Coherence for 3-dualizable objects

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Abstract

A fully extended framed topological field theory with target in a symmetric monoidal n-catgeory $\mathcal{C}$ is a symmetric monoidal functor $Z : \mathrm{Bord}_n^{fr} \to \mathcal{C}$, where Bord_n^{fr} is the symmetric monoidal n-category of n-framed bordisms. The cobordism hypothesis says that such field theories are classified by fully dualizable objects in $\mathcal{C}$.

Given a fully dualizable object $X \in \mathcal{C}$, we are interested in computing the values of the corresponding field theory on specific framed bordisms. This leads to the question of finding a presentation for Bord_n^{fr}. In view of the cobordism hypothesis, this can be rephrased in terms of finding *coherence data* for fully dualizable objects in a symmetric monoidal n-category.

We prove a characterization of full dualizability of an object X in terms of existence of a dual of X and existence of adjoints for a finite number of higher morphisms. This reduces the problem of finding coherence data for fully dualizable objects to that of finding coherence data for duals and adjoints. For $n = 3$, and in the setting of strict symmetric monoidal 3-categories, we find this coherence data, and we prove the corresponding coherence theorems. The proofs rely on extensive use of a graphical calculus for strict monoidal 3-categories.

Contents

Chapter 1

Introduction

1.1 Motivation

Consider the n-category Bord_n^{fr} whose objects are n-framed 0-manifolds and whose k-morphisms are n-framed k-manifolds, viewed as bordisms between n-framed $(k-1)$-manifolds, for $k = 1, \cdots, n$. Here by n-framed k-manifold we mean a manifold M together with a choice of trivialization of the stabilization of its tangent bundle to dimension n,

$$TM \oplus \mathbb{R}^{n-k} \simeq \mathbb{R}^n.$$

The operation of taking disjoint unions of manifolds turns Bord_n^{fr} into a symmetric monoidal n-category. We call this the *fully extended framed n-dimensional bordism category.* A detailed construction of Bord_n^{fr} as a symmetric monoidal (∞, n)-category appears in [4]. Here we denote by Bord_n^{fr} only the symmetric monoidal n-category obtained from this one by identifying isomorphic n-morphisms. A symmetric monoidal functor $F : \mathrm{Bord}_n^{fr} \to \mathcal{C}$ to some target symmetric monoidal n-category $\mathcal{C}$ is called a *fully extended framed n-dimensional field theory.* The cobordism hypothesis ([1],[9]) states that such functors are classified by so called *fully dualizable* objects in $\mathcal{C}$. More precisely the functor $\mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \to \mathcal{C}$, defined by evaluating on the object $+$ (the point with the standard n-framing), factors through the inclusion $\mathrm{Obj}^{fd}(\mathcal{C}) \hookrightarrow \mathcal{C}$ of the n-groupoid of fully dualizable objects in $\mathcal{C}$ and it induces an equivalence of n-groupoids

$$\mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Obj}^{fd}(\mathcal{C}).$$

Suppose we are given a fully dualizable object $X \in \mathcal{C}$. By the cobordism hypothesis there exists an essentially unique symmetric monoidal functor $F_X : \mathrm{Bord}_n^{fr} \to \mathcal{C}$ with $F(+) \simeq X$. Suppose we would like to compute the invariant $F_X(M)$ for M a closed n-manifold. One idea is to decompose M into certain generating bordisms on which we can compute the value of the field theory and then recover $F_X(M)$ by composition. So the question is what are these generating bordisms and how do we compute the value of F_X on them. Since the field theory F_X is only unique in a weak sense, meaning the fibre of $\mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \to \mathcal{C}$ over X is contractible, the precise question is not exactly about *computing* the value of F_X on the generating bordisms but really about *prescribing* these values in such a way that there exists a field theory F_X whose value on each generating bordism is this prescribed value. It will only be possible to find such an F_X if these prescribed values satisfy certain relations. So we are looking for a list of generating k-morphisms ($k = 0, \cdots, n$) and a list of relations between n-morphisms so that defining a functor out of Bord_n^{fr} amounts to picking images for the generating morphisms, which satisfy the relations. We can reformulate the problem as follows.

Problem 1.1.1. *Find a presentation $\mathcal{P}_n$ for the fully extended framed bordism category* Bord_n^{fr}.

By a presentation, we mean a list of k-morphisms for $k = 0, \cdots, n$ and a list of relations between the n-morphisms, together with a symmetric monoidal equivalence $\mathcal{F}^{\mathrm{sym}}(\mathcal{P}_n) \to \mathrm{Bord}_n^{fr}$, where $\mathcal{F}^{\mathrm{sym}}$ denotes the symmetric monoidal n-category generated by $\mathcal{P}_n$.

By the cobordism hypothesis, Bord_n^{fr} is the free symmetric monoidal n-category on a single fully dualizable object, so finding a presentation for it is equivalent to finding all the coherence data for a fully dualizable object in a symmetric monoidal n-category $\mathcal{C}$. What we mean by coherence data is the following. Being fully dualizable is characterized by the existence of a number of morphisms satisfying certain equations, up to isomorphism when they are not n-morphisms. But in reality, there is a much larger collection of morphisms which can be constructed from a fully dualizable object, with the n-morphisms satisfying certain relations. Moreover, such a collection of choices of morphisms is essentially unique, in the sense that the n-groupoid formed by all choices of such data is equivalent to the n-groupoid of fully dualizable objects in $\mathcal{C}$. More precisely, we want to find a presentation FD_n with a distinguished object X such that for any symmetric monoidal n-category $\mathcal{C}$ the functor $\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_n), \mathcal{C}) \to \mathcal{C}$ defined by evaluating at X factors through the inclusion $\mathrm{Obj}^{fd}(\mathcal{C}) \hookrightarrow \mathcal{C}$, inducing an equivalence of n-groupoids

$$\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_n), \mathcal{C}) \xrightarrow{\sim} \mathrm{Obj}^{fd}(\mathcal{C}).$$

Each generator in FD_n should have a natural interpretation as a framed bordism, inducing a symmetric monoidal functor $\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_n) \to \mathrm{Bord}_n^{fr}$ to the framed bordism n-category. One can then use the cobordism hypothesis to show that this functor is an equivalence.

For example, one can take FD_1 to have two objects X, Y and two 1-morphisms $\mathrm{ev} : XY \overset{\supset}{\to} 1$ and $\mathrm{coev} : 1 \overset{\subset}{\to} YX$ subject to the two snake relations  $=$ — and  $=$ — . The proof that $\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_1), \mathcal{C}) \to \mathrm{Obj}^{fd}(\mathcal{C})$ is an equivalence comes down to familiar statements about uniqueness of duals in monoidal categories. This means we can do computations with a 1-dimensional field theory F corresponding to a fully dualizable object $X \in \mathcal{C}$ by decomposing bordisms into composites of cups and caps. We can pick the value $F(-)$ on the negatively framed point to be any object dual to $X = F(+)$ and we are free to pick the value of the field theory on the cup and the cap, as long as the snake relations are satisfied.

Pstrągowski has given a description of FD_2 in [10]. The main goal of the project described in this thesis is to find a description of FD_3.

1.2 Results

The first step is to find a finite set of conditions which imply full dualizability, which is accomplished by the following 3-dimensional analog of a 2-dimensional statement proved by Lurie (in [9]).

Proposition 1.2.1. *Let X be an object in a symmetric monoidal 3-category $\mathcal{C}$. Suppose that X has a dual and that the evaluation and coevaluation 1-morphisms have right adjoints. Moreover, suppose the four unit and counit 2-morphisms witnessing these two adjunctions have right adjoints. Then X is fully dualizable.*

This means that it is easy to find a finite presentation $\tilde{\mathrm{FD}}_3$ with a distinguished 0-cell X such that for every functor $F : \mathcal{F}^{\mathrm{sym}}(\tilde{\mathrm{FD}}_3) \to \mathcal{C}$ the object $F(X)$ is fully dualizable. In fact, one can build such a presentation by the following procedure. Include in this presentation a dual for X, together with evaluation and coevaluation 1-morphisms $\mathrm{ev}, \mathrm{coev}$ and invertible 2-morphisms implementing the snake equations. Then add right adjoints for ev and coev together with unit and counit 2-morphisms $\eta_{\mathrm{ev}}, \epsilon_{\mathrm{ev}}, \eta_{\mathrm{coev}}, \epsilon_{\mathrm{coev}}$ witnessing the adjunctions, as well as invertible 3-morphisms implementing the snake equations for these adjunctions. Finally, add right adjoints to these four unit and counit 2-morphisms, together with unit and counit 3-morphisms

$u_{\eta_{\mathrm{ev}}}, v_{\eta_{\mathrm{ev}}}, u_{\epsilon_{\mathrm{ev}}}, v_{\epsilon_{\mathrm{ev}}}, u_{\eta_{\mathrm{coev}}}, v_{\eta_{\mathrm{coev}}}, u_{\epsilon_{\mathrm{coev}}}, v_{\epsilon_{\mathrm{coev}}}$ witnessing these adjunctions, and relations implementing the snake equations.

However, such a presentation won't satisfy the coherence theorem, meaning an equivalence $\mathrm{Fun}^{\mathrm{sym}}(\tilde{\mathrm{FD}}_3, \mathcal{C}) \to \mathrm{Obj}^{fd}(\mathcal{C})$. In order to obtain such an equivalence one must add extra coherence data to $\tilde{\mathrm{FD}}_3$.

The main result is as follows. For technical reasons (an appropriate theory of weak 3-categories with a string diagram calculus remains in development) we restrict attention to strictly symmetric, strictly monoidal, strict 3-categories, but all statements and proofs are done such that they will apply without modification in the weak case, once the appropriate formalism is in place.

Given strictly symmetric, strictly monoidal, strict 3-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Fun}^{\mathrm{sym}}(\mathcal{C}, \mathcal{D})$ the 3-category of strict symmetric monoidal functors, strict symmetric monoidal natural transformations, strict symmetric monoidal modifications and strict symmetric monoidal perturbations. See 2.1 for defintions.

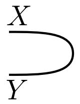
Theorem 1.2.2. *Let* FD_3 *be the presentation described below. Then, for any strictly symmetric strictly monoidal strict* 3*-category* $\mathcal{C}$*, the functor* $\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3), \mathcal{C}) \to \mathcal{C}$ *given by evaluation at* X *factors through the inclusion* $\mathrm{Obj}^{fd}(\mathcal{C}) \hookrightarrow \mathcal{C}$*, and induces an equivalence of strict* 3*-groupoids*

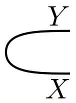
$$\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3), \mathcal{C}) \xrightarrow{\sim} \mathrm{Obj}^{fd}(\mathcal{C}).$$

The presentation FD_3**:**

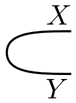
(0) objects X and Y;

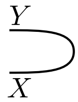
1. 1-morphisms

   ev = 

   $: XY \to 1$ and coev = 

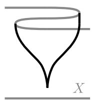
   $: 1 \to YX$;

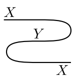
   ev^R = 

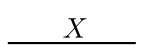
   $: 1 \to XY$ and coev^R = 

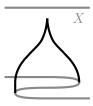
   $: YX \to 1$;

2. 2-morphisms

C_X = 

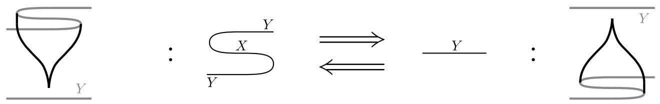
: 

$\overset{\Longrightarrow}{\Longleftarrow}$ 

: 

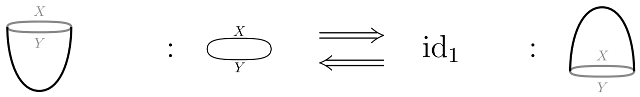
$= C_X^{-1}$

$C_Y =$



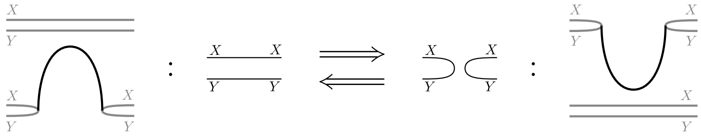
$= C_Y^{-1}$

$\varepsilon_{\mathrm{ev}} =$



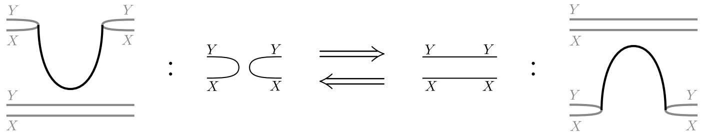
$= \varepsilon_{\mathrm{ev}}^R$

$\eta_{\mathrm{ev}} =$



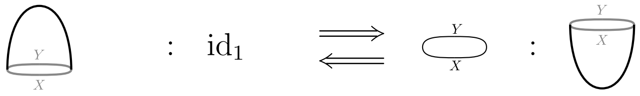
$= \eta_{\mathrm{ev}}^R$

$\varepsilon_{\mathrm{coev}} =$



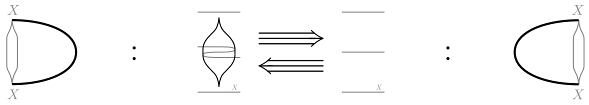
$= \varepsilon_{\mathrm{coev}}^R$

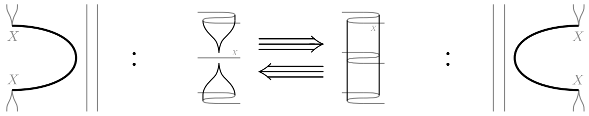
$\eta_{\mathrm{coev}} =$

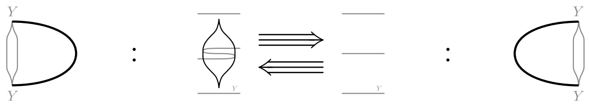


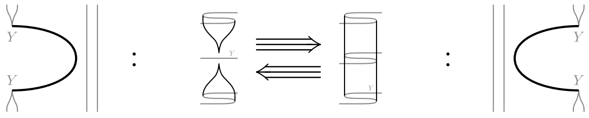
$= \eta_{\mathrm{coev}}^R$

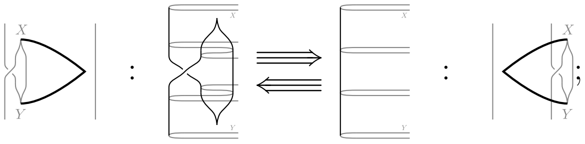
3. 3-morphisms

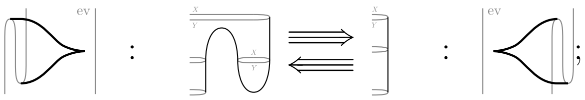


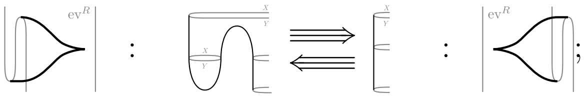


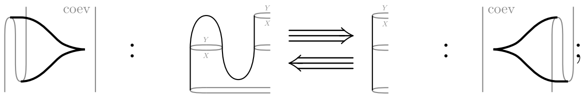


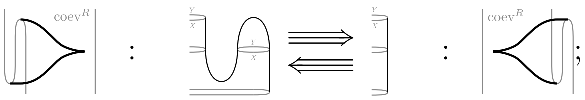




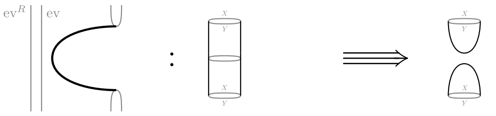




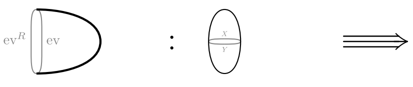




$u_{\varepsilon_{\mathrm{ev}}} =$



$v_{\varepsilon_{\mathrm{ev}}} =$



$\mathrm{Id}_{\mathrm{id}_1}$

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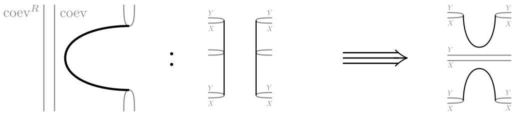
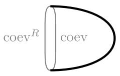
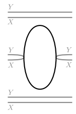
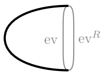
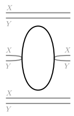


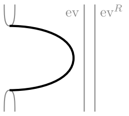
$v_{\varepsilon_{\mathrm{coev}}} =$ 

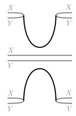
: 

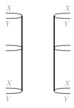
$\Longrightarrow \mathrm{Id}_{\mathrm{id}_{Y\otimes X}}$

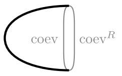
$u_{\eta_{\mathrm{ev}}} =$ 

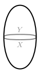
: $\mathrm{Id}_{\mathrm{id}_{X\otimes Y}} \Longrightarrow$ 

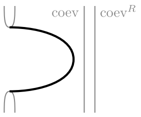
$v_{\eta_{\mathrm{ev}}} =$ 

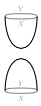
: 

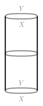
$\Longrightarrow$ 

$u_{\eta_{\mathrm{coev}}} =$ 

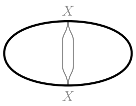
: $\mathrm{Id}_{\mathrm{id}_1} \Longrightarrow$ 

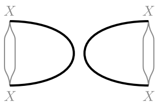
$v_{\eta_{\mathrm{coev}}} =$ 

: 

$\Longrightarrow$ 

4. relations



$=$ Id ; 

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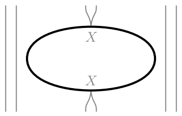
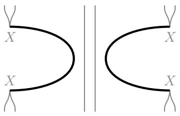


$=$ Id ; 

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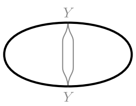
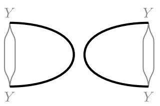


$=$ Id ; 

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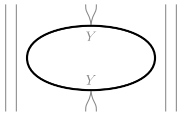

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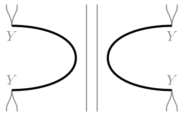

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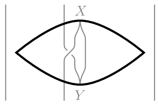
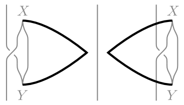


= Id ;



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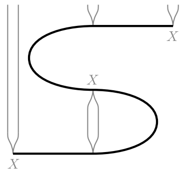

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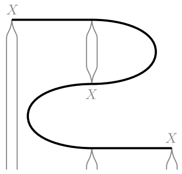
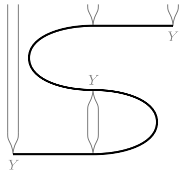
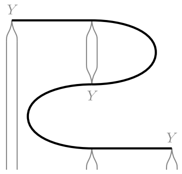


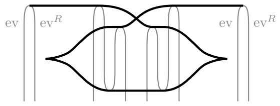
= Id;



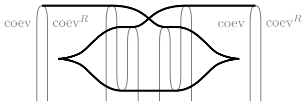
= Id ;



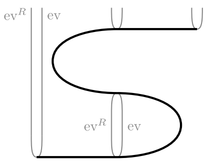
= Id;



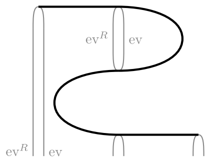
= Id ;



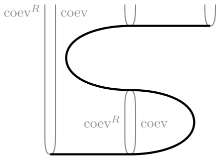
= Id;



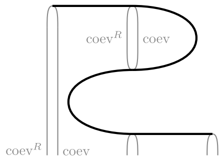
= Id ;



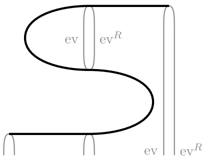
= Id;



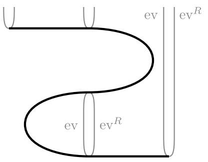
= Id ;



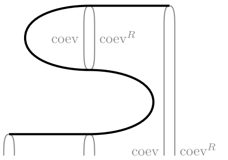
= Id;



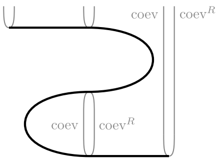
$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;

Each generator in FD_3 can be interpreted as a framed bordism, in such a way that each relation corresponds to a framed diffeomorphism, relative to the boundary (see 4.3). In order to conclude that FD_3 is actually a presentation of Bord_3^{fr}, one would need to reformulate the above Theorem in the setting of fully weak symmetric monoidal 3-categories, where the cobordism hypothesis holds. This would require a model for fully weak monoidal 3-categories where one can use a graphical calculus, as this is what is needed for the proof.

1.3 Future work

So far, all the presentations of bordism categories in the literature are obtained by geometric methods ([7],[12],[10],[3]), namely different versions of parameterized Morse theory, so the lists of generators come from classifying singularities of certain types of maps. However, these classifications of singularities become extremely complicated in higher dimensions, with infinite dimensional moduli spaces of nonequivalent singularities replacing the simple finite classifications which arise in low dimensions. This raises the question of whether bordism categories can still have finite presentations in higher dimensions.

The idea behind the work presented in this thesis is that one can use the cobordism hypothesis to reduce the problem of finding a presentation of the symmetric monoidal n-category Bord_n^{fr} to that of finding coherence data for fully dualizable objects in a symmetric monoidal n-category. We show in Chapter 4 that the full dualizablity of an object can be characterized in terms of existence of a dual and of a finite number of adjunctions. So in principle, if one understands the coherence data for adjunctions between k-morphisms in an n-category and duals in a monoidal n-category, it should be possible to understand the coherence data for fully dualizable objects in a symmetric monoidal n-category. And as long as there is a finite list of coherence data for duals and adjoints, the same should be true for fully dualizable objects. It is therefore plausible that the symmetric monoidal n-category Bord_n^{fr} has a finite presentation.

In this thesis we carry out part of this program in the case $n = 3$ and only in the context of strict categories. Specifically, we work out two main points: the first one is finding the coherence data for duals and adjoints; the second one is explaining how to assemble this data together to provide coherence data for fully dualizable objects. The second point seems amenable to generalization, both to weak categorical settings and to higher dimensions. The first one depends heavily on the use of a graphical calculus and for this reason seems more resistant to generalization.

Chapter 2

Preliminaries

In this chapter we set out some definitions and basic facts about strict n-categories. We also only consider strict monoidal structures and strictly symmetric strict monoidal structures on these strict n-categories. For this reason, we omit the word strict everywhere, since all categorical structures are assumed to be strict. We go on to discuss presentations of these types of categories, using the theory of computads, as presented in Schommer-Pries [12]. Finally we explain a graphical notation for doing computations in monoidal 3-categories.

2.1 Strict n-categories via monads

Let gSet_n be the category of n-globular sets and consider the monad

$$T_n : \mathrm{gSet}_n \to \mathrm{gSet}_n$$

where $T_n(X)$ has

$$T_n(X)_m = \coprod_{\pi \in pd(m)} \mathrm{Hom}_{\mathrm{gSet}_n}(\hat{\pi}, X)$$

as its set of m-cells. Here $pd(m)$ is the set of m-dimensional pasting diagrams and $\hat{\pi}$ is the globular set associated to the pasting diagram π (see Chapter 8 in Leinster [8]). This globular set $\hat{\pi}$ is such that a map of globular sets $\hat{\pi} \to X$ corresponds to a labelling of the pasting diagram π by X.

Definition 2.1.1. *An n-category is an algebra over the monad T_n. A functor between n-categories is a morphism of algebras over T_n.*

This agrees with the standard definitions of strict n-categories and strict functors between them.

Define a monad

$$T_n^{\otimes} : \mathrm{gSet}_n \to \mathrm{gSet}_n$$

by

$$T_n^{\otimes}(X)_m = \coprod_{k \geq 0} T_n(X)_m^{\times k}.$$

Definition 2.1.2. *A monoidal n-category is an algebra over the monad $T_n^{\otimes}$. A monoidal functor between monoidal n-categories is a morphism of algebras over $T_n^{\otimes}$.*

This agrees with the standard definitions of strict monoidal strict n-categories and strict monoidal strict functors between them. Define a monad

$$T_n^{sym} : \mathrm{gSet}_n \to \mathrm{gSet}_n$$

by

$$T_n^{sym}(X)_m = \coprod_{k \geq 0} T_n(X)_m^{\times k} / \Sigma_k,$$

where Σ_k acts by permuting the k factors.

Definition 2.1.3. *A symmetric monoidal n-category is an algebra over the monad T_n^{sym}. A symmetric monoidal functor between symmetric monoidal n-categories is a morphism of algebras over T_n^{sym}.*

According to this definition, a symmetric monoidal n-category is a monoidal n-category where the monoidal product of objects and morphisms is strictly commutative. Note that being symmetric is just a property imposed on a monoidal n-category, not a structure, so a symmetric monoidal functor is the same thing as a monoidal functor. Now we specialize to the case of 3-categories. Given 3-categories $\mathcal{C}$ and $\mathcal{D}$, we want to define the functor 3-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 2.1.4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be strict functors of strict 3-categories.*

A natural transformation $\alpha : F \implies G$ consists of the following data.

0. *For each object X of $\mathcal{C}$ a 1-morphism $\alpha_X : F(X) \to G(X)$.*

1. *For each 1-morphism $f : X \to Y$ in $\mathcal{C}$ an invertible 2-morphism*

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\end{tikzcd}$$

2. *For each 2-morphism*

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

an invertible 3-morphism

$$\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"{name=M}] \arrow[d, bend right=70, "F(f)"'{name=L}] & G(X) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y)
\arrow[Rightarrow, from=L, to=M, "F(\eta)"]
\end{tikzcd}
\overset{\alpha_\eta}{\Rrightarrow}
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'{name=N}] \arrow[d, bend left=70, "G(g)"{name=P}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=N, to=P, "G(\eta)"]
\end{tikzcd}$$

3. *For each 3-morphism*

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}
\overset{t}{\Rrightarrow}
\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\theta"] \end{tikzcd}$$

a commuting diagram of 3-morphisms

$$\begin{array}{ccc}
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"{name=M}] \arrow[d, bend right=70, "F(f)"'{name=L}] & G(X) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y)
\arrow[Rightarrow, from=L, to=M, "F(\eta)"]
\end{tikzcd}
& \overset{\alpha_\eta}{\Rrightarrow} &
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'{name=N}] \arrow[d, bend left=70, "G(g)"{name=P}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=N, to=P, "G(\eta)"]
\end{tikzcd} \\
{\scriptstyle F(t)}\Downarrow & & \Downarrow{\scriptstyle G(t)} \\
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"{name=M}] \arrow[d, bend right=70, "F(f)"'{name=L}] & G(X) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y)
\arrow[Rightarrow, from=L, to=M, "F(\theta)"]
\end{tikzcd}
& \overset{\alpha_\theta}{\Rrightarrow} &
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'{name=N}] \arrow[d, bend left=70, "G(g)"{name=P}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=N, to=P, "G(\theta)"]
\end{tikzcd}
\end{array}$$

This data is subject to conditions of compatibility with composition:
For 1*-morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *we have*

$$\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g\circ f)"'] & G(X) \arrow[d, "G(g\circ f)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_{g\circ f}"] & G(Z)
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"] \arrow[d, "F(g)"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y) \arrow[d, "G(g)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_g"] & G(Z)
\end{tikzcd}$$

For 2-morphisms

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f", ""'{name=F}] \arrow[r, "g" description, ""{name=G1}, ""'{name=G2}] \arrow[r, bend right=60, "h"', ""{name=H}] & Y
\arrow[Rightarrow, from=F, to=G1, "\lambda"] \arrow[Rightarrow, from=G2, to=H, "\eta"]
\end{tikzcd}$$

we have

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(h)", ""'{name=A}] \arrow[d, bend right=80, "F(f)"', ""{name=B}] & G(X) \arrow[d, "G(h)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_h"'] & G(Y)
\arrow[Rightarrow, from=B, to=A, "F(\eta\circ\lambda)"]
\end{tikzcd}
\overset{\alpha_{\eta\circ\lambda}}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=C}] \arrow[d, bend left=80, "G(h)", ""'{name=D}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=C, to=D, "G(\eta\circ\lambda)"]
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, ""'{name=A}] \arrow[d, bend right=60, ""{name=B1}, ""'{name=B2}] \arrow[d, bend right=90, ""{name=C}] & G(X) \arrow[d] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_h"'] & G(Y)
\arrow[Rightarrow, from=C, to=B2] \arrow[Rightarrow, from=B1, to=A]
\end{tikzcd}
\overset{\alpha_\eta}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, ""'{name=A}] \arrow[d, bend right=80, ""{name=B}] & G(X) \arrow[d, ""{name=C}] \arrow[d, bend left=80, ""'{name=D}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"] & G(Y)
\arrow[Rightarrow, from=B, to=A, "F(\lambda)"] \arrow[Rightarrow, from=C, to=D, "G(\eta)"]
\end{tikzcd}
\overset{\alpha_\lambda}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d] & G(X) \arrow[d, ""{name=A}] \arrow[d, bend left=60, ""'{name=B1}, ""{name=B2}] \arrow[d, bend left=90, ""'{name=C}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=A, to=B1] \arrow[Rightarrow, from=B2, to=C]
\end{tikzcd}$$

For 2-morphisms

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f", ""'{name=F}] \arrow[r, bend right=60, "h"', ""{name=H}] & Y \arrow[r, bend left=60, "g", ""'{name=G}] \arrow[r, bend right=60, "k"', ""{name=K}] & Z
\arrow[Rightarrow, from=F, to=H, "\lambda"] \arrow[Rightarrow, from=G, to=K, "\eta"]
\end{tikzcd}$$

we have

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(k\circ h)", ""'{name=A}] \arrow[d, bend right=80, "F(g\circ f)"', ""{name=B}] & G(X) \arrow[d, "G(k\circ h)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_{k\circ h}"'] & G(Z)
\arrow[Rightarrow, from=B, to=A, "F(\lambda*\eta)"]
\end{tikzcd}
\overset{\alpha_{\lambda*\eta}}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(g\circ f)"', ""{name=C}] \arrow[d, bend left=80, "G(k\circ h)", ""'{name=D}] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_{g\circ f}"] & G(Z)
\arrow[Rightarrow, from=C, to=D, "G(\lambda*\eta)"]
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(h)", ""'{name=A}] \arrow[d, bend right=80, "F(f)"', ""{name=B}] & G(X) \arrow[d, "G(h)"] \\
F(Y) \arrow[r, "\alpha_Y"] \arrow[ur, Rightarrow, "\alpha_h"] \arrow[d, "F(k)", ""'{name=C}] \arrow[d, bend right=80, "F(g)"', ""{name=D}] & G(Y) \arrow[d, "G(k)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_k"] & G(Z)
\arrow[Rightarrow, from=B, to=A, "F(\lambda)"] \arrow[Rightarrow, from=D, to=C, "F(\eta)"]
\end{tikzcd}
\overset{\alpha_\lambda}{\underset{\alpha_\eta}{\Rrightarrow}}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=A}] \arrow[d, bend left=80, "G(h)", ""'{name=B}] \\
F(Y) \arrow[r, "\alpha_Y"] \arrow[ur, Rightarrow, "\alpha_f"] \arrow[d, "F(g)"'] & G(Y) \arrow[d, "G(g)"', ""{name=C}] \arrow[d, bend left=80, "G(k)", ""'{name=D}] \\
F(Z) \arrow[r, "\alpha_Z"] \arrow[ur, Rightarrow, "\alpha_g"] & G(Z)
\arrow[Rightarrow, from=A, to=B, "G(\lambda)"] \arrow[Rightarrow, from=C, to=D, "G(\eta)"]
\end{tikzcd}$$

If $\mathcal{C}, \mathcal{D}, F$ and G are monoidal, then we say that α is monoidal if $\alpha_{x \otimes y} = \alpha_x \otimes \alpha_y$ for any k-morphisms x, y $(0 \leq k \leq 2)$.

Definition 2.1.5. *Given functors $F, G : \mathcal{C} \to \mathcal{D}$ and natural transformations $\alpha, \beta : F \Rightarrow G$, a modification $m : \alpha \Rrightarrow \beta$ consists of the following data.*

0. For each object X of $\mathcal{C}$, a 2-morphism

$$\begin{tikzcd}
F(X) \arrow[r, bend left=50, "\alpha_X", ""{name=U, below}] \arrow[r, bend right=50, "\beta_X"', ""{name=D}] & G(X) \arrow[Rightarrow, from=U, to=D, "m_X"]
\end{tikzcd}$$

1. For each 1-morphism $f : X \to Y$ in $\mathcal{C}$ an invertible 3-morphism

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=A}] \arrow[r, bend left=70, "\beta_X", ""{name=B, below}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] & & F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] & G(Y) & \overset{m_f}{\Rrightarrow} & F(Y), \arrow[r, "\beta_Y", ""{name=C, below}] \arrow[r, bend right=70, "\alpha_Y"', ""{name=E}] \arrow[ur, Rightarrow, "\beta_f"] & G(Y)
\arrow[Rightarrow, from=A, to=B, "m_X"]
\arrow[Rightarrow, from=E, to=C, "m_Y"]
\end{tikzcd}$$

2. For each 2-morphism

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D}] & Y \arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

a commutative diagram of 3-morphisms

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=aX}] \arrow[r, bend left=80, "\beta_X", ""'{name=bX}] \arrow[d, "F(g)", ""'{name=Fg}] \arrow[d, bend right=80, "F(f)"', ""{name=Ff}] & G(X) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y)
\arrow[Rightarrow, from=aX, to=bX, "m_X"]
\arrow[Rightarrow, from=Ff, to=Fg, "F(\eta)"]
\end{tikzcd}
\quad \overset{m_g}{\Rrightarrow} \quad
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(g)", ""'{name=Fg}] \arrow[d, bend right=80, "F(f)"', ""{name=Ff}] & G(X) \arrow[d, "G(g)"] \\
F(Y), \arrow[r, "\beta_Y", ""'{name=bY}] \arrow[r, bend right=80, "\alpha_Y"', ""{name=aY}] \arrow[ur, Rightarrow, "\beta_g"] & G(Y)
\arrow[Rightarrow, from=Ff, to=Fg, "F(\eta)"]
\arrow[Rightarrow, from=aY, to=bY, "m_Y"]
\end{tikzcd}$$

$$\alpha_\eta \Ddownarrow \qquad\qquad\qquad\qquad \Ddownarrow \beta_\eta$$

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=aX}] \arrow[r, bend left=80, "\beta_X", ""'{name=bX}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=Gf}] \arrow[d, bend left=80, "G(g)", ""'{name=Gg}] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\arrow[Rightarrow, from=aX, to=bX, "m_X"]
\arrow[Rightarrow, from=Gf, to=Gg, "G(\eta)"]
\end{tikzcd}
\quad \underset{m_f}{\Rrightarrow} \quad
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=Gf}] \arrow[d, bend left=80, "G(g)", ""'{name=Gg}] \\
F(Y), \arrow[r, "\beta_Y", ""'{name=bY}] \arrow[r, bend right=80, "\alpha_Y"', ""{name=aY}] \arrow[ur, Rightarrow, "\beta_f"] & G(Y)
\arrow[Rightarrow, from=aY, to=bY, "m_Y"]
\arrow[Rightarrow, from=Gf, to=Gg, "G(\eta)"]
\end{tikzcd}$$

This data is subject to conditions of compatibility with composition: for 1*-morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z\,,$$

we have

$$\begin{tikzcd}
F(X) \arrow[r, bend left=70, "\beta_X", ""{name=U1, below}] \arrow[r, "\alpha_X"', ""{name=D1}] \arrow[d, "F(g\circ f)"'] & G(X) \arrow[d, "G(g\circ f)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_{g\circ f}"'] & G(Z)
\arrow[Rightarrow, from=D1, to=U1, "m_X"]
\end{tikzcd}
\overset{m_{g\circ f}}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(g\circ f)"'] & G(X) \arrow[d, "G(g\circ f)"] \\
F(Z), \arrow[r, "\beta_Z", ""{name=U2, below}] \arrow[r, bend right=70, "\alpha_Z"', ""{name=D2}] \arrow[ur, Rightarrow, "\beta_{g\circ f}"] & G(Z)
\arrow[Rightarrow, from=D2, to=U2, "m_Z"]
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
F(X) \arrow[r, bend left=70, "\beta_X", ""{name=U3, below}] \arrow[r, "\alpha_X"', ""{name=D3}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] \arrow[d, "F(g)"'] & G(Y) \arrow[d, "G(g)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Z)
\arrow[Rightarrow, from=D3, to=U3, "m_X"]
\end{tikzcd}
\overset{m_f}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, bend left, "\beta_Y", ""{name=U4, below}] \arrow[r, bend right, "\alpha_Y"', ""{name=D4}] \arrow[ur, Rightarrow, "\beta_f"] \arrow[d, "F(g)"'] & G(Y) \arrow[d, "G(g)"] \\
F(Z) \arrow[r, "\alpha_Z"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Z)
\arrow[Rightarrow, from=D4, to=U4, "m_Y"]
\end{tikzcd}
\overset{m_g}{\Rrightarrow}
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\beta_Y"] \arrow[ur, Rightarrow, "\beta_f"] \arrow[d, "F(g)"'] & G(Y) \arrow[d, "G(g)"] \\
F(Z) \arrow[r, "\beta_Z", ""{name=U5, below}] \arrow[r, bend right=70, "\alpha_Z"', ""{name=D5}] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Z)
\arrow[Rightarrow, from=D5, to=U5, "m_Z"]
\end{tikzcd}$$

If $\mathcal{C}, \mathcal{D}, F, G, \alpha$ and β are monoidal, then we say that m is monoidal if $m_{x\otimes y} = m_x \otimes m_y$ for any k-morphisms x, y $(0 \le k \le 1)$.

Definition 2.1.6. *Given functors $F, G : \mathcal{C} \to \mathcal{D}$, natural transformations $\alpha, \beta : F \Rightarrow G$ and modifications $l, m : \alpha \Rrightarrow \beta$, a perturbation $\mathcal{A} : l \to m$ consists of the following data.*

0. *For each object X of $\mathcal{C}$, a 3-morphism $\mathcal{A}_X : l_X \Rrightarrow m_X$;*
1. *For each 1-morphism $f : X \to Y$ in $\mathcal{C}$, a commuting diagram of 3-morphisms*

$$\begin{tikzcd}
F(X) \arrow[rr, "\alpha_X"', ""{name=a1}] \arrow[rr, bend left=60, "\beta_X", ""'{name=b1}] \arrow[d, "F(f)"'] & {} & G(X) \arrow[d, "G(f)"] & {} \arrow[r, Rrightarrow, "\mathcal{A}_X"] & {} & F(X) \arrow[rr, "\alpha_X"', ""{name=a2}] \arrow[rr, bend left=60, "\beta_X", ""'{name=b2}] \arrow[d, "F(f)"'] & {} & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[rr, "\alpha_Y"'] \arrow[urr, Rightarrow, "\alpha_f"'] & {} \arrow[d, Rrightarrow, "l_f"'] & G(Y) & & & F(Y) \arrow[rr, "\alpha_Y"'] \arrow[urr, Rightarrow, "\alpha_f"'] & {} \arrow[d, Rrightarrow, "m_f"] & G(Y) \\
F(X) \arrow[rr, "\beta_X"] \arrow[d, "F(f)"'] & {} & G(X) \arrow[d, "G(f)\; e"] & {} \arrow[r, Rrightarrow, "\mathcal{A}_Y"] & {} & F(X) \arrow[rr, "\beta_X"] \arrow[d, "F(f)"'] & {} & G(X) \arrow[d, "G(f)"] \\
F(Y), \arrow[rr, "\beta_Y", ""'{name=c1}] \arrow[rr, bend right=60, "\alpha_Y"', ""{name=d1}] \arrow[urr, Rightarrow, "\beta_f"] & & G(Y) & & & F(Y), \arrow[rr, "\beta_Y", ""'{name=c2}] \arrow[rr, bend right=60, "\alpha_Y"', ""{name=d2}] \arrow[urr, Rightarrow, "\beta_f"] & & G(Y)
\arrow[Rightarrow, from=a1, to=b1, "l_X"]
\arrow[Rightarrow, from=a2, to=b2, "m_X"]
\arrow[Rightarrow, from=d1, to=c1, "l_Y"]
\arrow[Rightarrow, from=d2, to=c2, "m_Y"]
\end{tikzcd}$$

If $\mathcal{C}, \mathcal{D}, F, G, \alpha, \beta, l$ and m are monoidal, then we say that $\mathcal{A}$ is monoidal if

$$\mathcal{A}_{X \otimes Y} = \mathcal{A}_X \otimes \mathcal{A}_Y$$

for any objects X, Y.

There are standard composition operations organising functors, natural transformations, modifications and perturbations into a 3-category, which we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. If $\mathcal{C}$ and $\mathcal{D}$ are monoidal, then the 3-category of monoidal functors, monoidal transformations, monoidal modifications and monoidal perturbations is denoted by $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$. Just like for functors, there isn't a new notion of a symmetric monoidal transformation, modification or perturbation, since being symmetric is just a property of a monoidal category, so the 3-category of symmetric monoidal functors $\mathrm{Fun}^{\mathrm{sym}}(\mathcal{C}, \mathcal{D})$ between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is just the same as $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

Note that these notions of morphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ correspond to lax morphisms, with appropriate components declared invertible. We denote by $\overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ the 3-category where we instead take oplax morphisms and declare the appropriate components to be invertible. Similarly, we denote the monoidal version by $\overline{\mathrm{Fun}}^{\otimes}(\mathcal{C}, \mathcal{D})$. When working with 3-categories as algebras over a monad T_3 on gSet_3, the natural object to consider is just the category Alg_{T_3} of algebras over the monad, which only has information about the set $\mathrm{Hom}_{\mathrm{Alg}_{T_3}}(\mathcal{C}, \mathcal{D})$ of functors between 3-categories $\mathcal{C}$ and $\mathcal{D}$. So in order to easily prove statements about the 3-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ it's useful to have descriptions of transformations, modifications and perturbations as functors into another 3-category.

