Indirect control and power in mutual control structures☆

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Abstract

In a mutual control structure (mcs) agents exercise control over each other. Typical examples occur in the area of corporate governance: firms and investment companies exercise mutual control, in particular by owning each others' stocks. We represent such situations in two equivalent ways: by a function assigning to each coalition the set of controlled players, and by a simple game structure in which for each player a simple game describes who controls that player. These concepts are similar to authority distributions and command games in Hu and Shapley (2003a,b). An mcs is invariant if it incorporates all indirect control relations. We axiomatically develop a class of power indices for invariant mcs. We impose four axioms with a plausible interpretation in this framework, which together characterize a broad class of power indices based on dividends resulting both from exercising and from undergoing control. Extra conditions can further refine this broad class.

Keywords:
Mutual control structure, simple games, power index
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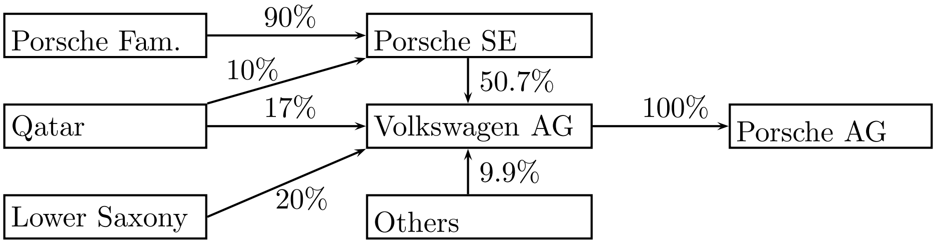


Figure 1: Porsche and VW voting rights by the end of 2012, based on the annual reports 2012 of Volkswagen AG and Porsche Automobil Holding SE GmbH.

1. Introduction

A mutual control structure describes a situation in which agents exercise control over each other. Typically, such situations occur in the area of corporate governance, when a conglomerate of firms and investment companies control each other, specifically by possessing shares or stocks. The purpose of this paper is to study a general abstract model for this kind of situations and to develop power indices, meant to represent the real power of the involved players. We first describe an example which will be revisited throughout the paper.

1.1. The Porsche-Volkswagen case

From 2008 onwards, the Porsche group started buying common stocks of Volkswagen AG on a grand scale. On January 5, 2009, Porsche announced to own more than 50% of the common stocks. However, the original plan to take over Volkswagen failed and the founding of an integrated corporation was announced. Figure 1 is a simplified organization chart of this corporation. *Porsche Families* is an aggregation of several companies and holdings which are held by members of the families Porsche and Piëch; it holds 90% of *Porsche SE*. *Qatar* is an aggregation of several holdings which are held by the Qatar Investment Authority, Doha. *Lower Saxony* includes Hannoversche Beteiligungsgesellschaft mbH which is held by Lower Saxony. *Volkswagen AG* holds 100% of Porsche Holding Stuttgart GmbH which in turn holds 100% of *Porsche AG* – therefore Porsche Holding Stuttgart GmbH has been left out. *Others* means investors which hold less than 3% of the shares and are therefore not mentioned in any reports.

Although Porsche SE has the majority of voting rights of Volkswagen AG, Lower Saxony has a veto power according to two laws.[1] These laws specify in particular that important decisions can be made only with 80% of the voting rights (of which Lower Saxony owns slightly more than 20%). Figure 1 describes the situation as it was before mid June, 2013. Meanwhile, Qatar has sold its shares in Porsche SE back to Porsche Families. For the sake of the example, however, we stick to the situation before mid June, 2013.

1.2. Mutual control structures

Interesting as the historical development of the Porsche-Volkswagen case may be, even until very recently, in this paper we will be mainly interested in the resulting organization chart as represented in Figure 1. This will serve as a recurring example (Example 2.3 in the next section, and its continuation further on). Here, and in similar situations, the question arises: Who is, ultimately, in control, and how much power do the involved parties have? The purpose of the present paper is to answer these questions and contribute to the literature by developing a general game-theoretic model.

Formally, a mutual control structure C will be a map assigning to each nonempty coalition – i.e., a subset of a given finite set of players N – another coalition. The interpretation of $C(T) = S$ is, that each player of S is controlled by the coalition T. For instance, $i \in S$ is a firm, and the coalition T of firms or investment companies has a majority of the shares of firm i. We impose the natural condition of monotonicity: if T controls S, then any coalition containing T also controls S. While the mutual control structure C thus captures direct control, it does not necessarily capture indirect control. The latter means that whenever T controls S, and S and T jointly control R, then T indirectly controls R. Thus, if j is a firm in R and S and T jointly have a majority of the shares of j, then T controls j if it is the case that T controls all firms in S. A mutual control structure will be called invariant if it satisfies this condition. In the paper we describe a procedure, similar to the one in Hu and Shapley (2003a), which assigns to each mutual control structure its unique minimal invariant extension.

Alternatively, a mutual control structure can be described by a vector of simple games, to be called a simple game structure in the paper. For each

[1] Namely, §111 AktG (Aktiengesetz) and §4 VWGmbHÜG (Gesetz über die Überführung der Anteilsrechte der Volkswagen Gesellschaft mit beschränkter Haftung in private Hand).

player, there is a (monotonic) simple game whose winning coalitions are exactly those that control that player. There is a one-to-one correspondence between mutual control structures and simple game structures.

1.3. Related literature

Our approach is closely related to the work of Hu and Shapley (2003a,b), in particular our Section 2. If player i is controlled by coalition S, i.e., $i \in C(S)$, then S is called a 'boss set' for player i in Hu and Shapley (2003b), but next to boss sets they also consider 'approval sets'. Our procedure to update mutual control structures in order to incorporate indirect control is quite similar to the one in Hu and Shapley (2003a), but, as mentioned, their assumptions on such a structure are different. In this respect, our approach is simpler and focusses on control ('boss sets' in their terminology). Hu and Shapley (2003a,b) also study command games, which are equivalent to our simple game structures. They propose a power index (Hu and Shapley, 2003b), which, however, is quite different from the power indices that we arrive at, see below.

Similar to the articles of Hu and Shapley (2003a,b) is the work of Grabisch and Rusinowska (2011). The authors introduce influence and follower functions and relate them to normal command games of Hu and Shapley (2003a,b), i.e., command games in which a player cannot be controlled by two disjoint coalitions.

A relatively early approach to the problem of indirect control in the literature is Gambarelli and Owen (1994). This approach explicitly distinguishes between firms and investors. In what is called a 'reduction', all power is reduced to power of the investors, i.e., the firms leave the scene. The proposed reduction operation bears some resemblance to our procedure of making a mutual control structure invariant. Gambarelli and Owen (1994) end up with so-called consistent reductions which, however, are not necessarily unique, in contrast to our minimal invariant extensions. Denti and Prati (2004) focus on the determination of winning coalitions among direct and indirect stockholders of corporations. Also Driessen and Sun (2006) distinguish between firms and investors. They consider such a situation as an application of a so-called 'set game'. The 'worth' of a coalition (of investors) is a set (of firms), meaning that the coalition of investors controls the firms in the set. In their framework a 'value' assigns to each investor a set of firms.

There are some other approaches in the literature aiming at establishing indirect control relations: see, for instance Crama and Leruth (2007). For a recent overview of the theoretical and empirical literature in this area see Crama and Leruth (2013).

Additionally, there is strand of literature which considers cooperative games respecting a specific graph theoretic structure imposed on the player set, and which is to some extent related to our approach. For instance, the 'permission structure' considered in Gilles et al. (1992) can be seen as a special instance of a mutual control structure (though with a somewhat different interpretation) in which all minimal controlling coalitions are singletons.

1.4. Power indices

In the main part of the paper we consider invariant mutual control structures and develop a class of power indices, intended to capture the 'true' power of individual players. We impose four axioms which have a plausible interpretation in the present framework. First, we set the power of null players equal to zero – a null player is a player who neither contributes to controlling any other player, nor is controlled by any other player or coalition himself. Second, we impose that the sum of all assigned powers is the same over all invariant mutual control structures. This axiom replaces the usual 'efficiency' condition; it will easily follow that this axiom together with the null player axiom actually makes this sum equal to zero. Third, we impose anonymity: the names of the players should not matter. Fourth, we impose a so-called transfer property, which says the following. For every player, the change in power when extending a mutual control structure C' to C should be equal to the change in power when extending a mutual control structure D' to D, whenever exactly the same control relations are added going from C' to C as when going from D' to D. This condition is called transfer property because it is related to the transfer property used by Dubey (1975) to characterize the Shapley value or Shapley-Shubik index Shapley (1953); Shapley and Shubik (1954) for monotonic simple games. See also Dubey et al. (2005).

