Supplemental Material: A proposal to use accelerated electrons to probe the axion-electron coupling

Georgios Vacalis[1,*] Atsushi Higuchi[2], Robert Bingham[3,4], and Gianluca Gregori[1]
[1] Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, UK
[2] Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
[3] Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 OQX, UK, and
[4] Department of Physics, University of Strathclyde, Glasgow G4 0NG, UK
(Dated: October 16, 2025)

In this supplemental material, we present the derivations of some results shown in the main text. We also show that the interaction amplitude in the weak-field limit can be found using the standard perturbation theory using Feynman diagrams.

I. WKB SOLUTIONS WITH GENERAL ELECTROMAGNETIC FIELD

The Dirac equation in an external electromagnetic potential A_μ is

$$i\gamma^\mu(\hbar\partial_\mu - ieA_\mu)\psi - m\psi = 0\,. \tag{1}$$

The solutions to this equation in the WKB approximation for a purely time-dependent potential was found in Ref. [1]. We seek an approximate solution in the form

$$\psi = \check{\Psi}\exp\left(-\frac{i}{\hbar}S\right)\,, \tag{2}$$

where S is a scalar function. Substituting eq. (2) into eq. (1), one finds

$$[\gamma^\mu(\partial_\mu S + eA_\mu) - m]\check{\Psi} + i\hbar\gamma^\mu\partial_\mu\check{\Psi} = 0\,. \tag{3}$$

Letting $\check{\Psi} = \Psi + \Psi^{(h)}$, where Ψ and $\Psi^{(h)}$ are of zeroth order and of higher order, respectively, in $\hbar$, we can write eq. (3) *to first order* in $\hbar$ as follows:

$$[\gamma^\mu(\partial_\mu S + eA_\mu) - m]\Psi = 0\,, \tag{4}$$

$$[\gamma^\mu(\partial_\mu S + eA_\mu) - m]\Psi^{(h)} + i\hbar\gamma^\mu\partial_\mu\Psi = 0\,. \tag{5}$$

By multiplying these equations by $\gamma^\mu(\partial_\mu S + eA_\mu) + m$ and using $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, we find

$$(\partial^\mu S + eA^\mu)(\partial_\mu S + eA_\mu) - m^2 = 0\,, \tag{6}$$

$$[\gamma^\mu(\partial_\mu S + eA_\mu) + m]\gamma^\nu\partial_\nu\Psi = 0\,. \tag{7}$$

The WKB solution to the Dirac equation (1) to lowest order in $\hbar$ is

$$\psi^{(\mathrm{WKB})} = \Psi\exp\left(-\frac{i}{\hbar}S\right)\,, \tag{8}$$

where the scalar function S and the spinor Ψ satisfy eqs. (4), (6) and (7).

* georgios.vacalis@chch.ox.ac.uk

Let us find the scalar function S satisfying eq. (6). We first describe the congruence of classical world lines used to construct the function S. These world lines satisfy the Lorentz-force equation,

$$\frac{d^2x^\mu}{d\tau^2} = -\frac{e}{m}F^{\mu\nu}\frac{dx_\nu}{d\tau}\,, \tag{9}$$

where τ is the proper time along these world lines. We assume $A_\mu = 0$ on the hypersurface $t = t_i$ in the past where v^μ is constant. (Thus, the world lines are parallel to each other when they emanate, with $\tau = 0$, from this hypersurface.) We assume that these world lines do not cross each other. If

$$\partial_\mu S + eA_\mu = mv_\mu\,, \tag{10}$$

where $v^\mu = dx^\mu/d\tau$, then eq. (6) is satisfied because $v^\mu v_\mu = 1$. Equation (10) can be solved for S if and only if

$$\partial_\mu(eA_\nu - mv_\nu) - \partial_\nu(eA_\mu - mv_\mu) = 0\,, \tag{11}$$

i.e.,

$$\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{e}{m}F_{\mu\nu} = 0\,, \tag{12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Observe that eq. (9) follows from this equation by contracting it by v^ν:

$$v^\nu\partial_\mu v_\nu - v^\nu\partial_\nu v_\mu - \frac{e}{m}F_{\mu\nu}v^\nu = 0\,. \tag{13}$$

Since $v^\nu\partial_\mu v_\nu = 0$ and $v^\nu\partial_\nu = d/d\tau$, we find eq. (9).

First we note that eq. (12) is satisfied at $t = t_i$, i.e., at $\tau = 0$, because $F_{\mu\nu} = 0$ and v^μ is constant there. Now, the τ-derivative of the left-hand side of eq. (13) is found, by using eq. (9) and the Bianchi identity for $F_{\mu\nu}$, as

$$\begin{aligned}&\frac{d}{d\tau}\left(\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{e}{m}F_{\mu\nu}\right)\\ &= (\partial_\mu v^\lambda)\left(\partial_\nu v_\lambda - \partial_\lambda v_\nu - \frac{e}{m}F_{\nu\lambda}\right)\\ &\quad - (\partial_\nu v^\lambda)\left(\partial_\mu v_\lambda - \partial_\lambda v_\mu - \frac{e}{m}F_{\mu\lambda}\right)\,.\end{aligned} \tag{14}$$

The unique solution to this equation with the initial condition (12) at $\tau = 0$ is eq. (12) for all τ. Thus, we have constructed the scalar function S satisfying (6). Then, if we define $p^\mu = mv^\mu$, eq. (4) becomes

$$(\gamma^\mu p_\mu - m)\Psi = 0\,, \tag{15}$$

and the solution is

$$\Psi \propto \begin{pmatrix} s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m} s \end{pmatrix}\,, \tag{16}$$

where s is a two-component spinor, as is well known.

The differential equation describing the time evolution of Ψ can be found from eq. (7) as follows. This equation can be written, using eq. (10) as

$$(\gamma^\mu v_\mu + 1)\gamma^\nu \partial_\nu \Psi = 0\,. \tag{17}$$

We add this equation and

$$\gamma^\mu \partial_\mu [(\gamma^\nu v_\nu - 1)\Psi] = 0\,, \tag{18}$$

which follows immediately from eq. (4), to find

$$\frac{d\Psi}{d\tau} + \frac{1}{2}\left(\partial_\mu v^\mu + \frac{e}{2m}\gamma^{\mu\nu}F_{\mu\nu}\right)\Psi = 0\,, \tag{19}$$

where we used eq. (12).

Let

$$\Psi = \widetilde{\Psi}\exp\left(-\frac{1}{2}\int_0^\tau \partial_\mu v^\mu(\xi)\,d\xi\right)\,. \tag{20}$$

Then,

$$\frac{d\widetilde{\Psi}}{d\tau} = -\frac{e}{4m}\gamma^{\mu\nu}F_{\mu\nu}\widetilde{\Psi}\,. \tag{21}$$

By recalling that

$$\begin{aligned} F_{0i} &= E_i\,, \\ F_{ij} &= -\frac{1}{2}\epsilon_{ijk}B_k\,, \end{aligned} \tag{22}$$

we find

$$-\frac{e}{4m}\gamma^{\mu\nu}F_{\mu\nu} = -\frac{e}{2m}\begin{pmatrix} i\boldsymbol{\sigma}\cdot\mathbf{B} & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & i\boldsymbol{\sigma}\cdot\mathbf{B}\end{pmatrix}\,. \tag{23}$$

We also note that the Lorentz-force equation eq. (9) can be written in terms of the electric and magnetic fields as

$$\begin{aligned} \frac{dp_0}{d\tau} &= -\frac{e}{m}\mathbf{p}\cdot\mathbf{E}\,, \\ \frac{d\mathbf{p}}{d\tau} &= -\frac{e}{m}(p_0\mathbf{E} + \mathbf{p}\times\mathbf{B})\,. \end{aligned} \tag{24}$$

Let

$$\widetilde{\Psi} = \begin{pmatrix} \sqrt{p_0+m}\,s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{p_0+m}} s \end{pmatrix}\,, \tag{25}$$

which satisfies eq. (16). Then, by using eqs. (23) and (24), we find that eq. (21) is satisfied if

$$\frac{ds}{d\tau} = -\frac{ie}{2m}\left(\mathbf{B}(\tau) - \frac{\mathbf{p}(\tau)\times\mathbf{E}(\tau)}{p_0(\tau)+m}\right)\cdot\boldsymbol{\sigma}\,s\,. \tag{26}$$

This is the Thomas-BMT equation for relativistic spin dynamics [2, 3]. Thus, the WKB solution to first order can be written as

$$\begin{aligned} \psi^{(\mathrm{WKB})} = \sqrt{p_0+m}\begin{pmatrix} s \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}\,s\end{pmatrix}\exp\left(-\frac{1}{2}\int_0^\tau \partial_\mu v^\mu(\xi)\,d\xi\right) \\ \times\exp\left(-\frac{i}{\hbar}S\right)\,, \end{aligned} \tag{27}$$

where S satisfies eq. (10), $p_\mu = mv_\mu$ and s satisfies eq. (26), whose solution is given by

$$s_\alpha(\tau) = T\exp\left[-i\int_0^\tau \mathbf{F}(\xi)\cdot\boldsymbol{\sigma}\,d\xi\right]s_\alpha(0)\,, \tag{28}$$

where T denotes time-ordering and where we defined

$$\mathbf{F} = \frac{e}{2m}\left(\mathbf{B} - \frac{\mathbf{p}\times\mathbf{E}}{p_0+m}\right)\,. \tag{29}$$

It is interesting to note that

$$\overline{\psi}\gamma^\mu\psi = 2mv^\mu\exp\left(-\int_0^\tau \partial_\mu v^\mu(\xi)\,d\xi\right)\,, \tag{30}$$

and that the current conservation equation $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$ is satisfied.

The WKB approximation would break down if the congruence of world lines of the classical electron had a focal point where $\partial_\mu v^\mu \to -\infty$ as can be seen in Eq. (27). We have observed numerically that these world lines tend not to have focal points in the background electromagnetic field we assume in this paper though we cannot exclude them. Even if focal points occur, we expect that the WKB approximation will be valid away from them as we argue below.