Definition 2.1.7. *Let* C_0 *be the* 3*-category with one object and only identity morphisms.*

Definition 2.1.8. *Let* C_1 *be the* 3*-category with two objects* A, B*, a* 1*-morphism* $a : A \to B$ *and only identity morphisms in dimensions two and three.*

Definition 2.1.9. *Let* C_2 *be the* 3*-category with two objects* A, B*, two* 1*-morphisms* $a, b : A \to B$*, a* 2*-morphism* $\alpha : a \to b$ *and only identity* 3*-morphisms.*

Definition 2.1.10. *Let* C_3 *be the* 3*-category with two objects* A, B*, two* 1*-morphisms* $a, b : A \to B$*, two* 2*-morphisms* $\alpha, \beta : a \to b$ *and a* 3*-morphism* $\mathcal{A} : \alpha \to \beta$*.*

Lemma 2.1.11. *Let* $\mathcal{C}, \mathcal{D}$ *be* 3*-categories. For* $1 \leq k \leq 3$*, the data of a* k*-morphism in* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is the same as the data of a functor* $\mathcal{C} \to \overline{\mathrm{Fun}}(C_k, \mathcal{D})$*.*

Proof. One just unravels the definitions. ☐

Note that when $\mathcal{C}$ is a 3-category and $\mathcal{D}$ is a monoidal 3-category, the 3-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\overline{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ have monoidal structures, defined by using the monoidal structure in $\mathcal{D}$ pointwise.

Lemma 2.1.12. *Let* $\mathcal{C}, \mathcal{D}$ *be monoidal* 3*-categories and* $1 \leq k \leq 3$*. The data of a* k*-morphism in* $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ *is the same as the data of a monoidal functor* $\mathcal{C} \to \overline{\mathrm{Fun}}(C_k, \mathcal{D})$*.*

Proof. One just unravels the definitions. ☐

One can also see the source and target maps in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ appearing naturally in this form. Consider the diagram of 3-categories

$$\begin{tikzcd} C_0 \arrow[r, shift left] \arrow[r, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] & C_2 \arrow[r, shift left] \arrow[r, shift right] & C_3 \end{tikzcd}$$

where C_0, C_1, C_2 and C_3 are defined above and the maps are the inclusions of the k-cell as source and target of the $(k+1)$-cell. Apply the functor $\mathrm{Hom}_{\mathrm{Alg}_{T_3}}(\mathcal{C}, \overline{\mathrm{Fun}}(-, \mathcal{D}))$ to get the following globular set, where we use the notation $C_k(\mathcal{D}) := \overline{\mathrm{Fun}}(C_k, \mathcal{D})$ and $\mathrm{Hom} := \mathrm{Hom}_{\mathrm{Alg}_{T_3}}$.

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}, C_3(\mathcal{D})) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(\mathcal{C}, C_2(\mathcal{D})) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(\mathcal{C}, C_1(\mathcal{D})) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}(\mathcal{C}, C_0(\mathcal{D})). \end{tikzcd}$$

It is easy to check that this is the globular set underlying $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The T_3-algebra structure can also be described as coming from that of $\mathcal{D}$.

Definition 2.1.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be 3-categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if it is essentially surjective on objects, 1-morphisms and 2-morphisms and bijective on 3-morphisms.*

2.2 Presentations via computads

In this section, we discuss presentations of n-categories, monoidal n-categories and symmetric monoidal n-categories, closely following [12].

Intuitively, a presentation $\mathcal{P}$ for an n-category consists of a set $\mathcal{P}_k$ of generating k-cells for each $0 \leq k \leq n+1$. Each generating k-cell has a source and a target, which should be $(k-1)$-morphisms in the free $(k-1)$-category generated by the ℓ-cells, for $\ell \leq k-1$. The n-category generated by $\mathcal{P}$ is then obtained by taking the free n-category generated by the k-cells for $k \leq n$ and then taking the quotient of the set of n-morphisms by the equivalence relation induced by declaring the source of every $(n+1)$-cell to be equivalent to its target.

In a presentation for a monoidal n-category, each generating k-cell has source and target $(k-1)$-morphisms in the free monoidal $(k-1)$-category generated by the ℓ-cells, for $\ell \leq k-1$.

These notions can be made precise by using the theory of computads, as in [12]. We now give a brief summary of the facts about computads. See Section 2.7 in [12] for the details.

Given a a monad T one denotes by Alg_T its category of algebras. If T is finitary (e.g. T_n, $T_n^{\otimes}$, T_n^{sym}) one can inductively define categories Comp_k^T of k-computads with respect to T, for $0 \leq k \leq n+1$, together with adjunctions

$$\begin{tikzcd} F_k : \mathrm{Comp}_k^T \arrow[r, shift left] & \mathrm{Alg}_T : V_k. \arrow[l, shift left] \end{tikzcd}$$

We also write F_k^T and V_k^T if we need to be clear about which monad is being used.

When $T = T_n$ a k-computad is a presentation with no cells in dimensions larger than k. The functor F_k takes a k-computad and produces the n-category generated by it. The functor V_k takes an n-category $\mathcal{C}$ and produces a k-computad whose k-cells are all k-morphisms in $\mathcal{C}$ together with choices of ways of writing down their source and target as composites of $(k-1)$-morphisms. An $(n+1)$-cell in $V_{n+1}(\mathcal{C})$ consists of a choice of an n-morphism f in $\mathcal{C}$ together with a choice of two n-morphisms in $F_n(V_n(\mathcal{C}))$ whose image in $\mathcal{C}$ is f. See [12] for precise definitions.

We are only interested in n-categories presented by $(n+1)$-computads, as this allows for generating cells in every dimension as well as relations between composites of n-cells, so we make the following definitions.

Definition 2.2.1. *An n-presentation is an $(n+1)$-computad with respect to the monad T_n. Given an n-presentation $\mathcal{P}$ we denote by $\mathcal{F}_n(\mathcal{P}) := F_{n+1}^{T_n}(\mathcal{P})$ the n-category generated by it. We omit n when it is clear from context, and write $\mathcal{F}(\mathcal{P})$.*

Definition 2.2.2. *A monoidal n-presentation is an $(n+1)$-computad with respect to the monad $T_n^{\otimes}$. Given a monoidal n-presentation $\mathcal{P}$ we denote $\mathcal{F}_n^{\otimes}(\mathcal{P}) := F_{n+1}^{T_n^{\otimes}}(\mathcal{P})$. We omit n when it is clear from context, and write $\mathcal{F}^{\otimes}(\mathcal{P})$.*

Definition 2.2.3. *A symmetric monoidal n-presentation is an $(n+1)$-computad with respect to the monad T_n^{sym}. Given a symmetric monoidal n-presentation $\mathcal{P}$ we denote $\mathcal{F}_n^{\mathrm{sym}}(\mathcal{P}) := F_{n+1}^{T_n^{sym}}(\mathcal{P})$. We omit n when it is clear from context, and write $\mathcal{F}^{\mathrm{sym}}(\mathcal{P})$.*

Given a 3-presentation $\mathcal{P}$ and a 3-category $\mathcal{C}$, the adjunction $F_k \dashv V_k$ makes it possible to give an explicit description of the 3-category $\mathrm{Fun}(\mathcal{F}(\mathcal{P}), \mathcal{C})$ in terms of data associated to the cells in $\mathcal{P}$.

Definition 2.2.4. *Given a presentation $\mathcal{P}$ and a 3-category $\mathcal{C}$, we define a 3-category* $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$ *as follows.*

0. *An object F is a choice of k-morphism $F(x)$ in $\mathcal{C}$ for each k-cell x in $\mathcal{P}$, for $0 \leq k \leq 3$, subject to source and target compatibility and satisfying appropriate relations between 3-morphisms, given by requiring that the image in $\mathcal{C}$ of the source of each 4-cell in $\mathcal{P}$ is equal to the image of its target.*

1. *A 1-morphism $\alpha : F \to G$ consists of the following data.*

   For each 0-cell X in $\mathcal{P}$ a 1-morphism $\alpha_X : F(X) \to G(X)$.

   For each 1-cell $f : X \to Y$ in $\mathcal{P}$ an invertible 2-morphism

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\end{tikzcd}$$

   For each 2-cell

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \eta"] & Y
\end{tikzcd}$$

   in $\mathcal{P}$ an invertible 3-morphism

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"] \arrow[d, bend right=70, "F(f)"'] \arrow[d, phantom, shift right=4ex, "\overset{F(\eta)}{\Rightarrow}"] & G(X) \arrow[d, "G(g)"] & {} \arrow[d, phantom, "\overset{\alpha_\eta}{\Rrightarrow}"] & F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'] \arrow[d, bend left=70, "G(g)"] \arrow[d, phantom, shift left=4ex, "\overset{G(\eta)}{\Rightarrow}"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y) & {} & F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\end{tikzcd}$$

For each 3*-cell*

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, phantom, "\overset{t}{\Rrightarrow}"] & X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & Y
\end{tikzcd}$$

in $\mathcal{P}$ *a commuting diagram of* 3*-morphisms*

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"] \arrow[d, bend right=70, "F(f)"'] \arrow[d, phantom, shift right=4ex, "\overset{F(\eta)}{\Rightarrow}"] & G(X) \arrow[d, "G(g)"] & {} \arrow[d, phantom, "\overset{\alpha_\eta}{\Rrightarrow}"] & F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'] \arrow[d, bend left=70, "G(g)"] \arrow[d, phantom, shift left=4ex, "\overset{G(\eta)}{\Rightarrow}"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] \arrow[dd, Rightarrow, "F(t)"'] & G(Y) & {} & F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] \arrow[dd, Rightarrow, "G(t)"] & G(Y) \\
{} & & & {} & \\
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(g)"] \arrow[d, bend right=70, "F(f)"'] \arrow[d, phantom, shift right=4ex, "\overset{F(\theta)}{\Rightarrow}"] & G(X) \arrow[d, "G(g)"] & {} \arrow[d, phantom, "\overset{\alpha_\theta}{\Rrightarrow}"] & F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'] \arrow[d, bend left=70, "G(g)"] \arrow[d, phantom, shift left=4ex, "\overset{G(\theta)}{\Rightarrow}"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_g"'] & G(Y) & {} & F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"] & G(Y)
\end{tikzcd}$$

2. *Given* 1*-morphisms* $\alpha, \beta : F \to G$*, a* 2*-morphism* $m : \alpha \Rightarrow \beta$ *consists of the following data.*

   For each 0*-cell* X *in* $\mathcal{P}$*, a* 2*-morphism*

$$\begin{tikzcd}
F(X) \arrow[r, bend left=50, "\alpha_X"] \arrow[r, bend right=50, "\beta_X"'] \arrow[r, phantom, "\Downarrow m_X"] & G(X)
\end{tikzcd}$$

   For each 1*-cell* $f : X \to Y$ *in* $\mathcal{P}$ *an invertible* 3*-morphism*

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"'] \arrow[r, bend left=70, "\beta_X"] \arrow[r, phantom, shift left=3ex, "\Uparrow m_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] & {} \arrow[d, phantom, "\overset{m_f}{\Rrightarrow}"] & F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] & G(Y) & {} & {F(Y),} \arrow[r, "\beta_Y"] \arrow[r, bend right=70, "\alpha_Y"'] \arrow[r, phantom, shift right=3ex, "\Uparrow m_Y"] \arrow[ur, Rightarrow, "\beta_f"] & G(Y)
\end{tikzcd}$$

For each 2*-cell*

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D}] & Y
\arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

a commutative diagram of 3*-morphisms*

$$\begin{array}{ccc}
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=AX}] \arrow[r, bend left=70, "\beta_X", ""{name=BX, below}] \arrow[d, "F(g)", ""{name=FG, left}] \arrow[d, bend right=70, "F(f)"', ""{name=FF, right}] & G(X) \arrow[d, "G(g)"] \\
F(Y) \arrow[r, "\alpha_Y"'] & G(Y)
\arrow[Rightarrow, from=AX, to=BX, "m_X"]
\arrow[Rightarrow, from=FF, to=FG, "F(\eta)"]
\arrow[Rightarrow, from=2-1, to=1-2, shorten=12pt, "\alpha_g"']
\end{tikzcd}
& \overset{m_g}{\Rrightarrow} &
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(g)", ""{name=FG, left}] \arrow[d, bend right=70, "F(f)"', ""{name=FF, right}] & G(X) \arrow[d, "G(g)"] \\
F(Y), \arrow[r, "\beta_Y", ""{name=BY, below}] \arrow[r, bend right=70, "\alpha_Y"', ""{name=AY}] & G(Y)
\arrow[Rightarrow, from=AY, to=BY, "m_Y"]
\arrow[Rightarrow, from=FF, to=FG, "F(\eta)"]
\arrow[Rightarrow, from=2-1, to=1-2, shorten=12pt, "\beta_g"]
\end{tikzcd}
\\
\alpha_\eta \Downarrow & & \Downarrow \beta_\eta
\\
\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=AX}] \arrow[r, bend left=70, "\beta_X", ""{name=BX, below}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=GF, right}] \arrow[d, bend left=70, "G(g)", ""{name=GG, left}] \\
F(Y) \arrow[r, "\alpha_Y"'] & G(Y)
\arrow[Rightarrow, from=AX, to=BX, "m_X"]
\arrow[Rightarrow, from=GF, to=GG, "G(\eta)"]
\arrow[Rightarrow, from=2-1, to=1-2, shorten=12pt, "\alpha_f"]
\end{tikzcd}
& \underset{m_f}{\Rrightarrow} &
\begin{tikzcd}
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"', ""{name=GF, right}] \arrow[d, bend left=70, "G(g)", ""{name=GG, left}] \\
F(Y), \arrow[r, "\beta_Y", ""{name=BY, below}] \arrow[r, bend right=70, "\alpha_Y"', ""{name=AY}] & G(Y)
\arrow[Rightarrow, from=AY, to=BY, "m_Y"]
\arrow[Rightarrow, from=GF, to=GG, "G(\eta)"]
\arrow[Rightarrow, from=2-1, to=1-2, shorten=12pt, "\beta_f"]
\end{tikzcd}
\end{array}$$

3. *Given* 1*-morphisms* $\alpha, \beta : F \to G$ *and* 2*-morphisms* $l, m : \alpha \Rightarrow \beta$*, a* 3*-morphism* $\mathcal{A} : l \Rrightarrow m$ *consists of the following data.*

   For each 0*-cell* X *in* $\mathcal{P}$*, a* 3*-morphism* $\mathcal{A}_X : l_X \Rrightarrow m_X$*.*

   For each 1*-cell* $f : X \to Y$ *in* $\mathcal{P}$*, a commuting diagram of* 3*-morphisms*

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"', ""{name=AX}] \arrow[r, bend left=50, "\beta_X", ""{name=BX, below}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] & & F(X) \arrow[r, "\alpha_X"', ""{name=AX2}] \arrow[r, bend left=50, "\beta_X", ""{name=BX2, below}] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] & G(Y) & & F(Y) \arrow[r, "\alpha_Y"'] \arrow[ur, Rightarrow, "\alpha_f"'] & G(Y) \\
 & & & & \\
F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)\ e"] & & F(X) \arrow[r, "\beta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y), \arrow[r, "\beta_Y", ""{name=BY, below}] \arrow[r, bend right=50, "\alpha_Y"', ""{name=AY}] \arrow[ur, Rightarrow, "\beta_f"] & G(Y) & & F(Y), \arrow[r, "\beta_Y", ""{name=BY2, below}] \arrow[r, bend right=50, "\alpha_Y"', ""{name=AY2}] \arrow[ur, Rightarrow, "\beta_f"] & G(Y)
\arrow[Rightarrow, from=AX, to=BX, "l_X"]
\arrow[Rightarrow, from=AX2, to=BX2, "m_X"]
\arrow[Rightarrow, from=AY, to=BY, "l_Y"]
\arrow[Rightarrow, from=AY2, to=BY2, "m_Y"]
\arrow[from=1-3, to=2-3, phantom, "\overset{\mathcal{A}_X}{\Rrightarrow}" description]
\arrow[from=4-3, to=5-3, phantom, "\overset{\mathcal{A}_Y}{\Rrightarrow}" description]
\arrow[from=2-1, to=4-2, phantom, "l_f\ \rotatebox{-90}{$\Rrightarrow$}" description]
\arrow[from=2-4, to=4-5, phantom, "\rotatebox{-90}{$\Rrightarrow$}\ m_f" description]
\end{tikzcd}$$

Lemma 2.2.5. *Let $\mathcal{P}$ be a 3-presentation, $\mathcal{C}$ a 3-category and $1 \leq k \leq 3$. Then an object in* $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$ *is the same thing as an element of*

$$\mathrm{Hom}_{Comp_4^{T_3}}(\mathcal{P}, V_4(\mathcal{C}))$$

and a k-morphism in $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$ *is the same thing as an element of*

$$\mathrm{Hom}_{Comp_4^{T_3}}(\mathcal{P}, V_4(\overline{\mathrm{Fun}}(C_k, \mathcal{C}))).$$

Proof. One just unravels the definitions. □

Proposition 2.2.6. *Let $\mathcal{P}$ be a 3-presentation and $\mathcal{C}$ a 3-category. Then the canonical functor* $\mathrm{Fun}(\mathcal{F}(\mathcal{P}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{P}, \mathcal{C})$ *is an isomorphism of 3-categories.*

Proof. We identify k-morphisms in $\mathrm{Fun}(\mathcal{P}, \mathcal{C})$ with elements of

$$\mathrm{Hom}_{Comp_4^{T_3}}(\mathcal{P}, V_4(\overline{\mathrm{Fun}}(C_k, \mathcal{C})))$$

and k-morphisms in $\mathrm{Fun}(\mathcal{F}(\mathcal{P}), \mathcal{C})$ with elements of

$$\mathrm{Hom}_{\mathrm{Alg}_{T_3}}(\mathcal{F}(\mathcal{P}), \overline{\mathrm{Fun}}(C_k, \mathcal{C})).$$

By the adjunction $F_4 \dashv V_4$, we have

$$\mathrm{Hom}_{\mathrm{Alg}_{T_3}}(\mathcal{F}(\mathcal{P}), \overline{\mathrm{Fun}}(C_k, \mathcal{C})) = \mathrm{Hom}_{Comp_4^{T_3}}(\mathcal{P}, V_4(\overline{\mathrm{Fun}}(C_k, \mathcal{C}))),$$

which is what's required.

□

Now we study the relation between algebras and computads over the monads T_3, $T_3^{\otimes}$ and T_3^{sym}. There are maps of monads $T_3 \to T_3^{\otimes} \to T_3^{sym}$. The first one is given by inclusion of the $k = 0$ term in

$$\coprod_{k \geq 0} T_3(X)^{\times k}$$

and the second one is given by the quotient maps

$$T_3(X)^{\times k} \to T_3(X)^{\times k}/\Sigma_k.$$

These induce functors $\mathrm{Alg}_{T_3^{sym}} \to \mathrm{Alg}_{T_3^{\otimes}} \to \mathrm{Alg}_{T_3}$. The first functor takes a symmetric monoidal 3-category and forgets the fact that it is symmetric. The second functor takes a monoidal 3-category and forgets the monoidal structure. There are also induced functors $\mathrm{Comp}_k^{T_3} \to \mathrm{Comp}_k^{T_3^{\otimes}} \to \mathrm{Comp}_k^{T_3^{sym}}$. Given a k-computad D with respect to T_3, we denote the corresponding computads over $T_3^{\otimes}$ and T_3^{sym} by D as well. In general, given finitary monads T, T' and a map $f : T \to T'$, we denote the induced functors by $f^* : \mathrm{Alg}_{T'} \to \mathrm{Alg}_T$ and $f_* : \mathrm{Comp}_k^T \to \mathrm{Comp}_k^{T'}$. As proved in Schommer-Pries [12], there is an adjunction

$$F_k^{T'} \circ f_* \dashv V_k^T \circ f^*.$$

So in particular, given a monoidal 3-category $\mathcal{C}$, and a 3-presentation $\mathcal{P}$ (i.e. a 4-computad with respect to T_3), we have

$$\mathrm{Hom}_{\mathrm{Alg}_{T_3^{\otimes}}}(\mathcal{F}^{\otimes}(\mathcal{P}), \mathcal{C}) = \mathrm{Hom}_{\mathrm{Comp}_4^{T_3}}(\mathcal{P}, V_4^{T_3}(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Alg}_{T_3}}(\mathcal{F}(\mathcal{P}), \mathcal{C}).$$

This means a monoidal functor out of $\mathcal{F}^{\otimes}(\mathcal{P})$ is the same thing as a functor out of $\mathcal{F}(\mathcal{P})$. By applying the same procedure with $\mathcal{C}$ replaced by $\overline{\mathrm{Fun}}(C_k, \mathcal{C})$, we get the same statement for transformations, modifications and perturbations. In the end, we get the following result.

Proposition 2.2.7. *Let $\mathcal{P}$ be a 3-presentation and $\mathcal{C}$ a monoidal 3-category. Then the canonical functor $\mathrm{Fun}^{\otimes}(\mathcal{F}^{\otimes}(\mathcal{P}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{F}(\mathcal{P}), \mathcal{C})$ is an isomorphism of 3-categories.*

Definition 2.2.8. *Let $\mathcal{C}, \mathcal{D}$ be 3-categories and $\mathcal{P}$ a presentation. We denote by $\mathrm{Map}(\mathcal{P}, \mathcal{D})$ the maximal 3-subgroupoid of $\mathrm{Fun}(\mathcal{P}, \mathcal{D})$ and by $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ the maximal 3-subgroupoid of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

Definition 2.2.9. *Let $\mathcal{C}, \mathcal{D}$ be monoidal 3-categories. We denote by $\mathrm{Map}^{\otimes}(\mathcal{C}, \mathcal{D})$ the maximal 3-subgroupoid of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.*

Now we discuss how it is possible to glue presentations together, in certain particular situations.

Definition 2.2.10. *Let $\mathcal{P}$ be a presentation. We say that every 0-cell in $\mathcal{P}$ is simple. We say that a k-cell is simple, if its source and target are in $\mathcal{P}$ and are distinct and simple.*

Definition 2.2.11. *Let $\mathcal{P}$ be a presentation. The span $\langle f \rangle$ of a simple k-cell f in $\mathcal{P}$ is defined inductively as follows. Given a 0-cell X, we write $\langle X \rangle := \{X\}$. Given a simple k-cell $f : X \to Y$, we write*

$$\langle f \rangle = \{f\} \cup \langle X \rangle \cup \langle Y \rangle .$$

We denote by $\langle f \rangle_k$ the set of k-cells in $\langle f \rangle$.

Definition 2.2.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations. Suppose f is a k-cell in $\mathcal{P}$ and g is a simple k-cell in $\mathcal{Q}$. We define a new presentation $\mathcal{M} = \mathcal{P} *_{f \sim g} \mathcal{Q}$ as follows. First define $\mathcal{M}_k = \mathcal{P}_k \coprod (\mathcal{Q}_k \setminus \langle g \rangle_k)$. Then, for any cell in $\mathcal{Q} \setminus \langle g \rangle$, we define its new source and target by replacing any occurence of a cell in $\langle g \rangle$, which is obtained by applying a sequence of source and target maps to g, by the element obtained by applying the same sequence to f.*

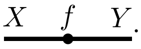
Remark 2.2.13. *Note that in the above definition, if one chooses different sequences α and β of letters in $\{s, t\}$ such that $\alpha(g) = \beta(g)$, then $\alpha(f) = \beta(f)$, so the definition makes sense. This follows from the assumption that g is simple.*

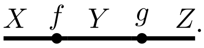
2.3 Graphical notation

In [2] the authors develop a theory of quasistrict n-categories generated by certain structures which the author calls *signatures* (for $1 \leq n \leq 4$). An m-signature consists of sets of cells in dimensions $0 \leq k \leq m$, with each k-cell having source and target $(k-1)$-diagrams in this signature. Intuitively, a k-diagram is a composite of k-cells in the signature using the usual operations in a k-category.

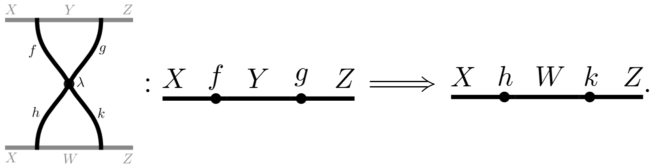
In the end, the n-globular set generated by an $(n+1)$-signature has as k-morphisms the k-diagrams in the signature, for $k \leq n-1$. For $k = n$ one takes the quotient of the set of n-diagrams by the equivalence relation induced by declaring the source of each $(n+1)$-cell to be equivalent to its target. If the signature includes certain standard cells called *homotopy generators*, the resulting globular set is called a quasistrict n-category. These categories have strictly associative and unital composition operations.

However, the interchange laws hold only up to coherent isomorphism, implemented by these homotopy generators. See [2] for the details.

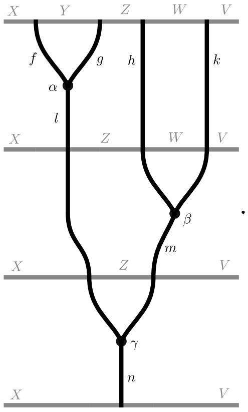
These categories are convenient for computations, since the diagrams which represent their morphisms have useful graphical depictions. A 1-cell $X\xrightarrow{f}Y$ is denoted by . Composing 1-cells, one forms 1-diagrams, which look like



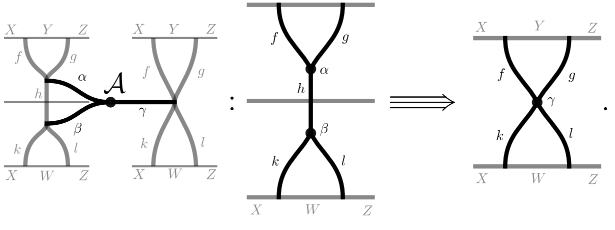
A 2-cell $g\circ f\xRightarrow{\lambda}k\circ h$ looks like



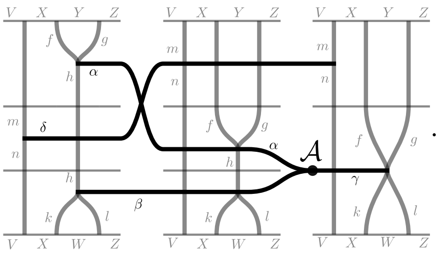
Composing 2-cells, one forms 2-diagrams, which look like



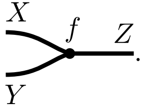
An example of a 3-cell is



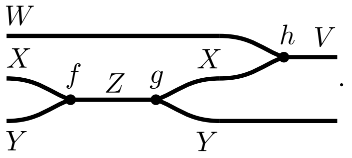
Composing 3-cells, one forms 3-diagrams, which look like



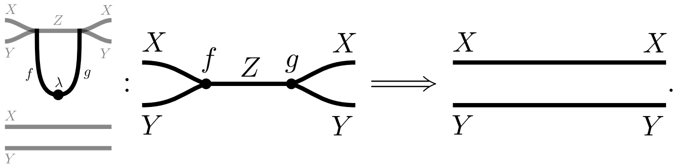
Note that this 3-diagram includes a type I_2 homotopy generator, interchanging the 2-cells α and δ (see [2]).

We model monoidal 3-categories as 4-categories with only one object. A 1-cell $X \otimes Y \xrightarrow{f} Z$ is denoted by 

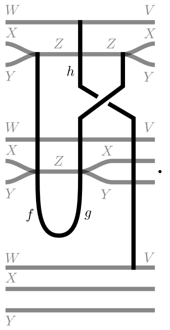
. An example of a 1-diagram is



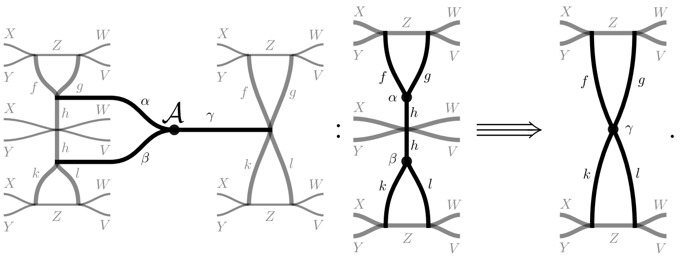
A 2-cell $g \circ f \xRightarrow{\lambda} \mathrm{Id}_{X\otimes Y}$ looks like



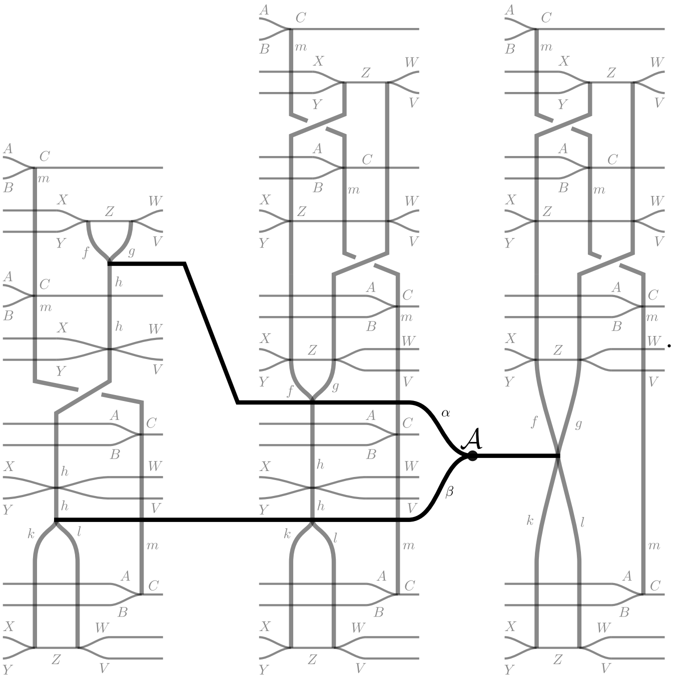
An example of a 2-diagram looks like



An example of a 3-cell is



Composing 3-cells, one forms 3-diagrams, which look like



Note that this 3-diagram includes a type II_3 homotopy generator, which moves the 2-cell α past the interchanger between 1-cells m and h (see [2]).

It turns out that there are monads T_n^{qs} whose k-computads are precisely the k-signatures for $0 \le k \le n+1$ and $1 \le n \le 4$. We call their algebras quasistrict n-categories. The quasistrict n-categories discussed in [2] are the ones generated by $(n+1)$-computads. The homotopy generators are already built into the monad T_n^{qs}. These quasistrict n-categories are certain kinds of weak n-categories where all composition operations are strictly associative and unital, but where the interchange laws are implemented by coherent isomorphism.

Consider the n-signature Σ_n having only one nontrivial k-cell x_k in each dimension, with source and target x_{k-1}, plus the associated homotopy generators. So a diagram over Σ_n is just a diagram with no labels and where each cell only has one input and one output wire. Given a k-diagram D over Σ_n and an n-globular set X, define $\mathrm{Hom}(D, X)$ to be the set of labellings of D where each occurrence of the k-cell x_k is labelled by a k-cell in X, in a way that respects source and target maps. Note that it is possible to define an n-globular set $\hat{D}$ with no cells in dimensions $> k$ such that $\mathrm{Hom}(D, X) = \mathrm{Hom}_{\mathrm{gSet}_n}(\hat{D}, X)$. Let $\mathrm{Diag}_k(\Sigma_n)$ be the set of k-diagrams over Σ_n. Now define

$$T_n^{qs}(X) = \coprod_{D \in \mathrm{Diag}_k(\Sigma_n)} \mathrm{Hom}(D, X).$$

The monad structure is just given by taking a diagram of diagrams over Σ_n and recognising it as a diagram over Σ_n.

There is a map $\phi : T_n^{qs} \to T_n$ given by associating to a diagram over Σ_n the corresponding pasting diagram. This induces a map

$$\phi^* : \mathrm{Alg}_{T_n} \to \mathrm{Alg}_{T_n^{qs}}$$

viewing a strict n-category as a quasistrict category, where the interchangers happen to be identities, and a map

$$\phi_* : \mathrm{Comp}_k^{T_n^{qs}} \to \mathrm{Comp}_k^{T_n} .$$

Recall that we have adjunctions $F_k^{T_n} \circ \phi_* \dashv V_k^{T_n^{qs}} \circ \phi^*$ and $F_k^{T_n^{qs}} \dashv V_k^{T_n^{qs}}$. So given an $(n+1)$-computad C with respect to T_n^{qs} and a strict n-category $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathrm{Alg}_{T_n}}(\mathcal{F}(\phi_*(C)), \mathcal{C}) = \mathrm{Hom}_{\mathrm{Comp}_{n+1}^{T_n^{qs}}}(C, V_{n+1}^{T_n^{qs}}(\phi^*(\mathcal{C}))) = \mathrm{Hom}_{\mathrm{Alg}_{T_n^{qs}}}(\mathcal{F}^{qs}(C), \phi^*(\mathcal{C})).$$

Now we explain how one can use this to do computations in strict n-categories (for $1 \leq n \leq 4$). Suppose we have a finite collection of morphisms in a strict n-category $\mathcal{C}$, which satisfy certain equations and suppose this data can be arranged into a computad C with respect to T_n^{qs}, together with a map of strict n-categories $F_{n+1}^{T_n}((\phi_n)_*(C)) \to \mathcal{C}$, or equivalently $(\phi_n)_*(C) \to V_{n+1}^{T_n}(\mathcal{C})$. Then we can use the graphical notation in $F_{n+1}^{T_n^{qs}}(C)$ to specify new morphisms as diagrams and to establish relations between n-morphisms. By the above argument using the adjunctions, the map $F_{n+1}^{T_n}((\phi_n)_*(C)) \to \mathcal{C}$ induces a map $F_{n+1}^{T_n^{qs}}(C) \to \phi_n^*(\mathcal{C}) = \mathcal{C}$ so we can transport the newly defined morphisms and relations between them back into $\mathcal{C}$.

Chapter 3

Coherence for adjoints and duals

In this chapter we prove coherence results for adjoint 1-morphisms and adjoint 2-morphisms in a 3-category, and duals in a monoidal 3-category. These will be used in the following chapter to prove a coherence result for fully dualizable objects in a symmetric monoidal 3-category.

Before we proceed, we need to fix some definitions.

Definition 3.0.1. *Let $\mathcal{C}$ be an n-category and let $f : X \to Y$ and $g : Y \to X$ be k-morphisms. If $k = n$ then we say that g is inverse to f if $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In this case we say that f is an isomorphism and write $X \simeq Y$. If $1 \leq k \leq n - 1$ then we say that g is inverse to f if $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Again we say that f is an isomorphism and write $x \simeq Y$.*

Definition 3.0.2. *Let $\mathcal{C}$ be an n-category and let $f : X \to Y$ and $g : Y \to X$ be k-morphisms. We say that f is left adjoint to g, and write $f \dashv g$, when there exist unit and counit $(k+1)$-morphisms u and c satisfying the snake equations, up to isomorphism.*

3.1 Statement of results

In the following sections we will define presentations $\mathrm{Adj}_{(3,1)}$, $\mathrm{Adj}_{(3,2)}$ and Dual_3, which consist of coherence data for adjunctions of 1-morphisms in 3-category, adjunctions of 2-morphisms in a 3-category and duals in a monoidal 3-category respectively, and we will prove the corresponding coherence statements, which we now state.

Definition 3.1.1. *The presentation* Arr_1 *consists of*

0. *two objects A and B;*

1. *a* 1*-morphism* $f : A \to B$

and no higher morphisms or relations.

Given a 3-category $\mathcal{C}$, we denote by $\mathrm{Arr}_1(\mathcal{C})$ the 3-groupoid of 1-morphisms of $\mathcal{C}$, defined as

$$\mathrm{Arr}_1(\mathcal{C}) := \mathrm{Map}(\mathrm{Arr}_1, \mathcal{C})$$

and we denote by $\mathrm{Arr}_1^L(\mathcal{C})$ (respectively $\mathrm{Arr}_1^R(\mathcal{C})$) the full subgroupoid whose objects are left (respectively right) adjoint 1-morphisms.

The presentation $\mathrm{Adj}_{(3,1)}$ will have two 1-cells l and r and so we have two restriction functors

$$E_l : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1(\mathcal{C})$$

and

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1(\mathcal{C}).$$

Proposition 3.1.2. *The restriction functor* $E_l : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1(\mathcal{C})$ *factors through* $\mathrm{Arr}_1^L(\mathcal{C})$ *and the induced functor*

$$E_l : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^L(\mathcal{C})$$

is an equivalence.

Proposition 3.1.3. *The restriction functor* $E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1(\mathcal{C})$ *factors through* $\mathrm{Arr}_1^R(\mathcal{C})$ *and the induced functor*

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

is an equivalence.

So we can think of the images of the cells in $\mathrm{Adj}_{(3,1)}$ under a functor

$$F : \mathrm{Adj}_{(3,1)} \to \mathcal{C}$$

as coherence data for the adjunction $F(l) \dashv F(r)$ and we refer to such a functor as a *coherent adjunction.*

There is an analogous coeherence statement for adjunctions between 2-morphisms.

Definition 3.1.4. *The presentation* Arr_2 *consists of*

0. *Objects* X *and* Y*;*

1. 1*-morphisms* $x, y : X \to Y$*;*

2. a 2*-morphism* $\phi : x \Longrightarrow y$

and no higher morphisms or relations.

Given a 3-category $\mathcal{C}$, we denote by $\mathrm{Arr}_2(\mathcal{C})$ the 3-groupoid of 2-morphisms of $\mathcal{C}$, defined as

$$\mathrm{Arr}_2(\mathcal{C}) := \mathrm{Map}(\mathrm{Arr}_2, \mathcal{C})$$

and we denote by $\mathrm{Arr}_2^L(\mathcal{C})$ (respectively $\mathrm{Arr}_2^R(\mathcal{C})$) the full subgroupoid whose objects are left (respectively right) adjoint 2-morphisms.

The presentation $\mathrm{Adj}_{(3,2)}$ will have two 2-cells λ and ρ and so we have two restriction functors

$$E_\lambda : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2(\mathcal{C})$$

and

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2(\mathcal{C}).$$

Proposition 3.1.5. *The restriction functor* $E_\lambda : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2(\mathcal{C})$ *factors through* $\mathrm{Arr}_2^L(\mathcal{C})$ *and the induced functor*

$$E_\lambda : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^L(\mathcal{C})$$

is an equivalence.

Proposition 3.1.6. *The restriction functor* $E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2(\mathcal{C})$ *factors through* $\mathrm{Arr}_2^R(\mathcal{C})$ *and the induced functor*

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

is an equivalence.

There is also a statement about coherence for duals in a monoidal 3-category.

Definition 3.1.7. *The presentation* Obj *has one* 0*-cell and no cells in higher dimensions.*

Given a monoidal 3-category $\mathcal{C}$, we denote by $\mathrm{Obj}(\mathcal{C})$ the 3-groupoid of objects of $\mathcal{C}$, defined as

$$\mathrm{Obj}(\mathcal{C}) := \mathrm{Map}(\mathrm{Obj}, \mathcal{C})$$

and we denote by $\mathrm{Obj}^L(\mathcal{C})$ (respectively $\mathrm{Obj}^R(\mathcal{C})$) the full subgroupoid whose objects are left (respectively right) duals in $\mathcal{C}$.

The presentation Dual_3 will have two 0-cells X and Y and so we have two restriction functors

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$$

and

$$E_Y : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}.$$

Proposition 3.1.8. *The restriction functor* $E_Y : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ *factors through* $\mathrm{Obj}^L(\mathcal{C})$ *and the induced functor*

$$E_Y : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^L(\mathcal{C})$$

is an equivalence.

Proposition 3.1.9. *The restriction functor* $E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ *factors through* $\mathrm{Obj}^R(\mathcal{C})$ *and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

is an equivalence.

Note that, in particular, we have $\mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) = \mathrm{Map}(\mathrm{Dual}_3, \mathcal{C})$, which is to say all natural transformations, modifications and perturbations are invertible.

Finally, all of these results have relative versions, which we will use in the proof of coherence for fully dualizable objects.

Definition 3.1.10. *Given a presentation* $\mathcal{P}$ *and a* k*-cell* $f \in \mathcal{P}$*,* $1 \leq k \leq 2$*, we denote by* $\mathrm{Map}^{f\dashv}(\mathcal{P}, \mathcal{C})$ *the full* 3*-subgroupoid of* $\mathrm{Map}(\mathcal{P}, \mathcal{C})$ *whose objects are those functors sending* f *to a left adjoint* k*-morphism in* $\mathcal{C}$*.*

Similarly, we denote by $\mathrm{Map}^{\dashv f}(\mathcal{P}, \mathcal{C})$ *the full* 3*-subgroupoid of* $\mathrm{Map}(\mathcal{P}, \mathcal{C})$ *whose objects are those functors sending* f *to a right adjoint* k*-morphism in* $\mathcal{C}$*.*

Proposition 3.1.11. *Let* $\mathcal{P}$ *be a presentation and* $f \in \mathcal{P}_1$ *a* 1*-cell. Then the restriction functor*

$$E_l : \mathrm{Map}(\mathcal{P} *_{f\sim l} \mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Map}(\mathcal{P}, \mathcal{C})$$

factors through $\mathrm{Map}^{f\dashv}(\mathcal{P}, \mathcal{C})$ *and the induced functor*

$$E_l : \mathrm{Map}(\mathcal{P} *_{f\sim l} \mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Map}^{f\dashv}(\mathcal{P}, \mathcal{C})$$

is an equivalence.

Proposition 3.1.12. *Let $\mathcal{P}$ be a presentation and $f \in \mathcal{P}_1$ a 1-cell. Then the restriction functor*

$$E_r : \mathrm{Map}(\mathcal{P} *_{f\sim r} \mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Map}(\mathcal{P}, \mathcal{C})$$

factors through $\mathrm{Map}^{\dashv f}(\mathcal{P}, \mathcal{C})$ *and the induced functor*

$$E_r : \mathrm{Map}(\mathcal{P} *_{f\sim r} \mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Map}^{\dashv f}(\mathcal{P}, \mathcal{C})$$

is an equivalence.

Proposition 3.1.13. *Let $\mathcal{P}$ be a presentation and $\phi \in \mathcal{P}_2$ a 2-cell. Then the restriction functor*

$$E_\lambda : \mathrm{Map}(\mathcal{P} *_{\phi\sim\lambda} \mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Map}(\mathcal{P}, \mathcal{C})$$

factors through $\mathrm{Map}^{\phi\dashv}(\mathcal{P}, \mathcal{C})$ *and the induced functor*

$$E_\lambda : \mathrm{Map}(\mathcal{P} *_{\phi\sim\lambda} \mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Map}^{\phi\dashv}(\mathcal{P}, \mathcal{C})$$

is an equivalence.

