An Additive-Noise Approximation to Keller–Segel–Dean–Kawasaki Dynamics

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Für Helmut und Gertrude „Tante Gerti“ Hohlstein

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original. The results presented in Chapters 2 & 3 have been obtained in a collaboration with Avi Mayorcas. Chapter 2 is based on our preprint arXiv:2207.10711, Chapter 3 is based on arXiv:2410.17022. No part of the thesis has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. No generative artificial intelligence tools were used in the conception or preparation of this work.

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Contents

Abstract

The theory of fluctuating hydrodynamics aims to describe density fluctuations of interacting particle systems as so-called Dean–Kawasaki stochastic partial differential equations (SPDEs). However, those Dean–Kawasaki equations are ill-posed and recent focus has shifted towards finding well-posed approximations that retain the statistical properties of the particle system. In this thesis we consider the fluctuating hydrodynamics of a system in which particles interact with one another through the potential gradient induced by the Green function of the Laplacian (Keller–Segel dynamics).

We propose an additive-noise approximation, which is given by a non-linear, non-local, parabolic-elliptic SPDE with a heterogeneous space-time noise, and show that it retains the same law of large numbers and central limit theorem as (conjectured for) the particle system. We further consider the large deviation principle and show that the approximation error lies in the skeleton equation that drives the rate function.

The results above depend on the relative scaling between two parameters that appear in our equation: the noise intensity, which governs the amplitude of our fluctuations, and the correlation length, which describes their spatial correlations. To loosen the required relationship between the noise intensity and the correlation length, one needs to assume that the approximation lies in a space of lower regularity. Consequently, we consider three different scenarios:

- The *regular* setting, in which the approximation converges in a Bessel potential space of regularity greater than $-1/2$, at the expense of a restrictive relative scaling.

- The *rough* setting, in which the approximation converges in a Hölder–Besov space of regularity less than -1, under a general relative scaling.

- The *renormalised* setting, in which one can make sense of the approximation even for zero correlation lengths, if one is willing to subtract a formally infinite counterterm.

Depending on the specific setting, we use different tools to analyse the approximation. For example, to deduce a large deviation principle in the regular setting

we rely on the weak-convergence approach, whereas in the rough setting we use the paracontrolled decomposition and a generalisation of Freidlin and Wentzell's theory due to Hairer and Weber.

In comparing these different approaches we demonstrate how the range of suitable scaling relations, and the regularity of the approximation, are determined by the asymptotic behaviour of certain stochastic objects for vanishing correlation lengths.

1
Introduction

Contents

1.1 Motivation and Main Results

1.1.1 Motivation

Non-trivial macroscopic structures commonly emerge in biological systems through a process known as *chemotaxis* where individuals respond with directed movement to an external chemical gradient. Examples of chemotaxis include the formation of pseudoplasmodia in the life cycle of *Dictyostelium discoideum* [BS47], the emergence of aggregates of *Escherichia coli* [BLB98], the onset of *amoebic dysentery* caused by *Entamoeba histolytica* [ZAI06], the collectiveness of neural crest groups during embryo development [SM16] and the immune evasion, invasion, angiogenesis and dissemination of tumours [CS93; RCP11]. An excellent survey on models of chemotaxis in biology is given in [Pai19].

Consider a chemotactic population in which immortal, non-reproducing particles attract one another. To model such a system it is common to consider

$$\begin{cases} (\partial_t - \Delta)\rho = -\nabla \cdot (\rho \nabla c), \\ (\tau\partial_t - \Delta)c = \rho, \end{cases} \tag{1.1.1}$$

where individuals, given by their density ρ, disperse a chemical c and move towards higher chemical concentrations through advection. The parameter $\tau \in [0, \infty)$ captures the separation of time scales between ρ and c, and upon rescaling, is given by the inverse diffusivity of the chemical. This model was introduced by Keller and Segel in [KS70] and is known as the parabolic–parabolic Keller–Segel model.

Assuming in (1.1.1) that the chemical diffuses much faster than the particles, we may set $\tau = 0$ so that c solves the Poisson equation $-\Delta c = \rho$, whose solution is, up to harmonic functions, explicitly given by $c = K * \rho$, where K denotes the Green function of the Laplacian on the full space $\mathbb{R}^d$,

$$K(x) := \begin{cases} -\dfrac{1}{2}|x|, & d = 1, \\ -\dfrac{1}{2\pi}\log|x|, & d = 2, \\ \dfrac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}\dfrac{1}{|x|^{d-2}}, & d \geq 3, \end{cases} \qquad \nabla K(x) = -\frac{\Gamma(d/2+1)}{d\pi^{d/2}}\frac{x}{|x|^d}. \tag{1.1.2}$$

It is further customary to parametrize the strength of the interaction by the *chemotactic sensitivity* $\chi \in \mathbb{R}$, which results in the PDE

$$(\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho(\nabla K * \rho)) \tag{1.1.3}$$

known as the parabolic–elliptic (Patlak–) Keller–Segel model, named after Keller and Segel [KS70] and Patlak [Pat53], who considered a similar equation in the context of random walks with persistence of direction and external bias. The population is auto-attractive (gravitational) for $\chi > 0$, non-interactive (diffusive) for $\chi = 0$ and auto-repulsive (Coulombian) for $\chi < 0$.

Acting on the population is a competition between the dispersal of mass and the formation of aggregates. In one space-dimension, the diffusion will always dominate aggregation so that solutions to (1.1.3) exist globally in time [BT96]. However, in higher dimensions, one observes the following dichotomy. Assume that the initial condition is non-negative and of total mass 1. In the two-dimensional model, the equation (1.1.3) is globally well-posed if the chemotactic sensitivity satisfies $\chi < 8\pi$, see [JL92; BDP06; Nag11]. On the other hand if $\chi > 8\pi$, all solutions with initial data of finite variance blow up in finite time (i.e. they leave the space of measures with $L^1(\mathbb{R}^2)$-density, see [BDP06, Sec. 2.1].) The critical case $\chi = 8\pi$ is more subtle and has been studied in [BCM08; LNY13; NS13]. Similar (although less simple) thresholds also exist in higher dimensions, see [CPZ04]. For results on the parabolic-parabolic model (1.1.1), we refer to [OY01; HP04] (one dimension), [CC08; Miz13] (two dimensions, $\chi \leq 8\pi$), [HV97; BCD11b] (two dimensions, $\chi > 8\pi$), and [CP08] (higher dimensions). For an overview of the mathematical literature dedicated to the analysis of Keller–Segel models, we refer to [Hor03; Hor04], [Per04], [HP09] and [Bil18].

The Keller–Segel equation (1.1.3) is a specific member of the general family of McKean–Vlasov equations

$$(\partial_t - \Delta)u = -\nabla \cdot (u(b * u)) \tag{1.1.4}$$

where $b\colon \mathbb{R}^d \to \mathbb{R}^d$ denotes the interaction. To each McKean–Vlasov equation (1.1.4) we can associate a stochastic particle approximation given by weakly interacting, mean-field, overdamped Langevin diffusions: Let $N \in \mathbb{N}$ be the finite population size, we define an individual-based model through the system

$$\mathrm{d}Y_t^i = \sqrt{2}\,\mathrm{d}B_t^i + \frac{1}{N}\sum_{j=1,\, j\neq i}^{N} b(Y_t^i - Y_t^j)\,\mathrm{d}t, \qquad i = 1, \ldots, N, \tag{1.1.5}$$

where $(B^i)_{i=1}^N$ denotes a sequence of independent, $d \in \mathbb{N}$-dimensional Brownian motions. In particular, for appropriate initial data and interactions b that are assumed to be bounded and Lipschitz continuous, one can show propagation of chaos [Mél96; Szn91], that is, the phenomenon in which the empirical measures $t \mapsto \nu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}$ converge in distribution as $N \to \infty$ to a solution of (1.1.4). Individual-based models as in (1.1.5) are ubiquitous in modelling and applications, see [FHM14; Ste00; Hel+21] for further examples.

To find a particle approximation to the Keller–Segel dynamics (1.1.3), we compare (1.1.3) with (1.1.4), choose for b the potential gradient $\chi\nabla K$ and consider

$$\mathrm{d}X_t^i = \sqrt{2}\,\mathrm{d}B_t^i + \frac{\chi}{N}\sum_{j=1,\, j\neq i}^{N} \nabla K(X_t^i - X_t^j)\,\mathrm{d}t, \qquad i = 1,\dots,N, \tag{1.1.6}$$

for appropriate initial data. The mathematical challenge is then due to the singularity of ∇K combined with its non-incompressibility. This combination places (1.1.6) outside the classical mean-field theory as presented in [Szn91] as well as recent extensions due to [FHM14]. In the one-dimensional case, where ∇K is bounded but discontinuous, propagation of chaos was derived in [BT96]. In two dimensions, Fournier, Jourdain and Tardy [FJ17; Tar22] show the tightness of the empirical measures $t \mapsto \mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ for $\chi \le 8\pi$ and Bresch–Jabin–Wang [BJW19b; BJW19a; BJW23] establish propagation of chaos for $\chi < 8\pi$. For parabolic-parabolic models, interacting particle systems were introduced in [JTT18; FT23b].

The influence of finite-number fluctuations on chemotactic populations is non-trivial[1], yet we ought to consider fluctuations since in biological systems population sizes are far from infinite. For example, in *Dictyostelium discoideum*, each aggregation centre consists of 10^4–10^6 individuals [Bon09, p. 12 & pp. 61–62] and has a volume of 0.01–0.05 mm^3 [BKP53]. In contrast, the same volume of water contains 10^{17}–10^{18} molecules; hence to understand biological phenomena, as opposed to many physical ones, one needs to consider finite-number fluctuations.

One approach to non-equilibrium diffusive systems is *macroscopic fluctuation theory* [Ber+15], a physical theory whose starting point is the large deviations rate function for the particle system as $N \to \infty$, which it applies to predict phenomena such as non-equilibrium phase transitions and long-range correlations.

The large deviation principle (LDP) for (1.1.6) is, to our knowledge, still largely open, owing to the singularity of ∇K. In the one-dimensional, repulsive case ($\chi < 0$), Fontbona [Fon04] established an LDP for (1.1.6) by using hydrodynamics techniques. On the other hand, in the case of smooth interactions b, Dawson and Gärtner [DG87]

[1] For instance in two dimensions and $\chi \ge 8\pi$, Fournier and Tardy [FT21] showed that (1.1.6) exhibits a sequence of particle collisions before the solution ceases to exist.

proved an LDP for (1.1.5) in arbitrary dimensions by using a projective limit approach. Let us also mention the recent paper [Hoe+24] in which Hoeksema, Holding, Maurelli and Tse established an LDP for (1.1.5) with interactions $b \in L^p(\mathbb{R}^d)$ of integrability $p > d$ by extending the Varadhan integral lemma.

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of probability measures equipped with the weak topology and $\mathcal{P}_1(\mathbb{R}^d)$ be the space of probability measures of finite first moment. Assume that the initial data $(X_0^i)_{i=1}^N$ are i.i.d. with common, non-atomic law $X_0^i \sim \rho_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and let T be smaller than the blow-up time of (1.1.3) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$. The above-mentioned results [DG87; Fon04; Hoe+24] lead us to seek an LDP for the empirical measure μ^N of (1.1.6) in $C([0,T];\mathcal{P}(\mathbb{R}^d))$ with rate $N/2$ and good rate function $\mathscr{A}$, which after additional (formal) reformulations takes the form

$$\begin{aligned}\mathscr{A}(\rho) := \inf\Big\{ &\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^d)} : \\ &(\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho(\nabla K * \rho)) - \nabla\cdot(\sqrt{\rho}h),\ \rho|_{t=0} = \rho_0\Big\}.\end{aligned} \tag{1.1.7}$$

Another approach to predict the LDP of (1.1.6) builds on a formal physical theory known as *fluctuating hydrodynamics*, whose starting point is not the particle system but rather the associated *Dean–Kawasaki equation* [Dea96; Kaw94],

$$(\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho(\nabla K * \rho)) - \sqrt{2/N}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}), \tag{1.1.8}$$

where $\boldsymbol{\xi}$ is a vector-valued space-time white noise (see Section C.1). To motivate (1.1.8) and the *Dean–Kawasaki noise* $\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi})$ appearing therein, see Subsection A.1.1 for a derivation of (1.1.8) from the particle system (1.1.6) and Subsection A.1.2 for a derivation of (1.1.8) as a fluctuating gradient flow in Wasserstein space. The idea of fluctuating hydrodynamics is then to apply Freidlin and Wentzell's theory [FW12] to (1.1.8) in order to predict the LDP of the particle system (1.1.6) and the particular form (1.1.7) of its rate function $\mathscr{A}$.

A different motivation for (1.1.8) is given by the problem of simulating the particle system (1.1.6). A naïve simulation of (1.1.6) would require us to compute N^2 interactions for every time step due to the mean-field term, which is expensive for population sizes $N \gg 1$. On the other hand in the case of effectively infinite populations, fluctuations are negligible and one may consider the deterministic mean-field limit (1.1.3). This leaves open the case of large but finite populations $1 \ll N \ll \infty$. Here, (1.1.8) provides a mesoscopic model that is efficient to simulate but still retains finite-number fluctuations.

The principal issue with fluctuating hydrodynamics is that the stochastic PDE

$$(\partial_t - \Delta)u = -\nabla \cdot (u(b * u)) - \sqrt{2/N}\nabla \cdot (\sqrt{u}\boldsymbol{\xi}), \tag{1.1.9}$$

associated to (1.1.5) is ill-posed. Not only is it supercritical in the sense of [Hai14] (see Subsection A.2.2), it was also shown by Konarovskyi, Lehmann and von Renesse [KLvR19; KLvR20] that the only martingale solution to (1.1.9) with smooth interaction b is given by the particle system (1.1.5). They further showed that (1.1.9) is unstable: If we replace the noise intensity $2/N$ by an arbitrary $\varepsilon > 0$ or if we replace the initial condition by a measure that is not of the form $\frac{1}{N}\sum_{i=1}^{N}\delta_{y^i}$ with $y^i \in \mathbb{R}^d$ for every $i \in \{1, \dots, N\}$, then the martingale problem no longer admits a solution. Even though there exist singular interactions b for which the above dichotomy breaks down (see e.g. the references in [KLvR20]), we do not expect (1.1.8) to be well-posed either.

Hence, instead of exactly solving equations that are subject to Dean–Kawasaki noise, recent focus has shifted towards finding good approximations. In the context of non-linear diffusions (e.g. the porous medium equation), Fehrman, Gess and Dirr studied the fluctuating hydrodynamics associated to the zero range process [FG23] and the symmetric simple exclusion process [DFG20] by using their theory of stochastic kinetic solutions and established the same LDP as for the respective particle system. This was extended by Wang, Wu and Zhang [WWZ22; WZ22] to the Dean–Kawasaki equation (1.1.9) with interactions b satisfying a Ladyzhenskaya–Prodi–Serrin condition along with an integrability condition on $\nabla \cdot b$. All of the above works [FG23; DFG20; WWZ22; WZ22] assume Stratonovich (Dean–Kawasaki) noise instead of Itô (Dean–Kawasaki) noise. Cornalba, Fischer, Ingmanns, Raithel and Shardlow [CF23b; CS23; Cor+23; CF23a] considered (1.1.9) with smooth interactions b and Itô noise; and established weak-error estimates between numerical (finite difference, finite element, multilevel Monte Carlo) discretizations of the Dean–Kawasaki equation and the particle system. Djurdjevac, Kremp and Perkowski [DKP24] considered (1.1.9) without interactions ($b = 0$) and with Itô noise; and used a duality argument to establish weak-error estimates for an SPDE approximation.

The works mentioned above have made great progress in the rigorous interpretation of several Dean–Kawasaki equations. However, a major limitation is that none of these works address interactions that are singular beyond the Ladyzhenskaya–Prodi–Serrin condition and non-incompressible as in (1.1.2). Taking a first step in this direction, we introduce in the present thesis an additive-noise approximation to the Keller–Segel–Dean–Kawasaki dynamics (1.1.8).

1.1.2 Main Results

In this work we restrict[2] our attention to the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, on which the Keller–Segel–Dean–Kawasaki equation (1.1.8) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$ takes the form

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla\Phi_\rho) - \sqrt{2/N}\nabla \cdot (\sqrt{\rho}\boldsymbol{\xi}), & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad -\Delta\Phi_\rho = \rho - \langle \rho, 1\rangle_{L^2(\mathbb{T}^2)}, & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad\ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{1.1.10}$$

where $T > 0$ is a time horizon and $\boldsymbol{\xi}$ a vector-valued space-time white noise on $[0,T] \times \mathbb{T}^2$ (see (C.1.1) and Definition C.1.1). We consider an additive-noise approximation to (1.1.10) given by

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla \cdot (\rho\nabla\Phi_\rho) - \varepsilon^{1/2}\nabla \cdot (\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}^\delta), & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad -\Delta\Phi_\rho = \rho - \langle \rho, 1\rangle_{L^2(\mathbb{T}^2)}, & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad\ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{1.1.11}$$

where $\varepsilon > 0$ is the noise intensity, $\delta > 0$ the correlation length, $(\psi_\delta)_{\delta>0}$ a sequence of mollifiers at scale δ (cf. (1.2.7)), $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ and ρ_{det} the solution (cf. Section 3.2)

[2] In the theory of singular SPDEs it is common to consider bounded domains to exclude the possibility of infrared divergences. Here, tori are especially convenient due to the availability of the Fourier transform. Subsection A.2.1 demonstrates that our approximation (1.1.11) is subcritical in dimensions 1, 2 and 3.

to the deterministic, periodic Keller–Segel equation

$$\begin{cases} (\partial_t - \Delta)\rho_{\det} = -\chi\nabla\cdot(\rho_{\det}\nabla\Phi_{\rho_{\det}}), & \text{in } (0,T]\times\mathbb{T}^2, \\ \quad -\Delta\Phi_{\rho_{\det}} = \rho_{\det} - \langle\rho_{\det},1\rangle_{L^2(\mathbb{T}^2)}, & \text{in } (0,T]\times\mathbb{T}^2, \\ \quad \rho_{\det}|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2. \end{cases} \tag{1.1.12}$$

The approximation (1.1.11) can be derived from the periodic Keller–Segel–Dean–Kawasaki equation (1.1.10) by replacing $2/N$ by ε, mollifying the noise, Taylor-expanding the square root,

$$\varepsilon^{1/2}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}^\delta) = \varepsilon^{1/2}\nabla\cdot(\sqrt{\rho_{\det}}\boldsymbol{\xi}^\delta) + \varepsilon^{1/2}\nabla\cdot\left(\frac{1}{2\sqrt{\rho_{\det}}}(\rho-\rho_{\det})\boldsymbol{\xi}^\delta\right) + \dots, \tag{1.1.13}$$

and noting that the fluctuations $\rho - \rho_{\det}$ should be of order $\varepsilon^{1/2}$, so that the term $\varepsilon^{1/2}\nabla\cdot(\sqrt{\rho_{\det}}\boldsymbol{\xi}^\delta)$ is of leading order in (1.1.13).

Comparing (1.1.11) and (1.1.10), it is clear that the noise intensity ε should be related to the population size N by $\varepsilon(N) = 2/N$. To identify the correlation length δ in terms of N, we consider the local neighbourhood size in two dimensions[3] and set $\delta(N) = N^{-1/2}$. Consequently, we obtain the *natural* relative scaling

$$\lim_{N\to\infty}\varepsilon(N) = 0, \quad \lim_{N\to\infty}\delta(N) = 0, \quad \lim_{N\to\infty}\varepsilon(N)^{1/2}\delta(N)^{-1} = \sqrt{2}. \tag{1.1.14}$$

In the following, we consider the correlation length as a function of the noise intensity and establish small-noise results for solutions $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (1.1.11) under various relative scalings.

Our first result concerns an analogue of the law of large numbers (LLN) in the *regular setting*, that is, in the periodic Sobolev space $\mathcal{H}^\gamma(\mathbb{T}^2)$ of regularity $\gamma\in(-1/2,0]$. We further denote by $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ the space of continuous functions $f\colon[0,T]\to\mathcal{H}^\gamma(\mathbb{T}^2)$ that may blow up in $\mathcal{H}^\gamma(\mathbb{T}^2)$ before or at time $T>0$ (cf. Section B.8). The following result paraphrases Theorem 3.3.3, which, along with its full proof, can be found in Chapter 3.

[3]To ascertain the local neighbourhood size, one can imagine placing N particles uniformly in the unit square $[0,1]^2$ such that all particles have maximum displacement from one another. Each particle will then need to sit at the centre of a sub-square of side length $N^{-1/2}$, which yields the local neighbourhood size. By setting $\delta(N) = N^{-1/2}$ we ensure that well-distributed particles are uncorrelated. It is not clear, however, if this is a good choice for systems that model aggregation, as is the case here (cf. (1.1.16) & (1.1.10)).

Theorem 1.1.1. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in (-1/2, 0]$. Then for all $T < T^*$, it follows that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$.*

Sketch of Proof. We first pass to the mild formulation,

$$\rho^{(\varepsilon)}_\delta = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho^{(\varepsilon)}_\delta \nabla\Phi_{\rho^{(\varepsilon)}_\delta}] - \varepsilon^{1/2}\boldsymbol{\uparrow}^{\delta(\varepsilon)}, \tag{1.1.15}$$

where $P_t := \mathrm{e}^{t\Delta}$ denotes the heat semigroup, $\mathcal{I}[f]_t := \int_0^t P_{t-s}f_s \,\mathrm{d}s$ the resolution of the heat equation and $\boldsymbol{\uparrow}^\delta := \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}^\delta]$ the noise, which enters into (1.1.15) additively. Using the relative scaling of $(\varepsilon, \delta(\varepsilon))$, we can deduce that $\varepsilon^{1/2}\boldsymbol{\uparrow}^{\delta(\varepsilon)} \to 0$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in probability as $\varepsilon \to 0$ (Lemma 3.3.2). The claim then follows by the continuous mapping theorem. For the full proof see Theorem 3.3.3.

□

Consider the periodic version of the Keller–Segel particle system (1.1.6) defined as

$$\mathrm{d}\bar{X}^i_t = \sqrt{2}\,\mathrm{d}\bar{B}^i_t + \frac{\chi}{N}\sum_{j=1,\, j\neq i}^N \nabla\mathscr{G}(\bar{X}^i_t - \bar{X}^j_t)\,\mathrm{d}t, \qquad i = 1,\dots,N, \tag{1.1.16}$$

where the diffusion is driven by $\bar{B}^i_t := B^i_t - \lfloor B^i_t\rfloor$, i.e. Brownian motions whose generator is the Laplace operator on $\mathbb{T}^2$, and the interaction is induced by the Green function $\mathscr{G}$ of the mean-free Laplacian on $\mathbb{T}^2$, that is,

$$\mathscr{G}(x) = \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\frac{1}{|2\pi\omega|^2}.$$

In particular, it follows that

$$\mathscr{G} * f = \Phi_f$$

for every distribution $f \in \mathcal{S}'(\mathbb{T}^2)$ and

$$\mathscr{G}(x) = K(x) + K_0(x), \tag{1.1.17}$$

where K is the Green function of the Laplacian on the full space (cf. (1.1.2)) and K_0 a smooth correction to periodize K. Theorem 1.1.1 is then consistent with the propagation of chaos for (1.1.16) which was shown to converge to the mean-field limit (1.1.12) for appropriate initial data, see [BJW19b; BJW19a; BJW23]. Hence, we refer to Theorem 1.1.1 as the regular LLN.

Next we show an LDP for our approximation. The following result paraphrases Theorem 3.3.5.

Theorem 1.1.2. *Let* $\rho_0 \in C(\mathbb{T}^2)$ *be such that* $\rho_0 > 0$ *and* $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} *be the weak solution to* (1.1.12) *with initial data* ρ_0 *and chemotactic sensitivity* $\chi \in \mathbb{R}$, *and* T^* *be its maximal time of existence. Assume* $\delta(\varepsilon) \to 0$ *and* $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ *as* $\varepsilon \to 0$ *for some* $\gamma \in (-1/2, 0)$. *Then for all* $T < T^*$, *it follows that the sequence* $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ *satisfies an LDP in* $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ *with rate* ε *and good rate function*

$$\mathscr{I}(\rho) \coloneqq \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h]\Big\}. \quad (1.1.18)$$

Sketch of Proof. An application of the weak-convergence approach to large deviations, the Boué–Dupuis formula [BDM08] and the relative scaling yields an LDP for $(\varepsilon^{1/2}\xi^{\delta(\varepsilon)})_{\varepsilon>0}$, which we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle. For the full proof see Theorem 3.3.5. $\square$

In formal analogy to the (conjectured) rate function (1.1.7) of (1.1.6), we expect that the periodic particle system (1.1.16) satisfies an LDP in $C([0,T];\mathcal{P}(\mathbb{T}^2))$ with rate $N/2$ and good rate function given by

$$\bar{\mathscr{A}}(\rho) \coloneqq \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho}h]\Big\}. \quad (1.1.19)$$

Comparing (1.1.18) with (1.1.19), we find that the approximation error lies in the so-called skeleton equation of (1.1.18),

$$\rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h],$$

which is additive and thereby differs from the skeleton equation that appears in (1.1.19).

Neither Theorem 1.1.1 nor 1.1.2 allows us to assume the natural relative scaling (1.1.14). Further, it is not possible to establish a central limit theorem (CLT) in $\mathcal{H}^\gamma(\mathbb{T}^2)$ for $\gamma > -1/2$ since the leading-order fluctuation $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\mathrm{det}})$ are driven by $\mathbf{I}^{\delta(\varepsilon)}$, which does not converge in $\mathcal{H}^\gamma(\mathbb{T}^2)$ as $\varepsilon \to 0$. Hence, to establish that our approximation (1.1.11) is precise up to first order in the noise intensity $\varepsilon^{1/2}$, we need to pass to a space of lower regularity. Denote by $\mathcal{C}^{-1-}(\mathbb{T}^2)$ the space of Hölder–Besov distributions of regularity $-1-\vartheta$ for some arbitrarily small but fixed $\vartheta > 0$ (for a definition, see Section B.2). It then follows that $\mathbf{I}^{\delta(\varepsilon)}$ converges in $\mathcal{C}^{-1-}(\mathbb{T}^2)$ in probability as $\varepsilon \to 0$ (see Lemma 2.2.7). Consequently, if we can make sense of (1.1.11) in $\mathcal{C}^{-1-}(\mathbb{T}^2)$, we can hope to establish a CLT and analogues of Theorems 1.1.1 & 1.1.2 under a more general relative scaling.

A formal argument (see Subsection A.2.1) shows that (1.1.11) is subcritical in the sense of [Hai14], so that we can use the techniques of singular SPDEs [Hai14; GIP15; Kup16; Duc21] to solve (1.1.11) in $\mathcal{C}^{-1-}(\mathbb{T}^2)$, which is what we refer to as the *rough setting*. Specifically, we apply the paracontrolled decomposition[4] of [GIP15] to derive the next four results (Theorems 1.1.3–1.1.6). The following analogue of the LLN paraphrases Theorem 3.4.4.

Theorem 1.1.3. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$.*

[4] Let us address why we chose paracontrolled calculus rather than the theory of regularity structures [Hai14]. Our approximation (1.1.11) involves parabolic and elliptic operators, and while there are works that study such anisotropic equations and associated regularity structures [BK16; BK19; HM18] the analysis requires a significant amount of bespoke machinery to be developed. Furthermore, (1.1.11) involves a noise that is inhomogeneous in space time, hence needs to be renormalised by a sequence of diverging fields rather than constants (cf. Theorem 1.1.6). Again, while there are similar works in regularity structures [GH19], it seemed more straightforward to rely on the paracontrolled decomposition.

The reason why we did not use the theory of flow equations [Kup16; Duc21] is simply because it wasn't as well developed when we started this project. It would be interesting to see how well flow equations can deal with situations involving anisotropic regularities or heterogeneous noises.

Sketch of Proof. Passing to the paracontrolled decomposition, we can show that the solution map is locally Lipschitz continuous in a vector of stochastic objects $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)}$ known as the noise enhancement. Using the relative scaling, we can deduce that $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} \to 0$ in probability as $\varepsilon \to 0$. The claim then follows by the continuous mapping theorem. For the full proof see Theorem 3.4.4. □

The same continuities of the solution map also allow us to establish a CLT. The following result paraphrases Theorem 3.4.6.

Theorem 1.1.4. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\mathrm{det}}) \to v$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$, where $v \in C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ is the unique mild solution to*

$$\begin{cases} (\partial_t - \Delta)v = -\chi\nabla\cdot(v\nabla\Phi_{\rho_{\mathrm{det}}}) - \chi\nabla\cdot(\rho_{\mathrm{det}}\nabla\Phi_v) - \nabla\cdot(\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}), \\ \qquad v|_{t=0} = 0. \end{cases} \tag{1.1.20}$$

We refer to this v as the generalized Ornstein–Uhlenbeck process.

Proof. For a proof see Theorem 3.4.6. □

While the CLT for the particle system (1.1.16) is to our knowledge still open, there are related systems for which a CLT is known. For example, replacing $\nabla\mathscr{G}$ in (1.1.16) by the Biot–Savart law $\nabla^\perp\mathscr{G}$, one obtains the point-vortex approximation to the two-dimensional incompressible Navier–Stokes equation, which was considered in [WZZ23] and therein shown to satisfy a CLT with leading-order fluctuations converging to a generalized Ornstein–Uhlenbeck process (driven by $\nabla^\perp\Phi$). Similarly if the particle system (1.1.16) allows a CLT (possibly for a certain range of small χ), we would expect the leading-order fluctuations to converge to the generalized Ornstein–Uhlenbeck process (1.1.20). Consequently, the combination of Theorems 1.1.3 & 1.1.4 leads us to expect that our additive-noise approximation is precise up to first order.

Finally, we can also show an LDP in the rough setting by adapting the techniques of [HW15] to our noise enhancement $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$, combined with the contraction principle and the local Lipschitz continuity of the solution map. The following result paraphrases Theorem 3.4.8.

Theorem 1.1.5. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence. Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies an LDP in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ with rate ε and good rate function $\mathscr{I}$.*

Sketch of Proof. An adaptation of [HW15] combined with the relative scaling allows us to deduce an LDP for the noise enhancement $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$, which we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle. For the full proof see Theorem 3.4.8. □

Another first-order approximation to (1.1.10) is given by the *linear fluctuating hydrodynamics* $\hat{\rho}^{(\varepsilon)} := \rho_{\mathrm{det}} + \varepsilon^{1/2} v$. Our additive-noise approximation $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ is more accurate than $\hat{\rho}^{(\varepsilon)}$, which one may observe by considering higher-order effects: for example, if we choose uniform initial data $\rho_0 \equiv 1$, it follows that $\hat{\rho}^{(\varepsilon)}$ solves the equation

$$\begin{cases} (\partial_t - \Delta)\rho = \chi(\rho - \langle \rho, 1\rangle_{L^2}) - \varepsilon^{1/2}\nabla \cdot \boldsymbol{\xi}, \\ \qquad \rho|_{t=0} \equiv 1, \end{cases}$$

which is linear and hence unable to model aggregation. In contrast, the additive-noise approximation started from uniform initial data still aggregates due to its non-linear advection, see Section 5.2 for a discussion. For a numerical comparison between the weak error induced by the linear fluctuating hydrodynamics and the one induced by the Dean–Kawasaki equation, see [CF23b].

Using the paracontrolled approach, we can also make sense of a suitably renormalised version of (1.1.11) even for zero correlation lengths. The following result paraphrases Theorem 4.0.2.

Theorem 1.1.6 (Renormalised Solution). *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence. Then for all $T < T^*$ there exists a sequence of deterministic functions $\text{⚲}^{\delta} \colon [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ indexed by the correlation length δ and given by the expectation*

$$\text{⚲}^{\delta} := \mathbb{E}[\nabla \cdot \mathcal{I}[\text{❙}^{\delta} \nabla \Phi_{\text{❙}^{\delta}}]],$$

such that the solutions $(\rho^\delta)_{\delta>0}$ to

$$\rho^\delta = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho^\delta \nabla \Phi_{\rho^\delta}] + \chi \text{⚲}^{\delta} - \text{❙}^{\delta}$$

converge in probability as $\delta \to 0$ to a non-trivial limit $\rho \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ for all $\alpha \in (-9/4, -2)$.

Proof. The claim follows by combining Theorem 2.3.18 of Chapter 2 with Lemma 3.2.5 of Chapter 3. For the full proof see Theorem 4.0.2 of Chapter 4. □

The sequence $(\text{⚲}^{\delta})_{\delta>0}$ is called the *renormalisation*. Informal power counting suggests that it diverges as δ^{-1} for $\delta \to 0$; however by using symmetries in the fundamental solution of the elliptic problem and non-trivial cancellations, we can show that

$$\|\text{⚲}^{\delta}\|_{C_T \mathcal{C}^{0-}} \lesssim (1 \vee \log(\delta^{-1})) \|\sqrt{\rho_{\mathrm{det}}}\|^2_{C_T \mathcal{H}^2},$$

and by the same symmetries also obtain $\text{⚲}^{\delta} \equiv 0$ if $\rho_0 \equiv 1$ (see Lemma 2.2.19 & Remark 2.2.21). The divergence of the renormalisation informs the relative scaling of the rough LLN (Theorem 1.1.3), CLT (Theorem 1.1.4) and rough LDP (Theorem 1.1.5)

While our additive-noise approximation $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ preserves the mean of the initial data ρ_0, it does not necessarily preserve its sign, which is problematic since $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ is supposed to approximate the density of a population. We aim to show that the probability of '$\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ becoming negative before it blows up' is exponentially small in $1/\varepsilon$. Keeping in mind that our model populations are of size $N = 10^6$ and that

$\varepsilon = 2/N$, we can expect this probability to be small. While we leave that part of the analysis to future work (see Subsection 5.1.2), we have included a similar argument based on [HW15, Rem. 4.6] which controls the possible blow-up time of $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ (Theorem 5.1.2).

1.2 Notations and Conventions

For a glossary of the function and distribution spaces used in this work, see Section J.1.

We write $\mathbb{N}$ for the natural numbers excluding zero, $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{-1} \coloneqq \mathbb{N}_0 \cup \{-1\}$. We define the two-dimensional torus by $\mathbb{T}^2 \coloneqq \mathbb{R}^2/\mathbb{Z}^2$. Throughout, $|\cdot|$ will indicate the norm $|x| = \left(\sum_{i=1}^2 |x_i|^2\right)^{1/2}$ on $\mathbb{T}^2$ or $\mathbb{R}^2$. For $r > 0$ we use the notation $B(0,r) \coloneqq \{x \in \mathbb{R}^2 : |x| < r\}$. From now on we will write $\langle a, b\rangle$ to denote the inner product on each Hilbert space which we either specify or leave clear from the context. For $k, n \in \mathbb{N}$ we denote by $C^k(\mathbb{T}^2;\mathbb{R}^n)$ (resp. $C^\infty(\mathbb{T}^2;\mathbb{R}^n)$) the space of k-times continuously differentiable (resp. smooth) functions on the torus with values in $\mathbb{R}^n$. We write $\mathcal{S}'(\mathbb{T}^2;\mathbb{R})$ for the dual of $C^\infty(\mathbb{T}^2;\mathbb{R})$ and $\mathcal{S}'(\mathbb{T}^2;\mathbb{R}^n)$ for vectors of elements in $\mathcal{S}'(\mathbb{T}^2;\mathbb{R})$. Denote by $\mathcal{S}'_{\mathrm{per}}(\mathbb{R}^2;\mathbb{R}^n)$ the space of periodic distributions, $\mathcal{S}'_{\mathrm{per}}(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^n)$ the space of tempered distributions that are periodic in the space variable (with an abuse of notation), and $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^2;\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}\times\mathbb{T}^2$.

For $f \in C^\infty(\mathbb{T}^2;\mathbb{R})$ (resp. complex sequences $(\zeta(\omega))_{\omega\in\mathbb{Z}^2}$ with $\overline{\zeta(-\omega)} = \zeta(\omega)$ that decay faster than each polynomial) we define its Fourier transform (resp. inverse Fourier transform) by the expression,

$$\mathscr{F} f(\omega) \coloneqq \int_{\mathbb{T}^2} \mathrm{e}^{-2\pi\mathrm{i}\langle\omega,x\rangle} f(x)\,\mathrm{d}x, \qquad \mathscr{F}^{-1}\zeta(x) \coloneqq \sum_{\omega\in\mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\zeta(\omega).$$

This is extended componentwise to vector-valued functions, by density to $f \in L^p(\mathbb{T}^2;\mathbb{R}^n)$ for $p \in [1,\infty)$ and by duality to $f \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}^n)$. Where convenient we use the shorthand $\widehat{f}(\omega) \coloneqq \mathscr{F} f(\omega)$.

We define the Bessel potential space $\mathcal{H}^\alpha(\mathbb{T}^2;\mathbb{R}^n)$ of regularity $\alpha \in \mathbb{R}$ as the space of distributions $u \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{H}^\alpha} := \Big(\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\alpha|\widehat{u}(\omega)|^2\Big)^{1/2} < \infty. \tag{1.2.1}$$

We denote the mean $\overline{u} := \langle u, 1\rangle_{L^2(\mathbb{T}^2)}$ for each $u \in L^2(\mathbb{T}^2;\mathbb{R}) = \mathcal{H}^0(\mathbb{T}^2;\mathbb{R})$.

We often work in the scale of Besov and Hölder–Besov spaces whose definitions we summarize, for their basic properties see Section B.2. We let:

$\varrho_{-1}, \varrho_0 \in C^\infty(\mathbb{R}^2;[0,1])$ be radially symmetric and such that

$$\begin{aligned} &\operatorname{supp}(\varrho_{-1}) \subset B(0,1/2), \qquad \operatorname{supp}(\varrho_0) \subset \{x\in\mathbb{R}^2 : 9/32 \le |x| \le 1\},\\ &\sum_{k=-1}^{\infty}\varrho_k(x) = 1 \quad \text{for all } x \in \mathbb{R}^2 \quad \text{where } \varrho_k(x) := \varrho_0(2^{-k}x) \quad \text{for each } k\in\mathbb{N}. \end{aligned} \tag{1.2.2}$$

This defines a dyadic partition of unity. Given $k \ge -1$ we write $\Delta_k u := \mathscr{F}^{-1}(\varrho_k \mathscr{F}u)$ for the associated Littlewood–Paley block and given $\alpha\in\mathbb{R}$, $p,q\in[1,\infty]$, we define the Besov-norm $\|u\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2;\mathbb{R}^n)} := \|(2^{k\alpha}\|\Delta_k u\|_{L^p(\mathbb{T}^2;\mathbb{R}^n)})_{k\in\mathbb{N}_{-1}}\|_{\ell^q}$ for $u \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}^n)$. We use $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2;\mathbb{R}^n)$ to denote the completion of $C^\infty(\mathbb{T}^2;\mathbb{R}^n)$ under $\|\cdot\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2;\mathbb{R}^n)}$ and use the shorthand $\mathcal{C}^\alpha(\mathbb{T}^2;\mathbb{R}^n) := \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^2;\mathbb{R}^n)$. The space $\mathcal{B}^\alpha_{2,2}(\mathbb{T}^2;\mathbb{R}^n)$ coincides with $\mathcal{H}^\alpha(\mathbb{T}^2;\mathbb{R}^n)$. For $\alpha\in\mathbb{R}$ and $p,q\in[1,\infty]$ we use the notation $\mathcal{B}^{\alpha-}_{p,q}(\mathbb{T}^2;\mathbb{R}^n) := \cap_{\alpha'<\alpha}\mathcal{B}^{\alpha'}_{p,q}(\mathbb{T}^2;\mathbb{R}^n)$. When the context is clear we will remove the target space so as to lighten notation.

For $u,v\in C^\infty(\mathbb{T}^2;\mathbb{R})$ we define the paraproduct $\varolessthan$ and resonant product $\odot$ by

$$u \varolessthan v := \sum_{k\ge -1}\sum_{l=-1}^{k-2}\Delta_l u\Delta_k v, \qquad u\odot v := \sum_{\substack{k,l\in\mathbb{N}_{-1}\\ |k-l|\le 1}}\Delta_l u \Delta_k v \tag{1.2.3}$$

and extend them to the scale of Besov distributions by Bony's lemma [BCD11a, Thm. 2.82 & Thm. 2.85] and generalizations thereof (see Section B.4).

We define the action of the heat semigroup on $f\in L^1(\mathbb{T}^2;\mathbb{R})$ by the flow,

$$[0,\infty) \ni t \mapsto P_t f := \mathscr{F}^{-1}(\mathrm{e}^{-t|2\pi\,\cdot\,|^2}\widehat{f}(\,\cdot\,)) = \mathscr{H}_t * f$$

where the convolution against the heat kernel $\mathscr{H}_t$ on $\mathbb{T}^2$ is defined by

$$\mathscr{H}_t * f = \int_{\mathbb{T}^2}\mathscr{H}_t(\,\cdot\,-y)f(y)\,\mathrm{d}y = \int_{(0,1)^2}\mathscr{H}^{\mathrm{per}}_t(\,\cdot\,-y)f^{\mathrm{per}}(y)\,\mathrm{d}y$$

with $f^{\mathrm{per}}\colon \mathbb{R}^2 \to \mathbb{R}$ the periodization of $f\colon \mathbb{T}^2 \to \mathbb{R}$ and $\mathscr{H}_t^{\mathrm{per}}$ given by

$$\mathscr{H}_t^{\mathrm{per}}(x) \coloneqq \frac{1}{4\pi t}\sum_{n\in\mathbb{Z}^2} \mathrm{e}^{-\frac{|x-n|^2}{4t}} = \sum_{\omega\in\mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\mathrm{e}^{-t|2\pi\omega|^2} \qquad \text{for } t>0,$$
$$\mathscr{H}_0^{\mathrm{per}} \coloneqq \sum_{n\in\mathbb{Z}^2}\delta_n, \qquad \text{and} \qquad \mathscr{H}_t^{\mathrm{per}}(x) \coloneqq 0 \qquad \text{for } t<0.$$

For $f\colon [0,T]\times\mathbb{T}^2 \to \mathbb{R}$, we define the resolution of the heat equation as

$$\mathcal{I}[f]_t \coloneqq \int_0^t P_{t-s}f_s\,\mathrm{d}s = \int_0^t \mathscr{H}_{t-s} * f_s\,\mathrm{d}s.$$

We define the solution to the elliptic equation $-\Delta\Phi_f = f - \overline{f}$ by

$$\Phi_f \coloneqq \mathscr{G} * f, \qquad \text{where} \quad f \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}) \quad \text{and}$$
$$\mathscr{G}(x) \coloneqq \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\frac{1}{|2\pi\omega|^2}, \qquad (\mathscr{G}*f)(x) = \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\frac{1}{|2\pi\omega|^2}\widehat{f}(\omega).$$

We denote for $\omega = (\omega^1,\omega^2) \in \mathbb{Z}^2$, $t\in\mathbb{R}$ and $j=1,2$ the Fourier multipliers $H_t^j(\omega) \coloneqq 2\pi\mathrm{i}\omega^j \exp(-t|2\pi\omega|^2)\mathbb{1}_{t\geq 0}$ and $G^j(\omega) \coloneqq 2\pi\mathrm{i}\omega^j|2\pi\omega|^{-2}\mathbb{1}_{\omega\neq 0}$.

Given a Banach space E and an interval $I\subseteq[0,\infty)$ we write $C_IE \coloneqq C(I;E)$ for the space of continuous maps $f\colon I\to E$. For compact I we equip C_IE with the norm $\|f\|_{C_IE} \coloneqq \sup_{t\in I}\|f_t\|_E$. For $\kappa\in(0,1)$, we define $C_I^\kappa E \coloneqq C^\kappa(I;E)$ as the completion of $C^\infty(I;E)$ under the norm $\|f\|_{C_I^\kappa E} \coloneqq \|f\|_{C_IE} + \sup_{s\neq t\in I}\frac{\|f_t-f_s\|_E}{|t-s|^\kappa}$, where $C^\infty(I;E)$ denotes the space of smooth functions in the interior of I, which, along with all of their derivatives, can be continuously extended to functions in C_IE. For $T>0$, we use the shorthand $C_TE = C_{[0,T]}E$, $C_T^\kappa E = C_{[0,T]}^\kappa E$ and understand $C_T^0E = C_TE$. Given $\eta\geq 0$, we denote $f_{\eta\text{-wt}}(t) \coloneqq (1\wedge t)^\eta f_t$ for each $f\colon (0,T]\to E$ and $t\in(0,T]$. We define the Banach space

$$C_{\eta;T}E \coloneqq \Big\{f\colon (0,T]\to E:\ f_{\eta\text{-wt}}(0) \coloneqq \lim_{t\to 0} f_{\eta\text{-wt}}(t) \in E,\ f_{\eta\text{-wt}} \in C_{(0,T]}E\Big\}$$

equipped with the norm

$$\|f\|_{C_{\eta;T}E} \coloneqq \sup_{t\in(0,T]}(1\wedge t)^\eta\|f_t\|_E$$

and refer to η as the weight at 0. For $\kappa\in(0,1]$ we let $C_{\eta;T}^\kappa E \coloneqq C_\eta^\kappa((0,T];E)$ denote the Banach space of functions $f\colon (0,T]\to E$ which are finite under the norm

$$\|f\|_{C_{\eta;T}^\kappa E} \coloneqq \|f\|_{C_{\eta;T}E} + \sup_{s\neq t\in(0,T]}(1\wedge s\wedge t)^\eta\frac{\|f_t-f_s\|_E}{|t-s|^\kappa}.$$

We also make use of the notation $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\alpha}(\mathbb{T}^2) \coloneqq C^{\kappa}_{\eta;T}\mathcal{C}^{\alpha-2\kappa}(\mathbb{T}^2) \cap C_{\eta;T}\mathcal{C}^{\alpha}(\mathbb{T}^2)$ to denote a weighted interpolation space, on which we define

$$\|u\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\alpha}} \coloneqq \max\{\|u\|_{C^{\kappa}_{\eta;T}\mathcal{C}^{\alpha-2\kappa}}, \|u\|_{C_{\eta;T}\mathcal{C}^{\alpha}}\}.$$

We set $\mathscr{L}^{\kappa}_{T}\mathcal{C}^{\alpha}(\mathbb{T}^2) \coloneqq C^{\kappa}_{T}\mathcal{C}^{\alpha-2\kappa}(\mathbb{T}^2) \cap C_{T}\mathcal{C}^{\alpha}(\mathbb{T}^2)$ and understand $\mathscr{L}^{0}_{\eta;T}\mathcal{C}^{\alpha}(\mathbb{T}^2) = C_{\eta;T}\mathcal{C}^{\alpha}(\mathbb{T}^2)$.

Let $C^2_1((0,T] \times \mathbb{T}^2)$ be the space of functions $f \in C((0,T] \times \mathbb{T}^2)$ such that $\partial_t f$, $\partial_{x_i} f$ and $\partial_{x_i}\partial_{x_j} f$ exist in $C((0,T) \times \mathbb{T}^2)$ and can be extended to functions in $C((0,T] \times \mathbb{T}^2)$ for all $i, j \in \{1,2\}$.

We do not distinguish between the spaces $L^2([0,T]^n \times \mathbb{T}^{2n} \times \{1,2\}^n; \mathbb{R})$, $L^2([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})$ and $L^2([0,T]^n; L^2(\mathbb{T}^{2n}; \mathbb{R}^{2n}))$, which are identical up to isometric isomorphisms.

We further denote $L^2_T\mathcal{H}^{\alpha}(\mathbb{T}^2) \coloneqq L^2([0,T]; \mathcal{H}^{\alpha}(\mathbb{T}^2;\mathbb{R}))$ and $W^{1,2}_T\mathcal{H}^{-1}(\mathbb{T}^2) = W^{1,2}([0,T]; \mathcal{H}^{-1}(\mathbb{T}^2))$ the space of functions in $L^2_T\mathcal{H}^{-1}(\mathbb{T}^2)$ with weak derivative (in time) belonging to $L^2_T\mathcal{H}^{-1}(\mathbb{T}^2)$, cf. [Eva98, Subsec. 5.9.2]. On the intersection $C_T\mathcal{H}^{\alpha}(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\alpha+1}(\mathbb{T}^2)$ we define the norm $\|\cdot\|_{C_T\mathcal{H}^{\alpha} \cap L^2_T\mathcal{H}^{\alpha+1}} \coloneqq \max\{\|\cdot\|_{C_T\mathcal{H}^{\alpha}}, \|\cdot\|_{L^2_T\mathcal{H}^{\alpha+1}}\}$.

We write $\lesssim$ to indicate that an inequality holds up to a constant depending on quantities that we do not keep track of or are fixed throughout. When we do wish to emphasise the dependence on certain quantities α, p, d, we either write $\lesssim_{\alpha,p,d}$ or define $C \coloneqq C(\alpha,p,d) > 0$ and write $\le C$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that is large enough to support two independent space-time white noises ξ^1, ξ^2 on $[0,T] \times \mathbb{T}^2$ as specified in Section C.1. Given $\boldsymbol{\xi} \coloneqq (\xi^1, \xi^2)$, we can define a family of complex Brownian motions by $W^j(t,\omega) \coloneqq \boldsymbol{\xi}(\mathbb{1}_{[0,t]}(\,\cdot\,) \otimes \mathrm{e}^{-2\pi\mathrm{i}\langle \omega, \,\cdot\, \rangle} \otimes \delta_{j,\,\cdot\,})$ where $t \ge 0$, $\omega \in \mathbb{Z}^2$ and $j = 1,2$, see Lemma C.3.1. This probability space will be fixed, so that whenever a property holds almost surely, it will do so with respect to $\mathbb{P}$.

For two vectors $\omega_1, \omega_2 \in \mathbb{Z}^2$, we use $(\omega_1 \perp \omega_2)$ to denote the formula $\langle \omega_1, \omega_2 \rangle = 0$ and $\neg(\omega_1 \perp \omega_2)$ to denote its logical negation.

Recall the dyadic partition of unity $(\varrho_k)_{k\in\mathbb{N}_{-1}}$ introduced in (1.2.2). Let $u, v \in \mathcal{S}'(\mathbb{T}^2)$, we define the truncated sums

$$\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega_1\sim\omega_2}} \widehat{u}(\omega_1)\widehat{v}(\omega_2) \coloneqq \sum_{\omega_1,\omega_2\in\mathbb{Z}^2} \widehat{u}(\omega_1)\widehat{v}(\omega_2) \sum_{\substack{k,l\in\mathbb{N}_{-1}\\ |k-l|\le 1}} \varrho_l(\omega_1)\varrho_k(\omega_2) \tag{1.2.4}$$

and

$$\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega_1\precsim\omega_2}} \widehat{u}(\omega_1)\widehat{v}(\omega_2) \coloneqq \sum_{\omega_1,\omega_2\in\mathbb{Z}^2} \widehat{u}(\omega_1)\widehat{v}(\omega_2) \sum_{k=1}^{\infty}\sum_{l=-1}^{k-2} \varrho_l(\omega_1)\varrho_k(\omega_2), \tag{1.2.5}$$

where we implicitly assume that those sums converge absolutely. This is a discrete analogue of the usual paraproducts (1.2.3). If $\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}$, then $\omega_1 \sim \omega_2$ implies $9/64|\omega_1| \le |\omega_2| \le 64/9|\omega_1|$ and $\omega_1 \precsim \omega_2$ implies $|\omega_1| \le 8/9|\omega_2|$. The constants $9/64$ and $8/9$ are invariant under dilations of the chosen partition of unity $(\varrho_k)_{k\in\mathbb{N}_{-1}}$, see e.g. [MWX17, (64)].

We define the cut-off

$$\begin{aligned} &\varphi \in C^\infty(\mathbb{R}^2) \text{ to be of compact support,}\\ &\operatorname{supp}(\varphi) \subset B(0,1),\ \text{even and such that } \varphi(0) = 1. \end{aligned} \tag{1.2.6}$$

Given φ satisfying (1.2.6), we define a sequence of mollifiers $(\psi_\delta)_{\delta>0}$ on $\mathbb{T}^2$ by

$$\psi_\delta(x) \coloneqq \sum_{\omega\in\mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\varphi(\delta\omega). \tag{1.2.7}$$

For a glossary of more specialized notions introduced throughout Chapter 2, see Section J.2.

2

Well-Posedness of Paracontrolled Solutions

Contents

2.1 Introduction

In this chapter we are concerned with the local well-posedness of singular stochastic partial differential equations of the kind

$$\begin{cases} (\partial_t - \Delta)\rho = \nabla \cdot (\rho \nabla \Phi_\rho) + \nabla \cdot (\sigma \boldsymbol{\xi}), & \text{in } (0,T] \times \mathbb{T}^2, \\ -\Delta \Phi_\rho = \rho - \langle \rho, 1 \rangle_{L^2}, & \text{in } (0,T] \times \mathbb{T}^2, \\ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{2.1.1}$$

where $\boldsymbol{\xi} = (\xi^1, \xi^2)$ is a two-dimensional vector of i.i.d. space-time white noises (see Section C.1), $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the two-dimensional torus, $T \in (0, \infty)$ is a time horizon, $\sigma \in C([0,T];\mathcal{H}^2(\mathbb{T}^2))$ is a space-time heterogeneity and ρ_0 is some suitable initial data which we specify later. Motivated by the theory of fluctuating hydrodynamics, our main example is given by the choice $\sigma = \sqrt{\rho_{\mathrm{det}}}$ where ρ_{det} solves the deterministic PDE (1.1.12). This choice will be applied in Chapter 3 to the additive-noise approximation (1.1.11) of the Dean–Kawasaki equation associated to the Keller–Segel model.

Due to the singularity of the noise, defining a suitable notion of solution to (2.1.1) is non-trivial and we will implement a paracontrolled approach [GIP15] to obtain local well-posedness (Theorem 2.1.1). To see why such an approach is necessary we consider the terms of (2.1.1) under a formal power counting argument. The proper definition of all function spaces used below can be found in Appendix B.

