

# Disjoint paths in tournaments

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### **Abstract**

Given  $k$  pairs of vertices  $(s_i, t_i)$  ( $1 \leq i \leq k$ ) of a digraph  $G$ , how can we test whether there exist  $k$  vertex-disjoint directed paths from  $s_i$  to  $t_i$  for  $1 \leq i \leq k$ ? This is NP-complete in general digraphs, even for  $k = 2$  [2], but for  $k = 2$  there is a polynomial-time algorithm when  $G$  is a tournament (or more generally, a semicomplete digraph), due to Bang-Jensen and Thomassen [1]. Here we prove that for all fixed  $k$  there is a polynomial-time algorithm to solve the problem when  $G$  is semicomplete.

# 1 Introduction

Let  $s_1, t_1, \dots, s_k, t_k$  be vertices of a graph or digraph  $G$ . The  $k$  *vertex-disjoint paths problem* is to determine whether there exist vertex-disjoint paths  $P_1, \dots, P_k$  (directed paths, in the case of a digraph) such that  $P_i$  is from  $s_i$  to  $t_i$  for  $1 \leq i \leq k$ . For undirected graphs, this problem is solvable in polynomial time for all fixed  $k$ ; this was one of the highlights of the Graph Minors project of Robertson and the third author [4]. The directed version is a natural and important question, but it was shown by Fortune, Hopcroft and Wyllie [2] that, without further restrictions on the input  $G$ , this problem is NP-complete for digraphs, even for  $k = 2$ . This motivates the study of subclasses of digraphs for which the problem is polynomial-time solvable.

In this paper, all graphs and digraphs are finite, and without loops or parallel edges; thus if  $u, v$  are distinct vertices of a digraph then there do not exist two edges both from  $u$  to  $v$ , although there may be edges  $uv$  and  $vu$ . Also, by a “path” in a digraph we always mean a directed path. A digraph is a *tournament* if for every pair of distinct vertices  $u, v$ , exactly one of  $uv, vu$  is an edge; and a digraph is *semicomplete* if for all distinct  $u, v$ , at least one of  $uv, vu$  is an edge. It was shown by Bang-Jensen and Thomassen [1] that

- the  $k$  vertex-disjoint paths problem (for digraphs) is NP-complete if  $k$  is not fixed, even when  $G$  is a tournament;
- the two vertex-disjoint paths problem is solvable in polynomial time if  $G$  is semicomplete.

We shall show:

**1.1** *For all fixed  $k \geq 0$ , the  $k$  vertex-disjoint paths problem is solvable in polynomial time if  $G$  is semicomplete.*

In fact we will prove a result for a wider class of digraphs, that we define next. Let  $P$  be a path of a digraph  $G$ , with vertices  $v_1, \dots, v_n$  in order. We say  $P$  is *minimal* if  $j \leq i + 1$  for every edge  $v_i v_j$  of  $G$  with  $1 \leq i, j \leq n$ . Let  $d \geq 1$ ; we say that a digraph  $G$  is *d-path-dominant* if for every minimal path  $P$  of  $G$  with  $d$  vertices, every vertex of  $G$  either belongs to  $V(P)$  or has an out-neighbour in  $V(P)$  or has an in-neighbour in  $V(P)$ . Thus a digraph is 1-path-dominant if and only if it is semicomplete; and 2-path-dominant if and only if its underlying simple graph is complete multipartite. We will show:

**1.2** *For all fixed  $d, k \geq 1$ , the  $k$  vertex-disjoint paths problem is solvable in polynomial time if  $G$  is  $d$ -path-dominant.*

We stress here that we are looking for vertex-disjoint paths. One can ask the same for edge-disjoint paths, and that question has also been recently solved for tournaments, and indeed for digraphs with bounded independence number [3], but the solution is completely different. We do not know a polynomial-time algorithm for the two vertex-disjoint paths problem for digraphs with independence number two.

But we can extend 1.2 in a different way:

**1.3** *For all  $d, k \geq 1$ , there is a polynomial-time algorithm as follows:*

- **Input:** Vertices  $s_1, t_1, \dots, s_k, t_k$  of a  $d$ -path-dominant digraph  $G$ , and integers  $x_1, \dots, x_k \geq 1$ .
- **Output:** Decides whether there exist pairwise vertex-disjoint directed paths  $P_1, \dots, P_k$  of  $G$  such that for  $1 \leq i \leq k$ ,  $P_i$  is from  $s_i$  to  $t_i$  and has at most  $x_i$  vertices.

Let  $s_1, t_1, \dots, s_k, t_k$  be vertices of a digraph  $G$ . We call  $(G, s_1, t_1, \dots, s_k, t_k)$  a *problem instance*. A *linkage* in a digraph  $G$  is a sequence  $L = (P_i : 1 \leq i \leq k)$  of vertex-disjoint paths, and  $L$  is a linkage for a problem instance  $(G, s_1, t_1, \dots, s_k, t_k)$  if  $P_i$  is from  $s_i$  to  $t_i$  for each  $i$ . (With a slight abuse of notation, we shall call  $k$  the “cardinality” of  $L$ , and  $P_1, \dots, P_k$  its “members”. Also, every subsequence of  $(P_i : 1 \leq i \leq k)$  is a linkage  $L'$ , and we say  $L$  “includes”  $L'$ .) If  $x = (x_1, \dots, x_k)$  is a  $k$ -tuple of integers, we say a linkage  $(P_i : 1 \leq i \leq k)$  is an  $x$ -linkage if each  $P_i$  has  $x_i$  vertices. We say a  $k$ -tuple of integers  $x = (x_1, \dots, x_k)$  is a *quality* of  $(G, s_1, t_1, \dots, s_k, t_k)$  if there is an  $x$ -linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . If  $x = (x_1, \dots, x_k)$  and  $y = (y_1, \dots, y_k)$ , we say  $x \leq y$  if  $x_i \leq y_i$  for  $1 \leq i \leq k$ ; and  $x < y$  if  $x \leq y$  and  $x \neq y$ . We say a quality  $x$  of  $(G, s_1, t_1, \dots, s_k, t_k)$  is *key* if there is no quality  $y$  with  $y < x$ . Our main result is the following:

**1.4** For all  $d, k$ , there is an algorithm as follows:

- **Input:** A problem instance  $(G, s_1, t_1, \dots, s_k, t_k)$  where  $G$  is  $d$ -path-dominant.
- **Output:** The set of all key qualities of  $(G, s_1, t_1, \dots, s_k, t_k)$ .
- **Running time:**  $O(n^t)$  where  $t = 6k^2d(k + d) + 13k$ .

The idea of the algorithm for 1.2 is easily described. We define an auxiliary digraph  $H$  with two special vertices  $s_0, t_0$ , and prove that there is a path in  $H$  from  $s_0$  to  $t_0$  if and only if there is a linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . Thus to solve the problem of 1.2 it suffices to construct  $H$  in polynomial time. The more general question of 1.4 is solved similarly, by assigning appropriate weights to the edges of  $H$ .

Recently we have been able to extend 1.1 to a more general class of digraphs, namely the digraphs whose vertex set can be partitioned into a bounded number of subsets such that each subset induces a semicomplete digraph. The proof is by a modification of the method of this paper, but it is considerably more difficult and not included here.

