

WHO CAN WIN A SINGLE-ELIMINATION TOURNAMENT?*

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Abstract. A single-elimination (SE) tournament is a popular way to select a winner both in sports competitions and in elections. A natural and well-studied question is the tournament fixing problem (TFP): given the set of all pairwise match outcomes, can a tournament organizer rig an SE tournament by adjusting the initial seeding so that the organizer’s favorite player wins? We prove new sufficient conditions on the pairwise match outcome information and the favorite player, under which there is guaranteed to be a seeding where the player wins the tournament. Our results greatly generalize previous results. We also investigate the relationship between the set of players that can win an SE tournament under some seeding (so-called SE winners) and other traditional tournament solutions. In addition, we generalize and strengthen prior work on probabilistic models for generating tournaments. For instance, we show that *every* player in an n player tournament generated by the Condorcet random model will be an SE winner even when the noise is as small as possible, $p = \Theta(\ln n/n)$; prior work only had such results for $p \geq \Omega(\sqrt{\ln n/n})$. We also establish new results for significantly more general generative models.

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1. Introduction. A single-elimination (SE) tournament, also known as a sudden death tournament, an Olympic system tournament, or a binary-cup election, is a popular way to select a winner among multiple candidates/players. In an SE tournament, pairs of players are matched according to an initial seeding, the winners of these pairs advance to the next round, and the losers are eliminated after a single loss. Play continues according to the seeding until a single player, the winner, remains. SE tournaments are popular among sports fans due to their exciting “do-or-die” nature and among tournament organizers due to their efficiency. In contrast to a round-robin tournament, which requires $\Theta(n^2)$ matches to be played between n players, the winner of an SE tournament is decided after a total of $n - 1$ matches. In tournaments such as the NCAA March Madness and the U.S. Open Tennis Championships, each of which involves more than 64 teams, the difference between a linear and quadratic number of matches is quite pronounced.

While the efficiency of SE tournaments is desirable, the winner of a given SE tournament can depend significantly on the initial seeding. A series of works [1, 5, 6, 7, 9, 11, 13, 14, 15, 16] has investigated how easily the winner of an SE tournament

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can be manipulated simply by adjusting the seeding of the tournament. Formally, the problem is called the *tournament fixing problem* (TFP) or the *agenda control problem for balanced knockout tournaments*. In TFP, we are given a set of players V , information for each pair of players (u, w) about whether u or w would win in a head-to-head matchup, and a player of interest v ; then we are asked the following question: is there a seeding for a balanced SE tournament where v wins? TFP is known to be NP-hard [1] with the best-known algorithm running in $2^n \text{poly}(n)$ time [7]. Thus, unless $P = NP$, it is in general intractable to determine which players can win an SE tournament. Nevertheless, a number of works on TFP have produced “structural results” which argue that for certain classes of instances, one can find a winning seeding for v in polynomial (and often linear) time [7, 14, 15]. These structural results suggest that in many practical settings, the winner of an SE tournament is susceptible to manipulation, because many players have winning seedings that can be found efficiently. Furthermore, under reasonable probabilistic models for generating tournaments, these structural results have been shown to occur with high probability [13, 15], further suggesting that the worst-case hardness results may not apply to real-world instances. In other words, in many actual tournaments, it is completely feasible for SE tournament organizers to rig the outcome of the competition. Experimental results [12] investigate this finding in practical settings.

While TFP can be seen as a way to understand manipulation in competition and elections, studying conditions under which players can and cannot win SE tournaments can also be seen as part of a larger study of *tournament solutions*: different ways to define the winners of a round-robin tournament. The input to TFP can be viewed as a *tournament* $T = (V, E)$, or a complete, oriented graph where for all pairs of nodes $u, w \in V$, exactly one of (u, w) and (w, u) is an element of E ; u points to w if u would win in the match between u and w . The study of tournaments is central to social choice theory; they provide a general framework for representing the outcomes between players in a round-robin tournament or, more generally, pairwise preferences between alternatives, often generated from voter information. As such, an essential question of social choice theory asks: given a tournament, how should we select a set of winners? SE tournaments provide one way of answering this question: we say that a player $v \in V$ is an *SE winner* if there is some seeding under which v wins the resulting SE tournament. The study of tournament solutions includes many other well-studied concepts (see, e.g., [2, 10]). One classical example is the Copeland set, consisting of the players with the maximum number of wins in the tournament. A natural question to investigate is how these traditional notions of strength in round-robin tournaments relate to the notion of strength in an SE tournament.

1.1. Results. In this work, we improve our understanding of conditions on the input tournament and player of interest that are sufficient for the player to be an SE winner. Many previous structural results involve the notion of a *king*, or a player v , such that for every other player $u \in V \setminus \{v\}$, either v beats u directly, or v beats some w who beats u . We present a vast generalization of many of the known structural results involving kings, showing that essentially any “combination” of the known sufficient conditions for a king to be an SE winner is also sufficient for the king to be a winner.