The white noise almost surely takes values in $\mathcal{C}_{\mathfrak{s}}^{-2-}([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ (see Theorem C.1.2), where $\mathcal{C}_{\mathfrak{s}}^{\alpha}([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ denotes the set of space-time Hölder-regular distributions of parameter $\alpha\in\mathbb{R}$ equipped with the parabolic scaling $\mathfrak{s}=(2,1,1)$, i.e. regularity in time counts twice as much as regularity in space (see Section B.9), and the shorthand $\alpha\pm$ is used to denote $\alpha\pm\varepsilon$ for some $\varepsilon>0$ arbitrarily small but fixed. Let us assume for now that we can define the product $\sigma\boldsymbol{\xi}$ intrinsically and that it is no more regular than the white noise itself. Due to the regularising effect of the heat equation (Lemma B.5.2), it follows that the solution $\mathbf{I}$ to the linear equation,

$$\begin{cases}(\partial_t-\Delta)\mathbf{I} = \nabla\cdot(\sigma\boldsymbol{\xi}), & \text{in } (0,T]\times\mathbb{T}^2,\\ \qquad \mathbf{I}|_{t=0}=0, & \text{on } \mathbb{T}^2,\end{cases}$$

may be no more regular than $C_T\mathcal{C}^{-1-}(\mathbb{T}^2)$ and certainly we cannot, in general, expect ρ to be any more regular than this. Consequently, upon applying the regularising effect of the elliptic equation (Lemma B.5.5), we would have $\nabla\Phi_\rho\in C_T\mathcal{C}^{0-}(\mathbb{T}^2;\mathbb{R}^2)$. However, by Bony's estimate the product fg is only a priori well-defined for $f\in\mathcal{C}^\alpha(\mathbb{T}^2)$ and $g\in\mathcal{C}^\beta(\mathbb{T}^2)$ with $\alpha+\beta>0$, which does not hold for any coordinate of the vector $\rho\nabla\Phi_\rho$.

The theories of regularity structures, paracontrolled distributions, flow equations and various recent extensions and adaptations thereof have revolutionised the study of singular SPDEs [Hai14; GIP15; Kup16; Ott+21; Lin+24; Duc21]. The common thread throughout these theories is to notice that the factors of the ill-defined products are not generic distributions but inherit structure from the noise. This inheritance allows one to define renormalised products, which excise the singular part, allowing one to give meaning to a renormalised equation which is continuous in a finite tuple of *diagrams* built from the noise. These diagrams are referred to as an enhancement.

The theory of paracontrolled distributions was first developed by Gubinelli, Imkeller and Perkowski in [GIP15]. The central idea is to use harmonic analysis to construct regular commutators, which allow us to decompose the equation into exogenous noise terms and terms that can be constructed as fixed points. One can then mollify the noise and, using this decomposition, obtain estimates that are uniform in the mollification parameter, whereupon limits in probability can be taken. Paracontrolled distributions have been successfully applied to analyse a range of singular SPDEs and operators including; the parabolic Anderson model (PAM) [GIP15; KPvZ22], the Anderson Hamiltonian [AC15; GUZ20; CvZ21], the Φ^4_3 model [MW17b; CC18], the Kardar–Parisi–Zhang equation [GP17], the stochastic Burgers and Navier–Stokes equations [ZZ15; GP17] and the stochastic non-linear wave equation [GKO23].

Let $(\psi_\delta)_{\delta>0}$ be a sequence of symmetric spatial mollifiers. In our case we find that there exists a family of deterministic fields $(f^\delta)_{\delta>0}$ satisfying

$$f^\delta \in L^\infty_T\mathcal{C}^{-1-}(\mathbb{T}^2;\mathbb{R}^2), \qquad \|f^\delta\|_{L^\infty_T\mathcal{C}^{-1-}} \lesssim (1\vee\log(\delta^{-1}))\|\sigma\|^2_{C_T\mathcal{H}^2} \quad \text{for every } \delta>0$$

such that the sequence of solutions $(\rho^\delta)_{\delta>0}$ each solving,

$$\begin{cases} (\partial_t-\Delta)\rho = \nabla\cdot(\rho\nabla\Phi_\rho - f^\delta) + \nabla\cdot(\sigma(\psi_\delta * \boldsymbol{\xi})), & \text{in } (0,T]\times\mathbb{T}^2, \\ -\Delta\Phi_\rho = \rho - \langle\rho,1\rangle_{L^2}, & \text{in } (0,T]\times\mathbb{T}^2, \\ \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{2.1.2}$$

converges in probability to a unique limit ρ, which we identify as the renormalised solution to (2.1.1). For a definition of the renormalised solution, see Definition 2.3.17; for its existence, see Theorem 2.3.18, Part 1; and for the convergence, see Theorem 2.3.18, Part 2; see also Subsection 2.1.1 for a formal discussion of these results.

We remark that the eventual application to fluctuating hydrodynamics motivates us to consider mollified noise terms of the form $\sigma\boldsymbol{\xi}^\delta := \sigma(\psi_\delta * \boldsymbol{\xi})$, rather than mollifying the whole product $(\sigma\boldsymbol{\xi})^\delta := \psi_\delta * (\sigma\boldsymbol{\xi})$, see [Cor+23, Sec. 3.2] and [FG23; DFG20]. Further, we mollify the noise in space only so that the Markov property of the underlying particle system (1.1.16) is retained.

Three points of interest arise from (2.1.2). Firstly, in the case where σ is genuinely heterogeneous the field f^δ is in general also heterogeneous. This has been observed elsewhere, having been pointed out as a possibility in [Hai14] and seen explicitly in the renormalisation of singular SPDEs on bounded domains, [GH19]. Secondly, if σ is a constant, so that our noise agrees with that of the stochastic Burgers' equation, then f^δ is zero. In this case the renormalised equation agrees exactly with the singular equation, i.e. the products are not explicitly renormalised when $\delta = 0$. This phenomenon has also been observed in [DDT94; DD02; ZZ15; GP17]. Thirdly, using the informal power counting described above, one might expect the singular product $\rho^\delta\nabla\Phi_{\rho^\delta}$ to diverge at the order of δ^{-1}, since this is the gap in regularity between the singular factors. However, (2.1.2) shows that this divergence is at most logarithmic. This improvement arises from symmetries in the fundamental solution of the elliptic problem, leading to non-trivial cancellations in our stochastic estimates. In the case of constant σ it is exactly these cancellations which show that no explicit renormalisation in (2.1.2) is necessary.

To demonstrate the underlying principle, let us consider a one-dimensional example. We assume that $u^\delta \to u$ as $\delta \to 0$ in a space of regularity $-1/2-$. The product rule gives the identity,

$$u^\delta \partial_x \partial_x^{-2} u^\delta = \frac{1}{2}\Big(\partial_x^2(\partial_x^{-1} u^\delta \partial_x^{-2} u^\delta) - \partial_x(u^\delta \partial_x^{-2} u^\delta)\Big), \tag{2.1.3}$$

where we write ∂_x^{-1} as a shorthand for integration in x, with the (arbitrary) normalisation that the primitive is mean-free. While the product on the left-hand side, between an object converging in $-1/2-$ and an object converging in $1/2-$ looks ill-posed, the right-hand side is in fact classically well-posed; the first term is the second derivative of a product between objects in $1/2-$ and $3/2-$, while the second is the derivative of a product between an object in $-1/2-$ and one in $3/2-$. Hence the anticipated logarithmic divergence of the left-hand side is removed by expanding as on the right-hand side. This basic observation extends to our higher-dimensional case through the symmetry of the Green's function for Poisson's equation. We see that the symmetry alleviates divergences by one order. Linear divergences of δ^{-1} are improved to logarithmic, and logarithmic divergences are improved to well-posedness. The heterogeneity σ makes these improvements visible, as when σ is constant the same symmetries lead to perfect cancellations removing the need for renormalising counterterms all together. Similar observations have also been made in the context of the KPZ equation, [GP17, Lem. 9.5].

We observe that (2.1.1) is an example of a singular SPDE involving anisotropic regularity since the regularising effect of the elliptic equation only takes place in the spatial variable. It is for this reason that we choose to work with the theory of paracontrolled distributions, since it naturally treats space and time separately. This is in contrast to the theory of regularity structures which naturally treats space and time simultaneously. While there are examples of works which study anisotropic equations and associated regularity structures, [BK16; HM18; BK19], the analysis is somewhat ad hoc. It is for this reason that we opt for a paracontrolled approach which more natively applies to the anisotropic regularisation present in (2.1.1).

Structure of the Chapter: In the following Subsection 2.1.1 we present an outline of the general strategy and the main result of this chapter. Section 2.2 contains a detailed proof of the existence and regularity of the various stochastic objects which we are required to construct and constitute our enhanced noise. The careful analysis of these stochastic objects and control over the diverging fields is the main contribution of this chapter. In Section 2.3 we show the local well-posedness of paracontrolled solutions given a suitable enhancement of the noise. Finally let us point out three appendices relevant to the present chapter: Appendix B recalls some useful and well-known results concerning Besov spaces and paraproducts; Appendix D provides various estimates on the so-called shape coefficients which we introduce in Section 2.2 and Appendix E contains a number of summation and discrete convolution estimates that we make repeated use of throughout.

2.1.1 Strategy and Main Result

We first outline the paracontrolled approach to (2.1.1) in a relatively loose manner, identifying the main steps of the method and the diagrams that we will need to give meaning to. Recall that we wish to define a sufficiently robust notion of solution to (2.1.1) which in particular is stable under regular approximations to the noise. To do so we first write (2.1.1) in mild form, setting

$$\rho = P\rho_0 + \nabla \cdot \mathcal{I}[\rho \nabla \Phi_\rho] + \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}]. \tag{2.1.4}$$

Note that ρ may blow up before time T due to the non-linearity on the right-hand side of (2.1.4). For the purpose of this discussion, we will assume that ρ exists until time T; see Subsection 2.3.2 for results on the maximal time of existence. In the remainder of this section we will assume that all terms on the right-hand side of (2.1.4) are continuous in time while taking values in a Hölder–Besov space $\mathcal{C}^\alpha(\mathbb{T}^2)$, where α is possibly negative.

Working, for now, with smooth initial data, we may assume that the final term on the right-hand side is the least regular component of ρ. Using the same stochastic estimates alluded to in the introduction, along with the regularising effect of the heat kernel and the effect of the derivative, we will work under the assumption that $\text{\textexclamdown} := \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}] \in C_T\mathcal{C}^{-1-}(\mathbb{T}^2)$. Passing this regularity to ρ and applying the regularising effect of the elliptic equation (Lemma B.5.5) we expect to have $\nabla \Phi_\rho \in C_T\mathcal{C}^{0-}(\mathbb{T}^2)$. Therefore, as discussed in the introduction, the product $\rho\nabla\Phi_\rho$ is not a priori well-defined. Our first step is to employ the so called Da Prato–Debussche trick, [DD03], to remove the most singular term by defining $u := \rho - \text{\textexclamdown}$ so that if ρ is a solution to (2.1.4),

$$u = P\rho_0 + \nabla \cdot \mathcal{I}[u\nabla\Phi_u] + \nabla \cdot \mathcal{I}[u\nabla\Phi_{\text{\textexclamdown}}] + \nabla \cdot \mathcal{I}[\text{\textexclamdown}\nabla\Phi_u] + \nabla \cdot \mathcal{I}[\text{\textexclamdown}\nabla\Phi_{\text{\textexclamdown}}].$$

We notice that the product $\treeI \nabla \Phi_{\treeI}$ is not classically well-posed, however it can be renormalised and replaced with the symbol $\treeY := \nabla \cdot \mathcal{I}[\treeI \nabla \Phi_{\treeI}] - \treeO$, where $\treeO := \mathbb{E}[\nabla \cdot \mathcal{I}[\treeI \nabla \Phi_{\treeI}]]$ denotes the singular part of this product. The term $\treeY$ is well-defined and will have the regularity that one would formally expect of $\nabla \cdot \mathcal{I}[\treeI \nabla \Phi_{\treeI}]$, namely $\treeY \in C_T\mathcal{C}^{0-}(\mathbb{T}^2)$ (see Subsection 2.2.3). From now on we continue with our expansion, replacing the singular product by its renormalised counterpart $\treeY$ so that we have in fact changed the equation solved by ρ.

We are now in better shape, as we may now work with $u \in C_T\mathcal{C}^{0-}(\mathbb{T}^2)$, which renders the first product on the right-hand side classically well-posed. However, the second and third products remain ill-defined. We may repeat the same trick, defining $w := u - \treeY$, which should solve,

$$w = P\rho_0 + \nabla \cdot \mathcal{I}[w \nabla \Phi_{\treeI}] + \nabla \cdot \mathcal{I}[\treeI \nabla \Phi_w] + \nabla \cdot \mathcal{I}[\nabla \Phi_{\treeY} \otimes \treeI] + \nabla \cdot \mathcal{I}[\treeW] + Q(w, \treeI, \treeY),$$

where $Q(w, \treeI, \treeY)$ denotes a finite sum of classically well-posed terms involving w, $\treeI$ and $\treeY$. The formal definition of Bony's decomposition into para and resonant products is given in Appendix B, however, for now we simply recall the rules that for $f \in \mathcal{C}^{\alpha}(\mathbb{T}^2)$, $g \in \mathcal{C}^{\beta}(\mathbb{T}^2)$, one has

$$f \otimes g \in \mathcal{C}^{\beta \wedge (\alpha+\beta)}(\mathbb{T}^2) \text{ for all } \alpha \in \mathbb{R} \setminus \{0\},\ \beta \in \mathbb{R} \qquad \text{and}$$
$$f \odot g \in \mathcal{C}^{\alpha+\beta}(\mathbb{T}^2) \text{ if } \alpha + \beta > 0.$$

The new symbol appearing on the right-hand side for w is a shorthand for $\treeW := \treeY \odot \nabla \Phi_{\treeI} + \nabla \Phi_{\treeY} \odot \treeI$. Although those resonant products are not classically well-defined, further stochastic arguments show that they can in fact be defined as objects finite in $C_T\mathcal{C}^{0-}(\mathbb{T}^2; \mathbb{R}^2)$ without the subtraction of any infinite counterterms. The full definition of $\treeW$, through stochastic calculus, is contained in Subsection 2.2.2 and we show in Subsections 2.2.3 and 2.2.4 that $\treeW \in C_T\mathcal{C}^{0-}(\mathbb{T}^2; \mathbb{R}^2)$. So even though it requires significant work to define, it is not the least regular term on the right-hand side.

Instead this is given by the paraproduct term, $\nabla \cdot \mathcal{I}[\nabla \Phi_{\treeY} \otimes \treeI]$, which using Bony's estimate (Lemma B.4.1) is only finite in $C_T\mathcal{C}^{0-}(\mathbb{T}^2)$, and the formal product term $\nabla \cdot \mathcal{I}[\tI \nabla \Phi_w]$, which is not even a priori well-defined. Hence, as before we can only expect to find $w \in C_T\mathcal{C}^{0-}(\mathbb{T}^2)$ which is not regular enough to define the products $w\nabla\Phi_{\tI}$ and $\tI\nabla\Phi_w$ a priori.

One sees that further applications of the Da Prato–Debussche trick will not improve the situation. Instead we employ the core idea that solutions should resemble the noise at small scales. This is formalised through the *paracontrolled Ansatz*, that is, we only look for solutions such that for every coordinate $j = 1, 2$,

$$w = \sum_{k=1}^{2} \partial_k \Phi_{w+\tY} \varolessthan \partial_k \mathcal{I}[\tI] + w^{\#}, \qquad \partial_j \Phi_w = \sum_{k=1}^{2} \partial_k \Phi_{w+\tY} \varolessthan \partial_{k,j}\mathcal{I}[\Phi_{\tI}] + (\nabla\Phi_w)^{\#}_j,$$

or in vector notation,

$$w = \nabla\Phi_{w+\tY} \varolessthan \nabla\mathcal{I}[\tI] + w^{\#}, \qquad \nabla\Phi_w = \nabla\Phi_{w+\tY} \varolessthan \nabla^2\mathcal{I}[\Phi_{\tI}] + (\nabla\Phi_w)^{\#}, \quad (2.1.5)$$

where the *paracontrolled remainders* $w^{\#}$ and $(\nabla\Phi_w)^{\#}$ are terms to be fixed by the equation which we stipulate to be finite in $C_T\mathcal{C}^{0+}(\mathbb{T}^2)$ and $C_T\mathcal{C}^{1+}(\mathbb{T}^2;\mathbb{R}^2)$ respectively.[1] This ensures that the products $w^{\#}\nabla\Phi_{\tI}$ and $\tI(\nabla\Phi_w)^{\#}$ are classically well-defined. Rearranging, using the linearity of the map $f \mapsto \nabla\Phi_f$ and applying Bony's decomposition to the products $w\nabla\Phi_{\tI}$ and $\tI\nabla\Phi_w$, we find the identity

$$\begin{aligned} w^{\#} &= P\rho_0 + \nabla\cdot\mathcal{I}[w \odot \nabla\Phi_{\tI}] + \nabla\cdot\mathcal{I}[\tI \odot \nabla\Phi_w] \\ &\qquad + \nabla\cdot\mathcal{I}[\nabla\Phi_{w+\tY} \varolessthan \tI] - \nabla\Phi_{w+\tY} \varolessthan \nabla\mathcal{I}[\tI] + \tilde{Q}(w, \tI, \tY, \tW), \end{aligned}$$

where $\tilde{Q}(w, \tI, \tY, \tW)$ is a new polynomial of its arguments and can be expected to be of strictly positive regularity. Hence, the regularity of $w^{\#}$ is governed by that of the commutator and that of the resonant products $\nabla\cdot\mathcal{I}[w\odot\nabla\Phi_{\tI}]$ and $\nabla\cdot\mathcal{I}[\tI\odot\nabla\Phi_w]$. The commutator can be controlled by Lemmas B.7.1 and B.7.2, which show that

$$\nabla\cdot\mathcal{I}[\nabla\Phi_{w+\tY} \varolessthan \tI] - \nabla\Phi_{w+\tY} \varolessthan \nabla\mathcal{I}[\tI] \in C_T\mathcal{C}^{1-}(\mathbb{T}^2).$$

To treat the resonant products we make use of the *Ansatz* again, writing for every coordinate $j = 1, 2$,

$$w \odot \partial_j\Phi_{\tI} = \sum_{k=1}^{2} (\partial_k\Phi_{w+\tY} \varolessthan \partial_k\mathcal{I}[\tI]) \odot \partial_j\Phi_{\tI} + w^{\#} \odot \partial_j\Phi_{\tI},$$

[1] The remainder $(\nabla\Phi_w)^{\#}$ can be expressed in terms of $\nabla\Phi_{w^{\#}}$ and a more regular commutator, cf. Lemma B.7.1. Hence, the paracontrolled Ansatz for w implies the Ansatz for $\nabla\Phi_w$. In fact in Section 2.3 we make use of an equivalent Ansatz which makes certain technical steps easier but is less clear to present—see Remark 2.3.5.

and

$$\mathbf{I} \odot \partial_j \Phi_w = \sum_{k=1}^{2} \mathbf{I} \odot (\partial_k \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \partial_{k,j}\mathcal{I}[\Phi_{\mathbf{I}}]) + \mathbf{I} \odot (\nabla \Phi_w)^{\#}_j;$$

or again in vector notation,

$$w \odot \nabla \Phi_{\mathbf{I}} = (\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla \mathcal{I}[\mathbf{I}]) \odot \nabla \Phi_{\mathbf{I}} + w^{\#} \odot \nabla \Phi_{\mathbf{I}},$$

and

$$\mathbf{I} \odot \nabla \Phi_w = \mathbf{I} \odot (\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}]) + \mathbf{I} \odot (\nabla \Phi_w)^{\#}.$$

Under our stipulation that $w^{\#} \in C_T\mathcal{C}^{0+}(\mathbb{T}^2)$ and $(\nabla\Phi_w)^{\#} \in C_T\mathcal{C}^{1+}(\mathbb{T}^2;\mathbb{R}^2)$, the final two resonant products are classically well-defined and so it only remains to check that the first term of each expansion is finite. To achieve this last step we consider a commutator for the triple product,

$$\begin{aligned}(\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla \mathcal{I}[\mathbf{I}]) \odot \nabla \Phi_{\mathbf{I}} &= (\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla \mathcal{I}[\mathbf{I}]) \odot \nabla \Phi_{\mathbf{I}} - \nabla \Phi_{w+\boldsymbol{\Upsilon}}(\nabla \mathcal{I}[\mathbf{I}] \odot \nabla \Phi_{\mathbf{I}}) \\ &\quad + \nabla \Phi_{w+\boldsymbol{\Upsilon}}(\nabla \mathcal{I}[\mathbf{I}] \odot \nabla \Phi_{\mathbf{I}}).\end{aligned}$$

Lemma B.7.3 shows that the commutator lies in $C_T\mathcal{C}^{1-}(\mathbb{T}^2;\mathbb{R}^2)$. We apply a similar trick to the resonant product $\mathbf{I} \odot \nabla \Phi_w$, writing

$$\begin{aligned}\mathbf{I} \odot (\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}]) &= \mathbf{I} \odot (\nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}]) - \nabla \Phi_{w+\boldsymbol{\Upsilon}}(\nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}] \odot \mathbf{I}) \\ &\quad + \nabla \Phi_{w+\boldsymbol{\Upsilon}}(\nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}] \odot \mathbf{I}).\end{aligned}$$

Again, the regularity of the commutator follows from Lemma B.7.3. Taken together the last two exogenous terms produce the final diagram we are required to construct,

$$\boldsymbol{\checkmark} := \nabla \mathcal{I}[\mathbf{I}] \odot \nabla \Phi_{\mathbf{I}} + \nabla^2 \mathcal{I}[\Phi_{\mathbf{I}}] \odot \mathbf{I}.$$

Note that the first resonant product above should be read as a vector outer product so that $\boldsymbol{\checkmark}$ is matrix valued. We would naively expect both summands of $\boldsymbol{\checkmark}$ to diverge logarithmically if we replace $\boldsymbol{\xi}$ by $\boldsymbol{\xi}^{\delta}$ and let $\delta \to 0$. However, the symmetry of the Green's function allows us to show that after summing both terms, $\boldsymbol{\checkmark}$ is well-defined in a sufficiently strong topology even for $\delta = 0$.

Reversing all of the above steps we find the modified equation solved by our paracontrolled object,

$$\rho = \mathbf{I} + \boldsymbol{\Upsilon} + \nabla \Phi_{w+\boldsymbol{\Upsilon}} \varolessthan \nabla \mathcal{I}[\mathbf{I}] + w^{\#}, \tag{2.1.6}$$

with $w^{\#}$ a solution to

$$\begin{aligned} w^{\#} &= P\rho_0 + \nabla\cdot\mathcal{I}[w^{\#}\odot\nabla\Phi_{\mathbf{I}}] + \nabla\cdot\mathcal{I}[\mathbf{I}\odot(\nabla\Phi_w)^{\#}] \\ &\quad + \nabla\cdot\mathcal{I}[\nabla\Phi_{w+\mathbf{Y}}\,\mathbf{V}] + \bar{Q}(w,\mathbf{I},\mathbf{Y},\boldsymbol{\Psi}), \end{aligned} \tag{2.1.7}$$

for $\bar{Q}(w,\mathbf{I},\mathbf{Y},\boldsymbol{\Psi})$ a third polynomial of its arguments, their paraproducts and commutators. We call this ρ a paracontrolled solution to (2.1.1) with enhancement $\mathbb{X}$. For a rigorous definition, see Definition 2.3.10.

In the paracontrolled decomposition (2.1.6) of ρ, the first three terms lie in spaces of negative regularity. Hence, the singular parts of the product $\rho\nabla\Phi_\rho$ will be determined by non-linear combinations of the first three terms. Since $\mathbf{I}$ and $\mathbf{Y}$ will be supplied as data these terms can be handled directly. However, as w also carries information from ρ, products involving $\nabla\Phi_{w+\mathbf{Y}}\varolessthan\nabla\mathcal{I}[\mathbf{I}]$ cannot be handled in the same way. Instead we make use of the commutator estimates above. To see this in practice and to identify the possibly diverging field f^δ alluded to in the introduction, we recall our notion of a mollified noise by setting $\mathbf{I}^\delta := \nabla\cdot\mathcal{I}[\sigma(\psi_\delta*\boldsymbol{\xi})]$, where ψ_δ is a standard mollifier. We use the notations $\mathbf{Y}^\delta$, $\boldsymbol{\Psi}^\delta$, $\mathbf{V}^\delta$ to denote the same diagrams now constructed from $\mathbf{I}^\delta$, and use ρ^δ to denote the solution of the mild equation

$$\rho^\delta = P\rho_0 + \nabla\cdot\mathcal{I}[\rho^\delta\nabla\Phi_{\rho^\delta}] - \mathbf{P}^\delta + \mathbf{I}^\delta. \tag{2.1.8}$$

We further denote the second Da Prato–Debussche remainder by $w^\delta := \rho^\delta - \mathbf{I}^\delta - \mathbf{Y}^\delta$ and define $\mathbf{Y}^\delta_{\mathrm{can}} := \nabla\cdot\mathcal{I}[\mathbf{I}^\delta\nabla\Phi_{\mathbf{I}^\delta}] = \mathbf{Y}^\delta + \mathbf{P}^\delta$.

We have the identity,

$$\begin{aligned} \rho^\delta\nabla\Phi_{\rho^\delta} &= \mathbf{I}^\delta\nabla\Phi_{\mathbf{I}^\delta} + \mathbf{Y}^\delta_{\mathrm{can}}\odot\nabla\Phi_{\mathbf{I}^\delta} - \mathbf{P}^\delta\odot\nabla\Phi_{\mathbf{I}^\delta} \\ &\quad + \nabla\Phi_{\mathbf{Y}^\delta_{\mathrm{can}}}\odot\mathbf{I}^\delta - \nabla\Phi_{\mathbf{P}^\delta}\odot\mathbf{I}^\delta + \nabla\Phi_{w^\delta\mathbf{Y}^\delta}\,\mathbf{V}^\delta + \dots. \end{aligned} \tag{2.1.9}$$

Here we have only kept track of terms that are either not classically well-defined or contain stochastic diagrams which require construction. The final term involving $\mathbf{V}^\delta$ arises from applying commutators to the paraproduct term in the expansion of ρ^δ where the more regular parts have been left implicit above. Since we only expect to have $\rho^\delta\to\rho$ in $C_T\mathcal{C}^{-1-}(\mathbb{T}^2)$ we do not expect (2.1.9) to converge directly. We have already identified the possibly diverging field which renormalises the first term, since

$$\nabla\cdot\mathcal{I}[\mathbf{I}^\delta\nabla\Phi_{\mathbf{I}^\delta}] - \mathbf{P}^\delta = \mathbf{Y}^\delta \to \mathbf{Y} \in C_T\mathcal{C}^{0-}(\mathbb{T}^2).$$

As discussed in the introduction, formal power counting would lead one to expect P^{δ} to diverge at order δ^{-1}, however, exploiting the symmetry of the elliptic Green's function we have that $\|\mathsf{P}^{\delta}\|_{C_T\mathcal{C}^{0-}} \lesssim (1 \vee \log(\delta^{-1}))\|\sigma\|^2_{C_T\mathcal{H}^2}$.

The diverging diagram P^{δ} is also contained in the terms $\mathsf{P}^{\delta} \odot \nabla\Phi_{\mathsf{I}^{\delta}}$ and $\nabla\Phi_{\mathsf{P}^{\delta}} \odot \mathsf{I}^{\delta}$. However, since P^{δ} is of regularity $0-$ and I^{δ} of regularity $-1-$, it is not directly clear how to make sense of those products. Note that if P^{δ} were a diverging constant rather than a field this would simply be scalar multiplication and we would have no trouble. Nevertheless, using that P^{δ} is deterministic, one can define the products $\mathsf{P}^{\delta} \odot \nabla\Phi_{\mathsf{I}^{\delta}}$ and $\nabla\Phi_{\mathsf{P}^{\delta}} \odot \mathsf{I}^{\delta}$ directly as Itô objects and show that they diverge at a rate no worse than P^{δ}. We refer to Subsection 2.2.5 for this argument.

Since $\mathsf{Y}^{\delta}_{\mathrm{can}} = \mathsf{Y}^{\delta} + \mathsf{P}^{\delta}$, we may expand the product $\mathsf{Y}^{\delta}_{\mathrm{can}} \odot \nabla\Phi_{\mathsf{I}^{\delta}} + \nabla\Phi_{\mathsf{Y}^{\delta}_{\mathrm{can}}} \odot \mathsf{I}^{\delta}$ to cancel the diverging terms $\mathsf{P}^{\delta} \odot \nabla\Phi_{\mathsf{I}^{\delta}}$ and $\nabla\Phi_{\mathsf{P}^{\delta}} \odot \mathsf{I}^{\delta}$ in (2.1.9). Hence, we can construct the renormalised product $\lim_{\delta\to 0}(\nabla\cdot\mathcal{I}[\rho^{\delta}\nabla\Phi_{\rho^{\delta}}] - \mathsf{P}^{\delta})$ without further modifications. In particular, this identifies the renormalising sequence $(f^{\delta})_{\delta>0}$ of (2.1.2) as $f^{\delta} = \mathbb{E}[\mathsf{I}^{\delta}\nabla\Phi_{\mathsf{I}^{\delta}}]$ and (2.1.8) as the corresponding mild formulation.

We have therefore identified both the solution ρ and the non-linear term in (2.1.4) as trilinear functions of a suitable enhancement of the noise. To conclude this section we paraphrase the main result of this chapter (Theorem 2.3.18). In the following, $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ denotes a tuple of exponents which determines the regularity and integrability of each object in our paracontrolled decomposition. We assume that those exponents satisfy assumption (2.3.1) in Section 2.3, see also Examples 2.3.1–2.3.3 for various special cases of interest.

Theorem 2.1.1. *Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ be a tuple of exponents satisfying (2.3.1), $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$ be some initial data, $T > 0$ be a time horizon, $\boldsymbol{\xi} = (\xi^1, \xi^2)$ be a vector-valued space-time white noise on $[0,T] \times \mathbb{T}^2$ (see Section C.1), $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$ be a heterogeneity and $(\psi_\delta)_{\delta>0}$ be a family of symmetric mollifiers as in (1.2.7). Then there exist enhancements $\mathbb{X} = (\mathsf{I}, \mathsf{Y}, \mathsf{W}, \mathsf{V})$, $\mathbb{X}^{\delta} = (\mathsf{I}^{\delta}, \mathsf{Y}^{\delta}, \mathsf{W}^{\delta}, \mathsf{V}^{\delta})$ as described above (in particular $\mathbb{X}^{\delta}$ is built from $\sigma\boldsymbol{\xi}^{\delta}$ with*

$\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$) and a random variable $T_{\max} \in (0, T]$ (the maximal time of existence) with the following properties:

1. *There exists a unique paracontrolled solution ρ to (2.1.1) on $[0, T_{\max})$ with enhancement $\mathbb{X}$ and initial data ρ_0 (in the sense of Definition 2.3.10), which we call the renormalised solution. The maximal time of existence $T_{\max}$ satisfies*

$$T_{\max} = T \qquad \textit{or} \qquad \lim_{t \uparrow T_{\max}} \|(1 \wedge t)^\eta \rho(t)\|_{\mathcal{C}^{\alpha+1}} = \infty \qquad \textit{almost surely.}$$

2. *For each $\lambda > 0$,*

$$\lim_{\delta \to 0} \mathbb{P}\Big(\sum_{L=1}^\infty \frac{2^{-L}}{L} \frac{\|\rho - \rho^\delta\|_{C_{\eta;T_L}\mathcal{C}^{\alpha+1}}}{1 + \|\rho - \rho^\delta\|_{C_{\eta;T_L}\mathcal{C}^{\alpha+1}}} > \lambda \Big) = 0,$$

where ρ^δ is the solution to (2.1.8) and

$$T_L \coloneqq T \wedge L \wedge \inf\Big\{ t \in [0, T] : \|(1 \wedge t)^\eta \rho(t)\|_{\mathcal{C}^{-1-}} > L \textit{ or } \|(1 \wedge t)^\eta \rho^\delta(t)\|_{\mathcal{C}^{-1-}} > L \Big\}$$

for every $L \in \mathbb{N}$.

Furthermore $\mathbf{P}^\delta = \nabla \cdot \mathcal{I}[\mathbf{I}^\delta \nabla \Phi_{\mathbf{I}^\delta}] - \mathbf{\Upsilon}^\delta = \mathbb{E}[\nabla \cdot \mathcal{I}[\mathbf{I}^\delta \nabla \Phi_{\mathbf{I}^\delta}]]$. If σ is a constant then $\mathbf{P}^\delta \equiv 0$ while in general one has the bound $\|\mathbf{P}^\delta\|_{C_T \mathcal{C}^{2\alpha+4}} \lesssim (1 \vee \log(\delta^{-1})) \|\sigma\|^2_{C_T \mathcal{H}^2}$ (see Remark 2.2.21 and Lemma 2.2.19).

Remark 2.1.2. In the case of constant σ it also holds that $\mathbb{E}[\mathbf{I}^\delta \nabla \Phi_{\mathbf{I}^\delta}] = 0$. This is due to the symmetry of the elliptic Green's function, see the discussion of (2.1.3).

Remark 2.1.3. Since the submission of this thesis, it has been brought to our attention that in the particular case of the Keller–Segel non-linearity $\rho \nabla \Phi_\rho$, one may in fact simplify the analysis of the equation (2.1.4) in the following manner: Using that Φ denotes the resolution of the mean-free Laplacian on $\mathbb{T}^2$, one may cast the Keller–Segel non-linearity in the following form

$$\rho \nabla \Phi_\rho = -\Delta \Phi_\rho \nabla \Phi_\rho + \langle \rho, 1 \rangle_{L^2} \nabla \Phi_\rho,$$

where the first term can be written by the product rule as

$$\Delta \Phi_\rho \nabla \Phi_\rho = \nabla \cdot (\nabla \Phi_\rho)^{\otimes 2} - \frac{1}{2} \nabla \operatorname{Tr}(\nabla \Phi_\rho)^{\otimes 2},$$

with

$$(\nabla\Phi_\rho)^{\otimes 2}_{i,j} := (\nabla\Phi_\rho \otimes \nabla\Phi_\rho)_{i,j} := \partial_i\Phi_\rho\partial_j\Phi_\rho, \qquad i,j = 1,2.$$

Hence, we may re-write (2.1.4) as

$$\begin{aligned}\rho = P\rho_0 - \nabla\cdot\mathcal{I}[\nabla\cdot(\nabla\Phi_\rho)^{\otimes 2}] + \frac{1}{2}\nabla\cdot\mathcal{I}[\nabla\operatorname{Tr}(\nabla\Phi_\rho)^{\otimes 2}] \\ + \nabla\cdot\mathcal{I}[\langle\rho,1\rangle_{L^2}\nabla\Phi_\rho] + \dagger.\end{aligned} \tag{2.1.10}$$

Using the bilinearity of each operator on the right-hand side of (2.1.10), it follows that the Da Prato–Debussche remainder $u = \rho - \dagger$ satisfies the equation

$$u = P\rho_0 + \nabla\cdot\mathcal{I}[\dagger\nabla\Phi_{\dagger}] + \hat{Q}(u,\dagger),$$

where $\hat{Q}(u,\dagger)$ denotes a finite sum of classically well-posed terms involving $\nabla\Phi_u$ and $\nabla\Phi_{\dagger}$ rather than u and $\dagger$. Consequently, it suffices to renormalise the product $\nabla\cdot\mathcal{I}[\dagger\nabla\Phi_{\dagger}]$ by subtracting $\ddagger$, without the need to perform a paracontrolled decomposition. Nevertheless our methods are still of independent interest, as they demonstrate how one may construct a noise enhancement that involves heterogeneities and anisotropic regularities. We thank M. Gubinelli and H. Weber for these insights.

2.2 Noise Enhancement

In this section we construct the enhancements required in Theorem 2.1.1 and establish their regularities.

2.2.1 Outline and Regularities

Our enhancements will lie in the following space of enhanced rough noises, which quantifies their regularities in interpolation spaces (see Section 1.2 and Section B.3).

Definition 2.2.1 (Enhanced rough noise)**.** *For all $T > 0$, $\alpha \in (-5/2, -2)$ and $\kappa \in (0, 1/2)$ let the map*

$$\begin{aligned}&\Theta\colon (\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+2}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+5}(\mathbb{T}^2;\mathbb{R})) \\ &\quad\to \mathscr{L}^\kappa_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_T\mathcal{C}^{3\alpha+6}(\mathbb{T}^2;\mathbb{R}^2) \times \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^{2\times 2}),\end{aligned}$$

$(v,f) \mapsto \Theta(v,f),$

be given by

$$\begin{aligned} Y &:= \nabla\cdot\mathcal{I}[v\nabla\Phi_v] - f,\\ \Theta(v,f) &:= (v, Y, Y\odot\nabla\Phi_v + \nabla\Phi_Y\odot v, \nabla\mathcal{I}[v]\odot\nabla\Phi_v + \nabla^2\mathcal{I}[\Phi_v]\odot v)\end{aligned}$$

and define the space $\mathcal{X}_T^{\alpha,\kappa}$ *to be the closure of the subset*

$$\begin{aligned}&\{\Theta(v,f) : (v,f)\in\mathscr{L}_T^{\kappa}\mathcal{C}^{\alpha+2}(\mathbb{T}^2;\mathbb{R})\times\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+5}(\mathbb{T}^2;\mathbb{R}),\ v_0=f_0=0\}\\ &\qquad\subset\mathscr{L}_T^{\kappa}\mathcal{C}^{\alpha+1}(\mathbb{T}^2;\mathbb{R})\times\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R})\times\mathscr{L}_T^{\kappa}\mathcal{C}^{3\alpha+6}(\mathbb{T}^2;\mathbb{R}^2)\times\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^{2\times2}).\end{aligned}$$

We shall denote a generic element of this closure by $\mathbb{X}=(\mathbf{I},\mathbf{Y},\mathbf{W},\mathbf{V})\in\mathcal{X}_T^{\alpha,\kappa}$ *and equip it with the metric induced by the norm*

$$\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}} := \max\{\|\mathbf{I}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{\alpha+1}}, \|\mathbf{Y}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}}, \|\mathbf{W}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{3\alpha+6}}, \|\mathbf{V}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}}\}.$$

The main result of this section is the following theorem reminiscent of [GP17, Thm. 9.1].

Theorem 2.2.2. *Let* $T>0$, $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$, $\boldsymbol{\xi}$ *be a two-dimensional vector of space-time white noises on* $[0,T]\times\mathbb{T}^2$ *(see Section C.1),* $(\psi_\delta)_{\delta>0}$ *be a sequence of mollifiers as in* (1.2.7), $\boldsymbol{\xi}^\delta := \psi_\delta * \boldsymbol{\xi} := (\psi_\delta*\xi^1, \psi_\delta*\xi^2)$ *and* $\mathbb{X}^\delta := (\mathbf{I}^\delta, \mathbf{Y}^\delta, \mathbf{W}^\delta, \mathbf{V}^\delta)$ *be given by*

$$\begin{aligned}&\mathbf{I}^\delta := \nabla\cdot\mathcal{I}[\sigma\boldsymbol{\xi}^\delta],\quad \mathbf{Y}^\delta := \nabla\cdot\mathcal{I}[\mathbf{I}^\delta\nabla\Phi_{\mathbf{I}^\delta}] - \mathbb{E}[\nabla\cdot\mathcal{I}[\mathbf{I}^\delta\nabla\Phi_{\mathbf{I}^\delta}]],\\ &\mathbf{W}^\delta := \mathbf{Y}^\delta\odot\nabla\Phi_{\mathbf{I}^\delta} + \nabla\Phi_{\mathbf{Y}^\delta}\odot\mathbf{I}^\delta,\quad \mathbf{V}^\delta := \nabla\mathcal{I}[\mathbf{I}^\delta]\odot\nabla\Phi_{\mathbf{I}^\delta} + \nabla^2\mathcal{I}[\Phi_{\mathbf{I}^\delta}]\odot\mathbf{I}^\delta.\end{aligned}$$

Then for all $\alpha\in(-5/2,-2)$ *and* $\kappa\in(0,1/2)$ *the following hold*

1. *Almost surely,* $\mathbb{X}^\delta\in\mathcal{X}_T^{\alpha,\kappa}$ *and*

$$\mathbb{X}^\delta\in\mathscr{L}_T^{\kappa}\mathcal{C}^{\alpha+2}(\mathbb{T}^2;\mathbb{R})\times\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+5}(\mathbb{T}^2;\mathbb{R})\times\mathscr{L}_T^{\kappa}\mathcal{C}^{3\alpha+8}(\mathbb{T}^2;\mathbb{R}^2)\times\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+6}(\mathbb{T}^2;\mathbb{R}^{2\times2}).$$

2. *Almost surely, there exists some* $\mathbb{X}=(\mathbf{I},\mathbf{Y},\mathbf{W},\mathbf{V})\in\mathcal{X}_T^{\alpha,\kappa}$*, such that for all* $p\in[1,\infty)$ *we have* $\lim_{\delta\to0}\mathbb{E}[\|\mathbb{X}-\mathbb{X}^\delta\|^p_{\mathcal{X}_T^{\alpha,\kappa}}]^{1/p}=0$ *and* $\mathbb{E}[\|\mathbb{X}\|^p_{\mathcal{X}_T^{\alpha,\kappa}}]^{1/p}<\infty$.

3. *Defining*

$$\mathsf{P}^{\delta} := \mathbb{E}[\nabla\cdot\mathcal{I}[\mathsf{I}^{\delta}\nabla\Phi_{\mathsf{I}^{\delta}}]], \qquad \mathsf{Y}^{\delta}_{\mathrm{can}} := \nabla\cdot\mathcal{I}[\mathsf{I}^{\delta}\nabla\Phi_{\mathsf{I}^{\delta}}] = \mathsf{Y}^{\delta} + \mathsf{P}^{\delta},$$

$$\mathsf{W}^{\delta}_{\mathrm{can}} := \mathsf{Y}^{\delta}_{\mathrm{can}} \odot \nabla\Phi_{\mathsf{I}^{\delta}} + \nabla\Phi_{\mathsf{Y}^{\delta}_{\mathrm{can}}} \odot \mathsf{I}^{\delta},$$

it holds that

$$\|\mathsf{P}^{\delta}\|_{\mathscr{L}^{\kappa}_{T}\mathcal{C}^{2\alpha+4}} \lesssim (1\vee\log(\delta^{-1}))\|\sigma\|^2_{C_T\mathcal{H}^2}$$

and for all $p\in[1,\infty)$*,*

$$\mathbb{E}[\|\mathsf{Y}^{\delta}_{\mathrm{can}}\|^p_{\mathscr{L}^{\kappa}_{T}\mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim (1\vee\log(\delta^{-1}))\|\sigma\|^2_{C_T\mathcal{H}^2},$$

$$\mathbb{E}[\|\mathsf{W}^{\delta}_{\mathrm{can}}\|^p_{\mathscr{L}^{\kappa}_{T}\mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim (1\vee\log(\delta^{-1}))\|\sigma\|^3_{C_T\mathcal{H}^2}.$$

An explicit definition of the limit $\mathbb{X} = (\mathsf{I}, \mathsf{Y}, \mathsf{W}, \mathsf{V})$ can be found in Subsection 2.2.2. We call $\mathbb{X}$ the *renormalised enhancement* and $\mathbb{X}^{\delta}_{\mathrm{can}} = (\mathsf{I}^{\delta}, \mathsf{Y}^{\delta}_{\mathrm{can}}, \mathsf{W}^{\delta}_{\mathrm{can}}, \mathsf{V}^{\delta})$ the *canonical enhancement*.

The result will be shown in several parts, namely in Lemma 2.2.7 (I), Lemma 2.2.9 (Y), Lemma 2.2.12 and Lemma 2.2.17 (W), Lemma 2.2.13 and Lemma 2.2.18 (V) and Lemma 2.2.19 (P^{δ}, $\mathsf{Y}^{\delta}_{\mathrm{can}}$, $\mathsf{W}^{\delta}_{\mathrm{can}}$).

Remark 2.2.3. Different aspects of Theorem 2.2.2 require different assumptions on the heterogeneity σ. For example the regularity of I and Y only requires $\sigma \in C_TL^{\infty}(\mathbb{T}^2)$ (Lemma 2.2.7 and Lemma 2.2.9) while the regularities of the contractions contained in W, V require that uniformly over $t\in[0,T]$ one has $\sup_{\omega\in\mathbb{Z}^2}|\widehat{\sigma}(t,\omega)|(1+|\omega|^2) < \infty$. The assumption $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$ implies both of these conditions and provides a convenient norm and well-studied space that controls the latter quantity; hence we choose to work with this simpler, if sub-optimal restriction. Furthermore, with a view to setting $\sigma = \sqrt{\rho_{\mathrm{det}}}$ (cf. (1.1.11)) the condition $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$ is more straightforward to check.

Remark 2.2.4. As discussed in the introduction, we build our *prelimiting* enhancement from $\sigma\boldsymbol{\xi}^{\delta} = \sigma(\psi_{\delta} * \boldsymbol{\xi})$ instead of $(\sigma\boldsymbol{\xi})^{\delta} = \psi_{\delta} * (\sigma\boldsymbol{\xi})$ and with only a spatial convolution. Hence, for I^{δ} to be more regular, we need to assume some regularity on σ. We again make use of the condition $\sup_{t\in[0,T]}\sup_{\omega\in\mathbb{Z}^2}|\widehat{\sigma}(t,\omega)|(1+|\omega|^2) < \infty$ (see the proof of Lemma 2.2.7), though other choices are possible (see Appendix I).

Remark 2.2.5. Our methods also allow us to establish that $\Upsilon \in \mathscr{L}_T^{1-}\mathcal{C}^{0-}(\mathbb{T}^2)$. However, since we do not make use of the additional time regularity, we omit the proof.

The following Kolmogorov criterion provides an efficient method for establishing the regularity of stochastic processes in Hölder–Besov spaces. The presentation of this lemma is reminiscent of [Per20, Prop. 4.1].

Lemma 2.2.6. *Let $\alpha \in \mathbb{R}$, $p \in (1,\infty)$, $\gamma \in (1/p, 1]$, $\gamma' \in (0, \gamma - 1/p)$ and $X\colon [0,T] \to \mathcal{S}'(\mathbb{T}^2)$ be a stochastic process such that $X(0) = 0$. Assume there exists some $K > 0$ such that*

$$\int_0^T \int_0^T |t-s|^{-2-p\gamma'} \sum_{q\in\mathbb{N}_{-1}} 2^{pq\alpha} \int_{\mathbb{T}^2} \mathbb{E}[|\Delta_q X(t,x) - \Delta_q X(s,x)|^p]\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \leq K < \infty. \tag{2.2.1}$$

Then there exists a modification of X (which we do not relabel) such that

$$\mathbb{E}\Big[\|X\|^p_{C_T^{\gamma'}\mathcal{C}^{\alpha-2/p}}\Big] \lesssim \mathbb{E}\Big[\|X\|^p_{C_T^{\gamma'}\mathcal{B}^{\alpha}_{p,p}}\Big] \lesssim_{p,\gamma'} K.$$

Assume that

$$M := \sup_{s\neq t\in[0,T]} |t-s|^{-p\gamma} \sum_{q\in\mathbb{N}_{-1}} 2^{pq\alpha} \int_{\mathbb{T}^2} \mathbb{E}[|\Delta_q X(t,x) - \Delta_q X(s,x)|^p]\,\mathrm{d}x < \infty,$$

then for each $\gamma' \in (0, \gamma - 1/p)$, (2.2.1) is satisfied with

$$K = M \int_0^T \int_0^T |t-s|^{-2+p(\gamma-\gamma')}\,\mathrm{d}s\,\mathrm{d}t < \infty$$

and in particular $X \in C_T^{\gamma'}\mathcal{C}^{\alpha-2/p}(\mathbb{T}^2)$ almost surely.

Proof. The bound follows by the definition of $\mathcal{B}^{\alpha}_{p,p}(\mathbb{T}^2)$ (Definition B.2.1), the Besov–Hölder embedding in time [FV10, Proof of Thm. A.10] and the Besov embedding in space (B.2.1). To show $X \in C_T^{\gamma'}\mathcal{B}^{\alpha}_{p,p}(\mathbb{T}^2) \hookrightarrow C_T^{\gamma'}\mathcal{C}^{\alpha-2/p}(\mathbb{T}^2)$, it suffices to apply Lemma B.3.3 and the Besov embedding (B.2.1). $\square$

In the remainder of this subsection we outline the basic arguments involved in proving Theorem 2.2.2 by motivating the definition of I and establishing its existence via an application of Lemma 2.2.6.

Let $\boldsymbol{\xi}$ be a two-dimensional vector of space-time white noises (see Section C.1), then (see Lemma C.3.1) there exists a family $(W^j(\,\cdot\,,m))_{m\in\mathbb{Z}^2,j=1,2}$ of complex-valued Brownian motions on $\mathbb{R}_+$ starting from 0 that satisfy $\overline{W^j(\,\cdot\,,m)} = W^j(\,\cdot\,,-m)$ and are otherwise independent. It is well-known that the Fourier frequencies of the stochastic heat equation are given by Ornstein–Uhlenbeck processes and by a similar argument we can find an expression for the Fourier transform of $\Psi = \nabla\cdot\mathcal{I}[\sigma\boldsymbol{\xi}]$. For all $\omega\in\mathbb{Z}^2$ and $t\in\mathbb{R}$, let $H^j_t(\omega) \coloneqq 2\pi\mathrm{i}\omega^j\exp(-t|2\pi\omega|^2)\mathbb{1}_{t\geq 0}$ be the multiplier appearing in $\partial_j\mathcal{I}$. We define Ψ by applying the inverse Fourier transform to the sequence

$$\widehat{\Psi}(t,\omega) \coloneqq \sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega-m_1)H^{j_1}_{t-u_1}(\omega). \tag{2.2.2}$$

We also introduce the Fourier transform of $\tau \coloneqq \nabla\cdot\mathcal{I}[\boldsymbol{\xi}]$ by

$$\widehat{\tau}(t,\omega) \coloneqq \sum_{j_1=1}^{2}\int_0^t \mathrm{d}W^{j_1}(u_1,\omega)H^{j_1}_{t-u_1}(\omega). \tag{2.2.3}$$

Lemma 2.2.7. *Let $T>0$, $\alpha<-2$, $\kappa\in(0,1/2)$ and $\sigma\in C_TL^\infty(\mathbb{T}^2)$. Then for all $p\in[1,\infty)$ we have $\mathbb{E}[\|\Psi\|^p_{\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+1}}]^{1/p}\lesssim\|\sigma\|_{C_TL^\infty}$ and in particular $\Psi\in\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ a.s. Assume in addition $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$ and $\delta>0$, then it holds that $\mathbb{E}[\|\Psi^\delta\|^p_{\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+2}}]^{1/p}\lesssim(1+\delta^{-2})^{1/2}\|\sigma\|_{C_T\mathcal{H}^2}$ and in particular $\Psi^\delta\in\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+2}(\mathbb{T}^2)$ a.s. What is more, $\lim_{\delta\to0}\mathbb{E}[\|\Psi-\Psi^\delta\|^p_{\mathscr{L}^\kappa_T\mathcal{C}^{\alpha+1}}]^{1/p}=0$ for all $p\in[1,\infty)$.*

Proof. Let $\gamma\in(0,1]$, $\varepsilon\in(0,\gamma/2)$ and $\max\{1/\varepsilon,2\}<p<\infty$. To establish the existence and regularity of Ψ in a Besov space, we apply Kolmogorov's continuity criterion (Lemma 2.2.6), Nelson's estimate (Lemma C.3.8) and the Besov embedding (B.2.1). Therefore, in order to establish that $\Psi\in C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-\varepsilon-2/p}(\mathbb{T}^2)\hookrightarrow C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-3\varepsilon}(\mathbb{T}^2)$ almost surely, it suffices to show that the quantity

$$\sup_{s\neq t\in[0,T]}|t-s|^{-p\gamma/2}\sum_{q\in\mathbb{N}_{-1}}2^{pq(-1-\gamma-\varepsilon)}\int_{\mathbb{T}^2}\mathbb{E}[|\Delta_q\Psi(t,x)-\Delta_q\Psi(s,x)|^2]^{p/2}\,\mathrm{d}x$$

is finite. Using that $\sigma\in C_TL^\infty(\mathbb{T}^2)$ is bounded, we can pass to real space with (C.3.1) to deduce for τ as in (2.2.3),

$$\mathbb{E}[|\Delta_q\Psi(t,x)-\Delta_q\Psi(s,x)|^2]\leq\|\sigma\|^2_{C_TL^\infty}\mathbb{E}[|\Delta_q\tau(t,x)-\Delta_q\tau(s,x)|^2].$$

Using that $\mathbb{E}[\widehat{\tau}(t,\omega)\overline{\widehat{\tau}(s,\omega')}] = 0$ if $\omega \neq \omega' \in \mathbb{Z}^2$, we obtain

$$
\begin{aligned}
&\mathbb{E}[|\Delta_q\tau(t,x) - \Delta_q\tau(s,x)|^2] \\
&= \sum_{\omega\in\mathbb{Z}^2}\sum_{\omega'\in\mathbb{Z}^2} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle\omega',x\rangle}\varrho_q(\omega)\varrho_q(\omega')\mathbb{E}[(\widehat{\tau}(t,\omega)-\widehat{\tau}(s,\omega))\overline{(\widehat{\tau}(t,\omega')-\widehat{\tau}(s,\omega'))}] \\
&= \sum_{\omega\in\mathbb{Z}^2}\varrho_q(\omega)^2\mathbb{E}[|\widehat{\tau}(t,\omega)-\widehat{\tau}(s,\omega)|^2].
\end{aligned}
$$

It follows by Itô's isometry and interpolation (B.5.1),

$$
\mathbb{E}[|\widehat{\tau}(t,\omega)-\widehat{\tau}(s,\omega)|^2] \leq \sum_{j_1=1}^{2}\int_{-\infty}^{\infty}\mathrm{d}u_1|H^{j_1}_{t-u_1}(\omega) - H^{j_1}_{s-u_1}(\omega)|^2 \lesssim |t-s|^{\gamma}|\omega|^{2\gamma}
$$

and therefore uniformly in $x\in\mathbb{T}^2$,

$$
\mathbb{E}[|\Delta_q\tau(t,x) - \Delta_q\tau(s,x)|^2] \lesssim |t-s|^{\gamma}2^{q(2+2\gamma)}.
$$

To summarize, we obtain by Lemma C.3.8, Lemma 2.2.6 and the arguments above for each $\max\{1/\varepsilon,2\} < p < \infty$,

$$
\begin{aligned}
&\mathbb{E}[\|\mathbf{I}\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-3\varepsilon}}] \\
&\lesssim \sup_{s\neq t\in[0,T]}|t-s|^{-p\gamma/2}\sum_{q\in\mathbb{N}_{-1}}2^{pq(-1-\gamma-\varepsilon)}\int_{\mathbb{T}^2}\mathbb{E}[|\Delta_q\mathbf{I}(t,x)-\Delta_q\mathbf{I}(s,x)|^2]^{p/2}\,\mathrm{d}x \\
&\leq \sup_{s\neq t\in[0,T]}|t-s|^{-p\gamma/2}\|\sigma\|^p_{C_TL^\infty}\sum_{q\in\mathbb{N}_{-1}}2^{pq(-1-\gamma-\varepsilon)}\sup_{x\in\mathbb{T}^2}\mathbb{E}[|\Delta_q\tau(t,x)-\Delta_q\tau(s,x)|^2]^{p/2} \\
&\lesssim \|\sigma\|^p_{C_TL^\infty}\sum_{q\in\mathbb{N}_{-1}}2^{-pq\varepsilon} \\
&\lesssim \|\sigma\|^p_{C_TL^\infty},
\end{aligned}
\tag{2.2.4}
$$

which we can generalize to $p\in[1,\infty)$ by Hölder's inequality. In particular it follows by Lemma 2.2.6 that $\mathbf{I}\in C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-3\varepsilon}(\mathbb{T}^2)$ almost surely, where $\varepsilon\in(0,\gamma/2)$ can be chosen arbitrarily small.