The WKB approximation is expected to be accurate if the first-order correction $\Psi^{(h)}$ to Ψ is much smaller than Ψ. It can be found from Eq. (5) as

$$\Psi^{(h)} = \frac{i\hbar}{2m}\gamma^\mu\partial_\mu\Psi\,, \tag{31}$$

with the help of Eq. (7). Hence, the WKB approximation is expected to be valid if

$$\frac{\hbar}{2m}\|\partial_\mu\Psi\| \ll \|\Psi\|\,. \tag{32}$$

The derivative of Ψ can be estimated using Eq. (19). Equation (13) indicates that $\partial_\mu v_\nu$ are of the same order as $(e/2m)F_{\mu\nu}$ if they do not diverge. Since $(e/m)F_{\mu\nu} \sim a_0\omega_0$, we find

$$\left\|\frac{d\Psi}{d\tau}\right\| \sim a_0\omega_0\|\Psi\|\,. \tag{33}$$

We note that, since the coordinate time is $\sim a_0\tau$, the change in $\|\Psi\|$ in one cycle is of the same order as $\|\Psi\|$ itself beacause $a_0\tau\times\omega_0 \sim 1$ in one cycle.

Next, if λ is a coordinate on the τ-constant hypersurface, then from Eq. (19) we find

$$\frac{d}{d\tau}\left(\frac{\partial\Psi}{\partial\lambda}\right) = -\frac{1}{2}\left(\partial_\mu v^\mu + \frac{e}{2m}\gamma^{\mu\nu}F_{\mu\nu}\right)\frac{\partial\Psi}{\partial\lambda} - \frac{\partial}{\partial\lambda}\left(\partial_\mu v^\mu + \frac{e}{2m}\gamma^{\mu\nu}F_{\mu\nu}\right)\Psi\,. \tag{34}$$

If we choose λ to be the proper length between the λ-constant surfaces, we have

$$\frac{\partial}{\partial\lambda}\left(\partial_\mu v^\mu + \frac{e}{2m}\gamma^{\mu\nu}F_{\mu\nu}\right) \sim a_0\omega_0^2\,, \tag{35}$$

since $F_{\mu\nu}$ oscillates with angular frequency ω_0. Hence, in one cycle we have at most

$$\left\|\frac{\partial\Psi}{\partial\lambda}\right\| \sim a_0\omega_0\|\Psi\|\,, \tag{36}$$

with possible Lorentz-boost factor of the order of a_0 incorporated. [Any enhancement of $\|\partial\Psi/\partial\lambda\|$ due to the first term on the right-hand side of Eq. (34) in one cycle is estimated to be of the same order as $\|\partial\Psi/\partial\lambda\|$ (see the comment after Eq. (33)). Also, if this term enhances $\|\partial\Psi/\partial\lambda\|$, then $\|\Psi\|$ should be enhanced in the same way. Hence we expect that it will not affect the ratio $\|\partial\Psi/\partial\lambda\|/\|\Psi\|$ significantly.] Then, the derivative $\partial_\mu\Psi$ with respect to a Cartesian coordinate is estimated as

$$\|\partial_\mu\Psi\| \sim a_0^2\omega_0\|\Psi\|\,, \tag{37}$$

by taking any Lorentz-boost factor, which is of the order of a_0, into account. Thus, the condition for the validity of the WKB approximation is

$$\frac{\hbar}{m}a_0^2\omega_0 \approx \left(\frac{a_0}{700}\right)^2 \ll 1\,, \tag{38}$$

where we have let $\hbar\omega_0 = 1\text{eV}$ and $m = 511\text{keV}$. This is satisfied if $a_0 \approx 30$.

II. EMISSION PROBABILITY AMPLITUDE

We expand the electron field as

$$\psi(t,\mathbf{x}) = \sum_{\mathbf{p},\alpha}\left[u_{(\mathbf{p},\alpha)}(x)b_{(\mathbf{p},\alpha)} + v_{(\mathbf{p},\alpha)}(x)d^\dagger_{(\mathbf{p},\alpha)}\right]\,, \tag{39}$$

where

$$\begin{aligned}\int d^3\mathbf{x}\, u^\dagger_{(\mathbf{p},\alpha)}u_{(\mathbf{p},\beta)} &= \int d^3\mathbf{x}\, v^\dagger_{(\mathbf{p},\alpha)}v_{(\mathbf{p},\beta)} = \delta_{\alpha\beta}\,,\\ \int d^3\mathbf{x}\, u^\dagger_{(\mathbf{p},\alpha)}v_{(\mathbf{p},\beta)} &= 0\,,\end{aligned} \tag{40}$$

and

$$\left\{b_{(\mathbf{p},\alpha)}, b^\dagger_{(\mathbf{p},\beta)}\right\} = \left\{d_{(\mathbf{p},\alpha)}, d^\dagger_{(\mathbf{p},\beta)}\right\} = \delta_{\alpha\beta}\,, \tag{41}$$

with all other anti-commutators vanishing. Then, the equal-time anti-commutation relation,

$$\{\psi_a(t,\mathbf{x}), \psi_b^\dagger(t,\mathbf{x}')\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta_{ab}\,, \tag{42}$$

is satisfied. We choose a wave-packet solution, $u_{(\mathbf{p},\alpha)}(t,\mathbf{x})$, peaked near a classical world line but approximately with a definite momentum and spin as one of these basis states. (Thus, this wave packet behaves like a classical electron.) The index α specifies the spin. The WKB solutions constructed in the previous subsection satisfies $\psi^\dagger\psi = 2p_0$. Thus, this wave-packet solution can be written approximately as

$$u_{(\mathbf{p},\alpha)}(t,\mathbf{x}) = \sqrt{\frac{p_0+m}{2p_0}}\begin{pmatrix} s_\alpha \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}s_\alpha\end{pmatrix}G(t,\mathbf{x})\,, \tag{43}$$

where s_α satisfies the Thomas-BMT equation, eq. (26). The function $G(t,\mathbf{x})$ is peaked about a classical world line and satisfies

$$\int d^3\mathbf{x}|G(t,\mathbf{x})|^2 = 1\,. \tag{44}$$

That is, although $G(t,\mathbf{x})$ is a smooth function, we may assume that $|G(t,\mathbf{x})|^2$ is well approximated by $\delta^{(3)}(\mathbf{x}-\mathbf{x}(t))$

The axion field is expanded as

$$\phi(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\left(a_\mathbf{k}e^{-ik\cdot x} + a^\dagger_\mathbf{k}e^{ik\cdot x}\right)\,, \tag{45}$$

where

$$\left[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}\right] = (2\pi)^3 2\hbar k_0\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,. \tag{46}$$

Although the axion mass m_a is nonzero, we let $m_a = 0$, assuming that it is very small. The term describing the interaction between the electron field and the axion field, ϕ, in the Lagrangian density is

$$\mathcal{L}_{\text{int}} = -\frac{\hbar g_{ae}}{2m}\partial_\mu\phi\bar\psi\gamma_5\gamma^\mu\psi\,. \tag{47}$$

The interaction Hamiltonian density can be found following the standard procedure as

$$\mathcal{H}_{\text{int}} = \frac{\hbar g_{ae}}{2m}\partial_\mu\phi\bar\psi\gamma_5\gamma^\mu\psi + \frac{1}{2}\left(\frac{\hbar g_{ae}}{2m}\right)^2(\bar\psi\gamma_5\gamma^0\psi)^2\,. \tag{48}$$

To lowest order in perturbation theory we may neglect the four-fermi interaction term. For this reason we adopt the first term as the interaction Hamiltonian density from now on.

Let the annihilation operator corresponding to the wave-packet solution $u_{(\mathbf{p},\alpha)}(t,\mathbf{x})$ be denoted by b_α. (We omit "$\mathbf{p}$" to simplify the notation.) The initial and final states are $b^\dagger_\alpha|0\rangle$, and $a^\dagger_\mathbf{k}b^\dagger_\beta|0\rangle$, where $a^\dagger_\mathbf{k}$ is the creation operator for the axion with momentum $\mathbf{k}$. We assume that the solutions corresponding to $b^\dagger_\alpha|0\rangle$ and $b^\dagger_\beta|0\rangle$ are approximately the same except possibly for the spin states. That is, s_α and s_β may or may not be the same, but they have approximately the same spatial wave function. If the initial state is $b^\dagger_\alpha|0\rangle$, then the final one-axion-emission state is, with $:\cdots:$ denoting normal-ordering,

$$\begin{aligned}
|f\rangle &= -\frac{i}{\hbar}\int d^4x\ :\mathcal{H}_{\rm int}(x):\, b_\alpha^\dagger\,|0\rangle \\
&= \frac{g_{ae}}{2m}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\sum_\beta\int d^4x\,\overline{u_\beta}\gamma_5\not{k}u_\alpha e^{ik\cdot x}a_{\mathbf{k}}^\dagger b_\beta^\dagger\,|0\rangle \\
&= \frac{g_{ae}}{2m}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\sum_\beta\int\frac{dt}{2p_0}\,\overline{\Phi_\beta}\gamma_5\not{k}\Phi_\alpha\, e^{ik_0t-i\mathbf{k}\cdot\mathbf{x}(t)}a_{\mathbf{k}}^\dagger b_\beta^\dagger\,|0\rangle \\
&= \frac{g_{ae}}{4m^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\sum_\beta\int d\tau\overline{\Phi_\beta}\gamma_5\not{k}\Phi_\alpha\, e^{ik\cdot x(\tau)}a_{\mathbf{k}}^\dagger b_\beta^\dagger\,|0\rangle\ ,
\end{aligned} \tag{49}$$

where

$$\Phi_\alpha = \sqrt{p_0+m}\begin{pmatrix} s_\alpha \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}\,s_\alpha\end{pmatrix}\ . \tag{50}$$