We characterize the class of power indices satisfying these four conditions. Each power index in this class corresponds to a weight vector of dimension $2n-2$ (where n is the number of players) and assigns to a player i a weighted sum of dividends obtained in the simple games capturing the control undergone by the other players, diminished by dividends gathered by the other players in the game describing the control undergone by player i. The number $2n-2 = 2(n-1)$ is twice the number of possible cardinalities of a (nonempty proper) coalition: for each cardinality there are two weights for a coalition with that cardinality and associated with control exercised by that coalition on a player, depending on whether or not that player is in the coalition. By imposing the so-called controlled player axiom we obtain more scaling and, in fact, a unique power index with all weights equal to 1. This means that each player i obtains the sum of all his Shapley-Shubik values in the games in which he contributes to controlling the other players, minus the sum of all Shapley-Shubik values of the other players in the game describing the control undergone by i.

Hu and Shapley (2003b) also propose a power index for their command games, which, as already mentioned, are equivalent to our simple games in a simple game structure. If the number of players is n, then for each of the n command games they take the Shapley-Shubik power index and put these together in an $n \times n$ Markov matrix. Then the overall power index, called 'authority distribution' is defined to be the stationary state of this matrix. Thus, this authority distribution can be obtained by starting with an arbitrary distribution and 'updating' it using the Shapley-Shubik power indices of the separate 'command' (i.e., control) games. Such a dynamic approach to obtain a power index has been applied by several other authors, see for instance Herings et al. (2004) and the references therein. The latter work applies to a more limited model of a directed graph, but takes both exercising and undergoing control (dominance) into account, and in that sense bears some similarity to the results of our approach.

The approach in this paper can be applied whenever elementary, direct control relations can be retrieved from the data – for instance by considering simple majority share holdings within a corporate structure or taking into account special voting rights (as in the Porsche-Volkswagen case). Next, indirect control relations can be determined, and to the resulting invariant mutual control structure a power index can be applied. It should be noted, however, that it is probably impossible to capture all potential situations within an area of application in a completely satisfactory way. See the final section for some discussion.

As emphasized by Crama and Leruth (2013), in corporate governance and finance it is important to distinguish between ownership and control. Also this point will be discussed in the final section of the paper.

In Section 2 we introduce and study mutual control structures and their relation with simple games. In Section 3 we develop our class of power indices. Section 4 concludes with further discussion.

2. Mutual control structures and simple games

We start with some notations. For a set A we denote by $P(A)$ the set of all subsets of A. By $|A|$ we denote the number of elements of A.

Let $N = \{1, \ldots, n\}$ with $n \geq 2$ denote the set of *players*. Elements of $P(N)$ are called *coalitions*.

We first introduce and study mutual control structures, and next relate them to simple games.

2.1. Mutual control structures

Definition 2.1. A *mutual control structure* (mcs) is a map $C : P(N) \rightarrow P(N)$ satisfying

(i) $C(\emptyset) = \emptyset$,
(ii) *monotonicity*: $C(S) \subseteq C(T)$ for all $S, T \in P(N)$ with $S \subseteq T$.

The set of all mutual control structures is denoted by $\mathcal{C}$. □

If $i \in C(S)$ for some $i \in N$ and $S \in P(N)$ then we say that player i is *controlled* by coalition S. (For instance, the firms in S hold a majority of the shares of firm i.) Similarly, we say that S controls $C(S)$. Thus, the empty coalition controls no one, and a player controlled by a coalition S is also controlled by any coalition T containing S.

Remark 2.2. The concept of a mutual control structure is quite close to the concept of a command function ω in Hu and Shapley (2003a). A command function is derived from so-called boss sets and approval sets and has additionally the property that $\omega(N) = N$ (Proposition 2.1 in Hu and Shapley, 2003a). We start from a higher level of abstraction and do not impose the condition $C(N) = N$ on an mcs, although this is a natural condition in many applications. □

Example 2.3. Consider the Porsche-Volkswagen example in Figure 2, reproduced from Figure 1, and augmented with player numbers. We can describe the situation by a mutal control structure C. The players are Porsche Families (1), Qatar (2), Lower Saxony (3), Porsche SE (4), Volkswagen AG (5), Porsche AG (6), and Others (7). For any coalition $S \subseteq N = \{1, \ldots, 7\}$ we have:

$$4 \in C(S) \Leftrightarrow 1 \in S, \ 5 \in C(S) \Leftrightarrow \{2,3,4\} \subseteq S, \ \text{and } 6 \in C(S) \Leftrightarrow 5 \in S.$$

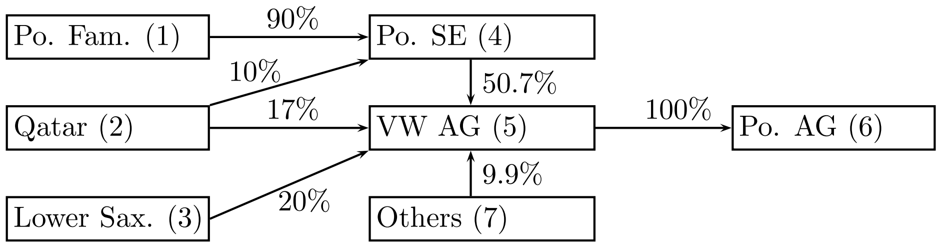


Figure 2: The Porsche-VW case from Figure 1, augmented with player numbers.

Clearly, C satisfies the conditions of Definition 2.1. Note that player 7 (Others) is controlled by no coalition and can be left out from any controlling coalition without changing control. Such a player will be called a 'null player' (Section 3) and normally speaking can be discarded from the model without essential changes. □

A mutual control structure does not necessarily capture 'indirect' control: in Example 2.3 Porsche Families controls Porsche SE ($4 \in C(\{1\}) \subseteq C(\{1,2,3\})$) which, together with Qatar and Lower Saxony, controls Volkswagen ($5 \in C(\{2,3,\ 4\}) \subseteq C(\{1,2,3,4\})$). But $5 \notin C(\{1,2,3\}$, that is, Volkswagen is not controlled by $\{1,2,3\}$. However, this coalition controls Volkswagen 'indirectly' as it can enforce Porsche SE to control Volkswagen. This property of 'indirect control' is captured by the following definition.

Definition 2.4. The mutual control structure C is *invariant* if it satisfies the following condition.

> *Indirect control:* For all $R, S, T \in P(N)$ with $S \subseteq C(T)$ and $R \subseteq C(S \cup T)$ we have $R \subseteq C(T)$.

The set of all invariant mutual control structures (imcs) is denoted by $\mathcal{C}^*$. □

Clearly, the control structure C in Example 2.3 violates indirect control with $T = \{1,2,3\}$, $S = \{4\}$, and $R = \{5\}$.

The term 'invariant' reflects the fact that such a mutual control structure does not change if we add further control relations in the sense of the indirect control property: if S is controlled by T and R is controlled by S and T jointly, then R is already controlled directly by T and, thus, adding this control relation does not change the mcs.

Remark 2.5. Note that an invariant mutual control structure C is *transitive*: for all $R, S, T \in P(N)$ with $S \subseteq C(T)$ and $R \subseteq C(S)$ we have $R \subseteq C(T)$. This follows since $R \subseteq C(S)$ implies $R \subseteq C(S \cup T)$ by monotonicity of C. Transitivity is strictly weaker than indirect control. For instance, for $N = \{1, 2, 3\}$ the mcs C defined by $C(\{2\}) = C(\{2, 3\}) = \{1\}$, $C(\{1, 2\}) = \{1, 3\}$, $C(N) = N$, and $C(S) = \emptyset$ otherwise, is (trivially) transitive but does not satisfy indirect control: $\{1\} \subseteq C(\{2\})$, $\{3\} \subseteq C(\{1, 2\})$, but $\{3\} \not\subseteq C(\{2\})$. Thus, transitivity is too weak for what we have in mind. □

We now turn to the question how an arbitrary mutual control structure (mcs) can be turned into an invariant mutual control structure (imcs). To discuss this question some additional terminology will be useful. For $C, D \in \mathcal{C}$, we write $C \subseteq D$ if $C(S) \subseteq D(S)$ for all $S \in P(N)$. We call $D \in \mathcal{C}^*$ an *invariant extension* of $C \in \mathcal{C}$ if $C \subseteq D$. For instance, $D \in \mathcal{C}^*$ defined by $D(S) = N$ for all $S \in P(N)$ is an invariant extension of any mutual control structure. Call an invariant extension D of C *minimal* if $D \subseteq D'$ for every invariant extension D' of C. Obviously, minimal invariant extensions are unique.

Now turning an mcs into an invariant mcs requires, in particular, that indirect control relations are incorporated explicitly into the mcs. Let C be an arbitrary mcs. We define $C^1, C^2, \ldots$ recursively by

$$C^k(S) = \begin{cases} C(S) & \text{if } k = 1 \\ C\left(C^{k-1}(S) \cup S\right) & \text{if } k > 1 \end{cases}$$

for each $S \in P(N)$. Observe that in each step of this algorithm the new coalition controlled by S is the coalition controlled by S jointly with the coalition already controlled by S according to the previous step. Thus, this definition is very much in the spirit of the definition of indirect control. Clearly, by monotonicity of C, there must be a natural number $p \geq 1$ such that for each $S \in P(N)$ we have[2]

$$C(S) = C^1(S) \subseteq C^2(S) \subseteq \ldots \subseteq C^p(S) = C^{p+1}(S) = C^{p+2}(S) = \ldots \quad (1)$$

Denote C^p by C^*. We have the following result.