Next we show that the approximating sequence $(\mathbf{I}^\delta)_{\delta>0}$ is more regular if $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$. In Fourier space,

$$
\widehat{\mathbf{I}^\delta}(t,\omega) = \sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega-m_1)\varphi(\delta m_1)H^{j_1}_{t-u_1}(\omega). \tag{2.2.5}
$$

We apply Itô's isometry, the triangle inequality and interpolation (B.5.1) to estimate

$$
\begin{aligned}
&\Big|\mathbb{E}\Big[\Big(\widehat{\mathbf{I}^\delta}(t,\omega)-\widehat{\mathbf{I}^\delta}(s,\omega)\Big)\overline{\Big(\widehat{\mathbf{I}^\delta}(t,\omega')-\widehat{\mathbf{I}^\delta}(s,\omega')\Big)}\Big]\Big| \\
&\lesssim |t-s|^{\gamma}|\omega|^{\gamma}|\omega'|^{\gamma}\|\sigma\|^2_{C_T\mathcal{H}^2}\sum_{m_1\in\mathbb{Z}^2}(1+|\omega-m_1|^2)^{-1}(1+|\omega'-m_1|^2)^{-1}(1+|\delta m_1|^2)^{-1},
\end{aligned}
$$

where we used that $|\widehat{\sigma}(u,\omega)| \lesssim (1+|\omega|^2)^{-1}\|\sigma\|_{C_T\mathcal{H}^2}$ uniformly in $u \in [0,T]$ and $\omega \in \mathbb{Z}^2$, and that $(1+x^2)^{1/2}|\varphi(x)| \lesssim \|\varphi\|_{L^\infty}$ uniformly in $x \in \mathbb{R}^2$ (the latter of which we leave implicit.) We may assume $\omega, \omega' \in \mathbb{Z}^2 \setminus \{0\}$, since $\widehat{\text{\textexclamdown}^\delta}(t,0) = 0$. We decompose the sum over $m_1 \in \mathbb{Z}^2$ into the domains $m_1 = 0$, $m_1 = \omega$, $m_1 = \omega'$ and $m_1 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\}$,

$$\begin{aligned}
&\sum_{m_1 \in \mathbb{Z}^2} (1+|\omega-m_1|^2)^{-1}(1+|\omega'-m_1|^2)^{-1}(1+|\delta m_1|^2)^{-1} \\
&\leq |\omega|^{-2}|\omega'|^{-2} + \delta^{-2}(1+|\omega-\omega'|^2)^{-1}(|\omega|^{-2}+|\omega'|^{-2}) \\
&\quad + \delta^{-2} \sum_{m_1 \in \mathbb{Z}^2\setminus\{0,\omega,\omega'\}} |\omega-m_1|^{-2}|\omega'-m_1|^{-2}|m_1|^{-2}.
\end{aligned}$$

We estimate the sum over $m_1 \in \mathbb{Z}^2\setminus\{0,\omega,\omega'\}$. Assume $\omega = \omega'$, then by Lemma E.1.5 and (E.1.3),

$$\sum_{m_1 \in \mathbb{Z}^2\setminus\{0,\omega,\omega'\}} |\omega-m_1|^{-4}|m_1|^{-2} \lesssim |\omega|^{-2}.$$

Assume $\omega \neq \omega'$, we apply Lemma E.1.4 to estimate for $\varepsilon < 1$,

$$\sum_{m_1 \in \mathbb{Z}^2\setminus\{0,\omega,\omega'\}} |\omega-m_1|^{-2}|\omega'-m_1|^{-2}|m_1|^{-2} \lesssim |\omega-\omega'|^{-2+\varepsilon}(|\omega|^{-2+2\varepsilon}+|\omega'|^{-2+2\varepsilon}).$$

Having established the decay of the Fourier coefficients, we can bound the Littlewood–Paley blocks. Let $q \in \mathbb{N}_{-1}$, we obtain uniformly in $x \in \mathbb{T}^2$,

$$\begin{aligned}
&\mathbb{E}[|\Delta_q \text{\textexclamdown}^\delta(t,x) - \Delta_q \text{\textexclamdown}^\delta(s,x)|^2] \\
&\leq \sum_{\omega,\omega' \in \mathbb{Z}^2\setminus\{0\}} \varrho_q(\omega)\varrho_q(\omega')\Big|\mathbb{E}\Big[\Big(\widehat{\text{\textexclamdown}^\delta}(t,\omega) - \widehat{\text{\textexclamdown}^\delta}(s,\omega)\Big)\overline{\Big(\widehat{\text{\textexclamdown}^\delta}(t,\omega') - \widehat{\text{\textexclamdown}^\delta}(s,\omega')\Big)}\Big]\Big| \\
&\lesssim (1+\delta^{-2})\|\sigma\|^2_{C_T\mathcal{H}^2}|t-s|^\gamma 2^{q(2\gamma+3\varepsilon)}.
\end{aligned}$$

Consequently, by Lemma C.3.8 and Lemma 2.2.6 for all $p \in [1,\infty)$ and $\varepsilon \in (0,\gamma/2)$,

$$\mathbb{E}[\|\text{\textexclamdown}^\delta\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-\gamma-4\varepsilon}}]^{1/p} \lesssim (1+\delta^{-2})^{1/2}\|\sigma\|_{C_T\mathcal{H}^2}$$

and in particular, $\text{\textexclamdown}^\delta \in C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-\gamma-4\varepsilon}(\mathbb{T}^2)$ almost surely, where $\varepsilon \in (0,\gamma/2)$ can be chosen arbitrarily small.

To deduce $\lim_{\delta\to 0}\mathbb{E}[\|\text{\textexclamdown} - \text{\textexclamdown}^\delta\|^p_{\mathscr{L}_T^\kappa\mathcal{C}^{\alpha+1}}]^{1/p} = 0$ for all $p \in [1,\infty)$, $\alpha < -2$ and $\kappa \in (0,1/2)$, let $\gamma \in (0,1]$, $\varepsilon \in (0,\gamma/2)$, $\max\{1/\varepsilon, 2\} < p < \infty$ and $0 < \vartheta' < \vartheta \leq 1$.

We use Nelson's estimate (Lemma C.3.8) and Kolmogorov's continuity criterion (Lemma 2.2.6) to bound

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{I} - \mathbf{I}^\delta\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-\vartheta-3\varepsilon}}] \\
&\lesssim \int_0^T \int_0^T |t-s|^{-2-p(\gamma/2-\varepsilon)} \sum_{q\in\mathbb{N}_{-1}} 2^{pq(-1-\gamma-\vartheta-\varepsilon)} \\
&\quad\times \int_{\mathbb{T}^2} \mathbb{E}[|\Delta_q(\mathbf{I} - \mathbf{I}^\delta)(t,x) - \Delta_q(\mathbf{I} - \mathbf{I}^\delta)(s,x)|^2]^{p/2} \,\mathrm{d}x \,\mathrm{d}s \,\mathrm{d}t.
\end{aligned}$$

We then use Lemma C.3.6 to bound

$$\begin{aligned}
&\mathbb{E}[|\Delta_q(\mathbf{I} - \mathbf{I}^\delta)(t,x) - \Delta_q(\mathbf{I} - \mathbf{I}^\delta)(s,x)|^2] \\
&\lesssim \delta^{2\vartheta'} \|1-\varphi\|^2_{C^1} \|\sigma\|^2_{C_T\mathcal{H}^1} \sum_{\omega\in\mathbb{Z}^2} \varrho_q(\omega)^2 \sum_{j_1=1}^2 \int_{-\infty}^\infty \mathrm{d}u_1 (1+|2\pi\omega|^2)^\vartheta |H^{j_1}_{t-u_1}(\omega) - H^{j_1}_{s-u_1}(\omega)|^2.
\end{aligned}$$

Using the same estimates as in the derivation of (2.2.4), we obtain

$$\begin{aligned}
&\mathbb{E}[|\Delta_q(\mathbf{I} - \mathbf{I}^\delta)(t,x) - \Delta_q(\mathbf{I} - \mathbf{I}^\delta)(s,x)|^2] \\
&\lesssim \delta^{2\vartheta'} |t-s|^\gamma \|1-\varphi\|^2_{C^1} \|\sigma\|^2_{C_T\mathcal{H}^1} \sum_{\omega\in\mathbb{Z}^2} \varrho_q(\omega)^2 (1+|2\pi\omega|^2)^\vartheta |\omega|^{2\gamma} \\
&\lesssim \delta^{2\vartheta'} |t-s|^\gamma \|1-\varphi\|^2_{C^1} \|\sigma\|^2_{C_T\mathcal{H}^1} 2^{2q(1+\gamma+\vartheta)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{I} - \mathbf{I}^\delta\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-\vartheta-3\varepsilon}}] \\
&\lesssim \delta^{p\vartheta'} \|1-\varphi\|^p_{C^1} \|\sigma\|^p_{C_T\mathcal{H}^1} \int_0^T \int_0^T |t-s|^{-2+p\varepsilon} \,\mathrm{d}s \,\mathrm{d}t \sum_{q\in\mathbb{N}_{-1}} 2^{-pq\varepsilon} \\
&\lesssim \delta^{p\vartheta'} \|1-\varphi\|^p_{C^1} \|\sigma\|^p_{C_T\mathcal{H}^1},
\end{aligned}$$

which implies $\lim_{\delta\to 0} \mathbb{E}[\|\mathbf{I} - \mathbf{I}^\delta\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-1-\gamma-\vartheta-3\varepsilon}}] = 0$. An application of Hölder's inequality allows us to deduce the convergence for all $p \in [1,\infty)$, where ε and ϑ can be chosen arbitrarily small. This yields the claim. □

The convergence of the other approximations in $\mathbb{X}^\delta$ is analogous. The only crucial observation here is that we can choose ϑ' and ϑ arbitrarily small when applying Lemma C.3.6, which ensures that the various applications of Lemma E.1.3 in the proof of Theorem 2.2.2 carry over. Hence, we will only show the existence of $\mathbb{X}$ and $\mathbb{X}^\delta_{\mathrm{can}}$, the second part of Theorem 2.2.2, that is, the convergence $\mathbb{X}^\delta \to \mathbb{X}$, then follows as above.

2.2.2 Feynman Diagrams

As demonstrated by (2.2.2), we may construct objects derived from white noise as (iterated) stochastic integrals. However, as we continue to multiply terms to build the enhancements of Theorem 2.2.2, we need to apply Itô's product rule to increasingly complicated expressions. To implement this procedure efficiently, we use an extension of a graphical representation that was developed by [MWX17; GP17], which relates our stochastic objects to Feynman diagrams.

We make use of several types of vertices, which will be introduced throughout this subsection. A root $\bullet_{(t,\omega)}$ represents the argument (t,ω). A circle $\circ$ denotes an instance of stochastic integration in time against a two-dimensional Brownian field with heterogeneity σ. Graphically this integrator is given by

$$\overset{(u_1,\omega_1,m_1,j_1)}{\circ} = \sum_{\omega_1\in\mathbb{Z}^2}\sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^\infty \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega_1-m_1)\ldots,$$

where the placeholder $\ldots$ stands for an integrand in u_1, ω_1, m_1, j_1, which is to be determined from the remaining diagram.

Generally, circles are enumerated by $k\in\mathbb{N}$ and equipped with tuples of dummy variables (u_k,ω_k,m_k,j_k), where $u_k\in(0,\infty)$, $\omega_k\in\mathbb{Z}^2$, $m_k\in\mathbb{Z}^2$ and $j_k=1,2$, which we point out explicitly for the reader's convenience. We refer to the ω, ω_k as the *frequencies* and the m_k as the *modes*. We denote coordinates of those by superscripts.

Vertices are connected by different types of directed edges, i.e. arrows, that represent integrands. Black arrows $(t,\omega)\longrightarrow(u_k,\omega_k,m_k,j_k)$ are associated to the integrand $H^{j_k}_{t-u_k}(\omega_k)$, which is the Fourier multiplier appearing in $\nabla\cdot\mathcal{I}$. Highlighted arrows $(t,\omega)\xrightarrow{j}(u_k,\omega_k,m_k,j_k)$, for $j=1,2$, are associated to $G^j(\omega_k)H^{j_k}_{t-u_k}(\omega_k)$, where

$$G^j(\omega_k)\coloneqq 2\pi\mathrm{i}\omega_k^j|2\pi\omega_k|^{-2}\mathbb{1}_{\omega_k\neq 0}$$

is the multiplier for $\partial_j\Phi$. The direction of an arrow indicates the smaller time variable u_k in the integration. Furthermore, if a single arrow emerges from the root $\bullet_{(t,\omega)}$, we multiply the integrand by $\mathbb{1}_{\omega=\omega_1}$.

For example by applying those rules, we can represent (2.2.2) as

$$\widehat{\uparrow}(t,\omega) = \overset{(u_1,\omega,m_1,j_1)}{\underset{(t,\omega)}{\uparrow}} := \sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega-m_1)H^{j_1}_{t-u_1}(\omega),$$

and $\nabla\Phi_{\uparrow}$ as

$$\widehat{\partial_j\Phi_{\uparrow}}(t,\omega) = \overset{(u_1,\omega,m_1,j_1)}{\underset{(t,\omega)}{\uparrow j}} := \sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega-m_1)G^j(\omega)H^{j_1}_{t-u_1}(\omega).$$

In particular if $\omega = 0$, then $H^j_t(\omega) = G^j(\omega) = 0$, hence we may assume $\omega \neq 0$ whenever it appears in either multiplier.

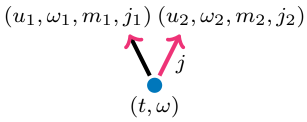
As a general rule, **black** objects in our diagrams are associated to scalars and **highlighted** objects are associated to vectors. Our arrows have highlighted arrowheads, indicating that they act on vector-valued objects. On the other hand, the type of object they return is determined by the arrow shaft. Accordingly, $(t,\omega)\longrightarrow(u_k,\omega_k,m_k,j_k)$ produces a scalar and $(t,\omega)\longrightarrow(u_k,\omega_k,m_k,j_k)$ a vector.

The existence of $\nabla\cdot\mathcal{I}[\uparrow\nabla\Phi_{\uparrow}]$ is not guaranteed by Bony's estimates (Lemma B.4.1) since $\uparrow\in C_T\mathcal{C}^{-1-}(\mathbb{T}^2)$ and hence $\nabla\Phi_{\uparrow}\in C_T\mathcal{C}^{0-}(\mathbb{T}^2;\mathbb{R}^2)$. In order to construct such non-linear objects, we formally apply Itô's product rule to identify the candidate Fourier transform. Let $n\in\mathbb{N}$ and assume $a_1,\dots,a_n\in\mathbb{N}$ are distinct. We denote by $\Sigma(a_1,\dots,a_n)$ the permutation group of $\{a_1,\dots,a_n\}$. Let $\omega_1,\omega_2\in\mathbb{Z}^2$, we compute

$$\begin{aligned}
&\mathscr{F}(\uparrow\nabla\Phi_{\uparrow})(t,\omega,j) = \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \widehat{\uparrow}(t,\omega_1)\widehat{\partial_j\Phi_{\uparrow}}(t,\omega_2)\\
&= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}}\sum_{j_1,j_2=1}^{2}\sum_{m_1,m_2\in\mathbb{Z}^2}\int_0^t \mathrm{d}W^{j_2}(u_2,m_2)\int_0^{u_2}\mathrm{d}W^{j_1}(u_1,m_1)\sum_{\varsigma\in\Sigma(1,2)}\\
&\quad\widehat{\sigma}(u_{\varsigma(1)},\omega_{\varsigma(1)}-m_{\varsigma(1)})\widehat{\sigma}(u_{\varsigma(2)},\omega_{\varsigma(2)}-m_{\varsigma(2)})H^{j_{\varsigma(1)}}_{t-u_{\varsigma(1)}}(\omega_{\varsigma(1)})G^j(\omega_{\varsigma(2)})H^{j_{\varsigma(2)}}_{t-u_{\varsigma(2)}}(\omega_{\varsigma(2)})\\
&+\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}}\sum_{j_1=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}u_1\widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_1,\omega_2+m_1)H^{j_1}_{t-u_1}(\omega_1)G^j(\omega_2)H^{j_1}_{t-u_1}(\omega_2)\\
&=: \widehat{\vee}(t,\omega,j)+\widehat{\circ}(t,\omega,j).
\end{aligned}\tag{2.2.6}$$

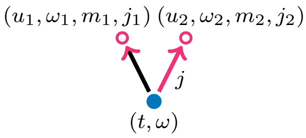
The symmetrization of the first integrand is a direct consequence of the Itô product rule. Note that in the second term, $j_1=j_2$, $u_1=u_2$ but $m_1=-m_2$, which is a consequence of the Hermitean structure of complex Brownian motion. Such a decomposition of stochastic products into iterated (stochastic) integrals is often called a *Wiener chaos decomposition* after [Wie38], see [Hai16; MWX17] for more details.

To represent , let us extend our graphical rules. Two arrows emerging from a common vertex represent a convolution in Fourier space. Their integrands are multiplied, but are related by the *Kirchhoff rule* [MWX17]: each vertex v has a frequency ω or ω_k which is part of its variables. This frequency will be called *ingoing* at the vertex v. An ingoing frequency at a vertex v is *outgoing* for the vertex w, if there exists an arrow pointing from w to v. A vertex is called *internal*, if there exists an arrow emerging from it. The rule states that at each internal vertex, the ingoing frequency (e.g. ω above) equals the sum of the outgoing frequencies (e.g. ω_1, ω_2 above). In graphical notation,



$$:= \mathbb{1}_{\omega=\omega_1+\omega_2} H^{j_1}_{t-u_1}(\omega_1) G^j(\omega_2) H^{j_2}_{t-u_2}(\omega_2).$$

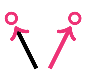
Those arrows will target the integrators $(u_1, \omega_1, m_1, j_1)$ and $(u_2, \omega_2, m_2, j_2)$ which will be multiplied and integrated over. The integral is then restricted to the simplex $u_1 < u_2$ to ensure that the integrand is adapted. To obtain the integral over the full domain, we symmetrize the integrand by permuting the indices that appear in the simplex. For example,



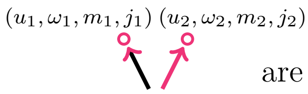
$$= \widehat{}(t, \omega, j)$$

is the first object in the decomposition (2.2.6).

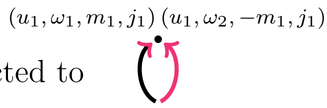
Next, let us discuss . As can be seen in (2.2.6), instances of Lebesgue integration arise through Itô correction terms. Itô corrections will be denoted by *contractions*, i.e. two arrows pointing at different vertices and are merged at a common vertex •. Graphically,

 are contracted to  .

Using the orthogonality of $(W^j(u,m))_{u\geq 0, m\in\mathbb{Z}^2, j=1,2}$, we can identify some of the dummy variables of the two vertices that are being merged. Indeed as in (2.2.6) we set $j_1 = j_2$, $u_1 = u_2$, $m_1 = -m_2$, but leave ω_1, ω_2 as they are. To make it easier for the reader to discern the multipliers attached to each arrow, we give both tuples of dummy variables, even after the contraction. Graphically,

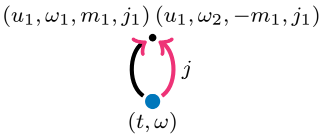


are contracted to



.

This results in the diagram



$$= \widehat{\mathsf{O}}(t,\omega,j),$$

which is the Itô correction in the decomposition (2.2.6) and coincides with the mean, $\mathsf{O} = \mathbb{E}[\uparrow\nabla\Phi_{\uparrow}]$. Diagrams carrying contractions are often called *Wick contractions* after [Wic50], see [GP17] for more details.

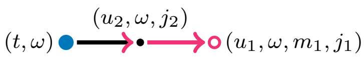
Depending on σ, O may be infinite. Hence, we consider the *renormalised enhancement*, where we only keep the first term $\vee$ of the decomposition (2.2.6) to define Υ. Formally, this is equivalent to subtracting the mean as a counterterm,

$$\vee = \uparrow\nabla\Phi_{\uparrow} - \mathbb{E}[\uparrow\nabla\Phi_{\uparrow}].$$

This identity can be made rigorous with suitable regularization and limiting procedures, see the discussion of the canonical enhancement at the end of this section.

Another source of Lebesgue integrals is the concatenation of the $\mathcal{I}$ operation. We obtain arrows pointing at other arrows, connected through a vertex $\bullet$. We multiply their integrands and make sure to respect the Kirchhoff rule. The multiplier of the incoming arrow will be determined by a tuple of dummy variables (u_k,ω_k,j_k) at the connecting vertex. For example, $\nabla\cdot\mathcal{I}[\nabla\Phi_{\uparrow}]$ can be expressed as

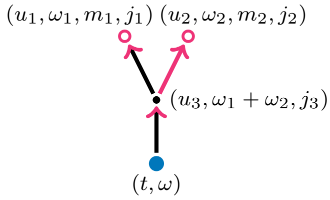
$$\mathscr{F}(\nabla\cdot\mathcal{I}[\nabla\Phi_{\uparrow}])(t,\omega) =$$



$$:= \sum_{j_1,j_2=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}u_2 \int_0^{u_2} \mathrm{d}W^{j_1}(u_1,m_1)\widehat{\sigma}(u_1,\omega-m_1)H^{j_2}_{t-u_2}(\omega)G^{j_2}(\omega)H^{j_1}_{u_2-u_1}(\omega).$$

The renormalised stochastic object $= \nabla \cdot \mathcal{I}[\nabla\Phi] - \mathbb{E}[\nabla \cdot \mathcal{I}[\nabla\Phi]]$ can then be expressed as

$\widehat{}(t,\omega) =$

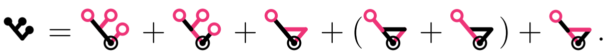


$$:= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_2,j_3=1}^{2} \sum_{m_1,m_2\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}W^{j_2}(u_2,m_2) \int_0^{u_2} \mathrm{d}W^{j_1}(u_1,m_1) \sum_{\varsigma\in\Sigma(1,2)}$$
$$\widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_2,\omega_2-m_2)H^{j_3}_{t-u_3}(\omega_1+\omega_2)H^{j_{\varsigma(1)}}_{u_3-u_{\varsigma(1)}}(\omega_{\varsigma(1)})G^{j_3}(\omega_{\varsigma(2)})H^{j_{\varsigma(2)}}_{u_3-u_{\varsigma(2)}}(\omega_{\varsigma(2)}).$$

In fact, we will not construct in itself. As we will see in Lemma 2.2.9, can be constructed as a continuous function in time that takes values in a space of distributions. On the other hand, we do not expect to admit pointwise-in-time values; instead we expect it to exist as a proper space-time distribution, which resembles the situation discussed in [MWX17, pp. 23–24 & pp. 32–33] and [CC18; HM18].

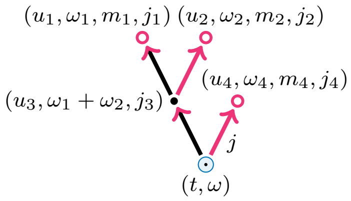
A particular variant of the root $\bullet$ is the vertex $\odot$ which arises through applications of the resonant product. The vertex $\odot$ relates the frequencies of the arrows that it joins through the $\sim$-relation defined in (1.2.4), see also [MWX17, (64)].

Let us consider more complicated objects. We have the Wiener chaos decomposition



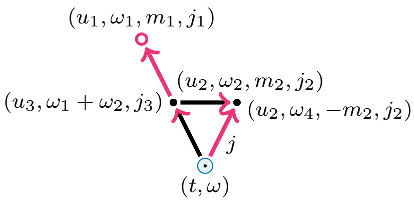
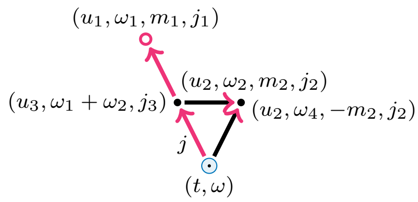
The contractions and that one might expect are absent in the renormalised enhancement due to our definition of $= \nabla \cdot \mathcal{I}[\nabla\Phi] - $.

We express the third-order Wiener chaos term  as follows. For $j = 1, 2$,

 $(t,\omega,j) =$ 

$$
\begin{aligned}
:= &\sum_{\substack{\omega_1,\omega_2,\omega_4\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2+\omega_4}} \sum_{j_1,j_2,j_3,j_4=1}^{2} \sum_{m_1,m_2,m_4\in\mathbb{Z}^2} \\
&\int_0^t \mathrm{d}u_3 \int_0^t \mathrm{d}W^{j_4}(u_4,m_4) \int_0^{u_4} \mathrm{d}W^{j_1}(u_1,m_1) \int_0^{u_1} \mathrm{d}W^{j_2}(u_2,m_2) \sum_{\varsigma\in\Sigma(1,2,4)} \\
&\widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_2,\omega_2-m_2)\widehat{\sigma}(u_4,\omega_4-m_4)H^{j_3}_{t-u_3}(\omega_{\varsigma(1)}+\omega_{\varsigma(2)})G^{j}(\omega_{\varsigma(4)})G^{j_3}(\omega_{\varsigma(2)}) \\
&\times H^{j_{\varsigma(4)}}_{t-u_{\varsigma(4)}}(\omega_{\varsigma(4)})H^{j_{\varsigma(1)}}_{u_3-u_{\varsigma(1)}}(\omega_{\varsigma(1)})H^{j_{\varsigma(2)}}_{u_3-u_{\varsigma(2)}}(\omega_{\varsigma(2)}) \sum_{\substack{k,l\in\mathbb{N}_{-1}\\ |k-l|\le 1}} \varrho_k(\omega_{\varsigma(1)}+\omega_{\varsigma(2)})\varrho_l(\omega_{\varsigma(4)}).
\end{aligned}
$$

The diagrams  and  may not exist in themselves, but the summed object  $:=$  $+$  does. Its iterated integral representation is given by

 $(t,\omega,j) =$  $+$ 

$$
\begin{aligned}
:= &\sum_{\substack{\omega_1,\omega_2,\omega_4\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2+\omega_4\\ (\omega_1+\omega_2)\sim\omega_4}} \sum_{j_1,j_2,j_3=1}^{2} \sum_{m_1,m_2\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_2 \int_0^{u_3} \mathrm{d}W^{j_1}(u_1,m_1) \\
&\widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_2,\omega_2-m_2)\widehat{\sigma}(u_2,\omega_4+m_2)(G^j(\omega_4)+G^j(\omega_1+\omega_2)) \\
&\times H^{j_3}_{t-u_3}(\omega_1+\omega_2)H^{j_2}_{t-u_2}(\omega_4)G^{j_3}(\omega_1)H^{j_1}_{u_3-u_1}(\omega_1)H^{j_2}_{u_3-u_2}(\omega_2).
\end{aligned}
$$

We will show in Lemma E.1.1 that the resulting factor $G^j(\omega_4) + G^j(\omega_1+\omega_2)$ has better decay in ω_4 than $G^j(\omega_4)$. This is due to the symmetry of G^j, which allows us to write $G^j(\omega_1+\omega_2) + G^j(\omega_4) = G^j(\omega-\omega_4) - G^j(-\omega_4)$. The improved decay leads to the well-posedness of  and is a higher-dimensional analogue of the product rule discussed in (2.1.3).

Remark 2.2.8. One could simplify the contraction by using the identity

$$
\sum_{m_2\in\mathbb{Z}^2} \widehat{\sigma}(u_2,\omega_2-m_2)\widehat{\sigma}(u_2,\omega_4+m_2) = \widehat{\sigma^2}(u_2,\omega_2+\omega_4). \tag{2.2.7}
$$

This idea would allow us to derive bounds in terms of $\|\sigma^2\|_{C_T\mathcal{H}^2}$ rather than $\|\sigma\|^2_{C_T\mathcal{H}^2}$. However, (2.2.7) is no longer applicable in our prelimiting enhancement due to the cut-off $\varphi(\delta m_2)$ (cf. (2.2.5)). Instead we use direct estimates that do not rely on (2.2.7).

The remaining diagrams , and are similar to the ones given above.

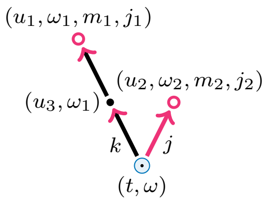
We extend our graphical rules to incorporate the operator $\partial_l \mathcal{I}$ for $l = 1, 2$. An indexed black arrow pointing at a scalar object $(t,\omega) \xrightarrow{l} (u_k,\omega_k)$ is associated to the multiplier $H^l_{t-u_k}(\omega_k)$. On the other hand, a doubly-indexed, highlighted arrow pointing at a scalar object $(t,\omega) \xrightarrow{l,j} (u_k,\omega_k)$ is associated to $G^j(\omega_k)H^l_{t-u_k}(\omega_k)$.

The remaining object in the enhancement is $_{k,j}$ for $k, j = 1, 2$. We consider the Wiener chaos decomposition

$$ {}_{k,j} = (k,j) + (k,j) + \big((k,j) + (k,j) \big). $$

The first term is given by

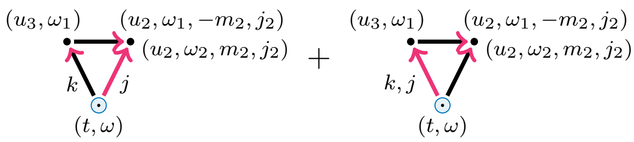
$$ \widehat{}(t,\omega,k,j) = $$



$$:= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2\\ \omega_1\sim\omega_2}} \sum_{j_1,j_2=1}^{2} \sum_{m_1,m_2\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^t \mathrm{d}W^{j_1}(u_1,m_1) \int_0^{u_1} \mathrm{d}W^{j_2}(u_2,m_2) \sum_{\varsigma\in\Sigma(1,2)} $$
$$ \widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_2,\omega_2-m_2)H^k_{t-u_3}(\omega_{\varsigma(1)})H^{j_{\varsigma(1)}}_{u_3-u_{\varsigma(1)}}(\omega_{\varsigma(1)})G^j(\omega_{\varsigma(2)})H^{j_{\varsigma(2)}}_{t-u_{\varsigma(2)}}(\omega_{\varsigma(2)}) $$

and the second term is again similar. We consider the contractions as a summed object $:=$ $+$. We obtain

$$ \widehat{}(t,\omega,k,j) = $$

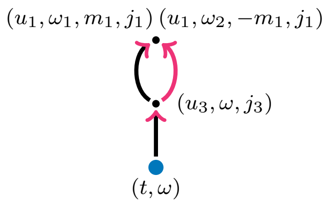


$$:= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2\\ \omega_1\sim\omega_2}} \sum_{j_2=1}^{2} \sum_{m_2\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_2 \widehat{\sigma}(u_2,\omega_1+m_2)\widehat{\sigma}(u_2,\omega_2-m_2) $$
$$ \times \big(G^j(\omega_2)+G^j(\omega_1)\big)H^k_{t-u_3}(\omega_1)H^{j_2}_{u_3-u_2}(\omega_1)H^{j_2}_{t-u_2}(\omega_2). $$

We can define approximate diagrams as in (2.2.5) by multiplying the cut-off $\varphi(\delta m_k)$ (cf. (1.2.6)) to each instance of the noise $\circ_{(u_k,\omega_k,m_k,j_k)}$. In general, we denote the regularization of a diagram by a superscript δ. The canonical enhancement $\mathbb{X}^\delta_{\text{can}} = (\;^\delta, \;^\delta_{\text{can}}, \;^\delta_{\text{can}}, \;^\delta)$ is then built from regularized noise terms, but retains the diverging sequences that are removed in the renormalised enhancement $\mathbb{X}$. Repeating (2.2.6), we may consider the decomposition of the diagram with cut-off,

$$\;^\delta_{\text{can}} = \nabla \cdot \mathcal{I}[\;^\delta \nabla \Phi_{\;^\delta}] = \;^\delta + \;^\delta,$$

where $\;^\delta = \mathbb{E}[\nabla \cdot \mathcal{I}[\;^\delta \nabla \Phi_{\;^\delta}]]$. In addition to $\;^\delta$, we also have to control the mean,

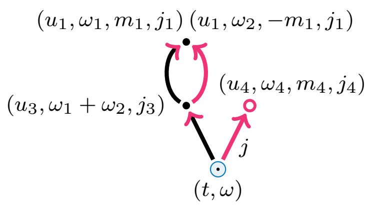
$\widehat{\;^\delta}(t,\omega) =$ 

$$:= \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_3=1}^{2} \sum_{m_1\in\mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_1 \widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_1,\omega_2+m_1)|\varphi(\delta m_1)|^2$$
$$\times H^{j_3}_{t-u_3}(\omega) H^{j_1}_{u_3-u_1}(\omega_1) G^{j_3}(\omega_2) H^{j_1}_{u_3-u_1}(\omega_2).$$

Including $\;^\delta$ in $\;^\delta_{\text{can}}$ generates additional terms in the decomposition of $\;^\delta_{\text{can}}$,

$$\;^\delta_{\text{can}} = \;^\delta + \;^\delta + \;^\delta.$$

The diagram $\;^\delta$ is given by

$\widehat{\;^\delta}(t,\omega,j) =$ 

$$:= \sum_{\substack{\omega_1,\omega_2,\omega_4\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2+\omega_4\\ (\omega_1+\omega_2)\sim\omega_4}} \sum_{j_1,j_3,j_4=1}^{2} \sum_{m_1,m_4\in\mathbb{Z}^2} \int_0^t \mathrm{d}W^{j_4}(u_4,m_4) \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_1$$
$$\widehat{\sigma}(u_1,\omega_1-m_1)\widehat{\sigma}(u_1,\omega_2+m_1)\widehat{\sigma}(u_4,\omega_4-m_4)|\varphi(\delta m_1)|^2\varphi(\delta m_4)$$
$$\times H^{j_3}_{t-u_3}(\omega_1+\omega_2) G^{j}(\omega_4) H^{j_4}_{t-u_4}(\omega_4) H^{j_1}_{u_3-u_1}(\omega_1) G^{j_3}(\omega_2) H^{j_1}_{u_3-u_1}(\omega_2)$$

and $\;^\delta$ is again similar. Here, we have implicitly changed our graphical rules to include the cut-off.

2.2.3 Diagrams of Order 2 and 3

In this section we construct the second-order diagrams Υ, $\color{magenta}\Upsilon$ and $\color{magenta}\Upsilon$ and the third-order diagrams $\color{magenta}\Upsilon$ and $\color{magenta}\Upsilon$.

Lemma 2.2.9. *Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T L^\infty(\mathbb{T}^2)$. Then for all $p \in [1, \infty)$ we have $\mathbb{E}[\| \Upsilon \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim \|\sigma\|^2_{C_T L^\infty}$ and in particular $\Upsilon \in \mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$ a.s.*

From now on we denote ${\color{magenta}\Upsilon} := \Upsilon$ to emphasize the separate rôles of colour and shape. We first derive a useful upper bound on the second moments of $\color{magenta}\Upsilon$ in terms of an explicit, time-dependent function $\mathsf{S}_{s,t}\Upsilon$. We call this function the *shape coefficient*.

Definition 2.2.10. *Let $s, t \geq 0$ and $\omega_1, \omega_2 \in 2\pi\mathbb{Z}^2 \setminus \{0\}$. We define the* shape coefficient

$$\mathsf{S}_{s,t}\Upsilon(\omega_1, \omega_2) := \int_0^t \mathrm{d}u_3 \int_0^s \mathrm{d}u_3' \int_{-\infty}^{u_3 \wedge u_3'} \mathrm{d}u_2 \int_{-\infty}^{u_3 \wedge u_3'} \mathrm{d}u_1 \\ \mathrm{e}^{-|t+s-(u_3+u_3')||\omega_1+\omega_2|^2} \mathrm{e}^{-|u_3+u_3'-2u_1||\omega_1|^2} \mathrm{e}^{-|u_3+u_3'-2u_2||\omega_2|^2}, \tag{2.2.8}$$

and the increment shape coefficient

$$\mathsf{D}_{s,t}\Upsilon := \mathsf{S}_{t,t}\Upsilon + \mathsf{S}_{s,s}\Upsilon - \mathsf{S}_{s,t}\Upsilon - \mathsf{S}_{t,s}\Upsilon. \tag{2.2.9}$$

Here, the letter S stands for shape and the letter D for difference. It will be clear from the proof of Lemma 2.2.11 that $\mathsf{D}_{s,t}\Upsilon \geq 0$.

Shape coefficients play a central rôle in our bounds, as they capture the iterated applications of $\nabla \cdot \mathcal{I}$; they fundamentally depend on the shape of the diagram, as opposed to the additional colouring induced by $\nabla\Phi$. Using this notation, we obtain the following bound.

Lemma 2.2.11. *Let $s, t \in [0, T]$, $x \in \mathbb{T}^2$ and $q \in \mathbb{N}_{-1}$. It holds that*

$$\mathbb{E}[|\Delta_q {\color{magenta}\Upsilon}(t,x) - \Delta_q {\color{magenta}\Upsilon}(s,x)|^2] \\ \lesssim \|\sigma\|^4_{C_T L^\infty} \sum_{\omega \in \mathbb{Z}^2} \varrho_q(\omega)^2 \sum_{\substack{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\} \\ \omega = \omega_1 + \omega_2}} |\omega_1|^2 |\omega_2|^{-2} |\langle \omega_2, \omega_1 + \omega_2 \rangle|^2 \mathsf{D}_{s,t}\Upsilon(2\pi\omega_1, 2\pi\omega_2). \tag{2.2.10}$$

We refer to the prefactor $|\omega_1|^2|\omega_2|^{-2}|\langle\omega_2,\omega_1+\omega_2\rangle|^2$ as the *colouring* of Υ.

Before we give the proof of Lemma 2.2.11, let us comment on its general strategy. In Υ, Υ and Υ, it does not suffice to apply the triangle inequality to push the absolute value past the integral sign. This is related to the appearance of the sub-diagram $\vee$, which we do not expect to be pointwise evaluable. Instead, we rely on bilinearity and expand the integrand according to the identity

$$(f(t)-f(s))\overline{(g(t)-g(s))} = f(t)\overline{g(t)} + f(s)\overline{g(s)} - f(s)\overline{g(t)} - f(t)\overline{g(s)},$$

which leads to the common equation for this type of shape coefficient,

$$\mathsf{D}_{s,t} = \mathsf{S}_{t,t} + \mathsf{S}_{s,s} - \mathsf{S}_{s,t} - \mathsf{S}_{t,s}.$$

We refer to Lemmas 2.2.15–2.2.19 for instances where we can simplify our calculations by applying the triangle inequality.

Proof of Lemma 2.2.11. Let $s,t\in[0,T]$, $x\in\mathbb{T}^2$ and $q\in\mathbb{N}_{-1}$. An application of (C.3.1) yields

$$\mathbb{E}[|\Delta_q\Upsilon(t,x)-\Delta_q\Upsilon(s,x)|^2] \le \|\sigma\|^4_{C_TL^\infty}\mathbb{E}[|\Delta_q\Upsilon(t,x)-\Delta_q\Upsilon(s,x)|^2]$$

where Υ is defined by

$$\begin{aligned}\widehat{\Upsilon}(t,\omega) \coloneqq \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_2,j_3=1}^{2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}W^{j_2}(u_2,\omega_2) \int_0^{u_2} \mathrm{d}W^{j_1}(u_1,\omega_1)\\ \times \sum_{\varsigma\in\Sigma(1,2)} H^{j_3}_{t-u_3}(\omega_{\varsigma(1)}+\omega_{\varsigma(2)}) H^{j_{\varsigma(1)}}_{u_3-u_{\varsigma(1)}}(\omega_{\varsigma(1)}) G^{j_3}(\omega_{\varsigma(2)}) H^{j_{\varsigma(2)}}_{u_3-u_{\varsigma(2)}}(\omega_{\varsigma(2)}).\end{aligned}$$

Using that

$$\mathbb{E}\Big[\widehat{\Upsilon}(t,\omega)\overline{\widehat{\Upsilon}(s,\omega')}\Big] = 0 \quad \text{if} \quad \omega\neq\omega'\in\mathbb{Z}^2,$$

we obtain

$$\mathbb{E}[|\Delta_q\Upsilon(t,x)-\Delta_q\Upsilon(s,x)|^2] = \sum_{\omega\in\mathbb{Z}^2}\varrho_q(\omega)^2\mathbb{E}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big].$$

It follows by an application of Itô's isometry and Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big]\\
&\leq \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}} \sum_{j_1,j_2,j_3,j_3'=1}^{2} 2! \int_{-\infty}^{\infty} \mathrm{d}u_2 \int_{-\infty}^{\infty} \mathrm{d}u_1 \int_0^{\infty} \mathrm{d}u_3 \int_0^{\infty} \mathrm{d}u_3'\\
&\quad (H^{j_3}_{t-u_3}(\omega_1+\omega_2)-H^{j_3}_{s-u_3}(\omega_1+\omega_2))H^{j_1}_{u_3-u_1}(\omega_1)G^{j_3}(\omega_2)H^{j_2}_{u_3-u_2}(\omega_2)\\
&\quad \times \overline{(H^{j_3'}_{t-u_3'}(\omega_1+\omega_2)-H^{j_3'}_{s-u_3'}(\omega_1+\omega_2))H^{j_1}_{u_3'-u_1}(\omega_1)G^{j_3'}(\omega_2)H^{j_2}_{u_3'-u_2}(\omega_2)},
\end{aligned}$$

where we used that the complex absolute value of $z\in\mathbb{C}$ is given by $|z|^2=z\overline{z}$. Recalling the definition of $\mathsf{D}_{s,t}\Upsilon$ from (2.2.9), we obtain

$$\mathbb{E}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big] \lesssim \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\}\\ \omega=\omega_1+\omega_2}} |\omega_1|^2|\omega_2|^{-2}|\langle\omega_2,\omega_1+\omega_2\rangle|^2\mathsf{D}_{s,t}\Upsilon(2\pi\omega_1,2\pi\omega_2).$$

This yields the claim. $\square$

To evaluate the integrals in $\mathsf{S}_{s,t}\Upsilon$, we use a case distinction over $(\omega_1\perp\omega_2)$ and $\neg(\omega_1\perp\omega_2)$. We can then find explicit expressions for $\mathsf{D}_{s,t}\Upsilon$ via (2.2.9), which can be used to derive the necessary bounds. This is the content of Lemma D.0.1. We can now give the proof of Lemma 2.2.9.

Proof of Lemma 2.2.9. Let $T>0$ and $\gamma\in[0,1]$. To decompose the right-hand side of (2.2.10), we introduce the orthogonal sum

$$\mathbf{E}^{\perp}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big] := \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\}\\ \omega=\omega_1+\omega_2\\ \omega_1\perp\omega_2}} |\omega_1|^2|\omega_2|^{-2}|\langle\omega_2,\omega_1+\omega_2\rangle|^2\mathsf{D}_{s,t}\Upsilon(2\pi\omega_1,2\pi\omega_2)$$

and the non-orthogonal sum

$$\mathbf{E}^{\neg}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big] := \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\}\\ \omega=\omega_1+\omega_2\\ \neg(\omega_1\perp\omega_2)}} |\omega_1|^2|\omega_2|^{-2}|\langle\omega_2,\omega_1+\omega_2\rangle|^2\mathsf{D}_{s,t}\Upsilon(2\pi\omega_1,2\pi\omega_2).$$

We obtain the decomposition

$$\begin{aligned}
&\mathbb{E}[|\Delta_q\Upsilon(t,x)-\Delta_q\Upsilon(s,x)|^2]\\
&\lesssim \|\sigma\|^4_{C_TL^\infty}\sum_{\omega\in\mathbb{Z}^2\setminus\{0\}}\varrho_q(\omega)^2\Big(\mathbf{E}^{\perp}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big]+\mathbf{E}^{\neg}\Big[\Big|\widehat{\Upsilon}(t,\omega)-\widehat{\Upsilon}(s,\omega)\Big|^2\Big]\Big),
\end{aligned}$$

where we used that $\widehat{\Upsilon}(t,0) = 0$. In the orthogonal sum $\mathbf{E}^{\perp}$, we obtain by Lemma D.0.1,

$$\mathsf{D}_{s,t}\Upsilon(2\pi\omega_1, 2\pi\omega_2) \lesssim |t-s|^{\gamma}|\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-4+2\gamma},$$

so that

$$\mathbf{E}^{\perp}\Big[\Big|\widehat{\Upsilon}(t,\omega) - \widehat{\Upsilon}(s,\omega)\Big|^2\Big] \lesssim |t-s|^{\gamma}|\omega|^{-4+2\gamma} \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\} \\ \omega=\omega_1+\omega_2 \\ \omega_1\perp\omega_2}} 1,$$

where we used the orthogonality ($\omega_1 \perp \omega_2$) to identify $|\langle\omega_2, \omega_1+\omega_2\rangle|^2 = |\omega_2|^4$. Using the orthogonality again, we obtain the bound $|\omega_1|^2 \leq |\omega_1|^2 + |\omega_2|^2 = |\omega|^2$. By applying (E.1.1) to the finite sum over $\omega_1 \in \mathbb{Z}^2 \setminus \{0\}$, $|\omega_1| \leq |\omega|$, we arrive at

$$\mathbf{E}^{\perp}\Big[\Big|\widehat{\Upsilon}(t,\omega) - \widehat{\Upsilon}(s,\omega)\Big|^2\Big] \lesssim |t-s|^{\gamma}|\omega|^{-2+2\gamma}.$$

Next we consider the non-orthogonal sum $\mathbf{E}^{\neg}$. Lemma D.0.1 implies

$$\begin{aligned}\mathsf{D}_{s,t}\Upsilon(2\pi\omega_1, 2\pi\omega_2) \lesssim{}& |t-s|^{\gamma}|\omega_1|^{-4+2\gamma}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2} \\ &+ |t-s|^{\gamma}|\omega_1|^{-4}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2+2\gamma},\end{aligned}$$

and an application of Lemma E.1.3 yields for each $\gamma \in (0,1)$ and $\varepsilon \in (0, (2-2\gamma\wedge 1))$,

$$\begin{aligned}\mathbf{E}^{\neg}\Big[\Big|\widehat{\Upsilon}(t,\omega) - \widehat{\Upsilon}(s,\omega)\Big|^2\Big] &\lesssim |t-s|^{\gamma} \sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\} \\ \omega=\omega_1+\omega_2 \\ \neg(\omega_1\perp\omega_2)}} (|\omega_1|^{-2+2\gamma}|\omega_2|^{-2} + |\omega|^{2\gamma}|\omega_1|^{-2}|\omega_2|^{-2}) \\ &\lesssim |t-s|^{\gamma}|\omega|^{-2+2\gamma+2\varepsilon},\end{aligned}$$

where we also used the Cauchy–Schwarz inequality to control $|\langle\omega_2, \omega_1+\omega_2\rangle|^2 \lesssim |\omega_2|^2|\omega_1+\omega_2|^2$. Applying these results to the bound (2.2.10), we arrive at

$$\mathbb{E}[|\Delta_q\Upsilon(t,x) - \Delta_q\Upsilon(s,x)|^2] \lesssim |t-s|^{\gamma}\|\sigma\|^4_{C_TL^{\infty}}2^{q(2\gamma+2\varepsilon)},$$

which is uniform in $x \in \mathbb{T}^2$. Assume in addition $\varepsilon < \gamma/2$. Using that $\Delta_q\Upsilon$ denotes an iterated Itô integral, we obtain by Lemma C.3.8 and Lemma 2.2.6 for all $p \in [1,\infty)$,

$$\mathbb{E}[\|\Upsilon\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-\gamma-4\varepsilon}}]^{1/p} \lesssim \|\sigma\|^2_{C_TL^{\infty}}$$

and therefore $\Upsilon \in \mathscr{L}^{\kappa}_T\mathcal{C}^{0-}(\mathbb{T}^2)$ a.s. for each $\kappa \in (0,1/2)$. $\square$

Next we consider the third-order diagrams and .

Lemma 2.2.12. *Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T L^\infty(\mathbb{T}^2)$. Then for all $p \in [1, \infty)$ we have*

$$\mathbb{E}[\| \, \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}}]^{1/p} + \mathbb{E}[\| \, \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim \|\sigma\|^3_{C_T L^\infty}$$

and in particular , $\in \mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}(\mathbb{T}^2; \mathbb{R}^2)$ *a.s.*

Proof. The proof of this lemma is similar to the one for Lemma 2.2.9, so we only provide a sketch. The key idea is to consider the shape coefficient

$$\mathsf{S}_{s,t}(\omega_1, \omega_2, \omega_4) := \mathsf{S}_{s,t}(\omega_1, \omega_2)\mathsf{S}_{s,t}(\omega_4), \qquad s, t \geq 0, \quad \omega_1, \omega_2, \omega_4 \in 2\pi\mathbb{Z}^2 \setminus \{0\},$$

where the factor $\mathsf{S}_{s,t}$ was already defined in (2.2.8), and $\mathsf{S}_{s,t}$ is given by

$$\mathsf{S}_{s,t}(\omega_4) := \int_{-\infty}^{s \wedge t} \mathrm{d}u_4 \mathrm{e}^{-|t-u_4||\omega_4|^2} \mathrm{e}^{-|s-u_4||\omega_4|^2} = \frac{1}{2}|\omega_4|^{-2} \mathrm{e}^{-|t-s||\omega_4|^2}.$$

The factorization $\mathsf{S}_{s,t} = \mathsf{S}_{s,t} \mathsf{S}_{s,t}$ follows since there is no arrow *pointing* at the common root between the vertices labelled by u_3 and u_4.

We can then find explicit expressions for $\mathsf{D}_{s,t}$, which we use to bound the second moments of and . The claim then follows by (C.3.1), Lemma C.3.8 and Lemma 2.2.6. □

We can also show the existence of the diagrams and .

Lemma 2.2.13. *Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T L^\infty(\mathbb{T}^2)$. Then for all $p \in [1, \infty)$ we have*

$$\mathbb{E}[\| \, \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}}]^{1/p} + \mathbb{E}[\| \, \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim \|\sigma\|^2_{C_T L^\infty}$$

and in particular , $\in \mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}(\mathbb{T}^2; \mathbb{R}^{2\times 2})$ *a.s.*

We define a shape coefficient for and . Since those do not contain the problematic sub-diagram , it suffices to push the absolute value past the integral sign. We denote this fact by the letter A for absolute value. In particular, we may bound every integral over $[0, \infty)$ by $(-\infty, \infty)$, which simplifies our calculations.

Definition 2.2.14. *Let $s,t \geq 0$, $\omega_1, \omega_1', \omega_2 \in \mathbb{Z}^2 \setminus \{0\}$ and $k,k' = 1,2$. We set*

$$\mathsf{A}^{k,k'}_{s,t}(\omega_1, \omega_1', \omega_2)$$
$$:= \sum_{j=1}^{2} \int_{-\infty}^{\infty} \mathrm{d}u_2 \int_{-\infty}^{\infty} \mathrm{d}u_1 \int_{-\infty}^{\infty} \mathrm{d}u_1' |H^k_{t-u_1}(\omega_1) H^j_{t-u_2}(\omega_2) - H^k_{s-u_1}(\omega_1) H^j_{s-u_2}(\omega_2)|$$
$$\times |H^{k'}_{t-u_1'}(\omega_1') H^j_{t-u_2}(\omega_2) - H^{k'}_{s-u_1'}(\omega_1') H^j_{s-u_2}(\omega_2)|.$$

We can then bound the second moment of in terms of this object.

Lemma 2.2.15. *Let $s,t \in [0,T]$, $x \in \mathbb{T}^2$, $k,j = 1,2$ and $q \in \mathbb{N}_{-1}$. It holds that*

$$\mathbb{E}[|\Delta_q (t,x,k,j) - \Delta_q (s,x,k,j)|^2]$$
$$\lesssim \|\sigma\|^4_{C_T L^\infty} \sum_{\omega \in \mathbb{Z}^2} \varrho_q(\omega)^2 \sum_{\substack{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\} \\ \omega = \omega_1 + \omega_2 \\ \omega_1 \sim \omega_2}} |\omega_2|^{-2} \mathsf{A}^{k,k}_{s,t}(\omega_1, \omega_1, \omega_2).$$

Proof. The claim follows by an application of (C.3.1), Itô's isometry (Lemma C.3.3), the triangle inequality and direct computation. □

Proof of Lemma 2.2.13. We provide a sketch. The shape coefficient is controlled in Lemma D.0.3, we can then use fairly direct estimates and apply Lemma C.3.8 and Lemma 2.2.6 as before. We observe that differs from only in its colouring of the ω_1 and ω_2 arrows, with the sum of the exponents preserved. Consequently by the same arguments as for , we can also construct . □

2.2.4 Wick Contractions

In this section we construct the contractions , , $=$ $+$ and $=$ $+$.

The diagrams , differ from , and , , despite their similarity in structure. The first two are well-defined, since two applications of $\nabla\Phi$ appear inside the -shaped sub-diagram. This is not the case for , and , as one may tell by the distribution of highlighted arrows.

However, by adding the problematic diagrams, $=$ $+$ and $=$ $+$ we can make use of the symmetry $G^j(\omega_4) = -G^j(-\omega_4)$, for all $j = 1,2$ and $\omega_4 \in \mathbb{Z}^2$, to establish the existence of the summed objects.

