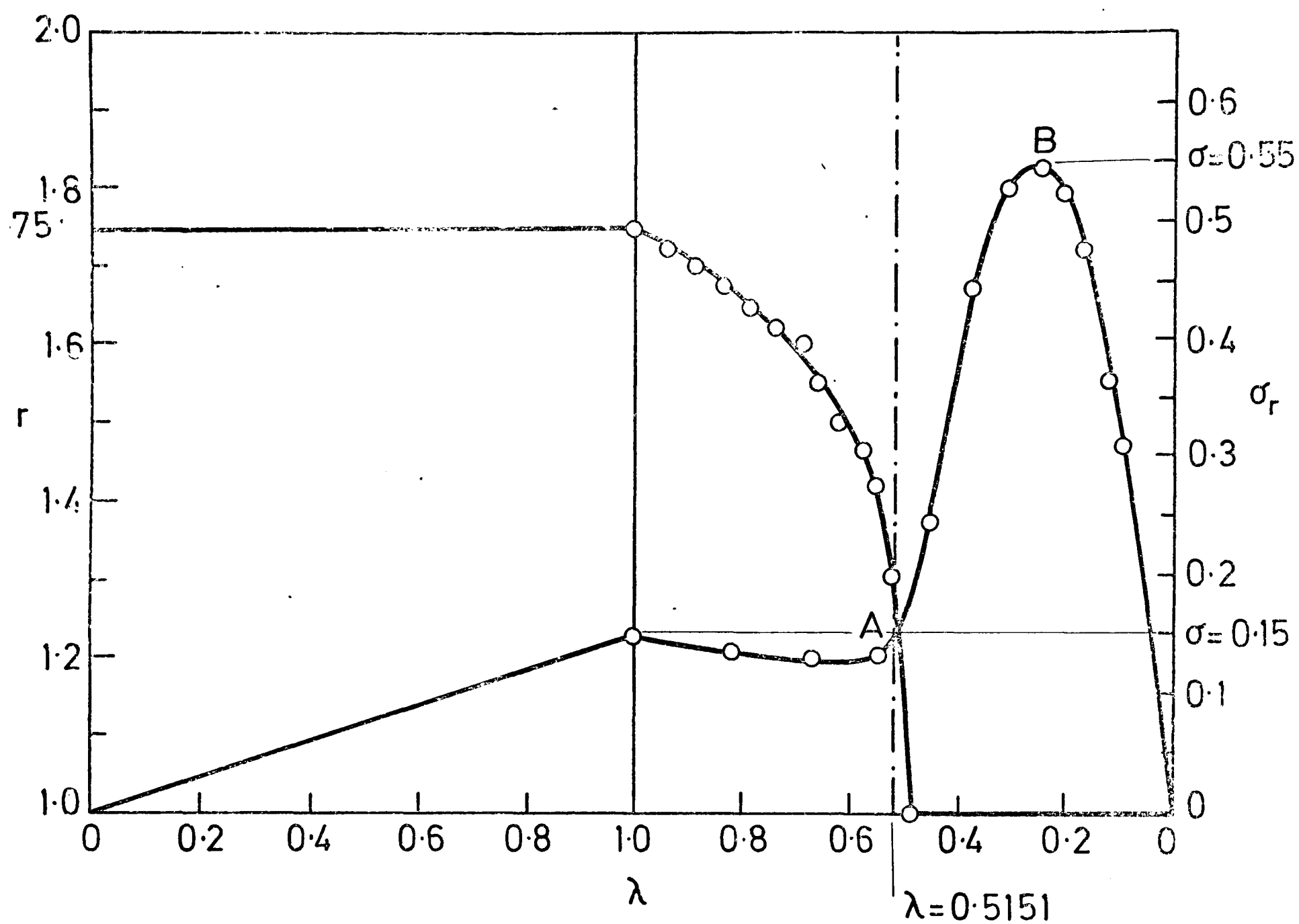


Fig. 5.2 Position, r , and value, σ_r , of the maximum tensile radial stress against the loading and unloading parameters, λ .

The Hertz value is $\sigma_r(r) = 1.000$

PART TWO
Chapter Six

ON THE DETAILS OF THE
NUMERICAL METHOD

§6.1 The Finite-Dimensional Forms for $\bar{p}(\rho)$, $\bar{q}(\rho)$ and P^* .

It will be recalled from §4.1, that the functions $p(r) \in H_1$, $q(r) \in H_2$, where H_1 and H_2 were defined in that section, are to be approximated by the elements of two finite dimensional subspaces, $\hat{p}(r) \in H_1^m$, $\hat{q}(r) \in H_2^m$, $r \in [0,1]$. The finite element approximation, in which H_1^m , H_2^m are spanned by a basis of splines, was used for this work. The line $[0,1]$ is divided into n elements, separated by $m = n+1$ nodes, the functions $\hat{p}(r)$, $\hat{q}(r)$ are required to be continuous at each node, and are given in the j th element by

$$\begin{aligned} \hat{p}(r) &= \alpha_1^j + \alpha_2^j r^2 & \hat{q}(r) &= \beta_1^j r + \beta_2^j r^3 & j=1, n-1 \\ \hat{p}(r) &= \frac{\alpha_1^n + \alpha_2^n r^2}{\sqrt{1-r^2}} & \hat{q}(r) &= \frac{\beta_1^n r + \beta_2^n r^3}{\sqrt{1-r^2}} & j=n \end{aligned} \quad 6.1$$

Stress vectors $\underline{p} = \{p_j\} \in \mathcal{R}^m$, $\underline{q} = \{q_j\} \in \mathcal{R}^m$ are defined such that

$$\begin{aligned} q_1 &= r_2 q'(0) \\ p_j &= p(r_j) \quad j=1, n & q_j &= q(r_j) \quad j=2, n \\ p_m &= \lim_{r \rightarrow 1} \frac{\sqrt{1-r^2} p(r)}{\sqrt{1-a^2}} & q_m &= \lim_{r \rightarrow 1} \frac{\sqrt{1-r^2} q(r)}{\sqrt{1-a^2}} \end{aligned} \quad 6.2$$

where r_j is the radius of the j th node, and $a = r_n$. Then, using the notation

$$\underline{\alpha}^j = \begin{pmatrix} \alpha_1^j \\ \alpha_2^j \end{pmatrix}; \quad \underline{\beta}^j = \begin{pmatrix} \beta_1^j \\ \beta_2^j \end{pmatrix}; \quad \underline{p}^j = \begin{pmatrix} p_1^j \\ p_2^j \end{pmatrix} = \begin{pmatrix} p_j \\ p_{j+1} \end{pmatrix}; \quad \underline{q}^j = \begin{pmatrix} q_1^j \\ q_2^j \end{pmatrix} = \begin{pmatrix} q_j \\ q_{j+1} \end{pmatrix}; \quad \underline{r}^j = \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} = \begin{pmatrix} r_j \\ r_{j+1} \end{pmatrix}$$

expressions for $\underline{\alpha}^j$ and $\underline{\beta}^j$ can be obtained in the j th element in terms of the nodal stress values $\underline{p}^j$, $\underline{q}^j$. It was shown in §4.1, that these expressions

are, after dropping the j superscripts,

$$\begin{aligned}
 \underline{\alpha} &= T \underline{p} = \frac{1}{(r_2^2 - r_1^2)} \begin{pmatrix} r_2^2 & -r_1^2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} & j=1, n-1 \\
 \underline{\alpha} &= T \underline{p} = \frac{1}{(1-a^2)} \begin{pmatrix} 1 & -a^2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-a^2} p_n \\ p_{n+1} \end{pmatrix} & j=n \\
 \underline{\beta} &= S \underline{q} = \frac{1}{r_2^3} \begin{pmatrix} r_2^2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} & j=1 \\
 \underline{\beta} &= S \underline{q} = \frac{1}{r_1 r_2 (r_2^2 - r_1^2)} \begin{pmatrix} r_2^3 & -r_1^3 \\ -r_2 & r_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} & j=2, n-1 \\
 \underline{\beta} &= T \underline{q} = \frac{1}{(1-a^2)} \begin{pmatrix} 1 & -a^2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-a^2} q_n / a \\ q_{n+1} \end{pmatrix} & j=n
 \end{aligned} \tag{6.3}$$

6.4

The above defines an isomorphism between $\mathcal{R}^m$ and H_1^m, H_2^m ,

$$\begin{aligned}
 \hat{p}(r) &= \sum_{j=1}^n (1-r^2) \underline{\alpha}^j = \sum_{j=1}^n (1-r^2) T^j \underline{p}^j & \hat{p}(r) \in H_1^m \\
 \hat{q}(r) &= \sum_{j=1}^n (r-r^3) \underline{\beta}^j = \sum_{j=1}^n (r-r^3) S^j \underline{q}^j & \hat{q}(r) \in H_2^m
 \end{aligned} \tag{6.5}$$

The Hankel transforms of the approximate functions, $\hat{p}(r)$, $\hat{q}(r)$, can be found by substituting (6.5) into (4.11)

$$\begin{aligned}
 \bar{p}(\rho) &= \int_0^1 p(r) J_0(\rho r) r dr \\
 &= \sum_{j=1}^{n-1} \int_{r_1}^{r_2} (1-r^2) J_0(\rho r) r dr \underline{\alpha}^j + \int_a^1 (1-r^2) J_0(\rho r) \frac{r dr}{\sqrt{1-r^2}} \underline{\alpha}^n \\
 &= \sum_{j=1}^n \underline{L}^j(\rho) \underline{\alpha}^j = \sum_{j=1}^n \underline{K}^j(\rho) \underline{p}^j = \underline{K}^T(\rho) \underline{p} \\
 \bar{q}(\rho) &= \int_0^1 q(r) J_1(\rho r) r dr \\
 &= \sum_{j=1}^{n-1} \int_{r_1}^{r_2} (r-r^3) J_1(\rho r) r dr \underline{\beta}^j + \int_a^1 (r-r^3) J_1(\rho r) \frac{r dr}{\sqrt{1-r^2}} \underline{\beta}^n \\
 &= \sum_{j=1}^n \underline{m}^j(\rho) \underline{\beta}^j = \sum_{j=1}^n \underline{n}^j(\rho) \underline{q}^j = \underline{n}^T(\rho) \underline{q}
 \end{aligned} \tag{6.6}$$

The terms in the above summation for $j=1, n-1$ contain indefinite integrals of the form

$$I_{\mu\nu} = \int z^\mu J_\nu(z) dz \quad \mu, \nu \text{ integers}$$

This integral can be found on integrating by parts, using a standard formula for Bessel functions,

$$I_{\mu\nu} = \int z^{\mu-\nu-1} (z^{\nu+1} J_\nu(z)) dz$$

$$= z^{\mu-\nu-1} z^{\nu+1} J_{\nu+1}(z) - (\mu-\nu-1) \int z^{\mu-\nu-2} z^{\nu+1} J_{\nu+1}(z) dz$$

$$= z^{\mu} J_{\nu+1}(z) - (\mu-\nu-1) I_{\mu-\nu-1, \nu+1}$$

If $\mu = 2n+\nu+1$, this recurrence relation can be repeated until $\mu_n = \nu+n+1$, $\nu_n = \nu+n$, in which case the coefficient of the integral on the RHS is zero. This gives an expansion for $I_{\mu\nu}$ in terms of powers of z , and $J_{\mu\nu}(z)$, $k = 1, n+1$, which when substituted into (6.6) gives

$$\underline{L}^j(\rho) = \frac{r_2 J_1(r_2 \rho)}{\rho} \begin{pmatrix} 1 \\ r_2^2 \end{pmatrix} - \frac{r_1 J_1(r_1 \rho)}{\rho} \begin{pmatrix} 1 \\ r_1^2 \end{pmatrix} - \frac{2r_2^2 J_2(r_2 \rho)}{\rho^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2r_1^2 J_2(r_1 \rho)}{\rho^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 6.7$$

$$\underline{m}^j(\rho) = \frac{r_2^2 J_2(r_2 \rho)}{\rho} \begin{pmatrix} 1 \\ r_2^2 \end{pmatrix} - \frac{r_1^2 J_2(r_1 \rho)}{\rho} \begin{pmatrix} 1 \\ r_1^2 \end{pmatrix} - \frac{2r_2^3 J_3(r_2 \rho)}{\rho^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2r_1^3 J_3(r_1 \rho)}{\rho^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Further defining

$$A(r, m; \sigma) = \frac{r^m J_m(r\sigma)}{\sigma^m}; \quad B(r, m; \sigma) = \frac{r^m J_{m+1}(r\sigma)}{\sigma^m}$$

$$\alpha_j = \frac{2}{\sigma_{j+1}^2 - \sigma_j^2}; \quad \gamma_j = \frac{\sigma_{j+1}}{\sigma_j}; \quad z_j = \frac{\sigma_j}{\sigma_{j+1}} \quad 6.8$$

then with (6.3), (6.4), it is found that for $j = 1, n-1$,

$$R_1^j(\rho) = -A(\sigma_j, 1; \rho) + \alpha_j A(\sigma_{j+1}, 2; \rho) - \alpha_j A(\sigma_j, 2; \rho)$$

$$R_2^j(\rho) = A(\sigma_{j+1}, 1; \rho) - \alpha_j A(\sigma_{j+1}, 2; \rho) + \alpha_j A(\sigma_j, 2; \rho)$$

$$n_1^j(\rho) = -B(\sigma_j, 1; \rho) + \alpha_j \gamma_j B(\sigma_{j+1}, 2; \rho) - \alpha_j B(\sigma_j, 2; \rho)$$

$$n_2^j(\rho) = B(\sigma_{j+1}, 1; \rho) - \alpha_j B(\sigma_{j+1}, 2; \rho) + \alpha_j z_j B(\sigma_j, 2; \rho) \quad 6.9$$

This equation holds even for $n_1^1(\rho)$, $n_2^1(\rho)$, in spite of the different definition for q_1 , because $r_1 = 0$.

The final terms of the summations are not as simple however, since they contain the integrals

$$\underline{L}^n(\rho) = \int_a^1 (r_2) J_0(\rho r) \frac{dr}{\sqrt{1-r^2}}; \quad \underline{m}^n(\rho) = \int_a^1 (r_2^2) J_1(\rho r) \frac{dr}{\sqrt{1-r^2}} \quad 6.10$$

which cannot be found exactly, there being no corresponding indefinite integral. To overcome this problem, it was decided to represent $(1-r^2)^{-1/2}$ as a polynomial of the form $A+Br^2$, with the coefficients chosen to make

$$\frac{1}{\sqrt{1-a^2}} = A + Ba^2 ; \quad \int_a^1 \frac{r dr}{\sqrt{1-r^2}} = \int_a^1 (A + Br^2) r dr \quad 6.11$$

the appropriate values being $A = -(3a^2 - 1)(1 - a^2)^{-1/2}$ and $B = 2(1 - a^2)^{-1/2}$.

Though it may appear, that this introduces a further approximation into the numerical method, it is, in fact, only equivalent to choosing $\hat{p}(r)$, $\hat{q}(r)$ in the n th element to have the form

$$\hat{p}(r) = (\alpha_1 r + \alpha_2 r^2)(A + Br^2) ; \quad \hat{q}(r) = (\beta_1 r + \beta_2 r^3)(A + Br^2) \quad 6.12$$

This cannot give infinite values for $\hat{p}(r)$, $\hat{q}(r)$ at $r = 1$, the value of the polynomial at that point being $A + B = 3(1 - a^2)^{-1/2}$, but using a polynomial of this form gave better results than taking no account of the singularity at all, $A = 1$, $B = 0$. It would be possible to take higher order polynomials, $A + Br^2 + Cr^4 \dots$, matching integrals with higher powers of r , arising in a Taylor series expansion of $J_0(\rho r)$, $J_1(\rho r)$, but to do so would prove needlessly cumbersome, considering the good results achieved with the quadratic. With the above polynomial, the final terms in the summations become

$$\underline{L}^n(\rho) = \int_a^1 \left(\frac{r}{r^3} \right) (A + Br^2) J_0(\rho r) dr ; \quad \underline{M}^n(\rho) = \int_a^1 \left(\frac{r^2}{r^4} \right) (A + Br^2) J_1(\rho r) dr$$

both of which can be calculated as before. It is found that

$$\underline{L}^n(\rho) = \frac{1}{\sqrt{1-a^2}} \left\{ \frac{3J_1(\rho)}{\rho} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{aJ_1(a\rho)}{\rho} \begin{pmatrix} 1 \\ a^2 \end{pmatrix} - \frac{2J_2(\rho)}{(1-a^2)\rho^2} \begin{pmatrix} 2 \\ 5-3a^2 \end{pmatrix} + \frac{2a^2J_2(a\rho)}{(1-a^2)\rho^2} \begin{pmatrix} 2 \\ 1+a^2 \end{pmatrix} \right. \\ \left. + \frac{16}{(1-a^2)} \left(\frac{J_3(\rho)}{\rho^3} - \frac{a^3J_3(a\rho)}{\rho^3} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\underline{M}^n(\rho) = \frac{1}{\sqrt{1-a^2}} \left\{ \frac{3J_2(\rho)}{\rho} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{a^2J_2(a\rho)}{\rho} \begin{pmatrix} 1 \\ a^2 \end{pmatrix} - \frac{2J_3(\rho)}{(1-a^2)\rho^2} \begin{pmatrix} 2 \\ 5-3a^2 \end{pmatrix} + \frac{2a^2J_3(a\rho)}{(1-a^2)\rho^2} \begin{pmatrix} 2 \\ 1+a^2 \end{pmatrix} \right. \\ \left. + \frac{16}{(1-a^2)} \left(\frac{J_4(\rho)}{\rho^3} - \frac{a^4J_4(a\rho)}{\rho^3} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

or on substituting from (6.3), (6.4) into (6.6)

$$\begin{aligned} R_1^n(\rho) &= -A(r_n, 1) + 3\lambda_n A(r_n, 2) + 2\lambda_n A(r_n, 2) - 4\lambda_n^2 (A(r_n, 3) - A(r_n, 3)) \\ R_2^n(\rho) &= 3A(r_n, 1) - 5\lambda_n A(r_n, 2) + 2\lambda_n A(r_n, 2) + 4\lambda_n^2 (A(r_n, 3) - A(r_n, 3)) \end{aligned} \quad 6.13a$$

$$\begin{aligned} n_1^*(\rho) &= -B(r_n, 1) + 3\lambda_n \gamma_n B(r_n, 2) + \lambda_n B(r_n, 2) - 4\lambda_n^2 \gamma_n (B(r_n, 3) - 2\lambda_n B(r_n, 3)) \\ n_2^*(\rho) &= 3B(r_n, 1) - 5\lambda_n B(r_n, 2) + \lambda_n 2\lambda_n B(r_n, 2) - 4\lambda_n^2 (B(r_n, 3) - 2\lambda_n B(r_n, 3)) \end{aligned} \quad 6.13$$

The beauty is that in both (6.9), (6.13), k_i^* can be obtained from n_i^* by changing $B(r_j, m)$ to $A(r_j, m)$, ($A(r_j, m; \rho)$ understood), and by letting $y_j = z_j = 1$.

Now finally inspecting (6.6), it can be seen that $k_j(\rho)$ is the coefficient of p_j , but this contains a contribution from the $(j-1)$ th element and from the j th element, that is

$$\begin{aligned} k_1(\rho) &= k_1^*(\rho) & ; & & n_1(\rho) &= n_1^*(\rho) \\ k_j(\rho) &= k_{j-1}^*(\rho) + k_j^*(\rho) & ; & & n_j(\rho) &= n_{j-1}^*(\rho) + n_j^*(\rho) \quad j=2, n \\ k_n(\rho) &= k_n^*(\rho) & ; & & n_n(\rho) &= n_n^*(\rho) \end{aligned} \quad 6.14$$

Substituting (6.9), (6.13) into (6.14) gives the expressions for $k_i(\rho)$, $n_i(\rho)$

$$\begin{aligned} k_1(\rho) &= \lambda_1 A(r_2, 2) \\ k_j(\rho) &= \lambda_j (A(r_{j+1}, 2) - A(r_j, 2)) + \lambda_{j-1} (A(r_{j-1}, 2) - A(r_j, 2)) \quad j=2, n-1 \\ k_n(\rho) &= \lambda_n (3A(r_n, 2) + A(r_n, 2)) + \lambda_{n-1} (A(r_n, 2) - A(r_n, 2)) - 4\lambda_n^2 (A(r_n, 3) - A(r_n, 3)) \\ k_n(\rho) &= 3A(r_n, 1) + \lambda_n (A(r_n, 2) - 5A(r_n, 2)) + 4\lambda_n^2 (A(r_n, 3) - A(r_n, 3)) \\ n_1(\rho) &= \lambda_1 B(r_2, 2) \\ n_j(\rho) &= \lambda_j (\gamma_j B(r_{j+1}, 2) - B(r_j, 2)) + \lambda_{j-1} (2\lambda_{j-1} B(r_{j-1}, 2) - B(r_j, 2)) \quad j=2, n-1 \\ n_n(\rho) &= \lambda_n (3\gamma_n B(r_n, 2) + B(r_n, 2)) + \lambda_{n-1} (2\lambda_{n-1} B(r_n, 2) - B(r_n, 2)) - 4\lambda_n^2 \gamma_n (B(r_n, 3) - 2\lambda_n B(r_n, 3)) \\ n_n(\rho) &= 3B(r_n, 1) + \lambda_n (2\lambda_n B(r_n, 2) - 5B(r_n, 2)) + 4\lambda_n^2 (B(r_n, 3) - 2\lambda_n B(r_n, 3)) \end{aligned} \quad 6.15$$

It is shown in the next section how these expressions can be used to calculate the matrices A_{11} , A_{12} , A_{22} , appearing in the quadratic part of the energy functionals $J_1 - J_4$, and in §6.3 to calculate the finite dimensional form of the in plane surface stresses.

In a similar fashion, expressions can be found for other linear terms

of the form (4.25). Only the compliance, P^* , is needed for the flat punch, so the finite dimensional form is derived here.

$$\begin{aligned} P^* &= \int_0^1 p(r) r dr = \sum_{j=1}^{n-1} \int_{r_1}^{r_2} (r r^3) dr \underline{\alpha}^j + \int_a^1 (r r^3) (A + B r^2) dr \underline{\alpha}^n \\ &= \sum_{j=1}^n \underline{b}^{jT} \underline{\alpha}^j = \sum_{j=1}^n \underline{b}^{jT} p_j = \underline{b}^T \underline{p} \end{aligned} \quad 6.16$$

It is necessary to use the polynomial expansion for $p(r)$ in the last term of the summation, to maintain consistency with the above, even though it is now possible to calculate the integral involved.

$$\int_a^1 (r r^3) \frac{dr}{\sqrt{1-r^2}} \quad 6.17$$

The difference between the integral (6.17), and the corresponding one in (6.16) is of order $(1-a)$. This was found to produce an appreciable change in the final solution, especially in the test runs which had only ten elements, though this error does vanish as the number of elements is increased.