Proposition 3.1.14. *Let $\mathcal{P}$ be a presentation and $\phi \in \mathcal{P}_2$ a 2-cell. Then the restriction functor*

$$E_\rho : \mathrm{Map}(\mathcal{P} *_{\phi\sim\rho} \mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Map}(\mathcal{P}, \mathcal{C})$$

factors through $\mathrm{Map}^{\dashv\phi}(\mathcal{P}, \mathcal{C})$ *and the induced functor*

$$E_\rho : \mathrm{Map}(\mathcal{P} *_{\phi\sim\rho} \mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Map}^{\dashv\phi}(\mathcal{P}, \mathcal{C})$$

is an equivalence.

Recall from Definition 2.1.13 that a strict functor between 3-categories is said to be an equivalence when it is essentially surjective on objects, 1-morphisms and 2-morphisms and bijective on 3-morphisms. We will actually prove that all the functors in question are surjective on objects, 1-morphisms and 2-morphisms and bijective on 3-morphisms.

3.1.1 Related work

Pstrągowski proved similar coherence results in the context of (monoidal) bicategories in [10]. Since here we are dealing with (monoidal) 3-categories, our presentations will include one extra layer of coherence data. Higher coherence data for adjunctions has also appeared in the work of Gurski [6]. In [11] Riehl and Verity construct an $(\infty, 2)$-category $\underline{\mathrm{Adj}}$ with the property that, given any $(\infty, 2)$-category $\mathcal{C}$, functors $\underline{\mathrm{Adj}} \to \mathcal{C}$

classify adjunctions between 1-morphisms in $\mathcal{C}$. The $(3,2)$-category $\mathcal{F}(\mathrm{Adj}_{(3,1)})$ should therefore be a strict version of a 3-truncation of $\underline{\mathrm{Adj}}$. From this perspective, finding the presentation $\mathrm{Adj}_{(3,1)}$ can be thought of as being analogous to finding a finite cell structure for a truncation of space defined by a universal property. It would be interesting to see whether one can find presentations $\mathrm{Adj}_{(n,1)}$ by understanding the truncations of $\underline{\mathrm{Adj}}$.

3.2 Coherence for adjunctions

In this section we prove the coherence statement for adjunctions between 1-morphisms and 2-morphisms in a 3-category. We prove only Propositions 3.1.3 and 3.1.6, the proof of 3.1.2 and 3.1.5 being analogous. The relative versions 3.1.11, 3.1.12, 3.1.13, 3.1.14 also have the same proof.

3.2.1 The presentations $\mathrm{Adj}_{(3,1)}$ and $\mathrm{Adj}_{(3,2)}$

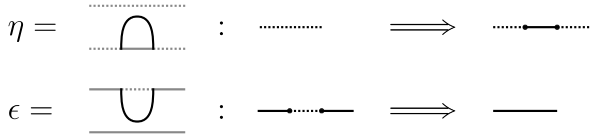
We now describe the presentations $\mathrm{Adj}_{(3,1)}$ and $\mathrm{Adj}_{(3,2)}$, which consist of coherence data for adjunctions between 1-morphisms and 2-morphisms in a 3-category.

Definition 3.2.1. *The presentation* $\mathrm{Adj}_{(3,1)}$ *consists of*

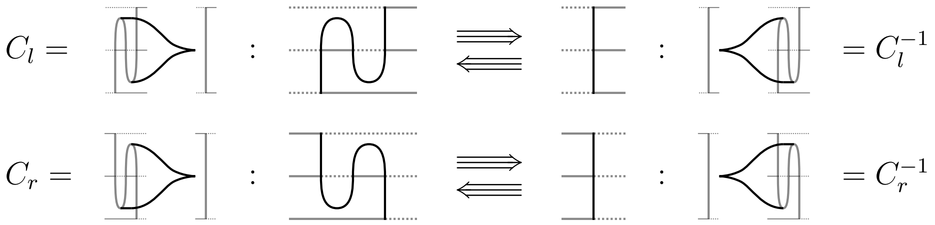
0. objects $X =$  *and* $Y =$ 

1. 1*-morphisms* $l =$  $: X \rightleftarrows Y :$  $= r$

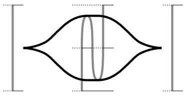
2. 2*-morphisms*

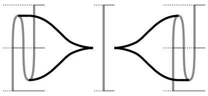
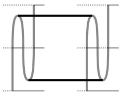


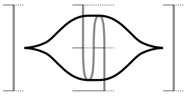
3. 3*-morphisms*

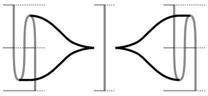
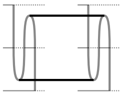


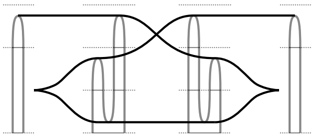
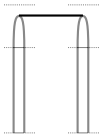
4. *relations*

 $=$ Id_l

 $=$ 

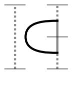
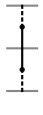
 $=$ Id_r

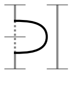
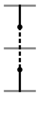
 $=$ 

 $=$ 

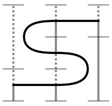
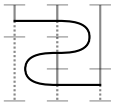
Definition 3.2.2. *The presentation* $\mathrm{Adj}_{(3,2)}$ *consists of*

0. *Objects* X *and* Y
1. 1-*morphisms* $x, y : X \to Y$
2. 2-*morphisms* $\lambda =$  $: x \mathrel{\substack{\Longrightarrow \\ \Longleftarrow}} y :$  $= \rho$
3. 3-*morphisms*

$u =$  $:$  $\Longrightarrow$ 

$v =$  $:$  $\Longrightarrow$ 

4. relations

 $=\ \mathrm{Id}_\lambda$ *and*  $=\ \mathrm{Id}_\rho$

Lemma 3.2.3. *The restriction functor* $E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1(\mathcal{C})$ *factors through* $\mathrm{Arr}_1^R(\mathcal{C})$*.*

Proof. We have to show that, given a functor $F : \mathcal{F}(\mathrm{Adj}_{(3,1)}) \to \mathcal{C}$, the 1-morphism $F(r)$ is a right adjoint. This is clear, because $l \dashv r$ in $\mathcal{F}(\mathrm{Adj}_{(3,1)})$.

□

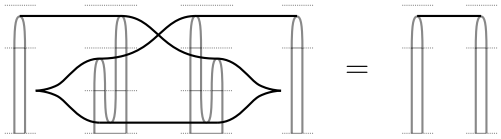
Lemma 3.2.4. *The restriction functor* $E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2(\mathcal{C})$ *factors through* $\mathrm{Arr}_2^R(\mathcal{C})$*.*

Proof. We have to show that, given a functor $F : \mathcal{F}(\mathrm{Adj}_{(3,2)}) \to \mathcal{C}$, the 2-morphism $F(\rho)$ is a right adjoint. This is clear, because $\lambda \dashv \rho$ in $\mathcal{F}(\mathrm{Adj}_{(3,2)})$.

□

3.2.2 Additional swallowtail relations in $\mathcal{F}(\mathrm{Adj}_{(3,1)})$

We now identify and prove some relations that hold in $\mathcal{F}(\mathrm{Adj}_{(3,1)})$, as they follow from the relations in $\mathrm{Adj}_{(3,1)}$. These will be useful in some of the proofs in the rest of this Chapter. The relation

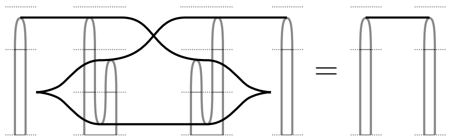


in $\mathrm{Adj}_{(3,1)}$ is called a *swallowtail relation.* There are three other similar swallowtail relations which hold in $\mathcal{F}(\mathrm{Adj}_{(3,1)})$. It would seem be natural to include these other swallowtail relations in the presentation $\mathrm{Adj}_{(3,1)}$, but as we now show they follow from the one swallowtail relation above.

Notation 3.2.5. *We use* $\overset{h}{=}$ *to denote an equality of* 3*-morphisms which follows by* *applying interchanger moves which are valid in any 3-category. In the quasistrict categories described in Section 2.3, these moves are called homotopy generators. We will also use* $\overset{r}{=}$ *to denote an equality that follows from using relations in a relevant presentation or previously established relations, and we will use* $\overset{d}{=}$ *to denote an equality which follows from applying a definition.*

We use $\overset{h}{\Longrightarrow}$, $\overset{r}{\Longrightarrow}$ *and* $\overset{d}{\Longrightarrow}$ *with analogous meaning.*

Lemma 3.2.6. *The following swallowtail relation holds in* $\mathcal{F}(\mathrm{Adj}_{(3,1)})$.

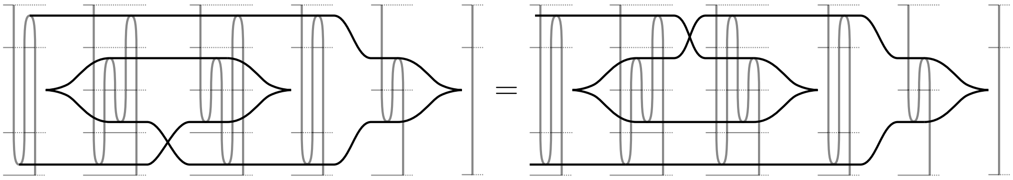


Proof. The 3-morphism on the left hand side is inverse to the 3-morphism on the left hand side of the swallowtail relation in $\mathrm{Adj}_{(3,1)}$.

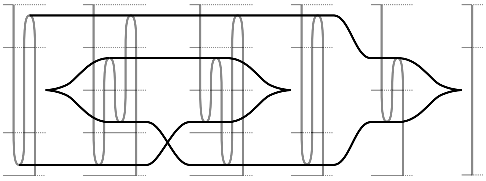
□

For the proof of the next swallowtail relation, we will need an auxiliary relation.

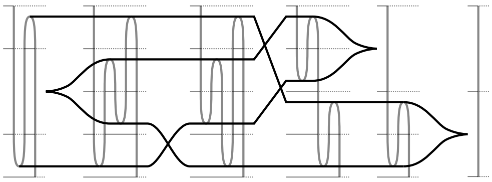
Lemma 3.2.7. *The following relation holds in* $\mathcal{F}(\mathrm{Adj}_{(3,1)})$.



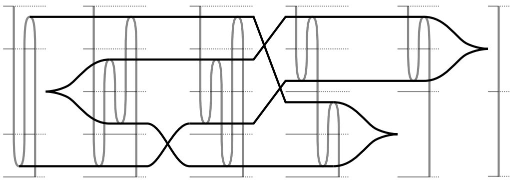
Proof. The proof is the following sequence of equalities.



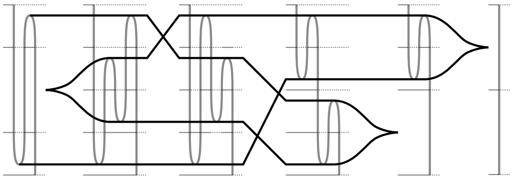
$\overset{h}{=}$



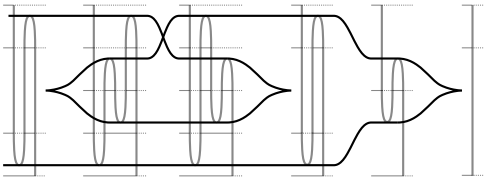
$\overset{h}{=}$



$\overset{h}{=}$

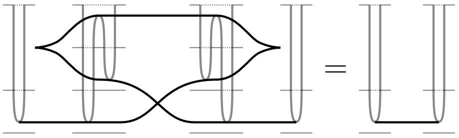


$\overset{h}{=}$

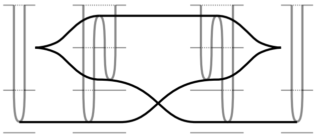


□

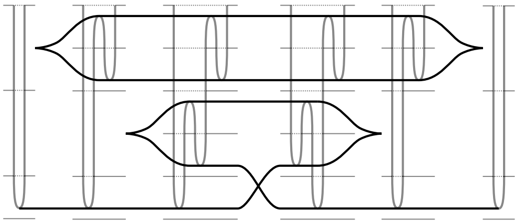
Lemma 3.2.8. *The following swallowtail relation holds in* $\mathcal{F}(\mathrm{Adj}_{(3,1)})$.



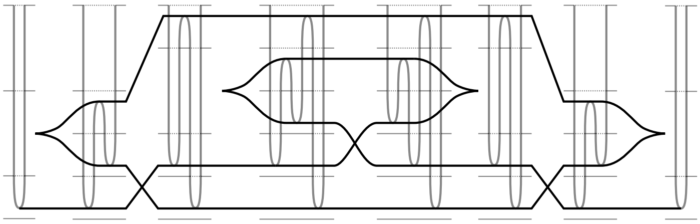
Proof. The proof is the following sequence of equalities.



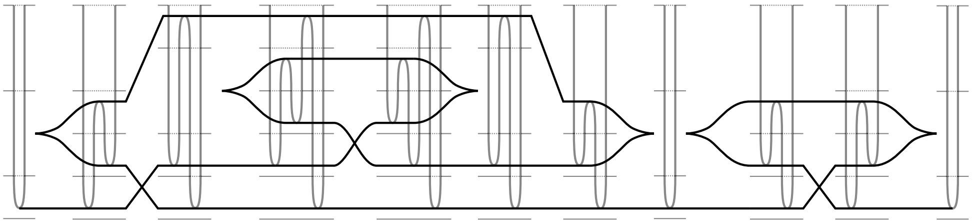
$\overset{r}{=}$



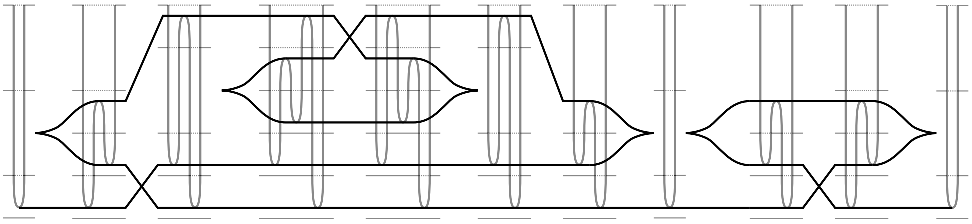
$\overset{h}{=}$



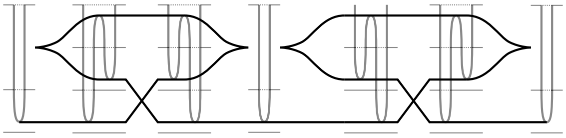
$\overset{r}{=}$



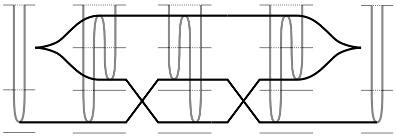
$\overset{(*)}{=}$



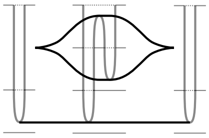
$\overset{(**)}{=}$



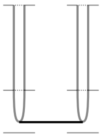
$\overset{r}{=}$



$\overset{h}{=}$



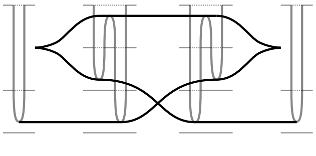
$\overset{r}{=}$



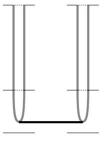
where in $(*)$ we use the previous Lemma and in $(**)$ we use the swallowtail relation from $\mathrm{Adj}_{(3,1)}$.

□

Lemma 3.2.9. *The following relation holds in* $\mathcal{F}(\mathrm{Adj}_{(3,1)})$.



$=$



Proof. The 3-morphism on the left hand side is inverse to the one on the left hand side of the equation in the previous Lemma.

□

3.2.3 Surjective on objects

Now we prove that the functors

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

and

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

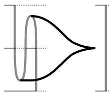
are surjective on objects.

Lemma 3.2.10. *The restriction functor*

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

is surjective on objects.

Proof. Given a functor $F : \mathrm{Arr}_1 \to \mathcal{C}$ where $F(f) : F(A) \to F(B)$ is a right adjoint, we want to extend this to a functor $\tilde{F} : \mathrm{Adj}_{(3,1)} \to \mathcal{C}$ with $\tilde{F}(r) = F(f)$. Since $F(f)$ is a right adjoint, we can a pick a left adjoint for it, together with unit and counit 2-morphisms and two inverse pairs of cusp 3-morphisms implementing the snake equations. All that is left is guaranteeing that these cusp 3-morphisms satisfy the swallowtail equation. This can be accomplished by redefining one of the inverse pairs of cusp 3-morphisms. We denote by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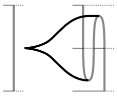
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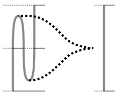
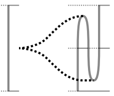
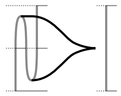

and


the initial choices of cusp 3-morphisms. We use dashed lines for the cusp 3-morphisms that will be redefined. We define a new pair of cusp 3-morphisms as follows.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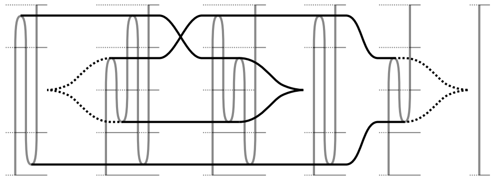
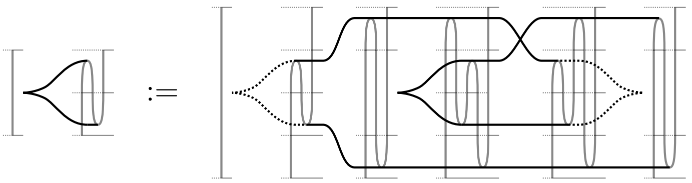
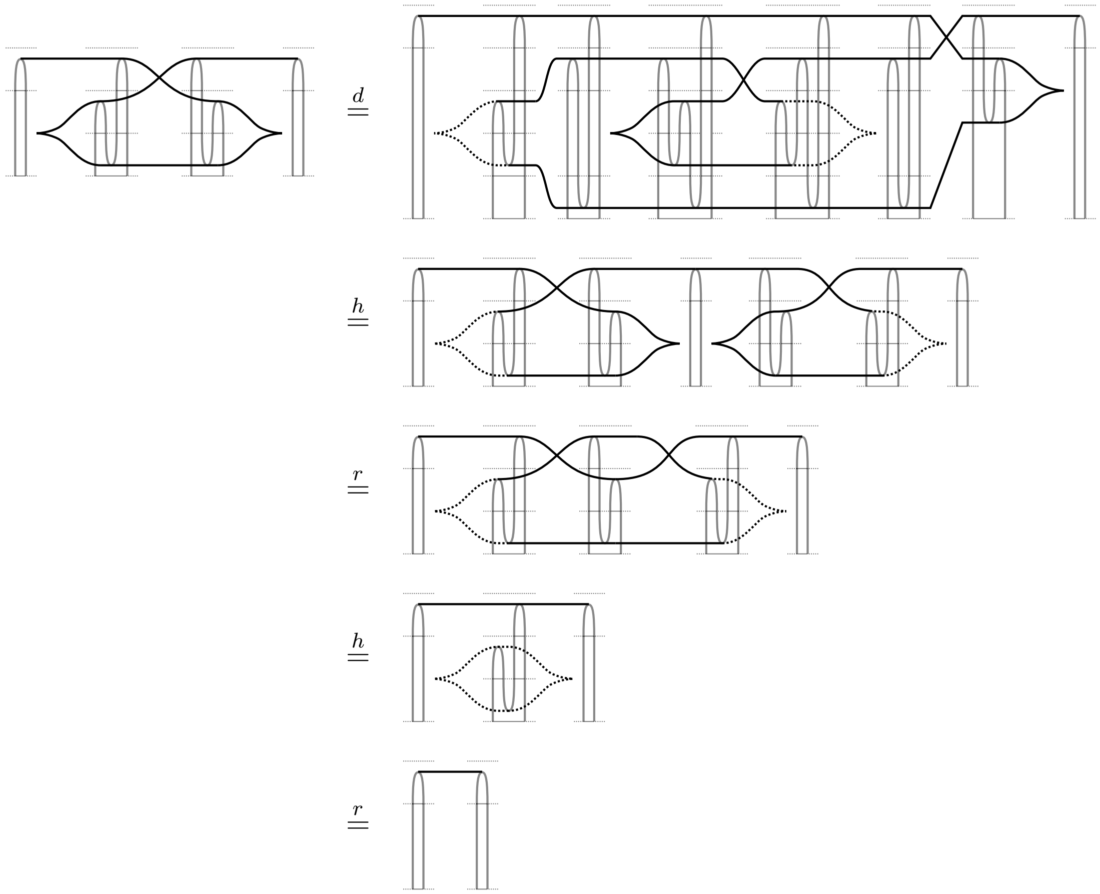




where the dashed cusps on the right hand side are the initial choices which we are redefining.

It is now easy to check that these new cusp 3-morphisms satisfy the swallowtail relation in the presentation $\mathrm{Adj}_{(3,1)}$:



where in the second step we pushed the swallowtail in the middle to the right.

□

Lemma 3.2.11. *The restriction functor*

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

is surjective on objects.

Proof. Given a right adjoint 2-morphism ϕ in $\mathcal{C}$, we can always pick a left adjoint, together with unit and counit 3-morphisms satisfying the snake equations. This data corresponds to a functor $F : \mathrm{Adj}_{(3,2)} \to \mathcal{C}$ with $E_\rho(F) = \phi$.

□

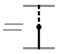
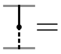
3.2.4 Partial coherence for isomorphisms

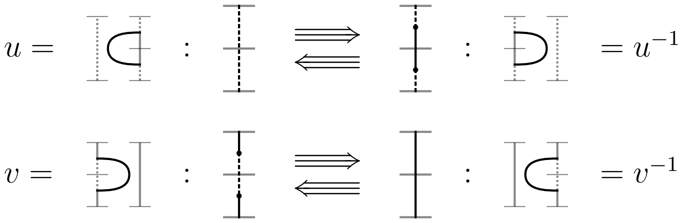
In order to prove that E_r and E_ρ are surjective on 1-morphisms, we will need to pick coherent inverses for certain invertible morphisms, so we need some partial coherence results. These are phrased in terms of surjectivity on objects of certain restriction functors. These functors are actually equivalences, but the proof of this fact requires the coherence results for adjunctions.

Definition 3.2.12. *The presentation* $\mathrm{Iso}_{(3,3)}$ *consists of*

0. *two objects* X, Y*;*
1. *two* 1*-morphisms* $x, y : X \to Y$*;*
2. *two* 2*-morphisms* $\chi, \upsilon : x \Longrightarrow y$*;*
3. *two* 3*-morphisms* $f : \chi \underset{\Longleftarrow}{\overset{\Longrightarrow}{}} \upsilon : f^{-1}$*;*
4. *two relations* $f^{-1} \circ f = \mathrm{Id}_\chi$ *and* $f \circ f^{-1} = \mathrm{Id}_\upsilon$*.*

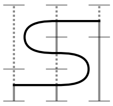
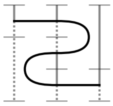
Definition 3.2.13. *The presentation* $\mathrm{Iso}_{(3,2)}$ *consists of*

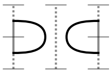
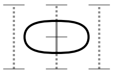
0. *Objects* X *and* Y*;*
1. 1*-morphisms* $x, y : X \to Y$*;*
2. 2*-morphisms* $\phi =$  $: x \underset{\Longleftarrow}{\overset{\Longrightarrow}{}} y :$  $= \phi^{-1}$*;*
3. 3*-morphisms*

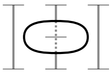
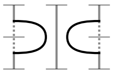


;

4. *relations*

 $= \mathrm{Id}_\phi$ *and*  $= \mathrm{Id}_{\phi^{-1}}$;

 $= \mathrm{Id}_{\phi^{-1}\circ\phi}$ *and*  $= \mathrm{Id}_{\mathrm{id}_x}$;

 $= \mathrm{Id}_{\mathrm{id}_y}$ *and*  $= \mathrm{Id}_{\phi\circ\phi^{-1}}$;

Note that there is an isomorphism

$$\left(\mathrm{Adj}_{(3,2)} *_{u\sim f} \mathrm{Iso}_{(3,3)}\right) *_{v\sim f} \mathrm{Iso}_{(3,3)} \to \mathrm{Iso}_{(3,2)}$$

sending $\lambda \mapsto \phi$ and $\rho \mapsto \phi^{-1}$.

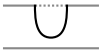
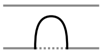
Definition 3.2.14. *The presentation* $\mathrm{Iso}_{(3,1)}$ *consists of*

0. *objects* $X =$ ┈ *and* $Y =$ ━;

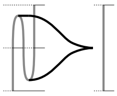
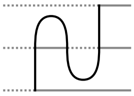
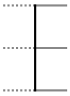
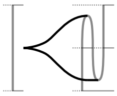
1. 1-*morphisms* $f =$ ┈┈•━━ $: X \underset{}{\rightleftarrows} Y :$ ━━•┈┈ $= f^{-1}$

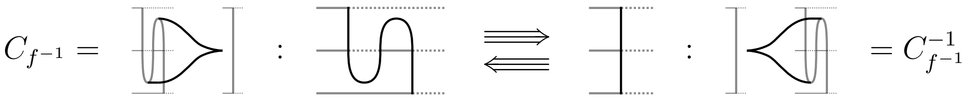
2. 2-*morphisms*

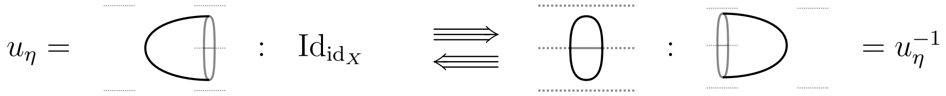
$\eta =$  : ┈┈┈ $\substack{\Longrightarrow \\ \Longleftarrow}$ ┈•━•┈ :  $= \eta^{-1}$

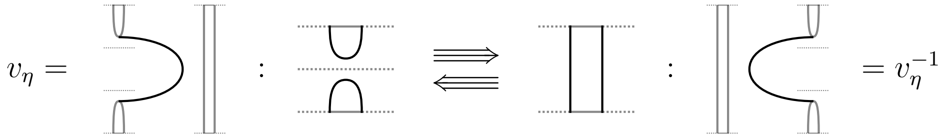
$\epsilon =$  : ━•┈•━ $\substack{\Longrightarrow \\ \Longleftarrow}$ ━━ :  $= \epsilon^{-1}$

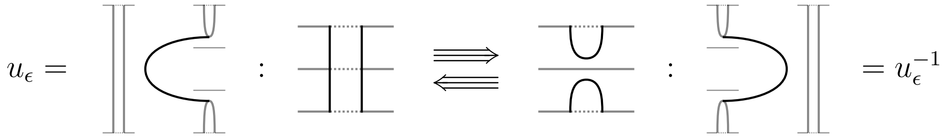
3. 3-*morphisms*

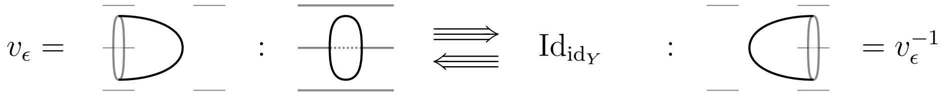
$C_f =$  :  $\substack{\Rrightarrow \\ \Lleftarrow}$  :  $= C_f^{-1}$


$C_{f^{-1}} =$
:
$\Rrightarrow$ / $\Lleftarrow$
:
$= C_{f^{-1}}^{-1}$

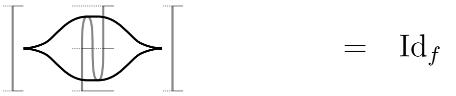

$u_\eta =$
:
$\mathrm{Id}_{\mathrm{id}_X}$
$\Rrightarrow$ / $\Lleftarrow$
:
$= u_\eta^{-1}$


$v_\eta =$
:
$\Rrightarrow$ / $\Lleftarrow$
:
$= v_\eta^{-1}$


$u_\epsilon =$
:
$\Rrightarrow$ / $\Lleftarrow$
:
$= u_\epsilon^{-1}$


$v_\epsilon =$
:
$\Rrightarrow$ / $\Lleftarrow$
$\mathrm{Id}_{\mathrm{id}_Y}$
:
$= v_\epsilon^{-1}$

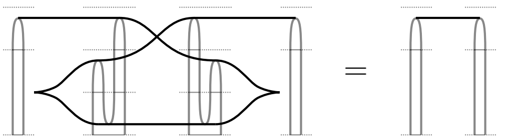
4. relations

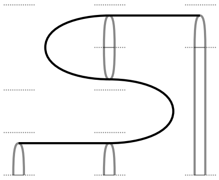

$= \quad \mathrm{Id}_f$


$=$

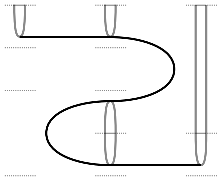

$= \quad \mathrm{Id}_{f^{-1}}$


$=$

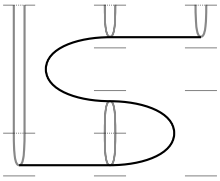

$=$



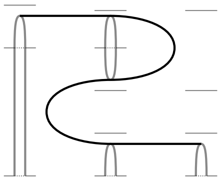
$= \mathrm{Id}_{\eta}$;



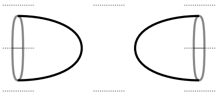
$= \mathrm{Id}_{\eta^{-1}}$



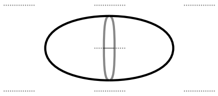
$= \mathrm{Id}_{\epsilon}$;



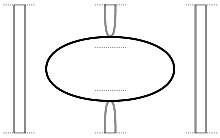
$= \mathrm{Id}_{\epsilon^{-1}}$



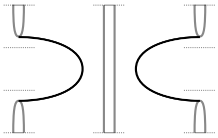
$= \mathrm{Id}$;



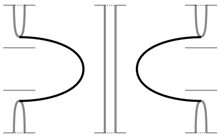
$= \mathrm{Id}$



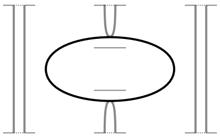
$= \mathrm{Id}$;



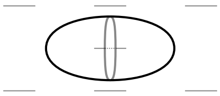
$= \mathrm{Id}$



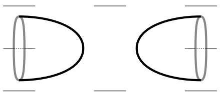
$= \mathrm{Id}$;



$= \mathrm{Id}$



$= \mathrm{Id}$;



$= \mathrm{Id}$

Note that there is an isomorphism

$$\left(\mathrm{Adj}_{(3,1)} *_{\eta\sim\phi} \mathrm{Iso}_{(3,2)}\right) *_{\epsilon\sim\phi} \mathrm{Iso}_{(3,2)} \to \mathrm{Iso}_{(3,1)}$$

sending $l \mapsto f$ and $r \mapsto f^{-1}$. By isomorphism we mean an isomorphism in the category $\mathrm{Comp}_4^{T_3}$, which simply means a bijection between the sets of k-cells, for each k, compatible with source and target maps.

Lemma 3.2.15. *Let $f : X \to Y$ be an invertible k-morphism in an n-category. Then any inverse $g : Y \to X$ for f is also a right adjoint. Moreover, if $f \dashv h$ is an adjunction, with unit and counit u and v, then u and v are invertible.*

Proof. This is well known. □

Lemma 3.2.16. *Given an invertible* 2*-morphism* $\bar{\phi}$ *in a* 3*-category* $\mathcal{C}$*, there exists a functor* $F : \mathrm{Iso}_{(3,2)} \to \mathcal{C}$ *with* $F(\phi) = \bar{\phi}$*.*

Proof. We have

$$\mathrm{Iso}_{(3,2)} = \left(\mathrm{Adj}_{(3,2)} *_{u\sim f} \mathrm{Iso}_{(3,3)}\right) *_{v\sim f} \mathrm{Iso}_{(3,3)}$$

where we chose to identify ϕ with the left adjoint $\lambda \in \mathrm{Adj}_{(3,2)}$ and ϕ^{-1} with the right adjoint ρ. Now $\bar{\phi}$ is an invertible 2-morphism in $\mathcal{C}$ and by Lemma 3.2.15 any inverse for $\bar{\phi}$ is also a right adjoint. So $\bar{\phi}$ is a left adjoint and we can use the fact that E_λ is surjective on objects to define a functor $\mathrm{Adj}_{(3,2)} \to \mathcal{C}$ sending λ to $\bar{\phi}$. We denote the image of ρ by $\bar{\phi}^{-1}$. Now the images of $u, v \in \mathrm{Adj}_{(3,2)}$ in $\mathcal{C}$ are a unit and counit witnessing the adjunction $\bar{\phi} \dashv \bar{\phi}^{-1}$, therefore by Lemma 3.2.15 they are invertible, which allows us to extend the functor $\mathrm{Adj}_{(3,2)} \to \mathcal{C}$ to $\mathrm{Adj}_{(3,2)} *_{u\sim f} \mathrm{Iso}_{(3,3)} *_{v\sim f} \mathrm{Iso}_{(3,3)}$.

□

Definition 3.2.17. *A functor as in Lemma 3.2.16 is called a choice of coherent inverse for* $\bar{\phi}$*.*

Lemma 3.2.18. *Given an invertible* 1*-morphism* $\bar{f}$ *in a* 3*-category* $\mathcal{C}$*, there exists a functor* $F : \mathrm{Iso}_{(3,1)} \to \mathcal{C}$ *with* $F(f) = \bar{f}$*.*

Proof. We have

$$\mathrm{Iso}_{(3,1)} = \left(\mathrm{Adj}_{(3,1)} *_{\eta\sim\phi} \mathrm{Iso}_{(3,2)}\right) *_{\epsilon\sim\phi} \mathrm{Iso}_{(3,2)}$$

where we chose to identify f with the left adjoint $l \in \mathrm{Adj}_{(3,1)}$ and f^{-1} with the right adjoint r. By Lemma 3.2.15, any inverse to $\bar{f}$ is also a right adjoint to $\bar{f}$, which means $\bar{f}$ is a left adjoint. So we can use the fact that $E_l : \mathrm{Fun}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^L(\mathcal{C})$ is surjective on objects, to define a functor $\mathrm{Adj}_{(3,1)} \to \mathcal{C}$ sending l to $\bar{f}$. We denote the image of r by $\bar{f}^{-1}$. Now the images of η and ϵ in $\mathcal{C}$ are unit and counit for an adjunction $\bar{f} \dashv \bar{f}^{-1}$, so by Lemma 3.2.15 they are invertible. Then, by Lemma 3.2.16, we can extend this functor $\mathrm{Adj}_{(3,1)} \to \mathcal{C}$ to $\mathrm{Adj}_{(3,1)} *_{\eta\sim\phi} \mathrm{Iso}_{(3,2)} *_{\epsilon\sim\phi} \mathrm{Iso}_{(3,2)}$.

□

Definition 3.2.19. *A functor as in Lemma 3.2.18 is called a choice of coherent inverse for* $\bar{f}$*.*

3.2.5 Surjective on 1-morphisms

Lemma 3.2.20. *The functor*

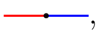
$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

is surjective on 1*-morphisms.*

Proof. Given functors $F, G \in \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C})$ and a natural isomorphism

$$\alpha : E_r(F) \to E_r(G)$$

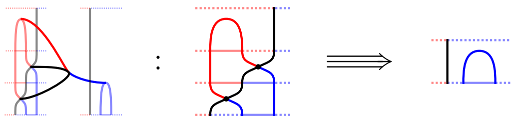
between their restrictions to Arr_1, we want to extend α to a natural isomorphism $\alpha : F \to G$. Throughout this proof, we use red to denote the images of cells under F, blue for the images under G and black for the components of the natural isomorphism α. The invertible natural transformation between the restricted functors consists of the following data.

0. Invertible 1-morphisms $\alpha_X : F(X) \to G(X)$ and $\alpha_Y : F(Y) \to G(Y)$, which we denote by  and , respectively, and
1. an invertible 2-morphism $\alpha_r : \alpha_X \circ F(r) \Rightarrow G(r) \circ \alpha_Y$, which we denote by

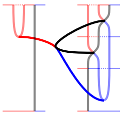
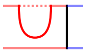
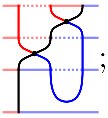
We want to extend this to a natural isomorphism $\alpha : F \to G$, which has the following additional data and relations.

1. An invertible 2-morphism $\alpha_l : \alpha_Y \circ F(l) \Rightarrow G(l) \circ \alpha_X$, which we denote by

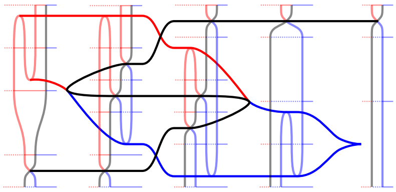
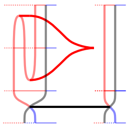
2. invertible 3-morphisms

α_η = 

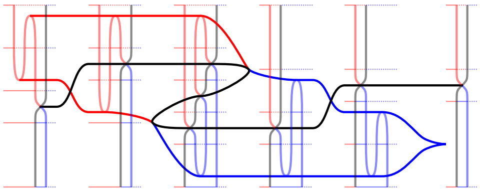
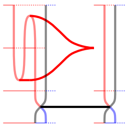
and

$\alpha_\epsilon \quad =$  :  $\Longrightarrow$  ;

3. relations

 $\overset{\alpha_{C_l}}{=\!=}$ 

and

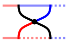
 $\overset{\alpha_{C_r}}{=\!=}$  .

Note that there are also relations $\alpha_{C_l^{-1}}$ and $\alpha_{C_r^{-1}}$ corresponding to the inverse cusp 3-morphisms, but these follow from α_{C_l} and α_{C_r}. We start by picking coherent inverses

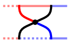
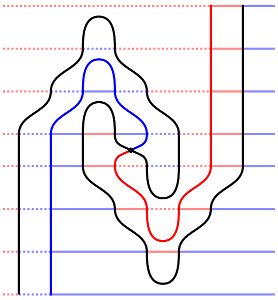
$\alpha_X^{-1} =$  $: G(X) \to F(X),$

$\alpha_Y^{-1} =$  $: G(Y) \to F(Y)$

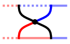
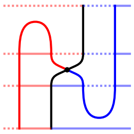
and

$\alpha_r^{-1} =$  :  $\Longrightarrow$  .

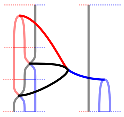
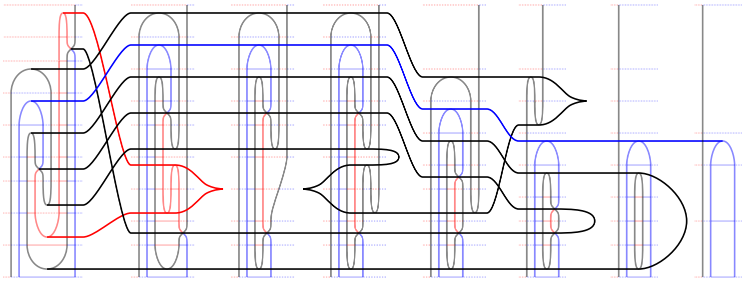
We will use the coherence data for these inverses to define the extension of α to $\mathrm{Adj}_{(3,1)}$. Define $\alpha_l : \alpha_Y \circ F(l) \Rightarrow G(l) \circ \alpha_X$ as follows.

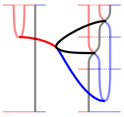
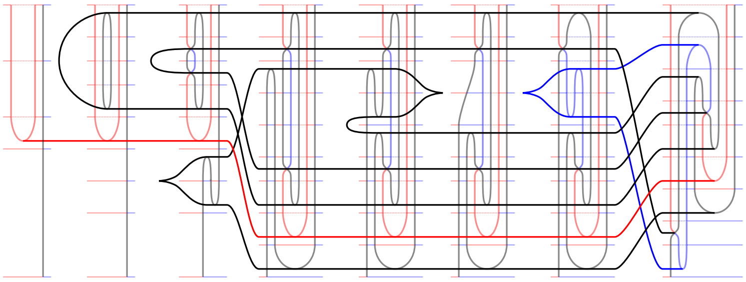
$\alpha_l \quad =$  $:=$ 

This is invertible, with inverse defined as follows.

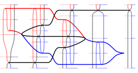
$\alpha_l^{-1} \quad =$  $:=$ 

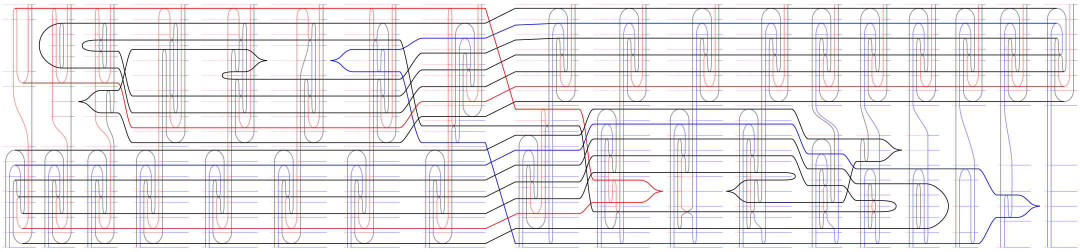
Define invertible 3-morphisms α_η and α_ϵ as follows.

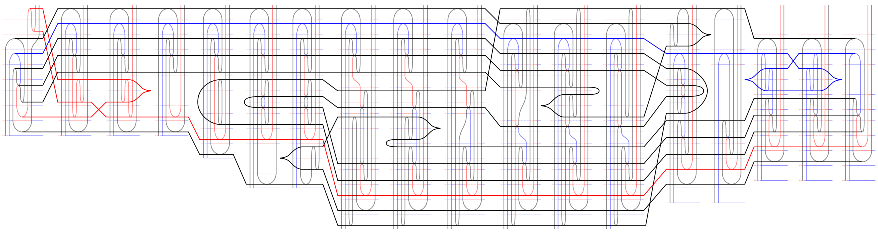
$\alpha_\eta \quad =$  $:=$ 

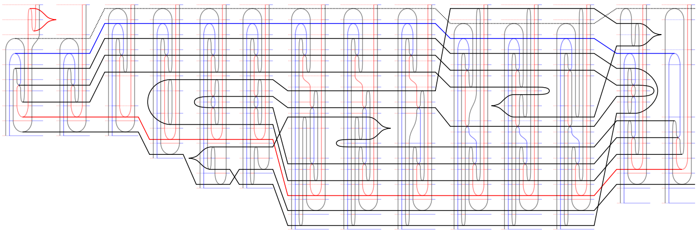
$\alpha_\epsilon \quad =$  $:=$ 

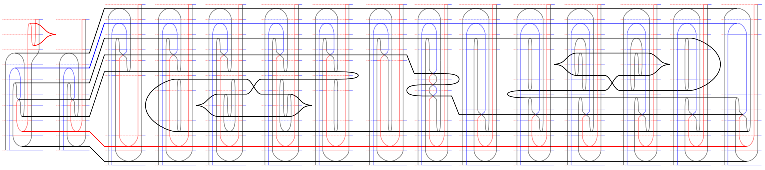
Finally we need to check that these satisfy the relations corresponding to the cusp 3-morphisms. We prove the relation α_{C_l} corresponding to the cusp 3-morphism C_l as follows.