Proposition 2.6. *Let $C \in \mathcal{C}$. Then C^* is the (unique) minimal invariant extension of C.*

[2]It is not difficult to see that p does not have to exceed the number of players n.

The updating procedure to obtain C^* is identical to the so-called command chain in Hu and Shapley (2003a). Therefore, Proposition 2.6 can be obtained from their results. Alternatively, see Karos and Peters (2013).

Applying the procedure to the Porsche-Volkswagen case yields the following result.

Example 2.7. For C as in Example 2.3 we have for all $S \in P(N)$:

$$\begin{aligned} 4 \in C^*(S) &\Leftrightarrow 1 \in S \\ 5 \in C^*(S) &\Leftrightarrow \{2,3,4\} \subseteq S \text{ or } \{1,2,3\} \subseteq S \\ 6 \in C^*(S) &\Leftrightarrow 5 \in S \text{ or } \{2,3,4\} \subseteq S \text{ or } \{1,2,3\} \subseteq S\ . \end{aligned}$$

In particular, in $\mathcal{C}^*$, Volkswagen AG (5) is controlled by the coalition of Porsche Families, Qatar, and Lower Saxony (1, 2, and 3, respectively). □

The next example shows that different mutual control structures may result in the same invariant mcs.

Example 2.8. Let $N = \{1,2,3\}$ and let $C, D \in \mathcal{C}$ be defined by monotonicity and $C(\{1\}) = \{2\}$, $C(\{2\}) = \{3\}$, $C(\{3\}) = \{1\}$, $D(\{1\}) = \{3\}$, $D(\{2\}) = \{1\}$, $D(\{3\}) = \{2\}$. Then $C^*(S) = D^*(S) = N$ for each nonempty coalition S. □

We conclude this subsection with a remark on the union and intersection of invariant mutual control structures. This will be relevant in particular when we discuss axioms for power indices later on.

Remark 2.9. We define $\cup$ and $\cap$ on $\mathcal{C}$ by

$$\begin{aligned} (C \cup D)(S) &= C(S) \cup D(S), \\ (C \cap D)(S) &= C(S) \cap D(S), \end{aligned}$$

for all $S \in P(N)$. Clearly, $C \cup D$, $C \cap D \in \mathcal{C}$ for all $C, D \in \mathcal{C}$. Let $C, D \in \mathcal{C}^*$ and consider $C \cap D$. If $S \subseteq (C \cap D)(T)$ and $R \subseteq (C \cap D)(S \cup T)$, then $R \subseteq C(T)$ and $R \subseteq D(T)$ by indirect control. Hence, $R \subseteq (C \cap D)(T)$. Thus, $C \cap D \in \mathcal{C}^*$.

However, $C \cup D$ does not have to be invariant. Consider the following invariant mutual control structures:

$$C(S) = \begin{cases} \{2\} & \text{if } 1 \in S, \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$D(S) = \begin{cases} \{3\} & \text{if } 2 \in S, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $(C \cup D)(\{1\}) = \{2\}$, $(C \cup D)(\{1,2\}) = \{2,3\}$, but $\{2,3\} \not\subseteq (C \cup D)(\{1\})$. So $C \cup D$ is not invariant. □

2.2. Simple games

A *simple game* is a map $v : P(N) \to \{0,1\}$ with $v(\emptyset) = 0$ and with $v(S) \leq v(T)$ for all $S, T \in P(N)$ with $S \subseteq T$. If $v(S) = 1$ then coalition S is *winning*, otherwise it is *losing*. The set of all simple games is denoted by Σ. Thus, in our paper, a simple game – like a mutual control structure – is monotonic by definition. Also observe that $v(N) = 0$ – and, thus, $v(S) = 0$ for every coalition S – is allowed.

Let C be an mcs. Instead of writing for each coalition all players it controls, we may equivalently write for each player all coalitions by which he is controlled. Formally, for each $i \in N$ we define a simple game w_i^C by

$$w_i^C(S) = \begin{cases} 1 & \text{if } i \in C(S) \\ 0 & \text{if } i \notin C(S). \end{cases}$$

This way with every $C \in \mathcal{C}$ a vector of simple games $w^C \in \Sigma^N$ is associated. We call an element of Σ^N a *simple game structure.*

Conversely, for a simple game structure $w \in \Sigma^N$ we can define an mcs C^w by $i \in C^w(S) :\Leftrightarrow w_i(S) = 1$ for all $i \in N$ and $S \in P(N)$. Clearly, these definitions determine a bijection between $\mathcal{C}$ and Σ^N, and all statements and results for $\mathcal{C}$ can be equivalently formulated for Σ^N and conversely.

Remark 2.10. The simple games in a simple game structure are called command games in Hu and Shapley (2003a,b). □

For an arbitrary $w \in \Sigma^N$ it is not necessarily the case that C^w is invariant, i.e., $C^w \in \mathcal{C}^*$. In Karos and Peters (2013) an updating procedure for a simple game structure w is provided, leading to a simple game structure w^* such that C^{w^*} is invariant, and $C^{w^*} = (C^w)^*$. Details are omitted here.

3. Power indices for invariant mutual control structures

In this section we try to establish some ways in which the power of each player in a mutual control structure can be measured. We do this by considering power indices. As before, the player set is denoted by $N = \{1, \ldots, n\}$, and $\mathcal{C}^*$ is the set of all imcs with this player set. A *power index* is a map $\varphi : \mathcal{C}^* \to \mathbb{R}^N$. We note that a power index can be extended to non-invariant mcs simply by applying it to their minimal invariant extensions; attempts to define and characterize power indices directly on the class of all mutual control structures have not been fruitful so far.

In the first part of this section we axiomatically develop a large class of power indices. In later parts, we refine this class, consider applications, and show that the axioms are independent.

3.1. A class of power indices

For an imcs C the *marginal control* of player $i \in N$ with respect to a coalition $S \subseteq N$ is defined as $\Delta_i^C(S) = C(S) \setminus C(S \setminus \{i\})$. In words, the marginal control of a player with respect to a coalition is the set of players that would not be under control by that coalition without player i.

We say that $i \in N$ is a *null player* (in C) if $\Delta_i^C(S) = \emptyset$ for all $S \subseteq N$ and $i \notin C(N)$. That is, player i is a null player if i is never needed by any coalition to exercise its control, and i is also not controlled by any coalition. (Observe, indeed, that by monotonicity of C the latter condition is equivalent to player i not being controlled by the grand coalition.) The imcs in which every player is a null player, is denoted by O, i.e., $O(S) = \emptyset$ for all $S \subseteq N$.

Let $\pi : N \to N$ be a permutation. Then we define $\pi C \in \mathcal{C}^*$ by

$$(\pi C)(S) = \pi(C(\pi^{-1}(S))).$$

The first two axioms that we impose on a power index, are as follows.

Null Player (NP) $\varphi_i(C) = 0$ for every null player i in C, for every $C \in \mathcal{C}^*$.

Constant Sum (CS) $\sum_{i \in N} \varphi_i(C) = \sum_{i \in N} \varphi_i(D)$ for all $C, D \in \mathcal{C}^*$.

The null player axiom sets the power of a player who neither controls nor is controlled, equal to 0. The constant sum axiom normalizes the power index over different mutual control structures, and plays the role of the usual efficiency axiom. The following observation is almost immediate.

Lemma 3.1. *Let φ be a power index satisfying NP and CS. Then $\sum_{i \in N} \varphi_i(C) = 0$ for every $C \in \mathcal{C}^*$.*

Proof. By NP, $\varphi_i(O) = 0$ for every $i \in N$. Hence, by CS, $\sum_{i \in N} \varphi_i(C) = \sum_{i \in N} \varphi_i(O) = 0$ for every $C \in \mathcal{C}^*$. ■

The other two basic axioms are the following.

Anonymity (AN) $\varphi_{\pi(i)}(\pi C) = \varphi_i(C)$ for every player $i \in N$, every permutation π of N, and every $C \in \mathcal{C}^*$.

Transfer Property (TP) $\varphi(C) - \varphi(C') = \varphi(D) - \varphi(D')$ for all $C, C', D, D' \in \mathcal{C}^*$ such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$.

The anonymity axiom needs no further explanation. The transfer property requires that if extending C' to C involves exactly the same increase in control as extending D' to D, then the power of each player should change by the same amount when extending C' to C as when extending D' to D. The transfer property is related to a property with the same name, used to characterize the Shapley value (Shapley, 1953) for (monotonic) simple games (Dubey, 1975). The form in which we present it is closely related to a version of the axiom discussed in Dubey et al. (2005) and Einy and Haimanko (2011). The following lemma shows that TP is equivalent to a condition closely related to the original format of the transfer axiom as introduced in Dubey (1975).