We define a shape coefficient for those diagrams.

Definition 2.2.16. *Let $s,t \geq 0$, $\omega_1, \omega_2, \omega_3 \in \mathbb{Z}^2 \setminus \{0\}$ and $k = 1,2$. We set*

$$\begin{aligned}&\mathsf{A}^k_{s,t}(\omega_1,\omega_2,\omega_3)\\ &:= \sum_{j=1}^{2} \int_{-\infty}^{t} \mathrm{d}u_1 \int_{-\infty}^{u_1} \mathrm{d}u_2 |(H^k_{t-u_1}(\omega_1) H^j_{t-u_2}(\omega_2) - H^k_{s-u_1}(\omega_1) H^j_{s-u_2}(\omega_2)) H^j_{u_1-u_2}(\omega_3)|.\end{aligned}$$

In Lemma 2.2.17 we establish the existence of , , and in Lemma 2.2.18, we establish the existence of .

Lemma 2.2.17. *Let $T > 0$, $\alpha < -2$, $\kappa \in (0,1/2)$ and $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$. Then for all $p \in [1,\infty)$ we have*

$$\mathbb{E}[\| \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}}]^{1/p} + \mathbb{E}[\| \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}}]^{1/p} + \mathbb{E}[\| \|^p_{\mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim \|\sigma\|_{C_T L^\infty} \|\sigma\|^2_{C_T\mathcal{H}^2}$$

and in particular , , $\in \mathscr{L}^\kappa_T \mathcal{C}^{3\alpha+6}(\mathbb{T}^2;\mathbb{R}^2)$ a.s.

Lemma 2.2.18. *Let $T > 0$, $\alpha < -2$, $\kappa \in (0,1/2)$ and $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$. Then it holds that $\in \mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^{2\times 2})$ and*

$$\| \|_{\mathscr{L}^\kappa_T \mathcal{C}^{2\alpha+4}} \lesssim \|\sigma\|^2_{C_T\mathcal{H}^2}.$$

We first show Lemma 2.2.17. Let us focus on , the derivation for and is similar, but easier.

Proof of Lemma 2.2.17. Let $s,t \in [0,T]$, $x \in \mathbb{T}^2$, $j = 1,2$ and $q \in \mathbb{N}_{-1}$. An application of (C.3.1) yields

$$\mathbb{E}[|\Delta_q (t,x,j) - \Delta_q (s,x,j)|^2] \leq \|\sigma\|^2_{C_T L^\infty} \mathbb{E}[|\Delta_q (t,x,j) - \Delta_q (s,x,j)|^2],$$

where is defined by

$$\begin{aligned}&\widehat{}(t,\omega,j)\\ &:= \sum_{\substack{\omega_1,\omega_2,\omega_4 \in \mathbb{Z}^2 \\ \omega = \omega_1+\omega_2+\omega_4 \\ (\omega_1+\omega_2) \sim \omega_4}} \sum_{j_1,j_2,j_3=1}^{2} \sum_{m_2 \in \mathbb{Z}^2} \int_0^t \mathrm{d}u_3 \int_0^{u_3} \mathrm{d}u_2 \int_0^{u_3} \mathrm{d}W^{j_1}(u_1,\omega_1)\\ &\quad \widehat{\sigma}(u_2,\omega_2 - m_2) \widehat{\sigma}(u_2,\omega_4 + m_2)(G^j(\omega_4) + G^j(\omega_1+\omega_2)) H^{j_3}_{t-u_3}(\omega_1+\omega_2) H^{j_2}_{t-u_2}(\omega_4)\\ &\quad \times G^{j_3}(\omega_1) H^{j_1}_{u_3-u_1}(\omega_1) H^{j_2}_{u_3-u_2}(\omega_2).\end{aligned}$$

Using the definition of the Littlewood–Paley block Δ_q, uniformly in $x \in \mathbb{T}^2$,

$$
\begin{aligned}
&\mathbb{E}[|\Delta_q (t,x,j) - \Delta_q (s,x,j)|^2] \\
&\leq \sum_{\omega,\omega' \in \mathbb{Z}^2} \varrho_q(\omega)\varrho_q(\omega')\Big|\mathbb{E}\Big[\Big(\widehat{\ }(t,\omega,j) - \widehat{\ }(s,\omega,j)\Big)\overline{\Big(\widehat{\ }(t,\omega',j) - \widehat{\ }(s,\omega',j)\Big)}\Big]\Big|.
\end{aligned}
$$

We apply Itô's isometry and a decay estimate (Lemma E.1.1) for the symmetrized elliptic multiplier $G^j(\omega_1+\omega_2) + G^j(\omega_4) = G^j(\omega-\omega_4) - G^j(-\omega_4)$. We obtain the bound

$$
\begin{aligned}
&\mathbb{E}[|\Delta_q (t,x,j) - \Delta_q (s,x,j)|^2] \\
&\lesssim \|\sigma\|^2_{C_T L^\infty}\|\sigma\|^4_{C_T\mathcal{H}^2} \sum_{\omega,\omega'\in\mathbb{Z}^2} \varrho_q(\omega)\varrho_q(\omega')|\omega||\omega'| \sum_{\substack{\omega_1,\omega_2,\omega_4\in\mathbb{Z}^2\setminus\{0\}\\ \omega_1+\omega_2\in\mathbb{Z}^2\setminus\{0\}\\ \omega=\omega_1+\omega_2+\omega_4\\ (\omega_1+\omega_2)\sim\omega_4}} \sum_{\substack{\omega_2',\omega_4'\in\mathbb{Z}^2\setminus\{0\}\\ \omega_1+\omega_2'\in\mathbb{Z}^2\setminus\{0\}\\ \omega'=\omega_1+\omega_2'+\omega_4'\\ (\omega_1+\omega_2')\sim\omega_4'}} \sum_{j_3,j_3'=1}^{2} \sum_{m_2,m_2'\in\mathbb{Z}^2} \\
&\quad (1+|\omega_2-m_2|^2)^{-1}(1+|\omega_2'-m_2'|^2)^{-1}(1+|\omega_4+m_2|^2)^{-1}(1+|\omega_4'+m_2'|^2)^{-1} \\
&\quad \times |\omega_1|^{-2}|\omega-\omega_4|^{-2}|\omega'-\omega_4'|^{-2}(1+|\omega||\omega_4|^{-1})(1+|\omega'||\omega_4'|^{-1}) \\
&\quad \times \mathsf{A}^{j_3}_{s,t}\blacktriangledown(\omega_1+\omega_2,\omega_4,\omega_2)\mathsf{A}^{j_3'}_{s,t}\blacktriangledown(\omega_1+\omega_2',\omega_4',\omega_2').
\end{aligned}
\tag{2.2.11}
$$

We control the shape coefficient with Lemma D.0.2 and obtain for all $\gamma \in [0,1]$,

$$
\mathsf{A}^{j_3}_{s,t}\blacktriangledown(\omega_1+\omega_2,\omega_4,\omega_2) \lesssim |t-s|^\gamma|\omega_4|^{2\gamma}|\omega_2|^{-1}.
$$

We can then plug this expression into (2.2.11) and apply Lemma E.2.1 to control the sums over ω_4 and $m_2 \in \mathbb{Z}^2$. Let $\gamma \in (0,1/2)$ and $\varepsilon \in (0, 1-2\gamma)$, it follows that

$$
\begin{aligned}
&\mathbb{E}[|\Delta_q (t,x,j) - \Delta_q (s,x,j)|^2] \\
&\lesssim \|\sigma\|^2_{C_T L^\infty}\|\sigma\|^4_{C_T\mathcal{H}^2}|t-s|^{2\gamma} \sum_{\omega,\omega'\in\mathbb{Z}^2}\varrho_q(\omega)\varrho_q(\omega')|\omega|^{2\gamma+\varepsilon}|\omega'|^{2\gamma+\varepsilon} \\
&\quad \times \sum_{\omega_1\in\mathbb{Z}^2\setminus\{0\}} |\omega_1|^{-2}(1\vee|\omega-\omega_1|)^{-2+\varepsilon}(1\vee|\omega'-\omega_1|)^{-2+\varepsilon}.
\end{aligned}
\tag{2.2.12}
$$

Hence it suffices to control the remaining sum over $\omega_1 \in \mathbb{Z}^2\setminus\{0\}$,

$$
\sum_{\omega_1\in\mathbb{Z}^2\setminus\{0\}} |\omega_1|^{-2}(1\vee|\omega-\omega_1|)^{-2+\varepsilon}(1\vee|\omega'-\omega_1|)^{-2+\varepsilon}.
$$

We distinguish the cases $\omega = \omega'$ and $\omega \neq \omega'$. In the case $\omega = \omega'$, we decompose the sum into the regions $\omega_1 = \omega$ and $\omega_1 \in \mathbb{Z}^2\setminus\{0,\omega\}$. We then estimate by Lemma E.1.3,

$$
\sum_{\omega_1\in\mathbb{Z}^2\setminus\{0\}} |\omega_1|^{-2}(1\vee|\omega-\omega_1|)^{-4+2\varepsilon} = |\omega|^{-2} + \sum_{\omega_1\in\mathbb{Z}^2\setminus\{0,\omega\}} |\omega_1|^{-2}|\omega-\omega_1|^{-4+2\varepsilon} \lesssim |\omega|^{-2+2\varepsilon}.
$$

In the case $\omega \neq \omega'$, we decompose the sum into the regions $\omega_1 = \omega$, $\omega_1 = \omega'$ and $\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}$,

$$\begin{aligned} &\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_1|^{-2}(1 \vee |\omega - \omega_1|)^{-2+\varepsilon}(1 \vee |\omega' - \omega_1|)^{-2+\varepsilon} \\ &= (|\omega|^{-2} + |\omega'|^{-2})|\omega - \omega'|^{-2+\varepsilon} + \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\}} |\omega_1|^{-2}|\omega - \omega_1|^{-2+\varepsilon}|\omega' - \omega_1|^{-2+\varepsilon} \end{aligned}$$

and apply Lemma E.1.4 to bound

$$\begin{aligned} &\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\}} |\omega_1|^{-2}|\omega - \omega_1|^{-2+\varepsilon}|\omega' - \omega_1|^{-2+\varepsilon} \\ &\lesssim |\omega - \omega'|^{-2+\varepsilon}|\omega|^{-2+2\varepsilon} + |\omega - \omega'|^{-2+\varepsilon}|\omega'|^{-2+2\varepsilon}. \end{aligned}$$

Assume $q \in \mathbb{N}_0$ and $\omega, \omega' \in \operatorname{supp}(\varrho_q)$, it follows that $2^q \lesssim |\omega| \lesssim 2^q$, $2^q \lesssim |\omega'| \lesssim 2^q$ and if $\omega \neq \omega'$ then $2^q \lesssim |\omega - \omega'| \lesssim 2^q$ as well. We obtain by (2.2.12),

$$\mathbb{E}[|\Delta_q (t, x, j) - \Delta_q (s, x, j)|^2] \lesssim \|\sigma\|^2_{C_T L^\infty} \|\sigma\|^4_{C_T \mathcal{H}^2} |t - s|^{2\gamma} 2^{q(4\gamma + 5\varepsilon)}.$$

Assume in addition $\varepsilon < \gamma$, we obtain by Lemma C.3.8 and Lemma 2.2.6 for all $p \in [1, \infty)$,

$$\mathbb{E}[\| \|^p_{C_T^{\gamma-\varepsilon}\mathcal{C}^{-2\gamma-5\varepsilon}}]^{1/p} \lesssim \|\sigma\|_{C_T L^\infty} \|\sigma\|^2_{C_T \mathcal{H}^2}$$

and therefore $\in \mathscr{L}^\kappa_T \mathcal{C}^{0-}(\mathbb{T}^2; \mathbb{R}^2)$ a.s. for each $\kappa \in (0, 1/2)$. □

Next we prove the existence of .

Proof of Lemma 2.2.18. Let $0 \leq s \leq t \leq T$, $\omega \in \mathbb{Z}^2$ and $k, j = 1, 2$. We can bound the increment

$$\begin{aligned} &|\widehat{\ }(t, \omega, k, j) - \widehat{\ }(s, \omega, k, j)| \\ &\lesssim \|\sigma\|^2_{C_T \mathcal{H}^2} \sum_{\substack{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\} \\ \omega = \omega_1 + \omega_2 \\ \omega_1 \sim \omega_2}} \sum_{m_2 \in \mathbb{Z}^2} \frac{|G^j(\omega_2) + G^j(\omega_1)|}{(1 + |\omega_1 + m_2|^2)(1 + |\omega_2 - m_2|^2)} \mathsf{A}^k_{s,t} \blacktriangledown (\omega_1, \omega_2, \omega_1). \end{aligned}$$

We control the shape coefficient with Lemma D.0.2 and the elliptic multiplier with Lemma E.1.1. Let $\kappa \in [0, 1]$, we arrive at

$$\begin{aligned} &|\widehat{\ }(t, \omega, k, j) - \widehat{\ }(s, \omega, k, j)| \\ &\lesssim \|\sigma\|^2_{C_T \mathcal{H}^2} |t - s|^\kappa |\omega| \sum_{\substack{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\} \\ \omega = \omega_1 + \omega_2 \\ \omega_1 \sim \omega_2}} \sum_{m_2 \in \mathbb{Z}^2} \frac{(1 + |\omega||\omega_2|^{-1})|\omega_1|^{-3+2\kappa}}{(1 + |\omega_1 + m_2|^2)(1 + |\omega_2 - m_2|^2)}. \end{aligned}$$

Let $\varepsilon \in (0,1)$, we bound by Lemma E.1.3,

$$\begin{aligned}
&\sum_{m_2\in\mathbb{Z}^2}(1+|\omega_1+m_2|^2)^{-1}(1+|\omega_2-m_2|^2)^{-1}\\
&= 2(1+|\omega|^2)^{-1} + \sum_{m_2\in\mathbb{Z}^2\setminus\{-\omega_1,\omega_2\}}(1+|\omega_1+m_2|^2)^{-1}(1+|\omega_2-m_2|^2)^{-1}\\
&\lesssim |\omega|^{-2+2\varepsilon}
\end{aligned}$$

and obtain

$$\begin{aligned}
&|\widehat{\triangledown}(t,\omega,k,j) - \widehat{\triangledown}(s,\omega,k,j)|\\
&\lesssim \|\sigma\|^2_{C_T\mathcal{H}^2}|t-s|^{\kappa}|\omega|^{-1+2\varepsilon}\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\setminus\{0\}\\ \omega=\omega_1+\omega_2\\ \omega_1\sim\omega_2}}(1+|\omega||\omega_2|^{-1})|\omega_1|^{-3+2\kappa}.
\end{aligned}$$

For each $\kappa \in (0,1/2)$, we bound by Lemma E.1.3,

$$|\widehat{\triangledown}(t,\omega,k,j) - \widehat{\triangledown}(s,\omega,k,j)| \lesssim \|\sigma\|^2_{C_T\mathcal{H}^2}|t-s|^{\kappa}|\omega|^{-2+2\kappa+2\varepsilon}.$$

It follows directly that

$$\|\triangledown\|_{C_T^{\kappa}\mathcal{C}^{-2\kappa-2\varepsilon}} \lesssim \|\sigma\|^2_{C_T\mathcal{H}^2}$$

and we obtain $\triangledown \in \mathscr{L}_T^{\kappa}\mathcal{C}^{0-}(\mathbb{T}^2;\mathbb{R}^{2\times 2})$ for each $\kappa \in (0,1/2)$. $\square$

2.2.5 Construction of the Canonical Enhancement

In this section we construct $\mathring{\imath}^{\delta}$ and $\mathcal{Q}_1^{\delta}$, $\mathcal{Q}_2^{\delta}$ for correlation lengths $\delta > 0$. Additionally, we bound their speeds of divergence as $\delta \to 0$ by a logarithmic rate using the symmetry of the elliptic equation.

Lemma 2.2.19. *Let $T>0$, $\alpha<-2$, $\kappa\in(0,1)$, $\delta>0$, $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$ and φ be as in* (1.2.6)*. Then it holds that*

$$\begin{aligned}
\|\mathring{\imath}^{\delta}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}} &\lesssim (1\vee\log(\delta^{-1}))\|\varphi\|^2_{L^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2},\\
\|\mathring{\imath}^{\delta}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+5}} &\lesssim (1\vee(\delta^{-1}\log(\delta^{-1})))\|\varphi\|^2_{L^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2},
\end{aligned}$$

and for all $\kappa\in(0,1/2)$, $p\in[1,\infty)$,

$$\mathbb{E}[\|\mathcal{Q}_1^{\delta}\|^p_{\mathscr{L}_T^{\kappa}\mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim (1\vee\log(\delta^{-1}))\|\varphi\|^3_{L^\infty}\|\sigma\|_{C_TL^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2},$$

and

$$\mathbb{E}[\|\mathcal{Q}_2^{\delta}\|^p_{\mathscr{L}_T^{\kappa}\mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim (1\vee\log(\delta^{-1}))\|\varphi\|^3_{L^\infty}\|\sigma\|_{C_TL^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2}.$$

Proof. We first establish $\Upsilon^\delta \in \mathscr{L}_T^{1-}\mathcal{C}^{0-}(\mathbb{T}^2)$. Let $s,t\in[0,T]$, $\omega\in\mathbb{Z}^2$ and $\kappa\in(0,1)$. It suffices to consider $\omega\in\mathbb{Z}^2\backslash\{0\}$, since $\widehat{\Upsilon^\delta}(t,0)=0$. We symmetrize the contraction. Changing the rôles of ω_1, ω_2 in the definition of $\widehat{\Upsilon^\delta}(t,\omega)$ and using that φ is even, we obtain

$$\widehat{\Upsilon^\delta}(t,\omega)=\frac{1}{2}\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\\ \omega=\omega_1+\omega_2}}\sum_{j_1,j_3=1}^{2}\sum_{m_1\in\mathbb{Z}^2}\int_0^t \mathrm{d}u_3\int_0^{u_3}\mathrm{d}u_1\hat{\sigma}(u_1,\omega_1-m_1)\hat{\sigma}(u_2,\omega_2+m_1)$$
$$\times|\varphi(\delta m_1)|^2H^{j_3}_{t-u_3}(\omega)H^{j_1}_{u_3-u_1}(\omega_1)H^{j_1}_{u_3-u_1}(\omega_2)(G^{j_3}(\omega_1)+G^{j_3}(\omega_2)).$$

We apply the triangle inequality, Lemma E.1.1 and (B.5.1) to bound

$$|\widehat{\Upsilon^\delta}(t,\omega)-\widehat{\Upsilon^\delta}(s,\omega)|$$
$$\lesssim\|\varphi\|^2_{L^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2}|t-s|^\kappa|\omega|^{2\kappa}\sum_{\substack{m_1\in\mathbb{Z}^2\\|m_1|\le\delta^{-1}}}\sum_{\substack{\omega_1\in\mathbb{Z}^2\\ \omega_1\notin\{0,\omega\}}}\frac{|\omega_1|^{-2}(1+|\omega||\omega-\omega_1|^{-1})}{(1+|\omega_1-m_1|^2)(1+|\omega-\omega_1+m_1|^2)}.$$

We can now apply Lemma E.2.2 to control the sums over $m_1\in\mathbb{Z}^2$, $|m_1|\le\delta^{-1}$, and $\omega_1\in\mathbb{Z}^2\setminus\{0,\omega\}$ for every $\varepsilon\in(0,1/2)$,

$$|\widehat{\Upsilon^\delta}(t,\omega)-\widehat{\Upsilon^\delta}(s,\omega)|\lesssim(1\vee\log(\delta^{-1}))\|\varphi\|^2_{L^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2}|t-s|^\kappa|\omega|^{-2+2\kappa+3\varepsilon}$$

and so

$$\|\Upsilon^\delta\|_{\mathscr{L}_T^\kappa\mathcal{C}^{-3\varepsilon}}\lesssim(1\vee\log(\delta^{-1}))\|\varphi\|^2_{L^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2}.$$

The proof that $\Upsilon^\delta\in\mathscr{L}_T^\kappa\mathcal{C}^{1-}(\mathbb{T}^2)$ with a divergence of order $1\vee(\delta^{-1}\log(\delta^{-1}))$ follows by a similar derivation, where we skip the symmetrization and use that $|\varphi(\delta m_1)|\lesssim(1+|\delta m_1|^2)^{-1/4}\|\varphi\|_{L^\infty}$.

Next we establish that $\Psi^\delta,\Psi'^\delta\in\mathscr{L}_T^{1/2-}\mathcal{C}^{0-}(\mathbb{T}^2;\mathbb{R}^2)$. We first consider Ψ^δ. We apply Itô's isometry, (C.3.2) and Lemma E.1.1 to bound uniformly in $x\in\mathbb{T}^2$,

$$\mathbb{E}[|\Delta_q\Psi^\delta(t,x,j)-\Delta_q\Psi^\delta(s,x,j)|^2]$$
$$\lesssim\|\varphi\|^6_{L^\infty}\|\sigma\|^2_{C_TL^\infty}\|\sigma\|^4_{C_T\mathcal{H}^2}\sum_{\omega,\omega'\in\mathbb{Z}^2}\varrho_q(\omega)\varrho_q(\omega')$$
$$\times\sum_{\substack{\omega_4\in\mathbb{Z}^2\backslash\{0,\omega,\omega'\}\\(\omega-\omega_4)\sim\omega_4\\(\omega'-\omega_4)\sim\omega_4}}\sum_{\substack{\omega_1,\omega_2\in\mathbb{Z}^2\backslash\{0\}\\ \omega-\omega_4=\omega_1+\omega_2}}\sum_{\substack{\omega_1',\omega_2'\in\mathbb{Z}^2\backslash\{0\}\\ \omega'-\omega_4=\omega_1'+\omega_2'}}\sum_{j_3,j_3'=1}^{2}\sum_{\substack{m_1\in\mathbb{Z}^2\\|m_1|\le\delta^{-1}}}\sum_{\substack{m_1'\in\mathbb{Z}^2\\|m_1'|\le\delta^{-1}}}$$
$$(1+|\omega_1-m_1|^2)^{-1}(1+|\omega_2+m_1|^2)^{-1}(1+|\omega_1'-m_1'|^2)^{-1}(1+|\omega_2'+m_1'|^2)^{-1}$$
$$\times|\omega_4|^{-2}|\omega-\omega_4||\omega'-\omega_4||\omega_2|^{-2}|\omega_2'|^{-2}(1+|\omega-\omega_4||\omega_1|^{-1})(1+|\omega'-\omega_4||\omega_1'|^{-1})$$
$$\times\mathsf{A}^{j_3,j_3'}_{s,t}\boldsymbol{\vee}(\omega_1+\omega_2,\omega_1'+\omega_2',\omega_4).$$

Assume $s \leq t$, $\gamma \in [0,1]$, $\delta > 0$ and $\varepsilon \in (0, 1/2)$ whose range we continue to restrict throughout the proof. We apply Lemma D.0.3 to control the shape coefficient and subsequently Lemma E.2.2 to obtain

$$\begin{aligned}
&\mathbb{E}[|\Delta_q \Psi^{\delta}(t,x,j) - \Delta_q \Psi^{\delta}(s,x,j)|^2] \\
&\lesssim (1 \vee \log(\delta^{-1}))^2 \|\varphi\|_{L^\infty}^6 \|\sigma\|_{C_T L^\infty}^2 \|\sigma\|_{C_T \mathcal{H}^2}^4 |t-s|^\gamma \sum_{\omega,\omega' \in \mathbb{Z}^2} \varrho(\omega)\varrho(\omega') \\
&\quad \times \sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\} \\ (\omega-\omega_4)\sim\omega_4 \\ (\omega'-\omega_4)\sim\omega_4}} |\omega_4|^{-2} |\omega-\omega_4|^{-2+\gamma+3\varepsilon} |\omega'-\omega_4|^{-2+\gamma+3\varepsilon}.
\end{aligned}$$

Assume in addition $6\varepsilon < 4 - 2\gamma$, we apply Hölder's inequality,

$$\begin{aligned}
&\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\} \\ (\omega-\omega_4)\sim\omega_4 \\ (\omega'-\omega_4)\sim\omega_4}} |\omega_4|^{-2} |\omega-\omega_4|^{-2+\gamma+3\varepsilon} |\omega'-\omega_4|^{-2+\gamma+3\varepsilon} \\
&\leq \Bigg(\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0,\omega\} \\ (\omega-\omega_4)\sim\omega_4}} |\omega_4|^{-2} |\omega-\omega_4|^{-4+2\gamma+6\varepsilon} \Bigg)^{1/2} \Bigg(\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0,\omega'\} \\ (\omega'-\omega_4)\sim\omega_4}} |\omega_4|^{-2} |\omega'-\omega_4|^{-4+2\gamma+6\varepsilon} \Bigg)^{1/2} \\
&\lesssim (1 \vee |\omega|)^{-2+\gamma+3\varepsilon} (1 \vee |\omega'|)^{-2+\gamma+3\varepsilon},
\end{aligned} \tag{2.2.13}$$

which implies

$$\begin{aligned}
&\mathbb{E}[|\Delta_q \Psi^{\delta}(t,x,j) - \Delta_q \Psi^{\delta}(s,x,j)|^2] \\
&\lesssim (1 \vee \log(\delta^{-1}))^2 \|\varphi\|_{L^\infty}^6 \|\sigma\|_{C_T L^\infty}^2 \|\sigma\|_{C_T \mathcal{H}^2}^4 |t-s|^\gamma \\
&\quad \times \sum_{\omega,\omega' \in \mathbb{Z}^2} \varrho_q(\omega)\varrho_q(\omega') (1 \vee |\omega|)^{-2+\gamma+3\varepsilon} (1 \vee |\omega'|)^{-2+\gamma+3\varepsilon}.
\end{aligned}$$

Assume in addition $\gamma \in (0,1)$ and $\varepsilon \in (0, \gamma/2)$, we obtain by Lemma C.3.8 and Lemma 2.2.6 for all $p \in [1, \infty)$,

$$\mathbb{E}[\|\Psi^{\delta}\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-\gamma-6\varepsilon}}]^{1/p} \lesssim (1 \vee \log(\delta^{-1})) \|\varphi\|_{L^\infty}^3 \|\sigma\|_{C_T L^\infty} \|\sigma\|_{C_T \mathcal{H}^2}^2$$

and therefore $\Psi^{\delta} \in \mathscr{L}_T^{\kappa} \mathcal{C}^{0-}(\mathbb{T}^2; \mathbb{R}^2)$ a.s. for each $\kappa \in (0, 1/2)$.

The only difference between $\widehat{}^{\delta}$ and $\widehat{}^{\delta}$ is that the factor $G^j(\omega_1+\omega_2)$ replaces $G^j(\omega_4)$. Instead of (2.2.13), we estimate by Hölder's inequality,

$$\begin{aligned}
&\sum_{\substack{\omega_4\in\mathbb{Z}^2\setminus\{0,\omega,\omega'\}\\(\omega-\omega_4)\sim\omega_4\\(\omega'-\omega_4)\sim\omega_4}} |\omega-\omega_4|^{-3+\gamma+3\varepsilon}|\omega'-\omega_4|^{-3+\gamma+3\varepsilon}\\
&\leq \Bigg(\sum_{\substack{\omega_4\in\mathbb{Z}^2\setminus\{\omega\}\\(\omega-\omega_4)\sim\omega_4}} |\omega-\omega_4|^{-6+2\gamma+6\varepsilon}\Bigg)^{1/2}\Bigg(\sum_{\substack{\omega_4\in\mathbb{Z}^2\setminus\{\omega'\}\\(\omega'-\omega_4)\sim\omega_4}} |\omega'-\omega_4|^{-6+2\gamma+6\varepsilon}\Bigg)^{1/2}\\
&\lesssim (1\vee|\omega|)^{-2+\gamma+3\varepsilon}(1\vee|\omega'|)^{-2+\gamma+3\varepsilon}.
\end{aligned}$$

As before, we obtain by (C.3.2), Lemma C.3.8 and Lemma 2.2.6 for all $\gamma\in(0,1)$, $6\varepsilon<4-2\gamma$, $\varepsilon<\gamma/2$, $\delta>0$ and $p\in[1,\infty)$,

$$\mathbb{E}[\|\;{}^{\delta}\|^p_{C_T^{\gamma/2-\varepsilon}\mathcal{C}^{-\gamma-6\varepsilon}}]^{1/p} \lesssim (1\vee\log(\delta^{-1}))\|\varphi\|^3_{L^\infty}\|\sigma\|_{C_TL^\infty}\|\sigma\|^2_{C_T\mathcal{H}^2}$$

and therefore ${}^{\delta}\in\mathscr{L}_T^{\kappa}\mathcal{C}^{0-}(\mathbb{T}^2;\mathbb{R}^2)$ a.s. for each $\kappa\in(0,1/2)$. □

Remark 2.2.20. We may also construct ${}^{\delta}=\mathbb{E}[\;{}^{\delta}\nabla\Phi_{\delta}]$, which is similar to ${}^{\delta}$ but without the lower stem (cf. (2.2.6)). However, to obtain κ-time regularity, we need to trade 2κ-space regularity in the parabolic multipliers $H^{j_1}_{t-u_1}(\omega_1)H^{j_1}_{t-u_1}(\omega_2)$. We then sum over $\omega_1,\omega_2\in\mathbb{Z}^2$, hence we will be stuck with a divergence of $\delta^{-2\kappa}$, where κ is arbitrarily small but positive.

Remark 2.2.21. We can show that ${}^{\delta}\equiv 0$, if $\sigma\equiv 1$. Indeed, if we choose $\sigma\equiv 1$, then $\widehat{\sigma}(u,\omega)=\mathbb{1}_{\omega=0}$. Consequently on the right-hand side of (2.2.6), $\omega_1=m_1$ and $\omega_2=-m_1$. By the symmetrization, we obtain the factor $G^j(\omega_1)+G^j(\omega_2)$. Using that G^j is odd and that $\omega_1=-\omega_2$, we see that this term is zero, so that ${}^{\delta}\equiv 0$. Similarly, ${}^{\delta}={}^{\delta}={}^{\delta}\equiv 0$, if $\sigma\equiv 1$.

2.3 Existence of Paracontrolled Solutions

In this section we show the existence and uniqueness of a paracontrolled solution to (2.1.1) given an abstract enhancement, where the notion of paracontrolled solution has previously been motivated in (2.1.6)–(2.1.7) and will be made precise in Definition 2.3.10.

In Subsection 2.3.1 we consider the solution on small time intervals (Lemma 2.3.12), which we then extend in Subsection 2.3.2 to a maximal time of existence (Lemma 2.3.16). In Subsection 2.3.3 we then combine the deterministic solution theory above with the stochastic existence of the renormalised enhancement (Theorem 2.2.2) to construct the renormalised solution to (2.1.1) as a random variable (Theorem 2.3.18, Part 1). We can then show that solutions to (2.1.8) converge in probability to the renormalised solution (Theorem 2.3.18, Part 2).

2.3.1 Local Well-Posedness

Throughout we fix exponents satisfying the assumptions below. To explain their usage: p and β_0 will be the integrability and regularity exponents of the admissible initial condition in the Besov scale $\mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$, where q is the microscopic parameter; α will be the regularity of the space-time white noise, so that almost surely $\mathbf{I} \in C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$; β measures the regularity of the second Da Prato–Debussche remainder, w, in the Hölder scale and η the allowed blow-up of w at $t = 0$; β' measures the regularity of the Gubinelli derivative w'; $\beta^{\#}$ measures the maximal spatial regularity of the paracontrolled remainder and finally κ will be used to denote time regularity.

From now on we fix $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ satisfying

$$\begin{aligned} \alpha &\in (-9/4, -2), & q &\in [1, \infty], \\ p &\in (2/(\alpha+3), \infty], & \beta &\in (-1/2, \alpha+2), \\ \beta' &\in (-2\alpha-4, (\beta+1) \wedge (2\alpha+5)], & \beta^{\#} &\in (-\alpha-2, \alpha+\beta'+2), \\ \beta_0 &\in (\beta+2/p-(\alpha+3), \beta^{\#}], & \kappa &\in ((\beta^{\#}-\alpha-2)/2, 1/2), \end{aligned}$$
$$\eta \in \Big[\Big(\Big(\frac{\beta-\beta_0}{2} \vee 0\Big) + \frac{1}{p}\Big) \vee \Big(\frac{\beta^{\#}-\beta_0}{4} + \frac{1}{2p}\Big), \frac{\alpha+3}{2}\Big). \tag{2.3.1}$$

One can confirm that the intervals in (2.3.1) are non-empty.

Particular choices of exponents allow us to consider regular and irregular initial data:

Example 2.3.1. By taking $p = \infty$, $\beta_0 = \beta^{\#}$ and $\eta = 0$, we can choose as initial data each $\rho_0 \in \mathcal{B}^{\beta^{\#}}_{\infty,q}(\mathbb{T}^2)$, without incurring a blow-up at 0.

Example 2.3.2. By taking $p = q = \infty$ and $\beta < 2\alpha + 4$, we can choose as initial data each $\rho_0 \in \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$, which is the natural regularity of $\mathbf{I}$ and the state space of $\rho = \mathbf{I} + \mathbf{Y} + w$.

Example 2.3.3. Using that $L^p(\mathbb{T}^2) \hookrightarrow \mathcal{B}^0_{p,\infty}(\mathbb{T}^2)$ (see Section B.2), we can choose as initial data each $\rho_0 \in L^p(\mathbb{T}^2)$ with $p > 2$.

Let us fix a $T > 0$, we define the space of paracontrolled distributions.

Definition 2.3.4. *Let $\mathbb{X} \in \mathcal{X}^{\alpha,\kappa}_T$ and $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$. We define the space*

$$\mathscr{D}_T \subset \mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_{\eta;T}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2) \times (\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta(\mathbb{T}^2;\mathbb{R}) \cap \mathscr{L}^\kappa_{2\eta;T}\mathcal{C}^{\beta^\#}(\mathbb{T}^2;\mathbb{R}))$$

of distributions paracontrolled by $\mathbb{X}$ as those triples $\boldsymbol{w} \coloneqq (w, w', w^\#)$ such that

$$w = \nabla \cdot \mathcal{I}[w' \varolessthan \mathbf{I}] + w^\# \tag{2.3.2}$$

as well as $\lim_{t\to 0} w_t = \lim_{t\to 0} w^\#_t = \rho_0$ in $\mathcal{S}'(\mathbb{T}^2;\mathbb{R})$ and $\lim_{t\to 0} w'_t = \nabla\Phi_{\rho_0}$ in $\mathcal{S}'(\mathbb{T}^2;\mathbb{R}^2)$. We equip this space with the metric induced by the norm

$$\|\boldsymbol{w}\|_{\mathscr{D}_T} \coloneqq \max\{\|w\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta}, \|w'\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^{\beta'}}, \|w^\#\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta}, \|w^\#\|_{\mathscr{L}^\kappa_{2\eta;T}\mathcal{C}^{\beta^\#}}\}.$$

Remark 2.3.5. The Ansatz (2.3.2) allows us to write the mild equation for $w^\#$ as $w^\# = P\rho_0 + \nabla \cdot \mathcal{I}[\Omega^\#(\boldsymbol{w})]$ with some $\Omega^\#(\boldsymbol{w})$ determined by (2.1.4). We can hence simplify our estimates by using the space-time regularization of $\mathcal{I}$. Note that (2.3.2) is equivalent to the Ansatz (2.1.5) discussed in the introduction, up to commutators.

Lemma 2.3.6. *Given $\mathbb{X} \in \mathcal{X}^{\alpha,\kappa}_T$ and $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$, the space $\mathscr{D}_T$ is a non-empty, complete metric space.*

Proof. To show that $\mathscr{D}_T$ is non-empty, we can choose $w' = \nabla\Phi_{P\rho_0}$, $w^\# = P\rho_0$ and set $w \coloneqq \nabla \cdot \mathcal{I}[w' \varolessthan \mathbf{I}] + w^\#$. The initial condition is satisfied, since $\lim_{t\to 0} w'_t = \nabla\Phi_{\rho_0}$ in $\mathcal{S}'(\mathbb{T}^2;\mathbb{R}^2)$ and $\lim_{t\to 0} w^\#_t = \rho_0$ in $\mathcal{S}'(\mathbb{T}^2;\mathbb{R})$. By Lemma B.5.2 and Lemma B.5.5, using that $\beta' \leq \beta + 1$, and $((\beta - \beta_0)/2 \vee 0) + 1/p \leq \eta$, we obtain $w' = \nabla\Phi_{P\rho_0} \in \mathscr{L}^\kappa_{\eta;T}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2)$, $w^\# = P\rho_0 \in \mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta(\mathbb{T}^2;\mathbb{R})$ and

$$\|\nabla\Phi_{P\rho_0}\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^{\beta'}} \lesssim \|\nabla\Phi_{P\rho_0}\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^{\beta+1}} \lesssim \|P\rho_0\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta} \lesssim_T \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}}.$$

Using that $\beta_0 \leq \beta^{\#}$ and $(\beta^{\#} - \beta_0)/2 + 1/p \leq 2\eta$, we obtain $w^{\#} \in \mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R})$ and

$$\|P\rho_0\|_{\mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}} \lesssim_T \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}}.$$

Since $w = \nabla \cdot \mathcal{I}[w' \varolessthan \boldsymbol{!}] + w^{\#}$, we find by an application of the triangle inequality, Lemma B.5.2 and Lemma B.4.1, that $w \in \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$ and

$$\|w\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}} \lesssim_T \|w'\|_{C_{\eta;T}\mathcal{C}^{\beta'}}\|\boldsymbol{!}\|_{C_T\mathcal{C}^{\alpha+1}} + \|w^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}},$$

where we used that $\beta < \alpha + 2$ and $\beta' > 0$. To show completeness, let $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{D}_T$. By the completeness of $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2) \times (\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R}) \cap \mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R}))$, we obtain $w_n \to w \in \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$, $w_n' \to w' \in \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2)$, $w_n^{\#} \to w^{\#} \in \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$ and $w_n^{\#} \to w^{\#} \in \mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R})$ all with the correct initial conditions. It suffices to show $w = \nabla \cdot \mathcal{I}[w' \varolessthan \boldsymbol{!}] + w^{\#}$ so that $\boldsymbol{w} \in \mathscr{D}_T$. Then again by Lemma B.5.2 and Lemma B.4.1,

$$w_n - \nabla \cdot \mathcal{I}[w' \varolessthan \boldsymbol{!}] - w^{\#} = \nabla \cdot \mathcal{I}[(w_n' - w') \varolessthan \boldsymbol{!}] + w_n^{\#} - w^{\#} \to 0 \qquad \text{in} \quad \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R}),$$

which combined with $w_n \to w \in \mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$ yields $w = \nabla \cdot \mathcal{I}[w' \varolessthan \boldsymbol{!}] + w^{\#}$. $\square$

In the next lemma we show that $w \odot \nabla\Phi_{\boldsymbol{!}} + \boldsymbol{!} \odot \nabla\Phi_w$ is well-defined in $\mathscr{D}_T \times \mathcal{X}^{\alpha,\kappa}_T$.

Lemma 2.3.7. *There exists a continuous operator $\mathscr{P} \colon \mathscr{D}_T \times \mathcal{X}^{\alpha,\kappa}_T \to C_{2\eta;T}\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^2)$, such that when all objects are smooth,*

$$\mathscr{P}(\boldsymbol{w}, \mathbb{X}) = w \odot \nabla\Phi_{\boldsymbol{!}} + \boldsymbol{!} \odot \nabla\Phi_w.$$

Proof. Using the notation $\mathscr{C}(f,g,h) = (f \varolessthan g) \odot h - f(g \odot h)$ (see Lemma B.7.3) and recalling that $\boldsymbol{\vee} = \nabla\mathcal{I}[\boldsymbol{!}] \odot \nabla\Phi_{\boldsymbol{!}} + \nabla^2\mathcal{I}[\Phi_{\boldsymbol{!}}] \odot \boldsymbol{!}$ we can expand the product into

$$\begin{aligned}\mathscr{P}(\boldsymbol{w}, \mathbb{X}) \coloneqq{}& \mathscr{C}(w', \nabla\mathcal{I}[\boldsymbol{!}], \nabla\Phi_{\boldsymbol{!}}) + \mathscr{C}(w', \nabla^2\mathcal{I}[\Phi_{\boldsymbol{!}}], \boldsymbol{!}) \\ &+ (w^{\#} + \nabla \cdot \mathcal{I}[w' \varolessthan \boldsymbol{!}] - w' \varolessthan \nabla\mathcal{I}[\boldsymbol{!}]) \odot \nabla\Phi_{\boldsymbol{!}} \\ &+ (\nabla\Phi_{w^{\#}} + \nabla\nabla \cdot \Phi_{\mathcal{I}[w' \varolessthan \boldsymbol{!}]} - w' \varolessthan \nabla^2\mathcal{I}[\Phi_{\boldsymbol{!}}]) \odot \boldsymbol{!} + w'\boldsymbol{\vee},\end{aligned}$$

where $\boldsymbol{w} \in \mathscr{D}_T$ and $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$. In what follows, we establish the bound

$$\|\mathscr{P}(\boldsymbol{w},\mathbb{X})\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \lesssim_T (\|w'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}} + \|w^{\#}\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}})(1+\|\boldsymbol{\uparrow}\|_{C_T\mathcal{C}^{\alpha+1}}+\|\boldsymbol{\vee}\|_{C_T\mathcal{C}^{2\alpha+4}})^2 \tag{2.3.3}$$

by various applications of our commutator results. The regularity $\mathscr{P}(\boldsymbol{w},\mathbb{X}) \in C_{2\eta;T}\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^2)$ follows by the same arguments.

Using Lemma B.7.3 and that $\beta' \in (0,1)$, $2\alpha+4 < 0 < 2\alpha+4+\beta'$, we obtain

$$\|\mathscr{C}(w', \nabla\mathcal{I}[\boldsymbol{\uparrow}], \nabla\Phi_{\boldsymbol{\uparrow}})\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \lesssim_T \|w'\|_{C_{\eta;T}\mathcal{C}^{\beta'}}\|\nabla\mathcal{I}[\boldsymbol{\uparrow}]\|_{C_T\mathcal{C}^{\alpha+2}}\|\nabla\Phi_{\boldsymbol{\uparrow}}\|_{C_T\mathcal{C}^{\alpha+2}}$$

and

$$\|\mathscr{C}(w', \nabla^2\mathcal{I}[\Phi_{\boldsymbol{\uparrow}}], \boldsymbol{\uparrow})\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \lesssim_T \|w'\|_{C_{\eta;T}\mathcal{C}^{\beta'}}\|\nabla^2\mathcal{I}[\Phi_{\boldsymbol{\uparrow}}]\|_{C_T\mathcal{C}^{\alpha+3}}\|\boldsymbol{\uparrow}\|_{C_T\mathcal{C}^{\alpha+1}}.$$

Further, by Lemma B.4.1, using that $2\alpha+4 < 0 < \beta^{\#}+\alpha+2$,

$$\begin{aligned}&\|(w^{\#} + \nabla\cdot\mathcal{I}[w' \otimes \boldsymbol{\uparrow}] - w' \otimes \nabla\mathcal{I}[\boldsymbol{\uparrow}]) \odot \nabla\Phi_{\boldsymbol{\uparrow}}\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \\ &\lesssim \|w^{\#} + \nabla\cdot\mathcal{I}[w' \otimes \boldsymbol{\uparrow}] - w' \otimes \nabla\mathcal{I}[\boldsymbol{\uparrow}]\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}}\|\nabla\Phi_{\boldsymbol{\uparrow}}\|_{C_T\mathcal{C}^{\alpha+2}}.\end{aligned}$$

To control the remainder, we apply Lemma B.7.1 and Lemma B.7.2, using that $\kappa \in ((\beta^{\#}-\alpha-2)/2, 1/2)$ and $\beta^{\#} < \alpha+\beta'+2$,

$$\|w^{\#} + \nabla\cdot\mathcal{I}[w' \otimes \boldsymbol{\uparrow}] - w' \otimes \nabla\mathcal{I}[\boldsymbol{\uparrow}]\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}} \lesssim_T \|w^{\#}\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}} + \|w'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}}\|\boldsymbol{\uparrow}\|_{C_T\mathcal{C}^{\alpha+1}}.$$

Similarly, by Lemma B.4.1, Lemma B.7.1 and Lemma B.7.2, using that $2\alpha+4 < 0 < \beta^{\#}+\alpha+2$,

$$\begin{aligned}&\|(\nabla\Phi_{w^{\#}} + \nabla\nabla\cdot\Phi_{\mathcal{I}[w'\otimes\boldsymbol{\uparrow}]} - w' \otimes \nabla^2\mathcal{I}[\Phi_{\boldsymbol{\uparrow}}]) \odot \boldsymbol{\uparrow}\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \\ &\lesssim_T (\|w^{\#}\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}} + \|w'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}}\|\boldsymbol{\uparrow}\|_{C_T\mathcal{C}^{\alpha+1}})\|\boldsymbol{\uparrow}\|_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned}$$

Finally, by Lemma B.4.1, using that $2\alpha+4 < 0 < 2\alpha+4+\beta'$,

$$\|w'\boldsymbol{\vee}\|_{C_{2\eta;T}\mathcal{C}^{2\alpha+4}} \lesssim_T \|w'\|_{C_{\eta;T}\mathcal{C}^{\beta'}}\|\boldsymbol{\vee}\|_{C_T\mathcal{C}^{2\alpha+4}}.$$

This yields the claim. □

We can now derive a priori bounds for our solution map.

Lemma 2.3.8. *For every $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$, let $\mathbf{\Psi}$, acting on $\boldsymbol{u} = (u, u', u^{\#}) \in \mathscr{D}_T$, be given by $\mathbf{\Psi}(\boldsymbol{u}) := (w, w', w^{\#})$, where*

$$\begin{cases} w := \nabla \cdot \mathcal{I}[w' \varolessthan \mathsf{I}] + w^{\#}, \quad w' := \nabla \Phi_u + \nabla \Phi_{\mathsf{Y}}, \\ w^{\#} := P\rho_0 + \nabla \cdot \mathcal{I}[\Omega^{\#}(\boldsymbol{u})], \end{cases}$$

and

$$\Omega^{\#}(\boldsymbol{u}) := u\nabla\Phi_u + u\nabla\Phi_{\mathsf{Y}} + \mathsf{Y}\nabla\Phi_u + \mathsf{Y}\nabla\Phi_{\mathsf{Y}} + \mathsf{V} + \nabla\Phi_{\mathsf{I}} \varolessthan \mathsf{Y} + \mathsf{Y} \varolessthan \nabla\Phi_{\mathsf{I}}$$
$$+ \mathsf{I} \varolessthan \nabla\Phi_{\mathsf{Y}} + u \varolessthan \nabla\Phi_{\mathsf{I}} + \nabla\Phi_{\mathsf{I}} \varolessthan u + \mathsf{I} \varolessthan \nabla\Phi_u + \mathscr{P}(\boldsymbol{u}, \mathbb{X}).$$

Then $\mathbf{\Psi}(\boldsymbol{u}) \in \mathscr{D}_T$ and there exists some $\theta > 0$ depending only on the chosen exponents and the dimension, such that for every $T \leq 1$,

$$\max\{\|w\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}}, \|w^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}}, \|w^{\#}\|_{\mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}}\}$$
$$\lesssim (1 + T^{\theta}\|\boldsymbol{u}\|_{\mathscr{D}_T})^2(1 + \|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}} + \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}})^2, \tag{2.3.4}$$

$$\|w'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}} \lesssim \|u\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}} + \|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}}. \tag{2.3.5}$$

Proof. We derive bounds for our solution map in several steps. By the same arguments it also follows that $\mathbf{\Psi}(\boldsymbol{u}) \in \mathscr{D}_T$. In the remainder of the proof, we will assume $T \leq 1$.

Step 1. The $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$-regularity of w. As in the proof of Lemma 2.3.6, but this time keeping track of the dependency on T, we see that

$$\|w\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}} \lesssim (T^{1-\frac{\beta-\alpha}{2}} \vee T^{1-\kappa})\|w'\|_{C_{\eta;T}\mathcal{C}^{\beta'}}\|\mathsf{I}\|_{C_T\mathcal{C}^{\alpha+1}} + \|w^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}}.$$

Step 2. The $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$ and $\mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R})$-regularity of $w^{\#}$. To establish the $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R})$-regularity of $w^{\#}$, we apply Lemma B.5.2, using that $((\beta-\beta_0)/2 \vee 0) + 1/p \leq \eta$, $\eta < 1/2$, $\beta + 1 < \alpha + \beta + 4$ and $-(\alpha+1)/2 \vee \kappa < 1 - \eta$,

$$\|w^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}} \lesssim T^{\eta - (\frac{\beta-\beta_0}{2} \vee 0) - \frac{1}{p}}\|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}} + (T^{\frac{\alpha+3}{2}-\eta} \vee T^{1-\kappa-\eta})\|\Omega^{\#}(\boldsymbol{u})\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}.$$

To establish the $\mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R})$-regularity of $w^{\#}$, we apply Lemma B.5.2, using that $\beta_0 \leq \beta^{\#}$, $(\beta^{\#} - \beta_0)/2 + 1/p \leq 2\eta$, $\eta < 1/2$, $\beta^{\#} + 1 < \alpha + \beta + 4$ and $(\beta^{\#} - \alpha - \beta - 1)/2 \vee \kappa < 1$,

$$\|w^{\#}\|_{\mathscr{L}^{\kappa}_{2\eta;T}\mathcal{C}^{\beta^{\#}}} \lesssim T^{2\eta - \frac{\beta^{\#}-\beta_0}{2} - \frac{1}{p}}\|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}} + (T^{1-\frac{\beta^{\#}-\alpha-\beta-1}{2}} \vee T^{1-\kappa})\|\Omega^{\#}(\boldsymbol{u})\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}.$$

Step 3. *The $C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}(\mathbb{T}^2;\mathbb{R}^2)$-regularity of $\Omega^{\#}(\boldsymbol{u})$.* We obtain by various applications of Lemma B.4.1, using in particular that $4\alpha+9>0$ and $\beta>-1/2$,

$$\|u\nabla\Phi_u\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}} \lesssim \|u\|^2_{C_{\eta;T}\mathcal{C}^{\beta}},$$

$$\begin{aligned}\max\Big\{&\|u\nabla\Phi_{\mathbf{Y}}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|\mathbf{Y}\nabla\Phi_u\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|u \olessthan \nabla\Phi_{\mathbf{I}}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}},\\ &\qquad \|\nabla\Phi_{\mathbf{I}} \olessthan u\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|\mathbf{I} \olessthan \nabla\Phi_u\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}\Big\}\\ &\qquad\qquad \lesssim T^{\eta}\|u\|_{C_{\eta;T}\mathcal{C}^{\beta}}\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}},\end{aligned}$$

and

$$\begin{aligned}\max\Big\{&\|\mathbf{Y}\nabla\Phi_{\mathbf{Y}}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|\mathbf{\Psi}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|\nabla\Phi_{\mathbf{I}} \olessthan \mathbf{Y}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}},\\ &\qquad \|\mathbf{Y} \olessthan \nabla\Phi_{\mathbf{I}}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}, \|\mathbf{I} \olessthan \nabla\Phi_{\mathbf{Y}}\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}}\Big\}\\ &\qquad\qquad \lesssim T^{2\eta}(1+\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}})^2.\end{aligned}$$

By (2.3.3) of Lemma 2.3.7, using that $\alpha+\beta+2\leq 2\alpha+4$,

$$\|\mathscr{P}(\boldsymbol{u},\mathbb{X})\|_{C_{2\eta;T}\mathcal{C}^{\alpha+\beta+2}} \lesssim (\|u'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}} + \|u^{\#}\|_{C_{2\eta;T}\mathcal{C}^{\beta^{\#}}})(1+\|\mathbf{I}\|_{C_T\mathcal{C}^{\alpha+1}} + \|\mathbf{V}\|_{C_T\mathcal{C}^{2\alpha+4}})^2.$$

Step 4. *The $\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2)$-regularity of w'.* By definition, $w'=\nabla\Phi_u+\nabla\Phi_{\mathbf{Y}}$. Using that $\beta'\leq\beta+1$ and $\beta'\leq 2\alpha+5$, we obtain

$$\|w'\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta'}} \lesssim \|u\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{C}^{\beta}} + T^{\eta}\|\mathbf{Y}\|_{\mathscr{L}^{\kappa}_{T}\mathcal{C}^{2\alpha+4}}.$$

Step 5. *Closing the bounds.* Using that $T\leq 1$, we can collect all of the terms above and cast them in the form (2.3.4)–(2.3.5). This yields the claim. □

While Lemma 2.3.8 shows that $\boldsymbol{\Psi}$ is a map from $\mathscr{D}_T$ to itself, it is not a contraction for small T, since there is no small time parameter on the right-hand side of (2.3.5). The remedy is to apply $\boldsymbol{\Psi}$ twice and argue that a fixed point of $\boldsymbol{\Psi}^{\circ 2}$ is also a fixed point of $\boldsymbol{\Psi}$ itself.

Proposition 2.3.9. *Let $\mathbb{X}\in\mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0\in\mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$. Then there exists some $\bar{T}\in(0,T\wedge 1]$ such that there is a unique solution $\boldsymbol{w}=(w,w',w^{\#})\in\mathscr{D}_{\bar{T}}$ to the equation*

$$\begin{cases} w=\nabla\cdot\mathcal{I}[w'\olessthan\mathbf{I}]+w^{\#}, \quad w'=\nabla\Phi_w+\nabla\Phi_{\mathbf{Y}},\\ w^{\#}=P\rho_0+\nabla\cdot\mathcal{I}[\Omega^{\#}(\boldsymbol{w})]. \end{cases} \tag{2.3.6}$$

The time of existence $\bar{T}$ depends only on $\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}}$, $\|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}}$, the chosen exponents and the dimension.