For the first $(n-1)$ terms in the summation (6.16),

$$\underline{b}^j = \int_{r_1}^{r_2} \begin{pmatrix} r \\ r^3 \end{pmatrix} dr = \begin{pmatrix} \frac{1}{2}(r_2^2 - r_1^2) \\ \frac{1}{4}(r_2^4 - r_1^4) \end{pmatrix} \quad j=1, n-1 \quad 6.18$$

whence using (6.3)

$$\underline{b}^j = \frac{r_2^2 - r_1^2}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and for the final term

$$\begin{aligned} \underline{b}^n &= \int_a^1 (A + B r^2) \begin{pmatrix} r \\ r^3 \end{pmatrix} dr = \frac{\sqrt{1-a^2}}{12} \begin{pmatrix} 12 \\ 7 + 5a^2 \end{pmatrix} \\ \underline{b}^n &= \left(\frac{1-a^2}{12} \right) \begin{pmatrix} 5 \\ 7 \end{pmatrix} \end{aligned} \quad 6.19$$

Again b_j is the coefficient of p_j in the summation, which has a contribution from both the $(j-1)$ th and the j th elements, as before.

$$b_j = b_{j-1} + b_j^j \quad 6.20$$

In programming, it proved easier here to calculate $\underline{b}^j$ in a loop, and add b_i^j and b_i^j into the appropriate b_i . This vector was compiled in the subroutine NOBLE1. Subroutines NOBLE2 and NOBLE3 were also written to calculate the approximation to linear terms of the form

$$\int_0^1 p(r) r^3 dr ; \int_0^1 q(r) r^3 dr$$

respectively. These are used in the solution of problems involving parabolic (spherical), indentors.

§6.2 The Matrices A_{11} , A_{22} , A_{12} .

Having calculated the Hankel transforms for the discrete stress distributions, They can be used to find the matrices A_{11} , A_{12} , A_{22} , contained in the corresponding quadratic terms in the functionals $J_1 - J_4$. It will be recalled from (4.24) that the matrices are given by

$$\begin{aligned} A_{11} &= \int_0^\infty R_i(\rho) R_i^T(\rho) d\rho & A_{11ij} &= \int_0^\infty R_i(\rho) R_j(\rho) d\rho \\ A_{22} &= \int_0^\infty n_i(\rho) n_i^T(\rho) d\rho & A_{22ij} &= \int_0^\infty n_i(\rho) n_j(\rho) d\rho \\ A_{12} &= \int_0^\infty R_i(\rho) n_i^T(\rho) d\rho & A_{12ij} &= \int_0^\infty R_i(\rho) n_j(\rho) d\rho \end{aligned} \quad 6.21$$

A complete list of all the different terms in the above matrices is given in Appendix C, and can be found in the programme listing, (Appendix D), as well. There are sixteen types of term for each matrix, for the values of i , $i=1$, $i=2, n-1$, $i=n$, $i=m$, and likewise j . A general term can be written for $A_{kl ij}$, with k, l taking either of the values 1, 2, but with i, j as each of the four separate cases, because as mentioned earlier, $k_i(\rho)$ can be obtained from $n_i(\rho)$, by changing $B(r_i, m; \rho)$ to $A(r_i, m; \rho)$, and setting $y_j = z_j = 1$, $\forall j = 1, n$. Some reduction was also obtained in the programme using the fact that A_{11} , A_{22} are symmetric. The matrices were calculated by the subroutine NOBLE, which is separated into two parts, the first to calculate A_{11} , A_{22} , and the second A_{12} .

The important point is that the integration in (6.21) involves only the following Weber-Schafheitlin integrals.

$$\begin{aligned}
I(r,s,m,n) &= \int_0^\infty A(r,m;\rho) A(s,n;\rho) d\rho = r^m s^n \int_0^\infty \frac{J_m(\rho r) J_n(\rho s)}{\rho^{m+n}} d\rho \\
J(r,s,m,n) &= \int_0^\infty B(r,m;\rho) B(s,n;\rho) d\rho = r^m s^n \int_0^\infty \frac{J_{m+1}(\rho r) J_{n+1}(\rho s)}{\rho^{m+n}} d\rho \\
K(r,s,m,n) &= \int_0^\infty A(r,m;\rho) B(s,n;\rho) d\rho = r^m s^n \int_0^\infty \frac{J_m(\rho r) J_{n+1}(\rho s)}{\rho^{m+n}} d\rho
\end{aligned} \tag{6.22}$$

for which exact expressions can be found, MAGNUS ET AL (1966), ppg 99.

Thus no approximations need be made to calculate the terms of the functionals $J_1 - J_4$ for the assumed stress distributions.

The Weber-Scharfheitlin integrals are discontinuous at $r = s$, though from the definitions, the above integrals obviously obey $I(r,s,m,n) = I(s,r,n,m)$, and $J(r,s,m,n) = J(s,r,n,m)$. It can be shown, (ibid), that

$$\begin{aligned}
I(r,s,m,n) &= G(m,n,1,1) r^{2m+2n-1} & r = s \\
&= G(m,n,2,1) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-m; n+1; z^2\right) r^{2m-1} s^{2n} & r > s \\
&= G(m,n,3,1) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-n; m+1; y^2\right) r^{2m} s^{2n-1} & r < s
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
J(r,s,m,n) &= G(m,n,1,2) r^{2m+2n-1} & r = s \\
&= G(m,n,2,2) {}_2F_1\left(\frac{3}{2}, \frac{1}{2}-m; n+2; z^2\right) r^{2m-2} s^{2n+1} & r > s \\
&= G(m,n,3,2) {}_2F_1\left(\frac{3}{2}, \frac{1}{2}-n; m+2; y^2\right) r^{2m+1} s^{2n-2} & r < s
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
K(r,s,m,n) &= G(m,n,1,3) r^{2m+2n-1} & r = s \\
&= G(m,n,2,3) {}_2F_1(1, 1-m; n+2; z^2) r^{2m-2} s^{2n+1} & r > s \\
&= G(m,n,3,3) {}_2F_1(1, -n; m+1; y^2) r^{2m} s^{2n-1} & r < s
\end{aligned} \tag{6.25}$$

where (a) $z = s/r$, $y = r/s$,

$$\begin{aligned}
(b) \quad G(m,n,1,1) &= \frac{\Gamma(\frac{1}{2})\Gamma(m+n)}{2^{m+n}\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})} \\
G(m,n,2,1) &= G(m,n,3,1) = \frac{\Gamma(\frac{1}{2})}{2^{m+n}\Gamma(m+\frac{1}{2})\Gamma(n+1)} \\
G(m,n,1,2) &= \frac{G(m,n,1,1)}{2^{m+2n+1}} \\
G(m,n,2,2) &= G(m,n,3,2) = \frac{G(m,n,2,1)}{2(n+1)} \\
G(m,n,1,3) &= \frac{G(m,n)}{m+n}
\end{aligned}$$

$$\begin{aligned}
 G(m,n,2,3) &= \frac{G(m,n)}{n+1} \\
 G(m,n,3,3) &= \frac{G(m,n)}{n} \\
 G(m,n) &= \frac{1}{2^{m+n} \Gamma(n+1) \Gamma(m)}
 \end{aligned}$$

These constants were calculated in the subroutine SONI1, and were stored in COMMON so that they were on hand whenever one of the equations (6.23) - (6.25) was called.

(c) ${}_2F_1(a,b;c;z)$ is the hypergeometric function, ABRAMOWITZ AND STEGUN (1966), Chapter 15, for which two cases can be recognised above.

In (6.23), (6.24), $F(\frac{1}{2}+1, \frac{1}{2}-m; 1+n; z)$ can be calculated using Gauss's relations for contiguous functions, (ibid) (15.2.10) - (15.2.27), and the fact that

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k\right) = K(k) ; \quad {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k\right) = E(k)$$

the complete elliptic integrals of the first and second kind. These in turn can be calculated using the process of arithmetic-geometric means, described in (ibid), §17.6. As an aside, this method could also be used to calculate the corresponding incomplete integrals, the Zeta function, and the Heumann lambda function required for the analytic solution of the shear in the frictional loading problem, (3.27). In the programme, $K(k)$, $E(k)$ were calculated by the subroutine MILNE, $K(k)$, $E(k)$, $F(k/\psi)$, $E(k/\psi)$, $Z(k/\psi)$ by MNTAA1, and $H(k,\psi)$ by MNTBA1. The hypergeometric function, ${}_2F_1(\frac{1}{2}+1, \frac{1}{2}-m; 1+n; k)$ was then calculated by the subroutine GAUSS2.

In (6.25), if $b = 0$, ${}_2F_1(a, 0; c; z) = 1$, $\forall a, c, z$, and if $b = -m$, where m is an integer, ${}_2F_1(a, -m; c, c)$ is a polynomial of degree $-m$, and was calculated by F21AA1.

The integrals themselves were calculated in the subroutine AMGNS1, which has five parameters, r, s, m, n and k , where $k = 1$ for A_{11} , $k = 2$ for A_{12} ,

and $k = 3$ for A_{12} . AMGNS1 calls the constants $G(m,n,p,k)$, where it can be seen that p takes the values 1,2,3, if $r = s$, $r > s$, $r < s$, respectively, and then GAUSS2 if $k = 1,2$, or F21AA1 if $k = 3$. Thus writing NOBLE, the subroutine could be cast solely in terms of k .

This concludes the numerical implementation of the minimization of the problems described in Chapter Four. The only approximation involved, other than the truncation of the iterative calculation of the elliptic integrals, is the limitation of the stress to a finite dimensional subspace of the original solution space. Convergence of this method would appear to be obvious, because as higher dimensional subspaces are taken, each containing the previous one, the energy norm must be non-increasing. This is demonstrated in the next chapter, though problems with stability arise because terms enter the solution which, while allowing the energy norm to decrease, actually increase the L_1 norm.

§6.3 The Finite Dimensional Form of the In-Plane Surface Stresses.

It was shown in the last chapter, equation (5.7), that outside the contact region, $r > 1$, the in-plane surface stresses are $O(\chi)$, and given by

$$\sigma_r(r, 0) = \sigma_1(r) - \sigma_2(r) + 2\sigma_3(r) ; \quad \sigma_\theta(r, 0) = -\sigma_1(r) + \sigma_2(r) + \sigma_3(r) \quad 6.26$$

where

$$\begin{aligned} \sigma_1(r) &= \frac{1}{r} \int_0^\infty \bar{p}(\rho) J_1(\rho r) d\rho \\ \sigma_2(r) &= \frac{1}{r} \int_0^\infty \bar{q}(\rho) J_1(\rho r) d\rho \\ \sigma_3(r) &= \int_0^\infty \bar{q}(\rho) J_0(\rho r) \rho d\rho \end{aligned} \quad 6.27$$

The above stresses can be calculated quite simply for finite dimensional functions, $\hat{p}(r)$, $\hat{q}(r)$ of the type (6.5). Substituting (6.6) into (6.27) gives

$$\sigma_1(r) = \underline{K}^T(r) \underline{p} \quad ; \quad \sigma_2(r) = \underline{N}^T(r) \underline{q} \quad ; \quad \sigma_3(r) = \underline{M}^T(r) \underline{q} \quad 6.28$$

$$\begin{aligned} \text{where} \quad \underline{K}(r) &= \frac{1}{r} \int_0^\infty \underline{k}(\rho) J_1(\rho r) d\rho & K_i(r) &= \frac{1}{r} \int_0^\infty k_i(\rho) J_1(\rho r) d\rho \\ \underline{N}(r) &= \frac{1}{r} \int_0^\infty \underline{n}(\rho) J_1(\rho r) d\rho & N_i(r) &= \frac{1}{r} \int_0^\infty n_i(\rho) J_1(\rho r) d\rho \\ \underline{M}(r) &= \int_0^\infty \underline{n}(\rho) J_0(\rho r) \rho d\rho & M_i(r) &= \int_0^\infty n_i(\rho) J_0(\rho r) \rho d\rho \end{aligned}$$

The vectors $\underline{K}(r)$, $\underline{N}(r)$, $\underline{M}(r)$ are found from (6.15). $\underline{K}(r)$ is derived from $\underline{k}(\rho)$ by replacing $A(s, m; \rho)$ with $A(s, m, r)$, $\underline{N}(r)$ from $\underline{n}(\rho)$, by replacing $B(s, m; \rho)$ with $B(s, m, r)$, and $\underline{M}(r)$ from the latter by replacing $B(s, m; \rho)$ with $C(s, m, r)$, where $A(s, m, r)$, $B(s, m, r)$, and $C(s, m, r)$ are the Weber-Scharfheitlin integrals defined below

$$(1) \quad A(s, m, r) = \frac{1}{r} \int_0^\infty A(s, m; \rho) J_1(\rho r) d\rho = \frac{s^m}{r} \int_0^\infty \frac{J_m(\rho s) J_1(\rho r)}{\rho^m} d\rho \quad 6.29$$

$$\begin{aligned} A(s, m, r) &= H(m, 1, 1) s^{2m-2} & r &= s \\ &= H(m, 1, 2) {}_2F_1(1, 1-m; 2; z^2) s^{2m-2} & r < s \\ &= H(m, 1, 3) s^{2m} r^{-2} & r > s \end{aligned}$$

$$(2) \quad B(s, m, r) = \frac{1}{r} \int_0^\infty B(s, m; \rho) J_1(\rho r) d\rho = \frac{s^m}{r} \int_0^\infty \frac{J_{m+1}(\rho s) J_1(\rho r)}{\rho^m} d\rho \quad 6.30$$

$$\begin{aligned} B(s, m, r) &= H(m, 2, 1) s^{2m-2} & r &= s \\ &= H(m, 2, 2) {}_2F_1\left(\frac{3}{2}, \frac{1}{2}-m; 2; z^2\right) s^{2m-2} & r < s \\ &= H(m, 2, 3) {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; m+2; y^2\right) s^{2m+1} r^{-3} & r > s \end{aligned}$$

$$(3) \quad C(s, m, r) = \int_0^\infty B(s, m; \rho) J_0(\rho r) \rho d\rho = s^m \int_0^\infty \frac{J_{m+1}(\rho s) J_0(\rho r)}{\rho^{m-1}} d\rho \quad 6.31$$

$$\begin{aligned} C(s, m, r) &= H(m, 3, 1) s^{2m-2} & r &= s \\ &= H(m, 3, 2) {}_2F_1\left(\frac{3}{2}, \frac{1}{2}-m; 1; z^2\right) s^{2m-2} & r < s \\ &= H(m, 3, 3) {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; m+2; y^2\right) s^{2m+1} r^{-3} & r > s \end{aligned}$$

and (a) $z = r/s$, $y = s/r$ as before

$$(b) \quad H(m, 1, 1) = H(m, 1, 3) = \frac{1}{H1}$$

$$H(m, 1, 2) = m H(m, 1, 1)$$

$$H(m, 2, 1) = \frac{H1}{m(2m+1)\pi H2^2}$$

$$20 H(m, 2, 2) = H(m, 3, 2) = \frac{1}{H2}$$

$$H(m, 2, 3) = -H(m, 3, 3) = \frac{H(m, 1, 1)}{2(m+1)}$$

$$H(m, 3, 1) = \frac{-20(2m-4)!!}{\pi H2^2}$$

$$H1 = 2m!! = 2^m m!$$

$$H2 = (2m-1)!! = \frac{2^m \Gamma(m+\frac{1}{2})}{\pi^{1/2}}$$

These constants were calculated by SONI2, and again stored in COMMON, to be available when any of the equations (6.29) - (6.31) were called.

(c) The hypergeometric function is as discussed in the last section.

Inspecting the above integrals for $r > s$, it is known that for vanishing argument the hypergeometric function tends to unity, thus the integrals in $\sigma_1(r)$ vanish like $(1/r^2)$, and those in $\sigma_2(r)$, $\sigma_3(r)$ vanish like $(1/r^3)$, giving the expected stress behaviour in the far field. Note also that $C(s,1,s)$ is unbounded. In fact $m = 1$, only for $s = 1.0$, and represents a stress arising from the approximation to the stress singularity at the edge of the contact area.

The above integrals were calculated in AMGNS2, with parameters r, s, m and $k = 1, 2, 3$, where k is the subscript of the corresponding stress, $\sigma_k(r)$. Equation (6.29) was called when $k = 1$, (6.30) when $k = 2$, and (6.31) when $k = 3$.

A subroutine SSTAA1 was written to calculate $K(r)$, $N(r)$, $M(r)$ for a number of different radii. Because as already mentioned $k(\rho)$ is obtained from $n(\rho)$ by replacing $B(s, m; \rho)$ with $A(s, m; \rho)$, and letting $y_j = z_j = 1$, this subroutine could be cast in terms of the above parameter k only. It was called from a programme STRESS, which also calculated the stress, $\sigma_1(r)$ associated with a given normal stress distribution, $\hat{p}(r) = \{p\}$. A second subroutine SSTBA1 was written to take the above $\sigma_1(r)$, $K(r)$, $N(r)$, $M(r)$, and calculate $\sigma_2(r)$, $\sigma_3(r)$, as well as the radial stress, $\sigma_r(r, 0)$, and hoop stress, $\sigma_\theta(r, 0)$, associated with a given shear distribution. This was then called at each step of the loading and unloading, to give a complete description of the surface stress distribution.

Before proceeding to the programme structure, it should be mentioned that the horizontal surface displacement, with suitable scaling, is given by equation (3.19) as

$$\begin{aligned} u(r) &= -\int_0^\infty \bar{p}(\rho) J_1(\rho r) d\rho + \int_0^\infty \bar{q}(\rho) J_1(\rho r) d\rho \\ &= -r (\sigma_1(r) - \sigma_2(r)) \end{aligned}$$

Thus the above can be used to find the actual surface displacement, $u(r)$, as opposed to the linear functionals of the type (4.25), found earlier from the derivative of the energy functionals. The normal displacement could be found in a similar fashion, though at the time of writing the actual programming was not available.

§6.4 The Programme Structure.

The programme was written as four steps, each solving one of the basic problems outlined. The required results were both displayed, and written into the filestore to be used in subsequent steps. The programme was structured in the following way.

STEP1; The subroutine SONI1 was called to calculate the constants $G(m,n,k,l)$, and the subroutine RDATA to read the mesh points r_j from the filestore, and calculate the x_j . It was originally intended to write both $G(m,n,k,l)$ and x_j into the filestore at this point, to be read from STEP2, but it was found that the calculation of A_{11} , and A_{22} is so sensitive to the input data, involving the subtraction of large numbers, that truncating these constants to as many as eight digits still caused noticeable errors. It was not necessary to use double precision, but in choosing the error in truncation of the calculation of elliptic integrals, it was found that at least ten significant digits were needed to cause no noticeable change in the results.

Command was passed to HERTZ1, which called NOBLE to calculate A_{11} , NOBLE1 to calculate the vector $\underline{b}$, and THINBA1 the solution, $\underline{p} = A_{11}^{-1} \underline{b}$. Because A_{11} is very nearly singular, the solution being very sensitive to oscillatory errors in $\underline{b}$, it was found, by comparing the results with the known analytical results, that the most accurate method of inversion was to calculate the eigenvalues, λ^i , and the eigenvectors, u_j^i , and then write

A_{11} in the form

$$A_{11} = U D U^T \quad D = \{\lambda^i \delta_{ij}\} \quad U = \{U_j^i\}$$

whence
$$\underline{p} = A_{11}^{-1} \underline{b}_1 = U D^{-1} U^T \underline{b}_1$$

The NAG (1976) library contains routines to find U , D which perform corrections to ensure $A_{11} \underline{u}^i = \lambda^i \underline{u}^i$, $\underline{u}^i \cdot \underline{u}^j = \delta^{ij}$. This method of solution was chosen in preference to the Cholesky method because, in the solution of STEP 1, it was found to give results less sensitive to numerical errors in $\underline{b}_1$.

The vector $\underline{p}$ so obtained was compared to (3.21), the results discussed in Chapter Four, and written into filestore from the MASTER segment.

STRESS: The programme STRESS was then run to calculate the vectors $\underline{K}(r)$, $\underline{N}(r)$, $\underline{M}(r)$, and the stress $\sigma_1(r)$, at 30 points outside the contact area. The MASTER segment first called SONI2 to calculate the constants $H(m,k,l)$, then read the nodal points, s_j , from the filestore, and calculated the corresponding x_j . The normal stress distribution was also retrieved from filestore. The subroutine SSTAA1 was called to calculate $\underline{K}(r)$ at each of the 30 required points, and the value of $\sigma_1(r)$ at each of these 30 points found. SSTAA1 was called again to find $\underline{N}(r)$, $\underline{M}(r)$. The vectors $\underline{K}(r)$, $\underline{N}(r)$, $\underline{M}(r)$, were written into filestore, as was $\sigma_1(r)$, for each of the 30 points, and the last was also output to be compared with the analytical result, (5.7). This result is discussed in Chapter Five.

STEP2: The input data, $G(m,n,k,l)$, r_j , and x_j were again calculated by SONI1 and RDATA, and the command passed to HERTZ2. NOBLE was called to calculate A_{12} , $\underline{p}$ read from filestore, and the new vector $\underline{b}_2 = A_{12}^T \underline{p}$ found. NOBLE was again called to calculate A_{22} and the solution

$$\underline{q} = A_{22}^{-1} \underline{b}_2 = A_{22}^{-1} A_{12}^T \underline{p}$$

found as in STEP 1 by THNBA1. This result was compared with (3.23), as

discussed in Chapter Four, and the vectors $\underline{p}$, $\underline{q}$, $\underline{b} = A_{11}^T \underline{p}$, $\underline{r}$, and the matrix A_{11} were written into the filestore. It was found that subsequent results were not so sensitive to these, so they could be truncated to eight significant digits. At the end of the programme, the subroutine SSTBA1 was called to read $\underline{K}(r)$, $\underline{N}(r)$, $\underline{M}(r)$, and $\sigma_1(r)$ back from filestore, and calculate the associated in-plane surface stresses, $\sigma_1(r)$, $\sigma_3(r)$, $\sigma_4(r)$, $\sigma_6(r)$, the results discussed in Chapter Five.