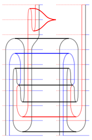


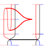
$\overset{d}{=}$ 

$\overset{h}{=}$ 

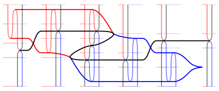
$\overset{r}{=}$ 

$\overset{h}{=}$ 

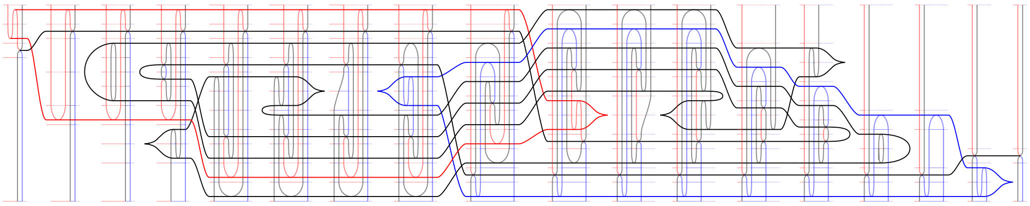
$\overset{r}{=}$ 

$\overset{d}{=}$ .

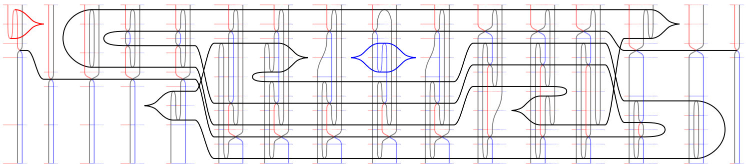
Note that on the third step we use a blue swallowtail relation and we use another relation which is easily deduced from a red swallowtail relation. The relation α_{C_r} corresponding to the cusp 3-morphism C_r is proved as follows.



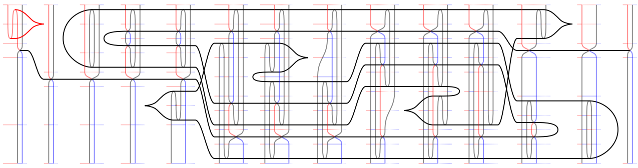
$\overset{d}{=}$



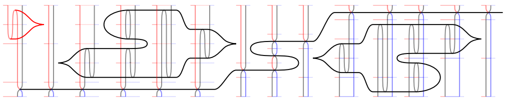
$\overset{h}{=}$



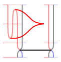
$\overset{r}{=}$



$\overset{h}{=}$



$\overset{r}{=}$



.

This finishes the proof of surjectivity on 1-morphisms ☐

Lemma 3.2.21. *The functor*

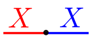
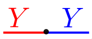
$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

is surjective on 1*-morphisms.*

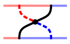
Proof. Given functors $F, G \in \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C})$ and a natural isomorphism

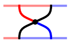
$$\alpha : E_\rho(F) \to E_\rho(G)$$

between their restrictions to Arr_2, we want to extend α to a natural isomorphism $\alpha : F \to G$. Throughout this proof, we use red to denote the images of cells under F, blue for the images under G and black for the components of the natural isomorphism α. The invertible natural transformation between the restricted functors consists of the following data.

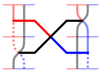
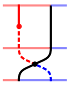
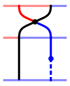
0. Invertible 1-morphisms $\alpha_X : F(X) \to G(X)$ and $\alpha_Y : F(Y) \to G(Y)$, which we denote by  and , respectively;

1. invertible 2-morphisms

$\alpha_x =$  $: \alpha_Y \circ F(x) \Longrightarrow G(x) \circ \alpha_X$

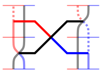
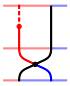
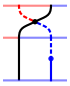
$\alpha_y =$  $: \alpha_Y \circ F(y) \Longrightarrow G(y) \circ \alpha_X;$

2. an invertible 3-morphism

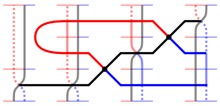
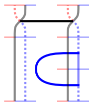
$\alpha_\rho =$  $:$  $\Rrightarrow$ 

We want to extend this to a natural isomorphism $\alpha : F \to G$, which has of the following additional data and relations.

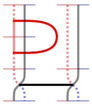
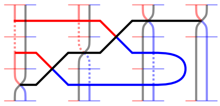
2. an invertible 3-morphism

$\alpha_\lambda =$  $:$  $\Rrightarrow$ 

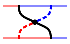
3. relations

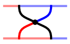
 $\overset{\alpha_u}{=\!=}$ 

and

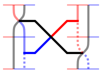
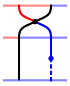
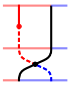
 $\overset{\alpha_v}{=\!=}$ .

We pick coherent inverses

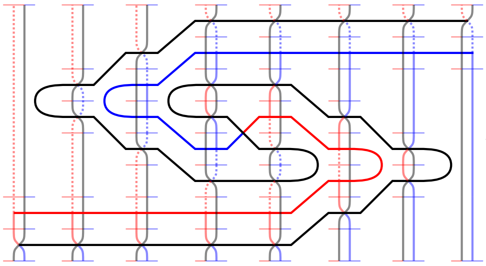
$\alpha_x^{-1} =$  $: G(x) \circ \alpha_X \Longrightarrow \alpha_Y \circ F(x)$

$\alpha_y^{-1} =$  $: G(y) \circ \alpha_X \Longrightarrow \alpha_Y \circ F(y);$

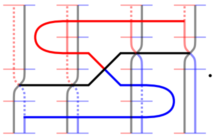
and

$\alpha_\rho^{-1} =$  $:$  $\Longrightarrow$ .

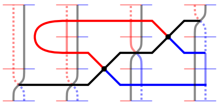
We will use the coherence data for these inverses to define the extension of α to $\mathrm{Adj}_{(3,2)}$. Define α_λ as follows.

$\alpha_\lambda \quad :=$ .

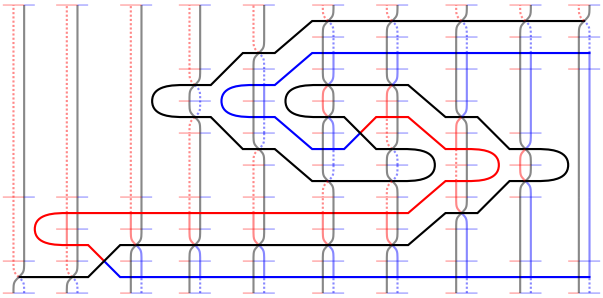
This is invertible, with inverse defined as follows.

$\alpha_\lambda^{-1} \quad :=$ .

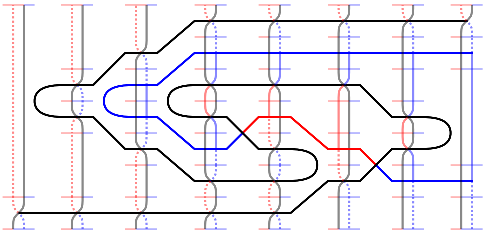
Now we need to check that these satisfy the relations corresponding to the 3-cells u and v in $\mathrm{Adj}_{(3,2)}$. We prove the relation α_u as follows.



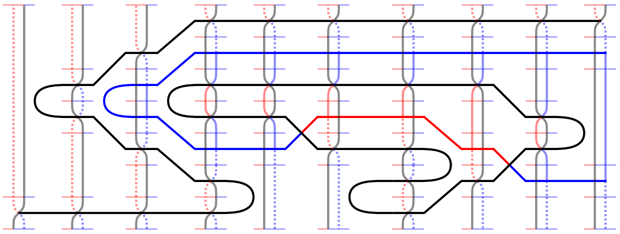
$\overset{d}{=}$



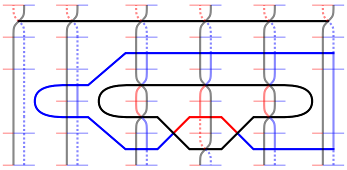
$\overset{r}{=}$

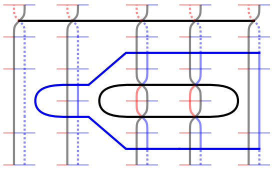


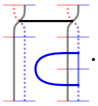
$\overset{r}{=}$



$\overset{r}{=}$



$\overset{r}{=}$ 

$\overset{r}{=}$  .

And the proof of the relation α_v is similar. $\square$

3.2.6 Surjective on 2-morphisms

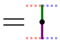
Lemma 3.2.22. *The functor*

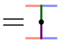
$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

is surjective on 2*-morphisms.*

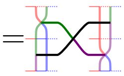
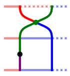
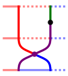
Proof. Consider two functors $F, G : \mathrm{Adj}_{(3,1)} \to \mathcal{C}$ and two invertible natural transformations $\alpha, \beta : F \to G$. Suppose we have an invertible modification $m : E_r(\alpha) \Rightarrow E_r(\beta)$ between their restrictions to Arr_1. We want to extend m to a modification $m : \alpha \Rightarrow \beta$. We use red for F, blue for G, green for α, purple for β and black for m. The invertible modification between the restricted natural transformations consists of the following data:

0. invertible 2-morphisms $m_X : \alpha_X \Rightarrow \beta_X$ and $m_Y : \alpha_Y \Rightarrow \beta_Y$, which we denote by

$m_X =$  $:$  $\Longrightarrow$ 

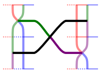
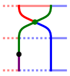
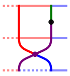
$m_Y =$  $:$  $\Longrightarrow$  ;

1. an invertible 3-morphism

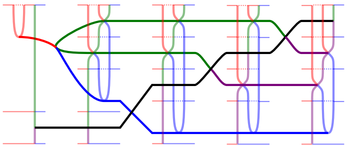
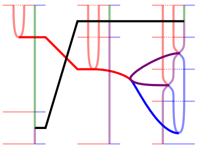
$m_r =$  $:$  $\Longrightarrow$  .

We want to extend this to an invertible modification $m : \alpha \Rrightarrow \beta$ over all of $\mathrm{Adj}_{(3,1)}$, which has the following additional data and relations.

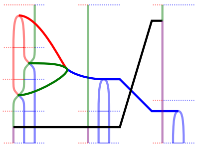
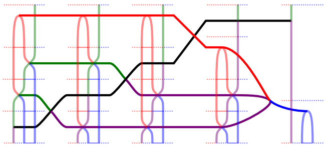
1. An invertible 3-morphism

$m_l =$  :  $\Longrightarrow$ ;

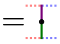
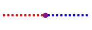
2. relations

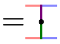
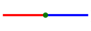
 $\overset{m_\epsilon}{=\!=}$ 

and

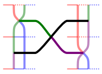
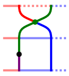
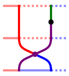
 $\overset{m_\eta}{=\!=}$  .

Pick coherent inverses

$m_X^{-1} =$  :  $\Longrightarrow$ 

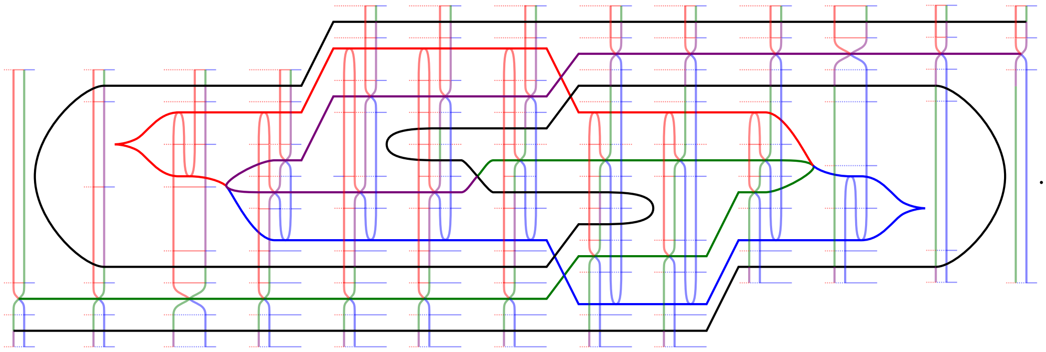
$m_Y^{-1} =$  :  $\Longrightarrow$ ;

and

$m_r =$  :  $\Longrightarrow$ .

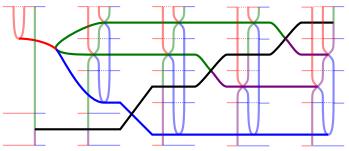
Define

$m_l :=$

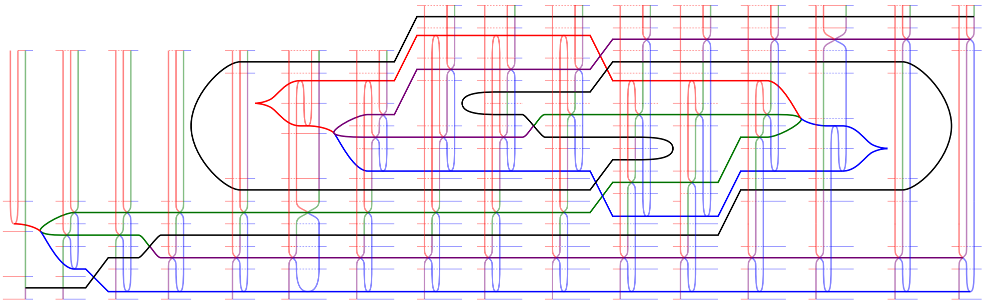


.

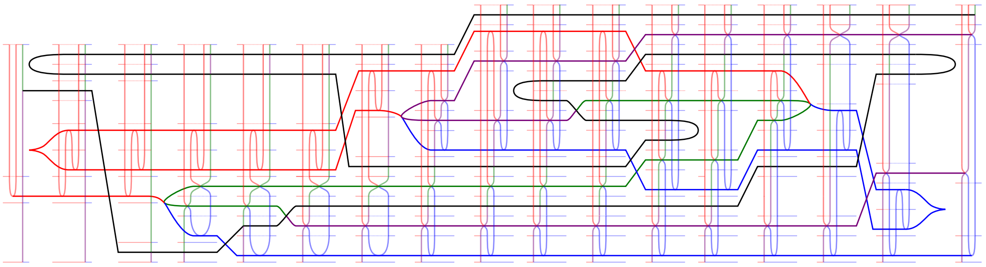
Now we need to check that this data satisfies the relations m_η and m_ϵ corresponding to the 2-morphisms in $\mathrm{Arr}_{(3,1)}$. We prove the relation m_ϵ corresponding to the counit 2-morphism as follows.



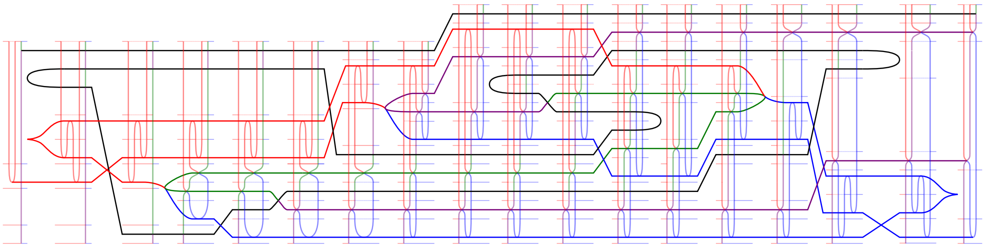
$\overset{d}{=}$



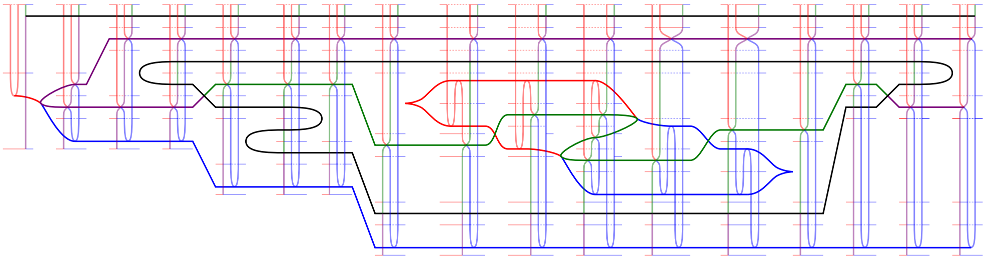
$\overset{h}{=}$



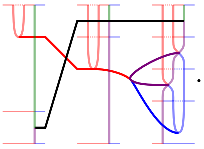
$\overset{r}{=}$



$\overset{h}{=}$



$\overset{r}{=}$



.

The proof for m_η is similar.

$\square$

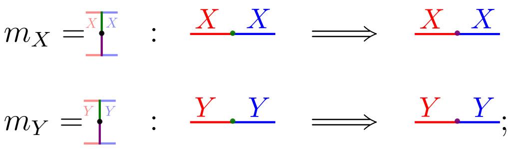
Lemma 3.2.23. *The functor*

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

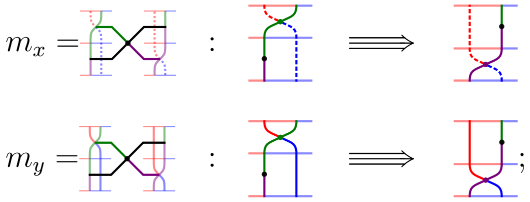
is surjective on 2*-morphisms.*

Proof. Consider two functors $F, G : \mathrm{Adj}_{(3,2)} \to \mathcal{C}$ and two invertible natural transformations $\alpha, \beta : F \to G$. Suppose we have an invertible modification $m : E_\rho(\alpha) \Rightarrow E_\rho(\beta)$ between their restrictions to Arr_2. We want to extend m to a modification $m : \alpha \Rightarrow \beta$. We use red for F, blue for G, green for α, purple for β and black for m. The invertible modification between the restricted natural transformations consists of the following data:

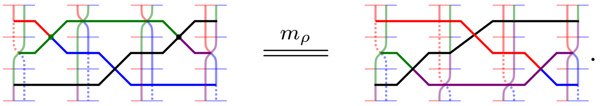
0. invertible 2-morphisms $m_X : \alpha_X \Rightarrow \beta_X$ and $m_Y : \alpha_Y \Rightarrow \beta_Y$, which we denote by



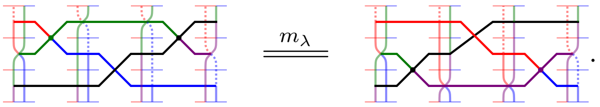
1. invertible 3-morphisms



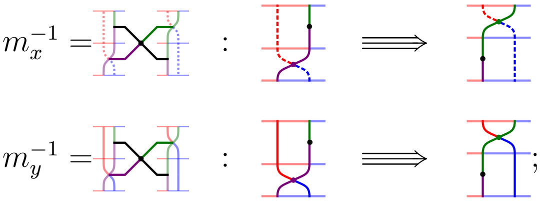
2. a relation



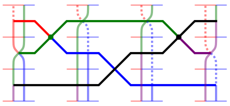
We want to extend this to an invertible modification $m : \alpha \Rightarrow \beta$ over all of $\mathrm{Adj}_{(3,2)}$, which amounts to checking the following additional relation.



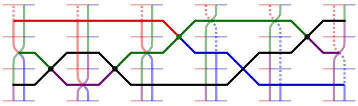
Pick inverses



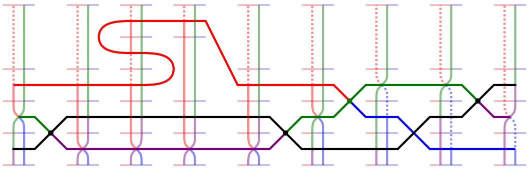
The proof of the additional relation is as follows.



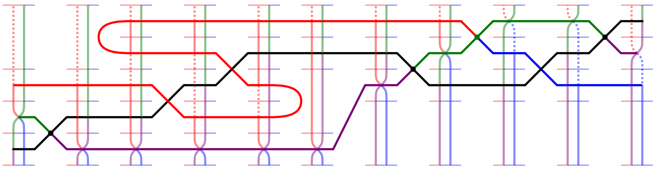
$\overset{r}{=}$



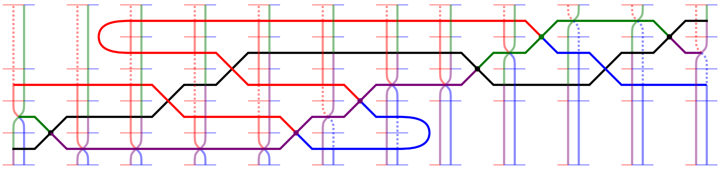
$\overset{r}{=}$



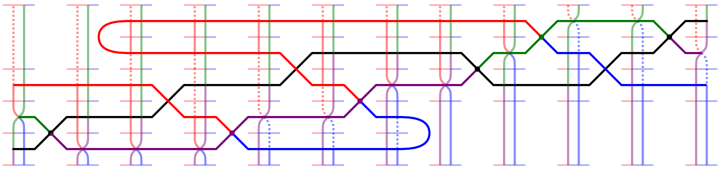
$\overset{h}{=}$



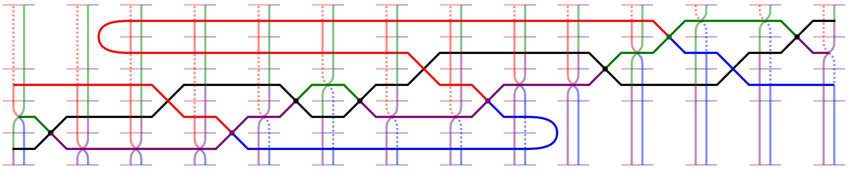
$\overset{r}{=}$



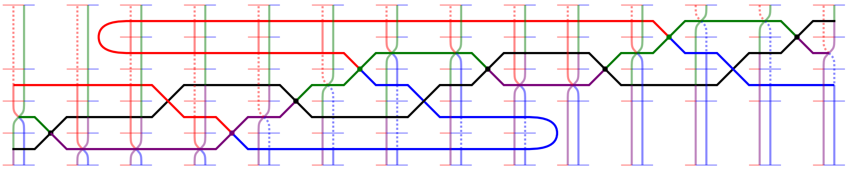
$\overset{h}{=}$



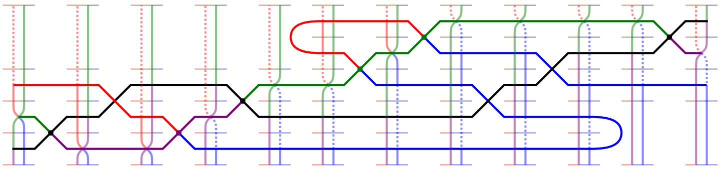
$\overset{r}{=}$



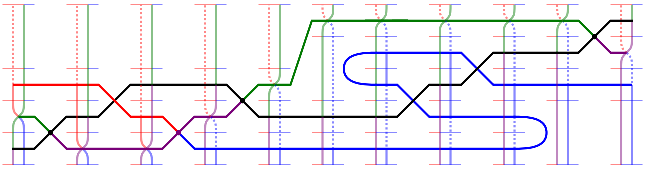
$\overset{m_\rho}{=}$



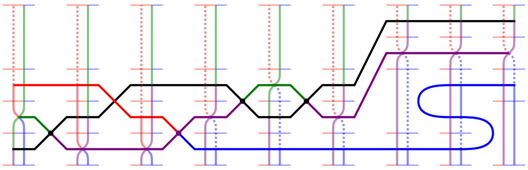
$\overset{r}{=}$



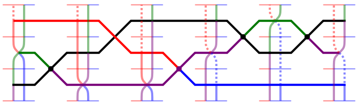
$\overset{r}{=}$



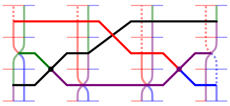
$\overset{h}{=}$



$\overset{r}{=}$



$\overset{r}{=}$


.

□

3.2.7 Bijective on 3-morphisms

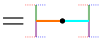
Lemma 3.2.24. *The functor*

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

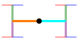
is surjective on 3*-morphisms.*

Proof. Consider $F, G \in \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C})$, two invertible natural transformations $\alpha, \beta : F \to G$ and two invertible modifications $\ell, m : \alpha \Rightarrow \beta$. Suppose we have an invertible perturbation $\mathcal{A} : E_r(\ell) \Rrightarrow E_r(m)$ between their restrictions to Arr_1. We want to extend $\mathcal{A}$ to a perturbation $\mathcal{A} : \ell \Rrightarrow m$. We use red for F, blue for G, green for α, purple for β, orange for ℓ, light blue for m and black for $\mathcal{A}$. The invertible perturbation between the restricted modifications consists of the following data and relations.

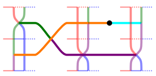
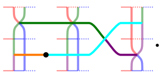
0. Invertible 3-morphisms $\mathcal{A}_X : \ell_X \Rrightarrow m_X$ and $\mathcal{A}_Y : \ell_Y \Rrightarrow m_Y$, which we denote by

$\mathcal{A}_X =$  $:$  $\Longrightarrow$ 

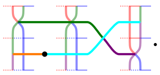
and

$\mathcal{A}_Y =$  $:$  $\Longrightarrow$ ;

1. a relation

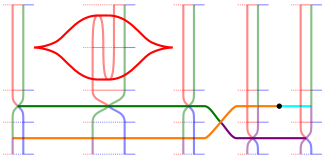
 $\overset{\mathcal{A}_r}{=\!=}$ .

In order to get a perturbation $\mathcal{A} : \ell \to m$ we just need to see that this data also satisfies the following relation, corresponding to the 1-cell $l : X \to Y$ in $\mathrm{Adj}_{(3,1)}$.

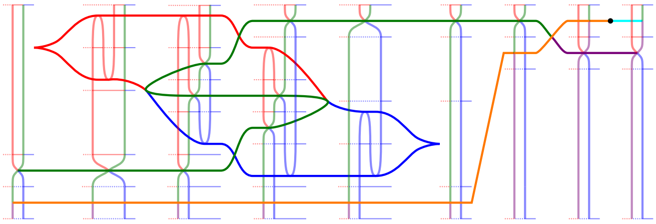
 $\overset{\mathcal{A}_l}{=\!=}$ .

The proof that this relation holds is the following.

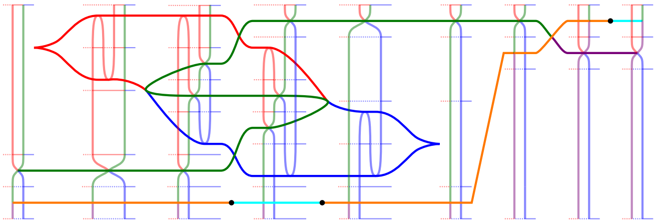
$\overset{r}{=}$



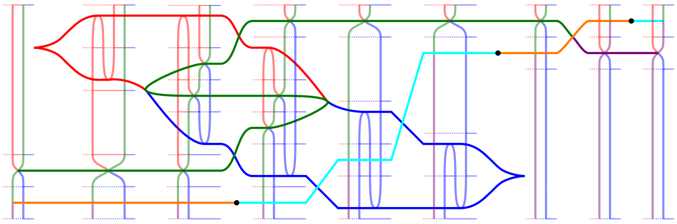
$\overset{\alpha_{C_l}}{=}$



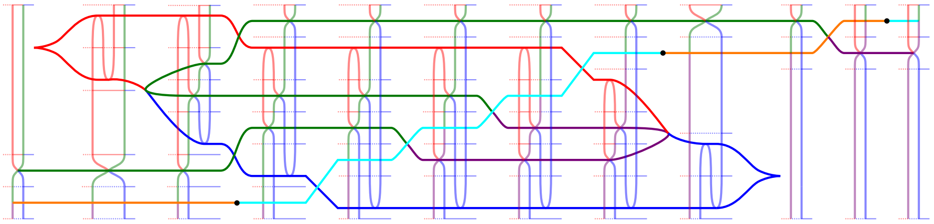
$\overset{r}{=}$



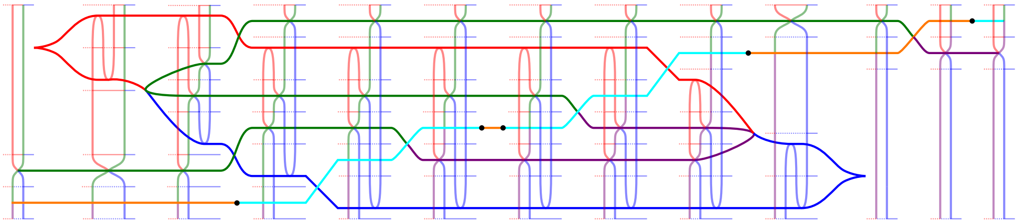
$\overset{h}{=}$



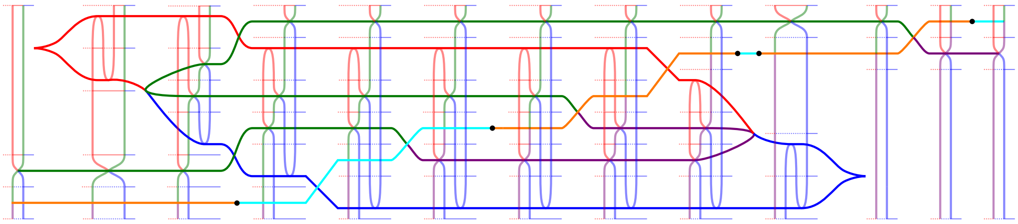
$\overset{m_\eta}{=}$



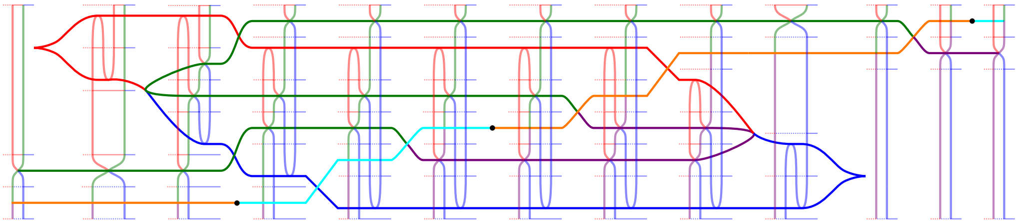
$\overset{r}{=}$



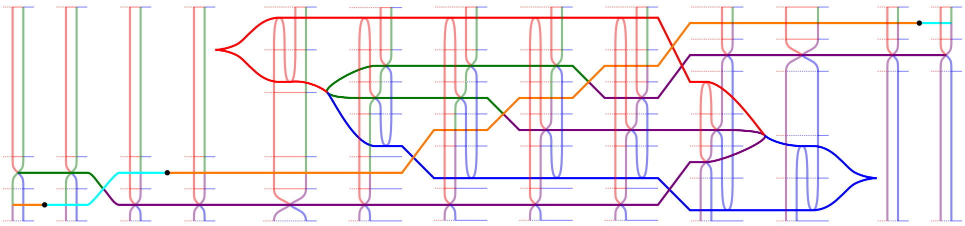
$\overset{\mathcal{A}_r}{=}$



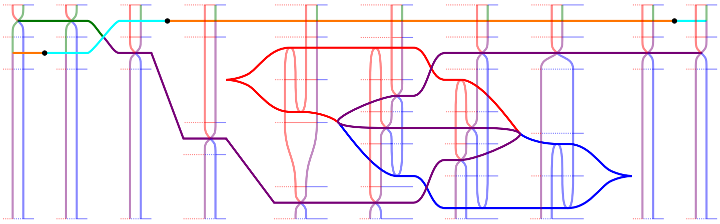
$\overset{r}{=}$



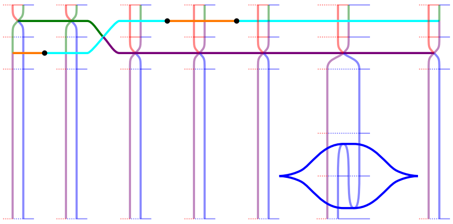
$\overset{h}{=}$



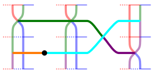
$\overset{l_\epsilon}{=}$



$\overset{\beta_{C_l^{-1}}}{=}$



$\overset{r}{=}$



□

Lemma 3.2.25. *The functor*

$$E_r : \mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C}) \to \mathrm{Arr}_1^R(\mathcal{C})$$

is injective on 3*-morphisms.*

Proof. This is obvious from the description of the 3-morphisms in both 3-groupoids given in the previous proof. Specifically, this follows from the fact that both consist of the same data, the difference being only the presence of an extra relation for 3-morphisms in $\mathrm{Map}(\mathrm{Adj}_{(3,1)}, \mathcal{C})$.

□

Lemma 3.2.26. *The functor*

$$E_\rho : \mathrm{Map}(\mathrm{Adj}_{(3,2)}, \mathcal{C}) \to \mathrm{Arr}_2^R(\mathcal{C})$$

is bijective on 3*-morphisms.*

Proof. There is nothing to prove, since there is no difference between perturbations defined over $\mathrm{Adj}_{(3,2)}$ and perturbations defined over Arr_2, when the target is a 3-category.

□

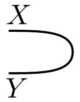
3.3 Coherence for duals

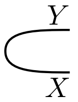
In this section we prove the coherence statement for duals in a monoidal 3-category. This means we will define a monoidal presentation Dual_3 and show that for any monoidal 3-category $\mathcal{C}$, the evaluation at X functor $E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ factors through the inclusion $\mathrm{Obj}^R(\mathcal{C}) \hookrightarrow \mathcal{C}$ of the 3-groupoid of objects which are right duals and induces an equivalence $\mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \xrightarrow{\sim} \mathrm{Obj}^R(\mathcal{C})$. Similarly for E_Y and $\mathrm{Obj}^L(\mathcal{C})$.

3.3.1 The presentation Dual$_3$

Definition 3.3.1. *The presentation* Dual_3 *consists of*

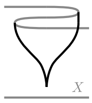
0. objects X *and* Y*;*

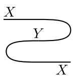
1. 1-*morphisms* ev $=$ 

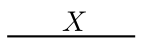
$: XY \to 1$ *and* coev $=$ 

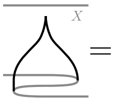
$: 1 \to YX$*;*

2. 2-*morphisms*

$C_X =$ 

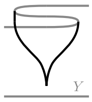
$:$ 

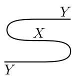
$\Rrightarrow$ / $\Lleftarrow$ 

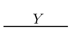
$:$ 

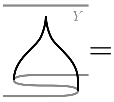
$= C_X^{-1}$

and

$C_Y =$ 

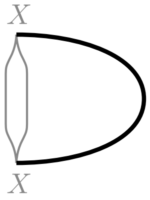
$:$ 

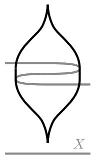
$\Rrightarrow$ / $\Lleftarrow$ 

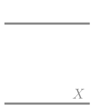
$:$ 

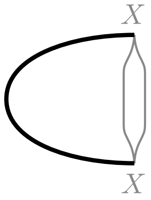
$= C_Y^{-1}$*;*

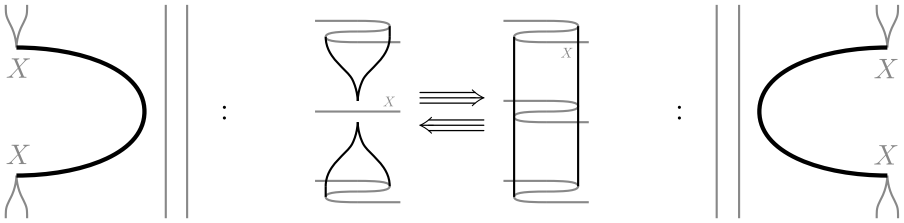
3. 3-*morphisms*

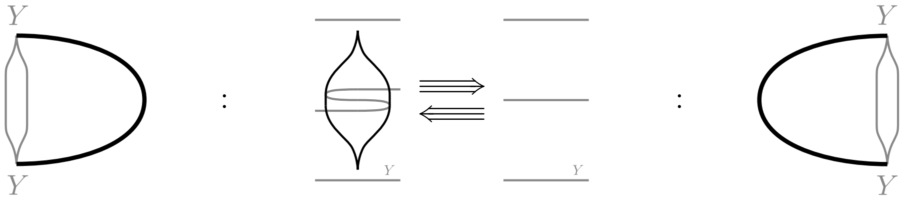


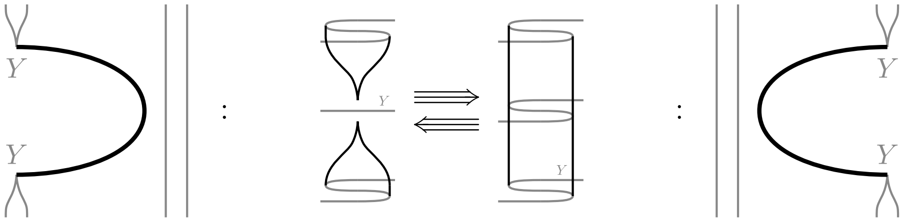
$:$ 

$\Rrightarrow$ / $\Lleftarrow$ 

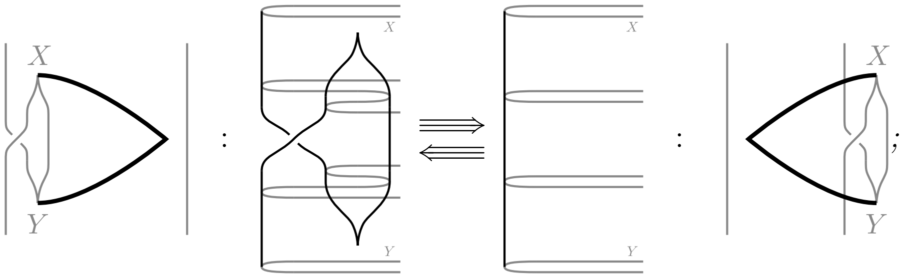
$:$ 



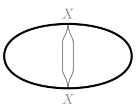
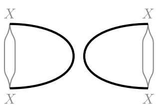


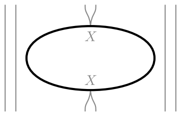
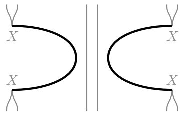


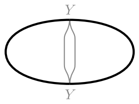
and



4. relations

 = Id ;  = Id;

 = Id ;  = Id;



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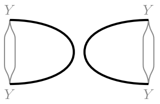

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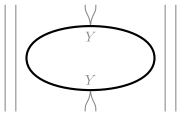

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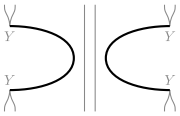

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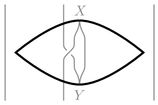

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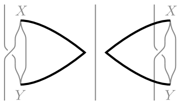
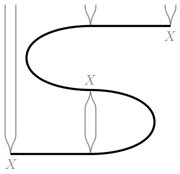


= Id;



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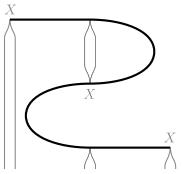

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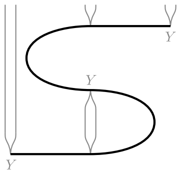
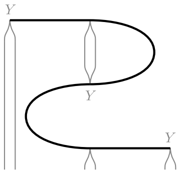


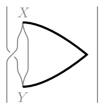
= Id ;



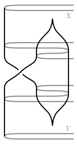
= Id;

3.3.2 Additional swallowtail 3-morphisms and butterfly relations

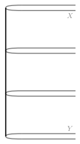
We now construct some 3-morphisms in $\mathcal{F}^{\otimes}(\mathrm{Dual}_3)$ and show that they satisfy certain relations. These will be useful in the rest of this Chapter. The 3-cell



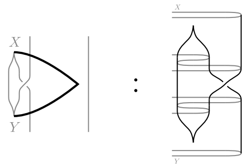
:



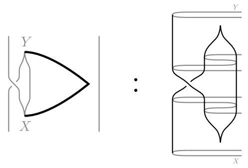
$\Longrightarrow$



in Dual_3 is called a *swallowtail* 3-morphism. We now construct three other invertible swallowtail 3-morphisms

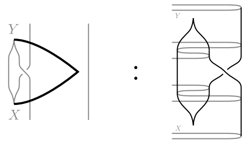


$$\Longrightarrow \quad \mathrm{Id}_{\mathrm{ev}}$$



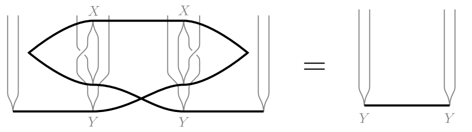
$$\Longrightarrow \quad \mathrm{Id}_{\mathrm{coev}}$$

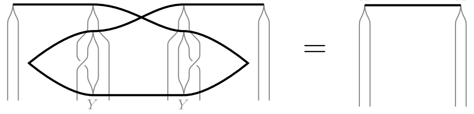
and

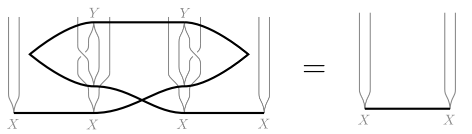


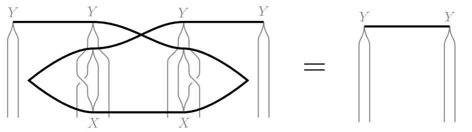
$$\Longrightarrow \quad \mathrm{Id}_{\mathrm{ev}}$$

in $\mathcal{F}^{\otimes}(\mathrm{Dual}_3)$, which satisfy the following *butterfly relations*.



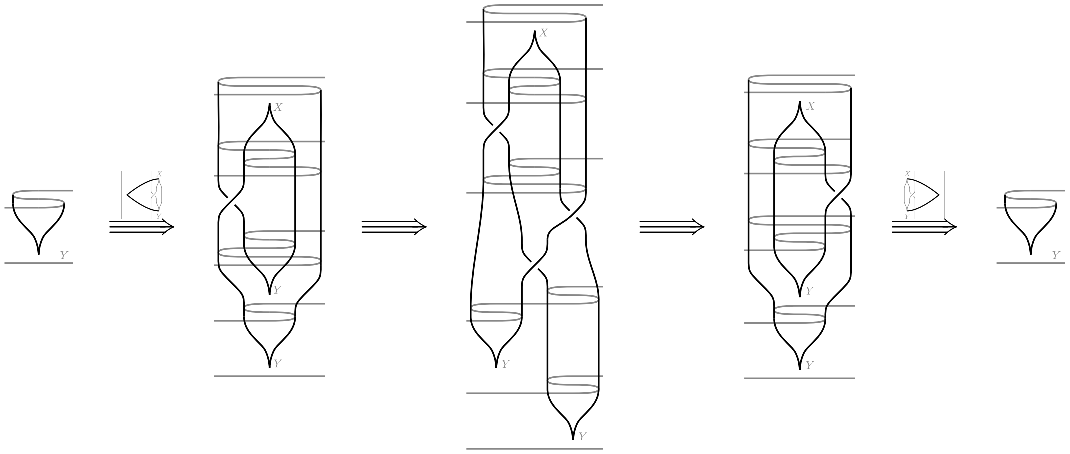






.