## 2 A useful enumeration

If  $P$  is a path of a digraph  $G$ , its *length* is  $|E(P)|$  (every path has at least one vertex); and  $s(P), t(P)$  denote the first and last vertices of  $P$ , respectively. If  $F$  is a subdigraph of  $G$ , a vertex  $v$  of  $G \setminus V(F)$  is *F-outward* if no vertex of  $F$  is adjacent from  $v$  in  $G$ ; and *F-inward* if no vertex of  $F$  is adjacent to  $v$  in  $G$ . If  $F$  is a digraph and  $v \in V(F)$ ,  $F \setminus v$  denotes the digraph obtained from  $F$  by deleting  $v$ ; if  $X \subseteq V(F)$ ,  $F|X$  denotes the subdigraph of  $F$  induced on  $X$ ; and  $F \setminus X$  denotes the subdigraph obtained by deleting all vertices in  $X$ .

Now let  $L = (P_i : 1 \leq i \leq k)$  be a linkage in  $G$ . We define  $V(L)$  to be  $V(P_1) \cup \dots \cup V(P_k)$ . A vertex  $v$  is an *internal vertex* of  $L$  if  $v \in V(L)$ , and  $v$  is not an end of any member of  $L$ . A linkage  $L$  is *internally disjoint* from a linkage  $L'$  if no internal vertex of  $L$  belongs to  $V(L')$  (note that this does not imply that  $L'$  is internally disjoint from  $L$ ); and we say that  $L, L'$  are *internally disjoint*

if each of them is internally disjoint from the other (and thus all vertices in  $V(L) \cap V(L')$  must be ends of paths in both  $L$  and  $L'$ )

Let  $Q, R$  be vertex-disjoint paths of a digraph  $G$ . A *planar  $(Q, R)$ -matching* is a linkage  $(M_j : 1 \leq j \leq n)$  for some  $n \geq 0$ , such that

- $M_1, \dots, M_n$  each have either two or three vertices;
- $s(M_1), \dots, s(M_n)$  are vertices of  $Q$ , in order in  $Q$ ; and
- $t(M_1), \dots, t(M_n)$  are vertices of  $R$ , in order in  $R$ .

Fix  $d, k \geq 1$ , and let  $L = (P_1, \dots, P_k)$  be a linkage in a  $d$ -path-dominant digraph  $G$ . A subset  $B \subseteq V(L)$  is said to be *acceptable* (for  $L$ ) if

- for  $1 \leq j \leq k$ , if  $uv$  is an edge of  $P_j$  and  $v \in B$  then  $u \in B$  (and so  $Q_j = P_j|B$  and  $R_j = P_j|(V(G) \setminus B)$  are paths if they are non-null);
- for  $1 \leq i, j \leq k$ , there is no planar  $(Q_i, R_j)$ -matching of cardinality  $(k-1)d + k^2 + 2$  internally disjoint from  $L$ .

Thus  $\emptyset$  and  $V(L)$  are acceptable.

**2.1** *Let  $d \geq 1$ , let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance, where  $G$  is  $d$ -path-dominant, let  $x$  be a key quality, and let  $L = (P_1, \dots, P_k)$  be an  $x$ -linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . Suppose that  $B \subseteq V(L)$  is acceptable for  $L$  and  $B \neq V(L)$ . Then there exists  $v \in V(L) \setminus B$  such that  $B \cup \{v\}$  is acceptable for  $L$ .*

**Proof.** Let  $A = V(G) \setminus B$ . For  $1 \leq j \leq k$ , let  $Q_j = P_j|B$  and  $R_j = P_j|A$ . Let  $q_j, r_j$  be the last vertex of  $Q_j$  and the first vertex of  $R_j$ , respectively (if they exist).

(1) *For  $1 \leq j \leq k$ ,  $P_j$  is a minimal path of  $G$ . In particular, the only edge of  $G$  from  $V(Q_j)$  to  $V(R_j)$  (if there is one) is  $q_j r_j$ . Moreover, every three-vertex path from  $V(Q_j)$  to  $V(R_j)$  with internal vertex in  $V(G) \setminus V(L)$  uses at least one of  $q_j, r_j$ . Consequently, there is no planar  $(Q_j, R_j)$ -matching of cardinality three internally disjoint from  $L$ .*

For suppose there is an edge  $uv$  of  $G$  such that  $u, v \in V(P_j)$  and  $u$  is before  $v$  in  $P_j$ , and there is at least one vertex of  $P_j$  between  $u$  and  $v$ . If we delete from  $P_j$  the vertices of  $P_j$  strictly between  $u$  and  $v$ , and add the edge  $uv$ , we obtain a path from  $s_j$  to  $t_j$  disjoint from every member of  $L$  except  $P_j$ , and with strictly fewer vertices than  $P_j$ , contradicting that  $x$  is key. Thus  $P_j$  is induced. Similarly there is no three-vertex path from  $V(Q_j)$  to  $V(R_j)$  with internal vertex in  $V(G) \setminus V(L)$  containing neither of  $q_j, r_j$ . The final assertion follows. This proves (1).

From (1), the theorem holds if  $k = 1$ , so we may assume that  $k \geq 2$ .

(2) *We may assume that for all  $i \in \{1, \dots, k\}$ , if  $R_i$  is non-null then for some  $j \in \{1, \dots, k\}$  with  $j \neq i$ , there is a planar  $(Q_i, R_j \setminus r_j)$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ .*

For suppose that some  $i$  does not satisfy the statement of (2). Thus  $R_i$  is non-null, and there is no  $j$  as in (2). Since  $R_i$  is non-null, it follows that  $r_i$  exists. We may assume that  $B \cup \{r_i\}$  is not acceptable. Consequently, one of the two conditions in the definition of “acceptable” is not satisfied by  $B \cup \{r_i\}$ . The first is satisfied since  $r_i$  is the first vertex of  $R_i$ . Thus the second is false, and so for some  $i', j \in \{1, \dots, k\}$ , there is a planar  $(P_{i'}|(B \cup \{r_i\}), P_j|(A \setminus \{r_i\}))$ -matching of cardinality  $(k-1)d + k^2 + 2$  internally disjoint from  $L$ . Since there is no planar  $(Q_{i'}, R_j)$ -matching of cardinality  $(k-1)d + k^2 + 2$  internally disjoint from  $L$ , and  $P_j|(A \setminus \{r_i\})$  is a subpath of  $R_j$ , it follows that  $P_{i'}|(B \cup \{r_i\}) \neq Q_{i'}$ , and so  $i' = i$ . Since only one vertex of  $P_i|(B \cup \{r_i\})$  does not belong to  $Q_i$ , it follows that there is a planar  $(Q_i, R_j \setminus r_j)$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ . Since  $(k-1)d + k^2 \geq 4$  (because  $k \geq 2$ ), (1) implies that  $j \neq i$ . This proves (2).

(3) *We may assume that for some  $p \geq 2$ , and for all  $i$  with  $1 \leq i < p$ , there is a planar  $(Q_i, R_{i+1} \setminus r_{i+1})$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ , and there is a planar  $(Q_p, R_1 \setminus r_1)$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ .*

For by hypothesis, there exists  $i \in \{1, \dots, k\}$  such that  $R_i$  is non-null. By repeated application of (2), there exist distinct  $h_1, \dots, h_p \in \{1, \dots, k\}$  such that for  $1 \leq i \leq p$  there is a planar  $(Q_{h_i}, R_{h_{i+1}} \setminus r_{h_{i+1}})$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ , where  $h_{p+1} = h_1$ ; and  $p \geq 2$  by (1). Without loss of generality, we may assume that  $h_i = i$  for  $1 \leq i \leq p$ . This proves (3).