Lemma 3.2. *Let φ be a power index. Then φ satisfies TP if and only if*

$$\varphi(C \cap D) + \varphi(C \cup D) = \varphi(C) + \varphi(D) \tag{2}$$

for all $C, D \in \mathcal{C}^*$ with $C \cup D \in \mathcal{C}^*$.

Remark 3.3. Note that $C \cap D \in \mathcal{C}^*$, but not necessarily $C \cup D \in \mathcal{C}^*$, see Remark 2.9. A proof of this lemma for simple games can be found in Dubey et al. (2005). See also Laruelle and Valenciano (2001). The proof in our case is more involved since the 'difference' mcs E, see below, is not necessarily invariant. □

Proof of Lemma 3.2. First, let φ satisfy TP and let $C, D \in \mathcal{C}^*$ with $C \cup D \in \mathcal{C}^*$. Clearly,

$$(C(S) \cup D(S)) \setminus C(S) = D(S) \setminus (C(S) \cap D(S))$$

for all $S \subseteq N$. Hence by TP, $\varphi(C \cup D) - \varphi(C) = \varphi(D) - \varphi(C \cap D)$, implying (2).

Next, let φ satisfy (2) for all $C, D \in \mathcal{C}^*$ with $C \cup D \in \mathcal{C}^*$. We show that φ satisfies TP. Let $C, C', D, D' \in \mathcal{C}^*$ such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$. Define

$$E(S) = \bigcup_{T \subseteq S} C(T) \setminus C'(T) = \bigcup_{T \subseteq S} D(T) \setminus D'(T).$$

Clearly, $E \in \mathcal{C}$. As before, denote the minimal invariant extension of E by E^*. We first prove:

$$C = C' \cup E^*, \ D = D' \cup E^* \ . \tag{3}$$

We only show the first equality, the proof of the second one is analogous. First, suppose $S \in P(N)$ and $j \in C(S)$. If $j \notin C'(S)$, then $j \in E(S) \subseteq E^*(S)$. Hence, $j \in C'(S) \cup E^*(S)$, and we conclude that $C \subseteq C' \cup E^*$. Second, suppose $S \in P(N)$ and $j \in C'(S) \cup E^*(S)$. If $j \in C'(S)$, then $j \in C(S)$. Suppose $j \in E^*(S)$. Since $E \subseteq C$, we have by invariance of C and Proposition 2.6(c) that $E^* \subseteq C$. So $j \in C(S)$ as well, and we conclude that $C \supseteq C' \cup E^*$. This completes the proof of (3).

Next, we prove:

$$C' \cap E^* = D' \cap E^* \ . \tag{4}$$

It is sufficient to show $C' \cap E^* \subseteq D' \cap E^*$, the other direction is analogous. Suppose $S \in P(N)$ and $j \in C'(S) \cap E^*(S)$. Then $j \notin C(S) \setminus C'(S)$, hence $j \notin D(S) \setminus D'(S)$. Since, by a similar argument as before, $E^*(S) \subseteq D(S)$, we obtain $j \in D(S)$. Thus, we must have $j \in D'(S)$, and therefore, $j \in D'(S) \cap E^*(S)$. This completes the proof of (4).

As $C' \cap E^*$ and $D' \cap E^*$ are invariant by Remark 2.9, we find with (2), (3), and (4) that

$$\begin{aligned}
\varphi(C) - \varphi(C') &= \varphi(C' \cup E^*) - \varphi(C') \\
&= \varphi(E^*) - \varphi(C' \cap E^*) \\
&= \varphi(E^*) - \varphi(D' \cap E^*) \\
&= \varphi(D' \cup E^*) - \varphi(D') \\
&= \varphi(D) - \varphi(D') \ ,
\end{aligned}$$

hence, φ satisfies TP. ■

Our main theorem will be a characterization of all power indices satisfying NP, CS, AN, and TP. A crucial result for this is the proposition below, which expresses the power index values assigned to an imcs C as sums of values assigned to so-called unanimity mcs. For every nonempty $S \subseteq N$ and $j \in N$ we define the *unanimity* mutual control structure $U^{S,\{j\}}$ by

$$U^{S,\{j\}}(T) = \begin{cases} \{j\} & \text{if } S \subseteq T \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to check that $U^{S,\{j\}}$ is an imcs. Similarly, it is easy to check that $\bigcap_{S \in M} U^{S,\{j\}} \in \mathcal{C}^*$ for any subset $M \subseteq P(N)$ (this follows also from Remark 2.9).

Let $C \in \mathcal{C}^*$. For a coalition $S \subseteq N$ we define the *excess* of S by

$$E^C(S) = C(S) \setminus \bigcup_{S' \subsetneq S} C(S').$$

Coalition S is *minimal controlling* in C if $E^C(S) \neq \emptyset$. Note that $i \in E^C(S)$ if and only if S is a minimal winning coalition in w_i^C (i.e., S is winning and all proper subsets of S are losing). Let $M(C)$ denote the set of minimal controlling coalitions in C.

Example 3.4. Consider the invariant mutual control structure from Example 2.7 (Porsche-Volkswagen). The minimal controlling coalitions are $\{1\}$, $\{5\}$, $\{1,2,3\}$, and $\{2,3,4\}$. Note that $C^*(\{1,5\}) = \{4,6\}$, so that both $C^*(\{1\})$ and $C^*(\{5\})$ are proper subsets of $C^*(\{1,5\})$. Still, $\{1,5\}$ is not minimal controlling since $E^{C^*}(\{1,5\}) = \emptyset$. □

The announced proposition is as follows.

Proposition 3.5. *Let φ be a power index satisfying satisfy TP and NP, and let $C \in \mathcal{C}^*$. For every $j \in N$ let $M^j = \{S \in M(C) \mid j \in E^C(S)\}$. Then*

$$\varphi(C) = \sum_{j \in N} \sum_{t=1}^{|M^j|} (-1)^{t+1} \sum_{M \subseteq M^j, |M|=t} \varphi\left(\bigcap_{S \in M} U^{S,\{j\}}\right). \tag{5}$$

Proof. We claim that

$$C = \bigcup_{S \in M(C)} \bigcup_{j \in E^C(S)} U^{S,\{j\}} = \bigcup_{j \in N} \bigcup_{S \in M^j} U^{S,\{j\}}. \tag{6}$$

The second equality is obvious. For the first equality, suppose that $T \subseteq N$ and $j \in C(T)$. Then there is a $T' \subseteq T$ with $T' \in M(C)$ and $j \in E^C(T')$, so that $U^{T',\{j\}}(T) = \{j\}$. Conversely, suppose that $T \subseteq N$, $S \in M(C)$ and $j \in E^C(S)$ with $U^{S,\{j\}}(T) = \{j\}$. Then $S \subseteq T$ and $j \in C(S)$, so that $j \in C(T)$. This completes the proof of (6).

For each $j \in N$ we define the control structures C_j and C_{-j} by

$$\begin{aligned} C_j &= \bigcup_{S \in M^j} U^{S,\{j\}} \\ C_{-j} &= \bigcup_{k \in N \setminus \{j\}} C_k \,. \end{aligned}$$

Then $C = C_j \cup C_{-j}$ by (6). Clearly, both C_j and C_{-j} are mcs. Note that $C_j(T) \subseteq \{j\}$ for every $T \subseteq N$. Hence, C_j is (trivially) invariant. Also C_{-j} is invariant. To see this, let $T \subseteq N$, and suppose $\ell \in N$ with $\ell \notin C_{-j}(T)$. It is sufficient to show that $\ell \notin C_{-j}\left(T \cup C_{-j}(T)\right)$. If $\ell = j$, then clearly $\ell \notin C_{-j}\left(T \cup C_{-j}(T)\right)$. Now suppose $\ell \neq j$. Since $\ell \notin C_j(T)$ and $\ell \notin C_{-j}(T)$, we have $\ell \notin C_j(T) \cup C_{-j}(T) = C(T)$. Since C is invariant, this implies $\ell \notin C(T \cup C(T))$. Now

$$\begin{aligned} C_{-j}\left(T \cup C_{-j}(T)\right) &\subseteq C_{-j}\left(T \cup C_{-j}(T) \cup C_j(T)\right) \\ &\subseteq C_{-j}(T \cup C(T)) \cup C_j(T \cup C(T)), \\ &= C(T \cup C(T)) \end{aligned}$$

so that $\ell \notin C_{-j}\left(T \cup C_{-j}(T)\right)$.