We find

$$\overline{\Phi_\beta}\gamma_5\not{k}\Phi_\alpha = 2s_\beta^\dagger\left(m\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p+mk}{p_0+m}\mathbf{p}\cdot\boldsymbol{\sigma}\right)s_\alpha\,, \tag{51}$$

Since $\langle 0|\, b_\beta b_\beta^\dagger\,|0\rangle = 1$, using eq. (46) we find the probability for emission to be

$$P_{\rm em} = \hbar\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\sum_\beta\left|\mathcal{A}_{(\mathbf{p},\mathbf{k},\beta,\alpha)}\right|^2\ , \tag{52}$$

where

$$\begin{aligned}
\mathcal{A}_{(\mathbf{p},\mathbf{k},\beta,\alpha)} &= \frac{g_{ae}}{2m}\int d\tau\, s_\beta^\dagger\left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p+mk}{m(p_0+m)}\mathbf{p}\cdot\boldsymbol{\sigma}\right]s_\alpha\, e^{ik\cdot x(\tau)} \\
&= \frac{ig_{ae}}{2m}\int d\tau\frac{d}{d\tau}\left\{(k\cdot v)^{-1}s_\beta^\dagger\left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p+mk}{m(p_0+m)}\mathbf{p}\cdot\boldsymbol{\sigma}\right]s_\alpha\right\}e^{ik\cdot x(\tau)} \\
&= \frac{ig_{ae}}{2m}\int d\tau\,\frac{e^{ik\cdot x(\tau)}}{(k\cdot v)^2}s_\beta^\dagger(\tau)\mathbf{Q}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)\,,
\end{aligned} \tag{53}$$

where in the second step we integrated by parts and defined

$$\mathbf{Q} = \mathbf{V} - (k\cdot a)\mathbf{k} - [V_0 - (k\cdot a)k_0]\,\frac{\mathbf{p}}{p_0+m}\,, \tag{54}$$

with

$$V^\mu = \frac{e}{m}(k\cdot v)F^{\mu\nu}k_\nu\,, \tag{55}$$

which can be re-expressed using eq. (22) as

$$V_0 = \frac{e}{m}(k\cdot v)\mathbf{k}\cdot\mathbf{E}\,, \tag{56}$$

$$\mathbf{V} = \frac{e}{m}(k\cdot v)(k_0\mathbf{E}+\mathbf{k}\times\mathbf{B})\,. \tag{57}$$

Here, $a^\mu = dv^\mu/d\tau$ is the proper acceleration. A useful formula for finding $\mathbf{Q}$ is

$$\frac{d}{d\tau}\left(\frac{\mathbf{p}}{p_0+m}\right) = \frac{2\mathbf{F}\times\mathbf{p} - e\mathbf{E}}{p_0+m}\,, \tag{58}$$

which can be derived from the Lorentz-force equations (24). It is also useful to note that the Lorentz-force equations can be given as

$$a^\mu = -\frac{e}{m}F^{\mu\nu}v_\nu\,, \tag{59}$$

or

$$a_0 = -\frac{e}{m}\mathbf{v}\cdot\mathbf{E}\,, \tag{60}$$

$$\mathbf{a} = -\frac{e}{m}(v_0\mathbf{E}+\mathbf{v}\times\mathbf{B})\,. \tag{61}$$

A. One-dimensional motion

Suppose that the electric field $\mathbf{E}$ is parallel to the z-direction and $\mathbf{B}=0$. Then, the motion can occur along the z-axis. From eq. (26) we find that the spin is time independent. Let the initial spin s_α be pointing in the

z-direction. If the final spin s_β is pointing in the same direction, then the corresponding amplitude, i.e., the spin-non-flip amplitude, is

$$\mathcal{A}^{\rm nf}_{(p_z,\mathbf{k})} = \frac{ig_{ae}}{2m}\int d\tau\, \frac{e^{ik\cdot x}}{(k\cdot v)^2} Q_z(\tau)\,, \tag{62}$$

where [4]

$$Q_z(\tau) = -k_\perp^2 a(\tau)\,, \tag{63}$$

with the acceleration $a(\tau)$ defined by

$$a(\tau) = -\frac{e}{m}E_z(\tau)\,. \tag{64}$$

Hence, the spin non-flip axion emission probability is

$$P^{\rm nf}_{\rm em} = \frac{\hbar g_{ae}^2}{4m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_\perp^4 \left|\int d\tau\, \frac{a}{(k\cdot v)^2}e^{ik\cdot x}\right|^2\,. \tag{65}$$

If the final spin is in the opposite direction, then

$$\mathcal{A}^{\rm f}_{(p_z,\mathbf{k})} = \frac{ig_{ae}}{2m}\int d\tau\, \frac{e^{ik\cdot x}}{(k\cdot v)^2}[Q_x(\tau)+iQ_y(\tau)]\,, \tag{66}$$

where

$$Q_x(\tau)+iQ_y(\tau) = -(k\cdot a)(k_x+ik_y)\,. \tag{67}$$

Hence the spin-flip emission probability is

$$P^{\rm f}_{\rm em} = \frac{\hbar g_{ae}^2}{4m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\, k_\perp^2 \left|\int d\tau\, \frac{k\cdot a}{(k\cdot v)^2}e^{ik\cdot x}\right|^2\,. \tag{68}$$

B. Two-dimensional motion

If the electric field is on the xz-plane and the magnetic field is in the y-direction, then the electron motion can occur on the xz-plane. In this case it is convenient to choose the initial spin in the y-direction since eq. (26) reads

$$\frac{ds_\pm}{d\tau} = \mp\frac{ie}{2m}\left(B_y - \frac{v_zE_x - v_xE_z}{v_0+1}\right)s_\pm\,, \tag{69}$$

where $\sigma_y s_\pm = \pm s_\pm$, so that

$$s_\pm(\tau) = \exp\left[\mp\frac{ie}{2m}\int_{\tau_i}^{\tau}\left(B_y - \frac{v_zE_x - v_xE_z}{v_0+1}\right)d\xi\right]s_\pm(\tau_i)\,. \tag{70}$$

Then, the spin-non-flip emission amplitude is

$$P^{\rm nf}_{\rm em} = \frac{\hbar g_{ae}^2}{4m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_y^2 \left|\int d\tau\, e^{ik\cdot x}\frac{k\cdot a}{(k\cdot v)^2}\right|^2\,. \tag{71}$$

The spin-flip amplitude is

$$P^{\rm f}_{\rm em} = \frac{\hbar g_{ae}^2}{4m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \left|\int d\tau\, e^{ik\cdot x}e^{\mp if(\tau)}\frac{Q_z \pm iQ_x}{(k\cdot v)^2}\right|^2 \tag{72}$$

where

$$\begin{aligned} Q_z \pm iQ_x = & -\frac{e}{m}(k\cdot v)[k_0(E_z \pm iE_x) + (k_x \mp ik_z)B_y] \\ & -(k\cdot a)(k_z \pm ik_x) \\ & -\left[\frac{e}{m}(k\cdot v)\mathbf{k}\cdot\mathbf{E} - (k\cdot a)k_0\right]\frac{v_z \pm iv_x}{v_0+1}\,, \end{aligned} \tag{73}$$

$$f(\tau) = \frac{e}{m}\int_{\tau_0}^{\tau}\left[B_y(\xi) - \frac{(\mathbf{v}(\xi)\times\mathbf{E}(\xi))_y}{v_0(\xi)+1}\right]d\xi\,. \tag{74}$$

III. THE FEYNMAN-DIAGRAM DERIVATION OF THE EMISSION PROBABILITY IN THE WEAK-FIELD LIMIT

Here we investigate the emission process in the weak-field limit but without assuming that the energy of the axion emitted is of lower order in $\hbar$ compared to the energy of the electron. We let $\hbar = 1$.

Assume that the external electromagnetic potential A_μ is nonzero only in a finite spacetime region and smooth. Then, the spacetime Fourier transform,

$$\widetilde{A}_\mu(q) = \int e^{iq\cdot x}A_\mu(x)\,d^4x\,, \tag{75}$$

exists. The electron propagator is

$$D(p) = \frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}\,. \tag{76}$$

The relevant Feynman diagram is shown in Fig. 1. With

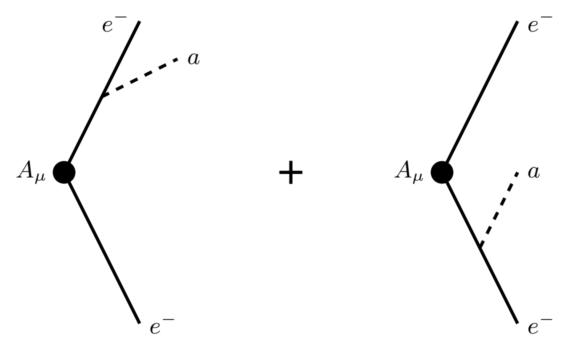

FIG. 1. The Feynman diagrams for the axion emission: The solid line represents the electron and the dashed line represents the axion. The dot represents the interaction of the electron with the external potential.

the initial spinor $u^{(i)}(p_i)e^{-ip_i\cdot x}$ and the final spinor $u^{(f)}(p_f)e^{-ip_f\cdot x}$, the amplitude for the emission of the axion with 4-momentum k^μ is

$$\mathcal{M} = \frac{eg_{ae}}{2m}\overline{u^{(f)}(p_f)}[\gamma_5 \not{k} D(p_f+k)\gamma^\mu - \gamma^\mu D(p_i-k)\gamma_5\not{k}]u^{(i)}(p_i)\widetilde{A}_\mu(q)\,, \quad (77)$$

where $u^{(i)}(p)$ and $u^{(f)}(p)$ are normalized so that

$$\sum_{\text{spin}} u^{(i)}(p)\overline{u^{(i)}(p)} = \not{p} + m\,, \quad (78)$$

and similarly for $u^{(f)}(p)$. We have defined $q = p_f + k - p_i$. Formally, the initial state, $b^\dagger_{(\mathbf{p}_i,\alpha)}|0\rangle$, is normalized as $\langle 0| b_{(\mathbf{p}_i,\alpha)} b^\dagger_{(\mathbf{p}_i,\alpha)}|0\rangle = (2\pi)^3 2p_i^0 \delta^{(3)}(\mathbf{0}) = 2p_i^0 V$, where V is the (infinite) volume of the space. The flux is $2p_i^0[\|\mathbf{p}_i\|/p_i^0] = 2\|\mathbf{p}_i\|$. Define

$$\Sigma = \frac{1}{2p_i^0}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{p}_f}{(2\pi)^3 2p_f^0}|\mathcal{M}|^2\,, \quad (79)$$

which we call the *interaction volume*. If the electron beam has the number density ρ, then the emission probability is given by $\rho\Sigma$.