Let us say a planar  $(Q, R)$ -matching is  $s$ -spaced if no subpath of  $Q$  with at most  $s$  vertices meets more than one member of the matching, and no subpath of  $R$  with at most  $s$  vertices meets more than one member of the matching.

(4) *We may assume that for some  $p \geq 2$ , and for all  $i$  with  $1 \leq i < p$ , there is a planar  $(Q_i, R_{i+1} \setminus r_{i+1})$ -matching  $L_i$ , and there is a planar  $(Q_p, R_1 \setminus r_1)$ -matching  $L_p$ , such that*

- $L_1, \dots, L_p$  all have cardinality  $k$ ;
- they are pairwise internally disjoint;
- each of  $L_1, \dots, L_p$  is internally disjoint from  $L$ ; and
- each of  $L_1, \dots, L_p$  is  $(d+1)$ -spaced.

For let  $L'_i$  be a planar  $(Q_i, R_{i+1} \setminus r_{i+1})$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ , for  $1 \leq i < p$ , and let  $L'_p$  be a planar  $(Q_p, R_1 \setminus r_1)$ -matching of cardinality  $(k-1)d + k^2$  internally disjoint from  $L$ . We choose  $L_i \subseteq L'_i$  inductively. Suppose that for some  $h < p$ , we have chosen  $L_1, \dots, L_h$ , such that

- $L_1, \dots, L_h$  all have cardinality  $k$ ;
- they are pairwise internally disjoint;
- each of  $L_1, \dots, L_h$  is internally disjoint from  $L$ ; and
- each of  $L_1, \dots, L_h$  is  $(d+1)$ -spaced.

We define  $L_{h+1}$  as follows. The union of the sets of internal vertices of  $L_1, \dots, L_h$  has cardinality at most  $hk \leq k(k-1)$ , and so  $L'_{h+1}$  includes a planar  $(Q_{h+1}, R_{h+2} \setminus r_{h+2})$ -matching (or  $(Q_p, R_1 \setminus r_1)$ -matching, if  $h = p-1$ ) of cardinality  $(k-1)d + k^2 - k(k-1) = 1 + (k-1)(d+1)$ , internally disjoint from each of  $L_1, \dots, L_h$ . By ordering the members of this matching in their natural order, and taking only the  $i$ th terms, where  $i = 1, 1 + (d+1), 1 + 2(d+1) \dots$ , we obtain a  $(d+1)$ -spaced matching of cardinality  $k$ . Let this be  $L_{h+1}$ . This completes the inductive definition of  $L_1, \dots, L_p$ , and so proves (4).

For  $1 \leq i \leq p$ , let  $L_i = \{M_i^1, \dots, M_i^k\}$ , numbered in order; thus, if  $q_i^h$  and  $r_{i+1}^h$  denote the first and last vertices of  $M_i^h$ , then  $q_i^1, \dots, q_i^k$  are distinct and in order in  $Q_i$ , and  $r_{i+1}^1, r_{i+1}^2, \dots, r_{i+1}^k$  are distinct and in order in  $R_{i+1}$  (or in  $R_1$  if  $i = p$ ). For  $1 \leq i \leq p$  and  $2 \leq h \leq k$ , let  $Q_i^h$  be the subpath of  $P_i$  with  $d$  vertices and with last vertex  $q_i^h$ . (Thus  $q_i^{h-1}$  does not belong to  $Q_i^h$  since  $L_i$  is  $d$ -spaced, and indeed  $(d+1)$ -spaced.) Since  $P_i$  and hence  $Q_i^h$  is a minimal path of  $G$ , and  $G$  is  $d$ -path-dominant, it follows that for  $1 \leq i \leq p$  and  $2 \leq h \leq k$ ,  $r_i^{h-1}$  is adjacent to or from some vertex  $v$  of  $Q_i^h$ . Since  $r_i^{h-1} \neq r_i$ , (1) implies that  $r_i^{h-1}$  is not adjacent from any vertex of  $Q_i^h$ ; and so there is a path  $R_i^{h-1}$  from  $r_i^{h-1}$  to  $q_i^h$  of length at most  $d$ , such that all its internal vertices belong to  $Q_i^h$ . For  $1 \leq i \leq p$ , and  $1 \leq h < k$ , let  $S_i^h$  be the path

$$q_i^h - M_i^h - r_{i+1}^h - R_{i+1}^h - q_{i+1}^{h+1},$$

or

$$q_p^h - M_p^h - r_1^h - R_1^h - q_1^{h+1}$$

if  $i = p$ ; then  $S_i^h$  is a path from  $q_i^h$  to  $q_{i+1}^{h+1}$  (or to  $q_1^{h+1}$  if  $i = p$ ), of length at most  $d+2$ . Thus (reading subscripts modulo  $p$ ) concatenating  $S_i^1, S_{i+1}^2, \dots, S_{i+p-2}^{p-1}$  and  $M_{i-1}^p$  gives a path  $T'_i$  from  $q_i^1$  to  $r_i^p$  of length at most  $(p-1)(d+2) + 2$ . The subpath  $T_i$  of  $P_i$  from  $q_i^1$  to  $r_i^p$  has length at least  $(p+k-2)(d+1) + 2$ , since  $L_{i-1}, L_i$  are  $(d+1)$ -spaced and  $r_i$  is different from  $r_i^1$ ; and since  $p+k-2 \geq 2(p-1)$  and  $d+1 > (d+2)/2$ , it follows that  $T_i$  has length strictly greater than that of  $T'_i$ . Let  $P'_i$  be obtained from  $P_i$  by replacing the subpath  $T_i$  by  $T'_i$ , for  $1 \leq i \leq p$ , and let  $P_{i'} = P_i$  for  $p+1 \leq i \leq k$ . Then  $\{P'_1, \dots, P'_k\}$  is a linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ , contradicting that  $x$  is key. This proves 2.1. ■

We deduce:

**2.2** Let  $d \geq 1$ , let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance where  $G$  is  $d$ -path-dominant, let  $x$  be a key quality, and let  $L = (P_1, \dots, P_k)$  be an  $x$ -linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . Let  $c = (k-1)d + k^2 + 2$ . Then there is an enumeration  $(v_1, \dots, v_n)$  of  $V(L)$ , such that

- for  $1 \leq h \leq k$  and  $1 \leq p, q \leq n$ , if  $v_p v_q$  is an edge of  $P_h$  then  $p < q$ ;
- for  $1 \leq h, i \leq k$  and  $1 \leq p \leq n-1$ , and every  $cd$ -vertex subpath  $Q$  of  $P_h \setminus \{v_1, \dots, v_p\}$ , and every  $cd$ -vertex subpath  $R$  of  $P_i \setminus \{v_{p+1}, \dots, v_n\}$ , there are at most  $c(2k+1)$  vertices of  $G$  that are both  $Q$ -outward and  $R$ -inward.

**Proof.** Since  $\emptyset$  is acceptable for  $L$ , by repeated application of 2.1 implies that there is an enumeration  $(v_1, \dots, v_n)$  of  $V(L)$ , such that  $\{v_1, \dots, v_p\}$  is acceptable for  $0 \leq p \leq n$ . We claim that this enumeration satisfies the theorem. For certainly the first bullet holds; we must check the second.

