

# LARGE GAPS BETWEEN PRIMES

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ABSTRACT. We discuss our recent joint proof of a conjecture of Erdős on the size of gaps between primes.

## 1. INTRODUCTION

How large can gaps between primes be? This is a very basic question in the study of the distribution of primes, which could have been studied by the ancient Greeks (see figure 1), but also has some direct relevance to the modern world. Various computer programs generate prime numbers of a given approximate size  $X$  by starting at  $X$  and then sequentially testing  $X$ ,  $X + 1$ ,  $X + 2$  etc in turn until one finds a prime. It is quick to test an individual number to see if it is prime, but it would take a long time to find a prime if you have to test a very large number of integers. How many numbers would you need to test before you found a prime? This is exactly the problem of how large gaps between primes are!

The prime number theorem shows that the *average* gap between prime numbers of size  $X$  is approximately  $\log X$ . Thus *typically* one would need to test about  $\log X$  numbers for primality, which is not too many.

**Conjecture** (Cramér). *Amongst primes less than  $X$ , all gaps are smaller than  $C(\log X)^2$  (for some absolute constant  $C > 0$ ).*

If Cramér's conjecture is true, then we would *never* have to test more than about  $(\log X)^2$  integers. This would enable one to quickly find a prime of any given size. In fact, this would give a simple *deterministic* way of generating prime numbers of any given size very quickly - something that we don't know how to do.

Our current knowledge of this problem is very limited. The best upper bound is the following.

**Theorem** (Baker, Harman, Pintz). *Amongst primes less than  $X$ , all gaps are smaller than  $CX^{0.525}$  (for some absolute constant  $C > 0$ ).*

Unfortunately this doesn't rule out the possibility of there being some pairs of consecutive primes very far apart. Testing  $X^{0.525}$  consecutive integers when  $X$  has a hundred digits would take much longer than a lifetime.

In the other direction, there is a very easy high school method of showing that there are arbitrarily large gaps between prime numbers. For  $j$  between 2 and  $n$ , the integer  $n! + j$  is a multiple of  $j$ , and so cannot be prime. This gives  $n - 1$  consecutive

composite numbers which are all of size approximately  $n!$ . Put another way, this constructs gaps between primes smaller  $X$  of size at least  $c \log X / \log \log X$  (for some constant  $c > 0$ ). This is worse than the average gap from the prime number theorem, but this argument has proven easier to generalize. A series of papers in the 1930s adapted this high-school method to show that there are gaps which can be arbitrarily large compared with the average gap.

**Theorem** (Erdős, Rankin, Westzynthius). *There exists consecutive primes less than  $X$  which differ by more than  $c \log X \cdot \log \log X \cdot \log \log \log X / (\log \log \log X)^2$  (for some absolute constant  $c > 0$ ).*

This (rather ugly) expression is only slightly larger than the average gap of size  $\log X$ , and well off Cramér's prediction of  $(\log X)^2$ , but the underlying method is the only way we currently have of showing that there are arbitrarily large gaps compared with the average size. Unfortunately, progress was slow at improving this bound - subsequent improvements over the next 75 years were only in the value of the constant  $c$ . Paul Erdős, who liked to offer cash prizes for math problems, offered his largest ever cash prize for the problem of showing that the constant  $c$  above could be made arbitrarily large as  $X \rightarrow \infty$ , and popularized the problem by mentioning it in several lectures and letters (see figures 2 and 3). This was because any noticeable improvement in the bound would need to use new arithmetic information about prime numbers.

In 2014 this challenge was solved independently by the author and by Ford, Green, Konyagin and Tao using rather different methods. Combining these approaches, the final result was

**Theorem** (Ford, Green, Konyagin, Maynard, Tao). *There exists consecutive primes less than  $X$  which differ by more than  $c \log X \cdot \log \log X \cdot \log \log \log X / \log \log \log X$  (for some absolute constant  $c > 0$ ).*

The improvement here (by a factor of  $\log \log \log X$ ) is quantitatively quite modest and we are still well off Cramér's conjecture. The key interest is that these approaches used stronger knowledge of the distribution of prime numbers to get these improvements of old questions. In my talk I will give an overview of this problem and recent developments, which involve a pleasing mixture of probability, combinatorics, analysis and number theory.

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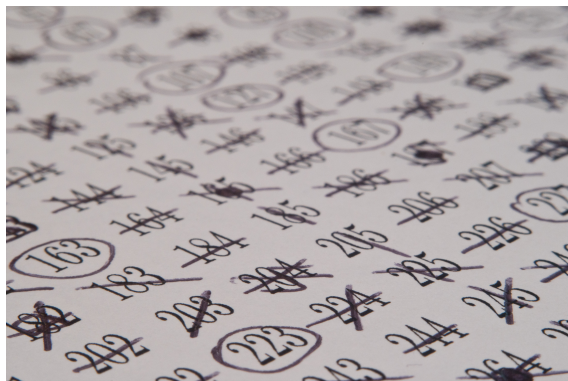


FIGURE 1. Calculations performing the sieve of Eratosthenes. This is a simple way of finding prime numbers dating back to the ancient Greeks. Generalizations of such sieves play a major role in modern work on large gaps between primes.

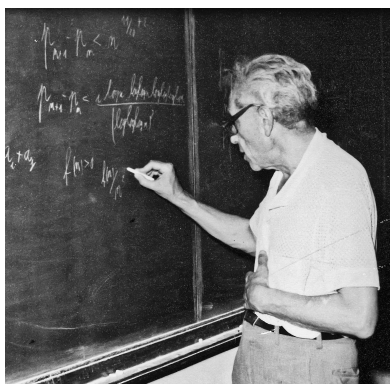


FIGURE 2. Paul Erdős explaining the large gaps between primes problem in a lecture in Madras, 1984.

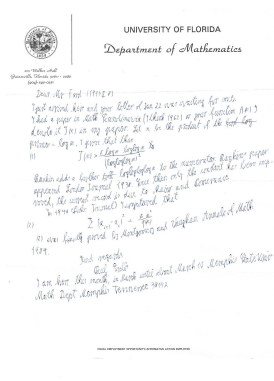


FIGURE 3. A letter from Paul Erdős to Kevin Ford when Ford was a graduate student explaining the large gaps problem. Ford later was one of the authors to solve Erdős' problem several years later.



FIGURE 4. James Maynard is a Clay Research Fellow at Magdalen College, Oxford. His research is in analytic number theory, particularly the distribution of prime numbers.