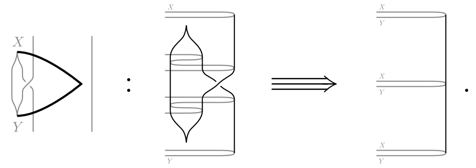
The left hand side of the first relation is the following composite 3-morphism.



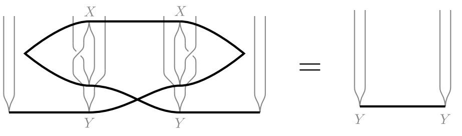
The left hand sides of the three remaining relations are similar composites of 3-morphisms.

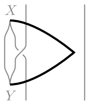
Now we construct the swallowtail 3-morphisms and prove that they satisfy the butterfly relations.

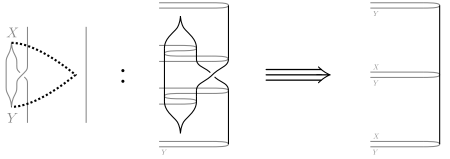
We start by constructing



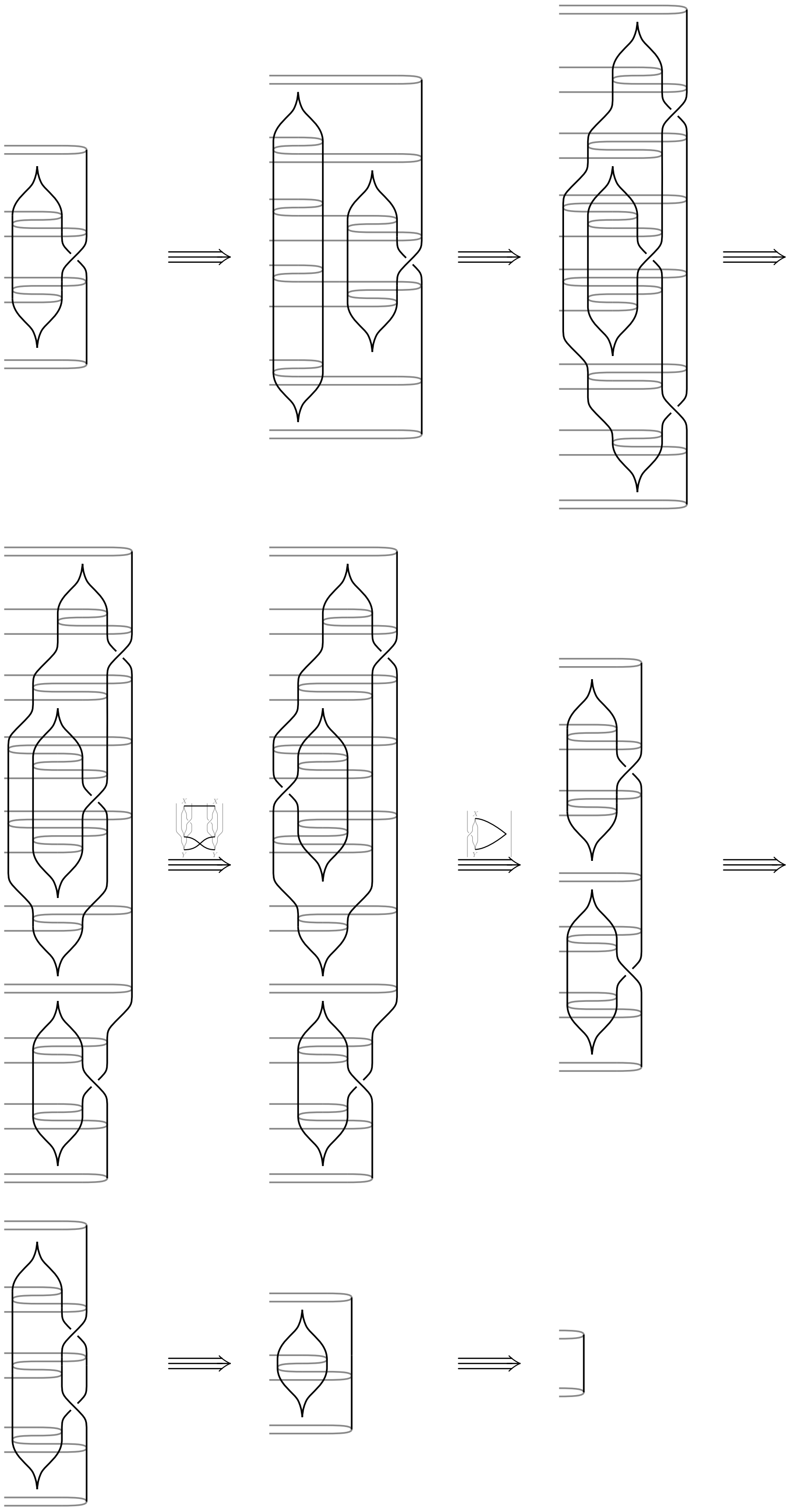
The goal is to have it satisfy the relation



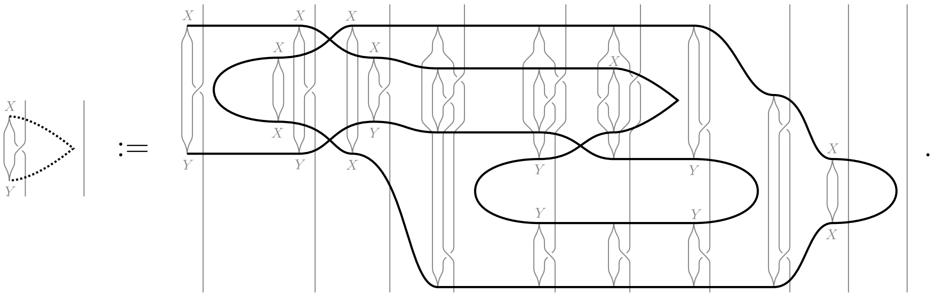
We construct  in two steps. First we define the 3-morphism

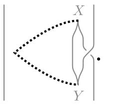


as the following composite.

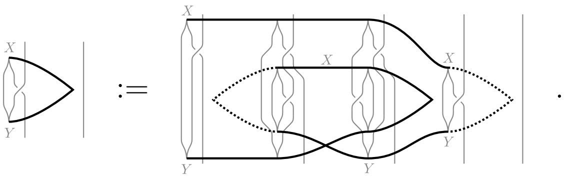


We can also denote this composite as

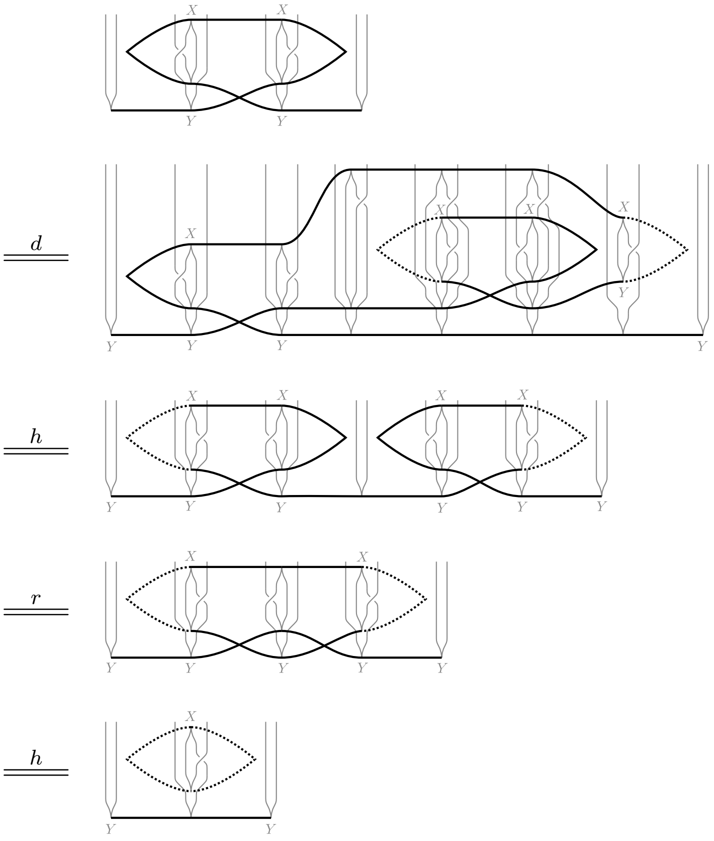


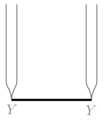
This has an obvious inverse, which we denote by .

Then we define

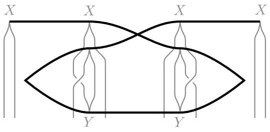


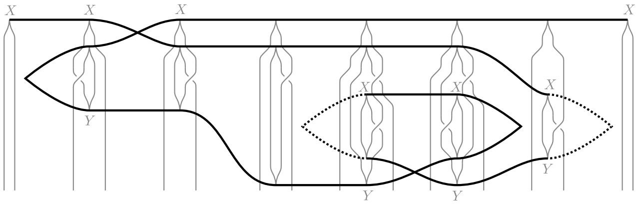
Now this satisfies two butterfly relations.

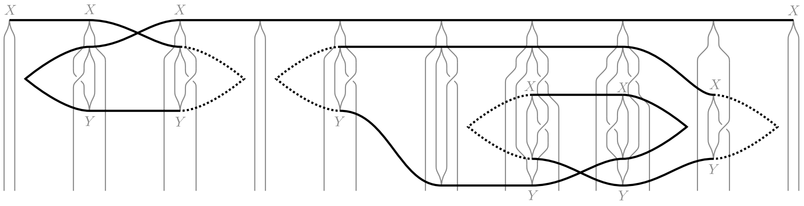


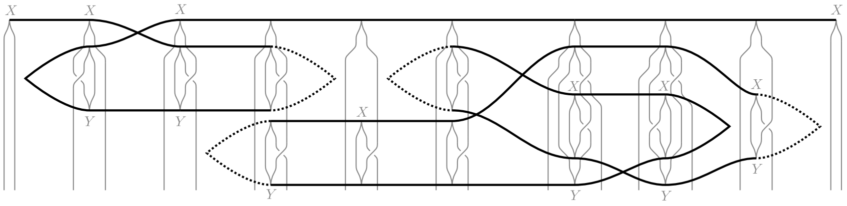
$\overset{r}{=\!=}$ 

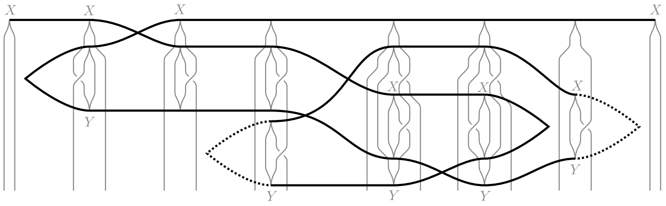
and

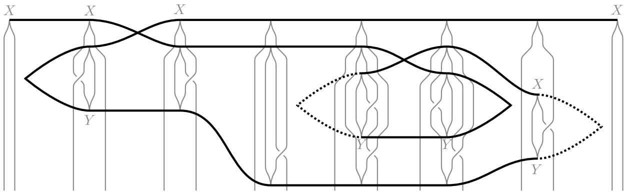


$\overset{d}{=\!=}$ 

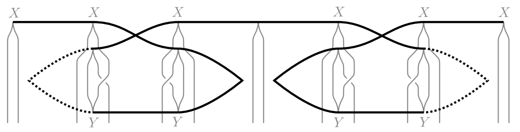
$\overset{r}{=\!=}$ 

$\overset{h}{=\!=}$ 

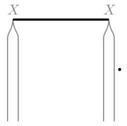
$\overset{r}{=\!=}$ 

$\overset{h}{=\!=}$ 

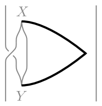
$\overset{h}{=\!=}$



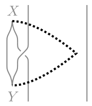
$\overset{r}{=\!=}$



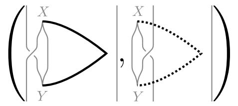
Notice that the first step takes as input an invertible 3-morphism



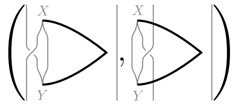
and constructs an invertible 3-morphism



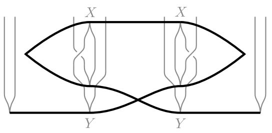
with the right source and target, whereas the second step takes as input any pair of invertible 3-morphisms



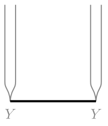
and modifies the second one to produce a pair



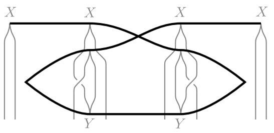
that satisfies the butterfly relations



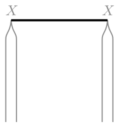
$=$



and

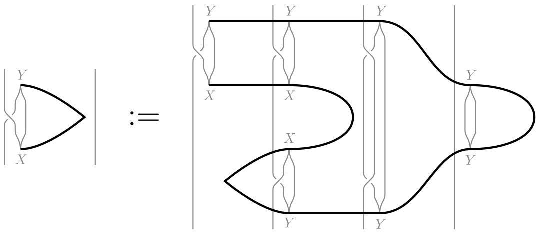


$=$

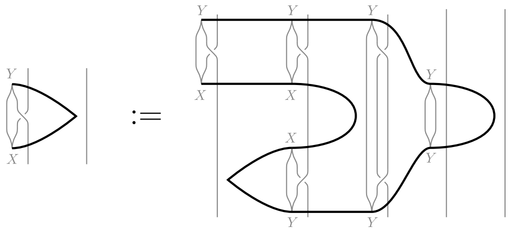


.

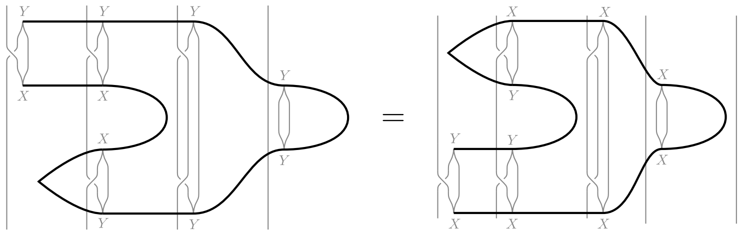
Finally, we define

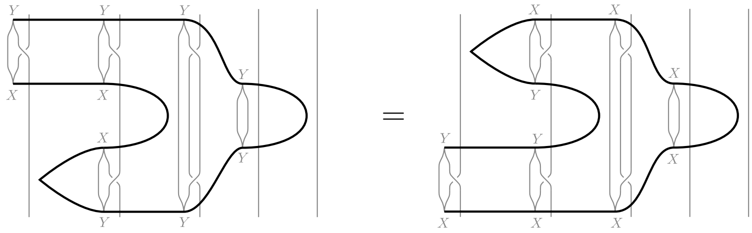


and



Lemma 3.3.2. *The following identities hold.*

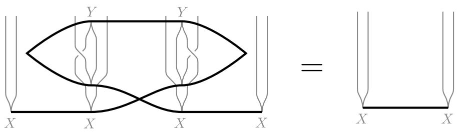


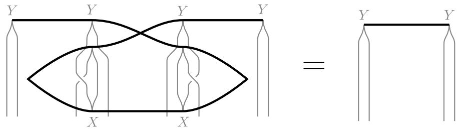


Proof. Each identity is easily proved by composing the left hand side with the inverse of the right hand side and checking the composite is equal to the identity.

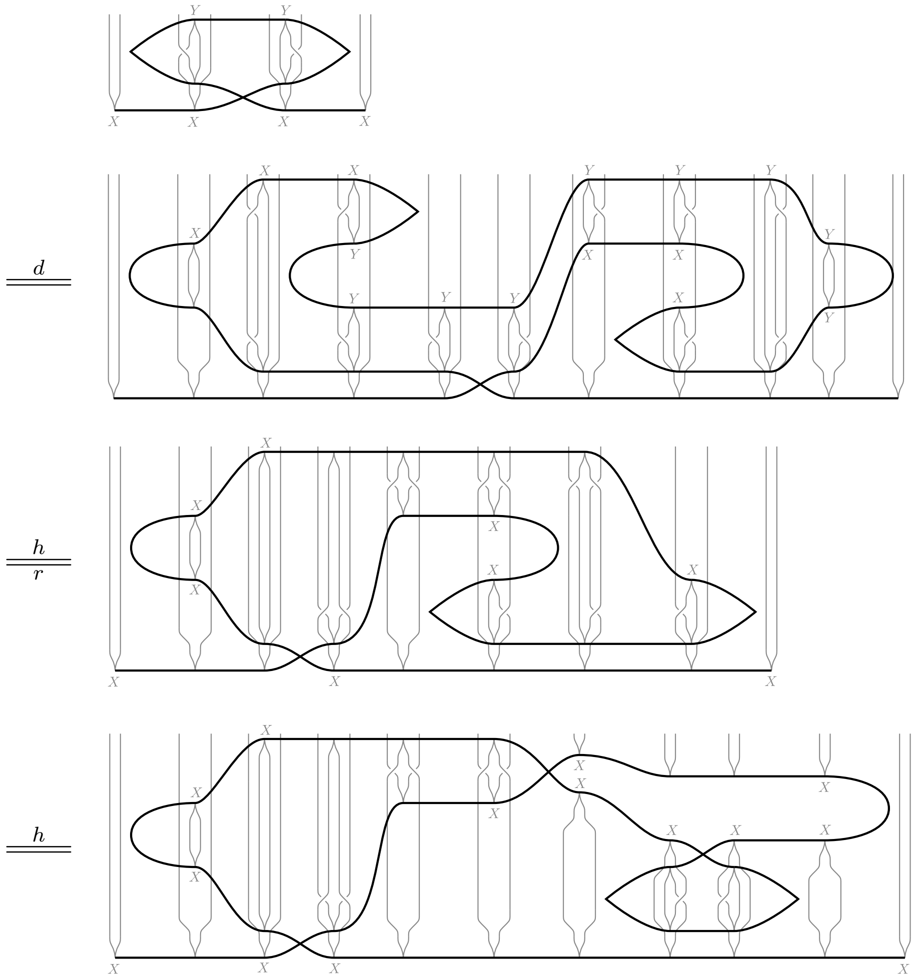
□

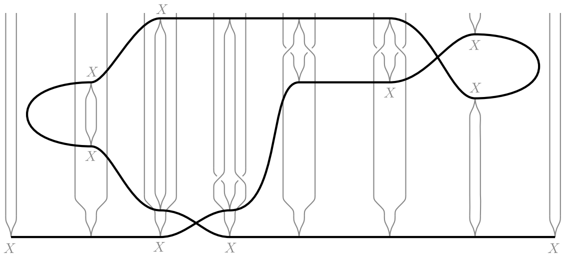
Finally, we have two butterfly identities involving these 3-morphisms.

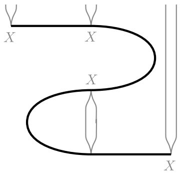


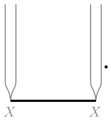


We prove only the first one. The proof of the second one is similar.



$\overset{r}{=}$ 

$\overset{h}{=}$ 

$\overset{r}{=}$  .

3.3.3 Evaluation factors through the 3-groupoid of dual objects

Now we show that the functor $E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ factors through the inclusion $\mathrm{Obj}^R(\mathcal{C}) \hookrightarrow \mathcal{C}$. This means the X component of any functor $\mathrm{Dual}_3 \to \mathcal{C}$ is a right dual and the X component of every natural transformation, modification and perturbation between functors $\mathrm{Dual}_3 \to \mathcal{C}$ is invertible. Similarly, the functor $E_Y : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ factors through the inclusion $\mathrm{Obj}^L(\mathcal{C}) \hookrightarrow \mathcal{C}$. This means the Y component of any functor $\mathrm{Dual}_3 \to \mathcal{C}$ is a left dual and the Y component of every natural transformation, modification and perturbation between functors $\mathrm{Dual}_3 \to \mathcal{C}$ is invertible. Note that this already implies that $\mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C})$ is actually a 3-groupoid, that is $\mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) = \mathrm{Map}(\mathrm{Dual}_3, \mathcal{C})$.

Lemma 3.3.3. *The restriction functor* $E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathcal{C}$ *factors through* $\mathrm{Obj}^R(\mathcal{C})$.

Proof. Given $F \in \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C})$, the object $E_X(F) = F(X)$ has $F(Y)$ as a left dual.

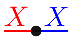
Given a natural transformation $\alpha : F \to G$, we need to show that

$$E_X(\alpha) = \alpha_X : F(X) \to G(X)$$

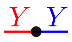
is an equivalence. We will denote the images of cells in Dual_3 under F and G by coloring them red and blue, respectively. We denote

$$\alpha_X : F(X) \to G(X) \text{ and } \alpha_Y : F(Y) \to G(Y)$$

by



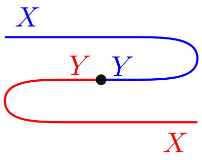
and



respectively.

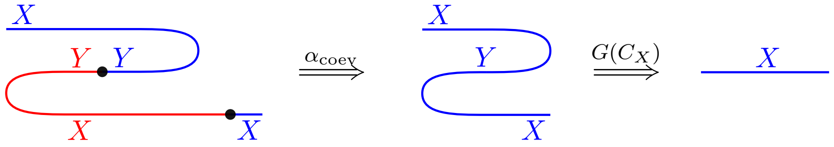
Then we can define an inverse $\alpha_X^{-1} : G(X) \to F(X)$ by setting

$\alpha_X^{-1} \quad :=$

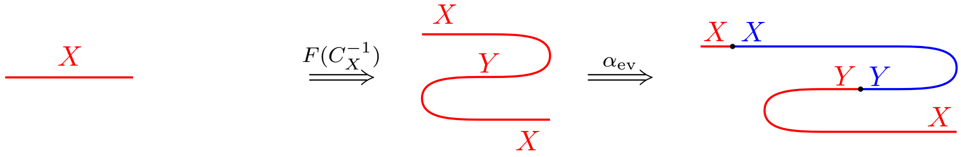


.

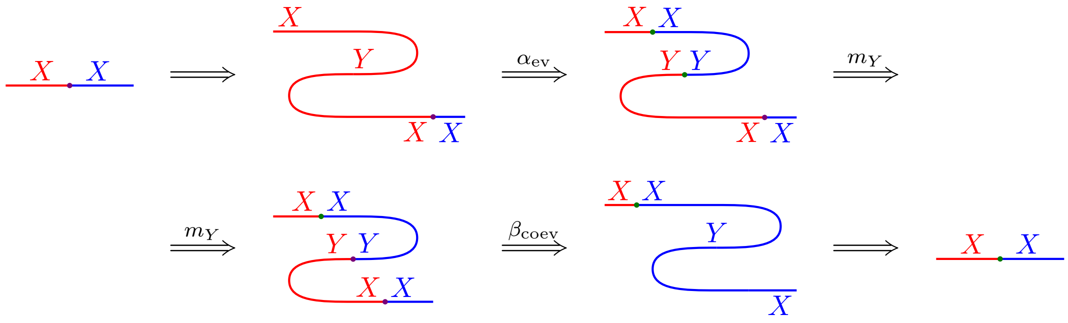
The fact that this is an inverse to α_X follows from the existence of invertible 2-morphisms



and

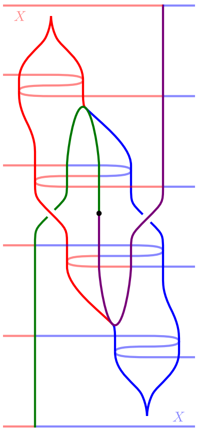


Now let $m : \alpha \Rightarrow \beta$ be a modification. We define an inverse $m_X^{-1} : \beta_X \Rightarrow \alpha_X$ as the following composite.

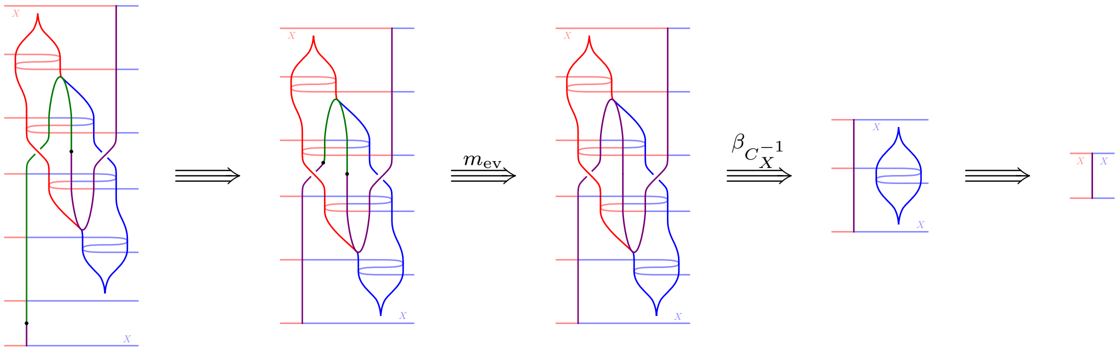


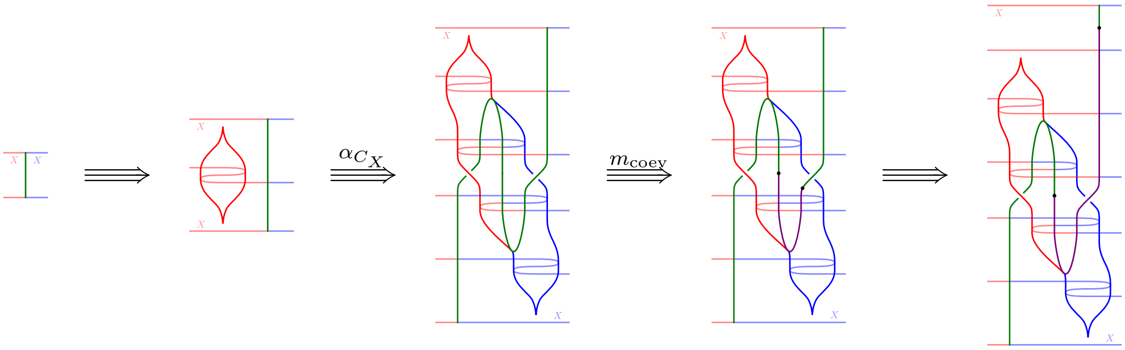
We can also denote this composite as follows.

$$m_X^{-1} \quad :=$$

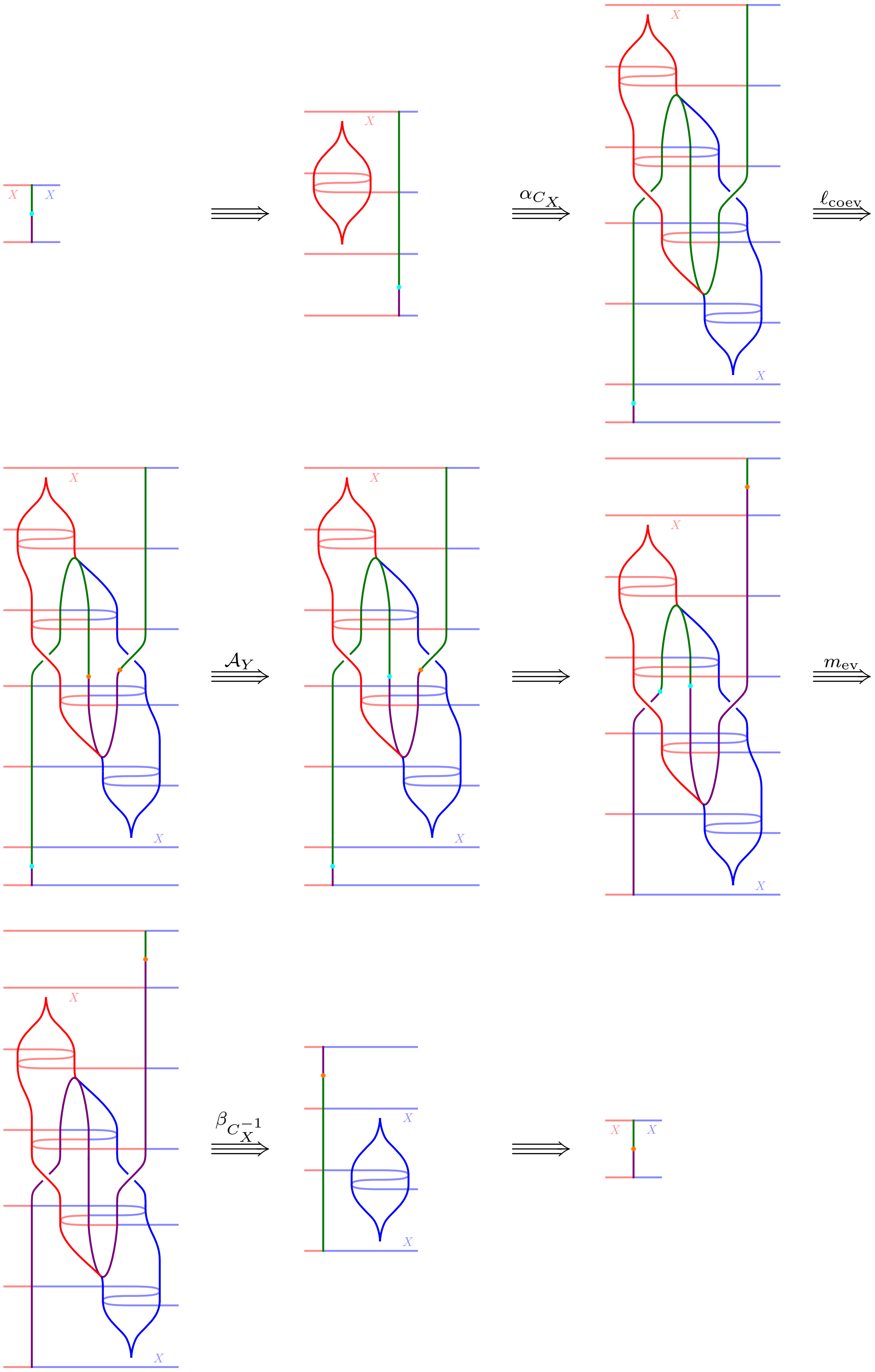


The fact that this is inverse to m_X follows from the existence of the following invertible 3-morphisms.



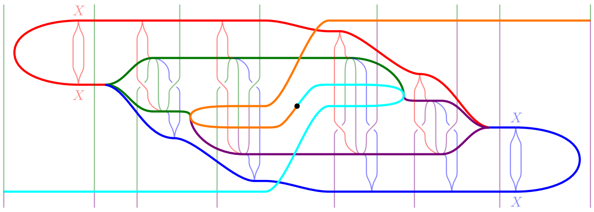


Now let $\mathcal{A} : \ell \Rrightarrow m$ be a perturbation. We define an inverse $\mathcal{A}_X^{-1} : m_X \Rrightarrow \ell_X$ as the following composite.

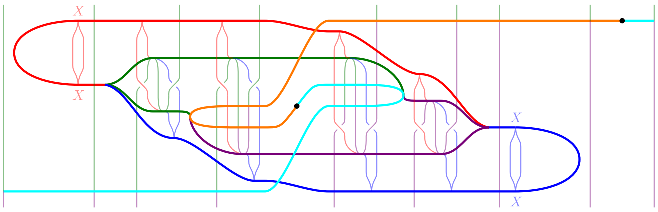


We can also denote this composite as follows.

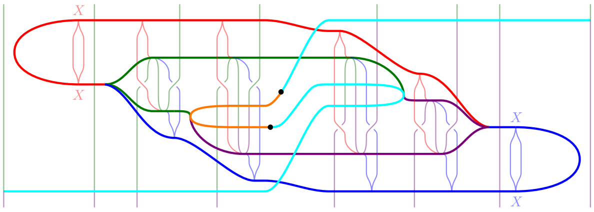
$$\mathcal{A}_X^{-1} \quad :=$$



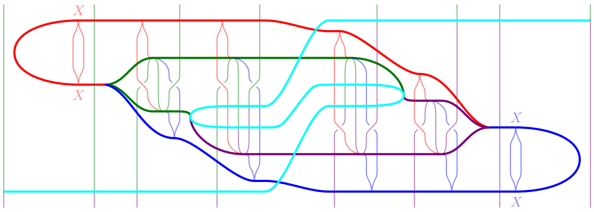
The fact that this is an inverse to $\mathcal{A}_X$ follows from the following identities.



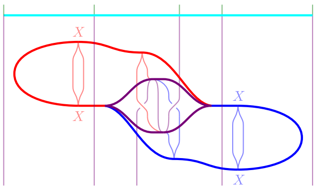
$$\overset{h}{=\!=}$$



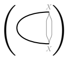
$$\overset{\mathcal{A}_{\mathrm{coev}}}{=\!=}$$



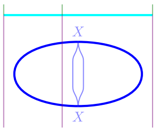
$$\overset{m_{C_X}}{=\!=}$$



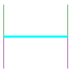
β



$$=\!=$$



$$\overset{r}{=\!=}$$



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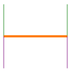
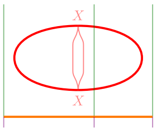
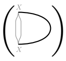
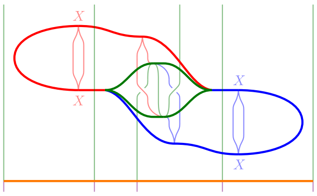
and



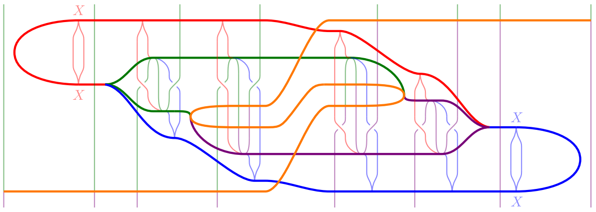
$$\overset{r}{=}$$



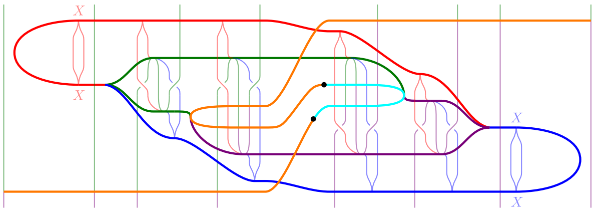
α() $=$



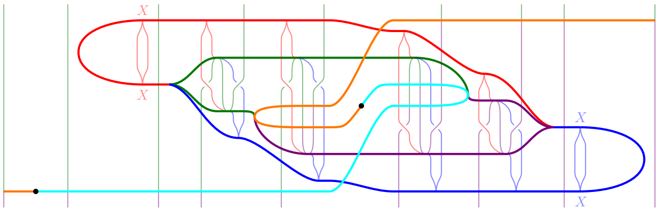
$$\overset{\ell_{C_X^{-1}}}{=}$$



$$\overset{\mathcal{A}_{\mathrm{ev}}}{=}$$



$$\overset{h}{=}$$



.

□

3.3.4 Surjective on objects

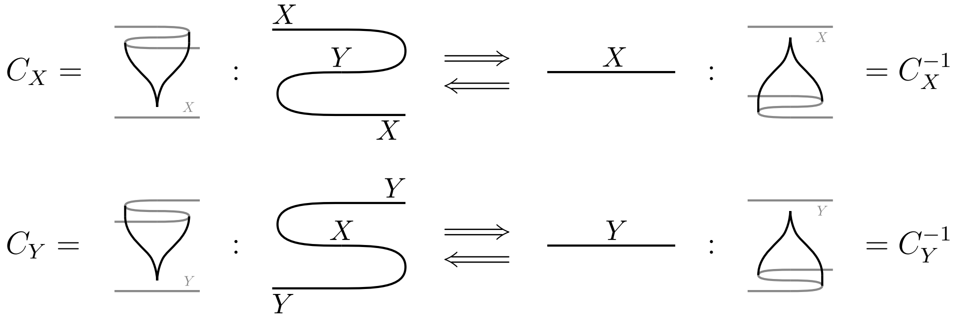
Lemma 3.3.4. *The restriction functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

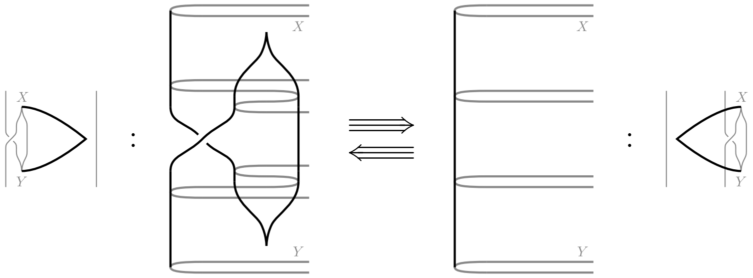
is surjective on objects.

Proof. Given an object $x \in \mathcal{C}$ with a left dual, we need to find a functor $F : \mathcal{F}(\mathcal{P}_{(3,1)}) \to \mathcal{C}$ such that $F(X) = x$. Pick a dual object y, together with evaluation and coevaluation morphisms $\mathrm{ev} : xy \to 1$ and $\mathrm{coev} : 1 \to yx$ satisfying the snake equations up to 2-isomorphism.

Pick cusp 2-morphisms with inverses, implemeting the snake equations.

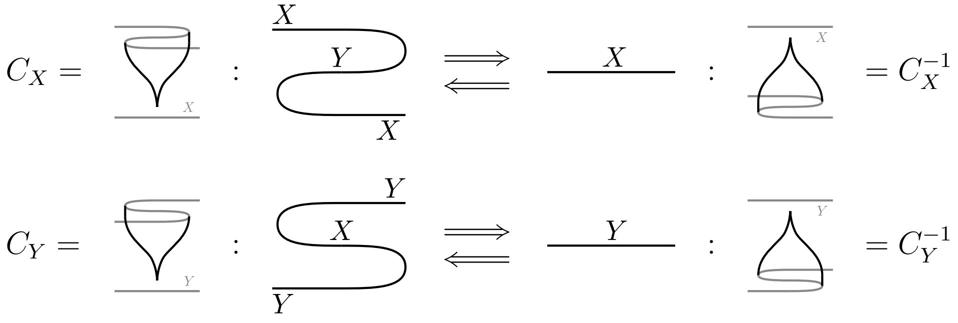


Now we must pick a swallowtail cancellation 3-morphism and its inverse

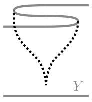


The following Lemma implies that this can be done, possibly after after replacing C_Y and C_Y^{-1} by a different pair of inverse 2-morphisms.

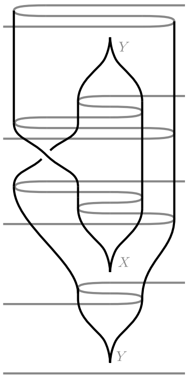
Lemma 3.3.5. *Suppose we have chosen* 2*-morphisms*



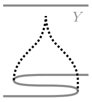
Define



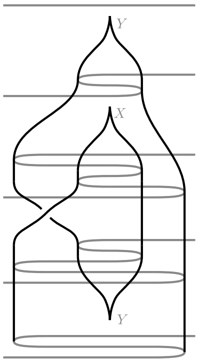
$:=$



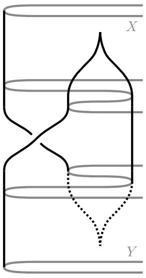
and



$:=$

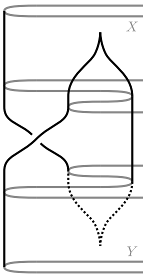


These are inverse to each other and there exists an invertible 3*-morphism*

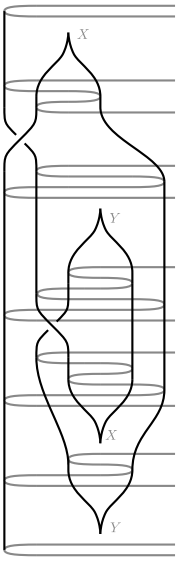


$\overset{\sim}{\Longrightarrow} \quad \mathrm{Id}$

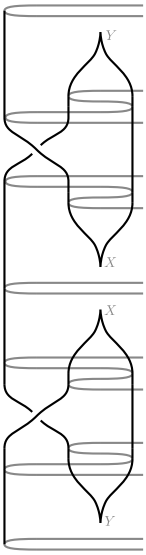
Proof. The fact that the new cusp 2-morphisms are inverse to each other is clear. We define the required 3-morphism as follows.



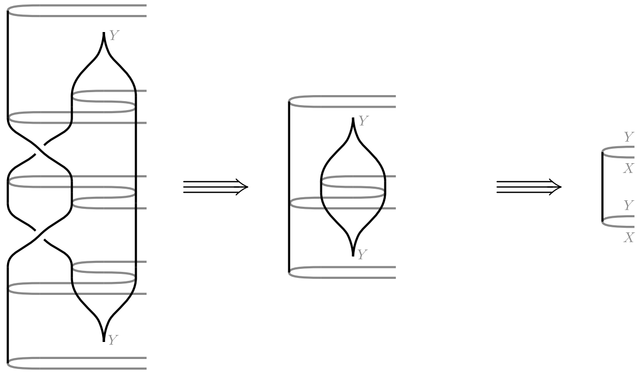
$\overset{d}{=\!=}$



$\overset{h}{\Longrightarrow}$

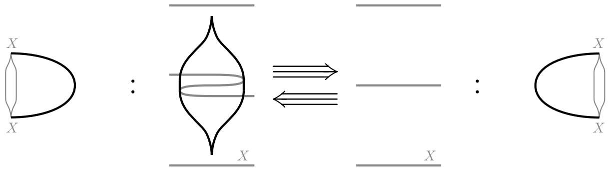


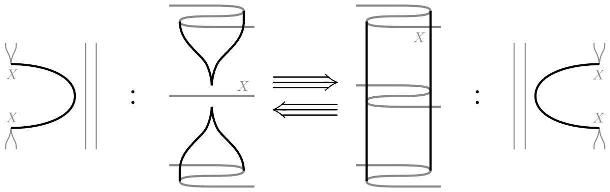
$\Longrightarrow$

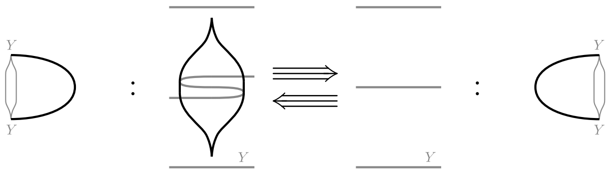


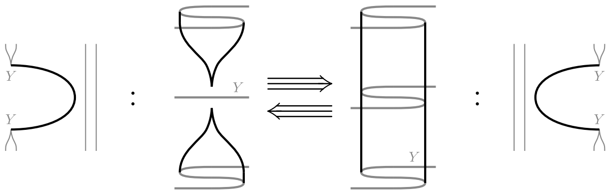
□

Finally, we need to pick cusp cancellation and creation 3-morphisms









which are inverse to each other and satisfy appropriate snake relations. This can be done, by Lemma 3.2.15.