Thus, let  $1 \leq p \leq n$ , and let  $B = \{v_1, \dots, v_p\}$  and  $A = \{v_{p+1}, \dots, v_n\}$ . For  $1 \leq h \leq k$ , let  $Q_h = P_h|B$  and  $R_h = P_h|A$ . Now let  $1 \leq h, i \leq k$ , and let  $Q, R$  be  $cd$ -vertex subpaths of  $Q_h, R_i$  respectively. Let  $X$  be the set of all vertices of  $G$  that are both  $Q$ -outward and  $R$ -inward. We must show that  $|X| \leq c(2k+1)$ .

(1) *If  $x_1, \dots, x_c \in X$  are distinct, then there exist  $y_1, \dots, y_c \in V(Q)$ , distinct and in order in  $Q$ , such that  $y_j x_j$  is an edge for  $1 \leq j \leq c$ .*

For  $Q$  has  $cd$  vertices; let its vertices be  $q_1, \dots, q_{cd}$  in order. Let  $1 \leq j \leq c$ . The subpath of  $Q$  induced on  $\{q_s : (j-1)d < s \leq jd\}$  has  $d$  vertices, and since  $Q$  is a minimal path of  $G$  and  $G$  is  $d$ -path-dominant, and  $X \cap V(Q) = \emptyset$ , it follows that  $x_j$  is in- or out-adjacent to a vertex of this subpath, say  $y_j$ . Since  $x_j \in X$  and hence is  $Q$ -outwards, it follows that  $x_j y_j$  is not an edge, and so  $y_j x_j$  is an edge. But then  $y_1, \dots, y_c$  satisfy (1). This proves (1).

(2) *The sets  $X \setminus V(L)$ ,  $X \cap V(Q_g)$  ( $1 \leq g \leq k$ ) and  $X \cap V(R_g)$  ( $1 \leq g \leq k$ ) all have cardinality at most  $c-1$ , and hence  $|X| \leq (2k+1)(c-1)$ .*

For suppose that there exist distinct  $x_1, \dots, x_c \in X \setminus V(L)$ . By (1) there exist distinct  $y_1, \dots, y_c \in V(Q)$ , in order in  $Q$ , such that  $y_j x_j$  is an edge for  $1 \leq j \leq c$ ; and similarly there exist  $z_1, \dots, z_c \in V(R)$ , in order in  $R$ , such that  $x_j z_j$  is an edge for  $1 \leq j \leq c$ . But then the  $c$  paths  $y_j x_j z_j$  ( $1 \leq j \leq c$ ) form a planar  $(Q_h, R_i)$ -matching of cardinality  $c$ , internally disjoint from  $L$ , contradicting that  $\{v_1, \dots, v_p\}$  is acceptable. Thus  $|X \setminus V(L)| \leq c-1$ . Now suppose that for some  $g \in \{1, \dots, k\}$ , there exist distinct  $x_1, \dots, x_c$  in  $X \cap V(R_g)$ , numbered in order in  $R_g$ . Choose  $y_1, \dots, y_c$  as in (1); then the paths  $y_j x_j$  ( $1 \leq j \leq c$ ) form a planar  $(Q_h, R_g)$ -matching of cardinality  $c$ , internally disjoint from  $L$ , contradicting that  $\{v_1, \dots, v_p\}$  is acceptable. Thus  $|X \cap V(R_g)| \leq c-1$ , and similarly  $|X \cap V(Q_g)| \leq c-1$ , for  $1 \leq g \leq k$ . This proves (2).

From (2), the theorem follows. ■

### 3 Confusion and the auxiliary digraph

Let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance, and let  $L = (M_1, \dots, M_k)$  be a linkage in  $G$  (not necessarily a linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ ). Let  $A(L)$  be the set of all vertices in  $V(G) \setminus V(L)$  that are  $M_j \setminus t(M_j)$ -inward for some  $j \in \{1, \dots, k\}$  such that  $t(M_j) \neq t_j$  and let  $B(L)$  be the set of all vertices in  $V(G) \setminus V(L)$  that are  $M_j \setminus s(M_j)$ -outward for some  $j \in \{1, \dots, k\}$  such that  $s(M_j) \neq s_j$ . We call  $|A(L) \cap B(L)|$  the *confusion* of  $L$ ; and it is helpful to keep the confusion small, as we shall see.

A  $(k, m, c)$ -rail in a problem instance  $(G, s_1, t_1, \dots, s_k, t_k)$  is a triple  $(L, X, Y)$ , where

- $L$  is a linkage in  $G$  consisting of  $k$  paths  $(M_1, \dots, M_k)$  (but not necessarily a linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ );
- for  $1 \leq j \leq k$ ,  $M_j$  has at most  $2m$  vertices, and if it has fewer than  $2m$  vertices then  $M_j$  either has first vertex  $s_j$  or last vertex  $t_j$ ;



- $L$  has confusion at most  $c$ ;
- $X, Y$  are disjoint subsets of  $V(G) \setminus V(L)$ ; and
- $X \subseteq A(L)$ ,  $Y \subseteq B(L)$ , and  $X \cup Y = A(L) \cup B(L)$ .

**3.1** For all  $k, m, c \geq 0$ , if  $(G, s_1, t_1, \dots, s_k, t_k)$  is a problem instance and  $G$  has  $n$  vertices then there are at most  $2^c n^{2km} (2km)^k$   $(k, m, c)$ -rails in  $(G, s_1, t_1, \dots, s_k, t_k)$ . Moreover, for all fixed  $k, m, c \geq 0$ , there is an algorithm which, with input a problem instance  $(G, s_1, t_1, \dots, s_k, t_k)$ , finds all its  $(k, m, c)$ -rails in time  $O(n^{2km+1})$ , where  $n = |V(G)|$ .

**Proof.** First, if  $L$  is a linkage with  $k$  paths each with at most  $2m$  vertices, then  $|V(L)| \leq 2km$ , and so the number of such linkages is at most  $n^{2km} (2km)^k$ , as is easily seen. Now fix a linkage  $L$  satisfying the first two bullets in the definition of  $(k, m, c)$ -rail; let us count the number of pairs  $(X, Y)$  such that  $(L, X, Y)$  is a  $(k, m, c)$ -rail. There are none unless  $|A(L) \cap B(L)| \leq c$ ; and in that case, there are at most  $2^c$  possibilities for the pair  $(X, Y)$ , since  $X$  consists of  $A(L) \setminus B(L)$  together with some subset of  $A(L) \cap B(L)$ , and  $Y = (A(L) \cup B(L)) \setminus X$ .

For the algorithm, we first find all linkages  $L$  with  $k$  paths each with at most  $2m$  vertices, by examining all ordered  $2km$ -tuples of distinct vertices of  $G$ . For each such  $L$ , we check whether it satisfies the first three bullets in the definition of  $(k, m, c)$ -rail (this takes time  $O(n)$ ); if not we discard it and otherwise we partition  $A(L) \cap B(L)$  into two subsets in all possible ways, and output the corresponding  $(k, m, c)$ -rails. The result follows.  $\blacksquare$

Let  $(L, X, Y)$  and  $(L', X', Y')$  be distinct  $(k, m, c)$ -rails in  $G$ , and let  $L = (P_1, \dots, P_k)$  and  $L' = (P'_1, \dots, P'_k)$ . We write  $(L, X, Y) \rightarrow (L', X', Y')$  if the following hold:

- for  $1 \leq i \leq k$ ,  $P_i \cup P'_i$  is a path from the first vertex of  $P_i$  to the last vertex of  $P'_i$ ;
- for  $1 \leq i \leq k$ ,  $V(P'_i) \subseteq V(P_i) \cup X$ , and  $V(P_i) \subseteq V(P'_i) \cup Y'$ ; and
- $X' \subseteq X$ , and  $Y \subseteq Y'$ .