Proof. We let $\bar{T} \in (0, T]$, $\bar{T} \le 1$, $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$, $\boldsymbol{u} \in \mathscr{D}_{\bar{T}}$ and define

$$(\Psi(\boldsymbol{u}), \Psi(\boldsymbol{u})', \Psi(\boldsymbol{u})^{\#}) \coloneqq \boldsymbol{\Psi}(\boldsymbol{u}).$$

By Lemma 2.3.8 there exists some $\theta > 0$ such that

$$\begin{aligned} &\max\{\|\Psi(\boldsymbol{u})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}, \|\Psi(\boldsymbol{u})^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}, \|\Psi(\boldsymbol{u})^{\#}\|_{\mathscr{L}^{\kappa}_{2\eta;\bar{T}}\mathcal{C}^{\beta^{\#}}}\} \\ &\qquad \lesssim (1 + \bar{T}^{\theta}\|\boldsymbol{u}\|_{\mathscr{D}_{\bar{T}}})^2 (1 + \|\mathbb{X}\|_{\mathcal{X}_{\bar{T}}^{\alpha,\kappa}} + \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}})^2, \end{aligned} \tag{2.3.7}$$

and

$$\|\Psi(\boldsymbol{u})'\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta'}} \lesssim \|u\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}} + \|\mathbb{X}\|_{\mathcal{X}_{\bar{T}}^{\alpha,\kappa}}. \tag{2.3.8}$$

Now denote

$$(\Psi^{\circ 2}(\boldsymbol{u}), \Psi^{\circ 2}(\boldsymbol{u})', \Psi^{\circ 2}(\boldsymbol{u})^{\#}) \coloneqq \boldsymbol{\Psi}^{\circ 2}(\boldsymbol{u}).$$

By iterating the bounds (2.3.7)–(2.3.8), using $\bar{T} \le 1$ to streamline exponents, we obtain

$$\|\boldsymbol{\Psi}^{\circ 2}(\boldsymbol{u})\|_{\mathscr{D}_{\bar{T}}} \lesssim (1 + \bar{T}^{\theta/2}\|\boldsymbol{u}\|_{\mathscr{D}_{\bar{T}}})^4 (1 + \|\mathbb{X}\|_{\mathcal{X}_{\bar{T}}^{\alpha,\kappa}} + \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}})^6. \tag{2.3.9}$$

Let $C > 0$ be larger than the implicit constants of the inequalities (2.3.7) and (2.3.9) above. Assume that $M, R > 0$ are sufficiently large such that

$$(1 + \|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}} + \|\rho_0\|_{\mathcal{B}^{\beta_0}_{p,q}}) < M, \qquad 2CM^6 < R.$$

Assume further that $\|\boldsymbol{u}\|_{\mathscr{D}_{\bar{T}}} < R$. Using the bound (2.3.9), we can choose $\bar{T} = \bar{T}(R, \theta) \le 1$ smaller, if necessary, such that

$$\|\boldsymbol{\Psi}^{\circ 2}(\boldsymbol{u})\|_{\mathscr{D}_{\bar{T}}} \le C(1 + \bar{T}^{\theta/2}R)^4 M^6 \le 2CM^6 < R.$$

Consequently, $\boldsymbol{\Psi}^{\circ 2}$ is a self-mapping on the ball

$$\mathfrak{B}_{R;\bar{T}} \coloneqq \{\boldsymbol{u} \in \mathscr{D}_{\bar{T}} : \|\boldsymbol{u}\|_{\mathscr{D}_{\bar{T}}} < R\}.$$

Upon choosing $R > 0$ sufficiently large, we can ensure that $\mathfrak{B}_{R;\bar{T}} \subset \mathscr{D}_{\bar{T}}$ is non-empty.

To achieve contractivity, we use the bilinearity of the equation. Let $\boldsymbol{v} = (v, v', v^{\#}), \boldsymbol{w} = (w, w', w^{\#}) \in \mathscr{D}_{\bar{T}}$ and denote

$$(\Psi(\boldsymbol{v}), \Psi(\boldsymbol{v})', \Psi(\boldsymbol{v})^{\#}) \coloneqq \boldsymbol{\Psi}(\boldsymbol{v}), \qquad (\Psi(\boldsymbol{w}), \Psi(\boldsymbol{w})', \Psi(\boldsymbol{w})^{\#}) \coloneqq \boldsymbol{\Psi}(\boldsymbol{w}).$$

We see that

$$\begin{aligned}
\Psi(\boldsymbol{v}) - \Psi(\boldsymbol{w}) &= \nabla \cdot \mathcal{I}[(\Psi(\boldsymbol{v})' - \Psi(\boldsymbol{w})') \otimes \dot{\mathbf{I}}] + \Psi(\boldsymbol{v})^{\#} - \Psi(\boldsymbol{w})^{\#}, \\
\Psi(\boldsymbol{v})' - \Psi(\boldsymbol{w})' &= \nabla \Phi_{v-w}, \\
\Psi(\boldsymbol{v})^{\#} - \Psi(\boldsymbol{w})^{\#} &= \nabla \cdot \mathcal{I}[\Omega^{\#}(\boldsymbol{v}) - \Omega^{\#}(\boldsymbol{w})],
\end{aligned}$$

where

$$\begin{aligned}
\Omega^{\#}(\boldsymbol{v}) - \Omega^{\#}(\boldsymbol{w}) = {} & v \nabla \Phi_{v-w} + (v-w) \nabla \Phi_w + (v-w) \nabla \Phi_{\mathbf{Y}} + \mathbf{Y} \nabla \Phi_{v-w} \\
& + (v-w) \otimes \nabla \Phi_{\dot{\mathbf{I}}} + \nabla \Phi_{\dot{\mathbf{I}}} \otimes (v-w) + \dot{\mathbf{I}} \otimes \nabla \Phi_{v-w} \\
& + \mathscr{P}(\boldsymbol{v}, \mathbb{X}) - \mathscr{P}(\boldsymbol{w}, \mathbb{X}).
\end{aligned}$$

The difference of the renormalised products is given by

$$\begin{aligned}
& \mathscr{P}(\boldsymbol{v}, \mathbb{X}) - \mathscr{P}(\boldsymbol{w}, \mathbb{X}) \\
& = \mathscr{C}(v' - w', \nabla \mathcal{I}[\dot{\mathbf{I}}], \nabla \Phi_{\dot{\mathbf{I}}}) + \mathscr{C}(v' - w', \nabla^2 \mathcal{I}[\Phi_{\dot{\mathbf{I}}}], \dot{\mathbf{I}}) \\
& \quad + (v^{\#} - w^{\#} + \nabla \cdot \mathcal{I}[(v' - w') \otimes \dot{\mathbf{I}}] - (v' - w') \otimes \nabla \mathcal{I}[\dot{\mathbf{I}}]) \odot \nabla \Phi_{\dot{\mathbf{I}}} \\
& \quad + (\nabla \Phi_{v^{\#} - w^{\#}} + \nabla \nabla \cdot \Phi_{\mathcal{I}[(v' - w') \otimes \dot{\mathbf{I}}]} - (v' - w') \otimes \nabla^2 \mathcal{I}[\Phi_{\dot{\mathbf{I}}}]) \odot \dot{\mathbf{I}} + (v' - w') \mathbf{V}.
\end{aligned}$$

Using the same bounds as before, we obtain, for some $\theta > 0$,

$$\begin{aligned}
\|\Psi(\boldsymbol{v}) - \Psi(\boldsymbol{w})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}} \mathcal{C}^{\beta}} \lesssim {} & \bar{T}^{\theta} \|\Psi(\boldsymbol{v})' - \Psi(\boldsymbol{w})'\|_{C_{\eta;\bar{T}} \mathcal{C}^{\beta'}} \|\dot{\mathbf{I}}\|_{C_{\bar{T}} \mathcal{C}^{\alpha+1}} \\
& + \|\Psi(\boldsymbol{v})^{\#} - \Psi(\boldsymbol{w})^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}} \mathcal{C}^{\beta}},
\end{aligned}$$

and

$$\begin{aligned}
& \max\{\|\Psi(\boldsymbol{v})^{\#} - \Psi(\boldsymbol{w})^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}} \mathcal{C}^{\beta}}, \|\Psi(\boldsymbol{v})^{\#} - \Psi(\boldsymbol{w})^{\#}\|_{\mathscr{L}^{\kappa}_{2\eta;\bar{T}} \mathcal{C}^{\beta^{\#}}}\} \\
& \qquad \lesssim \bar{T}^{\theta} \|\Omega^{\#}(\boldsymbol{v}) - \Omega^{\#}(\boldsymbol{w})\|_{C_{2\eta;\bar{T}} \mathcal{C}^{\alpha+\beta+2}},
\end{aligned}$$

as well as

$$\|\Psi(\boldsymbol{v})' - \Psi(\boldsymbol{w})'\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}} \mathcal{C}^{\beta'}} \lesssim \|v - w\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}} \mathcal{C}^{\beta}}.$$

For the right-hand side,

$$
\begin{aligned}
&\|\Omega^{\#}(\boldsymbol{v})-\Omega^{\#}(\boldsymbol{w})\|_{C_{2\eta;\bar{T}}\mathcal{C}^{\alpha+\beta+2}}\\
&\lesssim \|v\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}\|v-w\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}+\|v-w\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}\|w\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}+\|v-w\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}\|\mathbf{Y}\|_{C_{\bar{T}}\mathcal{C}^{2\alpha+4}}\\
&\quad+\|v-w\|_{C_{\eta;\bar{T}}\mathcal{C}^{\beta}}\|\mathbf{\uparrow}\|_{C_{\bar{T}}\mathcal{C}^{\alpha+1}}+\|\mathscr{P}(\boldsymbol{v},\mathbb{X})-\mathscr{P}(\boldsymbol{w},\mathbb{X})\|_{C_{2\eta;\bar{T}}\mathcal{C}^{2\alpha+4}}.
\end{aligned}
$$

By the same arguments as in the proof of Lemma 2.3.7 we have

$$
\begin{aligned}
&\|\mathscr{P}(\boldsymbol{v},\mathbb{X})-\mathscr{P}(\boldsymbol{w},\mathbb{X})\|_{C_{2\eta;\bar{T}}\mathcal{C}^{2\alpha+4}}\\
&\lesssim (\|v'-w'\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta'}}+\|v^{\#}-w^{\#}\|_{C_{2\eta;\bar{T}}\mathcal{C}^{\beta\#}})(1+\|\mathbf{\uparrow}\|_{C_{\bar{T}}\mathcal{C}^{\alpha+1}}+\|\mathbf{\vee}\|_{C_{\bar{T}}\mathcal{C}^{2\alpha+4}})^2.
\end{aligned}
$$

Combining the bounds above, we obtain

$$
\begin{aligned}
&\max\{\|\Psi(\boldsymbol{v})-\Psi(\boldsymbol{w})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}},\|\Psi(\boldsymbol{v})^{\#}-\Psi(\boldsymbol{w})^{\#}\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}},\|\Psi(\boldsymbol{v})^{\#}-\Psi(\boldsymbol{w})^{\#}\|_{\mathscr{L}^{\kappa}_{2\eta;\bar{T}}\mathcal{C}^{\beta\#}}\}\\
&\qquad\lesssim \bar{T}^{\theta}\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathscr{D}_{\bar{T}}}(1+\|\mathbb{X}\|_{\mathcal{X}^{\alpha,\kappa}_{\bar{T}}}+\|v\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}+\|w\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}})^2, \qquad (2.3.10)\\
&\|\Psi(\boldsymbol{v})'-\Psi(\boldsymbol{w})'\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta'}}\lesssim\|v-w\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}. \qquad (2.3.11)
\end{aligned}
$$

Next we consider

$$
(\Psi^{\circ 2}(\boldsymbol{v}),\Psi^{\circ 2}(\boldsymbol{v})',\Psi^{\circ 2}(\boldsymbol{v})^{\#})\coloneqq\mathbf{\Psi}^{\circ 2}(\boldsymbol{v}),\qquad(\Psi^{\circ 2}(\boldsymbol{w}),\Psi^{\circ 2}(\boldsymbol{w})',\Psi^{\circ 2}(\boldsymbol{w})^{\#})\coloneqq\mathbf{\Psi}^{\circ 2}(\boldsymbol{w}).
$$

Iterating (2.3.10)–(2.3.11), we arrive at

$$
\begin{aligned}
\|\mathbf{\Psi}^{\circ 2}(\boldsymbol{v})-\mathbf{\Psi}^{\circ 2}(\boldsymbol{w})\|_{\mathscr{D}_{\bar{T}}}\lesssim\bar{T}^{\theta}\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathscr{D}_{\bar{T}}}\Big(&1+\|\mathbb{X}\|_{\mathcal{X}^{\alpha,\kappa}_{\bar{T}}}+\|v\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}+\|w\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}\\
&+\|\Psi(\boldsymbol{v})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}+\|\Psi(\boldsymbol{w})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}\Big)^4.
\end{aligned}
$$

Assume $\boldsymbol{v},\boldsymbol{w}\in\mathfrak{B}_{R;\bar{T}}$, it follows by (2.3.7), the definition of $\bar{T}$ and $\mathfrak{B}_{R;\bar{T}}$, that

$$
\|v\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}<R,\qquad\|w\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}<R,\qquad\|\Psi(\boldsymbol{v})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}<R,\qquad\|\Psi(\boldsymbol{w})\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}<R.
$$

Choosing $\bar{T}$ still smaller, if necessary, we can arrange that for some $c<1$,

$$
\|\mathbf{\Psi}^{\circ 2}(\boldsymbol{v})-\mathbf{\Psi}^{\circ 2}(\boldsymbol{w})\|_{\mathscr{D}_{\bar{T}}}<c\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathscr{D}_{\bar{T}}},
$$

showing that $\mathbf{\Psi}^{\circ 2}$ is a contraction on $\mathfrak{B}_{R;\bar{T}}$.

By Banach's fixed-point theorem there exists a unique fixed point for $\mathbf{\Psi}^{\circ 2}$ in $\mathfrak{B}_{R;\bar{T}}$. It suffices to argue that a fixed point to $\mathbf{\Psi}^{\circ 2}$ is also a fixed point to $\mathbf{\Psi}$. The following is due to [Per20, Thm. 5.15]. Denote for the sake of notation, $\boldsymbol{w} = \boldsymbol{\Psi}^{\circ 2}(\boldsymbol{w})$ and $\boldsymbol{v} \coloneqq \boldsymbol{\Psi}(\boldsymbol{w})$. We have $\boldsymbol{\Psi}^{\circ 2}(\boldsymbol{v}) = \boldsymbol{\Psi}(\boldsymbol{w}) = \boldsymbol{v}$, hence $\boldsymbol{v}$ is itself a fixed point to $\boldsymbol{\Psi}^{\circ 2}$, yielding by uniqueness that $\boldsymbol{v} = \boldsymbol{w}$. One can furthermore show that this fixed point is in fact unique in all of $\mathscr{D}_{\bar{T}}$; it suffices to compare two putative solutions in $\mathscr{D}_{\bar{T}}$ and similar estimates to those above show that they must be equal on a small time interval. Continuity then gives equality in all of $[0, \bar{T}]$. □

The utility of Proposition 2.3.9 is that it allows us to show existence and uniqueness of a suitable notion of solution to (2.1.1) by setting $\rho \coloneqq \mathsf{I} + \mathsf{Y} + w$. We call this solution paracontrolled and are now in a position to give a rigorous definition (cf. (2.1.6)–(2.1.7) for a formal motivation.)

Definition 2.3.10. *Let $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$. For every $S \leq T$, we call $\rho\colon [0, S] \to \mathcal{S}'(\mathbb{T}^2)$ a paracontrolled solution to* (2.1.1) *on* $[0, S]$ *with enhancement* $\mathbb{X}$ *and initial data ρ_0, if*

- *the triple $\boldsymbol{w} \coloneqq (w, w', w^{\#})$ of distributions*

$$w \coloneqq \rho - \mathsf{I} - \mathsf{Y}, \qquad w' \coloneqq \nabla\Phi_w + \nabla\Phi_{\mathsf{Y}}, \qquad w^{\#} \coloneqq w - \nabla \cdot \mathcal{I}[w' \otimes \mathsf{I}];$$

  satisfies $\boldsymbol{w} \in \mathscr{D}_S$ (see Definition 2.3.4);

- *the solution ρ satisfies $\rho = P\rho_0 + \nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] + \mathsf{I}$, where the product $\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho]$ needs to be interpreted in the renormalised sense, that is*

$$\nabla \cdot \mathcal{I}[\rho\nabla\Phi_\rho] \coloneqq \mathsf{Y} + \nabla \cdot \mathcal{I}[w' \otimes \mathsf{I}] + \nabla \cdot \mathcal{I}[\Omega^{\#}(\boldsymbol{w})]$$

  with $\Omega^{\#}(\boldsymbol{w})$ defined as in Lemma 2.3.8.

In particular it follows by Definition 2.3.4, that $\lim_{t\to 0} \rho_t = \rho_0$ in $\mathcal{S}'(\mathbb{T}^2)$. We call $\rho\colon [0, S) \to \mathcal{S}'(\mathbb{T}^2)$ a paracontrolled solution on $[0, S)$, if ρ is a paracontrolled solution on $[0, S']$ for every $S' < S$.

Using the uniqueness of the fixed point in Proposition 2.3.9, one can show that paracontrolled solutions are unique.

Lemma 2.3.11. *Let $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$, $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$ and $S \le T$. Then there exists at most one paracontrolled solution $\rho\colon [0,S] \to \mathcal{S}'(\mathbb{T}^2)$ to* (2.1.1) *on* $[0,S]$ *with enhancement* $\mathbb{X}$ *and initial data* ρ_0*.*

Proof. Consider two paracontrolled solutions ρ, $\widetilde{\rho}$, which by definition are associated to two triples $\boldsymbol{w}$, $\widetilde{\boldsymbol{w}}$. However, these triples must both satisfy by definition the same fixed-point equation (2.3.6), which has a unique solution by Proposition 2.3.9. Therefore both paracontrolled solutions must agree. □

In the following lemma, we prove the existence of a paracontrolled solution given an abstract enhancement.

Lemma 2.3.12. *Let $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$. Let $\bar{T} \in (0, T \wedge 1]$ be as in Proposition 2.3.9 and $\boldsymbol{w} = (w, w', w^{\#}) \in \mathscr{D}_{\bar{T}}$ be the fixed point of* (2.3.6) *constructed therein. Then $\rho = \mathbf{I} + \mathbf{Y} + w \in \mathscr{L}_{\eta;\bar{T}}^{\kappa}\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ is the unique paracontrolled solution to* (2.1.1) *on* $[0,\bar{T}]$ *with enhancement* $\mathbb{X}$ *and initial data* ρ_0*.*

Proof. Let $\boldsymbol{w} = (w, w', w^{\#}) \in \mathscr{D}_{\bar{T}}$ be the fixed point to (2.3.6) and $\rho \coloneqq \mathbf{I} + \mathbf{Y} + w$. It follows by definition,

$$w = \rho - \mathbf{I} - \mathbf{Y}, \qquad w' = \nabla \Phi_w + \nabla \Phi_{\mathbf{Y}}, \qquad w^{\#} = w - \nabla \cdot \mathcal{I}[w' \varolessthan \mathbf{I}]$$

and

$$\rho = P\rho_0 + \mathbf{Y} + \nabla \cdot \mathcal{I}[w' \varolessthan \mathbf{I}] + \nabla \cdot \mathcal{I}[\Omega^{\#}(\boldsymbol{w})] + \mathbf{I}.$$

Hence, ρ is a paracontrolled solution to (2.1.1) on $[0,\bar{T}]$ with enhancement $\mathbb{X}$ and initial data ρ_0, which is unique by Lemma 2.3.11. □

The next lemma shows that paracontrolled solutions are locally Lipschitz continuous in the noise enhancement and the initial data.

Lemma 2.3.13. *Let $R > 0$, $\mathbb{X} = (\mathbf{I}_X, \mathbf{Y}_X, \mathbf{\Psi}_X, \mathbf{V}_X), \mathbb{Y} = (\mathbf{I}_Y, \mathbf{Y}_Y, \mathbf{\Psi}_Y, \mathbf{V}_Y) \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0^X, \rho_0^Y \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$ be such that*

$$\max\{\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}}, \|\mathbb{Y}\|_{\mathcal{X}_T^{\alpha,\kappa}}, \|\rho_0^X\|_{\mathcal{B}_{p,q}^{\beta_0}}, \|\rho_0^Y\|_{\mathcal{B}_{p,q}^{\beta_0}}\} < R.$$

Then there exists some $\bar{T} = \bar{T}(R) \in (0,T]$ with the following properties

1. *There exist solutions* $\boldsymbol{w}_X \coloneqq (w_X, w'_X, w^{\#}_X)$ *and* $\boldsymbol{w}_Y \coloneqq (w_Y, w'_Y, w^{\#}_Y)$ *to*
$$\begin{cases} w_X \coloneqq \nabla \cdot \mathcal{I}[w'_X \varolessthan \mathbf{I}_X] + w^{\#}_X, \quad w'_X \coloneqq \nabla\Phi_{w_X} + \nabla\Phi_{\mathbf{\Upsilon}_X}, \\ w^{\#}_X \coloneqq P\rho_0^X + \nabla \cdot \mathcal{I}[\Omega^{\#}_X(\boldsymbol{w}_X)], \end{cases}$$
and
$$\begin{cases} w_Y \coloneqq \nabla \cdot \mathcal{I}[w'_Y \varolessthan \mathbf{I}_Y] + w^{\#}_Y, \quad w'_Y \coloneqq \nabla\Phi_{w_Y} + \nabla\Phi_{\mathbf{\Upsilon}_Y}, \\ w^{\#}_Y \coloneqq P\rho_0^Y + \nabla \cdot \mathcal{I}[\Omega^{\#}_Y(\boldsymbol{w}_Y)], \end{cases}$$
on $[0, \bar{T}]$ *by an application of Proposition 2.3.9. Here,* $\Omega^{\#}_X(\boldsymbol{w}_X)$ *and* $\Omega^{\#}_Y(\boldsymbol{w}_Y)$ *are defined as in Lemma 2.3.8 with noises* $\mathbb{X}$, $\mathbb{Y}$ *respectively.*

2. *Setting*
$$\rho_X \coloneqq \mathbf{I}_X + \mathbf{\Upsilon}_X + w_X, \qquad \rho_Y \coloneqq \mathbf{I}_Y + \mathbf{\Upsilon}_Y + w_Y,$$
one has
$$\max\Big\{ \|w_X - w_Y\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}, \|w^{\#}_X - w^{\#}_Y\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\beta}}, \\ \|w^{\#}_X - w^{\#}_Y\|_{\mathscr{L}^{\kappa}_{2\eta;\bar{T}}\mathcal{C}^{\beta^{\#}}}, \|\rho_X - \rho_Y\|_{\mathscr{L}^{\kappa}_{\eta;\bar{T}}\mathcal{C}^{\alpha+1}} \Big\} \\ \lesssim \|\rho_0^X - \rho_0^Y\|_{\mathcal{B}^{\beta_0}_{p,q}} + \|\mathbb{X} - \mathbb{Y}\|_{\mathcal{X}^{\alpha,\kappa}_{\bar{T}}}(\|\boldsymbol{w}_X\|_{\mathscr{D}_{\bar{T}}} + \|\boldsymbol{w}_Y\|_{\mathscr{D}_{\bar{T}}} + \|\mathbb{X}\|_{\mathcal{X}^{\alpha,\kappa}_{\bar{T}}} + \|\mathbb{Y}\|_{\mathcal{X}^{\alpha,\kappa}_{\bar{T}}} + 1)^2.$$

Proof. The claim follows as in Lemma 2.3.8, using the trilinearity of the equation. □

2.3.2 Maximal Time of Existence

To establish a maximal time of existence for paracontrolled solutions to (2.1.1), we iterate the construction of Subsection 2.3.1 (Theorem 2.3.15).

Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ be a tuple of exponents satisfying (2.3.1). Assume we have constructed a solution $\boldsymbol{w} \in \mathscr{D}_{T_1}$ to (2.3.6) until time $T_1 \in (0, T)$. Given the initial data $\rho_{T_1} = \mathbf{I}_{T_1} + \mathbf{\Upsilon}_{T_1} + w_{T_1} \in \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ and an enhancement $\mathbb{X} \in \mathcal{X}^{\alpha,\kappa}_T$, our paracontrolled triple for the continuation is given by $\widetilde{\boldsymbol{w}} = (\widetilde{w}, \widetilde{w}', \widetilde{w}^{\#})$, where for $t \geq 0$,

$$\widetilde{w}_t = \nabla \cdot \mathcal{I}[\widetilde{w}' \varolessthan \mathbf{I}_{T_1 + \cdot}]_t + \widetilde{w}^{\#}_t, \tag{2.3.12}$$

with Gubinelli derivative

$$\widetilde{w}'_t = \nabla\Phi_{\widetilde{w}_t} + \nabla\Phi_{\mathbf{\Upsilon}_{T_1+t}} \tag{2.3.13}$$

and paracontrolled remainder

$$\begin{aligned}\widetilde{w}^{\#}_t = {} & P_t(\rho_{T_1} - \mathbf{I}_{T_1} - \mathbf{Y}_{T_1}) + \nabla\cdot\mathcal{I}[\widetilde{w}\nabla\Phi_{\widetilde{w}}]_t + \nabla\cdot\mathcal{I}[\widetilde{w}\nabla\Phi_{\mathbf{Y}_{T_1+\cdot}}]_t + \nabla\cdot\mathcal{I}[\mathbf{Y}_{T_1+\cdot}\nabla\Phi_{\widetilde{w}}]_t \\ & + \nabla\cdot\mathcal{I}[\mathbf{Y}_{T_1+\cdot}\nabla\Phi_{\mathbf{Y}_{T_1+\cdot}}]_t + \nabla\cdot\mathcal{I}[\boldsymbol{\Psi}_{T_1+\cdot}]_t + \nabla\cdot\mathcal{I}[\nabla\Phi_{\mathbf{I}_{T_1+\cdot}} \varolessthan \mathbf{Y}_{T_1+\cdot}]_t \\ & + \nabla\cdot\mathcal{I}[\mathbf{Y}_{T_1+\cdot} \varolessthan \nabla\Phi_{\mathbf{I}_{T_1+\cdot}}]_t + \nabla\cdot\mathcal{I}[\mathbf{I}_{T_1+\cdot} \varolessthan \nabla\Phi_{\mathbf{Y}_{T_1+\cdot}}]_t + \nabla\cdot\mathcal{I}[\widetilde{w} \varolessthan \nabla\Phi_{\mathbf{I}_{T_1+\cdot}}]_t \\ & + \nabla\cdot\mathcal{I}[\nabla\Phi_{\mathbf{I}_{T_1+\cdot}} \varolessthan \widetilde{w}]_t + \nabla\cdot\mathcal{I}[\mathbf{I}_{T_1+\cdot} \varolessthan \nabla\Phi_{\widetilde{w}}]_t + \nabla\cdot\mathcal{I}[\mathscr{P}_{T_1}(\widetilde{\boldsymbol{w}},\mathbb{X})]_t,\end{aligned} \tag{2.3.14}$$

where the renormalised product $\mathscr{P}_{T_1}(\widetilde{\boldsymbol{w}},\mathbb{X})_t$ is given by

$$\begin{aligned}\mathscr{P}_{T_1}(\widetilde{\boldsymbol{w}},\mathbb{X})_t := {} & \mathscr{C}(\widetilde{w}'_t, \nabla\mathcal{I}[\mathbf{I}_{T_1+\cdot}]_t, \nabla\Phi_{\mathbf{I}_{T_1+t}}) + \mathscr{C}(\widetilde{w}'_t, \nabla^2\mathcal{I}[\Phi_{\mathbf{I}_{T_1+\cdot}}]_t, \mathbf{I}_{T_1+t}) \\ & + (\widetilde{w}^{\#}_t + \nabla\cdot\mathcal{I}[\widetilde{w}' \varolessthan \mathbf{I}_{T_1+\cdot}]_t - \widetilde{w}'_t \varolessthan \nabla\mathcal{I}[\mathbf{I}_{T_1+\cdot}]_t) \odot \nabla\Phi_{\mathbf{I}_{T_1+t}} \\ & + (\nabla\Phi_{\widetilde{w}^{\#}_t} + \nabla\nabla\cdot\Phi_{\mathcal{I}[\widetilde{w}'\varolessthan\mathbf{I}_{T_1+\cdot}]_t} - \widetilde{w}'_t \varolessthan \nabla^2\mathcal{I}[\Phi_{\mathbf{I}_{T_1+\cdot}}]_t) \odot \mathbf{I}_{T_1+t} \\ & + \widetilde{w}'_t \mathbf{V}_{T_1+t} - \widetilde{w}'_t(\nabla P_t\mathcal{I}[\mathbf{I}]_{T_1} \odot \nabla\Phi_{\mathbf{I}_{T_1+t}} + \nabla^2 P_t\mathcal{I}[\Phi_{\mathbf{I}}]_{T_1} \odot \mathbf{I}_{T_1+t}).\end{aligned} \tag{2.3.15}$$

Hence, the only difference between the first iteration (Proposition 2.3.9) and subsequent steps is the initial data $\rho_{T_1} - \mathbf{I}_{T_1} - \mathbf{Y}_{T_1}$ in (2.3.14) and the additional term

$$\widetilde{w}'_t(\nabla P_t\mathcal{I}[\mathbf{I}]_{T_1} \odot \nabla\Phi_{\mathbf{I}_{T_1+t}} + \nabla^2 P_t\mathcal{I}[\Phi_{\mathbf{I}}]_{T_1} \odot \mathbf{I}_{T_1+t})$$

in the renormalised product (2.3.15).

Remark 2.3.14. The paracontrolled triple $\widetilde{\boldsymbol{w}}$ of the second iteration step is related to the paracontrolled triple $\boldsymbol{w}$ of the first iteration step by

$$\widetilde{w}_t = w_{T_1+t}, \qquad \widetilde{w}'_t = w'_{T_1+t}, \qquad \widetilde{w}^{\#}_t = w^{\#}_{T_1+t} + P_t(w_{T_1} - w^{\#}_{T_1})$$

(for those t such that both $\widetilde{\boldsymbol{w}}_t$ and $\boldsymbol{w}_{T_1+t}$ exist.) The additional terms in $\widetilde{w}^{\#}$ and $\mathscr{P}_{T_1}(\widetilde{\boldsymbol{w}},\mathbb{X})$ ensure that the initial data in (2.3.14) can be expressed in terms of ρ_{T_1}, $\mathbf{I}_{T_1}$ and $\mathbf{Y}_{T_1}$. Since we assumed $\mathbb{X} \in \mathcal{X}^{\alpha,\kappa}_T$, this allows us to restart the equation as long as $\rho_{T_1} \in \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$. The price to pay is that the dynamics of the paracontrolled remainder changes at T_1, hence the triple will no longer be continuous on the full time interval of existence. However, this will not be an issue, as we are primarily interested in the dynamics of the solution

$$\begin{aligned}\widetilde{\rho}_t &:= \mathbf{I}_{T_1+t} + \mathbf{Y}_{T_1+t} + \nabla\cdot\mathcal{I}[\widetilde{w}' \varolessthan \mathbf{I}_{T_1+\cdot}]_t + \widetilde{w}^{\#}_t \\ &= \mathbf{I}_{T_1+t} + \mathbf{Y}_{T_1+t} + \nabla\cdot\mathcal{I}[w' \varolessthan \mathbf{I}]_{T_1+t} + w^{\#}_{T_1+t} = \rho_{T_1+t}\end{aligned}$$

which will still be continuous on the full time interval.

We can now use a technique inspired by [MW17b, Derivation of (1.27)] to establish a maximal time of existence $T_{\max} \leq T$ and a paracontrolled solution on $[0, T_{\max})$.

Theorem 2.3.15. *Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ satisfy (2.3.1), $T > 0$, $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$. Then there exists a $T_{\max} \in (0, T]$ and a unique paracontrolled solution $\rho\colon [0, T_{\max}) \to \mathcal{S}'(\mathbb{T}^2)$ to (2.1.1) on $[0, T_{\max})$ with enhancement $\mathbb{X}$ and initial data ρ_0, such that*

$$T_{\max} = T \qquad \text{or} \qquad \lim_{t \uparrow T_{\max}} \|\rho_t\|_{\mathcal{C}^{\alpha+1}} = \infty.$$

Proof. Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ satisfy (2.3.1). Since $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$, we can apply Proposition 2.3.9 and run the equation for some small time $\bar{t}_0 = \bar{t}_0(\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}}, \|\rho_0\|_{\mathcal{B}_{p,q}^{\beta_0}}) \leq T$. By construction, we obtain $\rho_{t_0} \in \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ for each $0 < t_0 \leq \bar{t}_0$.

For the second iteration step, we choose a new tuple of exponents given by $(\alpha, \infty, \infty, \beta_1, \beta_1', \beta_1^{\#}, \alpha+1, \kappa, \eta_1)$ that satisfy the assumptions (2.3.1) and

$$\eta_1 \in \Big[\Big(\frac{\beta_1 - (\alpha+1)}{2}\Big) \vee \Big(\frac{\beta_1^{\#} - (\alpha+1)}{4}\Big), \frac{\alpha+3}{2}\Big) \cap \Big(\frac{-2\alpha-4}{2}, \frac{\alpha+3}{2}\Big). \tag{2.3.16}$$

Using the assumption (2.3.16), we can control the additional term appearing in (2.3.15) via Bony's estimates (Lemma B.4.1) and Schauder's estimates (Lemma B.5.2).

Let $R > 0$ be such that $\|\mathbb{X}\|_{\mathcal{X}_T^{\alpha,\kappa}} \leq R$ and assume that $0 < t_0 < \bar{t}_0$ satisfies $\|\rho_{t_0}\|_{\mathcal{C}^{\alpha+1}} \leq R$. We can then use the decomposition (2.3.12)–(2.3.15) and an argument analogous to Proposition 2.3.9 to re-start the equation from ρ_{t_0} and run it until time $t_0 + T(R)$ for some $T(R) > 0$.

Next, assume $\|\rho_{t_0+T(R)/2}\|_{\mathcal{C}^{\alpha+1}} \leq R$, we can then re-start the equation at time $t_0 + T(R)/2$ from $\rho_{t_0+T(R)/2}$ and run it until $t_0 + T(R)/2 + T(R)$ with the same exponents. We can then continue this procedure on the intervals

$$[0, \bar{t}_0], \quad [t_0, t_0 + T(R)], \quad \Big[t_0 + \frac{T(R)}{2}, t_0 + \frac{T(R)}{2} + T(R)\Big], \quad \Big[t_0 + T(R), t_0 + 2T(R)\Big], \dots \tag{2.3.17}$$

until the first time $t \in [t_0, T]$ such that $\|\rho_t\|_{\mathcal{C}^{\alpha+1}} > R$, or, if such a t does not exist, until T. We denote this time horizon by $T_{\max}^{(R)}$. This yields a paracontrolled solution $\rho \colon [0, T_{\max}^{(R)}] \to \mathcal{S}'(\mathbb{T}^2)$ to (2.1.1) with enhancement $\mathbb{X}$ and initial data $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$, where we used that the intervals (2.3.17) had some overlap as in [MW17b, Derivation of (1.27)] to ensure that ρ is independent of the choice of iteration used in its construction. Furthermore, the paracontrolled solution is unique by an application of Lemma 2.3.11. The sequence $(T_{\max}^{(R)})_{R>0}$ is increasing, hence we may define $T_{\max} \coloneqq \sup_{R>0} T_{\max}^{(R)} = \lim_{R\to\infty} T_{\max}^{(R)} \in (0, T]$. Assume $T_{\max} < T$, then

$$\lim_{t \uparrow T_{\max}} \|\rho_t\|_{\mathcal{C}^{\alpha+1}} = \lim_{R\uparrow\infty} \|\rho_{T_{\max}^{(R)}}\|_{\mathcal{C}^{\alpha+1}} > \lim_{R\uparrow\infty} R = \infty.$$

This yields the claim. □

Let $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_T^{\mathrm{sol}}$ be the space of continuous functions $f \colon [0, T] \to \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ that may blow up in time $T^{\skull}_{\mathcal{C}^{\alpha+1}}[f] \in [0, T]$ (see Section B.8) and denote by $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_{\eta;T}^{\mathrm{sol}}$ the corresponding η-weighted space (Definition B.8.4). In the following we show that each paracontrolled solution constructed in Theorem 2.3.15 is an element of $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_{\eta;T}^{\mathrm{sol}}$.

Lemma 2.3.16. *Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ satisfy (2.3.1), $T > 0$, $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$, $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}(\mathbb{T}^2)$ and ρ be the paracontrolled solution to (2.1.1) on $[0, T_{\max})$ with enhancement $\mathbb{X}$ and initial data ρ_0, as constructed in Theorem 2.3.15. Then $\rho \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_{\eta;T}^{\mathrm{sol}}$ and, in particular, $T_{\max} = T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho_{\eta\text{-wt}}]$.*

Proof. The paracontrolled solution constructed in Theorem 2.3.15 is continuous and exists until it blows up in $\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$, from which the claim follows. □

2.3.3 The Renormalised Solution

In this subsection we combine the deterministic solution theory (Lemma 2.3.16) with the stochastic existence of the renormalised enhancement (Theorem 2.2.2) to construct the renormalised solution to (2.1.1) as a random variable (Theorem 2.3.18, Part 1). We then show that the renormalised solution is the limit in probability of solutions to (2.1.8) (Theorem 2.3.18, Part 2).

Definition 2.3.17. *Let* $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ *satisfy* (2.3.1), $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$ *be some initial data,* $T > 0$, $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$ *and* $\mathbb{X}$ *be the renormalised enhancement of Theorem 2.2.2. We call the paracontrolled solution* $\rho \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_{\eta;T}$ *to* (2.1.1) *on* $[0, T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho_{\eta\text{-wt}}])$ *with enhancement* $\mathbb{X}$ *and initial data* ρ_0 *(in the sense of Definition 2.3.10) the renormalised solution.*

We can now prove the main result of this section, which is similar to [GIP15, Cor. 4.7 & Cor. 5.9], [GP17, Thm. 3.7] and [CC18, Cor. 3.13].

Theorem 2.3.18. *Let* $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ *satisfy* (2.3.1), $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}(\mathbb{T}^2)$ *be some initial data,* $T > 0$, $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$ *and* $\mathbb{X}$ *(resp.* $(\mathbb{X}^\delta)_{\delta>0}$*) be the renormalised (resp. prelimiting renormalised) enhancement of Theorem 2.2.2.*

1. *There exists a paracontrolled solution* ρ *(resp.* ρ^δ*) in* $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_{\eta;T}$ *to* (2.1.1) *on* $[0, T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho_{\eta\text{-wt}}])$ *(resp. on* $[0, T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^\delta_{\eta\text{-wt}}])$*) with enhancement* $\mathbb{X}$ *(resp.* $\mathbb{X}^\delta$*) and initial data* ρ_0 *in the sense of Definition 2.3.10 (for every* $\delta > 0$*). The random variables* $(\rho^\delta)_{\delta>0}$ *coincide with the mild solutions* (2.1.8)*.*

2. *It holds that* $\rho^\delta \to \rho$ *in* $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_{\eta;T}$ *in probability as* $\delta \to 0$*. In particular for each* $\lambda > 0$ *one has that*

$$\lim_{\delta\to 0} \mathbb{P}\Big(\sum_{L=1}^{\infty} \frac{2^{-L}}{L} \frac{\|\rho - \rho^\delta\|_{C_{\eta;T_L}\mathcal{C}^{\alpha+1}}}{1 + \|\rho - \rho^\delta\|_{C_{\eta;T_L}\mathcal{C}^{\alpha+1}}} > \lambda\Big) = 0, \tag{2.3.18}$$

where for every $L \in \mathbb{N}$ *we denote*

$$T_L \coloneqq T \wedge L \wedge \inf\{t \in [0,T] : \|\rho_{\eta\text{-wt}}(t)\|_{\mathcal{C}^{\alpha+1}} > L \text{ or } \|\rho^\delta_{\eta\text{-wt}}(t)\|_{\mathcal{C}^{\alpha+1}} > L\}.$$

Proof. Let $(\alpha, p, q, \beta, \beta', \beta^{\#}, \beta_0, \kappa, \eta)$ satisfy (2.3.1). The enhancements $\mathbb{X}$ and $(\mathbb{X}^\delta)_{\delta>0}$ exist almost surely by an application of Theorem 2.2.2. It then follows by Lemma 2.3.16 that ρ and $(\rho^\delta)_{\delta>0}$ are random variables in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_{\eta;T}$, which is a metric space whose distance we denote by $D^{\mathcal{C}^{\alpha+1}}_{\eta;T}$ (cf. Section B.8). By Lemma B.8.3 and the local Lipschitz continuity of the solution map in the enhancement (Lemma 2.3.13) we deduce that for every $\lambda > 0$ there exists some $\nu > 0$

such that $\|\mathbb{X} - \mathbb{X}^\delta\|_{\mathcal{X}_T^{\alpha,\kappa}} \le \nu$ implies $D_{\eta;T}^{\mathcal{C}^{\alpha+1}}(\rho, \rho^\delta) \le \lambda$. Hence, by an application of Theorem 2.2.2,

$$\mathbb{P}\Big(D_{\eta;T}^{\mathcal{C}^{\alpha+1}}(\rho, \rho^\delta) > \lambda\Big) \le \mathbb{P}(\|\mathbb{X} - \mathbb{X}^\delta\|_{\mathcal{X}_T^{\alpha,\kappa}} > \nu) \to 0 \quad \text{as } \delta \to 0,$$

which yields that $\rho^\delta \to \rho$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_{\eta;T}$ in probability. The convergence (2.3.18) then follows by the explicit form of $D_{\eta;T}^{\mathcal{C}^{\alpha+1}}$, see [Cha+22, Sec. 1.5.1]. $\square$

Example 2.3.19. The same arguments as in the proof of Theorem 2.3.18, Part 1, also allow us to construct the mild solution to (1.1.11) as a paracontrolled solution to (2.1.1) with enhancement $\mathbb{X}^{(\varepsilon)}_\delta = (\varepsilon^{1/2} \mathbf{I}^\delta, \varepsilon \mathbf{Y}^\delta_{\mathrm{can}}, \varepsilon^{3/2} \mathbf{\Psi}^\delta_{\mathrm{can}}, \varepsilon \mathbf{V}^\delta)$ (and heterogeneity $\sigma = \sqrt{\rho_{\mathrm{det}}}$). To obtain a limit for vanishing correlation lengths δ, one then needs to assume that the noise intensity ε decays sufficiently quickly depending on the behaviour of the renormalisation. This will be the focus of Chapter 3.

3 Small-Noise Results

Contents

3.1 Introduction

Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} \coloneqq \langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to the deterministic, periodic Keller–Segel equation (1.1.12) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (in $L^2(\mathbb{T}^2)$.) For every $T < T^*$ we can then apply the regularity theory for ρ_{det} to deduce $\sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{H}^2(\mathbb{T}^2)$ (see Section 3.2, in particular Lemma 3.2.5), so that we may choose the heterogeneity $\sigma := \sqrt{\rho_{\mathrm{det}}}$.

It follows by the methods developed in Chapter 2 (see Theorem 2.3.18) that there exists a sequence of counterterms $(\mathbf{\Upsilon}^\delta)_{\delta>0}$ (the renormalisation) given by the expectation

$$\mathbf{\Upsilon}^\delta := \mathbb{E}[\nabla \cdot \mathcal{I}[\mathbf{I}^\delta \nabla \Phi_{\mathbf{I}^\delta}]], \quad \mathbf{I}^\delta := \nabla \cdot \mathcal{I}[\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}^\delta], \quad \boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}, \quad \psi_\delta \text{ as in (1.2.7)}$$

such that the solutions $(\rho^\delta)_{\delta>0}$ to the *renormalised equation*

$$\rho^\delta = P\rho_0 - \chi\nabla \cdot \mathcal{I}[\rho^\delta \nabla \Phi_{\rho^\delta}] + \chi\,\mathbf{\Upsilon}^\delta - \mathbf{I}^\delta \tag{3.1.1}$$

converge as $\delta \to 0$ to a non-trivial limit. For a self-contained statement of this result and a full proof, see Theorem 4.0.2 in Chapter 4. It is clear that the renormalisation depends on the initial data ρ_0 and the chemotactic sensitivity χ through the heterogeneity $\sqrt{\rho_{\mathrm{det}}}$ and we further showed that it diverges at most logarithmically as $\delta \to 0$, i.e. for every $\alpha < -2$, $\kappa \in (0,1)$ and cut-off φ satisfying (1.2.6), we obtain the bound

$$\|\mathbf{\Upsilon}^\delta\|_{\mathscr{L}_T^\kappa \mathcal{C}^{2\alpha+4}} \lesssim (1 \vee \log(\delta^{-1}))\|\varphi\|_{L^\infty}^2 \|\sqrt{\rho_{\mathrm{det}}}\|_{C_T\mathcal{H}^2}^2;$$

see Lemma 2.2.19.

In the present chapter we consider another method for ensuring the existence of a well-posed limit for vanishing correlation lengths δ. Instead of renormalising the equation as in (3.1.1), we assume that the noise is scaled by a small parameter $\varepsilon > 0$ called the noise intensity and consider solutions $\rho_\delta^{(\varepsilon)}$ to the *canonical equation*

$$\rho = P\rho_0 - \chi\nabla \cdot \mathcal{I}[\rho\nabla\Phi_\rho] - \varepsilon^{1/2}\mathbf{I}^\delta. \tag{3.1.2}$$

The key question is then how one should choose δ relative to ε to obtain a non-trivial limit for $\rho_{\delta(\varepsilon)}^{(\varepsilon)}$ as $\varepsilon \to 0$ and $\delta = \delta(\varepsilon) \to 0$.

As alluded to in Subsection 1.1.1, there is a trade-off between the relative scaling of $(\varepsilon, \delta(\varepsilon))$ and the topology in which such asymptotic results hold. To show convergence in the Bessel potential space $\mathcal{H}^\gamma(\mathbb{T}^2)$ of regularity $\gamma > -1/2$, we need to assume the restrictive relative scaling $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ (see Section 3.3); to show convergence in the Hölder–Besov space $\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ of regularity $\alpha+1 < -1$ it suffices to assume the general relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ (see Section 3.4). We refer to these two situations as the *regular* and *rough* setting respectively.

We will show in the rough setting (Section 3.4) that the relative scaling assumption $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ is directly informed by the behaviour of the renormalisation $(\varepsilon \mathbf{Q}^{\delta(\varepsilon)})_{\varepsilon>0}$, connecting the solution ρ^δ of the renormalised equation (3.1.1) and the solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ of the canonical equation (3.1.2). We prove an analogue of the law of large numbers, i.e. $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ as $\varepsilon \to 0$, by relying on the continuity of the solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ in the noise enhancement $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} = (\varepsilon^{1/2}\mathbf{I}^\delta, \varepsilon \mathbf{Y}^\delta_{\mathrm{can}}, \varepsilon^{3/2}\mathbf{W}^\delta_{\mathrm{can}}, \varepsilon \mathbf{V}^\delta)$ combined with the relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ to deduce $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} \to 0$ (Theorem 3.4.4). We further show a large deviation principle for $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ by applying a generalization of Freidlin and Wentzell's theory [FW12] due to Hairer and Weber [HW15] to obtain a large deviation principle for the noise enhancement $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ (Theorem 3.4.7), which we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle (Theorem 3.4.8). See also [FV07] for a similar argument in the theory of rough paths.

In the regular setting it suffices to assume $\rho_0 \in C(\mathbb{T}^2)$ is such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$. We prove an analogue of the law of large numbers by relying on the continuity of the solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ in the noise $\varepsilon^{1/2}\mathbf{I}^{\delta(\varepsilon)}$ combined with the relative scaling $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ (Theorem 3.3.3; $\gamma \in (-1/2, 0]$). We then show a large deviation principle for $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ by applying the weak-convergence approach based on the Boué–Dupuis formula [BDM08] to obtain a large deviation principle for $(\varepsilon^{1/2}\mathbf{I}^{\delta(\varepsilon)})_{\varepsilon>0}$ (Theorem 3.3.4), which we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle (Theorem 3.3.5; $\gamma \in (-1/2, 0)$). Let us also point out the eventual application of these results in Subsection 5.1.2 to control the probability of the event that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ becomes negative.

What distinguishes the rough setting from the regular setting is the availability of a central limit theorem. The leading-order fluctuations $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\mathrm{det}})$ involve the term $\mathbf{I}^{\delta(\varepsilon)}$, which does not converge in $\mathcal{H}^\gamma(\mathbb{T}^2)$ for any $\gamma > -1/2$. However, as shown in Lemma 2.2.7, it follows that $\Psi^{\delta(\varepsilon)} \to \Psi$ in $\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ as $\varepsilon \to 0$. Consequently, in the rough setting we can establish a central limit theorem (Theorem 3.4.6) under the relative scaling $\varepsilon^{1/2} \log(\delta(\varepsilon)^{-1}) \to 0$ (where we lose one order of $\varepsilon^{1/2}$ in the scale of the leading-order fluctuations.)

By choosing the heterogeneity $\sqrt{\rho_{\mathrm{det}}}$, we build a bridge between the canonical model (3.1.2) and the theory of fluctuating hydrodynamics discussed in Subsection 1.1.1. There, (3.1.2) is introduced as an additive-noise approximation (1.1.11) to the Keller–Segel–Dean–Kawasaki equation (1.1.10). The small-noise results above suggest that our additive-noise approximation is precise up to first order and that the approximation error lies in its large deviations, see Subsection 1.1.2 for more details.

Structure of the Chapter: Section 3.2 contains the existence, uniqueness and regularity of solutions ρ_{det} to the deterministic, periodic Keller–Segel equation (1.1.12); in particular Lemma 3.2.5 ensures that $\sqrt{\rho_{\mathrm{det}}}$ is suitably regular to be chosen as heterogeneity. Section 3.3 concerns the regular setting and establishes a law of large numbers (Theorem 3.3.3) and a large deviation principle (Theorem 3.3.5). Section 3.4 concerns the rough setting and establishes a law of large numbers (Theorem 3.4.4), a central limit theorem (Theorem 3.4.6) and a large deviation principle (Theorem 3.4.8). Finally let us point out four appendices relevant to the present chapter: Appendix F concerns the positivity of generic diffusion-advection equations, which will be applied in Subsection 3.2.2. Appendix G contains an adaptation of Hairer and Weber's generalization of Freidlin–Wentzell theory, which is used in Theorem 3.4.7. Appendix H contains the existence of several objects driven by Cameron–Martin elements that appear in the rate functions of Theorems 3.3.4 & 3.4.7. Appendix I concerns the regularity of Ψ^{δ} in $C_T\mathcal{H}^{\gamma}(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$, which plays an important rôle in Section 3.3.

3.2 Analysis of the Deterministic Keller–Segel Equation

In this section we show the existence, uniqueness and regularity of the solution f to the deterministic Keller–Segel equation,

$$\begin{cases} (\partial_t - \Delta)f = -\chi\nabla \cdot (f\nabla\Phi_f), & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad -\Delta\Phi_f = f - \overline{f}, & \text{in } (0,T] \times \mathbb{T}^2, \\ \quad\quad f|_{t=0} = f_0, & \text{on } \mathbb{T}^2, \end{cases} \tag{3.2.1}$$

with initial data $f_0 \in L^2(\mathbb{T}^2)$ and chemotactic sensitivity $\chi \in \mathbb{R}$, as well as the regularity of $\sqrt{f}$ in a suitable Bessel potential space (Theorem 3.2.1 & Lemma 3.2.5), which will be applied throughout Sections 3.3 & 3.4 to ensure that the heterogeneity $\sqrt{f}$ is sufficiently regular.

3.2.1 Existence and Regularity

We assume that the initial data f_0 to (3.2.1) lies in $L^2(\mathbb{T}^2)$, is non-negative and of unit mass. We refer to [Per04; HP09; Hor03; Hor04] and Remark 3.2.2 for a wider ranging survey of the derivation and properties of the Cauchy problem (3.2.1) and related PDE.

Our main emphasis is to recall that the global well-posedness of (3.2.1) depends on the sign and size of χ. For small χ we can employ the periodic Gagliardo–Nirenberg–Sobolev inequality (B.1.4) to show that the solution will be globally well-posed (cf. Theorem 3.2.1), whereas for large χ it is known that the solution may blow up in finite time [KX16, Thm. 8.1].

In stating the main result of this subsection (Theorem 3.2.1) we use the notation $C_{\text{gns-per}}(2,3,1)$ to denote the optimal constant such that the periodic Gagliardo–Nirenberg–Sobolev inequality (B.1.4) holds, which is properly elucidated in Section B.1 below.

Theorem 3.2.1. *Let $f_0 \in L^2(\mathbb{T}^2)$ be such that $f_0 \geq 0$ and $\overline{f_0} = 1$, and $\chi \in \mathbb{R}$. Then the following hold:*

1. *There exists a* $T^* = T^*(\chi, \|f_0\|_{L^2}) \in (0, \infty]$ *and a unique map* $f\colon [0, T^*) \to L^2(\mathbb{T}^2)$ *such that* $f \in C_T L^2(\mathbb{T}^2) \cap L^2_T \mathcal{H}^1(\mathbb{T}^2)$ *for every* $T \in (0, T^*)$, $f|_{t=0} = f_0$ *and* f *solves* (3.2.1) *in the weak sense. Furthermore for each* $t \in [0, T^*)$ *this solution is non-negative almost everywhere and* $\|f_t\|_{L^1} = 1$. *Finally* T^* *is such that either* $T^* = \infty$ *or* $\lim_{t \nearrow T^*} \|f_t\|_{L^2} = \infty$, *and we refer to it as the maximal time of existence (for* f *in* $L^2(\mathbb{T}^2)$.*)*

2. *If* $\chi \leq \chi^* \coloneqq C_{\text{gns-per}}(2, 3, 1)^{-3}$, *then one has* $T^* = \infty$.

3. *It holds that* $f \in C^2_1((0, T] \times \mathbb{T}^2)$ *for every* $T \in (0, T^*)$. *Further if* $f_0 \in \mathcal{H}^\gamma(\mathbb{T}^2)$ *for some* $\gamma > 0$, *then* $f \in C_T \mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T \mathcal{H}^{\gamma+1}(\mathbb{T}^2)$.

Proof. Since these statements are mostly amalgams of known results we only provide a sketched proof with references for full details.