□

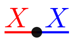
3.3.5 Surjective on 1-morphisms

Lemma 3.3.6. *The restriction functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

is surjective on 1*-morphisms.*

Proof. Let $\alpha_X : F(X) \to G(X)$ be an equivalence in $\mathcal{C}$, where F, G are functors $\mathcal{F}(\mathrm{Dual}_3) \to \mathcal{C}$. We want to extend this to a natural transformation $\alpha : F \to G$. We use red for F, blue for G and black for α. Denote

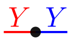
$\alpha_X =$ 

$: F(X) \to G(X).$

.

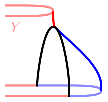
A natural transformation $\alpha : F \to G$ consists of the following additional data and relations.

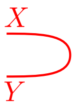
0. A 1-morphism

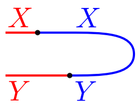
$\alpha_Y =$ 

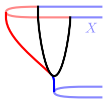
$: F(Y) \to G(Y);$

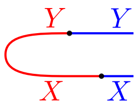
1. invertible 2-morphisms

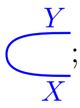
$\alpha_{\mathrm{ev}} :=$ 

$:$ 

$\Longrightarrow$ 

$\alpha_{\mathrm{coev}} :=$ 

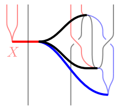
$:$ 

$\Longrightarrow$ 

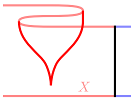
;

2. invertible 3-morphisms

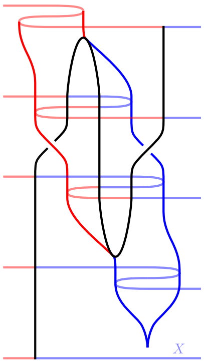
$\alpha_{C_X} :=$



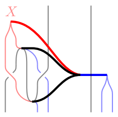
$:$



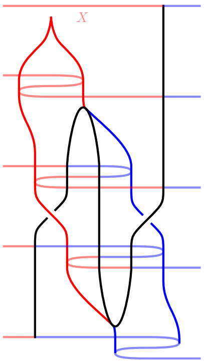
$\Longrightarrow$



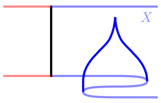
$\alpha_{C_X^{-1}} :=$



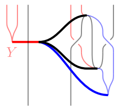
$:$



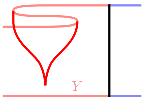
$\Longrightarrow$



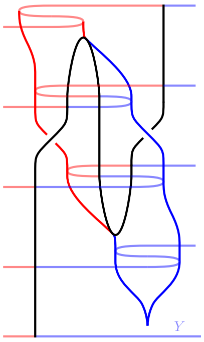
$\alpha_{C_Y} :=$



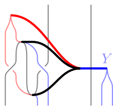
$:$



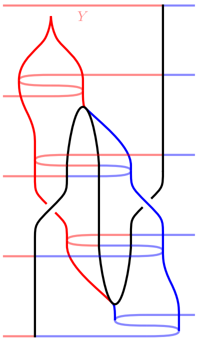
$\Longrightarrow$



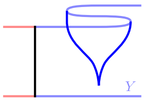
$\alpha_{C_Y^{-1}} :=$



$:$

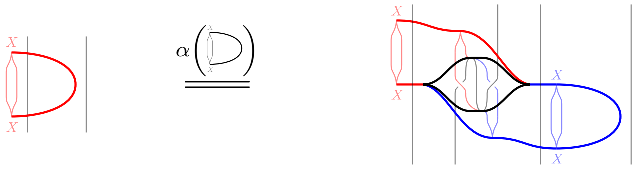


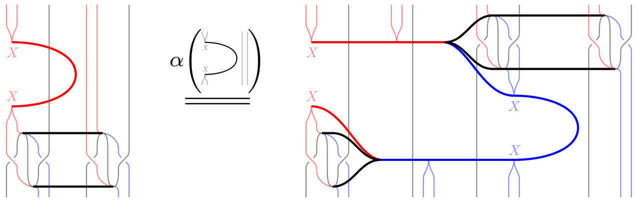
$\Longrightarrow$

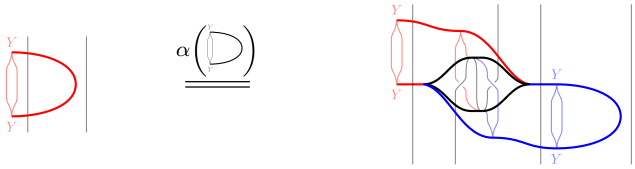


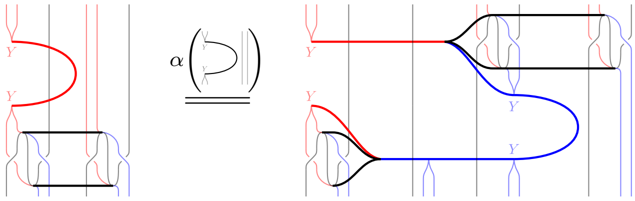
;

3. relations

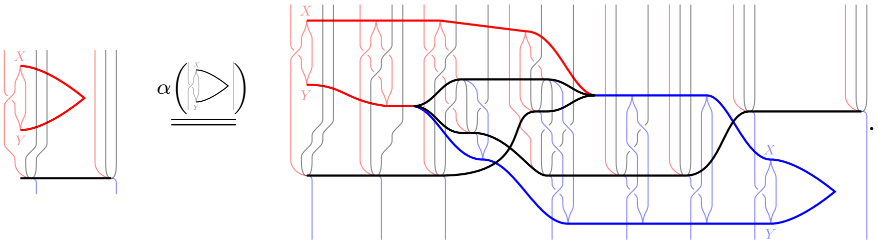


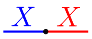






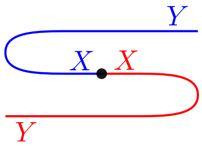
and



We pick a coherent inverse α_X^{-1} for α_X and denote it by 

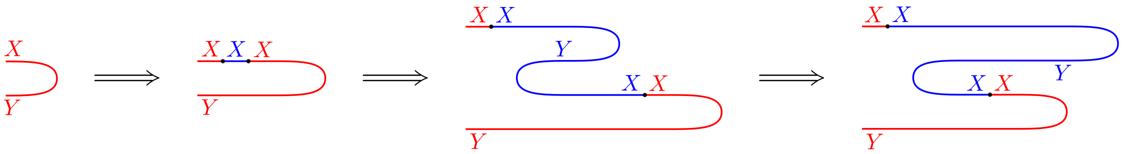
.
Define α_Y as follows.

$\alpha_Y \quad :=$



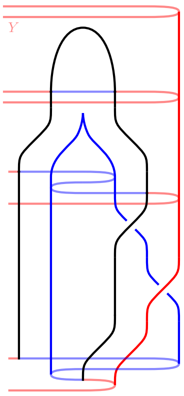
.

Define α_{ev} as the following composite.

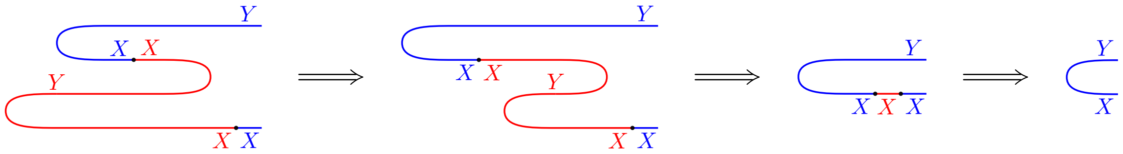


Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$\alpha_{\mathrm{ev}} \quad :=$

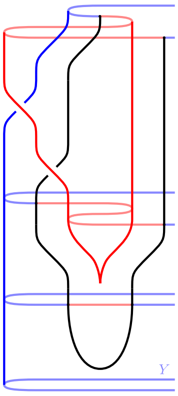


Define α_{coev} as the following composite.

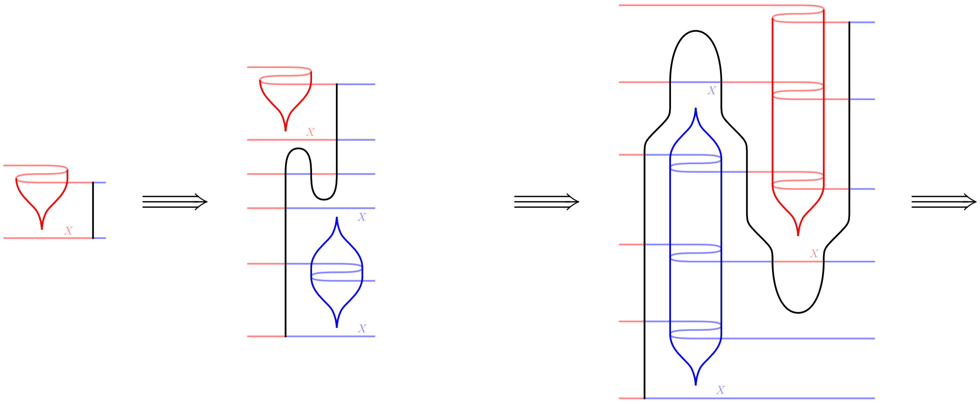


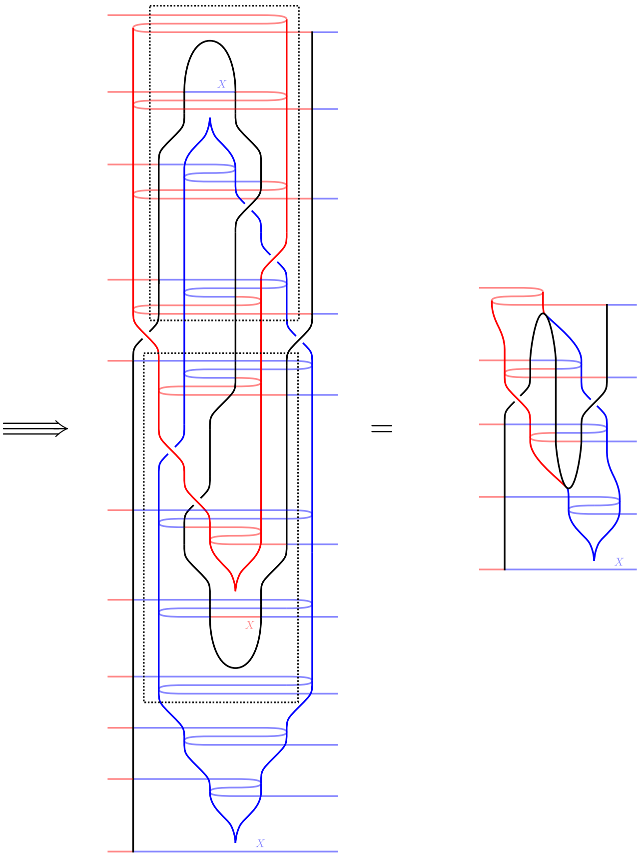
Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$$\alpha_{\mathrm{coev}} \quad :=$$



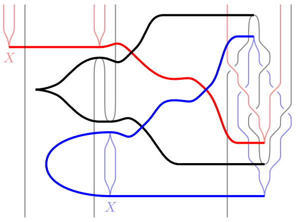
Define α_{C_X} as the following composite. We occasionally use doted boxes to indicate the parts of the diagram which are going to change in the following step.



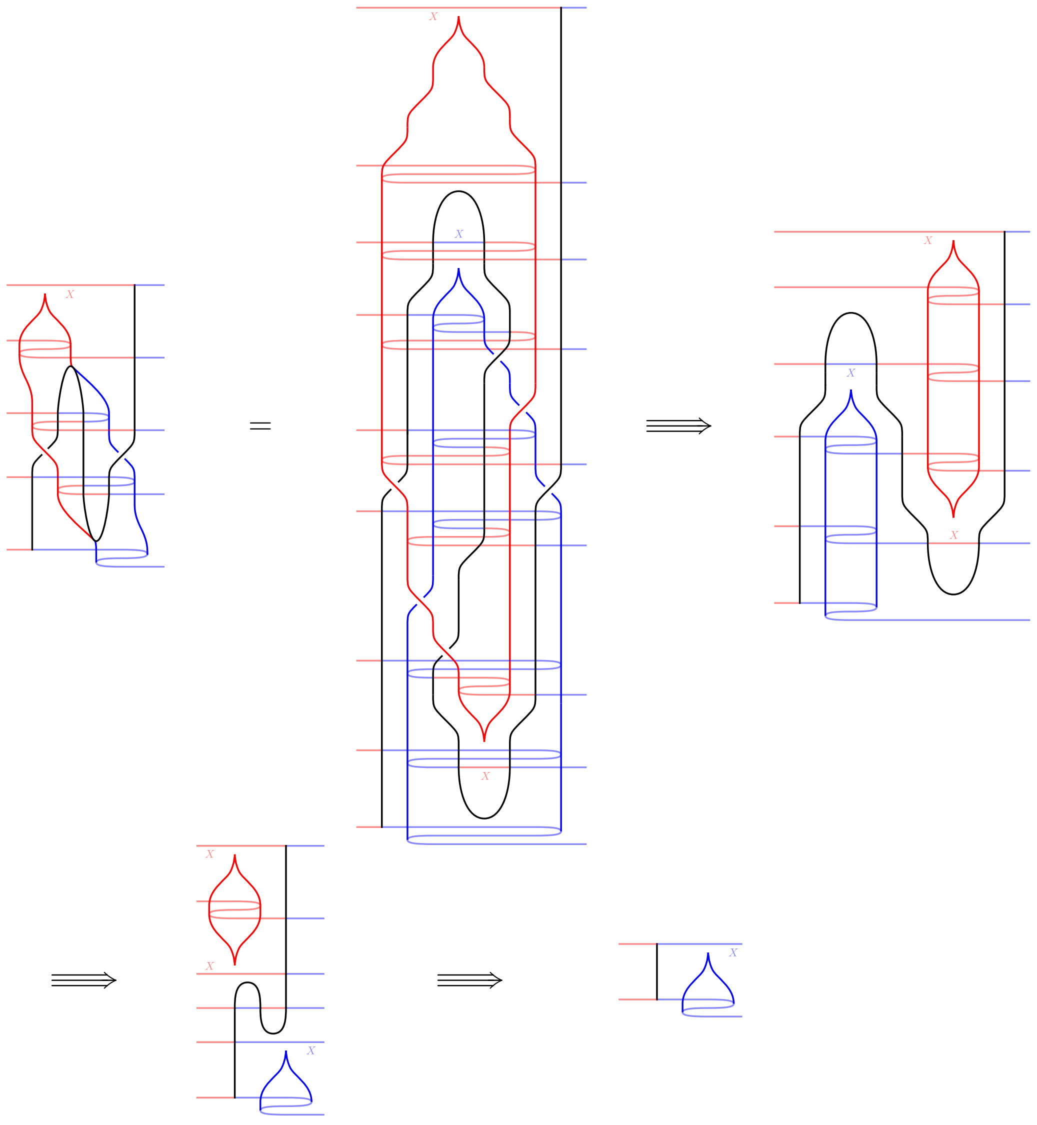


Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$$\alpha_{C_X} \quad :=$$

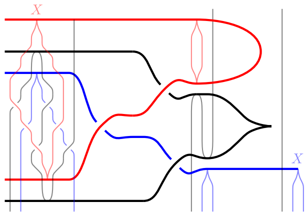


Define $\alpha_{C_X^{-1}}$ as the following composite.

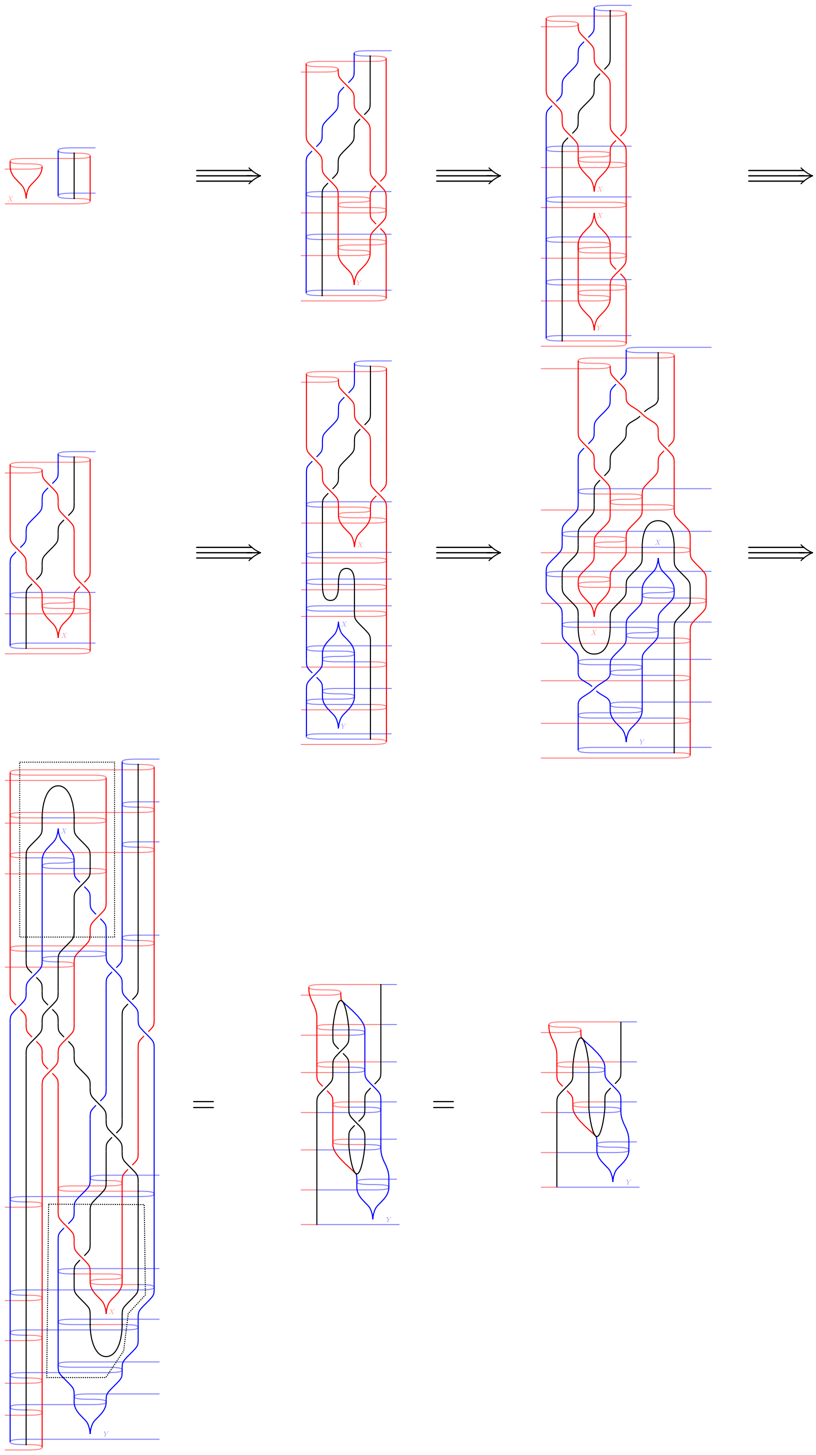


Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$$\alpha_{C_X^{-1}} :=$$

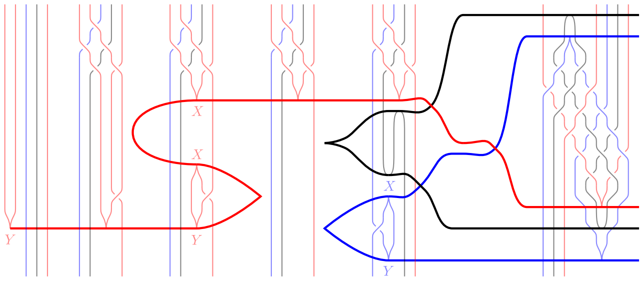


Define α_{C_Y} as the following composite.



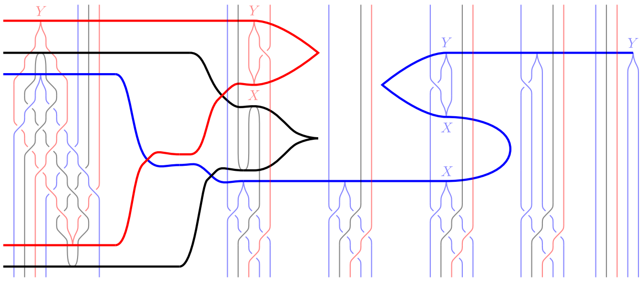
Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$$\alpha_{C_Y} \quad :=$$

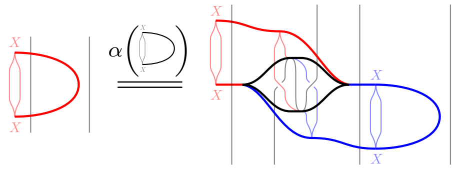


We define $\alpha_{C_Y^{-1}}$ similarly.

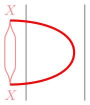
$$\alpha_{C_Y^{-1}} \quad :=$$



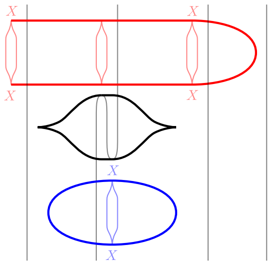
Now for each 3-morphism in Dual_3 we need to show that the above data satisfies an appropriate relation. We prove the relation



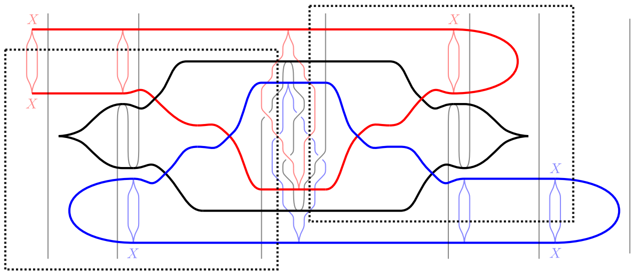
as follows.



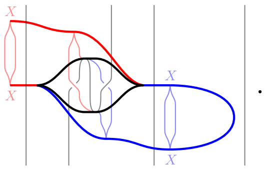
$$\overset{r}{=}$$



$\overset{h}{=\!=}$

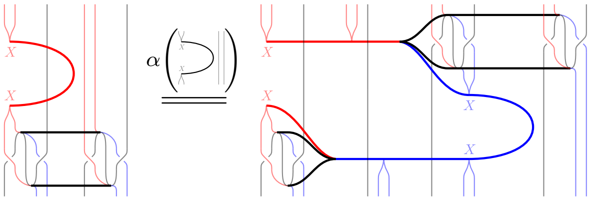


$\overset{d}{=\!=}$

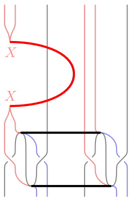


.

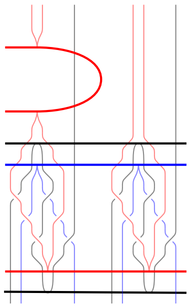
We prove the relation

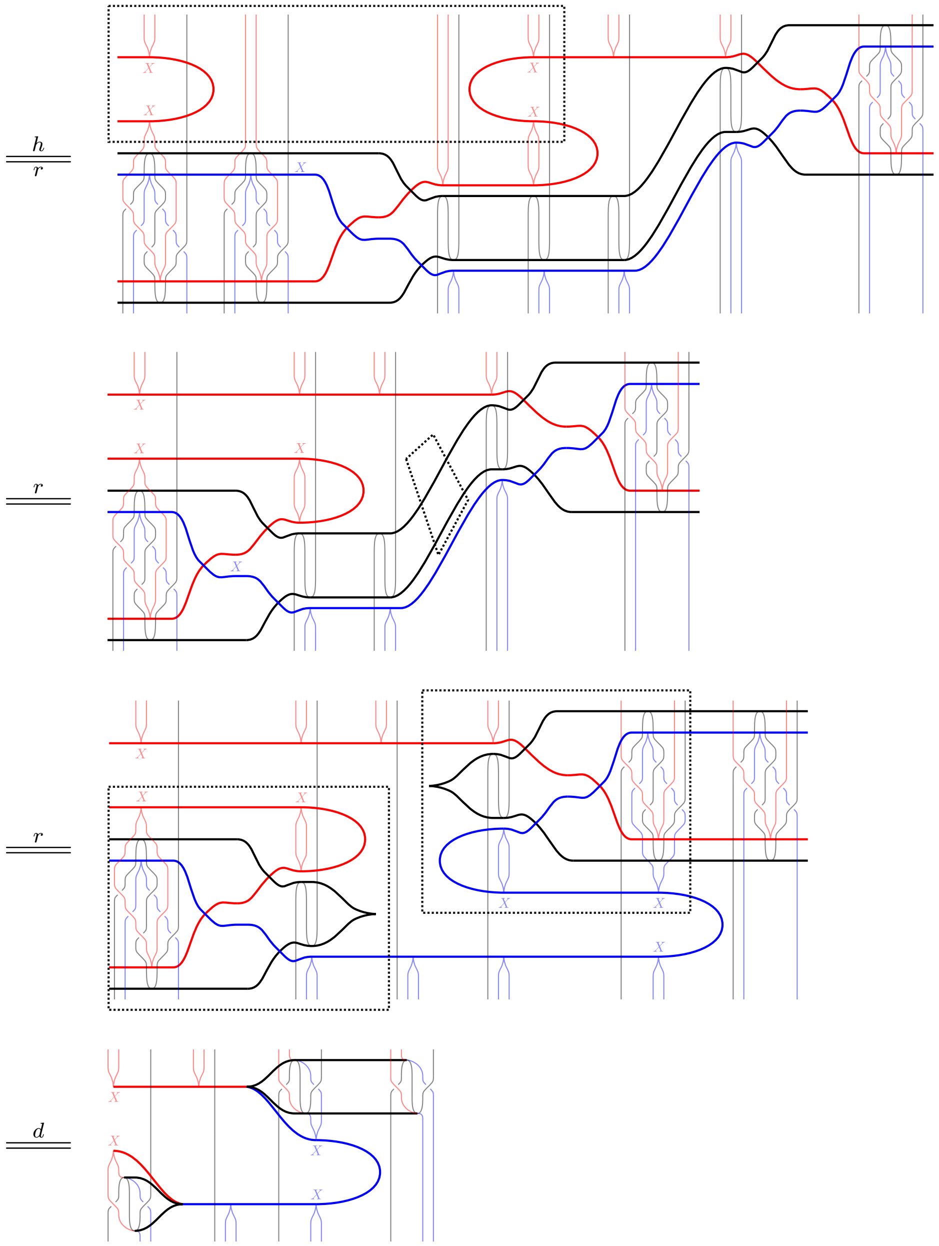


as follows.

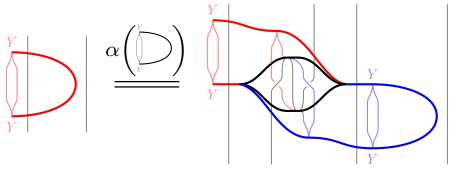


$\overset{d}{=\!=}$

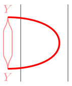




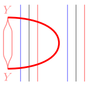
We prove the relation



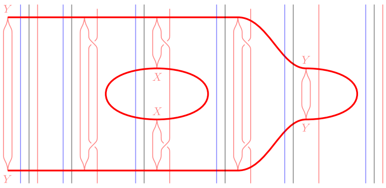
as follows.



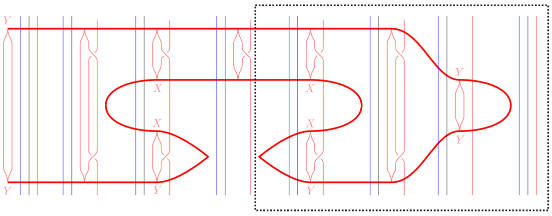
$$\overset{d}{=\!=}$$



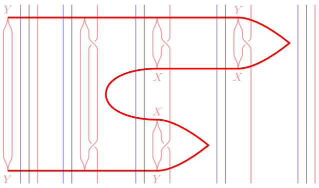
$$\overset{r}{=\!=}$$



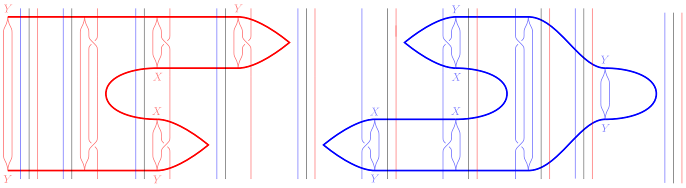
$$\overset{r}{=\!=}$$



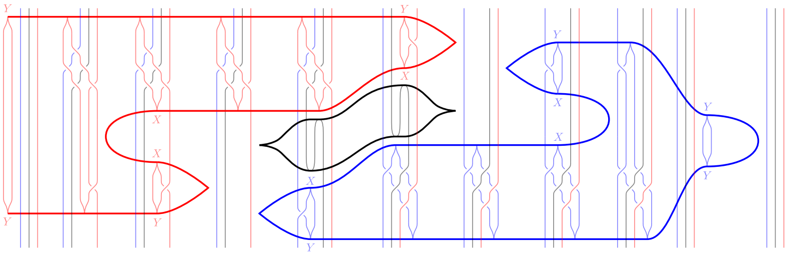
$$\overset{r}{=\!=}$$



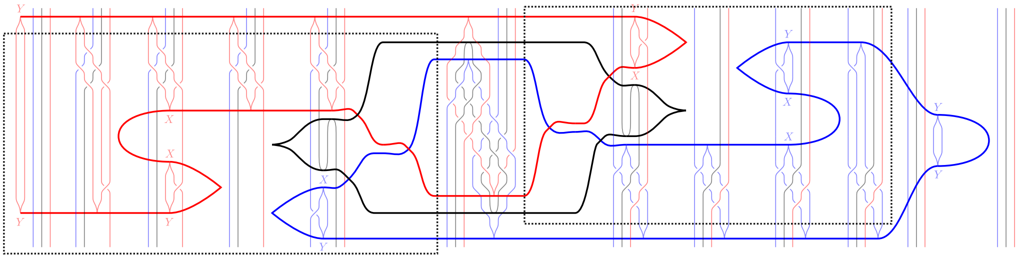
$$\overset{r}{=\!=}$$



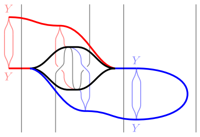
$$\overset{r}{\underset{h}{=\!=}}$$



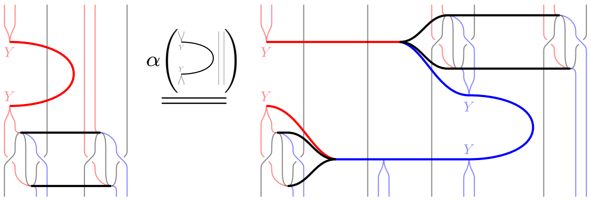
$$\overset{h}{=\!=}$$



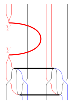
$\overset{d}{=}$



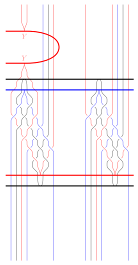
We prove the relation



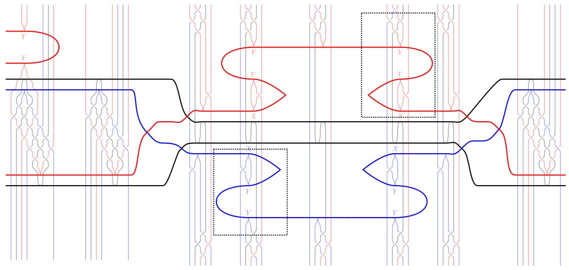
as follows.



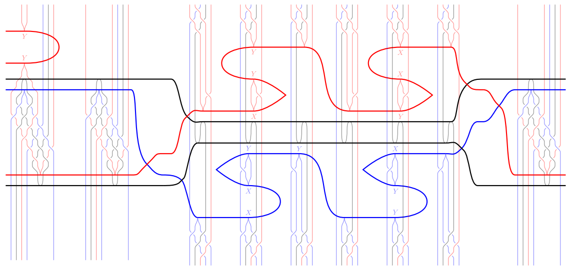
$\overset{d}{=}$



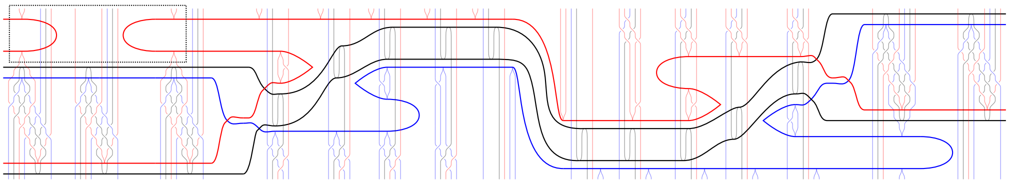
$\underset{r}{\overset{h}{=}}$



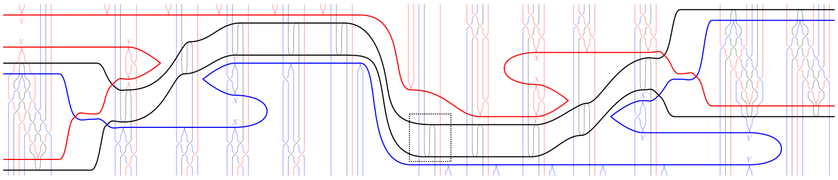
$\overset{r}{=}$



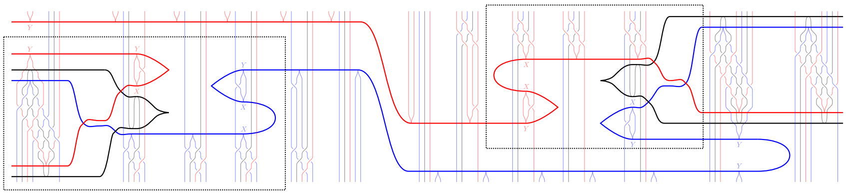
$\overset{h}{=\!=}$



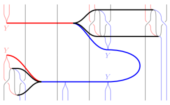
$\overset{r}{=\!=}$



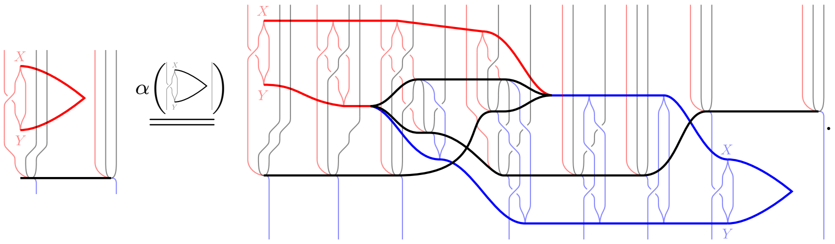
$\overset{r}{=\!=}$



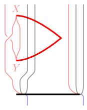
$\overset{d}{=\!=}$



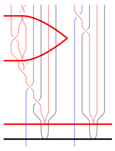
We prove the relation



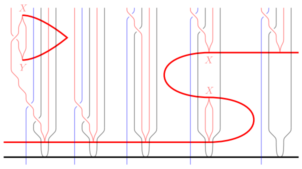
as follows.



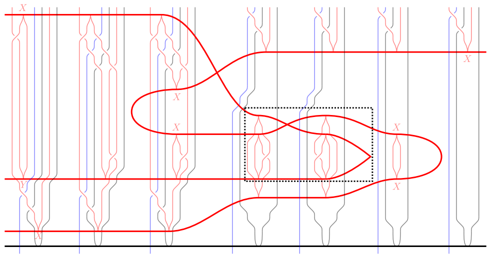
$\overset{d}{=\!=}$



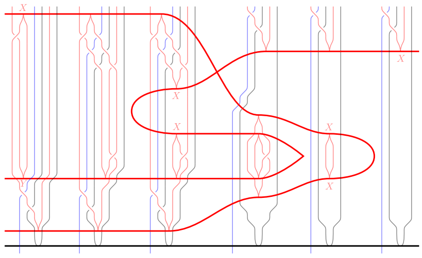
$\overset{r}{=\!=}$



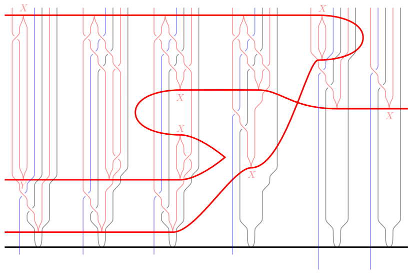
$\overset{h}{=\!=}$



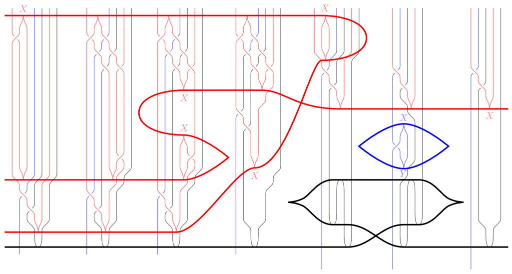
$\overset{r}{=\!=}$



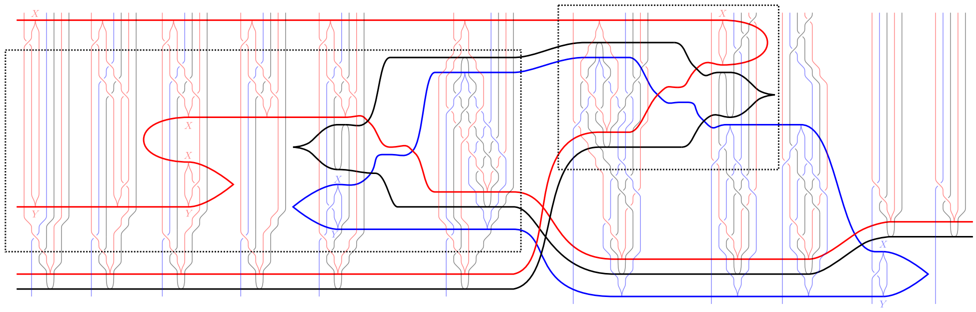
$\overset{h}{=\!=}$



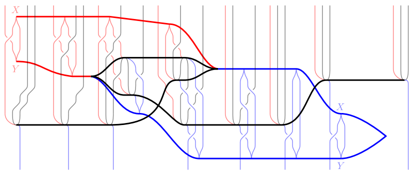
$\overset{r}{=\!=}$



$\overset{h}{=\!=}$



$\overset{d}{=\!=}$



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□

3.3.6 Surjective on 2-morphisms

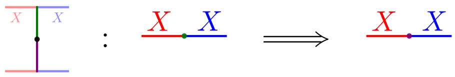
Lemma 3.3.7. *The restriction functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

is surjective on 2-morphisms.

Proof. Now let $\alpha, \beta : F \Rightarrow G$ be natural transformations and $m_X : \alpha_X \Rightarrow \beta_X$ an invertible 2-morphism. We use red for F, blue for G, green for α, purple for β and black for m. Denote

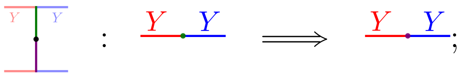
$m_X =$



We want to extend m_X to a modification $m : \alpha \Rightarrow \beta$, which consists of the following data.

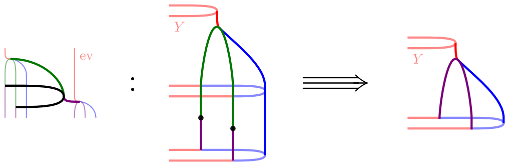
0. A 2-morphism

$m_Y =$

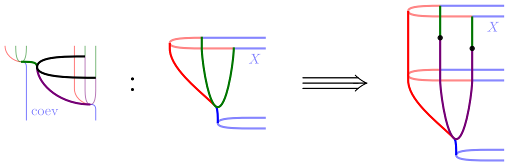


1. invertible 3-morphisms

$m_{\mathrm{ev}} =$

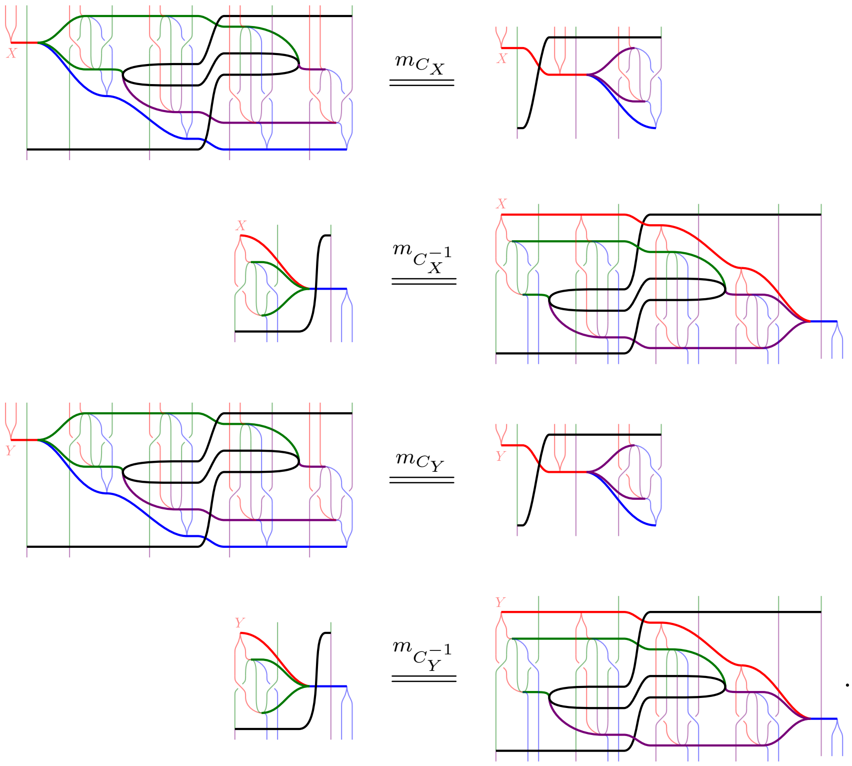


$m_{\mathrm{coev}} =$

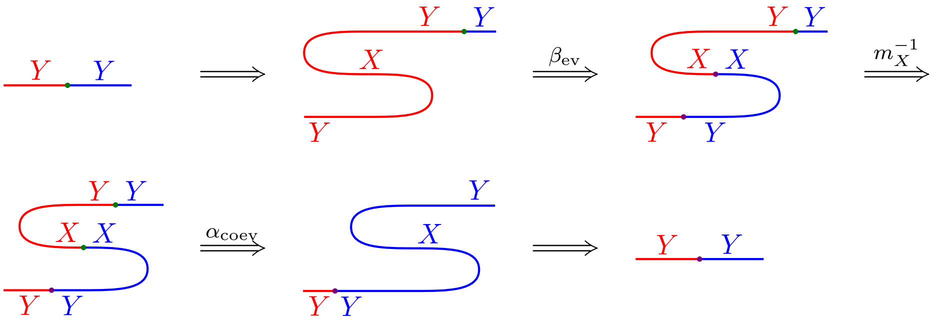


;

2. relations

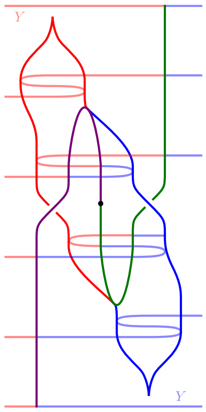


Pick a coherent inverse m_X^{-1} for m_X. Define m_Y as the following composite.

