Łojasiewicz Inequalities for the Harmonic Map Energy

James Kohout

Keble College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2021

Acknowledgements

I would first like to thank my supervisor Melanie Rupflin. Thank you for teaching me so much about mathematics and research, and for always being there to read and give input on my work. More than that, thank you for your guidance, patience and constant support over the last four years.

I have been lucky enough to benefit from a wonderful environment in which to do mathematics through the PDE CDT. I am deeply indebted to all of the faculty in the OxPDE group who I have learned so much from, especially Luc Nguyen and Jason Lotay who have been so generous with their time and support. I am also extremely grateful to Peter Topping for his encouragement and collaboration from the very beginnings of my research career.

I want to thank all of the friends in the mathematical institute for making my time here so enjoyable, especially my office mates Asad, John and Jonah for always being there as a welcome distraction. My time in Oxford would not have been the same either without the many friends I met in the Keble College MCR, not least Javi who brings the whole community together.

Thank you to my whole family for your love, especially my mother whose encouragement has always pushed me forward. Finally, thank you Célia for always being there for me, all I can say is that it would not have been the same without you.

Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. The work in Chapter 4 is the product of a collaboration with Peter M. Topping and my supervisor Melanie Rupflin, which has been published in the journal Advances in Calculus of Variations [21].

Contents

Abstract

In this thesis we study the Łojasiewicz-Simon inequality, a fundamental tool in studying the asymptotic behaviour of gradient flows and of almost-critical points. We focus on the harmonic map energy of maps from surfaces into closed Riemannian target manifolds, viewed as a functional of both map and a domain metric.

In the first part of this thesis we establish new results for the Teichmüller harmonic map flow, an L^2 gradient flow for the harmonic map energy that was introduced by Rupflin and Topping. Previous theory established that in the absence of singularities the flow converges to a branched minimal immersion, but only after passing to a subsequence of times $t_i \to \infty$ and pulling back by a sequence of diffeomorphisms. We show that in general both passing to a sub-sequence and pulling back by diffeomorphisms are actually necessary to obtain convergence of the flow by constructing several "winding" solutions of the flow from the torus. Next, when the target manifold is real-analytic we establish a Łojasiewicz-Simon inequality and use it to show that the flow converges as $t \to \infty$, that is without the need to pass to a subsequence or pull back by diffeomorphisms, provided it does not encounter singularities. These results are joint work with Rupflin and Topping.

In the second part of this thesis we study the harmonic map energy of maps from degenerating hyperbolic cylinders into general closed Riemannian target manifolds, motivated by the singular behaviour that can occur for the metric component of the Teichmüller harmonic map flow. We prove a Łojasiewicz-Simon inequality with optimal exponent for sequences of almost-critical points which have fixed boundary values and which do not form bubbles, provided their limit satisfies a non-degeneracy condition on the second variation of the energy. Such sequences undergo a change of topology in the limit meaning that the powerful techniques introduced by Simon do not apply. We instead use an approach introduced by Malchiodi, Rupflin and Sharp and obtain a Łojasiewicz-Simon inequality in the neighbourhood of a carefully constructed family of "adapted critical points".

Chapter 1

Introduction

In geometric analysis one is often concerned with finding certain "special" geometric objects, frequently as minimizers or critical points of a functional. A natural approach is to look at the gradient flows of these functionals, or consider solutions of an approximate functional, and study the resulting asymptotic convergence behaviour. In this thesis we will be studying related questions for the harmonic map energy of maps from surfaces in situations where the conformal structure on the domain changes. We answer questions about a natural gradient flow of the harmonic map energy, the Teichmüller harmonic map flow, as well as study almost-critical points of the harmonic map energy in a situation where the domain degenerates. A particular focus of this thesis will be so-called Łojasiewicz-Simon inequalities for the harmonic map energy.

We start this introduction by giving an overview of relevant existing results on Łojasiewicz-Simon inequalities and some of their wider impacts in the field of geometric analysis. After this, we will introduce the harmonic map energy and related gradient flows, before moving onto the statements of our results for the Teichmüller harmonic map flow. We then outline some properties of almost harmonic maps from degenerating cylinders and give the statements of our results for almost-critical points of the harmonic map energy on degenerating cylinders.

1.1 The Łojasiewicz-Simon Inequality

There is a stark contrast in the asymptotic behaviour of gradient flows of smooth and of real analytic functions, as can already be seen in finite dimensions.

Example 1.1.1 (from [25])**.** Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates by

$$f(r,\theta) = \begin{cases} e^{\frac{1}{r^2-1}} & \text{if } r<1 \\ 0 & \text{if } r=1 \\ e^{-1/(r^2-1)}\sin(1/(r-1)-\theta) & \text{if } r>1 \end{cases} \tag{1.1.1}$$

and observe that this function is smooth. Its set of critical points is made up of the unit circle and the origin. We can pick an initial point with $r > 1$ whose trajectory under the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ remains bounded and hence must limit to the unit circle. The trajectory continually spirals around this circle, never settling down.

This example shows that for trajectories of the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ of a smooth function $f\colon U \to \mathbb{R}$, defined on an open subset $U \subset \mathbb{R}^k$, the asymptotic limit $x_\infty = \lim_{t_n\to\infty} x(t_n)$ along a sequence of times $t_n \to \infty$ can depend on the chosen sequence $t_n \to \infty$. In contrast, Łojasiewicz [22] proved that, when the function f is real analytic, the asymptotic limit is unique in the sense that it does not depend on the sequence of times $t_n \to \infty$ (some authors say convergence is uniform as $t \to \infty$). Their key observation was the following so-called Łojasiewicz inequality: given $\bar{x} \in U$ there is a neighbourhood V of $\bar{x}$, $C < \infty$ and $\alpha \in (0, \frac{1}{2}]$ such that for every $y \in V$

$$|f(y) - f(\bar{x})|^{1-\alpha} \leq C|\nabla f(y)|. \tag{1.1.2}$$

Note that such an inequality always holds in the neighbourhood of a regular point $\bar{x}$ of f. While their proof of (1.1.2) for a general real analytic function is not simple and uses techniques from real algebraic geometry, the proof of convergence only makes use of the Łojasiewicz inequality and so does not need real analyticity specifically, see Proposition 2.1.3 below. The flexibility of this approach means that in many significantly more complicated scenarios the convergence of the associated gradient flow often quickly follows if one can obtain an appropriate Łojasiewicz inequality.

Simon [43] extended the Łojasiewicz inequality to the setting of functionals of the form $\mathcal{F}(u) = \int_\Sigma F(x, u, \nabla u)dx$ where Σ is a compact domain and F is real analytic and suitably convex in the last variable. He proved the inequality

$$|\mathcal{F}(u) - \mathcal{F}(\bar{u})|^{1-\alpha} \leq C\|\nabla\mathcal{F}(u)\|_{L^2}$$

for u in $C^{2,\beta}$ neighbourhood of a critical point $\bar{u}$, where $\nabla\mathcal{F}$ denotes the L^2 gradient of $\mathcal{F}$. His method is based on a Lyapunov-Schmidt reduction to reduce to the finite dimensional case where Łojasiewicz's original inequality holds. This result allowed him to in particular give a statement of uniqueness of asymptotic limits for the harmonic map flow, whose definition we recall in Section 1.2 below. Crucially the fact that this inequality holds only on a $C^{2,\beta}$ neighbourhood (and for compact domains) means that it cannot be applied when the flow has singular behaviour at infinite time.

Łojasiewicz-Simon inequalities have seen a wide variety of applications, notably to the asymptotic and stability properties of geometric evolution equations, for example the Ricci flow [15], the Willmore flow [5] and the elastic curve flow [24]. They have also proved useful tools in studying the singular behaviour of many elliptic and parabolic PDEs by understanding uniqueness of limits. One often "blows up" a singular point, zooming in at finer and finer scales, to extract a limiting object and obtain information about the structure of the singular point. Knowing that the limit is independent of the scaling sequence gives stronger information about the singularity. Important examples in the elliptic setting include results about uniqueness of blow ups for harmonic maps and minimal surfaces due to Simon [43]. In the parabolic setting one finds applications to blow ups of the Willmore flow [5] and to questions for the mean curvature flow [6,7,42]. Łojasiewicz-Simon inequalities have also proved useful in understanding the properties of energy spectra of critical points of the H-surface energy [23].

Simon's approach used obtain his Łojasiewicz-Simon inequality, which is based on the Lyapunov-Schmidt reduction, is at the heart of the proof of most versions of the Łojasiewicz-Simon inequality in the literature. The idea is that the inequality is easy to obtain in the neighbourhood of a *non-degenerate* critical point, and hence it suffices to deal with the directions in the kernel of the Hessian of the functional. Some form of ellipticity (given by convexity in the statement above) is then used to ensure that the kernel of the Hessian is finite dimensional and hence the finite dimensional Łojasiewicz inequality (1.1.2) can be applied there. One also expects the inequality to hold, without any assumption of analyticity, if the critical point is "integrable" – that is every element of the kernel of the Hessian arises from varying a family of critical points. It was later proved by Chill [4] that all that is needed for this method to go through is the assumption that a certain "reduced" functional coming from the Lyapunov-Schmidt reduction satisfies a Łojasiewicz-Simon inequality (which in Simon's case was guaranteed by analyticity and Łojasiewicz's original inequality).

While powerful, this method of Simon has limitations: principally that it cannot deal with *change in topology*. For example, the $C^{2,\beta}$ closeness (or similar stringent hypothesis that will arise) means that his technique does not yield an inequality that can be used to analyse singular behaviour of flows or almost-critical points, such as bubbling or domain degeneration for the harmonic map energy described in Sections 1.2 and 1.3 below. Similar issues arise when attempting to use these inequalities to analyse the blow up of a singular point when the limit is non-compact.

There are very few instances where alternative approaches have been successfully used to obtain Łojasiewicz-Simon inequalities applicable in such settings. Topping [46,47] proves a Łojasiewicz-Simon inequality for the harmonic map energy of maps between spheres which are near bubble trees using delicate analysis specific to the sphere S^2 and uses this to analyse the harmonic map flow between these spheres in the presence of bubbling. Meanwhile, versions of the Łojasiewicz-Simon inequality have been established to show uniqueness of tangent flows to the mean curvature flow in settings where blow ups are non-compact. Colding-Minicozzi [7] prove a Łojasiewicz-Simon inequality for the Gaussian area functional on (possibly non-compact) hypersurfaces near to cylinders using strongly the structure of these cylinders. Chodosh-Schulze [6] obtain a Łojasiewicz-Simon inequality for the same functional near an asymptotically conical self-shrinker, using an abstract approach along the lines of Simon but in suitable weighted spaces. The application of both of these inequalities are complicated by the fact that the evolving hypersurface cannot be written as an entire graph over the limit, meaning they need to cut-off their inequalities at carefully controlled scales.

A wholly different technique was introduced by Malchiodi-Rupflin-Sharp [23] to prove a Łojasiewicz-Simon for the H-surface energy near a simple bubble tree, based on an abstract Łojasiewicz-Simon near a suitable finite dimensional submanifold of a Hilbert space. This method was then used by Rupflin [32] to prove a Łojasiewicz-Simon inequality for the harmonic map energy of maps into general targets near simple bubble trees.

In this thesis we will prove two different results concerning Łojasiewicz-Simon inequalities. In one instance this will be in the context of the Teichmüller harmonic map flow where the result builds upon the method of Simon. In the second instance we prove a Łojasiewicz-Simon inequality in a singular setting using an approach based on the one introduced in [23]. Both of these results will be Łojasiewicz-Simon inequalities for the harmonic map energy. We also provide further background on the Łojasiewicz-Simon inequality in Chapter 2.

1.2 The Harmonic Map Energy

The harmonic map energy of a map $u\colon (M,g) \to (N,g_N)$ between Riemannian manifolds is defined by

$$E(u,g) = \frac{1}{2}\int_M |du|_g^2 \, \mathrm{d}\mu_g. \tag{1.2.1}$$

Maps which are critical points of this energy $E(\cdot,g)$ for a fixed domain metric g are known as *harmonic maps* and are characterized by the vanishing of the L^2 gradient in the map direction $\nabla_u E(u,g) = 0$. The negative L^2 gradient is given by the so called tension field $-\nabla_u E(u,g) = \tau_g(u)$ and hence harmonic maps satisfy the equation

$$\tau_g(u) = \Delta_g u + A(u)(du,du) = 0. \tag{1.2.2}$$

The tension is written here in extrinsic form which uses that (N,g_N) can be isometrically embedded into some Euclidean space $\mathbb{R}^n$ by Nash's embedding theorem and A is the second fundamental form of this embedding. The notion of harmonic maps encompasses other familiar notions, for example if $M = S^1$ then harmonic maps are geodesics while if $N = \mathbb{R}^n$ then each component of a harmonic map will be a harmonic function on (M, g).

The direct method of the calculus of variations can fail when one attempts to find harmonic maps in certain situations. To address this, Eells and Sampson [10] introduced the harmonic map heat flow, the negative L^2 gradient flow of $E(\cdot, g)$,

$$\frac{\partial u}{\partial t} = \tau_g(u), \tag{1.2.3}$$

in order to smoothly deform an initial mapping $u_0 \colon M \to N$ to a harmonic mapping. They were able to show that when M, N are closed manifolds and (N, g_N) has non-positive sectional curvature then, given smooth initial data, (1.2.3) admits a unique smooth solution existing for all times and smoothly sub-converging as $t \to \infty$ (that is, converging along a sequence of times $t_i \to \infty$) to a harmonic map. It was proved by Hartman [14] that the limit is independent of the sub-sequence $t_i \to \infty$, using specifically the non-positive sectional curvature. Without this curvature assumption the result of Eells-Sampson does not hold in general since there are homotopy classes of maps which do not contain a harmonic map, for example maps from the torus T^2 to the sphere S^2 with degree ± 1 [11]. This forces the flow with such initial data to become singular, either at finite or infinite time. However, the results of Struwe [44] (and Chen-Struwe [3] in dimensions greater than 2) ensure the existence of a global weak solution to the harmonic map flow for arbitrary closed manifolds M, N and arbitrary initial data in H^1. On two dimensional domains it was first shown by Chang-Ding-Ye [1] that there can indeed be finite time singularities. In the case of two dimensional domains all singularities, whether they are at finite or infinite time, are caused by energy concentrating onto smaller and smaller regions of the domain and the subsequent "bubbling off" of harmonic spheres, see [44].

In this thesis we will only look at the harmonic map energy E on domains M which are two dimensional. From the geometric perspective the case of surfaces is interesting because E is invariant under conformal changes of the domain metric. Moreover, the energy E is intimately connected to the area functional, written in isothermal coordinates where the metric takes the form $\rho^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ as $A(u) = \int_M (|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle^2)^{\frac{1}{2}} \mathrm{d}x\mathrm{d}y$. We have the inequality $A(u) \leq E(u, g)$, with equality if and only if $|u_x|^2 - |u_y|^2 = \langle u_x, u_y \rangle = 0$ i.e. $u \colon (M, g) \to (N, g_N)$ is conformal. Hence a map u is a critical point for the area (a minimal surface) if it is a conformal, harmonic map.

It turns out that if a metric g is a critical point for $E(u, \cdot)$ then $u \colon (M, g) \to (N, g_N)$ is a (weakly) conformal map, that is $u^* g_N = fg$ for some non-negative function f. The negative L^2 gradient of E in directions corresponding to variations of the metric is given by the real part of the Hopf differential $-\nabla_g E(u, g) = \frac{1}{4}\mathrm{Re}\Phi(u, g)$, so this Hopf differential measures the lack of conformality of maps $u\colon (M,g) \to (N,g_N)$. In isothermal coordinates, with $z = x + iy$, the Hopf differential can be written

$$\Phi(u,g) = (|u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle)\mathrm{d}z^2. \tag{1.2.4}$$

It turns out that the Hopf differential of a harmonic map is a *holomorphic* quadratic differential, see (3.1.7) below. This is the basis upon which it can be seen that harmonic maps from the sphere S^2 – and from the disc D with appropriate boundary conditions – are automatically conformal and hence are minimal (away from so-called branch points, see (3.1.8)). More generally, Sacks-Uhlenbeck [40] and Schoen-Yau [41] used a variational approach where they extremized over conformal structures on the domain to find branched minimal immersions of higher genus surfaces, assuming a topological condition on (N, g_N).

We finish this section by mentioning that the harmonic map energy on fixed domains satisfies a Łojasiewicz-Simon inequality when the target (N, g_N) is real analytic by the work of Simon [43]:

Theorem 1.2.1 (Follows from Theorem 3, [43])**.** *Let (M,g) be a closed Riemannian manifold, (N,g_N) a closed, real analytic manifold and denote by $E(u)$ the harmonic map energy of maps $u\colon (M,g) \to (N,g_N)$. Fix some $\beta \in (0,1)$ and a smooth harmonic map $\bar{u}\colon (M,g) \to (N,g_N)$. Then there are constants $\sigma > 0$, $C < \infty$ and $\alpha \in (0,\frac{1}{2})$ such that*

$$|E(u) - E(\bar{u})|^{1-\alpha} \leq C\|\tau_g(u)\|_{L^2(M,g)}$$

for every $u \in C^{2,\beta}$ with $\|u - \bar{u}\|_{C^{2,\beta}} < \sigma$.

Simon used this result to conclude that, given a solution to the harmonic map flow (1.2.3) into a real analytic target (N, g_N) which sub-converges smoothly at infinite time, the solution actually converges as $t \to \infty$ to a unique limit, extending the work of Hartman [14] by removing the curvature restriction on the target. In this thesis we will extend this approach to a related flow, the Teichmüller harmonic map flow.

1.3 The Teichmüller Harmonic Map Flow

Motivated by using a flow approach to find minimal immersions of higher genus surfaces, in [34] Rupflin and Topping introduced the Teichmüller Harmonic Map Flow, an L^2 gradient flow of E evolving both the map and domain metric on a closed, oriented surface M of genus $\gamma \geq 0$ (a formulation was also later given by Rupflin [31] for maps from cylinders into Euclidean space). Since the energy E is conformally invariant they were able to restrict the metrics to the space $\mathcal{M}_c$ of constant curvature metrics $c \in \{1, 0, -1\}$ when $\gamma = 0, 1$ or $\gamma \geq 2$ respectively. They defined the Teichmüller harmonic map flow

$$\begin{aligned} \frac{\partial u}{\partial t} &= \tau_g(u), \\ \frac{\partial g}{\partial t} &= \frac{\eta^2}{4} \mathrm{Re}(P_g(\Phi(u, g))), \end{aligned} \tag{1.3.1}$$

where $\tau_g(u)$ and $\Phi(u, g)$ are the tension and Hopf differential defined above, P_g is the projection onto the space of holomorphic quadratic differentials and η is an arbitrary coupling constant. The addition of the projection P_g should be thought of as ensuring the flow moves orthogonally to the action of diffeomorphisms, which can be imposed since $E(u \circ f, f^*g) = E(u, g)$ for any diffeomorphism $f \colon M \to M$. We note that an equivalent flow on the torus T^2 was first introduced by Ding-Li-Liu [9]. Moreover, when the domain is the sphere S^2, and hence all holomorphic quadratic differentials are trivial, the flow (1.3.1) agrees with the harmonic map flow (1.2.3).

Rupflin and Topping [37] (which further built upon their work with Zhu [38] and Huxol [18]) proved that, starting from any initial map $u \in H^1(M, N)$ and initial metric $g \in \mathcal{M}_c$, there exists a global weak solution of the Teichmüller harmonic map flow (1.3.1) which is smooth away from finitely many singular times. There are two distinct types of singularities that can occur. Just as for the harmonic map flow mentioned above the map can become singular when energy concentrates on smaller and smaller regions of the domain and there is bubbling off of finitely many harmonic spheres. The second is a singularity in the domain metric when the injectivity radius converges to zero and one or more simple closed geodesics in the domain shrink to a point. At finite times this singularity for the domain metric can only happen when the genus $\gamma \geq 2$. Note that by the Collar Lemma [28] every short closed geodesic in a hyperbolic surface has a neighbourhood isometric to a so-called collar $(-X(\ell), X(\ell)) \times S^1$ with the metric $g = \rho_\ell^2(s)(\mathrm{d}s^2 + \mathrm{d}\theta^2)$ where

$$\rho_\ell(s) = \frac{\ell}{2\pi \cos \frac{\ell s}{2\pi}}, \quad X(\ell) = \frac{2\pi}{\ell} \left(\frac{\pi}{2} - \arctan \sinh \frac{\ell}{2} \right). \tag{1.3.2}$$

This means that when analysing possible degeneration of the flow (1.3.1) one can work with this explicit family of metrics.

Both of these types of singular behaviour can occur for the Teichmüller harmonic map flow at infinite time. Nevertheless the flow still sub-converges, now after pulling back the map and metric components of the flow by diffeomorphisms, to a collection of branched minimal immersions defined over simpler domains [38]. To be precise, the following asymptotic convergence statement was proved by Rupflin and Topping [34] in the absence of singular behaviour of the metric at infinite time. Note in this setting Rupflin [30] proved that given smooth initial data, a unique weak solution to (1.3.1) exists in the class of weak solutions with non-increasing energy, see Theorem 3.4.2 below.

Theorem 1.3.1 (Theorem 1.4, [34])**.** *For initial data $(u_0, g_0) \in C^\infty(M,N) \times \mathcal{M}_c$ let (u, g) denote the unique solution weak solution of (1.3.1) with non-increasing energy with this initial data that is given by [30]. Suppose that the length $\ell(t)$ of the shortest closed geodesic in $(M, g(t))$ is uniformly bounded below by a positive constant. Then there exists a sequence of times $t_i \to \infty$, a sequence of orientation preserving diffeomorphisms $f_i \colon M \to M$, a hyperbolic metric $\bar{g}$ on M, a weakly conformal harmonic map $\bar{u} \colon (M, \bar{g}) \to (N, g_N)$ and finite set of points $S \subset M$ such that*

1. *$f_i^* g(t_i) \to \bar{g}$ smoothly*
2. *$u(t_i) \circ f_i \rightharpoonup \bar{u}$ weakly in $H^1(M)$*
3. *$u(t_i) \circ f_i \to \bar{u}$ strongly in $W^{1,p}_{loc}(M \setminus S)$ for any $p \in [1, \infty)$.*

This result leaves open the questions of whether this sub-convergence can be upgraded to full convergence and whether pulling back by these diffeomorphisms is necessary for convergence. In this thesis we will answer both of these questions by constructing suitable target manifolds for which the flow exhibits winding behaviour and moreover consider the situation when the target manifold is real analytic. In the next section we will give an overview of these results.

1.4 Obtained Results I: Uniqueness and Non-uniqueness of Limits of the Teichmüller harmonic map flow

One of the main problems we consider in this thesis is that of uniqueness of asymptotic limits for the Teichmüller Harmonic Map Flow. The results here are from Chapter 4 and come from joint work with Melanie Rupflin and Peter Topping appearing in [21].

Theorem 1.3.1 leaves open the following questions:

- Does the flow converge along every sequence of times $t \to \infty$ to the same limit?
- Do we need to pull back by diffeomorphisms to ensure the flow converges?

The situation for the Harmonic Map Flow (1.2.3) is instructive in this case. It was shown by Hartman [14] that if the sectional curvature (N, g_N) is non-positive and a solution of the harmonic map flow (1.2.3) converges smoothly along a sequence of times $t_i \to \infty$ then it converges along every sequence of times $t \to \infty$ to the same limit. Simon [43] proved the same result with the assumption that the target (N, g_N) is real analytic using his Łojasiewicz-Simon inequality. Later on, Topping [45,48] showed that this uniqueness of asymptotic limits is false for general smooth targets. He constructed smooth targets (N, g_N) as warped product manifolds to force the solution to exhibit "winding behaviour" and limit to a non-trivial circle of harmonic maps (analogous to the behaviour described for the gradient flow of the function (1.1.1)).

Our first main result for the Teichmüller harmonic map flow is along the lines of Topping's construction [45]. We construct smooth (but not analytic) settings where winding behaviour does indeed occur for both the map and metric components of the flow – even with initial for which the map component remains smooth for all times, and at infinite time, and for which we have a uniform lower bound on the injectivity radius of the domain.

We consider the flow on tori where, by pulling back the initial data and the whole flow by a fixed diffeomorphism, it suffices to consider the flow of metrics in the explicit two-parameter family of flat unit area metrics

$$\mathcal{M}^* := \{g_{a,b} = T_{a,b}^* g_E : (a,b) \in \mathbb{H}\} \tag{1.4.1}$$

on $T^2 := \mathbb{R}^2/\mathbb{Z}^2$, where $T_{a,b} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map sending $(1,0) \mapsto \frac{1}{\sqrt{b}}(1,0)$ and $(0,1) \mapsto \frac{1}{\sqrt{b}}(a,b)$, g_E is the Euclidean metric and $\mathbb{H}$ is the upper-half plane. The Weil-Petersson metric on $\mathcal{M}^*$ then corresponds, up to a scaling factor, to the hyperbolic distance on $\mathbb{H}$. See Section 3.5 for further details of the flow on the torus T^2.

Our first main result can then be stated as follows:

Theorem (Theorem 4.1.1)**.** *Let $(G_s)_{s\in[0,\infty)}$ be any smooth curve in $\mathcal{M}^*$ whose projection to moduli space is 1-periodic, i.e. for which there exists a diffeomorphism $\varphi \colon T^2 \to T^2$ so that $G_s = \varphi^* G_{s+1}$ for every $s \in [0,\infty)$.*

Then there exists a smooth closed target manifold (N, g_N) and initial data $(u_0, g_0) \in C^\infty(T^2, N) \times \mathcal{M}^*$ such that the corresponding solution $(u(t), g(t))$ of Teichmüller harmonic map flow has $\sup_t \|\nabla u(t)\|_{L^\infty(T^2,g(t))} < \infty$ and $\inf_t \mathrm{inj}(T^2, g(t)) > 0$, as well as the following asymptotic behaviour:

- *There exists a smooth function $z \colon [0,\infty) \to [0,\infty)$ satisfying $z(t) \to \infty$ as $t \to \infty$ so that*

$$d_{\mathrm{WP}}(g(t), G_{z(t)}) \to 0 \text{ as } t \to \infty \tag{1.4.2}$$

where d_{WP} is the Weil-Petersson distance on $\mathcal{M}^*$.

- *For every $z \in [0,1)$ there exists a sequence of times $t_i^z \to \infty$ so that after pulling back by the i^{th} iterate φ^i the maps $u(t_i^z)$ converge smoothly,*

$$u(t_i^z) \circ \varphi^i \to u_z, \tag{1.4.3}$$

to a minimal immersion $u_z : T^2 \to N$. The resulting minimal immersions u_{z_1} and u_{z_2} with $z_1 \neq z_2$ parametrise different minimal surfaces in (N, g_N). After pulling back by φ^i the metrics $g(t_i^z)$ also converge

$$(\varphi^i)^* g(t_i^z) \to G_z \text{ in } C^\infty.$$

Theorem 4.1.1 immediately yields the following corollary about the possible behaviour of Teichmüller harmonic map flow into suitable smooth closed targets, which answers the first question above:

Corollary (Corollary 4.1.2)**.** *The limit of solutions of Teichmüller harmonic map flow as $t \to \infty$ can be non-unique, even after pull-back by diffeomorphisms.*

The diffeomorphisms φ^i that we use to pull back are iterated compositions of the diffeomorphism φ obtained from G_s. If G_s is a periodic curve in $\mathcal{M}^*$ then φ is the identity map, for example. A quite different situation arises in the case that G_s leaves any compact subset of Teichmüller space. By taking the family corresponding to twisting a torus around a simple closed geodesic, a twist by angle 2π representing a Dehn twist, we show the following, which answers the second question above:

Corollary (Corollary 4.1.3)**.** *There exist a closed target manifold and a solution of Teichmüller harmonic map flow from T^2 into that target whose metric component leaves any compact subset of Teichmüller space even though the injectivity radius of the domain remains bounded away from zero and hence the metric component remains in a compact subset of moduli space.*

When the curve G_s is a non-trivial closed curve we obtain the following:

Corollary (Corollary 4.1.4)**.** *There exist a closed target manifold and a solution of Teichmüller harmonic map flow from T^2 into that target for which the metric component stays in a compact subset of Teichmüller space but for which the limit as $t \to \infty$ is not unique. In particular we can choose sequences of times $t_i, \tilde{t}_i \to \infty$ so that $(u(t_i), g(t_i))$ and $(u(\tilde{t}_i), g(\tilde{t}_i))$ converge (without having to pull back by diffeomorphisms) to limits which parametrize different minimal surfaces.*

Next we investigate whether the convergence of the flow can be improved when the target (N, g_N) is real analytic, keeping in mind that the target constructed in Theorem 4.1.1 is smooth but *not real analytic*. We show that when (N, g_N) is real analytic solutions of Teichmüller harmonic map flow into smooth closed targets cannot exhibit winding behaviour in situations where the energy density is bounded uniformly from above and the injectivity radius from below.

We note that one starts the analysis of the asymptotics of the flow by selecting a sequence of times $t_j \to \infty$ with $\|\partial_t(u,g)(t_j)\|_{L^2(M,g(t_j))} \to 0$ and then analyses the bubbling and/or collar degeneration singularities that might occur in the limit [18, 34, 38]. If there exists any such sequence for which no singularities occur, when the target is analytic, then we obtain uniform convergence:

Theorem (Theorem 4.1.5)**.** *Let (N, g_N) be a closed analytic manifold of any dimension and let M be a closed oriented surface of genus $\gamma \geq 1$. Let (u,g) be any global weak solution of Teichmüller harmonic map flow (3.2.3), with nonincreasing energy, for which there is a sequence $t_j \to \infty$ such that*

$$\lim_{j\to\infty} \|\partial_t(u,g)(t_j)\|_{L^2(M,g(t_j))} = 0, \quad \sup_j \|\nabla u(t_j)\|_{L^\infty(M,g(t_j))} < \infty \ \text{and} \ \inf_j \operatorname{inj}(M, g(t_j)) > 0. \tag{1.4.4}$$

Then $(u,g)(t)$ converges smoothly as $t \to \infty$ to a limiting pair (u_∞, g_∞) consisting of a metric $g_\infty \in \mathcal{M}_c$ and a weakly conformal harmonic map $u_\infty \colon (M, g_\infty) \to (N, g_N)$.

This theorem is proven by establishing a suitable Łojasiewicz-Simon inequality for the harmonic map energy on a space of maps and metrics, Theorem 4.3.1, and then applying this inequality to well behaved solutions of the flow. We write $\mathcal{M}_c^s$ for the space of metrics of constant curvature $c \in \{-1, 0\}$ with coefficients in the Sobolev space H^s (with respect to an arbitrary, fixed metric). The Łojasiewicz-Simon inequality that we prove and subsequently apply to the flow can be stated as follows:

Theorem (Theorem 4.3.1)**.** *Let (N, g_N) be a closed real analytic manifold, let M be a closed oriented surface of genus $\gamma \geq 1$, and let $(\bar{u}, \bar{g})$, $\bar{g} \in \mathcal{M}_c$, be a critical point of the harmonic map energy $E(u,g)$. Then for any $s > 3$ there is a neighbourhood $\mathcal{O}$ of $(\bar{u}, \bar{g})$ in $H^s(M,N) \times \mathcal{M}_c^s$, $\alpha \in (0, \frac{1}{2})$ and $C < \infty$ such that for any $(u,g) \in \mathcal{O}$ we have*

$$|E(u,g) - E(\bar{u},\bar{g})|^{1-\alpha} \leq C \left(\|\tau_g(u)\|^2_{L^2(M,g)} + \|P_g(\Phi(u,g))\|^2_{L^2(M,g)} \right)^{\frac{1}{2}}, \tag{1.4.5}$$

where P_g is the $L^2(M,g)$-orthogonal projection from the space of quadratic differentials to the space $\mathcal{H}(g)$ of holomorphic quadratic differentials.

This Łojasiewicz-Simon inequality also gives the rate of convergence in Theorem 4.1.5, see Remark 4.1.6

1.5 The Harmonic Map Energy on Degenerating Cylinders

When the Teichmüller harmonic map flow encounters singularities of the metric the domain degenerates and one or more simple closed geodesics in the domain collapse to a point. The regions on which the domain degenerates have an explicit description in terms of collar neighbourhoods by the Collar Lemma [28], see (1.3.2) above. A good model for these degenerating collars in hyperbolic surfaces is given by the hyperbolic cylinders

$$(\mathcal{C}_{Y(\ell)}, g_\ell) = ([-Y(\ell), Y(\ell)] \times S^1, \rho_\ell^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)) \tag{1.5.1}$$

where ρ_ℓ is the conformal factor defined above in (1.3.2) and $Y(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan(d^{-1}\ell)\right)$ for $d > 0$. In particular the asymptotic behaviour as $\ell \to 0$ for $Y(\ell)$ and the function $X(\ell)$ from (1.3.2) is the same, and furthermore for the metric g_ℓ defined in (1.5.1) it is the case that $\partial_\ell g_\ell$ is orthogonal to the action of diffeomorphisms on cylinders, compare with (3.2.2).

Given $\ell > 0$ and a map $u \colon \mathcal{C}_{Y(\ell)} \to N$ consider the following version of the harmonic map energy,

$$E(u, \ell) = \frac{1}{2} \int_{\mathcal{C}_{Y(\ell)}} |du|_{g_\ell}^2 \, \mathrm{d}\mu_{g_\ell}. \tag{1.5.2}$$

The infimum of this energy, considered as a function of both a map and the parameter $\ell > 0$, even among maps satisfying a fixed Dirichlet boundary condition, is not attained in general. One possibility is that a minimizing sequence undergoes bubbling, but the infimum of the energy may not be attained even in situations where the target supports no bubbles. To see an illustration of this phenomenon, consider for now the case where $N = \mathbb{R}^n$ and where the boundary values are fixed to be the "identity maps" from the circle to two co-axial unit circles in $\mathbb{R}^n$ which are a large distance apart. Then the infimum of the energy is 2π, the sum of the energies of standard embeddings of these discs, but this energy is not attained among maps from the cylinder. One can approach this minimal energy through a sequence of maps obtained by gluing in a thin cylinder of circumference $\ell > 0$ between the discs, where one can arrange that in the limit as $\ell \to 0$ the maps converge in a suitable sense to the discs on either half of the cylinder and becomes close to a curve away from the ends of the domain.

A compactness result for sequences of almost harmonic maps from these degenerating cylinders into general targets was given by Huxol, Rupflin and Topping in [18], building upon the work of Rupflin, Topping and Zhu [38]. We consider here the case where there can be no bubbles, in which case the results of [18] imply the following:

Theorem 1.5.1 (Follows from [18], Theorem 1.9). *Fix $\ell_i > 0$ such that $\ell_i \to 0$ and set $Y_i = Y(\ell_i) \to \infty$. Let $u_i \colon \mathcal{C}_{Y_i} \to N$ be a sequence of maps with uniformly bounded energy $E(u_i, \ell_i) \le E_0 < \infty$ which are almost harmonic in the sense that*

$$\|\tau_{g_{\ell_i}}(u_i)\|_{L^2(\mathcal{C}_{Y_i}, g_{\ell_i})} \to 0. \tag{1.5.3}$$

Suppose further that the energy density is uniformly bounded,

$$\|du_i\|_{L^\infty(\mathcal{C}_{Y_i}, g_{\ell_i})} \leq C < \infty. \tag{1.5.4}$$

Then, after passing to a subsequence, the shifted maps $u_{i,\pm}(s,\theta) = u_i(Y_i \mp s, \theta)$ converge smoothly locally on $[0,\infty) \times S^1$ to finite energy harmonic maps $\bar{v}_\pm \colon [0,\infty) \times S^1 \to N$ and the cylinders $(-Y_i + \lambda, Y_i - \lambda) \times S^1$ for λ large are mapped near curves in the sense that

$$\lim_{\lambda \to \infty} \lim_{i \to \infty} \sup_{s \in (-Y_i + \lambda, Y_i - \lambda)} \operatorname{osc}(u_i, \{s\} \times S^1) = 0. \tag{1.5.5}$$

Remark 1.5.2. In the setting of Theorem 1.5.1, the removable singularity theorem of Sacks-Uhlenbeck [39] allows one to fill in the punctures to obtain smooth harmonic maps $v_\pm \colon D \to N$. Moreover, (1.5.5) means one can define the connecting curves $\gamma_i(s) = \Pi(\fint_{\{s\} \times S^1} u_i \, \mathrm{d}\theta)$, where Π is the nearest point projection onto N, which in the limit are near u_i in the sense that

$$\lim_{\lambda \to \infty} \limsup_{i \to \infty} \|u_i - \gamma_i\|_{L^\infty(-Y_i + \lambda, Y_i - \lambda)} = 0. \tag{1.5.6}$$

Remark 1.5.3. The full strength of the results in [18] is probably not needed to make this statement. The first part of the compactness result essentially already follows from [38], which in turn uses ideas from [34] and [44]. Meanwhile, the estimate (1.5.5) follows from angular energy estimates which appear often in the study of almost-harmonic maps and the harmonic map heat flow, see for example [47], [27].

Without further assumptions very little can be said about the convergence behaviour of these connecting curves; indeed any C^2 curve $\alpha \colon [-1,1] \to N$ can be obtained as a limit since $E(\alpha(\frac{\cdot}{Y_i}), \ell_i) \leq C Y_i^{-1} \to 0$ and $\|\tau_{g_{\ell_i}}(\alpha(\frac{\cdot}{Y_i}))\|_{L^2(\mathcal{C}_{Y_i}, g_{\ell_i})} \to 0$, see [18]. Still, there are many cases where more is known. If u_i are harmonic maps with respect to the metrics g_{ℓ_i} then the connecting curves sub-converge to geodesics, which possibly have infinite length [2]; more generally this also holds if the tension $\tau_{g_{\ell_i}}(u_i)$ converges to zero fast enough compared to ℓ_i [29]. In the situation of harmonic maps from fixed domains it is known that there can be no necks forming between bubbles [26], as is also the case for solutions of the harmonic map flow [27]. Also, for sequences (u_i, ℓ_i) as above which are obtained from solutions of the Teichmüller harmonic map flow these connecting curves will in general have a non-trivial limit [18], and in contrast to the situation that the maps u_i are harmonic it is unknown whether the limiting curves must be geodesics – the only case in which this is known to be true is in the situation of the flow on the torus when the total energy converges to zero [9].

1.6 Obtained results II: Łojasiewicz-Simon Inequalities for the Harmonic Map Energy on Degenerating Cylinders

The second main problem we consider in this thesis is that of obtaining a Łojasiewicz-Simon inequality for the harmonic map energy on degenerating cylinders. The results here are from Chapter 5.

We will study almost-critical points of the Harmonic Map Energy $E(u, \ell)$, see (1.5.2), with a prescribed Dirichlet boundary condition. These are in particular almost-harmonic maps and so Theorem 1.5.1 applies. We will consider the case where the connecting curves additionally sub-converge to a finite length *geodesic* $\gamma\colon [-1,1] \to N$. Since our maps will not be close to a critical point of the energy on the same domain on which they are defined this is a setting where the powerful techniques introduced by Simon in [43] to obtain Łojasiewicz-Simon inequalities do not apply. These techniques would give a Łojasiewicz-Simon inequality for maps from the disc near one of the maps $v_\pm$ or curves near the geodesic γ if the target N was real analytic. This would also be the case if the real analyticity assumption was dropped as long as the critical objects satisfy a certain non-degeneracy condition; furthermore the exponent obtained would be the optimal one in this case. We will not make any real analyticity assumption on the target N here, instead we will assume that our limiting objects satisfy stability conditions which suffice to obtain Łojasiewicz-Simon inequalities in the usual setting of maps from fixed domains.

Our assumptions can be stated as follows:

(A) The curve $\gamma\colon [-1,1] \to N$ is a geodesic of length $L(\gamma) \in (0, \infty)$ connecting the points $v_\pm(0)$.

(B) The second variation of the energy $E_{[-1,1]}(\eta) = \frac{1}{2}\int_{-1}^{1} |\eta'|^2$ at γ is positive definite, that is there exists $c_\gamma \in (0,1]$ such that

$$d^2 E_{[-1,1]}(\gamma)(w,w) \geq c_\gamma \|w\|^2_{\dot{H}^1([-1,1])} \tag{1.6.1}$$

for every $w \in \Gamma^{H^1_0([-1,1])}(\gamma^* TN)$.

(C) The second variation of the energy $E_D(v) = \frac{1}{2}\int_D |\nabla v|^2$ at $v_\pm$ is positive definite, that is there exists $c_\pm \in (0,1]$ such that

$$d^2 E_D(v_\pm)(w,w) \geq c_\pm \|w\|^2_{\dot{H}^1(D)} \tag{1.6.2}$$

for every $w \in \Gamma^{H^1_0(D)}(v_\pm^* TN)$.

Given a map $u\colon D \to N$ we say $w \in \Gamma^{H^1(D)}(u^*TN)$ if and only if $w \in H^1(D;\mathbb{R}^n)$ is such that $w(p) \in T_{u(p)}N$ for almost every $p \in D$. We use the corresponding notation for maps from different domains and other spaces of functions.

Our results can be stated most clearly in a slightly different setting, although they can be translated to the language given above. We will work on a fixed domain $C_0 = [-1,1] \times S^1$ equipped with a family of hyperbolic metrics $G_\ell = f_\ell^*(\rho_\ell^2(s)(\mathrm{d}s^2 + \mathrm{d}\theta^2))$, introduced in [31], obtained by pulling back the aforementioned hyperbolic collars by a specific family of diffeomorphisms $f_\ell\colon C_0 \to \mathcal{C}_{Y(\ell)}$ in an almost canonical way (in the sense that the metrics G_ℓ are a horizontal curve of metrics), see (5.2.3) below.

Write $H^1_{v\pm}(C_0, N)$ for the space of H^1 functions $u\colon C_0 \to \mathbb{R}^n$ for which $u \in N$ a.e. and $u|_{\{\pm1\}\times S^1}(\theta) = v_\pm(e^{i\theta})$. We will then set $H = H^1_{v\pm}(C_0, N) \times (0,\infty)$ and define the energy

$$E(u,\ell) = \frac{1}{2}\int_{C_0} |\mathrm{d}u|^2_{G_\ell}\, \mathrm{d}\mu_{G_\ell} \tag{1.6.3}$$

for $(u,\ell) \in H$. We have a natural inner product on $T_{(u,\ell)}H = \Gamma^{H^1_0}(u^*TN) \times \mathbb{R}$, inherited from the L^2 inner product, that is characterized by

$$\|(w,0)\|_*^2 = \|w\|^2_{L^2(C_0,G_\ell)}, \quad \|(0,1)\|_*^2 = \|\partial_\ell G_\ell\|^2_{L^2(C_0,G_\ell)} \tag{1.6.4}$$

and the orthogonality $\langle (w,0),(0,1)\rangle_* = 0$ for every $w \in \Gamma^{H^1_0}(u^*TN)$. This inner product gives rise to the L^2 gradient $\nabla E(u,\ell) = (\nabla_u E(u,\ell), \nabla_\ell E(u,\ell))$ which is characterized by

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} E(u+\varepsilon w, \ell+\varepsilon) = \langle \nabla E(u,\ell),(w,1)\rangle_* \tag{1.6.5}$$

and therefore satisfies

$$\nabla_u E(u,\ell) = -\tau_{G_\ell}(u), \tag{1.6.6}$$

$$\nabla_\ell E(u,\ell) = \|\partial_\ell G_\ell\|^{-2}_{L^2(C_0,G_\ell)} \partial_\ell E(u,\ell) \in \mathbb{R}. \tag{1.6.7}$$

Given $\ell_i > 0$ and $u_i\colon C_0 \to N$ we say that a sequence (u_i,ℓ_i) are almost-critical points of $E(u,\ell)$ if they have uniformly bounded energy, $E(u_i,\ell_i) \le E_0 < \infty$, and if they satisfy $\|\nabla E(u_i,\ell_i)\|_* \to 0$. If (u_i,ℓ_i) is a sequence of almost-critical points of $E(u,\ell)$ such that $\ell_i \to 0$ and such that the maps u_i have uniformly bounded energy density, $\|du_i\|_{L^\infty(C_0,G_{\ell_i})} \le A < \infty$, then Theorem 1.5.1 can be applied to $\tilde{u}_i = u_i \circ f_\ell$ and consequently the results of Theorem 1.5.1 and Remark 1.5.2 hold for (u_i,ℓ_i).

Our first main theorem is a Łojasiewicz-Simon inequality for sequences of almost-critical points of $E(u,\ell)$ and can be stated as follows:

Theorem (Theorem 5.1.1). *Let $\ell_i > 0$ be such that $\ell_i \to 0$ and $u_i \colon C_0 \to N$ be a sequence of maps with fixed boundary values which are almost-critical points of E in the sense that $E(u_i, \ell_i) \leq E_0 < \infty$ and $\|\nabla E(u_i, \ell_i)\|_* \to 0$. Suppose further that the energy density is uniformly bounded, $\|du_i\|_{L^\infty(C_0, G_{\ell_i})} \leq A < \infty$.*

As a result of Theorem 1.5.1 the maps u_i sub-converge smoothly locally on $C^\pm := \{0 < \pm s < 1\} \times S^1$ to maps $v_\pm \colon C^\pm \to N$ which, extending across the punctures, yield smooth harmonic maps $v_\pm \colon D \to N$ defined on the unit disc. Moreover, there exist curves $\gamma_i \colon [-1, 1] \to N$ such that

$$\lim_{\lambda \to \infty} \limsup_{i \to \infty} \|u_i - \gamma_i\|_{L^\infty(f_{\ell_i}^{-1}([-Y(\ell_i)+\lambda, Y(\ell_i)-\lambda] \times S^1))} = 0. \tag{1.6.8}$$

Assume that the curves γ_i sub-converge in L^∞ to a limit $\gamma \colon [-1, 1] \to N$ and that this limit γ satisfies (A), (B) while the maps $v_\pm$ satisfy (C). Then for i sufficiently large we have a bound on the scale ℓ_i,

$$\ell_i \leq C \|\nabla E(u_i, \ell_i)\|_*^2, \tag{1.6.9}$$

and a Lojasiewicz-Simon inequality with optimal exponent for the harmonic map energy,

$$|E(u_i, \ell_i) - E^*|^{\frac{1}{2}} \leq C \|\nabla E(u_i, \ell_i)\|_* \tag{1.6.10}$$

where $E^* := E(v_+) + E(v_-)$.

The general method of proof we will use was introduced by Malchiodi-Rupflin-Sharp [23] in the context of the H-surfaces and later used by Rupflin [32] in the context of almost harmonic maps into general targets converging to simple bubble trees. The basic idea is to compare a general sequence of almost critical points of the functional with a specific family of "adapted" critical points. These adapted critical points are obtained by modifying the limiting objects, in this case defined on a different domain, so that they are defined on the same domain as the sequence. In [23] this was done in the situation of maps from a surface Σ of genus at least 1 which are near to a simple bubble tree – the "adapted" bubbles are obtained by gluing a map from S^2 to a constant map from Σ to obtain suitable maps from Σ.

In our situation we will glue together the geodesic γ and harmonic maps $v_\pm$ to obtain a sequence of maps $z_\ell \colon C_0 \to N$ defining a set of adapted critical points as

$$\mathcal{Z} = \{(z_\ell, \ell) \mid \ell \in (0, \ell_*)\}, \tag{1.6.11}$$

where $\ell_* > 0$ is some fixed number. We recall that we are always thinking of the second component as parametrizing the metrics G_ℓ and are interested in the degenerating case where $\ell > 0$ is small. The maps z_ℓ are constructed by taking a portion of the ends of C_0

and using this to parametrize most of the maps $\bar{v}_\pm$ and then using the middle region to parametrize most of the curve γ. For the precise definition of these adapted critical points see Section 5.2.

Our second main theorem of this section, which will give Theorem 5.1.1 as a consequence, is a Łojasiewicz-Simon inequality for $(u, \ell) \in H$ which are near to this set of adapted critical points:

Theorem (Theorem 5.1.2)**.** *Suppose that $\gamma\colon [-1,1] \to N$ is a curve satisfying (A), (B) and $v_\pm\colon D \to N$ are harmonic maps satisfying (C). Then there exists $\varepsilon > 0$, $\bar{\ell} \in (0, \ell_*)$ and $C < \infty$ such that the following holds. Let $(u, \lambda) \in H = H^1_{v_\pm}(C_0, N) \times (0, \infty)$ be such that*

$$\inf_{\ell > 0} \left[\|u - z_\ell\|_{\dot{H}^1(C_0, G_\ell)} + \ell^{-1}(\lambda - \ell) \right] \leq \varepsilon \tag{1.6.12}$$

and that the infimum is attained by a pair (z_ℓ, ℓ) with $0 < \ell \leq \bar{\ell}$ and $\|u - z_\ell\|_{L^\infty(C_0)} \leq \varepsilon$, where $(z_\ell, \ell) \in \mathcal{Z}$ are the adapted critical points. Then we have a bound on the scale ℓ,

$$\ell \leq \|\nabla E(u, \lambda)\|_*^2 \tag{1.6.13}$$

and Łojasiewicz-Simon inequalities with optimal exponent,

$$\|u - z_\ell\|_{\dot{H}^1(C_0, G_\ell)} + \ell^{-1}(\lambda - \ell) \leq C \|\nabla E(u, \lambda)\|_*, \tag{1.6.14}$$

$$|E(u, \lambda) - E^*|^{\frac{1}{2}} \leq C \|\nabla E(u, \lambda)\|_*, \tag{1.6.15}$$

where $E^* = E(v_+) + E(v_-)$.

1.7 Outline of Thesis

In Chapter 2 we give background on the Łojasiewicz-Simon inequality and also provide an abstract formulation of the inequality. In Chapter 3 we give background on the harmonic map energy and Teichmüller harmonic map flow. Chapter 4 and Chapter 5 then consist of new results.

In Chapter 4 we present our new results on the Teichmüller harmonic map flow. In Section 4.2 we construct our winding examples to give the proof of Theorem 4.1.1. In Section 4.3 we prove a Łojasiewicz-Simon inequality for the harmonic map energy on closed surfaces, Theorem 4.3.1. Finally in Section 4.4 we apply Theorem 4.3.1 to the Teichmüller harmonic map flow to prove Theorem 4.1.5.

Chapter 5 is dedicated to the proof of our new Łojasiewicz-Simon inequality, Theorem 5.1.2. In Section 5.2 we construct the set of adapted critical points $\mathcal{Z}$ and in Section 5.3 and Section 5.4 we study the energy and its variations on the set of adapted critical points. The main argument of the proof of Theorem 5.1.2 is then given in Section 5.5.

Chapter 2

Background: Łojasiewicz-Simon Inequalities

This section consists of background on the Łojasiewicz-Simon inequality. We start with an exposition of well known results, given in the context of finite dimensional gradient flows. We then give an abstract formulation of the Łojasiewicz-Simon inequality.

2.1 Finite dimensional gradient flows

Let $U \subset \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a C^1 function. A solution to the (negative) gradient flow of f is a curve $x\colon [0,\infty) \to U$ which satisfies the system of ordinary differential equations

$$\dot{x}(t) = -\nabla f(x(t)) \tag{2.1.1}$$

for $t \in [0,\infty)$. The function f is non-increasing along the solutions to the flow since

$$\tfrac{d}{dt} f(x(t)) = \nabla f(x(t)) \cdot \dot{x}(t) = -|\nabla f(x(t))|^2. \tag{2.1.2}$$

This implies that if a solution $x(t)$ has a limit point $\bar{x} \in U$ then $\nabla f(\bar{x}) = 0$, that is $\bar{x}$ is a critical point of f. Furthermore, in this finite dimensional case any bounded trajectory is precompact: there is a sequence of times $t_i \to \infty$ and a critical point $\bar{x} \in U$ such that $x(t_i) \to \bar{x}$ as $t_i \to \infty$.

The following example shows that we cannot expect convergence to a *unique* limit as $t \to \infty$.

Example 2.1.1 (Example 1.3, [25])**.** Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates by

$$f(r,\theta) = \begin{cases} e^{\frac{1}{r^2-1}} & \text{if } r < 1 \\ 0 & \text{if } r = 1 \\ e^{-1/(r^2-1)} \sin(1/(r-1) - \theta) & \text{if } r > 1 \end{cases}$$

and observe that this function is smooth. The set of critical points of this function is made up of the unit circle and the origin. We can pick an initial point with $r > 1$ whose trajectory remains bounded and hence the corresponding flow must limit to the circle $r = 1$. The trajectory continually spirals around this circle, never converging.

There are a variety of smooth functions demonstrating this behaviour (another example can be found in [46]) but all such functions share the property that they are not real analytic. There is a stark contrast between the case where f is smooth and where f is real analytic. When f is real analytic it is necessary that bounded trajectories converge to a unique limit, this was proved by Łojasiewicz [22] as a consequence of the following inequality which he established:

Theorem 2.1.2 (from [22])**.** *Let $f\colon U \to \mathbb{R}$ be a real analytic function and $\bar{x} \in \mathbb{R}^n$. Then there are constants $\alpha \in (0, \frac{1}{2}]$, $C < \infty$ and $\sigma > 0$ such that*

$$|f(\bar{x}) - f(y)|^{1-\alpha} \leq C|\nabla f(y)| \tag{2.1.3}$$

for any y such that $|y - \bar{x}| < \sigma$.

The proof of this inequality is quite deep, drawing on ideas from real algebraic geometry, and will not be discussed here. The main concern here is that the validity of such an inequality implies that limit points of trajectories of the corresponding gradient flow are unique. Indeed, this inequality implies the desired convergence:

Proposition 2.1.3. *Let $x\colon [0, \infty) \to \mathbb{R}^n$ be a trajectory of the gradient flow (2.1.1). Suppose that x has a limit point $\bar{x}$ for which the conclusion of Theorem 2.1.2 holds. Then the length of the curve x is finite and $\lim_{t\to\infty} x(t) = \bar{x}$.*

Proof. Let $\bar{x}$ be a limit point of x, that is there are times $t_j \to \infty$ for which $x(t_j) \to \bar{x}$. Let $\sigma > 0$, $C < \infty$ and $\alpha \in (0, \frac{1}{2}]$ be as in Theorem 2.1.2. This convergence implies that there is some j_0 such that $j \geq j_0$ implies that $x(t_j) \in B_{\sigma/2}(\bar{x})$. Hence for $j \geq j_0$ choose the maximal time $T_j \in (0, \infty]$ such that $x(t) \in B_\sigma(\bar{x})$ for every $t \in [t_j, T_j)$. We will show that $T_j = \infty$ for j large enough.

For $t \in [t_j, T_j)$ we have, crucially using (2.1.3),

$$\tfrac{d}{dt}[f(x(t)) - f(\bar{x})]^\alpha = -\alpha(f(x(t)) - f(\bar{x}))^{\alpha-1}|\nabla f(x(t))|^2 \leq -C^{-1}\alpha|\dot{x}|. \tag{2.1.4}$$

Integrating over the interval $[t_j, T_j)$ gives

$$\int_{t_j}^{T_j} |\dot{x}| \,\mathrm{d}t \leq C[f(x(t_j)) - f(\bar{x})]^\alpha \to 0 \tag{2.1.5}$$

as $j \to \infty$. This implies that

$$|x(T_j) - \bar{x}| \leq |x(t_j) - \bar{x}| + C[f(x(t_j)) - f(\bar{x})]^\alpha$$

and hence picking j large enough we can ensure that $x(T_j) \in B_{3\sigma/4}(\bar{x})$ implying that $T_j = \infty$.

Finally, due to (2.1.5) for $t > t_j$ we have

$$\int_t^\infty |\dot{x}|\,\mathrm{d}t \to 0 \text{ as } t \to \infty.$$

Hence the curve x has finite length and so must converge as $t \to \infty$ to the limit point $\bar{x}$. $\square$

Remark 2.1.4. The same proof also yields the rate of convergence, which will depend on the exponent α. This follows from the differential inequality

$$\tfrac{d}{dt}(f(x(t)) - f(\bar{x})) \leq C(f(x(t)) - f(\bar{x}))^{2\alpha}$$

which can be integrated to yield exponential convergence when $\alpha = \frac{1}{2}$ and polynomial convergence when $\alpha < \frac{1}{2}$.

2.2 An abstract formulation of the Łojasiewicz-Simon inequality

A word of caution: this section is the only section in which E does not denote the harmonic map energy.

The purpose of this section is to state and prove an abstract formulation of the Łojasiewicz-Simon inequality along the lines of Simon's original approach [43], where he considered functionals of the form $\mathcal{F}(u) = \int_\Sigma F(x, u, \nabla u)dx$ where Σ is a compact domain and F is real analytic and suitably convex in the last variable. He proved the inequality

$$|\mathcal{F}(u) - \mathcal{F}(\bar{u})|^{1-\alpha} \leq C\|\nabla \mathcal{F}(u)\|_{L^2}$$

for u in a $C^{2,\beta}$ neighbourhood of a critical point $\bar{u}$, where $\nabla\mathcal{F}$ denotes the L^2 gradient of $\mathcal{F}$. We will prove a Łojasiewicz-Simon inequality for suitable analytic operators on Hilbert and Banach spaces (rather than restricting to certain function spaces) essentially using Simon's argument. The formulation given here is with later applications to the Teichmüler harmonic map flow in mind. Other abstract formulations can also be found for example in [4, 12, 17].