By TP and Lemma 3.2 we have

$$\varphi(C) = \varphi\left(C_{-1}\right) + \varphi\left(C_1\right) - \varphi\left(C_{-1} \cap C_1\right) = \varphi\left(C_{-1}\right) + \varphi\left(C_1\right),$$

where the last equality follows by NP, noting that $C_{-1} \cap C_1 = O$. Next, note that we can write

$$C_{-1} = C_2 + \left(C_{-1}\right)_{-2} = C_2 + \bigcup_{j \in N \setminus \{1,2\}} \bigcup_{S \in M^j} U^{S,\{j\}}$$

where, again, $\left(C_{-1}\right)_{-2}$ is an imcs. Applying Lemma 3.2 once more, we obtain

$$\varphi(C) = \varphi\left(C_1\right) + \varphi\left(C_2\right) + \varphi\left(\left(C_{-1}\right)_{-2}\right).$$

Repeating this argument, we obtain

$$\varphi(C) = \sum_{j \in N} \varphi\left(C_j\right).$$

Let $j \in N$. It remains to show that

$$(\varphi(C_j) =)\ \varphi\left(\bigcup_{S\in M^j} U^{S,\{j\}}\right) = \sum_{t=1}^{\left|M^j\right|}(-1)^{t+1} \sum_{M\subseteq M^j, |M|=t} \varphi\left(\bigcap_{S\in M} U^{S,\{j\}}\right).$$

For $|M^j| = 1$ there is nothing to show. For $|M^j| = 2$, say $M^j = \{S_1, S_2\}$, we have by TP and Lemma 3.2 that

$$\varphi\left(U^{S_1,\{j\}} \cup U^{S_2,\{j\}}\right) = \varphi\left(U^{S_1,\{j\}}\right) + \varphi\left(U^{S_2,\{j\}}\right) - \varphi\left(U^{S_1,\{j\}} \cap U^{S_2,\{j\}}\right)$$

which results in the desired expression. Now suppose $|M^j| \geq 3$ and let $S^* \in M^j$. By induction we have

$$\begin{aligned}
\varphi\left(\bigcup_{S\in M^j\setminus\{S^*\}} U^{S,\{j\}} \cap U^{S^*,\{j\}}\right) &= \varphi\left(\bigcup_{S\in M^j\setminus\{S^*\}} U^{S\cup S^*,\{j\}}\right) \\
&= \sum_{t=1}^{\left|M^j\right|-1} (-1)^{t+1} \sum_{M\subseteq M^j\setminus\{S^*\}, |M|=t} \varphi\left(\bigcap_{S\in M} U^{S\cup S^*,\{j\}}\right) \\
&= \sum_{t=1}^{\left|M^j\right|-1} (-1)^{t+1} \sum_{M\subseteq M^j\setminus\{S^*\}, |M|=t} \varphi\left(\bigcap_{S\in M\cup\{S^*\}} U^{S,\{j\}}\right). \qquad (7)
\end{aligned}$$

Now

$$\begin{aligned}
\varphi\left(C_j\right) &= \varphi\left(\bigcup_{S\in M^j\setminus\{S^*\}} U^{S,\{j\}}\right)+\varphi\left(U^{S^*,\{j\}}\right) \\
&\quad -\varphi\left(\bigcup_{S\in M^j\setminus\{S^*\}} U^{S,\{j\}}\cap U^{S^*,\{j\}}\right) \\
&= \sum_{t=1}^{\left|M^j\right|-1}(-1)^{t+1}\sum_{M\subseteq M^j\setminus\{S^*\},|M|=t}\varphi\left(\bigcap_{S\in M}U^{S,\{j\}}\right)+\varphi\left(U^{S^*,\{j\}}\right) \\
&\quad -\sum_{t=1}^{\left|M^j\right|-1}(-1)^{t+1}\sum_{M\subseteq M^j\setminus\{S^*\},|M|=t}\varphi\left(\bigcap_{S\in M\cup\{S^*\}}U^{S,\{j\}}\right) \\
&= \sum_{t=1}^{\left|M^j\right|}(-1)^{t+1}\sum_{M\subseteq M^j,|M|=t}\varphi\left(\bigcap_{S\in M}U^{S,\{j\}}\right)
\end{aligned}$$

where the first equality follows by TP and Lemma 3.2, the second by (7) and induction, and the third by rewriting. This completes the proof of the proposition. ■

We now introduce our class of power indices, based on the concept of dividends. Recall that the *dividends* $d(S)$ of a simple game v with player set N are defined, recursively, by

$$d(S)=\begin{cases}0 & \text{if } S=\emptyset\\ v(S)-\sum_{T\subsetneq S}d(T) & \text{otherwise}\end{cases}$$

for all $S\subseteq N$. For an mcs C and $i\in N$, we write d_i^C for the dividends of w_i^C.

Remark 3.6. Let v be a simple game with player set N and dividends $d(S)$, $S\subseteq N$. We collect some facts about these dividends.

(a) For every coalition S, $v(S)=\sum_{T\subseteq S}d(T)$. This follows directly from the definition of dividends.

(b) Recall that the *Shapley value* (Shapley, 1953) of v is given by

$$Sh_k(v)=\sum_{S:k\in S}\frac{d(S)}{|S|}$$

for every $k \in N$. This restriction of the Shapley value to simple games is also called the *Shapley-Shubik index* (Shapley and Shubik, 1954).

(c) Player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. If, additionally, $v(\{i\}) = 0$, then i is a *null player*. It is well-known that $d(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ for every $S \subseteq N$. For a dummy i and $S \subseteq N$ with $i \in S$ and $|S| \geq 2$, we have $d(S) = 0$. This follows from:

$$\begin{aligned} d(S) &= \sum_{T \subseteq S: i \notin T} (-1)^{|S|-|T|} v(T) + \sum_{T \subseteq S: i \in T} (-1)^{|S|-|T|} v(T) \\ &= \sum_{T \subseteq S: i \notin T} (-1)^{|S|-|T|} v(T) + \sum_{T \subseteq S: i \notin T} (-1)^{|S|-|T|-1} v(T \cup \{i\}) \\ &= \sum_{T \subseteq S: i \notin T} \left((-1)^{|S|-|T|} \right) + (-1)^{|S|-|T|-1} \big) v(T) \\ &= 0. \end{aligned}$$

□

For every weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$, we now define the power index Φ^ω by

$$\begin{aligned} \Phi_i^\omega(C) &= \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \in S, k \notin S} \frac{d_k^C(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_k^C(S)}{|S|} \beta_{|S|} \right) \\ &\quad - \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \notin S, k \in S} \frac{d_i^C(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_i^C(S)}{|S|} \beta_{|S|} \right) \end{aligned} \tag{8}$$

for all $C \in \mathcal{C}^*$ and $i \in N$. The expression in brackets in the first line of (8) says that player i receives a weighted sum of dividends in the game w_k^C; this expresses the power player i derives from his role in controlling player k. The weights depend, both on the size of the coalition of whose dividend player i receives a share, and on whether or not the controlled player k is a member of that coalition. Thus, the first line in (8) represents the total power player i acquires from his role in controlling the other players. In the second line, the total (similarly weighted) power that all other players acquire from controlling player i, is subtracted.

Remark 3.7. In (8) the weight of a dividend $d_i^C(S)$ depends on whether or not i is contained in S. That distinction is related but not equal to the distinction between a boss set and an approval set in Hu and Shapley (2003b). Let S be winning in w_i. If $i \notin S$, S is for sure a boss set. However, if $i \in S$, S is an approval set for i if $S \setminus \{i\}$ is not winning, and a boss set otherwise. Hence, the dividend $d_i^C(S)$ of an approval set S is always weighted with $\beta_{|S|}$ whereas the dividend of a boss set is weighted with $\alpha_{|S|}$ or $\beta_{|S|}$, depending on whether or not $i \in S$. □

The central result of this section is the following theorem, in which the class of power indices of the form Φ^ω is characterized.

Theorem 3.8. *Let φ be a power index. Then φ satisfies NP, CS, AN, and TP if and only if there is a weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ such that $\varphi = \Phi^\omega$.*

Proof. First, let $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$. We show that Φ^ω satisfies the four axioms. For AN, this is obvious. When i is a null player in $C \in \mathcal{C}^*$, then $w_i^C(S) = 0$ for every S, so $d_i^C(S) = 0$ for every S. Moreover, for $k \neq i$ we have (see Remark 3.6(c)) that $d_k^C(S) = 0$ for every S with $i \in S$. Hence, $\Phi_i^\omega(C) = 0$, which shows NP. CS, more specifically, $\sum_{i \in N} \Phi_i^\omega(C) = 0$ for every $C \in \mathcal{C}^*$, follows almost directly from the definition of Φ^ω. Finally, for TP, let $C' \subseteq C$ and $D' \subseteq D$ satisfy $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every S. We claim that for every player j and every coalition S, $d_j^C(S) - d_j^{C'}(S) = d_j^D(S) - d_j^{D'}(S)$. This follows from:

$$\begin{aligned} d_j^C(S) - d_j^{C'}(S) &= \sum_{T \subseteq S} (-1)^{|S|-|T|} \left(w_j^C(T) - w_j^{C'}(T) \right) \\ &= \sum_{T \subseteq S : j \in C(T) \setminus C'(T)} (-1)^{|S|-|T|} \\ &= \sum_{T \subseteq S : j \in D(T) \setminus D'(T)} (-1)^{|S|-|T|} \\ &= \sum_{T \subseteq S} (-1)^{|S|-|T|} \left(w_j^D(T) - w_j^{D'}(T) \right) \\ &= d_j^D(S) - d_j^{D'}(S) \,. \end{aligned}$$

By using the definition of Φ^ω and this claim, we obtain that $\Phi^\omega(C) - \Phi^\omega(C') = \Phi^\omega(D) - \Phi^\omega(D')$, showing TP.