In particular, recall the following structural results from [15], where given a tournament T and a player v , we can find a winning seeding for v in polynomial time. One class of tractable instances is that in which every player w who beats v wins against at most as many players as v beats. It is also known that if v is a king and wins against at least half of the players, or is a “superking” (meaning that every w whom

v beats indirectly loses to at least $\log n$ players whom v beats directly), then v will be able to win an SE tournament. While these results have been useful on their own for showing that tournaments generated by certain random models are likely to have many players who can win [13, 15], it is natural to wonder how robust these results are to changes in the exact sufficient conditions. Recent results of [7] seem to suggest that the parameters for these structural results are brittle; namely, when the exact parameters of the conditions are relaxed, finding a winning seeding for v (if it exists) becomes NP-hard. In Theorem 2.1, we provide a broad generalization of the three structural results stated above. We show that these conditions are actually flexible in the sense that if the players who beat some king v can be partitioned into groups that satisfy these sufficient conditions, then v can win an SE tournament. Additionally, we extend the work on 3-kings (or players who have win-distance ≤ 3 to every other player) introduced in [7] and give a new set of sufficient conditions for a 3-king to win an SE tournament.

In section 3, we apply these and other known structural results to understand the relationship between SE winners and the winners according to other tournament solutions. In particular, Theorem 3.1 shows that the players selected by a number of well-studied tournament solutions are also SE winners, including the Copeland set described above. Another class of tournament solutions of interest was introduced in [10] as a natural extension of the Copeland set. In these “iterative matrix solutions,” we consider the tournament matrix A (corresponding to the adjacency matrix of the underlying tournament graph); a player is included in the k th iterative matrix solution if they have the maximum number of “wins” in A^k . We give a new interpretation of this solution and use it to show that for sufficiently large tournaments, the players in the iterative matrix solutions will also be SE winners.

Finally, in section 4 we investigate probabilistic models for generating random tournaments and the resulting structure of such tournaments. In particular, we start by giving an improved result for tournaments generated by the Condorcet random (CR) model. The CR model assumes an underlying order to players, where stronger players generally win against weaker players and are only upset with some small probability p . We demonstrate that in tournaments generated by the CR model, even when the probability of upset p is $\Theta(\ln n/n)$, with high probability every player in the tournament will have a winning seeding that can be discovered efficiently. This upset rate p is optimal (up to constant factors) because a player needs to win $\log n$ matches in order to win an SE tournament. Our result greatly improves on the previous best result from [15], which proves an analogous claim for $p \geq \Omega(\sqrt{\ln n/n})$. In light of this optimal result for the CR model, we propose a new generative model for tournaments that aims to remove the structure that arises from assuming an underlying order of players and a consistent noise parameter. Despite the fact that the model may produce tournaments with largely arbitrary structure, we are able to prove a nontrivial result similar to the previous results on the CR model. The details of the model and our theorem statement are given in section 4.

All of our results are constructive. In particular, we demonstrate that certain conditions are sufficient for a player v to be an SE winner by giving algorithms, running in polynomial time, that output a seeding where v will win.

1.2. Preliminaries and notation. We will assume throughout that all SE tournaments are balanced and played among a power of two, $n = 2^k$ for some $k \geq 0$, players. Table 1 provides a summary of the notation that is used to refer to players and their neighborhood in the underlying tournament. For subsets $A, B \subseteq V$, we say

TABLE 1
Summary of the notation used in this paper.

Notation
$N_{out}(v) = \{u : (v, u) \in E\},$ $N_{in}(v) = \{u : (u, v) \in E\}$
$out(v) = N_{out}(v) , out_S(v) = N_{out}(v) \cap S $ $in(v) = N_{in}(v) , in_S(v) = N_{in}(v) \cap S $

that A dominates B , denoted $A \succ B$, if for all $a \in A$ and all $b \in B$, $(a, b) \in E$. We will abuse this notation slightly, allowing individual players, rather than subsets, to be related to other players or subsets.

Recall that we can define the notions of king and 3-king of a tournament in terms of the underlying tournament graph. A *king* is a player v who has distance at most 2 to every other player $u \in V \setminus \{v\}$. A *3-king* is the generalization of kings to players who have distance at most 3 to every other player.

In section 3, we consider some tournament solutions. We provide brief descriptions of these solutions; for more detail, we refer the interested reader to [2]. The *uncovered set* refers to the set of kings in the tournament.¹ The *Copeland set* is the set of players of maximum out-degree in the tournament.

A tournament is *transitive* if we can label the players with labels from $\{1, \dots, n\}$ such that $\forall i, j, i < j$ implies $i \succ j$. Given a tournament T , consider flipping edges in T to produce a transitive tournament T' , while minimizing the number of edges flipped. The *Slater set* of T is the set of players who can be labeled 1 in such a T' .

The *Markov set* can be thought of as the set of players who win the most matches, in expectation, in a “winner-stays” tournament, where play proceeds by repeatedly selecting a random player to play the previous winner.

The *bipartisan set* is the support of the maximal lottery (i.e., the Nash equilibrium of the symmetric zero-sum game formed by the tournament matrix) for the tournament.