Given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we denote by $\mathcal{B}((X, \|\cdot\|_X), (Y, \|\cdot\|_Y))$ the space of bounded linear operators from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ and we write $\mathcal{B}(X, Y)$ for short. We write $\|\cdot\|_{\mathcal{B}(X,Y)}$ for the operator norm on $\mathcal{B}(X, Y)$. We need a notion of real analyticity for a map $X \to Y$.

Definition 2.2.1. A map $T\colon X \to Y$ is analytic at a point $u \in X$ if there is $r > 0$ such that for each $n \in \mathbb{N}$ there is a symmetric, continuous multilinear map $L_n\colon X^n \to Y$ with

$$\sum_{n \geq 1} \|L_n\|_{\mathcal{B}(X^n,Y)} r^n < \infty$$

and

$$T(u+h) = T(u) + \sum_{n \geq 1} L_n(h, \dots, h)$$

for $h \in X$ with $\|h\|_X < r$.

Here we are assuming that the Banach spaces are over the field $\mathbb{R}$. We remark that if the field is $\mathbb{C}$ then this follows as soon as T is Fréchet differentiable (as is the case for complex valued functions). We will use the following version of the inverse function theorem in Banach spaces.

Theorem 2.2.2. *Let X, Y be Banach spaces, $U \subset X$ an open neighbourhood of a point $\bar{u} \in X$ and $T\colon U \to Y$ a real analytic map. If the Fréchet derivative $d_{\bar{u}}T$ is invertible in $\mathcal{B}(X,Y)$ then there is a neighbourhood $W_1 \subset U$ of $\bar{u}$, a convex neighbourhood $W_2 \subset Y$ of $T\bar{u}$ and a real analytic map $S\colon W_2 \to W_1$ such that $S = T|_{W_1}^{-1}$. Moreover, we have*

$$d_y S = [d_{S(y)}T]^{-1}$$

for each $y \in W_2$.

The statement of the implicit function theorem and a discussion of analytic functions between Banach spaces can be found in, for example, [8].

Before proceeding, we recall the following definition:

Definition 2.2.3. A bounded linear operator $L\colon X \to Y$ is called Fredholm if it has a finite dimensional kernel and a finite dimensional cokernel. The index of L is the number $\dim \ker L - \dim \operatorname{coker} L$.

We can now give the formulation of the inequality.

Theorem 2.2.4. *Suppose we are given the following data:*

1. *A Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ together with subspaces*

   $$X \subset Y \subset H, \quad X \subset Z \subset H$$

   such that X, Y, Z are complete when equipped with norms $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$ respectively and such that the above inclusions are continuous.

2. *An analytic functional $E\colon U \to \mathbb{R}$ which is defined on an open subset $U \subset X$.*

3. *There is an analytic map $\nabla E\colon U \to Y$ so that for every $u \in U$ and $x \in X$*

$$\tfrac{d}{dt}E(u+tx)|_{t=0} = \langle \nabla E(u), x\rangle_H. \tag{2.2.1}$$

4. *Suppose that we have $\bar{u} \in U$ such that $\nabla E(\bar{u}) = 0$ and $L := d_{\bar{u}}\nabla E$ is a Fredholm operator of index 0 from X to Y. For every $u \in U$ the element $d_u\nabla E \in \mathcal{B}(X,Y)$ has an extension to $L_u \in \mathcal{B}(Z,H)$ such that the map $U \ni u \mapsto L_u \in \mathcal{B}(Z,H)$ is continuous. Finally, $L_{\bar{u}} \in \mathcal{B}(Z,H)$ (an extension of L) is Fredholm of index 0.*

Then there is $\alpha \in (0, \frac{1}{2})$, $A < \infty$ and a neighbourhood $\mathcal{O} \subset U$ of $\bar{u}$ so that

$$|E(u) - E(\bar{u})|^{1-\alpha} \leq A\|\nabla E(u)\|_H$$

for each $u \in \mathcal{O}$.

For applications, and indeed in the proof, it is helpful to think about a much more concrete setting in which one obtains Simon's result. For his result, the spaces considered are

$$C^{2,\alpha} \subset C^{0,\alpha} \subset L^2, \quad C^{2,\alpha} \subset H^2 \subset L^2.$$

Convexity leads to the Fredholm property for the linearisation L and analyticity of the integrand leads to analyticity of the gradient operator.

Remark 2.2.5.

1. In condition 3 we are assuming that the object $x \mapsto \frac{d}{dt}E(u+tx)|_{t=0}$, which in general only lies in X^*, can be represented by an element of Y (using the inclusion $Y \subset X^*$).

2. If we further assume that the inclusion $X \subset (Y, \|\cdot\|_H)$ is dense then for each $u \in U$ there is a unique element $\nabla E(u) \in Y$ satisfying (2.2.1).

Remark 2.2.6. The operator $L \in \mathcal{B}(X,Y)$ can be thought of as a Hessian of E and so is formally self adjoint with respect to H in the sense that $\langle Lx, x'\rangle_H = \langle x, Lx'\rangle_H$ for every $x, x' \in X$. It is not automatic that an operator $T \in \mathcal{B}(X,Y)$ that is Fredholm and is formally self-adjoint in this sense has Fredholm index 0. If, with the spaces X, Y, Z, H as above, we have an extension $\hat{L} \in \mathcal{B}(Z,H)$ which is Fredholm and formally self-adjoint (now $\langle Lz, z'\rangle_H = \langle z, Lz'\rangle_H$ for every $z, z' \in Z$) and we furthermore impose the requirement that $(\hat{L}(Z))^\perp \subset \ker \hat{L}$ then we have that $\hat{L}\colon Z \to H$ is Fredholm of index 0. Finally, if we have the implication $\hat{L}z \in Y \implies z \in X$ then $L\colon X \to Y$ is index 0 also.

We will prove Theorem 2.2.4 with the help of a few lemmas. In what follows the constants are allowed to depend on the following data: the spaces and their norms, the functional E, the gradient ∇E, the critical point $\bar{u}$ and the extensions L_u for $u \in U$.

Lemma 2.2.7. *After possibly reducing U, there is a constant $M < \infty$ such that*

$$\|\nabla E(u_1) - \nabla E(u_2)\|_H \leq M\|u_1 - u_2\|_Z \tag{2.2.2}$$

for every $u_1, u_2 \in U$.

Proof. By reducing U we can assume it is convex. Therefore, given $u_1, u_2 \in U$ we have $tu_1 + (1-t)u_2 \in U$ for each $t \in [0,1]$. We have

$$\nabla E(u_1) - \nabla E(u_2) = \int_0^1 \tfrac{d}{dt}\nabla E(tu_1 + (1-t)u_2)dt = \int_0^1 [d_{tu_1+(1-t)u_2}\nabla E](u_1 - u_2)dt.$$

Since $u_1 - u_2 \in X \subset Z$ we have $[d_{tu_1+(1-t)u_2}\nabla E](u_1 - u_2) = L_{tu_1+(1-t)u_2}(u_1 - u_2)$ where L_u is the extension of $d_u\nabla E$ to $\mathcal{B}(Z,H)$. We can therefore estimate

$$\|\nabla E(u_1) - \nabla E(u_2)\|_H \leq \sup_{u \in U}[\|L_u\|_{\mathcal{B}(Z,H)}]\|u_1 - u_2\|_Z.$$

By continuity of $U \ni u \mapsto L_u \in \mathcal{B}(Z,H)$ we can reduce U further so that $\|L_u - L_{\bar{u}}\|_{\mathcal{B}(Z,H)} \leq 1$ for $u \in U$. Therefore we obtain the result by setting $M = 1 + \|L_{\bar{u}}\|_{\mathcal{B}(Z,H)}$. □

Since the linear operator $L \in \mathcal{B}(X,Y)$ is Fredholm it has finite dimensional kernel $K := \ker L$. Therefore, the H-orthogonal projection $\Pi\colon H \to K$ is well defined and bounded from $(H, \|\cdot\|_H)$ to K equipped with any norm, since K is finite dimensional. It is therefore also bounded from $(X, \|\cdot\|_X)$ to K also.

Lemma 2.2.8. *Let $K = \ker L$ and $\Pi\colon H \to K$ denote the H-orthogonal projection onto K. There is a neighbourhood $W_1 \subset X$ of $\bar{u}$, a convex neighbourhood $W_2 \subset Y$ of $\Pi\bar{u}$ and a real analytic map $\Psi\colon W_2 \to W_1$ satisfying*

$$\begin{cases} (\nabla E + \Pi) \circ \Psi(y) = y & \text{for } y \in W_2 \\ \Psi \circ (\nabla E + \Pi)(u) = u & \text{for } u \in W_1 \end{cases} \tag{2.2.3}$$

and there is $C_1 < \infty$ such that

$$\sup_{y \in W_2} \|d_y\Psi\|_{\mathcal{B}((Y,\|\cdot\|_H),(X,\|\cdot\|_Z))} \leq C_1 < \infty \tag{2.2.4}$$

and therefore

$$\|\Psi(y_1) - \Psi(y_2)\|_Z \leq C_1\|y_1 - y_2\|_H \tag{2.2.5}$$

for $y_1, y_2 \in W_2$.

Proof. We consider the operator $\nabla E + \Pi\colon U \to Y$ which is again analytic (indeed, the projection Π is bounded and linear and so is analytic). We will apply the inverse function theorem at the point $\bar{u}$ to conclude that such an inverse exists.

We need to ensure that the Fréchet derivative at $\bar{u}$ which is given by $d_{\bar{u}}(\nabla E + \Pi) = L + \Pi\colon X \to Y$ is a bounded linear operator which is invertible. Indeed, L is bounded by assumption and $\Pi\colon (X, \|\cdot\|_X) \to (K, \|\cdot\|_Y)$ is bounded as described above. We have that L is formally self-adjoint with respect to H, that is for every $x, x' \in X$ we have

$$\begin{aligned}
\langle Lx, x'\rangle_H &= \tfrac{d}{dt}\langle \nabla E(\bar{u} + tx), x'\rangle_H|_{t=0} \\
&= \tfrac{d}{dt}\tfrac{d}{ds}E(\bar{u} + tx + sx')|_{s=0}|_{t=0} \\
&= \tfrac{d}{ds}\langle \nabla E(\bar{u} + sx'), x\rangle_H|_{s=0} = \langle Lx', x\rangle_H.
\end{aligned}$$

To see that $\Pi + L$ is injective we pick $x \in X$ such that $(\Pi + L)x = 0$. Then since $\langle Lx, \Pi x\rangle_H = \langle x, L \circ \Pi x\rangle_H = 0$ we have

$$\|\Pi x\|_H^2 = \langle \Pi x + Lx, \Pi x\rangle_H = 0.$$

From this we conclude that $\Pi x = 0$ and hence also $Lx = 0$. Then since $x \in K$ and $\Pi|_K = \mathrm{id}$ we have $x = 0$. To see that $L + \Pi$ is surjective we note that this operator is also Fredholm with the same index (since Π is a compact operator and L is Fredholm; the sum of a compact operator and a Fredholm operator is Fredholm with the same index). As the Fredholm index of this operator is 0 and it is injective it must also be surjective.

The inverse function theorem yields a neighbourhood $W_1 \subset U$ of $\bar{u}$ and a convex neighbourhood $W_2 \subset Y$ of $(\nabla E + \Pi)\bar{u} = \Pi\bar{u}$ such that $\Psi = (\nabla E + \Pi)|_{W_1}^{-1}\colon W_2 \to W_1$ exists (i.e. (2.2.3) holds) and is real analytic. Furthermore, we have

$$d_y\Psi = [d_{\Psi(y)}(\nabla E + \Pi)]^{-1} \tag{2.2.6}$$

for every $y \in W_2$.

We now wish to show the estimates for Ψ. Notice that using the same proof above $L_{\bar{u}} + \Pi \in \mathcal{B}(Z, H)$ is invertible since $L_{\bar{u}}$ is Fredholm with index 0. As the map taking $u \in W_1$ to $(L_u + \Pi) \in \mathcal{B}(Z, H)$ is continuous, we can assume that W_1 was chosen small enough that

$$\|(L_u + \Pi) - (L_{\bar{u}} + \Pi)\|_{\mathcal{B}(Z,H)} \le \tfrac{1}{2}\|(L_{\bar{u}} + \Pi)^{-1}\|_{\mathcal{B}(H,Z)}^{-1} \text{ for all } u \in W_1.$$

Then, using the Neumann series, we have that $L_u + \Pi$ is invertible for each $u \in W_1$ with estimate

$$\|(L_u + \Pi)^{-1}\|_{\mathcal{B}(H,Z)} \le 2\|(L_{\bar{u}} + \Pi)^{-1}\|_{\mathcal{B}(H,Z)}.$$

Observe from (2.2.6) that $(L_{\Psi(y)}+\Pi)^{-1}$ is an extension of $d_y\Psi$ which means we have for every $y \in W_2 = \Psi^{-1}(W_1)$,

$$\|d_y\Psi\|_{\mathcal{B}((Y,\|\cdot\|_H),(X,\|\cdot\|_Z))} \leq \|(L_{\Psi(y)}+\Pi)^{-1}\|_{\mathcal{B}(H,Z)} \leq C_1, \tag{2.2.7}$$

where we have set $C_1 = 2\|(L_{\bar{u}}+\Pi)^{-1}\|_{\mathcal{B}(H,Z)}$.

We now prove the Lipschitz estimate. For $y_1, y_2 \in W_2$ we have that, since W_2 is convex,

$$\Psi(y_1) - \Psi(y_2) = \int_0^1 \tfrac{d}{dt}\Psi(y_2 + t(y_1-y_2))dt = \int_0^1 [d_{y_2+t(y_1-y_2)}\Psi](y_1-y_2)dt$$

and then using (2.2.7) that

$$\begin{aligned}\|\Psi(y_1)-\Psi(y_2)\|_Z &\leq \sup_{y\in W_2}\left[\|d_y\Psi\|_{\mathcal{B}((Y,\|\cdot\|_H),(X,\|\cdot\|_Z))}\right]\|y_1-y_2\|_H \\ &\leq C_1\|y_1-y_2\|_H.\end{aligned}$$

□

We now define a new functional $F\colon W_2 \to \mathbb{R}$ by $F(y) = E(\Psi(y))$, which is well defined since $\Psi(W_2) = W_1 \subset U$. The corresponding H-gradient of F exists, that is there is an element $\nabla F(y) \in X$ such that

$$\tfrac{d}{dt}F(y+t\xi)|_{t=0} = \langle \nabla F(y), \xi\rangle_H \text{ for every } \xi \in Y. \tag{2.2.8}$$

Indeed, we have

$$\begin{aligned}\tfrac{d}{dt}|_{t=0}F(y+t\xi) &= \tfrac{d}{dt}|_{t=0}E(\Psi(y+t\xi)) \\ &= \langle \nabla E(\Psi(y)), \tfrac{d}{dt}|_{t=0}\Psi(y+t\xi)\rangle_H = \langle \nabla E(\Psi(y)), [d_y\Psi](\xi)\rangle_H.\end{aligned}$$

so we can choose $\nabla F(y) = [d_y\Psi](\nabla E(\Psi(y)))$ since $d_y\Psi = [d_{\Psi(y)}(\nabla E + \Pi)]^{-1}$ is formally self-adjoint with respect to H. This definition of ∇F makes sense since $\nabla E(y) \in Y$ and $d_y\Psi\colon Y \to X$. In the following lemma we will compare the functional F with E and also compare their gradients.

Lemma 2.2.9. *If we set $W = \{u \in W_1 \colon \Pi u \in W_2\}$ then W is open, non-empty and we have $C_2 < \infty$ such that*

$$\|\nabla F(\Pi u)\|_H \leq C_2\|\nabla E(u)\|_H, \tag{2.2.9}$$

$$|E(u) - F(\Pi u)| \leq C_2\|\nabla E(u)\|_H^2 \tag{2.2.10}$$

for every $u \in W$.

Proof. First we recall that for $y \in W_2$ we have $\nabla F(y) = [d_y\Psi](\nabla E(\Psi(y)))$ and so by (2.2.4)

$$\|\nabla F(y)\|_H \leq C_{Z,H} C_1 \|\nabla E(\Psi(y))\|_H \tag{2.2.11}$$

where $C_{Z,H}$ denotes the operator norm of the embedding $(Z, \|\cdot\|_Z) \hookrightarrow (H, \|\cdot\|_H)$.

We have $\bar{u} \in W$ and W is open since $\Pi\colon X \to Y$ is continuous. Now for $u \in W$ we use the Lipschitz bounds (2.2.2) and (2.2.5), for ∇E and Ψ respectively, together with the fact that $u = \Psi((\nabla E + \Pi)(u))$ to give

$$\begin{aligned}
\|\nabla E(\Psi(\Pi u))\|_H &\leq \|\nabla E(u)\|_H + \|\nabla E(u) - \nabla E(\Psi(\Pi u))\|_H \\
&\leq \|\nabla E(u)\|_H + M\|u - \Psi(\Pi u)\|_Z \\
&= \|\nabla E(u)\|_H + M\|\Psi((\nabla E + \Pi)(u)) - \Psi(\Pi u)\|_Z \\
&\leq \|\nabla E(u)\|_H + C_1 M \|\nabla E(u)\|_H.
\end{aligned}$$

This estimate together with (2.2.11), applied for $y = \Pi u \in W_2$, gives the result (2.2.9) if we set $C_2 = C_{Z,H} C_1 (1 + C_1 M)$.

We now proceed with the proof of (2.2.10). If $u \in W$ then $\Pi u \in W_2$ and $(\nabla E + \Pi)u \in \Psi^{-1}(W) \subset W_2$. Therefore, by convexity of W_2 we have $y_t = \Pi u + t\nabla E(u) \in W_2$ for every $t \in [0, 1]$. Thus we can estimate

$$\begin{aligned}
|E(u) - F(\Pi u)| = |F(y_1) - F(y_0)| &= \left| \int_0^1 \tfrac{d}{dt} F(y_t) dt \right| \\
&= \left| \int_0^1 \langle \nabla F(y_t), \nabla E(u) \rangle_H dt \right| \leq \sup_{t \in [0,1]} \|\nabla F(y_t)\|_H \|\nabla E(u)\|_H.
\end{aligned}$$

Finally using (2.2.11) we have since $y_t \in W_2$

$$\begin{aligned}
\|\nabla F(y_t)\|_H &\leq C_{Z,H} C_1 \|\nabla E(\Psi(y_t))\|_H \\
&\leq C_{Z,H} C_1 (\|\nabla E(\Psi(y_t)) - \nabla E(\Psi(y_1))\|_H + \|\nabla E(\Psi(y_1))\|_H) \\
&\leq C_{Z,H} C_1 (C_1 M \|y_t - y_1\|_H + \|\nabla E(u)\|_H) \\
&\leq C_2 \|\nabla E(u)\|_H
\end{aligned}$$

which proves the result. □

Proof of Theorem 2.2.4. Let $\psi_1, \ldots, \psi_k$ be an H orthonormal basis for $K = \ker L$ and write $\Pi\bar{u} = \sum_{i=1}^k \bar{\xi}_i \psi_i$. We can choose $\sigma_1 > 0$ so that $\xi \in B(\bar{\xi}, \sigma_1)$ implies that $\sum_i \xi_i \psi_i \in W_2$ where W_2 is the neighbourhood in Y of $\Pi\bar{u}$ given by the lemmas above.

We define a new function $f\colon B(\bar{\xi}, \sigma_1) \subset \mathbb{R}^k \to \mathbb{R}$ by

$$f(\xi) := F(\sum_{i=1}^k \xi_i \psi_i) = E(\Psi(\sum_{i=1}^k \xi_i \psi_i)).$$

Since E and Ψ are real analytic so is this new function f. The function f is defined on a finite dimensional domain so we can apply the classical Łojasiewicz inequality, Theorem 2.1.2, which yields $\sigma \in (0, \sigma_1)$, $C_Ł < \infty$ and $\alpha \in (0, \frac{1}{2})$ so that

$$|f(\xi) - f(\bar{\xi})|^{1-\alpha} \leq C_Ł |\nabla f(\xi)| \text{ for all } \xi \in B(\bar{\xi}, \sigma). \tag{2.2.12}$$

Now since $\bar{u}$ is a critical point of ∇E we have $\bar{u} = \Psi(\Pi \bar{u})$ and hence for any $u \in W$,

$$\begin{aligned} |E(u) - E(\bar{u})| &= |E(u) - F(\Pi \bar{u})| \\ &\leq |E(u) - F(\Pi u)| + |F(\Pi u) - F(\Pi \bar{u})| \\ &\leq C_2 \|\nabla E(u)\|_H^2 + |F(\Pi u) - F(\Pi \bar{u})|, \end{aligned}$$

where we have used (2.2.10). We finally choose $\mathcal{O} \subset W$ small enough that $\|\Pi(u - \bar{u})\|_H < \sigma$ for every $u \in \mathcal{O}$. This implies that if $u \in \mathcal{O}$ and $\Pi u = \sum_i \xi_i \psi_i$ then $\xi \in B(\bar{\xi}, \sigma)$. Since we have $F(\Pi u) = f(\xi)$ we can use the Łojasiewicz inequality (2.2.12) for f and then the estimate (2.2.9) to obtain

$$\begin{aligned} |F(\Pi u) - F(\Pi \bar{u})|^{1-\alpha} = |f(\xi) - f(\bar{\xi})|^{1-\alpha} &\leq C_Ł |\nabla f(\xi)| \\ &\leq C_Ł \|\nabla F(\Pi u)\|_H \leq C_Ł C_2 \|\nabla E(u)\|_H. \end{aligned}$$

Therefore we can conclude that for every $u \in \mathcal{O}$ we have

$$|E(u) - E(\bar{u})|^{1-\alpha} \leq A \|\nabla E(u)\|_H,$$

where $A < \infty$ is a constant that can be computed explicitly from α, C_2 and $C_Ł$. $\square$

Chapter 3

Background: Teichmüller Harmonic Map Flow

The purpose of this chapter is to introduce the existing background needed to understand our later results on the Teichmüller harmonic map flow. We start by discussing the harmonic map energy as a functional of a map and domain metric. We then give an outline of a derivation of the Teichmüller harmonic map flow as a gradient flow of the harmonic map energy on a suitable space. Then we discuss some basic properties of solutions to the flow which will be needed later. After this, we give a brief account of the known results on the theory of the flow in relation to existence, singularity formation and asymptotics. Finally, we discuss the simplifications that can be made for the flow from the torus T^2.

3.1 Minimal immersions and the harmonic map energy

Let M be a closed surface of genus $\gamma \geq 0$ and let (N, g_N) be a closed Riemannian manifold of arbitrary dimension. Write $\mathcal{M}$ for the set of Riemannian metrics on M. The harmonic map energy energy of a smooth map $u\colon (M, g) \to (N, g_N)$, $g \in \mathcal{M}$, is

$$E(u, g) = \frac{1}{2} \int_M |du|_g^2 \, \mathrm{d}\mu_g. \tag{3.1.1}$$

Fixing a metric $g \in \mathcal{M}$ recall that a C^2 critical point of the Dirichlet energy $E(\cdot, g)$ is called a harmonic map and satisfies the equation $\tau_g(u) = 0$ where $\tau_g(u) := \mathrm{tr}_g \nabla_g du$ is the tension field. The tension can be written in local coordinates as

$$\tau_g(u)^\gamma = \Delta_g u^\gamma + {}^N\Gamma^\gamma_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} g^{ij}, \tag{3.1.2}$$

where ${}^N\Gamma$ denotes the Christoffel symbols of (N, g_N). By Nash's embedding theorem (N, g_N) can be isometrically embedded into Euclidean space $\mathbb{R}^n$, in which case the tension takes the following extrinsic form

$$\tau_g(u) = \Delta_g u + A(u)(du, du), \tag{3.1.3}$$

where A denotes the second fundamental form of this embedding. For more details on harmonic maps, see [20].

The connection with minimal surfaces comes from studying variations of this functional with respect to both the map and domain metric. The following is a well known result giving the first variation of the energy:

Proposition 3.1.1. *Fix $(u, g) \in C^\infty(M, N) \times \mathcal{M}$. For any smooth variation $(u_\varepsilon, g_\varepsilon)$ of (u, g) we have*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(u_\varepsilon, g_\varepsilon) = -\int_M \langle \tau_g(u), \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} u_\varepsilon\rangle + \tfrac{1}{4}\langle \mathrm{Re}[\Phi(u,g)], \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} g_\varepsilon\rangle_g \,\mathrm{d}\mu_g \tag{3.1.4}$$

where $\langle \cdot, \cdot \rangle_g$ is the extension of g to $(0, 2)$-tensors, $\mathrm{Re}[\Phi(u, g)] := u^* g_N - |du|_g^2 g$ is the real part of the Hopf differential and $\tau_g(u) := \mathrm{tr}_g \nabla_g du$ is the tension field.

The reason for the notation is that the object appearing as the variation of the energy on the space of metrics is the real part of a well known object, the Hopf differential. The Hopf differential can be defined as a multiple of the $(2, 0)$-part of the tensor $u^* g_N$ extended to the complexified tangent bundle

$$\Phi(u, g) = 4(u^* g_N)^{(2,0)}. \tag{3.1.5}$$

In local complex coordinates $z = x + \mathrm{i}y$ for M where the metric takes the form $\rho^2(dx^2 + dy^2)$ it is given by

$$\begin{aligned} \Phi(u, g) = \phi dz^2 \text{ where } \phi &= 4(u^* g_N)(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z}) \\ &= |u_x|^2 - |u_y|^2 - 2\mathrm{i}\langle u_x, u_y\rangle. \end{aligned} \tag{3.1.6}$$

This object appears, in particular, in the study of harmonic maps since if $u: (M, g) \to (N, g_N)$ is harmonic then ϕ is holomorphic (and so Φ is a holomorphic quadratic differential). This follows directly from the computation

$$\partial_{\bar{z}} \phi = \rho^2 \langle \tau_g(u), \partial_z u\rangle, \tag{3.1.7}$$

see for example [20].

A map $u: (M, g) \to (N, g_N)$ is *weakly conformal* if there is a non-negative function $f: M \to [0, \infty)$ such that $u^* g_N = fg$. It is immediate that $\Phi(u, g) = 0$ if and only if $u: (M, g) \to (N, g_N)$ is weakly conformal.

The Proposition 3.1.1 implies that critical points of the energy are weakly conformal harmonic maps. One can compute that for a conformal immersion the tension field coincides with the mean curvature vector of its image [10], hence giving the following:

Proposition 3.1.2. *Let $u\colon M \to N$ be an immersion and $g \in \mathcal{M}$. Then (u, g) is a critical point of $E(u, g)$ (i.e. a conformal harmonic map) if and only if it is a* minimal *immersion.*

A result of [13] says that a weakly conformal harmonic map (not necessarily an immersion) is either a constant map or a branched minimal immersion, that is u is a minimal immersion away from isolated points where $du = 0$. Around the image of each of these points there are normal coordinates for N where the map takes the form,

$$\begin{aligned} u^1(z) &= \mathrm{Re}(z^k) + o(|z|^k); \\ u^2(z) &= \mathrm{Im}(z^k) + o(|z|^k); \\ u^\alpha(z) &= o(|z|^k) \text{ for } \alpha \geq 3, \end{aligned} \tag{3.1.8}$$

where $z = x + iy$ is a local complex coordinate for M and $k \in \{2, 3, \ldots\}$. This then explains the terminology in the following standard definition.

Definition 3.1.3. Critical points of the harmonic map energy with respect to variations of both the map and the metric are called *branched minimal immersion.*

The previous discussion can also be viewed from the point of view of the area functional

$$A(u) = \int_M (|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2)^{\frac{1}{2}} dxdy. \tag{3.1.9}$$

It is immediate that $A(u) \leq E(u, g)$ (as can be seen by computing in isothermal coordinates) with equality if and only if $u\colon (M, g) \to (N, g_N)$ is conformal. It then follows that a conformal, harmonic immersion is minimal (a critical point of the area). Conversely, given a minimal immersion it will be conformal and harmonic when the domain is equipped with the metric u^*g_N.

3.2 Definition of the Teichmüller harmonic map flow

We will now introduce the Teichmüller harmonic map flow as an L^2 gradient flow of the harmonic map energy on an appropriate space of maps and metrics, following Rupflin and Topping [34].

The Teichmüller harmonic map flow is not simply the L^2 gradient flow of the harmonic map energy on the whole space of maps and metrics, the key point here is that they were able to exploit the invariance of the problem to yield equations with better behaviour. These invariances are the conformal invariance of the energy and also the invariance under pullback of the map and metric by any diffeomorphism. Strictly speaking the derivation given is purely formal but nevertheless can be made precise.

As a first step one can restrict the space of metrics under consideration using conformal invariance of the energy. Denote by $\mathcal{M}_c$ the space of metrics of constant curvature $c \in$

$\{0,1,-1\}$, with the additional restriction that they have unit area if $c = 0$. By the uniformization theorem any metric $g \in \mathcal{M}$ is conformally related to a unique metric in $\mathcal{M}_{-1}$ when $\gamma \geq 2$, in $\mathcal{M}_0$ when $\gamma = 1$ or in $\mathcal{M}_{+1}$ when $\gamma = 0$. Therefore, since the energy is conformally invariant, it suffices to restrict attention to metrics in the class $\mathcal{M}_c$ (c will of course depend on the genus of surface being considering as described above).

Next one can exploit the fact that the energy is invariant under pullback in the following sense: $E(u,g) = E(u \circ f, f^*g)$ for any diffeomorphism $f\colon M \to M$. Define

$$\mathcal{A} := (C^\infty(M,N) \times \mathcal{M}_c)/\sim, \tag{3.2.1}$$

identifying pairs $(u,g) \sim (u \circ f, f^*g)$ when $f\colon M \to M$ is a diffeomorphism which is homotopic to the identity. The requirement that the diffeomorphisms be homotopic to the identity is natural as in some sense it means the topology of the "map component" is well defined.

To define a gradient flow on the space $\mathcal{A}$ a Riemannian metric is needed, that is a way to measure the length of tangent vectors. A first step is to understand the tangent space to $\mathcal{M}_c$, see for example [49]:

Lemma 3.2.1. *Given $g \in \mathcal{M}_c$ we have an $L^2(M,g)$ orthogonal splitting*

$$T_g\mathcal{M}_c = \mathrm{Re}(\mathcal{H}(g)) \oplus \{\mathcal{L}_X(g)\colon X \in \mathfrak{X}(M)\} \tag{3.2.2}$$

where $\mathcal{L}$ denotes the Lie derivative and $\mathcal{H}(g)$ denotes the space of holomorphic (with respect to the conformal structure determined by g) quadratic differentials.

This Lie derivative part represents the action of diffeomorphisms in the sense that $\mathcal{L}_X g = \frac{d}{dt}|_{t=0}(f_t^* g)$, where f_t is a family of diffeomorphisms generating a vector field X on M.

Therefore, given a representative $(u(t), g(t))$ of a path $[(u(t), g(t))] \in \mathcal{A}$ one can find a holomorphic quadratic differential Ψ and a vector field X on M such that $\partial_t g(0) = \mathrm{Re}(\Psi) + \mathcal{L}_X g(0)$. If f_t is the family of diffeomorphisms generating $-X$ then

$$\partial_t(f_t^* g(t))|_{t=0} = -\mathcal{L}_X g(0) + \partial_t g(0) = \mathrm{Re}(\Psi).$$

Hence the path $[(u(t), g(t))]$ is represented by $(\tilde{u}(t), \tilde{g}(t)) = (u(t) \circ f_t, f_t^* g(t))$ which has $\partial_t \tilde{g}(0) \in \mathrm{Re}(\mathcal{H}(g))$. Then an inner product can be defined on $\mathcal{A}$, induced by the norm

$$\begin{aligned}\|\,[\partial_t(\tilde{u},\tilde{g})(0)]\,\|_{\mathcal{A}} &= \|\partial_t \tilde{u}(0)\|_{L^2(M,\tilde{g})} + \eta^{-2}\|\partial_t \tilde{g}(0)\|_{L^2(M,\tilde{g})} \\ &= \|\partial_t \tilde{u}(0)\|_{L^2(M,\tilde{g})} + \eta^{-2}\|\mathrm{Re}(\Psi)\|_{L^2(M,\tilde{g})}\end{aligned}$$

where $\eta > 0$ is a coupling constant that can be chosen freely and then considered fixed.

Now one can make sense of the gradient flow of E on $\mathcal{A}$ with respect to this inner product. Note that by the above discussion a path in $\mathcal{A}$ can be represented by $(u(t), g(t)) \in C^\infty(M,N) \times \mathcal{M}_c$ with $\partial_t g(t) = \mathrm{Re}(\Psi(t)) \in \mathcal{H}(g(t))$. Set $(u,g) := (u(0), g(0))$, $v = \partial_t u(0)$ and $\Psi = \Psi(0)$ so that by Proposition 3.1.1

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} E(u(t), g(t)) &= -\int_M \langle \tau_g(u), v\rangle + \tfrac{1}{4}\langle \mathrm{Re}(\Phi(u,g)), \mathrm{Re}(\Psi)\rangle \, \mathrm{d}\mu_g \\
&= -\int_M \langle \tau_g(u), v\rangle + \tfrac{1}{4}\langle \mathrm{Re}(P_g(\Phi(u,g)), \mathrm{Re}(\Psi)\rangle \, \mathrm{d}\mu_g \\
&= -\langle (\tau_g(u), \frac{\eta^2}{4}\mathrm{Re}(P_g(\Phi(u,g))), (v, \mathrm{Re}(\Psi))\rangle_{\mathcal{A}}.
\end{aligned}$$

where P_g denotes the $L^2(M,g)$ orthogonal projection onto the space of holomorphic quadratic differentials. Therefore the directions in $\mathcal{A}$ in which E decreases the fastest are represented by $(\tau_g(u), \frac{\eta^2}{4}\mathrm{Re}(P_g(\Phi(u,g)))$. This yields the flow equations

$$\frac{\partial u}{\partial t} = \tau_g(u); \qquad \frac{\partial g}{\partial t} = \frac{\eta^2}{4}\mathrm{Re}(P_g(\Phi(u,g))) \tag{3.2.3}$$

which are taken as the definition of the Teichmüller harmonic map flow.

At this point it is worth noting how the flow differs depending on the genus γ of M. When $M = S^2$, i.e. $\gamma = 0$, it can be shown that there are no non-trivial holomorphic quadratic differentials (see e.g. Lemma 8.2.4 of [20] for a proof) and therefore the metric component is static; the flow reduces to the harmonic map flow. When $M = T^2$, i.e. $\gamma = 1$, the flow simplifies as the metric component can be constrained to a two dimensional submanifold of the whole space $\mathcal{M}_0$. In this case the basic theory was given by [9]. We will describe some aspects of the flow on the torus in Section 3.5.

3.3 Basic properties

In this section we collect some known properties of smooth solutions (u,g) of the Teichmüller harmonic map flow (3.2.3) defined on some time interval $[0,T)$ which will be used later on.

The first results are elementary consequences of the flow equations (3.2.3). The flow decreases the energy $E(t) = E(u(t), g(t))$ according to

$$\frac{dE}{dt} = -\|\tau_g(u)\|^2_{L^2(M,g)} - \left(\frac{\eta}{4}\right)^2 \|\mathrm{Re}(P_g(\Phi(u,g)))\|^2_{L^2(M,g)} \tag{3.3.1}$$

$$= -\|\tau_g(u)\|^2_{L^2(M,g)} - \frac{\eta^2}{32}\|P_g(\Phi(u,g))\|^2_{L^2(M,g)}. \tag{3.3.2}$$

Integrating over an interval $(t_1, t_2) \subset [0,T)$ gives

$$\int_{t_1}^{t_2} \|\tau_g(u)\|^2_{L^2(M,g)} + \left(\frac{\eta}{4}\right)^2 \|\mathrm{Re}(P_g(\Phi(u,g)))\|^2_{L^2(M,g)} dt = E(t_1) - E(t_2) < \infty. \tag{3.3.3}$$

It is also useful to note that, by (3.3.1) and the Cauchy-Schwarz inequality, the L^2 length of the metric component of the flow is bounded by

$$\int_{t_1}^{t_2} \|\partial_t g\|_{L^2(M,g)} dt \leq C(\eta)(t_2 - t_1)^{\frac{1}{2}} E(0)^{\frac{1}{2}}. \tag{3.3.4}$$

It is important that the metric component g of the flow moves in the direction of (real parts of) holomorphic quadratic differentials. Thus it is an instance of what is known as a *horizontal curve* of Riemannian metrics. These objects are well controlled when the injectivity radius is bounded away from zero as $\mathrm{inj}(M, g(t)) \geq \delta > 0$ and their length is finite (as is provided by (3.3.4) for solutions of the flow). More details on the following results can be found in [35], there they also prove stronger results to understand the properties of a horizontal curve as $\mathrm{inj}(M, g(t)) \to 0$.

The following results from [35] are only valid for surfaces M of genus $\gamma \geq 2$, the situation of the torus is considered in Section 3.5. As remarked in the proof of Lemma 2.6 in [35], since $\partial_t g$ is the real part of a holomorphic object, when $\mathrm{inj}_{g(t)}(x) \geq \delta > 0$ one can choose coordinates in a small ball so that its components are harmonic functions and thus elliptic regularity gives that for each $k \in \mathbb{N}$ the C^k norm of $\partial_t g$ is controlled by its L^1 norm,

$$|\partial_t g|_{C^k(g)}(x) \leq C\delta^{-1} \|\partial_t g\|_{L^1(B^g_\delta(x),g)} \leq C\delta^{-\frac{1}{2}} \|\partial_t g\|_{L^2(B^g_\delta(x),g)} \tag{3.3.5}$$

where C depends only on the genus of M and k. In fact one can be more precise; for every $x \in M$

$$|\partial_t g|_{C^k(M,g)}(x) \leq C[\mathrm{inj}_{g(t)}(x)]^{-\frac{1}{2}} \|\partial_t g\|_{L^2(M,g)} \tag{3.3.6}$$

for a constant C depending only on the genus of M and k, see Lemma 2.6 of [35]. It is also true that the evolution of the injectivity radius is controlled by the L^2 velocity of g; for each $x \in M$

$$|\tfrac{d}{dt}\mathrm{inj}_{g(t)}(x)| \leq C[\mathrm{inj}_{g(t)}(x)]^{-\frac{1}{2}} \|\partial_t g\|_{L^2(M,g)} \tag{3.3.7}$$

where C depends only on the genus of M, see Lemma 2.1 of [35]. Finally, if $g(t) \to \bar{g} \in \mathcal{M}_{-1}$ smoothly as $t \to T$ then the C^k norms with respect to $g(t)$ for $t \in [0, T)$ and with respect to $\bar{g}$ are equivalent,

$$C^{-1} \|\partial_t g\|_{C^k(M,g(t))} \leq \|\partial_t g\|_{C^k(M,\bar{g})} \leq C \|\partial_t g\|_{C^k(M,g(t))}, \tag{3.3.8}$$

see Lemma 3.2 of [35].

The following is a standard H^2 estimate, see for example [37], [44].

Lemma 3.3.1. *There exists constants $\varepsilon_0 = \varepsilon_0(N, g_N)$ and $C < \infty$ such that for every $g \in \mathcal{M}_c$, $r \in (0, \mathrm{inj}(M, g))$ and $u \in H^2(M, N)$ satisfying*

$$E(u, g; B_r) \leq \varepsilon_0$$

on some ball B_r we have

$$\int_{B_r} |\nabla_g^2 u|^2 dv_g \le C\left(r^{-2}E(u,g;B_r) + \|\tau_g(u)\|_{L^2(B_r,g)}\right).$$

Later on we will use the previous result together with the following which controls the evolution of the local energy. This result is from [37], the analogous results for the harmonic map flow can be found in [44] and [48].

Lemma 3.3.2. *Let (u,g) be a smooth solution of (3.2.3) on $[0,T)$ and let $\varphi \in C^\infty(M,[0,1])$ be a given function. The evolution of the cut-off energy*

$$E_\varphi(t) = \frac{1}{2}\int_M \varphi^2 |du(t)|^2_{g(t)} dv_{g(t)}$$

is controlled by

$$\begin{aligned} \left|\tfrac{d}{dt}E_\varphi + \int_M \varphi^2|\tau_g(u)|^2 d\mu_g\right| \le & 2\sqrt{2}E(u,g)^{\frac{1}{2}}\|d\varphi\|_{L^\infty(M,g)}\left(\int_M \varphi^2|\tau_g(u)|^2 d\mu_g\right)^{\frac{1}{2}} \\ & + \|\partial_t g\|_{L^\infty(supp(\varphi),g)} E_\varphi. \end{aligned} \tag{3.3.9}$$

3.4 General theory

In this section we discuss what is known about the Teichmüller harmonic map flow with respect to existence, uniqueness and singularity formation. These results are from a series of papers [18, 30, 34, 36–38] in which the theory for the flow was established.

The following notion of weak solution to the Teichmüller harmonic map flow (3.2.3) was given in [30]:

Definition 3.4.1. We call $(u,g) \in H^1_{\text{loc}}(M \times [0,T)) \times C^0([0,T), \mathcal{M}_c)$ a weak solution of (3.2.3) on an interval $[0,T)$, $T \le \infty$, if u satisfies the first equation in the sense of distributions and g is piecewise C^1 (as a map from $[0,T)$ into the space of symmetric $(0,2)$-tensors) and satisfies the second equation away from times where it is not differentiable.

The existence of weak solutions, at least until such time as the metric degenerates, was established in [30]:

Theorem 3.4.2. *Given initial data $(u_0,g_0) \in C^\infty(M,N) \times \mathcal{M}_c$ there exists a weak solution (u,g) of (3.2.3) defined on a maximal time interval $[0,T)$, $T \le \infty$. If $T < \infty$ then the flow degenerates in moduli space in the sense that $\lim_{t\nearrow T} \operatorname{inj}(M,g(t)) = 0$. The solution is smooth away from finitely many singular times $T_i \in (0,T)$ where energy concentrates at finitely many points $S(T_i) \subset M$. As $t \nearrow T_i$ the maps converge weakly in H^1 and smoothly away from the points $S(T_i)$. The metrics converge smoothly to an element $g(T_i) \in \mathcal{M}_c$. The energy $t \mapsto E(u(t),g(t))$ is non-increasing. Furthermore, the solution is uniquely determined by its initial data in the class of weak solutions with non-increasing energy.*

The structure of the proof is as follows. Given smooth initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_c$ one finds that a smooth solution exists at least on a small time interval $[0, T_1)$, $T_1 = T_1(u_0, g_0) > 0$. If $T_1 < \infty$ then let $T \geq T_1$ denote the maximal time for which a smooth solution exists. In the case that $\limsup_{t \nearrow T} \operatorname{inj}(M, g(t)) = 0$, i.e. when the metric degenerates, the maximal time of existence in Theorem 3.4.2 is finite. If $\limsup_{t \nearrow T} \operatorname{inj}(M, g(t)) > 0$ then the metrics $g(t)$ converge smoothly to a limit $g(T_1) \in \mathcal{M}_c$. In this case one can modify the arguments of Struwe [44] to give a bubbling analysis for the map component of the flow yielding a weak solution of the flow until such time as the metric degenerates.

It was shown in [37] that the flow can be extended beyond a finite time degeneration of the metric as a finite collection of flows, each from a domain of simpler topology. The major difficulty of their work is showing that this continuation can be done in such a way that reflects the flow just before the degeneration and they need a very refined description of the flow as it approaches the singular time.

Remark 3.4.3. In the spirit of the result of Eells-Sampson for the harmonic map flow [10], if the target (N, g_N) has non-positive sectional curvature then given smooth initial data there exists a smooth solution of the Teichmüller harmonic map flow existing for all times [36]. This result does not rule out degeneration of the metric at infinite time, however.

With the aforementioned existence theory in mind, the next natural question is about the asymptotics of the flow. The long time behaviour of the flow was analysed in [9] when $\gamma = 1$ and in [18, 34, 38] when $\gamma \geq 2$. For solutions to the flow which do not degenerate at infinite time the following result holds:

Theorem 3.4.4 (Follows from [9], Theorem 4.1 and [34], Theorem 1.4)**.** *For initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_c$ let (u, g) denote the unique solution given by Theorem 3.4.2 (or by Theorem 3.2 of [9] when $\gamma = 1$). Suppose that the length $\ell(t)$ of the shortest closed geodesic in $(M, g(t))$ is uniformly bounded below by a positive constant. Then there exists a sequence of times $t_i \to \infty$, a sequence of orientation preserving diffeomorphisms $f_i \colon M \to M$, a metric $\bar{g} \in \mathcal{M}_c$, a weakly conformal harmonic map $\bar{u} \colon (M, \bar{g}) \to (N, g_N)$ and finite set of points $S \subset M$ such that*

1. *$f_i^* g(t_i) \to \bar{g}$ smoothly*
2. *$u(t_i) \circ f_i \rightharpoonup \bar{u}$ weakly in $H^1(M)$*
3. *$u(t_i) \circ f_i \to \bar{u}$ strongly in $W^{1,p}_{loc}(M \setminus S)$ for any $p \in [1, \infty)$.*

Since it is relevant for our later work, we will give a brief sketch of the proof of this result. We focus on the case that $\gamma \geq 2$ and follow the proof of [34]. The case that $\gamma = 1$ uses some slightly different ingredients, see [9].

In the proof the following version of the Mumford Compactness theorem will be used, see Appendix C of [49]:

Theorem 3.4.5. *Let M be a closed oriented surface of genus $\gamma \geq 2$ and $(g_i)_{i=1}^{\infty} \subset \mathcal{M}_{-1}$ be a sequence of metrics such that*

$$\inf_{i \in \mathbb{N}} \operatorname{inj}(M, g_i) > 0.$$

Then there exists a sequence of orientation preserving diffeomorphisms $f_i \colon M \to M$ such that $f_i^* g_i$ converge smoothly along a subsequence to a limit $\bar{g} \in \mathcal{M}_{-1}$.

The following uniform Poincaré estimate for quadratic differentials due to Rupflin and Topping [33] will also be used (see also [34] for a simple proof for metrics in a compact subset of moduli space):

Theorem 3.4.6. *Let M be a closed oriented surface of genus $\gamma \geq 2$. There exists a constant $C < \infty$ depending only on γ such that for every $g \in \mathcal{M}_{-1}$ and any quadratic differential Ψ on M,*

$$\|\Psi - P_g(\Psi)\|_{L^1(M,g)} \leq C \|\bar{\partial} \Psi\|_{L^1(M,g)}, \tag{3.4.1}$$

where P_g denotes the projection onto the space of holomorphic quadratic differentials with respect to g and $\bar{\partial}$ is the anti-holomorphic derivative which, if $\Psi = \psi \mathrm{d}z^2$ in local complex coordinates z is given by $\bar{\partial}\Psi = \partial_{\bar{z}} \psi \mathrm{d}\bar{z} \otimes \mathrm{d}z^2$.

Proof of Theorem 3.4.4. We follow the proof given in [34]. Integrating the energy decay formula (3.3.1) gives

$$\int_0^\infty \int_M |\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |\mathrm{Re}(P_g(\Phi(u,g)))|^2 dv_g < \infty. \tag{3.4.2}$$

Therefore there exists a sequence of times $t_i \to \infty$ such that

$$\|\tau_g(u)(t_i)\|_{L^2(M,g(t_i))} \to 0, \quad \|P_g(\Phi(u,g))(t_i)\|_{L^2(M,g(t_i))} \to 0. \tag{3.4.3}$$

The lower bound on the injectivity radius and the Mumford compactness theorem, Theorem 3.4.5, imply that one can pass to a further subsequence (not relabelled) and find a sequence of orientation preserving diffeomorphisms $f_i \colon M \to M$ such that

$$f_i^* g(t_i) \to \bar{g} \text{ smoothly} \tag{3.4.4}$$

to an element $g \in \mathcal{M}_c$. Write $(u_i, g_i) = (u(t_i) \circ f_i, f_i^* g(t_i))$. Convergence of the metrics implies that the maps u_i are bounded in $H^1(M)$ (with respect to $\bar{g}$) since $E(u_i, \bar{g}) \leq CE(u_i, g_i) \leq$

$CE(u_0, g_0)$ and therefore that they converge (again, after passing to a subsequence) weakly in $H^1(M)$ to a limit $\bar{u} \in H^1(M, N)$.

Despite only having convergence of the projected Hopf differential to zero it can be shown that the whole Hopf differential converges to zero in L^1. Indeed, the Poincaré estimate Theorem 3.4.6, together with the formula (3.1.7) gives

$$\begin{aligned}\|\Phi(u_i, g_i)\|_{L^1(M,g_i)} &\leq C\|\bar{\partial}\Phi(u_i, g_i)\|_{L^1(M,g_i)} + \|P_{g_i}(\Phi(u_i, g_i))\|_{L^1(M,g_i)} \\ &\leq C\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)} E(u_i, g_i)^{\frac{1}{2}} + C\|P_{g_i}(\Phi(u_i, g_i))\|_{L^2(M,g_i)}.\end{aligned}$$

This implies that the whole Hopf differential converges to zero in L^1 using (3.4.3). This will also imply with the convergence obtained below that $\Phi(\bar{u}, \bar{g}) = 0$.

The final ingredient is to show that the limit $(\bar{u}, \bar{g})$ is indeed a weakly conformal harmonic map. In light of the H^2 estimate from Lemma 3.3.1 define the concentration set

$$S = \{x \in M \colon \text{ for any neighbourhood } U \text{ of } x, \limsup_{i\to\infty} E(u_i, g_i; U) \geq \varepsilon_0\},$$

possibly passing to a subsequence to ensure that S is finite. By the H^2 estimate and convergence of the tension (3.4.3) each u_i is uniformly bounded in H^2 in a small ball around each point in $M \setminus S$. Repeating this for each point gives weak $H^2_{\text{loc}}(M \setminus S, N)$ convergence along a subsequence of u_i to a limit which is weakly conformal with respect to $\bar{g}$ and also coincides with $\bar{u}$. The convergence also gives that the limit $\bar{u} \colon (M, \bar{g}) \to (N, g_N)$ is harmonic on $M \setminus S$ and therefore harmonic on all of M by the removable singularity theorem of Sacks-Uhlenbeck [39]. $\square$

Remark 3.4.7. If there is no concentration of energy along this sequence of times then the convergence of the map is smooth.

An important class of initial maps u_0 for which there can be no degeneration of the metric are *incompressible maps.*

Definition 3.4.8. A map $u_0 \colon M \to N$ is called *incompressible* if it is continuous, homotopically non-trivial and its action on the fundamental group is injective.

That there can be no degeneration of the Teichmüller harmonic map flow for such initial data was established in [34]. They show this using the following bound, combined with the fact the energy is non-decreasing: If $u \colon M \to N$ is incompressible and $g \in \mathcal{M}_{-1}$ then

$$E(u, g) \geq \frac{\varphi(\ell)}{\ell} L_{g_N}^2$$

where ℓ is the length of the shortest closed geodesic in (M, g), L_{g_N} is the length of the shortest closed geodesic in (N, g_N) and $\varphi(\ell) := (\pi/2 - \arctan\sinh\frac{\ell}{2}) \to \pi/2$ as $\ell \to 0$.

Finally, it is natural to ask what the asymptotic behaviour is like when the metric degenerates at infinite time, $\liminf_{t\to\infty} \mathrm{inj}(M, g(t)) = 0$. It was shown in [38] that one also obtains convergence to a minimal object after passing to a subsequence and suitably modifying by diffeomorphisms. The surfaces (M, g) converge to a hyperbolic punctured surface obtained by collapsing some number of simple closed geodesics. The maps converge weakly in H^1_{loc} and strongly away from points where energy concentrates and the limiting map extends to a branched minimal immersion from the surface obtained from filling in the punctures. Furthermore it was shown in [18] that, accounting for bubbles lost down degenerating collars, one can obtain an energy identity and a form of bubble tree convergence.

3.5 The Teichmüller harmonic map flow on T^2

In this section we give some background for the Teichmüller harmonic map flow in the case that the domain is the torus T^2. In this case the flow is equivalent to one which was first studied by Ding-Li-Liu [9].

The situation in this case simplifies and the flow can be viewed as the harmonic map flow coupled with an ODE for two extra parameters. Part of the reason for this simplification is the simple structure of the Teichmüller space of the torus. Recall that the Teichmüller space of the torus (or more generally, any closed surface) is the quotient of the set of conformal structures by the action of pullback by diffeomorphisms which are homotopic to the identity. Each conformal class is represented by a metric $g \in \mathcal{M}_0$ and so the Teichmüller space can be thought of as a quotient of the space $\mathcal{M}_0$. The Weil-Petersson metric on Teichmüller space is the metric inherited from the L^2 inner product on $\mathcal{M}_0$.

The Teichmüller space of the torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$ can be identified with the upper half plane $\mathbb{H} := \{(a, b) \in \mathbb{R}^2 \colon b > 0\}$ as follows. Denote by $g_E \in \mathcal{M}_0$ the metric inherited from the Euclidean metric on $\mathbb{R}^2$. Given $(a, b) \in \mathbb{H}$ define $T_{a,b} \colon \mathbb{R}^2 \to \mathbb{R}^2$ to be the unique linear map which sends

$$(1, 0) \mapsto \frac{1}{\sqrt{b}}(1, 0), \quad (0, 1) \mapsto \frac{1}{\sqrt{b}}(a, b).$$

Set $g_{a,b} = T^*_{a,b} g_E \in \mathcal{M}_0$ and note that in the standard coordinates on T^2

$$g_{a,b} = \frac{1}{b}\begin{pmatrix} 1 & a \\ a & a^2 + b^2 \end{pmatrix}. \tag{3.5.1}$$

It can be shown that given any $g \in \mathcal{M}_0$ there is a unique $f \colon T^2 \to T^2$ and $(a, b) \in \mathbb{H}$ such that $f^* g = g_{a,b}$. This identifies the Teichmüller space of the torus with $\mathbb{H}$ using the subset of metrics

$$\mathcal{M}^* := \{g_{a,b} = T^*_{a,b} g_E \colon (a, b) \in \mathbb{H}\}. \tag{3.5.2}$$

Moreover, a short computation of the L^2 metric on $\mathcal{M}^*$ actually shows that $d_{\mathrm{WP}} = 2d_{\mathbb{H}}$ where $d_{\mathbb{H}}$ is the distance coming from the Riemannian metric $g_{\mathbb{H}} = \frac{1}{b^2}(da^2 + db^2)$.

In [9] they define their flow as a modified gradient flow of the energy on the space of pairs of maps and metrics in $\mathcal{M}^*$. Their modification amounts to considering the gradient flow of the energy on $\mathcal{M}^*$ with the Weil-Petersson metric. By direct computation, the energy of a map $u\colon (T^2, g_{a,b}) \to (N, g_N)$ is given by

$$E(u, g_{a,b}) = \frac{1}{2b}\int_{T^2}(a^2 + b^2)|u_x|^2 + |u_y|^2 - 2a\langle u_x, u_y\rangle dxdy. \tag{3.5.3}$$

A short computation gives that the (negative) L^2 gradient flow of E with respect to the Weil-Petersson metric is

$$\partial_t u = \tau_{g_{a,b}}(u), \tag{3.5.4}$$
$$\dot{a} = -b\int_{T^2} a|u_x|^2 - \langle u_x, u_y\rangle dxdy, \tag{3.5.5}$$
$$\dot{b} = -\frac{1}{2}\int_{T^2}(b^2 - a^2)|u_x|^2 + 2a\langle u_x, u_y\rangle - |u_y|^2 dxdy. \tag{3.5.6}$$

The above equations are of course connected with the Teichmüller harmonic map flow, this is described in detail in [34]. The space $\mathcal{M}^*$ is a *horizontal submanifold*: its tangent space at each points consists of the space of real parts of holomorphic quadratic differentials. Thus, the metric component $g(t)$ of a solution to the Teichmüller harmonic map flow, being a horizontal curve (so its tangent is always given by the real part of a holomorphic quadratic differential), is constrained to $\mathcal{M}^*$ as long as $g(0) \in \mathcal{M}^*$. Thus one can view a solution to the Teichmüller harmonic map flow with $g(0) \in \mathcal{M}^*$ as a map $u\colon T^2 \to (N, g_N)$ together with a metric $g(t) = g_{a(t),b(t)}$ for $(a, b) \in \mathbb{H}$. It only remains to see that the flow equations (3.2.3) coincide with those obtained above for the gradient flow of the energy on $\mathcal{M}^*$ with respect to the Weil-Petersson metric. This follows from a short computation, the easiest way to see this is to pull back the map u and metric $g_{a,b}$ by $T_{a,b}^{-1}$ so that the metric is always given by g_E which is diagonal.