For the converse, let φ be a power index satisfying the four axioms. By NP, CS, AN, and Lemma 3.1, there are numbers $\alpha_{|S|}$ for $|S| = 1, \ldots, n-1$ and $\beta_{|S|}$ for $|S| = 2, \ldots, n$ such that for every $\emptyset \neq S \subseteq N$ and $j \in N$ we have

$$\varphi_i\left(U^{S,\{j\}}\right) = \begin{cases} 0 & \text{if } i \notin S \text{ and } i \neq j \\ -\alpha_{|S|} & \text{if } i = j \text{ and } j \notin S \\ \frac{1}{|S|}\alpha_{|S|} & \text{if } i \in S \text{ and } j \notin S \\ -\frac{|S|-1}{|S|}\beta_{|S|} & \text{if } i = j \text{ and } j \in S \\ \frac{1}{|S|}\beta_{|S|} & \text{if } i \in S \setminus \{j\} \text{ and } j \in S. \end{cases} \tag{9}$$

By Proposition 3.5 and the first part of the proof it is sufficient to prove that with $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$, (9) holds for Φ^ω as well. Let w be the simple game structure associated with the unanimity control structure $U^{S,\{j\}}$. Then $d_i(T) = 0$ for every $i \neq j$ and every $T \subseteq N$, whereas $d_j(T) = 1$ for $T = S$ and $d_j(T) = 0$ otherwise. It is now straightforward to verify (9) for Φ^ω. ■

We regard the transfer property as a basic axiomatic choice, in the sense that one should drop this condition in order to obtain essentially different power indices. Of course, also the null player and constant sum conditions have a considerable impact but they can be seen as scaling conditions, which are needed anyway in some form or another. The anonymity condition is natural and standard.

Indeed, the class of power indices characterized in Theorem 3.8 is large and leaves much room for further specification, as we will consider in the next subsections.

3.2. Controlled player

The axioms in Theorem 3.8 put no restrictions whatsoever on the weights in ω. A complete determination of the weights is obtained by adding the following condition.

Controlled player (CP) For all $C \in \mathcal{C}^*$, $j \in C(N)$, and $i \in N \setminus C(N)$,

$$\varphi_j(C) = \begin{cases} -1 & \text{if } \Delta_j^C(S) = \emptyset \text{ for all } S \subseteq N \\ \varphi_i(C) - 1 & \text{if } \Delta_i^C(S) = \Delta_j^C(S) \text{ for all } S \subseteq N. \end{cases}$$

The controlled player condition says that if j is a 'controlled player', i.e., controlled by at least one coalition and, thus, by N, but does not exercise any control himself, then the power of j is fixed at -1. Hence, the power of a least powerful player is fixed at -1. Further, if i is an uncontrolled player, i.e., controlled by no coalition at all, but i and j exercise the same marginal control with respect to any coalition, then their difference in power is fixed at 1, that is, i gets assigned 1 more than j. We now have the following corollary.

Corollary 3.9. *There is a unique power index satisfying NP, CS, AN, TP, and CP, namely the power index* Φ^ω *with* $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$.

Proof. We first show that Φ^ω with $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$ satisfies CP. Let $C \in \mathcal{C}^*$ with $j \in C(N)$ and $i \in N \setminus C(N)$. First suppose that $\Delta_j^C(S) = \emptyset$ for all $S \subseteq N$. Then player j is a dummy player in every simple game w_k^C, $k \in N$, so that by Remark 3.6(c), $d_k^C(S) = 0$ for all $k \in N$ and $S \subseteq N$ with $j \in S$. Therefore,

$$\Phi_j^\omega(C) = -\sum_{k \in N \setminus \{j\}} \sum_{S \subseteq N, k \in S} \frac{d_j^C(S)}{|S|} = -\sum_{S \subseteq N} d_j^C(S) = -1,$$

where the last equality follows since the sum of the dividends of all subcoalitions of a coalition is equal to the worth of that coalition. Next, suppose that $\Delta_i^C(S) = \Delta_j^C(S)$ for all $S \subseteq N$. Then $Sh_i\left(w_k^C\right) = Sh_j\left(w_k^C\right)$ for all $k \in N$, so that

$$\begin{aligned}
\Phi_j^\omega(C) &= \sum_{k \in N \setminus \{j\}} Sh_j\left(w_k^C\right) - \sum_{k \in N \setminus \{j\}} Sh_k\left(w_j^C\right) \\
&= \sum_{k \in N \setminus \{j\}} Sh_j\left(w_k^C\right) - \left(1 - Sh_j\left(w_j^C\right)\right) \\
&= \sum_{k \in N} Sh_j\left(w_k^C\right) - 1 \\
&= \sum_{k \in N} Sh_i\left(w_k^C\right) - 1 \\
&= \Phi_i^\omega(C) - 1
\end{aligned}$$

where the last equality follows since $i \notin C(N)$, i.e., w_i^C is the zero game.

For the converse, suppose that Φ^ω satisfies CP. Let S be a coalition and let $i, j \in S$. By using (9), we obtain

$$\Phi_i^\omega\left(U^{S,\{j\}}\right) = \frac{1}{|S|}\beta_{|S|},\ \Phi_j^\omega\left(U^{S,\{j\}}\right) = -\frac{|S|-1}{|S|}\beta_{|S|}\,.$$

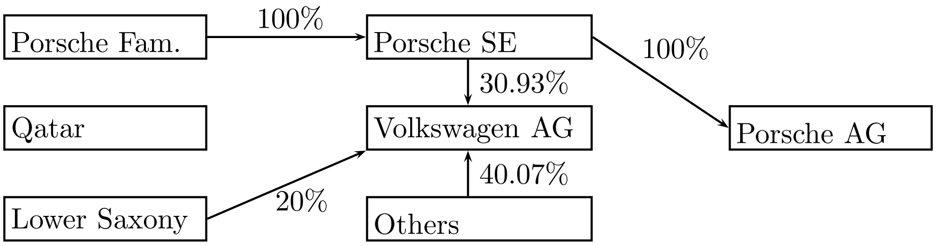


Figure 3: Porsche and VW voting rights by the end of 2007, based on the 2007 annual report of Volkswagen AG and the 2007/2008 annual report of Porsche Automobil Holding SE GmbH.

By CP, $\Phi_j^\omega = \Phi_i^\omega - 1$, which in turn implies $\beta_{|S|} = 1$.

For $|S| \leq n-2$ with $j \notin S$, take $i \in N \setminus (S \cup \{j\})$ and consider $U^{S,\{j\}}$. Clearly, i is null player in $U^{S,\{j\}}$. Hence, by using the second part of CP and NP, we obtain $\alpha_{|S|} = 1$. Finally, $\alpha_{n-1} = 1$ by the first part of CP. ■

Our proof of this corollary reveals that the first part of CP is only needed for the case where there is no null player.

The power index Φ^ω with $\omega = (1, \ldots, 1)$ takes the form

$$\begin{aligned}\Phi_i^\omega(C) &= \sum_{k\in N\setminus\{i\}} \left(\sum_{S:i\in S} \frac{d_k^C(S)}{|S|} - \sum_{S:k\in S} \frac{d_i^C(S)}{|S|} \right) \\ &= \sum_{k\in N\setminus\{i\}} \left(Sh_i\left(w_k^C\right) - Sh_k\left(w_i^C\right) \right) \\ &= \sum_{k\in N} Sh_i\left(w_k^C\right) - w_i^C(N)\end{aligned}$$

for every $i \in N$.

We apply this unique power index to the Porsche-Volkswagen case.