STEP3: Command was passed immediately to SEWEL1. The vector $\underline{b} = A_{11}^T \underline{p}$, and the matrix A_{11} , contained in the functional J , and known from STEP 2, and the vector $\underline{p}$, contained in the constraint, known from STEP 1, were read from the filestore. Because the theoretical solution (3.27), (3.28) was known, the coefficient of friction, ρ , was chosen to give a theoretical adhesive radius for each of the three cases, $c = 0.6, 0.7, 0.8$, respectively.

The resulting quadratic programming problem was solved by the subroutine, THNCA1, which calls a NAG (ibid) library routine that finds the vector $\underline{x}$ which

$$\min_{\underline{x} \leq \underline{x}_0} J = \underline{x}^T A \underline{x} - 2 \underline{b}^T \underline{x}$$

$\underline{x}_0$ given, using the quadratic Simplex algorithm. Since it was expected that no q would be negative, nor were any found to be so, the routine could be used as such. The resulting shear distribution was compared to (3.27).

The slip vector

$$\underline{e}^0 = A_{11} \underline{q} - \underline{b}$$

was also calculated, and this with $\underline{b}$, $\underline{c} = \rho \underline{p}$, and the matrix A_{11} , were written into the filestore to be used in STEP 4. At the end of the programme SSTBA1 was called to calculate the in plane surface stresses, as before.

STEP4: It is assumed that at each stage the pressure is to be reduced from $(1/\lambda) \underline{p}$ to $\underline{p}$, so it is required to minimize

$$J_4 = \underline{q}^T A_{22} \underline{q} - 2 \underline{\hat{b}}^j{}^T \underline{q}$$

subject to $-\underline{c} \leq \underline{q} \leq \underline{c}$, where the linear term is

$$\underline{\hat{b}}^j = \underline{b} + \underline{u}_0^j$$

In the j th unloading stage, $\underline{u}_0^j$ is the initial displacement, and is given by

$$\underline{u}_0^j = (1/\lambda) (\underline{u}_0^{j-1} + \underline{e}^{j-1}) \quad 6.33$$

$\underline{e}^{j-1}$ being the slip in the $(j-1)$ th stage

$$\underline{e}^{j-1} = A_{22} \underline{q}^{j-1} - \underline{\hat{b}}^{j-1} \quad 6.34$$

It is known that $\underline{u}_0^0$ is zero, and $\underline{e}^0$ is read from filestore with $\underline{b}$, $\underline{c}$, A_{22} .

With the above, $\underline{\hat{b}}^j$ can be found from $\underline{\hat{b}}^{j-1}$ in the following manner

$$\begin{aligned} \underline{\hat{b}}^j &= \underline{b} + \underline{u}_0^j \\ &= \underline{\hat{b}}^{j-1} - \underline{u}_0^{j-1} + (1/\lambda) (\underline{u}_0^{j-1} + \underline{e}^{j-1}) \\ &= \underline{\hat{b}}^{j-1} + (1/\lambda) [(1-\lambda) \underline{u}_0^{j-1} + \underline{e}^{j-1}] \end{aligned} \quad 6.35$$

Thus at each stage, $\underline{b}$ was calculated from $\underline{b}$, $\underline{u}$, and $\underline{e}$ given from the previous stage by (6.35), and $\underline{u}$ from $\underline{u}$ and $\underline{e}$ by (6.33). The minimization could then be performed.

It was not however quite as simple as before, because the algorithm can only handle non-negative variables, whereas in this case the quadratic programming problem is one of the form

$$\begin{aligned} \text{Min} \quad J &= \underline{x}^T A \underline{x} - 2 \underline{b}^T \underline{x} \\ -2\lambda_0 &\leq \underline{x} \leq 2\lambda_0 \end{aligned}$$

But making the change of variables $\underline{y} = \underline{x} + \underline{x}_0$, the above becomes

$$\begin{aligned} \text{Min} \quad J &= \underline{y}^T A \underline{y} - 2 (\underline{b} + \underline{x}_0)^T \underline{y} \\ 0 &\leq \underline{y} \leq 2\lambda_0 \end{aligned}$$

which being of the previous form, can be handled by the NAG (ibid)

routine. This minimization was performed by the subroutine THNCA2, giving the results discussed in Chapter Four. The surface stresses were calculated by SSTBA1, and the programme then advanced to the next stage.

As has been stated earlier, there is no previous numerical or analytic results to be compared with, the only justification for the obtained distributions being that they were predicted by a physical argument, Chapter One, or that they are expected from making a small change in the cost functional, of the form (6.35). However the results for the surface stress calculated at the end of each stage give some credence to the unloading regime obtained.

PART TWO
Chapter Seven

ON STABILITY AND CONVERGENCE
OF THE NUMERICAL METHOD

§7.1 Convergence of STEP 1.

The proof of convergence of unconstrained finite element problems is fairly standard, and is done by showing that the strain energy of the error, namely $S(p-\hat{p}|K_{11}(p-\hat{p}))$, which equals the error of the strain energy $S(p|K_{11}p) - S(\hat{p}|K_{11}\hat{p})$, vanishes with some representative element length.

The proof is slightly complicated for the problems considered here by the presence of constraints, and the singularity, but that for the unconstrained problems is still relatively straight forward. However the presence of the singularity results in $S(p-\hat{p}|K_{11}(p-\hat{p}))$ being proportional to two terms, one of which vanishes with the length of the singular element, and also as the order of the polynomial approximating $(1-r^2)^{1/2}$ is increased, but the other of which, though vanishing with the length of some standard element, becomes unbounded as the length of the singular element is reduced. Thus convergence is proved only for a given length of singular element as the length of a standard element vanishes, and the order of the approximating polynomial increases. The proof for STEP 1 will be reviewed in this section, and the modifications then required for STEP 2, and the problems with constraints, STEP 3 and 4, discussed in later sections.

Following STRANG AND FIX (1973) §1.6, the proof breaks into two parts, the first showing that the strain energy of the difference between the exact solution, $p(r) \in H_1$, and some element, $\tilde{p}(r) \in H_1^m$, is minimized at the approximate solution, $\hat{p}(r)$, and the second showing that this strain energy

associated with the spline interpolating the true solution vanishes under the conditions stated above. Consider the complementary energy functional to be minimized

$$J_1(p) = \frac{1}{2} S(p | K_{11} p) - S(w_0 | p) \quad 7.1$$

where $w_0(r)$ is the prescribed normal displacement. The true solution minimizes $J_1(p)$ over all $p(r) \in H_1$, and the approximate solution over all $\hat{p}(r) \in H_1^m$. Consider further elements $\pi(r) \in H_1$, $\hat{\pi}(r) \in H_1^m$. For all such elements $\pi(r)$, $\hat{\pi}(r)$, it is obvious that $p(r) + \epsilon \pi(r) \in H_1$, and $\hat{p}(r) + \epsilon \hat{\pi}(r) \in H_1^m$, $\forall \epsilon$, and thus the minimization can be expressed as

$$J_1(p) \leq J_1(p + \epsilon \pi) ; \quad J_1(\hat{p}) \leq J_1(\hat{p} + \epsilon \hat{\pi}) \quad 7.2$$

If the RHS expressions in (7.2) are expanded, then using the fact that ϵ can be of either sign and arbitrarily small, it can be deduced that the first variation of J_1 at p in H_1 , and at $\hat{p}$ in H_1^m , must be zero, that is

$$S(K_{11} p | \pi) - S(w_0 | \pi) = 0 ; \quad S(K_{11} \hat{p} | \hat{\pi}) - S(w_0 | \hat{\pi}) = 0 \quad 7.3$$

Because $H_1^m \subset H_1$, then $\hat{\pi}(r) \in H_1$, and so the first of (7.3) holds with $\hat{\pi} = \pi$. Making this substitution, and subtracting the two equations gives

$$S(K_{11} (p - \hat{p}) | \hat{\pi}) = 0 \quad \forall \hat{\pi} \in H_1^m \quad 7.4$$

There are two immediate consequences of (7.4). The first obtained by letting $\hat{\pi} = \hat{p}$, expanding $S(K_{11} (p - \hat{p}) | p - \hat{p})$, and recalling that K_{11} is self-adjoint, is that the error of the strain energy is the strain energy of the error.

$$\begin{aligned} S(K_{11} (p - \hat{p}) | p - \hat{p}) &= S(K_{11} p | p) - 2 S(K_{11} p | \hat{p}) + S(K_{11} \hat{p} | \hat{p}) \\ &= S(K_{11} p | p) - 2 S(K_{11} \hat{p} | \hat{p}) + S(K_{11} \hat{p} | \hat{p}) \\ &= S(K_{11} p | p) - S(K_{11} \hat{p} | \hat{p}) \\ &= S(w_0 | p) - S(w_0 | \hat{p}) \\ &= p^* - b^T p \end{aligned} \quad 7.5$$

Thus the difference between the true compliance, P^* , and the numerical value, $\underline{b}^T \underline{p}$, is an indication of the error of the solution. Further because the strain energy is a positive definite functional, $P^* \geq \underline{b}^T \underline{p}$.

The second consequence follows from expanding $S(K_{11} (p - \hat{p} + \epsilon \hat{\eta}) | p - \hat{p} + \epsilon \hat{\eta})$, $\forall \hat{\eta}(r) \in H_1^m$, from which, again using the fact that ϵ is arbitrarily small and of either sign, it is deduced that the strain energy of the difference between the true solution $p(r)$, and some element $\tilde{p}(r) \in H_1^m$, obtains a minimum at $\hat{p}(r)$.

$$S(K_{11} (p - \hat{p}) | p - \hat{p}) = \min_{\tilde{p} \in H_1^m} S(K_{11} (p - \tilde{p}) | p - \tilde{p}) \quad 7.6$$

This completes the first part of the proof. It remains only to show that $S(K_{11} (p - \tilde{p}) | p - \tilde{p})$ vanishes as $m \rightarrow \infty$, for some $\tilde{p}(r) \in H_1^m$, whence convergence will be proved because

$$0 \leq S(K_{11} (p - \hat{p}) | p - \hat{p}) \leq S(K_{11} (p - \tilde{p}) | p - \tilde{p}) \rightarrow 0 \quad 7.7$$

and thus $\hat{p}(r) \rightarrow \tilde{p}(r) \rightarrow p(r)$, in the strain energy norm.

The usual choice for $\tilde{p}(r)$ is the function that interpolates $p(r)$, such that $p(r_j) = \tilde{p}(r_j)$, $j = 1, n$. The second part of the proof follows by considering the error, $\pi(r) = p(r) - \tilde{p}(r)$, for which the above choice of $p(r)$ gives $\pi(r_j) = 0$, $j = 1, n$. An expansion can be found for $\pi(r)$ in terms of the singular functions, (3.55).

$$\pi(r) = \sum_{R=0}^{\infty} a_R \frac{P_{2R}(\sqrt{1-r^2})}{\sqrt{1-r^2}} \quad 7.8$$

and so by (3.48), the strain energy associated with $\pi(r)$ is

$$S(K_{11} \pi | \pi) = \sum_{R=0}^{\infty} (\lambda^R K^R) a_R^2 \quad 7.9$$

This will vanish, if it can be shown that each a_R vanishes under the conditions stated, that is with the representative element length, $h \sim (1/m)$, and the order of the polynomial in the singular element.

From the orthogonality relation for Legendre polynomials

$$\int_0^1 \frac{P_m(\sqrt{1-r^2}) P_n(\sqrt{1-r^2})}{\sqrt{1-r^2}} r dr = \begin{cases} 0 & m \neq n \\ (4n+1)^{-1} & m=n \end{cases}$$

it can be deduced that

$$a_k = (4k+1) \int_0^1 \pi(r) P_{2k}(\sqrt{1-r^2}) r dr \quad 7.10$$

It is through equation (7.10), that problems with stability arise. It will be shown below, that the integral in (7.10) is of order h^2 , and thus since λ^k , κ^k , are both $O(1/k)$, $S(K_{11} \pi | \pi)$ is $O(h^4)$, proving convergence in this strain energy norm. But because the $P_{2k}(\sqrt{1-r^2})$ are oscillatory functions, the integral may be $O(h^2)$ even for a_k , with $k = m$, where m is the number of nodes, and hence a_m will be $O(h)$. Thus even though the strain energy may be converging as h^4 , the L_1 norm for π , representing the magnitude of the error in the stress distribution, will be vanishing only as h , giving 4 figure accuracy in the compliance P^* , but only one figure accuracy in the stress values $p(r)$. This point will be discussed further in the last section of this chapter.

Because both λ^k , κ^k , are $O(1/k)$, the functional $S(K_{11} \pi | \pi)$ is of order I^2 , where I is the integral in (7.10)

$$I = \int_0^1 \pi(r) P_{2k}(\sqrt{1-r^2}) r dr$$

Legendre polynomials are normalised such that $\max_{x \in [0,1]} P_k(x)$ is unity, and hence

$$\begin{aligned} |I| &\leq \int_0^1 |\pi(r)| r dr \\ &= \int_0^a |\pi(r)| r dr + \int_a^1 |\pi(r)| r dr \\ &\leq \max_{r \in [0,a]} |\pi(r)| + \int_a^1 |\pi(r)| r dr \end{aligned} \quad 7.11$$

The two terms of (7.11) are considered separately, the first is shown to be $O(h^2)$, and the second $O((1-a)^{3/2})$, for the polynomial, $(A+Br^2)$.

The maximum value of $|\pi(r)|$ on $[0, a]$ is the maximum value it can obtain in an element. In an element, $\pi(r_1) = \pi(r_2) = 0$, and thus by Rolle's theorem $\pi'(r)$ must vanish at some point $r = \xi$ in the element, if the first derivative is continuous, and $|\pi(r)|$ obtain its maximum value, $|\pi(\xi)|$ at that point. Expanding in a Taylor series about $r = \xi$

$$\pi(r) = \pi(\xi) + 0 + \frac{(r-\xi)^2}{2} \pi''(\xi) + R_3$$

Since both $\pi(r_1)$, $\pi(r_2)$ are zero, $\pi(r)$ is zero in an element for $|r-\xi| \leq \frac{1}{2}h$, where h is the element length, that is

$$|\pi(\xi)| = \max |\pi(r)| \leq \frac{h^2}{8} \max \pi''(r) \quad 7.12$$

Further, because in an element $r \leq r_2$, it follows immediately that for $r \in [0, a]$

$$|\pi(r)| \leq \max_{j=1, n-1} \left\{ \frac{h^2}{8} |\pi''(r)| \right\} \quad \text{for the element } j.$$

Now in the j th element, $\pi(r) = p(r) - \tilde{p}(r)$, where $\tilde{p}(r) = \alpha_1 + \alpha_2 r^2$, and hence $\pi''(r) = p''(r) + 2\alpha_2$. Thus

$$|\pi''(r)| \leq \max |p''(r)| + 2|\alpha_2|$$

But $p_1 = \alpha_1 + \alpha_2 r_1^2$, and $p_2 = \alpha_1 + \alpha_2 r_2^2$, and thus subtracting

$$(p_2 - p_1) = 2(r_1 + r_2)\alpha_2 h \quad ; \quad 2r_2|\alpha_2| = \frac{|p_2 - p_1|}{h} \leq \max |p'(r)|$$

by the mean value theorem. Hence for $r \in [0, a]$

$$|\pi(r)| \leq \max_{j=1, n-1} \left(\frac{h^2}{8} m_j \right) = \frac{h^2}{8} m \quad 7.13$$

where $m_j = \max_{r \in [r_1, r_2]} |p''(r)| + \max_{r \in [r_1, r_2]} |p'(r)|$; $m = \max_{j=1, n-1} m_j$

proving the first of the bounds above, if both $p'(r)$, and $p''(r)$ are smooth and bounded, for $r \in [0, a]$.

It will of course be realised that, given the square-root singularity, M

becomes $O((1-a)^{5/2})$, towards $r = a$, so, as mentioned earlier, (7.13) only proves convergence if a remains constant. Should $(1-a)$ vanish with h , convergence is not proved at all, and in fact the method diverges. Thus a must be set, and to match the $O(h^2)$ convergence above, higher order polynomials introduced in the singular element, as discussed in §6.1.

Consider then the second term

$$\int_a^1 \pi(r) r dr = \int_a^1 p(r) r dr - \int_a^1 \tilde{p}(r) r dr$$

If $p(r)$ has a square root singularity, it can be expressed as $p(r) = g(r)(1-r^2)^{1/2}$ where $g(r)$ is analytic for $r \in [0,1]$, and so has a Taylor expansion, $g(r) = g(1) - (1-r) g'(\xi)$. Hence

$$\begin{aligned} \int_a^1 p(r) r dr &= g(1) \int_a^1 \frac{r dr}{\sqrt{1-r^2}} - g'(\xi) \int_a^1 \sqrt{\frac{1-r}{1+r}} r dr \\ &= g(1) \sqrt{1-a^2} + O((1-a)^{3/2}) \end{aligned}$$

In this singular element, the interpolating polynomial ceases to have any meaning. Thus $\tilde{p}(r)$ is free to be chosen, within the given form, to give the desired convergence rate, remembering that $\hat{p}(r)$ must always converge at least as fast. The only constraint is that $\tilde{p}(r)$ be continuous at $r = a$, that is $\tilde{p}(a) = g(a)(1-a^2)$. For any $\tilde{p}(r)$ of the desired form

$$\begin{aligned} \int_a^1 \tilde{p}(r) r dr &= \int_a^1 (\alpha_1 + \alpha_2 r^2) (A + B r^2) r dr \\ &= \sqrt{1-a^2} [\alpha_1 + \alpha_2 (7+5a^2)/12] \end{aligned}$$

It is possible to choose two numbers α_1, α_2 , of order 1, such that $(\alpha_1 + \alpha_2 a^2) = g(a)$, $(\alpha_1 + \alpha_2 (7+5a^2)/12) = g(1)$, preserving not only the continuity of $p(r)$ at $r = a$, but also making

$$\int_a^1 \tilde{p}(r) r dr = g(1) \sqrt{1-a^2}$$

or
$$\int_a^1 \pi(r) r dr = O((1-a)^{3/2})$$

7.14

If higher order polynomials were taken, faster convergence would be

obtained, in fact, if the polynomial is order $2n$, this second term is $O((1-a)^{h+n})$.

This leads to a balance in the convergence of the strain energy norm, between the two terms in I. The second converges independently like $(1-a)^{3/2}$, but the first only for fixed a like h^2 . If however the convergence in the second term were to be improved by reducing $(1-a)$, the convergence would be lost in the first. Thus an optimal a must be chosen, and if it were required to further improve the convergence of the second term, a higher order polynomial should be introduced.

In the above the actual numerical solution must have smaller strain energy norm than the interpolating spline, but because the eigenfunctions of this norm are highly oscillatory, $\hat{p}(r)$ may be able to manoeuvre itself 'closer' to the true solution by wildly increasing its L_1 norm. This leads to instabilities which can only be overcome by increasing h , leading to a second balance, giving an optimal mesh size of m about 40.

§7.2 Convergence of STEP 2.

STEP 2 is also an unconstrained problem, and so the proof of convergence follows much the same lines, that is showing that the strain energy of the error, $w(r) = q(r) - \tilde{q}(r)$, the difference between the true solution and its interpolating spline, vanishes with h and $(1-a)$.

$$S(K_{22}w|w) \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad 7.15$$

This reduces to, proving that an integral, similar to (7.10) involving ultraspherical polynomials, vanishes, the proof for the first term following through as before. However the second term, which is now of the form

$$\int_a^1 w(r) r \, dr = \int_a^1 q(r) r \, dr - \int_a^1 (\beta_1^* r + \beta_2^* r^3)(A + Br^2) r \, dr \quad 7.16$$

does not have the same convergence rate as the corresponding term in STEP 1. For STEP 2, the shear $q(r)$ has a stronger singularity at $r = 1$, namely

$$q(r) = h(r) \frac{\ln \frac{1}{1-r}}{\sqrt{1-r}}$$

and so β_1^*, β_2^* , cannot be chosen to match the first integral. Thus this term, rather than being of order $(1-a)^{3/2}$, will in fact be $O((1-a)^{1/2} \ln(1-a))$, which vanishes with $(1-a)$, but obviously at a much slower rate than for STEP 1. This problem does not occur with STEP 3 and 4, because the square root singularity is regained, the corresponding integral being $O((1-a)^{3/2})$

It was found in Chapter Four, that this weaker convergence manifested itself in the inaccuracy of the results.

§7.3 Convergence of the Constrained Problems.

The proof for the constrained case is not a trivial extension of the unconstrained case for three reasons, first the constraints must be approximated, secondly, the presence of the constraints means that the first variation of the functional is not zero at the true or approximate solution, invalidating (7.3) - (7.7), and finally the derivatives of the true solution are neither bounded, nor continuous. However all three of these objections can be overcome, the proof being demonstrated here for STEP 3, that for STEP 4 being similar.

Let $p(r)$ be the true constraint, and $\bar{p}(r)$ an approximate constraint found from STEP 1 say. Three functions, $q(r) \in H_2$, $\bar{q}(r) \in H_2$, $\hat{q}(r) \in H_2^m$, can then be defined as those functions which minimize

$$J_3 = \frac{1}{2} S'(K_{22} q | q) - S(u_0 | q)$$

over the following subsets of H_2

$$q(r) \in \{ q(r) \in H_2 \mid q(r) \leq \rho p(r) \} = H_2'$$

$$\bar{q}(r) \in \{ \bar{q}(r) \in H_2 \mid \bar{q}(r) \leq \rho \bar{p}(r) \} = \bar{H}_2'$$

$$\hat{q}(r) \in \{ \hat{q}(r) \in H_2^m \mid \hat{q}(r) \leq \rho \bar{p}(r) \} = \bar{H}_2'^m$$

7.17

The first difficulty has been overcome, because it has been shown that $\bar{p}(r)$ converges pointwise to $p(r)$, and so $\bar{q}(r)$ must converge pointwise to $q(r)$. It remains to show that $\hat{q}(r)$ converges to $\bar{q}(r)$, and hence to $q(r)$.