To understand the metric component of the flow (3.2.3) on tori it is enough to understand the flow on $\mathcal{M}^*$. Indeed, given any initial data $(u_0, g_0) \in C^\infty(T^2, N) \times \mathcal{M}_0$ we can always find some $f\colon T^2 \to T^2$ such that $f^*g_0 \in \mathcal{M}^*$, as stated above. Then the pulled back flow $f^*(u, g)$ has metric component constrained to $\mathcal{M}^*$ since the space $\mathcal{M}^*$ is a horizontal submanifold.

A solution to the Teichmüller harmonic map flow from the torus which has finite energy cannot degenerate in finite time, this was observed already in [9]. This is an instance of the completeness of the Teichmüller space of the torus: the metric component of the flow has finite length due to (3.3.4) and so must remain in a compact subset of $\mathcal{M}^*$.

Proposition 3.5.1 (Corollary 2.3, [9]). *Let (u, g) be a solution to the flow (3.2.3) from the torus T^2. Then $\mathrm{inj}(T^2, g(t))$ is positive on any finite time interval.*

As is the case for surfaces of genus at least 2 there can be no degeneration of the Teichmüller harmonic map flow if the initial map $u\colon T^2 \to N$ is incompressible. Note that this is valid as long as there is a positive lower bound on the length of homotopically non-trivial closed curves in the target (N, g_N).

We will need later some elementary facts about the identity map $\mathrm{id}\colon (T^2, g) \to (T^2, G)$ between tori equipped with metrics $g, G \in \mathcal{M}^*$. for completeness we include some ideas of the proofs.

Lemma 3.5.2. *If $g, G \in \mathcal{M}^*$ then the following hold:*

1. *The identity map $\mathrm{id}\colon (T^2, g) \to (T^2, G)$ is always harmonic.*
2. *The identity map $\mathrm{id}\colon (T^2, g) \to (T^2, G)$ is conformal if and only if $g = G$.*

Proof. The first statement follows since we can pull-back both metrics by appropriate linear maps so that they are given by the Euclidean metric (on possibly different domains). The identity map is pulled back to an affine map which is certainly a harmonic map between Euclidean domains.

It is immediate that the identity map between different flat unit area tori is conformal. For the converse, we note that metrics in $\mathcal{M}^*$ are unique in their conformal class. $\square$

Moreover, we can control the Weil-Petersson distance between two elements of $\mathcal{M}^*$ by the energy of the identity map between them:

Lemma 3.5.3. *There exists $C < \infty$ such that for any $g, G \in \mathcal{M}^*$ we have*

$$d_{\mathrm{WP}}(g, G) \leq C(\mathcal{E}(g, G) - 1)^{\frac{1}{2}} \tag{3.5.7}$$

where $\mathcal{E}(g, G) = E(\mathrm{id}\colon (T^2, g) \to (T^2, G))$ and d_{WP} is the Weil-Petersson distance.

Proof. Since $g, G \in \mathcal{M}^*$ we can write $g = g_{a,b}$ and $G = g_{\alpha,\beta}$ for some $(a, b), (\alpha, \beta) \in \mathbb{H}$, as in (3.5.2). The energy of the identity map written in terms of these parameters is given by

$$\mathcal{E}(g_{a,b}, g_{\alpha,\beta}) = \frac{1}{2}\int_{T^2} \mathrm{tr}_{g_{a,b}}(g_{\alpha,\beta}) dv_{g_{a,b}} = 1 + \frac{1}{2b\beta}(a-\alpha)^2 + \frac{1}{2b\beta}(b-\beta)^2, \tag{3.5.8}$$

see (3.5.3). This formula connects the quantity $\mathcal{E}$ to the hyperbolic distance $d_{\mathbb{H}}((a, b), (\alpha, \beta))$ on the upper half plane $\mathbb{H}$, indeed we find that $\mathcal{E}(g_{a,b}, g_{\alpha,\beta}) = \cosh(d_{\mathbb{H}}(g_{a,b}, g_{\alpha,\beta}))$. The Weil-Petersson metric on $\mathcal{M}^*$ is a multiple of the hyperbolic distance, $d_{\mathrm{WP}} = 2d_{\mathbb{H}}$, and so

$$d_{\mathrm{WP}}(g, G) = 2\mathrm{arcosh}(\mathcal{E}) \leq 2\sqrt{2}(\mathcal{E} - 1)^{\frac{1}{2}}$$

using that $\mathrm{arcosh}(x) \leq \sqrt{2}(x-1)^{\frac{1}{2}}$ for all $x > 1$. $\square$

Chapter 4

Uniqueness and Non-uniqueness of Limits of Teichmüller Harmonic Map Flow

In this chapter we present our original results from joint with with Rupflin and Topping [21]. We answer a number of questions about the Teichmüller Harmonic Map Flow (1.3.1) which is a gradient flow for the Harmonic Map Energy of maps $u\colon (M,g) \to (N,g_N)$ where both the map and domain metric evolve. We explain the background of the flow in Chapter 3.

4.1 Statement of results

The starting point is the following result of [34], stated in Theorem 3.4.4 above. It states that in situations where the metric does not degenerate as $t \to \infty$ there exists a sequence of times $t_i \to \infty$, orientation preserving diffeomorphisms $f_i\colon M \to M$, a metric $\bar{g} \in \mathcal{M}_c$ and a weakly conformal harmonic map $\bar{u}\colon (M,\bar{g}) \to (N,g_N)$ such that $f_i^* g(t_i) \to \bar{g}$ smoothly and $u(t_i) \circ f_i \rightharpoonup \bar{u}$ weakly in H^1 and strongly away from finitely many points. We ask here whether can have winding behaviour in this situation. Precisely, does the limit $(\bar{u},\bar{g})$ depend on the sequence of times above? Secondly, is it necessary to pullback by diffeomorphisms, or does the flow converge without them? After this, we will investigate the situation when (N,g_N) is real analytic.

We first construct smooth (but not analytic) settings where winding behaviour of the domain metric does indeed occur even for initial data for which the map component remains smooth for all time, and at infinite time, and for which we have a uniform lower bound on the injectivity radius of the domain.

Here we consider the flow on tori, see Section 3.5. Recall that in this case the analysis of the flow is simplified by the fact that not only does the velocity $\frac{\partial g}{\partial t}$ lie in a two-dimensional subspace of the infinite dimensional space $T\mathcal{M}_0$, but also the distribution defined by these "horizontal" subspaces is integrable. Indeed, by pulling back the initial data and the whole flow by a fixed diffeomorphism, it suffices to consider the flow of metrics in the explicit two-parameter family of flat unit area metrics

$$\mathcal{M}^* := \{g_{a,b} = T_{a,b}^* g_E : (a,b) \in \mathbb{H}\} \tag{4.1.1}$$

on $T^2 := \mathbb{R}^2/\mathbb{Z}^2$, where $T_{a,b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map sending $(1,0) \mapsto \frac{1}{\sqrt{b}}(1,0)$ and $(0,1) \mapsto \frac{1}{\sqrt{b}}(a,b)$, g_E is the Euclidean metric and $\mathbb{H}$ is the upper-half plane. The Weil-Petersson metric on $\mathcal{M}^*$ then corresponds, up to a scaling factor, to the hyperbolic distance on $\mathbb{H}$.

In our first main result of the Chapter we construct smooth targets so that the flow of the metrics $g(t)$ has a prescribed asymptotic behaviour:

Theorem 4.1.1. *Let $(G_s)_{s\in[0,\infty)}$ be any smooth curve in $\mathcal{M}^*$ whose projection to moduli space is 1-periodic, i.e. for which there exists a diffeomorphism $\varphi\colon T^2 \to T^2$ so that $G_s = \varphi^* G_{s+1}$ for every $s \in [0,\infty)$.*

Then there exists a smooth closed target manifold (N, g_N) and initial data $(u_0, g_0) \in C^\infty(T^2, N) \times \mathcal{M}^*$ such that the corresponding solution $(u(t), g(t))$ of Teichmüller harmonic map flow has $\sup_t \|\nabla u(t)\|_{L^\infty(T^2,g(t))} < \infty$ and $\inf_t \operatorname{inj}(T^2, g(t)) > 0$, as well as the following asymptotic behaviour:

- *There exists a smooth function $z\colon [0,\infty) \to [0,\infty)$ satisfying $z(t) \to \infty$ as $t \to \infty$ so that*

$$d_{\mathrm{WP}}(g(t), G_{z(t)}) \to 0 \ \text{as } t \to \infty \tag{4.1.2}$$

  where d_{WP} is the Weil-Petersson distance on $\mathcal{M}^*$.

- *For every $z \in [0,1)$ there exists a sequence of times $t_i^z \to \infty$ so that after pulling back by the i^{th} iterate φ^i the maps $u(t_i^z)$ converge smoothly,*

$$u(t_i^z) \circ \varphi^i \to u_z, \tag{4.1.3}$$

  to a minimal immersion $u_z : T^2 \to N$. The resulting minimal immersions u_{z_1} and u_{z_2} with $z_1 \neq z_2$ parametrise different minimal surfaces in (N, g_N). After pulling back by φ^i the metrics $g(t_i^z)$ also converge

$$(\varphi^i)^* g(t_i^z) \to G_z \ \text{in } C^\infty.$$

Since $\mathcal{M}^*$ is a horizontal submanifold of the space of metrics, any diffeomorphism will pull it back to another horizontal submanifold. The specific diffeomorphism φ pulls back $G_1 \in \mathcal{M}^*$ to $G_0 \in \mathcal{M}^*$, and $\mathcal{M}^*$ is connected and complete, so φ must pull back $\mathcal{M}^*$ to $\mathcal{M}^*$

itself. In particular, given a solution of the flow as above, we know that $\varphi^* g(t) \in \mathcal{M}^*$ for each $t \in [0, \infty)$.

Theorem 4.1.1 immediately yields the following corollary about the possible behaviour of Teichmüller harmonic map flow into suitable smooth closed targets:

Corollary 4.1.2. *The limit of solutions of Teichmüller harmonic map flow as $t \to \infty$ can be non-unique, even after pull-back by diffeomorphisms.*

The diffeomorphisms φ^i that we use to pull back in (4.1.3) are iterated compositions of the diffeomorphism φ obtained from G_s. If G_s is a periodic curve in $\mathcal{M}^*$ then φ is the identity map, for example. A quite different situation arises in the case that G_s leaves any compact subset of Teichmüller space. By taking the family corresponding to twisting a torus around a simple closed geodesic, a twist by angle 2π representing a Dehn twist, we show the following, which answers the second question above:

Corollary 4.1.3. *There exist a closed target manifold and a solution of Teichmüller harmonic map flow from T^2 into that target whose metric component leaves any compact subset of Teichmüller space even though the injectivity radius of the domain remains bounded away from zero and hence the metric component remains in a compact subset of moduli space.*

When the curve G_s is a non-trivial closed curve we obtain the following:

Corollary 4.1.4. *There exist a closed target manifold and a solution of Teichmüller harmonic map flow from T^2 into that target for which the metric component stays in a compact subset of Teichmüller space but for which the limit as $t \to \infty$ is not unique. In particular we can choose sequences of times $t_i, \tilde{t}_i \to \infty$ so that $(u(t_i), g(t_i))$ and $(u(\tilde{t}_i), g(\tilde{t}_i))$ converge (without having to pull back by diffeomorphisms) to limits which parametrize different minimal surfaces.*

While the above results show that solutions of Teichmüller harmonic map flow into smooth closed targets can exhibit winding behaviour even in situations where the energy density is bounded uniformly from above and the injectivity radius from below, our second main result excludes this behaviour for analytic targets. Recall that one starts the analysis of the asymptotics of the flow by selecting a sequence of times $t_j \to \infty$ with $\|\partial_t(u, g)(t_j)\|_{L^2(M, g(t_j))} \to 0$ and then analyses the bubbling and/or collar degeneration singularities that might occur in the limit [18, 34, 38]. If there exists any such sequence for which no singularities occur, when the target is analytic, then we obtain uniform convergence:

Theorem 4.1.5. *Let (N, g_N) be a closed analytic manifold of any dimension and let M be a closed oriented surface of genus $\gamma \geq 1$. Let (u, g) be any global weak solution of Teichmüller* *harmonic map flow* (3.2.3)*, with nonincreasing energy, for which there is a sequence* $t_j \to \infty$ *such that*

$$\lim_{j\to\infty} \|\partial_t(u,g)(t_j)\|_{L^2(M,g(t_j))} = 0, \quad \sup_j \|\nabla u(t_j)\|_{L^\infty(M,g(t_j))} < \infty \text{ and } \inf_j \operatorname{inj}(M,g(t_j)) > 0. \tag{4.1.4}$$

Then $(u,g)(t)$ *converges smoothly as* $t \to \infty$ *to a limiting pair* (u_∞, g_∞) *consisting of a metric* $g_\infty \in \mathcal{M}_c$ *and a weakly conformal harmonic map* $u_\infty\colon (M,g_\infty) \to (N,g_N)$.

Remark 4.1.6. This result is proved using the Łojasiewicz-Simon inequality obtained in Theorem 4.3.1. This also yields a rate of convergence depending on the exponent $\alpha \in (0, \frac{1}{2}]$ in exactly the same way as described in Remark 2.1.4: the rate of convergence is exponential if $\alpha = \frac{1}{2}$ and a polynomial rate depending on α otherwise. Theorem 4.3.1 however does not give any information other than $\alpha \in (0, \frac{1}{2}]$ and so nothing more can be concluded than polynomial convergence in general.

Remark 4.1.7. The assumption in (4.1.4) can be weakened. It is enough to ask that there exist $r_1 > 0$ and $t_j \to \infty$ so that $\inf_j \operatorname{inj}(M, g(t_j)) > 0$ and so that we have uniform control on the energy on balls of radius r_1 of

$$\frac{1}{2}\int_{B_{r_1}^{g(t_j)}(x)} |du(t_j)|^2_{g(t_j)} dv_{g(t_j)} \leq \varepsilon_0 \text{ for every } x \in M \text{ and } j \in \mathbb{N} \tag{4.1.5}$$

where $\varepsilon_0 = \varepsilon_0(N, g_N) > 0$, since parabolic regularity theory and (3.3.1) would allow us to choose nearby times for which (4.1.4) holds, as can be derived from the proof of Theorem 4.1.5.

4.2 Winding behaviour: Proof of Theorem 4.1.1

Our set-up is inspired by the construction of Topping [48] for which solutions of harmonic map flow exhibit winding behaviour. Our first step here will be to construct a non-compact target manifold N_0 for which the metric component of the flow (3.2.3) closely follows the prescribed curve of metrics G_s as the image of the map drifts off to infinity. We will then wrap this non-compact target around a cylinder to obtain a smooth, closed target for which solutions wind around the cylinder and approach a circle of critical points.

The first, non-compact, target is given by $N_0 = \mathbb{R} \times T^2$ equipped with the metric

$$g_{N_0} = dz^2 + f_0(z) G_z \tag{4.2.1}$$

where G_s is the prescribed curve of metrics in $\mathcal{M}^*$ (extended to $(-\infty, 0]$ via $G_s = \varphi^* G_{s+1}$) and $f_0 \in C^\infty(\mathbb{R}, [1, \infty))$ is bounded with $-C \leq f_0' < 0$ and $\lim_{z\to\infty} f_0(z) = 1$.

We will consider maps $u\colon T^2 \to N_0$ whose first component is constant in space and whose second component is given by the identity map of the torus. The energy of such maps $u = (z, \mathrm{id})$ is given by

$$E(u, g) = f_0(z)\mathcal{E}(g, G_z) \text{ where } \mathcal{E}(g, G) := E(\mathrm{id}\colon (T^2, g) \to (T^2, G)), \tag{4.2.2}$$

for $g, G \in \mathcal{M}^*$ and we always have $\mathcal{E}(g, G) \geq \mathrm{Area}(T^2, G) = 1$, with equality if and only if $\mathrm{id} : (T^2, g) \to (T^2, G)$ is conformal. As metrics in $\mathcal{M}^*$ are unique in their conformal class we thus have

$$E((z, \mathrm{id}), g) \geq f_0(z) \text{ with equality if and only if } g = G_z. \tag{4.2.3}$$

Since the identity map $\mathrm{id}\colon (T^2, g) \to (T^2, G)$ is harmonic for any metrics $g, G \in \mathcal{M}^*$ (see Lemma 3.5.2), we have that for each $z \in \mathbb{R}$ the T^2 component of the tension of any map $u = (z, \mathrm{id})$ from (T^2, g) to (N_0, g_{N_0}) vanishes, while the first component of the tension is given by

$$\tau_g(u)^{\mathbb{R}} = -f_0(z) D_G \mathcal{E}_{(g,G_z)}(\tfrac{dG_z}{dz}) - f_0'(z)\mathcal{E}(g, G_z). \tag{4.2.4}$$

Remark 4.2.1. If $\bar{g} \in \mathcal{M}^*$ and $u = (z, \mathrm{id})\colon (T^2, \bar{g}) \to (N_0, g_{N_0})$ is conformal, then we have $\bar{g} = G_z$. Hence the first term on the right-hand side of (4.2.4) vanishes while the second term is always non-zero as $f_0' \neq 0$. We thus observe that there are no maps u of this form that are both harmonic and conformal.

Although our target N_0 is non-compact, and the existing theory for this flow is phrased for compact targets, the short-time existence for an initial map of the form $u_0 = (z_0, \mathrm{id})$ and any initial metric $g_0 \in \mathcal{M}^*$ will follow from simple ODE theory. Recall that the equation for the metric component $g(t) = g_{(a,b)(t)}$ reduces to a system of ODEs for $(a, b) \in \mathbb{H}$ given by (3.5.5), (3.5.6). Standard ODE theory gives short-time existence of solutions to the system consisting of the ODEs for $(a, b)(t)$ coupled with the equation for $z(t)$

$$\dot{z} = \tau_g(u)^{\mathbb{R}} = -f_0(z) D_G \mathcal{E}_{(g,G_z)}(\tfrac{dG_z}{dz}) - f_0'(z)\mathcal{E}(g, G_z), \tag{4.2.5}$$

where $g = g_{(a,b)}$. Therefore setting $u(t) = (z(t), \mathrm{id})$ and $g(t) = g_{(a,b)(t)}$ we obtain that (u, g) is a solution to the flow.

We will now show that the solution exists for all times and does not degenerate, as well as control the Weil-Petersson distance between the metric component of the flow and the prescribed curve of metrics G_s.

Lemma 4.2.2. *Let (N_0, g_{N_0}) be as in (4.2.1) with G_s the prescribed curve of metrics as in Theorem 4.1.1 and let $f_0 \in C^\infty(\mathbb{R}, [1, \infty))$ be bounded with $-C \leq f_0' < 0$ and $\lim_{z\to\infty} f_0(z) = 1$.*

Given any $z_0 \in \mathbb{R}$ and any $g_0 \in \mathcal{M}^*$ we have that the solution $(u, g) = ((z, \mathrm{id}), g)$ of the flow (3.2.3) *with initial map $u_0 = (z_0, \mathrm{id})$ and initial metric g_0 exists and is smooth for all times, and does not degenerate at infinite time in the sense that*

$$\mathrm{inj}(T^2, g(t)) \geq c > 0 \text{ for all } t \in [0, \infty). \tag{4.2.6}$$

Here c depends only on an upper bound for the initial energy and on the curve G_s. Furthermore, the Weil-Petersson distance between the metric component and the given curve of metrics G_s is controlled by

$$d_{\mathrm{WP}}(g(t), G_{z(t)}) \leq C(E(t) - 1)^{\frac{1}{2}} \text{ for every } t \in [0, \infty), \tag{4.2.7}$$

where $E(t) := E(u(t), g(t))$ and $C > 0$ is a universal constant.

Remark 4.2.1 excludes the possibility that this flow has stationary points $(\bar{u}, \bar{g})$ of the above form. This will later allow us to show that the map component of the flow drifts off to infinity while the energy tends to 1 as $t \to \infty$. By the above lemma this then yields that $g(t)$ and $G_{z(t)}$ have the same limiting behaviour.

Proof of Lemma 4.2.2. Let (u, g) be a solution of the flow as in the lemma and let $[0, T)$ be the maximal time interval on which the solution is defined and smooth. There can be no finite-time degeneration of the metric component owing to the completeness of the Teichmüller space of the torus T^2, see Proposition 3.5.1. To show that $T = \infty$ it hence remains to exclude the possibility that $|z(t)|$ goes to infinity in finite time, which we shall do at the end of the proof.

Since the initial map is incompressible and since G_s is smooth and periodic in moduli space, we have a uniform positive lower bound on the lengths of homotopically non-trivial closed curves in the target. Therefore, as remarked in Section 3.5, even though the target is non-compact we can conclude that the injectivity radius of (T^2, g) remains bounded from below by a positive constant $c > 0$ that only depends on the initial energy and the curve G_s.

We also note that (4.2.7) holds true on the maximal time interval $[0, T)$ as an immediate consequence of Lemma 3.5.3, the structure of the energy (4.2.2) and the fact that $f_0 \geq 1$.

It finally remains to exclude that $z(t)$ goes to infinity in finite time. By (3.3.4), we see that on any finite time interval the metric can only travel a finite distance. Therefore for any finite $T_1 \leq T$, we can use (4.2.7) to obtain a compact subset K of $\mathcal{M}^*$ so that $g(t) \in K$ and $G_{z(t)} \in K$ for every $t \in [0, T_1)$.

Hence (4.2.5), combined with the assumed bounds on f_0, yields that $|\dot{z}|$ is bounded on $[0, T_1)$ for any finite $T_1 \leq T$ and thus that the solution cannot blow up in finite time. $\square$

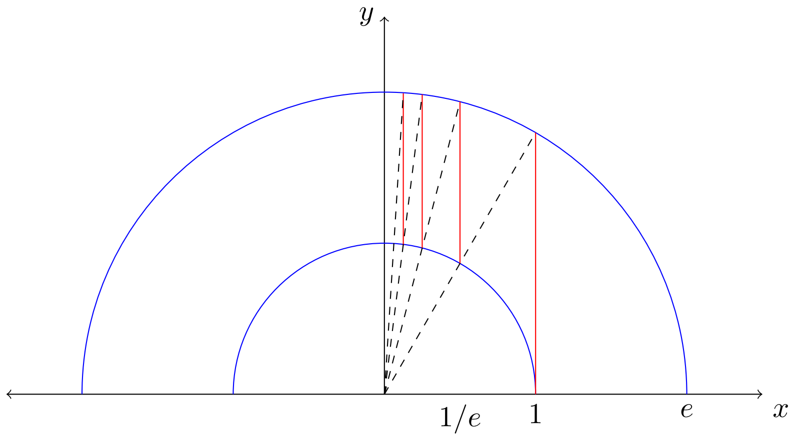

Figure 4.1: The action of Γ on the $\mathbb{H}$ component of N_1: points with the same angle on the blue circles are identified. The red lines represent the image of the geodesic L which spirals towards the set $x = 0$.

We now construct a new target which contains the previously constructed (N_0, g_{N_0}) as a totally geodesic submanifold. We first consider an auxiliary non-compact target defined as

$$N_1 = \mathbb{H} \times T^2, \quad g_{N_1} = g_{\mathbb{H}} + f(x,y)G_{\log y}, \tag{4.2.8}$$

where $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ is the upper half plane equipped with the standard hyperbolic metric $g_{\mathbb{H}} = \frac{1}{y^2}(dx^2 + dy^2)$, G_s is the prescribed curve of metrics from Theorem 4.1.1 (extended to $(-\infty, 0)$ by periodicity) and $f\colon \mathbb{H} \to [1,\infty)$ is a smooth function that will be determined below.

We will in particular choose f with the symmetry $f(ex, ey) = f(x,y)$. As $(x,y) \mapsto (ex, ey)$ is an isometry of the hyperbolic plane and as G_s satisfies $G_s = \varphi^* G_{s+1}$ this ensures that

$$\Psi\big((x,y),p\big) = \big((ex,ey),\varphi(p)\big) \tag{4.2.9}$$

is an isometry of (N_1, g_{N_1}). Denoting by $\Gamma = \langle \Psi \rangle$ the generated group of isometries of (N_1, g_{N_1}) we consider the target $(N_1, g_{N_1})/\Gamma$. The final target will be a slight modification of this in order to make it compact.

For appropriate f, the function $f_0(z) := f(1, e^z)$ will induce an associated manifold (N_0, g_{N_0}) as in (4.2.1). We can then see this as a submanifold of (N_1, g_{N_1}) by identifying $(z,p) \in \mathbb{R} \times T^2$ with $((1,y),p) \in \mathbb{H} \times T^2$ for $z = \log y$. As $L := \{1\} \times (0,\infty) \subset \mathbb{H}$ is the image of a geodesic in the hyperbolic plane we obtain that the submanifold is totally geodesic provided $\partial_x f(1,y) = 0$ for every $y \in (0,\infty)$.

We therefore choose our coupling function f as follows. Using the embedding and identification described above the function $1 + e^{-y/x}$ leads to a function f_0 to which Lemma 4.2.2 applies. We will modify the function f so that the x-derivative vanishes on the line

L, without changing the corresponding function f_0. Choose a function $\rho \in C^\infty(\mathbb{R})$ with $\rho(n) = n$ for each $n \in \mathbb{Z}$, that is constant in a neighbourhood of each integer and satisfies $\rho(s+1) = \rho(s) + 1$; we think of ρ as an approximation of the function $s \mapsto s$. Now we define $f : \mathbb{H} \to [1, \infty)$ by

$$f(x,y) = \begin{cases} 1 + \exp(-ye^{-\rho(\log x)}) & \text{if } x > 0 \\ 1 & \text{if } x \leq 0. \end{cases} \tag{4.2.10}$$

The properties of ρ imply that f has the invariance $f(ex, ey) = f(x, y)$. A computation of the derivatives of f shows that it is smooth and satisfies $\partial_x f(1, y) = 0$ for every $y \in (0, \infty)$.

Given initial data $y_0 \in (0, \infty)$ and $g_0 \in \mathcal{M}^*$ we set $z_0 = \log y_0$ and consider the corresponding solution of the flow into (N_0, g_{N_0}) defined by (4.2.1) for $f_0(z) := f(1, e^z)$. As (N_0, g_{N_0}) is a totally geodesic submanifold of (N_1, g_{N_1}) this induces a solution of the flow with target (N_1, g_{N_1}). This solution can be projected down to a solution of the flow with target $(N_1, g_{N_1})/\Gamma$ and initial data (u_0, g_0) where $u_0 = (1, y_0, \mathrm{id})$. We therefore find that all of the conclusions of Lemma 4.2.2 hold in this setting, in particular the solution $(u(t), g(t))$ does not degenerate, has $u(t) = (1, y(t), \mathrm{id})$ with $y(t)$ constant in space and the metric component satisfies the estimate

$$d_{\mathrm{WP}}(g(t), G_{\log y(t)}) \leq C(E(t) - 1)^{\frac{1}{2}}. \tag{4.2.11}$$

By restricting to initial data (u_0, g_0) with energy $E(u_0, g_0) \leq f_0(0)$, e.g. by considering

$$(u_0, g_0) = ((1, y_0, \mathrm{id}), G_{\log y_0}), \quad y_0 \geq 1, \tag{4.2.12}$$

we can ensure that $y(t) \geq 1$ along the flow, see (4.2.3), and hence that all solutions of the flow under consideration remain in a fixed compact subset of $(N_1, g_{N_1})/\Gamma$. Hence we can modify $(N_1, g_{N_1})/\Gamma$ away from this compact subset to obtain a closed target manifold (N, g_N) without changing the solutions of the flow from these initial data.

We now show that these solutions have the winding properties claimed in the theorem.

Proof of Theorem 4.1.1. Let (N, g_N) be the closed target as defined above and let (u, g), $u = ((1, y), \mathrm{id})$ be a solution of the flow from initial data (u_0, g_0) as in (4.2.12). We are free to view this flow as mapping into $(N_1, g_{N_1})/\Gamma$ or (N_1, g_{N_1}) as is convenient. Because of (3.3.3) we can choose a sequence $t_j \to \infty$ so that

$$\|\partial_t(u, g)(t_j)\|_{L^2(M, g(t_j))} \to 0 \quad \text{as } j \to \infty. \tag{4.2.13}$$

We first claim that $y(t_j) \to \infty$. If this were not true then $y(t_j)$ would have a bounded subsequence as $y(t) \geq 1$, so by passing to a further subsequence we could ensure that $y(t_j) \to \tilde{y}$ for some finite $\tilde{y}$. In particular, the map $\tilde{u} := ((1, \tilde{y}), \mathrm{id})$ would be a smooth limit of the maps $u(t_j)$. Next, by (4.2.11) we know that the metrics $g(t_j)$ must remain a bounded distance from $G_{\log(y(t_j))} \to G_{\log(\tilde{y})}$, and are hence contained in a compact subset of $\mathcal{M}^*$. Thus we may pass to a further subsequence and obtain smooth convergence of $g(t_j)$ to some limit metric $\tilde{g}$. By (4.2.13), we can deduce that $(\tilde{u}, \tilde{g})$ is a stationary point, which contradicts Remark 4.2.1, completing the proof of the claim.

For each $t \in [0, \infty)$, let $n(t)$ be the unique nonnegative integer so that $\hat{y}(t) = y(t)e^{-n(t)} \in [1, e)$. By the claim above, we have $n(t_j) \to \infty$ and after passing to a subsequence, we may assume that $\hat{y}(t_j) \to \bar{y}$ for some $\bar{y} \in [1, e]$. As the map component of the flow can be represented as

$$u(t) = ((1, y(t)), \mathrm{id}) \sim ((e^{-n(t)}, \hat{y}(t)), \varphi^{-n(t)})$$

we thus obtain that the pulled back maps $u(t_j) \circ \varphi^{n(t_j)} : T^2 \to (N_1, g_{N_1})/\Gamma$ converge smoothly to $\bar{u} = ((0, \bar{y}), \mathrm{id})$.

According to (4.2.11) the pulled-back metrics $(\varphi^{n(t_j)})^* g(t_j) \in \mathcal{M}^*$ remain at a finite distance from $(\varphi^{n(t_j)})^* G_{\log(y(t_j))} = (\varphi^{n(t_j)})^* G_{\log(\hat{y}(t_j)) + n(t_j)} = G_{\log(\hat{y}(t_j))} \to G_{\log(\bar{y})}$ and are hence contained in a compact subset of $\mathcal{M}^*$. After passing to a further subsequence, we thus obtain that $(\varphi^{n(t_j)})^* g(t_j) \to \bar{g}$ for some $\bar{g} \in \mathcal{M}^*$.

By (4.2.13), the obtained limit $(\bar{u}, \bar{g})$ must be a stationary point of the flow. In particular $\bar{u}$ must be conformal and hence we must have $\bar{g} = G_{\log \bar{y}}$. As the energy decreases along the flow we conclude that

$$\lim_{t \to \infty} E(t) = E(\bar{u}, \bar{g}) = f(0, \bar{y}) = 1. \tag{4.2.14}$$

As $E(t) \geq f(1, y(t)) \geq 1$ we must therefore have that $f(1, y(t)) \to 1$ and so $y(t) \to \infty$ as $t \to \infty$. Combining (4.2.14) with (4.2.11) also yields the claimed convergence of metrics (4.1.2) for $z(t) = \log y(t)$, which tends to infinity as $t \to \infty$.

It remains to prove the second part of the theorem. So let $z \in [0, 1)$ be any fixed number. As $z(t) \to \infty$ as $t \to \infty$ we can pick times $t_j^z \to \infty$ (for sufficiently large j) such that $z(t_j^z) = z + j$. As above we conclude that $(\varphi^j)^*(u(t_j^z), g(t_j^z)) \to (u_z, G_z)$ where $u_z = (0, e^z, \mathrm{id})$. We finally observe that for $z_1 \neq z_2$ these maps are minimal immersions with disjoint images and so parametrise different minimal surfaces in (N, g_N). □

4.3 A Łojasiewicz-Simon inequality for the harmonic map energy on closed surfaces

The key step in proving uniqueness of limits of the flow for analytic targets will be establishing a suitable Łojasiewicz-Simon inequality, recalling that the flow is the gradient flow of the functional E on the set of equivalence classes of maps and metrics under the action of pulling back by diffeomorphisms homotopic to the identity, see Section 3.2. In this section we will prove a Łojasiewicz-Simon inequality for the Harmonic Map Energy as a functional of maps and metrics. We gave a discussion of Łojasiewicz-Simon inequalities in Chapter 2. We will use the general formulation of these inequality stated in Theorem 2.2.4, but we need to do some extra work related to the coupling between the map and metric. We will use the structure of the Banach manifold $\mathcal{M}^s_{-1}$ of hyperbolic metrics with coefficients in the Sobolev space of H^s as discussed in [49] and described below. We write $\mathcal{M}^s_c$ for the space of metrics of constant curvature $c \in \{-1, 0\}$ with coefficients in the Sobolev space H^s (with respect to an arbitrary, fixed metric).

The statement is as follows:

Theorem 4.3.1. *Let (N, g_N) be a closed real analytic manifold, let M be a closed oriented surface of genus $\gamma \geq 1$, and let $(\bar{u}, \bar{g})$, $\bar{g} \in \mathcal{M}_c$, be a critical point of the Dirichlet energy $E(u, g)$. Then for any $s > 3$ there is a neighbourhood $\mathcal{O}$ of $(\bar{u}, \bar{g})$ in $H^s(M, N) \times \mathcal{M}^s_c$, $\alpha \in (0, \frac{1}{2})$ and $C < \infty$ such that for any $(u, g) \in \mathcal{O}$ we have*

$$|E(u,g) - E(\bar{u}, \bar{g})|^{1-\alpha} \leq C \left(\|\tau_g(u)\|^2_{L^2(M,g)} + \|P_g(\Phi(u,g))\|^2_{L^2(M,g)} \right)^{\frac{1}{2}}, \tag{4.3.1}$$

where P_g is the $L^2(M, g)$-orthogonal projection from the space of quadratic differentials to the space $\mathcal{H}(g)$ of holomorphic quadratic differentials.

Remark 4.3.2. Since we work in dimension 2 the assumption of H^s closeness to a critical point, $s > 3$, implies the usual assumption of $C^{2,\beta}$ closeness for some $\beta > 0$ that appears in Theorem 1.2.1 and many versions of the Łojasiewicz-Simon inequality.

The simpler, analogous result for $\gamma = 0$, where the final term of (4.3.1) is always zero because all holomorphic quadratic differentials are trivial, is already included in the work of Simon.

For the proof we will focus on the case of surfaces of genius $\gamma \geq 2$, the case of $\gamma = 1$ will be discussed later in Remark 4.3.4. For surfaces of genus $\gamma \geq 2$ we will first consider the special case of (not necessarily hyperbolic) metrics that are contained in a finite-dimensional affine space, then conclude that the desired inequality holds for metrics in a finite-dimensional slice of hyperbolic metrics as considered e.g. in [49] and finally use the structure of the space of hyperbolic metrics, in particular the so-called *Slice Theorem*, to lift the inequality to a neighbourhood of $\bar{g}$ in $\mathcal{M}^s_{-1}$.

Proposition 4.3.3. *Let (N, g_N) be a closed real analytic manifold and let M be a closed oriented surface of genus $\gamma \geq 2$. Fix $s > 3$ and a smooth critical point $(\bar{u}, \bar{g})$ of the Dirichlet energy $E(u, g)$. Then there exist numbers $\alpha \in (0, \frac{1}{2})$ and $C < \infty$, a neighbourhood $\hat{U}$ of $\bar{u}$ in*

$H^s(M,N)$ and a neighbourhood $\hat{V}$ of 0 in $\mathrm{Re}(\mathcal{H}(M,\bar{g}))$, with the property that each element of $\bar{g}+\hat{V}$ is a metric, such that for any $u\in\hat{U}$ and any $g\in\bar{g}+\hat{V}$ we have

$$|E(u,g)-E(\bar{u},\bar{g})|^{1-\alpha}\le C\left(\|\tau_g(u)\|_{L^2(M,g)}^2+\|\Phi(u,g)\|_{L^1(M,g)}^2\right)^{\frac{1}{2}}. \tag{4.3.2}$$

Proof. Given a closed analytic manifold (N,g_N) we can first use Nash's embedding theorem to isometrically embed (N,g_N) into a suitable Euclidean space $(\mathbb{R}^n,g_{\mathbb{R}^n})$ in an analytic manner. We can then modify the metric on $\mathbb{R}^n$ in a tubular neighbourhood of N to obtain a new metric h on $\mathbb{R}^n$, which is analytic in a neighbourhood of N, such that N is a totally geodesic submanifold of $(\mathbb{R}^n,h)$ and so that at each point $p\in N$ we have $h(p)=g_{\mathbb{R}^n}(p)$. Indeed, letting Π denote the (analytic) nearest point projection from this tubular neighbourhood onto N the above can be arranged as follows. Let $p\in\mathbb{R}^n$ lie in the tubular neighbourhood and $v\in T_p\mathbb{R}^n$. Translate v to lie at the point $\Pi(p)$ and let $v_T\in T_{\Pi(p)}N$ denote the component tangent to N. If we then define

$$h(p)(v,v)=g_N(\Pi(p))(v_T,v_T)+g_{\mathbb{R}^n}(v-v_T,v-v_T)$$

we obtain a metric h with the aforementioned properties. A similar procedure is described in detail in the proof of Lemma 4.1.2 in [16].

If we view any map $u:(M,g)\to(N,g_N)$ as a map $u:(M,g)\to(\mathbb{R}^n,h)$ using this embedding, then the Hopf-differential remains unchanged and, as N is totally geodesic, the tension fields can be identified.

The claim of the proposition hence follows immediately if we prove the inequality (4.3.2) for maps in an H^s neighbourhood $\hat{U}$ of $\bar{u}$ in the larger space of maps from (M,g) to $(\mathbb{R}^n,h)$. From now on we hence consider the Dirichlet energy of maps $u:M\to(\mathbb{R}^n,h)$.

We also identify the $6(\gamma-1)$ dimensional affine space $\bar{g}+\mathrm{Re}(\mathcal{H}(\bar{g}))$ with $\mathbb{R}^{6(\gamma-1)}$ by fixing an $L^2(M,\bar{g})$-orthonormal basis $\{k_i\}_{i=1}^{6(\gamma-1)}$ of $\mathrm{Re}(\mathcal{H}(\bar{g}))$ and setting $\hat{g}(\mu)=\bar{g}+\sum_i\mu_ik_i$. In the following we restrict μ to a neighbourhood V of $0\in\mathbb{R}^{6(\gamma-1)}$ so that $\frac{1}{2}\bar{g}\le\hat{g}(\mu)\le2\bar{g}$ for all $\mu\in V$.

We now define $F\colon H^s(M,\mathbb{R}^n)\times V\to\mathbb{R}$ by

$$F(u,\mu):=E(u:(M,\hat{g}(\mu))\to(\mathbb{R}^n,h)).$$

We want to consider the gradient of F with respect to the fixed inner product

$$\langle(v,\mu),(\tilde{v},\tilde{\mu})\rangle=\langle v,\tilde{v}\rangle_{L^2((M,\bar{g}),(\mathbb{R}^n,g_{\mathbb{R}^n}))}+\langle\mu,\tilde{\mu}\rangle_{\mathbb{R}^{6(\gamma-1)}}, \tag{4.3.3}$$

i.e. $\nabla F(u,\mu)=:(\nabla_uF(u,\mu),\nabla_\mu F(u,\mu))$ satisfies

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}F(u+\varepsilon v,\mu+\varepsilon\xi)=\langle\nabla F(u,\mu),(v,\xi)\rangle \tag{4.3.4}$$

for every $(v,\xi) \in H^s(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)}$.

By (3.1.4) we have $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}F(u+\varepsilon v,\mu) = -\int_M \langle \tau_{\hat{g}(\mu)}(u), v\rangle_{h\circ u}\, dv_{\hat{g}(\mu)}$ and so

$$\nabla_u F(u,\mu) = -\sum_{i,j=1}^{n} (h\circ u)_{ij}(\tau_{\hat{g}(\mu)}(u))_j \psi(\mu) e_i \tag{4.3.5}$$

where $\tau_{\hat{g}(\mu)}(u)_i$ are the components of $\tau_{\hat{g}(\mu)}(u)$ with respect to the standard basis $\{e_i\}$ of $\mathbb{R}^n$ and $\psi(\mu)\colon M\to(0,\infty)$ is characterised by the relation $dv_{\hat{g}(\mu)} = \psi(\mu)dv_{\bar{g}}$ between the volume forms.

As the $L^2(M,\hat{g}(\mu))$-gradient at $g=\hat{g}(\mu)$ of $g\mapsto E(u,g)$ on the space of all metrics is given by $-\frac{1}{4}\mathrm{Re}(\Phi(u,\hat{g}(\mu))) = -\frac{1}{4}\big[u^*h - \frac{1}{2}\mathrm{tr}_{\hat{g}(\mu)}(u^*h)\hat{g}(\mu)\big]$, see (3.1.4), we can furthermore see that $\nabla_\mu F = (\partial_{\mu_j}F)_{j=1,\dots,6(\gamma-1)}$ is given by

$$\partial_{\mu_j}F(u,\mu) = -\tfrac{1}{4}\langle \mathrm{Re}(\Phi(u,\hat{g}(\mu))), k_j\rangle_{L^2(M,\hat{g}(\mu))}. \tag{4.3.6}$$

Hence

$$\nabla F\colon H^s(M,\mathbb{R}^n)\times V \to H^{s-2}(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)}$$

is well defined. We now collect some properties of F and its gradient which will allow us to prove a Łojasiewicz inequality for F.

Keeping in mind that $\nabla F(\bar{u},0)=0$, we now consider the linearisation

$$L\colon H^s(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)} \to H^{s-2}(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)}$$

of ∇F at $(\bar{u},0)$. Note that L is the Hessian of F and so, as observed in [43], is formally self-adjoint in the sense that $\langle L(v,\mu),(\tilde{v},\tilde{\mu})\rangle = \langle (v,\mu), L(\tilde{v},\tilde{\mu})\rangle$ for the inner product defined in (4.3.3). In practice, this will follow from a more abstract machinery that will be invoked later.

We recall that for $p\in N$ we have $h(p)=g_{\mathbb{R}^n}$, in particular $h\circ\bar{u}=g_{\mathbb{R}^n}$, that $\psi=1$ at $\mu=0$ and that $\tau_{\bar{g}}(\bar{u})=0$. Thus the linearisation $L^{1,1}$ of $v\mapsto\nabla_u F(\bar{u}+v,0)$ at $v=0$ agrees with the linearisation of $v\mapsto -\tau_{\bar{g}}(\bar{u}+v)$ at $v=0$, and is thus Fredholm of index 0 from $H^s(M,\mathbb{R}^n)$ to $H^{s-2}(M,\mathbb{R}^n)$ since it is a linear second-order elliptic differential operator with smooth coefficients that can be written as $-\Delta_{\bar{g}}$ plus lower order terms which constitute a compact perturbation. At the same time, the linearisations $L^{1,2}$, $L^{2,2}$ of $\mu\mapsto\nabla_u F(\bar{u},\mu)$, $\mu\mapsto\nabla_\mu F(\bar{u},\mu)$ at $\mu=0$ and the linearisation $L^{2,1}$ of $v\mapsto\nabla_\mu F(\bar{u}+v,0)$ at $v=0$ are bounded linear operators with finite dimensional domain respectively range and are thus compact. It follows that viewed as maps $H^s(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)}\to H^{s-2}(M,\mathbb{R}^n)\times\mathbb{R}^{6(\gamma-1)}$ the operator $\begin{pmatrix} L^{1,1} & 0\\ 0 & 0\end{pmatrix}$ is Fredholm of index 0 and the operator $\begin{pmatrix} 0 & L^{1,2}\\ L^{2,1} & L^{2,2}\end{pmatrix}$ is compact. Hence the operator $L = \begin{pmatrix} L^{1,1} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & L^{1,2}\\ L^{2,1} & L^{2,2}\end{pmatrix}$ is a compact perturbation of an operator that is Fredholm of index 0, so L itself is Fredholm of index 0.

Similarly, we consider the linearisation $L_{(u,\mu)}$ of ∇F around different points $(u,\mu) \in H^s(M,\mathbb{R}^n) \times V$. For $k \in \mathbb{N}$ we denote by $\mathcal{B}_k$ the space of bounded linear operators $H^k(M,\mathbb{R}^n) \times \mathbb{R}^{6(\gamma-1)} \to H^{k-2}(M,\mathbb{R}^n) \times \mathbb{R}^{6(\gamma-1)}$ and by $\|\cdot\|_{\mathcal{B}_k}$ the operator norm on this space. We observe from the formulae (4.3.5), (4.3.6) that for each $k \in \{2,\dots,s\}$ the linearisation $L_{(u,\mu)} \in \mathcal{B}_s$ has a (unique) extension to an element of $\mathcal{B}_k$ and the map $(H^s(M,\mathbb{R}^n) \times V, \|\cdot\|_{H^s \times \mathbb{R}^{6(\gamma-1)}}) \to (\mathcal{B}_k, \|\cdot\|_{\mathcal{B}_k})$ taking $(u,\mu) \mapsto L_{(u,\mu)}$ is continuous. Furthermore, the extension of L to $\mathcal{B}_k$ is Fredholm with index 0.

We choose an $H^s(M,\mathbb{R}^n)$ neighbourhood U of $\bar{u}$ so that the image of each map $u \in U$ is contained in the neighbourhood of $N \subset \mathbb{R}^n$ where the metric h is analytic and hence that the functional F and its gradient ∇F are analytic.

We can now apply Theorem 2.2.4 to obtain a Łojasiewicz inequality for F, as F and its gradient ∇F have the required properties. Namely we have an $H^s(M,\mathbb{R}^n)$ neighbourhood U of $\bar{u}$ and a neighbourhood V of $0 \in \mathbb{R}^{6(\gamma-1)}$ such that

1. The functional $F\colon U \times V \to \mathbb{R}$ and its gradient $\nabla F\colon U \times V \to H^{s-2}(M,\mathbb{R}^n) \times \mathbb{R}^{6(\gamma-1)}$ are both analytic.

2. For each $(u,\mu) \in U \times V$ the linearisation $L_{(u,\mu)}$ of ∇F around (u,μ) is an element of $\mathcal{B}_s$ and has an extension to an element of $\mathcal{B}_2$. Furthermore the map $(U \times V, \|\cdot\|_{H^s \times \mathbb{R}^{6(\gamma-1)}}) \to (\mathcal{B}_2, \|\cdot\|_{\mathcal{B}_2})$ is continuous.

3. The linearisation $L = L_{(\bar{u},0)} \in \mathcal{B}_s$ of ∇F around the critical point $(\bar{u},0)$ is Fredholm with index 0 and its extension to $\mathcal{B}_2$ is also Fredholm with index 0.

We consequently obtain the following Łojasiewicz inequality for F. There exists $\alpha \in (0,\frac{1}{2})$ and $\sigma > 0$ so that both $\hat{U} := \{u \in H^s(M,\mathbb{R}^n) \colon \|u-\bar{u}\|_{H^s} < \sigma\} \subset U$ and $\{\mu \in \mathbb{R}^{6(\gamma-1)} \colon |\mu| < \sigma\} \subset V$ and we have the estimate

$$|F(u,\mu) - F(\bar{u},0)|^{1-\alpha} \leq \|\nabla F(u,\mu)\| \tag{4.3.7}$$

for all $u \in H^s(M,\mathbb{R}^n)$ and $\mu \in \mathbb{R}^{6(\gamma-1)}$ with $\|u-\bar{u}\|_{H^s} < \sigma$ and $|\mu| < \sigma$. We finally set $\hat{V} = \{h \in \mathrm{Re}(\mathcal{H}(\bar{g})) : \|h\|_{L^2(M,\bar{g})} < \sigma\}$.

To explain how (4.3.7) implies (4.3.2) we now need to relate ∇F to the quantities appearing on the right-hand side of (4.3.2).

The formula (4.3.5) for $\nabla_u F$, combined with equivalence of the metrics $\bar{g}$ and $\hat{g}(\mu)$ (and boundedness of h) immediately implies that

$$\|\nabla_u F(u,\mu)\|_{L^2(M,\bar{g})} \leq C\|\tau_{\hat{g}(\mu)}(u)\|_{L^2(M,\hat{g}(\mu))}. \tag{4.3.8}$$

Similarly, we can use (4.3.6) to bound

$$\begin{aligned}\left|\partial_{\mu_j} F(u,\mu)\right| &= \tfrac{1}{4}\left|\langle \operatorname{Re}(\Phi(u,\hat g(\mu))), k_j\rangle_{L^2(M,\hat g(\mu))}\right| \le \tfrac{1}{4}\|k_j\|_{L^\infty(M,\hat g(\mu))}\|\operatorname{Re}(\Phi(u,\hat g(\mu))\|_{L^1(M,\hat g(\mu))}\\ &\le C\|k_j\|_{L^\infty(M,\bar g)}\|\Phi(u,\hat g(\mu))\|_{L^1(M,\hat g(\mu))} \le C\|\Phi(u,\hat g(\mu))\|_{L^1(M,\hat g(\mu))}\end{aligned}$$

where we use the equivalence of the metrics $\bar g$ and $\hat g(\mu)$ in the penultimate step, while the last step follows as the $L^2(M,\bar g)$-norm is equivalent to the $L^\infty(M,\bar g)$-norm on the finite-dimensional space $\operatorname{Re}(\mathcal{H}(\bar g))$. Hence

$$\|\nabla F(u,\mu)\| \le C(\|\tau_{\hat g(\mu)}(u)\|^2_{L^2(M,\hat g(\mu))} + \|\Phi(u,\hat g(\mu))\|^2_{L^1(M,\hat g(\mu))})^{\frac{1}{2}}$$

and the claim of the proposition follows from (4.3.7). □

Before proceeding, we give a brief outline of the slice theorem, following [49]. Given $\bar g \in \mathcal{M}_{-1}$ and $k \in V \subset \mathcal{H}(\bar g)$, V a small neighbourhood, there is a unique function $\lambda(k)\colon V \to C^\infty(M,(0,\infty))$ such that $\lambda(k)(\bar g + k)$ has constant curvature -1. We define a slice around $\bar g$ by

$$\mathcal{S} = \{\lambda(k)(\bar g + k)\colon k \in V\} \subset \mathcal{M}_{-1}.$$

Let $\mathcal{D}_0^s$ denote the space of H^s diffeomorphisms of M which are homotopic to the identity. The slice theorem states that, after possibly reducing $\mathcal{S}$, there are neighbourhoods $W \subset \mathcal{M}_{-1}^s$ of $\bar g$ and $Q \subset \mathcal{D}_0^{s+1}$ of the identity such that the map

$$\Xi : \mathcal{S} \times Q \to W, \quad (\hat g, f) \mapsto f^*\hat g$$

is a diffeomorphism.

We will now explain how, for surfaces of genus $\gamma \ge 2$, Proposition 4.3.3 implies that the Łojasiewicz inequality (4.3.1) holds on a H^s neighbourhood of $(\bar u, \bar g)$ as claimed in Theorem 4.3.1.

Proof of Theorem 4.3.1. As above we consider the slice of hyperbolic metrics

$$\mathcal{S} = \left\{\lambda(k)(\bar g + k)\colon k \in \hat V\right\} \tag{4.3.9}$$

associated with the neighbourhood $\hat V$ obtained in Proposition 4.3.3 above, which we can assume is chosen small enough so that all metrics g in $\bar g + \hat V$ satisfy $\frac{1}{2}\bar g \le g \le 2\bar g$. We note that the functions $\lambda(k)$ are uniformly bounded for $k \in \hat V$ as the neighbourhood $\hat V$ was chosen small enough so that the metrics in $\hat V$ are uniformly equivalent. As the energy is conformally invariant we immediately deduce from Proposition 4.3.3 that there exists $C > 0$ so that

$$|E(\hat u,\hat g) - E(\bar u,\bar g)|^{1-\alpha} \le C\left(\|\tau_{\hat g}(\hat u)\|^2_{L^2(M,\hat g)} + \|\Phi(\hat u,\hat g)\|^2_{L^1(M,\hat g)}\right)^{\frac{1}{2}} \text{ for } (\hat u,\hat g) \in \hat U \times \mathcal{S} \tag{4.3.10}$$

where $\hat{U}$ is the H^s-neighbourhood of $\bar{u}$ obtained in Proposition 4.3.3.

The Poincaré inequality for quadratic differentials on hyperbolic surfaces, see Theorem 3.4.6, implies that

$$\|\Phi(\hat{u},\hat{g}) - P_{\hat{g}}(\Phi(\hat{u},\hat{g}))\|_{L^1(M,\hat{g})} \le C\|\bar{\partial}\Phi(\hat{u},\hat{g})\|_{L^1(M,\hat{g})} \le C\|\tau_{\hat{g}}(\hat{u})\|_{L^2(M,\hat{g})}.$$

As the area of $(M,\hat{g})$ is determined by the genus of M we can thus bound

$$\|\Phi(\hat{u},\hat{g})\|_{L^1(M,\hat{g})} \le C\|P_{\hat{g}}(\Phi(\hat{u},\hat{g}))\|_{L^2(M,\hat{g})} + C\|\tau_{\hat{g}}(\hat{u})\|_{L^2(M,\hat{g})}.$$

Inserting this into (4.3.10) yields the claimed inequality (4.3.1) in the special case of maps $\hat{u} \in \hat{U}$ and metrics $\hat{g} \in \mathcal{S}$.

As (4.3.1) is invariant under pullback by diffeomorphisms we thus know that the inequality (4.3.1) holds for any pair $(u,g) = f^*(\hat{u},\hat{g})$ that is obtained by pulling back an element $(\hat{u},\hat{g}) \in \hat{U} \times \mathcal{S}$ by an arbitrary H^{s+1} diffeomorphism. Hence the theorem follows once we prove that there is a neighbourhood $\mathcal{O}$ of $(\bar{u},\bar{g})$ in $H^s \times \mathcal{M}^s_{-1}$ so that every pair in $\mathcal{O}$ can be written in this form.

Let $\mathcal{D}^{s+1}_0$ be the set of H^{s+1} diffeomorphisms of M that are homotopic to the identity, recall we are assuming that $s > 3$. We recall that the map $H^s \times \mathcal{D}^{s+1}_0 \to H^s$ given by $(u,f) \mapsto u \circ f^{-1}$ is continuous (see for example Theorem 1.2 of [19]). We can hence choose a neighbourhood Q of the identity in $\mathcal{D}^{s+1}_0$ and a neighbourhood U of $\bar{u}$ in H^s so that $u \circ f^{-1}$ lies in the H^s neighbourhood $\hat{U}$ for every $f \in Q$ and every $u \in U$.

The slice theorem above ensures that, after possibly reducing Q and $\mathcal{S}$,

$$\Xi : \mathcal{S} \times Q \to W, \quad (\hat{g}, f) \mapsto f^*\hat{g} \tag{4.3.11}$$

is a diffeomorphism onto a neighbourhood W of $\bar{g}$ in $\mathcal{M}^s_{-1}$. We hence conclude that (4.3.1) holds on the $H^s \times \mathcal{M}^s_{-1}$ neighbourhood $\mathcal{O} := U \times W$ of $(\bar{u},\bar{g})$ as claimed. □

Remark 4.3.4. For genus one surfaces we can modify the above proof of Theorem 4.3.1 as follows. As we can pull back any given critical point $(\bar{u},\bar{g})$, $\bar{g} \in \mathcal{M}_0$, of E by a smooth diffeomorphism to obtain a critical point of E for which the metric component is in $\mathcal{M}^*$, it suffices to consider $(\bar{u},\bar{g})$ with $\bar{g} \in \mathcal{M}^*$. By considering the function $F : H^s(M,\mathbb{R}^n) \times \mathbb{H} \to \mathbb{R}$ given by $F(u,(a,b)) = E(u : (T^2, g_{a,b}) \to (\mathbb{R}^n, h))$ and arguing as in the proof of Proposition 4.3.3 for higher genus surfaces we obtain the analogue to Proposition 4.3.3, now for metrics in a $\mathcal{M}^*$ neighbourhood of $\bar{g} \in \mathcal{M}^*$, which we may assume to be contained in a compact set K.

To obtain Theorem 4.3.1, we note that even though the Poincaré estimate for quadratic differentials is not uniform for flat tori, it is still valid for metrics in the compact set K, now with a constant that also depends on K. Hence we can argue as above, now using that the map $(g, f) \mapsto f^* g$ is a surjective map from $\mathcal{M}^* \times \mathcal{D}_0^{s+1}$ to $\mathcal{M}_0^s$ which takes a small neighbourhood in $\mathcal{M}^* \times \mathcal{D}_0^{s+1}$ to a neighbourhood in $\mathcal{M}_0^s$ (in place of the slice theorem).