Let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance, and let  $\mathcal{T}$  be the set of all  $(k, m, c)$ -rails in  $(G, s_1, t_1, \dots, s_k, t_k)$ . Take two new vertices  $s_0, t_0$ , and let us define a digraph  $H$  with vertex set  $\mathcal{T} \cup \{s_0, t_0\}$  as follows. Let  $u, v \in V(H)$ . If  $u, v \in \mathcal{T}$  are distinct, then  $uv \in E(H)$  if and only if  $u \rightarrow v$ . If  $u = s_0$  and  $v \in \mathcal{T}$ , let  $v = (L, X, Y)$  where  $L = (M_1, \dots, M_k)$ ; then  $uv \in E(H)$  if and only if  $M_j$  has first vertex  $s_j$  for all  $j \in \{1, \dots, k\}$ . Similarly, if  $u \in \mathcal{T}$  and  $v = t_0$ , let  $u = (L, X, Y)$  where  $L = (M_1, \dots, M_k)$ ; then  $uv \in E(H)$  if and only if  $M_j$  has last vertex  $t_j$  for all  $j \in \{1, \dots, k\}$ . This defines  $H$ . We call  $H$  the  $(k, m, c)$ -tracker of  $(G, s_1, t_1, \dots, s_k, t_k)$ .

We shall show that with an appropriate choice of  $m, c$ , when  $G$  is  $d$ -path-dominant we can reduce our problems about linkages for  $(G, s_1, t_1, \dots, s_k, t_k)$  to problems about paths from  $s_0$  to  $t_0$  in the  $(k, m, c)$ -tracker. Let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance, let  $(P_1, \dots, P_k)$  be a linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ , and let  $P$  be a path from  $s_0$  to  $t_0$  in the  $(k, m, c)$ -tracker. Let  $P$  have vertices

$$s_0, (L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n), t_0$$

in order, and let  $L_p = (M_{p,1}, \dots, M_{p,k})$  for  $1 \leq p \leq n$ . We say that  $P$  traces  $(P_1, \dots, P_k)$  if  $P_j$  is the union of  $M_{1,j}, \dots, M_{n,j}$  for all  $j \in \{1, \dots, k\}$ .

**3.2** Let  $k, m, c \geq 0$  be integers, and let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance, with  $(k, m, c)$ -tracker  $H$ . Every path in  $H$  from  $s_0$  to  $t_0$  traces some linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ .

**Proof.** Let  $P$  be a path of  $H$ , with vertices

$$s_0, (L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n), t_0$$

in order, and let  $L_p = (M_{p,1}, \dots, M_{p,k})$  for  $1 \leq p \leq n$ . For  $1 \leq p \leq n$  and  $1 \leq j \leq k$ , let  $P_{p,j}$  be the union of  $M_{1,j}, \dots, M_{p,j}$ .

(1) For  $1 \leq p \leq n$  and  $1 \leq j \leq k$ , every vertex of  $P_{p,j}$  belongs to  $Y_p \cup V(M_{p,j})$ .

We prove this by induction on  $p$ . If  $p = 1$  the claim is true, since then  $P_{1,j} = M_{1,j}$ . We assume then that  $p > 1$  and the result holds for  $p - 1$ . Let  $v \in V(P_{p,j})$ . If  $v \in V(M_{p,j})$  then the claim is true, so we assume not. Since  $v \in V(P_{p,j})$ , and  $P_{p,j} = P_{p-1,j} \cup M_{p,j}$ , it follows that  $v \in V(P_{p-1,j})$ , and so from the inductive hypothesis,  $v \in Y_{p-1} \cup V(M_{p-1,j})$ . But since  $(L_{p-1}, X_{p-1}, Y_{p-1}) \rightarrow (L_p, X_p, Y_p)$ , we deduce that  $Y_{p-1} \subseteq Y_p$ , and  $V(M_{p-1,j}) \subseteq V(M_{p,j}) \cup Y_p$ , and so  $v \in V(M_{p,j}) \cup Y_p$ . This proves (1).

(2) For  $1 \leq p \leq n$  and  $1 \leq j \leq k$ ,  $P_{p,j}$  is a path from  $s_j$  to the last vertex of  $M_{p,j}$ .

The claim holds if  $p = 1$ ; so we assume that  $p > 1$  and the claim holds for  $p - 1$ . Thus  $P_{p-1,j}$  is a path from  $s_j$  to the last vertex of  $M_{p-1,j}$ ; and also,  $M_{p-1,j} \cup M_{p,j}$  is a path, from the first vertex of  $M_{p-1,j}$  to the last vertex of  $M_{p,j}$ , since  $(L_{p-1}, X_{p-1}, Y_{p-1}) \rightarrow (L_p, X_p, Y_p)$ . We claim that every vertex  $v$  that belongs to both of  $P_{p-1,j}, M_{p,j}$  also belongs to  $M_{p-1,j}$ . For suppose not; then by (1),  $v \in Y_{p-1}$  since  $v \in V(P_{p-1,j}) \setminus V(M_{p-1,j})$ , and  $v \in X_{p-1}$ , since  $v \in V(M_{p,j}) \setminus V(M_{p-1,j})$ . This is impossible since  $X_{p-1} \cap Y_{p-1} = \emptyset$ . This proves that every vertex that belongs to both of  $P_{p-1,j}, M_{p,j}$  also belongs to  $M_{p-1,j}$ . Since  $M_{p-1,j}$  is non-null, we deduce that  $P_{p-1,j} \cup M_{p,j}$  is a path from  $s_j$  to the last vertex of  $M_{p,j}$ . This proves (2).

(3) For  $1 \leq p \leq n$ , the paths  $P_{p,1}, \dots, P_{p,k}$  are pairwise vertex-disjoint.

For again we proceed by induction on  $p$ , and may assume that  $p > 1$  and the result holds for  $p - 1$ . Suppose that  $v$  belongs to two of the paths  $P_{p,1}, \dots, P_{p,k}$ , say to  $P_{p,1}$  and  $P_{p,2}$ . From the inductive hypothesis,  $v$  does not belong to both of  $P_{p-1,1}$  and  $P_{p-1,2}$ , so we may assume that  $v \in V(M_{p,1})$ . Now  $v \notin V(M_{p,2})$ , because  $L_p$  is a linkage, and so  $v \in V(P_{p-1,2})$ . From (1) we deduce that  $v \in Y_{p-1} \cup V(M_{p-1,2})$ . But  $Y_{p-1} \subseteq Y_p$ , and  $V(M_{p-1,2}) \setminus V(M_{p,2}) \subseteq Y_p$ , and so  $v \in Y_p$ ; but  $Y_p \cap V(L_p) = \emptyset$  since  $(L_p, X_p, Y_p)$  is a  $(k, m, c)$ -rail, a contradiction. This proves (3).

From (2) and (3) we deduce that  $(P_{n,1}, \dots, P_{n,k})$  is a linkage  $L$  for  $(G, s_1, t_1, \dots, s_k, t_k)$ . Thus  $P$  traces  $L$ . This proves 3.2. ■

The next result is a kind of partial converse; but we have to choose  $m, c$  carefully, and we need  $G$  to be  $d$ -path-dominant, and the proof only works for linkages that realize a key quality.