The proof of Claim 1 is carried out through a Galerkin approximation and is classical. To give a brief sketch, local existence for general $\chi \in \mathbb{R}$ can be deduced by the readily obtained a priori energy identity

$$\|f_t\|^2_{L^2} = \|f_0\|^2_{L^2} - 2 \int_0^t \|\nabla f_s\|^2_{L^2} \, \mathrm{d}s + \chi \int_0^t \|f_s\|^3_{L^3} \, \mathrm{d}s - \chi \int_0^t \|f_s\|^2_{L^2} \, \mathrm{d}s \tag{3.2.2}$$

so that applying the Sobolev embedding $\mathcal{H}^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$ (cf. (B.1.1)), real interpolation and Young's inequality, one finds that there exists a constant $C > 0$ such that

$$\|f_t\|^2_{L^2} \leq \|f_0\|^2_{L^2} - \int_0^t \|\nabla f_s\|^2_{L^2} \, \mathrm{d}s + C\chi^2 \int_0^t \|f_s\|^4_{L^2} \, \mathrm{d}s + \int_0^t \|f_s\|^2_{L^2} \, \mathrm{d}s.$$

A comparison argument to the ODE $\dot{x}(t) = C\chi^2 x(t)^2 + x(t)$ shows that there exists a time horizon T^* such that $t \mapsto \|f_t\|^2_{L^2}$ and $t \mapsto \int_0^t \|\nabla f_s\|^2_{L^2} \, \mathrm{d}s$ are locally bounded on $[0, T^*)$.

To prove uniqueness, let g be another weak solution with $g_0 = f_0$. By a similar calculation, one finds a priori that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_t - g_t\|_{L^2}^2 + \|\nabla(f_t - g_t)\|_{L^2}^2 \\
&\quad = \chi\langle f_t\nabla\Phi_{f_t-g_t} + (f_t - g_t)\nabla\Phi_{g_t}, \nabla(f_t - g_t)\rangle_{L^2} \\
&\quad \le |\chi|(\|f_t\nabla\Phi_{f_t-g_t}\|_{L^2} + \|(f_t - g_t)\nabla\Phi_{g_t}\|_{L^2})\|\nabla(f_t - g_t)\|_{L^2} \\
&\quad \lesssim |\chi|(\|f_t\|_{\mathcal{H}^1}\|\nabla\Phi_{f_t-g_t}\|_{\mathcal{H}^1} + \|f_t - g_t\|_{L^2}\|\nabla\Phi_{g_t}\|_{\mathcal{H}^2})\|\nabla(f_t - g_t)\|_{L^2} \\
&\quad \lesssim |\chi|(\|f_t\|_{\mathcal{H}^1} + \|g_t\|_{\mathcal{H}^1})\|f_t - g_t\|_{L^2}\|\nabla(f_t - g_t)\|_{L^2},
\end{aligned}$$

where in the second inequality we used that $uv \in L^2(\mathbb{T}^2)$ for each $(u, v) \in \mathcal{H}^1(\mathbb{T}^2) \times \mathcal{H}^1(\mathbb{T}^2)$ or $(u, v) \in L^2(\mathbb{T}^2)\times\mathcal{H}^2(\mathbb{T}^2)$, see [BH21, Thm. 7.4]. A combination of Young's and Gronwall's inequalities then implies that there exists some $C > 0$ such that

$$\|f_t - g_t\|_{L^2}^2 \le \|f_0 - g_0\|_{L^2}^2 \exp\Big(C\chi^2\int_0^t \|f_s\|_{\mathcal{H}^1}^2 + \|g_s\|_{\mathcal{H}^1}^2\,\mathrm{d}s\Big),$$

from which uniqueness follows on the interval $[0, T^*)$.

Taking the constant function 1 as a test function shows that the mean of the solution is preserved. The almost everywhere non-negativity can be shown by testing with (a smooth approximant to) the negative part of the solution $f_t^- := \min\{0, f_t\}$ and taking limits by the dominated convergence theorem. For a similar proof in the two-dimensional plane, see [MT23, Prop. 3.4]. A stronger notion of positivity of solutions given initial data that are continuous and bounded away from 0 is shown in Lemma 3.2.4.

To prove Claim 2 we use that for $T < T^*$ and weak solutions $f \in C_TL^2(\mathbb{T}^2) \cap L_T^2\mathcal{H}^1(\mathbb{T}^2)$ one has for each $t \le T$ the identities (3.2.2) and

$$\|f_t-\overline{f_t}\|_{L^2}^2 = \|f_0-\overline{f_0}\|_{L^2}^2-2\int_0^t \|\nabla f_s\|_{L^2}^2\,\mathrm{d}s+\chi\int_0^t \|f_s-\overline{f_s}\|_{L^3}^3\,\mathrm{d}s+2\chi\int_0^t \|f_s-\overline{f_s}\|_{L^2}^2\,\mathrm{d}s. \tag{3.2.3}$$

We first note that if $\chi \le 0$ then by Gronwall's inequality applied to (3.2.2) we have a direct bound for $\|f_t\|_{L^2}^2$ which is sufficient to show that $\|f_t\|_{L^2} < \infty$ for all $t < \infty$.

In the case $\chi > 0$ we apply the periodic Gagliardo–Nirenberg–Sobolev inequality (B.1.4) with $d = 2$, $q = 3$ and $r = 1$ to obtain

$$\|f_s - \overline{f_s}\|_{L^3}^3 \le C_{\text{gns-per}}(2,3,1)^3 \|\nabla f_s\|_{L^2}^2 \|f_s - \overline{f_s}\|_{L^1} \le 2C_{\text{gns-per}}(2,3,1)^3 \|\nabla f_s\|_{L^2}^2,$$

where in the second step we used Minkowski's inequality to estimate $\|f_s - \overline{f_s}\|_{L^1} \le 2$. Inserting this into (3.2.3) we find the inequality

$$\begin{aligned}\|f_t - \overline{f_t}\|_{L^2}^2 \le \|f_0 - \overline{f_0}\|_{L^2}^2 - 2\Big(1 - \chi C_{\text{gns-per}}(2,3,1)^3\Big) \int_0^t \|\nabla f_s\|_{L^2}^2 \,\mathrm{d}s \\ + 2\chi \int_0^t \|f_s - \overline{f_s}\|_{L^2}^2 \,\mathrm{d}s.\end{aligned}$$

If $\chi \le \chi^* := C_{\text{gns-per}}(2,3,1)^{-3}$ then

$$\|f_t - \overline{f_t}\|_{L^2}^2 \le \|f_0 - \overline{f_0}\|_{L^2}^2 + 2\chi \int_0^t \|f_s - \overline{f_s}\|_{L^2}^2 \,\mathrm{d}s$$

and we can apply Gronwall's inequality to deduce

$$\|f_t - \overline{f_t}\|_{L^2}^2 \le \|f_0 - \overline{f_0}\|_{L^2}^2 \exp(2\chi t),$$

which implies $T^* = \infty$.

To prove Claim 3 we first use the fact that weak solutions to (3.2.1) are mild,

$$f = Pf_0 - \chi \nabla \cdot \mathcal{I}[f \nabla \Phi_f], \tag{3.2.4}$$

which can be shown by testing the equation against the time-reversed heat kernel. Using the mild formulation (3.2.4) we can then bootstrap the space regularity of f at positive times to establish $f \in C((0,T];\mathcal{H}^4(\mathbb{T}^2))$, which by Sobolev's embedding $\mathcal{H}^4(\mathbb{T}^2) \hookrightarrow C^2(\mathbb{T}^2)$ is enough to deduce $\partial_{x_i}\partial_{x_j} f \in C((0,T] \times \mathbb{T}^2)$ for every $i,j \in \{1,2\}$. Furthermore, we can use the same space regularity to deduce that the weak time derivative $\partial_t f$ lies in $C((0,T];\mathcal{H}^2(\mathbb{T}^2))$, which together with the embedding $\mathcal{H}^2(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ (cf. (B.1.2)) yields $\partial_t f \in C((0,T] \times \mathbb{T}^2)$. The fundamental theorem of calculus then allows us to argue that the weak time derivative coincides with the (usual) time derivative, which yields $f \in C_1^2((0,T] \times \mathbb{T}^2)$.

To establish $f \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ for every $\gamma > 0$, $f_0 \in \mathcal{H}^\gamma(\mathbb{T}^2)$ and $T < T^*$, it suffices to estimate each term in the mild formulation (3.2.4) by using

Schauder's estimates (Lemma B.5.3) combined with product estimates in Bessel potential spaces. This yields Claim 3. □

Remark 3.2.2 (Sharp a priori bounds). On the whole space instead of the torus, one can make use of the sharp Gagliardo–Nirenberg–Sobolev inequality found in [Wei83] to show that the entropy is strictly non-increasing and the solution is globally well-posed provided $\chi < 4\pi(1.86225\ldots)$, see [BDP06, Sec. 2.2]. Still on the whole space, this range can be further improved to allow for all $\chi < 8\pi$ by instead analysing the evolution of the free energy (A.1.2) and replacing the Gagliardo–Nirenberg–Sobolev inequality with the logarithmic Hardy–Littlewood–Sobolev inequality, see [BDP06]. This result is in fact sharp. A straightforward argument, see [BDP06, Sec. 2.1], analysing the evolution of the second moment of solutions demonstrates that weak and very weak solutions to (3.2.1) must blow up in finite time provided $\chi > 8\pi$ with a blow-up time converging to $+\infty$ as χ approaches 8π. See also more recent results that treat more general initial data [Wei18; FT23a]. The picture on the torus is more complicated [KX16, Thm. 8.1]; however, for our purposes we do not require a sharp result, only well-posedness of the PDE (3.2.1) for some range of χ.

Next, we show that the solution to (3.2.1) started from continuous initial data is continuous everywhere.

Lemma 3.2.3. *Let $f_0 \in C(\mathbb{T}^2)$ be such that $f_0 \geq 0$ and $\overline{f_0} = 1$, f be the weak solution to* (3.2.1) *with initial data f_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Then $f \in C([0,T] \times \mathbb{T}^2)$ for every $T < T^*$ and T^* coincides with the maximal time of existence in $C(\mathbb{T}^2)$.*

Proof. Using that $f_0 \in C(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$, we can apply Theorem 3.2.1 to find the unique weak solution $f \in C((0,T] \times \mathbb{T}^2)$; hence it suffices to establish the continuity of f at time 0, i.e. on a small time interval $[0,\bar{T}]$ where $\bar{T} \in (0, T \wedge 1]$. By the proof of Theorem 3.2.1 (cf. (3.2.4)), it follows that the weak solution is also mild, hence we can use Young's convolution estimate and Schauder's lemma to bound for every

$\vartheta \in (0,1)$,

$$\|f_t\|_{C(\mathbb{T}^2)} \leq \|f_0\|_{C(\mathbb{T}^2)} + C|\chi|\|f\|_{C_{\bar{T}}L^2} \int_0^t |t-s|^{-\frac{1+\vartheta}{2}} \|f_s\|_{C(\mathbb{T}^2)}\, \mathrm{d}s,$$

where C is some constant. Denote $\beta := \frac{2}{1-\vartheta}$, we can now apply the fractional Gronwall estimate [Web19, Rem. 3.7] to deduce

$$\|f_t\|_{C_{\bar{T}}C(\mathbb{T}^2)} \leq \beta\|f_0\|_{C(\mathbb{T}^2)} \exp\Big(C_\beta|\chi|^\beta\|f\|^\beta_{C_{\bar{T}}L^2}t\Big),$$

where $C_\beta > 0$ is a constant that only depends on β. This a priori estimate allows us to use an approximation argument to show that f is continuous at time 0 which implies $f \in C([0,T]\times\mathbb{T}^2)$. $\square$

3.2.2 Boundedness Away From Zero

In this subsection we show that the solution to (3.2.1) constructed in Theorem 3.2.1 is bounded away from zero uniformly in $[0,T]\times\mathbb{T}^2$ for every $T < T^*$, if its initial data is assumed to be continuous and positive.

Lemma 3.2.4. *Let $f_0 \in C(\mathbb{T}^2)$ be such that $f_0 > 0$ and $\overline{f_0} = 1$, f be the weak solution to* (3.2.1) *with initial data f_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$ there exists a $c > 0$ such that $f(t,x) \geq c$ for all $(t,x) \in [0,T]\times\mathbb{T}^2$.*

Proof. It follows by Theorem 3.2.1 and Lemma 3.2.3 that $f \in C^2_1((0,T]\times\mathbb{T}^2) \cap C([0,T]\times\mathbb{T}^2)$ is a classical solution to (3.2.1). Denote $b := -\chi\nabla\Phi_f$, it then follows by elliptic regularity (Lemma B.5.5) that $b \in C([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ and $\nabla\cdot b = \chi(f-\overline{f}) \in C([0,T]\times\mathbb{T}^2;\mathbb{R})$. Hence we can apply Proposition F.0.1 to deduce that there exist some $c > 0$ such that $f(t,x) \geq c$ for all $(t,x) \in [0,T]\times\mathbb{T}^2$. $\square$

3.2.3 Regularity of the Square Root

In this subsection we establish the regularity of the square root of the solution to the deterministic Keller–Segel equation (3.2.1) (Lemma 3.2.5).

Lemma 3.2.5. *Let $f_0 \in C(\mathbb{T}^2)$ be such that $f_0 \geq 0$ and $\overline{f_0} = 1$, f be the weak solution to* (3.2.1) *with initial data f_0 and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$ it holds that $\sqrt{f} \in C([0,T] \times \mathbb{T}^2)$. Assuming further $f_0 > 0$ and $f_0 \in \mathcal{H}^\gamma(\mathbb{T}^2) \cap C(\mathbb{T}^2)$ for some $\gamma \geq 0$, then it holds that $\sqrt{f} \in C_{\eta;T}\mathcal{H}^{\gamma+2\eta}(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ for every $\eta \in [0, 1/2)$.*

Proof. Given $f_0 \in C(\mathbb{T}^2)$ such that $f_0 \geq 0$ and $\overline{f_0} = 1$, we can apply Lemma 3.2.3 to construct $f \in C([0,T] \times \mathbb{T}^2)$ for all $T < T^*$. Using the continuity of the square root in the non-negative reals, it follows that $\sqrt{f} \in C([0,T] \times \mathbb{T}^2)$.

If further $f_0 \in \mathcal{H}^\gamma(\mathbb{T}^2) \cap C(\mathbb{T}^2)$ for some $\gamma \geq 0$, we can apply Theorem 3.2.1, the mild formulation (3.2.4), a bootstrapping argument and Lemma 3.2.3 to construct $f \in C_{\eta;T}\mathcal{H}^{\gamma+2\eta}(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2) \cap C([0,T] \times \mathbb{T}^2)$ for all $T < T^*$; if in addition $f_0 > 0$, it also follows by Lemma 3.2.4 that there exists a constant $c > 0$ such that $f \geq c$ on $[0,T] \times \mathbb{T}^2$. To prove $\sqrt{f} \in C_{\eta;T}\mathcal{H}^{\gamma+2\eta}(\mathbb{T}^2)$, it then suffices to apply [BCD11a, Cor. 2.91] combined with a smooth cut-off. To prove $\sqrt{f} \in L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$, we apply Lemma B.6.1 with $F = \sqrt{\cdot}$ to show the existence of a constant $C \coloneqq C(\gamma + 1, F, \|f\|_{C_TL^\infty}, c^{-1}) > 0$ such that

$$\left\|\sqrt{f}\right\|^2_{L^2_T\mathcal{H}^{\gamma+1}} = \int_0^T \left\|\sqrt{f_t}\right\|^2_{\mathcal{H}^{\gamma+1}} \mathrm{d}t \leq C^2 \int_0^T \|f_t\|^2_{\mathcal{H}^{\gamma+1}} \mathrm{d}t = C^2\|f\|^2_{L^2_T\mathcal{H}^{\gamma+1}},$$

which yields the claim. $\square$

3.3 Regular Theory for Small Noise Intensities

In this section we first show the well-posedness of the solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) in the Bessel potential space $\mathcal{H}^\gamma(\mathbb{T}^2)$ of regularity $\gamma \in (-1/2, 0]$ (Lemma 3.3.1), which we then use to establish a law of large numbers (Theorem 3.3.3) and a large deviation principle (Theorem 3.3.5, if $\gamma < 0$) under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$.

Throughout this section we consider initial data $\rho_0 \in C(\mathbb{T}^2)$ such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$. In particular, this ensures that $\sqrt{\rho_{\mathrm{det}}} \in C([0,T] \times \mathbb{T}^2)$ (see Lemma 3.2.5) and that $\text{⫯}^\delta \in C_TL^2(\mathbb{T}^2) \cap L^2_T\mathcal{H}^1(\mathbb{T}^2)$ (see Proposition I.0.4).

3.3.1 Well-Posedness

In this subsection we construct solutions to (3.1.2) with initial data in $C(\mathbb{T}^2)$ taking values in the Bessel potential space $\mathcal{H}^\gamma(\mathbb{T}^2)$ of regularity $\gamma \in (-1/2, 0]$ (Lemma 3.3.1). The following lemma is conceptually similar to the results of Chapter 2, although since we don't take $\delta \to 0$ here, the proof is far simpler and does not require techniques of singular SPDEs [GIP15; Hai14; Duc21].

Recall that $(\mathcal{H}^\gamma(\mathbb{T}^2))_T^{\text{sol}}$ denotes the space of continuous functions that may blow up in finite time (see Section B.8).

Lemma 3.3.1. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for every $T < T^*$, $\varepsilon > 0$, $\delta > 0$ and $\gamma \in (-1/2, 0]$, there exists a unique solution $\rho_\delta^{(\varepsilon)}$ to* (3.1.2) *such that, almost surely, $\rho_\delta^{(\varepsilon)} \in (\mathcal{H}^\gamma(\mathbb{T}^2))_T^{\text{sol}}$ and $\rho_\delta^{(\varepsilon)} \in C_S\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_S^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ for every $S < T_{\mathcal{H}^\gamma}^{\text{☠}}[\rho_\delta^{(\varepsilon)}]$ (where $T_{\mathcal{H}^\gamma}^{\text{☠}}[\rho_\delta^{(\varepsilon)}] \in (0, T]$ denotes the maximal time of existence of $\rho_\delta^{(\varepsilon)}$.) In particular, the solution is almost surely locally Lipschitz continuous in the noise $\varepsilon^{1/2}\text{⫯}^{\delta(\varepsilon)} \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$.*

Proof. An application of Proposition I.0.4 yields $\text{⫯}^\delta \in C_T L^2(\mathbb{T}^2) \cap L_T^2\mathcal{H}^1(\mathbb{T}^2)$ almost surely for each $\delta > 0$. Given a realisation of the noise, $\varepsilon > 0$, $\gamma \in (-1/2, 0]$ and $\bar{T}$ sufficiently small, we can construct a local solution $\rho_\delta^{(\varepsilon)}$ to (3.1.2) as a fixed point in $C_{\bar{T}}\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_{\bar{T}}^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$. Re-starting the equation, we may then iterate the construction to obtain a solution in $(\mathcal{H}^\gamma(\mathbb{T}^2))_T^{\text{sol}}$. Since the noise is additive in (3.1.2), it follows that the solution is continuous in $\varepsilon^{1/2}\text{⫯}^{\delta(\varepsilon)} \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$. $\square$

3.3.2 Law of Large Numbers

In this subsection we establish a (weak) law of large numbers for solutions $\rho_{\delta(\varepsilon)}^{(\varepsilon)}$ to (3.1.2) in $(\mathcal{H}^\gamma(\mathbb{T}^2))_T^{\text{sol}}$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in (-1/2, 0]$ (Theorem 3.3.3). It is convenient to first prove a law of large numbers for $(\varepsilon^{1/2}\text{⫯}^{\delta(\varepsilon)})_{\varepsilon>0}$ (Lemma 3.3.2), which we can then transfer to $(\rho_{\delta(\varepsilon)}^{(\varepsilon)})_{\varepsilon>0}$ by the continuous mapping theorem (Theorem 3.3.3).

Lemma 3.3.2. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in [-1, 0]$. Then for all $T < T^*$, it follows that $\varepsilon^{1/2} \uparrow^{\delta(\varepsilon)} \to 0$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in probability as $\varepsilon \to 0$.*

Proof. An application of Proposition I.0.4 combined with the scaling assumption $\varepsilon^{1/2}\delta(\varepsilon)^{\gamma-2} \to 0$ yields

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}\mathbb{E}\Big[\| \uparrow^{\delta} \|^2_{C_T\mathcal{H}^\gamma \cap L_T^2\mathcal{H}^{\gamma+1}}\Big]^{1/2} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|_{C_TL^2\cap L_T^2\mathcal{H}^1} \lim_{\varepsilon\to 0} \varepsilon^{1/2}(1 + \delta(\varepsilon)^{-\gamma-2}) = 0,$$

which implies convergence in probability by Markov's inequality. ☐

The following theorem demonstrates an analogue of the law of large numbers for $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$. Note that in contrast to Lemma 3.3.2 we have to restrict the range of γ to $\gamma \in (-1/2, 0]$ so that we have enough regularity to make sense of the non-linearity in (3.1.2) (see also Lemma 3.3.1).

Theorem 3.3.3. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in (-1/2, 0]$. Then for all $T < T^*$, it follows that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$.*

Proof. It follows by Lemma 3.3.2 and the relative scaling $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ that $\varepsilon^{1/2} \uparrow^{\delta(\varepsilon)} \to 0$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in probability as $\varepsilon \to 0$. Using the local Lipschitz continuity of $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ in $\varepsilon^{1/2} \uparrow^{\delta(\varepsilon)}$ (Lemma 3.3.1), it follows by the continuous mapping theorem that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$. ☐

3.3.3 Large Deviation Principle

In this subsection we we apply the weak-convergence approach to large deviations [BDM08] to establish a large deviation principle for solutions $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$

for some $\gamma \in (-1/2, 0)$ (Theorem 3.3.5). As in Subsection 3.3.2, we first consider the noise $(\varepsilon^{1/2}\boldsymbol{\uparrow}^{\delta(\varepsilon)})_{\varepsilon>0}$ driving $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ (cf. Lemma 3.3.1) for which we establish a large deviation principle (Theorem 1.1.2) that we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle (Theorem 3.3.5).

Theorem 3.3.4. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in [-1, 0)$. Then for all $T < T^*$, it follows that the sequence $(\varepsilon^{1/2}\boldsymbol{\uparrow}^{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $C_T\mathcal{H}^{\gamma}(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ with rate ε and good rate function*

$$\mathscr{I}_{\boldsymbol{\uparrow}}(s) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \boldsymbol{\uparrow}^h = s\Big\}, \qquad \boldsymbol{\uparrow}^h := \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h].$$

For the existence and regularity of $\boldsymbol{\uparrow}^h$, see also Lemmas H.0.1–H.0.2.

Proof. It follows from (I.0.2) that $\boldsymbol{\uparrow}^\delta$ is a functional of finitely many Brownian motions with the number of Brownian drivers increasing to ∞ as $\delta \to 0$. Hence to deduce a large deviation principle we apply [BDM08, Thm. 6], which is based on an infinite-dimensional version of the Boué–Dupuis formula.

First let us introduce some notation. We define for every $M < \infty$ the closed ball

$$S^M := \{h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2) : \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} \le M\}$$

equipped with the weak topology, which is metrizable (resp. compact) by the separability (resp. reflexivity) of $L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ (see [Bre11, Thm. 3.29 & Thm. 3.18]). We further denote

$$L^2_{\mathrm{pred}} := \Big\{\phi\colon \Omega\times[0,T] \to L^2(\mathbb{T}^2;\mathbb{R}^2) \text{ predictable} : \|\phi\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} < \infty \text{ a.s.}\Big\}$$

and

$$L^{2,M}_{\mathrm{pred}} := \{\phi \in L^2_{\mathrm{pred}} : \phi \in S^M \text{ a.s.}\}.$$

For every sequence $(u^{(\varepsilon)})_{\varepsilon>0}$ such that $u^{(\varepsilon)} \in L^{2,M}_{\mathrm{pred}}$ for each $\varepsilon > 0$ and $u^{(\varepsilon)} \rightharpoonup u$ in S^M in law as $\varepsilon \to 0$ (i.e. as S^M-valued random variables), define $u^{(\varepsilon)}_\delta := u^{(\varepsilon)} * \psi_\delta$, where ψ_δ is as in (1.2.7).

To apply [BDM08, Thm. 6] in our context, we need to show that for every $M > 0$, $(u^{(\varepsilon)})_{\varepsilon>0}$ and $\gamma \in [-1, 0)$:

1. The set $\Gamma_M \coloneqq \{\boldsymbol{\uparrow}^h : h \in S^M\} \subset C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ is compact in the strong topology induced by the norm $\|\cdot\|_{C_T\mathcal{H}^\gamma \cap L^2_T\mathcal{H}^{\gamma+1}}$.
2. It holds that $\varepsilon^{1/2}\boldsymbol{\uparrow}^{\delta(\varepsilon)} + \boldsymbol{\uparrow}^{u^{(\varepsilon)}_{\delta(\varepsilon)}} \to \boldsymbol{\uparrow}^u$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in law as $\varepsilon \to 0$.

Let us first show 1. For the existence and regularity of $\boldsymbol{\uparrow}^h = \nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}}h]$, see Appendix H. The weak time derivative of $\boldsymbol{\uparrow}^h$ is given by $\partial_t\boldsymbol{\uparrow}^h \coloneqq \Delta\boldsymbol{\uparrow}^h + \nabla\cdot(\sqrt{\rho_{\det}}h)$; hence Lemma H.0.2 implies for every $h \in S^M$ the bound

$$
\begin{aligned}
\|\boldsymbol{\uparrow}^h\|_{L^2_T\mathcal{H}^1} + \|\partial_t\boldsymbol{\uparrow}^h\|_{L^2_T\mathcal{H}^{-1}} &\lesssim \|\sqrt{\rho_{\det}}\|_{C([0,T]\times\mathbb{T}^2)}\|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} \\
&\leq \|\sqrt{\rho_{\det}}\|_{C([0,T]\times\mathbb{T}^2)}M.
\end{aligned} \tag{3.3.1}
$$

For every $R > 0$ define the set

$$
A_R \coloneqq \Big\{u \in L^2_T\mathcal{H}^1(\mathbb{T}^2) \cap W^{1,2}_T\mathcal{H}^{-1}(\mathbb{T}^2) : \|u\|_{L^2_T\mathcal{H}^1} + \|\partial_t u\|_{L^2_T\mathcal{H}^{-1}} \leq R\Big\},
$$

it then follows by (3.3.1) that $\Gamma_M \subset A_R$ for R sufficiently large depending on $\|\sqrt{\rho_{\det}}\|_{C([0,T]\times\mathbb{T}^2)}M$. An application of [Eva98, Sec. 5.9, Thm. 3] combined with the Aubin–Lions lemma yields that the set A_R is totally bounded in $C_T\mathcal{H}^\gamma(\mathbb{T}^2)\cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$, where we used the compact embeddings $L^2(\mathbb{T}^2) \hookrightarrow \mathcal{H}^\gamma(\mathbb{T}^2)$ and $\mathcal{H}^1(\mathbb{T}^2) \hookrightarrow \mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ (see Lemma B.2.5). Using that every subset of a totally bounded set is again totally bounded, it follows that Γ_M is itself totally bounded in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$. Hence to show that Γ_M is compact it suffices to show that it is closed.

Let $(\boldsymbol{\uparrow}^{h_n})_{n\in\mathbb{N}}$ be a sequence such that $\boldsymbol{\uparrow}^{h_n} \in \Gamma_M$ for every $n \in \mathbb{N}$ and $\boldsymbol{\uparrow}^{h_n} \to x \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ as $n \to \infty$. Using the compactness of S^M, we can extract a subsequence of $(h_n)_{n\in\mathbb{N}}$ (without re-labelling) such that $h_n \rightharpoonup h \in S^M$. The linear map $h \mapsto \boldsymbol{\uparrow}^h \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ is weak-strong continuous by Lemma H.0.2 and the compactness of the embedding $L^2_T\mathcal{H}^1(\mathbb{T}^2) \cap W^{1,2}_T\mathcal{H}^{-1}(\mathbb{T}^2) \subset C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$. Therefore $x = \boldsymbol{\uparrow}^h \in \Gamma_M$, which shows that Γ_M is closed hence compact in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$, proving 1.

Next we show 2. Let $M < \infty$ and $(u^{(\varepsilon)})_{\varepsilon>0}$ be a sequence such that $u^{(\varepsilon)} \in L^{2,M}_{\text{pred}}$ for every $\varepsilon > 0$ and $u^{(\varepsilon)} \rightharpoonup u$ in S^M in law as $\varepsilon \to 0$. An application of Skorokhod's representation theorem combined with classical continuity properties of the weak dual pairing [Bre11, Prop. 3.5] yields that $u^{(\varepsilon)}_{\delta(\varepsilon)} = u^{(\varepsilon)} * \psi_{\delta(\varepsilon)} \rightharpoonup u$ in S^M in law as $\varepsilon \to 0$. This combined with the weak-strong continuity of $S^M \ni h \to \upharpoonleft^h \in \Gamma_M$ and the continuous mapping theorem yields $\upharpoonleft^{u^{(\varepsilon)}_{\delta(\varepsilon)}} \to \upharpoonleft^u$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in law.

Proposition I.0.4 and the relative scaling $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ yield $\varepsilon^{1/2}\upharpoonleft^{\delta(\varepsilon)} \to 0$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in $L^p(\mathbb{P})$ as $\varepsilon \to 0$ for every $p \in [1, \infty]$, which implies convergence in probability by Markov's inequality. Therefore we can deduce by Slutsky's theorem that $\varepsilon^{1/2}\upharpoonleft^{\delta(\varepsilon)} + \upharpoonleft^{u^{(\varepsilon)}_{\delta(\varepsilon)}} \to \upharpoonleft^u$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ in law, which shows 2.

Having shown 1 and 2, we can apply [BDM08, Thm. 6] to deduce a large deviation principle for $(\varepsilon^{1/2}\upharpoonleft^{\delta(\varepsilon)})_{\varepsilon>0}$ in $C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ with good rate function $\mathscr{I}_\upharpoonleft$. □

We can now apply the contraction principle and Lemma 3.3.1 to deduce a large deviation principle for solutions to (3.1.2) in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\text{sol}}_T$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in (-1/2, 0)$

Theorem 3.3.5. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-\gamma-2} \to 0$ as $\varepsilon \to 0$ for some $\gamma \in (-1/2, 0)$. Then for all $T < T^*$, it follows that the sequence $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\text{sol}}_T$ with rate ε and good rate function*

$$\mathscr{I}(\rho) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\text{det}}}h]\Big\}.$$

Proof. The solution constructed in Lemma 3.3.1 is locally Lipschitz continuous in the noise $\varepsilon^{1/2}\upharpoonleft^{\delta(\varepsilon)} \in C_T\mathcal{H}^\gamma(\mathbb{T}^2) \cap L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$, hence we can apply Theorem 3.3.4

and the contraction principle [DZ10, Thm. 4.2.1] to deduce that $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $(\mathcal{H}^{\gamma}(\mathbb{T}^2))^{\mathrm{sol}}_T$ with rate ε and good rate function $\mathscr{I}$. $\square$

3.4 Rough Theory for Large Noise Intensities

In this section we first show the well-posedness of the solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) in the Hölder–Besov space $\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ of regularity $\alpha+1<-1$ (Theorem 3.4.2), which we then use to establish a law of large numbers (Theorem 3.4.4) and a large deviation principle (Theorem 3.4.8) under the relative scaling $\delta(\varepsilon)\to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1})\to 0$ as $\varepsilon\to 0$. Furthermore, we also establish a central limit theorem under the scaling $\delta(\varepsilon)\to 0$ and $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1})\to 0$ as $\varepsilon\to 0$ (Theorem 3.4.6).

Throughout this section we consider initial data $\rho_0\in\mathcal{H}^2(\mathbb{T}^2)$ such that $\rho_0>0$ and $\overline{\rho_0}=1$, which ensures that $\sqrt{\rho_{\mathrm{det}}}\in C_T\mathcal{H}^2(\mathbb{T}^2)$ (see Lemma 3.2.5).

3.4.1 Well-Posedness

In this subsection we use the arguments of Chapter 2 to show that (3.1.2) is well-posed in a space of Hölder–Besov distributions $\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ of regularity $\alpha+1<-1$ (Theorem 3.4.2). Our technique relies on the paracontrolled decomposition [GIP15] (cf. (3.4.2)), which, compared to the mild formulation (3.1.2), introduces more structure to achieve the well-posedness of (3.1.2) in a larger space.

The paracontrolled decomposition of (3.1.2) has been carried out in Subsection 2.1.1 for the equation with generic heterogeneity $\sigma\in C_T\mathcal{H}^2(\mathbb{T}^2)$ driven by space-time white noise (correlation length zero; suitably renormalised). Here, we recap the decomposition for positive correlation lengths $\delta=\delta(\varepsilon)>0$ without renormalisation and further specialize the heterogeneity to $\sqrt{\rho_{\mathrm{det}}}\in C_T\mathcal{H}^2(\mathbb{T}^2)$ (see Lemma 3.2.5).

We first apply the so-called Da Prato–Debussche trick and consider the remainder $u:=\rho+\varepsilon^{1/2}\text{¡}^{\delta}$, which then satisfies the equation

$$u=P\rho_0-\chi\nabla\cdot\mathcal{I}[u\nabla\Phi_u]+\chi\varepsilon^{1/2}\nabla\cdot\mathcal{I}[u\nabla\Phi_{\text{¡}^{\delta}}]+\chi\varepsilon^{1/2}\nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_u]-\chi\varepsilon\nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_{\text{¡}^{\delta}}].$$

The regularity of u is consequently determined by the second-order stochastic object $\mathbf{Y}^{\delta}_{\mathrm{can}}:=\nabla\cdot\mathcal{I}[\text{¡}^{\delta}\nabla\Phi_{\text{¡}^{\delta}}]$. While this improves the situation, this is not yet enough to make sense of the equation under the assumption $\mathbf{I}^{\delta} \in C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$, so we apply the Da Prato–Debussche trick again and define the remainder $w \coloneqq u + \chi\varepsilon \mathbf{Y}^{\delta}_{\mathrm{can}}$. This is still not enough and subsequent applications of the Da Prato–Debussche trick no longer yield any improvements.

To proceed further, we apply the paracontrolled Ansatz of [GIP15], which allows us to make sense of (3.1.2) in a space of distributions, if we can define a vector of stochastic objects $\mathbb{X}^{(\varepsilon)}_{\delta} = (\varepsilon^{1/2}\mathbf{I}^{\delta}, \varepsilon\mathbf{Y}^{\delta}_{\mathrm{can}}, \varepsilon^{3/2}\mathbf{\Psi}^{\delta}_{\mathrm{can}}, \varepsilon\mathbf{V}^{\delta})$ that satisfy the relations

$$
\begin{aligned}
&\mathbf{I}^{\delta} \coloneqq \nabla\cdot\mathcal{I}[\sqrt{\rho_{\det}}\boldsymbol{\xi}^{\delta}], \quad \mathbf{Y}^{\delta}_{\mathrm{can}} \coloneqq \nabla\cdot\mathcal{I}[\mathbf{I}^{\delta}\nabla\Phi_{\mathbf{I}^{\delta}}],\\
&\mathbf{\Psi}^{\delta}_{\mathrm{can}} \coloneqq \mathbf{Y}^{\delta}_{\mathrm{can}} \odot \nabla\Phi_{\mathbf{I}^{\delta}} + \nabla\Phi_{\mathbf{Y}^{\delta}_{\mathrm{can}}} \odot \mathbf{I}^{\delta}, \quad \mathbf{V}^{\delta} \coloneqq \nabla\mathcal{I}[\mathbf{I}^{\delta}] \odot \nabla\Phi_{\mathbf{I}^{\delta}} + \nabla^2\mathcal{I}[\Phi_{\mathbf{I}^{\delta}}] \odot \mathbf{I}^{\delta},
\end{aligned}
\tag{3.4.1}
$$

and are suitably regular. Recall that $\odot$ denotes the resonant product, see e.g. Section B.4 or [GIP15] for a definition. We call this vector the (canonical) *enhancement*. For the existence and regularity of $\mathbb{X}^{(\varepsilon)}_{\delta}$, see Theorem 3.4.1.

The fully decomposed equation and its solution take the following form (cf. Subsection 2.1.1): We call $\boldsymbol{w} = (w, w', w^{\#})$ a paracontrolled triple to (3.1.2), if

$$
\begin{cases}
\;\; w \coloneqq \chi\varepsilon^{1/2}\nabla\cdot\mathcal{I}[w' \otimes \mathbf{I}^{\delta}] + w^{\#}, \quad w' \coloneqq \nabla\Phi_w - \chi\varepsilon\nabla\Phi_{\mathbf{Y}^{\delta}_{\mathrm{can}}},\\
w^{\#} \coloneqq P\rho_0 + \chi\nabla\cdot\mathcal{I}[\Omega^{(\varepsilon)}_{\delta}(\boldsymbol{w})],
\end{cases}
\tag{3.4.2}
$$

with

$$
\begin{aligned}
\Omega^{(\varepsilon)}_{\delta}(\boldsymbol{w}) \coloneqq{}& -w\nabla\Phi_w + \chi\varepsilon(w\nabla\Phi_{\mathbf{Y}^{\delta}_{\mathrm{can}}}) + \chi\varepsilon(\mathbf{Y}^{\delta}_{\mathrm{can}}\nabla\Phi_w) - \chi^2\varepsilon^2(\mathbf{Y}^{\delta}_{\mathrm{can}}\nabla\Phi_{\mathbf{Y}^{\delta}_{\mathrm{can}}})\\
&- \chi\varepsilon^{3/2}\mathbf{\Psi}^{\delta}_{\mathrm{can}} - \chi\varepsilon^{3/2}(\nabla\Phi_{\mathbf{I}^{\delta}} \otimes \mathbf{Y}^{\delta}_{\mathrm{can}}) - \chi\varepsilon^{3/2}(\mathbf{Y}^{\delta}_{\mathrm{can}} \otimes \nabla\Phi_{\mathbf{I}^{\delta}})\\
&- \chi\varepsilon^{3/2}(\mathbf{I}^{\delta} \otimes \nabla\Phi_{\mathbf{Y}^{\delta}_{\mathrm{can}}}) + \varepsilon^{1/2}(w \otimes \nabla\Phi_{\mathbf{I}^{\delta}}) + \varepsilon^{1/2}(\nabla\Phi_{\mathbf{I}^{\delta}} \otimes w)\\
&+ \varepsilon^{1/2}(\mathbf{I}^{\delta} \otimes \nabla\Phi_w) + \mathscr{P}(\boldsymbol{w}, \mathbb{X}^{(\varepsilon)}_{\delta})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{P}(\boldsymbol{w}, \mathbb{X}^{(\varepsilon)}_{\delta}) \coloneqq{}& \varepsilon^{1/2}(w \odot \nabla\Phi_{\mathbf{I}^{\delta}}) + \varepsilon^{1/2}(\mathbf{I}^{\delta} \odot \nabla\Phi_w)\\
={}& \chi\varepsilon\mathscr{C}(w', \nabla\mathcal{I}[\mathbf{I}^{\delta}], \nabla\Phi_{\mathbf{I}^{\delta}}) + \chi\varepsilon\mathscr{C}(w', \nabla^2\mathcal{I}[\Phi_{\mathbf{I}^{\delta}}], \mathbf{I}^{\delta})\\
&+ \varepsilon^{1/2}\Big(w^{\#} + \chi\varepsilon^{1/2}\nabla\cdot\mathcal{I}[w' \otimes \mathbf{I}^{\delta}] - \chi\varepsilon^{1/2}(w' \otimes \nabla\mathcal{I}[\mathbf{I}^{\delta}])\Big) \odot \nabla\Phi_{\mathbf{I}^{\delta}}\\
&+ \varepsilon^{1/2}\Big(\nabla\Phi_{w^{\#}} + \chi\varepsilon^{1/2}\nabla\nabla\cdot\Phi_{\mathcal{I}[w' \otimes \mathbf{I}^{\delta}]} - \chi\varepsilon^{1/2}(w' \otimes \nabla^2\mathcal{I}[\Phi_{\mathbf{I}^{\delta}}])\Big) \odot \mathbf{I}^{\delta}\\
&+ \chi\varepsilon(w'\mathbf{V}^{\delta}),
\end{aligned}
$$

where $\mathscr{C}(f,g,h)$ denotes the commutator $(f \varolessthan g) \odot h - f(g \odot h)$ of [GIP15, Lem. 2.4], see Lemma B.7.3. We frequently denote $w_\delta^{(\varepsilon)} = w$ to point out the dependence of w on (ε, δ). We call $\rho_\delta^{(\varepsilon)} := -\varepsilon^{1/2}\mathbf{I}^\delta - \chi\varepsilon \mathbf{Y}^\delta_{\mathrm{can}} + w_\delta^{(\varepsilon)}$ the paracontrolled solution to (3.1.2). Since the correlation length is non-zero, it follows that $\rho_\delta^{(\varepsilon)}$ solves (3.1.2) in the classical sense.

The next theorem recaps the existence and regularity of the noise enhancement $\mathbb{X}_\delta^{(\varepsilon)}$ in $\mathcal{X}_T^{\alpha,\kappa}$ (cf. Definition 2.2.1) along with its divergence as $\delta \to 0$.

Theorem 3.4.1. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, T^* be its maximal time of existence (see Theorem 3.2.1), $T < T^*$, $\boldsymbol{\xi}$ be a two-dimensional vector of space-time white noises on $[0,T]\times\mathbb{T}^2$ (see Section C.1), $(\psi_\delta)_{\delta>0}$ be a sequence of mollifiers as in* (1.2.7)*, $\boldsymbol{\xi}^\delta := \psi_\delta * \boldsymbol{\xi} := (\psi_\delta * \xi^1, \psi_\delta * \xi^2)$ and $\mathbb{X}_\delta^{(\varepsilon)} = (\varepsilon^{1/2}\mathbf{I}^\delta, \varepsilon\mathbf{Y}^\delta_{\mathrm{can}}, \varepsilon^{3/2}\mathbf{\Psi}^\delta_{\mathrm{can}}, \varepsilon\mathbf{V}^\delta)$ be given by* (3.4.1) *for every $\varepsilon > 0$ and $\delta > 0$. Then for all $\alpha \in (-5/2, -2)$, $\kappa \in (0, 1/2)$ and $p \in [1, \infty)$, it holds that*

$$\mathbb{E}[\|\mathbf{I}^\delta\|^p_{\mathscr{L}_T^\kappa\mathcal{C}^{\alpha+1}}]^{1/p} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|_{C_TL^\infty}, \quad \mathbb{E}[\|\mathbf{Y}^\delta_{\mathrm{can}}\|^p_{\mathscr{L}_T^\kappa\mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim (1 \vee \log(\delta^{-1}))\|\sqrt{\rho_{\mathrm{det}}}\|^2_{C_T\mathcal{H}^2},$$
$$\mathbb{E}[\|\mathbf{\Psi}^\delta_{\mathrm{can}}\|^p_{\mathscr{L}_T^\kappa\mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim (1 \vee \log(\delta^{-1}))\|\sqrt{\rho_{\mathrm{det}}}\|^3_{C_T\mathcal{H}^2}, \; \mathbb{E}[\|\mathbf{V}^\delta\|^p_{\mathscr{L}_T^\kappa\mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|^2_{C_T\mathcal{H}^2},$$

and in particular $\mathbb{X}_\delta^{(\varepsilon)} \in \mathcal{X}_T^{\alpha,\kappa}$ almost surely.

Proof. The proof follows by Theorem 2.2.2, Lemma 2.2.7, Lemma 2.2.13, Lemma 2.2.18 with the particular choice of heterogeneity $\sigma = \sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{H}^2(\mathbb{T}^2)$ (Lemma 3.2.5). □

The next theorem establishes the well-posedness of paracontrolled solutions to (3.1.2) (cf. Definition 2.3.10).

Theorem 3.4.2. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Then for all $T < T^*$, $\varepsilon > 0$, $\delta > 0$ and $\alpha \in (-9/4, -2)$ there exists a unique (paracontrolled) solution $\rho_\delta^{(\varepsilon)}$ to* (3.1.2) *such that $\rho_\delta^{(\varepsilon)} \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ almost surely. Furthermore, the solution is locally Lipschitz continuous in the noise enhancement $\mathbb{X}_\delta^{(\varepsilon)}$.*

Proof. The proof follows by Theorem 3.4.1 and Lemma 2.3.16, where we used that the arguments of Chapter 2 are agnostic as to changes in $\chi \in \mathbb{R}$. For the local Lipschitz continuity in the noise enhancement, see Lemma 2.3.13. $\square$

Remark 3.4.3. Following Remark 2.2.3, it should be possible to consider more general initial data ρ_0 such that $(1-\Delta)\sqrt{\rho_{\mathrm{det}}} \in C_T L^1(\mathbb{T}^2)$ (modulo the introduction of a suitable weight at 0 to solve (3.1.2), cf. Lemma 2.3.16.) This would suggest taking initial data in the Bessel potential space $\mathcal{H}_1^2(\mathbb{T}^2)$ of regularity 2 and integrability 1. Unfortunately, $\mathcal{H}_1^2(\mathbb{T}^2)$ seems to be highly pathological (see [Ste70, Chpt. V, Ex. 6.6] for a discussion in the full space) and it is not even clear if one can establish $\rho_{\mathrm{det}} \in C_T\mathcal{H}_1^2(\mathbb{T}^2)$. A suitable, albeit suboptimal, replacement is given by $\rho_0 \in \mathcal{B}_{1,1}^2(\mathbb{T}^2)$, which allows us to solve $\rho_{\mathrm{det}} \in C_T\mathcal{B}_{1,1}^2(\mathbb{T}^2)$. A generalization of Lemma 3.2.5 would then imply $\sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{B}_{1,1}^2(\mathbb{T}^2)$, so that in particular $(1-\Delta)\sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{B}_{1,1}^0(\mathbb{T}^2) \hookrightarrow C_T L^1(\mathbb{T}^2)$. However, this extension doesn't seem sufficiently interesting to warrant an extensive discussion.

3.4.2 Law of Large Numbers

In this subsection we establish a (weak) law of large numbers for solutions $\rho_{\delta(\varepsilon)}^{(\varepsilon)}$ to (3.1.2) in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_T^{\mathrm{sol}}$ for every $\alpha \in (-9/4,-2)$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$ (Theorem 3.4.4). See [GIP15, Cor. 4.7] and [CC18, Cor. 3.13] for similar results in the literature.

Theorem 3.4.4. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4,-2)$, it follows that $\rho_{\delta(\varepsilon)}^{(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_T^{\mathrm{sol}}$ in probability as $\varepsilon \to 0$.*

Proof. It follows by the relative scaling $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$, Theorem 3.4.1 and Markov's inequality that $\mathbb{X}_{\delta(\varepsilon)}^{(\varepsilon)} \to 0$ in $\mathcal{X}_T^{\alpha,\kappa}$ in probability for every $\alpha \in (-5/2,-2)$ and $\kappa \in (0,1/2)$. Further for every $\alpha \in (-9/4,-2)$ we can apply Theorem 3.4.2 to construct the paracontrolled solution $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) such that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ almost surely. Using that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ is locally Lipschitz continuous in $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)}$ (Theorem 3.4.2), we can apply the continuous mapping theorem to deduce $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho_{\mathrm{det}}$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability. $\square$

3.4.3 Central Limit Theorem

Having established a law of large numbers (Theorem 3.4.4), we now turn to a central limit theorem (Theorem 3.4.6). For this we need to specialize our relative scaling of $(\varepsilon, \delta(\varepsilon))$ to $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$ since we lose one order of $\varepsilon^{1/2}$ on the scale of the leading-order fluctuations $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\mathrm{det}})$.

We first show the well-posedness of the generalized Ornstein–Uhlenbeck process, which describes the limiting dynamics of the leading-order fluctuations.

Lemma 3.4.5. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Then for all $T < T^*$ and $\alpha \in (-3,-2)$ there exists a unique solution $v \in C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ to*

$$v = -\chi\nabla\cdot\mathcal{I}[v\nabla\Phi_{\rho_{\mathrm{det}}}] - \chi\nabla\cdot\mathcal{I}[\rho_{\mathrm{det}}\nabla\Phi_v] - \text{¡}, \qquad \text{¡} := \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}]. \tag{3.4.3}$$

We refer to this v as the generalized Ornstein–Uhlenbeck process.

Proof. It suffices to construct a fixed point u to the linear mild equation

$$u = -\chi\nabla\cdot\mathcal{I}[u\nabla\Phi_{\rho_{\mathrm{det}}}] - \chi\nabla\cdot\mathcal{I}[\rho_{\mathrm{det}}\nabla\Phi_u] + \chi\nabla\cdot\mathcal{I}[\text{¡}\nabla\Phi_{\rho_{\mathrm{det}}}] + \chi\nabla\cdot\mathcal{I}[\rho_{\mathrm{det}}\nabla\Phi_{\text{¡}}], \tag{3.4.4}$$

from which we may recover a suitable solution v to (3.4.3) via the Da Prato–Debussche decomposition $v = -\text{¡} + u$. In particular, the space regularity of v is determined by the regularity of ¡ which is given by $C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ (cf. Lemma 2.2.7). It follows by the linearity of (3.4.4) that the lifetimes of v and u are determined by the lifetimes of their coefficients. The lower bound on α ensures that the products $\text{¡}\nabla\Phi_{\rho_{\mathrm{det}}}$ and $\rho_{\mathrm{det}}\nabla\Phi_{\text{¡}}$ appearing in (3.4.4) are well-posed by Bony's estimates [GIP15, Lem. 2.1]. $\square$

Next we show the central limit theorem for $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$.

Theorem 3.4.6. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\text{det}}) \to v$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\text{sol}}_T$ in probability as $\varepsilon \to 0$.*

Proof. We use the Da Prato–Debussche decompositions for $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ and v, given by $\rho^{(\varepsilon)}_{\delta(\varepsilon)} = -\varepsilon^{1/2}\mathbf{I}^{\delta(\varepsilon)} - \chi\varepsilon\mathbf{Y}^{\delta(\varepsilon)}_{\text{can}} + w^{(\varepsilon)}_{\delta(\varepsilon)}$ (see Subsection 3.4.1) and $v = -\mathbf{I} + u$ (see the proof of Lemma 3.4.5), to deduce

$$\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\text{det}}) - v = \varepsilon^{-1/2}(w^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\text{det}}) - u - \mathbf{I}^{\delta(\varepsilon)} + \mathbf{I} - \chi\varepsilon^{1/2}\mathbf{Y}^{\delta(\varepsilon)}_{\text{can}}.$$

The convergence $\varepsilon^{-1/2}(w^{(\varepsilon)}_{\delta(\varepsilon)} - \rho_{\text{det}}) \to u$ follows by the paracontrolled decomposition, in particular the local Lipschitz continuity of the paracontrolled solution in the enhancement (Theorem 3.4.2); the convergence $\mathbf{I}^{\delta(\varepsilon)} \to \mathbf{I}$ follows by Lemma 2.2.7; the convergence $\varepsilon^{1/2}\mathbf{Y}^{\delta(\varepsilon)}_{\text{can}} \to 0$ follows by Theorem 3.4.1 combined with the relative scaling $\varepsilon^{1/2}\log(\delta(\varepsilon)^{-1}) \to 0$. □

3.4.4 Large Deviation Principle

In this subsection we apply a generalization of Freidlin and Wentzell's theory [FW12] (see Appendix G as well as [HW15]) to establish a large deviation principle for solutions $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\text{sol}}_T$ for every $\alpha \in (-9/4, -2)$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$ (Theorem 3.4.8). As in Subsection 3.4.2 we first consider the noise enhancement $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ driving $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ (cf. Theorem 3.4.2) for which we establish a large deviation principle (Theorem 3.4.7) that we can then transfer to $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ via the contraction principle (Theorem 3.4.8).

Recall that $\mathcal{C}^{\alpha}_{\mathfrak{s}}([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ denotes the space of vector-valued space-time Hölder distributions of regularity α equipped with the parabolic scaling $\mathfrak{s} = (2,1,1)$ (see Section B.9).

Theorem 3.4.7. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$, $\alpha \in (-5/2, -2)$ and $\kappa \in (0, 1/2)$, it follows that the sequence $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $\mathcal{X}^{\alpha,\kappa}_T$ (cf. Definition 2.2.1) with rate ε and good rate function*

$$\mathscr{I}_{\mathrm{Enh}}(\boldsymbol{s}) \coloneqq \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \mathbb{X}^h = \boldsymbol{s}\Big\},$$

where $\mathbb{X}^h$ is the enhancement driven by the Cameron–Martin element $h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ defined as in (H.0.3) *(see also Lemma H.0.3.)*

Proof. It follows by the Wiener chaos decomposition (Subsection 2.2.2), Theorem C.2.1 and Lemma C.3.9 that every enhancement

$$\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} = (\varepsilon^{1/2}\mathbf{I}^{\delta(\varepsilon)}, \varepsilon\mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}}, \varepsilon^{3/2}\mathbf{\Psi}^{\delta(\varepsilon)}_{\mathrm{can}}, \varepsilon\mathbf{V}^{\delta(\varepsilon)})$$

is of the form (G.0.3) with abstract Wiener space

$$B \coloneqq \mathcal{C}^{\alpha}_{\mathfrak{s}}([0,T]\times\mathbb{T}^2;\mathbb{R}^2), \qquad \mu \coloneqq \mathrm{Law}(\boldsymbol{\xi}), \qquad \mathcal{H}_\mu \coloneqq L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2),$$

index set

$$\mathcal{W} \coloneqq \{\mathbf{I}, \mathbf{Y}, \mathbf{\Psi}, \mathbf{V}\}, \qquad (K_{\mathbf{I}}, K_{\mathbf{Y}}, K_{\mathbf{\Psi}}, K_{\mathbf{V}}) \coloneqq (1,2,3,2)$$

and target space

$$\begin{aligned}\mathbf{E} &\coloneqq \bigoplus_{\tau\in\mathcal{W}} E_\tau \\ &\coloneqq \mathscr{L}^\kappa_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}^\kappa_T\mathcal{C}^{3\alpha+6}(\mathbb{T}^2;\mathbb{R}^2) \times \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2;\mathbb{R}^{2\times2}),\end{aligned}$$

which is separable by an application of Lemma B.3.4 combined with the fact that finite Cartesian products of separable spaces are again separable.