4.4 Convergence of the flow: Proof of Theorem 4.1.5

If (u, g) is a solution of the flow (3.2.3) then (modulo constants including η) the right-hand side of (4.3.1) is given by the L^2-norm of the velocity of (u, g). Arguing as in the proof of [43, Lemma 1] (similar to Proposition 2.1.3), we can thus control the L^2-length of any solution (u, g) of the flow on intervals $I = (s_1, s_2)$ for which $(u, g)|_I$ is contained in $\mathcal{O}$ and for which $E(u, g) \geq E(\bar{u}, \bar{g})$. To be more precise, setting $\Delta E(t) = E(u(t), g(t)) - E(\bar{u}, \bar{g}) \geq 0$ and combining the Łojasiewicz inequality with (3.3.1) and (3.2.3), gives

$$\frac{d}{dt}(\Delta E(t))^\alpha = -\alpha\,(\Delta E(t))^{\alpha-1}\left(\|\partial_t u\|_{L^2(M,g)}^2 + \eta^{-2}\|\partial_t g\|_{L^2(M,g)}^2\right) \leq -C^{-1}\|\partial_t(u, g)\|_{L^2(M,g)}$$

for C independent of t, so

$$\int_{s_1}^{s_2} \|\partial_t(u, g)\|_{L^2(M,g)} dt \leq C(\Delta E(s_1))^\alpha. \tag{4.4.1}$$

Proof of Theorem 4.1.5. Let (u, g) be as in the theorem. As there is no degeneration of the metric g along the sequence of times $t_j \to \infty$ we can pass to a subsequence (not relabelled) so that there are orientation preserving diffeomorphisms $\psi_j \colon M \to M$ for which

$$\psi_j^* g(t_j) \to \bar{g} \text{ smoothly and } \psi_j^* u(t_j) := u(t_j) \circ \psi_j \rightharpoonup \bar{u} \text{ weakly in } H^1, \tag{4.4.2}$$

where $(\bar{u}, \bar{g}) \in C^\infty(M, N) \times \mathcal{M}_{-1}$ is a critical point of E, see Theorem 3.4.4. Although here we have only claimed weak H^1 and hence strong L^2 convergence, in the present situation the first hypothesis of (4.1.4), combined with the convergence of the metrics, tells us that we have strong H^1 convergence. This is all we need for now, but later in the proof, parabolic regularity theory will yield uniform C^k bounds on $\psi_j^* u(t_j)$, see (4.4.13) below, allowing us to deduce that also the convergence of the map component in (4.4.2) is smooth. Note that because different smooth domain metrics all lead to the same notion of H^s (or C^k) convergence, here and in the following there is no need to specify the domain metric when talking about convergence.

By the convergence of the metric in (4.4.2), we may drop a finite number of terms in j in order to assume that

$$\frac{1}{2}\bar{g} \leq \psi_j^* g(t_j) \leq 2\bar{g} \tag{4.4.3}$$

for each j. It will also be convenient to drop finitely many terms so that $t_j \geq 1$.

Let $s \in \mathbb{N}_{>3}$ and let $\mathcal{O}$ be the neighbourhood of $(\bar{u}, \bar{g})$ on which the Łojasiewicz inequality (4.3.1) holds. Let $V = \{h \in \mathcal{M}^s_{-1} \colon \|h - \bar{g}\|_{H^s} < \sigma\}$ and $U = \{w \in H^s \colon \|w - \bar{u}\|_{H^s} < \sigma\}$ and assume that $\sigma > 0$ is chosen small enough so that $U \times V \subset \mathcal{O}$ and so that the projection of V onto moduli space lies in a compact subset.

For each j, if $\psi_j^*(u,g)(t_j) \in U \times V \subset \mathcal{O}$ we let $T_j > t_j$ be the maximal time so that $\psi_j^*(u,g)(t) \in U \times V \subset \mathcal{O}$ for all $t \in [t_j, T_j)$; otherwise we set $T_j = t_j$. As $(\tilde{u}_j, \tilde{g}_j) := \psi_j^*(u,g)$ is also a solution of the flow (3.2.3) we can apply (4.4.1) to conclude that for C independent of j,

$$\int_{t_j}^{T_j} \|\partial_t(u,g)\|_{L^2(M,g)} dt = \int_{t_j}^{T_j} \|\partial_t(\tilde{u}_j, \tilde{g}_j)\|_{L^2(M,\tilde{g}_j)} dt \le C(\Delta E(t_j))^\alpha \to 0 \tag{4.4.4}$$

as $j \to \infty$ since $\Delta E(t_j) = E((u,g)(t_j)) - E(\bar{u}, \bar{g}) \searrow 0$ as a result of the strong H^1 convergence of $\tilde{u}_j$ and smooth convergence of $\tilde{g}_j$.

The main part of the proof will be to show that for sufficiently large j we have $T_j = \infty$. Once we have established this, we can then use (4.4.4) to deduce that (u,g) converges to a critical point of E as $t \to \infty$. While (4.4.4) on its own would only indicate L^2-convergence we will later be able to combine it with parabolic regularity theory for the map and the properties of horizontal curves of hyperbolic metrics (described in Section 3.3) to conclude that the flow indeed converges smoothly as claimed.

We will have to ensure we have control of the metric and map on suitable intervals. The results of Section 3.3 yield appropriate C^k control on the metric as long as we have a positive lower bound on the injectivity radius. Such a bound on $\mathrm{inj}(M,g)$ holds trivially for metrics in the H^s neighbourhood V of $\bar{g}$ as the projection of V onto moduli space is assumed to be compact. Furthermore, as $\inf_j \mathrm{inj}(M, g(t_j)) > 0$ we use (3.3.7) and (3.3.4) to fix $\tau_0 \in (0,1)$ in a way that ensures that the injectivity radii of the surfaces $(M, g(t))$ are bounded away from zero for $t \in [t_j - \tau_0, t_j + \tau_0]$, uniformly in j. Combined we thus find that there is some $\delta_0 > 0$ so that

$$\mathrm{inj}(M, g(t)) \ge \delta_0 \text{ for every } t \in I_j^{\tau_0}, \text{ for every } j \in \mathbb{N}, \tag{4.4.5}$$

where we define

$$I_j^\tau := [t_j - \tau, \max(t_j + \tau, T_j)) \text{ for } \tau > 0. \tag{4.4.6}$$

In addition we also have a uniform upper bound on the L^2-length of $g|_{I_j^{\tau_0}}$ of the form $CE(0)^\alpha + C\sqrt{E(0)\tau_0} \le C$, for C independent of j, compare (4.4.4) and (3.3.4). This together with (4.4.5) and (3.3.6) allows us to conclude that the metrics $\tilde{g}_j(t)$ are uniformly equivalent on $I_j^{\tau_0}$, i.e. comparable by a factor that is independent of j. By (4.4.3), we find that there exists $C > 0$ independent of j so that

$$C^{-1}\bar{g} \le \tilde{g}_j(t) \le C\bar{g} \qquad \text{for every } t \in I_j^{\tau_0}. \tag{4.4.7}$$

We also use (3.3.6) to obtain that for every $k \in \mathbb{N}$, there exists $C < \infty$ independent of j such that for every $t \in I_j^{\tau_0}$ we have

$$\|\partial_t \tilde{g}_j(t)\|_{C^k(M,\bar{g})} \leq C\|\partial_t \tilde{g}_j(t)\|_{C^k(M,\tilde{g}_j(t_j))} \leq C\|\partial_t \tilde{g}_j(t)\|_{L^2(M,\tilde{g}_j(t))} \tag{4.4.8}$$

where for the first inequality we use that $\tilde{g}_j(t_j)$ converges smoothly to $\bar{g}$ and (3.3.8).

Having established this control on the metric component on the intervals $I_j^{\tau_0}$ we now turn to the analysis of $\tilde{u}_j(t) = u(t) \circ \psi_j$ on these intervals. We recall that the standard H^2-estimates given in Lemma 3.3.1 imply that there exist constants $\varepsilon_0 = \varepsilon_0(N) > 0$ and $C = C(N)$ so that for any hyperbolic metric g, any H^2 map $v : M \to N$ and any $r \in (0, \mathrm{inj}(M, g)]$ with

$$\sup_{x\in M} \frac{1}{2}\int_{B_r^g(x)} |dv|_g^2 dv_g \leq 2\varepsilon_0 \tag{4.4.9}$$

we have

$$\int_{B^g_{r/2}} |\nabla_g^2 v|_g^2 + |dv|_g^4 dv_g \leq C\left(r^{-2}E(v; B_r^g) + \|\tau_g(v)\|^2_{L^2(M,g)}\right). \tag{4.4.10}$$

We will show that we can choose $r \in (0, \delta_0]$, $\tau \in (0, \tau_0]$ and j_0 so that (4.4.9) holds true for $g = g(t)$ and $v = u(t)$ on I_j^τ for every $j \geq j_0$. Letting $r_1 > 0$ be as in Remark 4.1.7, this will follow from that remark for appropriate $r \in (0, r_1]$ if we can control the evolution of the local energy not only forward in time but also backwards in time.

To do this, we first note that by reducing $r_1 > 0$ if necessary we may assume that $r_1 \leq \delta_0$. Fix a smooth cut-off function $\sigma \colon \mathbb{R} \to [0, 1]$ with $\sigma(\rho) = 1$ for $\rho \leq r_1/2$ and $\sigma(\rho) = 0$ for $\rho \geq 3r_1/4$. For each $x \in M$, define $\varphi \in C_c^\infty(B_{r_1}^{g(t_j)}(x))$ by $\varphi(y) := \sigma(d_{g(t_j)}(y, x))$. Appealing to the lower injectivity radius bound for $g(t)$ on $I_j^{\tau_0}$, and the estimate (3.3.5) to deduce that $\|\partial_t g\|_{L^\infty(M,g)} \leq C\|\partial_t g\|_{L^1(M,g)} \leq CE(0)$. Consequently, Lemma 3.3.2 combined with (4.4.7) allows us to control the evolution of the local energy by

$$\begin{aligned}\left|\frac{d}{dt}\left(\frac{1}{2}\int \varphi^2 |dv|_g^2 dv_g\right) - \int \varphi^2|\tau_g(u)|^2 dv_g\right| &\leq C\left(\int \varphi^2|\tau_g(u)|^2 dv_g\right)^{\frac{1}{2}} + C \\ &\leq \int \varphi^2|\tau_g(u)|^2 dv_g + C\end{aligned} \tag{4.4.11}$$

for each $t \in I_j^{\tau_0}$, where C is independent of j and t. Therefore, for $t \in I_j^{\tau_0}$ with $t < t_j$ we obtain that

$$\begin{aligned}\frac{1}{2}\int_{B^{g(t_j)}_{r_1/2}(x)} |du(t)|^2_{g(t)} dv_{g(t)} \leq{}& \frac{1}{2}\int_{B^{g(t_j)}_{r_1}(x)} |du(t_j)|^2_{g(t_j)} dv_{g(t_j)} \\ &+ C(t_j - t) + 2\int_t^{t_j} \|\tau_g(u)\|^2_{L^2(M,g)} dt',\end{aligned} \tag{4.4.12}$$

while for $t \in I_j^{\tau_0}$ with $t > t_j$ we have

$$\frac{1}{2}\int_{B_{r_1/2}^{g(t_j)}(x)} |du(t)|^2_{g(t)} dv_{g(t)} \leq \frac{1}{2}\int_{B_{r_1}^{g(t_j)}(x)} |du(t_j)|^2_{g(t_j)} dv_{g(t_j)} + C(t-t_j),$$

As (3.3.1) implies that $t \mapsto \|\tau_g(u)\|^2_{L^2}$ is integrable in time, the tension term in (4.4.12) is less than $\frac{\varepsilon_0}{2}$ for all sufficiently large j and any $t \in I_j^{\tau_0}$. By (4.4.7), the metrics $g(t)$ for $t \in I_j^{\tau_0}$ are equivalent, by a factor independent of j, so setting $r := C^{-1}r_1 \in (0, \min(r_1, \delta_0))$ for a suitably large $C > 1$, independent of j, ensures that $B_r^{g(t)}(x) \subset B_{r_1/2}^{g(t_j)}(x)$ for every $j \in \mathbb{N}$, $t \in I_j^{\tau_0}$ and $x \in M$.

We can thus choose $\tau \in (0, \tau_0]$ and j_0 so that (4.4.9) holds for $v = u(t)$ and $g = g(t)$ on the intervals $[t_j - \tau, t_j + \tau]$, $j \geq j_0$. Of course, (4.4.9) also holds trivially for maps and metrics in the H^s neighbourhood $\mathcal{O}$ of $(\bar{u}, \bar{g})$ if σ is initially chosen small enough, after reducing r if necessary. Therefore we obtain that (4.4.8) and, for $v = u(t)$, $g = g(t)$, (4.4.9) and hence (4.4.10), hold on the interval I_j^{τ} for $j \geq j_0$. Standard parabolic theory, see for example [37], now yields that for every $k \in \mathbb{N}$ there exists a constant C independent of j so that

$$\|\tilde{u}_j(t)\|_{C^k(M,\bar{g})} \leq C \quad \text{for all } t \in [t_j, \max(t_j + \tau, T_j)). \tag{4.4.13}$$

This implies that $\tilde{u}_j(t_j)$ converges to $\bar{u}$ not only in L^2 but in C^k for every k as claimed earlier.

Let now $\varepsilon > 0$ be a constant to be determined below, independently of j. Then (4.4.4) and the smooth convergence of maps and metrics ensures that

$$\|\tilde{u}_j(t_j) - \bar{u}\|_{H^s(M,\bar{g})} + \|\tilde{g}_j(t_j) - \bar{g}\|_{C^s(M,\bar{g})} + \int_{t_j}^{T_j} \|\partial_t(\tilde{u}_j, \tilde{g}_j)\|_{L^2(M,\tilde{g}_j)} dt \leq \varepsilon, \tag{4.4.14}$$

for all sufficiently large j depending, in particular, on ε. We claim that if $\varepsilon > 0$ is chosen small enough then for every $j \geq j_0$ for which (4.4.14) holds we have that both $\|\tilde{g}_j(t) - \bar{g}\|_{H^s(M,\bar{g})} < \sigma/2$ and $\|\tilde{u}_j(t) - \bar{u}\|_{H^s(M,\bar{g})} < \sigma/2$ for every $t \in [t_j, T_j)$ and hence $T_j = \infty$. First we deal with the metric component; we have for every $t \in [t_j, T_j)$

$$\begin{aligned}\|\tilde{g}_j(t) - \bar{g}\|_{C^s(M,\bar{g})} &\leq \|\tilde{g}_j(t_j) - \bar{g}\|_{C^s(M,\bar{g})} + \int_{t_j}^{T_j} \|\partial_t \tilde{g}_j\|_{C^s(M,\bar{g})} dt \\ &\leq \|\tilde{g}_j(t_j) - \bar{g}\|_{C^s(M,\bar{g})} + C\int_{t_j}^{T_j} \|\partial_t \tilde{g}_j\|_{L^2(M,\tilde{g}_j(t))} dt \leq \varepsilon + C\varepsilon\end{aligned} \tag{4.4.15}$$

for C independent of j and ε, where we have used (4.4.8) and (4.4.14). This establishes the necessary claim on the metric component if $\varepsilon > 0$ is chosen small enough.

To deal with the map component we first use (4.4.14) and (4.4.7) to conclude that for every $t \in [t_j, T_j)$

$$\begin{aligned}\|\tilde{u}_j(t) - \bar{u}\|_{L^2(M,\bar{g})} &\leq \|\tilde{u}_j(t_j) - \bar{u}\|_{L^2(M,\bar{g})} + \int_{t_j}^{t} \|\partial_t \tilde{u}_j\|_{L^2(M,\bar{g})} dt \\ &\leq \|\tilde{u}_j(t_j) - \bar{u}\|_{H^s(M,\bar{g})} + C \int_{t_j}^{t} \|\partial_t \tilde{u}_j\|_{L^2(M,\tilde{g}_j(t))} dt \\ &\leq C\varepsilon,\end{aligned} \tag{4.4.16}$$

for C independent of j and ε. At the same time (4.4.13) implies the uniform bound $\|\tilde{u}_j(t) - \bar{u}\|_{C^{s+1}(M,\bar{g})} \leq C_1$ on $[t_j, T_j)$, where C_1 is independent of j and ε. Because C^{s+1} embeds compactly into H^s, which in turn embeds continuously into L^2, by Ehrling's lemma we know that for every $\delta > 0$ there exists a number C so that for every $w \in C^{s+1}(M, N)$ we have

$$\|w\|_{H^s(M,\bar{g})} \leq \delta \|w\|_{C^{s+1}(M,\bar{g})} + C\|w\|_{L^2(M,\bar{g})}.$$

Applied for $\delta = \frac{\sigma}{4C_1}$ this allows us to conclude that on $[t_j, T_j)$

$$\|\tilde{u}_j(t) - \bar{u}\|_{H^s(M,\bar{g})} \leq \frac{\sigma}{4} + C\varepsilon \leq \frac{\sigma}{2}$$

where the last inequality holds provided $\varepsilon > 0$ is initially chosen small enough. This concludes the proof of our claim that $T_j = \infty$ for every $j \geq j_0$ such that (4.4.14) holds for an $\varepsilon > 0$ that can now be considered fixed.

Let us fix $J \in \mathbb{N}$, $J \geq j_0$, large enough so that (4.4.14) holds with this ε for $j = J$ and hence so that $T_J = \infty$. Thus (4.4.1) can be applied on all of $[t_J, \infty)$ allowing us to conclude that

$$\int_t^\infty \|\partial_t(u, g)\|_{L^2(M,g)} dt \leq C(\Delta E(t))^\alpha \to 0 \text{ as } t \to \infty.$$

Using this we will be able to show that the original flow $(u, g)(t)$ converges in L^2 as $t \to \infty$ without having to restrict to a sequence of times $t_i \to \infty$ and without having to pull back by diffeomorphisms. Indeed, because

$$\int_t^\infty \|\partial_t(\tilde{u}_J, \tilde{g}_J)\|_{L^2(M,\tilde{g}_J)} dt = \int_t^\infty \|\partial_t(u, g)\|_{L^2(M,g)} dt \leq C(\Delta E(t))^\alpha \to 0 \tag{4.4.17}$$

we can compute

$$\begin{aligned}\int_t^\infty \|\partial_t(u, g)\|_{L^2(M,(\psi_J^{-1})^*\bar{g})} dt &= \int_t^\infty \|\partial_t(\tilde{u}_J, \tilde{g}_J)\|_{L^2(M,\bar{g})} dt \\ &\overset{(4.4.7)}{\leq} C \int_t^\infty \|\partial_t(\tilde{u}_J, \tilde{g}_J)\|_{L^2(M,\tilde{g}_J)} dt \to 0,\end{aligned} \tag{4.4.18}$$

which implies that $(u, g)(t)$ converges in L^2 to some limit (u_∞, g_∞) as $t \to \infty$.

Moreover, as the curve of metrics is horizontal and as we have a uniform lower bound on the injectivity radius on all of $[t_J, \infty)$ we have

$$\int_t^\infty \|\partial_t g\|_{C^k(M,(\psi_J^{-1})^*\bar{g})} dt = \int_t^\infty \|\partial_t \tilde{g}_J(t)\|_{C^k(M,\bar{g})} dt \overset{(4.4.8)}{\leq} C \int_t^\infty \|\partial_t \tilde{g}_J(t)\|_{L^2(M,\tilde{g}_J(t))} dt \to 0 \tag{4.4.19}$$

by (4.4.17), which can be integrated to find that the metrics converge to g_∞ *smoothly* as $t \to \infty$. In addition, (4.4.13) yields uniform bounds on $u(t)$ in $C^k(M, (\psi_J^{-1})^*\bar{g})$, $t \in [t_J, \infty)$, so also the map u converges smoothly to its limit u_∞.

Now that we have the smooth convergence, we see readily that the limit (u_∞, g_∞) is a critical point of E. $\square$

Remark 4.4.1. For surfaces of genus 1 this proof still applies but can be simplified. As observed previously we may assume that $g(0) \in \mathcal{M}^*$ and hence that the flow of metrics is constrained to $\mathcal{M}^*$. Also we may choose the diffeomorphisms ψ_j in the above proof so that $\psi_j^*\mathcal{M}^* = \mathcal{M}^*$ and hence $\tilde{g}_j(t) \in \mathcal{M}^*$ for any t and $\bar{g} \in \mathcal{M}^*$. Combining the convergence of the $\tilde{g}_j(t_j)$ with the uniform upper bound on the L^2-length of $\tilde{g}_j|_{I_j^{\tau_0}}$, say for $\tau_0 := 1$ and the completeness of $\mathcal{M}^*$ with respect to the Weil-Petersson distance yields that the metrics $\tilde{g}_j(t)$, $t \in I_j^{\tau_0}$, are contained in a (j independent) compact subset of $\mathcal{M}^*$. Hence (4.4.5) and (4.4.7) still hold and the C^k estimate (4.4.8) trivially follows from the explicit form of the metrics g_{ab}. For the rest of the proof we can then argue exactly as above.

Chapter 5

Łojasiewicz Inequalities for the Harmonic Map Energy on Degenerating Cylinders

5.1 Statement of results

We define the following version of the harmonic map energy

$$E(u,\ell) = \frac{1}{2}\int_{\mathcal{C}_{Y(\ell)}} |du|^2_{g_\ell}\, \mathrm{d}\mu_{g_\ell}$$

on hyperbolic cylinders $(\mathcal{C}_{Y(\ell)}, g_\ell) = ([-Y(\ell), Y(\ell)]\times S^1, \rho_\ell^2(\mathrm{d}s^2+\mathrm{d}\theta^2))$ where $\rho_\ell(s) = \frac{2\pi}{\ell\cos\frac{\ell s}{2\pi}}$ and $Y(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan(d^{-1}\ell)\right)$ for $d>0$, see Section 1.5. We consider almost critical points of $E(u,\ell)$ which satisfy a prescribed Dirichlet boundary condition. These are in particular almost-harmonic maps and so Theorem 1.5.1 and Remark 1.5.2 apply. We will consider the case where the connecting curves additionally sub-converge to a finite length *geodesic* $\gamma\colon [-1,1]\to N$. Since our maps will not be close to a critical point of the energy on the same domain on which they are defined this is a setting where the powerful techniques introduced by Simon in [43] to obtain Łojasiewicz-Simon inequalities do not apply. These techniques would give a Łojasiewicz-Simon inequality for maps from the disc near one of the maps $v_\pm$ or curves near the geodesic γ if the target N was real analytic. This would also be the case if the real analyticity assumption was dropped as long as the critical objects satisfy a certain non-degeneracy condition; furthermore the exponent obtained would be the optimal one in this case. We will not make any real analyticity assumption on the target N here, instead we will assume that our limiting objects satisfy a stronger condition of strict stability which in particular suffice to obtain Łojasiewicz-Simon inequalities in the usual setting of maps from fixed domains.

Our assumptions can be stated as follows:

(A) The curve $\gamma\colon[-1,1]\to N$ is a geodesic of length $L(\gamma)\in(0,\infty)$ connecting the points $v_\pm(0)$.

(B) The second variation of the energy $E_{[-1,1]}(\eta)=\frac{1}{2}\int_{-1}^{1}|\eta'|^2$ at γ is positive definite, that is there exists $c_\gamma\in(0,1]$ such that

$$d^2E_{[-1,1]}(\gamma)(w,w)\geq c_\gamma\|w\|^2_{\dot{H}^1([-1,1])}$$

for every $w\in\Gamma^{H^1_0([-1,1])}(\gamma^*TN)$.

(C) The second variation of the energy $E_D(v)=\frac{1}{2}\int_D|\nabla v|^2$ at $v_\pm$ is positive definite, that is there exists $c_\pm\in(0,1]$ such that

$$d^2E_D(v_\pm)(w,w)\geq c_\pm\|w\|^2_{\dot{H}^1(D)}$$

for every $w\in\Gamma^{H^1_0(D)}(v_\pm^*TN)$.

Given a map $u\colon D\to N$ we say $w\in\Gamma^{H^1(D)}(u^*TN)$ if and only if $w\in H^1(D;\mathbb{R}^n)$ is such that $w(p)\in T_{u(p)}N$ for almost every $p\in D$. We use the corresponding notation for maps from different domains and other spaces of functions.

Our results can be stated most clearly in the following setting, although they can be translated to the language given above. We work on a fixed domain $C_0=[-1,1]\times S^1$ equipped with a family of hyperbolic metrics $G_\ell=f_\ell^*(\rho_\ell^2(s)(\mathrm{d}s^2+\mathrm{d}\theta^2))$, introduced in [31], obtained by pulling back the aforementioned hyperbolic collars by a specific family of diffeomorphisms $f_\ell\colon C_0\to\mathcal{C}_{Y(\ell)}$ in an almost canonical way (in the sense that the metrics G_ℓ are a horizontal curve of metrics), see (5.2.3) below.

Write $H^1_{v_\pm}(C_0,N)$ for the space of H^1 functions $u\colon C_0\to\mathbb{R}^n$ for which $u\in N$ a.e. and $u|_{\{\pm1\}\times S^1}(\theta)=v_\pm(e^{i\theta})$. We will then set $H=H^1_{v_\pm}(C_0,N)\times(0,\infty)$ and define the energy

$$E(u,\ell)=\frac{1}{2}\int_{C_0}|\mathrm{d}u|^2_{G_\ell}\,\mathrm{d}\mu_{G_\ell}$$

for $(u,\ell)\in H$. We have a natural inner product on $T_{(u,\ell)}H=\Gamma^{H^1_0}(u^*TN)\times\mathbb{R}$, inherited from the L^2 inner product, that is characterized by

$$\|(w,0)\|_*^2=\|w\|^2_{L^2(C_0,G_\ell)},\quad \|(0,1)\|_*^2=\|\partial_\ell G_\ell\|^2_{L^2(C_0,G_\ell)} \tag{5.1.1}$$

and the orthogonality $\langle(w,0),(0,1)\rangle_*=0$ for every $w\in\Gamma^{H^1_0}(u^*TN)$. This inner product gives rise to the L^2 gradient $\nabla E(u,\ell)=(\nabla_uE(u,\ell),\nabla_\ell E(u,\ell))$ which is characterized by

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}E(u+\varepsilon w,\ell+\varepsilon)=\langle\nabla E(u,\ell),(w,1)\rangle_* \tag{5.1.2}$$

and therefore satisfies

$$\nabla_u E(u,\ell) = -\tau_{G_\ell}(u),$$
$$\nabla_\ell E(u,\ell) = \|\partial_\ell G_\ell\|^{-2}_{L^2(C_0,G_\ell)}\partial_\ell E(u,\ell) \in \mathbb{R}.$$

Given $\ell_i > 0$ and $u_i\colon C_0 \to N$ we say that a sequence (u_i,ℓ_i) are almost-critical points of $E(u,\ell)$ if they have uniformly bounded energy, $E(u_i,\ell_i) \leq E_0 < \infty$, and if they satisfy $\|\nabla E(u_i,\ell_i)\|_* \to 0$. If (u_i,ℓ_i) is a sequence of almost-critical points of $E(u,\ell)$ such that $\ell_i \to 0$ and such that the maps u_i have uniformly bounded energy density, $\|du_i\|_{L^\infty(C_0,G_{\ell_i})} \leq A < \infty$, then Theorem 1.5.1 can be applied to $\tilde{u}_i = u_i \circ f_\ell$ and consequently the results of Theorem 1.5.1 and Remark 1.5.2 hold for (u_i,ℓ_i).

Our first main theorem is a Łojasiewicz-Simon inequality for sequences of almost-critical points of $E(u,\ell)$ and can be stated as follows:

Theorem 5.1.1. *Let $\ell_i > 0$ be such that $\ell_i \to 0$ and $u_i\colon C_0 \to N$ be a sequence of maps with fixed boundary values which are almost-critical points of E in the sense that $E(u_i,\ell_i) \leq E_0 < \infty$ and $\|\nabla E(u_i,\ell_i)\|_* \to 0$. Suppose further that the energy density is uniformly bounded, $\|du_i\|_{L^\infty(C_0,G_{\ell_i})} \leq A < \infty$.*

As a result of Theorem 1.5.1 the maps u_i sub-converge smoothly locally on $C^\pm := \{0 < \pm s < 1\} \times S^1$ to maps $v_\pm\colon C^\pm \to N$ which, extending across the punctures, yield smooth harmonic maps $v_\pm\colon D \to N$ defined on the unit disc. Moreover, there exist curves $\gamma_i\colon [-1,1] \to N$ such that

$$\lim_{\lambda\to\infty} \limsup_{i\to\infty} \|u_i - \gamma_i\|_{L^\infty(f^{-1}_{\ell_i}([-Y(\ell_i)+\lambda, Y(\ell_i)-\lambda]\times S^1))} = 0. \tag{5.1.3}$$

Assume that the curves γ_i sub-converge in L^∞ to a limit $\gamma\colon [-1,1] \to N$ and that this limit γ satisfies (A), (B) while the maps $v_\pm$ satisfy (C). Then for i sufficiently large we have a bound on the scale ℓ_i,

$$\ell_i \leq C\|\nabla E(u_i,\ell_i)\|^2_* \tag{5.1.4}$$

and a Łojasiewicz-Simon inequality with optimal exponent for the harmonic map energy,

$$|E(u_i,\ell_i) - E^*|^{\frac{1}{2}} \leq C\|\nabla E(u_i,\ell_i)\|_*, \tag{5.1.5}$$

where $E^* := E(v_+) + E(v_-)$.

The general method of proof we will use was introduced by Malchiodi-Rupflin-Sharp [23] in the context of the H-surfaces and later used by Rupflin [32] in the context of almost harmonic maps into general targets converging to simple bubble trees. The basic idea is to compare a general sequence of almost critical points of the functional with a specific family of "adapted" critical points. These adapted critical points are obtained by modifying the limiting objects, which are defined on a different domain, so that they are defined on the same domain as the sequence. In [23] this was done in the situation of maps from a surface Σ of genus at least 1 which are near to a simple bubble tree – the "adapted" bubbles are obtained by gluing a map from S^2 to a constant map from Σ to obtain suitable maps from Σ.

In our situation we will glue together the geodesic γ and harmonic maps $v_\pm$ to obtain a sequence of maps $z_\ell \colon C_0 \to N$ defining a set of adapted critical points as

$$\mathcal{Z} = \{(z_\ell, \ell) \mid \ell \in (0, \ell_*)\},$$

where $\ell_* > 0$ is some fixed number. We recall that we are always thinking of the second component as parametrizing the metrics G_ℓ and are interested in the degenerating case where $\ell > 0$ is small. The maps z_ℓ are constructed by taking a portion of the ends of C_0 and using this to parametrize most of the maps $\bar{v}_\pm$ and then using the middle region to parametrize most of the curve γ. For the precise definition of these adapted critical points see Section 5.2.

Our second main theorem of this section, which will give Theorem 5.1.1 as a consequence, is a Łojasiewicz-Simon inequality for $(u, \ell) \in H$ which are near to this set of adapted critical points:

Theorem 5.1.2. *Suppose that $\gamma \colon [-1, 1] \to N$ is a curve satisfying (A), (B) and $v_\pm \colon D \to N$ are harmonic maps satisfying (C). Then there exists $\varepsilon > 0$, $\bar{\ell} \in (0, \ell_*)$ and $C < \infty$ such that the following holds. Let $(u, \lambda) \in H = H^1_{v_\pm}(C_0, N) \times (0, \infty)$ be such that*

$$\inf_{\ell > 0} \left[\|u - z_\ell\|_{\dot{H}^1(C_0, G_\ell)} + \ell^{-1}(\lambda - \ell) \right] \leq \varepsilon \tag{5.1.6}$$

and that the infimum is attained by a pair (z_ℓ, ℓ) with $0 < \ell \leq \bar{\ell}$ and $\|u - z_\ell\|_{L^\infty(C_0)} \leq \varepsilon$, where $(z_\ell, \ell) \in \mathcal{Z}$ are the adapted critical points. Then we have a bound on the scale ℓ,

$$\ell \leq \|\nabla E(u, \lambda)\|_*^2 \tag{5.1.7}$$

and Łojasiewicz-Simon inequalities with optimal exponent,

$$\|u - z_\ell\|_{\dot{H}^1(C_0, G_\ell)} + \ell^{-1}(\lambda - \ell) \leq C \|\nabla E(u, \lambda)\|_*, \tag{5.1.8}$$

$$|E(u, \lambda) - E^*|^{\frac{1}{2}} \leq C \|\nabla E(u, \lambda)\|_*, \tag{5.1.9}$$

where $E^* = E(v_+) + E(v_-)$.

Remark 5.1.3. An important example where the hypotheses of Theorem 5.1.2 hold, and indeed the result is new, is when $N = \mathbb{R}^n$, the harmonic maps $v_\pm$ are standard embeddings of flat discs in parallel planes and γ is a straight line connecting their centres.

Remark 5.1.4. The assumption that γ is a geodesic in (A) is essentially asking for the natural property that the sequence of almost-critical points actually converges to a collection of critical points each domain – this is not guaranteed by the other assumptions as discussed after Remark 1.5.3. In the proof its most significant role is in obtaining the rates in Remark 5.3.4 and therefore could likely be weakened as long as the rates obtained are appropriate in the sense described in Remark 5.1.6.

Remark 5.1.5. It would be interesting to replace the strict stability assumptions (B) and (C) with merely non-degeneracy conditions. We have chosen here to use the stability assumptions since they more straightforwardly yield the important ingredient of the proof, Lemma 5.4.7. In [23] and [32] non-degeneracy assumptions and a spectral argument are used to obtain the results to which Lemma 5.4.7 corresponds.

For the proof we will use a specifically defined inner product $\langle \cdot, \cdot \rangle_\ell$ related to the $\dot{H}^1$ inner product, see (5.2.19). A key element of the proof will be that the stability assumptions (B) and (C) can be used to show that the second variation of the energy on the set of adapted critical points is uniformly positive definite in directions orthogonal (with respect to the inner product $\langle \cdot, \cdot \rangle_\ell$) to the tangent space $T_{(z_\ell, \ell)}\mathcal{Z}$, Lemma 5.4.7. Together with certain scaling of the first and second variation of the energy along the adapted critical point set, Remark 5.3.4, we can establish a type of Łojasiewicz-Simon inequality with error terms depending on the scale ℓ, Lemma 5.5.2. The proof is then finished by bounding ℓ in terms of the gradient which uses the rate of decay of the energy along $\mathcal{Z}$ established in Remark 5.3.2.

Remark 5.1.6. What will be important for the proof to go through is a certain ratio quantities – including the rates appearing in Remark 5.3.2 and Remark 5.3.4, as well as the size of the element $\partial_\ell(z_\ell, \ell)$ with respect to the norm $\| \cdot \|_\ell$ computed in (5.2.41) – goes to zero. In the situations considered in [23] this idea is made explicit, but we avoid doing so in our case since the non-linearity of the target makes the resulting quantities less transparent.

The rest of this chapter will be structured as follows. In Section 5.2 we give the precise definition of the adapted critical point set $\mathcal{Z}$ and derive some basic properties that we will be needed later on. In Section 5.3 we study the first and second variation of the energy along the set of adapted critical points. In Section 5.4 we prove that the second variation of the energy is uniformly definite in directions orthogonal to the tangent space of $\mathcal{Z}$. Finally, we give the main argument to complete the proof in Section 5.5.

5.2 The set of adapted critical points $\mathcal{Z}$

5.2.1 Hyperbolic cylinders

Let us motivate our choice of metrics on the cylinder $C_0 = [-1,1] \times S^1$. As we are working with the Dirichlet energy which is conformally invariant one only wishes to consider conformal classes of metrics on C_0 (it is important to note that various quantities we will consider later will *not* be conformally invariant and so will depend on the choices we are about to make). Each of these conformal classes can be represented by $([-Y,Y] \times S^1, \mathrm{d}s^2 + \mathrm{d}\theta^2)$ for a unique $Y > 0$. Motivated by the Collar Lemma, each cylinder $\mathcal{C}_Y = [-Y,Y] \times S^1$ can be equipped with one of the collar metrics $g_\ell = \rho_\ell^2(s)(\mathrm{d}s^2 + \mathrm{d}\theta^2)$ for some $\ell > 0$, which are each conformal to the flat metric, where

$$\rho_\ell(s) = \frac{\ell}{2\pi \cos(\frac{\ell s}{2\pi})} \tag{5.2.1}$$

and then these metrics g_ℓ can be pulled back by an arbitrary smooth diffeomorphism to C_0. Hence given $Y > 0$ we wish to find $\ell > 0$ and a diffeomorphism $f \colon C_0 \to [-Y,Y] \times S^1$ which represent the corresponding conformal structure on C_0 in a natural way. Note that since we will be looking at a situation with Dirichlet boundary conditions, it is important that the diffeomorphisms fix the parametrization of the boundary curves. The choice of diffeomorphisms is given by the following Lemma:

Lemma 5.2.1 (Lemma 2.4, [31])**.** *Fix $d > 0$. For $\ell > 0$ define*

$$Y(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan(d^{-1}\ell)\right) \tag{5.2.2}$$

and $f_\ell \colon C_0 \to \mathcal{C}_{Y(\ell)}$ by

$$f_\ell(x,\theta) = (s_\ell(x),\theta), \quad s_\ell(x) = \tfrac{2\pi}{\ell}\arctan[\tfrac{\ell_0}{\ell}\tan(\tfrac{\ell_0 x}{2\pi})] \tag{5.2.3}$$

where ℓ_0 is determined by $Y(\ell_0) = 1$. Then the family $G_\ell := f_\ell^*(\rho_\ell^2(s)(\mathrm{d}s^2 + \mathrm{d}\theta^2))$ satisfies

$$\partial_\ell G_\ell = \frac{\ell}{2\pi^2} f_\ell^*(\mathrm{d}s^2 - \mathrm{d}\theta^2) \tag{5.2.4}$$

and so is horizontal *in the sense that $\partial_\ell G_\ell \in \mathrm{Re}(\mathcal{H}(C_0, G_\ell))$, where $\mathcal{H}(C_0, G_\ell)$ denotes the space of holomorphic quadratic differentials which are real on the boundary.*

The space of holomorphic quadratic differentials on C_0 which are real on the boundary is L^2 orthogonal to the action of diffeomorphisms, see Lemma 2.4 of [31]. This means that if instead of using the diffeomorphisms f_ℓ we pull back by a different family of diffeomorphisms $\hat{f}_\ell$ that fix the parametrization of the boundary curves and obtain metrics $\hat{G}_\ell = \hat{f}_\ell^* g_\ell$ we can still view almost critical points as fitting into the setting of Theorem 5.1.1.

Such a simple choice of these diffeomorphisms (in particular that we only need to consider a one parameter family) can be made since the space of holomorphic quadratic differentials which are real on the boundary of a cylinder $[-Y, Y] \times S^1$ with respect to its natural conformal structure is one dimensional, spanned by elements of the form $c dz^2$ for $c \in \mathbb{R}$ where $z = s + i\theta$ and $(s, \theta) \in [-Y, Y] \times S^1$.

We have the following uniform Poincaré inequality on the domains (C_0, G_ℓ), equivalently $(\mathcal{C}_{Y(\ell)}, g_\ell)$, for which we give a short proof in the Section 5.6.2 of the Appendix.

Lemma 5.2.2. *There exists $C < \infty$ such that for every $w \in H_0^1(C_0, G_\ell)$ we have*

$$\|w\|_{L^2(C_0,G_\ell)} \leq C\|w\|_{\dot{H}^1(C_0,G_\ell)}. \tag{5.2.5}$$

Later on we will frequently use that we can change perspectives between working on (C_0, G_ℓ) or on $(\mathcal{C}_{Y(\ell)}, g_\ell)$ since the objects we work with will be invariant under pullback by diffeomorphisms. It will be useful that when working on the the domain C_0 we can be very explicit and use the functions s_ℓ describing the transition from C_0 to $\mathcal{C}_{Y(\ell)}$. We can write G_ℓ in terms of the coordinates (x, θ) on C_0 as

$$G_\ell = (\rho_\ell \circ s_\ell)^2 \left[(\partial_x s_\ell)^{-2} \mathrm{d}x^2 + \mathrm{d}\theta^2\right] \tag{5.2.6}$$

and hence as a consequence of (5.2.4) we have

$$\partial_\ell (\rho_\ell \circ s_\ell)^2 = \frac{\ell}{2\pi^2}. \tag{5.2.7}$$

Another consequence of G_ℓ being a horizontal curve, and therefore in particular trace free, is that the volume form $d\mu_{G_\ell} = (\rho_\ell \circ s_\ell)^2 \frac{\partial s_\ell}{\partial x} \mathrm{d}x\mathrm{d}\theta$ is independent of ℓ. We will need the consequence that

$$\partial_\ell[(\rho_\ell \circ s_\ell)^2 (\partial_x s_\ell)] = 0 \tag{5.2.8}$$

and using (5.2.7) also

$$\partial_\ell \frac{\partial s_\ell}{\partial x} = -\frac{\ell}{2\pi^2 (\rho_\ell \circ s_\ell)^2} \frac{\partial s_\ell}{\partial x} \tag{5.2.9}$$

and

$$\partial_\ell[(\rho_\ell \circ s_\ell)^{-2} \partial_x s_\ell] = -\frac{\ell}{\pi^2} (\rho_\ell \circ s_\ell)^{-4} \partial_x s_\ell. \tag{5.2.10}$$

We also have the following estimates for the function s_ℓ whose proof is given in Section 5.6.2 of the Appendix.

Lemma 5.2.3. *The function $s_\ell \colon [-1, 1] \to [-Y(\ell), Y(\ell)]$, $\ell \in (0, 1)$, satisfies the following estimates. For $\pm s_\ell \geq 0$, or equivalently on $\{\pm x \in [0, 1]\}$, we have*

$$|\partial_\ell (Y(\ell) \mp s_\ell)| \leq C|Y(\ell) \mp s_\ell|^2 \tag{5.2.11}$$

and

$$\left|\partial_\ell \left(\frac{s_\ell}{Y(\ell)}\right)\right| \leq C|Y(\ell) \mp s_\ell|. \tag{5.2.12}$$

We emphasize that while we make heavy use of the specific form of the diffeomorphism $f_\ell(x,\theta) = (s_\ell(x),\theta)$ some key facts from this section remain in more general settings. In particular, for a general smooth curve of hyperbolic metrics $g(t)$ with $\partial_t g = P_g(\Psi(t))$ we have $\partial_t(\rho^2) = -\mathrm{Re}(b_0) + O(e^{-1/\rho}\|\Psi\|_{L^1})$, compare with (5.2.7), where b_0 is the principal part of $P_g(\Psi)$ on the collar, see [36].

5.2.2 Definition of the maps z_ℓ

The goal of this section will be to construct the maps z_ℓ needed to define our set of adapted critical points $\mathcal{Z}$. We will start by defining maps $z_Y\colon [-Y,Y]\times S^1 \to N$ for $Y\in(Y_*,\infty)$, $Y_*>1$ chosen below. Then given $\ell\in(0,\ell_*)$, where $\ell_*>0$ is chosen such that $Y(\ell_*)=Y_*$, we pull back using the diffeomorphisms $f_\ell\colon C_0\to[-Y(\ell),Y(\ell)]\times S^1$ to define $z_\ell = f_\ell^* z_{Y(\ell)}$. We partition the domain $[-Y,Y]\times S^1$ into different sub-cylinders and, roughly speaking, we will parametrize the geodesic γ on the middle of this domain, the harmonic maps $v_\pm$ on the outer parts and interpolate between these on a carefully chosen transition region. For this construction it will be easier to interpolate using the ambient Euclidean space $\mathbb{R}^n$ in which N is isometrically embedded to obtain maps $\hat{z}_Y\colon[-Y,Y]\times S^1\to\mathbb{R}^n$ and in the end project these maps onto N.

Let us now proceed with the details of the construction. Let $v_\pm\colon D\to N$, $\gamma\colon[-1,1]\to N$ be as in Theorem 5.1.2, in particular satisfying assumptions (A), (B) and (C). Fix a number $\alpha>1$. We partition the domain $[-Y,Y]\times S^1$, $Y\in(Y_*,\infty)$ for $Y_*>1$ defined below, into the following sub-cylinders:

- The central region $\Omega_Y = [-Y+2\alpha\log Y, Y-2\alpha\log Y]\times S^1$
- The end regions $\Sigma_Y^\pm = \{(s,\theta)\mid \pm s\in[Y-\alpha\log Y, Y]\}$ (5.2.13)
- The transition regions $T_Y^\pm = \{(s,\theta)\mid \pm s\in[Y-2\alpha\log Y, Y-\alpha\log Y]\}$.

We now define the auxiliary maps $\hat{z}_Y\colon[-Y,Y]\times S^1\to\mathbb{R}^n$ as follows. First note that we can view the maps $v_\pm\colon D\to N$ as maps $\bar{v}_\pm\colon[0,\infty)\times S^1\to N$ by setting $\bar{v}_\pm(s,\theta)=v_\pm(e^{-s}e^{i\theta})$. We choose a smooth cut-off function $\psi\colon[0,\infty)\to[0,1]$ such that $\psi\equiv0$ on $[0,1]$ and $\psi\equiv1$ on $[2,\infty)$. Define

$$\hat{z}_Y(s,\theta) = \begin{cases}(1-\psi(\frac{Y-|s|}{\alpha\log Y}))\bar{v}_-(Y+s,\theta)+\psi(\frac{Y-|s|}{\alpha\log Y})\gamma(\frac{s}{Y}) & \text{on } [-Y,0]\times S^1\\ (1-\psi(\frac{Y-|s|}{\alpha\log Y}))\bar{v}_+(Y-s,\theta)+\psi(\frac{Y-|s|}{\alpha\log Y})\gamma(\frac{s}{Y}) & \text{on } [0,Y]\times S^1.\end{cases} \tag{5.2.14}$$

We will now explain why, for $Y>0$ large, we can obtain a map z_Y taking values in N by projecting $\hat{z}_Y$. Observe that on the middle region Ω_Y and on the ends $\Sigma_Y^\pm$ the map $\hat{z}_Y$ already takes values in N. Meanwhile for $(s,\theta)\in T_Y^\pm$ we have, since $\gamma(\pm1)=v_\pm(0)$,

$$\begin{aligned}|\bar{v}_\pm(Y\mp s,\theta)-\gamma(\tfrac{s}{Y})| &\le |v_\pm(e^{-(Y\mp s)}e^{i\theta})-v_\pm(0)|+|\gamma(\pm1)-\gamma(\tfrac{s}{Y})|\\ &\le Ce^{-(Y\mp s)}+C|1-\tfrac{|s|}{Y}|\le CY^{-\alpha}+CY^{-1}\log Y\end{aligned} \tag{5.2.15}$$

where $C = \max(|\gamma'|, \|\nabla v_\pm\|_{L^\infty(D)})$. Recall that there exists $\delta_N > 0$ such that the nearest point projection Π to N is well defined and smooth on a δ_N tubular neighbourhood of N in $\mathbb{R}^n$. The estimate (5.2.15) lets us choose $Y_* > 1$ large enough that for $Y > Y_*$ we have $\operatorname{dist}(\hat{z}_Y, N) \le CY^{-1}\log Y < \delta_N$ and hence obtain a well-defined map

$$z_Y = \Pi(\hat{z}_Y)\colon [-Y, Y] \times S^1 \to N \text{ for } Y \ge Y_*. \tag{5.2.16}$$

Finally, slightly abusing notation, we define maps

$$z_\ell = z_{Y(\ell)} \circ f_\ell \colon C_0 \to N \text{ for } \ell \in (0, \ell_*) \tag{5.2.17}$$

where ℓ_* is determined by $Y(\ell_*) = Y_*$ and $f_\ell\colon C_0 \to \mathcal{C}_{Y(\ell)}$ are given in Lemma 5.2.1. This allows us to define

$$\mathcal{Z} = \{(z_\ell, \ell) \mid \ell \in (0, \ell_*)\} \tag{5.2.18}$$

which we call the set of "adapted critical points". We note that the elements of this set are not critical points of the energy $E(u, \ell)$, the name instead refers to the fact that they have been obtained by modifying critical points $v_\pm$ and γ of the harmonic map energy on their respective domains.

Remark 5.2.4. We always have two points of view available: we can work on the cylinder $\mathcal{C}_{Y(\ell)} = [-Y(\ell), Y(\ell)] \times S^1$ equipped with the metrics $g_\ell = \rho_\ell(s)^2(\mathrm{d}s^2 + \mathrm{d}\theta^2)$ or on the fixed cylinder $C_0 = [-1, 1] \times S^1$ equipped with the metrics $G_\ell = f_\ell^* g_\ell$. We will use both view points later on in different situations. In what follows we will denote objects defined on $\mathcal{C}_{Y(\ell)}$ with a subscript $Y(\ell)$ and the corresponding objects defined on C_0 obtained by pulling back by f_ℓ with a subscript ℓ, with an exception for the conformal factor ρ_ℓ and metrics g_ℓ.

In addition to the L^2 inner product $\langle \cdot, \cdot \rangle_*$ defined in (5.1.1) we will in addition need another inner product on the space $T_{(u,\ell)}H \subset H_0^1(C_0; \mathbb{R}^n) \times \mathbb{R}$, one which is related to the notion of closeness used in Theorem 5.1.2. We will consider the inner product $\langle \cdot, \cdot \rangle_\ell$ which is characterized by the orthogonality $\langle (w, 0), (0, 1) \rangle_\ell = 0$ and the associated norm $\| \cdot \|_\ell$ satisfying

$$\|(w, 0)\|_\ell = \|w\|_{\dot{H}^1(C_0, G_\ell)}, \quad \|(0, 1)\|_\ell = \ell^{-1}, \tag{5.2.19}$$

for every $w \in H_0^1(C_0; \mathbb{R}^n)$. We will see in the next sections that the norm $\| \cdot \|_\ell$ is a natural one with which to measure continuity properties of the first and second variation of the energy acting on the set of adapted critical points $\mathcal{Z}$. We also remark that this norm is related to the L^2 norm $\| \cdot \|_*$, defined in (5.1.1), by

$$\|(0, 1)\|_\ell \approx \ell^{-\frac{1}{2}} \|(0, 1)\|_*, \quad \|(w, 0)\|_* \le C\|(w, 0)\|_\ell$$

since we have $\|\partial_\ell G_\ell\|_{L^2(C_0,G_\ell)} \approx \ell^{-\frac{1}{2}}$ as well as the Poincaré inequality (5.2.5). Here and in the following we write $a(\ell) \approx b(\ell)$ if there exists $\ell_* > 0$ and $C < \infty$ such that for all $\ell \in (0, \ell_*)$, $C^{-1}a(\ell) \leq b(\ell) \leq Ca(\ell)$.

Remark 5.2.5. Given $\lambda, \ell > 0$ defining $\|\cdot\|_*$ using metrics G_ℓ or G_λ will lead to equivalent norms provided $\|(0, \lambda - \ell)\|_\ell \leq \frac{1}{2}$. The equivalence of the map component follows since the volume form $\mathrm{d}\mu_{G_\ell}$ is independent of ℓ, while the equivalence of the second component follows because $\|(0, \lambda - \ell)\|_\ell \leq \frac{1}{2}$ implies that $\frac{2}{3}\ell^{-1} \leq \lambda^{-1} \leq 2\ell^{-1}$ and hence $\|\partial_\ell G_\ell\|_{L^2(C_0,G_\ell)} \approx \|\partial_\lambda G_\lambda\|_{L^2(C_0,G_\lambda)}$. The same assumption implies also that the norms $\|\cdot\|_\ell$ and $\|\cdot\|_\lambda$ are equivalent, this is described in Remark 5.4.5.

5.2.3 Basic properties of elements of $\mathcal{Z}$

We will collect basic estimates for the maps $z_{Y(\ell)}\colon \mathcal{C}_{Y(\ell)} \to N$ and also $z_\ell = z_{Y(\ell)} \circ f_\ell\colon C_0 \to N$, defined in Section 5.2.2 above, which will be used later on. We recall the important subdivision of the domain $\mathcal{C}_{Y(\ell)}$ into sub-cylinders $\Omega_{Y(\ell)}$, $\Sigma^\pm_{Y(\ell)}$ and $T^\pm_\ell$ as described in (5.2.13).

Here and in the following ∇ will denote the Euclidean gradient on $\mathcal{C}_{Y(\ell)}$ and ∇_ℓ will denote the gradient on C_0 computed with respect to the metric $\widetilde{G}_\ell = f_\ell^*(\mathrm{d}s^2 + \mathrm{d}\theta^2)$. We recall that $f_\ell(x, \theta) = (s_\ell(x), \theta)$ is defined in (5.2.3) and note that the properties derived in Lemma 5.2.3 will be important in what follows. We will also frequently need that $Y(\ell) \approx \ell^{-1}$ and $|\partial_\ell Y(\ell)| \leq C\ell^{-2}$ for $\ell > 0$ small enough.

On $\Omega_{Y(\ell)}$ the map $z_{Y(\ell)} = \gamma(\frac{s}{Y(\ell)})$ is a parametrization of the geodesic γ and hence we have

$$|\nabla z_{Y(\ell)}| \circ f_\ell = Y(\ell)^{-1}|\gamma'| \leq C\ell \text{ on } \Omega_\ell, \tag{5.2.20}$$

where $\Omega_\ell = f_\ell^{-1}\Omega_{Y(\ell)}$ is the corresponding subset of C_0. On Ω_ℓ we can also write $z_\ell = z_{Y(\ell)} \circ f_\ell = \gamma(\frac{s_\ell}{Y(\ell)})$ giving

$$|\partial_\ell z_\ell| = |\gamma'||\partial_\ell \tfrac{s_\ell}{Y(\ell)}| \leq C|Y(\ell) - |s_\ell|| \text{ on } \Omega_\ell, \tag{5.2.21}$$

where we have used (5.2.12).

Meanwhile, on $\Sigma^\pm_{Y(\ell)}$ we have $z_{Y(\ell)}(s, \theta) = v_\pm(e^{-(Y(\ell)\mp s)}e^{i\theta})$ and so we obtain

$$|\nabla z_{Y(\ell)}| \circ f_\ell \leq Ce^{-(Y(\ell)\mp s_\ell)} \text{ on } \Sigma^\pm_\ell, \tag{5.2.22}$$

where $\Sigma^\pm_\ell = f_\ell^{-1}\Sigma^\pm_{Y(\ell)} \subset C_0$. On $\Sigma^\pm_\ell$ we also have $z_\ell(x, \theta) = v_\pm(e^{-(Y(\ell)\mp s_\ell(x))}e^{i\theta})$ and hence we obtain

$$|\partial_\ell z_\ell| \leq C|\partial_\ell[e^{-(Y(\ell)\mp s_\ell)}]| \leq C|Y(\ell) \mp s_\ell|^2 e^{-(Y(\ell)\mp s_\ell)} \text{ on } \Sigma^\pm_\ell, \tag{5.2.23}$$

where we have used the estimate (5.2.11).

For completeness we give the details of the corresponding estimates on the transition region in Section 5.6.3 of the Appendix: in (5.6.28) we show

$$|\nabla z_{Y(\ell)}|\circ f_\ell \leq C\ell \text{ on } T_\ell^\pm, \tag{5.2.24}$$

while in (5.6.31) we show

$$|\partial_\ell z_\ell| \leq C|\log \ell| \text{ on } T_\ell^\pm. \tag{5.2.25}$$

It will be useful to summarize the estimates we have derived so far in this section globally on C_0 by comparing them with the conformal factor ρ_ℓ. Combining the above estimates with (5.6.16), (5.6.17) and (5.6.18) we can see that

$$|\nabla_\ell z_\ell|_{\widetilde{G}_\ell} = |\nabla z_{Y(\ell)}|\circ f_\ell \leq C\rho_\ell(s_\ell), \tag{5.2.26}$$

$$|\partial_\ell z_\ell| \leq C\rho_\ell(s_\ell)^{-1}. \tag{5.2.27}$$

We note for later use that the conformal factor satisfies

$$\tfrac{\ell}{2\pi} \leq \rho_\ell(s_\ell) \leq C|\log \ell|^{-1} \qquad \text{on } \Omega_\ell \tag{5.2.28}$$

$$C^{-1}|\log \ell|^{-1} \leq \rho_\ell(s_\ell) \leq C|\log \ell|^{-1} \qquad \text{on } T_\ell^\pm \tag{5.2.29}$$

$$C^{-1}|\log \ell|^{-1} \leq \rho_\ell(s_\ell) \leq C \qquad \text{on } \Sigma_\ell^\pm. \tag{5.2.30}$$

In addition we also derive estimates for $\partial_\ell[(\partial_i z_{Y(\ell)})\circ f_\ell]$ where we write $\partial_1 = \partial_s$ and $\partial_2 = \partial_\theta$ on $\mathcal{C}_{Y(\ell)}$. We have

$$|\partial_\ell[(\partial_s z_{Y(\ell)})\circ f_\ell]| = |\partial_\ell[Y(\ell)^{-1}\gamma'(\tfrac{s_\ell}{Y(\ell)})]| \leq C|\partial_\ell Y(\ell)^{-1}| + CY(\ell)^{-1}|\partial_\ell \tfrac{s_\ell}{Y(\ell)}| \leq C \text{ on } \Omega_\ell \tag{5.2.31}$$

using (5.2.12), while $(\partial_\theta z_{Y(\ell)})\circ f_\ell = 0$ here. We have for $i = 1, 2$

$$|\partial_\ell[(\partial_i z_{Y(\ell)})\circ f_\ell]| \leq C|\partial_\ell[e^{-(Y(\ell)\mp s_\ell)}]| \leq C|Y(\ell)\mp s_\ell|^2 e^{-(Y(\ell)\mp s_\ell)} \text{ on } \Sigma_\ell \tag{5.2.32}$$

using (5.2.11). In Section 5.6.3 of the Appendix we show

$$|\partial_\ell[(\partial_i z_{Y(\ell)})\circ f_\ell]| \leq C \text{ on } T_\ell^\pm, \tag{5.2.33}$$

see (5.6.34). All together we obtain

$$|\partial_\ell[(\partial_i z_{Y(\ell)})\circ f_\ell]| \leq C \text{ on } C_0. \tag{5.2.34}$$

Let us observe how the estimates derived so far give bounds on the size of our adapted critical points with respect to various norms we are interested in.