Example 3.10. Consider the invariant mutual control structure C^* from Example 2.7. Then we find for $\omega = (1, \ldots, 1) \in \mathbb{R}^{2n-2}$ that $\Phi_1^\omega(C^*) = \frac{67}{60}$, $\Phi_2^\omega(C^*) = \Phi_3^\omega(C^*) = \frac{32}{60}$, $\Phi_4^\omega(C^*) = -\frac{53}{60}$, $\Phi_5^\omega(C^*) = -\frac{18}{60}$, $\Phi_6^\omega(C^*) = -1$, and $\Phi_7^\omega(C^*) = 0$. It is interesting to compare the power of Porsche Families with its power at the end of 2007. Figure 3 depicts the control structure between the same companies at the end of 2007. At that time, Volkswagen was not controlled by any group of main investors. Although Porsche SE has veto power in the game on Volkswagen AG, we ignore this fact, as it is not clear how this power can be exercised. We describe this situation by a mutual control structure D. Its minimal invariant extension is:

$$D^*(S) \quad = \quad \{4, 6\} \quad \text{for all } S \text{ with } 1 \in S.$$

Thus, even while ignoring the power of Porsche Families on Volkswagen, we still have $\Phi_1^\omega(D^*) = 2 > \frac{67}{60} = \Phi_1^\omega(C^*)$. Hence, according to this power index it had more power in 2007 than it has in the situation of Example 2.7 (June 2013). □

We end this part with a remark on simple games as special mcs.

Remark 3.11. Consider a simple game $v \in \Sigma$ with $v(N) = 1$. A natural way to associate an mcs with v is to define $C : P(N) \to P(N)$ by

$$C(S) = \begin{cases} N & \text{if } v(S) = 1 \\ \emptyset & \text{if } v(S) = 0 \end{cases}$$

for all $S \in P(N)$. It is easy to check that C is an invariant mcs, and that $w_i^C = v$ for each $i \in N$. It can be verified (see the Appendix for the computations) that $\Phi^\omega(C)$ for $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n)$ is independent of $(\beta_2, \ldots, \beta_n)$. If, in particular, we take $\alpha_1 = \ldots = \alpha_{n-1} = 1$, then we obtain $\Phi_i^\omega(C) = n\, Sh_i(v) - 1$ for each $i \in N$. Thus, in this case the power index Φ^ω is a rescaling of the Shapley-Shubik power index for simple games. □

3.3. Dummy players

In this section we consider a strengthening of the null player axiom. Call player $i \in N$ a *dummy player* in an imcs C if both (i) $\Delta_i^C(S) \subseteq \{i\}$ and (ii) $i \in C(S) \Rightarrow i \in S$, for all for all $S \in P(N)$. In other words, player i's marginal control consists of at most himself, and he is not controlled by any coalition to which he does not belong. Clearly, a null player in C is also a dummy player. We consider the following condition.

Dummy Player (DP) $\varphi_i(C) = 0$ for every dummy player i in C, for every $C \in \mathcal{C}^*$.

Since every null player is a dummy player, DP implies NP. Replacing NP by DP has the consequence that the β-weights in a power index Φ^ω vanish:

Corollary 3.12. *Let φ be a power index. Then φ satisfies DP, CS, AN, and TP if and only if there is a weight vector $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ with $\beta_2 = \ldots = \beta_n = 0$ such that $\varphi = \Phi^\omega$.*

Proof. Suppose φ satisfies the four axioms. Then, by Theorem 3.8, there is an $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{2n-2}$ such that $\varphi = \Phi^\omega$. Consider a coalition S with $|S| \geq 2$ and $i \in S$. Then i is a dummy player in $U^{S,\{i\}}$, hence, using (9), $\beta_{|S|} = 0$.

We still have to show that Φ^ω with $\omega = (\alpha_1, \ldots, \alpha_{n-1}, 0, \ldots, 0)$ satisfies DP. Let i be a dummy player in an imcs C, and let $k \in N$. Then, by Remark 3.6(c), $d_k^C(S) = 0$ for every S with $i \in S$ and $k \notin S$, and also $d_i^C(S) = 0$ for every S with $i \notin S$ and $k \in S$. This implies that $\Phi_i^\omega(C) = 0$. ■

Consider a unanimity mcs $U^{S,\{j\}}$ with $j \in S$. Then DP of a power index φ would imply $\varphi_j(U^{S,\{j\}}) = 0$. If φ additionally satisfies CS, AN, and TP, then by Corollaries 3.9 and 3.12 it cannot satisfy the controlled player property CP. We consider the following weakening of CP.

Weak controlled player (WCP) $\varphi_j(C) = -1$ for all $C \in \mathcal{C}^*$ and $j \in C(N)$ with $\Delta_j^C(S) = \emptyset$ for all $S \subseteq N$.

By using (9) and Corollary 3.12 we immediately obtain:

Corollary 3.13. *Let φ be a power index. Then φ satisfies DP, CS, AN, TP, and WCP, if and only if $\varphi = \Phi^\omega$ for $\omega = (\alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_n)$ with $\alpha_1 = \ldots = \alpha_{n-1} = 1$ and $\beta_2 = \ldots = \beta_n = 0$.*

The power index of Corollary 3.13 takes the simple form

$$\begin{aligned}
\Phi_i^\omega(C) &= \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \in S, k \notin S} \frac{d_k^C(S)}{|S|} - \sum_{S: i \notin S, k \in S} \frac{d_i^C(S)}{|S|} \right) \\
&= \sum_{k \in N \setminus \{i\}} \left(Sh_i\left(w^C_{k|N\setminus\{k\}}\right) - Sh_k\left(w^C_{i|N\setminus\{i\}}\right)\right) \\
&= \sum_{k \in N} Sh_i(w^C_{k|N\setminus\{k\}}) - w_i^C\left(N \setminus \{i\}\right),
\end{aligned}$$

where for every $k \in N$, $w^C_{k|N\setminus\{k\}}$ is the subgame of w_k^C defined by $w^C_{k|N\setminus\{k\}}(S) = w_k^C(S \setminus \{k\})$ for all $S \subseteq N$.

Remark 3.14. When we apply this power index to the Porsche-Volkswagen case (Example 2.7) we find the same result as in Example 3.10, since the terms involving the β-weights are zero anyway in this example. Also for the mcs associated with the simple game v as in Remark 3.11 we still find the same values, namely $n\,Sh_i(v) - 1$ for each player i. $\square$

3.4. Independence of the axioms in Theorem 3.8

We conclude with showing independence of the axioms in Theorem 3.8.

Null Player. Fix $\varepsilon > 0$ and for every nonempty $S \subseteq N$ and $j \in N$ define

$$\varphi_i\left(U^{S,\{j\}}\right) = \begin{cases} \varepsilon & \text{if } i \notin S \text{ and } i \neq j \\ -1 & \text{if } i = j \text{ and } j \notin S \\ \frac{1}{|S|} - \frac{|N|-|S|-1}{|S|}\varepsilon & \text{if } i \in S \text{ and } j \notin S \\ -\frac{|S|-1}{|S|} & \text{if } i = j \text{ and } j \in S \\ \frac{1}{|S|} - \frac{|N|-|S|}{|S|-1}\varepsilon & \text{if } i \in S \setminus \{j\} \text{ and } j \in S. \end{cases}$$

Extend φ to $\mathcal{C}^*$ by using Proposition 3.5. Then φ satisfies CS, AN, and TP, but not NP.

Constant Sum. Define the power index φ by $\varphi_i(C) = |C(\{i\})| - w_i^C(N)$ for all $C \in \mathcal{C}^*$ and $i \in N$. Then φ satisfies NP and AN. Also, for $C, D \in \mathcal{C}^*$,

$$\begin{aligned} \varphi_i(C \cup D) + \varphi_i(C \cap D) &= |(C(\{i\}) \cup D(\{i\}))| - w_i^{C \cup D}(N) \\ &\quad + |(C(\{i\}) \cap D(\{i\}))| - w_i^{C \cap D}(N) \\ &= |C(\{i\})| + |D(\{i\})| - \left(w_i^C(N) + w_i^D(N)\right) \\ &= \varphi_i(C) + \varphi_i(D) \end{aligned}$$

for all $i \in N$, so that φ satisfies TP by Lemma 3.2. Consider $D \in \mathcal{C}^*$ defined by $D(S) = N$ for all nonempty $S \subseteq N$. Then $\varphi_i(D) = |N| - 1$ for all $i \in N$, so that φ does not satisfy CS.

Anonymity. Let $N = \{1, 2\}$ and define φ by $\varphi(U^{\{1\},\{1\}}) = \varphi(U^{\{2\},\{2\}}) = \varphi(U^{\{2\},\{1\}}) = (0, 0)$, $\varphi(U^{\{1,2\},\{2\}}) = -\varphi(U^{\{1,2\},\{1\}}) = \varphi(U^{\{1\},\{2\}}) = (1, -1)$; and by extending φ to $\mathcal{C}^*$ using Proposition 3.5. Then φ satisfies NP, CS, and TP, but not AN.