Using the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$ combined with the bounds of Lemma 2.2.7 (for $\varepsilon^{1/2}\mathbf{I}^{\delta(\varepsilon)}$), Lemmas 2.2.9 & 2.2.19 (for $\varepsilon\mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}}$), Lemmas 2.2.12, 2.2.17 & 2.2.19 (for $\varepsilon^{3/2}\mathbf{\Psi}^{\delta(\varepsilon)}_{\mathrm{can}}$) and Lemmas 2.2.13 & 2.2.18 (for $\varepsilon\mathbf{V}^{\delta(\varepsilon)}$), one can show that $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)}$ converges as $\varepsilon \to 0$ in the sense of (G.0.3) to a non-trivial limit $\mathbb{X} = \bigoplus_{\tau\in\mathcal{W}} \mathbb{X}_{\tau,K_\tau}$ which has no contributions from lower-order Wiener chaoses, (G.0.5). In the notation of Subsection 2.2.2,

$$\mathbb{X} := (\mathbf{I}, \mathbf{Y}, \quad + \quad, \quad + \quad).$$

An application of Theorem G.0.4 then implies that $(\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $\mathbf{E}$ with rate ε and good rate function

$$\mathscr{I}_{\mathrm{Enh}}(\boldsymbol{s}) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \mathbb{X}_{\mathrm{hom}}(h) = \boldsymbol{s}\Big\},$$

where (cf. Definition G.0.3) the homogeneous part $\mathbb{X}_{\mathrm{hom}}(h)$ is given by

$$\mathbb{X}_{\mathrm{hom}}(h) = \bigoplus_{\tau\in\mathcal{W}} (\mathbb{X}_{\tau,K_\tau})_{\mathrm{hom}} = \bigoplus_{\tau\in\mathcal{W}} \int_B \mathbb{X}_{\tau,K_\tau}(\xi + h)\mu(\mathrm{d}\xi).$$

We can further apply Lemma C.3.10 to identify the homogeneous part $\mathbb{X}_{\mathrm{hom}}(h)$ with the enhancement $\mathbb{X}^h$ driven by the Cameron–Martin element h (cf. Appendix H), which yields the rate function of the claim.

The large deviation principle on $\mathcal{X}^{\alpha,\kappa}_T$ then follows by [DZ10, Lem. 4.1.5 (b)], using that $\mathcal{X}^{\alpha,\kappa}_T \subset \mathbf{E}$ is closed (cf. Definition 2.2.1) and that $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} \in \mathcal{X}^{\alpha,\kappa}_T$ almost surely for every $\varepsilon > 0$ (Theorem 3.4.1). □

We can now apply the contraction principle and Theorem 3.4.7 to deduce a large deviation principle for solutions to (3.1.2) in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ for every $\alpha \in (-9/4, -2)$ under the relative scaling $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$.

Theorem 3.4.8. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ with rate ε and good rate function*

$$\mathscr{I}(\rho) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h]\Big\}.$$

Proof. An application of Theorem 3.4.2 shows that $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ is locally Lipschitz continuous in $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)} \in \mathcal{X}^{\alpha,\kappa}_T$, which we can combine with the contraction principle [DZ10, Thm. 4.2.1] and Theorem 3.4.7 to deduce the claim. □

As the following proposition shows, it is possible to remove the relative scaling in Theorem 3.4.8, if the initial data ρ_0 is uniform, i.e. $\rho_0 \equiv 1$.

Proposition 3.4.9. *Let* $\rho_0 \equiv 1$ *and* ρ_{det} *be the weak solution to* (3.2.1) *with initial data* ρ_0 *and chemotactic sensitivity* $\chi \in \mathbb{R}$. *Assume* $\delta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Then for all* $T > 0$ *and* $\alpha \in (-9/4, -2)$, *it follows that* $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ *satisfies a large deviation principle in* $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ *with rate* ε *and good rate function* $\mathscr{I}$.

Proof. The deterministic Keller–Segel equation (3.2.1) preserves uniform initial data, that is, $\rho_0 \equiv 1$ yields $\rho_{\mathrm{det}} \equiv 1$ globally. In particular, $\sqrt{\rho_{\mathrm{det}}} \equiv 1$ which by Remark 2.2.21 implies that for every $\delta \geq 0$ and $p \in [1, \infty)$,

$$\mathbb{E}[\| \mathbf{Y}^{\delta}_{\mathrm{can}} \|^p_{\mathscr{L}^{\kappa}_T \mathcal{C}^{2\alpha+4}}]^{1/p} \lesssim 1, \qquad \mathbb{E}[\| \mathbf{\Psi}^{\delta}_{\mathrm{can}} \|^p_{\mathscr{L}^{\kappa}_T \mathcal{C}^{3\alpha+6}}]^{1/p} \lesssim 1. \tag{3.4.5}$$

The relative scaling of Theorem 3.4.7 (and, consequently, Theorem 3.4.8) was chosen such that $\mathbb{X}^{(\varepsilon)}_{\delta(\varepsilon)}$ converges as $\varepsilon \to 0$ in the sense of (G.0.4) to a limit that has no contributions from lower-order Wiener chaoses, (G.0.5). As demonstrated by (3.4.5), in the uniform case there is no such divergence as $\delta \to 0$, hence we can choose $\delta(\varepsilon) \to 0$ arbitrarily. □

Remark 3.4.10. In contrast to Proposition 3.4.9, which concerns the large deviations in the rough setting, one cannot improve the relative scaling in the regular setting (Theorem 3.3.5) by choosing uniform initial data.

4

The Renormalised Additive-Noise Approximation

Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to (3.2.1) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, T^* be its maximal time of existence (see Theorem 3.2.1), $T < T^*$, $\boldsymbol{\xi} = (\xi^1, \xi^2)$ be a vector-valued space-time white noise on $[0,T] \times \mathbb{T}^2$ (see Section C.1), $(\psi_\delta)_{\delta>0}$ be a family of symmetric mollifiers as in (1.2.7) and $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$.

The goal of this chapter is to show Theorem 1.1.6, which establishes that the renormalised, mollified additive-noise approximations

$$\begin{aligned} &\rho^\delta = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho^\delta\nabla\Phi_{\rho^\delta}] + \chi\,\Upsilon^\delta - \Psi^\delta, \\ &\Psi^\delta = \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}^\delta], \qquad \Upsilon^\delta = \mathbb{E}[\nabla\cdot\mathcal{I}[\Psi^\delta\nabla\Phi_{\Psi^\delta}]] \end{aligned} \tag{4.0.1}$$

converge in probability as $\delta \to 0$ to a non-trivial limit ρ. To write down the full statement and proof of Theorem 1.1.6, we need to combine Theorem 2.3.18 of Chapter 2 with Lemma 3.2.5 of Chapter 3. For the convenience of the reader, let us carry this out in detail.

We first define what we understand to be a renormalised, paracontrolled solution to

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho\nabla\Phi_\rho) - \nabla\cdot(\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}), & \text{in } (0,T]\times\mathbb{T}^2, \\ \quad -\Delta\Phi_\rho = \rho - \overline{\rho}, & \text{in } (0,T]\times\mathbb{T}^2, \\ \quad \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2. \end{cases} \tag{4.0.2}$$

The following is similar to Definition 2.3.10, modulo the introduction of the chemotactic sensitivity χ, several factors of -1 and the choice of of heterogeneity $\sigma := \sqrt{\rho_{\mathrm{det}}}$.

Definition 4.0.1. *Let $(\alpha, \infty, 1, \beta, \beta', \beta^{\#}, \beta^{\#}, \kappa, 0)$ be a tuple of exponents satisfying* (2.3.1)*, $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, T^* be its maximal time of existence (see Theorem 3.2.1), $T < T^*$ and $\mathbb{X} \in \mathcal{X}_T^{\alpha,\kappa}$ be the renormalised enhancement of Theorem 2.2.2 with heterogeneity $\sigma := \sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{H}^2(\mathbb{T}^2)$ (see Lemma 3.2.5). For every $S \le T$, we call $\rho\colon [0, S] \to \mathcal{S}'(\mathbb{T}^2)$ the renormalised, paracontrolled solution to* (4.0.2) *on $[0, S]$ with initial data ρ_0 and chemotactic sensitivity χ, if*

- *the triple $\boldsymbol{w} := (w, w', w^{\#})$ of distributions*

$$w := \rho + \mathsf{I} + \chi\,\mathsf{Y}, \quad w' := \nabla\Phi_w - \chi\nabla\Phi_{\mathsf{Y}}, \quad w^{\#} := w - \chi\nabla\cdot\mathcal{I}[w' \varolessthan \mathsf{I}]$$

  satisfies

$$\boldsymbol{w} \in \mathscr{L}_S^{\kappa}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R}) \times \mathscr{L}_S^{\kappa}\mathcal{C}^{\beta'}(\mathbb{T}^2;\mathbb{R}^2) \times (\mathscr{L}_S^{\kappa}\mathcal{C}^{\beta}(\mathbb{T}^2;\mathbb{R}) \cap \mathscr{L}_S^{\kappa}\mathcal{C}^{\beta^{\#}}(\mathbb{T}^2;\mathbb{R})),$$

$$\lim_{t\to 0} w_t = \lim_{t\to 0} w_t^{\#} = \rho_0 \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}) \quad \textit{and} \quad \lim_{t\to 0} w_t' = \nabla\Phi_{\rho_0} \in \mathcal{S}'(\mathbb{T}^2;\mathbb{R}^2);$$

- *the solution ρ satisfies $\rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \mathsf{I}$, where the product needs to be interpreted in the renormalised sense, that is*

$$-\chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] := -\chi\,\mathsf{Y} + \chi\nabla\cdot\mathcal{I}[w' \varolessthan \mathsf{I}] + \chi\nabla\cdot\mathcal{I}[\Omega_\chi^{\#}(\boldsymbol{w})]$$

  with $\Omega_\chi^{\#}(\boldsymbol{w})$ defined by

$$\begin{aligned}\Omega_\chi^{\#}(\boldsymbol{w}) := {}& -w\nabla\Phi_w + \chi(w\nabla\Phi_{\mathsf{Y}}) + \chi(\mathsf{Y}\nabla\Phi_w) - \chi^2(\mathsf{Y}\nabla\Phi_{\mathsf{Y}}) \\ & - \chi\,\mathsf{V} - \chi(\nabla\Phi_{\mathsf{I}} \varolessthan \mathsf{Y}) - \chi(\mathsf{Y} \varolessthan \nabla\Phi_{\mathsf{I}}) - \chi(\mathsf{I} \varolessthan \nabla\Phi_{\mathsf{Y}}) \\ & + (w \varolessthan \nabla\Phi_{\mathsf{I}}) + (\nabla\Phi_{\mathsf{I}} \varolessthan w) + (\mathsf{I} \varolessthan \nabla\Phi_w) + \mathscr{P}(\boldsymbol{w}, \mathbb{X})\end{aligned}$$

and $\mathscr{P}(\boldsymbol{w}, \mathbb{X})$ *by*

$$
\begin{aligned}
\mathscr{P}(\boldsymbol{w}, \mathbb{X}) &\coloneqq (w \odot \nabla \Phi_{\text{¡}}) + (\text{¡} \odot \nabla \Phi_{w}) \\
&= \chi \mathscr{C}(w', \nabla \mathcal{I}[\text{¡}], \nabla \Phi_{\text{¡}}) + \chi \mathscr{C}(w', \nabla^2 \mathcal{I}[\Phi_{\text{¡}}], \text{¡}) \\
&\quad + \Big(w^{\#} + \chi \nabla \cdot \mathcal{I}[w' \otimes \text{¡}] - \chi (w' \otimes \nabla \mathcal{I}[\text{¡}]) \Big) \odot \nabla \Phi_{\text{¡}} \\
&\quad + \Big(\nabla \Phi_{w^{\#}} + \chi \nabla \nabla \cdot \Phi_{\mathcal{I}[w' \otimes \text{¡}]} - \chi (w' \otimes \nabla^2 \mathcal{I}[\Phi_{\text{¡}}]) \Big) \odot \text{¡} \\
&\quad + \chi (w' \boldsymbol{\vee}).
\end{aligned}
$$

In particular it follows that $\lim_{t \to 0} \rho_t = \rho_0$ *in* $\mathcal{S}'(\mathbb{T}^2)$. *We call* $\rho \colon [0, S) \to \mathcal{S}'(\mathbb{T}^2)$ *a paracontrolled solution on* $[0, S)$, *if* ρ *is a paracontrolled solution on* $[0, S']$ *for every* $S' < S$.

We can now establish a rigorous version of Theorem 1.1.6.

Theorem 4.0.2. *Let* $(\alpha, \infty, 1, \beta, \beta', \beta^{\#}, \beta^{\#}, \kappa, 0)$ *be a tuple of exponents satisfying* (2.3.1), $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ *be such that* $\rho_0 > 0$ *and* $\overline{\rho_0} = 1$, ρ_{det} *be the weak solution to* (3.2.1) *with initial data* ρ_0 *and chemotactic sensitivity* $\chi \in \mathbb{R}$, T^* *be its maximal time of existence (see Theorem 3.2.1),* $T < T^*$, $\boldsymbol{\xi} = (\xi^1, \xi^2)$ *be a vector-valued space-time white noise on* $[0, T] \times \mathbb{T}^2$ *(see Section C.1),* $(\psi_\delta)_{\delta > 0}$ *be a family of symmetric mollifiers as in* (1.2.7), $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ *and* $\mathbb{X} \in \mathcal{X}_T^{\alpha, \kappa}$ *be the renormalised enhancement of Theorem 2.2.2 with heterogeneity* $\sigma \coloneqq \sqrt{\rho_{\mathrm{det}}} \in C_T \mathcal{H}^2(\mathbb{T}^2)$*(see Lemma 3.2.5).*

1. *There exists a renormalised, paracontrolled solution* ρ *in* $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_T^{\mathrm{sol}}$ *to* (4.0.2) *on* $[0, T^{\circledast}_{\mathcal{C}^{\alpha+1}}[\rho])$ *with initial data* ρ_0 *and chemotactic sensitivity* χ *in the sense of Definition 4.0.1.*

2. *For every* $\delta > 0$ *let* ρ^δ *be a solution to* (4.0.1). *It holds that* $\rho^\delta \to \rho$ *in* $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))_T^{\mathrm{sol}}$ *in probability as* $\delta \to 0$. *In particular for each* $\lambda > 0$ *one has that*

$$
\lim_{\delta \to 0} \mathbb{P} \Big(\sum_{L=1}^{\infty} \frac{2^{-L}}{L} \frac{\| \rho - \rho^\delta \|_{C_{T_L} \mathcal{C}^{\alpha+1}}}{1 + \| \rho - \rho^\delta \|_{C_{T_L} \mathcal{C}^{\alpha+1}}} > \lambda \Big) = 0,
$$

where for every $L \in \mathbb{N}$ *we denote*

$$
T_L \coloneqq T \wedge L \wedge \inf\{ t \in [0, T] : \| \rho(t) \|_{\mathcal{C}^{\alpha+1}} > L \text{ or } \| \rho^\delta(t) \|_{\mathcal{C}^{\alpha+1}} > L \}.
$$

Proof. Let $(\alpha, \infty, 1, \beta, \beta', \beta^\#, \beta^\#, \kappa, 0)$ be a tuple of exponents satisfying (2.3.1), and $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$. For every $T < T^*$ it then follows by Lemma 3.2.5 that $\sigma \coloneqq \sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{H}^2(\mathbb{T}^2)$. We further obtain by Besov's embedding (B.2.1) and $\beta^\# < 1$,

$$\rho_0 \in \mathcal{H}^2(\mathbb{T}^2) = \mathcal{B}^2_{2,2}(\mathbb{T}^2) \hookrightarrow \mathcal{B}^1_{\infty,2}(\mathbb{T}^2) \hookrightarrow \mathcal{B}^{\beta^\#}_{\infty,1}(\mathbb{T}^2).$$

The claim then follows by Theorem 2.3.18. □

5
Future Work

Contents

In this chapter we collect open questions around the additive-noise approximation (1.1.11) and possible avenues for future work. In Section 5.1 we discuss the quantitative behaviour of (1.1.11); in particular we give a conjecture on the probability that our approximation produces negative values (Conjecture 5.1.3). In Section 5.2 we propose a research idea on the metastability of (1.1.11). In Section 5.3 we consider the asymptotic behaviour of the renormalisation and possible generalisations of the relative scaling used in Section 3.4. In Section 5.4 we discuss approximations that go beyond (1.1.11).

5.1 Quantitative Estimates

In this section we explore how one may leverage the results of Chapter 3 to bound probabilities of certain behaviours of the additive-noise approximation $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$. In Subsection 5.1.1 we use the rough large deviation principle (Theorem 3.4.8) to show that the probability of blow-up in $\mathcal{C}^{-1-}(\mathbb{T}^2)$ is exponentially small in ε^{-1}. In Subsection 5.1.2 we conjecture how one can use similar techniques combined with the regular large deviation principle (Theorem 3.3.5) to control the event that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ produces negative values.

5.1.1 Controlling Blow-Ups

In this subsection we apply the rough large deviation principle (Theorem 3.4.8) to control the probability that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ blows up before time $T < T^*$ (Theorem 5.1.2). In particular, we show that the probability of a blow-up is asymptotically exponentially small in ε^{-1}, a result that was inspired by [HW15, Rem. 4.6].

We combine Lemma B.8.6 and Theorem 3.4.8 to deduce a provisional large deviations upper bound for $(T^{\skull}_{\mathcal{C}^{-1-}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}])_{\varepsilon>0}$, which will then be applied to prove the exponential unlikelihood of blow-up in Theorem 5.1.2 below.

Lemma 5.1.1. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4, -2)$, it follows that $(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}])_{\varepsilon>0}$ satisfies for every $S \in [0, T]$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}] \leq S\Big) \leq -\inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h] \leq S\Big\},$$

where ρ^h is the solution to the mild equation

$$\rho^h := P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho^h\nabla\Phi_{\rho^h}] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h].$$

Proof. It follows by the lower semicontinuity of $T^{\skull}_{\mathcal{C}^{\alpha+1}}$ (Lemma B.8.6) that the sublevel sets $\{\rho \in (\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T : T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho] \leq S\}$ are closed in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$, hence we can deduce from the large deviation principle of $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ (Theorem 3.4.8) the upper bound

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}] \le S\Big) \le - \inf_{\substack{\rho\in(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T \\ T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho]\le S}} \mathscr{I}(\rho).$$

We can simplify the right-hand side

$$\begin{aligned}
&\inf_{\substack{\rho\in(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T \\ T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho]\le S}} \mathscr{I}(\rho) \\
&= \inf_{\substack{\rho\in(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T \\ T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho]\le S}} \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h]\Big\} \\
&= \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h] \le S, \\
&\qquad\qquad \rho^h = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho^h\nabla\Phi_{\rho^h}] - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h]\Big\},
\end{aligned}$$

which yields the claim. $\square$

We can now use Lemma 5.1.1 to deduce that solutions $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ to (3.1.2) blow up before time $T < T^*$ with exponentially small probability in ε^{-1}.

Theorem 5.1.2. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi\in\mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon)\to 0$ and $\varepsilon\log(\delta(\varepsilon)^{-1}) \to 0$ as $\varepsilon\to 0$. Then for all $S \le T < T^*$ and $\alpha\in(-9/4,-2)$, there exists a constant $c>0$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}] \le S\Big) \le -c.$$

In particular for every $\vartheta\in(0,c)$ there exists some $\varepsilon_0(\vartheta) > 0$ such that for all $\varepsilon < \varepsilon_0(\vartheta)$,

$$\mathbb{P}\Big(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}] \le S\Big) \le \mathrm{e}^{(-c+\vartheta)/\varepsilon}.$$

Proof. Assume we are given constant $c > 0$ with the following property: All $h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ such that $T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h] \le S$ satisfy $\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} \ge 2c$. We can then apply Lemma 5.1.1 to deduce

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big(T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^{(\varepsilon)}_{\delta(\varepsilon)}] \le S\Big) \le -\inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h] \le S\Big\} \le -c,$$

which yields the claim. Hence it suffices to find such a c.

Using the lower semicontinuity of $T^{\skull}_{\mathcal{C}^{\alpha+1}}$ (Lemma B.8.6), it follows that for all $S \leq T$ there exists a constant $\nu > 0$ such that $D_T^{\mathcal{C}^{\alpha+1}}(\rho, \rho_{\mathrm{det}}) < \nu$ implies $S < T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho]$. Using the local Lipschitz continuity of the solution map in the noise enhancement (Theorem 3.4.2), we deduce that for every $\nu > 0$ there exists an $R > 0$ such that $\|\mathbb{X}^h\|_{\mathcal{X}_T^{\alpha,\kappa}} < R$ implies $D_T^{\mathcal{C}^{\alpha+1}}(\rho^h, \rho_{\mathrm{det}}) < \nu$, where $\mathbb{X}^h$ denotes the enhancement driven by the Cameron–Martin element h (cf. Lemma H.0.3). Hence, combining both we deduce that for all $S \leq T$ there exists an $R > 0$ such that $\|\mathbb{X}^h\|_{\mathcal{X}_T^{\alpha,\kappa}} < R$ implies $S < T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h]$.

By an application of Lemma H.0.3, we can find for every $R > 0$ a constant $c > 0$ such that $\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} < 2c$ implies $\|\mathbb{X}^h\|_{\mathcal{X}_T^{\alpha,\kappa}} < R$. Consequently, for all $S \leq T$ we can find a constant $c > 0$ such that $\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} < 2c$ implies $S < T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho^h]$, which yields the claim. $\square$

5.1.2 Controlling Negativity

Our SPDE (1.1.11) may blow up before time T, even if the deterministic Keller–Segel equation (1.1.12) is well-posed until $T^* > T$. In fact, this possibility of blow-up is due to the additive noise, which causes the sign of the initial data ρ_0 in (1.1.11) to no longer be preserved, which is especially problematic since our SPDE is supposed to approximate the continuum fluctuations of the mean-field particle system (1.1.16). In upcoming work we intend to control the probability of the event that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ becomes negative before it blows up in $L^2(\mathbb{T}^2)$. Here we use the $L^2(\mathbb{T}^2)$-theory of Section 3.3 (the regular setting) to ensure that the negative part of the solution is well-defined (see also Lemma 3.3.1).

For every $\rho \in (L^2(\mathbb{T}^2))^{\mathrm{sol}}_T$ define $T^-_\vartheta[\rho]$ to be the first time when the $L^2(\mathbb{T}^2)$-norm of the negative part ρ^- is greater than or equal to a tolerance $\vartheta > 0$, i.e.

$$T^-_\vartheta[\rho] := T^{\skull}_{L^2}[\rho] \wedge \inf\Big\{t \in [0, T^{\skull}_{L^2}[\rho]] : \|\rho^-_t\|_{L^2} \geq \vartheta\Big\}. \tag{5.1.1}$$

We conjecture that the event $T^-_\vartheta[\rho^{(\varepsilon)}_\delta] < T$ has exponentially small probability in ε^{-1}:

Conjecture 5.1.3. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon^{1/2}\delta(\varepsilon)^{-2} \to 0$ as $\varepsilon \to 0$. Then for all $S \le T < T^*$ and $\vartheta > 0$, there exists a constant $c > 0$ such that*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\Big(T_\vartheta^-[\rho_{\delta(\varepsilon)}^{(\varepsilon)}] \le S\Big) \le -c. \tag{5.1.2}$$

Idea of Proof. The idea is to show that T_ϑ^- is lower semicontinuous on (a suitable subspace of) $(L^2(\mathbb{T}^2))_T^{\mathrm{sol}}$, which we can then combine with the regular large deviation principle (Theorem 3.3.5) to deduce (5.1.2). See also Theorem 5.1.2, which rigorously establishes an analogous result for the blow-up time $T^{\skull}_{\mathcal{C}^{\alpha+1}}[\rho]$. □

A key issue in proving Conjecture 5.1.3 is the fact that the regular large deviation principle (Theorem 3.3.5), in contrast to the regular law of large numbers (Theorem 3.3.3), does not actually allow us to estimate subsets of $(L^2(\mathbb{T}^2))_T^{\mathrm{sol}}$, which is a consequence of the regularity that we had to trade in order to obtain the compactness required in the proof of Theorem 3.3.4.

To extend Conjecture 5.1.3 to the case of zero tolerance, we would have to consider the time

$$T_0^-[\rho] \coloneqq T^{\skull}_{L^2}[\rho] \wedge \inf\Big\{t \in [0, T^{\skull}_{L^2}[\rho]] : \|\rho_t^-\|_{L^2} > 0\Big\}, \tag{5.1.3}$$

since (5.1.1) is trivial for $\vartheta = 0$. While the proof of Lemma B.8.6 suggests that T_ϑ^- should be lower semicontinuous for every $\vartheta > 0$ (on a suitable subspace of $(L^2(\mathbb{T}^2))_T^{\mathrm{sol}}$), the same is not clear for (5.1.3).

5.2 Metastability

The Keller–Segel–Dean–Kawasaki equation (1.1.10) allows us to understand the influence of stochastic fluctuations on the process of aggregation: The advection in (1.1.10) coincides with the non-linearity of the Keller–Segel equation (1.1.12), is well understood and shown to form aggregates; while the Dean–Kawasaki noise provides a (non-linear) stochastic perturbation of these dynamics.

To demonstrate the effect of the noise, consider the case of uniform initial data $\rho_0 \equiv 1$ on a torus. The deterministic, periodic Keller–Segel equation (1.1.12) leaves ρ_0 invariant and no aggregation occurs. In contrast, the finite-number fluctuations of (1.1.10) lead to the formation of an aggregate for sufficiently large χ, for a simulation see Figure 5.1.

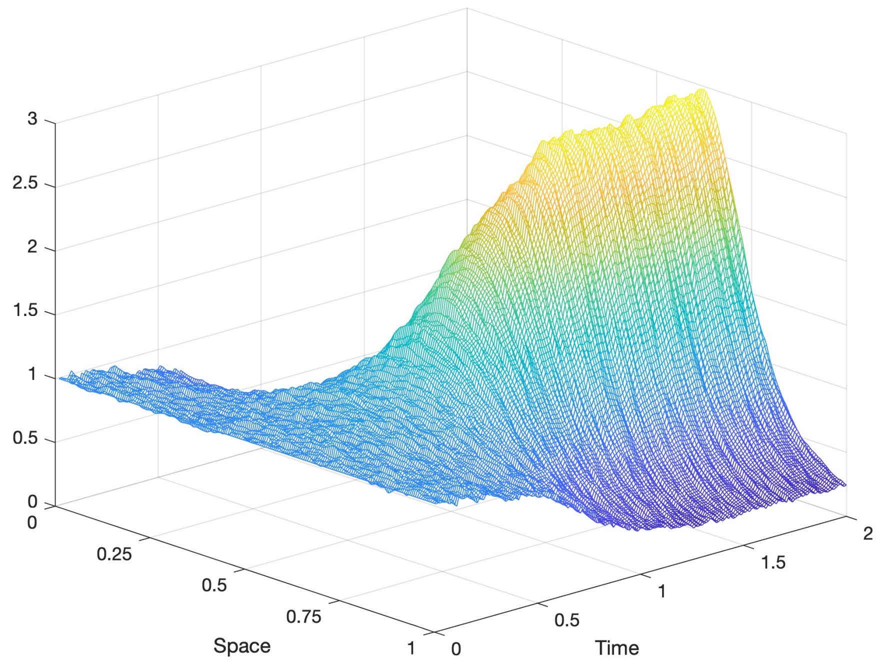


Figure 5.1: Finite-difference simulation of (1.1.10) in $d = 1$ with $\chi = 45$, $N = 10^4$ and initial data $\rho_0 \equiv 1$.

This is akin to a formal and numerical analysis carried out by Chavanis and Delfini [CD14] who showed that metastable transitions are possible in (1.1.10). On the other hand, as was shown by Konarovskyi, Lehmann and von Renesse [KLvR19; KLvR20], the corresponding (1.1.9) with smooth interactions on the full space has no solution beyond the particle system (1.1.5); hence we do not expect (1.1.10) to be well-posed either.[1]

[1]One might argue that this was addressed in [CD14] through coarse graining.

Instead, we propose to study the metastability of the now well-understood additive-noise approximation (1.1.11) to (1.1.10). An important ingredient of such an analysis would be a large deviation principle (LDP) that is uniform in the initial data. Our proof of the rough LDP (Theorem 3.4.8) is based on the techniques of [HW15], which were recently extended by Klose and Mayorcas [KM24] to yield locally uniform LDPs. Alternatively, our proof of the regular LDP (Theorem 3.3.5) is based on [BDM08], which even yields uniform LDPs.

5.3 Asymptotic Behaviour of the Renormalisation

Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to (1.1.12) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, T^* be its maximal time of existence (see Theorem 3.2.1), $T < T^*$ and choose the heterogeneity $\sigma \coloneqq \sqrt{\rho_{\mathrm{det}}} \in C_T\mathcal{H}^2(\mathbb{T}^2)$ (see Lemma 3.2.5). In Lemma 2.2.19 we showed that the renormalisation $(\overset{\bullet}{\mathbf{|}}^{\delta})_{\delta>0}$ satisfies the bound

$$\|\overset{\bullet}{\mathbf{|}}^{\delta}\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}} \lesssim (1 \vee \log(\delta^{-1}))\|\varphi\|_{L^\infty}^2 \|\sqrt{\rho_{\mathrm{det}}}\|_{C_T\mathcal{H}^2}^2 \tag{5.3.1}$$

for every $\alpha < -2$, $\kappa \in (0,1)$ and Fourier cut-off φ as in (1.2.6); which directly informed the relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ employed throughout Section 3.4. However, what happens if instead we assume $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2$ for some $\lambda > 0$?

In the context of the Φ^4_3-equation, Hairer and Weber [HW15, Thm. 4.7] showed that a large deviation principle still holds and that its skeleton equation carries an additional term depending on λ^2. To establish an analogous result in the case of (1.1.11), we would have to find the deterministic limit $\Upsilon_\lambda \coloneqq \lim_{\varepsilon\to 0} \varepsilon \overset{\bullet}{\mathbf{|}}^{\delta(\varepsilon)} \in \mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$ under the relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2$, which would imply for all $\kappa \in (0,1/2)$ and $p \in [1,\infty)$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{E}[\|\varepsilon \mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}} - \Upsilon_\lambda\|^p_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}}]^{1/p} \leq{}& \lim_{\varepsilon\to 0} \mathbb{E}[\|\varepsilon \mathbf{Y}^{\delta(\varepsilon)}\|^p_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}}]^{1/p} \\ &+ \lim_{\varepsilon\to 0} \|\varepsilon \overset{\bullet}{\mathbf{|}}^{\delta(\varepsilon)} - \Upsilon_\lambda\|_{\mathscr{L}_T^{\kappa}\mathcal{C}^{2\alpha+4}} = 0,\end{aligned}$$

where we used Lemma 2.2.9 and $\kappa \in (0, 1/2)$ to make sure that the first summand vanishes as $\varepsilon \to 0$. Even though neither shown nor used, the same also extends to $\kappa \in (0, 1)$, see Remark 2.2.5.

In the case $\lambda = 0$, it follows by $\varepsilon \log(\delta(\varepsilon)^{-1}) \to 0$ and (5.3.1) that $\Upsilon_0 = 0$; however it seems non-trivial to establish the limit $\Upsilon_\lambda := \lim_{\varepsilon \to 0} \varepsilon \Psi^{\delta(\varepsilon)}$ for general $\lambda > 0$. In particular, it is not even true that $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2$ is the ideal relative scaling for every initial data ρ_0. For example upon choosing $\rho_0 \equiv 1$, it follows that $\rho_{\mathrm{det}} \equiv 1$ and hence $\Psi^\delta = 0$ for every $\delta > 0$ (see Remark 2.2.21). Consequently no scaling assumption is required to deduce $0 = \lim_{\varepsilon \to 0} \varepsilon \Psi^{\delta(\varepsilon)}$ in the case of uniform initial data. See also Proposition 3.4.9 for the associated large deviation principle.

In the remainder of this section let us conjecture analogues of Theorems 1.1.3–1.1.5 under the relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2$. We first state an analogue of the law of large numbers.

Conjecture 5.3.1. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1 \rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2 \in [0, \infty)$ as $\varepsilon \to 0$. Then for all $T < T^*$, $\alpha \in (-9/4, -2)$ and $\kappa \in (0, 1/2)$, it follows that $\Upsilon_\lambda := \lim_{\varepsilon \to 0} \varepsilon \Psi^{\delta(\varepsilon)} \in \mathscr{L}_T^\kappa \mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$ deterministically and $\rho^{(\varepsilon)}_{\delta(\varepsilon)} \to \rho^\lambda_{\mathrm{det}}$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$, where $\rho^\lambda_{\mathrm{det}}$ solves*

$$\rho^\lambda_{\mathrm{det}} = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho^\lambda_{\mathrm{det}} \nabla \Phi_{\rho^\lambda_{\mathrm{det}}}] - \chi \Upsilon_\lambda.$$

Idea of Proof. Assuming that $\Upsilon_\lambda := \lim_{\varepsilon \to 0} \varepsilon \Psi^{\delta(\varepsilon)} \in \mathscr{L}_T^\kappa \mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$ exists, the conjecture follows by the paracontrolled decomposition (3.4.2) as in Theorem 3.4.4. □

Next we conjecture a large deviation principle under the relative scaling $\varepsilon \log(\delta(\varepsilon)^{-1}) \to \lambda^2$.

Conjecture 5.3.2. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle \rho_0, 1 \rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Theorem 3.2.1). Assume $\delta(\varepsilon) \to 0$ and*

$\varepsilon\log(\delta(\varepsilon)^{-1}) \to \lambda^2 \in [0,\infty)$ as $\varepsilon \to 0$. Then for all $T < T^*$ and $\alpha \in (-9/4,-2)$, it follows that $(\rho^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon>0}$ satisfies a large deviation principle in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ with rate ε and good rate function

$$\mathscr{I}_\lambda(\rho) := \inf\Big\{\frac{1}{2}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi\nabla\cdot\mathcal{I}[\rho\nabla\Phi_\rho] - \chi\Upsilon_\lambda - \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}h]\Big\}.$$

Idea of Proof. Assuming that $\Upsilon_\lambda := \lim_{\varepsilon\to 0}\varepsilon\,\mathring{\mathbf{I}}^{\delta(\varepsilon)} \in \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$ exists and we can generalize assumption (G.0.5) of Theorem G.0.4, the result should follow as in Theorem 3.4.8. □

Since $\Upsilon_0 = 0$, it is clear that Conjectures 5.3.1 and 5.3.2 specialize to Theorems 3.4.4 and 3.4.8 in the case $\lambda = 0$.

We further conjecture a central limit theorem under an additional assumption on the speed of convergence between $\varepsilon\,\mathring{\mathbf{I}}^{\delta(\varepsilon)}$ and Υ_λ as $\varepsilon \to 0$.

Conjecture 5.3.3. *Let $\rho_0 \in \mathcal{H}^2(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\langle\rho_0, 1\rangle_{L^2} = 1$, ρ_{det} be the weak solution to* (1.1.12) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, T^* be its maximal time of existence (see Theorem 3.2.1), $T < T^*$, $\alpha \in (-9/4,-2)$ and $\kappa \in (0,1/2)$. Assume $\delta(\varepsilon) \to 0$, $\varepsilon\log(\delta(\varepsilon)^{-1}) \to \lambda^2 \in [0,\infty)$ and*

$$\varepsilon^{-1/2}(\varepsilon\,\mathring{\mathbf{I}}^{\delta(\varepsilon)} - \Upsilon_\lambda) \to 0 \quad \textit{in} \quad \mathscr{L}^\kappa_T\mathcal{C}^{2\alpha+4}(\mathbb{T}^2) \quad \textit{as} \quad \varepsilon \to 0. \tag{5.3.2}$$

It then follows that $\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho^\lambda_{\mathrm{det}}) \to v_\lambda$ in $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$ in probability as $\varepsilon \to 0$, where $v_\lambda \in C_T\mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ is the unique solution to

$$v_\lambda = -\chi\nabla\cdot\mathcal{I}[v_\lambda\nabla\Phi_{\rho^\lambda_{\mathrm{det}}}] - \chi\nabla\cdot\mathcal{I}[\rho^\lambda_{\mathrm{det}}\nabla\Phi_{v_\lambda}] - \dot{\mathbf{I}}, \qquad \dot{\mathbf{I}} = \nabla\cdot\mathcal{I}[\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}].$$

Idea of Proof. The proof should follow by the paracontrolled decomposition (3.4.2) and an argument as in Theorem 3.4.6. To motivate the assumption (5.3.2), let us consider the Da Prato–Debussche decompositions $w^{(\varepsilon)}_{\delta(\varepsilon)} := \rho^{(\varepsilon)}_{\delta(\varepsilon)} + \varepsilon^{1/2}\dot{\mathbf{I}}^{\delta(\varepsilon)} + \chi\varepsilon\mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}}$ (cf. Subsection 3.4.1) and $w^\lambda_{\mathrm{det}} := \rho^\lambda_{\mathrm{det}} + \chi\Upsilon_\lambda$, which we can use to decompose the leading-order fluctuations into

$$\varepsilon^{-1/2}(\rho^{(\varepsilon)}_{\delta(\varepsilon)} - \rho^\lambda_{\mathrm{det}}) = -\dot{\mathbf{I}}^{\delta(\varepsilon)} - \chi\varepsilon^{1/2}\mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}} + \chi\varepsilon^{-1/2}\Upsilon_\lambda + \varepsilon^{-1/2}(w^{(\varepsilon)}_{\delta(\varepsilon)} - w^\lambda_{\mathrm{det}}).$$

It follows by Lemma 2.2.7 that $\mathbf{I}^{\delta(\varepsilon)} \to \mathbf{I}$ in $\mathscr{L}_T^\kappa \mathcal{C}^{\alpha+1}(\mathbb{T}^2)$ in probability as $\varepsilon \to 0$; and by Lemma 2.2.9 combined with (5.3.2) that

$$-\chi \varepsilon^{1/2} \mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}} + \chi \varepsilon^{-1/2} \Upsilon_\lambda = -\chi \varepsilon^{-1/2}(\varepsilon \mathbf{Y}^{\delta(\varepsilon)}_{\mathrm{can}} - \Upsilon_\lambda) \to 0 \quad \text{in} \quad \mathscr{L}_T^\kappa \mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$$

in probability as $\varepsilon \to 0$. It then suffices to apply the paracontrolled Ansatz for $w^{(\varepsilon)}_{\delta(\varepsilon)}$ (cf. (3.4.2)) to show

$$\varepsilon^{-1/2}(w^{(\varepsilon)}_{\delta(\varepsilon)} - w^\lambda_{\mathrm{det}}) \to -\chi \nabla \cdot \mathcal{I}[v_\lambda \nabla \Phi_{\rho^\lambda_{\mathrm{det}}}] - \chi \nabla \cdot \mathcal{I}[\rho^\lambda_{\mathrm{det}} \nabla \Phi_{v_\lambda}],$$

which would yield the claim. $\square$

Assume the relative scaling $\varepsilon^{1/2} \log(\delta(\varepsilon)^{-1}) \to 0$. We can then apply (5.3.1) to deduce that assumption (5.3.2) is satisfied with $\Upsilon_0 = 0$ and, in particular, that Conjecture 5.3.3 specializes to Theorem 3.4.6.

Of course, once Conjectures 5.3.1 and 5.3.3 have been shown, one can ask if a central limit theorem continues to hold if assumption (5.3.2) is replaced by the mere existence of a limit $\lim_{\varepsilon \to 0} \varepsilon^{-1/2}(\varepsilon \mathbf{\Psi}^{\delta(\varepsilon)} - \Upsilon_\lambda) \in \mathscr{L}_T^\kappa \mathcal{C}^{2\alpha+4}(\mathbb{T}^2)$. Needless to say, there are many opportunities for future research.

5.4 Improved Approximations

We expect our additive-noise approximation $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ (cf. (1.1.11)) to be precise up to first order (see the discussion of Theorems 1.1.3 & 1.1.4), but it does not satisfy the large deviation principle one would conjecture for the particle system $(\bar{X}^i)_{i=1}^N$ (cf. (1.1.16)). Indeed, under an appropriate relative scaling on $(\varepsilon, \delta(\varepsilon))$, it follows by Theorem 1.1.2 (resp. Theorem 1.1.5), that $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ satisfies a large deviation principle in $(\mathcal{H}^\gamma(\mathbb{T}^2))^{\mathrm{sol}}_T$ (resp. $(\mathcal{C}^{\alpha+1}(\mathbb{T}^2))^{\mathrm{sol}}_T$) with good rate function

$$\mathscr{I}(\rho) = \inf\Big\{ \frac{1}{2} \|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho \nabla \Phi_\rho] - \nabla \cdot \mathcal{I}[\sqrt{\rho_{\mathrm{det}}} h] \Big\},$$

whereas the particle system $(\bar{X}^i)_{i=1}^N$ is conjectured to satisfy a large deviation principle in $C_T \mathcal{P}(\mathbb{T}^2)$ with good rate function

$$\bar{\mathscr{I}}(\rho) = \inf\Big\{ \frac{1}{2} \|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} : \rho = P\rho_0 - \chi \nabla \cdot \mathcal{I}[\rho \nabla \Phi_\rho] - \nabla \cdot \mathcal{I}[\sqrt{\rho} h] \Big\},$$

see (1.1.19) for a discussion. Modulo the difference in state space, the main approximation error between $\rho^{(\varepsilon)}_{\delta(\varepsilon)}$ and $(\bar{X}^i)_{i=1}^N$ lies in the functional $\mathscr{I}$, which differs from $\bar{\mathscr{A}}$ by its additive-in-h skeleton equation.

Is it possible to find an approximation that is more precise? One possibility would be to generalize the methods introduced in [FG23; DFG20; WWZ22; WZ22] to interactions given by (1.1.17); though this seems challenging. Another avenue would be to continue the expansion (1.1.13), possibly along the lines of [GWZ24].

Finally, is it possible to establish weak-error estimates between such an approximation and the particle system (1.1.16)? Similar estimates were obtained in [Cor+23; DKP24] but are still open in the case of interactions governed by (1.1.17).

Appendices

A
Formal Intuitions

Contents

A.1 Formal Derivation of the Keller–Segel–Dean–Kawasaki Equation

We present two formal derivations of the Keller–Segel–Dean–Kawasaki equation (1.1.8). The first one (Subsection A.1.1) follows Dean's original argument [Dea96] as presented in [BGN16, Sec. 4.2] and derives (1.1.8) from the particle system (1.1.6). The second one (Subsection A.1.2) interprets (1.1.8) as a fluctuating gradient flow in Wasserstein space. Let us point out the results of [KLvR19; KLvR20], which imply that the following is purely formal (see Subsection 1.1.1 for a discussion.)

A.1.1 Dean's Argument

Let $(X^i)_{i=1}^N$ be given by (1.1.6) and consider the empirical density $\mu_t^N(x) = \frac{1}{N}\sum_{i=1}^N \delta(x - X_t^i)$. By a formal application of Itô's formula, we derive the dynamics

$$\mathrm{d}\mu_t^N(x) = \Delta\mu_t^N(x)\,\mathrm{d}t - \chi\nabla\cdot(\mu_t^N(\nabla K * \mu_t^N))(x)\,\mathrm{d}t - \frac{\sqrt{2}}{N}\nabla\cdot\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i, \quad \text{(A.1.1)}$$

where ∇K satisfies (1.1.2) and $(B^i)_{i=1}^N$ is a family of i.i.d. d-dimensional Brownian motions $B^i = (B^{i,1}, \dots, B^{i,d})$. The noise $\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i$ has the covariance

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \sum_{i=1}^N \delta(x - X_r^i)\,\mathrm{d}B_r^{i,k}\Big)\Big(\int_0^s \sum_{i=1}^N \delta(y - X_r^i)\,\mathrm{d}B_r^{i,l}\Big)\Big] \\
&= \delta_{k,l}\int_0^{t\wedge s} \sum_{i=1}^N \delta(x - X_r^i)\delta(y - X_r^i)\,\mathrm{d}r = \delta_{k,l}N\int_0^{t\wedge s} \delta(x-y)\rho_r(x)\,\mathrm{d}r
\end{aligned}$$

for every $k, l \in \{1, \dots, d\}$. On the other hand, let $\boldsymbol{\xi} = (\xi^1, \dots, \xi^d)$ be a vector-valued space-time white noise, i.e. a random, Gaussian, centred distribution such that formally $\mathbb{E}[\xi^k(t,x)\xi^l(s,y)] = \delta(t-s)\delta(x-y)\delta_{k,l}$ for every $s, t \geq 0$, $x, y \in \mathbb{R}^d$ and $k, l \in \{1, \dots, d\}$. It follows that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \sqrt{\mu_r^N(x)}\xi^k(r,x)\,\mathrm{d}r\Big)\Big(\int_0^s \sqrt{\mu_r^N(y)}\xi^l(r,y)\,\mathrm{d}r\Big)\Big] \\
&= \int_0^t\int_0^s \sqrt{\mu_r^N(x)}\sqrt{\mu_{r'}^N(y)}\mathbb{E}[\xi^k(r,x)\xi^l(r',y)]\,\mathrm{d}r\,\mathrm{d}r' \\
&= \delta_{k,l}\int_0^{t\wedge s} \delta(x-y)\mu_r^N(x)\,\mathrm{d}r,
\end{aligned}$$

hence $\sqrt{\mu_t^N(x)}\boldsymbol{\xi}(t,x)$ has the same covariance as $N^{-1/2}\sum_{i=1}^N \delta(x - X_t^i)\,\mathrm{d}B_t^i$ and upon replacing the latter by the former in (A.1.1), we obtain (1.1.8).

Dean's replacement can be seen as an analogue of the general result by Dambis, Dubins and Schwarz [KS91, Subsec. 3.4.B] that every continuous local martingale with respect to a right-continuous filtration is a time change of a Brownian motion (possibly on a larger probability space.) Indeed define the noise

$$M(\mathrm{d}s\,\mathrm{d}x) := \frac{\sqrt{2}}{N}\sum_{i=1}^N \delta(x - X_s^i)\,\mathrm{d}B_s^i\,\mathrm{d}x,$$

which is (formally) a martingale measure in the sense of [Wal86]. We obtain for every sufficiently nice $\phi\colon \mathbb{R}^d \to \mathbb{R}^d$ the martingale

$$\int_0^t\int_{\mathbb{R}^d} \phi(x)\cdot M(\mathrm{d}s\,\mathrm{d}x) = \frac{\sqrt{2}}{N}\sum_{i=1}^N\int_0^t \phi(X_s^i)\cdot\mathrm{d}B_s^i,$$

whose quadratic variation is given by

$$\left[\int_0^{\cdot} \int_{\mathbb{R}^d} \phi(x) \cdot M(\mathrm{d}s\,\mathrm{d}x)\right]_t = \frac{2}{N^2} \sum_{i=1}^N \int_0^t |\phi(X_s^i)|^2 \,\mathrm{d}s = \frac{2}{N} \int_0^t \int_{\mathbb{R}^d} |\phi(x)|^2 \mu_s^N(x) \,\mathrm{d}s\,\mathrm{d}x.$$

Upon defining

$$W_t(\phi) := \int_0^t \int_{\mathbb{R}^d} \frac{\sqrt{N}}{\sqrt{2}} \frac{\phi(x)}{\sqrt{\mu_s^N(x)}} \cdot M(\mathrm{d}s\,\mathrm{d}x),$$

we obtain a cylindrical Brownian motion, since

$$[W(\phi)]_t = \frac{2}{N} \int_0^t \int_{\mathbb{R}^d} \frac{N}{2} \frac{|\phi(x)|^2}{\rho_s(x)} \mu_s^N(x) \,\mathrm{d}x\,\mathrm{d}s = t\|\phi\|_{L^2}^2.$$

General theory [Daw93, Ex. 7.1.2] then implies that there exists a vector-valued space-time white noise $W(\mathrm{d}s\,\mathrm{d}x)$ such that

$$W_t(\phi) = \int_0^t \int_{\mathbb{R}^d} \phi(x) \cdot W(\mathrm{d}s\,\mathrm{d}x),$$

which we can relate back to $M(\mathrm{d}s\,\mathrm{d}x)$ by the formula

$$\frac{\sqrt{2}}{\sqrt{N}} \sqrt{\mu_s^N(x)} W(\mathrm{d}s\,\mathrm{d}x) = M(\mathrm{d}s\,\mathrm{d}x).$$

Consequently, we can identify the noise in (A.1.1) by

$$\frac{\sqrt{2}}{N} \nabla \cdot \sum_{i=1}^N \delta(x - X_s^i) \,\mathrm{d}B_s^i \,\mathrm{d}x = \nabla \cdot M(\mathrm{d}s\,\mathrm{d}x) = \frac{\sqrt{2}}{\sqrt{N}} \nabla \cdot \left(\sqrt{\mu_s^N(x)} W(\mathrm{d}s\,\mathrm{d}x)\right),$$

which is the same Dean–Kawasaki noise as in (1.1.8).

A.1.2 Derivation as a Fluctuating Gradient Flow

In this subsection we derive the Keller–Segel–Dean–Kawasaki equation (1.1.8) as a fluctuating gradient flow in Wasserstein space. The presentation of this theory follows talks given by Avi Mayorcas (Oxford '23) and Benjamin Gess (Delft '24), see also [JKO98; Ott01; Ada+13] for further references.

We cast the Keller–Segel equation (1.1.3) as a formal gradient flow in Wasserstein space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$, where $\mathcal{P}_2(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$

with finite second moments and $\mathcal{W}_2$ the 2-Wasserstein distance. For every $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ let g_ρ be the local metric tensor

$$g_\rho(\phi, \psi) = \int_{\mathbb{R}^d} \rho(x) \nabla \tilde{\phi}(x) \cdot \nabla \tilde{\psi}(x) \,\mathrm{d}x, \qquad \begin{cases} -\nabla \cdot (\rho \nabla \tilde{\phi}) = \phi, \\ -\nabla \cdot (\rho \nabla \tilde{\psi}) = \psi, \end{cases}$$

acting on smooth, mean-free $\phi, \psi \colon \mathbb{R}^d \to \mathbb{R}$, see [Ott01] for a rigorous discussion.

We define the free energy (also sometimes called the entropy) by

$$F(\rho) := \int_{\mathbb{R}^d} (\rho(x) \log \rho(x) - \frac{\chi}{2} \rho(x) c(x)) \,\mathrm{d}x, \tag{A.1.2}$$

where $c = K * \rho$ denotes the chemical concentration and K the Green function of the Laplacian on $\mathbb{R}^d$. We further define the Wasserstein gradient $\nabla_{\mathcal{W}_2} F(\rho)$ to be the unique vector such that for every smooth, mean-free $\psi \colon \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{h \to 0} \frac{F(\rho + h\psi) - F(\rho)}{h} = g_\rho(\nabla_{\mathcal{W}_2} F(\rho), \psi).$$

One can then show

$$-\nabla_{\mathcal{W}_2} F(\rho) = \Delta \rho - \chi \nabla \cdot (\rho(\nabla K * \rho)), \tag{A.1.3}$$

which implies that the Keller–Segel equation (1.1.3) is equivalent to the Otto–Wasserstein gradient flow

$$\partial_t \rho = -\nabla_{\mathcal{W}_2} F(\rho). \tag{A.1.4}$$

In finite dimensions we can associate to each gradient flow

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = -\nabla U(x_t), \qquad U \colon \mathbb{R}^d \to \mathbb{R}$$

a family of stochastic dynamics,

$$\mathrm{d}X_t = -a(X_t) \cdot \nabla U(X_t) \,\mathrm{d}t + N^{-1} \nabla \cdot a(X_t) \,\mathrm{d}t - \sqrt{2/N} \sigma(X_t) \,\mathrm{d}B_t \tag{A.1.5}$$

such that for every $N \in \mathbb{N}$, (A.1.5) is reversible with respect to the probability measure $\mu(\mathrm{d}x) = \frac{1}{Z} \mathrm{e}^{-NU(x)} \,\mathrm{d}x$, where Z denotes the normalisation (partition function), B a d-dimensional Brownian motion, $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $a = \sigma \sigma^T \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $(\nabla \cdot a(x))_i = \sum_{j=1}^d \partial_j a_{j,i}(x)$ (under suitable assumptions on U and σ.)

Returning to the infinite-dimensional setting of (A.1.4), we rewrite the Wasserstein gradient as $\nabla_{\mathcal{W}_2}F(\rho) = M(\rho)\frac{\partial F}{\partial \rho}(\rho)$, where $\frac{\partial F}{\partial \rho}(\rho)$ denotes the variational derivative

$$\lim_{h\to 0}\frac{F(\rho+h\psi)-F(\rho)}{h} = \Big\langle \frac{\partial F}{\partial \rho}(\rho),\psi\Big\rangle_{L^2(\mathbb{R}^d)},$$

and $M(\rho)$ the *mobility*,

$$M(\rho)\phi = -\nabla\cdot(\rho\nabla\phi), \qquad \phi\colon \mathbb{R}^d\to\mathbb{R},$$

which takes the rôle of the diffusion matrix a in (A.1.5). We identify its square root by $M^{1/2}(\rho)\psi = \nabla\cdot(\sqrt{\rho}\psi)$ (acting on $\psi\colon\mathbb{R}^d\to\mathbb{R}^d$) so that $M(\rho)\phi = M^{1/2}(\rho)(M^{1/2}(\rho))^*\phi$, where the adjoint $(M^{1/2}(\rho))^*$ is given by $(M^{1/2}(\rho))^*\phi = -\sqrt{\rho}\nabla\phi$. Let $\boldsymbol{\xi}$ be a vector-valued space-time white noise; then in analogy to (A.1.5) we define the stochastic dynamics

$$\partial_t\rho = -M(\rho)\frac{\partial F}{\partial\rho}(\rho) - \sqrt{2/N}M^{1/2}(\rho)\boldsymbol{\xi} = -\nabla_{\mathcal{W}_2}F(\rho) - \sqrt{2/N}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}), \quad \text{(A.1.6)}$$

where we dropped the term of order N^{-1} as it won't contribute to the large deviations of ρ for large N (cf. [FH20, Thm. 9.5]). Using (A.1.3), it follows that equation (A.1.6) is equivalent to the Keller–Segel–Dean–Kawasaki equation (1.1.8).

Furthermore, we can identify the noise $\zeta_\rho := \nabla\cdot(\sqrt{\rho}\boldsymbol{\xi})$ appearing in (A.1.6) as the natural white noise with respect to the local metric tensor g_ρ. For the sake of notation let us ignore the time component. We obtain for every $\rho\in\mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}[g_\rho(\zeta_\rho,\phi)g_\rho(\zeta_\rho,\psi)] &= \mathbb{E}\Big[\int_{\mathbb{R}^d}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi})(x)\tilde{\phi}(x)\,\mathrm{d}x\int_{\mathbb{R}^d}\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi})(y)\tilde{\psi}(y)\,\mathrm{d}y\Big]\\ &= \mathbb{E}\Big[\int_{\mathbb{R}^d}\sqrt{\rho(x)}\boldsymbol{\xi}(x)\cdot\nabla\tilde{\phi}(x)\,\mathrm{d}x\int_{\mathbb{R}^d}\sqrt{\rho(y)}\boldsymbol{\xi}(y)\cdot\nabla\tilde{\psi}(y)\,\mathrm{d}y\Big]\\ &= \int_{\mathbb{R}^d}\rho(x)\nabla\tilde{\phi}(x)\cdot\nabla\tilde{\psi}(x)\,\mathrm{d}x = g_\rho(\phi,\psi),\end{aligned}$$

which yields the claim.