Proposition 5.2.6. *There exists $\ell_* > 0$, $C < \infty$ such that for any $\ell \in (0, \ell_*)$ we have*

$$\|\partial_\ell z_\ell\|_{L^2(C_0,G_\ell)} \leq C\ell^{-\frac{1}{2}} \tag{5.2.35}$$

and

$$\|\partial_\ell z_\ell\|_{\dot{H}^1(C_0,G_\ell)} \leq C\ell^{-\frac{1}{2}}. \tag{5.2.36}$$

Proof. The estimate (5.2.27) immediately implies

$$\|\partial_\ell z_\ell\|_{L^2(C_0,G_\ell)} = \left(\int_{C_0} |\partial_\ell z_\ell|^2 \rho_\ell(s_\ell)^2 \, \mathrm{d}\mu_{\widetilde{G}_\ell}\right)^{\frac{1}{2}} \leq C\mathrm{vol}(C_0, \widetilde{G}_\ell)^{\frac{1}{2}} \leq C\ell^{-\frac{1}{2}}$$

since $\mathrm{vol}(C_0, \widetilde{G}_\ell) = \mathrm{vol}(\mathcal{C}_{Y(\ell)}, g_E) = 4\pi Y(\ell) \leq C\ell^{-1}$.

Meanwhile, we have the formula

$$\|\partial_\ell z_\ell\|^2_{\dot{H}^1(C_0,G_\ell)} = \int_{C_0} |d(\partial_\ell z_\ell)|^2_{\widetilde{G}_\ell} \, \mathrm{d}\mu_{\widetilde{G}_\ell} = \int_{C_0} \left((\tfrac{\partial s_\ell}{\partial x})^{-2}|\partial_x\partial_\ell z_\ell|^2 + |\partial_\theta\partial_\ell z_\ell|^2\right) \mathrm{d}\mu_{\widetilde{G}_\ell}. \tag{5.2.37}$$

We note that $\partial_\theta\partial_\ell z_\ell = \partial_\ell\partial_\theta z_\ell = \partial_\ell[(\partial_\theta z_{Y(\ell)}) \circ f_\ell]$, while using (5.2.9) implies

$$\begin{aligned}\partial_\ell[(\tfrac{\partial s_\ell}{\partial x})^{-1}\partial_x z_\ell] &= \partial_\ell[(\tfrac{\partial s_\ell}{\partial x})^{-1}]\partial_x z_\ell + (\tfrac{\partial s_\ell}{\partial x})^{-1}\partial_\ell\partial_x z_\ell \\ &= \tfrac{\ell}{2\pi^2(\rho_\ell \circ s_\ell)^2}(\tfrac{\partial s_\ell}{\partial x})^{-1}\partial_x z_\ell + (\tfrac{\partial s_\ell}{\partial x})^{-1}\partial_x\partial_\ell z_\ell\end{aligned}$$

and then gives

$$(\tfrac{\partial s_\ell}{\partial x})^{-1}\partial_x\partial_\ell z_\ell = \partial_\ell[(\partial_s z_{Y(\ell)}) \circ f_\ell] - \tfrac{\ell}{2\pi^2(\rho_\ell \circ s_\ell)^2}(\partial_s z_{Y(\ell)}) \circ f_\ell.$$

Therefore, using (5.2.33) and (5.2.26) we obtain

$$(\tfrac{\partial s_\ell}{\partial x})^{-2}|\partial_x\partial_\ell z_\ell|^2 + |\partial_\theta\partial_\ell z_\ell|^2 \leq C$$

and so returning to (5.2.37) we have

$$\|\partial_\ell z_\ell\|_{\dot{H}^1(C_0,G_\ell)} \leq C\mathrm{vol}(C_0, \widetilde{G}_\ell)^{\frac{1}{2}} \leq C\ell^{-\frac{1}{2}}$$

as desired. □

From the proof of this Proposition we also obtain the following pointwise bound

$$|\nabla_\ell[\partial_\ell z_\ell]|_{\widetilde{G}_\ell} = \left[(\tfrac{\partial s_\ell}{\partial x})^{-2}|\partial_x\partial_\ell z_\ell|^2 + |\partial_\theta\partial_\ell z_\ell|^2\right]^{\frac{1}{2}} \leq C \tag{5.2.38}$$

which we will make use of later.

We also note an estimate for $\partial_\ell G_\ell$ which is already observed in [31]. The formula (5.2.4) gives $|\partial_\ell G_\ell|^2_{G_\ell} = 2\left(\frac{\ell}{2\pi^2}\right)^2 \rho_\ell^{-4}(s_\ell)$ giving

$$\|\partial_\ell G_\ell\|_{L^2(C_0,G_\ell)} = \frac{\ell}{\sqrt{2}\pi^2}\left(\int_{\mathcal{C}_{Y(\ell)}} \rho_\ell^{-2}(s) \, \mathrm{d}s\mathrm{d}\theta\right)^{\frac{1}{2}} \approx C\ell^{-1}. \tag{5.2.39}$$

We hence obtain from (5.2.39) and (5.2.35) that

$$\|(\partial_\ell z_\ell, 1)\|_* = \left(\|\partial_\ell z_\ell\|_{L^2}^2 + \|\partial_\ell G_\ell\|_{L^2}^2\right)^{\frac{1}{2}} \approx \ell^{-\frac{1}{2}}. \tag{5.2.40}$$

and from (5.2.36) that

$$\|(\partial_\ell z_\ell, 1)\|_\ell = \left(\|\partial_\ell z_\ell\|_{\dot{H}^1}^2 + \ell^{-2}\right)^{\frac{1}{2}} \approx \ell^{-1}. \tag{5.2.41}$$

Furthermore, setting $y_z = \frac{\partial_\ell(z_\ell,\ell)}{\|\partial_\ell(z_\ell,\ell)\|_\ell}$ we have

$$\|y_z\|_* \approx \ell^{\frac{1}{2}}. \tag{5.2.42}$$

We finish this section by giving a technical lemma describing certain behaviour of integrals over the transition regions $T_\ell^\pm$ behave. The proof of this Lemma will be provided in Section 5.6.2 of the Appendix.

Lemma 5.2.7. *Let $T_\ell^\pm = f_\ell^{-1} T_{Y(\ell)}^\pm$ be the transition region defined in (5.2.13). Given $\omega \in L^1(C_0, G_\ell)$ we have*

$$\left| \frac{\partial}{\partial \ell} \int_{T_\ell^\pm} \omega \, \mathrm{d}\mu_{G_\ell} \right| \leq C\ell^{-1} |\log \ell|^{-1} \|\omega\|_{L^1(\partial T_\ell^\pm, G_\ell)} \tag{5.2.43}$$

5.3 Analysis of the energy and its variations on $\mathcal{Z}$

In this section our aim will be to prove various facts about the energy E on the set $\mathcal{Z}$ of adapted critical points that will be used in the proof of Theorem 5.1.2.

Given $(u,\ell) \in H$ we define the first variation of E at (u,ℓ) in direction $(w,p) \in T_{(u,\ell)}H = \Gamma^{H_0^1}(u^*TN) \times \mathbb{R}$ by

$$dE(u,\ell)(w,p) = \tfrac{d}{dt} E(u+tw, \ell+tp)\big|_{t=0}. \tag{5.3.1}$$

We have the familiar formulae for each component of this object since the energy E is just a restriction of the harmonic map energy,

$$dE(u,\ell)(w,0) = -\int_{C_0} \tau_{G_\ell}(u) \cdot w \, \mathrm{d}\mu_{G_\ell} \tag{5.3.2}$$

and

$$dE(u,\ell)(0,1) = -\frac{1}{4} \int_{C_0} \langle \mathrm{Re}\Phi(u, G_\ell), \partial_\ell G_\ell \rangle_{G_\ell} \, \mathrm{d}\mu_{G_\ell}. \tag{5.3.3}$$

We remark that we have the formula (5.2.4) for $\partial_\ell G_\ell$ and so can further write

$$\begin{aligned} dE(u,\ell)(0,1) &= -\int_{\mathcal{C}_{Y(\ell)}} \frac{\ell}{2\pi^2 \rho_\ell^2} \left(|\tilde{u}_s|^2 - |\tilde{u}_\theta|^2\right) \mathrm{d}s\mathrm{d}\theta \text{ where } \tilde{u} = u \circ f_\ell^{-1} \\ &= -\int_{C_0} \frac{\ell}{2\pi^2 (\rho_\ell \circ s_\ell)^2} \left((\partial_x s_\ell)^{-1} |u_x|^2 - (\partial_x s_\ell)|u_\theta|^2\right) \mathrm{d}x\mathrm{d}\theta. \end{aligned} \tag{5.3.4}$$

We can also write the first variation in map directions in terms of $\widetilde{G}_\ell = f_\ell^*(\mathrm{d}s^2 + \mathrm{d}\theta^2)$ using conformal invariance,

$$dE(u,\ell)(w,0) = \langle u, w\rangle_{\dot{H}^1(C_0,G_\ell)} = \int_{C_0} \langle \nabla_\ell u, \nabla_\ell w\rangle_{\widetilde{G}_\ell}\, \mathrm{d}\mu_{\widetilde{G}_\ell}. \tag{5.3.5}$$

Note that as a consequence of the above formula and that u takes values in N we always have $dE(u,\ell)(w,0) = dE(u,\ell)(P_u w, 0)$, where $P_a = d_a\Pi$ for $a \in N$, where Π is the nearest point projection onto N.

We have the following expansion of the energy $E(z_\ell,\ell)$ and its derivative.

Proposition 5.3.1. *There exists $\ell_* > 0$ such that for any $\ell \in (0,\ell_*)$ we have*

$$E(z_\ell,\ell) = E(v_+) + E(v_-) + \frac{L(\gamma)^2}{2\pi}\ell + O(\ell^2|\log \ell|) \tag{5.3.6}$$

and

$$\frac{\partial}{\partial \ell}E(z_\ell,\ell) = \frac{L(\gamma)^2}{2\pi} + O(\ell|\log \ell|). \tag{5.3.7}$$

Remark 5.3.2. We will need later a normalized version of the latter estimate above, keeping in mind that $\|\partial_\ell(z_\ell,\ell)\|_\ell \approx \ell^{-1}$. Precisely, there exists $c_0 > 0$ such that

$$c_0\ell \le dE(z_\ell,\ell)(y_z) \le c_0^{-1}\ell \tag{5.3.8}$$

where $y_z = \frac{\partial_\ell(z_\ell,\ell)}{\|\partial_\ell(z_\ell,\ell)\|_\ell}$. This gives a precise estimate on how fast the energy on $\mathcal{Z}$ goes to zero and the rate (or rather an upper bound on the rate) will be important later.

Proof of Proposition 5.3.1. The key here will be that the map z_ℓ has a simple form in terms of the geodesic $\gamma\colon [-1,1] \to N$ on Ω_ℓ and in terms of the harmonic maps $v_\pm\colon D \to N$ on $\Sigma_\ell^\pm$, see Section 5.2.3, from which we can deduce the behaviour of the energy.

For now we will work with the maps $z_{Y(\ell)}\colon \mathcal{C}_{Y(\ell)} = [-Y(\ell), Y(\ell)] \times S^1 \to N$, see Remark 5.2.4. Given $U \subset \mathcal{C}_{Y(\ell)}$ we write

$$E_U(\ell) := \frac{1}{2}\int_U |\nabla z_{Y(\ell)}|^2\, \mathrm{d}s\mathrm{d}\theta \tag{5.3.9}$$

so that the energy splits into its contributions from each of the regions $\Sigma^\pm_{Y(\ell)}$, $\Omega_{Y(\ell)}$, $T^\pm_{Y(\ell)}$ in our partition of $\mathcal{C}_{Y(\ell)}$ as

$$E(z_\ell,\ell) = E_{\Sigma^+_{Y(\ell)}} + E_{\Sigma^-_{Y(\ell)}} + E_{\Omega_{Y(\ell)}} + E_{T^-_{Y(\ell)}} + E_{T^+_{Y(\ell)}}.$$

We first show that the leading order term comes from region $\Sigma^\pm_{Y(\ell)} = \{\pm s \in [Y(\ell) - \alpha\log Y(\ell), Y(\ell)]\} \times S^1$ where we have $z_{Y(\ell)}(s,\theta) = \bar{v}_\pm(Y(\ell) \mp s, \theta)$. Making a change of variables $s \mapsto Y(\ell) \mp s$ in (5.3.9) we obtain

$$\begin{aligned} E_{\Sigma^\pm_\ell}(\ell) &= \frac{1}{2}\int_0^{\alpha\log Y(\ell)} \int_{S^1} |\nabla \bar{v}_\pm|^2\, \mathrm{d}\theta\mathrm{d}s = E(\bar{v}_\pm) - \int_{\alpha \log Y(\ell)}^\infty \int_{S^1} |\nabla \bar{v}_\pm|^2\, \mathrm{d}\theta\mathrm{d}s \\ &= E(v_\pm) - \int_0^{Y(\ell)^{-\alpha}} \int_{S^1} |\nabla v_\pm|^2\, r\mathrm{d}\theta\mathrm{d}r \end{aligned} \tag{5.3.10}$$

using conformal invariance of the energy and $\bar{v}_\pm(s,\theta) = v_\pm(e^{-s}e^{i\theta})$ in the final step. The maps $v_\pm \in C^1(D)$ have uniformly bounded gradient giving $\int_0^{Y^{-\alpha}} \int_{S^1} |\nabla v_\pm|^2 \, r\mathrm{d}\theta\mathrm{d}r = O(Y(\ell)^{-2\alpha})$ and since $Y(\ell) = \frac{2\pi}{\ell}(\frac{\pi}{2} - \arctan\frac{\ell}{d})$ we have $Y(\ell)^{-1} = \frac{\ell}{\pi^2} + O(\ell^2)$ and can conclude

$$E_{\Sigma^\pm_{Y(\ell)}}(\ell) = E(v_\pm) + O(\ell^{2\alpha}) = E(v_\pm) + O(\ell^2), \tag{5.3.11}$$

since $\alpha \geq 1$. Differentiating the final equality in (5.3.10) and using the Leibniz integral rule gives

$$\begin{aligned} \left|\frac{\partial}{\partial \ell} E_{\Sigma^\pm_\ell}\right| &= \frac{1}{2}\left|\frac{\partial}{\partial \ell}\int_0^{Y(\ell)^{-\alpha}} \int_{S^1} |\nabla v_\pm|^2 \, r\mathrm{d}r\mathrm{d}\theta\right| \\ &\leq C\partial_\ell(Y(\ell)^{-\alpha})Y(\ell)^{-\alpha}\|\nabla v_\pm\|^2_{L^\infty(D)} = O(\ell^{2\alpha-1}) = O(\ell), \end{aligned} \tag{5.3.12}$$

where for the final step we additionally use $\partial_\ell Y(\ell)^{-1} = \frac{1}{\pi^2} + O(\ell) = O(1)$.

Meanwhile on the region $\Omega_{Y(\ell)} = \{|s| \leq Y(\ell) - 2\alpha\log Y(\ell)\} \times S^1$, where $z_{Y(\ell)}$ is a parametrization of the geodesic γ, we have $z_{Y(\ell)}(s,\theta) = \gamma(\frac{s}{Y(\ell)})$. Hence $|\nabla z_{Y(\ell)}|^2 = Y(\ell)^{-2}|\gamma'|^2$ and then since $|\gamma'|^2 = \frac{1}{4}L(\gamma)^2$ we have

$$\begin{aligned} E_{\Omega_{Y(\ell)}}(\ell) &= \frac{1}{2}\int_{S^1}\int_{|s|\leq Y(\ell)-2\alpha\log Y(\ell)} \frac{L(\gamma)^2}{4} Y(\ell)^{-2}\,\mathrm{d}s\mathrm{d}\theta \\ &= \pi[2Y(\ell) - 4\alpha\log Y(\ell)]\frac{L(\gamma)^2}{4}Y(\ell)^{-2} \\ &= \frac{L(\gamma)^2}{2\pi}\ell + O(\ell^2|\log\ell|). \end{aligned} \tag{5.3.13}$$

Differentiating the middle line in the above formula gives

$$\begin{aligned} \frac{\partial}{\partial \ell}E_{\Omega_{Y(\ell)}} &= \frac{\partial}{\partial \ell}\left[\frac{\pi}{2}Y(\ell)^{-1} - \pi\alpha\frac{\log Y(\ell)}{Y(\ell)^2}\right]L(\gamma)^2 \\ &= \frac{L(\gamma)^2}{2\pi} + O(\ell) - \pi\alpha L(\gamma)^2\partial_\ell\frac{\log Y(\ell)}{Y(\ell)^2} \\ &= \frac{L(\gamma)^2}{2\pi} + O(\ell) - \pi\alpha L(\gamma)^2 Y(\ell)^{-3}(1 - 2\log Y(\ell))\partial_\ell Y(\ell) \\ &= \frac{L(\gamma)^2}{2\pi} + O(\ell|\log\ell|) \end{aligned} \tag{5.3.14}$$

since $\partial_\ell Y(\ell) = O(\ell^{-2})$.

We now look at the contribution from the transition region. We note that the estimate (5.2.24) immediately implies

$$E_{T^\pm_{Y(\ell)}} = \int_{T^\pm_{Y(\ell)}} |\nabla z_{Y(\ell)}|^2\,\mathrm{d}s\mathrm{d}\theta \leq C\ell^2\mathrm{vol}(T^\pm_{Y(\ell)}, g_E) \leq C\ell^2|\log\ell| \tag{5.3.15}$$

since $\mathrm{vol}(T^\pm_{Y(\ell)}, g_E) = 2\pi\alpha\log Y(\ell) \leq C|\log\ell|$.

For the derivative of $E_{T^{\pm}_{Y(\ell)}}$ we will change view points and work on the corresponding subset $T^{\pm}_\ell \subset C_0$ rather than $T^{\pm}_{Y(\ell)} \subset \mathcal{C}_{Y(\ell)}$, recalling again the convention described in Remark 5.2.4. We have that $E_{T^{\pm}_{Y(\ell)}}(\ell) = \frac{1}{2}\int_{T^{\pm}_\ell} |dz_\ell|^2_{G_\ell}\, \mathrm{d}\mu_{G_\ell}$. The estimate (5.2.24) gives

$$|dz_\ell|^2_{G_\ell} = (\rho_\ell \circ s_\ell)^{-2}|\nabla z_{Y(\ell)}|^2 \circ f_\ell \leq C\ell^2|\log \ell|^2 \tag{5.3.16}$$

since $(\rho_\ell \circ s_\ell) \approx |\log \ell|^{-1}$ on $T^{\pm}_\ell$ by (5.2.29). We also have by (5.2.33) and again (5.2.24)

$$\left|\partial_\ell[|\nabla z_{Y(\ell)}|^2 \circ f_\ell]\right| \leq C\ell \text{ on } T^{\pm}_\ell. \tag{5.3.17}$$

Therefore, using additionally (5.2.7) to give $\partial_\ell(\rho_\ell \circ s_\ell)^{-2} = -(\rho_\ell \circ s_\ell)^{-4}\partial_\ell(\rho_\ell \circ s_\ell)^2 = O(\ell|\log \ell|^4)$, we have

$$\begin{aligned}|\partial_\ell|dz_\ell|^2_{G_\ell}| &= (\rho_\ell \circ s_\ell)^{-2}|\partial_\ell[|\nabla z_{Y(\ell)}|^2 \circ f_\ell]| + \partial_\ell[(\rho_\ell \circ s_\ell)^{-2}]|\nabla z_{Y(\ell)}|^2 \circ f_\ell \\ &\leq C\ell|\log \ell|^2 + C\ell^2|\log \ell|^4 \leq C\ell|\log \ell|^2 \text{ on } T^{\pm}_\ell.\end{aligned} \tag{5.3.18}$$

To estimate the derivative of $E_{T^{\pm}_{Y(\ell)}}$ we split

$$\frac{\partial}{\partial \ell}E_{T^{\pm}_{Y(\ell)}} = \frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{T^{\pm}_{\ell+\varepsilon}} |dz_\ell|^2_{G_\ell}\, \mathrm{d}\mu_{G_{\ell+\varepsilon}} + \frac{1}{2}\int_{T^{\pm}_\ell} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|dz_{\ell+\varepsilon}|^2_{G_{\ell+\varepsilon}}\, \mathrm{d}\mu_{G_\ell}, \tag{5.3.19}$$

and then use (5.3.18) to estimate the second term

$$\left|\int_{T^{\pm}_\ell} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|dz_{\ell+\varepsilon}|^2_{G_{\ell+\varepsilon}}\, \mathrm{d}\mu_{G_\ell}\right| \leq \mathrm{vol}(T^{\pm}_\ell, G_\ell)\|\partial_\ell[|dz_\ell|^2_{G_\ell}]\|_{L^\infty(T^{\pm}_\ell)} \leq C\ell|\log \ell| \tag{5.3.20}$$

since now $\mathrm{vol}(T^{\pm}_\ell, G_\ell) = \int_{T^{\pm}_{Y(\ell)}} \rho^2_\ell(s)\, \mathrm{d}s\mathrm{d}\theta \leq C|\log \ell|^{-1}$. The first term of (5.3.19) can be bounded using Lemma 5.2.7 and (5.3.16),

$$\left|\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{T^{\pm}_{\ell+\varepsilon}} |dz_\ell|^2_{G_\ell}\, \mathrm{d}\mu_{G_{\ell+\varepsilon}}\right| \leq C\ell^{-1}|\log \ell|^{-1}\mathrm{vol}(\partial T^{\pm}_\ell, G_\ell)\||dz_\ell|^2_{G_\ell}\|_{L^\infty(\partial T^{\pm}_\ell)} \leq C\ell, \tag{5.3.21}$$

where now we use that $\mathrm{vol}(\partial T^{\pm}_\ell, G_\ell) \leq C|\log \ell|^{-1}$. Returning to (5.3.19) and using (5.3.20) and (5.3.21) yields

$$|\partial_\ell E_{T^{\pm}_{Y(\ell)}}| \leq C\ell|\log \ell|. \tag{5.3.22}$$

To conclude the proof we combine (5.3.11), (5.3.13) and (5.3.15) to give

$$\begin{aligned}E(z_\ell, \ell) &= E_{\Sigma^+_{Y(\ell)}} + E_{\Sigma^-_{Y(\ell)}} + E_{\Omega_{Y(\ell)}} + E_{T^-_{Y(\ell)}} + E_{T^+_{Y(\ell)}} \\ &= E(v_+) + E(v_-) + \frac{L(\gamma)^2}{2\pi}\ell + O(\ell^2|\log \ell|)\end{aligned}$$

and combine (5.3.12), (5.3.14) and (5.3.22) to give

$$\frac{\partial}{\partial \ell}E(z_\ell, \ell) = \partial_\ell E_{\Omega_\ell} + O(\ell|\log \ell|) = \frac{L(\gamma)^2}{2\pi} + O(\ell|\log \ell|),$$

as desired.

$\square$

Along with Proposition 5.3.1 above which describes the behaviour of the first variation of E in directions tangent to the specific set $\mathcal{Z}$, we will also need to understand the first variation of E in more general directions which is done in the following Proposition.

Proposition 5.3.3. *There exists $C < \infty$ and $\ell_* > 0$ such that for any $\ell \in (0, \ell_*)$ we have*

$$|dE(z_\ell,\ell)(0,1)| \leq C, \qquad \left|\frac{\partial}{\partial \ell} dE(z_\ell,\ell)(0,1)\right| \leq C|\log \ell| \tag{5.3.23}$$

and for $w \in \Gamma^{H_0^1}(z_\ell{}^*TN)$ we have

$$|dE(z_\ell,\ell)(w,0)| \leq C\ell|\log \ell|^{\frac{1}{2}}\|w\|_{L^2(C_0,G_\ell)}, \qquad \left|\frac{\partial}{\partial \ell} dE(z_\ell,\ell)(w,0)\right| \leq C|\log \ell|^{\frac{1}{2}}\|w\|_{L^2(C_0,G_\ell)}. \tag{5.3.24}$$

Remark 5.3.4. In the spirit of Remark 5.3.2 Proposition 5.3.3 yields the normalized estimate, keeping in mind that we are setting $y_z = \frac{\partial_\ell(z_\ell,\ell)}{\|\partial_\ell(z_\ell,\ell)\|_\ell}$ and $\|\partial_\ell(z_\ell,\ell)\|_\ell \approx \ell^{-1}$,

$$\left|d^2E(z_\ell,\ell)[y_z,W]\right| = \frac{1}{\|\partial_\ell(z_\ell,\ell)\|_\ell}\left|\frac{\partial}{\partial \ell} dE(z_\ell,\ell)(W)\right| \leq C\ell|\log \ell|^{\frac{1}{2}}\|W\|_\ell. \tag{5.3.25}$$

This further uses that by the Poincaré inequality (5.2.5) and that $1 \leq \ell^{-1} = \|(0,1)\|_\ell$ for $\ell \in (0,1)$. The Proposition also gives

$$|dE(z_\ell,\ell)(W)| \leq C\ell|\log \ell|^{\frac{1}{2}}\|W\|_\ell. \tag{5.3.26}$$

Proof of Proposition 5.3.3. We start by estimating $dE(z_\ell,\ell)(0,1)$ and its derivative, leaving the variations in the map direction to be considered after. Given $U \subset \mathcal{C}_{Y(\ell)}$ write

$$J_U(\ell) = -\int_U \frac{\ell}{2\pi^2\rho_\ell^2}(|\partial_s z_{Y(\ell)}|^2 - |\partial_\theta z_{Y(\ell)}|^2)\,\mathrm{d}s\mathrm{d}\theta \tag{5.3.27}$$

so that owing to (5.3.4) and (5.2.13) we have

$$dE(z_\ell,\ell)(0,1) = J_{\Omega_{Y(\ell)}} + J_{\Sigma^-_{Y(\ell)}} + J_{\Sigma^+_{Y(\ell)}} + J_{T^+_{Y(\ell)}} + J_{T^-_{Y(\ell)}}. \tag{5.3.28}$$

To prove our claimed estimates on $dE(z_\ell,\ell)(0,1)$ and its derivative it therefore suffices to estimate the contribution from each region and we will do each one in turn.

We start with the regions $\Sigma^\pm_{Y(\ell)} = \{\pm s \in [Y(\ell) - \alpha \log Y(\ell), Y(\ell)]\}$ where we have $z_{Y(\ell)}(s,\theta) = \bar{v}_\pm(Y(\ell) \mp s, \theta)$. We will see that there is no contribution from this region with a brief digression into the Hopf differential and holomorphic quadratic differentials.

Recall that $\bar{v}_\pm = v_\pm \circ f$ for $f(s,\theta) = e^{-s}e^{i\theta}$. Since $v_\pm\colon D \to N$ are harmonic maps their Hopf differentials $\Phi(v_\pm, \mathrm{d}x^2 + \mathrm{d}y^2)$ are holomorphic quadratic differentials and so can be written in terms of the complex coordinate $w = x + iy$ as $\Phi(v_\pm, \mathrm{d}x^2 + \mathrm{d}y^2) = \phi(w)\mathrm{d}w^2$ for a function ϕ that is holomorphic on D. The map f is a conformal diffeomorphism, which we note reverses orientation, and we can compute for $z = s + i\theta$ that

$$\bar{w} \circ f = e^{-z}, \quad f^*(\tfrac{\partial}{\partial \bar{w}}) = -e^{-z}\tfrac{\partial}{\partial z}, \quad f^*(\mathrm{d}\bar{w}^2) = e^{-2z}\mathrm{d}z^2$$

giving

$$\Phi(\bar{v}_\pm, \mathrm{d}s^2 + \mathrm{d}\theta^2) = 4(\bar{v}_\pm^* g_N)(\tfrac{\partial}{\partial z}, \tfrac{\partial}{\partial z})\mathrm{d}z^2 = f^*\left[\overline{4(v_\pm^* g_N)(\tfrac{\partial}{\partial w}, \tfrac{\partial}{\partial w})\mathrm{d}\bar{w}^2}\right] = f^*(\bar{\phi}(w)\mathrm{d}\bar{w}^2).$$

We have $\phi(w) = \sum_{n=0}^\infty b_n w^n$ for coefficients $b_n \in \mathbb{C}$ as ϕ is a holomorphic function on D and so we obtain

$$\Phi(\bar{v}_\pm, \mathrm{d}s^2 + \mathrm{d}\theta^2) = \left(\sum_{n=2}^\infty c_n e^{-nz}\right)\mathrm{d}z^2 = \left(\sum_{n=2}^\infty c_n e^{-ns}e^{-in\theta}\right)\mathrm{d}z^2, \tag{5.3.29}$$

where $c_n = \overline{b_{n-2}}$. Note that to observe such a series expansion starting from $n = 1$ it would have already been enough to use that $\Phi(\bar{v}_\pm, \mathrm{d}s^2 + \mathrm{d}\theta^2)$ is an L^1 holomorphic quadratic differential (as $\bar{v}_\pm$ are harmonic and $\|\Phi(\bar{v}_\pm, \mathrm{d}s^2 + \mathrm{d}\theta^2)\|_{L^1} \leq CE(\bar{v}_\pm) < \infty$) since being holomorphic on $[0, \infty) \times S^1$ means we can obtain a Laurent series expansion, and the L^1 norm being finite ensures there can be no non-decaying terms. In particular, the vanishing of the term corresponding to $n = 0$ in (5.3.29) together with the fact that each term is L^2 orthogonal yields, writing $g_E = \mathrm{d}s^2 + \mathrm{d}\theta^2$,

$$0 = \mathrm{Re}\int_{S^1} \langle \Phi(\bar{v}_\pm, g_E), \mathrm{d}z^2\rangle_{g_E}\,\mathrm{d}\theta = 2\int_{S^1} |\partial_s \bar{v}_\pm|^2 - |\partial_\theta \bar{v}_\pm|^2\,\mathrm{d}\theta. \tag{5.3.30}$$

Making the change of variables $t = Y(\ell) \mp s$ in the definition of $J_{\Sigma^\pm_{Y(\ell)}}$ we obtain

$$J_{\Sigma^\pm_{Y(\ell)}}(\ell) = -\int_0^{\alpha \log Y} \frac{\ell}{2\pi^2 \rho_\ell^2(Y(\ell) \mp t)} \int_{S^1} |\partial_t \bar{v}_\pm|^2 - |\partial_\theta \bar{v}_\pm|^2 \mathrm{d}\theta\,\mathrm{d}t = 0 \tag{5.3.31}$$

since the integral over S^1 vanishes by (5.3.30). It therefore also holds that $\partial_\ell J_{\Sigma^\pm_\ell} = 0$.

On $\Omega_{Y(\ell)} = [-Y(\ell) + 2\alpha \log Y(\ell), Y(\ell) - 2\alpha \log Y(\ell)]$ we have $z_{Y(\ell)}(s, \theta) = \gamma(\frac{s}{Y(\ell)})$ and so $|\partial_s z_{Y(\ell)}|^2 - |\partial_\theta z_{Y(\ell)}|^2 = Y(\ell)^{-2}|\gamma'|^2$. It is then already enough to use in the definition of $J_{\Omega_{Y(\ell)}}$ that $\rho_\ell^{-1} \leq 2\pi\ell^{-1}$ to give

$$|J_{\Omega_{Y(\ell)}}| \leq C\frac{Y(\ell) - 2\alpha \log Y(\ell)}{\ell Y(\ell)^2} \leq \frac{C}{\ell Y(\ell)} \leq C, \tag{5.3.32}$$

where we use in the final step that $\frac{2\pi}{\ell Y(\ell)} = \frac{1}{\frac{\pi}{2} - \arctan d^{-1}\ell} = O(1)$. Since we will also want to differentiate we will however also need an explicit expansion of this term. Setting

$\widetilde{Y}(\ell) = Y(\ell) - 2\alpha \log Y(\ell)$ we have

$$\begin{aligned} J_{\Omega_{Y(\ell)}} &= -2\pi \frac{|\gamma'|^2}{Y(\ell)^2} \int_{-\widetilde{Y}(\ell)}^{\widetilde{Y}(\ell)} \frac{\ell}{2\pi^2 \rho_\ell^2(s)} \,\mathrm{d}s \\ &= -2\pi \frac{|\gamma'|^2}{Y(\ell)^2} \frac{\ell}{2\pi^2} \int_{-\tilde{Y}(\ell)}^{\tilde{Y}(\ell)} \left(\tfrac{2\pi}{\ell}\right)^2 \cos^2(\tfrac{\ell s}{2\pi}) \mathrm{d}s \\ &= -2 \frac{|\gamma'|^2}{Y(\ell)^2} \left(\frac{2\pi}{\ell}\right)^2 \left[\tfrac{\ell \widetilde{Y}(\ell)}{2\pi} + \cos \tfrac{\ell \widetilde{Y}(\ell)}{2\pi} \sin \tfrac{\ell \widetilde{Y}(\ell)}{2\pi}\right]. \end{aligned} \tag{5.3.33}$$

To differentiate this we need some facts about the functions $Y(\ell)$ and $\widetilde{Y}(\ell)$. We have $\frac{2\pi}{\ell Y(\ell)} = \frac{1}{\frac{\pi}{2} - \arctan d^{-1}\ell} \approx 1$ and so $\partial_\ell(\frac{2\pi}{\ell Y(\ell)}) \approx 1$ also. In particular we have $\log Y(\ell) \approx |\log \ell|$ and $|\partial_\ell Y(\ell)| \approx \ell^{-2}$. Meanwhile, it holds that $\frac{\ell \widetilde{Y}(\ell)}{2\pi} = \frac{\ell Y(\ell)}{2\pi} - 2\alpha \frac{\ell}{2\pi} \log Y(\ell)$ giving

$$\left| \frac{\ell \widetilde{Y}(\ell)}{2\pi} \right| \leq C + C\ell |\log Y(\ell)| \leq C$$

and

$$\begin{aligned} \left| \partial_\ell \frac{\ell \widetilde{Y}(\ell)}{2\pi} \right| &\leq \left| \partial_\ell \frac{\ell Y(\ell)}{2\pi} \right| + \frac{\alpha}{\pi} |\partial_\ell [\ell \log Y(\ell)]| \\ &\leq C + C \log Y(\ell) + C\ell |\partial_\ell \log Y(\ell)| \\ &\leq C \log Y(\ell) + C\ell \frac{|\partial_\ell Y(\ell)|}{Y(\ell)} \leq C|\log \ell| + C \\ &\leq C|\log \ell|. \end{aligned}$$

We can then differentiate (5.3.33) to give

$$|\partial_\ell J_{\Omega_\ell}| \leq C|\log \ell|. \tag{5.3.34}$$

For the transition region we will change view points and work on the subsets $T_\ell^\pm \subset C_0$ obtained by pulling back by f_ℓ, see Remark 5.2.4. We can write

$$J_{T^\pm_{Y(\ell)}} = \int_{T_\ell^\pm} \varphi_\ell \,\mathrm{d}\mu_{G_\ell} \text{ where } \varphi_\ell := \frac{\ell}{2\pi^2 (\rho_\ell \circ s_\ell)^4} (|\partial_s z_{Y(\ell)}|^2 - |\partial_\theta z_{Y(\ell)}|^2) \circ f_\ell.$$

The estimate (5.2.24), together with the fact that $(\rho_\ell \circ s_\ell)^{-1} \approx |\log \ell|$ on $T_\ell^\pm$, immediately gives

$$|\varphi_\ell| \leq C\ell^3 |\log \ell|^4. \tag{5.3.35}$$

The derivative satisfies

$$\begin{aligned} \partial_\ell \varphi_\ell = &\left[\frac{1}{2\pi^2 (\rho_\ell \circ s_\ell)^4} + \frac{\ell}{2\pi^2} \partial_\ell \frac{1}{(\rho_\ell \circ s_\ell)^4}\right] (|\partial_s z_{Y(\ell)}|^2 - |\partial_\theta z_{Y(\ell)}|^2) \circ f_\ell \\ &+ \frac{\ell}{2\pi^2 (\rho_\ell \circ s_\ell)^4} \partial_\ell \left[(|\partial_s z_{Y(\ell)}|^2 - |\partial_\theta z_{Y(\ell)}|^2) \circ f_\ell\right] \end{aligned}$$

and so, since (5.2.7) implies $\partial_\ell(\rho_\ell\circ s_\ell)^{-4}=-2(\rho_\ell\circ s_\ell)^{-6}\partial_\ell(\rho_\ell\circ s_\ell)^2=O(\ell|\log\ell|^6)$ on $T_\ell^\pm$, we can use the estimate (5.3.17) to give

$$|\partial_\ell\varphi_\ell|\leq C\ell^2[|\log\ell|^4+\ell^2|\log\ell|^6]+\ell^2|\log\ell|^4\leq C\ell^2|\log\ell|^4. \tag{5.3.36}$$

A bound for $J_{T^\pm_{Y(\ell)}}$ immediately follows from (5.3.35),

$$|J_{T^\pm_{Y(\ell)}}|\leq \mathrm{vol}(T_\ell^\pm,G_\ell)\|\varphi_\ell\|_{L^\infty(T_\ell^\pm)}\leq C\ell^3|\log\ell|^3. \tag{5.3.37}$$

since $\mathrm{vol}(T_\ell^\pm,G_\ell)\leq C|\log\ell|^{-1}$. Meanwhile we split the derivative into two terms, immediately using Lemma 5.2.7 on the first term below, and using (5.3.36) on the second term,

$$\begin{aligned}|\partial_\ell J_{T_\ell^\pm}|&=\left|\tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{T^\pm_{\ell+\varepsilon}}\varphi_\ell\,\mathrm{d}\mu_{G_{\ell+\varepsilon}}+\int_{T_\ell^\pm}\tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\varphi_{\ell+\varepsilon}\,\mathrm{d}\mu_{G_\ell}\right|\\&\leq C\ell^{-1}|\log\ell|^{-1}\|\varphi_\ell\|_{L^1(\partial T_\ell^\pm,G_\ell)}+\mathrm{vol}(T_\ell^\pm,G_\ell)\|\partial_\ell\varphi_\ell\|_{L^\infty(T_\ell^\pm)}\\&\leq C\ell^{-1}|\log\ell|^{-1}\mathrm{vol}(\partial T_\ell^\pm,G_\ell)\|\varphi_\ell\|_{L^\infty(\partial T_\ell^\pm)}+C\ell^2|\log\ell|^3\\&\leq C\ell^2|\log\ell|^2+C\ell^2|\log\ell|^3\leq C\ell^2|\log\ell|^3,\end{aligned}\tag{5.3.38}$$

where we have used that $\mathrm{vol}(\partial T_\ell^\pm,G_\ell)\leq C|\log\ell|^{-1}$ and $\mathrm{vol}(T_\ell^\pm,G_\ell)\leq C|\log\ell|^{-1}$.

All together, inserting $J_{\Sigma_\ell^\pm}=0$ and the estimate (5.3.37) into (5.3.28) gives

$$dE(z_\ell,\ell)(0,1)=J_{\Omega_{Y(\ell)}}+O(\ell^3|\log\ell|^3).$$

Similarly, using now (5.3.38) gives

$$\partial_\ell[dE(z_\ell,\ell)(0,1)]=\partial_\ell J_{\Omega_{Y(\ell)}}+O(\ell^2|\log\ell|^3).$$

We can therefore conclude our first claim (5.3.23) by inserting (5.3.32) and (5.3.34) into these formulae.

To estimate $dE(z_\ell,\ell)(w,0)$ for $w\in\Gamma^{H^1_0}(z_\ell{}^*TN)$ we note that the maps z_ℓ are harmonic maps with respect to G_ℓ on the regions Ω_ℓ and $\Sigma_\ell^\pm$ and therefore we only need to look at the transition regions $T_\ell^\pm$.

We first derive L^2 bounds on the tension and its derivative on the transition region. We have $|\tau_{G_\ell}(z_\ell)|_{G_\ell}=(\rho_\ell\circ s_\ell)^{-2}|\tau_{g_E}(z_{Y(\ell)})|\circ f_\ell$ and we can compute

$$|\tau_{g_E}(z_{Y(\ell)})|\circ f_\ell\leq C\ell|\log\ell|^{-1} \tag{5.3.39}$$

and

$$|\partial_\ell[\tau_{g_E}(z_{Y(\ell)})\circ f_\ell]|\leq C|\log\ell|^{-1}. \tag{5.3.40}$$

We give the details of these two estimates in Section 5.6.3 of the Appendix, see (5.6.37) and (5.6.41). Using (5.3.39) together with $(\rho_\ell \circ s_\ell)^{-1} \approx |\log \ell|$ we have

$$\begin{aligned} \|\tau_{G_\ell}(z_\ell)\|_{L^2(T_\ell^\pm, G_\ell)} &= \left(\int_{T_\ell^\pm} (\rho_\ell \circ s_\ell)^{-4} (|\tau_{g_E}(z_{Y(\ell)})|^2) \circ f_\ell \, \mathrm{d}\mu_{G_\ell} \right)^{\frac{1}{2}} \\ &\leq C|\log \ell|^2 \ell |\log \ell|^{-1} \mathrm{vol}(T_\ell^\pm, G_\ell)^{\frac{1}{2}} \leq C\ell |\log \ell|^{\frac{1}{2}} \end{aligned} \tag{5.3.41}$$

where in the final step we use $\mathrm{vol}(T_\ell^\pm, G_\ell) \leq C|\log \ell|^{-1}$. We also have

$$\begin{aligned} |\partial_\ell[\tau_{G_\ell}(z_\ell)]| &= \left|\partial_\ell[(\rho_\ell \circ s_\ell)^{-2}](\tau_{g_E}(z_{Y(\ell)})) \circ f_\ell + (\rho_\ell \circ s_\ell)^{-2} \partial_\ell[(\tau_{g_E}(z_{Y(\ell)})) \circ f_\ell]\right| \\ &\leq C\ell^2 |\log \ell|^3 + C|\log \ell| \leq C|\log \ell| \end{aligned}$$

where we use the estimates (5.3.39) and (5.3.40) and additionally that $\partial_\ell(\rho_\ell \circ s_\ell)^{-2} = -(\rho_\ell \circ s_\ell)^{-4} \partial_\ell(\rho_\ell \circ s_\ell)^2 = O(\ell|\log \ell|^4)$ from (5.2.7). We therefore obtain

$$\|\partial_\ell[\tau_{G_\ell}(z_\ell)]\|_{L^2(T_\ell^\pm, G_\ell)} \leq C|\log \ell| \mathrm{vol}(T_\ell^\pm, G_\ell)^{\frac{1}{2}} \leq C|\log \ell|^{\frac{1}{2}}. \tag{5.3.42}$$

With these L^2 estimates in hand we then return to estimating the first variation, using (5.3.41) in (5.3.2) gives

$$\begin{aligned} |dE(z_\ell, \ell)(w, 0)| &\leq \|\tau_{G_\ell}(z_\ell)\|_{L^2(T_\ell^\pm, G_\ell)} \|w\|_{L^2(C_0, G_\ell)} \\ &\leq C\ell |\log \ell|^{\frac{1}{2}} \|w\|_{L^2(C_0, G_\ell)}, \end{aligned}$$

which is the first desired estimate from (5.3.24). To estimate the derivative we note that the volume form $\mathrm{d}\mu_{G_\ell}$ is independent of ℓ and also that $\partial_\ell[\tau_{G_\ell}(z_\ell)]$ vanishes away from $T_\ell^\pm$, giving

$$\begin{aligned} \left| \frac{\partial}{\partial \ell} [dE(z_\ell, \ell)(w, 0)] \right| &= \left| \int_{T_\ell^\pm} \partial_\ell[\tau_{G_\ell}(z_\ell)] \cdot w \, \mathrm{d}\mu_{G_\ell} \right| \\ &\leq \|\partial_\ell[\tau_{G_\ell}(z_\ell)]\|_{L^2(T_\ell^\pm, G_\ell)} \|w\|_{L^2(C_0, G_\ell)} \\ &\leq C|\log \ell|^{\frac{1}{2}} \|w\|_{L^2(C_0, G_\ell)} \end{aligned} \tag{5.3.43}$$

using (5.3.42) in the final step. This gives the second estimate from (5.3.24), completing the proof of Proposition 5.3.3. □

5.4 Properties of the second variation

In this section we will prove a number of facts about the second variation of the energy on the space H. Given $(u, \ell) \in H$ and $(w, p), (v, q) \in T_{(u,\ell)}H$ such that $w, v \in L^\infty$ we define the second variation

$$d^2E(u, \ell)[(w, p), (v, q)] = \tfrac{\partial^2}{\partial t_1 \partial t_2} E(\Pi(u + t_1 w + t_2 v), \ell + t_1 p + t_2 q)\big|_{t_1, t_2 = 0} \tag{5.4.1}$$

where Π denotes the nearest point projection of a tubular neighbourhood of N in $\mathbb{R}^n$ onto N. We remark now that the formulae (5.4.8), (5.4.7), (5.4.6) derived below make sense for any functions $v, w \in H^1$ and in what follows we use the standard abuse of notation writing $d^2E(u,\ell)$ for the corresponding bilinear form defined on H^1.

The second variation in the map directions is just given by the second variation of the Dirichlet energy,

$$d^2E(u,\ell)[(w,0),(v,0)] = \int_{C_0} \langle dw, dv\rangle_{G_\ell} - A_{G_\ell}(u)(\mathrm{d}u,\mathrm{d}u)A(u)(w,v)\,\mathrm{d}\mu_{G_\ell} \tag{5.4.2}$$

where $A_{G_\ell}(u)(\mathrm{d}u,\mathrm{d}u) = G_\ell^{ij}A(u)(\partial_i u, \partial_j u)$ and A denotes the second fundamental form of $(N, g_N) \hookrightarrow \mathbb{R}^n$. We can also express this working in the pulled back setting

$$d^2E(u,\ell)[(w,0),(v,0)] = \int_{\mathcal{C}_{Y(\ell)}} \nabla\tilde{w}\cdot\nabla\tilde{v} - A(\tilde{u})(\nabla\tilde{u},\nabla\tilde{u})A(\tilde{u})(\tilde{w},\tilde{v})\,\mathrm{d}s\mathrm{d}\theta \tag{5.4.3}$$

where $\tilde{u} = u \circ f_\ell^{-1}$, $\tilde{w} = w \circ f_\ell^{-1}$, $\tilde{v} = v \circ f_\ell^{-1}$ and ∇ denotes the standard Euclidean gradient on $\mathcal{C}_{Y(\ell)}$.

Taking a variation of the first expression in (5.3.4) with respect to the map gives a formula for the mixed terms

$$d^2E(u,\ell)[(w,0),(0,1)] = -\int_{\mathcal{C}_{Y(\ell)}} \frac{\ell}{2\pi^2\rho_\ell^2}\left(\tilde{u}_s\tilde{w}_s - \tilde{u}_\theta\tilde{w}_\theta\right)\mathrm{d}s\mathrm{d}\theta. \tag{5.4.4}$$

Similarly we can differentiate the second expression in (5.3.4) with respect to ℓ and use the formulae (5.2.8) and (5.2.10) to give

$$\begin{aligned} &d^2E(u,\ell)[(0,1),(0,1)] \\ &= -\frac{1}{2\pi^2}\int_{\mathcal{C}_{Y(\ell)}} \rho_\ell^{-2}(|\tilde{u}_s|^2 - |\tilde{u}_\theta|^2)\,\mathrm{d}s\mathrm{d}\theta - \frac{\ell}{2\pi^2}\int_{C_0} \partial_\ell[(\rho_\ell\circ s_\ell)^{-2}(\partial_x s_\ell)^{-1}]|u_x|^2\,\mathrm{d}x\mathrm{d}\theta \\ &\quad - \frac{\ell}{2\pi^2}\int_{C_0} \partial_\ell[(\rho_\ell\circ s_\ell)^{-2}\partial_x s_\ell]|u_\theta|^2\,\mathrm{d}x\mathrm{d}\theta \\ &= -\frac{1}{2\pi^2}\int_{\mathcal{C}_{Y(\ell)}} \rho_\ell^{-2}(|\tilde{u}_s|^2 - |\tilde{u}_\theta|^2)\,\mathrm{d}s\mathrm{d}\theta - \frac{\ell^2}{2\pi^4}\int_{\mathcal{C}_{Y(\ell)}} \rho_\ell^{-4}|\tilde{u}_\theta|^2\,\mathrm{d}s\mathrm{d}\theta. \end{aligned} \tag{5.4.5}$$

In particular we have using $\rho_\ell^{-1} \le 2\pi\ell^{-1}$ that

$$\begin{aligned} |d^2E(u,\ell)[(w,0),(0,1)]| &\le C\int_{\mathcal{C}_{Y(\ell)}} \rho_\ell^{-1}(s)|\nabla\tilde{u}||\nabla\tilde{w}|\,\mathrm{d}s\mathrm{d}\theta \\ &\le C\|\rho_\ell^{-1}\nabla\tilde{u}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)}\|\nabla\tilde{w}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)} \end{aligned} \tag{5.4.6}$$

and

$$|d^2E(u,\ell)[(0,1),(0,1)]| \le C\|\rho_\ell^{-1}\nabla\tilde{u}\|^2_{L^2(C_0,g_E)}. \tag{5.4.7}$$

We also remark that the variations in the map direction are bounded as follows

$$|d^2E(u,\ell)[(w,0),(v,0)]| \le \int_{\mathcal{C}_{Y(\ell)}} |\nabla\tilde{w}||\nabla\tilde{v}| + C|\nabla\tilde{u}|^2|\tilde{w}||\tilde{v}|\,\mathrm{d}s\mathrm{d}\theta \tag{5.4.8}$$

We are particularly interested in the behaviour of the second variation at (z_ℓ,ℓ) which is described in the following lemma.

Lemma 5.4.1. *There exists $\ell_* > 0$ and $C < \infty$ such that for every $\ell \in (0,\ell_*)$ and $v,w \in \Gamma^{H_0^1}({z_\ell}^*TN)$ we have*

$$|d^2E(z_\ell,\ell)[(w,0),(0,1)] \le C\ell^{\frac{1}{2}}\|(0,1)\|_\ell\|(w,0)\|_\ell \tag{5.4.9}$$

$$|d^2E(z_\ell,\ell)[(0,1),(0,1)]| \le C\ell\|(0,1)\|_\ell^2. \tag{5.4.10}$$

$$|d^2E(z_\ell,\ell)[(w,0),(v,0)] \le C\|(w,0)\|_\ell\|(v,0)\|_\ell \tag{5.4.11}$$

In particular, given $W,V \in T_{(z_\ell,\ell)}H$ we have

$$|d^2E(z_\ell,\ell)[W,V]| \le C\|W\|_\ell\|V\|_\ell. \tag{5.4.12}$$

Proof. In what follows we will again set $\tilde{w} = w \circ f_\ell^{-1}$ and note that this implies

$$\|\nabla\tilde{w}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)} = \|w\|_{\dot{H}^1(C_0,G_\ell)} = \|(w,0)\|_\ell.$$

Furthermore, by the Poincaré inequality (5.2.5) we have $\|\tilde{w}\|_{L^2(\mathcal{C}_{Y(\ell)},g_\ell)} \le C\|(w,0)\|_\ell$. We also recall that $\|(0,1)\|_\ell = \ell^{-1}$.

For $\ell > 0$ small enough, $\rho_\ell^{-1}|\nabla z_{Y(\ell)}|$ is uniformly bounded (see (5.2.26)) and therefore $\|\rho_\ell^{-1}\nabla z_{Y(\ell)}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)} \le C\mathrm{vol}(\mathcal{C}_{Y(\ell)},g_E)^{\frac{1}{2}} \le C\ell^{-\frac{1}{2}}$ and so we immediately obtain the first claim using (5.4.6),

$$\begin{aligned}|d^2E(z_\ell,\ell)[(w,0),(0,1)]| &\le C\|\rho_\ell^{-1}\nabla z_{Y(\ell)}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)}\|\nabla\tilde{w}\|_{L^2(\mathcal{C}_{Y(\ell)},g_E)}\\ &\le C\ell^{-\frac{1}{2}}\|(w,0)\|_\ell = C\ell^{\frac{1}{2}}\|(0,1)\|_\ell\|(w,0)\|_\ell\end{aligned}$$

and in the same way we obtain the second claim using (5.4.7),

$$|d^2E(z_\ell,\ell)[(0,1),(0,1)]| \le C\|\rho_\ell\nabla z_{Y(\ell)}\|^2_{L^2(\mathcal{C}_{Y(\ell)},g_E)} \le C\ell^{-1} = C\ell\|(0,1)\|_\ell^2.$$

For the third estimate we again use that $\rho_\ell^{-1}|\nabla z_{Y(\ell)}|$ is uniformly bounded and (5.4.8) to give

$$\begin{aligned}\left|d^2E(z_\ell,\ell)[(w,0),(v,0)]\right| &\le \int_{\mathcal{C}_{Y(\ell)}} |\nabla\tilde{w}||\nabla\tilde{v}| + C\rho_\ell^2|\tilde{w}||\tilde{v}|\,\mathrm{d}s\mathrm{d}\theta\\ &\le C\|\tilde{w}\|_{H^1(\mathcal{C}_{Y(\ell)},g_\ell)}\|\tilde{v}\|_{H^1(\mathcal{C}_{Y(\ell)},g_\ell)} \le C\|(w,0)\|_\ell\|(v,0)\|_\ell,\end{aligned}$$

using additionally the Poincaré inequality (5.2.5) to bound $\|\tilde{w}\|_{H^1(\mathcal{C}_{Y(\ell)},g_\ell)} = \|w\|_{H^1(C_0,G_\ell)} \le C\|w\|_{\dot{H}^1} = C\|(w,0)\|_\ell$. This concludes the proof since the final estimate (5.4.12) follows immediately from the first three. $\square$

We will now look at the second variation evaluated at points (u_t, ℓ_t), for $u_t = \Pi(z_\ell + tw)$ and $\ell_t = \ell + tp$, which interpolate between the pair (z_ℓ, ℓ) and $(u, \ell+p)$ for which $z_\ell + w$ takes values in N. Note that $d_a\Pi \colon \mathbb{R}^n \to T_pN$ is the orthogonal projection onto T_aN provided we only consider points $a \in N$, and we denote by P_a this orthogonal projection. In what follows we will often use properties of this orthogonal projection which are given in the Appendix 5.6.1.

The following is a technical lemma which will later be used to derive continuity properties of the second variation $d^2E(u_t, \ell_t)$ with respect to the norm $\|\cdot\|_\ell$. We note that the norms used on the right hand side of the forthcoming estimates (as well as the formulae for the second variation) are conformally invariant and therefore either the metric $\widetilde{G}_\ell = f_\ell^*(\mathrm{d}s^2 + \mathrm{d}\theta^2)$ or the metric $G_\ell = \rho_\ell(s_\ell)^2\widetilde{G}_\ell$ can be used; in the proof we will use $\widetilde{G}_\ell$.