By averaging over the initial spin and summing over the final spin, we find

$$\frac{1}{2}\sum_{\text{spin sum}} |\mathcal{M}|^2 = e^2 \mathcal{M}^{\mu\nu}\widetilde{A}^*_\mu(q)\widetilde{A}_\nu(q)\,, \quad (80)$$

where

$$\mathcal{M}^{\mu\nu} = -\frac{1}{2}\mathrm{Tr}\left\{(\not{p}_f + m)\left[\frac{\gamma_5\not{k}(\not{p}_f+m)\gamma^\mu}{2p_f\cdot k} - \frac{\gamma^\mu(\not{p}_i+m)\gamma_5\not{k}}{2p_i\cdot k}\right](\not{p}_i+m)\left[\frac{\gamma^\nu(\not{p}_f+m)\not{k}\gamma_5}{2p_f\cdot k} - \frac{\not{k}\gamma_5(\not{p}_i+m)\gamma^\nu}{2p_i\cdot k}\right]\right\}\,. \quad (81)$$

The overall minus sign is due to the insertion of γ_5 and $\gamma^0\gamma_5\gamma^0 = -\gamma^5$. After a tedious but straightforward calculation we find

$$\frac{1}{2}\sum_{\text{spin sum}} |\mathcal{M}|^2 = \frac{4e^2}{(k\cdot v_f)(k\cdot v_i)} k^\mu \widetilde{F}_{\mu\nu}(q)\widetilde{F}^{\nu\lambda}(q)k_\lambda\,, \quad (82)$$

where

$$\widetilde{F}_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x}[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)] = -i[q_\mu \widetilde{A}_\nu(q) - q_\nu\widetilde{A}_\mu(q)]\,. \quad (83)$$

Thus, we find

$$\Sigma = \frac{e^2 g_{ae}^2}{2p_i^0 m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{p}_f}{(2\pi)^3 2p_f^0}\frac{k^\mu \widetilde{F}^*_{\mu\nu}(q)\widetilde{F}^{\nu\lambda}(q)k_\lambda}{(k\cdot v_f)(k\cdot v_i)}\,. \quad (84)$$

Now, we consider the case where the electromagnetic field is only t-dependent, i.e., space-independent with the assumption that the characteristic frequency of the electromagnetic field is much smaller than the Compton wavelength of the electron so that, typically, $k_0 \ll \|\mathbf{p}_i\|, \|\mathbf{p}_f\|$. The aim is to show that the Feynman-diagram method gives the same result as our WKB result in this limit.

The Fourier transform of the electromagnetic field becomes

$$\widetilde{F}_{\mu\nu}(q) = (2\pi)^3\delta^{(3)}(\mathbf{p}_f + \mathbf{k} - \mathbf{p}_i)\hat{F}(p_f^0 - k_0 + p_i^0)\,, \quad (85)$$

where

$$\hat{F}(p_f^0 + k_0 - p_i^0) = \int e^{i(p_f^0 + k - p_i^0)t}F_{\mu\nu}(t)\,dt\,. \quad (86)$$

Then, by using the momentum conservation, we have

$$p_f^0 - p_i^0 = -\mathbf{k}\cdot\boldsymbol{\beta}, \quad (87)$$

to first order in k_0, where $\boldsymbol{\beta} = \mathbf{p}_i/p_i^0$. Thus, we find

$$\hat{F}(p_f^0 + k_0 - p_i^0) = v_0\int e^{ik\cdot x(\tau)}F_{\mu\nu}(\tau)\,d\tau\,, \quad (88)$$

where $v_0 = v_i^0 = v_f^0$ at lowest order in $\hbar$. Substituting eq. (85) into eq. (84) and using the formal equality $(2\pi)^3\delta^{(3)}(\mathbf{0}) = V$ we find

$$\Sigma = V\frac{e^2 g_{ae}^2}{4m^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{k^\mu \widetilde{F}^*_{\mu\nu}(q)\widetilde{F}^{\nu\lambda}(q)k_\lambda}{(k\cdot v)^2}\,, \quad (89)$$

where $v^\mu = v_f^\mu = v_i^\mu$ at this order. Then, since the number density of one electron in volume V is $\rho = 1/V$, the probability of axion emission is

$$\begin{aligned}P_{\rm em}^{\rm (av)} &= \Sigma/V \\ &= \frac{\hbar e^2 g_{ae}^2}{4m^4}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \\ &\times \int d\tau' \int d\tau \frac{e^{-ik\cdot[x(\tau')-x(\tau)]}}{(k\cdot v)^2} k_\mu F^{\mu\nu}(\tau')F_{\nu\lambda}(\tau)k^\lambda\,.\end{aligned} \tag{90}$$

Now we show that eq. (90) agrees with the weak-field limit of eq. (52). Note that the spin may be taken to be constant in the weak-field limit. Then

$$\begin{aligned}&\frac{1}{2}\sum_{\alpha,\beta} s_\beta^\dagger \mathbf{Q}(\tau')\cdot\boldsymbol{\sigma} s_\alpha s_\alpha^\dagger \mathbf{Q}(\tau)\cdot\boldsymbol{\sigma} s_\beta \\ &= \frac{1}{2}\mathrm{Tr}[\mathbf{Q}(\tau')\cdot\boldsymbol{\sigma}\mathbf{Q}(\tau)\cdot\boldsymbol{\sigma}] = \mathbf{Q}(\tau')\cdot\mathbf{Q}(\tau)\,.\end{aligned} \tag{91}$$

In the expression (54) for $\mathbf{Q}$ we may approximate the 4-momentum $p^\mu = mv^\mu$ to be constant and treat only the electromagnetic field to be time-dependent. Since

$$[V^\mu - (k\cdot a)k^\mu]v_\mu = 0\,, \tag{92}$$

i.e.,

$$[\mathbf{V}(\tau) - (k\cdot a(\tau))\mathbf{k}]\cdot\mathbf{v} = [V_0(\tau) - (k\cdot a(\tau))k_0]v_0\,, \tag{93}$$

and $\mathbf{p}^2 = p_0^2 - m^2$, we find

$$\begin{aligned}\mathbf{Q}(\tau')\cdot\mathbf{Q}(\tau) &= -V^\mu(\tau')V_\mu(\tau) \\ &= \frac{e^2}{m^2}(k\cdot v)^2 k_\mu F^{\mu\nu}(\tau')F_{\nu\lambda}(\tau)k^\lambda\,,\end{aligned} \tag{94}$$

in the weak-field limit, where we also used $V^\mu(\tau)k_\mu = 0$. Then, we find that the weak-field approximation of eq. (52) indeed agrees with eq. (90).

IV. ENERGY OF EMITTED AXIONS

Here we derive the formula for the energy of the emitted axions. As is the case with the Larmor radiation (see, e.g., Ref. [5]), the energy emitted takes a local form in the sense that it is a single integral, rather than a double integral, over the proper time. That is, there are no interference terms between different proper times. This is in contrast with the number of emitted axions, which has interference terms of this kind.

Define

$$\begin{aligned}&\widetilde{\mathbf{Q}} = k_0^{-2}\mathbf{Q}\,,\quad n^\mu = \frac{k^\mu}{k_0}\,, \\ &\widetilde{V}^\mu = F^{\mu\nu}n_\nu\,.\end{aligned} \tag{95}$$

Thus,

$$\begin{aligned}\widetilde{\mathbf{Q}} &= \frac{e}{m}(n\cdot v)\widetilde{\mathbf{V}} - (n\cdot a)\mathbf{n} \\ &+ \frac{\mathbf{p}}{p_0+m}\left[n\cdot a - \frac{e}{m}(n\cdot v)\widetilde{V}_0\right]\,.\end{aligned} \tag{96}$$

Then, the energy emitted is obtained from eq. (52) as

$$E_{\beta\alpha} = \frac{g_{ae}^2\hbar}{16\pi^2 m^2}\int\frac{d\Omega}{4\pi}\int_0^\infty dk\,k^2\int\frac{d\tau}{[n\cdot v(\tau)]^2}\int\frac{d\tau'}{[n\cdot v(\tau')]^2} s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)s_\alpha^\dagger(\tau')\widetilde{\mathbf{Q}}(\tau')\cdot\boldsymbol{\sigma}s_\beta(\tau')e^{ik\cdot[x(\tau)-x(\tau')]}\,. \tag{97}$$

We define $\xi = t - \mathbf{n}\cdot\mathbf{x}$ [6]. Then $d\xi/d\tau = n\cdot v$ so that $d\tau/(n\cdot v)^2 = d\xi/(n\cdot v)^3$. Thus

$$\begin{aligned}E_{\beta\alpha} &= \frac{g_{ae}^2\hbar}{16\pi^2 m^2}\int\frac{d\Omega}{4\pi}\int_0^\infty dk\int d\xi\int d\xi' \\ &\times \frac{d}{d\xi'}\left\{\frac{1}{[n\cdot v(\tau)]^3}s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)\right\}\frac{d}{d\xi'}\left\{\frac{1}{[n\cdot v(\tau')]^3}s_\alpha^\dagger(\tau')\widetilde{\mathbf{Q}}(\tau')\cdot\boldsymbol{\sigma}s_\beta(\tau')\right\}e^{ik(\xi-\xi')}\,,\end{aligned} \tag{98}$$

where we have integrated by parts.