The second difficulty arises because (7.3) no longer holds for the constrained optimization. However the Kuhn-Tucker theorem guarantees the existence of two Lagrange multipliers, $\bar{u}_1(r) \geq 0$, $\hat{u}_1(r) \geq 0$, such that

$$S(K_{12} \bar{q} | \bar{\omega}) - S(\bar{u}_0 | \bar{\omega}) + S(\bar{U}_1 | \bar{\omega}) = 0 ; S(K_{12} \hat{q} | \hat{\omega}) - S(\hat{u}_0 | \hat{\omega}) + S(\hat{U}_1 | \hat{\omega}) = 0 \quad 7.18$$

$\forall \bar{\omega}(r) \in H_2$, $\hat{\omega}(r) \in H_2^m$, or on letting $\omega(r) = \hat{\omega}(r)$, and subtracting as before

$$S(K_{12}(\bar{q} - \hat{q}) | \hat{\omega}) + S(\bar{U}_1 - \hat{U}_1 | \hat{\omega}) = 0 \quad \forall \hat{\omega}(r) \in H_2^m \quad 7.19$$

the equivalent of (7.4). The next step in §7.1 was to consider the positive definite strain energy functional

$$U(\bar{p}) = S(K_{11}(\bar{p} - \bar{p}) | \bar{p} - \bar{p}) \quad \bar{p} \in H_1^m \quad 7.20$$

and show that this was minimized at the approximate solution. However the presence of the constraints in the optimization means that this is no longer true. But considering instead the functional

$$W(\tilde{q}) = S(K_{12}(\tilde{q} - \tilde{q}) | \tilde{q} - \tilde{q}) + 2 S(\bar{U}_1 | \tilde{q} - \tilde{q}) \quad \tilde{q} \in \bar{H}_2^m \quad 7.21$$

because both $\bar{q}$ and $\tilde{q}$ obey the same constraints, namely $\tilde{q}(r) \leq \rho \bar{p}(r)$, and because $\bar{u}_1(r)$ is such that $\bar{u}_1(r) \geq 0$, $\bar{u}_1(r)(\bar{q}(r) - \rho \bar{p}(r)) = 0$, it can be shown as in §2.5 that $W(\tilde{q})$ is positive definite. Further letting $\tilde{q}(r)$ be $\tilde{q}(r) = \hat{q}(r) + \delta \hat{q}(r)$, expanding (7.21), and using (7.19) with $\hat{\omega}(r) = \delta \hat{q}(r)$, it is found that

$$\begin{aligned} W(\tilde{q}) &= S(K_{12}(\bar{q} - \hat{q} - \delta \hat{q}) | \bar{q} - \hat{q} - \delta \hat{q}) + 2 S(\bar{U}_1 | \bar{q} - \hat{q} - \delta \hat{q}) \\ &= S(K_{12}(\bar{q} - \hat{q} | \bar{q} - \hat{q}) + 2 S(\bar{U}_1 | \bar{q} - \hat{q}) \\ &\quad - 2 S(K_{12}(\bar{q} - \hat{q}) | \delta \hat{q}) - 2 S(\bar{U}_1 | \delta \hat{q}) + S(K_{12} \delta \hat{q} | \delta \hat{q}) \quad 7.22 \\ &= W(\hat{q}) + S(\bar{U}_1 - \hat{U}_1 | \delta \hat{q}) - 2 S(\bar{U}_1 | \delta \hat{q}) + S(K_{12} \delta \hat{q} | \delta \hat{q}) \\ &= W(\hat{q}) - S(\bar{U}_1 | \delta \hat{q}) + S(K_{12} \delta \hat{q} | \delta \hat{q}) \end{aligned}$$

Again as in §2.5 $S(\hat{u}_1 | \delta \hat{q}) \leq 0$, and since the strain energy is positive definite, $S(K_{11} \delta \hat{q} | \delta \hat{q}) \geq 0$, it follows that

$$0 \leq W(\hat{q}) = \min_{\tilde{q} \in \tilde{H}_1^m} W(\tilde{q}) \quad 723$$

Thus if $W(\tilde{q})$ vanishes for some $\tilde{q} \in \tilde{H}_1^m$ as $m \rightarrow \infty$, then convergence is proved.

Again this amounts to proving that the interpolating spline, $\tilde{q}(r)$, converges to $\bar{q}(r)$, but the earlier proof is no longer entirely valid, because the first and second derivatives of the true solution are discontinuous, and unbounded on a domain of measure zero in $[0,1)$. It will be recalled that the nodal points were not placed at these singularities, and thus the Lipschitz conditions, (7.13), do not hold as such. However $q(r)$ is the boundary value of a function that is C^∞ on $\bar{W}$, and continuous onto ∂W , and so the Lipschitz condition holds with the constant M representing derivatives close to the boundary. (This is mathematically not very rigorous, and indeed it can not be, because such a Lipschitz condition would be a proof of compactness, §2.6, and the resulting convergence proof would be a proof of existence, but it is sufficient for all practical purposes). Further $q(r)$ has a squareroot singularity at $r = 1$, and hence the integral over the last element converges as $(1-a)^{1/2}$ as in §7.1, and not $(1-a)^{1/2} \ln(1-a)$ as in §7.2. Hence for fixed a the quadratic term in (7.21) converges as h^4 , and the linear term as h^2 , and because the constraint in (7.17) also converges at this rate, the total convergence is $O(h^2)$.

Finally there is for the constrained problem, an expression equivalent to (7.5). Expanding $W(\hat{q})$, using (7.19) with $\hat{w}(r) = \hat{q}(r)$ gives

$$\begin{aligned} W(\hat{q}) &= S(K_{11}(\bar{q} - \hat{q}) | \bar{q} - \hat{q}) + 2S(\bar{U} | \bar{q} - \hat{q}) \\ &= S(K_{11} \bar{q} | \bar{q}) + 2S(\bar{U} | \bar{q}) - 2S(K_{11} \bar{q} | \hat{q}) - 2S(\bar{U} | \hat{q}) + S(K_{11} \hat{q} | \hat{q}) \\ &= S(K_{11} \bar{q} | \bar{q}) + 2S(\bar{U} | \bar{q}) - 2S(K_{11} \hat{q} | \hat{q}) - 2S(\bar{U} | \hat{q}) + S(K_{11} \hat{q} | \hat{q}) \quad 724 \\ &= \{S(K_{11} \bar{q} | \bar{q}) + 2S(\bar{U} | \bar{q})\} - \{S(K_{11} \hat{q} | \hat{q}) + 2S(\bar{U} | \hat{q})\} \end{aligned}$$

Thus again, the energy of the difference is the difference of the energy, the latter converging at the same rate, and thus not only must $\hat{q}(r)$ converge to $q(r)$, but also $\hat{u}(r)$ to $\bar{u}(r)$, and hence to $u(r)$.

§7.4 The Stability of the Numerical Method and Choice of Element Mesh.

Instabilities in the numerical method arise from two sources, both of which affect the choice of element mesh. The first, as discussed in §7.1, is caused by the contribution of high order, and hence oscillatory, singular functions to the solution, which contribute negligible error to the strain energy, but appreciable errors to the actual solution. The second arises in the calculation of the stiffness matrix, which involves the subtraction of approximately equal numbers, thus causing accumulation of round-off errors. The two instabilities will be discussed in the order in which they were discovered and tackled. Fortunately, because both are intimately related to the position of the zeroes of the singular functions, the method of minimizing the effects of the two instabilities were consistent.

It was shown in §7.1, that even though the strain energy of the solution may converge as $O(h^4)$, one of the higher order Legendre polynomials may enter the solution at $O(h)$. Thus the error as judged by the compliance may be very small, but since Legendre polynomials are oscillatory, the numerical solution may oscillate about the actual solution with quite large errors. It was found empirically that most of the oscillations died away by about $r = 0.3 - 0.4$, for meshes of the size being used, that is $m < 50$. So the most direct way to smooth these oscillations was to absorb them into the first element, by making this fairly large. Because the true solutions are smooth functions, (except on a set of

measure zero), and because $p(r)$ is even, and very nearly parabolic near the origin, and $q(r)$ odd and linear, the first node could be placed at about $r_1 = 0.2$, introducing very little absolute error between the true solutions, and the interpolating polynomials. For the meshes used this simple device reduced the absolute error of the nodal values to about the same order of magnitude as the error in the compliance, for up to the first half of the mesh points, that is $r < 0.8$ approximately.

The explanation of this empirical observation lies in the distribution of the zeroes of $P_{2k}(\cos \theta)$, and the fact that it has its greatest value, 1, at $\theta = 0$, while at $\theta = \pi/2$, the value is roughly inversely proportional to $k^{1/2}$. The zeroes of $P_{2k}(\cos \theta)$ are evenly distributed in θ , ABRAMOWITZ AND STEGUN (1966) equation (22.16.6), and thus tend to be gathered more towards $r = \sin \theta = 1$, than $r = 0$. Now the integral (7.10) will be small, and hence the corresponding a_k small for one of the following three reasons, the integrals over successive elements cancel, if a number of elements lie between two zeroes, or the integrals between successive zeroes cancel, if a number of zeroes lie in a given element, or the absolute error, $\pi(r)$, is small. The third can only be guaranteed if the nodes and zeroes of the highest contributing polynomial approximately correspond, but it is more important that, toward $r = 0$ where P_{2k} has its greatest contribution, there be a couple of zeroes between each node, rather than vice-versa. It was simply found that to keep the absolute error small on the large scale, and overcome the first instability, the position of the first node had to be at $r_1 \doteq 0.1\sqrt{0.1m}$, giving a maximum useful limit for the number of nodes of about $m = 50$, with $r_2 = 0.2$.

But having achieved this stability, the approximate inverse sine, or near enough parabolic, spacing of nodes had to be maintained for no local oscillations to occur. It was at one point decided, after a parabolic spacing of nodes had been settled on, to place some extra nodes in the

vicinity of the inner adhesive radius, to plot its course during unloading, but rather than improving the accuracy, as would be expected with extra nodes, it was found that this caused accumulation of truncation error, and thus decreased the accuracy.

Thus choosing a suitable mesh reduced to taking a set of equally spaced numbers, $s_j = ((j-1)/n)$, and finding a suitable function, $s = f(r)$, choosing the nodal points to be placed at $r_j = f^{-1}(s_j)$. Two suitable functions were the inverse sine, $s = (2/\pi) \arcsin r$, or as a close approximation the parabola, $s = r^2$, the latter being used. This gave reasonable results, and allowed some change in spacing of the numbers, s_j , within the contact area. In fact the s_j were chosen such that $\Delta s = 0.04$ for $0 \leq r \leq 0.6$, $\Delta s = 0.03$ for $0.6 \leq r \leq 0.8$, and $\Delta s = 0.02$ for $0.8 \leq r \leq 1.0$, thus placing even more nodes towards the singularity, but still maintaining the parabolic spacing, except at two nodes. There was no noticeable error introduced at $r = 0.6$, and only a small one at $r = 0.8$. An extra node was also placed at $r = 0.9950$ to fix the size of the final element, this mesh of 37 elements, and 38 nodes being used in Chapter Four.

One way the second instability showed itself was in A_{22} having a small negative eigenvalue, and therefore not being strictly positive definite. The mesh chosen gave all the eigenvalues positive, and also satisfied the requirement $r_2 = 0.2$, to overcome the first instability.

Chapter Eight

Conclusion

§8.1 A Summary of the Results and Contribution of this Thesis.

A numerical solution has been obtained for the frictional loading and unloading problem, of a rigid circular cylindrical indenter being pressed quasistatically into a homogeneous, isotropic, incompressible, linear elastic half-space, and then removed, with conditions of Coulomb friction acting over the interface.

Considering a variational formulation of the elasticity equations, it was shown that, at the solution, the complementary energy achieves a minimum within the subset of admissible stress states, that is those that obey the equilibrium equations, and the boundary conditions on stress. By further considering the general traction boundary value problem, it was shown that it is then possible to express the complementary energy as a quadratic functional of the surface stresses alone, the frictional boundary conditions acting as linear constraints on the minimization. Though the complementary energy principle is valid for any general non-homogeneous, anisotropic, linear elastic body, a fairly simple form can be obtained for axi-symmetric problems on a homogeneous, isotropic half-space. However it is only valid if the frictional constraint on the shear is a known function, which it is in the incompressible limit. Thus only incompressible problems were considered, but it was shown how the general solution could be obtained by iterating on this solution.

Because the primary concern of this thesis was to find the unloading regime, rather than studying the effect of anisotropy, inhomogeneity, or compressi-

bility on the contact, all the above assumptions were made, and the complementary energy principle obtained in the axi-symmetric case. Further the mathematics is inherently much simpler for cylindrical, rather than spherical indentors. Thus, because all the physical principles are preserved in either case, and the unloading regimes expected to be similar, only the former were considered. This led to a constrained minimization of a quadratic functional on an infinite dimensional Hilbert space, for which a numerical solution was found.

By limiting the surface stresses to finite dimensional subspaces, the complementary energy principle could be reduced to matrix form, without further approximation. This finite dimensional form of the energy principle could be solved, for both loading and unloading, by standard methods. The results obtained for the loading problem confirm those obtained by analytic techniques, giving some credence to the unloading solution, while those for the unloading problem form the major contribution of this thesis, and were as predicted by a physical argument.

The numerical method, though convergent, is somewhat unstable, and thus at times the actual nodal stress values were rather inaccurate. However they always indicated a consistent pattern, from which the unloading profile could be judged quite accurately. This is a general feature of the finite element method, in that it tends to indicate the overall behaviour of a solution, rather than giving the solution at a point. Thus the obtained unloading regime is presented as a solution to the frictional unloading problem.

During the derivation of the complementary energy principles, a solution to the general traction boundary value problem was obtained using the Airy stress function. This together with the energy principle could be used to find a complete solution to any traction or displacement boundary value problem, as well as a variety of mixed boundary value problems, a

number of which were discussed. But further, given any finite dimensional stress state of the type above, the solution could be used to find the entire surface stress/displacement state, again without further approximation. In an attempt to check the obtained unloading profile, the method was used to calculate the surface radial stress on loading and unloading, and results consistent with observed empirical behavior were obtained. Further experiments to check the numerical results were also suggested.

As yet the method has not been extended to spherical indentors. There is the problem of an unknown coefficient in the quadratic part of the displacement, for normal contact representing the relationship between the contact radius and the radius of the sphere, and in the tangential part representing the horizontal displacement at a point before the half-space comes in contact with the sphere. It can be shown in the frictionless and adhesive cases, that the first and second derivatives of the complementary energy with respect to these parameters is zero. But little, as yet, has been done along these lines in the frictional case. However, it can be proved analytically that the adhesive radius on loading must be the same for both spherical and cylindrical indentors, and thus as a first approximation it could be assumed that the various radii of slip and adhesion are the same in both cases, for the same value of the unloading parameter. Further because, for the cylindrical indenter, the shear profile is roughly the same shape in all regions of adhesion on loading and unloading, it could be assumed to be the same for spherical indentors, the curves given in SPENCE (1975) being extrapolated. In this way a complete shear profile can be deduced, but it will give no new information about the radial stress distribution, which is the result of practical interest. If it were desired to check the above heuristic deduction, loading and unloading could be done using the Mindlin beehive model, but the numerical errors would be enormous, and it can be seen before one starts that it will lead to a similar profile as that predicted. The author himself believes that

if all that is required is the behaviour on unloading, then sufficient information can be deduced from the results for the cylindrical indenter. If an actual solution is required for spherical indentors, then the method outlined at the beginning of this paragraph should be explored. The author has achieved some quite useful results with the frictionless case. If anything further is obtained for the adhesive or frictional loading problems, the results will be published separately.

Finally one should consider what further work is possible. Apart from following up the work on curved indentors, the author intends to study the effect of introducing inhomogeneities and anisotropy into the half-space, and also the effect of contact by a flexible raft, (Cosserat plate), rather than a rigid disc, (circular indenter), maintaining the frictional conditions in all cases. In conclusion, a numerical solution paves the way for the elusive analytical solution. The author has achieved a small amount of success with this, using a rough asymptotic analysis. The results agree with the numerical ones, with an accuracy that is too good to be true considering the approximations made. This is the subject of the final section.

§8.2 An Analytical Solution?

This, as with most analytic solutions of the contact equations, starts with the semi-inverse of (3.34). These are the Abel equations, one involving the normal displacement, and the other the horizontal displacement. The latter is obtained, in the incompressible limit, by operating on equation (3.38) with $\int_0^r (r^2 - t^2)^{-1/2} \frac{d}{dt} t() dt$, the resulting equation being, SPENCE (1975)

$$\int_r^1 \frac{q(t) dt}{\sqrt{1-t^2}} = \int_0^r \frac{t p(t) dt}{\sqrt{r^2-t^2}} + \int_0^r \frac{(t u(t))'}{\sqrt{r^2-t^2}} dt = A_1(r) \quad 0 \leq r \leq 1 \quad 8.1$$

Given $p(r)$, $u(r)$, as in the adhesive case, this equation gives $A_1(r)$, and hence $q(r)$ through the inverse

$$-q(r) = \frac{2}{\pi} \frac{d}{dx} \int_x^1 \frac{A_1(s) ds}{\sqrt{s^2-x^2}} \quad 8.2$$

Close to $r = 1$, the behaviour of the integrals in (8.1), (8.2) will be determined by the singularities of the respective functions. The singularities of the functions or integrals must balance to provide a solution with the required properties. In the incompressible limit, the normal stress $p(r)$ is determined by (3.21),

$$p(r) = \frac{\lambda}{\sqrt{1-r^2}} \quad 8.3$$

where λ is the unloading parameter.

In the frictionless problem, $q(r) = 0$, the $(t u(t))'$ term in (8.1) must balance the $p(t)$ term, which gives as in (3.21)

$$u(r) = \frac{-\lambda r}{1 + \sqrt{1-r^2}} \quad 8.4$$

Alternatively in the adhesive case, $u(r) \neq 0$, which gives $A_1(r) = \lambda \tanh^{-1} r$, and hence (3.32) through (8.2)

$$-q(r) = \frac{2\lambda}{\pi^2} \frac{\ln(1-r^2)}{r\sqrt{1-r^2}} \quad 8.5$$

This singularity is stronger than allowed in the frictional loading problem. In that case it is required that

$$|q(r)| \leq \rho p(r) = \frac{\rho\lambda}{\sqrt{1-r^2}} \quad 8.6$$

Thus $u(r)$ must contain a term to balance the $\tanh^{-1} r$ term in $A_1(r)$, the frictionless solution, as well as terms to produce this singularity.

Taking

$$(r u(r))' = -\frac{\lambda r}{\sqrt{1-r^2}} + \alpha_1 r + \alpha_2 r \sqrt{1-r^2} \quad 8.7$$

gives $A_1(r) \sim \alpha r$ as $r \rightarrow 1$, or substituting into (8.2)

$$q(r) \sim -\frac{2\alpha}{\pi} \frac{r}{\sqrt{1-r^2}} \sim \frac{\rho\lambda f(\lambda)}{\sqrt{1-r^2}} \quad \text{as } r \rightarrow 1 \quad 8.8$$

So studying the singularities close to $r = 1$, it is possible to say something about the behaviour of the functions at the edge of the contact region. At a stage of unloading, λ , $q(r)$ is given in terms of the shear profile, $\theta(r, \lambda)$ as

$$q(r) = \frac{\rho\lambda}{\sqrt{1-r^2}} \theta(r, \lambda) ; \quad \theta(r, \lambda) \rightarrow f(\lambda) \quad \text{as } r \rightarrow 1 \quad 8.9$$

Now it is possible to attempt some form of analytical solution in the early stages of unloading, by studying the behaviour of (8.1) in the two regions $0 \leq r \leq c_1$, $c_2 \leq r \leq 1$. It will be assumed that $(1 - c_2) \ll 1$, and thus integrals over the range $c_2 \leq r \leq 1$ can be determined primarily by the behaviour of the singularity.

In the range $c_1 \leq r \leq 1$, equation (8.1) becomes

$$r \int_r^1 \frac{q(t) dt}{\sqrt{t^2 - r^2}} = A_1(r) \sim \alpha r \quad 8.10$$

which gives a solution for $q(r)$ of the form (8.9). Later it will be necessary to calculate integrals over the outer adhesive region of the form

$$\int_{c_1}^1 \frac{q(t) dt}{\sqrt{t^2 - r^2}} \quad r \ll c_2 \quad 8.11$$

If λ is only just less than unity, so that it does not vary much over the outer adhesive region, $q(r)$ can be replaced by the approximate expression (8.8). Further if $r \ll c_2$, and $c_2 \leq t \leq 1$, then $\sqrt{t^2 - r^2} = \sqrt{1 - r^2}$, and hence

$$\begin{aligned} \int_{c_2}^1 \frac{q(t) dt}{\sqrt{t^2 - r^2}} &\sim \frac{\rho \lambda f(\lambda)}{\sqrt{1 - r^2}} \int_{c_2}^1 \frac{dt}{\sqrt{1 - t^2}} \\ &\sim \frac{\rho \lambda f(\lambda)}{\sqrt{1 - r^2}} \propto \cos^{-1} c_2 \end{aligned} \quad 8.12$$

Nominally $\alpha = 1$, but it is introduced now to represent the fact that $\theta(r, \lambda)$ may change rapidly over $c_2 \leq r \leq 1$. It will be exactly one when $\lambda f(\lambda) = 1$, that is at the commencement of unloading, but because $\theta(r, \lambda)$ is both positive and negative in the outer adhesive region, an order of magnitude lower when outward slip starts, $\lambda f(\lambda) = -1$.