Lemma 5.4.2. *There exists $\ell_* > 0$, $C < \infty$ such that the following holds for all $(z_\ell, \ell) \in \mathcal{Z}$ with $\ell \in (0, \ell_*)$ and $W = (w, p) \in H_0^1(C_0, \mathbb{R}^n) \times \mathbb{R}$ with $\|w\|_{L^\infty} < \delta_N$ and $\|W\|_\ell < \frac{1}{2}$.*

Define $u_t = \Pi(z_\ell + tw)$ and $\ell_t = \ell + tp$ for $t \in [0,1]$. Fix $v_1, v_2 \in H^1(C_0, \mathbb{R}^n) \cap L^\infty$. We then have the following estimates:

$$\left|d^2E(u_t, \ell_t)[(0,1),(0,1)] - d^2E(z_\ell, \ell)[(0,1),(0,1)]\right| \le C\|(0,1)\|_\ell^2\|W\|_\ell, \tag{5.4.13}$$

$$\begin{aligned}&\left|d^2E(u_t, \ell_t)[(P_{u_t}v_1, 0),(0,1)] - d^2E(z_\ell, \ell)[(P_{z_\ell}v_1, 0),(0,1)]\right| \\ &\quad \le C\|(0,1)\|_\ell\left(\|v_1\|_{H^1}\|W\|_\ell + \|v_1\|_{L^\infty}\|w\|_{\dot{H}^1}^2\right)\end{aligned} \tag{5.4.14}$$

and

$$\begin{aligned}&\left|d^2E(u_t, \ell_t)[(P_{u_t}v_1, 0),(P_{u_t}v_2, 0)] - d^2E(z_\ell, \ell)[(P_{z_\ell}v_1, 0),(P_{z_\ell}v_2, 0)]\right| \\ &\le C\left(\|w\|_{L^\infty} + \|(0,p)\|_\ell\right)\|v_1\|_{H^1}\|v_2\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{H^1}\|w\|_{H^1} \\ &\quad + C\|v_2\|_{L^\infty}\|v_2\|_{H^1}\|w\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{L^\infty}\|w\|_{\dot{H}^1}^2.\end{aligned} \tag{5.4.15}$$

Remark 5.4.3. In once instance later on we will need an alternative form of (5.4.15) which reads,

$$\begin{aligned}&\left|d^2E(u_t, \ell_t)[(P_{u_t}v_1, 0),(P_{u_t}v_2, 0)] - d^2E(z_\ell, \ell)[(P_{z_\ell}v_1, 0),(P_{z_\ell}v_2, 0)]\right| \\ &\le C\int_{C_0}|\nabla_\ell v_1||\nabla_\ell v_2||w| + (|v_1||\nabla_\ell v_2| + |\nabla_\ell v_1||v_2|)\left[|w|(\rho_\ell \circ s_\ell) + |\nabla_\ell w|\right]\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\quad + C\int_{C_0}|v_1||v_2|\left[|w|(\rho_\ell \circ s_\ell)^2 + (\rho_\ell \circ s_\ell)|\nabla_\ell w| + |\nabla_\ell w|^2\right]\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\quad + C|p|\int_{C_0}\rho_\ell(s_\ell)^{-1}\left[|\nabla_\ell v_1||\nabla_\ell v_2| + (|v_1||\nabla_\ell v_2| + |\nabla_\ell v_1||v_2|)\rho_\ell(s_\ell) + |v_1||v_2|\rho_\ell(s_\ell)^2\right]\mathrm{d}\mu_{\widetilde{G}_\ell},\end{aligned} \tag{5.4.16}$$

where $\widetilde{G}_\ell = f_\ell^*(\mathrm{d}s^2 + \mathrm{d}\theta^2)$, $\nabla_\ell = \nabla_{\widetilde{G}_\ell}$ and we suppress the dependence of the norms on $\widetilde{G}_\ell$. This estimate is obtained in (5.4.24) below and we note that in particular it implies (5.4.15).

Before giving proof, we will need an auxiliary Lemma which will allow us to work with the metrics G_ℓ or G_{ℓ_t}.

Lemma 5.4.4. *Given $k \in \mathbb{N}$ there exists $C < \infty$ such that for all $\ell > 0$, $p \in \mathbb{R}$ with $\ell^{-1}|p| \leq \frac{1}{2}$ the following holds. Given $a \in L^\infty(C_0)$ and $w_1, w_2 \in H^1(C_0, \mathbb{R}^n)$ we have*

$$\int_{C_0} |a||\nabla_{\ell_t} w_1||\nabla_{\ell_t} w_2|(\rho_{\ell_t} \circ s_{\ell_t})^{-k}\, \mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \leq C \int_{C_0} |a||\nabla_\ell w_1||\nabla_\ell w_2|(\rho_\ell \circ s_\ell)^{-k}\, \mathrm{d}\mu_{\widetilde{G}_\ell} \quad (5.4.17)$$

where $\ell_t = \ell + tp$, $t \in [0, 1]$.

Remark 5.4.5. An important consequence, taking $k = 0$ and $a = 1$ above, will be that given $W \in H^1(C_0, \mathbb{R}^n) \times \mathbb{R}$ such that $\|W\|_\ell \leq \frac{1}{2}$ we have

$$\|W\|_{\ell_t} \leq C\|W\|_\ell. \quad (5.4.18)$$

Note that the scaling of the second component follows from $\|(0, 1)\|_{\ell_t} = {\ell_t}^{-1} \leq 2\ell^{-1} = 2\|(0, 1)\|_\ell$, see also Remark 5.2.5.

Proof of Lemma 5.4.4. First note that $\ell^{-1}|p| \leq \frac{1}{2}$ implies ${\ell_t}^{-1} \leq 2\ell^{-1}$. Owing to (5.2.7) we have

$$|\tfrac{d}{dt}(\rho_{\ell_t} \circ s_{\ell_t})^{-k}| = |\tfrac{d}{dt}((\rho_{\ell_t} \circ s_{\ell_t})^2)^{-k/2}| = \frac{k}{2}(\rho_{\ell_t} \circ s_{\ell_t})^{-k-2}\frac{\ell_t}{2\pi^2}|p| \leq C(\rho_{\ell_t} \circ s_{\ell_t})^{-k}, \quad (5.4.19)$$

where we use in the final step that $\rho_{\ell_t}^{-1} \leq 2\pi{\ell_t}^{-1}$ implies $\frac{\ell_t}{(\rho_{\ell_t} \circ s_{\ell_t})^2}|p| \leq C{\ell_t}^{-1}|p| \leq C$.

Recall that ∇_{ℓ_t} is the gradient with respect to the metric $\widetilde{G}_{\ell_t} = f_{\ell_t}^*(\mathrm{d}s^2 + \mathrm{d}\theta^2) = (\partial_x s_{\ell_t})^2 \mathrm{d}x^2 + \mathrm{d}\theta^2$. We will need that (5.2.9) implies

$$|\tfrac{d}{dt}(\partial_x s_{\ell_t})^{\pm 1}| \leq \frac{\ell_t}{2\pi^2(\rho_{\ell_t} \circ s_{\ell_t})^2}(\partial_x s_{\ell_t})^{\pm 1}|p| \leq C(\partial_x s_{\ell_t})^{\pm 1}. \quad (5.4.20)$$

We can write

$$\begin{aligned} &|\nabla_{\ell_t} w_1||\nabla_{\ell_t} w_2|\, \mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\ &\quad = \left[(\partial_x s_{\ell_t})^{-1}(\partial_x w_1)^2 + (\partial_x s_{\ell_t})(\partial_\theta w_1)^2\right]^{\frac{1}{2}} \left[(\partial_x s_{\ell_t})^{-1}(\partial_x w_2)^2 + (\partial_x s_{\ell_t})(\partial_\theta w_2)^2\right]^{\frac{1}{2}} \mathrm{d}x\mathrm{d}\theta \end{aligned}$$

and can hence differentiate using (5.4.19) and (5.4.20) giving

$$\begin{aligned} &\left|\tfrac{d}{dt}\int_{C_0} |a||\nabla_{\ell_t} w_1||\nabla_{\ell_t} w_2|(\rho_{\ell_t} \circ s_{\ell_t})^{-k}\, \mathrm{d}\mu_{\widetilde{G}_{\ell_t}}\right| \\ &\qquad \leq C\int_{C_0} |a||\nabla_{\ell_t} w_1||\nabla_{\ell_t} w_2|(\rho_{\ell_t} \circ s_{\ell_t})^{-k}\, \mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \end{aligned}$$

from which we conclude the desired result using Grönwall's inequality. $\square$

Proof of Lemma 5.4.2. We start with estimate (5.4.15). We will compute the difference as

$$
\begin{aligned}
&\left|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(P_{u_t}v_2,0)] - d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(P_{z_\ell}v_2,0)]\right| \\
&\leq \sup_{t\in(0,1)} \left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(P_{u_{t+\varepsilon}}v_1,0),(P_{u_{t+\varepsilon}}v_2,0)]\right| \\
&\leq \sup_{t\in(0,1)} \left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_t)[(P_{u_{t+\varepsilon}}v_1,0),(P_{u_{t+\varepsilon}}v_2,0)]\right| \\
&\quad + \sup_{t\in(0,1)} \left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_t,\ell_{t+\varepsilon})[(P_{u_t}v_1,0),(P_{u_t}v_2,0)]\right| \\
&= T_1 + T_2.
\end{aligned}
$$

The formula (5.4.2) for the second variation gives

$$
\begin{aligned}
&d^2E(u,\ell)[(P_uv_1,0),(P_uv_2,0)] \\
&\qquad\qquad = \int_{C_0} \nabla_\ell P_uv_1\cdot\nabla_\ell P_uv_2 - A_{\widetilde{G}_\ell}(u)(\nabla_\ell u,\nabla_\ell u)A(u)(P_uv_1,P_uv_2)\,\mathrm{d}\mu_{\widetilde{G}_\ell}
\end{aligned}
$$

and hence we can use the estimates (5.6.6)-(5.6.13) in the Appendix to expand

$$
\begin{aligned}
|T_1| &= \left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_t)[(P_{u_{t+\varepsilon}}v_1,0),(P_{u_{t+\varepsilon}}v_2,0)]\right| \\
&\leq \int_{C_0} |\nabla_{\ell_t}\tfrac{d}{dt}P_{u_t}v_1||\nabla_{\ell_t}P_{u_t}v_2| + |\nabla_{\ell_t}P_{u_t}v_1||\nabla_{\ell_t}\tfrac{d}{dt}P_{u_t}v_2|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad + C\int_{C_0} (|\tfrac{d}{dt}u_t||\nabla_{\ell_t}u_t|^2 + |\nabla_{\ell_t}\tfrac{d}{dt}u_t||\nabla_{\ell_t}u_t|)|v_1||v_2|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad + C\int_{C_0} |\nabla_{\ell_t}u_t|^2(|\tfrac{d}{dt}P_{u_t}v_1||v_2| + |v_1||\tfrac{d}{dt}P_{u_t}v_2|)\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\leq C\int_{C_0} |w||\nabla_{\ell_t}v_1||\nabla_{\ell_t}v_2|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad + C\int_{C_0} (|v_1||\nabla_{\ell_t}v_2| + |\nabla_{\ell_t}v_1||v_2|)\,(|w||\nabla_{\ell_t}z_\ell| + |\nabla_{\ell_t}w|)\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad + C\int_{C_0} |v_1||v_2|\left(|w||\nabla_{\ell_t}z_\ell|^2 + |\nabla_{\ell_t}w|^2 + |\nabla_{\ell_t}w||\nabla_{\ell_t}z_\ell|\right)\mathrm{d}\mu_{\widetilde{G}_{\ell_t}}.
\end{aligned}
\tag{5.4.21}
$$

Note that all terms appearing in the final estimate of (5.4.21) are of the form so that Lemma 5.4.4 can be applied and hence we can further bound by the same expression with ℓ_t replaced by ℓ. Using also (5.2.26) to bound $|\nabla_\ell z_\ell| \leq C(\rho_\ell\circ s_\ell)$ we obtain

$$
\begin{aligned}
|T_1| &\leq C\int_{C_0} |w||\nabla_\ell v_1||\nabla_\ell v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\quad + C\int_{C_0} (|v_1||\nabla_\ell v_2| + |\nabla_\ell v_1||v_2|)\,(|w|(\rho_\ell\circ s_\ell) + |\nabla_\ell w|)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\quad + C\int_{C_0} |v_1||v_2|\left(|w|(\rho_\ell\circ s_\ell)^2 + |\nabla_\ell w|^2 + |\nabla_\ell w|(\rho_\ell\circ s_\ell)\right)\mathrm{d}\mu_{\widetilde{G}_\ell}.
\end{aligned}
\tag{5.4.22}
$$

For T_2 we use that by (5.2.9) we have $|\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\partial_x s_{\ell_{t+\varepsilon}}| = \frac{\ell_t}{2\pi^2(\rho_{\ell_t}\circ s_{\ell_t})^2}|p| \leq C(\rho_{\ell_t}\circ s_{\ell_t})^{-1}|p|$, and hence we can write

$$
\begin{aligned}
|T_2| &= \left|\tfrac{d}{d\varepsilon}\right|_{\varepsilon=0} d^2E(u_t,\ell_{t+\varepsilon})[(P_{u_t}v_1,0),(P_{u_t}v_2,0)]\Big| \\
&= \Big|\int_{C_0} \tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\partial_x s_{\ell_{t+\varepsilon}})^{-1}\left[\partial_x P_{u_t}v_1\partial_x P_{u_t}v_2 - A(u_t)(\partial_x u_t,\partial_x u_t)A(u_t)(P_{u_t}v_1,P_{u_t}v_2)\right]\mathrm{d}x\mathrm{d}\theta \\
&\quad + \int_{C_0} \tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\partial_x s_{\ell_{t+\varepsilon}})\left[\partial_\theta P_{u_t}v_1\partial_\theta P_{u_t}v_2 - A(u_t)(\partial_\theta u_t,\partial_\theta u_t)A(u_t)(P_{u_t}v_1,P_{u_t}v_2)\right]\mathrm{d}x\mathrm{d}\theta\Big| \\
&\le C|p|\int_{C_0}(\rho_{\ell_t}\circ s_{\ell_t})^{-1}\left[|\nabla_{\ell_t}P_{u_t}v_1||\nabla_{\ell_t}P_{u_t}v_2| + C|\nabla_{\ell_t}u_t|^2|v_1||v_2|\right]\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\le C|p|\int_{C_0}(\rho_{\ell_t}\circ s_{\ell_t})^{-1}\Big[(|\nabla_{\ell_t}v_1| + |v_1|(|\nabla_{\ell_t}z_\ell| + |\nabla_{\ell_t}w|))(|\nabla_{\ell_t}v_2| + |v_2|(|\nabla_{\ell_t}z_\ell| + |\nabla_{\ell_t}w|)) \\
&\qquad\qquad + (|\nabla_{\ell_t}z_\ell| + |\nabla_{\ell_t}w|)^2|v_1||v_2|\Big]\mathrm{d}\mu_{\widetilde{G}_{\ell_t}}.
\end{aligned}
$$

where we additionally the estimates (5.6.6)-(5.6.13). Applying Lemma 5.4.4 and (5.2.26) we then obtain

$$
\begin{aligned}
|T_2| &\le C|p|\int_{C_0}(\rho_\ell\circ s_\ell)^{-1}\Big[|\nabla_\ell v_1||\nabla_\ell v_2| + (|\nabla_\ell v_1||v_2| + |v_1||\nabla_\ell v_2|)(|\nabla_\ell z_\ell| + |\nabla_\ell w|) \\
&\qquad\qquad + (|\nabla_\ell z_\ell|^2 + |\nabla_\ell w|^2)|v_1||v_2|\Big]\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\le C|p|\int_{C_0}(\rho_\ell\circ s_\ell)^{-1}\Big[|\nabla_\ell v_1||\nabla_\ell v_2| + (|\nabla_\ell v_1||v_2| + |v_1||\nabla_\ell v_2|)(\rho_\ell\circ s_\ell) + |v_1||v_2|(\rho_\ell\circ s_\ell)^2\Big] \\
&\quad + C\|(0,p)\|_\ell\int_{C_0}(|\nabla_\ell v_1||v_2| + |v_1||\nabla_\ell v_2|)|\nabla_\ell w| + |\nabla_\ell w|^2|v_1||v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell}
\end{aligned}
\tag{5.4.23}
$$

using also that (5.2.19) implies $\rho_\ell^{-1}|p| \le C\|(0,p)\|_\ell$.

Adding (5.4.22) and (5.4.23) and using that $\|(0,p)\|_\ell \le \|W\|_\ell \le \frac{1}{2}$ gives

$$
\begin{aligned}
&\left|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(P_{u_t}v_2,0)] - d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(P_{z_\ell}v_2,0)]\right| = |T_1 + T_2| \\
&\le C\int_{C_0}|\nabla_\ell v_1||\nabla_\ell v_2||w| + (|v_1||\nabla_\ell v_2| + |\nabla_\ell v_1||v_2|)\left[|w|(\rho_\ell\circ s_\ell) + |\nabla_\ell w|\right] \\
&\qquad + |v_1||v_2|\left[|w|(\rho_\ell\circ s_\ell)^2 + (\rho_\ell\circ s_\ell)|\nabla_\ell w| + |\nabla_\ell w|^2\right]\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\quad + C|p|\int_{C_0}\rho_\ell(s_\ell)^{-1}\left[|\nabla_\ell v_1||\nabla_\ell v_2| + (|v_1||\nabla_\ell v_2| + |\nabla_\ell v_1||v_2|)\rho_\ell(s_\ell) + |v_1||v_2|\rho_\ell(s_\ell)^2\right]\mathrm{d}\mu_{\widetilde{G}_\ell},
\end{aligned}
\tag{5.4.24}
$$

from which we immediately obtain the desired result (5.4.15) using Hölder's inequality and the fact that $\rho_\ell^{-1}|p| \le C\|(0,p)\|_\ell$.

We now turn to the estimate (5.4.14). As before our starting point will be to look at

$$
\begin{aligned}
&\left|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(0,1)] - d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(0,1)]\right| \\
&\le \sup_{t\in(0,1)}\left|\tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(P_{u_{\ell_{t+\varepsilon}}}v_1,0),(0,1)]\right|.
\end{aligned}
\tag{5.4.25}
$$

We note that (5.2.8) and (5.2.10) imply

$$\left|\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{\ell_{t+\varepsilon}}{2\pi^2(\rho_{\ell_{t+\varepsilon}}\circ s_{\ell_{t+\varepsilon}})^2}(\partial_x s_{\ell_{t+\varepsilon}})^{\pm 1}\right|\leq C|p|(\rho_{\ell_t}\circ s_{\ell_t})^{-2}(\partial_x s_{\ell_t})^{\pm 1}.$$

Hence writing (5.4.4) in coordinates on C_0 we have

$$\begin{aligned}
&\left|\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(P_{u_{\ell_{t+\varepsilon}}}v_1,0),(0,1)]\right|\\
&=\Bigg|\int_{C_0}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\frac{\ell_{t+\varepsilon}}{2\pi^2(\rho_{\ell_{t+\varepsilon}}\circ s_{\ell_{t+\varepsilon}})^2}(\partial_x s_{\ell_{t+\varepsilon}})^{-1}\right]\partial_x u_t\partial_x P_{u_t}v_1\,\mathrm{d}x\mathrm{d}\theta\\
&\quad+\int_{C_0}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\frac{\ell_{t+\varepsilon}}{2\pi^2(\rho_{\ell_{t+\varepsilon}}\circ s_{\ell_{t+\varepsilon}})^2}(\partial_x s_{\ell_{t+\varepsilon}})\right]\partial_\theta u_t\partial_\theta P_{u_t}v_1\,\mathrm{d}x\mathrm{d}\theta\\
&\quad+\int_{C_0}\frac{\ell_t}{2\pi^2(\rho_{\ell_t}\circ s_{\ell_t})^2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\partial_x s_{\ell_t})^{-1}\partial_x u_{t+\varepsilon}\partial_x P_{u_{t+\varepsilon}}v_1-(\partial_x s_{\ell_t})\partial_\theta u_{t+\varepsilon}\partial_\theta P_{u_{t+\varepsilon}}v_1\right]\,\mathrm{d}x\mathrm{d}\theta\Bigg|\\
&\leq C|p|\int_{C_0}(\rho_{\ell_t}\circ s_{\ell_t})^{-2}|\nabla_{\ell_t}u_t||\nabla_{\ell_t}P_{u_t}v_1|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}}\\
&\quad+C\int_{C_0}\frac{\ell_t}{(\rho_{\ell_t}\circ s_{\ell_t})^2}\left[|\nabla_{\ell_t}\tfrac{d}{dt}u_t||\nabla_{\ell_t}P_{u_t}v_1|+|\nabla_{\ell_t}u_t||\nabla_{\ell_t}\tfrac{d}{dt}P_{u_t}v_1|\right]\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}}.
\end{aligned}$$

Using the estimates (5.6.6)-(5.6.13) and then Lemma 5.4.4 we find

$$\begin{aligned}
&\left|\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(P_{u_{\ell_{t+\varepsilon}}}v_1,0),(0,1)]\right|\\
&\leq C|p|\int_{C_0}(\rho_\ell\circ s_\ell)^{-2}\left[|\nabla_\ell v_1|(|\nabla_\ell z_\ell|+|\nabla_\ell w|)+|v_1|(|\nabla_\ell z_\ell|^2+|\nabla_\ell w|^2)\right]\,\mathrm{d}\mu_{\widetilde{G}_\ell}\\
&\quad+C\int_{C_0}(\rho_\ell\circ s_\ell)^{-1}|\nabla_\ell v_1|(|w||\nabla_\ell z_\ell|+|\nabla_\ell w|)\,\mathrm{d}\mu_{\widetilde{G}_\ell}\\
&\quad+C\int_{C_0}(\rho_\ell\circ s_\ell)^{-1}\left[|v_1|(|w||\nabla_\ell z_\ell|^2+|\nabla_\ell z_\ell||\nabla_\ell w|+|\nabla_\ell w|^2)\right]\,\mathrm{d}\mu_{\widetilde{G}_\ell}\\
&\leq C|p|\int_{C_0}(\rho_\ell\circ s_\ell)^{-2}\left[|\nabla_\ell v_1||\nabla_\ell z_\ell|+|v_1||\nabla_\ell z_\ell|^2\right]\,\mathrm{d}\mu_{\widetilde{G}_\ell}\\
&\quad+C\int_{C_0}(\rho_\ell\circ s_\ell)^{-1}\Big[|\nabla_\ell v_1|(|w||\nabla_\ell z_\ell|+|\nabla_\ell w|)\\
&\qquad\qquad+|v_1|(|w||\nabla_\ell z_\ell|^2+|\nabla_\ell z_\ell||\nabla_\ell w|+|\nabla_\ell w|^2)\Big]\,\mathrm{d}\mu_{\widetilde{G}_\ell}
\end{aligned}\tag{5.4.26}$$

using $\rho_\ell^{-1}|p|\leq C\|(0,p)\|_\ell\leq C\|W\|_\ell\leq C$ to absorb terms into the final integral on the last step. We can finally use (5.2.26) and $\rho_\ell^{-1}\leq C\|(0,1)\|_\ell$ from (5.2.19) in (5.4.26) to give

$$\begin{aligned}
&\left|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(0,1)]-d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(0,1)]\right|\\
&\leq\sup_{t\in(0,1)}\left|\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(P_{u_{\ell_{t+\varepsilon}}}v_1,0),(0,1)]\right|\\
&\leq C\|(0,p)\|_\ell\int_{C_0}|\nabla_\ell v_1|+|v_1|\rho_\ell(s_\ell)\,\mathrm{d}\mu_{\widetilde{G}_\ell}\\
&\quad+\|(0,1)\|_\ell\int_{C_0}|\nabla_\ell v_1|[|w|\rho_\ell(s_\ell)+|\nabla_\ell w|]+|v_1|[|w|\rho_\ell(s_\ell)^2+\rho_\ell(s_\ell)|\nabla_\ell w|+|\nabla_\ell w|^2]\,\mathrm{d}\mu_{\widetilde{G}_\ell}
\end{aligned}\tag{5.4.27}$$

from which we conclude

$$
\begin{aligned}
&\left|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(0,1)] - d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(0,1)]\right| \\
&\le C\|(0,p)\|_\ell \mathrm{vol}(C_0,\widetilde{G}_\ell)^{\frac12}\|v_1\|_{H^1} + C\|(0,1)\|_\ell\|v_1\|_{H^1}\|w\|_{H^1} + C\|(0,1)\|_\ell\|v_1\|_{L^\infty}\|w\|_{\dot H^1}^2 \\
&\le C\|W\|_\ell\|(0,1)\|_\ell\|v_1\|_{H^1} + C\|(0,1)\|_\ell\|v_1\|_{L^\infty}\|w\|_{\dot H^1}^2,
\end{aligned}
$$

using $\mathrm{vol}(C_0,\widetilde{G}_\ell)^{\frac12} \le C\ell^{-\frac12} \le C\ell^{\frac12}\|(0,1)\|_\ell \le C\|(0,1)\|_\ell$, by (5.2.19), and using the Poincaré inequality (5.2.5) to bound $\|w\|_{H^1} \le C\|w\|_{\dot H^1} \le C\|W\|_\ell$. Hence we have finished the proof of estimate (5.4.14).

Finally for (5.4.13) we use, as before,

$$
\begin{aligned}
&\left|\left(d^2E(u_t,\ell_t) - d^2E(z_\ell,\ell)\right)[(0,1),(0,1)]\right| \\
&\qquad \le \sup_{t\in(0,1)}\left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(0,1),(0,1)]\right|.
\end{aligned}
\tag{5.4.28}
$$

We write out (5.4.5) in coordinates on C_0 and differentiate using (5.2.8), (5.2.10) to estimate

$$
\begin{aligned}
&\left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(0,1),(0,1)]\right| \\
&\le \left|\int_{C_0}\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}(\partial_x s_{\ell_{t+\varepsilon}})\left[\frac{1}{2\pi^2(\rho_{\ell_{t+\varepsilon}}\circ s_{\ell_{t+\varepsilon}})^2} - \frac{\ell_{t+\varepsilon}^2}{2\pi^4(\rho_{\ell_{t+\varepsilon}}\circ s_{\ell_{t+\varepsilon}})^4}\right]|\partial_\theta u_t|^2\,\mathrm{d}x\mathrm{d}\theta\right| \\
&\quad + \left|\int_{C_0}\frac{{\ell_t}^2}{2\pi^4(\rho_{\ell_t}\circ s_{\ell_t})^4}(\partial_x s_{\ell_t})\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}|\partial_\theta u_{t+\varepsilon}|^2\right. \\
&\qquad\qquad \left.+\frac{1}{2\pi^2(\rho_{\ell_t}\circ s_{\ell_t})^2}\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}((\partial_x s_{\ell_t})^{-1}|\partial_x u_{t+\varepsilon}|^2 - (\partial_x s_{\ell_t})|\partial_\theta u_{t+\varepsilon}|^2)\,\mathrm{d}x\mathrm{d}\theta\right| \\
&\le C|p|\int_{C_0}(\rho_{\ell_t}\circ s_{\ell_t})^{-3}|\nabla_{\ell_t}u_t|^2\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} + C\int_{C_0}(\rho_{\ell_t}\circ s_{\ell_t})^{-2}|\nabla_{\ell_t}u_t||\nabla_{\ell_t}\tfrac{d}{dt}u_t|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}}
\end{aligned}
$$

where the final step uses (5.6.7) and (5.6.13). Using Lemma 5.4.4 we then have

$$
\begin{aligned}
&\left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(0,1),(0,1)]\right| \\
&\quad\le C|p|\int_{C_0}(\rho_\ell\circ s_\ell)^{-3}(|\nabla_\ell z_\ell|^2 + |\nabla_\ell w|^2)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\qquad + C\int_{C_0}(\rho_\ell\circ s_\ell)^{-2}(|w||\nabla_\ell z_\ell|^2 + |\nabla_\ell z_\ell||\nabla_\ell w| + |\nabla_\ell w|^2)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\quad\le C\|(0,1)\|_\ell^2\|(0,p)\|_\ell\int_{C_0}(\rho_\ell\circ s_\ell)^2 + |\nabla_\ell w|^2\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\qquad + C\|(0,1)\|_\ell^2\int_{C_0}|w|\rho_\ell(s_\ell)^2 + |\nabla_\ell w|\rho_\ell(s_\ell) + |\nabla_\ell w|^2\,\mathrm{d}\mu_{\widetilde{G}_\ell}.
\end{aligned}
\tag{5.4.29}
$$

using also (5.2.26) and $\rho_\ell^{-1} \le C\|(0,1)\|_\ell$. We conclude by combining (5.4.29) and (5.4.28) giving

$$
\begin{aligned}
&\left|\left(d^2E(u_t,\ell_t) - d^2E(z_\ell,\ell)\right)[(0,1),(0,1)]\right| \\
&\quad \leq \sup_{t\in(0,1)} \left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} d^2E(u_{t+\varepsilon},\ell_{t+\varepsilon})[(0,1),(0,1)]\right| \\
&\quad \leq C\|(0,1)\|_\ell^2 \|(0,p)\|_\ell \left[\|\rho_\ell\circ s_\ell\|^2_{L^2(C_0,\widetilde{G}_\ell)} + \|w\|^2_{\dot{H}^1}\right] \\
&\qquad + C\|(0,1)\|_\ell^2 \left[\|w\|_{H^1}\|\rho_\ell\circ s_\ell\|_{L^2(C_0,\widetilde{G}_\ell)} + \|w\|^2_{\dot{H}^1}\right] \\
&\quad \leq C\|(0,1)\|_\ell^2\|(0,p)\|_\ell + C\|(0,1)\|_\ell^2\|w\|_{\dot{H}^1} \leq C\|(0,1)\|_\ell^2\|W\|_\ell,
\end{aligned}
$$

where for the penultimate inequality we have used $\|\rho_\ell\circ s_\ell\|_{L^2(C_0,\widetilde{G}_\ell)} = \mathrm{vol}(C_0,G_\ell)^{\frac{1}{2}} \leq C$ to bound $\|\rho_\ell\circ s_\ell\|^2_{L^2(C_0,\widetilde{G}_\ell)} + \|w\|^2_{\dot{H}^1} \leq C$ and the Poincaré inequality (5.2.5) to bound $\|w\|_{H^1}\|\rho_\ell\circ s_\ell\|_{L^2(C_0,\widetilde{G}_\ell)} + \|w\|^2_{\dot{H}^1} \leq C\|w\|_{\dot{H}^1}$. This concludes the proof of estimate (5.4.13) and hence of Lemma 5.4.2.

□

Lemma 5.4.6. *With the same assumptions as Lemma 5.4.2 we have the following estimates for the second variation of E. In the "mixed" directions we have*

$$
\begin{aligned}
&|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(0,1)]| \\
&\quad \leq C\|(0,1)\|_\ell \int_{C_0} |\nabla_\ell v_1|(\rho_\ell(s_\ell) + |\nabla_\ell w|) + |v_1|(\rho_\ell(s_\ell)^2 + |\nabla_\ell w|^2)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \qquad (5.4.30)\\
&\quad \leq C\|(0,1)\|_\ell \left[\|v_1\|_{\dot{H}^1} + \|v_1\|_{L^\infty}\|w\|^2_{\dot{H}^1}\right],
\end{aligned}
$$

while in the "map" directions we have

$$
\begin{aligned}
&|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(P_{u_t}v_2,0)]| \\
&\quad \leq C\int_{C_0} |\nabla_\ell v_1||\nabla_\ell v_2| + (|v_1||\nabla_\ell v_2| + |v_2||\nabla_\ell v_1|)\rho_\ell(s_\ell) + \rho_\ell(s_\ell)^2|v_1||v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\qquad + C\int_{C_0} (|v_1||\nabla_\ell v_2| + |v_2||\nabla_\ell v_1|)|\nabla_\ell w| + |\nabla_\ell w|^2|v_1||v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell} \qquad (5.4.31)\\
&\quad \leq C\|v_1\|_{H^1}\|v_2\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{H^1}\|w\|_{H^1} \\
&\qquad + C\|v_2\|_{L^\infty}\|v_1\|_{H^1}\|w\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{L^\infty}\|w\|^2_{\dot{H}^1}.
\end{aligned}
$$

In the directions that correspond to varying ℓ we have

$$
|d^2E(u_t,\ell_t)[(0,1),(0,1)]| \leq C\|(0,1)\|_\ell^2. \qquad (5.4.32)
$$

Proof. We recall that we have estimates for $d^2E(z_\ell,\ell)$ in Lemma 5.4.1 and estimates on certain differences between $d^2E(u_t,\ell_t)$ and $d^2E(z_\ell,\ell)$ in Lemma 5.4.2. To derive the integral versions (5.4.30) and (5.4.31) we will also need some estimates from the proof of Lemma 5.4.1 and Lemma 5.4.2.

Firstly by (5.4.10) and (5.4.13) we have

$$|d^2E(u_t,\ell_t)[(0,1),(0,1)]| \le C\ell\|(0,1)\|_\ell^2 + C\|W\|_\ell\|(0,1)\|_\ell^2 \le C\|(0,1)\|_\ell^2,$$

which yields the desired estimate (5.4.32).

To obtain the desired estimate (5.4.30) we note that by (5.6.2) and (5.2.26) we have

$$|\nabla_\ell P_{z_\ell} v_1| \le C(|\nabla_\ell v_1| + \rho_\ell(s_\ell)|v_1|) \tag{5.4.33}$$

and so (5.4.6) gives

$$\begin{aligned} |d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(0,1)]| &\le C\int_{C_0} \rho_\ell(s_\ell)^{-1}|\nabla_\ell z_\ell||\nabla_\ell P_{z_\ell}v_1|\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\le C\int_{C_0} |\nabla_\ell v_1| + \rho_\ell(s_\ell)|v_1|\,\mathrm{d}\mu_{\widetilde{G}_\ell}. \end{aligned} \tag{5.4.34}$$

where we have used additionally (5.2.26). Combining (5.4.34) with (5.4.27) (from the proof of Lemma 5.4.2) and $\|(0,p)\|_\ell \le \|W\|_\ell \le \frac{1}{2}$ we have

$$\begin{aligned} &|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(0,1)]| \\ &\le \int_{C_0} |\nabla_\ell v_1| + |v_1|\rho_\ell(s_\ell)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\quad + \|(0,1)\|_\ell \int_{C_0} |\nabla_\ell v_1|[|w|\rho_\ell(s_\ell) + |\nabla_\ell w|] + |v_1|[|w|\rho_\ell(s_\ell)^2 + \rho_\ell(s_\ell)|\nabla_\ell w| + |\nabla_\ell w|^2]\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\le C\|(0,1)\|_\ell \int_{C_0} |\nabla_\ell v_1|(\rho_\ell(s_\ell) + |\nabla_\ell w|) + |v_1|(\rho_\ell(s_\ell)^2 + |\nabla_\ell w|^2)\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\le C\|(0,1)\|_\ell \left[\|v_1\|_{H^1}\|\rho_\ell(s_\ell)\|_{L^2(C_0,\widetilde{G}_\ell)} + \|v_1\|_{H^1}\|w\|_{H^1} + \|v_1\|_{L^\infty}\|w\|_{\dot{H}^1}^2\right] \\ &\le C\|(0,1)\|_\ell \left[\|v_1\|_{H^1} + \|v_1\|_{L^\infty}\|w\|_{\dot{H}^1}^2\right] \end{aligned}$$

where we have also used $\|w\|_{L^\infty} \le C$ and $\rho_\ell^{-1} \le C\|(0,1)\|_\ell$, then $\|\rho_\ell(s_\ell)\|_{L^2(C_0,\widetilde{G}_\ell)} = \mathrm{vol}(C_0,G_\ell)^{\frac{1}{2}} \le C$ and used the Poincaré inequality (5.2.5) in the final step. This concludes the proof of the estimate (5.4.30).

Finally we turn to the estimate (5.4.31). Using (5.4.33) in (5.4.8) we have

$$\begin{aligned} &|d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(P_{z_\ell}v_2,0)]| \\ &\le \int_{C_0} |\nabla_\ell P_{z_\ell}v_1||\nabla_\ell P_{z_\ell}v_2| + C|\nabla_\ell z_\ell|^2|P_{z_\ell}v_1||P_{z_\ell}v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\le C\int_{C_0} |\nabla_\ell v_1||\nabla_\ell v_2| + \rho_\ell(s_\ell)(|v_1||\nabla_\ell v_2| + |v_2||\nabla_\ell v_1|) + \rho_\ell(s_\ell)^2|v_1||v_2|\,\mathrm{d}\mu_{\widetilde{G}_\ell}, \end{aligned} \tag{5.4.35}$$

using also (5.2.26). Combining (5.4.35) with (5.4.16), keeping in mind $\rho_\ell^{-1}|p| \le C\|(0,p)\|_\ell \le$

$C\|W\|_\ell \leq C$ and $\|w\|_{L^\infty} \leq C$, we have

$$
\begin{aligned}
&|d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(P_{u_t}v_2,0)]| \\
&\quad\leq |d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(P_{z_\ell}v_2,0)]| \\
&\qquad + |d^2E(u_t,\ell_t)[(P_{u_t}v_1,0),(P_{u_t}v_2,0)] - d^2E(z_\ell,\ell)[(P_{z_\ell}v_1,0),(P_{z_\ell}v_2,0)]| \\
&\quad\leq C\int_{C_0} |\nabla_\ell v_1||\nabla_\ell v_2| + \rho_\ell(s_\ell)(|v_1||\nabla_\ell v_2| + |v_2||\nabla_\ell v_1|) + \rho_\ell(s_\ell)^2|v_1||v_2| \, d\mu_{\widetilde{G}_\ell} \\
&\qquad + C\int_{C_0} (|v_1||\nabla_\ell v_2| + |v_2||\nabla_\ell v_1|)|\nabla_\ell w| + |v_1||v_2||\nabla_\ell w|^2 \, d\mu_{\widetilde{G}_\ell} \\
&\quad\leq C\|v_1\|_{H^1}\|v_2\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{H^1}\|w\|_{H^1} \\
&\qquad + C\|v_2\|_{L^\infty}\|v_1\|_{H^1}\|w\|_{H^1} + C\|v_1\|_{L^\infty}\|v_2\|_{L^\infty}\|w\|_{\dot{H}^1}^2,
\end{aligned}
$$

as desired. $\square$

5.4.1 Proof of the definiteness of the second variation

In this section we will show that the second variation of the energy at (z_ℓ, ℓ) is uniformly positive definite in directions orthogonal to $T_{(z_\ell,\ell)}\mathcal{Z}$.

Lemma 5.4.7. *There exists $\ell_* > 0$, $c_1 > 0$ such that for all $(z_\ell, \ell) \in \mathcal{Z}$ with $\ell \in (0, \ell_*)$ and $W \in (T_{(z_\ell,\ell)}\mathcal{Z})^\perp$ we have*

$$d^2E(z_\ell,\ell)[W,W] \geq c_1\|W\|_\ell^2, \tag{5.4.36}$$

where here we mean orthogonality in the space $T_{(z_\ell,\ell)}H$ with respect to $\langle\cdot,\cdot\rangle_\ell$.

Remark 5.4.8. This uniform definiteness does not hold for general $W = (w,p) \in T_{(z_\ell,\ell)}H$. Indeed we have $d^2E(z_\ell,\ell)[(0,1),(0,1)] \lesssim \ell\|(0,1)\|_\ell^2$, as shown in Lemma 5.4.1, and we also have $d^2E(z_\ell,\ell)[y_z,y_z] \lesssim \ell$ for the element $y_z = \frac{\partial_\ell(z_\ell,\ell)}{\|\partial_\ell(z_\ell,\ell)\|_\ell} \in T_{(z_\ell,\ell)}\mathcal{Z}$, as shown in Remark 5.3.4.

The main difficulty will be to deal with the non-linearity of the target (N, g_N) and establish a similar non-degeneracy statement for the second variation in map directions, Lemma 5.4.9 below. We remark that in the case that (N, g_N) is Euclidean Lemma 5.4.9 below is trivial and it would suffice to deal with the metric components.

We will deal with the contributions to $d^2E(z_\ell,\ell)$ coming from the metric component by showing they are lower order provided the second variation acts on the space $(T_{(z_\ell,\ell)}\mathcal{Z})^\perp$. To this end, fix $(w,p) \in (T_{(z_\ell,\ell)}\mathcal{Z})^\perp$. Using that $0 = \langle (w,p), (\partial_\ell z_\ell, 1)\rangle_\ell = \langle w, \partial_\ell z_\ell\rangle_{\dot{H}^1} + \ell^{-2}p$ and $\|\partial_\ell z_\ell\|_{\dot{H}^1} \leq C\ell^{-\frac{1}{2}}$ from (5.2.36) we have

$$\|(0,p)\|_\ell = \ell^{-1}|p| \leq C\ell|\langle w, \partial_\ell z_\ell\rangle_{\dot{H}^1}| \leq C\ell^{1/2}\|w\|_{\dot{H}^1}. \tag{5.4.37}$$

Then since $\|(w,p)\|_\ell^2 = \|w\|_{\dot H^1}^2 + \|(0,p)\|_\ell^2$ we have the key observation that

$$\|w\|_{\dot H^1} \geq \tfrac{1}{2}\|(w,p)\|_\ell \text{ for } (w,p) \in (T_{(z_\ell,\ell)}\mathcal{Z})^\perp \tag{5.4.38}$$

for $\ell > 0$ small enough. Of course, orthogonality is essential here as this estimate is false for $w = 0$ and $p \neq 0$ or for $(w,p) = y_z$, see also Remark 5.4.8.

We can combine (5.4.37) with Lemma 5.4.1 to give

$$d^2E(z_\ell,\ell)[(w,p),(w,p)] = d^2E(z_\ell,\ell)[(w,0),(w,0)] + O(\ell\|w\|_{\dot H^1}^2) \text{ for } (w,p) \in (T_{(z_\ell,\ell)}\mathcal{Z})^\perp.$$

It therefore suffices to prove positive definiteness $d^2E(z_\ell,\ell)[(w,0),(w,0)] \geq c\|w\|_{\dot H^1}^2$ for the map component, since using the above and our key observation (5.4.38) we would have

$$d^2E(z_\ell,\ell)[(w,p),(w,p)] \geq c\|w\|_{\dot H^1}^2 \geq \tilde c\|(w,p)\|_\ell^2. \tag{5.4.39}$$

This will be a consequence of the following Lemma 5.4.9 and hence we will have proved Lemma 5.4.7 once it has been established.

To state this lemma we use the following notation. Given $U \subset (-\infty,\infty)\times S^1$, $u\colon U \to N$ and $v_1, v_2 \in \Gamma^{H^1(U)}(u^*TN)$ we define

$$I_U(u)[v_1,v_2] = \int_U \nabla v_1 \cdot \nabla v_2 - A(u)(\nabla u, \nabla u)A(u)(v_1,v_2)\,\mathrm{d}s\mathrm{d}\theta.$$

In particular we have

$$d^2E(z_\ell,\ell)[(w,0),(w,0)] = I_{\mathcal{C}_{Y(\ell)}}(z_{Y(\ell)})[\tilde w,\tilde w] \tag{5.4.40}$$

where $\tilde w = w\circ f_\ell^{-1}$, see (5.4.3). We will write $\|w\|_{\dot H^1(\mathcal{C}_{Y(\ell)})} = \|w\|_{\dot H^1(\mathcal{C}_{Y(\ell)},g_E)} = \|w\|_{\dot H^1(\mathcal{C}_{Y(\ell)},g_\ell)}$, note that the second equality holds by conformal invariance of the $\dot H^1$ norm.

Lemma 5.4.9. *There exists $\ell_* > 0$ and $c_2 > 0$ such that for all $\ell \in (0,\ell_*)$ and $w \in \Gamma^{H_0^1(\mathcal{C}_{Y(\ell)})}(z_{Y(\ell)}^*TN)$ we have*

$$I_{\mathcal{C}_{Y(\ell)}}(z_{Y(\ell)})[w,w] \geq c_2\|w\|_{\dot H^1(\mathcal{C}_{Y(\ell)})}^2. \tag{5.4.41}$$

Proof of Lemma 5.4.9. Here we will be using the non-degeneracy of the geodesic γ and of the harmonic maps $v_\pm$ to give us uniform positive definiteness of $I_{\mathcal{C}_{Y(\ell)}}(z_{Y(\ell)})$. Precisely, the assumption (C) gives

$$\int_D |\nabla\eta|^2 - A(v_\pm)(\nabla v_\pm,\nabla v_\pm)A(v_\pm)(\eta,\eta) \geq c_\pm\|\nabla\eta\|_{L^2(D)}^2 \text{ for all } \eta \in \Gamma^{H_0^1(D)}(v_\pm^*TN) \tag{5.4.42}$$

and (B) gives

$$\int_{-1}^1 |\eta'|^2 - A(\gamma)(\gamma',\gamma')A(\gamma)(\eta,\eta) \geq c_\gamma\|\eta'\|_{L^2(-1,1)}^2 \text{ for all } \eta \in \Gamma^{H_0^1(-1,1)}(\gamma^*TN), \tag{5.4.43}$$

where $c_\pm, c_\gamma \in (0,1]$, see also (5.4.3) where we give the related formula for the second variation of E.

We will want corresponding positive-definiteness statements for appropriate maps on cylinders. Write $\mathcal{C}^+ = (-\infty, Y(\ell)] \times S^1$, $\mathcal{C}^- = [-Y(\ell), \infty) \times S^1$ and define $\bar{v}_{\pm,Y(\ell)} \colon \mathcal{C}^\pm \to N$ by $\bar{v}_{\pm,Y(\ell)}(s,\theta) = v_\pm(e^{-(Y(\ell)\mp s)}e^{i\theta})$. By conformal invariance of (5.4.42) we can see that

$$I_{\mathcal{C}^\pm}(\bar{v}_{\pm,Y(\ell)})[w,w] \geq c_\pm \|\nabla w\|^2_{L^2(\mathcal{C}^\pm, g_E)} \tag{5.4.44}$$

for every $w \in \Gamma^{H^1_0(\mathcal{C}^\pm)}(\bar{v}^*_{\pm,Y(\ell)}TN)$. Notice the implied boundary condition for admissibility in the positive case is that w vanishes on $\partial\mathcal{C}^+ = \{Y(\ell)\} \times S^1$ (vanishing on $\partial\mathcal{C}^- = \{-Y(\ell)\} \times S^1$ in the negative case).

Define also $\gamma_{Y(\ell)} \colon \mathcal{C}_{Y(\ell)} \to N$ by $\gamma_{Y(\ell)}(s,\theta) = \gamma(\frac{s}{Y(\ell)})$. Now given $w \in \Gamma^{H^1_0(\mathcal{C}_{Y(\ell)})}(\gamma^*_{Y(\ell)}TN)$ and applying (5.4.43) to $\eta(x) = w(Y(\ell)x, \theta)$ for each $\theta \in S^1$ we obtain

$$I_{\mathcal{C}_{Y(\ell)}}(\gamma_{Y(\ell)})[w,w] \geq c_\gamma \|\nabla w\|^2_{L^2(\mathcal{C}_{Y(\ell)}, g_E)} \tag{5.4.45}$$

for every such w.

Now fixing $w \in \Gamma^{H^1_0(\mathcal{C}_{Y(\ell)})}(z^*_{Y(\ell)}TN)$ we can write, keeping in mind the definitions (5.2.13) and (5.2.14),

$$\begin{aligned} I_{\mathcal{C}_{Y(\ell)}}(z_{Y(\ell)})[w,w] = I_\Omega(\gamma_{Y(\ell)})(w,w) & \\ + I_{\Sigma^+}(\bar{v}_{+,Y(\ell)})(w,w) + I_{\Sigma^-}(\bar{v}_{-,Y(\ell)})(w,w) & \\ + I_{T^+}(z_{Y(\ell)})(w,w) + I_{T^-}(z_{Y(\ell)})(w,w), & \end{aligned} \tag{5.4.46}$$

where for the remainder of the proof we abbreviate $\Omega = \Omega_{Y(\ell)}$, $\Sigma^\pm = \Sigma^\pm_{Y(\ell)}$, $T^\pm = T^\pm_{Y(\ell)}$ and $Y = Y(\ell)$. We also abbreviate $\|w\|_{\dot{H}^1} = \|w\|_{\dot{H}^1(\mathcal{C}_Y)}$. Note that taking $\ell > 0$ small corresponds to taking $Y = Y(\ell)$ large, see the definition of $Y(\ell)$ in Lemma 5.2.1.

The main step in our proof will be to control the terms I_Ω and $I_{\Sigma^\pm}$, which we do with the following two claims whose proofs we delay until the end. Note that in what follows we will only treat the term I_{Σ^+} since the corresponding results for I_{Σ^-} can be stated and proved in exactly the same way.

Claim 1. For any $w \in \Gamma^{H^1_0(\mathcal{C}_Y)}(z^*_Y TN)$ we can choose $s_2 \in [Y - \alpha \log Y, Y - \frac{\alpha}{2}\log Y]$ such that the following holds for $\tilde{\Sigma}^+ = (s_2, Y] \times S^1 \subset \Sigma^+$. There exists an extension $\tilde{w} \in \Gamma^{H^1_0}(\bar{v}^*_{+,Y}TN)$ of $w|_{\tilde{\Sigma}^+}$ to $\mathcal{C}^+ = (-\infty, Y] \times S^1$ such that

$$I_{\mathcal{C}^+ \setminus \tilde{\Sigma}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] = o(1)\|w\|^2_{\dot{H}^1}. \tag{5.4.47}$$

Claim 2. For any $w \in \Gamma^{H^1_0(\mathcal{C}_Y)}(z_Y^* TN)$ we can choose $s_- \in [-Y + 2\alpha \log Y, -Y + 3\alpha \log Y]$ and $s_+ \in [Y - 3\alpha \log Y, Y - 2\alpha \log Y]$ such that the following holds for $\tilde{\Omega} = [s_-, s_+] \times S^1 \subset \Omega$. There exists a map $\bar{w} \in \Gamma^{H^1_0(\mathcal{C}_Y)}(\gamma_Y^* TN)$ such that

$$\|\bar{w}\|^2_{\dot{H}^1(\mathcal{C}_Y)} = \|w\|^2_{\dot{H}^1(\tilde{\Omega})} + o(1)\|w\|^2_{\dot{H}^1} \tag{5.4.48}$$

and

$$I_{\mathcal{C}_Y}(\gamma_Y)[\bar{w}, \bar{w}] = I_{\tilde{\Omega}}(\gamma_Y)[w, w] + o(1)\|w\|^2_{\dot{H}^1}. \tag{5.4.49}$$

We can now give a proof of the result, assuming Claims 1 and 2. We start with the $I_{\Sigma^+}(\bar{v}_{+,Y})$ term. Using Claim 1 we have

$$\begin{aligned} I_{\tilde{\Sigma}^+}(\bar{v}_{+,Y})[w, w] = I_{\tilde{\Sigma}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] &= I_{\mathcal{C}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] - I_{\mathcal{C}^+ \setminus \tilde{\Sigma}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] \\ &= I_{\mathcal{C}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] + o(1)\|w\|^2_{\dot{H}^1} \end{aligned}$$

and hence splitting the domain Σ^+ we have

$$\begin{aligned} I_{\Sigma^+}(\bar{v}_{+,Y})[w, w] &= I_{\tilde{\Sigma}^+}(\bar{v}_{+,Y})[w, w] + I_{\Sigma^+ \setminus \tilde{\Sigma}^+}(\bar{v}_{+,Y})[w, w] \\ &= I_{\mathcal{C}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] + I_{\Sigma^+ \setminus \tilde{\Sigma}^+}(\bar{v}_{+,Y})[w, w] + o(1)\|w\|^2_{\dot{H}^1}. \end{aligned} \tag{5.4.50}$$

We can bound the first term of (5.4.50) using the positive definiteness (5.4.44), since $\tilde{w} \in \Gamma^{H^1_0}(\bar{v}^*_{+,Y} TN)$,

$$I_{\mathcal{C}^+}(\bar{v}_{+,Y})[\tilde{w}, \tilde{w}] \geq c_+ \|\tilde{w}\|^2_{\dot{H}^1(\mathcal{C}^+)} \geq c_+ \|w\|^2_{\dot{H}^1(\tilde{\Sigma}^+)}$$

while the second term of (5.4.50) can be bounded below by

$$\begin{aligned} I_{\Sigma^+ \setminus \tilde{\Sigma}^+}(\bar{v}_{+,Y})[w, w] &= \int_{\Sigma^+ \setminus \tilde{\Sigma}^+} |\nabla w|^2 - A(\bar{v}_{+,Y})(\nabla \bar{v}_{+,Y}, \nabla \bar{v}_{+,Y}) A(\bar{v}_{+,Y})(w, w) \, \mathrm{d}s\mathrm{d}\theta \\ &\geq \|w\|^2_{\dot{H}^1(\Sigma^+ \setminus \tilde{\Sigma}^+)} - C \int_{\Sigma^+ \setminus \tilde{\Sigma}^+} |\nabla \bar{v}_{+,Y}|^2 |w|^2 \, \mathrm{d}s\mathrm{d}\theta \\ &\geq \|w\|^2_{\dot{H}^1(\Sigma^+ \setminus \tilde{\Sigma}^+)} - C \|\rho_\ell^{-2} |\nabla \bar{v}_{+,Y}|^2\|_{L^\infty(\Sigma^+ \setminus \tilde{\Sigma}^+)} \|w\|^2_{L^2(\mathcal{C}_Y, g_\ell)} \\ &\geq \|w\|^2_{\dot{H}^1(\Sigma^+ \setminus \tilde{\Sigma}^+)} - C(\log Y)^2 Y^{-\alpha} \|w\|^2_{\dot{H}^1} \\ &= \|w\|^2_{\dot{H}^1(\Sigma^+ \setminus \tilde{\Sigma}^+)} + o(1)\|w\|^2_{\dot{H}^1}, \end{aligned}$$

where we have used (5.6.18) and $|\nabla \bar{v}_{\pm,Y}(s, \theta)| \leq C e^{-(Y \mp s)} \leq C Y^{-\alpha/2}$ on $\Sigma^+ \setminus \tilde{\Sigma}^+ \subset \{\pm s \in [Y - 2\alpha \log Y, Y - \frac{\alpha}{2} \log Y]\} \times S^1$ as well as the Poincaré inequality (5.2.5) in the second-to-last inequality. We can then conclude that

$$I_{\Sigma^+}(\bar{v}_{+,Y})[w, w] \geq c_+ \|w\|^2_{\dot{H}^1(\tilde{\Sigma}^+)} + \|w\|^2_{\dot{H}^1(\Sigma^+ \setminus \tilde{\Sigma}^+)} + o(1)\|w\|^2_{\dot{H}^1} \geq c_+ \|w\|^2_{\dot{H}^1(\Sigma^+)} + o(1)\|w\|^2_{\dot{H}^1} \tag{5.4.51}$$

since $c_+ \leq 1$.

Now we turn to the terms I_Ω. As before we split the domain Ω and apply Claim 2 to the first term

$$\begin{aligned} I_\Omega(\gamma_Y)[w,w] &= I_{\tilde\Omega}(\gamma_Y)[w,w] + I_{\Omega\setminus\tilde\Omega}(\gamma_Y)[w,w] \\ &= I_{\mathcal{C}_Y}(\gamma_Y)[\bar w,\bar w] + I_{\Omega\setminus\tilde\Omega}(\gamma_Y)[w,w] + o(1)\|w\|^2_{\dot H^1}. \end{aligned} \tag{5.4.52}$$

We can bound the first term using the positive definiteness (5.4.45), since $\bar w \in \Gamma^{H^1_0(\mathcal{C}_Y)}(\gamma_Y^* TN)$,

$$I_{\mathcal{C}_Y}(\gamma_Y)[\bar w,\bar w] \ge c_\gamma \|\bar w\|^2_{\dot H^1(\mathcal{C}_Y)} = c_\gamma \|w\|^2_{\dot H^1(\tilde\Omega)} + o(1)\|w\|^2_{\dot H^1}$$

where we have used (5.4.48) from Claim 2. We bound the second term of (5.4.52) below by

$$\begin{aligned} I_{\Omega\setminus\tilde\Omega}(\gamma_Y)[w,w] &= \int_{\Omega\setminus\tilde\Omega} |\nabla w|^2 - A(\gamma_Y)(\gamma_Y',\gamma_Y')A(\gamma_Y)(w,w)\,\mathrm{d}s\mathrm{d}\theta \\ &\ge \|w\|^2_{\dot H^1(\Omega\setminus\tilde\Omega)} - C\int_{\Omega\setminus\tilde\Omega} |\gamma_Y'|^2|w|^2\,\mathrm{d}s\mathrm{d}\theta \\ &\ge \|w\|^2_{\dot H^1(\Omega\setminus\tilde\Omega)} - CY^{-2}\|\rho_\ell^{-2}\|_{L^\infty(\Omega\setminus\tilde\Omega)}\|w\|^2_{L^2(\mathcal{C}_{Y(\ell)},g_\ell)} \\ &\ge \|w\|^2_{\dot H^1(\Omega\setminus\tilde\Omega)} - CY^{-2}(\log Y)^2\|w\|^2_{\dot H^1} \\ &= \|w\|^2_{\dot H^1(\Omega\setminus\tilde\Omega)} + o(1)\|w\|^2_{\dot H^1} \end{aligned}$$

using the bound (5.6.18) for the conformal factor and the Poincaré inequality (5.2.5). Hence we conclude

$$I_\Omega(\gamma_Y)[w,w] \ge c_\gamma\|w\|^2_{\dot H^1(\tilde\Omega)} + \|w\|^2_{\dot H^1(\Omega\setminus\tilde\Omega)} + o(1)\|w\|^2_{\dot H^1} \ge c_\gamma\|w\|^2_{\dot H^1(\Omega)} + o(1)\|w\|^2_{\dot H^1} \tag{5.4.53}$$

since $c_\gamma \le 1$.