2. Structural results. A number of results are known about conditions under which a player is guaranteed to be an SE winner [7, 14, 15]. Many of these results concern players who are kings. In particular, [15] showed that a “superking”—a king v where every player in $N_{in}(v)$ loses to at least $\log n$ players from $N_{out}(v)$ —is always an SE winner. On the other hand, [13] showed that a “king of high out-degree”—that is, a king with out-degree k who loses to fewer than k players that have out-degree greater than k —is always an SE winner. This result was the first to generalize the conditions on players who can win SE tournaments. In this section, we further generalize these results by combining their respective conditions. Moreover, we further explore the notion of 3-kings that was considered by [7] and present an alternative condition under which a 3-king can win an SE tournament.

Before we proceed to the results, we make some remarks on the strength of the king condition. While the ability to reach any other player in the tournament in at most two steps might seem like a strong condition (which would limit the usefulness of our results), it is in fact not as strong as it may first appear. Indeed, every tournament contains a king, and in particular any player with the maximum number of wins in the tournament is always a king. Moreover, if we generate a tournament by choosing

¹The name *uncovered set* stems from the *covering relation*. A player v is said to *cover* another player u if (i) v beats u , and (ii) any player who beats v also beats u . The uncovered set corresponds to the set of players that are not covered by any other player.

the direction of each edge independently at random, it is known that the set of kings is equal to the entire set of players with high probability [4].

THEOREM 2.1. *Consider a tournament $T = (V, E)$ where $\mathcal{K} \in V$ is a king. Let $A = N_{\text{out}}(\mathcal{K})$ and $B = V \setminus (A \cup \{\mathcal{K}\}) = N_{\text{in}}(\mathcal{K})$. Suppose that B is a disjoint union of three (possibly empty) sets H, I, J such that*

1. $|H| < |A|$,
2. $\text{in}_A(i) \geq \log |V|$ for all $i \in I$ (i.e., $\text{out}_A(i) \leq |A| - \log |V|$ for all $i \in I$),
3. $\text{out}(j) \leq |A|$ for all $j \in J$.

Then \mathcal{K} is an SE winner, and we can compute a winning seeding for \mathcal{K} in polynomial time.

Note that the superking result [15] corresponds to the special case where $H = J = \emptyset$, while the “king of high out-degree” result [13] corresponds to the special case where $I = \emptyset$. Hence Theorem 2.1 is much stronger than previous results in the sense that each player in B only has to satisfy one of the three “reasons” why it is not strong, instead of having to adhere to any particular reason.

Proof. We proceed by induction, arguing that we can construct a seeding where, in each round, the three properties listed above and the fact that \mathcal{K} is a king are maintained as invariants. We will first take care of the cases where the tournament is small. If $|V| = 1$ or 2 , B is empty and the result is clear.

Suppose that $|V| = 4$. If $|A| \geq 2$, the result follows from [13]. Otherwise $|A| = 1$, and it follows that $H = I = \emptyset$ and $|J| \leq 1$, which contradicts $|V| = 4$.

Suppose now that $|V| \geq 8$. If $|A| \leq 2$, then $|H| \leq 1$, $I = \emptyset$, and $|J| \leq 3$, which contradicts $|V| \geq 8$. If $I = \emptyset$, or $H \cup J = \emptyset$, or $|A| \geq |V|/2$, the result follows from [13] and [15]. Hence we may assume from now on that $|V| \geq 8$, $3 \leq |A| < |V|/2$, $I \neq \emptyset$, and $H \cup J \neq \emptyset$.

We will present an algorithm to compute a winning seeding for \mathcal{K} . The algorithm will match the players for the first round of the tournament in such a way that the leftover tournament after the first round also satisfies the conditions of the theorem. The description of the algorithm is as follows.

1. Perform a maximal matching M_1 from A to H .
2. Since $|H| < |A|$, we have $A \setminus M_1 \neq \emptyset$. Perform a maximal matching M_2 (which might be an empty matching) from $A \setminus M_1$ onto $I \cup J$.
3. If A was not fully used in the two matchings, match an arbitrary unused player in A with \mathcal{K} . Else, choose an arbitrary player $a \in A \cap M_2$ and rematch it to \mathcal{K} .
4. Perform arbitrary matchings within A, H , and $I \cup J$.
5. If there are leftover players, there must be exactly two of them; match them to each other.

We prove the correctness of the algorithm by showing that the four invariants are maintained by the algorithm. Let V', A', B', H', I', J' denote the subsets of V, A, B, H, I, J that remain after the iteration.

1. $|H'| < |A'|$. We will show that $|H'| \leq |H|/2$ and $|A'| \geq |A|/2$. The claim follows since $|H| < |A|$. If $H = \emptyset$, then $|H'| < |A'|$ holds trivially, so we may assume that H is nonempty. At least one player in H is used in the matching M_1 , so we have $|H'| \leq |H|/2$. We will show that the matching $M_1 \cup M_2$ consists of at least two pairs. Since there can be at most two pairs in the matching provided by the algorithm in which a player in A is beaten by a player outside of A (i.e., the pair in which a player in A is matched to \mathcal{K} and the pair in which a player in A is matched in the final step of the algorithm

for leftover nodes), it will follow that $|A'| \geq |A|/2$.

If M_1 consists of at least two pairs, we are done. Suppose that M_1 consists of exactly one pair. Since $|V| \geq 8$, each player in I is beaten by at least three players in A . (Recall that I is nonempty.) One of these players is possibly used in M_1 , and one is possibly used to match with \mathcal{K} , but that still leaves at least one player in A who beats a player in I . Hence $M_1 \cup M_2$ consists of at least two pairs, as desired.