Now consider the inner region of adhesion. In $0 \leq r \leq c_1$, $u(r) \equiv 0$, and thus (8.1) becomes

$$r \int_r^1 \frac{q(t) dt}{\sqrt{t^2 - r^2}} = \int_0^r \frac{t p(t) dt}{\sqrt{r^2 - t^2}} \quad 8.13$$

The L.H.S. can be split into three terms, the first over $r \leq t \leq c_1$, where $q(t)$ is an unknown function, the second over $c_1 \leq t \leq c_2$, where $q(t) = \rho p(t)$, and the third over $c_2 \leq t \leq 1$, which is given by (8.12). Thus (8.13) becomes

$$r \int_r^{c_1} \frac{q(t) dt}{\sqrt{t^2 - r^2}} = \int_0^r \frac{t p(t) dt}{\sqrt{r^2 - t^2}} - \rho r \int_{c_1}^{c_2} \frac{p(t) dt}{\sqrt{t^2 - r^2}} + \frac{\alpha \rho \lambda f'(\lambda) \cos^{-1} c_2 r}{\sqrt{1 - r^2}} \quad 8.14$$

$$= A_2(r)$$

where $A_2(r)$ is a known function. If $p(r)$ is given by (8.3) then every term in $A_2(r)$ depends linearly on λ , and hence $A_2(r)$ can be written $\lambda A_3(r)$. Further the second integral on the R.H.S. can be split into two terms, the second of which can be found by an approximation similar to (8.12), with $\alpha = 1$.

$$\int_{c_1}^{c_2} \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - r^2}} = \int_{c_1}^1 \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - r^2}} - \int_{c_2}^1 \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - r^2}} \quad 8.15$$

$$\doteq \int_{c_1}^1 \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - r^2}} - \frac{\cos^{-1} c_2}{\sqrt{1-r^2}}$$

Thus (8.14) becomes

$$r \int_r^{c_1} \frac{q(t) dt}{\sqrt{t^2 - r^2}} = \lambda A_3(r) \quad 8.16$$

where $A_3(r) = \int_0^r \frac{t dt}{\sqrt{1-t^2} \sqrt{r^2 - t^2}} - \rho r \int_{c_1}^1 \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - r^2}} + \rho \cos^{-1} c_2 (1 - \alpha f(\lambda)) \frac{r}{\sqrt{1-r^2}}$

The inverse of (8.16) could be found by the previous method, yielding a solution very close to the frictionless solution if $(1 - \alpha f(\lambda))$ is small, that is in the early stages of unloading. The important point is that the solution will only be bounded if $A_3(c_1) = 0$, from which an equation to determine c_1 is deduced.

$$A_3(c_1) = \int_0^{c_1} \frac{t dt}{\sqrt{1-t^2} \sqrt{c_1^2 - t^2}} - \rho c_1 \int_{c_1}^1 \frac{dt}{\sqrt{1-t^2} \sqrt{t^2 - c_1^2}} + \rho \cos^{-1} c_2 (1 - \alpha f(\lambda)) \frac{c_1}{\sqrt{1-c_1^2}} = 0$$

or $\frac{1}{2} \ln \left(\frac{1+c_1}{1-c_1} \right) - \rho c_1 K'(c_1) + \rho \cos^{-1} c_2 (1 - \alpha f(\lambda)) \frac{c_1}{\sqrt{1-c_1^2}} = 0 \quad 8.17$

which is similar to (1.6). There are a number of observations to be made. The first is that at the instant of unloading $\alpha f(\lambda) = 1$, whence (8.17) reduces to (1.6) as expected, but unexpectedly independently of c_2 . (A

consequence of the approximations made?). Secondly it was observed numerically, and predicted physically, that during unloading, c_1 will follow the loading profile, until the region of outward slip forms, that is from (1.5)

$$1 - \psi(c_1, c) = \lambda \quad ; \quad \psi(c_1, c) = 1 - \lambda$$

8 18.

Because it does so fairly closely, and from the form of $\psi(c_1, c)$, the author was able to convince himself that c_1 must be approximately constant during unloading, which it is, and also that $\beta = \rho \cos^{-1} c_2$ must be constant for different values of ρ . Unloading regimes were obtained for three values of ρ , and the corresponding values of β , in degrees, are tabulated in TABLE 8.1, taking as c_2 the last node at which adhesion occurred. Considering the large derivative of $\cos^{-1} x$ close to $x = 1$, and the limited number of nodes, the agreement is surprisingly good, lending further weight to the validity of the numerical solution.

Further β can be calculated from the numerical results independently of the above, for any value of ρ , by using the values of λ and c_1 at which outward slip starts. It was stated above that when $\lambda f(\lambda) = -1$, $\alpha f(\lambda) = 0$, that is outward slip starts when

$$\frac{1}{2} \ln \left(\frac{1+c_1}{1-c_1} \right) - \rho c_1 K'(c_1) + \frac{\beta c_1}{\sqrt{1-c_1^2}} = 0$$

8 19

For $\rho = 0.6652$, the value $c_1 = 0.5657$ is read from FIG 4.8 as being the relevant value c_1 , and the resulting value of β is 10.7, compared to 10.8 above. This agreement is independent of β being constant for different values of ρ , neither confirming nor denying it, but for a given value of ρ , two calculations of β , one using the numerical values for c_1 , and the other for c_2 , give answers to within 1%. This accuracy is absurd considering the approximations made, but the important point emerging is that outward slip cannot begin until λ has fallen to at least 0.515.

TABLE 8.1 $\beta = \rho \cos^{-1} c_2$

c	ρ	c	β
0.6	0.5780	0.9381	11.7°
0.7	0.6652	0.9592	10.8°
0.8	0.7844	0.9695	11.0°

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Appendices

APPENDIX A THE MINDLIN UNLOADING PROFILE

In a private communication, SPENCE has given a proof that the unloading profile cannot simply be one of an outer annulus of outward slip, with an inner region of adhesion. His proof applies only to the incompressible limit, $\nu \rightarrow \frac{1}{2}$, (though the theorem continues to hold for other values of Poissons ratio), and is summarised here. The result is shown by first deriving the solution to the frictional loading problem, with normal pressure, $p^0(r)$, shear stress, $q^0(r)$, and horizontal surface displacement, $u^0(r)$, and then trying to derive a solution of the required form, for a half-space unloaded from this initial state, to one with normal pressure $\lambda p^0(r)$, $0 \leq \lambda \leq 1$. It is shown that there is no shear profile of the proposed form, which is bounded in the contact region, except at $r = 1$, and obeys the conditions of the unloading problem.

The starting point for this is the Abel equation relating surface shear, $q(r)$, normal pressure, $p(r)$, and horizontal surface displacement, $u(r)$, derived by SPENCE (1975). This is equation (8.1), and is in the limit $\gamma \rightarrow 0$

$$r \int_r^1 \frac{q(t) dt}{\sqrt{t^2 - r^2}} = \int_0^r \frac{+p(t) dt}{\sqrt{r^2 - t^2}} + \int_0^r \frac{(t u(t))'}{\sqrt{r^2 - t^2}} dt \quad A1$$

Also in the limit $\gamma \rightarrow 0$, the normal stress, $p(r)$, is given by the Hertz solution. Thus if the total normal load reached during loading is $P^* = 2\pi$, the normal pressure $p^0(r)$ is

$$p^0(r) = \frac{1}{\sqrt{1 - r^2}} \quad A2$$

and at some later stage when the normal load has reduced to λP^* , the normal pressure will be $p(r) = \lambda p^0(r)$.

It is known from SPENCE (1975), that during the loading phase there is a constant region of adhesion, $0 \leq r \leq c$, surrounded by an annulus of

inward slip, $c \leq r \leq 1$. Therefore, as in TURNER (1975), the following boundary conditions can be prescribed in the contact region, $0 \leq r \leq 1$.

$$\begin{aligned} q^0(r) &\leq \rho p^0(r) ; & u^0(r) &= 0 & 0 \leq r \leq c \\ q^0(r) &= -\rho p^0(r) ; & u^0(r) &\leq 0 & c \leq r \leq 1 \end{aligned} \quad A3$$

where ρ is the coefficient of Coulomb friction. Given $p^0(r)$ from (A2), and the conditions (A3), $u^0(r)$ can be eliminated from (A1) to give an integral equation, (Volterra of the first kind), for $q^0(r)$, namely,

$$-r \int_r^c \frac{q^0(t) dt}{\sqrt{t^2 - r^2}} = \rho r \int_c^1 \frac{p^0(t) dt}{\sqrt{t^2 - r^2}} - \int_0^r \frac{t p^0(t) dt}{\sqrt{r^2 - t^2}} = A(r) \text{ say} \quad A4$$

$0 \leq r \leq c$

If $q^0(r)$ is to be bounded at $r = c$, $A(c)$ must vanish, from which the relation (1.6) between ρ and c can be deduced.

$$\rho = \frac{1}{2c k'(c)} \ln \left(\frac{1+c}{1-c} \right) \quad A5$$

Further inverting (A4) gives the required expression for $q^0(r)$

$$\frac{q^0(r)}{\rho p^0(r)} = 1 - \psi(r, c) \quad A6$$

where $\psi(r, c)$ is defined in (1.5). Since $0 \leq \psi(r, c) \leq 1$, $\forall r, c$, this solution satisfies all the conditions of the loading problem, and can therefore be accepted as the unique solution.

Now consider the unloading phase, in which the normal force is reduced to λP^* , that is the normal stress becomes $p(r) = \lambda p^0(r)$. The Mindlin profile assumes that the contact region divides into an inner region of adhesion, $0 \leq r \leq b(\lambda)$, $c \leq b(\lambda)$, and an annulus of outward slip, $b(\lambda) \leq r \leq 1$. Thus the boundary conditions corresponding to (A3) can be written for this assumed profile as

$$\begin{aligned} -\rho p(r) &\leq q(r) \leq \rho p(r) & u(r) &= u^0(r) & 0 \leq r \leq b(\lambda) \\ q(r) &= -\rho p(r) & u(r) &\geq u^0(r) & b(\lambda) \leq r \leq 1 \end{aligned} \quad A7$$

Equation (A1) gives relations between the initial variables, $p^0(r)$, $q^0(r)$, $u^0(r)$, and between the final variables, $p(r)$, $q(r)$, $u(r)$. If these two equations are subtracted, then equation (A7) can be used to eliminate $u(r) - u^0(r)$ from the resulting equation as before. On doing so and replacing $p(r)$ by $\lambda p^0(r)$, then the resulting equation is, in analogy to (A4)

$$-r \int_r^b \frac{q^0(t) - q(t)}{\sqrt{r^2 - t^2}} dt = \rho(1+\lambda) r \int_b^1 \frac{p^0(t) dt}{\sqrt{r^2 - t^2}} - (1-\lambda) \int_0^r \frac{t p^0(t) dt}{\sqrt{r^2 - t^2}} \quad A8$$

$$= B(r) \text{ say} \quad 0 \leq r \leq b(\lambda)$$

Now if $q^0(r) - q(r)$ is to be bounded at $r = b$, $B(b)$ must vanish, and hence one can find a relation between b , ρ and λ , namely

$$\left(\frac{1+\lambda}{1-\lambda}\right) \rho = \frac{1}{2b K(b)} \ln\left(\frac{1+b}{1-b}\right) \quad A9$$

Thus the Mindlin unloading profile requires that at the commencement of unloading, $\lambda = 1$, the entire region of contact is adhesive, but as unloading proceeds, the proposed region of outward slip expands, so that when the half-space is unloaded, $\lambda = 0$, it equals the original region of inward slip, $b(0) = c$. Inverting (A8) also gives the required expression for the shear

$$\frac{q(r)}{\rho p(r)} = -1 + \frac{1}{\lambda} [(1+\lambda) \psi(r, b) - \psi(r, c)] \quad A10$$

If $c < b \leq r \leq 1$, $\psi(r, b) = \psi(r, c) = 0$, and thus $q(r) = -\rho p(r)$, satisfying the second of (A7). The obtained solution (A10) will therefore satisfy the conditions of the unloading problem if

$$-1 \leq \frac{q(r)}{\rho p(r)} \leq 1 \quad \text{in } 0 \leq r \leq c \text{ and } c \leq r \leq b \quad A11$$

for all λ . Considering first the range $c \leq r \leq b$, it is still true that $\psi(r, c) = 0$, and because $\psi(r, b) > 0$, the lower limit is satisfied. The shear, $q(r)$, can then be rewritten as

$$\frac{q(r)}{\rho p(r)} = 1 - \frac{1}{\lambda} \phi(r, b)$$

A12

where $\phi(r, b) = 2\lambda - (1+\lambda) \psi(r, b)$, and hence the upper limit will be satisfied if $\phi(r, b) > 0$. Because $\phi(r, b)$ is a decreasing function of b , and an increasing function of r , $\phi(r, b) > 0$, $\forall r, b$, if $\phi(c, b) > 0$, as $b \rightarrow 1$. Splitting 2λ into $(1+\lambda) - (1-\lambda)$, substituting from (A9), and the definition of $\psi(r, b)$, (1.5), $\phi(r, b)$ becomes

$$\phi(r, b) = \left(\frac{1-\lambda}{\rho} \right) \left[\rho \left(\frac{1+\lambda}{1-\lambda} \right) \left(1 - \lambda_0 \left(\sin^{-1} \frac{\omega}{b}, \sqrt{1-b^2} \right) - \rho \right. \right. \\ \left. \left. + \frac{1}{\pi r} \left(\frac{1}{b} \ln \left(\frac{1+b}{1-b} \right) - \ln \left(\frac{1+\omega}{1-\omega} \right) \right) \right]$$

A13

where $\omega = \left(\frac{b^2 - r^2}{1 - r^2} \right)^{1/2}$

As $b \rightarrow 1$, so to does $\omega \rightarrow 1$, the Heumann lambda function then tending to 1 faster than λ , thus

$$\phi(r, b) \sim \left(\frac{1-\lambda}{\rho} \right) \left[\frac{1}{\pi r} \ln \left(\frac{1-\omega}{1-b} \right) - \rho \right] \quad \forall r$$

or

$$\phi(c, b) \rightarrow \left(\frac{1-\lambda}{\rho} \right) \left[\frac{1}{\pi c} \ln \frac{1}{1-c^2} - \frac{1}{2cK'(c)} \ln \frac{1+c}{1-c} \right]$$

A14

since $(1-\omega)/(1-b) \sim 1/(1-r^2)$ as $b \rightarrow 1$. This function $\phi(c, b)$ in (A14) is zero when $c = 0$, and decreases to $-(1/\pi) \ln 4$ when $c = 1$, and hence $\phi(r, b)$ becomes negative at $r = c$, at the commencement of unloading, the upper frictional constraint being violated. This means that as unloading proceeds some further inward slip must occur, and proves the theorem that the Mindlin profile is, in fact, invalid.

As a final note, an analytic solution of this form could be attempted, because the profile assumes a region of outward slip expanding into one of adhesion. Thus at any point of the contact area, either the displacement or the shear is known allowing the boundary conditions to be written in the form (A7), and hence $u(r)$ to be eliminated from the integral equation to

yield (A8). For the profile predicted in Chapter One, there is an adhesive region expanding into one of inward slip, and thus at a point in the adhesive region, the displacement that occurred before the point became adhesive is not known. Therefore the conditions cannot be written in the form (A7), that is a differentially incremental solution must be found. The numerical solution found in this thesis is an incremental one, but being based on an energy method, causes some smoothing of errors arising from a region of adhesion expanding into a region of slip, and thus allows fairly large time steps to be taken.

APPENDIX B DERIVATION OF EQUATION (3.50).

(1) Consider the Bateman Expansion, WATSON (1944) pg 370

$$\begin{aligned} \frac{1}{2} z J_\nu(z \cos \phi \cos \Phi) J_\nu(z \sin \phi \sin \Phi) \\ = \cos^\nu \phi \cos^\nu \Phi \sin^\nu \phi \sin^\nu \Phi \sum_{n=0}^{\infty} (-1)^n (\nu + \nu + 2n + 1) J_{\nu + \nu + 2n + 1}(z) \\ \times \frac{\Gamma(\nu + \nu + n + 1) \Gamma(\nu + n + 1)}{\Gamma(n + 1) \Gamma(\nu + n + 1) \{\Gamma(\nu + 1)\}^2} {}_2F_1(-n, \nu + \nu + n + 1; \nu + 1; \sin^2 \phi) \\ \times {}_2F_1(-n, \nu + \nu + n + 1; \nu + 1; \sin^2 \Phi) \end{aligned}$$

(2) Let $\nu = -\frac{1}{2}$, and use the fact that $J_{-\frac{1}{2}}(x) = (2/\pi x)^{1/2} \cos x$, then

$$\begin{aligned} \left(\frac{z}{2\pi}\right)^{1/2} J_\nu(z \cos \phi \cos \Phi) \cos(z \sin \phi \sin \Phi) \\ = \cos^\nu \phi \cos^\nu \Phi \sum_{n=0}^{\infty} (-1)^n (2n + \nu + \frac{1}{2}) J_{2n + \nu + \frac{1}{2}}(z) \\ \times \frac{\Gamma(\nu + n + \frac{1}{2}) \Gamma(n + \frac{1}{2})}{\pi \Gamma(n + 1) \Gamma(\nu + n + 1)} {}_2F_1(-n, \nu + n + \frac{1}{2}; \frac{1}{2}; \sin^2 \phi) \\ \times {}_2F_1(-n, \nu + n + \frac{1}{2}; \frac{1}{2}; \sin^2 \Phi) \end{aligned}$$

(3) Now let $\Phi \rightarrow 0$ and use the fact that $\lim_{\epsilon \rightarrow 0} {}_2F_1(a, b; c; \epsilon) = 1$

$$\begin{aligned} J_\nu(z \cos \phi) = \left(\frac{2\pi}{z}\right)^{1/2} \cos^\nu \phi \sum_{n=0}^{\infty} (2n + \nu + \frac{1}{2}) \frac{(2n-1)!!}{2^n \Gamma(\nu + n + 1)} J_{2n + \nu + \frac{1}{2}}(z) \\ \times (-1)^n \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(n + 1)} {}_2F_1(-n, \nu + n + \frac{1}{2}; \frac{1}{2}; \sin^2 \phi) \end{aligned}$$

(4) It is known that, MAGNUS ET AL (1966) pg 220

$$C_{2n}^{\nu + \frac{1}{2}}(z) = (-1)^n \frac{\Gamma(\nu + n + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \frac{1}{\Gamma(n + 1)} {}_2F_1(-n, \nu + n + \frac{1}{2}; \frac{1}{2}; z^2)$$

so letting $\theta = \pi/2 - \phi$

$$\begin{aligned} J_\nu(z \sin \theta) = \left(\frac{2\pi}{z}\right)^{1/2} \sin^\nu \theta \sum_{n=0}^{\infty} (2n + \nu + \frac{1}{2}) \frac{(2n-1)!!}{2^n \Gamma(\nu + n + 1)} J_{2n + \nu + \frac{1}{2}}(z) \\ \times \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2})} C_{2n}^{\nu + \frac{1}{2}}(\cos \theta) \end{aligned}$$

Equation (3.50) follows if μ is an integer. The special case $\mu = 0$ can be found in GRADSHTEYN AND RYZHIK (1965) 8.531(3).

APPENDIX C THE COMPLIANCE MATRICES

This appendix contains the general terms of the compliance matrix A_{Rij} , where $A_{Rij} = A(I,J)$. Where relevant only the upper diagonal term is given, the lower diagonal can be obtained from this using the symmetry $A_{Reij} = A_{Rij}$. Below I and J take the values $I = 2, N-1$, $J = 2, N-1$, and the following notations are used:

$$I1 = I + 1, \text{ so too } J1;$$

$$I2 = I - 1, \text{ so too } J2, N2;$$

$$X(I) = x_i, X(I,J) = x_i x_j;$$

$$YK(I) = y_{Ri} = 1 \quad k = 1, \text{ so too } YL(I), ZK(I), ZL(I);$$

$$y_i \quad k = 2$$

$$I(I,J,M,N) = I(r_i, r_j, m, n) \quad k = 1 = 1$$

$$J(r_i, r_j, m, n) \quad k = 1 = 2$$

$$K(r_i, r_j, m, n) \quad k = 1-1 = 1$$

$$AM(I,J) = I(I,J,2,2)$$

$$AI(I) = I(I,M,2,1)$$

$$AJ(I) = I(M,I,1,2)$$

$$AL(I) = 4.0 X(N,N) (I(I,M,2,3) - ZL(N) I(I,N,2,3))$$

$$AK(I) = 4.0 X(N,N) (I(M,I,3,2) - ZK(N) I(N,I,3,2))$$

$$A(1,1) = X(1,1) \text{ AM}(2,2)$$

$$A(1,J) = X(1,J) (Y_L(J) \text{ AM}(2,J1) - \text{AM}(2,J)) + X(1,J2) (Z_L(J2) * \text{AM}(2,J2) - \text{AM}(2,J))$$

$$A(1,N) = X(1,N) (3 Y_L(N) \text{ AM}(2,M) + \text{AM}(2,N)) + X(1,N2) (Z_L(N2) * \text{AM}(2,N2) - \text{AM}(2,N)) - X(1) Y_L(N) \text{ AL}(2)$$

$$A(1,M) = 3 X(1) \text{ AI}(2) + X(1,N) (Z_L(N) \text{ AM}(2,N) - 5 \text{ AM}(2,M)) + X(1) \text{ AL}(2)$$

$$\begin{aligned} A(I,J) = & X(I,J) (Y_K(I) Y_L(J) \text{ AM}(I1,J1) - Y_K(I) \text{ AM}(I1,J) - Y_L(J) * \\ & \text{AM}(I,J1) + \text{AM}(I,J)) \\ & + X(I,J2) (Y_K(I) Z_L(J2) \text{ AM}(I1,J2) - Y_K(I) \text{ AM}(I1,J) - \\ & Z_L(J2) \text{ AM}(I,J2) + \text{AM}(I,J)) \\ & + X(I2,J) (Z_K(I2) Y_L(J) \text{ AM}(I2,J1) - Z_K(I2) \text{ AM}(I2,J) - \\ & Y_L(J) \text{ AM}(I,J1) + \text{AM}(I,J)) \\ & + X(I2,J2) (Z_K(I2) Z_L(J2) \text{ AM}(I2,J2) - Z_K(I2) \text{ AM}(I2,J) - \\ & Z_K(J2) \text{ AM}(I,J2) + \text{AM}(I,J)) \end{aligned}$$

$$\begin{aligned} A(I,N) = & X(I,N) (3 Y_K(I) Y_L(N) \text{ AM}(I1,M) + Y_K(I) \text{ AM}(I1,N) - \\ & 3 Y_L(N) \text{ AM}(I,M) - \text{AM}(I,N)) \\ & + X(I,N2) (Y_K(I) Z_L(N2) \text{ AM}(I1,N2) - Y_K(I) \text{ AM}(I1,N) - \\ & Z_L(N2) \text{ AM}(I,N2) + \text{AM}(I,N)) \\ & + X(I2,N) (3 Z_K(I2) Y_L(N) \text{ AM}(I2,M) + Z_K(I2) \text{ AM}(I2,N) - \\ & 3 Y_L(N) \text{ AM}(I,M) - \text{AM}(I,N)) \\ & + X(I2,N2) (Z_K(I2) Z_L(N2) \text{ AM}(I2,N2) - Z_K(I2) \text{ AM}(I2,N) - \\ & Z_K(N2) \text{ AM}(I,N2) + \text{AM}(I,N)) \\ & - Y_L(N) (X(I) (Y_K(I) \text{ AL}(I1) - \text{AL}(I)) + X(I2) (Z_K(I2) * \\ & \text{AL}(I2) - \text{AL}(I))) \end{aligned}$$