If we sum over the final spin states and average over the initial spin states, then the integrand is symmetric under the exchange $\xi \leftrightarrow \xi'$. This symmetry can be used to extend the range of the k-integral to $(-\infty,\infty)$. Thus, we find the total energy emitted with the initial spin averaged over as

$$\begin{aligned}\langle E\rangle &= \frac{g_{ae}^2\hbar}{32\pi m^2}\int\frac{d\Omega}{4\pi}\int\frac{d\tau}{n\cdot v} \\ &\times\sum_{\alpha,\beta}\left|\frac{d}{d\tau}\left[\frac{1}{[n\cdot v(\tau)]^3}s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)\right]\right|^2\,.\end{aligned} \tag{99}$$

Now,

$$\frac{d}{d\tau}\left[\frac{1}{[n\cdot v(\tau)]^3}s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)\right]=s_\beta^\dagger\mathbf{S}\cdot\boldsymbol{\sigma}s_\alpha\,, \quad (100)$$

where

$$\mathbf{S}=-\frac{3n\cdot a}{(n\cdot v)^4}\widetilde{\mathbf{Q}}+\frac{1}{(n\cdot v)^3}[\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}]\,, \quad (101)$$

where the dot denotes the τ-derivative and $\mathbf{F}$ is defined in eq. (29). Then, since $\sum_\alpha s_\alpha(\tau)s_\alpha^\dagger(\tau)=\mathbb{1}$ for all τ, we find

$$\begin{aligned}\langle E\rangle &= \frac{\hbar^2 g_{ae}^2}{32\pi m^2}\int\frac{d\Omega}{4\pi}\int\frac{d\tau}{n\cdot v}\mathrm{Tr}\left[(\mathbf{S}\cdot\boldsymbol{\sigma})^2\right]\\ &= \frac{g_{ae}^2\hbar}{16\pi m^2}\int\frac{d\Omega}{4\pi}\int\frac{d\tau}{(n\cdot v)^7}\\ &\times\left[\frac{9(n\cdot a)^2}{(n\cdot v)^2}\widetilde{\mathbf{Q}}^2-\frac{3n\cdot a}{n\cdot v}\frac{d}{d\tau}\widetilde{\mathbf{Q}}^2+\|\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}\|^2\right].\end{aligned} \quad (102)$$

To simplify the integrand in this equation, it is useful to define

$$\begin{aligned}U^\mu &= (n\cdot a)n^\mu-\frac{e}{m}(n\cdot v)\widetilde{V}^\mu\\ &= -\frac{e}{m}\left(n_\alpha F^{\alpha\beta}v_\beta\, n^\mu-n_\alpha F^{\alpha\mu}n^\nu v_\nu\right).\end{aligned} \quad (103)$$

Then,

$$\widetilde{\mathbf{Q}}=-\mathbf{U}+\frac{m\mathbf{v}}{p_0+m}U_0\,. \quad (104)$$

Equation (94) is translated to

$$\widetilde{\mathbf{Q}}^2=-U^\mu U_\mu\,. \quad (105)$$

Next we examine $\|\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}\|^2$. We find

$$\begin{aligned}\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}} &= -\frac{d\mathbf{U}}{d\tau}+\frac{\mathbf{p}}{p_0+m}\frac{dU_0}{d\tau}+\frac{d}{d\tau}\left(\frac{\mathbf{p}}{p_0+m}\right)U_0\\ &+2\mathbf{F}\times\mathbf{U}-\frac{2\mathbf{F}\times\mathbf{p}}{p_0+m}U_0\,.\end{aligned} \quad (106)$$

By eq. (58) and the definition (29), we have

$$\begin{aligned}&\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}\\ &= -\frac{d\mathbf{U}}{d\tau}+\frac{\mathbf{p}}{p_0+m}\frac{dU_0}{d\tau}-\frac{eU_0\mathbf{E}}{p_0+m}\\ &-\frac{e}{m}\mathbf{U}\times\mathbf{B}-\frac{e}{m(p_0+m)}\left[(\mathbf{p}\cdot\mathbf{U})\mathbf{E}-(\mathbf{E}\cdot\mathbf{U})\mathbf{p}\right].\end{aligned} \quad (107)$$

Using $U^\mu v_\mu=0$, i.e., $\mathbf{p}\cdot\mathbf{U}=p_0U_0$, we find

$$\begin{aligned}\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}} &= -\left[\frac{d\mathbf{U}}{d\tau}+\frac{e}{m}(U_0\mathbf{E}+\mathbf{U}\times\mathbf{B})\right]\\ &+\frac{\mathbf{p}}{p_0+m}\left[\frac{dU_0}{d\tau}+\frac{e}{m}\mathbf{U}\cdot\mathbf{E}\right].\end{aligned} \quad (108)$$

Define

$$H^\mu=\frac{dU^\mu}{d\tau}+\frac{e}{m}F^{\mu\nu}U_\nu\,. \quad (109)$$

Then,

$$\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}=-\mathbf{H}+\frac{\mathbf{p}}{p_0+m}H_0\,. \quad (110)$$

The equation $dv^\mu/d\tau=-(e/m)F^{\mu\nu}v_\nu$ can be used to show that $H^\mu v_\mu=H_0v_0-\mathbf{H}\cdot\mathbf{v}=0$. Then, the same calculation as that demonstrated $\widetilde{\mathbf{Q}}^2=-U^\mu U_\mu$ shows that

$$\|\dot{\widetilde{\mathbf{Q}}}-2\mathbf{F}\times\widetilde{\mathbf{Q}}\|^2=-H^\mu H_\mu\,. \quad (111)$$

Thus, we find that

$$\begin{aligned}\langle E\rangle &= \frac{\hbar^2 g_{ae}^2}{16\pi m^2}\int\frac{d\Omega}{4\pi}\int\frac{d\tau}{(n\cdot v)^7}\\ &\times\left[-\frac{9(n\cdot a)^2}{(n\cdot v)^2}U_\mu U^\mu+\frac{3n\cdot a}{n\cdot v}\frac{d}{d\tau}(U_\mu U^\mu)-H_\mu H^\mu\right],\end{aligned} \quad (112)$$

where U^μ is given by eq. (103), and

$$\begin{aligned}H^\mu &= \frac{d(n\cdot a)}{d\tau}n^\mu-\frac{e}{m}(n\cdot v)\dot{F}^{\mu\nu}n_\nu\\ &-\frac{e^2}{m^2}(n\cdot v)F^{\mu\nu}F_{\nu\lambda}n^\lambda\,.\end{aligned}$$

When evaluating $U_\mu U^\mu$, we can use $n^\mu n_\mu=0$ and $n^\mu\widetilde{V}_\mu=0$. Thus,

$$U_\mu U^\mu=-\frac{e^2}{m^2}(n\cdot v)^2 n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda\,. \quad (113)$$

Similarly, one can simplify $H_\mu H^\mu$ as

$$H_\mu H^\mu = -\frac{2e^2}{m^2}(n\cdot v)\frac{d(n\cdot a)}{d\tau}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda + \frac{e^2}{m^2}(n\cdot v)^2\left(\dot{F}^{\mu\nu} + \frac{e}{m}F^{\mu\alpha}F_\alpha{}^\nu\right)n_\nu\left(\dot{F}_{\mu\lambda} + \frac{e}{m}F_{\mu\alpha}F^\alpha{}_\lambda\right)n^\lambda\,. \tag{114}$$

Hence, the expression inside the brackets of eq. (112) divided by $(n\cdot v)^7$ is

$$\begin{aligned}
&\frac{1}{(n\cdot v)^7}\left[-\frac{9(n\cdot a)^2}{(n\cdot v)^2}U_\mu U^\mu + \frac{3n\cdot a}{n\cdot v}\frac{d}{d\tau}(U_\mu U^\mu) - H_\mu H^\mu\right] \\
&= \frac{3e^2}{m^2}\frac{(n\cdot a)^2}{(n\cdot v)^7}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda - \frac{6e^2}{m^2}\frac{n\cdot a}{(n\cdot v)^6}n_\mu \dot{F}^{\mu\nu}F_{\nu\lambda}n^\lambda + \frac{2e^2}{m^2}\frac{n\cdot\dot{a}}{(n\cdot v)^6}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda \\
&\quad - \frac{e^2}{m^2}\frac{1}{(n\cdot v)^5}\left(\dot{F}^{\mu\nu} + \frac{e}{m}F^{\mu\alpha}F_\alpha{}^\nu\right)n_\nu\left(\dot{F}_{\mu\lambda} + \frac{e}{m}F_{\mu\beta}F^\beta{}_\lambda\right)n^\lambda\,.
\end{aligned} \tag{115}$$

A. Angular integral for the energy emitted

Next we carry out the angular integral in eq. (112). All integrals can be found from

$$\frac{1}{4\pi}\int d\Omega\,\frac{n_{\mu_1}n_{\mu_2}}{(n\cdot v)^5} = 2\frac{v^0 v_{\mu_1}v_{\mu_2}}{(v\cdot v)^4} - \frac{1}{3}\frac{\delta^0_{\mu_1}v_{\mu_2} + \delta^0_{\mu_2}v_{\mu_1} + v^0 g_{\mu_1\mu_2}}{(v\cdot v)^3}\,. \tag{116}$$

By letting $v\cdot v = 1$ we have, for any tensor $N_{\mu\nu}$,

$$\frac{1}{4\pi}\int d\Omega\,\frac{n^\mu n^\nu N_{\mu\nu}}{(n\cdot v)^5} = 2v^0 v^\mu v^\nu N_{\mu\nu} - \frac{1}{3}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\,. \tag{117}$$