2. $in_{A'}(i) \geq \log |V'|$ for all $i \in I$. Let $i \in I'$. Since M_2 is a maximal matching, every player that contributes to the in-degree of i in A survives the iteration, except possibly the player who is rematched to \mathcal{K} . Hence the in-degree of i in A' is at least $\log |V| - 1 = \log(|V|/2)$.
3. $out(j) \leq |A'|$ for all $j \in J'$. The condition is equivalent to $out_{B'}(j) < in_{A'}(j)$. Let $j \in J'$. We have either $in_{A'}(j) = in_A(j)$ or $in_{A'}(j) = in_A(j) - 1$, where the latter case occurs exactly when a player in A who beats j is rematched to \mathcal{K} . In the former case we immediately obtain $out_{B'}(j) < in_{A'}(j)$. In the latter case, A has been fully used in the two matchings before one player is rematched to \mathcal{K} . This means that j eliminates another player in B , and it follows that $out_{B'}(j) \leq out_B(j) - 1 < in_A(j) - 1 = in_{A'}(j)$.
4. \mathcal{K} is a king. Let $b \in B'$. If $b \in H'$, then since M_1 is a maximal matching, b is beaten by some player in A' . If $b \in I'$, then since the second invariant is maintained, b is beaten by some player in A' . Otherwise $b \in J'$. Since the third invariant is maintained, b beats at most $|A'| - 1$ players in A' , and hence b is also beaten by some player in A' in this case.

Hence the four invariants are maintained, and the algorithm correctly computes a winning seeding for \mathcal{K} . \square

Thus, we have shown a very general result about kings that holds in tournaments on n players for *any* power of two n , answering an open research problem posed in [14] to provide more general structural results that hold independently of the size of the tournament. (Some earlier results only hold for large n .)

Next, we consider the weaker notion of a 3-king. Prior work presented a set of conditions under which a 3-king is an SE winner [7]. One of their conditions is that there exists a perfect matching from the set of nodes that are reachable in exactly two steps from the 3-king \mathcal{K} onto the set of nodes that are reachable in exactly three steps from \mathcal{K} . Here, we present a different set of conditions that does not include the requirement of a perfect matching.

THEOREM 2.2. *Consider a tournament $T = (V, E)$ where $\mathcal{K} \in V$ is a 3-king. Let $A = N_{out}(\mathcal{K})$, $B = N_{out}(A) \cap N_{in}(\mathcal{K})$, and $C = N_{in}(\mathcal{K}) \setminus B$. Suppose that the following three conditions hold:*

1. $|A| \geq \frac{|V|}{2}$,
2. $A \succ B$,
3. $|B| \geq |C|$.

Then \mathcal{K} is an SE winner, and we can compute a winning seeding for \mathcal{K} in polynomial time.

Proof. If $|V| = 1, 2$, or 4 , the result is clear. For $|V| \geq 8$, first perform a maximal matching from B to C and match \mathcal{K} to an arbitrary player in A , and then pair off players within A . If $|A|$ is odd, then $A \cup \{\mathcal{K}\}$ matches evenly. Else, match the remaining $a \in A$ to some $b \in B$. We pair off players within each of B, C arbitrarily and match the remaining pair between B and C if needed. After the round, $|A| \geq \frac{|V|}{4}$.

Since the matching from B to C is nonempty, we still have that $|B| \geq |C|$ after the iteration. Moreover, since we applied a maximal matching, each player in C is still beaten by some player in B . Thus, the required conditions are maintained as invariant, and we can efficiently compute a winning seeding for \mathcal{K} . \square

It would be interesting to investigate the extent to which we can weaken the strong second condition that all players in A beat all players in B . It should be noted that if any of the three conditions is removed, the theorem no longer holds. In particular, if the second condition is dropped, a counterexample from [7] shows that for any constant $\varepsilon > 0$, there is a tournament on n players, where \mathcal{K} is a 3-king who wins against $(1 - \varepsilon)n$ players but cannot win an SE tournament. Given that the notion of a 3-king is significantly weaker than that of a king (recall that kings who beat at least $|V|/2$ players are SE winners), it seems reasonable to conjecture that a strong assumption such as the second condition (or the conditions seen in [7]) may be required to prove structural results for 3-kings.

3. SE winners and tournament solutions. Tournament solutions are functions that map each tournament graph to a subset of players, usually called the choice set. The choice set is often thought of as containing the stronger players, or “winners,” within the tournament. Many tournament solutions have been considered, including the Copeland set, the Slater set, the Markov set, and the bipartisan set [2, 10]. The ability to win an SE tournament provides us with another criterion with which we can distinguish between stronger and weaker players in a tournament. In this section, we investigate the relationship between the set of SE winners and some traditional tournament solutions.