$$\begin{aligned} A(I,M) = & 3 X(I) (Y_K(I) \text{ AI}(I1) - \text{AI}(I)) + 3 X(I2) (Z_K(I2) * \\ & \text{AI}(I2) - \text{AI}(I)) \\ & + X(I,N) (Y_K(I) Z_L(N) \text{ AM}(I1,N) - 5 Y_K(I) \text{ AM}(I1,M) - \\ & Z_L(N) \text{ AM}(I,N) + 5 \text{ AM}(I,M)) \\ & + X(I2,N) (Z_K(I2) Z_L(N) \text{ AM}(I2,N) - 5 Z_K(I2) \text{ AM}(I2,M) - \\ & Z_L(N) \text{ AM}(I,N) + 5 \text{ AM}(I,M)) \\ & + X(I) (Y_K(I) \text{ AL}(I1) - \text{AL}(I)) + X(I2) (Z_K(I2) * \\ & \text{AL}(I2) - \text{AL}(I)) \end{aligned}$$

$$\begin{aligned}
A(N,N) &= X(N,N) (9 YK(N) YL(N) AM(M,M) + 3 YK(N) AM(M,N) + \\
&\quad 3 YL(N) AM(N,M) + AM(N,N)) \\
+ X(N,N2) (3 YK(N) ZL(N2) AM(M,N2) + 3 YK(N) AM(M,N) + \\
&\quad ZK(N2) AM(N,N2) + AM(N,N)) \\
+ X(N2,N) (3 ZK(N2) YL(N) AM(N2,M) + ZK(N2) AM(N2,N) + \\
&\quad 3 YL(N) AM(N,M) + AM(N,N)) \\
+ X(N2,N2) (ZK(N2) ZL(N2) AM(N2,N2) + ZK(N2) AM(N2,N) + \\
&\quad ZL(N2) AM(N,N2) + AM(N,N)) \\
= YL(N) (X(N) (3 YK(N) AL(M) + AL(N)) + X(N2) (ZK(N2) * \\
&\quad AL(N2) - AL(N))) \\
= YK(N) (X(N) (3 YL(N) AK(M) + AK(N)) + X(N2) (ZL(N2) * \\
&\quad AK(N2) - AL(N))) \\
+ YK(N) YL(N) (16 X(N,N) X(N,N) (I(M,M,3,3) + ZK(N) * \\
&\quad I(N,M,3,3) + ZL(N) I(M,N,3,3) + ZK(N) ZL(N) * \\
&\quad I(N,N,3,3))
\end{aligned}$$

$$\begin{aligned}
A(N,M) &= 3 X(N) (YK(N) AI(M) + AI(N)) + 3 X(N2) (ZL(N2) AI(N2) + \\
&\quad AI(N)) \\
+ X(N,N) (3 YK(N) ZL(N) AM(N,N) + 15 YK(N) AM(M,N) + \\
&\quad ZL(N) AM(N,N) + 5 AM(N,M)) \\
+ X(N2,N) (ZK(N2) ZL(N) AM(N2,N) + 5 ZK(N2) AM(N2,M) + \\
&\quad ZL(N) AM(N,N) + 5 AM(N,M)) \\
+ X(N) (3 YK(N) AL(M) + AL(N)) + X(N2) (ZK(N2) AL(N2) + \\
&\quad AL(N)) = YK(N) X(N) (ZL(N) AK(N) + 5 AK(M)) \\
= YK(N) 12 X(N,N) (I(M,M,3,1) + ZK(N) I(N,M,3,1)) \\
= 16 YK(N) X(N,N) X(N,N) (I(M,M,3,3) + ZK(N) I(N,M,3,3) + \\
&\quad ZL(N) I(M,N,3,3) + ZK(N) ZL(N) I(N,N,3,3))
\end{aligned}$$

$$\begin{aligned}
A(M,M) &= 9 I(M,M,1,1) + 3 X(N) (ZL(N) AJ(N) + 5 AJ(M) + ZK(N) * \\
&\quad AI(N) + 5 AI(M)) \\
+ 12 X(N,N) (I(M,M,3,1) + ZK(N) I(N,M,3,1) + I(M,M,1,3) + \\
&\quad ZL(N) I(M,N,1,3)) \\
+ X(N,N) (ZK(N) ZL(N) AM(N,N) + 5 ZK(N) AM(N,M) + 5 ZL(N) * \\
&\quad AM(M,N) + 25 AM(M,M)) \\
+ X(N) (ZK(N) AL(N) + 5 AL(M) + ZL(N) AK(N) + 5 AK(M)) \\
+ 16 X(N,N) X(N,N) (I(M,M,3,3) + ZK(N) I(N,M,3,3) + ZL(N) * \\
&\quad I(M,N,3,3) + ZK(N) ZL(N) I(N,N,3,3))
\end{aligned}$$

APPENDIX D PROGRAMME LISTING

Contained in this appendix is a programme listing. The five programmes are listed first in the order in which they were run, followed by the subroutines in logical sequence. Also contained here are three subroutines not discussed in this thesis, but used in the solution of the problem of frictionless or adhesive contact by a sphere. The order of the listing is

PROGRAMMES:

STEP1
STRESS
STEP2
STEP3
STEP4

SUBROUTINES:

HERTZ1	MILNE
HERTZ2	MNTAA1
SEWEL1	MNTBA1
SEWEL2	F21AA1
SONI1	GAMMA1
SONI2	GAMMA2
RDATA	GAMMB1
NOBLE	SSTAA1
NOBLE1	SSTBA1
NOBLE2	THNAA1
NOBLE3	THNBA1
AMGNS1	THNBB1
AMGNS2	THNCA1
GAUSS	THNCA2
GAUSS1	
GAUSS2	

```

INPUT :TURNERR,STEP1IN
SP B,VALUE (%A+1)
INPUT LITTLEFRAN,PS,T$$$$
*FORTRAN
PROGRAM(STEP1)
INPUT 1=CR1
OUTPUT 2=LPO
OUTPUT 3=CP0
END
MASTER TURNERJR
DIMENSION A(%B,%B),AM(%B,%B)
DIMENSION B(%B),C(%B),D(%B),E(%B)
DIMENSION X(%B,%B),XI(%B),RN(%B),R(%B)
COMMON PI,ERR,G(3,3,3,3)
COMMON /PARAM/ NEL,MND,H
COMMON /IDIM/ IA
IA=%B
CALL SONI1
CALL RDATA(R,RN,X,XI,MERR)
WRITE(2,200)NEL,MND,PI,MERR
IFF=0
IFG=%C
CALL HERTZ1(A,B,C,D,E,AM,X,XI,RN,R,PMA,IFF,IFG)
WRITE(3,101)C
STOP
100 FORMAT(4I4)
101 FORMAT(4E20,8)
200 FORMAT(1H1,10X,'AXISYMMETRIC HERTZ CONTACT PROBLEM',/,11X,54(' '),
1'////,' SOLUTION FOR FLAT INDENTOR W(R,0)=1',/,/,
2' FINITE ELEMENT SOLUTION WITH A CONTACT REGION DIVIDED INTO',14,'
3 ELEMENTS AND',14,' NODES',/,/, ' THE VALUE OF PI IS',F15.10,/,/,
4' THE SOLUTION IS CORRECT TO NO MORE THAN',14,' SIGNIFICANT DIGIT
5S',/,/, ' ***RESULTS',/)
END
FINISH
*LINK LIB HERTZ
*LINK LIB NAGF
*END
&&&&
$$$$
OXCMP *CR LITTLEFRAN,*CP RESULTS1,/(AS *CR1,DATA0),T&&&
ENDJOB
****

```

```

INPUT :TURNERR,STRESSIN
SP A,VALUE (%A+1)
INPUT LITTLEFRAN,PS,T$$$$
*FORTRAN

```

```

PROGRAM(STRESS)
TRACE 2
INPUT 1=CR1
OUTPUT 2=LPO
INPUT 3=CR2
OUTPUT 6=CP0
END
MASTER TURNERR
DIMENSION VM(%A,30,3),ST(30),RN(30)
DIMENSION VK(%A),S(%A),X(%A)
DIMENSION A(%A),B(%A),C(%A),P(%A)
COMMON PI,ERR,H(3,3,3)
COMMON /A2/ K,L
CALL SON12
IA=%A

```

```

READ(1,100)NEL
READ(1,100)HERR
READ(1,101)RC
MND=NEL+1
ERR=10.0*(-MERR)
S(1)=0.

```

```

DO 1 I=1,NEL
READ(1,101)S(I+1)
1 X(I)=2.0/(S(I+1)*S(I+1)-S(I)*S(I))

```

```

WRITE(2,200)
READ(3,101)P
DO 4 IRA=1,30
READ(1,101)R
RN(IRA)=R

```

```

DO 4 K=1,3
CALL SSTAA1(VK,S,R,X,NEL,IA,B,C,A)

```

```

DO 2 I=1,MND
2 VM(I,IRA,K)=VK(I)
IF(K.GT.1) GO TO 4
ST(IRA)=0.

```

```

DO 3 I=1,MND
3 ST(IRA)=ST(IRA)+VM(I,IRA,1)*P(I)
WRITE(2,201)IRA,R,ST(IRA)

```

```

4 CONTINUE
WRITE(6,101)VM
WRITE(6,101)ST,RN

```

```

STOP

```

```

100 FORMAT(I4)

```

```

101 FORMAT(4E20,8)

```

```

200 FORMAT('1*+*+STRESS RESULTS FOR HETTZ PROBLEM',///,
1 ' RADIAL STRESS = HOOP STRESS, VALUES TO SIX DIAMETERS',///,
24X,'I',10X,'R',14X,'S',///)

```

```

201 FORMAT(I4,2F15,4)

```

```

END

```

```

FINISH

```

```

*LINK LIB HERTZ

```

```

*LINK LIB NAGF

```

```

*END

```

```

&&&&

```

```

$$$$

```

```

OXCMP *CR LITTLEFRAN,*CP STRESS%C,#(AS *CR1,DATA%B),#(AS *CR2,RESULTS%C)

```

```

T&&&&

```

```

ENDJOB

```

```

****

```

```

INPUT :TURNERR,STEP2IN
SP B,VALUE (%A*1)
INPUT LITTLEFRAN,PS,TSSSS$
*FORTRAN
PROGRAM(STEP2)
INPUT 1=CR1
OUTPUT 2=LPO
INPUT 3=CR2
OUTPUT 4=CP0
INPUT 6=CR3
END
MASTER TURNERR
DIMENSION A(%B,%B),AM(%B,%B)
DIMENSION B(%B),C(%B),D(%B),E(%B)
DIMENSION X(%B,%B),XI(%B),RN(%B),R(%B)
DIMENSION VM(%B,50,3)
COMMON PI,ERR,G(3,3,3,3)
COMMON /PARAM/ NEL,MND,H,RC
COMMON /IDIM/ IA
IA=%B
CALL SONI1
CALL RDATA(R,RN,X,XI,MERR)
WRITE(2,200)NEL,MND,PI,MERR
IFF=0
IFG=%C
CALL HERTZ2(A,B,C,D,E,AM,X,XI,RN,R,PMA,IFF,IFG,VM)
DO 1 I=1,NEL
1 C(I)=1.0/SQRT(1.0-R(I)*R(I))
C(MND)=C(NEL)
WRITE(4,400)NEL,MND,MERR,PI,RC
WRITE(4,401)A,B,C,D,R
STOP
400 FORMAT(3I4,2E20.12)
401 FORMAT(4E20.12)
200 FORMAT(1H1,10X,'AXISYMMETRIC HERTZ CONTACT PROBLEM',/,11X,54(' '),
1'///,' SOLUTION FOR FLAT INDENTOR W(R,0)=1',///,
2' FINITE ELEMENT SOLUTION WITH A CONTACT REGION DIVIDED INTO',14,'
3 ELEMENTS AND',14,' NODES',///,' THE VALUE OF PI IS',F15.10,///,
4' THE SOLUTION IS CORRECT TO NO MORE THAN',14,' SIGNIFICANT DIGIT
5S',///,' ****RESULTS',//)
END
FINISH
*LINK LIB HERTZ
*LINK LIB NAGF
*END
&&&&
$$$$
OXCOMP *CR LITTLEFRAN,*CP RESULTS2,#(AS *CR1,DATA0),#(AS *CR2,RESULTS1),
#(AS *CR3,STRESS1),T&&&&
ENDJOB
****

```

```

INPUT :TURNERR,STEP3IN
SP A,VALUE (%A+1)
SP B,VALUE (%A+1)
SP C,VALUE(2*%A)
SP D,VALUE (%C+%A+2)
SP E,VALUE (%D*%B)
INPUT LITTLEFRAN,PS,T$$$$
*FORTRAN
PROGRAM(STEP3)
OUTPUT 2=LP0
INPUT 3=CR2
INPUT 4=CR1
OUTPUT 5=CP0
INPUT 6=CR3
END
MASTER TURNERR
DIMENSION A(%A,%A)
DIMENSION AII(%E)
DIMENSION B(%A),C(%A),D(%C),E(%D)
DIMENSION P(%A),Q(%A),R(%A)
DIMENSION IV(%C),JV(%C)
DIMENSION VII(%A,30,3)
COMMON NEL,MND,MERR,RHO,RC
COMMON /IDIM/ IA,IB,IC,ID,IE
IA=%A
IB=%B
IC=%C
ID=%D
IE=%E
IF=%F
RC=%G
CALL SEWEL1(A,B,C,D,E,R,AH,IV,JV,P,Q,IF,VM)
WRITE(5,400)NEL,MND,MERR,RHO
WRITE(5,401)A,B,C,E,R
STOP
400 FORMAT(3I4,E20.12)
401 FORMAT(4E20.12)
END
FINISH
*LINK LIB HERTZ
*LINK LIB NAGF
*END
&&&&
$$$$
OXCOMP *CR LITTLEFRAN,*CP RESULTS3,#(AS *CR1,RESULTS2),-
#(AS *CR2,RESULTS1),#(AS *CR3,STRESS1)
ERASE LITTLEFRAN
ENDJOB
****

```

```

INPUT :TURNERR,STEP4IN
SP A,VALUE (%A+1)
SP B,VALUE (%A+1)
SP C,VALUE(2*%A)
SP D,VALUE (%C+%A+2)
SP E,VALUE (%D*%B)
INPUT LITTLEFRAN,PS,T$$$$
*FORTRAN

```

```

PROGRAM(STEP4)
INPUT 1=CR0
OUTPUT 2=LPO
INPUT 5=CR1
INPUT 6=CR2
END
MASTER TURNERR
DIMENSION A(%A,%A)
DIMENSION AM(%E)
DIMENSION B(%A),C(%A),D(%C),E(%D)
DIMENSION U(%A),R(%A)
DIMENSION IV(%C),JV(%C)
DIMENSION VH(%A,30,3)
COMMON /IDIM/ IA,IB,IC,ID,IE
IA=%A
IB=%B
IC=%C
ID=%D
IE=%E
CALL SEWEL2(A,B,C,D,E,R,AM,IV,JV,U,VM)
STOP
END
FINISH

```

```

*LINK LIB HERTZ
*LINK LIB NAGF
*END BIN STEP4PROG
&&&&
$$$$$
OXCMP *CR LITTLEFRAN,T&&&&
ERASE LITTLEFRAN
ENDJOB
****

```

```

INPUT :TURNERR,STEP4RUN
INPUT DATA1,PS,T&&&&

```

```

% A
&&&&
RUN PROGRAM STEP4PROG,DATA *CR DATA1,#(AS *CR1,RESULTS),#(AS *CR2,STRES:
*END
ERASE DATA1
ENDJOB
****

```

```

SUBROUTINE HERTZ1(A,B,C,D,E,AM,X,XI,RN,R,PMA,IFF,IW)
DIMENSION A(IA,IA),AM(IA,IA)
DIMENSION B(IA),C(IA),D(IA),E(IA)
DIMENSION X(IA,IA),XI(IA),RN(IA),R(IA)
COMMON PI,ERR,G(3,3,3,3)
COMMON /PARAM/ NEL,MND,H
COMMON /IDIM/ IA
WRITE(2,200)
CALL NOBLE(1,1,A,AM,X,XI,B,C,RN,D,E)
DO 1 I=1,MND
C(I)=0.
D(I)=0.
1 B(I)=0.0
IF(IFF.GT.0) GO TO 2
CALL NOBLE1(B,IA,RN)
CALL THNBA1(A,B,IA,AM,C,D,E,MND)
PMA=2.0/PI
PNA=1.0
GO TO 3
2 CALL NOBLE2(B,C,IA,RN)
CALL THNAA1(A,B,C,IA,AM,D,E,RN,MND)
PMA=4.0/PI
PNA=1.0/3.0
3 PM=C(1)
IF(IW.EQ.0) WRITE(2,201)PM,PMA
DO 4 I=1,MND
C(I)=C(I)/PMA
IF(IW.EQ.1) GO TO 4
IF(I.LE.NEL) W=SQRT(1.0-R(I)*R(I))
V=C(I)*W
WRITE(2,202)I,R(I),V
4 CONTINUE
PN=0.
DO 5 I=1,MND
5 PN=PN+C(I)*B(I)
PN=PN*H*H
WRITE(2,203)PN,PNA
RETURN
200 FORMAT(' HERTZ1 ENTERED',//)
201 FORMAT(' THE PRESSURE AT THE ORIGIN IS',F10.7,' COMPARED WITH TH
1E ACTUAL VALUE OF',F10.7,/,/, ' THE RESULTING PRESSURE DISTRIBUTION
2IS',/,/,6X,'I',10X,'R(I)',10X,'P(I)/PHTZ',/)
202 FORMAT(17,F16.4,F17.4)
203 FORMAT(/,/, ' THE NON-DIMENSIONALISED TOTAL LOAD IS PN=',F10.7,' C
- 1OMPARED WITH A TRUE VALUE OF',F10.7,/)
END

```

```

SUBROUTINE HERTZ2(A,B,C,D,E,AM,X,XI,RN,R,QMA,IFF,IW,VM)
DIMENSION A(IA,IA),AM(IA,IA)
DIMENSION B(IA),C(IA),D(IA),E(IA)
DIMENSION X(IA,IA),XI(IA),RN(IA),R(IA)
DIMENSION VH(IA,30,3)
COMMON PJ,ERR,G(3,3,3,3)
COMMON /PARAM/ NEL,MND,H
COMMON /IDIII/ IA
WRITE(2,200)
CALL NOBLE(1,2,A,AM,X,XI,B,C,RN,D,E)
READ(3,300)C
N2=NEL-1
DO 10 J=1,N2
J1=J+2
DO 10 I=J1,MND
10 A(I,J)=0.
DO 1 I=1,MND
B(I)=0.
DO 1 J=1,MND
1 B(I)=B(I)+A(J,I)*C(J)
CALL NOBLE(2,2,A,AM,X,XI,D,E,RN,C,R)
IF(IFF.GT.0) GO TO 2
CALL THNBA1(A,B,IA,AM,D,E,RN,MND)
GO TO 4
2 DO 3 I=1,MND
3 D(I)=0.
CALL NOBLE3(D,IA,RN)
CALL THNAA1(A,B,D,IA,AM,C,E,RN,MND)
4 QMA=1.0/PI
QM=D(1)/R(2)
IF(IW.EQ.1) GO TO 6
WRITE(2,201)QM,QMA
W=PI/R(2)
DO 5 I=1,MND
Z=R(I)
Y=1.0-Z*Z
IF(I.GT.1.AND.I.LE.NEL) W=-PI*Z*SQRT(Y)/ALOG(Y)
V=D(I)*W
5 WRITE(2,202)I,R(I),V
JR=0
ST=1.
CALL SSTBA1(D,VM,JR,ST,MND)
6 RETURN
-200 FORMAT(' HERTZ2 ENTERED',//)
201 FORMAT(' THE SLOPE Q'(0)/P(0) IS',F10.7,' COMPARED WITH TH
1E ACTUAL VALUE OF',F10.7,'//', ' THE RESULTING SHEAR DISTRIBUTION
2IS',//,6X,'I',10X,'R(I)',10X,'Q(I)/QHTZ',//)
202 FORMAT(17,F16.4,F17.4)
300 FORMAT(4E20,12)
END

```

```

SUBROUTINE SEWEL1(A,B,C,D,E,R,AM,IV,JV,P,Q,IF,VM)
DIMENSION A(IA,IA)
DIMENSION AM(IE)
DIMENSION B(IA),C(IA),D(IC),E(ID)
DIMENSION P(IA),Q(IA),R(IA)
DIMENSION IV(IC),JV(IC)
DIMENSION W(8),U(2)
DIMENSION VM(IA,30,3)
COMMON NEL,MND,MERR,RHO,RC
COMMON /IDIM/ IA,IB,IC,ID,IE
READ(4,400)NEL,MND,MERR,PI
READ(4,401)A,B,P,Q,R
IF(IF.EQ.1) READ(3,401)P
WRITE(2,200)NEL,MND,PI,MERR
ERR=10.0*(-MERR)
CS=RC*RC
RD=SQRT(1.0-CS)
CALL MILNE(D,RD,RC,ERR,IF)
PK=PI*0.5*D(1)
PL=ALOG((1.0+RC)/(1.0-RC))
RHO=PL/(2.0*RC*PK)
WRITE(2,201)RHO,RC
N1=MND+1
MN=2*MND
MV=N1+MN+1
DO 1 I=1,MND
C(I)=RHO*P(I)
1 D(I)=C(I)
CALL THNCA1(A,B,AM,D,E,IV,JV,MND,N1,MV,MN,IA)
WRITE(2,202)
W(5)=1.0/PI*R(2)
V=0
DO 2 I=1,MND
V=V+D(I)*E(I)
W(1)=R(1)
W(2)=P(1)
Z=W(1)*W(1)
IF(1.LE.NEL) Y=SQRT(1.0-Z)
W(3)=W(2)*Y
W(4)=Q(1)/P(1)
IF(1.GT.1) W(5)=-2.0*ALOG(Y)/(PI*W(1))
W(6)=D(1)/C(1)
W(8)=E(1)
W(7)=1.0
IF(W(1).EQ.0.0.OR.W(1).GT.RC) GO TO 2
S=SQRT(CS-Z)/Y
T=ALOG((1.0+S)/(1.0-S))/(RHO*PI*R(1))
PHI=ASIN(S/RC)
S=S/R(1)
CALL MNTBA1(HL,U,AM,RD,RC,PHI,PI,ERR,IF)
HL=HL-S*U(1)+T
W(7)=1.0-HL
2 WRITE(2,203)I,W
JR=0
ST=1
CALL SSTBA1(D,VM,JR,ST,MND)
WRITE(2,204)V
RETURN
100 FORMAT(I4)
200 FORMAT(1H1,10X,'AXISYMMETRIC HERTZ CONTACT PROBLEM',/,11X,54(' '),
1///,' SOLUTION FOR FLAT INDENTOR W(R,0)=1',///,
2' FINITE ELEMENT SOLUTION WITH A CONTACT REGION DIVIDED INTO',I4,'
3' ELEMENTS AND',I4,' NODES',///,' THE VALUE OF PI IS',F15.10,///,
4' THE SOLUTION IS CORRECT TO NO MORE THAN',I4,' SIGNIFICANT DIGIT
5S',///,' ***RESULTS',///)
201 FORMAT(///,' SOLUTION TO FRICTIONAL LOADING PROBLEM, RHO/GAMMA='
1,E13.4,///,' THE EXPECTED ADHESIVE RADIUS IS,RC=',F13.4,///)
202 FORMAT('
1,///,' THE RESULTING STRESS DISTRIBUTIONS ARE',/,6X,'I',8X,'RI',
212X,'PH',12X,'PE',12X,'QA',12X,'QE',12X,'QS',12X,'QE',12X,'US',/)
203 FORMAT(I7,F12.4,7F14.4)
204 FORMAT(///,' THE FRICTIONAL WORK DONE IN LOADING IS V=',E13.4)
400 FORMAT(3I4,E20,12)
401 FORMAT(4E20,12)
END