**3.3** Let  $d, k \geq 1$  be integers, and let

$$\begin{aligned} c &= ((k-1)d + k^2 + 2)(2k+1)k^2 \\ m &= ((k-1)d + k^2 + 2)d + 1. \end{aligned}$$

Let  $(G, s_1, t_1, \dots, s_k, t_k)$  be a problem instance where  $G$  is  $d$ -path-dominant, let  $x$  be a key quality, and let  $(P_1, \dots, P_k)$  be an  $x$ -linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . Let  $H$  be the  $(k, m, c)$ -tracker of  $(G, s_1, t_1, \dots, s_k, t_k)$ . Then there is a path in  $H$  from  $s_0$  to  $t_0$  tracing  $(P_1, \dots, P_k)$ .

**Proof.** Let  $L = (P_1, \dots, P_k)$ . By 2.2, there is an enumeration  $(v_1, \dots, v_n)$  of  $V(L)$ , such that

- for  $1 \leq j \leq k$  and  $1 \leq p, q \leq n$ , if  $v_p v_q$  is an edge of  $P_j$  then  $p < q$ ;
- for  $1 \leq i, j \leq k$  and  $1 \leq p \leq n-1$ , and every  $(m-1)$ -vertex subpath  $Q$  of  $P_i \setminus \{v_1, \dots, v_p\}$ , and every  $(m-1)$ -vertex subpath  $R$  of  $P_j \setminus \{v_{p+1}, \dots, v_n\}$ , there are at most  $((k-1)d + k^2 + 2)(2k+1)$  vertices of  $G$  that are both  $Q$ -outward and  $R$ -inward.

For each  $v \in V(L)$ , let  $\phi(v) = i$  where  $v = v_i$ ; thus  $\phi$  is a bijection from  $V(L)$  onto  $\{1, \dots, n\}$ .

For all  $p \in \{0, \dots, n\}$  and all  $j \in \{1, \dots, k\}$ , if  $\phi(s_j) \leq p$ , let  $Q_{p,j}$  be the maximal subpath of  $P_j$  with at most  $m$  vertices and with last vertex  $v_q$ , where  $q \leq p$  is maximum such that  $v_q \in V(P_j)$ . If  $\phi(s_j) > p$ , let  $Q_{p,j}$  be the null digraph. Similarly, if  $\phi(t_j) > p$ , let  $R_{p,j}$  be the maximal subpath of  $P_j$  with at most  $m$  vertices and with first vertex  $v_r$ , where  $r > p$  is minimum such that  $v_r \in V(P_j)$ . If  $\phi(t_j) \leq p$ , let  $R_{p,j}$  be the null digraph. Thus, if  $Q_{p,j}, R_{p,j}$  are both non-null, then  $t(Q_{p,j})$  and  $s(R_{p,j})$  are consecutive in  $P_j$ .

For all  $p \in \{0, \dots, n\}$  and all  $j \in \{1, \dots, k\}$ , let  $M_{p,j}$  be the subpath of  $P_j$  defined as follows: if both  $Q_{p,j}, R_{p,j}$  are non-null,  $M_{p,j}$  consists of  $Q_{p,j} \cup R_{p,j}$  together with the edge of  $P_j$  from  $t(Q_{p,j})$  to  $s(R_{p,j})$ , while if one of  $Q_{p,j}, R_{p,j}$  is null,  $M_{p,j}$  equals the other (not both can be null). We see that, for all  $p, j$ ,  $M_{p,j}$  has at most  $2m$  vertices; and either it has exactly  $2m$ , or its first vertex is  $s_j$ , or its last vertex is  $t_j$ . For all  $p \in \{0, \dots, n\}$ , let  $L_p$  be the linkage  $(M_{p,1}, \dots, M_{p,k})$ .

(1) For all  $p \in \{0, \dots, n\}$ ,  $L_p$  has confusion at most  $c$ .

Let  $v \in A(L_p) \cap B(L_p)$ , where  $A(L_p), B(L_p)$  are as in the definition of confusion. Thus there exists  $j \in \{1, \dots, k\}$  such that  $v$  is  $M_{p,j} \setminus t(M_{p,j})$ -inward and  $t(M_{p,j}) \neq t_j$ . Since  $t(M_{p,j}) \neq t_j$ , it follows from the choice of  $R_{p,j}$  that  $R_{p,j}$  has exactly  $m$  vertices. Moreover,  $v$  is  $R_{p,j} \setminus t(R_{p,j})$ -inward, since  $v$  is  $M_{p,j} \setminus t(M_{p,j})$ -inward. Similarly, there exists  $i \in \{1, \dots, k\}$  such that  $v$  is  $Q_{p,i} \setminus s(Q_{p,i})$ -outward and  $Q_{p,i}$  has  $m$  vertices. For each choice of  $i, j \in \{1, \dots, k\}$ , there are at most  $((k-1)d + k^2 + 2)(2k+1)$  vertices that are both  $Q_{p,i} \setminus s(Q_{p,i})$ -outward and  $R_{p,j} \setminus t(R_{p,j})$ -inward, from the choice of the enumeration  $(v_1, \dots, v_n)$ . Consequently in total there are only  $c$  possibilities for  $v$ , and so  $|A(L_p) \cap B(L_p)| \leq c$ . This proves (1).

(2) For  $0 \leq p \leq n$  and each  $v \in V(L) \setminus V(L_p)$ , if  $\phi(v) > p$  then  $v \in A(L_p)$ , and if  $\phi(v) \leq p$  then  $v \in B(L_p)$ .

For let  $v \in V(P_j)$  say. Assume first that  $\phi(v) > p$ . Since  $v \notin V(L_p)$ , it follows that  $M_{p,j}$  does not have last vertex  $t_j$ ; and since  $x$  is key,  $v$  is not adjacent from any vertex in  $M_{p,j}$  except possibly

$t(M_{p,j})$ . Consequently  $v$  is  $M_{p,j} \setminus t(M_{p,j})$ -inward, and hence belongs to  $A(L_p)$ . Similarly, if  $\phi(v) \leq p$  then  $v \in B(L_p)$ . This proves (2).

For all  $p \in \{0, \dots, n\}$ , define  $X_p, Y_p$  as follows:

$$\begin{aligned} X_p &= \{v \in V(L) \setminus V(L_p) : \phi(v) > p\} \cup (A(L_p) \setminus B(L_p)) \\ Y_p &= (A(L_p) \cup B(L_p)) \setminus X_p. \end{aligned}$$

(3) For all  $p \in \{0, \dots, n\}$ ,  $(L_p, X_p, Y_p)$  is a  $(k, m, c)$ -rail.

From (1), it suffices to check that

- $X_p, Y_p$  are disjoint subsets of  $V(G) \setminus V(L_p)$ ;
- $X_p \subseteq A(L_p), Y_p \subseteq B(L_p)$ ; and
- $X_p \cup Y_p = A(L_p) \cup B(L_p)$ .

Certainly they are disjoint, and have union  $A(L_p) \cup B(L_p)$ . Moreover, from (2),  $X_p \subseteq A(L_p)$ . It remains to show that  $Y_p \subseteq B(L_p)$ . Let  $v \in Y_p$ . Thus  $v \in A(L_p) \cup B(L_p)$ ; and  $v \notin A(L_p) \setminus B(L_p)$ , since  $v \notin X_p$ . Consequently  $v \in B(L_p)$  as required. This proves (3).