THEOREM 3.1. *A player chosen by the Copeland set, the Slater set, or the Markov set is an SE winner. A player in the bipartisan set with the highest Copeland score is also an SE winner.*

Proof. All four tournament solutions are contained in the uncovered set, meaning that a player from any of these sets is a king. Therefore, using a special case of Theorem 2.1 (or an earlier result of [15]), it suffices to show that such a player wins against at least half of the remaining players. For the Copeland set, this is trivial, while [10] and [8] show that any player in the Slater set, as well as any player in the bipartisan set with the highest Copeland score, beats at least half of the players. Next, we show that players from the Markov set win against at least half of the players.

Recall that the Markov set is defined to be the set of players of maximum probability in the stationary distribution of the Markov chain defined by the normalized Laplacian matrix $Q = (q_{ij})_{n \times n}$ of the Markov chain of the tournament, where $q_{ij} = 1/n$ if v_i beats v_j (0 otherwise) and $q_{ii} = \text{out}(v_i)/n$. Assume that the first player is in the Markov set. It follows that the probability associated with the first player in the eigenvector $p = (p_i)_{n \times 1}$ corresponding to the eigenvalue 1 is maximal. Assume for contradiction that $q_{11} < \frac{1}{2}$. We then have

$$\begin{aligned} p_1 &= q_{11}p_1 + q_{12}p_2 + \cdots + q_{1n}p_n \\ &\leq q_{11}p_1 + q_{12}p_1 + \cdots + q_{1n}p_1 \\ &= 2q_{11}p_1 \\ &< p_1, \end{aligned}$$

a contradiction. \square

It is not true that any player in the bipartisan set is always an SE winner. Indeed, consider a transitive tournament with the slight modification that the weakest player beats the strongest player. Then the former player is included in the bipartisan set even though this player only beats one player and cannot be an SE winner.

Another family of tournament solutions is introduced in [10] as “iterative matrix solutions.” Consider the tournament adjacency matrix $A = (a_{ij})$, in which $a_{ij} = 1$ if i beats j , and 0 otherwise. The Copeland score is given by $A\mathbf{1}$. For any positive integer k , we consider $A^k\mathbf{1}$ and include the player(s) with the maximum resulting score in our k th iterative tournament solution.

There is a natural interpretation of iterative matrix solutions as the number of paths of length k starting from each player. Any player in an iterative matrix solution belongs to the uncovered set. In other words, if the player v is covered by some w (i.e., $w \succ \{v\} \cup N_{\text{out}}(v)$), then v cannot be in the iterative matrix solution. Indeed, if v is covered by w , then the first steps of the paths starting from w form a superset of the first steps of the paths starting from v . On the other hand, it is not the case that any player in an iterative matrix solution always beats at least half of the remaining players, as shown by the following example.

Consider $k = 2$ and the tournament with player set $V = A \cup B \cup \{x\}$, where $A \approx rn$ and $B \approx (1-r)n$ with $\frac{1}{2} < r < \frac{1}{\sqrt{3}}$. Suppose that $A \succ x \succ B \succ A$ and that A and B are close to regular. The Copeland scores of $a \in A, b \in B, x$ are $\frac{rn}{2}, \frac{(1+r)n}{2}, (1-r)n$, respectively. It follows that the iterative matrix scores of a, b, x are $\frac{r^2n^2}{4}, \frac{(1+r^2)n^2}{4}, \frac{(1-r^2)n^2}{2}$. This implies that x has the maximum iterative matrix score but beats fewer than half of the remaining players.

Nevertheless, we will show that for large enough tournaments, players in an iterative matrix solution are always SE winners. First, we need the following lemma and the subsequent corollary.

LEMMA 3.2. *In a tournament with n players, the minimum possible number of k -paths is $\binom{n}{k+1}$.*

Proof. In a transitive tournament, each subset of size $k+1$ gives rise to exactly one k -path. On the other hand, by a simple inductive argument, each subset of size $k+1$ gives rise to at least one k -path that goes through all $k+1$ players. The result follows immediately. \square

COROLLARY 3.3. *In a tournament with n players, a player with the maximum number of k -paths originating from it is the origin of at least $\frac{1}{n} \binom{n}{k+1}$ k -paths.*

We are now ready to prove the theorem.

THEOREM 3.4. *For any fixed k , there exists a constant N_k such that for any tournament of size at least N_k , a player with the maximum number of k -paths originating from it is an SE winner.*

Proof. Let v be a player with the maximum number of k -paths originating from it, and let A and B be the sets of players who lose to v and who beat v , respectively. From Corollary 3.3, v is the origin of at least $\frac{1}{n} \binom{n}{k+1} \geq \frac{n^k}{2(k+1)!}$ k -paths for large enough n . Hence it must have out-degree at least $\frac{n}{2(k+1)!}$. In other words, $|A| \geq \frac{n}{2(k+1)!}$.

If the number of players in B with in-degree from A less than $\log n$ is less than $|A|$, we can apply Theorem 2.1. Otherwise, there are at least $|A| \geq \frac{n}{2(k+1)!}$ players in B with in-degree from A less than $\log n$. Call this set H , and consider a player $h \in H$. Since h beats all but at most $\log n$ players in A , we can compare the number of k -paths originating from v with the number of k -paths originating from h by removing

the common k -paths. The remaining number of k -paths originating from v is at most $\log n \cdot n^{k-1}$, while by Corollary 3.3 again, a player in H with the maximum number of k -paths within H is the origin of at least $O(n^k)$ k -paths, since $|H|$ is linear in n . This contradicts the assumption that v has the maximum number of k -paths originating from it. \square

3.1. The strength of kings. Since results concerning SE winners often involve the assumption that a player is a king in the given tournament, one might hope that there is a strong relation between SE winners and the uncovered set. For example, it could always be that a constant fraction of players in the uncovered set are SE winners or vice versa. This is not the case, however, as the following theorem shows.