```

```

SUBROUTINE SEWEL2(A,B,C,D,E,R,AM,IV,JV,U,VM)
C PROGRAMME TO SOLVE THE AXI-SYMMETRIC CONTACT PROBLEM
C FOR UNLOADING BY A FLAT PUNCH, GAMMA SMALL
C SOLUTION BY THE FINITE ELEMENT TECHNIQUE FOR A CONTACT REGION
C DIVIDED INTO A GIVEN NUMBER OF ELEMENTS
  DIMENSION A(IA,IA)
  DIMENSION AM(IE)
  DIMENSION B(IA),C(IA),D(IC),E(ID)
  DIMENSION U(IA),R(IA)
  DIMENSION IV(IC),JV(IC)
  DIMENSION W(5)
  DIMENSION VM(IA,30,3)
  COMMON /IDIM/ IA,IB,IC,ID,IE
  JR=0
  READ(1,100)XN
  READ(5,500)NEL,MND,MERR,RHO
  READ(5,501)A,B,C,E,R
  WRITE(2,200)NEL,MND,RHO
  T=0.99*(XN)
  RT=1.0/T
  ST=RT-1.0
  DO 1 I=1,MND
1  U(I)=0.
  S=1.
  V=0.
  DO 4 J=1,12
  S=S*T
  WRITE(2,201)T,J,S
  DO 2 I=1,MND
  B(I)=B(I)+ST*U(I)+RT*E(I)
  U(I)=RT*(U(I)+E(I))
2  D(I)=C(I)
  N1=MND+1
  MN=2*MND
  MV=MN+N1+1
  CALL THNCA2(A,B,C,AM,D,E,IV,JV,MND,N1,MV,MN,IA)
  P=0
  DO 3 I=1,MND
  P=P+D(I)*E(I)
  W(1)=R(I)
  W(2)=D(I)/C(I)
  W(3)=E(I)
  W(4)=U(I)
  W(5)=E(I)+U(I)
3  WRITE(2,202)I,W
  CALL SSTRAI(D,VM,JK,S,MND)
  P=P*S*S
  V=V+P
4  WRITE(2,203)P,V
  RETURN
100 FORMAT(F12.4)
200 FORMAT(1H1,10X,'AXI-SYMMETRIC HERTZ CONTACT PROBLEM',/,11X,S4(' '),
1  '/',/,1  ' SOLUTION FOR FLAT INDENTOR W(R,0)=1',/,/,
2  ' FINITE ELEMENT SOLUTION WITH A CONTACT REGION DIVIDED INTO',14,'
3  ' ELEMENTS AND',14,' NODES',/,/,1  ' THE VALUE OF RHO IS',F15.10,/,/
4  ' *****RESULTS',/,/)
201 FORMAT(/,/,1  ' UNLOADING SOLUTION WITH TIME STEP T=',E13.4,/,
1  16,'TH STEP',/,1  ' TOTAL UNLOADING TO DATE IS S=',E13.4,
2  ' ',6X,'I',8X,'RI',12X,'QU',12X,'VI',12X,'UO',12X,
3  ' UN',/,/)
202 FORMAT(17,F12.4,4F14.4)
203 FORMAT(/,/,1  ' FRICTIONAL WORK DONE ON THIS STEP P=',E15.4,
1  ' AND THE TOTAL TO DATE IS V=',E13.4)
500 FORMAT(3I4,E20.12)
501 FORMAT(4E20.12)
END

```

```

SUBROUTINE SONI1
COMMON PI,ERR,G(3,5,3,3)
PI=4.0*ATAN(1.0)
DO 1 M=1,3
DO 1 N=1,3
MN=M+N
XM=M
XN=N
XMN=MN
TM=2.0**M
TN=2.0**N
G(M,N,1,1)=TM*TN*GAMMA1(MN=1)/(GAMMA2(MN)*GAMMA2(M)*GAMMA2(N)*PI)
TN=TN*GAMMA1(N)
G(M,N,2,1)=1.0/(TN*GAMMA2(M))
G(M,N,3,3)=1.0/(TM*TN*GAMMA1(M=1))
G(N,M,3,1)=G(M,N,2,1)
G(M,N,1,2)=G(M,N,1,1)/(2.0*XMN+1.0)
G(M,N,2,2)=G(M,N,2,1)/(2.0*XN+2.0)
G(N,M,3,2)=G(M,N,2,2)
G(M,N,1,3)=G(M,N,3,3)/XIN
G(M,N,2,3)=G(M,N,3,3)/(XN+1.0)
1 G(M,N,3,3)=G(M,N,3,3)/XM
RETURN
END

```

```

SUBROUTINE SONI2
COMMON PI,ERR,H(3,5,3)
PI=4.0*ATAN(1.0)
DO 1 M=1,3
XM=M
H1=GAMMB1(M)
H2=GAMMA2(M)
H(M,1,1)=1.0/H1
H(M,3,2)=1.0/H2
H(M,1,2)=XM*H(M,1,1)
H(M,1,3)=H(M,1,1)
H(M,2,1)=H1/((2.0*XM+1.0)*XM*PI*H2*H2)
H(M,2,2)=0.5*H(M,3,2)
H(M,2,3)=0.5*H(M,1,1)/(XM+1.0)
H(M,3,3)=-H(M,2,3)
IF(M.EQ.1) GO TO 1
H(M,3,1)=-2.0*GAMMB1(M-2)/(PI*H2*H2)
1 CONTINUE
RETURN
END

```

```

SUBROUTINE RDATA(R,RN,X,XI,HERR)
DIMENSION X(IR,IR),XI(IR),R(IR),RN(IR)
COMMON PI,ERR,G(3,5,3,3)
COMMON /PARAM/ NEL,MND,H,RC
COMMON /IDIM/ IR
READ(1,100)NEL
READ(1,100)HERR
READ(1,101)RC
ERR=10.0*(-HERR)
MND=NEL+1
HN=NEL
H=1.0/HN
R(1)=0.
RN(1)=0.
DO 1 I=1,NEL
READ(1,101)R(I+1)
RN(I+1)=R(I+1)*HN
X(NEL,1)=2.0/(RN(I+1)*RN(I+1)-RN(I)*RN(I))
XI(I)=H*X(NEL,1)
DO 1 J=1,I
X(J,1)=XI(I)*X(NEL,J)
1 X(I,J)=X(J,I)
RETURN
100 FORMAT(4I4)
101 FORMAT(F10.5)
END

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SUBROUTINE NOBLE(IPQ,JPQ,A,AM,X,XI,YI,ZI,RN,AN,BN)
DIMENSION A(IA,IA),AM(IA,IA)
DIMENSION X(IA,IA),XI(IA),YI(IA),ZI(IA),RN(IA),AN(IA),BN(IA)
COMMON PI,ERR,G(3,3,3,3)
COMMON /PARAM/ NEL,MND,H
COMMON /IDIM/ IA
COMMON /AMG/ IP,JP,KP
IP=IPQ
JP=JPQ
AB=4.0*X(NEL,NEL)
AA=AB/H
ZI(1)=0.0
IF(JPQ-2)1,3,14
1 DO 2 I=2,NEL
  YI(1)=1.0
2 ZI(1)=1.0
  GO TO 5
3 DO 4 I=2,NEL
  YI(1)=RN(I+1)/RN(1)
4 ZI(1)=1.0/YI(1)
5 N2=NEL-1
  Y=YI(NEL)
  Z=ZI(NEL)
  ZN=ZI(N2)
  YN3=3.0*Y
  IF(IPQ-JPQ)10,6,14
6 KP=IPQ
  DO 7 I=1,MND
  AN(I)=AA*(AMGNS1(I,MND,2,3,RN)-Z*AMGNS1(I,NEL,2,5,RN))
  DO 7 J=1,MND
  AM(I,J)=AMGNS1(I,J,2,2,RN)
7 AM(J,I)=AM(I,J)
  A(1,1)=X(1,1)*AM(2,2)
  A1=AMGNS1(2,MND,2,1,RN)
  A2=0.0
  A0=XI(1)*AN(2)
  A(1,NEL)=X(1,NEL)*(Y*3.0*AM(2,MND)+AM(2,NEL))+
1 X(1,N2)*(ZN*AM(2,N2)-AM(2,NEL))-Y*A0
  A(NEL,1)=A(1,NEL)
  A(1,MND)=3.0*XI(1)*A1+X(1,NEL)*(Z*AM(2,NEL)-5.0*AM(2,MND))+A0
  A(MND,1)=A(1,MND)
  DO 9 J=2,N2
  J1=J+1
  J2=J-1
  Y1=YI(J)
  Z1=ZI(J2)
  Z2=ZI(J2)
  A(1,J)=X(1,J)*(Y1*AM(2,J1)-AM(2,J))+X(1,J2)*(Z2*AM(2,J2)-AM(2,J))
  A(J,1)=A(1,J)
  DO 8 I=2,J
  I1=I+1
  I2=I-1
  A(I,J)=X(I,J)*(YI(1)*YI(J)*AM(I1,J1)-YI(1)*AM(I1,J)-YI(J)*AM(I,J1)
1 +AM(I,J))+X(I,J2)*(YI(1)*ZI(J2)*AM(I1,J2)-YI(1)*AM(I1,J)-ZI(J2)*AM
2 (I,J2)+AM(I,J))+X(I2,J)*(ZI(I2)*YI(J)*AM(I2,J1)-ZI(I2)*AM(I2,J)-YI
3 (J)*AM(I,J1)+AM(I,J))+X(I2,J2)*(ZI(I2)*ZI(J2)*AM(I2,J2)-ZI(I2)*AM(I
4 I2,J)-ZI(J2)*AM(I,J2)+AM(I,J))
8 A(J,I)=A(I,J)
  A3=A2
  A2=A1
  A1=AMGNS1(J1,MND,2,1,RN)
  ANJ=XI(J)*(Y1*AN(J1)-AN(J))+XI(J2)*(Z1*AN(J2)-AN(J))
  A(J,NEL)=-Y*ANJ+
1 X(J,NEL)*(Y1*(YN3*AM(J1,MND)+AM(J1,NEL))-YN3*AM(J,MND)-AM(J,NEL))+
2 X(J,N2)*(Y1*(ZN*AM(J1,N2)-AM(J1,NEL))-ZN*AM(J,N2)+AM(J,NEL))+
3 X(J2,NEL)*(Z2*(YN3*AM(J2,MND)+AM(J2,NEL))-YN3*AM(J,MND)-AM(J,NEL))
4 +X(J2,N2)*(Z2*(ZN*AM(J2,N2)-AM(J2,NEL))-ZN*AM(J,N2)+AM(J,NEL))
  A(NEL,J)=A(J,NEL)
  A(J,MND)=3.0*(XI(J)*(Y1*A1-A2)+XI(J2)*(Z2*A3-A2))+ANJ-X(J,NEL)*(Y1
1 *(5.0*AM(J1,MND)-Z*AM(J1,NEL))-5.0*AM(J,MND)+Z*AM(J,NEL))-X(J2,NEL
2 *(Z2*(5.0*AM(J2,MND)-Z*AM(J2,NEL))-5.0*AM(J,MND)+Z*AM(J,NEL))
9 A(MND,J)=A(J,MND)
  A3=A2
  A2=A1
  A1=AMGNS1(MND,MND,2,1,RN)
  AI=AMGNS1(MND,MND,1,1,RN)
  AOM=3.0*AB*(AMGNS1(MND,MND,3,1,RN)-Z*AMGNS1(NEL,MND,5,1,RN))
  AP2=XI(NEL)*(5.0*AN(MND)-Z*AN(NEL))

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AQ=AA*AB*(AMGNS1(MND,MND,3,3,RN)-2.0*Z*AMGNS1(MND,NEL,3,3,RN)+2*Z*
1 AMGNS1(NEL,NEL,3,3,RN))
A(MND,MND)=9.0*H*AI-6.0*XI(NEL)*(5.0*A1-2*A2)+2.0*AOM-2.0*AP2+AQ*
1 X(NEL,NEL)*(25.0*AM(MND,MND)-10.0*Z*AM(MND,NEL)+2*Z*AM(NEL,NEL))
AQ=Y*AQ
AP1=XI(NEL)*(Y*3.0*AN(MND)+AN(NEL))+XI(N2)*(ZN*AN(N2)-AN(NEL))
YN4=5.0-YN3*Z
A(NEL,MND)=5.0*(XI(NEL)*(Y*
1 3.0*A1+A2)+XI(N2)*(ZN*A3-A2))-Y*AOM+AP1+Y*AP2-AQ-X(NEL,NEL)*(Y*15.
2 *AM(MND,MND)+YN4*AM(MND,NEL)-Z*AM(NEL,NEL))-X(NEL,N2)*(ZN*(5.0*
3 AM(N2,MND)-Z*AM(N2,NEL))-5.0*AM(NEL,MND)+Z*AM(NEL,NEL))
A(MND,NEL)=A(NEL,MND)
A(NEL,NEL)=X(NEL,NEL)*(Y*(Y*9.0*AM(MND,MND)+6.0*AM(MND,NEL))+AM(NE
1 L,NEL))+2.0*X(NEL,N2)*(ZN*(YN3*AM(N2,MND)+AM(N2,NEL))-YN3*AM(NEL,M
2 ND)-AM(NEL,NEL))+X(N2,N2)*(ZN*(ZN*AM(N2,N2)-2.0*AM(N2,NEL))+AM(NEL
3 ,NEL))+Y*(AQ-2.0*AP1)
RETURN
10 KP=3
DO 11 I=1,MND
AN(I)=AA*(AMGNS1(I,MND,2,3,RN)-Z*AMGNS1(I,NEL,2,3,RN))
BN(I)=AA*(AMGNS1(MND,I,3,2,RN)-AMGNS1(NEL,I,3,2,RN))
DO 11 J=1,MND
11 AM(I,J)=AMGNS1(I,J,2,2,RN)
A(1,1)=X(1,1)*AM(2,2)
A2=0.0
A1=AMGNS1(2,MND,2,1,RN)
B2=0.0
B1=AMGNS1(MND,2,1,2,RN)
AO=XI(1)*AN(2)
BO=XI(1)*BN(2)
A(1,NEL)=X(1,NEL)*(Y*3.0*AM(2,MND)+AM(2,NEL))+X(1,N2)
1 *(ZN*AM(2,N2)-AM(2,NEL))-Y*AO
A(NEL,1)=X(NEL,1)*(3.0*AM(MND,2)+AM(NEL,2))+X(N2,1)*(AM(N2,2)
1 -AM(NEL,2))-BO
A(1,MND)=3.0*XI(1)*A1+X(1,NEL)*(Z*AM(2,NEL)-5.0*AM(2,MND))+AO
A(MND,1)=3.0*XI(1)*B1-X(NEL,1)*(5.0*AM(MND,2)-AM(NEL,2))+BO
DO 13 J=2,N2
J1=J+1
J2=J-1
Y1=YI(J)
Z1=ZI(J2)
A(1,J)=X(1,J)*(Y1*AM(2,J1)-AM(2,J))+X(1,J2)*(Z1*AM(2,J2)-AM(2,J))
A(J,1)=X(J,1)*(AM(J1,2)-AM(J,2))+X(J2,1)*(AM(J2,2)-AM(J,2))
DO 12 I=2,N2
I1=I+1
I2=I-1
12 A(I,J)=X(I,J)*(YI(J)*AM(I1,J1)-AM(I1,J)-YI(J)*AM(I,J1)+AM(I,J))
1 +X(I,J2)*(ZI(J2)*AM(I1,J2)-AM(I1,J)-ZI(J2)*AM(I,J2)+AM(I,J))
2 +X(I2,J)*(YI(J)*AM(I2,J1)-AM(I2,J)-YI(J)*AM(I,J1)+AM(I,J))
3 +X(I2,J2)*(ZI(J2)*AM(I2,J2)-AM(I2,J)-ZI(J2)*AM(I,J2)+AM(I,J))
A3=A2
A2=A1
A1=AMGNS1(J1,MND,2,1,RN)
B3=B2
B2=B1
B1=AMGNS1(MND,J1,1,2,RN)
ANJ=XI(J)*(AN(J1)-AN(J))+XI(J2)*(AN(J2)-AN(J))
BNJ=XI(J)*(Y1*BN(J1)-BN(J))+XI(J2)*(Z1*BN(J2)-BN(J))
A(J,NEL)=Y*ANJ+X(J,NEL)*(Y*3.0*(
1 AM(J1,MND)-AM(J,MND))+AM(J1,NEL)-AM(J,NEL))+X(J2,NEL)*(Y*3.0*(AM(
2 J2,MND)-AM(J,MND))+AM(J2,NEL)-AM(J,NEL))+X(J,N2)*(ZN*(AM(J1,N2)-
3 AM(J,N2))-AM(J1,NEL)+AM(J,NEL))+X(J2,N2)*(ZN*(AM(J2,N2)-AM(J,N2))
4 -AM(J2,NEL)+AM(J,NEL))
A(NEL,J)=X(NEL,J)*(3.0*(Y1*AM(MND,J1)-AM(MND,J))+Y1*AM(NEL,J1)-AM(
1 NEL,J))+X(J2,NEL)*(3.0*(Z1*AM(MND,J2)-AM(MND,J))+Z1*AM(NEL,J2)-
2 AM(NEL,J))+X(N2,J)*(Y1*(AM(N2,J1)-AM(NEL,J1))-AM(N2,J)+AM(NEL,J))+
3 X(N2,J2)*(Z1*(AM(N2,J2)-AM(NEL,J2))-AM(N2,J)+AM(NEL,J))-BNJ
A(J,MND)=3.0*(XI(J)*(A1-A2)+XI(J2)*(A3-A2))+ANJ-X(J,NEL)*(5.0*(
1 AM(J1,MND)-AM(J,MND))-Z*(AM(J1,NEL)-AM(J,NEL))-X(J2,NEL)*(5.0*(
2 AM(J2,MND)-AM(J,MND))-Z*(AM(J2,NEL)-AM(J,NEL)))
13 A(MND,J)=3.0*(XI(J)*(Y1*B1-B2)+XI(J2)*(Z1*B3-B2))+BNJ-X(NEL,J)*(
15.0*(Y1*AM(MND,J1)-AM(MND,J))-Y1*AM(NEL,J1)+AM(NEL,J))-X(NEL,J2)*(
25.0*(Z1*AM(MND,J2)-AM(MND,J))-Z1*AM(NEL,J2)+AM(NEL,J))
A3=A2
A2=A1
A1=AMGNS1(MND,MND,4,1,RN)
B3=B2

```

```

B2=B1
B1=AMGNS1(MND,MND,1,2,RN)
A1=AMGNS1(MND,MND,1,1,RN)
AOM=3.0*AB*(AMGNS1(MND,MND,3,1,RN)-AMGNS1(NEL,MND,3,1,RN))
BOM=3.0*AB*(AMGNS1(MND,MND,1,3,RN)-2*AMGNS1(MND,NEL,1,3,RN))
AP2=XI(NEL)*(5.0*AN(MND)-AN(NEL))
BP2=XI(NEL)*(5.0*BN(MND)-2*BN(NEL))
AQ=AA*AB*(AMGNS1(MND,MND,3,3,RN)-2*AMGNS1(MND,NEL,3,3,RN))
1-AMGNS1(NEL,MND,3,3,RN)+2*AMGNS1(NEL,NEL,3,3,RN))
A(MND,MND)=9.0*H*A1-3.0*X1(NEL)*(5.0*(A1+B1)-A2-2*B2)+AOM+BOM+AQ
1+X(NEL,NEL)*(25.0*A1(MND,MND)-5.0*(2*AM(MND,NEL)+AM(NEL,MND))
2+2*AM(NEL,NEL))-AP2-BP2
AP1=XI(NEL)*(3.0*AN(MND)+AN(NEL))+X1(N2)*(AN(N2)-AN(NEL))
BP1=XI(NEL)*(Y*3.0*BN(MND)+BN(NEL))+X1(N2)*(ZN*BN(N2)-BN(NEL))
A(NEL,MND)=3.0*(X1(NEL)*(3.0*A1+A2)+X1(N2)*(A3-A2))-AOM+BP2+AP1-AQ
1-X(NEL,NEL)*(15.0*A1(MND,MND)-3.0*Z*AM(MND,NEL)+5.0*AM(NEL,MND)-2*
2AM(NEL,NEL))-X(NEL,N2)*(5.0*(AM(N2,MND)-AM(NEL,MND))-Z*AM(N2,NEL)+
3Z*AM(NEL,NEL))
A(MND,NEL)=3.0*(X1(NEL)*(Y*3.0*B1+B2)+
1X1(N2)*(ZN*B3-B2))-Y*BOM-X(NEL,NEL)*(Y*(15.0*AM(MND,MND)-
23.0*AM(NEL,MND))+5.0*AM(MND,NEL)-AM(NEL,NEL))-X(NEL,N2)*(5.0*(ZN*
3AM(MND,N2)-AM(MND,NEL))-ZN*AM(NEL,N2)+AM(NEL,NEL))+BP1+Y*(AP2-AQ)
A(NEL,NEL)=X(NEL,NEL)*(YN3*(3.0*AM(MND,MND)+AM(NEL,MND))+5.0*AM(MND,
1D,NEL)+AM(NEL,NEL))+X(NEL,N2)*(ZN*(3.0*AM(MND,N2)+AM(NEL,N2))-5.0*
2AM(MND,NEL)-AM(NEL,NEL))+X(N2,NEL)*(YN3*(AM(N2,MND)-AM(NEL,MND))+A
3M(N2,NEL)-AM(NEL,NEL))+X(N2,N2)*(ZN*(AM(N2,N2)-AM(NEL,N2))-AM(N2,N
4EL)+AM(NEL,NEL))+Y*(AQ-AP1)-BP1
RETURN
14 WRITE(2,200)IPQ,JPQ
STOP
200 FORMAT(' ****MESSAGE...AN ERROR HAS OCCURED IN NOBLE',2I4)
END