(4) For all  $p \in \{0, \dots, n-1\}$ , and all  $j \in \{1, \dots, k\}$ ,  $M_{p,j} \cup M_{p+1,j}$  is a path from the first vertex of  $M_{p,j}$  to the last vertex of  $M_{p+1,j}$ .

For  $M_{p,j}, M_{p+1,j}$  are both subpaths of  $P_j$ , and we may assume they are distinct, and so  $v_{p+1} \in V(P_j)$ . Hence, since  $m > 0$ ,  $v_{p+1}$  is the first vertex of  $R_{p,j}$ , and the last vertex of  $Q_{p+1,j}$ ; and so  $M_{p,j} \cup M_{p+1,j}$  is a path. Moreover, it follows from the definition of the paths  $M_{p,j}$  that  $M_{p,j} \cup M_{p+1,j}$  is a path from the first vertex of  $M_{p,j}$  to the last vertex of  $M_{p+1,j}$ . This proves (4).

(5) For all  $p \in \{0, \dots, n-1\}$ , and all  $j \in \{1, \dots, k\}$ ,  $A(L_{p+1}) \subseteq A(L_p) \cup V(L)$  and  $B(L_p) \subseteq B(L_{p+1}) \cup V(L)$ .

For let  $v \in A(L_{p+1})$ . We need to prove that  $v \in A(L_p) \cup V(L)$ , and so we may assume that  $v \notin V(L)$ . Choose  $j$  with  $1 \leq j \leq k$  such that  $v$  is  $M_{p+1,j} \setminus t(M_{p+1,j})$ -inward and  $t(M_{p+1,j}) \neq t_j$ . Consequently  $t(M_{p,j}) \neq t_j$ , and so if  $v$  is  $M_{p,j} \setminus t(M_{p,j})$ -inward then  $v \in A(L_p)$  as required, so we may assume that  $v$  is adjacent from some vertex of  $M_{p,j}$ . In particular,  $M_{p,j} \neq M_{p+1,j}$  and so  $v_{p+1} \in V(P_j)$ , and  $v_{p+1} = s(R_{p,j}) = t(Q_{p+1,j})$ . Moreover, since  $s(M_{p,j})$  is the only vertex of  $M_{p,j}$  that may not belong to  $M_{p+1,j}$ , we deduce that  $s(M_{p,j})$  is adjacent to  $v$ , and  $s(M_{p,j})$  does not belong to  $M_{p+1,j}$ . Consequently  $s(M_{p+1,j}) \neq s_j$ , and so  $Q_{p+1,j}$  has  $m$  vertices. Since  $v$  is  $M_{p+1,j} \setminus t(M_{p+1,j})$ -inward, and  $G$  is  $d$ -path-dominant, and  $M_{p+1,j} \setminus t(M_{p+1,j})$  is a minimal path of  $G$ , and it has  $m-1 \geq d+2$  vertices, there is a subpath of  $M_{p+1,j} \setminus t(M_{p+1,j})$  with  $d$  vertices, not containing the first or second vertex of  $M_{p+1,j} \setminus t(M_{p+1,j})$ ; and so  $v$  is adjacent to some vertex  $w$  of  $M_{p+1,j} \setminus t(M_{p+1,j})$  different from its first and second vertices. But  $v$  is adjacent from  $u$ , so by replacing the subpath of  $P_j$  between  $u$  and  $w$  by the path  $u-v-w$ , we contradict that  $x$  is key. This proves that  $v \in A(L_p)$ , and so  $A(L_{p+1}) \subseteq A(L_p) \cup V(L)$ . Similarly  $B(L_p) \subseteq B(L_{p+1}) \cup V(L)$ . This proves (5).

(6) For all  $p \in \{0, \dots, n-1\}$ ,  $X_{p+1} \subseteq X_p$ , and  $Y_p \subseteq Y_{p+1}$ .

Let  $v \in X_{p+1}$ . Suppose first that  $v \notin V(L)$ . Then  $v \in A(L_{p+1}) \setminus B(L_{p+1})$ . By (5),  $v \in A(L_p) \setminus B(L_p)$ , and so  $v \in X_p$  as required. Thus we may assume that  $v \in V(L)$ . Since  $v \in X_{p+1}$ , it follows that either  $\phi(v) > p+1$ , or  $v \notin B(L_{p+1})$ . If  $\phi(v) > p+1$ , then since  $v \notin V(L_{p+1})$ , it follows that  $v \notin V(L_p)$ , and hence  $v \in X_p$  from the definition of  $X_p$ . Thus we may assume that  $\phi(v) \leq p+1$  and  $v \notin B(L_{p+1})$ , contrary to (2). This proves that  $X_{p+1} \subseteq X_p$ .

For the second inclusion, let  $v \in Y_p$ . Suppose first that  $v \notin V(L)$ . Then  $v \in B(L_p)$ ; and so  $v \in B(L_{p+1})$  by (5), and hence  $v \in Y_{p+1}$  as required. Thus we may assume that  $v \in V(L)$ . Since  $v \in Y_p$ , it follows that  $\phi(v) \leq p$ . Now  $v \notin V(L_p)$ , and therefore  $v \notin V(L_{p+1})$ . But  $\phi(v) \leq p+1$ , and so by (2),  $v \in B(L_{p+1})$ , and consequently  $v \notin X_{p+1}$ . Thus  $v \in Y_{p+1}$ , as required. This proves that  $Y_p \subseteq Y_{p+1}$ , and so proves (6).

(7) For all  $p \in \{0, \dots, n-1\}$ , and all  $j \in \{1, \dots, k\}$ ,  $V(P_{p+1,j}) \subseteq V(P_{p,j}) \cup X_p$ , and  $V(P_{p,j}) \subseteq V(P_{p+1,j}) \cup Y_{p+1}$ .

To prove the first assertion, let  $v \in V(P_{p+1,j}) \setminus V(P_{p,j})$ . It follows that  $\phi(v) > p$ ; but then  $v \in X_p$  from the definition of  $X_p$ . For the second assertion, let  $v \in V(P_{p,j}) \setminus V(P_{p+1,j})$ ; then  $\phi(v) \leq p+1$ , and so  $v \in B(L_{p+1})$  by (2). Consequently  $v \notin X_{p+1}$ , and so  $v \in Y_{p+1}$  as required. This proves (7).

(8) For all  $p \in \{0, \dots, n-1\}$ ,  $(L_p, X_p, Y_p) \rightarrow (L_{p+1}, X_{p+1}, Y_{p+1})$ .

This is immediate from (4), (6) and (7).

Now  $(L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n)$  are not necessarily all distinct. But we have:

(9) For all  $p, r$  with  $0 \leq p \leq r \leq n$ , if  $(L_p, X_p, Y_p) = (L_r, X_r, Y_r)$ , then  $(L_p, X_p, Y_p) = (L_q, X_q, Y_q)$  for all  $q$  with  $p \leq q \leq r$ .

For (6) implies that  $X_q \subseteq X_p$ , and  $X_r \subseteq X_q$ , and so  $X_p = X_q$ , and similarly  $Y_p = Y_q$ . If some vertex  $v$  belongs to  $V(L_q) \setminus V(L_p)$ , then by (7) and (6),  $v \in X_p = X_q$ , a contradiction. Similarly, if  $v \in V(L_p) \setminus V(L_q)$  then  $v \in Y_q = Y_p$ , a contradiction. This proves (9).