THEOREM 3.5. *Let $r \in (0, 1)$. There exists a tournament such that the proportion of players in the uncovered set that are SE winners is less than r and the proportion of SE winners that are contained in the uncovered set is also less than r .*

Proof. Consider a tournament with player set $V = A \cup B \cup \{x, y\}$ such that

- $x \succ y, B$,
- $y \succ B, A$,
- $B \succ A$,
- $A \succ x$.

The uncovered set is $A \cup \{x, y\}$.

Let $|A| = k$ and $|B| = n$. If $k < \log n$, then players in A do not win enough matches to become SE winners. Hence the proportion of players in the uncovered set that are SE winners is at most $\frac{2}{k+2}$.

On the other hand, suppose that B is a regular tournament with all players isomorphic. By symmetry, if one player in B is an SE winner, then all of them are SE winners. In order for a player in B to be an SE winner, players x and y need to be eliminated. But this can easily be done in two rounds, with x beating y in the first round and a player in A beating x in the second round. Hence the proportion of SE winners that are contained in the uncovered set is at most $\frac{2}{n+2}$.

Taking k and n large enough with $k < \log n$, we obtain the desired result. \square

4. Generative models for tournaments. Recall the Condorcet random (CR) model studied in [3, 13, 15]. In the CR model, we assume that there is an underlying ordering to the players and that, in general, stronger players win against weaker players; however, with some small probability $p < 1/2$, the weaker player will upset the stronger player. In the corresponding tournament graph, we say that for two players i, j such that i occurs before j in the ordering, $(i, j) \in E$ with probability $1 - p$ and $(j, i) \in E$ otherwise. A number of results are known about which players are SE winners in tournaments drawn from a CR model. When $p \geq \Omega(\sqrt{\ln n/n})$, then with high probability, every player in the tournament will be a superking and therefore an SE winner [15]. In fact, even when $p \geq C \ln n/n$, roughly the first half of players will be SE winners, and more generally, if $p = C \cdot 2^i \ln n/n$, then roughly the first $1 - 1/2^{i+1}$ fraction of players are SE winners [13]. Previous work has also studied various generalizations of the CR model [7, 13].

In this section, we present improved results about tournaments generated by the standard CR model, showing that with high probability, every player in a CR tournament will be an SE winner, even with the noise $p = \Theta(\ln n/n)$ (with no dependence on the player's rank).

THEOREM 4.1. *Let $C \geq 64$ be a constant, and let $p \geq C \ln n/n$. Let T be a tournament generated by the CR model with noise parameter p on $n > n_C$ players*

(for some constant n_C). With probability $\geq 1 - 1/\Omega(n^2)$, every player has an efficiently computable winning seeding over T .

Note that this result is asymptotically optimal, as a player must have at least $\log n$ wins to be able to win an SE tournament. If $p = o(\ln n/n)$, then with high probability, the weakest player will not be able to win an SE tournament, regardless of the seeding. The case where $p \geq C\sqrt{\ln n/n}$ is covered in [15], which shows that every player in such a tournament is an SE winner.

The proof will use the following concentration bound, which can easily be derived from standard Chernoff–Hoeffding bounds.

LEMMA 4.2. *Let X_1, \dots, X_n be independent random variables with $X = \sum_i X_i$ and $E[X] = \mu$. Suppose $d \leq \mu$ and $\delta \in [0, 1]$. Then $\Pr[X < (1 - \delta)d] \leq \exp(-\delta^2 d/2)$.*

We give a sketch of the proof before proceeding to the full proof. First, we argue that the weakest player w will win against more than $k \log n$ players in the first half for some constant k . We will think of “swapping” $k \log n$ of these losers, which we call S , from the first half with some arbitrary set of players from the second half (so that these losers become some of the strongest players over the second half). Then we argue that at least one player v that w beats will be among the first $n/6$ players. This player, with high probability, will be a king over the first half of players who wins against more than half of the players; thus, by [15], this player will be an SE winner over the first half of players. Next, we argue that for some arbitrary player u in the weaker half of players, at least $\log n$ players from the $k \log n$ that were swapped with the second half will beat u . We then take a union bound over the players in the second half and argue that w will be a superking over the second half and, again by [15], an SE winner over the second half. Thus, w will be an SE winner over the entire tournament by winning over the weaker half, while v wins against the stronger half, and w wins against v in the final round. We take a union bound over all players to arrive at the desired result.

The detailed proof follows.