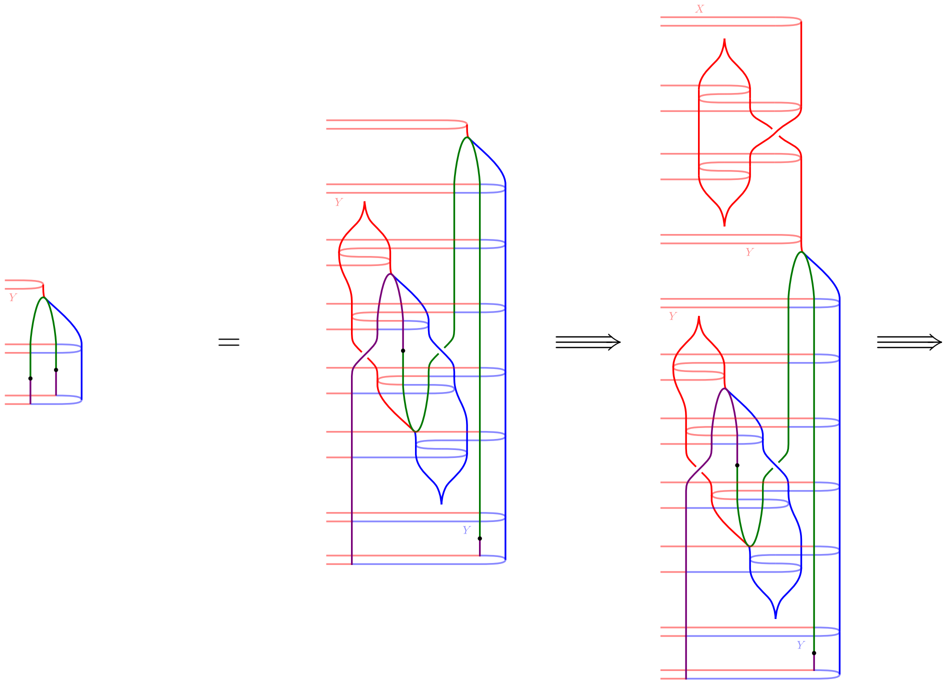


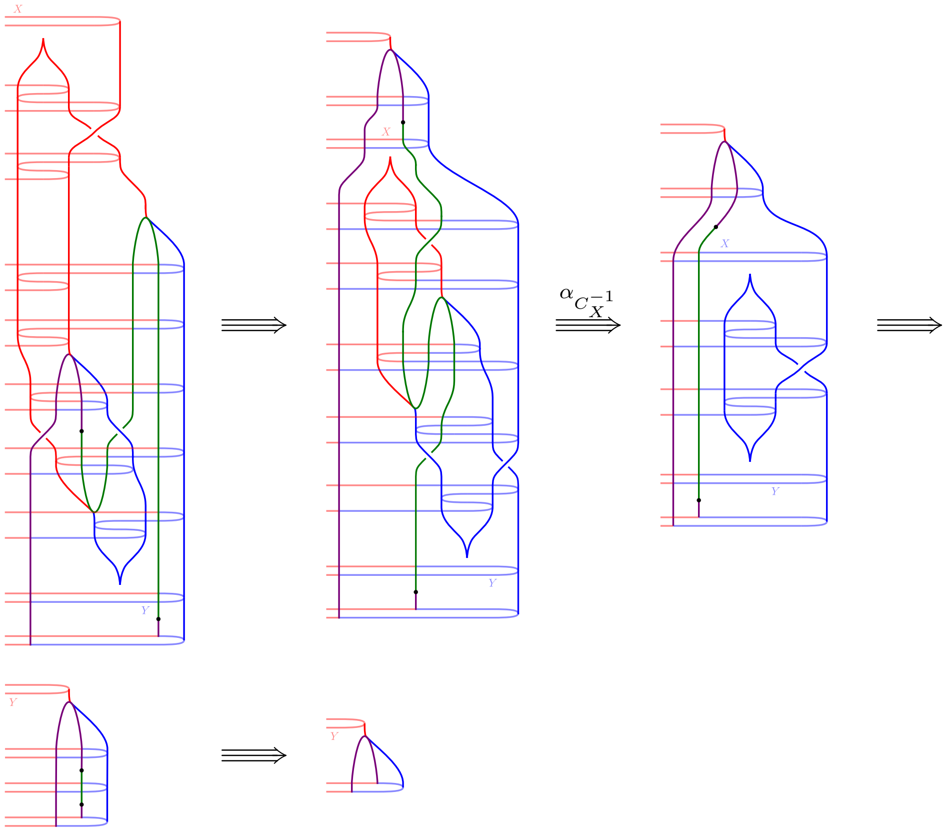
Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$m_Y \quad :=$



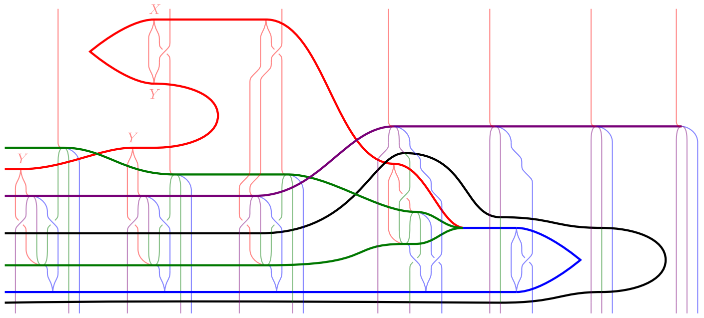
Define m_{ev} as the following composite.



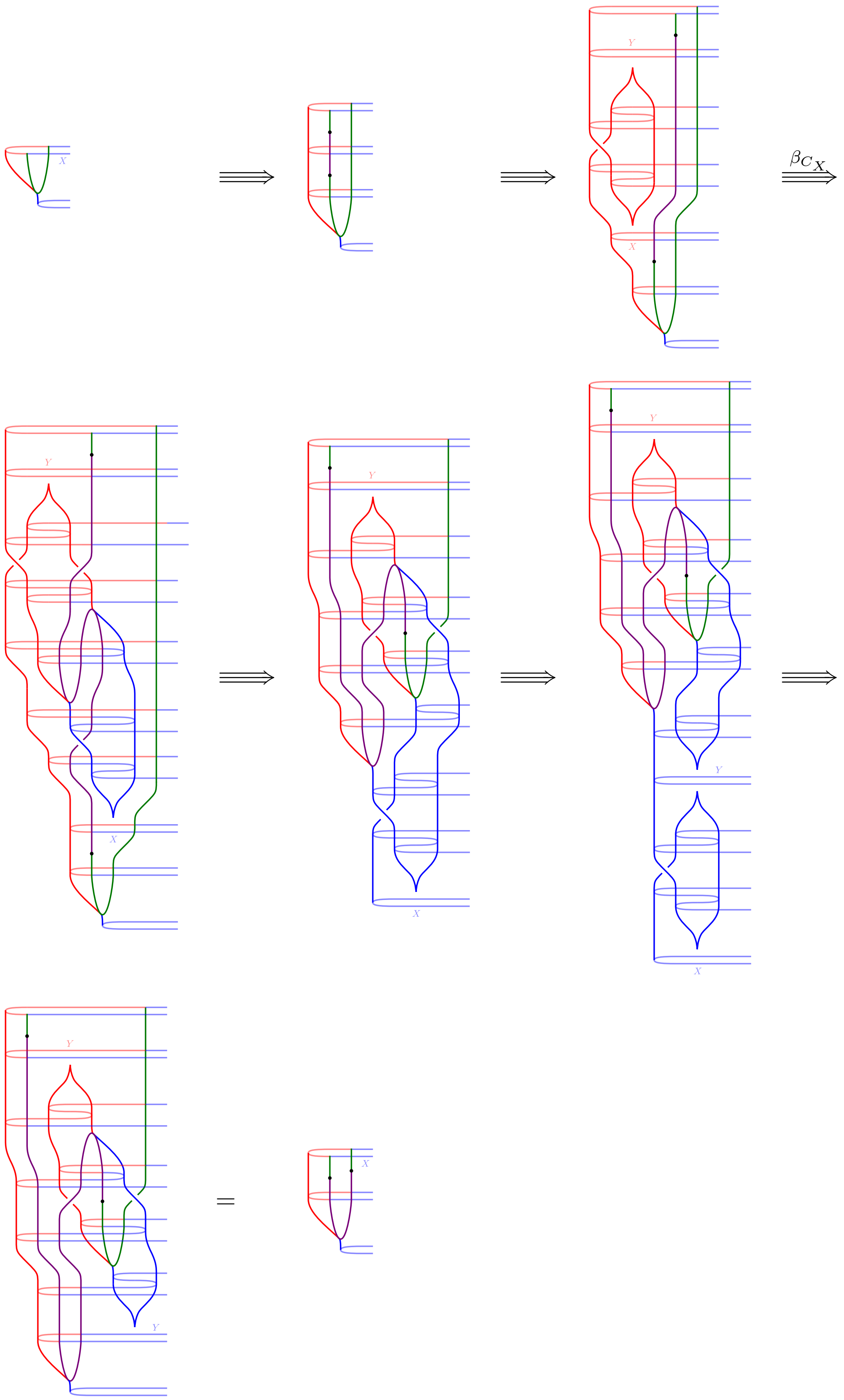


Note that this is invertible, being a composite of invertible 2-morphisms. We can denote this composite as follows.

$m_{\text{ev}} \quad :=$

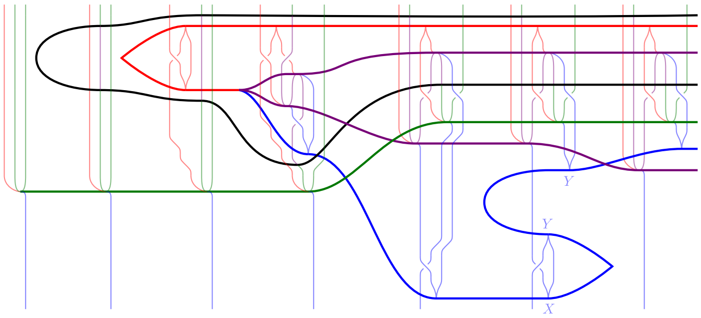


Define m_{coev} as the following composite.

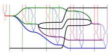

X
Y
X
β_{C_X}
Y
X
Y
X
Y
Y
X
Y
Y
X

Note that this is invertible, being a composite of invertible 3-morphisms. We can denote this composite as follows.

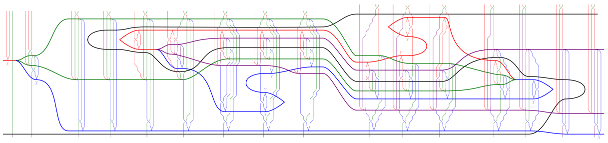
$m_{\text{coev}} \quad :=$



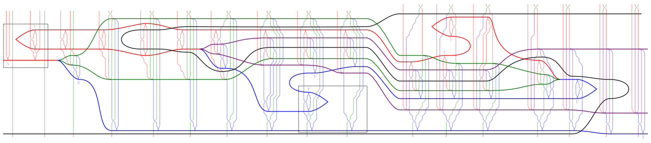
Now this data satisfies relations corresponding to cusp 3-morphisms. We show only the proof of the relation corresponding to C_X, as the other three are analogous.



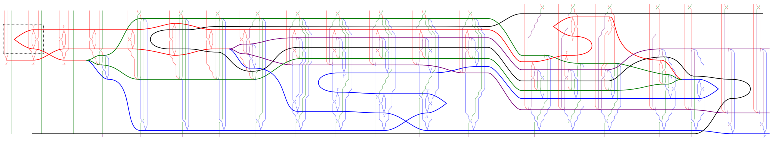
$\overset{d}{=}$



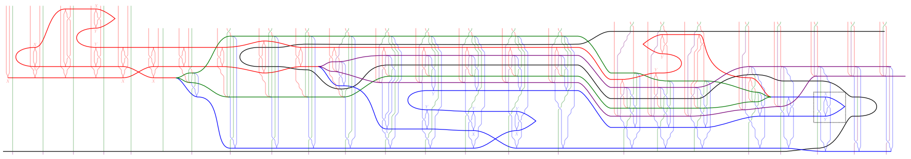
$\overset{h}{=}$



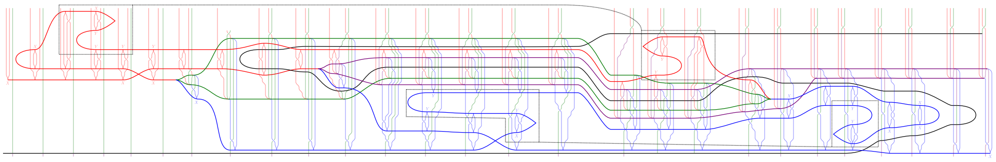
$\overset{r}{=}$



$\overset{r}{=}$



$\overset{r}{=}$



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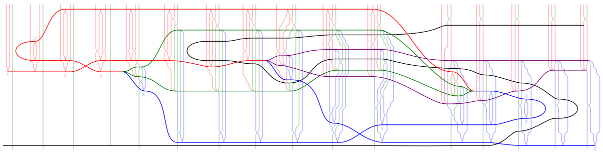
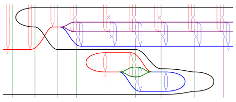
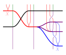


$\overset{h}{=}$



$\underset{r}{\overset{h}{=}}$



□

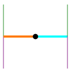
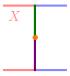
3.3.7 Bijective on 3-morphisms

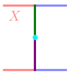
Lemma 3.3.8. *The restriction functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

is surjective on 3*-morphisms.*

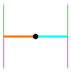
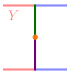
Proof. Let $\ell, m : \alpha \Rrightarrow \beta$ be modifications and $\mathcal{A}_X : \ell_X \Rrightarrow m_X$ an invertible 3-morphism. We use red for F, blue for G, green for α, purple for β, orange for ℓ, light blue for m and black for $\mathcal{A}$. Denote

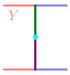
$\mathcal{A}_X =$  : 

$\Longrightarrow$ 

We want to extend this to a perturbation $\ell \to m$, which consists of the following data and relations.

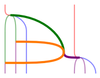
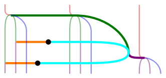
0. A 3-morphism

$\mathcal{A}_Y =$  : 

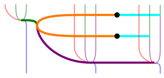
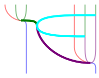
$\Longrightarrow$ 

;

1. relations

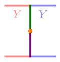
 $\overset{\mathcal{A}_{\mathrm{ev}}}{=\!=}$ 

and

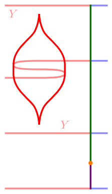
 $\overset{\mathcal{A}_{\mathrm{coev}}}{=\!=}$ .

Pick an inverse $\mathcal{A}_X^{-1}$ for $\mathcal{A}_X$.

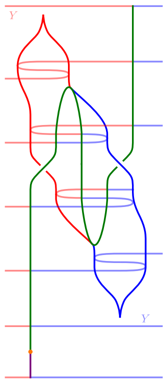
Define $\mathcal{A}_Y$ as the following composite.



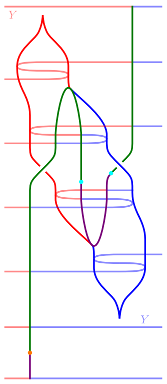
$\Rrightarrow$



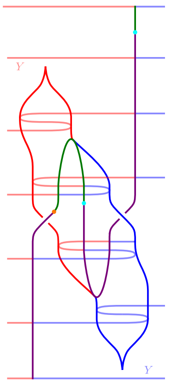
$\overset{\alpha_{C_Y}}{\Rrightarrow}$



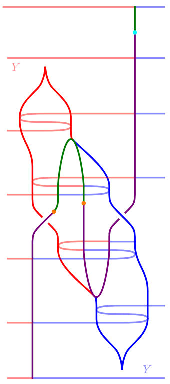
$\overset{m_{\mathrm{coev}}}{\Rrightarrow}$



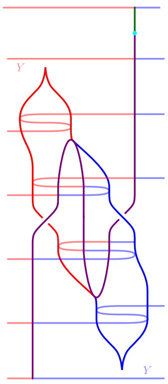
$\Rrightarrow$



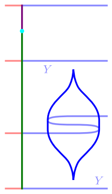
$\overset{\mathcal{A}_X^{-1}}{\Rrightarrow}$



$\overset{\ell_{\mathrm{ev}}}{\Rrightarrow}$



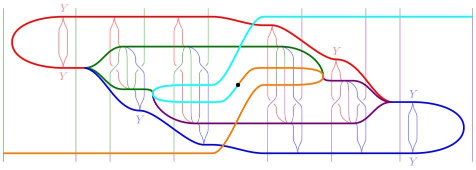
$\overset{\beta_{C_Y^{-1}}}{\Rrightarrow}$



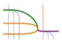
$\Rrightarrow$

Note that this is invertible, being a composite of invertible 3-morphisms. We can denote this composite as follows.

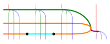
$$\mathcal{A}_Y \quad :=$$



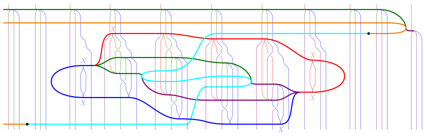
Now we need to show that this data satisfies relations associated to ev and coev. We do it only for ev, since the one for coev is similar.



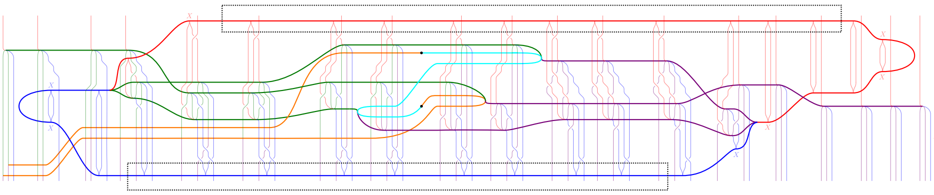
$$\overset{r}{=}$$



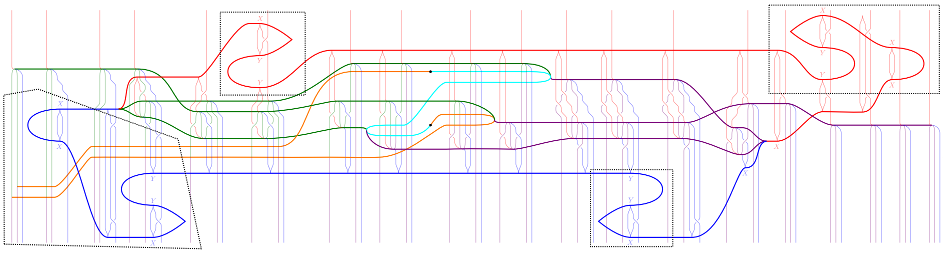
$$\overset{r}{=}$$



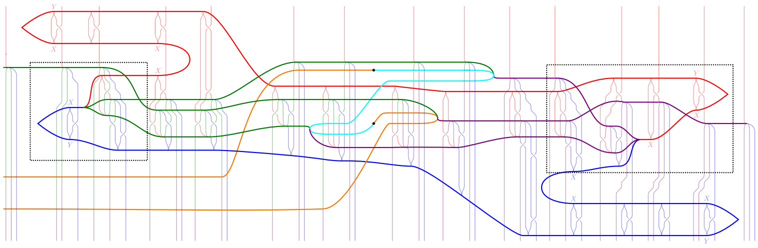
$$\overset{h}{=}$$

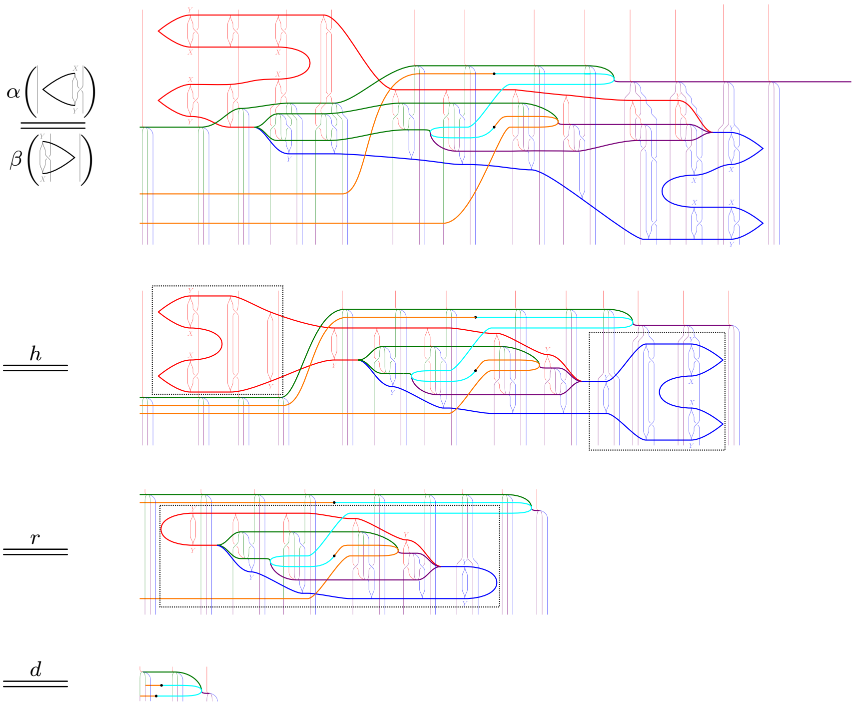


$$\overset{r}{=}$$



$$\overset{r}{=}$$





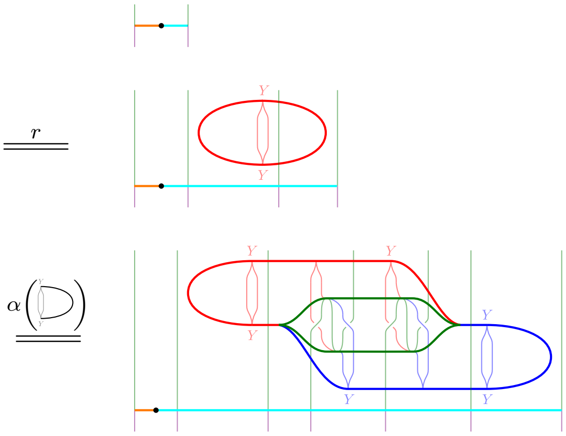
□

Lemma 3.3.9. *The restriction functor*

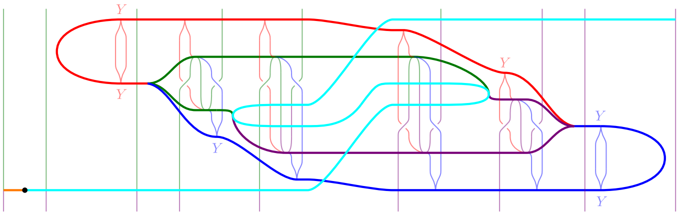
$$E_X : \mathrm{Fun}(\mathrm{Dual}_3, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C})$$

is injective on 3*-morphisms.*

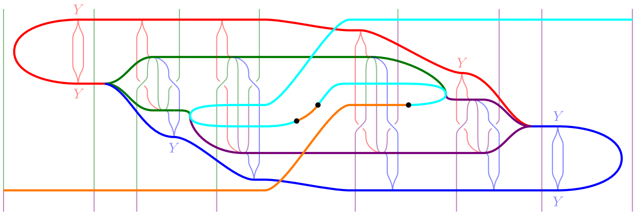
Proof. Let $\mathcal{A}, \mathcal{B} : \ell \to m$ be perturbations and suppose $\mathcal{A}_X = \mathcal{B}_X$. Then we need to show that $\mathcal{A}_Y = \mathcal{B}_Y$.



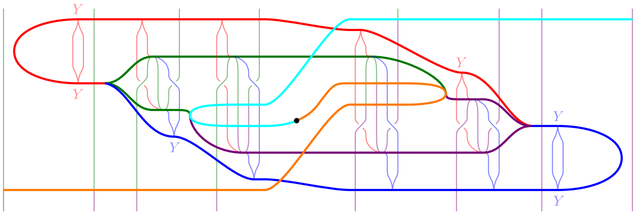
$\underset{=\!=}{m_{C_Y^{-1}}}$



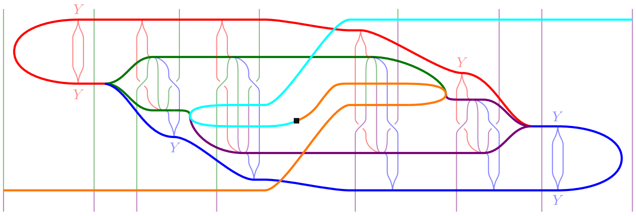
$\underset{=\!=}{r}$



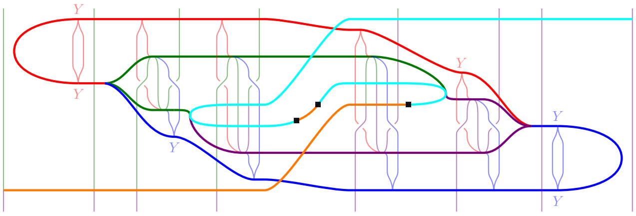
$\underset{=\!=}{\mathcal{A}_{\mathrm{ev}}}$



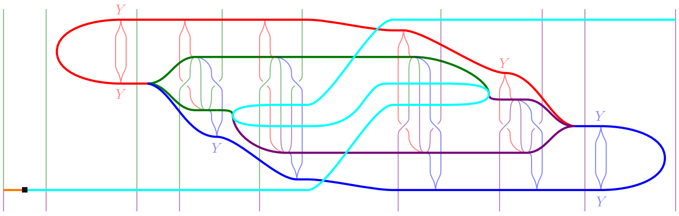
$\underset{=\!=}{\mathcal{A}_X=\mathcal{B}_X}$



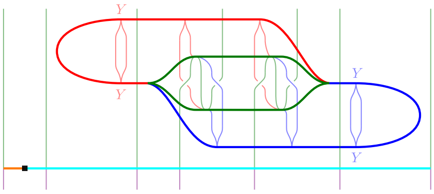
$\underset{=\!=}{\mathcal{B}_{\mathrm{ev}}}$

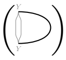
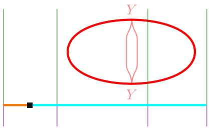


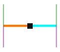
$\underset{=\!=}{r}$



$\underset{=\!=}{m_{C_Y^{-1}}}$



$\alpha\Big($  $\Big)$ $=$ 

$\overset{r}{=}$ 

□

Chapter 4

Fully dualizable objects

In this chapter, we combine the results of the previous chapters to obtain a coherence theorem for fully dualizable objects in a symmetric monoidal 3-category. Concretely, we define a presentation FD_3 with a distinguished 0-cell X and prove the following Theorem.

Definition 4.0.1. *Let $\mathcal{C}$ be a symmetric monoidal 3-category. We denote by $\mathrm{Obj}^{fd}(\mathcal{C})$ the full 3-subgroupoid of $\mathrm{Obj}(\mathcal{C})$ whose objects are the fully dualizable objects in $\mathcal{C}$.*

Theorem 4.0.2. *The restriction functor $E_X : \mathrm{Fun}(\mathrm{FD}_3, \mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Obj}^{fd}(\mathcal{C})$ and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{FD}_3, \mathcal{C}) \to \mathrm{Obj}^{fd}(\mathcal{C})$$

is an equivalence of 3-groupoids.

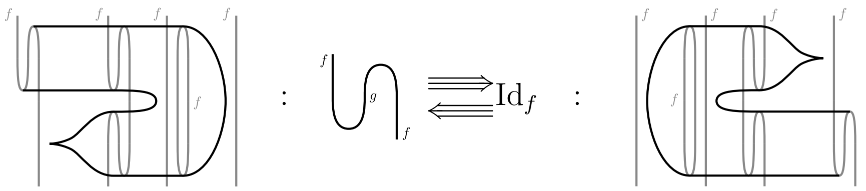
4.1 A characterization for full dualizability

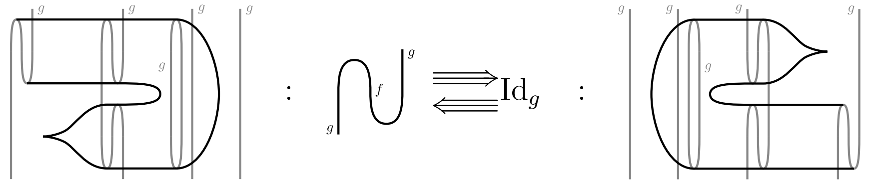
In this section, we give a characterization of full dualizability by a finite set of conditions. This will be essential in defining the presentation FD_3.

Definition 4.1.1. *Let $f : X \to Y$, $g : Y \to X$ be k-morphisms in an n-category $\mathcal{C}$ and suppose $U : \mathrm{id}_X \to g \circ f$ and $C : f \circ g \to \mathrm{id}_Y$ are $(k+1)$-morphisms in $\mathcal{C}$. Then we say that $(f \dashv g, U, C)$ is an adjunction if U and C satisfy the standard snake equations, up to $(k+2)$-isomorphism.*

Lemma 4.1.2. *Suppose $(f \dashv g, U, C)$ is an adjunction between k-morphisms in an n-category. Suppose U and C have right (resp left) adjoints U^R and C^R (resp U^L and C^L). Then $(g \dashv f, C^R, U^R)$ (resp $(g \dashv f, C^L, U^L)$) is an adjunction.*

Proof. We have adjunctions $(U \dashv U^R, \eta_U, \epsilon_U)$ and $(C \dashv C^R, \eta_C, \epsilon_C)$. Denote $U := {}_f\bigcap_g$, $C := {}^g\bigcup^f$, $U^R := {}^f\bigcup^g$ and $C^R := {}_g\bigcap_f$. The following two pairs of inverse $(k+2)$-morphisms show that $(g \dashv f, C^R, U^R)$ is an adjunction.



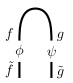


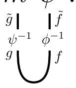
.

The proof for $(g \dashv f, C^L, U^L)$ is analogous.

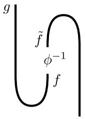
∎

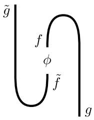
Lemma 4.1.3. *Suppose* $(f \dashv g, {}_f\bigcap_g, {}^g\bigcup^f)$ *and* $(\tilde f \dashv \tilde g, {}_{\tilde f}\bigcap_{\tilde g}, {}^{\tilde g}\bigcup^{\tilde f})$ *are adjunctions between k-morphisms in an n-category and* $\phi : f \to \tilde f$ *is a isomorphism. Then there exists an isomorphism* $\psi : g \to \tilde g$ *such that*



$\simeq {}_{\tilde f}\bigcap_{\tilde g}$ *and*


$\simeq {}^{\tilde g}\bigcup^{\tilde f}$.

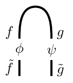
Proof. One defines $\psi :=$


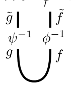
and $\psi^{-1} :=$


. It's easy to check that these are inverse to each other and have the desired properties.

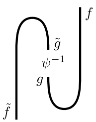
∎

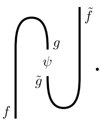
Lemma 4.1.4. *Suppose* $(f \dashv g, {}_f\bigcap_g, {}^g\bigcup^f)$ *and* $(\tilde f \dashv \tilde g, {}_{\tilde f}\bigcap_{\tilde g}, {}^{\tilde g}\bigcup^{\tilde f})$ *are adjunctions between k-morphisms in an n-category and* $\psi : g \to \tilde g$ *is an isomorphism. Then there exists an isomorphism* $\phi : f \to \tilde f$ *such that*



$\simeq {}_{\tilde f}\bigcap_{\tilde g}$ *and*


$\simeq {}^{\tilde g}\bigcup^{\tilde f}$.

Proof. One defines $\phi :=$


and $\phi^{-1} :=$


. It's easy to check that these are inverse to each other and have the desired properties.

∎

When we have isomorphisms as above, we write

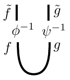
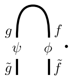
$$(f \dashv g, {}_f\bigcap_g, {}^g\bigcup^f) \cong (\tilde{f} \dashv \tilde{g}, {}_{\tilde{f}}\bigcap_{\tilde{g}}, {}^{\tilde{g}}\bigcup^{\tilde{f}}).$$

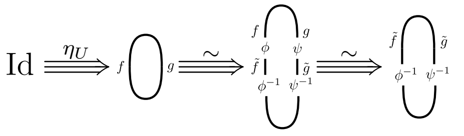
Lemma 4.1.5. *Suppose we have isomorphic adjunctions* $(f \dashv g, U, C) \cong (\tilde{f} \dashv \tilde{g}, \tilde{U}, \tilde{C})$ *and that* U *and* C *have right adjoints* $(U \dashv U^R, \eta_U, \epsilon_U)$ *and* $(C \dashv C^R, \eta_C, \epsilon_C)$*. Then there exist right adjoints* $(\tilde{U} \dashv \tilde{U}^R, \eta_{\tilde{U}}, \epsilon_{\tilde{U}})$ *and* $(\tilde{C} \dashv \tilde{C}^R, \eta_{\tilde{C}}, \epsilon_{\tilde{C}})$*. Moreover,* $\eta_{\tilde{U}}$*,* $\epsilon_{\tilde{U}}$*,* $\eta_{\tilde{C}}$ *and* $\epsilon_{\tilde{C}}$ *are composites of* η_U*,* ϵ_U*,* η_C*,* ϵ_C *and invertible morphisms.*

Proof. Pick isomorphisms $\phi : f \Rightarrow \tilde{f}$ and $\psi : g \Rightarrow \tilde{g}$ such that

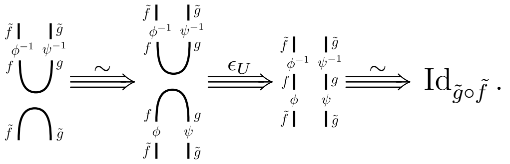
Denote

$$U := {}_f\bigcap_g,\ C := {}^g\bigcup^f,\ U^R := {}^f\bigcup^g,\ C^R := {}_g\bigcap_f,\ \tilde{U} := {}_{\tilde{f}}\bigcap_{\tilde{g}} \text{ and } \tilde{C} := {}^{\tilde{g}}\bigcup^{\tilde{f}}.$$

Define $\tilde{U}^R :=$  and $\tilde{C}^R :=$ . Define $\eta_{\tilde{U}}$ as the composite



and $\epsilon_{\tilde{U}}$ as the composite



It is easy to check that these satisfy the snake equations.

Then $\eta_{\tilde{C}}$ and $\epsilon_{\tilde{C}}$ are defined in a similar way.

□

Lemma 4.1.6. *Suppose we have isomorphic adjunctions* $(f \dashv g, U, C) \cong (\tilde{f} \dashv \tilde{g}, \tilde{U}, \tilde{C})$ *and that* U *and* C *have left adjoints* $(U^L \dashv U, \eta_U, \epsilon_U)$ *and* $(C^L \dashv C, \eta_C, \epsilon_C)$*. Then there exist left adjoints* $(\tilde{U}^L \dashv \tilde{U}, \eta_{\tilde{U}}, \epsilon_{\tilde{U}})$ *and* $(\tilde{C}^L \dashv \tilde{C}, \eta_{\tilde{C}}, \epsilon_{\tilde{C}})$*. Moreover,* $\eta_{\tilde{U}}$*,* $\epsilon_{\tilde{U}}$*,* $\eta_{\tilde{C}}$ *and* $\epsilon_{\tilde{C}}$ *are composites of* η_U*,* ϵ_U*,* η_C*,* ϵ_C *and invertible morphisms.*

Proof. The proof is analogous to that of the previous Lemma.

□

Lemma 4.1.7. *Suppose f is a k-morphism which is a composite of morphisms having left (respectively right) adjoints. Then f has a left (respectively right) adjoint, with unit and counit $(k+1)$-morphisms being composites of those for the original adjunctions.*

Proof. It is easy to check that if $f^L \dashv f$ and $g^L \dashv g$ then $f^L \circ g^L \dashv g \circ f$. Similarly, if $f \dashv f^R$ and $g \dashv g^R$ then $g \circ f \dashv f^R \circ g^R$.

□

Definition 4.1.8. *Let $\mathcal{C}$ be an n-category and f a k-morphism. We say that f has all adjoints in $\mathcal{C}$ if there exists a tower of adjunctions*

$$\cdots \dashv f^{LL} \dashv f^L \dashv f \dashv f^R \dashv f^{RR} \dashv \cdots$$

in $\mathcal{C}$. We denote by $\mathcal{C}^{(k)}$ the n-category obtained from $\mathcal{C}$ by discarding all k-morphisms which don't have all adjoints in $\mathcal{C}$.

Definition 4.1.9. *([9])*

Let X be an object in a symmetric monoidal n-category $\mathcal{C}$. We say that X is 1-dualizable in $\mathcal{C}$ if it has a dual in $\mathcal{C}$. For $k \geq 2$ we say that X is k-dualizable in $\mathcal{C}$ if it is $(k-1)$-dualizable as an object of $\mathcal{C}^{(k-1)}$.

A ***fully dualizable*** *object is an n-dualizable object.*

Remark 4.1.10. *Let X be an object in a symmetric monoidal n-category $\mathcal{C}$ and let $m > n$. Then X is m-dualizable if and only if it is invertible.*

Definition 4.1.11. *Let X be an object in a symmetric monoidal n-category $\mathcal{C}$. A complete set of 1-dualizability data for X is a choice of dual Y, together with evaluation and coevaluation 1-morphisms $\mathrm{ev} : XY \to 1$ and $\mathrm{coev} : 1 \to YX$. For $k \geq 2$, a complete set of k-dualizability data for X consists of a complete set of $(k-1)$-dualizablity data for X, together with the choice of a tower of left and right adjoints*

$$\cdots \dashv f^{LL} \dashv f^L \dashv f \dashv f^R \dashv f^R R \dashv \cdots$$

for each $(k-1)$-morphism f in this set, and choices of unit and counit k-morphisms witnessing these adjunctions.

A complete set of full dualizability data is a complete set of n-dualizability data.

Example 4.1.12. *For $n = 3$, a complete set of full dualizability data consists of*

1. *a choice of dual Y, together with evaluation and coevaluation 1-morphisms $\mathrm{ev} : XY \to 1$ and $\mathrm{coev} : 1 \to YX$;*

2. choices of towers of adjoints

$$\cdots \dashv \mathrm{ev}^{LL} \dashv \mathrm{ev}^{L} \dashv \mathrm{ev} \dashv \mathrm{ev}^{R} \dashv \mathrm{ev}^{RR} \dashv \cdots$$

and

$$\cdots \dashv \mathrm{coev}^{LL} \dashv \mathrm{coev}^{L} \dashv \mathrm{coev} \dashv \mathrm{coev}^{R} \dashv \mathrm{coev}^{RR} \dashv \cdots$$

together with unit and counit 2*-morphisms witnessing these adjunctions;*

3. choices of towers of adjoints for the 2*-morphisms in the previous step, together with unit and counit* 3*-morphisms witnessing these adjunctions.*

Given such a complete set of full dualizability data it is possible to produce a simpler one. Consider the chosen adjunctions

$$(\mathrm{ev} \dashv \mathrm{ev}^{R}, u_{\mathrm{ev}}, v_{\mathrm{ev}})$$

and

$$(\mathrm{coev} \dashv \mathrm{coev}^{R}, u_{\mathrm{coev}}, v_{\mathrm{coev}}).$$

By Lemma 4.1.2, we can use our choice of right adjoints to u_{ev}*,* v_{ev}*,* u_{coev} *and* v_{coev} *to produce adjunctions*

$$(\mathrm{ev}^{R} \dashv \mathrm{ev}, v_{\mathrm{ev}}^{R}, u_{\mathrm{ev}}^{R})$$

and

$$(\mathrm{coev}^{R} \dashv \mathrm{coev}, v_{\mathrm{coev}}^{R}, u_{\mathrm{coev}}^{R}).$$

So the towers of adjoints for ev *and* coev *collapse to*

$$\mathrm{ev}^{R} \dashv \mathrm{ev} \dashv \mathrm{ev}^{R}$$

and

$$\mathrm{coev}^{R} \dashv \mathrm{coev} \dashv \mathrm{coev}^{R}.$$

So in order to provide complete full dualizability data for X *we only need towers of adjoints for* u_{ev}*,* v_{ev}*,* u_{coev} *and* v_{coev}*.*

Definition 4.1.13. *A reduced set of* k*-dualizability data for an object* X *is a complete set of* k*-dualizability data for* X *where all the towers of adjunctions between* ℓ*-morphisms, with* $\ell \leq k-2$*, are two step towers, of the form* $g \dashv f \dashv g$ *where the unit and counit morphisms witnessing* $g \dashv f$ *are adjoint to the the ones witnessing* $f \dashv g$*, and* f *is part of the set of* ℓ*-dualizabilty data.*

Proposition 4.1.14. *Let X be an object in a symmetric monoidal n-category $\mathcal{C}$. Then X is k-dualizable if and only if there exists a complete set of k-dualizability data for X.*

Proof. We begin by showing that if there exists a complete set of k-dualizability data for X then X is k-dualizable. We prove this by induction on k. For $k = 1$ this is clear. For $k \geq 2$, we need to prove that X is k-dualizable in $\mathcal{C}$, which by definition amounts to showing it is $(k-1)$-dualizable in $\mathcal{C}^{k-1}$. Now the given complete set of k-dualizabilty data for X in $\mathcal{C}$ includes a complete set of $(k-1)$-dualizabilty data for X in $\mathcal{C}$ and moreover, all the $(k-1)$-morphisms in this set have all left and right adjoints in $\mathcal{C}$, since they belong to a complete set of k-dualizability data for X in $\mathcal{C}$. This way we obtain a complete set of $(k-1)$-dualizability data for X in $\mathcal{C}^{k-1}$ and therefore, by the induction hypothesis, X is $(k-1)$-dualizable in $\mathcal{C}^{k-1}$ as desired.