(10) For all  $j \in \{1, \dots, k\}$ ,  $M_{0,j}$  has first vertex  $s_j$ , and  $M_{n,j}$  has last vertex  $t_j$ .

This follows from the definitions of  $M_{0,j}$  and  $M_{n,j}$ .

We recall that  $H$  is the  $(k, m, c)$ -tracker, with two special vertices  $s_0, t_0$ . Now (10) implies that  $s_0$  is adjacent to  $(L_1, X_1, Y_1)$  in  $H$ , and  $(L_n, X_n, Y_n)$  is adjacent to  $t_0$ . From (8) and (9), there is a subsequence of the sequence

$$s_0, (L_1, X_1, Y_1), \dots, (L_n, X_n, Y_n), t_0,$$

which lists the vertex set in order of a path of  $H$  from  $s_0$  to  $t_0$ . By 3.2, this path traces some linkage  $L'$  for  $(G, s_1, t_1, \dots, s_k, t_k)$ . But for all  $j \in \{1, \dots, k\}$ ,  $M_{0,j}, M_{1,j}, \dots, M_{n,j}$  are all subpaths of  $P_j$ ;

and since their union is a path from  $s_j$  to  $t_j$ , it follows that their union is  $P_j$ . Hence  $L' = L$ . This proves 3.3. ■

## 4 The algorithm

Next, we need a polynomial algorithm to solve a kind of vector-valued shortest path problem. If  $n \geq 0$  is an integer,  $K_n$  denotes the set of all  $k$ -tuples  $(x_1, \dots, x_k)$  of nonnegative integers such that  $x_1 + \dots + x_k \leq n$ .

**4.1** *There is an algorithm as follows:*

- **Input:** A digraph  $H$ , and distinct vertices  $s_0, t_0 \in V(H)$ ; an integer  $n \geq 0$ ; and for each edge  $e$  of  $H$ , a member  $l(e)$  of  $K_n$ .
- **Output:** The set of all minimal (under component-wise domination) vectors  $l(P)$ , over all paths  $P$  of  $H$  from  $s_0$  to  $t_0$ ; where for a path  $P$  with edge set  $\{e_1, \dots, e_p\}$ ,  $l(P) = l(e_1) + \dots + l(e_p)$ .
- **Running time:**  $O(n^k |V(H)| |E(H)|)$ .

**Proof.** Let  $Q_0(s_0) = \{(0, \dots, 0)\}$ , and let  $Q_0(v) = \emptyset$  for every other vertex  $v$  of  $D$ . Inductively, for  $1 \leq i \leq |V(H)|$ , let  $Q_i(v)$  be the set of minimal vectors in  $K_n$  that either belong to  $Q_{i-1}(v)$  or are expressible in the form  $l(e) + x$  for some edge  $e = uv$  of  $H$  and some  $x \in Q_{i-1}(u)$ .

Now here is an algorithm for the problem:

- For  $i = 1, \dots, |V(H)|$  in turn, compute  $Q_i(v)$  for every  $v \in V(H)$ .
- Output  $Q_{|V(H)|}(t_0)$ .

It is easy to check that this output is correct, and we leave it to the reader. To compute  $Q_i(v)$  at the  $i$ th step takes time  $O(n^k d^-(v))$ , where  $d^-(v)$  is the in-degree of  $v$  in  $H$  (since  $K_n$  has at most  $(n+1)^k$  members), and so the  $i$ th step in total takes time  $O(n^k |E(H)|)$ . Thus the running time is  $O(n^k |V(H)| |E(H)|)$ . ■

Finally, we can give the main algorithm, 1.4, which we restate.

**4.2** *For all  $d, k \geq 1$ , there is an algorithm as follows:*

- **Input:** A problem instance  $(G, s_1, t_1, \dots, s_k, t_k)$  where  $G$  is  $d$ -path-dominant.
- **Output:** The set of all key qualities of  $(G, s_1, t_1, \dots, s_k, t_k)$ .
- **Running time:**  $O(n^t)$  where  $t = 6k^2d(k+d) + 13k$ .

**Proof.** Here is the algorithm.

- Compute the  $(k, m, c)$ -tracker  $H$ , where

$$\begin{aligned} c &= ((k-1)d + k^2 + 2)(2k+1)k^2 \\ m &= ((k-1)d + k^2 + 2)d + 1. \end{aligned}$$

- For each edge  $e = uv$  of  $H$ , define  $l(e)$  as follows:
  - if  $u = s_0$  and  $v = (L, X, Y)$  where  $L = (M_1, \dots, M_k)$ , let  $l(e) = (|V(M_1)|, \dots, |V(M_k)|)$ ;
  - if  $u = (L, X, Y)$  where  $L = (M_1, \dots, M_k)$ , and  $v = (L', X', Y')$  where  $L' = (M'_1, \dots, M'_k)$ , let  $l(e) = (|V(M'_1) \setminus V(M_1)|, \dots, |V(M'_k) \setminus V(M_k)|)$ ;
  - if  $v = t_0$  let  $l(e) = (0, \dots, 0)$ .
- Run the algorithm of 4.1 with input  $H, s_0, t_0, l$ .
- Output its output.

To see its correctness, we must check that every key quality is in the output, and everything in the output is a key quality. We show first that every vector in the output is a quality. For let  $x$  be in the output, and let  $P$  be a path in  $H$  from  $s_0$  to  $t_0$  with  $l(P) = x$ . By 3.2,  $P$  traces some linkage  $L = (P_1, \dots, P_k)$  for  $(G, s_1, t_1, \dots, s_k, t_k)$ ; and so  $(|V(P_1)|, |V(P_2)|, \dots, |V(P_k)|) = l(P) = x$ . Hence  $x$  is a quality.

Next, we show that every key quality is in the output. For let  $x$  be a key quality. Let  $L$  be an  $x$ -linkage for  $(G, s_1, t_1, \dots, s_k, t_k)$ . By 3.3, there is a path  $P$  of  $H$  from  $s_0$  to  $t_0$  tracing  $L$ ; and hence  $l(P) = x$  (where  $l(P)$  is defined as in the statement of 4.1). Thus the output of 4.1 contains a vector dominated by  $x$ . But  $x$  does not dominate any other quality, since it is key; and since every member of the output is a quality, it follows that  $x$  belongs to the output.

Third, we show that every member of the output is key. For let  $x$  be in the output, and suppose it is not key. Hence  $x$  dominates some other quality, and hence dominates some other key quality  $y$  say. Consequently  $y$  is in the output. But no two members of the output dominate one another, a contradiction. This proves that every member of the output is key, and so completes the proof that the output of the algorithm is as claimed.

Finally, for the running time: by 3.1, we can find all  $(k, m, c)$ -rails in time  $O(n^{2km+1})$ ; and since there are at most  $O(n^{2km})$  of them (by 3.1), we can compute  $H$  and the function  $l$  in time  $O(n^{4km})$ . Then running 4.1 takes time  $O(n^k |V(H)|^3)$ , and hence time at most  $O(n^{6km+k})$ . Thus the total running time is  $O(n^{6km+k})$ . Since  $m = ((k-1)d + k^2 + 2)d + 1$ , the running time is  $O(n^t)$  where

$$t = 6k(k-1)d^2 + 6k(k^2 + 2)d + 7k = 6k^2d^2 + 6k^3d + 12kd + 7k - 6kd^2 \leq 6k^2d(k+d) + 13k$$

as claimed. This proves 4.2. ■

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