Proof of Theorem 4.1. Let $C \geq 64$ be a constant, and let $C \ln n/n \leq p \leq C\sqrt{\ln n/n}$. First, note that we expect w will win against $\frac{C}{2} \ln n = \frac{C \ln 2}{2} \log n$ players in the first half. Next, we can show that with high probability w wins against greater than $\frac{C \ln 2}{4} \log n$ players. Let $k = \frac{C \ln 2}{4}$. We have

$$\begin{aligned} & \Pr[w \text{ wins against } > k \log n \text{ players in the first half}] \\ & \geq 1 - \exp\left(-\frac{(k \log n)^2}{4k \log n}\right) \\ & = 1 - \exp\left(-\frac{k \log n}{4}\right) \\ & = 1 - \exp\left(-\frac{C \ln 2 \log n}{16}\right) \\ & = 1 - 1/n^{C/16}. \end{aligned}$$

We can also argue that with probability at least $1 - 1/n^{C/6}$, w wins against some

player v in the first $n/6$ players:

$$\begin{aligned}
 \Pr[w \text{ wins against some } v \in [1, n/6]] & \\
 &= 1 - (1 - p)^{n/6} \\
 &\geq 1 - (1 - (C \ln n/6)/(n/6))^{n/6} \\
 &\geq 1 - \exp(-C \ln n/6) \\
 &= 1 - 1/n^{C/6},
 \end{aligned}$$

where the inequality follows from the approximation $(1 - a/x)^x \leq e^{-a}$ for $a > 0$.

In what follows, we will imagine swapping a set of $k \log n$ players, called S , whom w wins against from the first half (excluding v), with $k \log n$ arbitrary players from the second half. This allows us to argue about the “first half” and the “second half” of players independently. We will argue that v is an SE winner over the new “first half” of players and that the inclusion of $k \log n$ strong players whom w beats makes w a superking over the new “second half.”

First, we argue that it is likely that v , whose rank is at most $n/6$, will be an SE winner over the first half. In particular, with high probability, v will be a king over the first half of players who win against at least $n/4$ players. Note that we expect v to win against at least $n/3 \cdot (1 - p) + pn/6 - 1 \geq n/3 - C\sqrt{n \ln n}/6 - 1$ players from the first half. The out-degree of v is given by a random variable, which is the sum of independent random variables, so we can bound the probability that $\text{out}(v) < n/4$ using Lemma 4.2 as follows:

$$\begin{aligned}
 \Pr[\text{out}(v) \geq n/4] &\geq 1 - \exp\left(-\frac{(n/12 - \frac{C}{6}\sqrt{n \ln n} - 1)^2}{2(n/3 - \frac{C}{6}\sqrt{n \ln n} - 1)}\right) \\
 &> 1 - 1/n^4,
 \end{aligned}$$

where the last inequality is a very loose bound on this probability that takes effect for sufficiently large n .

Next, we consider the probability that v is a king over the first half, conditioned on its high out-degree. We take a union bound over all possible players who did not lose against v , and we show that it is unlikely that any of these players beats every single player whom v beat:

$$\begin{aligned}
 \Pr[v \text{ is a king over the first half} \mid \text{out}(v) \geq n/4] & \\
 &\geq 1 - \sum_{i=1}^{n/4-1} (1 - p)^{\text{out}(v)} \\
 &\geq 1 - n/4 \cdot (1 - p)^{n/4} \\
 &\geq 1 - n/4 \cdot \exp(-C \ln n/4) \\
 &\geq 1 - 1/4n^{C/4-1}.
 \end{aligned}$$

Finally, we argue that with high probability, w will be a superking over the second half of players. Consider some other u from the second half of players. The expected number of players from S who beat u is $\geq k \log n \cdot (1 - p) = k \log n - \frac{kC \log^{3/2} n}{\sqrt{n}} \geq (k-1) \log n$ for sufficiently large n . Applying Lemma 4.2 again, we obtain the following

bound:

$$\begin{aligned}
 & \Pr[u \text{ loses to fewer than } \log n \text{ players from } S] \\
 & \leq \exp\left(-\frac{((k-2)\log n)^2}{2(k-1)\log n}\right) \\
 & = \exp\left(-\frac{(k^2-4k+4)}{2(k-1)}\log n\right) \\
 & = n^{-\left(\frac{k^2-4k+4}{2\ln 2 \cdot (k-1)}\right)}.
 \end{aligned}$$

Then, to guarantee that every u in the second half loses to at least $\log n$ players whom w beats, we take a union bound over the $n/2$ players. For any $k > 11$, this probability will be $\leq 1/n^3$.

The overall probability that w beats a sufficiently strong king over the first half of players is at least the following:

$$1 - 1/n^{C/6} - 1/n^4 - 1/4n^{C/4-1} \geq 1 - 2/n^4.$$

Thus, the probability that w wins against $k \log n$ players from the (true) first half, wins against some strong king v over the first half, and is a superking over the second half, is at least the following:

$$1 - 1/n^{C/16} - 2/n^4 - 1/n^3 \geq 1 - 2/n^3.$$

Since w is the weakest player in the tournament, the probability that any other player is an SE winner can only be greater. Taking a union bound over all players, we conclude that with probability at least $1 - 1/\Omega(n^2)$, every player in the tournament will be an SE winner. \square

4.1. Generalizing the CR model for tournaments. As the prior claims demonstrate, in the standard CR model, every player is an SE winner with high probability, even when upsets occur at an asymptotically minimal rate. While this result indicates the depth of our understanding of conditions under which a player is an SE winner, it also suggests that the assumption that tournaments are drawn from a CR model—where the noise parameter p is fixed for all matchups—may be too rigid, incidentally providing structure that may not exist in practical settings. Prior work [13] proposes a generalized CR model, where for two players $i < j$, j upsets i with probability $p \leq p(i, j) \leq 1/2$ for some globally specified p . But even this model asserts that the probability of upsets for *every* edge must occur within the range of $[p, 1/2]$. We aim to relax our restrictions even further in order to disrupt this structure inherent in the CR model.