```

```

SUBROUTINE NOBLE1(B,IA,RN)
C TO CALCULATE THE VECTORS, B, ASSOCIATED WITH THE INTEGRAL OF THE
C BOUNDARY VALUE OF DISPLACEMENT
DIMENSION B(IA),RN(IA)
COMMON /PARAM/ NEL,MND,H
N2=NEL-1
DO 1 I=1,N2
R1S=RN(I)*RN(I)
R2S=RN(I+1)*RN(I+1)
XI1=(R2S-R1S)*0.25
B(I)=B(I)+XI1
1 B(I+1)=B(I+1)+XI1
R1S=RN(NEL)*RN(NEL)
R2S=RN(MND)*RN(MND)
XI1=(R2S-R1S)/12.0
B(NEL)=B(NEL)+5.0*XI1
B(MND)=B(MND)+7.0*XI1
RETURN
END

```

```

C TO CALCULATE THE VECTORS, R, C, ASSOCIATED WITH THE INTEGRAL OF THE
C BOUNDARY VALUE OF DISPLACEMENT
SUBROUTINE NORLE2(B,C,IA,RN)
DIMENSION B(IA),C(IA),RN(IA)
COMMON /PARAM/ NEL,MND,H
N2=NEL-1
HS=H*H
DO 1 I=1,N2
R1S=RN(I)*RN(I)
R2S=RN(I+1)*RN(I+1)
X11=(R2S-R1S)*0.25
B(I)=B(I)+X11
B(I+1)=B(I+1)+X11
X11=X11/3.0
RI=R1S+R2S
C(I)=C(I)+X11*(R1S+RI)
1 C(I+1)=C(I+1)+X11*(R2S+RI)
R1S=RN(NEL)*RN(NEL)
R2S=RN(MND)*RN(MND)
X11=(R2S-R1S)/12.0
B(NEL)=B(NEL)+5.0*X11
B(MND)=B(MND)+7.0*X11
RI=2.0*(R1S+R2S)
C(NEL)=C(NEL)+X11*(RI+R1S)
C(MND)=C(MND)+X11*(RI+3.0*R2S)
DO 2 I=1,MND
2 C(I)=C(I)*HS
RETURN
END

```

```

C TO CALCULATE THE VECTOR, D, ASSOCIATED WITH THE INTEGRAL OF THE
C BOUNDARY VALUE OF DISPLACEMENT
SUBROUTINE NORLE3(D,IA,RN)
DIMENSION D(IA),RN(IA)
DIMENSION D1(2),D2(2),S(2,2)
COMMON /PARAM/ NEL,MND,H
HS=H*H
N2=NEL-1
DO 3 IEL=2,N2
R1=RN(IEI)
R2=RN(IEI+1)
S(1,2)=-R2
S(2,2)=R1
R11=R1*R1
R22=R2*R2
DD=HS/(R1*R2*(R22-R11))
R111=R11*R1
R222=R22*R2
S(1,1)=R222
S(2,1)=-R111
R111=R111*R11
R222=R222*R22
D1(1)=(R222-R111)/5.0
R111=R111*R11
R222=R222*R22
D1(2)=(R222-R111)/7.0
DO 2 I=1,2
D2(I)=0
DO 1 J=1,2
1 D2(I)=D2(I)+S(I,J)*D1(J)
2 D2(I)=D2(I)*DD
3 D(IEI)=D(IEI)+D2(1)
D(IEI+1)=D(IEI+1)+D2(2)
D(1)=HS*(RN(2)**4)/7.0
D(2)=D(2)+D(1)
D(1)=0.4*D(1)
A=RN(NEL)*H
AS=A*A
AF=AS*AS
BS=1.0+A
BS=1.0/BS
BS=BS*BS*BS
DS=(RN(MND)*RN(MND)-RN(NEL)*RN(NEL))/315.
DA=38.0+60.0*AS+AF*BS*(66.0+135.0*A+65.0*AS)
DB=115.0+57.0*AS+AF*BS*(15.0+45.0*A+34.0*AS)
D(NEL)=D(NEL)+DS*DA/A
D(MND)=D(MND)+DS*DB
RETURN
END

```

```

FUNCTION AMGNS1(I,J,M,N,RN)
DIMENSION RN(1A)
COMMON P1,ERR,G(3,3,3,3)
COMMON /AMG/ IPQ,JPQ,KPQ
COMMON /IDIM/ 1A
IF(I.EQ.1.OR.J.EQ.1) GO TO 4
RI=RN(I)
RJ=RN(J)
IF(I-J)3,1,2
1 AMGNS1=G(M,N,1,KPQ)*RI** (M+M+N+N-1)
RETURN
2 Z=(RJ*RJ)/(RI*RI)
AMGNS1=G(M,N,2,KPQ)*GAUSS(M-KPQ/3,N+JPQ,KPQ,Z,ERR)*RI** (M+M-JPQ)*R
J** (N+N+JPQ-1)
RETURN
3 Z=(RI*RI)/(RJ*RJ)
AMGNS1=G(M,N,3,KPQ)*GAUSS(N,M+IPQ,KPQ,Z,ERR)*RI** (M+M+IPQ-1)*RJ** (
1 N+N-IPQ)
RETURN
4 AMGNS1=0.
RETURN
END

```

```

FUNCTION AMGNS2(S,M,R)
COMMON P1,ERR,H(3,3,3)
COMMON /A2/ K,L
IF(ABS(S-R).LT.0.001) GO TO 1
IF(S-R)3,1,2
1 AMGNS2=H(M,K,1)*S** (2*M-2)
RETURN
2 Z=R/S
AMGNS2=H(M,K,2)*GAUSS(M-L/3,2-K/3,L,Z*Z,ERR)*S** (2*M-2)
RETURN
3 Z=S/R
J=4-L
AMGNS2=H(M,K,3)*GAUSS(J-K,M+2,L,Z*Z,ERR)*S** (2*M-1+J)/R** (1+J)
RETURN
END

```

```

FUNCTION GAUSS(MA,NA,KPQ,Z,ERR)
IF(KPQ-2)1,1,4
1 IF(MA)3,3,2
2 GAUSS=GAUSS2(KPQ-1,MA,NA,0.5,0.5,0.0,Z,ERR)
RETURN
3 GAUSS=GAUSS1(KPQ-1,-MA,NA,0.5,0.5,0.0,Z,ERR)
RETURN
4 GAUSS=F21AA1(MA,NA,Z)
RETURN
END

```

```

FUNCTION GAUSS1(LA,MA,NA,A0,B0,C0,Z,ERR)
DIMENSION E(2)
ZI=1.0-Z
CALL MILNE(E,SQRT(Z),SQRT(ZI),ERR,IF)
A=A0
B=B0
C=C0+1.0
IF(LA.EQ.0) GO TO 2
CMB=C-B
DO 1 L=1,LA
EO=E(1)
E(1)=(CMB*E(2)-(CMB-A)*E(1))/(ZI*A)
E(2)=((A-B+1.0)*E(2)-(1.0-B)*EO)/A
1 A=A+1.0
2 IF(MA.EQ.0) GO TO 4
DO 3 M=1,MA
EO=E(1)
E(1)=((C-B)*E(2)+(2.0*B-C-Z*(B-A))*EO)/(B*ZI)
E(2)=EO
3 B=B+1.0
4 IF(NA.EQ.1) GO TO 6
EO=E(1)
E(1)=(C/(A-C))*((EO-E(2))/Z-EO)
E(2)=EO
IF(NA.EQ.2) GO TO 6
DO 5 N=1,NA
C=C+1.0
EO=E(1)
E(1)=(C/((C-A)*(C-B)))*((C-1.0)*(E(2)-EO)/Z-(C-1.0)*E(2)+(2.0*C-A-
1 B-1.0)*EO)
5 E(2)=EO
6 GAUSS1=E(1)
RETURN
END

```

```

FUNCTION GAUSS2(LA,MA,NA,A0,B0,C0,Z,ERR)
DIMENSION E(2)
ZI=1.0-Z
CALL MILNE(E,SQRT(Z),SQRT(ZI),ERR,IF)
A=A0
B=B0
C=C0+1.0
IF(LA.EQ.0) GO TO 2
CMB=C-B
DO 1 L=1,LA
EO=E(1)
E(1)=(CMB*E(2)-(CMB-A)*E(1))/(ZI*A)
E(2)=((A-B+1.0)*E(2)-(1.0-B)*EO)/A
1 A=A+1.0
2 B=B-1.0
IF(MA.EQ.1) GO TO 4
DO 3 M=2,MA
EO=E(2)
E(2)=((C-2.0*B-(A-B)*Z)*EO+B*ZI*E(1))/(C-B)
E(1)=EO
3 B=B-1.0
4 IF(NA.EQ.1) GO TO 6
EO=E(2)
E(2)=(C/((C-A)*(C-B)))*(B*(E(1)-EO)/Z+(C-A)*EO-B*E(1))
E(1)=EO
IF(NA.EQ.2) GO TO 6
DO 5 N=1,NA
C=C+1.0
EO=E(2)
E(2)=(C/((C-A)*(C-B)))*((C-1.0)*(E(1)-EO)/Z-(C-1.0)*E(1)+(2.0*C-A-
1 B-1.0)*EO)
5 E(1)=EO
6 GAUSS2=E(2)
RETURN
END

```

```

C  SUBROUTINE MILNE(E1,SA,CA,ERR,IF)
C  CALCULATION OF COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KIND
C  K(K), E(K), WITH MODULUS K=SA=SIN(ALPHA).
C  EI(1)=(2/PI)*K(K), EI(2)=(2/PI)*E(K)
C  SEP ABRAMOWITZ AND STEGUN (1965) S17.6, P598
  DIMENSION EI(2)
  A=1.
  B=CA
  C=SA
  SIGMA=C*C
  IF=0
  DO 1 I=1,100
    C=(A-B)/2.
    SIGMA=SIGMA+C*C*2.**I.
    B=SQRT(A*B)
    A=A-C
    IF (ABS(C).LT.ERR) GO TO 2
  1 CONTINUE
  IF=1
  WRITE(2,200)C,ERR
200 FORMAT(' ****MESSAGE',//,' MILNE DID NOT CONVERGE C=',E10.9,' ERR
  1=',E10.9,//)
  2 EI(1)=1./A
    EI(2)=(1.-SIGMA/2.)*EI(1)
  RETURN
  END

```

```

  SUBROUTINE MNTAA1(E,SA,CA,PHI,PI,ERR,IF)
  DIMENSION E(5)
  A=1.
  B=CA
  C=SA
  SIGMA=C*C
  E(5)=0.
  IF=0
  DO 1 I=1,100
    PSI=ATAN(B*TAN(PHI)/A)
    IF (PSI.LT.0.) PSI=PSI+PI
    XM=AIN(T(PHI/PI)
    PHI=PSI+XM*PI+PHI
    C=(A-B)/2.
    SIGMA=SIGMA+C*C*2.**I
    B=SQRT(A*B)
    A=A-C
    E(5)=E(5)+C*SIN(PHI)
    N=I
    IF (ABS(C).LT.ERR) GO TO 2
  1 CONTINUE
  IF=1
  WRITE(2,200)C,ERR
200 FORMAT(' ****MESSAGE',//,' MNTAA1 DID NOT CONVERGE C=',E10.9,' ER
  1 R=',E10.9,//)
  2 E(1)=0.5*PI/A
    E(2)=(1.-SIGMA/2.)*E(1)
    E(3)=PHI/(A*2.0**N)
    E(4)=E(5)+E(3)*E(2)/E(1)
  RETURN
  END

```

```

  SUBROUTINE MNTBA1(HL,E,F,SA,CA,PHI,PI,ERR,IF)
  DIMENSION E(2),F(6)
  CALL MILNE(E,SA,CA,ERR,IF)
  CALL MNTAA1(F,CA,SA,PHI,PI,ERR,IF)
  HL=F(3)/F(1)+E(1)*F(5)
  RETURN
  END

```

```

      FUNCTION F21AA1(M,N,Z)
      F=1
      IF(M)4,3,1
1     AF=1
      DO 2 I=1,M
      J=I-1
      MN=J-M
      NN=J+N
      XM=MN
      XN=NN
      AF=AF*XM*Z/XN
2     F=F+AF
3     F21AA1=F
      RETURN
4     F21AA1=0
      RETURN
      END

```

```

      FUNCTION GAMMA1(N)
      M=1
      IF(N,EQ,0) GO TO 2
      DO 1 I=1,N
1     M=M*I
2     GAMMA1=M
      RETURN
      END

```

```

      FUNCTION GAMMA2(N)
      M=1
      IF(N,EQ,0) GO TO 2
      DO 1 I=1,N
1     M=M*(2*I-1)
2     GAMMA2=M
      RETURN
      END

```

```

      FUNCTION GAMMB1(N)
      M=1
      IF(N,EQ,0) GO TO 2
      DO 1 I=1,N
1     M=2*M*I
2     GAMMB1=M
      RETURN
      END

```

```

SUBROUTINE SSTAA1(VK,S,R,X,NEL,IA,Y,Z,A)
DIMENSION VK(IA),S(IA)
DIMENSION X(IA),Y(IA),Z(IA),A(IA)
COMMON /A2/ K,L
N1=NEL-1
MND=NEL+1
L=3
IF(K.GT.1) L=2
Z(1)=0.
A(1)=0.
DO 1 I=2,MND
Y(I)=1.0
Z(I)=1.0
1 A(I)=AMGNS2(S(I),2,R)
IF(K-1)6,4,2
2 DO 3 I=2,NEL
Y(I)=S(I+1)/S(I)
3 Z(I)=1.0/Y(I)
4 VK(1)=X(1)*A(2)
DO 5 I=2,N1
I2=I-1
5 VK(I)=X(I)*(Y(I)*A(I+1)-A(I))+X(I2)*(Z(I2)*A(I2)-A(I))
VKM=4.0*X(NEL)*X(NEL)*(AMGNS2(S(MND),3,R)-Z(NEL)*AMGNS2(S(NEL),3,R)
1))
VK(NEL)=X(NEL)*(3.0*Y(NEL)*A(MND)+A(NEL))+X(N1)*(Z(N1)*A(N1)-
1A(NEL))-VKM*Y(NEL)
VK(MND)=3.0*AMGNS2(S(MND),1,R)+X(NEL)*(Z(NEL)*A(NEL)-5.0*A(MND))+V
1KM
6 RETURN
END

```

```

SUBROUTINE SSTBA1(Q,VM,JR,A,MND)
DIMENSION Q(IA),VM(IA,30,3),S1(30),R(30)
COMMON /IDIM/ IA
SMAX=0.
RMAX=0.
WRITE(2,200)A
IF(JR.EQ.1) GO TO 1
JR=1
READ(6,101)VM
READ(6,101)S1,R
1 DO 3 IRA=1,30
S2=0.
S3=0.
DO 2 I=1,MND
S2=S2+VM(I,IRA,2)*Q(I)
2 S3=S3+VM(I,IRA,3)*Q(I)
SA=-S2+2.0*S3
ST=-S1(IRA)+S2+S3
SR=-ST+3.0*S3
WRITE(2,201)IRA,R(IRA),S1(IRA),S2,S3,SR,ST,SA
IF(SR.LT.SMAX) GO TO 3
SMAX=SR
RMAX=R(IRA)
3 CONTINUE
PMA=SMAX*A
WRITE(2,202)RMAX,SMAX,PMA
RETURN
101 FORMAT(4E20,8)
200 FORMAT(///,' SURFACE STRESS DISTRIBUTION WITH ABOVE SHEAR PROFILE
1 UNLOADING PARAMETER LAMBDA =',F10.4,///,4X,'I',10X,'R',
214X,'S1',13X,'S2',13X,'S3',13X,'SR',13X,'ST',13X,'SA',///)
201 FORMAT(15,7F15.4)
202 FORMAT(///,' THE MAXIMUM RADIAL STRESS OCCURS AT A RADIUS OF, R=',
1F10.4,/, ' HAS A SCALED VALUE OF, SR=',F10.4,/, ' AND A TRUE VALUE C
2F, SR=',F10.4,///)
END

```

```

SUBROUTINE THNAA1(A,B,C,IA,AM,D,E,F,MND)
DIMENSION A(IA,IA),B(IA),C(IA)
DIMENSION AM(IA,IA),D(IA),E(IA),F(IA)
CALL THNBA1(A,B,IA,AM,D,E,F,MND)
CALL THNBB1(AM,C,E,F,MND,IA)
ALPHA=D(MND)/F(MND)
WRITE(2,200)ALPHA
A1=0.
A2=0.
A3=0.
DO 1 I=1,MND
A1=A1+C(I)*F(I)
A2=A2+C(I)*D(I)
1 A3=A3+B(I)*D(I)
A1=5.0*A1
ALPHA=3.0*A2/A1
A3=A3/A1
DELTA=ALPHA*ALPHA-A3
WRITE(2,201)ALPHA,DELTA
DO 2 I=1,MND
2 C(I)=D(I)-ALPHA*F(I)
RETURN
200 FORMAT(///,' THE VALUE OF ALPHA TO GIVE P(MND)=0 IS',F10.5)
201 FORMAT(///,' THE VALUE OF ALPHA TO GIVE A STATIONARY VALUE IN ENER
1GY IS',F10.5,' WITH DELTA =',F10.5,///)
END

```

```

SUBROUTINE THNBA1(A,B,IA,AM,D,E,F,N)
DIMENSION A(IA,IA),B(IA)
DIMENSION AM(IA,IA),D(IA),E(IA),F(IA)
ACC=2.0*(-218)
CALL F01AJF(N,ACC,A,IA,D,E,AM,IA)
IF=0
ACC=2.0*(-37)
CALL F02AMF(N,ACC,D,E,AM,IA,IF)
WRITE(2,200)(D(I),I=1,N)
DO 1 I=1,N
1 F(I)=D(I)
CALL THNBB1(AM,B,E,D,N,IA)
RETURN
200 FORMAT(' SUBROUTINE THNBA1 CALLED....THE MATRIX HAS EIGENVALUES',
1//,(10E12,4,/) )
END

```

```

SUBROUTINE THNBB1(A,B,C,D,N,IA)
DIMENSION A(IA,IA),B(IA),C(IA),D(IA)
DO 2 I=1,N
C(I)=0.
DO 1 J=1,N
1 C(I)=C(I)+A(J,I)*B(J)
2 C(I)=C(I)/D(I)
WRITE(2,200)(C(I),I=1,N)
DO 3 I=1,N
D(I)=0.
DO 3 J=1,N
3 D(I)=D(I)+A(I,J)*C(J)
RETURN
200 FORMAT(//,' THE VECTOR BETA IS',//,(10E12,4,/) )
END

```

```

SUBROUTINE THNCA1(A,B,P,RV1,RV2,IV1,IV2,N,N1,MV,MN,IA)
DIMENSION A(IA,IA),B(IA)
DIMENSION P(MV,N1),RV1(N1),RV2(MV),IV1(N1),IV2(MN)
N2=MV-1
DO 2 I=1,N
K1=N+I
K2=N+K1
DO 1 J=1,N
P(I,J)=0.
1 P(K1,J)=0.
P(K2,J)=A(I,J)
P(I,I)=1.
P(K1,I)=1.
P(I,N1)=0.
P(K1,N1)=RV1(I)
P(K2,N1)=-B(I)
P(N2,I)=-B(I)
2 IV2(I)=0.
P(N2,N1)=0.
P(MV,N1)=0.
IF1=0
IF2=0
CALL H02AAF(MV,N1,MN,IV2,P,IF1,IF2,IV1,RV2,RV1)
IF(IF1.EQ.0.AND. IF2.EQ.0) GO TO 3
WRITE(2,200)IF1,IF2
200 FORMAT(' ****H02AAF ERROR INDICATED, IF1=',I4,' IF2=',I4)
3 DO 4 I=1,N1
4 RV1(I)=0.
DO 5 I=1,MV
5 RV2(I)=0.
DO 6 I=1,N
6 RV1(I)=P(I,N1)
DO 7 I=1,N
RV2(I)=-B(I)
DO 7 J=1,N
7 RV2(I)=RV2(I)+A(I,J)*RV1(J)
RETURN
END

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SUBROUTINE THNCA2(A,B,C,P,RV1,RV2,IV1,IV2,N,N1,MV,MN,IA)
DIMENSION A(IA,IA),B(IA),C(IA)
DIMENSION P(MV,N1),RV1(N1),RV2(MV),IV1(N1),IV2(MN)
N2=MV-1
DO 2 I=1,N
RV2(I)=B(I)
K1=N+I
K2=N+K1
DO 1 J=1,N
RV2(I)=RV2(I)+A(I,J)*C(J)
P(I,J)=0.
1 P(K1,J)=0.
P(K2,J)=A(I,J)
P(I,I)=1.
P(K1,I)=1.
P(I,N1)=0.
P(K1,N1)=RV1(I)+C(I)
P(K2,N1)=-RV2(I)
P(N2,I)=-RV2(I)
2 IV2(I)=0.
P(N2,N1)=0.
P(MV,N1)=0.
IF1=0
IF2=0
CALL H02AAF(MV,N1,MN,IV2,P,IF1,IF2,IV1,RV2,RV1)
IF(IF1.EQ.0.AND. IF2.EQ.0) GO TO 3
WRITE(2,200)IF1,IF2
3 DO 4 I=1,N1
4 RV1(I)=0.
DO 5 I=1,MV
5 RV2(I)=0.
DO 6 I=1,N
6 RV1(I)=P(I,N1)-C(I)
DO 7 I=1,N
RV2(I)=-B(I)
DO 7 J=1,N
7 RV2(I)=RV2(I)+A(I,J)*RV1(J)
RETURN
200 FORMAT(' ****H02AAF ERROR INDICATED, IF1=',I4,' IF2=',I4)
END

```