Transfer Property. For a simple game v and $i \in N$ let $\sigma_i(v) = |\{S \subseteq N \setminus \{i\} \mid v(S \cup \{i\}) - v(S) = 1\}|$ and let

$$Bz_i(v) = \begin{cases} \sigma_i(v)/\sum_{j \in N} \sigma_j(v) & \text{if } \sigma_i(v) \neq 0 \\ 0 & \text{if } \sigma_i(v) = 0. \end{cases}$$

Thus, Bz is the normalized Banzhaf value (Banzhaf, 1965; Dubey et al., 2005). Define the power index φ by

$$\varphi_i(C) = \sum_{j \neq i} Bz_i\left(w_j^C\right) - \sum_{j \neq i} Bz_j\left(w_i^C\right)$$

for all $C \in \mathcal{C}^*$ and $i \in N$. It is straightforward to verify that φ satisfies NP, CS, and AN. We show that it does not satisfy TP by using Lemma 3.2. For $S, S' \subseteq N$ define $U^{S,S'} \in \mathcal{C}^*$ by $U^{S,S'}(T) = S'$ if $S \subseteq T$ and $U^{S,S'}(T) = \emptyset$ otherwise. Now take $N = \{1, 2, 3\}$ and let $C = U^{\{1,2\},N}$ and $D = U^{\{1,3\},N}$. Then $Bz_1\left(w_i^C\right) = \frac{1}{2}$ and $Bz_1\left(w_i^D\right) = \frac{1}{2}$ for $i = 1, 2, 3$. Hence, $\varphi_1(C) = \varphi_1(D) = \frac{1}{2}$. Further,

$$\begin{aligned} C \cup D &= U^{\{1,2\},N} \cup U^{\{1,3\},N}, \\ C \cap D &= U^{N,N}. \end{aligned}$$

Now $Bz_1\left(w_i^{C \cup D}\right) = \frac{3}{5}$ for $i = 1, 2, 3$. Hence, $\varphi_1(C \cup D) = \frac{4}{5}$. Further, by computation or by CS and AN, we have $\varphi(C \cap D) = 0$. Thus,

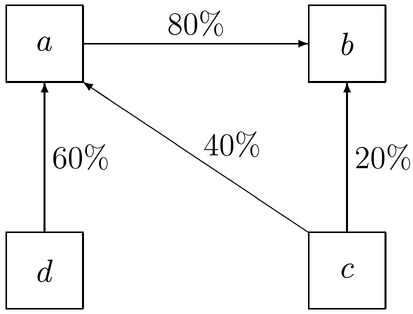
$$\varphi_1(C \cup D) + \varphi_1(C \cap D) = \frac{4}{5} \neq 1 = \varphi_1(C) + \varphi_1(D).$$

So, φ does not satisfy TP.

4. Concluding remarks

Control versus ownership. Crama and Leruth (2013) point out that the concepts of control – as studied in our paper – and ownership must be carefully distinguished. The following example illustrates this.

Example 4.1. Let a, b, c, d be firms such that a owns 80% of the shares of b and c owns the remaining 20% of b. Let further a's shares be distributed such that c owns 40% and d owns 60%. See the following diagram.



Assume that all decisions can be made by an absolute majority. From a financial point of view c owns 32% of b *indirectly* via a and 20% directly, so all together 52%, which is the majority. On the other hand d owns 48% of b indirectly via a. However, if we build a mutual control structure C to represent this situation, we find $C(\{a\}) = \{b\}$, $C(\{d\}) = \{a\}$ and $C(\{c\}) = \emptyset$. For the minimal invariant extension of C we have $C^*(\{d\}) = \{a, b\}$ and $C^*(\{c\}) = \emptyset$. We see that indirect ownership of firms may not lead to any control at all. In particular, while indirect ownership is proportional to the number of shares, indirect control follows a *winner takes all* principle: although firm c owns 52% of firm b its indirect votes are useless as it can not force firm a to use it. Of course, this is analogous to the observation in political situations that, rather than the absolute number of votes or seats, the possibility to be crucial in forming majorities is what matters. □

Importance of modelling. As we pointed out in the introduction, our theory can be used to evaluate control structures, provided we can model them as a mutual control structure (i.e., an element of $\mathcal{C}$). This modelling part is important and should be done with care, as the following example illustrates.

Example 4.2. Consider five players, $N = \{1, \ldots, 5\}$, and a mutual control structure C determined by $C(\{1\}) = \{2\}$ and $C(\{2\}) = \{3, 4, 5\}$. Then C^* is determined by $C^*(\{1\}) = \{2, 3, 4, 5\}$ and $C^*(\{2\}) = \{3, 4, 5\}$. One might argue here that player 2 should actually be left out. Indeed, if this player is merely an administrative entity (a mailbox, or an office), then it would be natural to leave it out and consider the mutual control structure C' with $C'(\{1\}) = \{3, 4, 5\}$ instead of C, and thus with player 2 as a null player. If, however, player 2 is a real player, then it should be left in. In that case, one might follow Gambarelli and Owen (1994) who distinguish between controlled firms and uncontrolled investors. One would argue that a mutual control structure C'' with $C''(\{1\}) = \{2, 3, 4, 5\}$ and $C''(\{2\}) = \emptyset$ would reflect the situation more adequately than C^* as player 1 is the only investor. But this is, indeed, subject to discussion: player 2 would be put at the same level of control as players 3, 4, and 5, which is questionable. Applying the power index φ with $\varphi = \Phi^\omega$, $\omega = (1, \ldots, 1) \in \mathbb{R}^8$, one obtains

$$\begin{aligned} \varphi(C^*) &= (5/2, 1/2, -1, -1, -1) \\ \varphi(C') &= (3, 0, -1, -1, -1) \\ \varphi(C'') &= (4, -1, -1, -1, -1)\ , \end{aligned}$$

reflecting each of the three choices above. In favor of C^* one could argue that, although player 2 is completely controlled by player 1, the fact that he has direct control over 3, 4, and 5, is reflected by its positive power index which, moreover, would increase if the number of players it directly controls, increases. In general, it seems doubtful whether it is possible to find an updating procedure, leading to 'invariant' mutual control structures, that would adequately cover each and every possible situation. In applications, the modelling phase, resulting in a formal mutual control structure C, should therefore be done with care. □

Variable number of players. In view of the area of application, it is not unnatural to assume that the number of players can vary. New firms or investment companies may enter the scene, while existing players may disappear. In the Porsche-Volkswagen case, we may choose to regard the small shareholders as a null player (player 7 above), or simply disregard them. In the same case, as noted before, at mid June 2013 Qatar left the scene. For a characterization of the power index Φ^ω for $\omega = (1, \ldots, 1)$ where the number of players may vary, see Karos and Peters (2013).

Further research. There are several interesting avenues for further theoretical research in this area. One obvious question is what becomes feasible in terms of power indices if we change the axioms. For instance, a Banzhaf value based power index becomes possible if the Transfer Property is dropped (cf. Section 3.4). Another direction of research is to consider the combination of mutual control structures with transferable utility games in the spirit of Myerson (1977). A mutual control structure can be seen as a hypergraph, and a value for transferable utility games combined with such a hypergraph can take the imposed control relations into consideration.

Appendix A. Appendix: Computations for Remark 3.11

We compute the power index Φ^ω for the mcs C, derived from the simple game v. Since $d_j^C(S) = d_v(S)$ for all $j \in N$ and $S \subseteq N$, we obtain

$$\begin{aligned}
\Phi_i^\omega(C) &= \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \in S, k \notin S} \frac{d_v(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_v(S)}{|S|} \beta_{|S|} \right) \\
&\quad - \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \notin S, k \in S} \frac{d_v(S)}{|S|} \alpha_{|S|} + \sum_{S: i \in S, k \in S} \frac{d_v(S)}{|S|} \beta_{|S|} \right) \\
&= \sum_{k \in N \setminus \{i\}} \alpha_{|S|} \left(\sum_{S: i \in S, k \notin S} \frac{d_v(S)}{|S|} - \sum_{S: i \notin S, k \in S} \frac{d_v(S)}{|S|} \right)
\end{aligned}$$

which shows that Φ^ω is independent of the β-weights. If we take $\alpha_1 = \ldots = \alpha_{n-1} = 1$ then we obtain

$$\Phi_i^\omega(C) = \sum_{k \in N \setminus \{i\}} \left(\sum_{S: i \in S, k \notin S} \frac{d_v(S)}{|S|} - \sum_{S: i \notin S, k \in S} \frac{d_v(S)}{|S|} \right) .$$

Now note that

$$\sum_{S: i \in S, k \notin S} \frac{d_v(S)}{|S|} = Sh_i(v) - \sum_{S: i \in S, k \in S} \frac{d_v(S)}{|S|}$$

and

$$\sum_{S: i \notin S, k \in S} \frac{d_v(S)}{|S|} = Sh_k(v) - \sum_{S: i \in S, k \in S} \frac{d_v(S)}{|S|} .$$

Hence,

$$\begin{aligned}
\Phi_i^\omega(C) &= \sum_{k \in N \setminus \{i\}} (Sh_i(v) - Sh_k(v)) \\
&= (n-1)\, Sh_i(v) - (v(N) - Sh_i(v)) \\
&= n\, Sh_i(v) - 1 .
\end{aligned}$$

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