Next, by differentiating eq. (116) with respect to τ and letting $v\cdot v = 1$ and $v\cdot a = 0$, we obtain

$$\frac{1}{4\pi}\int d\Omega\,\frac{n\cdot a}{(n\cdot v)^6}n_\mu N^{\mu\nu}n_\nu = -\frac{2}{5}\left[a^0 v^\mu v^\nu N_{\mu\nu} + v^0(a^\mu v^\nu + v^\mu a^\nu)N_{\mu\nu}\right] + \frac{1}{15}\left(a^\mu N_{\mu 0} + N_{0\nu}a^\nu + a^0 N^\mu{}_\mu\right)\,. \tag{118}$$

Next, by differentiating eq. (116) twice with respect to τ, equating the contribution containing $\dot{a}_\mu$ and that not containing $\dot{a}_\mu$ on both sides, and then letting $v\cdot v = 1$ and $v\cdot a = 0$, we find

$$\begin{aligned}
\int\frac{d\Omega}{4\pi}\frac{(n\cdot a)^2}{(n\cdot v)^7}n^\mu n^\nu N_{\mu\nu} &= (a\cdot a)\left[-\frac{8}{15}v^0 v^\mu v^\nu N_{\mu\nu} + \frac{1}{15}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\right] \\
&\quad + \frac{2}{15}[a^0 a^\mu v^\nu + a^0 v^\mu a^\nu + v^0 a^\mu a^\nu]N_{\mu\nu}\,,
\end{aligned} \tag{119}$$

$$\begin{aligned}
\int\frac{d\Omega}{4\pi}\frac{n\cdot\dot{a}}{(n\cdot v)^6}n^\mu n^\nu N_{\mu\nu} &= (v\cdot\dot{a})\left[\frac{16}{5}v^0 v^\mu v^\nu N_{\mu\nu} - \frac{2}{5}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\right] \\
&\quad - \frac{2}{5}[\dot{a}^0 v^\mu v^\nu + v^0\dot{a}^\mu v^\nu + v^0 v^\mu\dot{a}^\nu]N_{\mu\nu} + \frac{1}{15}(N^{0\nu}\dot{a}_\nu + \dot{a}_\mu N^{\mu 0} + \dot{a}^0 N_\mu{}^\mu)\,.
\end{aligned} \tag{120}$$

Let us denote the angular integrals with the measure $d\Omega/4\pi$ of the first, second, third and fourth terms in eq. (115) by I_1, I_2, I_3 and I_4, respectively. These are performed by using eqs. (119), (118), (120) and (117), respectively. Then, after using eq. (59) to express $F^{\mu\nu}v_\nu$ and $\dot{F}^{\mu\nu}v_\nu$ in terms of a^μ and $\dot{a}^\mu$, we find

$$I_1 = -3(a\cdot a)\left[-\frac{8}{15}v^0(a\cdot a) + \frac{2e}{15m}F^{0\mu}a_\mu + \frac{e^2}{15m^2}v^0F_{\mu\nu}F^{\mu\nu}\right] + \frac{2e^2}{5m^2}v^0a^\mu F_{\mu\nu}F^{\nu\lambda}a_\lambda\,, \tag{121}$$

$$I_2 = 6\left[-\frac{2}{5}a^0a^\mu\dot{a}_\mu + \frac{2e}{5m}v^0F^{\mu\nu}a_\nu\left(\dot{a}_\mu + \frac{e}{m}F_{\mu\lambda}a^\lambda\right) - \frac{e^2}{15m^2}F^{0\lambda}\dot{F}_{\lambda\mu}a^\mu - \frac{e^2}{15m^2}\dot{F}^{0\lambda}F_{\lambda\mu}a^\mu + \frac{e^2}{15m^2}a^0F^{\mu\nu}\dot{F}_{\mu\nu}\right], \tag{122}$$

$$I_3 = -2(a\cdot a)\left[-\frac{16}{5}v^0(a\cdot a) + \frac{4e}{5m}F^{0\mu}a_\mu + \frac{2e^2}{5m^2}v^0F_{\mu\nu}F^{\mu\nu}\right] + \frac{4}{5}\dot{a}^0(a\cdot a) - \frac{8e}{5m}v^0a_\nu F^{\nu\mu}\dot{a}_\mu$$
$$+ \frac{4e^2}{15m^2}F^{0\mu}F_{\mu\nu}\dot{a}^\nu - \frac{2e^2}{15m^2}\dot{a}^0F^{\mu\nu}F_{\mu\nu}\,, \tag{123}$$

$$I_4 = -2v^0\left(\dot{a}^\mu + \frac{2e}{m}F^{\mu\nu}a_\nu\right)\left(\dot{a}_\mu + \frac{2e}{m}F_{\mu\lambda}a^\lambda\right) + \frac{2e}{3m}\left(\dot{F}^{0\lambda} - \frac{e}{m}F^{0\alpha}F_\alpha{}^\lambda\right)\left(\dot{a}_\lambda + \frac{2e}{m}F_{\lambda\beta}a^\beta\right)$$
$$+ \frac{e^2}{3m^2}v^0\left(\dot{F}^{\mu\nu} + \frac{e}{m}F^{\mu\lambda}F_\lambda{}^\nu\right)\left(\dot{F}_{\mu\nu} + \frac{e}{m}F_{\mu\beta}F^\beta{}_\nu\right). \tag{124}$$

For I_3 we used the equality $v\cdot\dot{a} = -a\cdot a$. We can write the sum $I = I_1 + I_2 + I_3 + I_4$ as

$$I = K_0 + \frac{e}{m}K_1 + \frac{e^2}{m^2}K_2 - \frac{4e^3}{3m^3}F^{0\alpha}F_{\alpha\beta}F^{\beta\nu}a_\nu + \frac{e^4}{3m^4}v^0F_{\mu\nu}F^{\nu\lambda}F_{\lambda\alpha}F^{\alpha\mu}\,, \tag{125}$$

where

$$K_0 = 2v^0[4(a\cdot a)^2 - \dot{a}\cdot\dot{a}] - \frac{12}{5}a^0(a\cdot\dot{a}) + \frac{4}{5}\dot{a}^0(a\cdot a)\,, \tag{126}$$

$$K_1 = -4v^0\dot{a}^\mu F_{\mu\nu}a^\nu - 2(a\cdot a)F^{0\mu}a_\mu + \frac{2}{3}\dot{F}^{0\mu}\dot{a}_\mu\,, \tag{127}$$

$$K_2 = 6v^0a_\mu F^{\mu\nu}F_{\nu\lambda}a^\lambda - v^0(a\cdot a)F_{\mu\nu}F^{\mu\nu} - \frac{2}{5}F^{0\mu}F_{\mu\nu}\dot{a}^\nu + \frac{14}{15}\dot{F}^{0\lambda}F_{\lambda\mu}a^\mu$$
$$+ \frac{1}{3}v^0\dot{F}^{\mu\nu}\dot{F}_{\mu\nu} - \frac{2}{5}F^{0\lambda}\dot{F}_{\lambda\mu}a^\mu + \frac{2}{5}a^0F^{\mu\nu}\dot{F}_{\mu\nu} - \frac{2}{15}\dot{a}^0F^{\mu\nu}F_{\mu\nu}\,. \tag{128}$$

These can be given in terms of the electric and magnetic fields as

$$K_1 = -4v^0[\dot{a}^0\mathbf{a}\cdot\mathbf{E} - \dot{\mathbf{a}}\cdot(a^0\mathbf{E} + \mathbf{a}\times\mathbf{B})] - 2(a\cdot a)\mathbf{a}\cdot\mathbf{E} + \frac{2}{3}\dot{\mathbf{E}}\cdot\dot{\mathbf{a}}\,, \tag{129}$$

$$K_2 = 6v^0\left[\|a^0\mathbf{E} + \mathbf{a}\times\mathbf{B}\|^2 - (\mathbf{a}\cdot\mathbf{E})^2\right] + 2v^0(a\cdot a)(\mathbf{E}^2 - \mathbf{B}^2) - \frac{2}{5}\mathbf{E}\cdot(\dot{a}^0\mathbf{E} + \dot{\mathbf{a}}\times\mathbf{B}) + \frac{14}{15}\dot{\mathbf{E}}\cdot(a^0\mathbf{E} + \mathbf{a}\times\mathbf{B})$$
$$- \frac{2}{3}v^0(\dot{\mathbf{E}}^2 - \dot{\mathbf{B}}^2) - \frac{2}{5}\mathbf{E}\cdot(a^0\dot{\mathbf{E}} + \mathbf{a}\times\dot{\mathbf{B}}) - \frac{4}{5}a^0(\mathbf{E}\cdot\dot{\mathbf{E}} - \mathbf{B}\cdot\dot{\mathbf{B}}) + \frac{4}{15}\dot{a}^0(\mathbf{E}^2 - \mathbf{B}^2)\,, \tag{130}$$

where we have used

$$F^{\mu\nu}a_\nu = (\mathbf{a}\cdot\mathbf{E}, a^0\mathbf{E} + \mathbf{a}\times\mathbf{B})\,, \tag{131}$$

and

$$F^{0\alpha}F_{\alpha\beta}F^{\beta\nu}a_\nu = (\mathbf{a}\cdot\mathbf{E})(\mathbf{E}^2 - \mathbf{B}^2) + (\mathbf{a}\cdot\mathbf{B})(\mathbf{E}\cdot\mathbf{B})\,, \tag{132}$$

$$F_{\mu\nu}F^{\nu\lambda}F_{\lambda\alpha}F^{\alpha\mu} = 2(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2\,. \tag{133}$$

The energy emitted is given by

$$\langle E\rangle = \frac{\hbar^2 g_{ae}^2}{16\pi m^2}\int I\,d\tau\,. \tag{134}$$