Now we prove the converse, also by induction. For $k = 1$ this is clear. So suppose $k \geq 2$ and X is k-dualizable. Then X is $(k-1)$-dualizable in $\mathcal{C}^{k-1}$, so there is a complete set of $(k-1)$-dualizability data for X where all $(k-1)$-morphisms have all adjoints. Then we can construct towers of adjoints for all the $(k-1)$-morphisms in this set and we produce a complete set of k-dualizability data for X.

□

Definition 4.1.15. *Let X be an object in a monoidal n-category $\mathcal{C}$. A partial set of* 1*-dualizability data for X is a choice of dual Y, together with evaluation and coevaluation* 1*-morphisms* $\mathrm{ev} : XY \to 1$ *and* $\mathrm{coev} : 1 \to YX$*. For $k \geq 2$ a partial set of k-dualizability data for X consists of a partial set of $(k-1)$-dualizabily data for X, together with choices of left adjoints for all $(k-1)$-morphisms in this set or right adjoints for all $(k-1)$-morphisms in this set, together with unit and counit k-morphisms witnessing these adjucntions.*

Example 4.1.16. *For $n = 3$, the following data is a partial set of full dualizablity data:*

1. *a choice of dual Y, together with evaluation and coevaluation* 1*-morphisms* $\mathrm{ev} : XY \to 1$ *and* $\mathrm{coev} : 1 \to YX$*;*
2. *choices of right adjoints* $(\mathrm{ev} \dashv \mathrm{ev}^R, u_{\mathrm{ev}}, v_{\mathrm{ev}})$ *and* $(\mathrm{coev} \dashv \mathrm{coev}^R, u_{\mathrm{coev}}, v_{\mathrm{coev}})$*;*
3. *choices of right adjoints* $(u_{\mathrm{ev}} \dashv u_{\mathrm{ev}}^R, \eta_{u_{\mathrm{ev}}}, \epsilon_{u_{\mathrm{ev}}})$*,* $(v_{\mathrm{ev}} \dashv v_{\mathrm{ev}}^R, \eta_{v_{\mathrm{ev}}}, \epsilon_{v_{\mathrm{ev}}})$*,* $(u_{\mathrm{coev}} \dashv u_{\mathrm{coev}}^R, \eta_{u_{\mathrm{coev}}}, \epsilon_{u_{\mathrm{coev}}})$ *and* $(v_{\mathrm{coev}} \dashv v_{\mathrm{coev}}^R, \eta_{v_{\mathrm{coev}}}, \epsilon_{v_{\mathrm{coev}}})$*.*

Definition 4.1.17. *Given a partial set of k-dualizabilty data, we say that a reduced set of k-dualizability data is an extension of the partial set if all the two step towers of adjunctions consist of data in the partial set.*

Example 4.1.18. *For $n = 3$, a reduced set of full dualizability data extending a partial set consists of*

1. *a choice of dual Y, together with evaluation and coevaluation 1-morphisms $\mathrm{ev} : XY \to 1$ and $\mathrm{coev} : 1 \to YX$;*
2. *choices of adjunctions $(\mathrm{ev} \dashv \mathrm{ev}^R, u_{\mathrm{ev}}, v_{\mathrm{ev}})$, $(\mathrm{coev} \dashv \mathrm{coev}^R, u_{\mathrm{coev}}, v_{\mathrm{coev}})$, $(\mathrm{ev}^R \dashv \mathrm{ev}, v^R_{\mathrm{ev}}, u^R_{\mathrm{ev}})$, $(\mathrm{coev}^R \dashv \mathrm{coev}, v^R_{\mathrm{coev}}, u^R_{\mathrm{coev}})$*
3. *choices of towers of adjoints for u_{ev}, v_{ev}, u_{coev} and v_{coev}, with corresponding units and counits.*

Theorem 4.1.19. *Let X be an object in a symmetric monoidal n-category $\mathcal{C}$ and suppose we have a partial set of k-dualizability data for X. Then, there is an extension of this data to a reduced set of k-dualizability data for X where all morphisms are composites of morphisms in the given partial set and invertible morphisms.*

Proof. The proof is by induction on k. For $k = 1$, we are given $(Y \dashv X, \mathrm{coev}, \mathrm{ev})$ where $\mathrm{ev} : XY \to 1$ and $\mathrm{coev} : 1 \to YX$ satisfy the snake equations. This is already a complete set of 1-dualizability data, so there is nothing to prove. Notice that we have also $(X \dashv Y, \beta_{X,Y} \circ \mathrm{coev}, \mathrm{ev} \circ \beta_{Y,X})$ where $\beta_{X,Y} : XY \to YX$ and $\beta_{Y,X} : YX \to XY$ are components of the symmetric structure. So we have a tower of adjunctions

$$\cdots \dashv Y \dashv X \dashv Y \dashv X \dashv Y \dashv \cdots$$

where the unit and counit 1-morphisms are composites of ev, coev and the invertible 1-morphisms $\beta_{X,Y}$ and $\beta_{Y,X}$. This is useful for understanding the inductive procedure.

Now let $k \geq 2$. We assume, without loss of generality, that the partial set of k-dualizablity data consists of choices of right adjoints at every level (if this is not the case, we can pass to the category where we reverse the direction of the ℓ-morphisms for every ℓ at which the partial set adds left adjoints). The partial set of k-dualizability data restricts to a partial set of $(k-1)$-dualizability data and by the induction hypothesis this extends to a reduced set. Let f be a $(k-2)$-morphism in the partial set and consider its tower of adjunctions in the reduced set

$$\cdots \dashv f^{LL} \dashv f^L \dashv f \dashv f^R \dashv f^{RR} \dashv \cdots .$$

Now $(f \dashv f^R, u_f, v_f)$ is in the partial set of $(k-1)$-dualizability data, so u_f and v_f have right adjoints. Then we have $(f^R \dashv f, v_f^R, u_f^R)$, which makes the tower collapse to $f^R \dashv f \dashv f^R$. Now we construct towers of adjoints for u_f and v_f.

The unit and counit morphisms for the adjunction $(f \dashv f^R, u_f, v_f)$ have right adjoints. Now any adjunction of the form $(f \dashv g, u, v)$ is isomorphic to $(f \dashv f^R, u_f, v_f)$, by applying Lemma 4.1.3 with $\phi = \mathrm{id}_f$ and therefore by Lemma 4.1.5 there exist right adjoints $u \dashv u^R$ and $v \dashv v^R$ with unit and counit morphisms given by composites of the ones witnessing $u_f \dashv u_f^R$ and $v_f \dashv v_f^R$ and isomorphisms.

Moreover, the unit and counit $(k-1)$-morphisms in the adjunction $(f^L \dashv f, \tilde{u}_f, \tilde{v}_f)$ are composites of $(k-1)$-morphisms in the partial set of $(k-1)$-dualizability data and invertible morphisms. Every $(k-1)$-morphism in the partial set of $(k-1)$-dualizability data has a right adjoint in the set partial set of k-dualizability data. Therefore, by Lemma 4.1.7, $\tilde{u}_f$ and $\tilde{v}_f$ have right adjoints, whose units and counits are composites of morphisms in the partial set of k-dualizability data and invertible morphisms. So the above argument applies to show that for any adjunction of the form $(g \dashv f, u, v)$ there exist right adjoints for u and v witnessed by unit and counit k-morphisms which are composites of morphisms in the partial set of k-dualizability data and invertible morphisms.

Using the adjunction $(f^R \dashv f, v_f^R, u_f^R)$ and applying Lemma 4.1.6 one can show that given any adjunction of the form $(g \dashv f, u, v)$ there are left adjoints for u and v, with unit and counit being composites of morphisms in the partial set of k-dualizability data and invertible morphisms.

Using the adjunction $(f \dashv f^L, \tilde{v}_f^R, \tilde{u}_f^R)$ and applying Lemma 4.1.6 one can show that given any adjunction of the form $(f \dashv g, u, v)$ there are left adjoints for u and v, with unit and counit being composites of morphisms in the partial set of k-dualizability data and invertible morphisms.

So we have shown that, for any adjunction involving f, the unit and counit $(k-1)$-morphisms (u, v) have left and right adjoints, witnessed by unit and counit k-morphisms which are composites of morphisms in the partial set of k-dualizability data and invertible morphisms. Moreover, the adjoints to u and v are again unit and counit for an adjuntion involving f. So we can repeatedly apply this to $(f \dashv f^R, u_f, v_f)$ to obtain towers of left and right adjoints for u_f and v_f, with every adjunction witnessed by unit and counit k-morphisms which are composites of k-morphisms in the parial set and invertible k-morphisms. Doing this for every $(k-2)$-morphism f in the partial set of $(k-1)$-dualizability data, we produce a reduced set of k-dualizability data extending the orignal partial set and where all morphisms are composites of morphisms in the partial set and invertible morphisms.

□

Corollary 4.1.20. *Let X be an object in a symmetric monoidal n-category $\mathcal{C}$. Then X is fully dualizable if and only if there exists a partial set of full dualizability data for X.*

Proof. If there exists a partial set of full dualizability data, then by Theorem 4.1.19 there exists a reduced set and so by Proposition 4.1.14 X is fully dualizable.

Conversely, supposing that X is fully dualizable it's easy to construct a partial set of full dualizability data for X by induction, or by taking a complete set and forgetting most of the data.

□

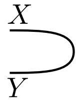
4.2 Coherence for fully dualizable objects

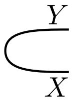
We start by defining a monoidal presentation $\mathrm{Dual}_{(3,2)}$ which consists of coherence data for 2-dualizable objects in a symmetric monoidal 3-category.

Definition 4.2.1. *The monoidal presentation* $\mathrm{Dual}_{(3,2)}$ *consists of*

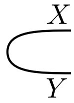
(0) objects X and Y;

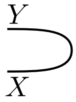
1. *1-morphisms*

   ev = 

   $: XY \to 1$ *and* coev = 

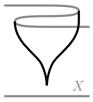
   $: 1 \to YX$*;*

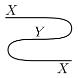
   ev^R = 

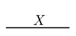
   $: 1 \to XY$ *and* coev^R = 

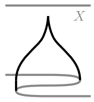
   $: YX \to 1$*;*

2. *2-morphisms*

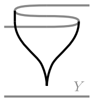
$C_X =$ 

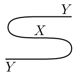
$:$ 

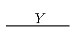
$\Longrightarrow$ $\Longleftarrow$ 

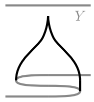
$:$ 

$= C_X^{-1}$

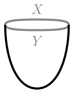
$C_Y =$ 

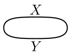
$:$ 

$\Longrightarrow$ $\Longleftarrow$ 

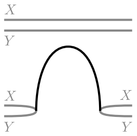
$:$ 

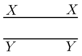
$= C_Y^{-1}$

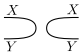
$\varepsilon_{\mathrm{ev}} =$ 

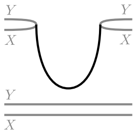
: 

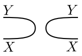
$\Longrightarrow \quad \mathrm{id}_1$

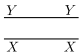
$\eta_{\mathrm{ev}} =$ 

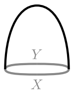
: 

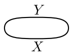
$\Longrightarrow$ 

$\varepsilon_{\mathrm{coev}} =$ 

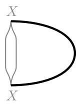
: 

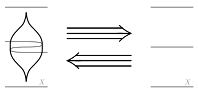
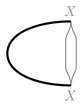
$\Longrightarrow$ 

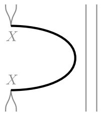
$\eta_{\mathrm{coev}} =$ 

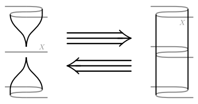
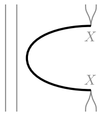
: $\mathrm{id}_1 \quad \Longrightarrow$ 

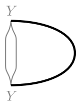
3. 3-*morphisms*

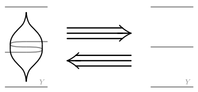
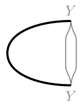


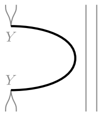
: 
: 

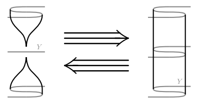
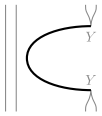


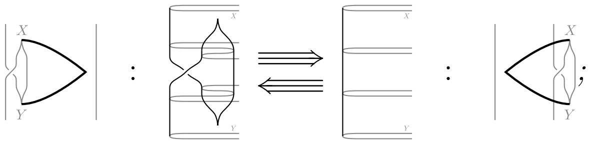
: 
: 

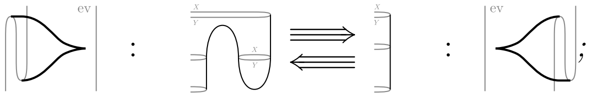


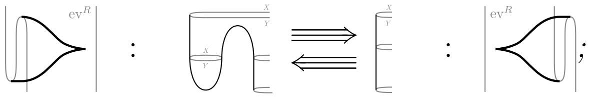
: 
: 

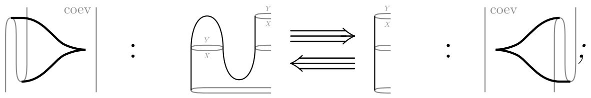


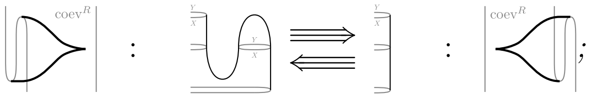
: 
: 



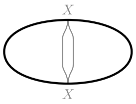








4. relations



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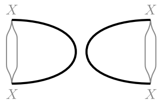

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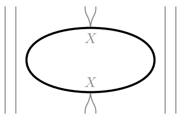

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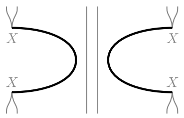

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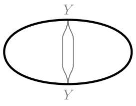

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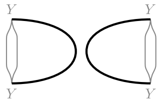
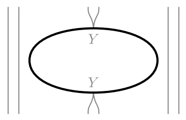
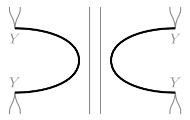


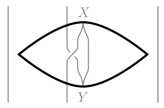
= Id;



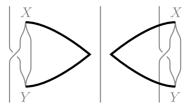
$= \mathrm{Id}$;



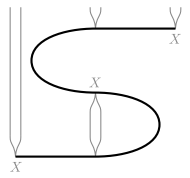
$= \mathrm{Id}$;



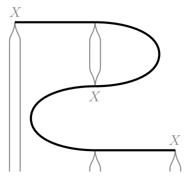
$= \mathrm{Id}$;



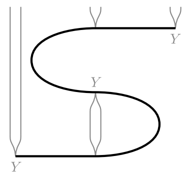
$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;



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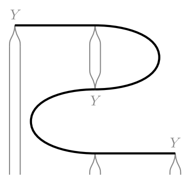
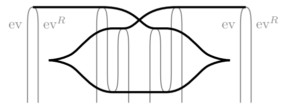
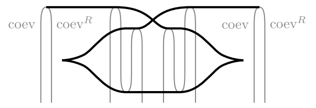


$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;

Note that there is an isomorphism $\big(\mathrm{Dual}_3 *_{\mathrm{ev}\sim l} \mathrm{Adj}_{(3,1)}\big) *_{\mathrm{coev}\sim l} \mathrm{Adj}_{(3,1)} \to \mathrm{Dual}_{(3,2)}$.

Definition 4.2.2. *Let $\mathcal{C}$ be a 3-category. We define $\mathcal{C}^{1L}$ to be the 3-category obtained by discarding all 1-morphisms which are not left adjoints.*

Lemma 4.2.3. *Let $\mathcal{C}$ be a monoidal 3-category. Then the restriction functor $E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Obj}^R(\mathcal{C}^{1L})$ and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) \to \mathrm{Obj}^R(\mathcal{C}^{1L})$$

is an equivalence of 3-groupoids.

Proof. Note that by the results in 3.3.3 we have $\mathrm{Fun}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) = \mathrm{Map}(\mathrm{Dual}_{(3,2)}, \mathcal{C})$. Moreover, we have

$$\mathrm{Dual}_{(3,2)} = \big(\mathrm{Dual}_3 *_{\mathrm{ev}\sim l} \mathrm{Adj}_{(3,1)}\big) *_{\mathrm{coev}\sim l} \mathrm{Adj}_{(3,1)} .$$

By applying Proposition 3.1.11 twice, we get an equivalence

$$\mathrm{Map}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) \to \mathrm{Map}^{\mathrm{ev}\dashv,\mathrm{coev}\dashv}(\mathrm{Dual}_3, \mathcal{C}).$$

Since ev and coev are the only 1-cells in Dual_3, this is the same as $\mathrm{Map}(\mathrm{Dual}_3, \mathcal{C}^{1L})$. Finally, we use Proposition 3.1.9 to get an equivalence $\mathrm{Map}(\mathrm{Dual}_3, \mathcal{C}^{1L}) \to \mathrm{Obj}^R(\mathcal{C}^{1L})$. □

Definition 4.2.4. *Let $\mathcal{C}$ be symmetric monoidal 3-category. We define $\mathrm{Obj}^{2d}(\mathcal{C})$ to be full 3-subgroupoid of $\mathrm{Obj}(\mathcal{C})$ whose objects are the 2-dualizable objects in $\mathcal{C}$.*

Now we can prove the coherence statement for 2-dualizable objects in a symmetric monoidal 3-category.

Theorem 4.2.5. *Let $\mathcal{C}$ be a symmetric monoidal 3-category. Then the restriction functor $E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Obj}^{2d}(\mathcal{C})$ and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,2)}, \mathcal{C}) \to \mathrm{Obj}^{2d}(\mathcal{C})$$

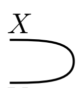
is an equivalence of 3-groupoids.

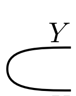
Proof. By the results of the previous section, we have $\mathrm{Obj}^R(\mathcal{C}^{1L}) = \mathrm{Obj}^{2d}(\mathcal{C})$, since $\mathcal{C}$ is symmetric monoidal. So the statement follows from the previous Lemma. □

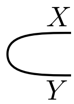
Now we define the monoidal presentation $\mathrm{FD}_3 = \mathrm{Dual}_{(3,3)}$, which consists of coherence data for a fully dualizable, i.e. 3-dualizable object in a symmetric monoidal 3-category.

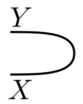
Definition 4.2.6. *The monoidal presentation $\mathrm{FD}_3 = \mathrm{Dual}_{(3,3)}$ consists of*

(0) *objects X and Y;*

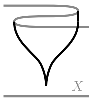
1. 1-*morphisms*
   ev = 

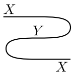
   : $XY \to 1$ *and* coev = 

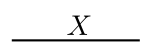
   : $1 \to YX$;
   ev^R = 

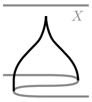
   : $1 \to XY$ *and* coev^R = 

   : $YX \to 1$;
2. 2-*morphisms*

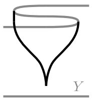
$C_X =$ 

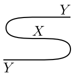
: 

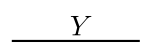
$\underset{\Longleftarrow}{\Longrightarrow}$ 

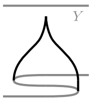
: 

$= C_X^{-1}$

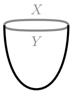
$C_Y =$ 

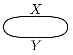
: 

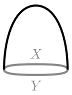
$\underset{\Longleftarrow}{\Longrightarrow}$ 

: 

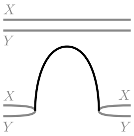
$= C_Y^{-1}$

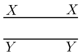
$\varepsilon_{\mathrm{ev}} =$ 

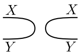
: 

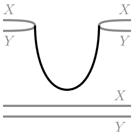
$\underset{\Longleftarrow}{\Longrightarrow}$ id_1 : 

$= \varepsilon_{\mathrm{ev}}^R$

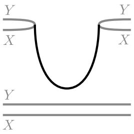
$\eta_{\mathrm{ev}} =$ 

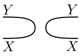
: 

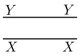
$\underset{\Longleftarrow}{\Longrightarrow}$ 

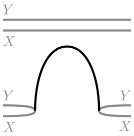
: 

$= \eta_{\mathrm{ev}}^R$

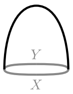
$\varepsilon_{\mathrm{coev}} =$ 

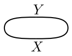
: 

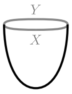
$\underset{\Longleftarrow}{\Longrightarrow}$ 

: 

$= \varepsilon_{\mathrm{coev}}^R$

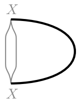
$\eta_{\mathrm{coev}} =$ 

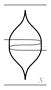
: id_1 $\underset{\Longleftarrow}{\Longrightarrow}$ 

: 

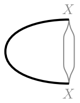
$= \eta_{\mathrm{coev}}^R$

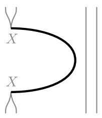
3. 3-morphisms

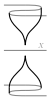


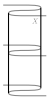
: 

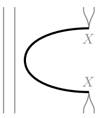
$\underset{\Lleftarrow}{\Rrightarrow}$ 

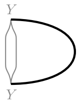
: 

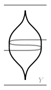


: 

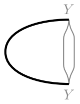
$\underset{\Lleftarrow}{\Rrightarrow}$ 

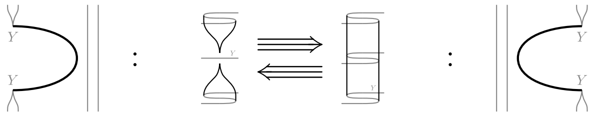
: 

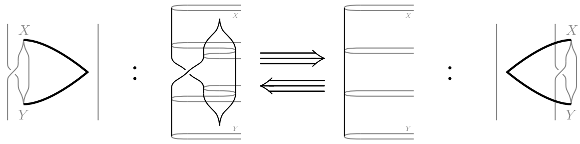


: 

$\underset{\Lleftarrow}{\Rrightarrow}$ 

: 





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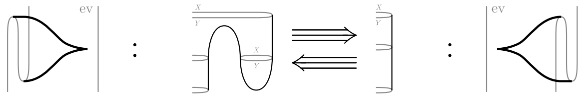

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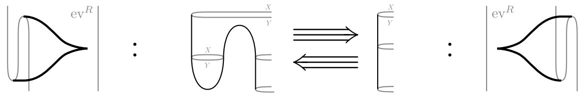

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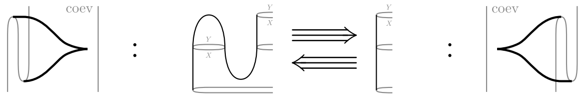

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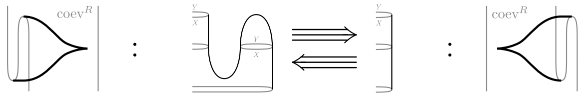
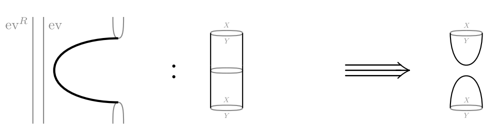
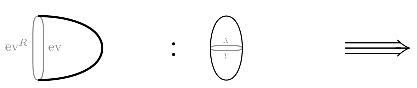


;

$u_{\varepsilon_{\mathrm{ev}}} =$

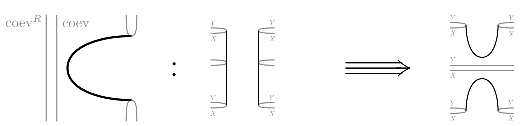


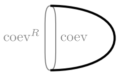
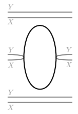
$v_{\varepsilon_{\mathrm{ev}}} =$

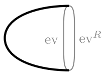
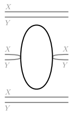


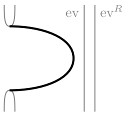
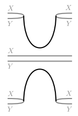
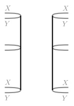
$\mathrm{Id}_{\mathrm{id}_1}$

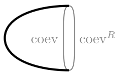
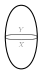
$u_{\varepsilon_{\mathrm{coev}}} =$

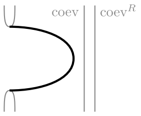
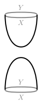
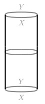


$v_{\varepsilon_{\mathrm{coev}}} =$  :  $\Longrightarrow \mathrm{Id}_{\mathrm{id}_{Y\otimes X}}$

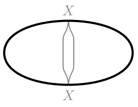
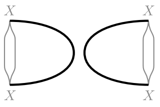
$u_{\eta_{\mathrm{ev}}} =$  : $\mathrm{Id}_{\mathrm{id}_{X\otimes Y}} \Longrightarrow$ 

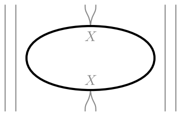
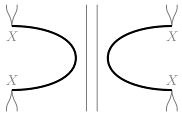
$v_{\eta_{\mathrm{ev}}} =$  :  $\Longrightarrow$ 

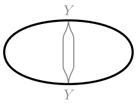
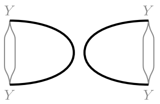
$u_{\eta_{\mathrm{coev}}} =$  : $\mathrm{Id}_{\mathrm{id}_1} \Longrightarrow$ 

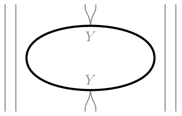
$v_{\eta_{\mathrm{coev}}} =$  :  $\Longrightarrow$ 

4. relations

 $= \mathrm{Id}$;  $= \mathrm{Id}$;

 $= \mathrm{Id}$;  $= \mathrm{Id}$;

 $= \mathrm{Id}$;  $= \mathrm{Id}$;



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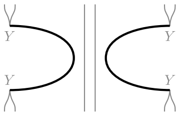

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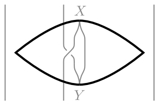

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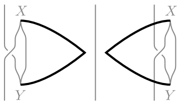

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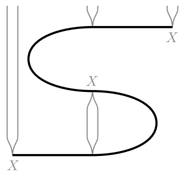

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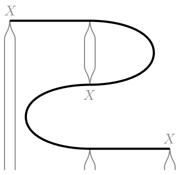

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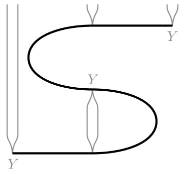

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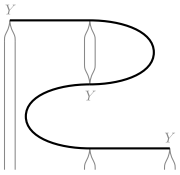

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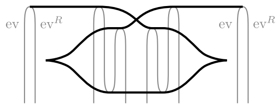

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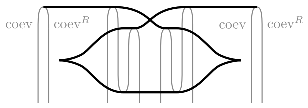

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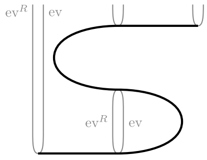

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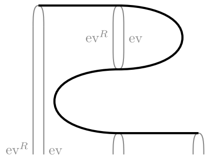

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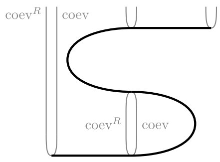

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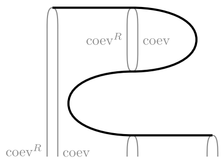

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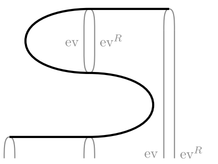
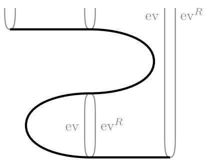
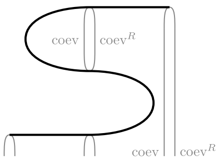


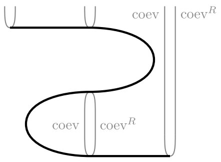
$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;



$= \mathrm{Id}$;

Note there is an isomorphism

$$\Big(\Big(\big(\mathrm{Dual}_{(3,2)} *_{\epsilon_{\mathrm{ev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\big) *_{\eta_{\mathrm{ev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\Big) *_{\epsilon_{\mathrm{coev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\Big) *_{\eta_{\mathrm{coev}}\sim\lambda} \mathrm{Adj}_{(3,2)} \to \mathrm{Dual}_{(3,3)}\,.$$

Definition 4.2.7. *Let $\mathcal{C}$ be a 3-category. We define $\mathcal{C}^{2L}$ to be the 3-category obtained by discarding all 2-morphisms which are not left adjoints.*

Lemma 4.2.8. *Let $\mathcal{C}$ be a monoidal 3-category. Then the restriction functor $E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Obj}^R((\mathcal{C}^{2L})^{1L})$ and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) \to \mathrm{Obj}^R((\mathcal{C}^{2L})^{1L})$$

is an equivalence of 3-groupoids.

Proof. First note that $\mathrm{Fun}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) = \mathrm{Map}(\mathrm{Dual}_{(3,3)}, \mathcal{C})$, by the results in 3.3.3. We have

$$\mathrm{Dual}_{(3,3)} = \Big(\Big(\big(\mathrm{Dual}_{(3,2)} *_{\epsilon_{\mathrm{ev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\big) *_{\eta_{\mathrm{ev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\Big) *_{\epsilon_{\mathrm{coev}}\sim\lambda} \mathrm{Adj}_{(3,2)}\Big) *_{\eta_{\mathrm{coev}}\sim\lambda} \mathrm{Adj}_{(3,2)}$$

and so we can apply Proposition 3.1.13 four times to get an equivalence

$$\mathrm{Map}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) \to \mathrm{Map}^{\epsilon_{\mathrm{ev}}\dashv, \eta_{\mathrm{ev}}\dashv, \epsilon_{\mathrm{coev}}\dashv, \eta_{\mathrm{coev}}\dashv}(\mathrm{Dual}_{(3,2)}, \mathcal{C}).$$

Now ϵ_{ev}, η_{ev}, ϵ_{coev} and η_{ev} are the only noninvertible 2-cells in $\mathrm{Dual}_{(3,2)}$, therefore this is equal to $\mathrm{Map}(\mathrm{Dual}_{(3,2)}, \mathcal{C}^{2L})$. Now the result follows from Lemma 4.2.3

□

Definition 4.2.9. *Let $\mathcal{C}$ be a symmetric monoidal 3-category. We define $\mathrm{Obj}^{3d}(\mathcal{C})$ to be the full 3-subgroupoid of $\mathrm{Obj}(\mathcal{C})$ whose objects are the 3-dualizable objects in $\mathcal{C}$.*

Now we can prove the coherence statement for 3-dualizable objects in a symmetric monoidal 3-category.

Theorem 4.2.10. *Let $\mathcal{C}$ be a symmetric monoidal 3-category. Then the restriction functor $E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) \to \mathcal{C}$ factors through $\mathrm{Obj}^{3d}(\mathcal{C})$ and the induced functor*

$$E_X : \mathrm{Fun}(\mathrm{Dual}_{(3,3)}, \mathcal{C}) \to \mathrm{Obj}^{3d}(\mathcal{C})$$

is an equivalence of 3-groupoids.

Proof. By the results of the previous section, we have $\mathrm{Obj}^R((\mathcal{C}^{2L})^{1L}) = \mathrm{Obj}^{3d}(\mathcal{C})$, since $\mathcal{C}$ is symmetric monoidal. So the statement follows from Lemma 4.2.8.

□

4.3 The generators as 3-framed bordisms

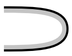
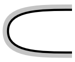
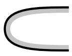
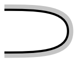
In this section we explain how to associate to each cell in FD_3 a 3-framed manifold and to each relation a framed diffeomorphism. Since we don't have in mind a specific model for the bordism category, we will necessarily be somewhat imprecise. For example, we don't bother specifying collars for the boundary of the bordisms, as is usually necessary in specific constructions of the bordism category. We specify k-framed manifolds using the method and notation in [5]. This means we exhibit a k-framing on an m-manifold M, for $k \geq m$ as an immersion of M into $\mathbb{R}^k$ together with a framing of the normal bundle. Note that a k-framing on M determines a $(k+1)$-framing on M by stabilization. The process of stabilization can be described as as follows. Given an m-manifold M and an immersion of M into $\mathbb{R}^k$ with a normal framing, we consider the immersion of M into $\mathbb{R}^{k+1}$ given by composing with the standard embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ as the first k coordinates. The normal framing for this immersion is obtained by adding the $(k+1)^{st}$ coordinate vector field as the last element of the framing.

Our conventions only differ from those of [5] in one aspect: the choice of induced normal framing on the incoming and outgoing boundary components of a bordism M. For N an incoming boundary component, we pick a vector field X_{in} on N pointing into M and define the normal framing of N to be that of M with X_{in} added at the end. For an outgoing boundary component we do the same with vector field pointing out of M. In [5], the authors take this vector field as the first vector in the normal framing.

The reason for our different choice of convention is the following. Suppose we immerse an m-manifold M in $\mathbb{R}^{k-1} \times [0,1] \subset \mathbb{R}^k$, with the incoming boundary components of M contained in $\mathbb{R}^{k-1} \times \{0\}$, the outgoing boundary components in $\mathbb{R}^{k-1} \times \{1\}$ and $M \backslash \partial M$ in $\mathbb{R}^{k-1} \times (0,1)$. Moreover, suppose we pick a normal framing of this immersion which is tangent to $\mathbb{R}^{k-1} \times \{0\}$ at the incoming boundary and tangent to $\mathbb{R}^{k-1} \times \{1\}$ at the outgoing boundary. Then we have induced immersions of the incoming and outgoing boundary components into $\mathbb{R}^{k-1}$ which come with normal framings, and the stabilization of these normal framings is the induced normal framing on the boundary of M. This makes it very easy to deduce the induced framing on the boundary from a picture of a normally framed immersed bordism.

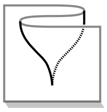
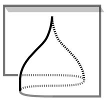
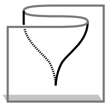
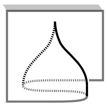
Note that there are only two 3-framings on the point and both are stabilizations of 1-framings. We associate to X the point with the standard positive framing, meaning we take a point in $\mathbb{R}$ and pick a normal vector pointing in the positive direction, and to Y the point with the opposite framing.

Now we move on to the 1-cells ev, coev, ev^R and coev^R. We will describe 2-framed 1-bordisms associated to these, by specifying immersions in $\mathbb{R}^2$ and normal framings and we take the 3-framings obtained by stabilizing. In the pictures, the first coordinate vector in $\mathbb{R}^2$ points down and the second one points right, so the 1-bordisms are read left to right, whereas their sources and targets are read top to bottom. We denote the normal framing by putting gray shading on the side of the manifold where the normal vector field points.

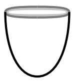
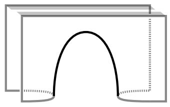
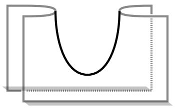
$\mathrm{ev} \mapsto$  $\mathrm{coev} \mapsto$  $\mathrm{ev}^R \mapsto$  $\mathrm{coev}^R \mapsto$ 

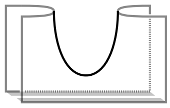
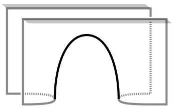
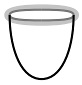
Now we describe the 3-framed 2-bordisms corresponding to the 2-cells in the presentation by giving 2-bordisms embedded in $\mathbb{R}^3$ with a normal framing. In the pictures, the first coordinate in $\mathbb{R}^3$ points out of the page, the second coordinate points to the right and the third one points down, so the incoming boundary will be on the top and the outgoing boundary on the bottom.

The four cusp morphisms correspond to the following normally framed bordisms. The darker lines are the singular locus of the projection to $\mathbb{R}^2$ given by forgetting the first coordinate.

$C_X \mapsto$  $C_X^{-1} \mapsto$  $C_Y \mapsto$  $C_Y^{-1} \mapsto$ 

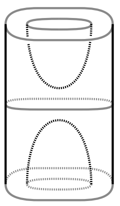
The noninvertible 2-morphims correspond to the following normally framed bordisms.

$\epsilon_{\mathrm{ev}} \mapsto$  $\eta_{\mathrm{ev}} \mapsto$  $\epsilon_{\mathrm{coev}} \mapsto$  $\eta_{\mathrm{coev}} \mapsto$ 

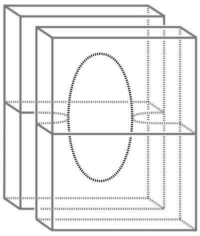
$\epsilon_{\mathrm{ev}}^R \mapsto$  $\eta_{\mathrm{ev}}^R \mapsto$  $\epsilon_{\mathrm{coev}}^R \mapsto$  $\eta_{\mathrm{coev}}^R \mapsto$ 

Now consider the invertible 3-cells. All of these have source and target 2-morphisms corresponding to 2-bordisms which are diffeomorphic as 3-framed manifolds, relative to the boundary. Therefore, there is a 3-framing on the mapping cylinder of such a diffeomorphism and the resulting 3-framed bordism is the one corresponding to the 3-cell in question.

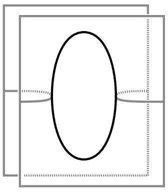
Now consider the eight noninvertible 3-cells. We will describe four 3-manifolds embedded in $\mathbb{R}^3$. For each of these we describe two different choices of incoming and outgoing boundary, corresponding to two different 3-cells. First consider a solid cylinder in $\mathbb{R}^3$, with a cup removed from the top and a cap removed from the bottom.



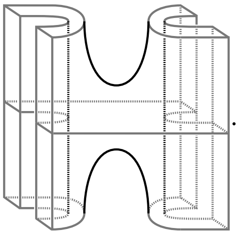
If we declare the cylinder to be incoming boundary and the cup and cap to be outgoing boundary, we get a 3-framed 3-bordism corresponding to $u_{\epsilon_{\mathrm{ev}}}$. If we declare the cylinder to be outgoing boundary and the cup and cap to be incoming boundary, we get a 3-framed 3-bordism corresponding to $v_{\eta_{\mathrm{coev}}}$. Now consider a solid 3-ball in $\mathbb{R}^3$. If we declare the boundary sphere to be incoming boundary, we get a 3-framed 3-bordism corresponding to $v_{\epsilon_{\mathrm{ev}}}$. If we declare the boundary sphere to be outgoing boundary, we get a 3-framed 3-bordism corresponding to $u_{\eta_{\mathrm{coev}}}$. Now consider two solid rectangular boxes connected by a solid tube, where we smooth out the circles where the tube meets the boxes.



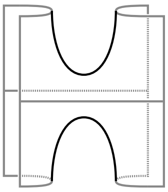
If we declare the front and back rectangles to be incoming boundary and



to be outgoing boundary, we get a 3-framed 3-bordism correponding to $u_{\eta_{\mathrm{ev}}}$. If we make the opposite choice, we get a 3-framed 3-bordism correponding to $v_{\epsilon_{\mathrm{coev}}}$. Finally, consider the embedded 3-manifold



If we declare the two half cylinders to be incoming boundary and



to be outgoing boundary, we get a 3-framed 3-bordism correponding to $u_{\epsilon_{\mathrm{coev}}}$. If we make the opposite choice, we get a 3-framed 3-bordism correponding to $v_{\eta_{\mathrm{ev}}}$.

If we were in the context of fully weak symmetric monoidal 3-categories, this would induce a symmetric monoidal functor $\phi : \mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3) \to \mathrm{Bord}_3^{fr}$ with $\phi(X) = +$, the positively framed point. One could then use the cobordism hypothesis to show that ϕ is an equivalence, as we now explain. Let $\mathcal{C}$ be a symmetric monoidal 3-category.

From the commuting diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3), \mathcal{C}) \arrow[r, "E_X"] & \mathrm{Obj}^{fd}(\mathcal{C}) \\
\mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_3^{fr}, \mathcal{C}) \arrow[r, "E_+"] \arrow[u, "\phi^*"] & \mathrm{Obj}^{fd}(\mathcal{C}) \arrow[u, "\mathrm{Id}"]
\end{tikzcd}$$

we deduce that

$$\phi^* : \mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_3^{fr}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3), \mathcal{C})$$

is an equivalence. Taking $\mathcal{C} = \mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3)$ we get an equivalence

$$\phi^* : \mathrm{Fun}^{\mathrm{sym}}(\mathrm{Bord}_3^{fr}, \mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3)) \to \mathrm{Fun}^{\mathrm{sym}}(\mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3), \mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3))$$

and therefore there exists $\psi : \mathrm{Bord}_3^{fr} \to \mathcal{F}^{\mathrm{sym}}(\mathrm{FD}_3)$ such that $\psi \circ \phi = \phi^*(\psi)$ is naturally isomorphic to the identity functor. Moreover, $\phi^*(\phi \circ \psi) = \phi \circ \psi \circ \phi \simeq \phi = \phi^*(\mathrm{id})$ and therefore $\phi \circ \psi \simeq \mathrm{id}$, so ψ is an inverse for ϕ.

Bibliography

[1] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. 1995.

[2] Krzysztof Bar and Jamie Vicary. Data structures for quasistrict higher categories, 2016.

[3] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category, 2015.

[4] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms, 2015.

[5] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. Dualizable tensor categories, 2013.

[6] Nick Gurski. Biequivalences in tricategories, 2011.

[7] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories.* Cambridge University Press, 2003.

[8] Tom Leinster. *Higher Operads, Higher Categories.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[9] Jacob Lurie. On the classification of topological field theories, 2009.

[10] Piotr Pstrągowski. On dualizable objects in monoidal bicategories, framed surfaces and the cobordism hypothesis, 2014.

[11] Emily Riehl and Dominic Verity. Homotopy coherent adjunctions and the formal theory of monads, 2013.

[12] Christopher Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories.* PhD thesis, 2009.