Finally, on the transition region we can use (5.2.24) and (5.6.18) to give

$$\begin{aligned} I_{T^\pm}(z_Y)[w,w] &= \int_{T^\pm} |\nabla w|^2 - A(z)(\nabla z_Y,\nabla z_Y)A(z_Y)(w,w)\,\mathrm{d}s\mathrm{d}\theta \\ &\ge \|w\|^2_{\dot H^1(T^\pm)} - C\int_{T^\pm} |\nabla z_Y|^2|w|^2\,\mathrm{d}s\mathrm{d}\theta \\ &\ge \|w\|^2_{\dot H^1(T^\pm)} - C\|\rho_\ell^{-1}\nabla z_Y\|^2_{L^\infty(T^\pm)}\|w\|^2_{L^2(\mathcal{C}_{Y(\ell)},g_\ell)} \\ &\ge \|w\|^2_{\dot H^1(T^\pm)} - C(\log Y)^2Y^{-2}\|w\|^2_{\dot H^1} \\ &= \|w\|^2_{\dot H^1(T^\pm)} + o(1)\|w\|^2_{\dot H^1}. \end{aligned} \tag{5.4.54}$$

Then returning to (5.4.46) and using the estimates (5.4.51), (5.4.53) and (5.4.54) we have

$$I_{\mathcal{C}_{Y(\ell)}}(z_{Y(\ell)})[w,w] \ge \min(c_\gamma, c_\pm, 1)\|w\|^2_{\dot H^1} + o(1)\|w\|^2_{\dot H^1} \ge c_2\|w\|^2_{\dot H^1},$$

for $\ell > 0$ small enough. Hence in order to finish the proof of Lemma 5.4.9, it remains to prove the Claims 1 and 2.

Proof of Claim 1. Denote by $\vartheta(s) = \int_{\{s\}\times S^1} |w_\theta|^2 \, \mathrm{d}\theta$ the angular energy on the circle $\{s\}\times S^1$. To establish the claim we first note that we can choose some $s_2 \in [Y - \alpha \log Y, Y - \frac{\alpha}{2}\log Y]$ where the angular energy ϑ is less than its average on this interval and so

$$\vartheta(s_2) \le \fint_{Y-\alpha\log Y}^{Y-\frac{\alpha}{2}\log Y} \vartheta(s)\,\mathrm{d}s \le \frac{\|w\|_{\dot{H}^1}^2}{\frac{\alpha}{2}\log Y}. \tag{5.4.55}$$

Before continuing we note some estimates which hold by this choice of s_2. Firstly, on the circle $\{s_2\}\times S^1$ the map w is close to its average $\mu_{s_2} = \fint_{\{s_2\}\times S^1} w \, \mathrm{d}\theta$ on this circle,

$$\|w(s_2,\cdot) - \mu_{s_2}\|_{L^\infty(S^1)} \le C\vartheta(s_2)^{\frac{1}{2}} \le C\frac{\|w\|_{\dot{H}^1}}{(\log Y)^{\frac{1}{2}}}. \tag{5.4.56}$$

Secondly, since $w|_{\{Y\}\times S^1} = 0$, we have

$$|\mu_{s_2}| \le C(Y-s_2)^{\frac{1}{2}} \left(\int_{S^1} \int_{s_2}^{Y} |\partial_s w|^2 \, \mathrm{d}s\mathrm{d}\theta \right)^{\frac{1}{2}} \le C(\log Y)^{\frac{1}{2}} \|w\|_{\dot{H}^1}. \tag{5.4.57}$$

We now obtain the desired extension of $w|_{\tilde{\Sigma}^+}$ to $\mathcal{C}^+$ by taking the orthogonal projection onto $T_{\bar{v}_{+,Y}}N$ of an interpolation, on the interval $[s_2-1, s_2]$, between μ_{s_2} and $w(s_2,\cdot)$. Precisely, let σ be the piecewise linear function with $\sigma(s) = 1$ for $s \ge s_2$ and $\sigma(s) = 0$ for $s \in (-\infty, s_2 - 1)$. We define

$$\tilde{w}(s,\cdot) = \begin{cases} P_{\bar{v}_{+,Y}}\left[\sigma(s)w(s_2,\cdot) + (1-\sigma(s))\mu_{s_2}\right] & \text{for } s \in (-\infty, s_2] \\ w(s,\cdot) & \text{for } s \in [s_2, Y] \end{cases}$$

and so in particular $\tilde{w} = w$ on $\tilde{\Sigma}^+$. Note this definition yields a continuous function $\tilde{w}$ since $w(s_2,\cdot)$ takes values in $T_{\bar{v}_{\pm,Y}(s_2,\cdot)}N$.

Now that we have defined our extension $\tilde{w}$ we can proceed with the proof of (5.4.47). We need estimates for $\tilde{w}$ on $\mathcal{C}^+ \setminus \tilde{\Sigma}^+ \subset (-\infty, Y - \frac{\alpha}{2}\log Y] \times S^1$. Using (5.4.57) and (5.4.56) we obtain

$$\|\tilde{w}\|_{L^\infty(\mathcal{C}^+\setminus\tilde{\Sigma}^+)} \le |\mu_{s_2}| + \|w(s_2,\cdot) - \mu_{s_2}\|_{L^\infty(S^1)} \le C(\log Y)^{\frac{1}{2}}\|w\|_{\dot{H}^1}$$

and so can estimate, using $|\nabla \bar{v}_{+,Y}(s,\theta)| \le Ce^{s-Y}$,

$$\begin{aligned} &\left| \int_{\mathcal{C}^+\setminus\tilde{\Sigma}^+} A(\bar{v}_{+,Y})(\nabla\bar{v}_{+,Y}, \nabla\bar{v}_{+,Y}) A(\bar{v}_{+,Y})(\tilde{w},\tilde{w}) \, \mathrm{d}s\mathrm{d}\theta \right| \le C \left| \int_{\mathcal{C}^+\setminus\tilde{\Sigma}^+} |\nabla\bar{v}_{+,Y}|^2 |\tilde{w}|^2 \, \mathrm{d}s\mathrm{d}\theta \right| \\ &\le C(\log Y)\|w\|_{\dot{H}^1}^2 \int_{-\infty}^{Y-\frac{\alpha}{2}\log Y} e^{2(s-Y)} \, \mathrm{d}s \le CY^{-\alpha}\log Y \|w\|_{\dot{H}^1}^2 = o(1)\|w\|_{\dot{H}^1}^2. \end{aligned} \tag{5.4.58}$$

Meanwhile, for $(s,\theta) \in \mathcal{C}^+ \setminus \tilde{\Sigma}^+$ we estimate the derivative of $\tilde{w}$ using (5.6.2) and also the bound $|\nabla \bar{v}_{+,Y}(s,\theta)| \le Ce^{s-Y}$,

$$\begin{aligned} |\nabla\tilde{w}|(s,\theta) &\le |\nabla[\sigma(s)w(s_2,\theta) - (1-\sigma(s))\mu_{s_2}]| + C|\nabla\bar{v}_{+,Y}||\sigma(s)w(s_2,\theta) - (1-\sigma(s))\mu_{s_2}| \\ &\le \left(|\partial_s\sigma| + Ce^{s-Y}|\sigma|\right)\|w(s_2,\cdot) - \mu_{s_2}\|_{L^\infty(S^1)} + |\sigma||(\partial_\theta w)(s_2,\theta)| + Ce^{s-Y}|\mu_{s_2}|, \end{aligned}$$

where we note the first two terms vanish unless $s \in [s_2 - 1, s_2]$. Integrating this estimate over $\mathcal{C}^+ \setminus \tilde{\Sigma}^+$ and using (5.4.56), (5.4.55) (5.4.57) gives

$$\begin{aligned}\int_{\mathcal{C}^+ \setminus \tilde{\Sigma}^+} |\nabla \tilde{w}|^2 \, \mathrm{d}s\mathrm{d}\theta &\leq C\|w(s_2, \cdot) - \mu_{s_2}\|^2_{L^\infty(S^1)} + C\vartheta(s_2) + C|\mu_{s_2}|^2 \int_{-\infty}^{Y - \frac{\alpha}{2}\log Y} e^{2(s-Y)} \, \mathrm{d}s \\ &\leq C\frac{\|w\|^2_{\dot{H}^1}}{\log Y} + CY^{-\alpha}\log Y \|w\|^2_{\dot{H}^1} = o(1)\|w\|^2_{\dot{H}^1}.\end{aligned} \tag{5.4.59}$$

Combining (5.4.58) and (5.4.59) yields the desired result (5.4.47) for $\tilde{w}$ and hence concludes the proof of Claim 1. □

Proof of Claim 2. Denote by $\vartheta(s) = \int_{\{s\}\times S^1} |w_\theta|^2 \, \mathrm{d}\theta$ the angular energy on the circle $\{s\}\times S^1$. We first choose some $s_+ \in [Y - 3\alpha \log Y, Y - 2\alpha \log Y]$ so that the angular energy ϑ is less than its average on this interval,

$$\vartheta(s_+) \leq \fint_{Y-3\alpha\log Y}^{Y-2\alpha\log Y} \vartheta(s)\, \mathrm{d}s \leq \frac{\|w\|^2_{\dot{H}^1}}{\alpha \log Y} \tag{5.4.60}$$

and we similarly choose $s_- \in [-Y + 3\alpha \log Y, -Y + 2\alpha \log Y]$ so that

$$\vartheta(s_-) \leq \frac{\|w\|^2_{\dot{H}^1}}{\alpha \log Y}. \tag{5.4.61}$$

Before continuing we note some estimates which hold by the choice of $s_\pm$. Firstly on the circle $\{s_\pm\} \times S^1$ the map w is close to its average $\mu_{s_\pm} = \fint_{\{s_\pm\}\times S^1} w \, \mathrm{d}\theta$,

$$\|w(s_\pm, \cdot) - \mu_{s_\pm}\|_{L^\infty(S^1)} \leq C\vartheta(s_\pm)^{\frac{1}{2}} \leq C\frac{\|w\|_{\dot{H}^1}}{(\log Y)^{\frac{1}{2}}}. \tag{5.4.62}$$

Secondly since $w|_{\{\pm Y\}\times S^1} = 0$ we obtain, exactly as in (5.4.57),

$$|\mu_{s_\pm}| \leq C(\log Y)^{\frac{1}{2}} \|w\|_{\dot{H}^1}. \tag{5.4.63}$$

In a first step towards the claim we will construct an extension $\hat{w} \in \Gamma^{H^1}(\gamma_Y^* TN)$ of $w|_{\tilde{\Omega}}$, where $\tilde{\Omega} = [s_-, s_+] \times S^1$, which satisfies

$$I_{\mathcal{C}_Y \setminus \tilde{\Omega}}(\gamma_Y)[\hat{w}, \hat{w}] = o(1)\|w\|^2_{\dot{H}^1} \tag{5.4.64}$$

but does not necessarily vanish on the boundary of $\mathcal{C}_Y$. After this we will show how to modify $\hat{w}$ to obtain a map satisfying all of the desired properties in Claim 2 – note in the claim we do not ask that the final map agrees with w on $\tilde{\Omega}$.

We extend $w|_{\tilde{\Omega}}$ to the right with the orthogonal projection onto $T_{\gamma_Y} N$ of an interpolation, on the interval $[s_+, s_+ + 1]$, between $w(s_+, \cdot)$ and the constant μ_{s_+} (and similarly to the left with s_-). Precisely we let σ_+ be the piecewise linear function with $\sigma_+(s) = 1$ for $s \leq s_+$

and $\sigma_+(s) = 0$ for $s \in (s_+ + 1, Y]$ and let σ_- the piecewise linear function such that $\sigma_- = 1$ for $s \geq s_-$ and $\sigma_- = 0$ for $s \in [-Y, s_- - 1)$. We define

$$\hat{w}(s, \cdot) = \begin{cases} P_{\gamma_Y}\left[\sigma_-(s) w(s_-, \cdot) + (1 - \sigma_-(s))\mu_{s_-}\right] & \text{for } s \in [-Y, s_-) \\ w(s, \cdot) & \text{for } s \in [s_-, s_+] \\ P_{\gamma_Y}\left[\sigma_+(s) w(s_+, \cdot) + (1 - \sigma_+(s))\mu_{s_+}\right] & \text{for } s \in (s_+, Y] \end{cases}$$

and note that this map is continuous in s since $w(s_\pm, \cdot)$ takes values in $T_{\gamma_Y(s_\pm)}N$.

Now that we have defined the extension $\hat{w}$ we proceed with the proof of (5.4.64). We desire estimates for $\hat{w}$ on the region $\mathcal{C}_Y \setminus \tilde{\Omega} \subset \{s : |s| \in [Y - 3\alpha \log Y, Y]\} \times S^1$. Using the estimates (5.4.63) and (5.4.62) we have

$$\|\hat{w}\|_{L^\infty(\mathcal{C}_Y \setminus \tilde{\Omega})} \leq |\mu_{s_+}| + \|w(s_+, \cdot) - \mu_{s_+}\|_{L^\infty(S^1)} \leq C(\log Y)^{\frac{1}{2}} \|w\|_{\dot{H}^1} \tag{5.4.65}$$

and so we can estimate the second fundamental form terms, using $\mathrm{vol}(\mathcal{C}_Y \setminus \tilde{\Omega}, g_E) \leq C \log Y$ and $|\gamma_Y'| \leq CY^{-1}$,

$$\begin{aligned} &\left| \int_{\mathcal{C}_Y \setminus \tilde{\Omega}} A(\gamma_Y)(\gamma_Y', \gamma_Y') A(\gamma_Y)(\hat{w}, \hat{w}) \, \mathrm{d}s \mathrm{d}\theta \right| \\ &\quad \leq C \int_{\mathcal{C}_Y \setminus \tilde{\Omega}} |\gamma_Y'|^2 |\hat{w}|^2 \, \mathrm{d}s \mathrm{d}\theta \leq C (\log Y)^2 Y^{-2} \|w\|^2_{\dot{H}^1} = o(1) \|w\|^2_{\dot{H}^1}. \end{aligned} \tag{5.4.66}$$

Meanwhile for $(s, \theta) \in (\mathcal{C}_Y \setminus \tilde{\Omega}) \cap \{\pm s \geq 0\}$ we have, using (5.6.2) and $|\gamma_Y'| \leq CY^{-1}$,

$$\begin{aligned} |\nabla \hat{w}|(s, \theta) &\leq |\nabla[\sigma_\pm(s) w(s_\pm, \theta) - (1 - \sigma_\pm(s))\mu_{s_\pm}]| + C|\gamma_Y'| |\sigma_\pm(s) w(s_\pm, \theta) - (1 - \sigma_\pm(s))\mu_{s_\pm}| \\ &\leq \left(|\partial_s \sigma_\pm| + CY^{-1} |\sigma_\pm|\right) \|w(s_\pm, \cdot) - \mu_{s_\pm}\|_{L^\infty(S^1)} + |\sigma_\pm| |(\partial_\theta w)(s_\pm, \theta)| + CY^{-1} |\mu_{s_\pm}| \end{aligned}$$

where we note that the first two terms vanish unless $s \in [s_+, s_+ + 1]$ (in the $+$ case) or $s \in [s_- - 1, s_-]$ (in the $-$ case). This bound for $\nabla \hat{w}$ implies that

$$\begin{aligned} \int_{(\mathcal{C}_Y \setminus \tilde{\Omega}) \cap \{\pm s \geq 0\}} |\nabla \hat{w}|^2 \, \mathrm{d}s \mathrm{d}\theta &\leq C \|w(s_\pm, \cdot) - \mu_{s_\pm}\|^2_{L^\infty(S^1)} + C\vartheta(s_\pm) + CY^{-2} |\mu_{s_\pm}|^2 \mathrm{vol}(\mathcal{C}_Y \setminus \tilde{\Omega}, g_E) \\ &\leq C \frac{\|w\|^2_{\dot{H}^1}}{\log Y} + C(\log Y)^2 Y^{-2} \|w\|^2_{\dot{H}^1} = o(1) \|w\|^2_{\dot{H}^1} \end{aligned} \tag{5.4.67}$$

where in the final inequality we have used (5.4.60), (5.4.61), (5.4.62) and (5.4.63). Combining (5.4.66) and (5.4.67) we conclude that the extension $\hat{w}$ satisfies (5.4.64).

We will now show how we can modify this extension $\hat{w}$ to obtain a map $\bar{w}$ vanishing on the boundary of $\mathcal{C}_Y$ and satisfying the desired properties (5.4.48) and (5.4.49). Notice that if we had $\hat{w}(-Y) = \hat{w}(Y)$ then we would be able to obtain a map vanishing on $\partial \mathcal{C}_Y$ and satisfying the desired properties by setting $\bar{w} = \hat{w} - \hat{w}(\pm Y)$. In general the values of $\hat{w}$ on either component of the boundary do not agree (and do not even lie in the same space), and so we instead subtract the projection of a linear interpolation between the two boundary values. Precisely, we set

$$\bar{w} = \hat{w} - \lambda \text{ where } \lambda(s) = P_{\gamma_Y(s)}\left[\tfrac{s+Y}{2Y}\mu_{s_+} + \tfrac{Y-s}{2Y}\mu_{s_-}\right].$$

We now gather some estimates for λ and $\hat{w}$ which are valid on the whole cylinder $\mathcal{C}_Y$ which we will need shortly. Using (5.4.63) we have

$$\|\lambda\|_{L^\infty(\mathcal{C}_Y)} \leq \max(|\mu_{s_-}|, |\mu_{s_+}|) \leq C|\log Y|^{\frac{1}{2}}\|w\|_{\dot{H}^1}. \tag{5.4.68}$$

In addition we can estimate the gradient of λ using (5.6.2) and $|\gamma_Y'| \leq CY^{-1}$,

$$|\nabla\lambda| \leq \tfrac{1}{2Y}(\mu_{s_+} - \mu_{s_-}) + C|\gamma_Y'|\max(|\mu_{s_+}|, |\mu_{s_-}|) \leq CY^{-1}|\log Y|^{\frac{1}{2}}\|w\|_{\dot{H}^1}, \tag{5.4.69}$$

using additionally (5.4.63), which leads to

$$\|\lambda\|_{\dot{H}^1(\mathcal{C}_Y)} \leq C\mathrm{vol}(\mathcal{C}_Y, g_E)^{\frac{1}{2}}Y^{-1}(\log Y)^{\frac{1}{2}}\|w\|_{\dot{H}^1} \leq CY^{-\frac{1}{2}}(\log Y)^{\frac{1}{2}}\|w\|_{\dot{H}^1}. \tag{5.4.70}$$

We can also use (5.4.65) to obtain

$$\begin{aligned}
\|\hat{w}\|_{L^\infty(\mathcal{C}_Y)} &\leq \|\hat{w}\|_{L^\infty(\tilde{\Omega})} + \|\hat{w}\|_{L^\infty(\mathcal{C}_Y\setminus\tilde{\Omega})} \\
&= \|w\|_{L^\infty(\tilde{\Omega})} + C(\log Y)^{\frac{1}{2}}\|w\|_{\dot{H}^1} \\
&\leq CY^{\frac{1}{2}}\|w\|_{\dot{H}^1} + C(\log Y)^{\frac{1}{2}}\|w\|_{\dot{H}^1} \leq CY^{\frac{1}{2}}\|w\|_{\dot{H}^1},
\end{aligned} \tag{5.4.71}$$

while (5.4.67) ensures the $\dot{H}^1$ norm of $\hat{w}$ is controlled

$$\|\hat{w}\|_{\dot{H}^1(\mathcal{C}_Y)} = \|\hat{w}\|_{\dot{H}^1(\tilde{\Omega})} + \|\hat{w}\|_{\dot{H}^1(\mathcal{C}_Y\setminus\tilde{\Omega})} = \|w\|_{\dot{H}^1(\tilde{\Omega})} + o(1)\|w\|_{\dot{H}^1} \leq C\|w\|_{\dot{H}^1}. \tag{5.4.72}$$

We are now ready to show that $\bar{w}$ satisfies the desired properties (5.4.48) and (5.4.49). Since $\bar{w} = \hat{w} - \lambda$ and $\hat{w}|_{\tilde{\Omega}} = w|_{\tilde{\Omega}}$ we can write

$$I_{\mathcal{C}_Y}(\gamma_Y)[\bar{w}, \bar{w}] = I_{\tilde{\Omega}}(\gamma_Y)[w, w] + \mathrm{err}_1 \tag{5.4.73}$$

with

$$\mathrm{err}_1 = I_{\mathcal{C}_Y\setminus\tilde{\Omega}}(\gamma_Y)(\hat{w}, \hat{w}) - 2I_{\mathcal{C}_Y}(\gamma_Y)[\hat{w}, \lambda] + I_{\mathcal{C}_Y}(\gamma_Y)[\lambda, \lambda].$$

Recall we have already shown (5.4.64) which says exactly that $I_{\mathcal{C}_{Y(\ell)}\setminus\tilde{\Omega}}(\gamma_Y)(\hat{w}, \hat{w}) = o(1)\|w\|^2_{\dot{H}^1}$. Meanwhile, the second term in err_1 is small owing to the estimates (5.4.68), (5.4.70) for λ and (5.4.71) and (5.4.72) for $\hat{w}$:

$$\begin{aligned}
|I_{\mathcal{C}_Y}(\gamma_Y)[\hat{w}, \lambda]| &\leq \int_{\mathcal{C}_Y} |\nabla\hat{w}||\nabla\lambda| + C|\gamma_Y'|^2|\hat{w}||\lambda| \, \mathrm{d}s\mathrm{d}\theta \\
&\leq \|\hat{w}\|_{\dot{H}^1}\|\lambda\|_{\dot{H}^1} + C\mathrm{vol}(\mathcal{C}_Y, g_E)Y^{-2}\|\hat{w}\|_{L^\infty}\|\lambda\|_{L^\infty} \\
&\leq CY^{-\frac{1}{2}}(\log Y)^{\frac{1}{2}}\|w\|^2_{\dot{H}^1} = o(1)\|w\|^2_{\dot{H}^1}.
\end{aligned} \tag{5.4.74}$$

Finally, the third term in err_1 needs only the aforementioned estimates (5.4.68), (5.4.70) for λ

$$
\begin{aligned}
|I_{\mathcal{C}_Y}(\gamma_Y)[\lambda,\lambda]| &\leq \int_{\mathcal{C}_Y} |\nabla\lambda|^2 + C|\gamma_Y'|^2|\lambda|^2 \, \mathrm{d}s\mathrm{d}\theta \\
&\leq \|\lambda\|_{\dot{H}^1}^2 + C\mathrm{vol}(\mathcal{C}_Y, g_E)Y^{-2}\|\lambda\|_{L^\infty}^2 \\
&\leq CY^{-1}\log Y\|w\|_{\dot{H}^1}^2 = o(1)\|w\|_{\dot{H}^1}^2.
\end{aligned} \tag{5.4.75}
$$

Therefore we have $\mathrm{err}_1 = o(1)\|w\|_{\dot{H}^1}^2$ which, returning to (5.4.73), concludes the proof of (5.4.49).

To establish (5.4.48) we argue in the same way, writing

$$
\|\bar{w}\|_{\dot{H}^1}^2 = \|w\|_{\dot{H}^1(\tilde{\Omega})}^2 + \mathrm{err}_2 \text{ with } \mathrm{err}_2 = \|\hat{w}\|_{\dot{H}^1(\mathcal{C}_Y\setminus\tilde{\Omega})}^2 - 2\langle\hat{w},\lambda\rangle_{\dot{H}^1} + \|\lambda\|_{\dot{H}^1}^2,
$$

and then noting that $|\mathrm{err}_2| = o(1)\|w\|_{\dot{H}^1}^2$ using (5.4.67), (5.4.70) and (5.4.72). This finishes the proof of Claim 2.

□

Having established both Claim 1 and Claim 2 we are done with the proof of Lemma 5.4.9.

□

5.5 Proof of Theorem 5.1.2

In this section we will give the proof of Theorem 5.1.2. Let $\gamma\colon [-1,1] \to N$ a curve satisfying assumptions (A), (B) and $v_\pm\colon D \to N$ be harmonic maps satisfying assumption (C). In Section 5.2 we described how to glue these objects to obtain "adapted critical points" $z_\ell\colon C_0 = [-1,1]\times S^1 \to N$ for $\ell \in (0,\ell_*)$, $\ell_* > 0$ a fixed number, and we write

$$
\mathcal{Z} = \{(z_\ell,\ell) \mid \ell \in (0,\ell_*)\}
$$

for the set of all adapted critical points, a subset of $H = H^1_{v_\pm}(C_0,N)\times(0,\infty)$. We are considering the harmonic map energy E defined on the space H, $E(u,\ell) = \frac{1}{2}\int_{C_0}|du|_{G_\ell}^2 \, \mathrm{d}\mu_{G_\ell}$ for $(u,\ell) \in H$. We equip the tangent space $T_{(u,\ell)}H$ with the inner product $\langle\cdot,\cdot\rangle_\ell$ defined in (5.2.19).

We are taking $(u,\lambda) \in H$ satisfying (5.1.6), which can be phrased as

$$
\inf_{(z_\ell,\ell)\in\mathcal{Z}} \|(u-u_\ell,\lambda-\ell)\|_\ell \leq \varepsilon,
$$

and this infimum is attained at some scale $\ell \in (0,\bar{\ell})$ for which $\|u - z_\ell\|_{L^\infty} \leq \varepsilon$ for suitably chosen $\bar{\ell} \in (0,\ell_*)$ and $\varepsilon > 0$. Hence we can always assume $\ell > 0$ is as small as needed and – setting $w = u - z_\ell$, $p = \lambda - \ell$, $W = (w,p)$ – that $\|W\|_\ell$, $\|w\|_{L^\infty}$ are as small as needed also.

Since the norm $\|\cdot\|_\ell$ being minimized depends on ℓ we cannot expect W to be orthogonal to $T_{(z_\ell,\ell)}\mathcal{Z}$. Our first step will instead to be show that W is "almost" orthogonal in the sense described in the following Lemma.

We write $V^\perp = V - \langle V, y_z\rangle_\ell y_z \in (T_{(z_\ell,\ell)}\mathcal{Z})^\perp$ for the orthogonal projection onto $(T_{(z_\ell,\ell)}\mathcal{Z})^\perp$, where $y_z = \frac{\partial_\ell(z_\ell,\ell)}{\|\partial_\ell(z_\ell,\ell)\|_\ell}$. Recall also that for Π the nearest point projection onto N that $d_a\Pi\colon \mathbb{R}^n \to T_aN$ is the orthogonal projection onto T_aN provided we only consider points $a \in N$, and we denote by P_a this orthogonal projection. We also write $P_aW = (P_aw, p)$.

Lemma 5.5.1. *There exists $\ell_* > 0$ such that for all $(u,\lambda) \in H$ and $\ell \in (0,\ell_*)$ satisfying*

$$\|(u - z_\ell, \lambda - \ell)\|_\ell = \inf_{\ell' \in (0,\ell_*)} \|(u - z_{\ell'}, \lambda - \ell')\|_{\ell'}, \qquad \|u - z_\ell\|_{L^\infty} < \delta_N,$$

we have, for $W = (u - z_\ell, \lambda - \ell)$,

$$\|P_{z_\ell}W - (P_{z_\ell}W)^\perp\|_\ell \le C\|W\|_\ell^2 + C\ell^{\frac{1}{2}}\|u - z_\ell\|_{L^\infty}\|W\|_\ell. \tag{5.5.1}$$

Proof. Write $W = (w,p)$ – that is $w = u - z_\ell$ and $p = \lambda - \ell$. As a first step we will estimate $\|W - W^\perp\|_\ell$ before later showing how to deal with the orthogonal projection P_{z_ℓ} onto $T_{z_\ell}N$.

The pair (z_ℓ, ℓ) is chosen so that it minimizes the quantity $\|W\|_\ell^2 = \|u - z_\ell\|^2_{\dot H^1(C_0,G_\ell)} + \ell^{-2}(\lambda-\ell)^2$ and hence the derivative with respect to ℓ vanishes, giving

$$\begin{aligned} 0 &= \frac{\partial}{\partial \ell}\left[\|u - z_\ell\|^2_{\dot H^1(C_0,G_\ell)} + \ell^{-2}(\lambda-\ell)^2\right] \\ &= \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\left[\|u - z_{\ell+\varepsilon}\|^2_{\dot H^1(C_0,G_\ell)} + \|u - z_\ell\|^2_{\dot H^1(C_0,G_{\ell+\varepsilon})}\right] - 2\ell^{-3}(\lambda-\ell)^2 - 2\ell^{-2}(\lambda-\ell) \\ &= -2\langle (u - z_\ell, \lambda-\ell), (\partial_\ell z_\ell, 1)\rangle_\ell + \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\|u - z_\ell\|^2_{\dot H^1(C_0,G_{\ell+\varepsilon})} - 2\ell^{-1}\|(0,\lambda-\ell)\|_\ell^2 \end{aligned} \tag{5.5.2}$$

using for the final step that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\|u - z_{\ell+\varepsilon}\|^2_{\dot H^1(C_0,G_\ell)} = -2\langle u - z_\ell, \partial_\ell z_\ell\rangle_{\dot H^1(C_0,G_\ell)}$ and keeping in mind the definition (5.2.19) of $\langle\cdot,\cdot\rangle_\ell$.

Note that for maps taking values in N the energy $2E(\cdot,\ell)$ coincides with the squared norm $\|\cdot\|^2_{\dot H^1(C_0,G_\ell)}$. In fact the formula (5.3.4) holds for maps not necessarily taking values in N and so estimating using $\rho_\ell^{-1} \le 2\pi\ell^{-1}$ we have

$$\left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\|u - z_\ell\|^2_{\dot H^1(C_0,G_{\ell+\varepsilon})}\right| \le C\ell^{-1}\|u - z_\ell\|^2_{\dot H^1(C_0,G_\ell)}.$$

Now using using (5.2.41) and rearranging (5.5.2) gives

$$\begin{aligned} |\langle W, y_z\rangle_\ell| &= \frac{1}{\|(\partial_\ell z_\ell, 1)\|_\ell}|\langle (u - z_\ell, \lambda-\ell), (\partial_\ell z_\ell, 1)\rangle_\ell| \\ &\le C\ell\left[\left|\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\|u - z_\ell\|^2_{\dot H^1(C_0,G_{\ell+\varepsilon})}\right| + \ell^{-3}(\lambda-\ell)^2\right] \\ &\le C\left[\|(u - z_\ell, 0)\|_\ell^2 + \|(0,\lambda-\ell)\|_\ell^2\right] = C\|W\|_\ell^2. \end{aligned}$$

Since $W - W^\perp = \langle W, y_z \rangle_\ell y_z$ we obtain

$$\|W - W^\perp\|_\ell = |\langle W, y_z \rangle_\ell| \le C\|W\|_\ell^2. \tag{5.5.3}$$

To deal with the projection P_{z_ℓ} onto $T_{z_\ell}N$ we use

$$\begin{aligned} \|(P_{z_\ell}W - W) - (P_{z_\ell}W - W)^\perp\|_\ell &= \|\langle (P_{z_\ell}w - w, 0), y_z\rangle_\ell y_z\|_\ell \\ &= \frac{1}{\|(\partial_\ell z_\ell, 1)\|_\ell} |\langle P_{z_\ell}w - w, \partial_\ell z_\ell \rangle_{\dot{H}^1(C_0,G_\ell)}| \\ &\le \frac{1}{\|(\partial_\ell z_\ell, 1)\|_\ell} \|P_{z_\ell}w - w\|_{\dot{H}^1} \|\partial_\ell z_\ell\|_{\dot{H}^1(C_0,G_\ell)} \\ &\le C\ell^{\frac{1}{2}} \|P_{z_\ell}w - w\|_{\dot{H}^1(C_0,G_\ell)} \\ &\le C\ell^{\frac{1}{2}} \|w\|_{L^\infty(C_0)} \|w\|_{\dot{H}^1(C_0,G_\ell)} \end{aligned} \tag{5.5.4}$$

using (5.2.41) and the estimate $\|P_{z_\ell}w - w\|_{\dot{H}^1(C_0,G_\ell)} \le C\|w\|_{L^\infty(C_0)}\|w\|_{\dot{H}^1(C_0,G_\ell)}$, see (5.5.9) below. This allows us to conclude

$$\begin{aligned} \|P_{z_\ell}W - (P_{z_\ell}W)^\perp\|_\ell &\le \|W - W^\perp\|_\ell + \|(P_{z_\ell}W - W) - (P_{z_\ell}W - W)^\perp\|_\ell \\ &\le C\|W\|_\ell^2 + C\ell^{\frac{1}{2}}\|w\|_{L^\infty(C_0)}\|w\|_{\dot{H}^1(C_0,G_\ell)} \end{aligned}$$

using (5.5.3) and (5.5.4), which gives the result since $w = u - z_\ell$ and $\|w\|_{\dot{H}^1(C_0,G_\ell)} \le \|W\|_\ell$. □

We can now use the "almost-orthogonality" of $W = (u - z_\ell, \lambda - \ell)$ together with the uniform definiteness of d^2E in directions orthogonal to $T_{(z_\ell,\ell)}\mathcal{Z}$, Lemma 5.4.7, to obtain the following key lemma.

Lemma 5.5.2. *There exists $\varepsilon > 0$ and $\ell_* > 0$ such that for all $(u, \lambda) \in H$ and $\ell \in (0, \ell_*)$ satisfying*

$$\|(u - z_\ell, \lambda - \ell)\|_\ell = \inf_{\ell' \in (0,\ell_*)} \|(u - z_{\ell'}, \lambda - \ell')\|_{\ell'} \le \varepsilon, \quad \|u - z_\ell\|_{L^\infty} \le \varepsilon, \tag{5.5.5}$$

we have, for $W = (u - z_\ell, \lambda - \ell)$,

$$\|W\|_\ell^2 \le C|dE(u,\lambda)((P_zW)^\perp)| + C|dE(z_\ell,\ell)((P_zW)^\perp)| \tag{5.5.6}$$

and

$$\|W\|_\ell \le C\|\nabla E(u,\lambda)\|_* + C\ell|\log \ell|^{\frac{1}{2}} \tag{5.5.7}$$

where ∇E, defined in (5.1.2), is the gradient of E with respect to the inner product $\langle \cdot, \cdot \rangle_*$ defined in (5.1.1).

In what follows we will need to interpolate between (u,λ) and (z_ℓ,ℓ) via $u_t=\Pi(z_\ell+tw)$, $\ell_t=\ell+tp$ where $w=u-z_\ell$ and $p=\lambda-\ell$. We derive various important estimates for u_t and P_{u_t} in Section 5.6.1 of the Appendix – all estimates there can be transferred directly to the situation here where the domain is C_0 by pulling back and working with ∇_ℓ, the gradient computed with the metric $\widetilde{G}_\ell=f_\ell^*(\mathrm{d}s^2+\mathrm{d}\theta^2)$. It will be important that we can write $\frac{d}{dt}u_t=P_{u_t}w+\mathrm{err}_t$, where the error term satisfies

$$|\mathrm{err}_t|\leq C|w|^2,\quad |\nabla_\ell \mathrm{err}_t|\leq C\rho_\ell(s_\ell)|w|^2+C|w||\nabla_\ell w|,\tag{5.5.8}$$

this uses (5.6.3), (5.6.4) and (5.2.26). We furthermore have, using (5.6.5) and (5.2.26),

$$\|w-P_{u_t}w\|_{\dot{H}^1}^2\leq C\|w\|_{L^\infty}^2\int_{C_0}|\nabla_\ell w|^2+\rho_\ell(s_\ell)^2|w|^2\,\mathrm{d}\mu_{\widetilde{G}_\ell}\leq C\|w\|_{L^\infty}^2\|w\|_{\dot{H}^1}^2,\tag{5.5.9}$$

using also the Poincaré inequality (5.2.5). We remark that here, and in the following, whenever we omit the domain and metric in our norms they will always be with respect to (C_0,G_ℓ).

Proof of Lemma 5.5.2. As before we write $W=(w,p)$ for $w=u-z_\ell$ and $p=\lambda-\ell$. We will choose $\ell_*\in(0,\bar{\ell})$ small enough so that Lemma 5.5.1 holds, giving

$$\|P_{z_\ell}W-(P_{z_\ell}W)^\perp\|_\ell\leq C\|W\|_\ell^2+C\ell^{\frac{1}{2}}\|w\|_{L^\infty}\|W\|_\ell\leq C\varepsilon\|W\|_\ell,\tag{5.5.10}$$

using additionally our assumptions that $\|W\|_\ell\leq\varepsilon$ and $\|w\|_{L^\infty}\leq\varepsilon$. At the same time (5.5.9) implies

$$\|W-P_{z_\ell}W\|_\ell=\|w-P_{z_\ell}w\|_{\dot{H}^1}\leq C\|w\|_{L^\infty}\|w\|_{\dot{H}^1}\leq C\varepsilon\|W\|_\ell\tag{5.5.11}$$

which immediately gives

$$\|P_{z_\ell}W\|_\ell\leq\|W\|_\ell+C\varepsilon\|W\|_\ell\leq C\|W\|_\ell$$

and hence

$$\|(P_{z_\ell}W)^\perp\|_\ell\leq\|P_{z_\ell}W\|_\ell\leq C\|W\|_\ell.\tag{5.5.12}$$

Using the triangle inequality and orthogonality gives

$$\begin{aligned}\|W\|_\ell^2&\leq 2\|P_{z_\ell}W\|_\ell^2+2\|W-P_{z_\ell}W\|_\ell^2\\&=2\|(P_{z_\ell}W)^\perp\|_\ell^2+2\|P_{z_\ell}W-(P_{z_\ell}W)^\perp\|_\ell^2+2\|W-P_{z_\ell}W\|_\ell^2\\&\leq 2\|(P_{z_\ell}W)^\perp\|_\ell^2+C\varepsilon^2\|W\|_\ell^2\end{aligned}\tag{5.5.13}$$

using (5.5.10) and (5.5.11). Hence using the positive definiteness of $d^2E(z_\ell,\ell)$ in directions orthogonal to $T_{(z_\ell,\ell)}\mathcal{Z}$, Lemma 5.4.7, and the boundedness of $d^2E(z_\ell,\ell)$, Lemma 5.4.1, we have

$$\begin{aligned}
&d^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp, P_{z_\ell}W]\\
&= d^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp,(P_{z_\ell}W)^\perp] + d^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp, P_{z_\ell}W-(P_{z_\ell}W)^\perp]\\
&\geq c_0\|(P_{z_\ell}W)^\perp\|_\ell^2 - C\|(P_{z_\ell}W)^\perp\|_\ell\|P_{z_\ell}W-(P_{z_\ell}W)^\perp\|\\
&\geq \tfrac{c_0}{2}\|W\|_\ell^2 - C\varepsilon^2\|W\|_\ell^2 - C\varepsilon\|W\|_\ell^2
\end{aligned}$$

using (5.5.13), (5.5.12) and (5.5.10) in the final inequality. Therefore, by taking $\varepsilon>0$ small enough, we have

$$\|W\|_\ell^2 \leq Cd^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp, P_{z_\ell}W]. \tag{5.5.14}$$

Next we observe that we can write

$$\begin{aligned}
&dE(u,\lambda)((P_{z_\ell}W)^\perp) - dE(z_\ell,\ell)((P_{z_\ell}W)^\perp)\\
&= \int_0^1 \tfrac{d}{dt}dE(u_t,\ell_t)((P_{z_\ell}W)^\perp)\,\mathrm{dt} = \int_0^1 \tfrac{d}{dt}\left[dE(u_t,\ell_t)(P_{u_t}(P_{z_\ell}W)^\perp)\right]\,\mathrm{dt}\\
&= \int_0^1 d^2E(u_t,\ell_t)[P_{u_t}(P_{z_\ell}W)^\perp, \tfrac{d}{dt}(u_t,\ell_t)] + dE(u_t,\ell_t)(\tfrac{d}{dt}P_{u_t}(P_{z_\ell}W)^\perp)\,\mathrm{dt}\\
&= \int_0^1 d^2E(u_t,\ell_t)[P_{u_t}(P_{z_\ell}W)^\perp, (P_{u_t}w+\mathrm{err}_t,p)] + dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}(P_{z_\ell}W)^\perp)\,\mathrm{dt}\\
&= d^2E(z_\ell,\ell)[(P_zW)^\perp, P_{z_\ell}W] + T_1+T_2+T_3,
\end{aligned}\tag{5.5.15}$$

where we recall the error term satisfies (5.5.8) and we are writing

$$\begin{aligned}
T_1 &:= \int_0^1 d^2E(u_t,\ell_t)[P_{u_t}(P_{z_\ell}W)^\perp, P_{u_t}W] - d^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp, P_{z_\ell}W]\,\mathrm{dt}\\
T_2 &:= \int_0^1 d^2E(u_t,\ell_t)[P_{u_t}(P_zW)^\perp, (\mathrm{err}_t,0)]\,\mathrm{dt}\\
T_3 &:= \int_0^1 dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}(P_zW)^\perp)\,\mathrm{dt}.
\end{aligned}$$

Claim 3. We have the estimates

$$|T_1|+|T_2|+|T_3| \leq C\varepsilon\|W\|_\ell^2. \tag{5.5.16}$$

Proof of Claim 3. We will write $w_{z_\ell} = P_{z_\ell}w$ and introduce the notation $(P_{z_\ell}W)^\perp = (w_{z_\ell}^\perp, p_{z_\ell}^\perp)$, that is we define

$$w_{z_\ell}^\perp = w_{z_\ell} - \langle P_{z_\ell}W, y_z\rangle_\ell \frac{\partial_\ell z_\ell}{\|(\partial_\ell z_\ell,1)\|_\ell} \tag{5.5.17}$$

and

$$p_{z_\ell}^\perp = p - \langle P_{z_\ell}W, y_z\rangle_\ell \frac{1}{\|(\partial_\ell z_\ell,1)\|_\ell}. \tag{5.5.18}$$

In particular we can estimate using (5.5.12) we have

$$\|w_{z_\ell}^\perp\|_{\dot H^1} \le \|(P_{z_\ell}W)^\perp\|_\ell \le C\|W\|_\ell \tag{5.5.19}$$

and

$$\|(0,p_{z_\ell}^\perp)\|_\ell \le \|(P_{z_\ell}W)^\perp\|_\ell \le C\|W\|_\ell. \tag{5.5.20}$$

We can also use (5.2.27) with $\rho_\ell^{-1} \le 2\pi\ell^{-1}$ and (5.2.41) to give

$$\begin{aligned}\|w_{z_\ell}^\perp\|_{L^\infty} &\le \|w_{z_\ell}\|_{L^\infty} + |\langle P_{z_\ell}W, y_z\rangle_\ell| \frac{\|\partial_\ell z_\ell\|_{L^\infty}}{\|(\partial_\ell z_\ell, 1)\|_\ell} \\ &\le \|w\|_{L^\infty} + C\|P_{z_\ell}W\|_\ell \le \|w\|_{L^\infty} + C\|W\|_\ell \le C\varepsilon.\end{aligned} \tag{5.5.21}$$

Finally we will repeatedly use below that the Poincaré inequality (5.2.5) gives

$$\|w\|_{H^1} \le C\|w\|_{\dot H^1}, \quad \|w_{z_\ell}^\perp\|_{H^1} \le C\|w_{z_\ell}^\perp\|_{\dot H^1} \tag{5.5.22}$$

since w (hence also w_{z_ℓ}) and $\partial_\ell z_\ell$ vanish on ∂C_0.

Estimate for T_1. We can bound $|T_1| \le \sup_{t\in[0,1]} |T_1^{(1)}| + |T_1^{(2)}| + |T_1^{(3)}| + |T_1^{(4)}|$ where

$$\begin{aligned}T_1^{(1)} &= d^2E(u_t,\ell_t)[(P_{u_t}w_{z_\ell}^\perp,0),(P_{u_t}w,0)] - d^2E(z_\ell,\ell)[(w_{z_\ell}^\perp,0),(P_{z_\ell}w,0)],\\ T_1^{(2)} &= d^2E(u_t,\ell_t)[(P_{u_t}w_{z_\ell}^\perp,0),(0,p)] - d^2E(z_\ell,\ell)[(w_{z_\ell}^\perp,0),(0,p)],\\ T_1^{(3)} &= d^2E(u_t,\ell_t)[(0,p_{z_\ell}^\perp),(P_{u_t}w,0)] - d^2E(z_\ell,\ell)[(0,p_{z_\ell}^\perp),(P_{z_\ell}w,0)],\\ T_1^{(4)} &= d^2E(u_t,\ell_t)[(0,p_{z_\ell}^\perp),(0,p)] - d^2E(z_\ell,\ell)[(0,p_{z_\ell}^\perp),(0,p)].\end{aligned}$$

To estimate these terms we use Lemma 5.4.2 in which we have computed the difference in the second variation at different points. We will first set $v_1 = w_{z_\ell}^\perp$ and $v_2 = w$ in (5.4.15) – notice that by (5.5.21), (5.5.5) we have $\|w_{z_\ell}^\perp\|_{L^\infty} + \|w\|_{L^\infty} \le C\varepsilon$ and (5.5.19), (5.5.22) that $\|w_{z_\ell}^\perp\|_{H^1} + \|w\|_{H^1} \le C\|w_{z_\ell}^\perp\|_{\dot H^1} + C\|w\|_{\dot H^1} \le C\|W\|_\ell$ – so that we obtain the estimate

$$\begin{aligned}|T_1^{(1)}| &\le C[\|w\|_{L^\infty} + \|(0,p)\|_\ell]\|w_{z_\ell}^\perp\|_{H^1}\|w\|_{H^1} + C\|w_{z_\ell}^\perp\|_{L^\infty}\|w\|_{H^1}^2\\ &\quad + C\|w\|_{L^\infty}\|w_{z_\ell}^\perp\|_{H^1}\|w\|_{\dot H^1} + C\|w_{z_\ell}^\perp\|_{L^\infty}\|w\|_{L^\infty}\|w\|_{\dot H^1}^2\\ &\le C\varepsilon\|W\|_\ell^2.\end{aligned}$$

Setting $v_1 = w_{z_\ell}^\perp$ in (5.4.14) – using that (5.5.21) implies $\|w_{z_\ell}^\perp\|_{L^\infty} \le C$ and (5.5.22), (5.5.19) imply $\|w_{z_\ell}^\perp\|_{H^1} \le C\|W\|_\ell \le C\varepsilon$ – gives

$$|T_1^{(2)}| \le \|(0,p)\|_\ell \left[\|w_{z_\ell}^\perp\|_{H^1}\|W\|_\ell + \|w_{z_\ell}^\perp\|_{L^\infty}\|w\|_{\dot H^1}^2\right] \le C\varepsilon\|W\|_\ell^2.$$

Setting $v_1 = w$ in (5.4.14) – using (5.5.20), that $\|w\|_{L^\infty} \le C$ and that (5.5.22) implies $\|w\|_{H^1} \le C\|W\|_\ell \le C\varepsilon$ – gives

$$|T_1^{(3)}| \le \|(0,p_{z_\ell}^\perp)\|_\ell \left[\|w\|_{H^1}\|W\|_\ell + \|w\|_{L^\infty}\|w\|_{\dot H^1}^2\right] \le C\varepsilon\|W\|_\ell^2.$$

Finally applying (5.4.13) with (5.5.20) gives

$$|T_1^{(4)}| \le C\|(0, p_{z_\ell}^\perp)\|_\ell \|(0,p)\|_\ell \|W\|_\ell \le C\varepsilon \|W\|_\ell^2.$$

Estimate for T_2. Before continuing we note that (5.5.8) implies

$$\|\mathrm{err}_t\|_{L^\infty} \le C\|w\|_{L^\infty}^2 \le C\varepsilon^2 \tag{5.5.23}$$

and

$$\begin{aligned}\|\mathrm{err}_t\|_{H^1}^2 &= \|\mathrm{err}_t\|_{L^2}^2 + \|\mathrm{err}_t\|_{\dot H^1}^2 \\ &\le C\|w\|_{L^\infty}^2 \|w\|_{L^2}^2 + C\|w\|_{L^\infty}^2 \int_{C_0} |\nabla_\ell w|^2 + |w|^2 \rho_\ell(s_\ell)^2 \,\mathrm{d}\mu_{\widetilde{G}_\ell} \\ &\le C\|w\|_{L^\infty}^2 \|w\|_{H^1}^2 \le C\varepsilon^2 \|w\|_{\dot H^1}^2,\end{aligned} \tag{5.5.24}$$

using the Poincaré inequality (5.2.5) in the final step.

Now to estimate T_2 we use Lemma 5.4.6. We set $v_1 = \mathrm{err}_t$ and $v_2 = w_{z_\ell}^\perp$ in (5.4.31) and (5.4.30) giving

$$\begin{aligned}|T_2| &\le \sup_{t\in[0,1]} |d^2E(u_t,\ell_t)[(w_{z_\ell}^\perp, 0), (\mathrm{err}_t, 0)]| + |d^2E(u_t,\ell_t)[(0, p_{z_\ell}^\perp), (\mathrm{err}_t, 0)]| \\ &\le C \sup_{t\in[0,1]} \Big\{ \|\mathrm{err}_t\|_{H^1} \|w_{z_\ell}^\perp\|_{H^1} + \|\mathrm{err}_t\|_{L^\infty} \|w_{z_\ell}^\perp\|_{H^1} \|w\|_{H^1} + \|w_{z_\ell}^\perp\|_{L^\infty} \|\mathrm{err}_t\|_{H^1} \|w\|_{H^1} \\ &\qquad + \|\mathrm{err}_t\|_{L^\infty} \|w_{z_\ell}^\perp\|_{L^\infty} \|w\|_{\dot H^1}^2 + \|(0, p_{z_\ell}^\perp)\|_\ell \left[\|\mathrm{err}_t\|_{H^1} + \|\mathrm{err}_t\|_{L^\infty} \|w\|_{\dot H^1}^2 \right] \Big\} \\ &\le C\varepsilon \|W\|_\ell^2 + C\varepsilon \|W\|_\ell \|(0, p_{z_\ell}^\perp)\|_\ell \le C\varepsilon \|W\|_\ell^2.\end{aligned}$$

Here we use that by (5.5.24), (5.5.19) we have $\|\mathrm{err}_t\|_{H^1} \le C\varepsilon \|W\|_\ell$ and $\|w_{z_\ell}^\perp\|_{H^1} \le C\|W\|_\ell$, that by (5.5.23), (5.5.21) we have $\|\mathrm{err}_t\|_{L^\infty} + \|w_{z_\ell}^\perp\|_{L^\infty} \le C\varepsilon$ and we also use (5.5.20) in the final step.

Estimate for T_3. Since it will be needed later, we will obtain a more general estimate valid for $v \in \Gamma^{H_0^1}(z_\ell{}^*TN)$ and then specify here to $v = w_{z_\ell}^\perp$. We will need that for maps $v \in H_0^1(C_0, \mathbb{R}^n)$ taking values in $T_{z_\ell}N$ the estimate (5.6.15) implies

$$|\nabla_{\ell_t} P_{u_t} \tfrac{d}{dt} P_{u_t} v| \le C |\nabla_{\ell_t} v| |w|^2 + C|v||w| \left[|w| |\nabla_{\ell_t} z_\ell| + |\nabla_{\ell_t} w| \right]$$

and using this together with (5.6.7) in (5.3.5) gives

$$\begin{aligned}
&\left|dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}v,0)\right| \\
&\quad\leq \int_{C_0} |\nabla_{\ell_t}u_t||\nabla_{\ell_t}P_{u_t}\tfrac{d}{dt}P_{u_t}v|\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad\leq C\int_{C_0} (|\nabla_{\ell_t}z_\ell|+|\nabla_{\ell_t}w|)\,|w|\,[|w||\nabla_{\ell_t}v|+|v|\,(|w||\nabla_{\ell_t}z_\ell|+|\nabla_{\ell_t}w|)]\,\mathrm{d}\mu_{\widetilde{G}_{\ell_t}} \\
&\quad\leq C\int_{C_0} |w|\Big[(\rho_\ell\circ s_\ell)|w||\nabla_\ell v|+(\rho_\ell\circ s_\ell)^2|v||w|+(\rho_\ell\circ s_\ell)|v||\nabla_\ell w| \qquad (5.5.25)\\
&\qquad\qquad +|w||\nabla_\ell v||\nabla_\ell w|+(\rho_\ell\circ s_\ell)|v||w||\nabla_\ell w|+|v||\nabla_\ell w|^2\Big]\,\mathrm{d}\mu_{\widetilde{G}_\ell} \\
&\quad\leq C\int_{C_0}((\rho_\ell\circ s_\ell)+|\nabla_\ell w|)|w|^2|\nabla_\ell v|\,\mathrm{d}\mu_{\widetilde{G}_\ell}+C\|v\|_{L^\infty}[\|w\|_{H^1}^2+\|w\|_{L^\infty}\|w\|_{\dot{H}^1}^2] \\
&\quad\leq C\int_{C_0}((\rho_\ell\circ s_\ell)+|\nabla_\ell w|)|w|^2|\nabla_\ell v|\,\mathrm{d}\mu_{\widetilde{G}_\ell}+C\|v\|_{L^\infty}\|w\|_{\dot{H}^1}^2.
\end{aligned}$$

Here we have applied Lemma 5.4.4 before expanding and using (5.2.26) to bound $|\nabla_\ell z_\ell|\leq(\rho_\ell\circ s_\ell)$. The final step uses $\|w\|_{L^\infty}\leq C$ and the Poincaré inequality (5.2.5).

We now specify $v=w_{z_\ell}^\perp$ in (5.5.25) giving

$$\begin{aligned}
|T_3|&\leq C\int_{C_0}(\rho_\ell\circ s_\ell)|w|^2|\nabla_\ell w_{z_\ell}^\perp|+|w|^2|\nabla_\ell w||\nabla_\ell w_{z_\ell}^\perp|\,\mathrm{d}\mu_{\widetilde{G}_\ell}+C\|w_{z_\ell}^\perp\|_{L^\infty}\|w\|_{\dot{H}^1}^2 \\
&\leq C\|w\|_{L^\infty}\left[\|w\|_{L^2}+\|w\|_{L^\infty}\|w\|_{\dot{H}^1}\right]\|w_{z_\ell}^\perp\|_{\dot{H}^1}+C\|w_{z_\ell}^\perp\|_{L^\infty}\|w\|_{\dot{H}^1}^2 \\
&\leq C\|w\|_{L^\infty}\|w\|_{H^1}\|W\|_\ell+C\varepsilon\|w\|_{\dot{H}^1}^2\leq C\varepsilon\|W\|_\ell^2,
\end{aligned}$$

by (5.5.21) and (5.5.19), as desired.

This completes the proof of Claim 3. □

We can now continue with the proof of Lemma 5.5.2. Using (5.5.14), (5.5.15) and then Claim 3 gives

$$\begin{aligned}
\|W\|_\ell^2&\leq Cd^2E(z_\ell,\ell)[(P_{z_\ell}W)^\perp,P_{z_\ell}W] \\
&=dE(u,\lambda)[(P_{z_\ell}W)^\perp]-dE(z_\ell,\ell)[(P_{z_\ell}W)^\perp]+T_1+T_2+T_3 \\
&\leq\left|dE(u,\lambda)[(P_{z_\ell}W)^\perp]\right|+\left|dE(z_\ell,\ell)[(P_{z_\ell}W)^\perp]\right|+C\varepsilon\|W\|_\ell^2
\end{aligned}$$

which by taking $\varepsilon>0$ small enough gives the first desired result (5.5.6).

We now proceed with the proof of the second desired result (5.5.7). First note that $\|(P_{z_\ell}W)^\perp\|_*\leq C\|(P_{z_\ell}W)^\perp\|_\ell\leq C\|W\|_\ell$ (it does not matter whether we use λ or ℓ for the norm $\|\cdot\|_*$, see Remark 5.2.5) and hence

$$\left|dE(u,\lambda)[(P_{z_\ell}W)^\perp]\right|\leq\|\nabla E(u,\lambda)\|_*\|(P_{z_\ell}W)^\perp\|_*\leq C\|\nabla E(u,\lambda)\|_*\|W\|_\ell. \qquad (5.5.26)$$

Second note that Remark 5.3.4 (a consequence of Proposition 5.3.3) gives

$$\left|dE(z_\ell,\ell)[(P_{z_\ell}W)^\perp]\right|\leq C\ell|\log\ell|^{\frac{1}{2}}\|(P_{z_\ell}W)^\perp\|_\ell\leq C\ell|\log\ell|^{\frac{1}{2}}\|W\|_\ell. \qquad (5.5.27)$$

Using (5.5.6) with (5.5.26) and (5.5.27) and dividing through by $\|W\|_\ell$ gives the second desired result (5.5.7) and hence finishes the proof of Lemma 5.5.2. $\square$

We are now ready to complete the proof of the main Theorem 5.1.2.