Consider the following generative model, which is parameterized by a noise factor $p < 1/2$ and a participation factor $\Delta \leq 1/2$. The tournament on n players is generated as follows: pick any set of pairs of players E' satisfying the condition that each player appears in at least $(1/2 + \Delta)n$ such pairs; then, for every pair $\{u, v\} \in E'$, pick (u, v) with probability $p_{u,v} \in [p, 1 - p]$, and (v, u) otherwise. The probabilities $p_{u,v}$ can be arbitrary as long as they are in $[p, 1 - p]$. The remaining edges between players may be set arbitrarily. In this new model, many typical arguments used in analyzing CR tournaments, including those used in the proof of Theorem 4.1 which hinge on the precise definition of the CR model, break down.

Note that unlike the CR model, the new model does not start with an underlying ordering of players; however, such an ordering can easily be emulated. For instance, to

emulate the CR model, simply choose an ordering σ , set $\Delta = 1/2$, and for all u, v such that $\sigma(u) < \sigma(v)$, sample (u, v) with probability $1 - p$. That said, because the model does not start with an explicit ordering, it is much more versatile. Moreover, because only a $(1/2 + \Delta)$ fraction of the edges is determined randomly, known structures can be (adversarially) hard-coded into the resulting graphs. In this sense, any results that we can state about tournaments generated from this model are extremely general and will apply broadly. Despite this generality, we are able to give a statement for our model mirroring that of [15] for the CR model.

THEOREM 4.3. *Let $p > c\sqrt{\frac{\log n}{2\Delta n}}$ for some $c > 5$. Then with probability $> 1 - 1/\Omega(n^{(c-5)/2\ln 2})$, every player in a tournament T sampled from the aforementioned model has an efficiently computable winning seeding over T .*

The proof of Theorem 4.3 is similar to the proof of the analogous statement about the CR model found in [15]. It argues that with high probability every player in the tournament will be a superking.

Proof of Theorem 4.3. Let $p = c\sqrt{\frac{\log n}{2\Delta n}}$. We will argue that with high probability all nodes in a randomly sampled tournament are superkings, so by [15] they will be SE winners. Let $T = (V, E)$ be a randomly sampled tournament. We will bound the probability that $v \in V$ is not a superking, namely, the probability that there exists some $u \in V \setminus \{v\}$ such that u loses to fewer than $\log n$ players whom v beats.

Let $u \in V \setminus \{v\}$. Let A_v be the set of players w for which the edge between v and w was sampled randomly with probability in the range $[p, 1 - p]$. Let A_u be defined analogously. We let $W = A_v \cap A_u$ be the players whose relation is sampled randomly for both v and u . Note that we can lower bound the size of this intersection as $|W| \geq (1/2 + \Delta)n - 1 + (1/2 + \Delta)n - 1 - (n - 2) = 2\Delta n$. Now, note that the expected number of edges from v into W is the sum of the probabilities that (v, w) is an edge for each $w \in W$, and thus is at least $2\Delta np$. Applying Lemma 4.2, we can bound the probability that this set of edges into W is smaller than $c \log n / p = 2\Delta np / c$ as follows:

$$\begin{aligned} \Pr \left[\text{number of edges from } v \text{ into } W \leq \frac{2\Delta np}{c} \right] \\ &\leq \exp \left(-(1 - 1/c)^2 \Delta np \right) \\ &= \exp \left(-(1 - 1/c)^2 c \sqrt{\Delta n \log n / 2} \right) \\ &= 2^{-\Omega(\sqrt{n \log n})}. \end{aligned}$$

Now, we will condition on the assumption that v beats at least $c \log n / p$ players from W . Note that each of these players beats u with probability $\geq p$, so we expect $\geq c \log n$ of these players to beat u . Thus, using Lemma 4.2 again, we can bound the probability that u does not lose to at least $\log n$ of these players as follows:

$$\begin{aligned} \Pr [\text{number of edges from } W \text{ into } u \leq \log n] \\ &\leq \exp \left(-(1 - 1/c)^2 c \log n / 2 \right) \\ &= n^{-(1-1/c)^2 c / 2 \ln 2}. \end{aligned}$$

Letting $C = (1 - 1/c)^2 c / 2 \ln 2 - 2$, by a union bound over v 's opponents, the probability that v is not a superking is at most $2^{-\Omega(\sqrt{n \log n})} + n^{-C-1}$. Applying

another union bound over all players, the probability that there is any player who is not a superking is at most $2^{\Omega(\sqrt{n \log n})} + n^{-C} \leq O(n^{-C})$. Hence with probability $1 - 1/\Omega(n^C)$, all nodes are superkings. The result follows from the fact that $C \geq (c - 5)/2 \ln 2$. \square

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