B. One-dimensional motion

Let $\mathbf{B} = 0$ and let $\mathbf{E}$ and $\mathbf{v}$ be in the z-direction. With the notation $v^z = w$, and with the prime dnoting the t-derivative, we find from eqs. (126), (129) and (130),

$$K_0 = v^0\left[\frac{26}{5}(w')^4 + \frac{8}{5}w(w')^2w'' + 2(1+w^2)(w'')^2\right],$$

$$\frac{e}{m}K_1 = v^0\left[2(w')^4 - \frac{2}{3}(1+w^2)(w'')^2 - \frac{2}{3}w(w')^2w''\right],$$

where we have used $(e/m)E_z = -w'$, and

$$- \frac{4e^3}{3m^3} F^{0\alpha} F_{\alpha\beta} F^{\beta\nu} a_\nu + \frac{e^4}{3m^4} v^0 F_{\mu\nu} F^{\nu\lambda} F_{\lambda\alpha} F^{\alpha\mu} = 2v^0 (w')^4 , \tag{136}$$

Then,

$$I = v^0 \left[\frac{16}{15}(w')^4 + \frac{2}{3}(1+w^2)(w'')^2 + \frac{8}{15} w w'' (w')^2 \right] . \tag{137}$$

For $w = 2a_0 \sin \omega_0 t$, we find the energy emitted in one cycle from eq. (134) as

$$E^{(1)}_{\text{linear}} = \frac{g_{ae}^2 \hbar^2}{6m^2} (7a_0^2 + 1) a_0^2 \omega_0^3 . \tag{138}$$

C. Circular motion with constant magnetic field

If the magnetic field $\mathbf{B}$ is constant, $\mathbf{E} = \mathbf{0}$, and if the motion is stationary, then

$$K_1 = 4v^0 \dot{\mathbf{a}} \cdot (\mathbf{a} \times \mathbf{B}) , \tag{139}$$

$$K_2 = 6v^0 \|\mathbf{a} \times \mathbf{B}\|^2 - 2v^0 (a \cdot a) \mathbf{B}^2 , \tag{140}$$

so that

$$\begin{aligned} I &= 2v^0 [4(a\cdot a)^2 - \dot{a}\cdot\dot{a}] + \frac{4e}{m} v^0 \dot{\mathbf{a}} \cdot (\mathbf{a}\times\mathbf{B}) \\ &+ \frac{e^2}{m^2} [6v^0 \|\mathbf{a}\times\mathbf{B}\|^2 - 2v^0 (a\cdot a)\mathbf{B}^2] \\ &+ \frac{2e^4}{3m^4} v^0 (\mathbf{B}^2)^2 . \end{aligned} \tag{141}$$

($a \cdot \dot{a} = 0$ because $a \cdot a$ is constant in this case.) From eqs. (60) and (61), we find $a^0 = 0$ and

$$\mathbf{a} = -\frac{e}{m} \mathbf{v} \times \mathbf{B} , \tag{142}$$

$$\dot{\mathbf{a}} = -\frac{e}{m} \mathbf{a} \times \mathbf{B} . \tag{143}$$

Using the fact that $\mathbf{a} \perp \mathbf{B}$, we find

$$I = 2v^0 \left[4(\mathbf{a}^2)^2 + \frac{3e^2}{m^2} \mathbf{B}^2 \mathbf{a}^2 + \frac{e^4}{3m^4} (\mathbf{B}^2)^2 \right] . \tag{144}$$

Now, suppose that the motion is circular with angular frequency ω_0. We choose the magnetic field to be in the y-direction and define

$$a_0 = \frac{eB_y}{m\omega_0} . \tag{145}$$

Then, the motion of the electron can be described by

$$z = \frac{\sqrt{1 - a_0^{-2}}}{\omega_0} \cos \omega_0 t , \tag{146}$$

$$x = \frac{\sqrt{1 - a_0^{-2}}}{\omega_0} \sin \omega_0 t . \tag{147}$$

Then,

$$\mathbf{a}^2 = a_0^2 (a_0^2 - 1) \omega_0^2 , \tag{148}$$

and

$$I = v^0 \left(8a_0^8 - 10a_0^6 + \frac{8}{3} a_0^4 \right) \omega_0^4 . \tag{149}$$

Thus, the energy of the axions emitted in one cycle is

$$E^{(1)}_{\text{circular}} = \frac{g_{ae}^2 \hbar}{4m^2} \left(4a_0^8 - 5a_0^6 + \frac{4}{3} a_0^4 \right) \omega_0^3 . \tag{150}$$

Notice that this is much larger than the energy emitted in the linear case, eq. (138) if $a_0 \gg 1$, i.e., if the electron is highly relativistic.

V. THE EMISSION SPECTRUM FOR THE CONSTANT MAGNETIC FIELD

Here we investigate the axion-emission spectrum for a circular motion with a constant magnetic field given by eqs. (146) and (147).

The motion is periodic, and the emission amplitude can be expressed as a Fourier series with period $2\pi/\omega_0$. The (infinite) emission probability is formally of the form

$$P = \int_0^\infty dk \int d\Omega g(k,\Omega) \left| \int_{-\infty}^\infty \sum_{n=-\infty}^\infty c_n e^{i(k-n\omega_0)\xi} d\xi \right|^2 , \tag{151}$$

where ξ is a variable we identify later and has the same period $2\pi/\omega_0$ as the time t. We formally find

$$P = 2\pi \int_0^\infty dk \int d\Omega g(k,\Omega) \sum_{n=-\infty}^\infty |c_n|^2 \delta(k - n\omega_0) \times 2\pi\delta(0) , \tag{152}$$

where $2\pi\delta(0)$ is formally the integral over ξ from $-\infty$ to ∞. Integration over ξ from $-T/2$ to $T/2$ is equivalent to that over t for the same period if $T/2$ is a multiple of their period $2\pi/\omega_0$. Thus, we can regard $2\pi\delta(0)$ as the time duration T. Hence, the emission rate, $R = P/T$, is

$$R = 2\pi \sum_{n=1}^\infty \int d\Omega\, g(n\omega_0, \Omega) |c_n|^2 . \tag{153}$$

To use this formula to find the emission rate, we need to find the Fourier coefficients c_n.

The periodicity of the motion also lets us use the first line of eq. (53), which is the form of the amplitude before the integration by parts to remove the contribution from the "axion cloud" around the electron.

A. The spin-non-flip case

The amplitude in the first line of eq. (53) in the spin-non-flip case with the spin in the positive or negative y-direction reads

$$\mathcal{A}_{(\mathbf{p},\mathbf{k},\alpha,\beta)} = \frac{g_{ae}}{2m} k_y \int e^{ik\cdot x(\tau)} d\tau \,. \tag{154}$$

Then, the (infinite) emission probability is

$$P^{\rm nf}_{\rm em} = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_y^2 \left| \frac{1}{a_0} \int e^{ik\xi} \frac{dt}{d\xi} d\xi \right|^2 , \tag{155}$$

where we have used $d\tau = a_0 dt$. The variable ξ is defined by $k\xi = k\cdot x = kt - \mathbf{k}\cdot\mathbf{x}$, where $k = k_0$. We define the spherical polar coordinates θ and φ by

$$(k_z, k_x, k_y) = (k\sin\theta\cos\varphi, k\sin\theta\sin\varphi, k\cos\theta)\,. \tag{156}$$

Using this and eqs. (146) and (147), we find

$$\xi = t - \frac{\sqrt{1-a_0^{-2}}}{\omega_0} \sin\theta\cos(\omega_0 t - \varphi)\,. \tag{157}$$

Comparing eqs. (152) and (155), we can make the following identification:

$$\int_0^\infty dk \int d\Omega\, g(k,\Omega) = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} k_y^2 \,, \tag{158}$$

$$\frac{1}{a_0}\frac{dt}{d\xi} = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0\xi}\,, \tag{159}$$

where

$$\frac{dt}{d\xi} = \frac{1}{1+\sqrt{1-a_0^{-2}}\sin\theta\sin(\omega_0 t - \varphi)}\,, \tag{160}$$

is a periodic function of ξ with period $2\pi/\omega_0$. The Fourier coefficients c_n can be found as

$$\begin{aligned} c_n &= \frac{\omega_0}{2\pi a_0} \int_0^{2\pi/\omega_0} e^{in\omega_0\xi} \frac{dt}{d\xi} d\xi \\ &= \frac{1}{2\pi a_0} \int_0^{2\pi} e^{in[s-\beta\sin\theta\cos(s-\varphi)]} ds \quad (s=\omega_0 t)\,, \end{aligned} \tag{161}$$

where $\beta = \sqrt{1-a_0^{-2}}$. Then, by the formula

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ins - z\cos s} ds = e^{-in\pi/2} J_n(z)\,, \tag{162}$$

which follows from eqs. 3.715.13 and 3.715.18 of Ref. [7], where $J_n(x)$ is the Bessel function of order n, we find

$$c_n = \frac{e^{in\varphi - in\pi/2}}{a_0} J_n(n\sqrt{1-a_0^{-2}}\sin\theta)\,. \tag{163}$$

Then, the rate of emission is

$$R^{\rm nf}_{\rm em} = \frac{\hbar g_{ae}^2\omega_0^3}{16\pi m^2} \sum_{n=1}^{\infty} \frac{n^3}{a_0^2} \int_0^\pi \left| J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2 \cos^2\theta\sin\theta\, d\theta\,, \tag{164}$$

and the power is

$$S^{\rm nf}_{\rm em} = \frac{\hbar^2 g_{ae}^2\omega_0^4}{16\pi m^2} \sum_{n=1}^{\infty} \frac{n^4}{a_0^2} \int_0^\pi \left| J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2 \cos^2\theta\sin\theta\, d\theta\,. \tag{165}$$

It is possible to carry out the summation and θ-integral in eq. (165). Define $u = s - \alpha\cos s$, where $0 < \alpha < 1$. then, the variable $du/ds > 0$. We write eq. (162) with $z = n\alpha$ as

$$e^{-in\pi/2} J_n(n\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{inu}}{1+\alpha\sin s} du\,. \tag{166}$$

Then,

$$e^{-in\pi/2} n^2 J_n(n\alpha) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{d^2}{du^2}\left(\frac{1}{1+\alpha\sin s}\right) e^{inu} du\,. \tag{167}$$