Proof of Theorem 5.1.2. We have already obtained a bound on $\|W\|_\ell$ in terms of the gradient of the energy in Lemma 5.5.2. We will now use the lower bound on the first variation in the direction y_z stated in Remark 5.3.2 to give

$$\begin{aligned}
c_0\ell &\leq dE(z_\ell,\ell)(y_z) \\
&= dE(u,\lambda)(y_z) + \left[dE(z_\ell,\ell) - dE(u,\lambda)\right](y_z) \\
&= dE(u,\lambda)(y_z) + \int_0^1 \tfrac{d}{dt} dE(u_t,\ell_t)(y_z)\,\mathrm{d}t \\
&= dE(u,\lambda)(y_z) + \int_0^1 \tfrac{d}{dt}\left[dE(u_t,\ell_t)(P_{u_t}y_z)\right]\mathrm{d}t \\
&= dE(u,\lambda)(y_z) + \int_0^1 d^2E(u_t,\ell_t)[P_{u_t}y_z, \tfrac{d}{dt}(u_t,\ell_t)] + dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}y_z)\,\mathrm{d}t \\
&\leq dE(u,\lambda)(y_z) + d^2E(z_\ell,\ell)[y_z, P_{z_\ell}W] + \sup_{t\in[0,1]} T_1 + T_2 + T_3,
\end{aligned} \tag{5.5.28}$$

where we have used that $\frac{d}{dt}(u_t,\ell_t) = P_{u_t}W + (\mathrm{err}_t, 0)$ and recall that err_t satisfies the estimates (5.5.8). We are setting

$$\begin{aligned}
T_1 &= d^2E(u_t,\ell_t)[P_{u_t}y_z, P_{u_t}W] - d^2E(z_\ell,\ell)[y_z, P_{z_\ell}W], \\
T_2 &= d^2E(u_t,\ell_t)[P_{u_t}y_z, (\mathrm{err}_t, 0)], \\
T_3 &= dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}y_z).
\end{aligned}$$

Claim 4. We have the estimates

$$|T_1| + |T_2| + |T_3| \leq C\|W\|_\ell^2. \tag{5.5.29}$$

Proof of Claim 4. In what follows we write $y_z = (y,q)$, that is $y = \frac{\partial_\ell z_\ell}{\|(\partial_\ell z_\ell, 1)\|_\ell}$ and $q = \frac{1}{\|(\partial_\ell z_\ell, 1)\|_\ell}$, and note that (5.2.27) and (5.2.38) as well as (5.2.41) means that on the whole domain C_0 we have

$$|y| \leq C\ell(\rho_\ell \circ s_\ell)^{-1} \leq C, \quad |\nabla_\ell y| \leq C\ell \leq C(\rho_\ell \circ s_\ell), \tag{5.5.30}$$

keeping in mind that $\rho_\ell^{-1} \leq 2\pi\ell^{-1}$. We also have $\|(0,q)\|_\ell \leq \|y_z\|_\ell = 1$ and recall that we have defined $P_{z_\ell}W = (P_{z_\ell}w, p)$.

Estimate for T_1. We split $T_1 = T_1^{(1)} + T_1^{(2)} + T_1^{(3)} + T_1^{(4)}$, where

$$
\begin{aligned}
T_1^{(1)} &= d^2E(u_t,\ell_t)[(P_{u_t}y,0),(P_{u_t}w,0)] - d^2E(z_\ell,\ell)[(y,0),(P_{z_\ell}w,0)],\\
T_1^{(2)} &= d^2E(u_t,\ell_t)[(P_{u_t}y,0),(0,p)] - d^2E(z_\ell,\ell)[(y,0),(0,p)],\\
T_1^{(3)} &= d^2E(u_t,\ell_t)[(0,q),(P_{u_t}w,0)] - d^2E(z_\ell,\ell)[(0,q),(P_{z_\ell}w,0)],\\
T_1^{(4)} &= \left(d^2E(u_t,\ell_t) - d^2E(z_\ell,\ell)\right)[(0,q),(0,p)],
\end{aligned}
$$

and treat each term with the corresponding estimate from Lemma 5.4.2.

Using (5.4.16) with $v_1 = y$ and $v_2 = w$, together with the estimates (5.5.30) for y, we obtain

$$
\begin{aligned}
|T_1^{(1)}| &\le C\int_{C_0} (\rho_\ell\circ s_\ell)^2|w|^2 + |\nabla_\ell w|^2 + |w||\nabla_\ell w|^2 \,d\mu_{\widetilde{G}_\ell}\\
&\quad + C|p|\int_{C_0} |\nabla_\ell w| + (\rho_\ell\circ s_\ell)|w| \,d\mu_{\widetilde{G}_\ell}\\
&\le C\|w\|_{H^1}^2 + C|p|\mathrm{vol}(C_0,\widetilde{G}_\ell)^{\frac12}\|w\|_{H^1}\\
&\le C\|w\|_{\dot H^1}^2 + C\ell^{\frac12}\|(0,p)\|_\ell\|w\|_{\dot H^1} \le C\|W\|_\ell^2
\end{aligned}
$$

where we have used the Poincaré inequality (5.2.5), $\mathrm{vol}(C_0,\widetilde{G}_\ell) = \mathrm{vol}(\mathcal{C}_{Y(\ell)}, g_E) \le C\ell^{-1}$ and the definition (5.2.19) of $\|\cdot\|_\ell$.

Setting $v_1 = y$ in (5.4.14) and using again (5.5.30) gives

$$
\begin{aligned}
|T_1^{(2)}| &\le C\|(0,p)\|_\ell\left[\|y\|_{\dot H^1}\|W\|_\ell + \|y\|_{L^\infty}\|w\|_{\dot H^1}^2\right]\\
&\le C\|(0,p)\|_\ell\left[\ell^{\frac12}\|W\|_\ell + \|w\|_{\dot H^1}^2\right] \le C\|W\|_\ell^2,
\end{aligned}
$$

furthermore using that $\|y\|_{\dot H^1} \le C\ell^{\frac12}$ from (5.2.36) and (5.2.41). Meanwhile setting $v_1 = w$ in (5.4.14) gives

$$
|T_1^{(3)}| \le C\|(0,q)\|_\ell\left[\|w\|_{\dot H^1}\|W\|_\ell + \|w\|_{L^\infty}\|w\|_{\dot H^1}^2\right] \le C\|W\|_\ell^2.
$$

Finally (5.4.13) yields

$$
|T_1^{(4)}| \le C\|(0,q)\|_\ell\|(0,p)\|_\ell\|W\|_\ell \le C\|W\|_\ell^2.
$$

Estimate for T_2. We will use here the estimates (5.4.31) and (5.4.30) with $v_1 = \mathrm{err}_t$ and $v_2 = y$. Using (5.5.8) and (5.5.30) we obtain

$$
\begin{aligned}
|T_2| &= |d^2E(u_t,\ell_t)[(P_{u_t}y,0),(\mathrm{err}_t,0)] + d^2E(u_t,\ell_t)[(0,q),(\mathrm{err}_t,0)]|\\
&\le C\int_{C_0} \rho_\ell(s_\ell)(|w||\nabla_\ell w| + \rho_\ell(s_\ell)|w|^2) + \rho_\ell(s_\ell)(|w||\nabla_\ell w| + \rho_\ell(s_\ell)|w|^2) + \rho_\ell(s_\ell)^2|w|^2 \,d\mu_{\widetilde{G}_\ell}\\
&\quad + C\int_{C_0} (|w||\nabla_\ell w| + \rho_\ell(s_\ell)|w|^2)|\nabla_\ell w| + |w|^2|\nabla_\ell w|^2 \,d\mu_{\widetilde{G}_\ell}\\
&\quad + C\|(0,q)\|_\ell\int_{C_0} (|w||\nabla_\ell w| + \rho_\ell(s_\ell)|w|^2)(\rho_\ell(s_\ell) + |\nabla_\ell w|) + |w|^2(\rho_\ell(s_\ell)^2 + |\nabla_\ell w|^2) \,d\mu_{\widetilde{G}_\ell}\\
&\le C\|w\|_{H^1}^2 \le C\|W\|_\ell^2
\end{aligned}
$$

where we have used $\|w\|_{L^\infty} \le C$ and $\|(0,q)\|_\ell \le \|y_z\|_\ell = 1$, as well as the Poincaré inequality (5.2.5) in the final step.

Estimate for T_3. We will use the estimate (5.5.25) with $v = y \in \Gamma^{H_0^1}(z_\ell{}^*TN)$ and estimate using (5.5.30) to give

$$\begin{aligned}|T_3| &= |dE(u_t,\ell_t)(P_{u_t}\tfrac{d}{dt}P_{u_t}y_z)|\\ &\le C\int_{C_0}(\rho_\ell(s_\ell)+|\nabla_\ell w|)|w|^2|\nabla_\ell y| + C\|y\|_{L^\infty}\|w\|_{\dot H^1}^2\\ &\le C\int_{C_0}\rho_\ell(s_\ell)^2|w|^2+\rho_\ell(s_\ell)|w|^2|\nabla_\ell w|\,\mathrm{d}\mu_{\widetilde{G}_\ell}+C\|w\|_{\dot H^1}^2\\ &\le C\|w\|_{H^1}^2 \le C\|W\|_\ell^2\end{aligned}$$

where we use $\|w\|_{L^\infty} \le C$ and in the final step the Poincaré inequality (5.2.5).

This completes the proof of Claim 4. □

Combining Claim 4 with (5.3.25) from Remark 5.3.4 we can bound the second term in (5.5.28),

$$\begin{aligned}c_0\ell &\le dE(u,\lambda)(y_z) + C\ell|\log\ell|^{\frac12}\|W\|_\ell + C\|W\|_\ell^2\\ &\le dE(u,\lambda)(y_z) + C\|W\|_\ell^2 + C\ell^2|\log\ell|\\ &\le dE(u,\lambda)(y_z) + C\|\nabla E(u,\lambda)\|_*^2 + C\ell^2|\log\ell|\end{aligned} \tag{5.5.31}$$

using also Lemma 5.5.2. We will estimate the first term using the L^2 gradient ∇E from (5.1.2) since the element y_z scales better with respect to the norm $\|\cdot\|_*$ than it does with respect to $\|\cdot\|_\ell$. Indeed, using (5.2.42) we have

$$|dE(u,\lambda)(y_z)| \le \|\nabla E(u,\lambda)\|_*\|y_z\|_* \le C\ell^{\frac12}\|\nabla E(u,\lambda)\|_* \le \tfrac{c_0}{2}\ell + C\|\nabla E(u,\lambda)\|_*^2,$$

see also Remark 5.2.5. Plugging this into (5.5.31) we obtain the estimate

$$\tfrac{c_0}{2}\ell \le C\|\nabla E(u,\lambda)\|_*^2 + C\ell^2|\log\ell|,$$

and by taking $\ell > 0$ small enough we can absorb the final term onto the left hand side to give the desired bound (5.1.7) on the collapsing scale.

From (5.1.7) and the result (5.5.7) of Lemma 5.5.2 we can immediately obtain the desired estimate (5.1.8), recalling $W = (u - z_\ell, \lambda - \ell)$ and (5.2.19),

$$\|W\|_\ell \le C\|\nabla E(u,\lambda)\|_* + C\ell|\log\ell|^{\frac12} \le C\|\nabla E(u,\lambda)\|_* + C\ell^{\frac12} \le C\|\nabla E(u,\lambda)\|_*,$$

by taking $\ell > 0$ small enough that $\ell|\log\ell|^{\frac12} \le C\ell^{\frac12}$.

We now turn to establishing the desired estimate (5.1.9). Using (5.3.6) from Proposition 5.3.1 we have

$$\begin{aligned}
|E(u,\lambda)-E^*| &\le |E(u,\lambda)-E(z_\ell,\ell)| + |E(z_\ell,\ell)-E^*| \\
&\le |E(u,\lambda)-E(z_\ell,\ell)| + C\ell \\
&= \left|\int_0^1 \tfrac{d}{dt}E(u_t,\ell_t)\,\mathrm{dt}\right| + C\ell \\
&= \left|dE(z_\ell,\ell)(W) + \int_0^1 dE(u_t,\ell_t)(\tfrac{d}{dt}(u_t,\ell_t)) - dE(z_\ell,\ell)(P_{z_\ell}W)\,\mathrm{dt}\right| + C\ell \\
&\le |dE(z_\ell,\ell)(W)| + \left|\int_0^1 \tfrac{d}{dt}\left[dE(u_t,\ell_t)(\tfrac{d}{dt}(u_t,\ell_t))\right]\,\mathrm{dt}\right| + C\ell \\
&\le |dE(z_\ell,\ell)(W)| + \sup_{t\in[0,1]} T_4 + T_5 + C\ell
\end{aligned} \tag{5.5.32}$$

where we recall $\frac{d}{dt}(u_t,\ell_t) = (\frac{d}{dt}u_t, p)$ and we set

$$\begin{aligned}
T_4 &= d^2E(u_t,\ell_t)[\tfrac{d}{dt}(u_t,\ell_t), \tfrac{d}{dt}(u_t,\ell_t)] \\
T_5 &= dE(u_t,\ell_t)(\tfrac{d^2}{dt^2}u_t, 0).
\end{aligned}$$

Claim 5. We have the estimates

$$|T_4| + |T_5| \le C\|W\|_\ell^2. \tag{5.5.33}$$

Proof of Claim 5. **Estimate for T_4.** We first note that by combining (5.6.8) with (5.2.26) we have $|\nabla_\ell \frac{d}{dt}u_t| \le C|\nabla_\ell w| + C\rho_\ell(s_\ell)|w|$, hence using also (5.6.6) we have

$$\|\tfrac{d}{dt}u_t\|_{H^1} \le C\|w\|_{H^1} \le C\|w\|_{\dot{H}^1} \tag{5.5.34}$$

using the Poincaré inequality (5.2.5) for the last inequality. Moreover the estimate (5.6.6) immediately implies that

$$\|\tfrac{d}{dt}u_t\|_{L^\infty} \le C\|w\|_{L^\infty} \le C. \tag{5.5.35}$$

Now for our desired estimate on T_4 we appeal directly to the estimates (5.4.31), (5.4.30) and (5.4.32) from Lemma 5.4.6. Recalling that $\frac{d}{dt}(u_t,\ell_t) = (\frac{d}{dt}u_t, p)$ we have

$$\begin{aligned}
|T_4| &= \left|d^2E(u_t,\ell_t)[(\tfrac{d}{dt}u_t,p),(\tfrac{d}{dt}u_t,p)]\right| \\
&\le C\|\tfrac{d}{dt}u_t\|_{H^1}^2 + C\|\tfrac{d}{dt}u_t\|_{L^\infty}\|\tfrac{d}{dt}u_t\|_{H^1}\|w\|_{H^1} + C\|\tfrac{d}{dt}u_t\|_{L^\infty}^2\|w\|_{\dot{H}^1}^2 \\
&\quad + C\|(0,p)\|_\ell\left[\|\tfrac{d}{dt}u_t\|_{H^1} + \|\tfrac{d}{dt}u_t\|_{L^\infty}\|w\|_{\dot{H}^1}^2\right] + C\|(0,p)\|_\ell^2 \\
&\le C\|w\|_{\dot{H}^1}^2 + C\|(0,p)\|_\ell\|w\|_{\dot{H}^1} + C\|(0,p)\|_\ell^2 \le C\|W\|_\ell^2,
\end{aligned}$$

additionally using (5.5.34), (5.5.35) above and (5.2.5).

Estimate for T_5. We use the formula (5.3.5) for the first variation of E and then the estimates (5.6.7), (5.6.10) to give

$$\begin{aligned}
|T_5| = |dE(u_t,\ell_t)(\tfrac{d^2u_t}{dt^2},0)| &\le \int_{C_0} |\nabla_{\ell_t}u_t||\nabla_{\ell_t}\tfrac{d^2u_t}{dt^2}|\, d\mu_{\widetilde{G}_{\ell_t}} \\
&\le C\int_{C_0} (|\nabla_{\ell_t}z_\ell| + |\nabla_{\ell_t}w|)(|\nabla_{\ell_t}w||w| + |\nabla_{\ell_t}z_\ell||w|^2)\, d\mu_{\widetilde{G}_{\ell_t}} \\
&\le C\int_{C_0} |w|^2|\nabla_\ell z_\ell|^2 + |w||\nabla_\ell z_\ell||\nabla_\ell w| + |w|^2|\nabla_\ell z_\ell||\nabla_\ell w| + |w|^2|\nabla_\ell w|\, d\mu_{\widetilde{G}_\ell} \\
&\le C\int_{C_0} |w|^2\rho_\ell(s_\ell)^2 + |w|\rho_\ell(s_\ell)|\nabla_\ell w| + |\nabla_\ell w|^2\, d\mu_{\widetilde{G}_\ell} \\
&\le C\|w\|_{H^1}^2 \le C\|W\|_\ell^2,
\end{aligned}$$

where we have used Lemma 5.4.4. We have additionally used $\|w\|_{L^\infty} \le C$, (5.2.26) and (5.2.5).

This concludes the proof of Claim 5. □

Finally, returning to (5.5.32), we can use (5.3.26) from Remark 5.3.4 and Claim 5 to give our desired Łojasiewicz inequality (5.1.9),

$$\begin{aligned}
|E(u,\lambda) - E^*| &\le C\ell|\log\ell|^{\frac12}\|W\|_\ell + C\|W\|_\ell^2 + C\ell \\
&\le C\|W\|_\ell^2 + C\ell \\
&\le C\|\nabla E(u,\lambda)\|_*^2 + C\ell \\
&\le C\|\nabla E(u,\lambda)\|_*^2
\end{aligned}$$

where we also use the estimates (5.1.8) and (5.1.7) that have we have already established. This completes the proof of Theorem 5.1.2. □

5.6 Appendix to Chapter 5

5.6.1 Formulae for projections

Let N denote a closed Riemannian manifold which, by Nash's embedding theorem, we assume is isometrically embedded into some Euclidean space $\mathbb{R}^n$. We choose $\delta_N > 0$ such that the nearest point projection Π to N is well defined and smooth on a δ_N tubular neighbourhood of N in $\mathbb{R}^n$. Given $a \in N$ we recall that $P_a = d_a\Pi\colon \mathbb{R}^n \to T_aN$ is the orthogonal projection onto T_aN. In what follows we will use that there exists an orthonormal basis $\{\nu_a^j\}$ for $T_a^\perp N \subset \mathbb{R}^n$ which varies smoothly in the neighbourhood of a point $a \in N \subset \mathbb{R}^n$.

Given any map $\hat{z}\colon \mathcal{C}_{Y(\ell)} \to \mathbb{R}^n$ such that $\mathrm{dist}(\hat{z}, N) < \delta_N$ we can define $z = \Pi(\hat{z})\colon \mathcal{C}_{Y(\ell)} \to N$. In what follows we collect some useful estimates for z, as well as on the projections $P_z w$ of a general vector field $w\colon \mathcal{C}_{Y(\ell)} \to \mathbb{R}^n$. We have

$$|\nabla z| = |d_{\hat{z}}\Pi(\nabla\hat{z})| \le C|\nabla\hat{z}|. \tag{5.6.1}$$

We can also compute the derivative of the projection $P_z(w) = w - \langle w, \nu_z^j\rangle \nu_z^j$, where repeated indices are summed over,

$$\begin{aligned} \nabla(P_z(w)) &= P_z(\nabla w) - \langle w, \nabla \nu_z^j\rangle \nu_z^j - \langle w, \nu_z^j\rangle \nabla \nu_z^j, \\ |\nabla(P_z(w))| &\leq |P_z(\nabla w)| + C|w||\nabla z|. \end{aligned} \tag{5.6.2}$$

using $|\nabla \nu_z^j| \leq C|\nabla z|$.

Next, given $u \in H^1(\mathcal{C}_{Y(\ell)}, N)$ such that $\|u - z\|_{L^\infty} < \delta_N$ we consider

$$u_t = \Pi(z + tw) \text{ for } w = u - z \text{ and } t \in [0,1]$$

and derive estimates for u_t and the orthogonal projections P_{u_t} onto $T_{u_t}N$.

We can write $\frac{d}{dt}u_t = d_{z+tw}\Pi(w) = P_{u_t}(w) + \mathrm{err}_t$ where

$$|\mathrm{err}_t| = |d_{z+tw}\Pi(w) - d_{u_t}\Pi(w)| \leq C|w|^2 \tag{5.6.3}$$

and

$$|\nabla \mathrm{err}_t| \leq C|w|^2|\nabla(z + tw)| + C|w||\nabla w| \leq C|w|^2|\nabla z| + C|w||\nabla w|. \tag{5.6.4}$$

We have $w = u - z = \int_0^1 \frac{d}{ds}u_s \,\mathrm{d}s = P_{u_t}w + \int_0^1 (P_{u_s} - P_{u_t})w + \mathrm{err}_s \,\mathrm{d}s$ and since

$$(P_{u_t} - P_{u_s})w = \langle w, \nu_{u_t}^j - \nu_{u_s}^j\rangle \nu_{u_t}^j + \langle w, \nu_{u_s}^j\rangle(\nu_{u_t}^j - \nu_{u_s}^j)$$

we arrive at

$$|w - P_{u_t}w| \leq C|w|^2, \quad |\nabla(w - P_{u_t}w)| \leq C|w||\nabla w| + C|w|^2|\nabla z|. \tag{5.6.5}$$

Furthermore, we have

$$|\tfrac{d}{dt}u_t| = |d_{z+tw}\Pi(w)| \leq C|w|, \tag{5.6.6}$$

$$|\nabla u_t| = |d_{z+tw}\Pi(\nabla z + t\nabla w)| \leq C(|\nabla z| + |\nabla w|), \tag{5.6.7}$$

$$|\nabla \tfrac{d}{dt}u_t| \leq C|\nabla w| + C|\nabla(z + tw)||w| \leq C(|\nabla w| + |w||\nabla z|). \tag{5.6.8}$$

We also have

$$|\tfrac{d^2 u_t}{dt^2}| = |d^2_{z+tw}\Pi[w, w]| \leq C|w|^2, \tag{5.6.9}$$

$$|\nabla \tfrac{d^2 u_t}{dt^2}| \leq C|\nabla w||w| + C|\nabla(z + tw)||w|^2 \leq C|\nabla w||w| + |\nabla z||w|^2. \tag{5.6.10}$$

Now given any function $v \in H^1(\mathcal{C}_{Y(\ell)}, \mathbb{R}^n)$ we have

$$|\tfrac{d}{dt}P_{u_t}v| \leq C|v||w|, \tag{5.6.11}$$

$$|\nabla P_{u_t}v| \leq |\nabla v| + |v|(|\nabla z| + |\nabla w|), \tag{5.6.12}$$

$$|\nabla \tfrac{d}{dt}P_{u_t}v| \leq |\nabla v||w| + C|v|(|w||\nabla z| + |\nabla w|). \tag{5.6.13}$$

We will also use that if additionally $v \in \Gamma^{H^1}(z^*TN)$ then we can write

$$P_{u_t}\tfrac{d}{dt}P_{u_t}v = -\langle v, \nu^j_{u_t}\rangle P_{u_t}(\tfrac{d}{dt}\nu^j_{u_t}) = -\langle v, \nu^j_{u_t} - \nu^j_z\rangle P_{u_t}(\tfrac{d}{dt}\nu^j_{u_t})$$

and thus obtain the estimates

$$|P_{u_t}\tfrac{d}{dt}P_{u_t}v| \leq C|v||w|^2, \tag{5.6.14}$$

$$|\nabla P_{u_t}\tfrac{d}{dt}P_{u_t}v| \leq C|\nabla v||w|^2 + C|v||w|(|w||\nabla z| + |\nabla w|). \tag{5.6.15}$$

5.6.2 Hyperbolic cylinders

Here we will give some facts about the hyperbolic cylinders,

$$(\mathcal{C}_{Y(\ell)}, g_\ell) = ([-Y(\ell), Y(\ell)] \times S^1, \rho_\ell^2(s)(ds^2 + d\theta^2)),$$

defined for $\ell > 0$. We are defining

$$\rho_\ell(s) = \frac{\ell}{2\pi\cos(\frac{\ell s}{2\pi})} \text{ and } Y(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan(d^{-1}\ell)\right)$$

for $d > 0$ fixed. In what follows we will always assume that $\ell > 0$ is bounded above, say $\ell \in (0,1)$.

Given $s \in (-Y(\ell), Y(\ell))$ we have

$$\frac{\ell}{2\pi} = \rho(0) \leq \rho(s) \leq \rho(Y(\ell)) \leq C. \tag{5.6.16}$$

Away from the ends $|s| = Y(\ell)$ we have an improved upper bound and near the ends we have an improved lower bound. Indeed, given $\Lambda \in (0, Y(\ell))$ we have

$$\rho(Y(\ell) - \Lambda) \leq C\Lambda^{-1} \tag{5.6.17}$$

and

$$\rho^{-1}(Y(\ell) - \Lambda) \leq \Lambda + C. \tag{5.6.18}$$

The remainder of this section of the Appendix will be dedicated to providing proofs of facts about these hyperbolic cylinders used in the main text.

Proof of Lemma 5.2.2. Pick $w \in H^1_0(C_0, G_\ell)$ and set $v = w \circ f_\ell^{-1} \in H^1_0(\mathcal{C}_{Y(\ell)}, g_\ell)$. We start from the observation that $\rho_\ell^2(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\ell}{2\pi}\tan(\frac{\ell s}{2\pi})\right]$ from which it follows

$$\begin{aligned}\|v\|^2_{L^2(\mathcal{C}_{Y(\ell)}, g_\ell)} &= \int_{\mathcal{C}_{Y(\ell)}} |v|^2\rho_\ell^2(s)\,\mathrm{d}s\mathrm{d}\theta \leq 2\int_{\mathcal{C}_{Y(\ell)}} |\partial_s v||v|\tfrac{\ell}{2\pi}\tan(\tfrac{\ell s}{2\pi})\,\mathrm{d}s\mathrm{d}\theta \\ &\leq 2\int_{\mathcal{C}_{Y(\ell)}} |\partial_s v||v|\rho_\ell(s)\,\mathrm{d}s\mathrm{d}\theta \leq 2\|v\|_{\dot{H}^1(\mathcal{C}_{Y(\ell)}, g_\ell)}\|v\|_{L^2(\mathcal{C}_{Y(\ell)}, g_\ell)}.\end{aligned}$$

The desired result (5.2.5) then follows since $\|w\|_{\dot{H}^1(C_0, G_\ell)} = \|v\|_{\dot{H}^1(\mathcal{C}_{Y(\ell)}, g_\ell)}$ and $\|w\|_{L^2(C_0, G_\ell)} = \|v\|_{L^2(\mathcal{C}_{Y(\ell)}, g_\ell)}$. □

Proof of Lemma 5.2.3. Define $x_\ell = s_\ell^{-1}\colon [-Y(\ell), Y(\ell)] \to [-1,1]$ and set $h = (\partial_\ell s_\ell) \circ x_\ell\colon [-Y(\ell), Y(\ell)] \to \mathbb{R}$. Notice that (5.2.9) implies $\partial_x \partial_\ell s_\ell = \partial_\ell(\partial_x s_\ell) = -\frac{\ell}{2\pi^2\rho_\ell^2}\partial_x s_\ell$ and hence

$$h' = \frac{\partial}{\partial x}(\partial_\ell s_\ell)(x_\ell)\frac{\partial x_\ell}{\partial s} = -\frac{\ell}{2\pi^2\rho_\ell^2}.$$

This, combined with (5.6.18), implies that for $s_\ell \geq 0$ we have

$$\begin{aligned}
|h(s_\ell) - h(Y(\ell))| &\leq \sup_{t\in[0,1]} \left|h'(Y(\ell) + t[s_\ell - Y(\ell)])\right| |Y(\ell) - s_\ell| \\
&\leq \tfrac{\ell}{2\pi^2} \sup_{t\in[0,1]} \rho_\ell^{-2}(Y(\ell) + t[s_\ell - Y(\ell)])|Y(\ell) - s_\ell| \\
&\leq C|Y(\ell) - s_\ell|^2,
\end{aligned}$$

where for the final inequality we use $\rho_\ell^{-1} \leq 2\pi\ell^{-1}$ when $|Y(\ell) - s_\ell| \leq \ell$ and we use (5.6.18) and $|Y(\ell) - s_\ell| \leq Y(\ell) \leq C\ell^{-1}$ when $|Y(\ell) - s_\ell| \geq \ell$. This concludes the proof of (5.2.11) since $h(s_\ell) - h(Y(\ell)) = \partial_\ell s_\ell - \partial_\ell Y(\ell)$ in the case that $s_\ell \geq 0$, and we simply remark that the case of $s_\ell < 0$ can be obtained in the same manner.

To obtain (5.2.12) we compute, again considering only $s_\ell \geq 0$,

$$\begin{aligned}
\left|\partial_\ell\left(\frac{s_\ell}{Y(\ell)}\right)\right| &= \left|\frac{Y(\ell)\partial_\ell s_\ell - s_\ell \partial_\ell Y(\ell)}{Y(\ell)^2}\right| \\
&\leq Y(\ell)^{-1}|\partial_\ell s_\ell - \partial_\ell Y(\ell)| + \frac{\partial_\ell Y(\ell)}{Y(\ell)^2}|Y(\ell) - s_\ell| \\
&\leq C|s_\ell - Y(\ell)|
\end{aligned}$$

where we have used (5.2.11) and $|s_\ell - Y(\ell)| \leq Y(\ell)$ on the first term and $\frac{\partial_\ell Y(\ell)}{Y(\ell)^2} = O(1)$ for the second. □

Proof of Lemma 5.2.7. It will be important to carry out this computation on the domain C_0 rather than on $\mathcal{C}_{Y(\ell)}$ since the boundary of the corresponding domain $T^{\pm}_{Y(\ell)}$ and the conformal factor can both change significantly, in contrast the volume form $\mathrm{d}\mu_{G_\ell}$ on C_0 is fixed.

We will only show the estimate for T_ℓ^+, the other estimate is similar. Set $a_\ell = s_\ell^{-1}(Y(\ell) - 2\alpha \log Y(\ell))$ and $b_\ell = s_\ell^{-1}(Y(\ell) - \alpha \log Y(\ell))$ so that $T_\ell^{\pm} = [a_\ell, b_\ell] \times S^1 \subset C_0$. We can write

$$\begin{aligned}
\frac{\partial}{\partial \ell}\int_{T_\ell^+} \omega \,\mathrm{d}\mu_{G_\ell} &= \frac{\partial}{\partial \ell}\int_{a_\ell}^{b_\ell}\int_{S^1} \omega\ \rho_\ell^2(s_\ell)\tfrac{\partial s_\ell}{\partial x}\mathrm{d}x\mathrm{d}\theta \\
&= \partial_\ell b_\ell \frac{\partial s_\ell}{\partial x}(b_\ell)\int_{S^1} \omega(b_\ell, \theta)\,\rho^2(b_\ell)\mathrm{d}\theta - \partial_\ell a_\ell \frac{\partial s_\ell}{\partial x}(a_\ell)\int_{S^1} \omega(a_\ell, \theta)\,\rho^2(a_\ell)\mathrm{d}\theta
\end{aligned}$$

where we use that $\mathrm{d}\mu_{G_\ell} = \rho_\ell^2(s_\ell)\frac{\partial s_\ell}{\partial x}\mathrm{d}x\mathrm{d}\theta$ is independent of ℓ. Since $\rho_\ell(s_\ell) \approx |\log \ell|^{-1}$ on T_ℓ^+ we can bound

$$\left|\frac{\partial}{\partial \ell}\int_{T_\ell^+} \omega \,\mathrm{d}\mu_{G_\ell}\right| \leq C|\log \ell|^{-1}\left(|\partial_\ell b_\ell \tfrac{\partial s_\ell}{\partial x}(b_\ell)| + |\partial_\ell a_\ell \tfrac{\partial s_\ell}{\partial x}(a_\ell)|\right)\|w\|_{L^1(\partial T_\ell^+, G_\ell)}. \tag{5.6.19}$$

Hence it suffices to bound the expressions $\partial_\ell b_\ell \frac{\partial s_\ell}{\partial x}(b_\ell)$ and $\partial_\ell a_\ell \frac{ds_\ell}{dx}(a_\ell)$ from above. We have

$$\begin{aligned}\partial_\ell b_\ell \frac{\partial s_\ell}{\partial x}(b_\ell) &= \partial_\ell[s_\ell(b_\ell)] - (\partial_\ell s_\ell)(b_\ell)\\ &= \partial_\ell[Y(\ell) - \alpha \log Y(\ell)] - (\partial_\ell s_\ell)(b_\ell)\\ &= -\alpha\partial_\ell[\log Y(\ell)] + \partial_\ell[Y(\ell) - s_\ell](b_\ell).\end{aligned}$$

We can compute $|\partial_\ell \log Y(\ell)| = |\frac{\partial_\ell Y(\ell)}{Y(\ell)}| \leq C\ell^{-1}$ and use (5.2.11) to give $|\partial_\ell(Y(\ell) - s_\ell)(b_\ell)| \leq C|Y(\ell) - b_\ell|^2 = C(\log Y(\ell))^2 \leq C|\log \ell|^2$ which gives

$$\left|\partial_\ell b_\ell \tfrac{\partial s_\ell}{\partial x}(b_\ell)\right| \leq C(\ell^{-1} + |\log \ell|^2) \leq C\ell^{-1}.$$

Note that the estimate for $\partial_\ell a_\ell \frac{\partial s_\ell}{\partial x}(b_\ell)$ follows in the same way. Hence combining the last estimate with (5.6.19) finishes the proof of Lemma 5.2.7. $\square$

5.6.3 Estimates for z_ℓ on the transition region

Here we will establish some estimates for the maps $z_{Y(\ell)}\colon \mathcal{C}_{Y(\ell)} \to N$ and $z_\ell = z_{Y(\ell)} \circ f_\ell$, defined in (5.2.14), on the transition region $T_\ell^\pm = f_\ell^{-1} T_{Y(\ell)}^\pm$ defined in (5.2.13) to complete the estimates in Section 5.2.3 and Section 5.3. We recall that we are writing ∇ for the Euclidean gradient on $\mathcal{C}_{Y(\ell)}$. We also recall $f_\ell(x,\theta) = (s_\ell(x),\theta)$ is defined in (5.2.3).

For what follows we introduce the notation

$$\begin{aligned}\gamma_{Y(\ell)}(s) &= \gamma(\tfrac{s}{Y(\ell)}),\\ \psi_{Y(\ell)}(s) &= \psi(\tfrac{Y(\ell)-|s|}{\alpha \log Y(\ell)}),\\ \bar{v}_{\pm,Y(\ell)}(s,\theta) &= v_\pm(e^{-(Y(\ell)\mp s)}e^{i\theta}),\end{aligned} \tag{5.6.20}$$

and continue to use the notational convention described in Remark 5.2.4 for these objects, so that $\gamma_\ell = \gamma_{Y(\ell)} \circ f_\ell$, $\psi_\ell = \psi_{Y(\ell)} \circ f_\ell$ and $\bar{v}_{\pm,\ell} = \bar{v}_{\pm,Y(\ell)} \circ f_\ell$. We have in particular that for $\pm s > 0$

$$\hat{z}_{Y(\ell)} = (1 - \psi_{Y(\ell)})\bar{v}_{\pm,Y(\ell)} + \psi_{Y(\ell)}\gamma_{Y(\ell)}, \quad z_{Y(\ell)} = \Pi(\hat{z}_{Y(\ell)}). \tag{5.6.21}$$

We also recall that on $T_{Y(\ell)}^\pm$ the estimate (5.2.15) implies

$$|\gamma_{Y(\ell)} - \bar{v}_{\pm,Y(\ell)}| \leq C(Y(\ell)^{-1}\log Y(\ell) + Y(\ell)^{-\alpha}) \leq C(\ell|\log \ell| + \ell^\alpha) \leq C\ell|\log \ell| \tag{5.6.22}$$

since $\alpha \geq 1$ implies that the second term above is $O(\ell)$. Here and in the following we use that, since $Y(\ell) = \frac{2\pi}{\ell}(\frac{\pi}{2} - \arctan \frac{\ell}{d})$, we have $Y(\ell) \approx \ell^{-1}$, and it also follows that

$$|\partial_\ell Y(\ell)| \leq C\ell^{-2}. \tag{5.6.23}$$

Furthermore, on $T_\ell^\pm$ we have $|Y(\ell) \mp s_\ell| \leq 2\alpha \log Y(\ell) \approx |\log \ell|$ and hence that

$$|\partial_\ell(Y(\ell) \pm s_\ell)| \leq C|Y(\ell) \mp s_\ell|^2 \leq C|\log \ell|^2, \tag{5.6.24}$$

$$|\partial_\ell(\tfrac{s_\ell}{Y(\ell)})| \leq C|Y(\ell) \mp s_\ell| \leq C|\log \ell| \tag{5.6.25}$$

using Lemma 5.2.3.

Proof of the estimate (5.2.24)

We note that by composing with f_ℓ it suffices to bound $|\nabla z_{Y(\ell)}|$ on $T^{\pm}_{Y(\ell)}$. By differentiating the first equation in (5.6.21) we have that on $T^{\pm}_{Y(\ell)}$

$$\nabla \hat{z}_{Y(\ell)} = \nabla\psi_{Y(\ell)}(\gamma_{Y(\ell)} - \bar{v}_{\pm,Y(\ell)}) + \psi_{Y(\ell)}\nabla\gamma_{Y(\ell)} + (1-\psi_{Y(\ell)})\nabla\bar{v}_{\pm,Y(\ell)}, \tag{5.6.26}$$

recalling the notation from (5.6.20). We can then compute

$$\begin{aligned} |\nabla\gamma_{Y(\ell)}| &= Y(\ell)^{-1}|\gamma'(\tfrac{s_\ell}{Y(\ell)})| = O(\ell) \\ |\nabla\bar{v}_{\pm,Y(\ell)}|(s,\theta) &= |e^{-(Y(\ell)\mp s)}(\nabla v_\pm)(e^{-(Y(\ell)\mp s)}e^{i\theta})| = O(\ell^\alpha) \\ |\nabla\psi_{Y(\ell)}|(s) &= |(\alpha\log Y(\ell))^{-1}\psi'(\tfrac{Y(\ell)-|s|}{\alpha\log Y(\ell)})| = O(|\log\ell|^{-1}) \end{aligned} \tag{5.6.27}$$

which, keeping in mind $\alpha \geq 1$, can be combined with (5.6.22) to give

$$|\nabla z_{Y(\ell)}| = |d_{\hat{z}_{Y(\ell)}}\Pi(\nabla\hat{z}_{Y(\ell)})| \leq C|\nabla\hat{z}_{Y(\ell)}| \leq C\ell \text{ on } T^{\pm}_{Y(\ell)}. \tag{5.6.28}$$

Proof of the estimate (5.2.25)

We pull back the equation (5.6.21) by f_ℓ and differentiate with respect to ℓ, keeping in mind the notation defined in (5.6.20) and immediately after, to give on $T^{\pm}_\ell$

$$\partial_\ell\hat{z}_\ell = \partial_\ell\psi_\ell(\gamma_\ell - \bar{v}_{\pm,\ell}) + \psi_\ell\partial_\ell\gamma_\ell + (1-\psi_\ell)\partial_\ell\bar{v}_{\pm,\ell}. \tag{5.6.29}$$

We then use (5.6.24) and (5.6.25) to estimate on $T^{\pm}_\ell$

$$\begin{aligned} &|\partial_\ell\gamma_\ell| = |\gamma'||\partial_\ell\tfrac{s_\ell}{Y(\ell)}| \leq C|Y(\ell)\mp s_\ell| \leq C|\log\ell| \\ &|\partial_\ell\bar{v}_{\pm,\ell}| \leq C|\partial_\ell[e^{-(Y(\ell)\mp s_\ell)}]| \leq C|Y(\ell)\mp s_\ell|^2 e^{-(Y(\ell)\mp s_\ell)} \leq C\ell^\alpha(\log\ell)^2 \\ &|\partial_\ell\psi_\ell| \leq C|\partial_\ell[\tfrac{Y(\ell)-|s_\ell|}{\alpha\log Y(\ell)}]| \leq C\frac{|Y(\ell)-|s_\ell||^2}{\log Y(\ell)} + \frac{|Y(\ell)-s_\ell|\partial_\ell\log Y(\ell)}{(\log Y(\ell))^2} \leq C\ell^{-1}|\log\ell|^{-1} \end{aligned} \tag{5.6.30}$$

where we have used also $\partial_\ell\log Y(\ell) = Y(\ell)^{-1}\partial_\ell Y(\ell) = O(\ell^{-1})$ by (5.6.23). Plugging (5.6.30) and (5.6.22) into (5.6.29) then gives

$$|\partial_\ell z_\ell| = |d_{\hat{z}_\ell}\Pi(\partial_\ell\hat{z}_\ell)| \leq C|\partial_\ell\hat{z}_\ell| \leq C|\log\ell|. \tag{5.6.31}$$

Proof of the estimate (5.2.33)

We differentiate (5.6.26) with respect to ℓ, keeping in mind the notation defined in (5.6.20) and immediately after, to give

$$\begin{aligned} \partial_\ell[(\nabla\hat{z}_{Y(\ell)})\circ f_\ell] = {}& \partial_\ell[(\nabla\psi_{Y(\ell)})\circ f_\ell](\gamma_\ell - \bar{v}_{\pm,\ell}) + (\nabla\psi_{Y(\ell)})\circ f_\ell\partial_\ell[\gamma_\ell - \bar{v}_{\pm,\ell}] \\ &+ \partial_\ell[\psi_\ell](\nabla\gamma_{Y(\ell)} - \nabla\bar{v}_{\pm,Y(\ell)})\circ f_\ell \\ &+ \psi_\ell\partial_\ell[(\nabla\gamma_{Y(\ell)})\circ f_\ell] + (1-\psi_\ell)\partial_\ell[(\nabla\bar{v}_{\pm,Y(\ell)})\circ f_\ell]. \end{aligned} \tag{5.6.32}$$

We clarify that here and in the following we slightly abuse notation using $\nabla \hat{z}_{Y(\ell)}$ to refer to the components of the gradient with respect to the fixed basis $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\}$. We can compute using (5.6.24), (5.6.25) and (5.6.23), see also (5.6.30),

$$\begin{aligned} |\partial_\ell[(\nabla \gamma_{Y(\ell)}) \circ f_\ell]| &= \left|\partial_\ell \left[Y(\ell)^{-1}\gamma'(\tfrac{s_\ell}{Y(\ell)})\right]\right| \leq C \\ |\partial_\ell[(\nabla \bar{v}_{\pm,Y(\ell)}) \circ f_\ell]| &\leq C|\partial_\ell[e^{-(Y(\ell)\mp s_\ell)}]| \leq C\ell^\alpha (\log \ell)^2 \\ |\partial_\ell[(\nabla \psi_{Y(\ell)}) \circ f_\ell]| &= \left|\partial_\ell \left[\tfrac{1}{\alpha \log Y(\ell)}\psi'(\tfrac{Y(\ell)-|s_\ell|}{\alpha \log Y(\ell)})\right]\right| \leq C\ell^{-1}|\log \ell|^{-2}. \end{aligned} \tag{5.6.33}$$

Using the three estimates from (5.6.33), together with (5.6.27) and (5.6.29), in (5.6.32) gives $|\partial_\ell[(\nabla \hat{z}_{Y(\ell)}) \circ f_\ell]| \leq C$ and hence

$$\begin{aligned} |\partial_\ell[(\nabla z_{Y(\ell)}) \circ f_\ell]| &= |\partial_\ell \left[d_{\hat{z}_\ell}\Pi((\nabla \hat{z}_{Y(\ell)}) \circ f_\ell)\right]| \\ &\leq C|\partial_\ell[(\nabla \hat{z}_{Y(\ell)}) \circ f_\ell]| + C|\partial_\ell \hat{z}_\ell||(\nabla \hat{z}_{Y(\ell)}) \circ f_\ell| \leq C, \end{aligned} \tag{5.6.34}$$

using also (5.6.31) and (5.6.28).

Proof of the estimate (5.3.39)

It suffices to estimate $\tau_{g_E}(z_{Y(\ell)})$ on $T^\pm_{Y(\ell)}$. We have $\tau_{g_E}(z_{Y(\ell)}) = P_{z_{Y(\ell)}}(\Delta z_{Y(\ell)}) = \Delta z_{Y(\ell)} + A(z_{Y(\ell)})(\nabla z_{Y(\ell)}, \nabla z_{Y(\ell)})$, where Δ denotes the Laplacian with respect to the Euclidean metric. We start by looking at the $\Delta \hat{z}_{Y(\ell)}$ using (5.6.21), keeping in mind the notation introduced in (5.6.20) and immediately after,

$$\begin{aligned} \Delta \hat{z}_{Y(\ell)} = &\Delta \psi_{Y(\ell)}(\gamma_{Y(\ell)} - \bar{v}_{\pm,Y(\ell)}) + 2\nabla \psi_{Y(\ell)} \cdot \nabla(\gamma_{Y(\ell)} - \bar{v}_{\pm,Y(\ell)}) \\ &+ \psi_{Y(\ell)}\Delta \gamma_{Y(\ell)} + (1-\psi_{Y(\ell)})\Delta \bar{v}_{\pm,Y(\ell)}, \end{aligned} \tag{5.6.35}$$

and note we have, as in (5.6.27), that on $T^\pm_\ell$

$$\begin{aligned} |\Delta \gamma_{Y(\ell)}|(s) &= |Y(\ell)^{-2}\gamma''(\tfrac{s}{Y(\ell)})| \leq C\ell^2 \\ |\Delta \bar{v}_{\pm,Y(\ell)}(s,\theta)| &= |e^{-2(Y(\ell)\mp s)}(\Delta v_\pm)(e^{-(Y(\ell)\mp s)}e^{i\theta})| \leq C\ell^{2\alpha} \\ |\Delta \psi_{Y(\ell)}|(s) &= |(\alpha \log Y(\ell))^{-2}\psi''(\tfrac{Y(\ell)-|s|}{\alpha \log Y(\ell)})| \leq C|\log \ell|^{-2}. \end{aligned} \tag{5.6.36}$$

Plugging (5.6.36), (5.6.27) and (5.6.22) into (5.6.35) gives $|\Delta \hat{z}_{Y(\ell)}| \leq C\ell|\log \ell|^{-1}$ and hence

$$\begin{aligned} |\tau_{g_E}(z_{Y(\ell)})| &= |P_{z_{Y(\ell)}}\Delta z_{Y(\ell)}| \leq |\Delta z_{Y(\ell)}| = |\mathrm{div}\left[d_{\hat{z}_{Y(\ell)}}\Pi(\nabla \hat{z}_{Y(\ell)})\right]| \\ &\leq C|\Delta \hat{z}_{Y(\ell)}| + C|\nabla \hat{z}_{Y(\ell)}|^2 \leq C\ell|\log \ell|^{-1} + C\ell^2 \leq C\ell|\log \ell|^{-1}, \end{aligned} \tag{5.6.37}$$

where we have also used (5.6.28).

Proof of the estimate (5.3.40)

As a first step we pull back the formula (5.6.35) by f_ℓ and differentiate with respect to ℓ, keeping in mind the notation introduced in (5.6.20) and immediately after, to obtain

$$\begin{aligned}\partial_\ell[(\Delta\hat{z}_{Y(\ell)})\circ f_\ell] &= \partial_\ell[(\Delta\psi_{Y(\ell)})\circ f_\ell](\gamma_\ell-\bar{v}_{\pm,\ell})+(\Delta\psi_{Y(\ell)})\circ f_\ell\partial_\ell[\gamma_\ell-\bar{v}_{\pm,\ell}]\\ &\quad+2\partial_\ell[(\nabla\psi_{Y(\ell)})\circ f_\ell]\cdot(\nabla\gamma_{Y(\ell)}-\nabla\bar{v}_{\pm,Y(\ell)})\circ f_\ell\\ &\quad+2(\nabla\psi_{Y(\ell)})\circ f_\ell\cdot\partial_\ell[(\nabla\gamma_{Y(\ell)}-\nabla\bar{v}_{\pm,Y(\ell)})\circ f_\ell]\\ &\quad+\partial_\ell[\psi_\ell](\Delta\gamma_\ell-\Delta\bar{v}_{\pm,Y(\ell)})\circ f_\ell+\psi_\ell\partial_\ell[(\Delta\gamma_{Y(\ell)})\circ f_\ell]+(1-\psi_\ell)\partial_\ell[(\Delta\bar{v}_{\pm,Y(\ell)})\circ f_\ell],\end{aligned} \tag{5.6.38}$$

where Δ is the Laplacian with respect to the Euclidean metric on $\mathcal{C}_{Y(\ell)}$. Note that using (5.6.36) and also (5.6.25), (5.6.24) and (5.6.23), see also (5.6.33) and (5.6.30), we have that on $T_\ell^\pm$

$$\begin{aligned}|\partial_\ell[(\Delta\gamma_{Y(\ell)})\circ f_\ell]| &= \left|\partial_\ell\left[Y(\ell)^{-2}\gamma''(\tfrac{s_\ell}{Y(\ell)})\right]\right|\leq C\ell\\ |\partial_\ell[(\Delta\bar{v}_{\pm,Y(\ell)})\circ f_\ell]| &\leq C|\partial_\ell[e^{-2(Y(\ell)\mp s_\ell)}]|\leq \ell^{2\alpha}|\log\ell|^2\\ |\partial_\ell[(\Delta\psi_{Y(\ell)})\circ f_\ell]| &= \left|\partial_\ell\left[(\alpha\log Y(\ell))^{-2}\psi''(\tfrac{Y(\ell)-|s_\ell|}{\alpha\log Y(\ell)})\right]\right|\leq C\ell^{-1}|\log\ell|^{-3}.\end{aligned} \tag{5.6.39}$$

Plugging (5.6.39), (5.6.36), (5.6.27), (5.6.33) and (5.6.22) into (5.6.38) gives $|\partial_\ell[\Delta\hat{z}_{Y(\ell)}]|\leq C|\log\ell|^{-1}$ and so

$$\begin{aligned}|\partial_\ell[(\Delta z_{Y(\ell)})\circ f_\ell]| &\leq C|\partial_\ell[(\Delta\hat{z}_{Y(\ell)})\circ f_\ell]|+C|\partial_\ell\hat{z}_\ell||(\nabla\hat{z}_{Y(\ell)})\circ f_\ell|^2\\ &\qquad+C|\partial_\ell[(\nabla z_{Y(\ell)})\circ f_\ell]||(\nabla\hat{z}_{Y(\ell)})\circ f_\ell|\\ &\leq C|\log\ell|^{-1},\end{aligned} \tag{5.6.40}$$

using additionally (5.6.28), (5.6.31) and (5.6.34). This allows us to conclude – using $\tau_{g_E}(z_{Y(\ell)})=P_{z_{Y(\ell)}}[\Delta z_{Y(\ell)}]$, (5.6.2) and (5.6.37) – that

$$|\partial_\ell[(\tau_{g_E}(z_{Y(\ell)}))\circ f_\ell]|\leq|\partial_\ell[(\Delta z_{Y(\ell)})\circ f_\ell]|+C|\partial_\ell z_\ell||\Delta z_{Y(\ell)}|\circ f_\ell\leq C|\log\ell|^{-1}. \tag{5.6.41}$$

Bibliography

[1] K.-C. Chang, W. Y. Ding, and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom., 36 (1992), pp. 507–515.

[2] J. Chen and G. Tian, *Compactification of moduli space of harmonic mappings*, Comment. Math. Helv., 74 (1999), pp. 201–237.

[3] Y. M. Chen and M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z., 201 (1989), pp. 83–103.

[4] R. Chill, *On the Łojasiewicz-Simon gradient inequality*, J. Funct. Anal., 201 (2003), pp. 572–601.

[5] R. Chill, E. Fašangová, and R. Schätzle, *Willmore blowups are never compact*, Duke Math. J., 147 (2009), pp. 345–376.

[6] O. Chodosh and F. Schulze, *Uniqueness of asymptotically conical tangent flows*, Duke Math. J., (2021).

[7] T. Colding and W. Minicozzi II, *Uniqueness of blowups and Łojasiewicz inequalities*, Ann. of Math., 182 (2015), pp. 221–285.

[8] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

[9] W. Ding, J. Li, and Q. Liu, *Evolution of minimal torus in Riemannian manifolds*, Invent. Math., 165 (2006), pp. 225–242.

[10] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), pp. 109–160.

[11] J. Eells and J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology, 15 (1976), pp. 263–266.

[12] P. M. Feehan and M. Maridakis, *Łojasiewicz-Simon gradient inequalities for analytic and Morse-Bott functions on Banach spaces*, J. Reine Angew. Math., 2020 (2020), pp. 35–67.

[13] R. D. Gulliver, R. Osserman, and H. L. Royden, *A theory of branched immersions of surfaces*, Amer. J. Math., 95 (1973), pp. 750–812.

[14] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math, 19 (1967), pp. 673–687.

[15] R. Haslhofer and R. Müller, *Dynamical stability and instability of Ricci-flat metrics*, Math. Ann., 360 (2014), pp. 547–553.

[16] F. Helein, *Harmonic maps, conservation laws and moving frames*, Cambridge University Press, Cambridge, 2002.

[17] S.-Z. Huang, *Gradient inequalities with applications to asymptotic behavior and stability of gradient-like systems*, Mathematical surveys and monographs, American Mathematical Society, Providence, 2006.

[18] T. Huxol, M. Rupflin, and P. M. Topping, *Refined asymptotics of the Teichmüller harmonic map flow into general targets*, Calc. Var. Partial Differential Equations, 55 (2016).

[19] H. Inci, T. Kappeler, and P. Topalov, *On the regularity of the composition of diffeomorphisms*, Mem. Amer. Math. Soc, 226 (2013), pp. 1–72.

[20] J. Jost, *Riemannian geometry and geometric analysis*, Universitext, Springer, 1995.

[21] J. Kohout, M. Rupflin, and P. M. Topping, *Uniqueness and nonuniqueness of limits of Teichmüller harmonic map flow*, Adv. Calc. Var., (2020).

[22] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES notes, 1965.

[23] A. Malchiodi, M. Rupflin, and B. Sharp, *Łojasiewicz inequalities near simple bubble trees*, 2020.

[24] C. Mantegazza and M. Pozzetta, *The Lojasiewicz-Simon inequality for the elastic flow*, Calc. Var. Partial Differential Equations, 60 (2021).

[25] J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems*, Springer US, New York, NY, 1982.

[26] T. H. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom., 44 (1996), pp. 595–633.

[27] J. Qing and G. Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math., 50 (1997), pp. 295–310.

[28] B. Randol, *Cylinders in Riemann surfaces*, Comment. Math. Helv., 54 (1979), pp. 1–5.

[29] M. Rupflin, *Convergence of almost harmonic maps to geodesic bubble branches*. In preparation.

[30] ——, *Flowing maps to minimal surfaces: Existence and uniqueness of solutions*, Ann. Inst. H. Poincare Anal. Non Lineaire, 31 (2014), pp. 349–368.

[31] ——, *Teichmüller harmonic map flow from cylinders*, Math. Ann., 368 (2017), pp. 1227–1276.

[32] ——, *Lojasiewicz inequalities for almost harmonic maps near simple bubble trees*, 2021.

[33] M. Rupflin and P. M. Topping, *A uniform Poincaré estimate for quadratic differentials on closed surfaces*, Calc. Var. Partial Differential Equations, 53 (2015), pp. 587–604.

[34] ——, *Flowing maps to minimal surfaces*, Amer. J. Math., 138 (2016), pp. 1095–1115.

[35] ——, *Horizontal curves of hyperbolic metrics*, Calc. Var. Partial Differential Equations, 57 (2018).

[36] ——, *Teichmüller harmonic map flow into nonpositively curved targets*, J. Differential Geom., 108 (2018), pp. 135–184.

[37] ——, *Global weak solutions of the Teichmüller harmonic map flow into general targets*, Anal. PDE, 12 (2019), pp. 815–842.

[38] M. Rupflin, P. M. Topping, and M. Zhu, *Asymptotics of the Teichmüller harmonic map flow*, Adv. Math., 244 (2013), pp. 874–893.

[39] J. Sacks and K. Uhlenbeck, *The Existence of Minimal Immersions of 2-Spheres*, Ann. of Math., 113 (1981), pp. 1–24.

[40] ——, *Minimal immersions of closed Riemann surfaces*, Trans. Amer. Math. Soc, 271 (1982), pp. 639–639.

[41] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math., 110 (1979), pp. 127–142.

[42] F. Schulze, *Uniqueness of compact tangent flows in Mean Curvature Flow*, J. Reine Angew. Math, (2014), pp. 163 – 172.

[43] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math., 118 (1983), pp. 525–571.

[44] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv., 60 (1985), pp. 558–581.

[45] P. M. Topping, *The harmonic map heat flow from surfaces*, PhD thesis, University of Warwick, 1996.

[46] ——, *Rigidity in the harmonic map heat flow*, J. Differential Geom., 45 (1997), pp. 593–610.

[47] ——, *Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow*, Ann. of Math., 159 (2004), pp. 465–534.

[48] ——, *Winding behaviour of finite-time singularities of the harmonic map heat flow*, Math. Z., 247 (2004), pp. 279–302.

[49] A. J. Tromba, *Teichmüller theory in Riemannian geometry*, Birkhauser Verlag, 1992.