Since $|J_n(n\alpha)|^2 = |J_{-n}(-n\alpha)|^2$, Parceval's theorem implies

$$\sum_{n=1}^{\infty} n^4 |J_n(n\alpha)|^2 = \frac{1}{4\pi}\int_0^{2\pi}\left|\frac{d^2}{du^2}\left(\frac{1}{1+\alpha\sin s}\right)\right|^2 du$$
$$= \frac{27\alpha^6 + 472\alpha^4 + 592\alpha^2 + 64)\alpha^2}{256(1-\alpha^2)^{13/2}}\,. \tag{168}$$

Then, we perform the θ-integral with $\alpha = \sqrt{1-a_0^{-2}}\sin\theta$. We find the energy emitted in one cycle as

$$E^{(1)\mathrm{nf}}_{\mathrm{circular}} = \frac{g_{ae}^2\hbar^2}{8m^2}\left(\frac{1}{3}a_0^8 - \frac{3}{5}a_0^6 + \frac{4}{15}a_0^4\right)\omega_0^3\,. \tag{169}$$

B. The spin-flip case

For the spin-flip case, the amplitude in the first line of eq. (53) reads

$$\mathcal{A}_{(\mathbf{p},\mathbf{k},\mp,\pm)} = \frac{g_{ae}}{2m}\int\left[k(n_z \pm i n_x) - \frac{k\cdot p + mk}{m(p_0+m)}(p_z \pm i p_x)\right] e^{\mp i\omega_0 t} e^{ik\cdot x}\, d\tau\,. \tag{170}$$

Since $n_z \pm i n_x = \sin\theta\, e^{\pm i\varphi}$ and $p_z \pm i p_x = m\sqrt{a_0^2-1}(-\sin\omega_0 t \pm i\cos\omega_0 t) = \pm i m\sqrt{a_0^2-1}e^{\pm i\omega_0 t}$, we find

$$\mathcal{A}_{(\mathbf{p},\mathbf{k},\mp,\pm)} = \frac{g_{ae}}{2m}\int\left[k\sin\theta e^{\mp i(\omega_0 t - \varphi)} \mp i\sqrt{a_0^2-1}\frac{k\cdot p + mk}{p_0+m}\right] e^{ik\cdot x}\, d\tau\,. \tag{171}$$

In our case, $p_0 = m a_0$ is constant, and

$$\int k\cdot p\, e^{ik\cdot x} d\tau = -im\int \frac{d}{d\tau}e^{ik\cdot x}\, d\tau = 0\,, \tag{172}$$

by periodicity. Thus, we find

$$\mathcal{A}_{(\mathbf{p},\mathbf{k},\mp,\pm)} = \mp i\frac{g_{ae}k}{2m}\int\left[\sqrt{\frac{a_0-1}{a_0+1}} \pm i\sin\theta\, e^{\mp i(\omega_0 t-\varphi)}\right] e^{ik\cdot x}\, d\tau\,. \tag{173}$$

Then, we can make the following identification in eq. (152):

$$\int_0^\infty dk\int d\Omega g(k,\Omega) = \frac{\hbar g_{ae}^2}{4m^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}k^2\,, \tag{174}$$

and

$$\left[\sqrt{\frac{a_0-1}{a_0+1}} \pm i\sin\theta e^{\mp i(\omega_0 t-\varphi)}\right]\frac{dt}{d\xi} = \sum_{n=-\infty}^{\infty} c_n^{\pm} e^{-in\omega_0\xi}\,, \tag{175}$$

where $dt/d\xi$ is given by eq. (160).

Proceeding as in the spin-non-flip case, we find

$$c_n^{\pm} = \frac{e^{in\varphi}}{2\pi a_0}\int_0^{2\pi}\left[\sqrt{\frac{a_0-1}{a_0+1}}e^{in(s-\beta\sin\theta\cos s)} \pm i\sin\theta e^{i(n\mp 1)s - \beta\sin\theta\cos s}\right] ds\,. \tag{176}$$

This integral is evaluated using eq. (162) again, and we obtain

$$a_0 c_n^{\pm} e^{-in\varphi} = e^{-i\frac{n\pi}{2}}\sqrt{\frac{a_0-1}{a_0+1}}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right) \pm i\sin\theta e^{-i(n\mp 1)\pi/2}J_{n\mp 1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\,. \tag{177}$$

Then, we obtain the rate of emission as

$$R^{\mathrm{f},\pm}_{\mathrm{em}} = \frac{\hbar g_{ae}^2\omega_0^3}{16\pi m^2}\sum_{n=1}^{\infty}\frac{n^3}{a_0^2}\int_0^\pi\left|\sqrt{\frac{a_0-1}{a_0+1}}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right) - \sin\theta J_{n\mp 1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2 \sin\theta\, d\theta\,, \tag{178}$$

and the power is

$$S_{\rm em}^{\rm f,\pm} = \frac{\hbar^2 g_{ae}^2 \omega_0^4}{16\pi m^2} \sum_{n=1}^{\infty} \frac{n^4}{a_0^2} \int_0^{\pi} \left| \sqrt{\frac{a_0-1}{a_0+1}} J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right) - \sin\theta J_{n\mp 1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2 \sin\theta\, d\theta\,. \tag{179}$$

Parceval's theorem can be used to find the average of the up-to-down and down-to-up spin-flip powers of emission because, although $|c_n^{(\pm)}| \neq |c_{-n}^{(\pm)}|$, we have $|c_{-n}^{(+)}| = |c_n^{(-)}|$. That is,

$$\sum_{n=1}^{\infty} n^4 (|c_n^{(+)}|^2 + |c_n^{(-)}|^2) = \sum_{n=-\infty}^{\infty} n^4 |c_n^{(+)}|^2\,. \tag{180}$$

Thus, Parceval's theorem implies

$$Y := \sum_{n=1}^{\infty} n^4 (|c_n^{(+)}|^2 + |c_n^{(-)}|^2) = \frac{1}{4\pi}\int_0^{2\pi} \left| \frac{d^2}{du^2}\left\{ \frac{1}{1+\alpha\sin s}\left[\sqrt{\frac{a_0-1}{a_0+1}} + i\sin\theta e^{-is}\right]\right\}\right|^2 (1+\alpha\sin s) ds\,. \tag{181}$$

This integral can be evaluated as

$$Y = \frac{(27\alpha^6 + 472\alpha^4 + 592\alpha^2 + 64)}{256(1-\alpha^2)^{13/2}} \left(\alpha^2 \frac{a_0-1}{a_0+1} - 2\alpha\sqrt{\frac{a_0-1}{a_0+1}}\sin\theta\right) - \frac{45\alpha^8 + 560\alpha^6 - 480\alpha^4 - 1152\alpha^2 - 128}{256(1-\alpha^2)^{13/2}} \sin^2\theta\,. \tag{182}$$

For $\alpha = \sqrt{1-a_0^{-2}}\sin\theta$ we find

$$\alpha^2 \frac{a_0-1}{a_0+1} - 2\alpha\sqrt{\frac{a_0-1}{a_0+1}}\sin\theta = -\alpha^2\,. \tag{183}$$

Hence

$$Y = -\frac{27\alpha^8 + 472\alpha^6 + 592\alpha^4 + 64\alpha^2}{256(1-\alpha^2)^{13/2}} - \frac{45\alpha^8 + 560\alpha^6 - 480\alpha^4 - 1152\alpha^2 - 128}{256(1-\alpha^2)^{13/2}} \sin^2\theta\,. \tag{184}$$

Then, by evaluating the angular integral in eq. (179), we find the spin-averaged axion energy emitted in one cycle as

$$E_{\rm circular}^{(1)\mathbf{f},\mathbf{av}} = \frac{\hbar^2 g_{ae}^2}{8m^2}\left(\frac{23}{3}a_0^8 - \frac{47}{5}a_0^6 + \frac{12}{5}a_0^4\right)\omega_0^4\,. \tag{185}$$

We find $E_{\rm circular}^{(1)\rm nf} + E_{\rm circular}^{(1)\mathbf{f},\mathbf{av}} = E_{\rm circular}^{(1)}$, where $E_{\rm circular}^{(1)}$ is given by eq. (150), as expected.

The ratio between power and rate (i.e. dividing eq. (165) by eq. (164) and eq. (179) by eq. (178)) can be used as a function of a_0 to find the typical axion energy. We find numerically that, in the spin-non-flip case, the ratio scales as $\sim 3a_0^3\omega_0$. In the spin-flip case, the ratio scales as $\sim 3.7a_0^3\omega_0$ and $\sim 2a_0^3\omega_0$ for initial spin in the positive or negative y-direction, respectively. We also find numerically that the largest part of the emitted energy comes from the spin-flip case for initial spin in the positive y-direction. We can therefore take the typical axion energy to be $\sim 4a_0^3\omega_0$. Although this result holds for a constant magnetic field only, we expect the same to be qualitatively true for axion emission from an electron accelerated by two laser fields, as described in the main text.

[1] G. D. R. Martin, *Classical and Quantum Radiation Reaction*, Ph.D. thesis, York U., England, Dept. Math. (2007), arXiv:0805.0666 [gr-qc].

[2] L. H. Thomas, The motion of the spinning electron, Nature **117**, 514 (1926).

[3] V. Bargmann, L. Michel, and V. L. Telegdi, Precession of the polarization of particles moving in a homogeneous electromagnetic field, Phys. Rev. Lett. **2**, 435 (1959).

[4] If we do not make the approximation $m_a \approx 0$, where m_a is the axion mass, then $k_\perp^2$ here is replaced by $k_\perp^2 + m_a^2/\hbar^2$.

[5] A. Higuchi and P. J. Walker, Quantum corrections to the Larmor radiation formula in scalar electrodynamics, Phys. Rev. D **80**, 105019 (2009), arXiv:0908.2723 [hep-th].

[6] A. Higuchi and G. D. R. Martin, Radiation reaction on charged particles in three-dimensional motion in classical and quantum electrodynamics, Phys. Rev. D **73**, 025019 (2006), arXiv:quant-ph/0510043.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products - eighth edition* (Academic Press, 2014).