A.2 Criticality

To ascertain whether a given non-linear stochastic PDE can be solved with the theories of regularity structures [Hai14], paracontrolled calculus [GIP15] or flow equations [Kup16; Duc21], one can first check if the non-linearity is dominated by the noise on small scales. If it is, then the equation is called (locally) subcritical (in the sense of [Hai14]) and there is—morally—a good chance that it can be solved with the techniques of singular SPDEs. Conversely, if an equation is not subcritical, it is unlikely that one can give meaning to it using the tools of singular SPDEs. The notion of subcriticality has been formalized in the (so-called) black-box papers [BHZ19; Bru+21; CH16; HS24], although here we choose to argue heuristically.

There are at least two back-of-the-envelope calculations one can perform to show subcriticality [Per20, Sec. 4.7]. The first is based on scaling and will be used in Subsection A.2.1 to establish the subcriticality of our additive-noise equation (1.1.11) in dimensions $d \leq 3$. The second is based on regularity (power counting) and will be used in Subsection A.2.2 to establish the failure of subcriticality for the Keller–Segel–Dean–Kawasaki equation (1.1.10) in every dimension. See [Per20] for further details.

A.2.1 Subcriticality of the Additive-Noise Approximation

To establish the subcriticality of (1.1.11), it suffices to consider the formal equation

$$\begin{cases} (\partial_t - \Delta)\rho = -\chi\nabla\cdot(\rho(\nabla K * \rho)) - \nabla\cdot(\sqrt{\rho_{\mathrm{det}}}\boldsymbol{\xi}) & \text{in } (0,T]\times\mathbb{R}^d, \\ \qquad \rho|_{t=0} = \rho_0 & \text{on } \mathbb{R}^d, \end{cases} \tag{A.2.1}$$

where ∇K is given by (1.1.2), ρ_{det} is a solution to (1.1.3) and $\boldsymbol{\xi}$ is a vector-valued space-time white noise on $[0,T]\times\mathbb{R}^d$. Equation (A.2.1) is the full-space analogue of (1.1.11) without mollification. In this subsection we show that (A.2.1) is subcritical in dimensions $d \leq 3$, critical in dimension $d = 4$ and supercritical in dimension $d \geq 5$.

We denote for $\lambda > 0$ the parabolic space-time scaling operator $\mathcal{S}_\lambda u(t,x) = u(\lambda^2 t, \lambda x)$ and define $\rho_\lambda = \lambda^a \mathcal{S}_\lambda \rho$ for some exponent $a \in \mathbb{R}$. Setting $\boldsymbol{\xi}_\lambda = \lambda^{\frac{d}{2}+1}\boldsymbol{\xi}$, it follows that $\boldsymbol{\xi}_\lambda$ is again a space-time white noise [Per20, Ex. 11.3]. We obtain by the chain rule,

$$\partial_t\rho_\lambda = \Delta\rho_\lambda - \chi\lambda^{-a+2}\nabla\cdot(\rho_\lambda(\nabla K * \rho_\lambda)) - \nabla\cdot\Big(\sqrt{\mathcal{S}_\lambda\rho_{\mathrm{det}}}\lambda^{a+1}\mathcal{S}_\lambda\boldsymbol{\xi}\Big),$$

which upon choosing $a = \frac{d}{2}$ yields

$$\partial_t\rho_\lambda = \Delta\rho_\lambda - \chi\lambda^{-\frac{d}{2}+2}\nabla\cdot(\rho_\lambda(\nabla K * \rho_\lambda)) - \nabla\cdot\Big(\sqrt{\mathcal{S}_\lambda\rho_{\mathrm{det}}}\boldsymbol{\xi}_\lambda\Big).$$

Note $\lim_{\lambda\to 0}\mathcal{S}_\lambda\rho_{\mathrm{det}} = \text{const}$, so that the heterogeneity $\sqrt{\mathcal{S}_\lambda\rho_{\mathrm{det}}}$ is harmless on small scales. In dimensions $d \leq 3$ the non-linearity formally vanishes as $\lambda \to 0$, which shows that the equation is subcritical. In dimension $d = 4$ the equation is scaling invariant, hence critical. In dimension $d \geq 5$ the non-linearity dominates small scales, so that the equation is supercritical.

A.2.2 Supercriticality of Dean–Kawasaki Equations

To deduce the supercriticality of the Keller–Segel–Dean–Kawasaki equation (1.1.10) (resp. the Dean–Kawasaki equation with smooth interactions (1.1.9)), it suffices to consider the non-interacting version

$$\begin{cases}(\partial_t - \Delta)\rho = -\nabla\cdot(\sqrt{\rho}\boldsymbol{\xi}), & \text{in } (0,\infty)\times\mathbb{T}^d,\\ \qquad \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^d,\end{cases}$$

which by the product rule is equivalent to

$$\begin{cases}(\partial_t - \Delta)\rho = -\nabla\sqrt{\rho}\cdot\boldsymbol{\xi} - \sqrt{\rho}\nabla\cdot\boldsymbol{\xi}, & \text{in } (0,\infty)\times\mathbb{T}^d,\\ \qquad \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^d.\end{cases} \tag{A.2.2}$$

Let us show that (A.2.2) is supercritical in every dimension $d \in \mathbb{N}$.

The vector-valued space-time white noise $\boldsymbol{\xi}$ has regularity $\alpha < -d/2 - 1$ (cf. e.g. Section C.1). Assuming that the least regular term on the right-hand side of (A.2.2) is given by $\nabla\cdot\boldsymbol{\xi}$ with regularity $\alpha - 1$, it follows by the parabolic equation that ρ has regularity $\alpha + 1$. Further if ρ is bounded away from 0, we may assume $\sqrt{\cdot}$ is smooth so that $\sqrt{\rho}$ also has regularity $\alpha + 1$. Consequently, we obtain the power counting

$$(\partial_t - \Delta)\underbrace{\rho}_{\alpha+1} = -\underbrace{\nabla\sqrt{\rho}}_{\alpha}\cdot\underbrace{\boldsymbol{\xi}}_{\alpha} - \underbrace{\sqrt{\rho}}_{\alpha+1}\underbrace{\nabla\cdot\boldsymbol{\xi}}_{\alpha-1}.$$

Formally, both products have regularity 2α. Hence, for the non-linearity to be dominated by the noise $\nabla\cdot\boldsymbol{\xi}$ on small scales, we need $2\alpha > \alpha - 1$, which is equivalent to $\alpha > -1$. This is impossible since $\alpha < -d/2 - 1$ and $d \in \mathbb{N}$. It follows that the Dean–Kawasaki equation (A.2.2) is supercritical in every dimension.

B
Analysis of Function Spaces

Contents

Throughout the following appendices all relevant properties are given for mappings or distributions on $\mathbb{T}^d$ for some $d \in \mathbb{N}$. The definitions introduced in Section 1.2 generalize to d dimensions *mutatis mutandis.*

B.1 Sobolev and Gagliardo–Nirenberg–Sobolev Inequalities

We apply the following Sobolev and Gagliardo–Nirenberg–Sobolev inequalities in the two-dimensional case to prove Theorem 3.2.1.

Lemma B.1.1 (Sobolev's inequality)**.** *The following hold:*

1. Let $s > 0$ and $2 < q < \infty$ be such that $\frac{s}{d} \geq \frac{1}{2} - \frac{1}{q}$; then $\mathcal{H}^s(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$ and

$$\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{\mathcal{H}^s(\mathbb{T}^d)}. \tag{B.1.1}$$

2. Let $s > d/2$; then $\mathcal{H}^s(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ and

$$\|u\|_{L^\infty(\mathbb{T}^d)} \lesssim \|u\|_{\mathcal{H}^s(\mathbb{T}^d)}. \tag{B.1.2}$$

Proof. See [BO13, Cor. 1.2 & Subsec. 2.1]. □

Combining Sobolev's inequality (Lemma B.1.1) with the Gagliardo–Nirenberg interpolation estimate [BM18] yields the following Gagliardo–Nirenberg–Sobolev inequality.

Lemma B.1.2. *Let $q, r \in [1, \infty]$; then given $u \in \mathcal{H}^1(\mathbb{T}^d) \cap L^r(\mathbb{T}^d)$ such that $\overline{u} = 0$ and $\theta \in (0, 1)$ satisfying*

$$\theta\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1-\theta}{r} = \frac{1}{q}, \tag{B.1.3}$$

there exists a constant $C \coloneqq C(d, q, r) > 0$ such that

$$\|u\|_{L^q(\mathbb{T}^d)} \leq C\|\nabla u\|^{\theta}_{L^2(\mathbb{T}^d)}\|u\|^{1-\theta}_{L^r(\mathbb{T}^d)}. \tag{B.1.4}$$

We define $C_{\text{gns-per}}(d, q, r)$ to be the optimal constant such that (B.1.4) *holds.*

Proof. Let $u \in \mathcal{H}^1(\mathbb{T}^d) \cap L^r(\mathbb{T}^d)$. Then upon passing to the full space, u is a periodic element of $W^{1,2}_{\text{loc}}(\mathbb{R}^d) \cap L^r_{\text{loc}}(\mathbb{R}^d)$, where the subspace of periodic functions in $W^{1,2}_{\text{loc}}(\mathbb{R}^d)$ (resp. $L^r_{\text{loc}}(\mathbb{R}^d)$) is equipped with the norm $\|\cdot\|_{W^{1,2}((0,1)^d)}$ (resp. $\|\cdot\|_{L^r((0,1)^d)}$).

Let $q, r \in [1, \infty]$ and $\theta \in (0, 1)$ satisfy (B.1.3). We apply the Gagliardo–Nirenberg–Sobolev inequality on $(0, 1)^d$ (see e.g. [BM19, Cor. 1]) to deduce

$$\|u\|_{L^q((0,1)^d)} \lesssim \|u\|^{\theta}_{W^{1,2}((0,1)^d)}\|u\|^{1-\theta}_{L^r((0,1)^d)}.$$

Passing back to the torus, we obtain

$$\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|^{\theta}_{\mathcal{H}^1(\mathbb{T}^d)}\|u\|^{1-\theta}_{L^r(\mathbb{T}^d)}.$$

Using that u is mean-free, the claim follows from the Poincaré inequality on $\mathbb{T}^d$. □

B.2 Besov Spaces

Applying essentially the same arguments as in the proof of [BCD11a, Prop. 2.10] there exists a *dyadic partition of unity*, i.e. a pair of non-negative, radially symmetric and compactly supported smooth functions $\varrho_{-1}, \varrho_0 \in C^\infty(\mathbb{R}^d; [0,1])$ such that $\operatorname{supp}(\varrho_{-1}) \subset B(0,1/2)$, $\operatorname{supp}(\varrho_0) \subset \{x \in \mathbb{R}^d : 9/32 \leq |x| \leq 1\}$ and $\sum_{k=-1}^\infty \varrho_k(x) = 1$ for all $x \in \mathbb{R}^d$, where we denote $\varrho_k(x) \coloneqq \varrho_0(2^{-k}x)$ for each $k \in \mathbb{N}$.

For every $k \geq -1$ we define the Littlewood–Paley block Δ_k to be the Fourier multiplier $\Delta_k u \coloneqq \mathscr{F}^{-1}(\varrho_k \mathscr{F} u)$ and set

$$\Delta_{<k} u \coloneqq \sum_{l=-1}^{k-1} \Delta_l u.$$

As with the Fourier transform, we initially define these operators on smooth functions and then extend them by duality to $\mathcal{S}'(\mathbb{T}^d; \mathbb{R}^n)$.

Definition B.2.1 (Besov spaces)**.** *Let $\alpha \in \mathbb{R}$ and $p, q \in [1,\infty]$. We define the non-homogeneous Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ to be the completion of the smooth functions $C^\infty(\mathbb{T}^d)$ under the norm*

$$\|u\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)} \coloneqq \|(2^{k\alpha}\|\Delta_k u\|_{L^p(\mathbb{T}^d)})_{k\in\mathbb{N}_{-1}}\|_{\ell^q},$$

which is extended to vector (resp. matrix-) valued functions in a natural componentwise manner. Here ℓ^q denotes the usual space of q-summable sequences (or bounded when $q = \infty$.) When $p = q = \infty$ we recall the shorthand $\mathcal{C}^\alpha(\mathbb{T}^d) \coloneqq \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^d)$ and call $\mathcal{C}^\alpha(\mathbb{T}^d)$ the Hölder–Besov space.

Remark B.2.2. Note that the dyadic partition of unity obtained by [BCD11a, Prop. 2.10] is built from $\tilde{\varrho}_{-1}$, $\tilde{\varrho}_0$ with $\operatorname{supp}(\tilde{\varrho}_{-1}) \subset B(0,4/3)$ and $\operatorname{supp}(\tilde{\varrho}_0) \subset \{x \in \mathbb{R}^d : 3/4 \leq |x| \leq 8/3\}$. However, for our purposes it is convenient to rescale these functions by a factor of 3/8 so that the only integer in the support of ϱ_{-1} is 0. Since the Besov spaces are independent of the chosen dyadic partition of unity [BCD11a, Cor. 2.70] this change is harmless.

Besov spaces enjoy a number of useful properties which we list below. Proofs of the following statements can be found in [BCD11a; GIP15].

1. Embeddings: There exists a constant $C > 0$ such that for all $\alpha \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$,
$$\|u\|_{\mathcal{B}^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}} \leq C\|u\|_{\mathcal{B}^{\alpha}_{p_1,q_1}}. \tag{B.2.1}$$
We also have the following continuous embeddings,
$$\|u\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha'}_{p,q}}, \quad \alpha < \alpha' \in \mathbb{R},$$
$$\|u\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha'}_{p,q'}}, \quad \alpha < \alpha' \in \mathbb{R}, \quad q < q' \in [1, \infty].$$

2. Relations to $L^p(\mathbb{T}^d)$-spaces: For all $p \in [1, \infty]$, one has,
$$\|f\|_{\mathcal{B}^0_{p,\infty}} \lesssim \|f\|_{L^p} \lesssim \|f\|_{\mathcal{B}^0_{p,1}}.$$

We can show that each $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ is separable, where we use that our Besov spaces are defined as closures of smooth functions (see Definition B.2.1).

Lemma B.2.3. *Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. It holds that*
$$\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d) = \left\{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{\mathcal{B}^{\alpha}_{p,q}} < \infty, \lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0\right\} \tag{B.2.2}$$
and each $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ is separable.

Proof. We first prove the characterization (B.2.2). Assume $u \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$, we need to show that $\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0$. By definition, there exist a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in C^\infty(\mathbb{T}^d)$ for every $n \in \mathbb{N}$ and $u_n \to u \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ as $n \to \infty$. It follows that for each $j \in \mathbb{N}_{-1}$,
$$2^{j\alpha}\|\Delta_j u\|_{L^p} \leq 2^{j\alpha}\|\Delta_j u - \Delta_j u_n\|_{L^p} + 2^{j\alpha}\|\Delta_j u_n\|_{L^p} \leq \|u - u_n\|_{\mathcal{B}^{\alpha}_{p,q}} + 2^{j\alpha}\|\Delta_j u_n\|_{L^p}.$$
Letting $j \to \infty$, we obtain $\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = \|u - u_n\|_{\mathcal{B}^{\alpha}_{p,q}}$, subsequently letting $n \to \infty$ yields $\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0$. Conversely assume that $u \in \mathcal{S}'(\mathbb{T}^d)$ satisfies $\|u\|_{\mathcal{B}^{\alpha}_{p,q}} < \infty$ and $\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0$. We need to show that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in C^\infty(\mathbb{T}^d)$ for every $n \in \mathbb{N}$ and $\|u - u_n\|_{\mathcal{B}^{\alpha}_{p,q}} \to 0$ as $n \to \infty$. We define $u_n \coloneqq \sum_{j=-1}^{n-1} \Delta_j u \in C^\infty(\mathbb{T}^d)$ and apply [GIP15, Lem. A.3]: For $q = \infty$ we deduce $\|u - u_n\|_{\mathcal{B}^{\alpha}_{p,\infty}} \lesssim \sup_{j\geq n}(2^{j\alpha}\|\Delta_j u\|_{L^p})$, which vanishes as $n \to 0$, since

$\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0$. For $q<\infty$ we deduce $\|u-u_n\|^q_{\mathcal{B}^\alpha_{p,q}} \lesssim \sum_{j=n}^\infty 2^{jq\alpha}\|\Delta_j u\|^q_{L^p}$, which vanishes as $n\to 0$, since $\|u\|_{\mathcal{B}^\alpha_{p,q}}<\infty$. This yields the characterization (B.2.2).

Next we show the separability of each $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$. If $p<\infty$, let D be a countable, dense subset of $L^p(\mathbb{T}^d)$; and if $p=\infty$ let D be a countable, dense subset of $C(\mathbb{T}^d)$. We define the countable set

$$H := \Big\{ \sum_{j=-1}^{J} \Delta_j h_j : J\in\mathbb{N}_{-1},\ h_j\in D,\ j=1,\dots,J\Big\}$$

and show that H is dense in $C^\infty(\mathbb{T}^d)\subset \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$.

Assume first $q=\infty$. Let $u\in C^\infty(\mathbb{T}^d)$, then for each $\varepsilon>0$ there exists a $J\in\mathbb{N}_{-1}$ such that $2^{j\alpha}\|\Delta_j u\|_{L^p}<\varepsilon$ for every $j>J$. For $j=-1,\dots,J$, let $h_j\in D$ be such that $\|u-h_j\|_{L^p}<\varepsilon 2^{-j\alpha}$, which combined with Poisson's summation formula and Young's convolution inequality yields

$$\|\Delta_j u - \Delta_j h_j\|_{L^p} \le (\|\mathscr{F}^{-1}_{\mathbb{R}^d}\varrho_0\|_{L^1} \vee \|\mathscr{F}^{-1}_{\mathbb{R}^d}\varrho_{-1}\|_{L^1})\|u-h_j\|_{L^p} \lesssim \varepsilon 2^{-j\alpha}.$$

It then follows that

$$\Big\|u-\sum_{j=-1}^{J}\Delta_j h_j\Big\|_{\mathcal{B}^\alpha_{p,\infty}} \lesssim \sup_{j=-1,\dots,J}\Big(2^{j\alpha}\|\Delta_j u-\Delta_j h_j\|_{L^p}\Big) \vee \sup_{j=J+1,\dots,\infty}\Big(2^{j\alpha}\|\Delta_j u\|_{L^p}\Big) \lesssim \varepsilon,$$

which shows that H is dense in $\mathcal{B}^\alpha_{p,\infty}(\mathbb{T}^d)$.

Assume next $q<\infty$. Let $u\in C^\infty(\mathbb{T}^d)$, then for each $\varepsilon>0$ there exists a $J\in\mathbb{N}_{-1}$ such that $2^{j\alpha}\|\Delta_j u\|_{L^p}<\varepsilon 2^{-j}$ for every $j>J$. For $j=-1,\dots,J$, let $h_j\in D$ be such that $\|u-h_j\|_{L^p}<\varepsilon 2^{-j(\alpha+1)}$, which combined with Poisson's summation formula and Young's convolution inequality yields

$$\|\Delta_j u - \Delta_j h_j\|_{L^p} \le (\|\mathscr{F}^{-1}_{\mathbb{R}^d}\varrho_0\|_{L^1} \vee \|\mathscr{F}^{-1}_{\mathbb{R}^d}\varrho_{-1}\|_{L^1})\|u-h_j\|_{L^p} \lesssim \varepsilon 2^{-j(\alpha+1)}.$$

It then follows that

$$\begin{aligned}\Big\|u-\sum_{j=-1}^{J}\Delta_j h_j\Big\|^q_{\mathcal{B}^\alpha_{p,q}} &\lesssim \sum_{j=-1}^{J} 2^{jq\alpha}\|\Delta_j u-\Delta_j h_j\|^q_{L^p} + \sum_{j=J+1}^{\infty} 2^{jq\alpha}\|\Delta_j u\|^q_{L^p}\\ &\lesssim \varepsilon^q\sum_{j=-1}^{\infty}2^{-jq}\lesssim\varepsilon^q,\end{aligned}$$

which shows that H is dense in $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$. This yields the claim. □

The next lemma presents a natural criterion that allows us to find elements of $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$.

Lemma B.2.4. *Let $p, q \in [1, \infty]$ and $\alpha < \alpha' \in \mathbb{R}$.*

- *If $q \in [1, \infty)$, then each distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ such that $\|u\|_{\mathcal{B}^{\alpha}_{p,q}} < \infty$ is an element of $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$.*
- *If $q = \infty$, then each distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ such that $\|u\|_{\mathcal{B}^{\alpha'}_{p,\infty}} < \infty$ is an element of $\mathcal{B}^{\alpha}_{p,\infty}(\mathbb{T}^d)$.*

Proof. Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. To prove the first claim, let $q < \infty$ and $u \in \mathcal{S}'(\mathbb{T}^d)$ be such that $\|u\|_{\mathcal{B}^{\alpha}_{p,q}} < \infty$. It follows immediately that $\lim_{j\to\infty} 2^{j\alpha}\|\Delta_j u\|_{L^p} = 0$, hence $u \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ by Lemma B.2.3. To prove the second claim, let $q = \infty$, $\alpha < \alpha' \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{T}^d)$ be such that $\|u\|_{\mathcal{B}^{\alpha'}_{p,\infty}} < \infty$. It follows that $\|u\|_{\mathcal{B}^{\alpha}_{p,\infty}} < \infty$ and

$$2^{j\alpha}\|\Delta_j u\|_{L^p} \leq 2^{j(\alpha-\alpha')}\|u\|_{\mathcal{B}^{\alpha'}_{p,\infty}} \to 0 \quad \text{as } j \to \infty,$$

hence $u \in \mathcal{B}^{\alpha}_{p,\infty}(\mathbb{T}^d)$ by Lemma B.2.3. This yields the claim. □

The next lemma shows that Besov spaces are compactly embedded upon passing to a larger space.

Lemma B.2.5. *Let $p, q \in [1, \infty]$ and $\alpha' < \alpha \in \mathbb{R}$. It then follows that the embedding $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d) \hookrightarrow \mathcal{B}^{\alpha'}_{p,q}(\mathbb{T}^d)$ is compact.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$, to prove the claim it suffices to show that it has a subsequence which converges in $\mathcal{B}^{\alpha'}_{p,q}(\mathbb{T}^d)$.

We follow the proof of [BCD11a, Thm. 2.94]. By the Fatou property of Besov spaces [BCD11a, Thm. 2.72], there exists a subsequence (which we do not re-label) such that $u_n \to u$ in $\mathcal{S}'(\mathbb{T}^d)$ and

$$\|u\|_{\mathcal{B}^{\alpha}_{p,q}} \leq \liminf_{n\to\infty}\|u_n\|_{\mathcal{B}^{\alpha}_{p,q}}.$$

Denote $v_n := u_n - u$, so that it suffices to show that $\lim_{n\to\infty}\|v_n\|_{\mathcal{B}^{\alpha'}_{p,q}} = 0$. First let us assume $q < \infty$, we estimate for every $K > 0$,

$$\|v_n\|^q_{\mathcal{B}^{\alpha'}_{p,q}} = \sum_{j\in\mathbb{N}_{-1}} 2^{jq\alpha'}\|\Delta_j v_n\|^q_{L^p} \leq \sum_{\substack{j\in\mathbb{N}_{-1}\\ 2^j\leq K}} 2^{jq\alpha'}\|\Delta_j v_n\|^q_{L^p} + K^{q(\alpha'-\alpha)} \sum_{\substack{j\in\mathbb{N}_{-1}\\ K<2^j}} 2^{jq\alpha}\|\Delta_j v_n\|^q_{L^p}. \tag{B.2.3}$$

To control the second summand in (B.2.3), we can use that $(\|v_n\|_{\mathcal{B}^{\alpha}_{p,q}})_{n\in\mathbb{N}}$ is bounded (by the boundedness of $(\|u_n\|_{\mathcal{B}^{\alpha}_{p,q}})_{n\in\mathbb{N}}$), to choose for every $\varepsilon > 0$ the constant K sufficiently large such that uniformly in $n \in \mathbb{N}$,

$$K^{q(\alpha'-\alpha)} \sum_{\substack{j\in\mathbb{N}_{-1}\\ K<2^j}} 2^{jq\alpha}\|\Delta_j v_n\|^q_{L^p} \leq K^{q(\alpha'-\alpha)}\|v_n\|^q_{\mathcal{B}^{\alpha}_{p,q}} < \varepsilon^q.$$

Given a K independent of n, we can then let $n \to \infty$ in the first summand of (B.2.3) to deduce

$$\lim_{n\to\infty} \sum_{\substack{j\in\mathbb{N}_{-1}\\ 2^j\leq K}} 2^{jq\alpha'}\|\Delta_j v_n\|^q_{L^p} = \sum_{\substack{j\in\mathbb{N}_{-1}\\ 2^j\leq K}} 2^{jq\alpha'} \lim_{n\to\infty}\|\Delta_j v_n\|^q_{L^p}.$$

Using that $v_n \to 0$ in $\mathcal{S}'(\mathbb{T}^d)$ as $n \to \infty$ and that $\|v_n\|_{\mathcal{B}^{\alpha}_{p,q}}$ is bounded uniformly in $n \in \mathbb{N}$, we can apply Bernstein's inequality [GIP15, Lem. A.1] and the dominated convergence theorem to deduce $\lim_{n\to\infty}\|\Delta_j v_n\|_{L^p} = 0$ for every $j \in \mathbb{N}_{-1}$. Hence, passing to the limit in (B.2.3) we obtain $\lim_{n\to\infty}\|v_n\|_{\mathcal{B}^{\alpha'}_{p,q}} < \varepsilon$ for every $\varepsilon > 0$, which yields the claim for $q < \infty$. The claim for $q = \infty$ follows by a similar argument. □

B.3 Interpolation Spaces

We regularly work in a scale of interpolation spaces which relate temporal and spatial regularity and are suitable for solutions to parabolic PDEs.

Definition B.3.1 (Interpolation spaces). *Let $T > 0$, $\eta \geq 0$, $p, q \in [1, \infty]$, $\alpha \in \mathbb{R}$ and $\kappa \in (0, 1]$. We define the norm*

$$\|u\|_{\mathscr{L}^{\kappa}_{\eta;T}\mathcal{B}^{\alpha}_{p,q}} := \max\{\|u\|_{C^{\kappa}_{\eta;T}\mathcal{B}^{\alpha-2\kappa}_{p,q}}, \|u\|_{C_{\eta;T}\mathcal{B}^{\alpha}_{p,q}}\}$$

and the spaces

$$\mathscr{L}^{\kappa}_{\eta;T}\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d) = C^{\kappa}_{\eta;T}\mathcal{B}^{\alpha-2\kappa}_{p,q}(\mathbb{T}^d) \cap C_{\eta;T}\mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d).$$

We set

$$\mathscr{L}_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d) := C_T^\kappa \mathcal{B}_{p,q}^{\alpha-2\kappa}(\mathbb{T}^d) \cap C_T \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$$

and by an abuse of notation understand $\mathscr{L}_{\eta;T}^0 \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d) = C_{\eta;T} \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.

Recall that the space $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is defined as the completion of $C_T^\infty \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ under the usual Hölder norm for $\kappa \in (0,1)$ and that $C_T^0 \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d) := C_T \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ (cf. Subsection 1.2). Next we show that each $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is separable.

Lemma B.3.2. *Let $T > 0$, $p, q \in [1,\infty]$, $\alpha \in \mathbb{R}$ and $\kappa \in [0,1)$. It holds that $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ consists of those $f\colon [0,T] \to \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ such that*

$$\|f\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha} < \infty \quad \textit{and} \quad \lim_{r\to 0} \sup\left\{ \frac{\|f(t)-f(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^\kappa} : s \neq t \in [0,T],\ |t-s| < r \right\} = 0 \tag{B.3.1}$$

and each $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is separable.

Proof. We first prove that $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is characterized by (B.3.1), that is, each $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ satisfies (B.3.1) and each f satisfying (B.3.1) lies in $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.

Assume $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$, we need to show that (B.3.1) holds. It is clear that $\|f\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha} < \infty$ and it suffices to prove the second condition of (B.3.1). By definition, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in C_T^\infty \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|f_n - f\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha} = 0$. Let $r > 0$ and $s \neq t \in [0,T]$ such that $|t-s| < r$, then

$$\frac{\|f(t)-f(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^\kappa} \leq \|f - f_n\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha} + \frac{\|f_n(t)-f_n(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^\kappa},$$

which yields

$$\lim_{r\to 0} \sup\left\{ \frac{\|f(t)-f(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^\kappa} : s \neq t \in [0,T],\ |t-s| < r \right\} = \|f - f_n\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha}.$$

Since $n \in \mathbb{N}$ is arbitrary, we can pass to the limit to show that each $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ satisfies (B.3.1).

Next we need to show that each $f\colon [0,T] \to \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ satisfying (B.3.1) lies in $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$, i.e. that it can be approximated by a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in$

$C_T^\infty \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ for every $n \in \mathbb{N}$. We extend f from $[0,T]$ to $\mathbb{R}$ by setting $f(t) = f(0)$ for $t < 0$ and $f(t) = f(T)$ for $t > T$. We define

$$\phi_n(r) \coloneqq n\left(1 - \frac{|r|^2}{n^2}\right)^{n^4} \pi^{-1/2}$$

and construct the polynomials

$$f_n(t) \coloneqq \int_{\mathbb{R}} f(r)\phi_n(t-r)\,\mathrm{d}r = \int_{\mathbb{R}} f(t-r)\phi_n(r)\,\mathrm{d}r, \tag{B.3.2}$$

which converge to f in $C_T\mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ as $n \to \infty$, see [EK86, Appendix, Prop. 7.1] for the case of real-valued functions. The extension to Banach space-valued functions follows by the discussion in [AE09, p. 177].

Next we show that $f_n \to f$ in $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$, where we argue as in [Haj19]. Let $\varepsilon > 0$ be arbitrary, by the definition of $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ we can find some $\tau > 0$ such that if $s \neq t \in [0,T]$ and $|t-s| < \tau$, then

$$\|f(t) - f(s)\|_{\mathcal{B}_{p,q}^\alpha} < \frac{1}{2}\varepsilon|t-s|^\kappa, \tag{B.3.3}$$

which also extends from $[0,T]$ to $\mathbb{R}$ by our definition of f. For every $n \geq T$ it follows by [EK86, Appendix, (7.3)] that $\int_{[-T,T]}|\phi_n(r)|\,\mathrm{d}r \leq 1$, which combined with (B.3.3) and the definition of f_n yields

$$\|f_n(t) - f_n(s)\|_{\mathcal{B}_{p,q}^\alpha} \leq \int_{[-T,T]} \|f(t-r) - f(s-r)\|_{\mathcal{B}_{p,q}^\alpha} |\phi_n(r)|\,\mathrm{d}r \leq \frac{1}{2}\varepsilon|t-s|^\kappa.$$

Consequently we obtain for all $|t-s| < \tau$, that

$$\|f(t) - f(s) - f_n(t) + f_n(s)\|_{\mathcal{B}_{p,q}^\alpha} < \varepsilon|t-s|^\kappa.$$

Let n be sufficiently large such that $\|f - f_n\|_{C_T\mathcal{B}_{p,q}^\alpha} < \varepsilon\tau^\kappa/2$. Assume $|t-s| > \tau$, then

$$\|f(t) - f(s) - f_n(t) + f_n(s)\|_{\mathcal{B}_{p,q}^\alpha} \leq 2\|f - f_n\|_{C_T\mathcal{B}_{p,q}^\alpha} < \varepsilon\tau^\kappa < \varepsilon|t-s|^\kappa.$$

This implies that for all $s \neq t \in [0,T]$ and n sufficiently large,

$$\|(f - f_n)(t) - (f - f_n)(s)\|_{\mathcal{B}_{p,q}^\alpha} < \varepsilon|t-s|^\kappa.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\lim_{n\to\infty}\|f - f_n\|_{C_T^\kappa \mathcal{B}_{p,q}^\alpha} = 0$, which shows that each $f\colon [0,T] \to \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ satisfying (B.3.1) can be approximated by a polynomial. In particular, $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$, which proves that (B.3.1) characterises the space $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.

To show that $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is separable, it suffices to approximate every $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ by a polynomial with coefficients in a countable, dense subset of $\mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$, which exists by (B.3.2) and the separability of $\mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ (Lemma B.2.3). This yields the claim. $\square$

The next lemma presents a natural criterion that allows us to find elements of $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.

Lemma B.3.3. *Let $T > 0$, $p, q \in [1,\infty]$, $\alpha < \alpha' \in \mathbb{R}$ and $0 \le \kappa < \kappa' < 1$.*

- *If $q \in [1,\infty)$, then each $f\colon [0,T] \to \mathcal{S}'(\mathbb{T}^d)$ such that $\|f\|_{C_T^{\kappa'} \mathcal{B}_{p,q}^\alpha} < \infty$ is an element of $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.*
- *If $q = \infty$, then each $f\colon [0,T] \to \mathcal{S}'(\mathbb{T}^d)$ such that $\|f\|_{C_T^{\kappa'} \mathcal{B}_{p,\infty}^{\alpha'}} < \infty$ is an element of $C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$.*

Proof. Let $T > 0$, $p, q \in [1,\infty]$, $\alpha \in \mathbb{R}$ and $0 \le \kappa < \kappa' < 1$. To prove the first claim, let $q < \infty$ and $f\colon [0,T] \to \mathcal{S}'(\mathbb{T}^d)$ be such that $\|f\|_{C_T^{\kappa'} \mathcal{B}_{p,q}^\alpha} < \infty$. By Lemma B.2.4 it follows that $f\colon [0,T] \to \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$. Let $r > 0$ and $s \neq t \in [0,T]$ with $|t - s| < r$, we obtain

$$\frac{\|f(t) - f(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^\kappa} \lesssim |t-s|^{\kappa'-\kappa} \frac{\|f(t) - f(s)\|_{\mathcal{B}_{p,q}^\alpha}}{|t-s|^{\kappa'}} \le r^{\kappa'-\kappa} \|f\|_{C_T^{\kappa'} \mathcal{B}_{p,q}^\alpha} \to 0 \quad \text{as } r \to 0,$$

hence f satisfies (B.3.1), which implies $f \in C_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ by Lemma B.3.3. The second claim follows by the same argument, using that $\|f\|_{C_T^{\kappa'} \mathcal{B}_{p,\infty}^{\alpha'}} < \infty$ implies $f\colon [0,T] \to \mathcal{B}_{p,\infty}^\alpha(\mathbb{T}^d)$ by Lemma B.2.4. $\square$

Next we show that each $\mathscr{L}_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is separable.

Lemma B.3.4. *Let $T > 0$, $\alpha \in \mathbb{R}$ and $\kappa \in (0,1)$. Then $\mathscr{L}_T^\kappa \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d) = C_T^\kappa \mathcal{B}_{p,q}^{\alpha-2\kappa}(\mathbb{T}^d) \cap C_T \mathcal{B}_{p,q}^\alpha(\mathbb{T}^d)$ is separable.*

Proof. The proof is classical: There exists an isometric isomorphism that maps $\mathscr{L}_T^\kappa \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d) = C_T^\kappa \mathcal{B}^{\alpha-2\kappa}_{p,q}(\mathbb{T}^d) \cap C_T \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ to the diagonal $\{(f,f) : f \in \mathscr{L}_T^\kappa \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)\} \subset C_T^\kappa \mathcal{B}^{\alpha-2\kappa}_{p,q}(\mathbb{T}^d) \times C_T \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$. The product space $C_T^\kappa \mathcal{B}^{\alpha-2\kappa}_{p,q}(\mathbb{T}^d) \times C_T \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ is a separable metric space by Lemma B.3.2, which implies that the diagonal is a separable subspace. Using that isometric isomorphisms preserve separability, it follows that each $\mathscr{L}_T^\kappa \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$ is separable. □

B.4 Paraproducts

For $u, v \in C^\infty(\mathbb{T}^d; \mathbb{R})$ we define the paraproduct $\varolessthan$ and resonant product $\odot$ by

$$u \varolessthan v := \sum_{k \geq -1} \Delta_{<k-1} u \Delta_k v, \qquad u \odot v := \sum_{|k-l| \leq 1} \Delta_l u \Delta_k v.$$

Formally one has the decomposition $uv = u \varolessthan v + v \varolessthan u + u \odot v$. Conditions under which this decomposition is valid for (time-dependent) distributions u and v are given by the following version of Bony's estimates. These operators naturally extend to vector-valued and matrix-valued objects as either inner or outer products. Where the precise meaning is not clear from context it will be specified in the text.

Lemma B.4.1 (Bony's estimates). *Let $T > 0$ and $\eta, \eta_1, \eta_2 \geq 0$ be such that $\eta = \eta_1 + \eta_2$.*

- *Let $\beta \in \mathbb{R}$, $u \in C_{\eta_1;T} L^\infty(\mathbb{T}^d; \mathbb{R})$ and $v \in C_{\eta_2;T} \mathcal{C}^\beta(\mathbb{T}^d; \mathbb{R})$. Then $u \varolessthan v \in C_{\eta;T} \mathcal{C}^\beta(\mathbb{T}^d; \mathbb{R})$ and*

$$\|u \varolessthan v\|_{C_{\eta;T}\mathcal{C}^\beta} \lesssim_\beta \|u\|_{C_{\eta_1;T}L^\infty} \|v\|_{C_{\eta_2;T}\mathcal{C}^\beta}.$$

- *Let $\beta \in \mathbb{R}$, $\alpha < 0$, $u \in C_{\eta_1;T} \mathcal{C}^\alpha(\mathbb{T}^d; \mathbb{R})$ and $v \in C_{\eta_2;T} \mathcal{C}^\beta(\mathbb{T}^d; \mathbb{R})$. Then $u \varolessthan v \in C_{\eta;T} \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d; \mathbb{R})$ and*

$$\|u \varolessthan v\|_{C_{\eta;T}\mathcal{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|u\|_{C_{\eta_1;T}\mathcal{C}^\alpha} \|v\|_{C_{\eta_2;T}\mathcal{C}^\beta}.$$

- *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$, $u \in C_{\eta_1;T} \mathcal{C}^\alpha(\mathbb{T}^d; \mathbb{R})$ and $v \in C_{\eta_2;T} \mathcal{C}^\beta(\mathbb{T}^d; \mathbb{R})$. Then $u \odot v \in C_{\eta;T} \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d; \mathbb{R})$ and*

$$\|u \odot v\|_{C_{\eta;T}\mathcal{C}^{\alpha+\beta}} \lesssim_{\alpha,\beta} \|u\|_{C_{\eta_1;T}\mathcal{C}^\alpha} \|v\|_{C_{\eta_2;T}\mathcal{C}^\beta}.$$

Proof. The result is a direct consequence of [GIP15, Lem. 2.1]. □

B.5 Parabolic and Elliptic Regularity Estimates

We will make use of the following interpolation inequality. Let $x \geq 0$ and $\gamma \in [0,1]$. Then

$$0 \leq 1 - \mathrm{e}^{-x} \leq x^{\gamma}. \tag{B.5.1}$$

We also apply the following rapid-decay inequality. For every $r > 0$ uniformly in $x \geq 0$,

$$x^r \mathrm{e}^{-x} \lesssim 1. \tag{B.5.2}$$

The operators P, $\mathcal{I}$ and Φ introduced in Subsection 1.2 and their accompanying kernels $\mathscr{H}$ and $\mathscr{G}$ can be generalized to $\mathbb{T}^d$ *mutatis mutandis*. For all $n \in \mathbb{N}$, we also apply $\mathcal{I}$ to $f = (f_1, \ldots, f_n) \colon [0,T] \times \mathbb{T}^d \to \mathbb{R}^n$ by setting $\mathcal{I}[f] = (\mathcal{I}[f_1], \ldots, \mathcal{I}[f_n])$.

Lemma B.5.1. *Let $\alpha \leq \beta \in \mathbb{R}$ and $p, q, p', q' \in [1,\infty]$ be such that $p \geq p'$ and $q \geq q'$. Then for all $t > 0$,*

$$\|P_t f\|_{\mathcal{B}^{\beta}_{p,q}} \lesssim (1 \vee t^{-\frac{\beta-\alpha}{2}})(1 \vee t^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})})\|f\|_{\mathcal{B}^{\alpha}_{p',q'}}. \tag{B.5.3}$$

Secondly, if $\alpha \leq \beta \leq \alpha + 2$ then for all $t > 0$,

$$\|(P_t - 1)f\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim t^{\frac{\beta-\alpha}{2}}\|f\|_{\mathcal{B}^{\beta}_{p,q}}. \tag{B.5.4}$$

Proof. We first show (B.5.3), which is an easy consequence of the Besov embedding and the regularising effect of the heat flow. By (B.2.1) for all $\beta \in \mathbb{R}$ and $p, q, p', q' \in [1,\infty]$ with $p \geq p'$ and $q \geq q'$, it holds that,

$$\|P_t f\|_{\mathcal{B}^{\beta}_{p,q}} \lesssim \|P_t f\|_{\mathcal{B}^{\beta+d(1/p'-1/p)}_{p',q'}}.$$

We now apply the semigroup property and the regularizing effect of the heat flow ([BCD11a, Lem. 2.4] & (B.5.2)),

$$\|P_t f\|_{\mathcal{B}^{\beta+d(1/p'-1/p)}_{p',q'}} \lesssim (1 \vee t^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})})\|P_{t/2} f\|_{\mathcal{B}^{\beta}_{p',q'}} \lesssim (1 \vee t^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})})(1 \vee t^{-\frac{\beta-\alpha}{2}})\|f\|_{\mathcal{B}^{\alpha}_{p',q'}},$$

which yields (B.5.3). The bound (B.5.4) can be found in [MW17b, Prop. A.13], who cite [MW17a, Prop. 6] for a proof in the full space. We provide a short argument. We consider the Littlewood–Paley blocks $\Delta_k(P_t-1)u=(P_t-1)\Delta_k u$ for $k\in\mathbb{N}_{-1}$. Since $(P_t-1)\Delta_{-1}u=0$, we may assume $k\in\mathbb{N}$. We apply [BCD11a, Lem. 2.4 & Lem. 2.1] to obtain the existence of some $c>0$ such that

$$\begin{aligned}\|(P_t-1)\Delta_k f\|_{L^p} &\leq \int_0^t \|\partial_s P_s \Delta_k f\|_{L^p}\,\mathrm{d}s = \int_0^t \|P_s\Delta\Delta_k f\|_{L^p}\,\mathrm{d}s\\ &\lesssim \|\Delta\Delta_k f\|_{L^p}\int_0^t \mathrm{e}^{-cs2^{2k}}\,\mathrm{d}s \lesssim \|\Delta_k f\|_{L^p}2^{2k}\int_0^t \mathrm{e}^{-cs2^{2k}}\,\mathrm{d}s\\ &\lesssim \|\Delta_k f\|_{L^p}(1-\mathrm{e}^{-ct2^{2k}}) \lesssim \|\Delta_k f\|_{L^p} t^{\frac{\beta-\alpha}{2}}2^{k(\beta-\alpha)}.\end{aligned}$$

In the last inequality, we applied (B.5.1) with $(\beta-\alpha)/2\in[0,1]$. This yields the claim using the definition of the $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)$-norm. □

We can now establish Schauder estimates similar to [GIP15, Lem. A.9] and [CC18, Prop. 2.7]. For the sake of readability let us distinguish Schauder estimates in interpolation spaces (Lemma B.5.2), which are mainly applied in Chapter 2, and Schauder estimates in Lebesgue spaces (Theorem B.5.3), which are mainly applied in Chapter 3.

Lemma B.5.2. *Let $T>0$, $\eta\geq 0$, $\alpha\leq\beta\in\mathbb{R}$, $\kappa\in[0,1)$ and $p,q\in[1,\infty]$. Then the following hold*

1. *If $\frac{\beta-\alpha}{2}+\frac{d}{2p}\leq\eta$, then*

$$\|Pf\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta} \lesssim (1\vee T^{-\frac{\beta-\alpha}{2}})(1\vee T^{-\frac{d}{2p}})(1\wedge T)^\eta\|f\|_{\mathcal{B}^\alpha_{p,q}} \tag{B.5.5}$$

and $P\colon \mathcal{B}^\alpha_{p,q}(\mathbb{T}^d)\to\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta(\mathbb{T}^d)$ is a continuous map.

2. *If $\eta'\in[0,1)$, $\eta\leq\eta'$ and $\beta<\alpha+2$ are such that $\frac{\beta-\alpha}{2}\vee\kappa<1-(\eta'-\eta)$, then*

$$\|\mathcal{I}[f]\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta} \lesssim_T \|f\|_{C_{\eta';T}\mathcal{C}^\alpha} \tag{B.5.6}$$

and $\mathcal{I}\colon C_{\eta';T}\mathcal{C}^\alpha(\mathbb{T}^d)\to\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta(\mathbb{T}^d)$ is a continuous map. In particular if $T\leq 1$, then

$$\|\mathcal{I}[f]\|_{\mathscr{L}^\kappa_{\eta;T}\mathcal{C}^\beta} \lesssim (T^{1-\frac{\beta-\alpha}{2}-(\eta'-\eta)}\vee T^{1-\kappa-(\eta'-\eta)})\|f\|_{C_{\eta';T}\mathcal{C}^\alpha}.$$

3. *Furthermore,* $\mathcal{I}\colon C_T\mathcal{C}^\alpha(\mathbb{T}^d) \to \mathscr{L}_T^\kappa \mathcal{C}^{\alpha+2}(\mathbb{T}^d)$ *is a continuous map.*

Proof. The proofs of (B.5.5) and (B.5.6) are simple consequences of Lemma B.5.1, the semigroup property and the definition of the interpolation spaces, Definition B.3.1. The continuity of $t \mapsto (1\wedge t)^\eta P_t u_0$ and $t \mapsto (1 \wedge t)^\eta \mathcal{I}[f]_t$ follows from the fact that the same statement is true for smooth u_0, f, and then by taking limits along a smooth approximating sequence. $\square$

The following Schauder estimates are applied in the proof of Lemma 3.3.1.

Lemma B.5.3. *Let* $T > 0$ *and* $\gamma \in \mathbb{R}$. *Then the following hold:*

1. *The map* $P\colon \mathcal{H}^\gamma(\mathbb{T}^d) \to C_T\mathcal{H}^\gamma(\mathbb{T}^d) \cap L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^d)$ *is continuous and*

$$\|Pf\|_{C_T\mathcal{H}^\gamma \cap L_T^2\mathcal{H}^{\gamma+1}} \lesssim_T \|f\|_{\mathcal{H}^\gamma}.$$

2. *If* $\gamma' \leq \gamma < \gamma' + 2$, *then the map* $\mathcal{I}\colon C_T\mathcal{H}^{\gamma'}(\mathbb{T}^d) \to C_T\mathcal{H}^\gamma(\mathbb{T}^d)$ *is continuous and*

$$\|\mathcal{I}[f]\|_{C_T\mathcal{H}^\gamma} \lesssim (T + T^{1-\frac{\gamma-\gamma'}{2}})\|f\|_{C_T\mathcal{H}^{\gamma'}}.$$

3. *If* $\gamma' \leq \gamma \leq \gamma' + 2$, *then the map* $\mathcal{I}\colon L_T^2\mathcal{H}^{\gamma'}(\mathbb{T}^d) \to L_T^2\mathcal{H}^\gamma(\mathbb{T}^d)$ *is continuous and*

$$\|\mathcal{I}[f]\|_{L_T^2\mathcal{H}^\gamma} \lesssim (T + T^{1-\frac{\gamma-\gamma'}{2}})\|f\|_{L_T^2\mathcal{H}^{\gamma'}}.$$

Proof. To show that P and $\mathcal{I}$ are continuous maps into $C_T\mathcal{H}^\gamma(\mathbb{T}^d)$, it suffices to argue as in the proof of Lemma B.5.2. To establish their regularities in $L_T^2\mathcal{H}^{\gamma+1}(\mathbb{T}^d)$ and $L_T^2\mathcal{H}^\gamma(\mathbb{T}^d)$ respectively, one can use the Fourier transform (in space), Young's convolution inequality and interpolation. $\square$

Assume $\theta\colon \mathbb{R}^d \to \mathbb{C}$ is smooth and such that $\partial^\nu\theta$ is of at most polynomial growth for all multi-indices $\nu \in \mathbb{N}_0^d$. Additionally assume that θ satisfies the *reality condition*

$$\overline{\theta(\omega)} = \theta(-\omega), \qquad \omega \in \mathbb{Z}^d. \tag{B.5.7}$$

We define the Fourier multiplier acting on $u \in \mathcal{S}'(\mathbb{T}^d;\mathbb{R})$ by the expression

$$\theta(D)u \coloneqq \mathscr{F}^{-1}(\theta\hat{u}).$$

The polynomial-growth condition on all partial derivatives and (B.5.7) ensure that $\theta(D)$ maps real-valued distributions to real-valued distributions, a result which we generalize in the following lemma.

Lemma B.5.4. *Let $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{B}^{\alpha}_{p,q}(\mathbb{T}^d)$ and $k = 2\lfloor 1 + d/2 \rfloor$. Assume that $\theta \colon \mathbb{R}^d \to \mathbb{C}$ satisfies $\theta \in C^k(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$, $\theta(0) = 0$, the reality condition* (B.5.7) *and that there exist some $m \in \mathbb{R}$, $C > 0$, such that for each multi-index $\nu \in \mathbb{N}_0^d$ with $|\nu| \leq k$,*

$$|\partial^{\nu} \theta(x)| \leq C |x|^{m - |\nu|}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

Then,

$$\|\theta(D) u\|_{\mathcal{B}^{\alpha - m}_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha}_{p,q}}.$$

Proof. Since u is periodic the only frequency contained in the support of ϱ_{-1} is $\omega = 0$, hence $\theta(D)\Delta_{-1} u = 0$. The remaining Littlewood–Paley blocks can then be addressed directly with [BCD11a, Lem. 2.2]. □

Lemma B.5.4 leads directly to a control on solutions to Poisson's equation and their derivatives.

Lemma B.5.5. *For all $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$, it follows that*

$$\|\mathscr{G} * u\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha-2}_{p,q}}, \qquad \|\nabla \mathscr{G} * u\|_{\mathcal{B}^{\alpha}_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha-1}_{p,q}}.$$

Proof. Simply apply Lemma B.5.4 to the multipliers $\theta_1(\omega) = \frac{1}{|2\pi\omega|^2} \mathbb{1}_{\omega \neq 0}$ and $\theta_2(\omega) = \frac{2\pi \mathrm{i} \omega}{|2\pi\omega|^2} \mathbb{1}_{\omega \neq 0}$. □

B.6 A Chain Rule for Bessel Potential Spaces

We use the following chain rule in Bessel potential spaces in the proof of Lemma 3.2.5.

Lemma B.6.1. *Let $F \in C^{\infty}(\mathbb{R} \setminus \{0\})$, $\gamma \geq 0$, $c > 0$ and $u \in \mathcal{H}^{\gamma}(\mathbb{T}^d) \cap L^{\infty}(\mathbb{T}^d)$ be such that $u \geq c$. Then there exists a constant $C(\gamma, F, \|u\|_{L^{\infty}}, c^{-1}) \geq 0$ satisfying*

$$\|F(u)\|_{\mathcal{H}^{\gamma}} \leq C(\gamma, F, \|u\|_{L^{\infty}}, c^{-1}) \|u\|_{\mathcal{H}^{\gamma}}.$$

For all γ and F, the function $(x, y) \mapsto C(\gamma, F, x, y)$ can be chosen to be non-decreasing.

Proof. Since u is bounded away from 0, we may use a smooth cut-off to assume without limitation of generality that F vanishes at 0 and that F' is locally bounded. If $\gamma = 0$, this yields the square integrability of $F(u)$. If $\gamma > 0$ we may apply [BCD11a, Thm. 2.87] to deduce the claim, see also [Con+20, Lem. B.2]. $\square$

B.7 Commutator Results

Many of the results presented below are analogues and simple extensions of similar results found in [Per13; CC18] to time-weighted spaces, $C_{\eta;T}\mathcal{C}^{\alpha}(\mathbb{T}^d; \mathbb{R})$.

Lemma B.7.1. *Let $T > 0$, $\eta, \eta_1, \eta_2 \geq 0$ be such that $\eta = \eta_1 + \eta_2$, $\alpha \in (-\infty, 1)$, $\beta \in \mathbb{R}$ and $k = 2\lfloor 1 + d/2 \rfloor$. Assume $\theta \colon \mathbb{R}^d \to \mathbb{C}$ satisfies $\theta \in C^{k+1}(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$, $\theta(0) = 0$,* (B.5.7) *and that there exist some $m \in \mathbb{R}$, $C > 0$, such that for each multi-index $\nu \in \mathbb{N}_0^d$ with $|\nu| \leq k + 1$,*

$$|\partial^{\nu}\theta(x)| \leq C|x|^{m-|\nu|}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

Then for every $u \in C_{\eta_1;T}\mathcal{C}^{\alpha}(\mathbb{T}^d; \mathbb{R})$ and $v \in C_{\eta_2;T}\mathcal{C}^{\beta}(\mathbb{T}^d; \mathbb{R})$, it holds that

$$\theta(D)(u \varolessthan v) - u \varolessthan \theta(D)v \in C_{\eta;T}\mathcal{C}^{\alpha+\beta-m}(\mathbb{T}^d; \mathbb{R})$$

and

$$\|\theta(D)(u \varolessthan v) - u \varolessthan \theta(D)v\|_{C_{\eta;T}\mathcal{C}^{\alpha+\beta-m}} \lesssim \|u\|_{C_{\eta_1;T}\mathcal{C}^{\alpha}} \|v\|_{C_{\eta_2;T}\mathcal{C}^{\beta}}.$$

Proof. The result is a simple extension of [Per13, Lem. 5.3.20] and [CC18, Lem. A.1] to functions with prescribed blow-up in $\mathcal{C}^{\alpha}(\mathbb{T}^d; \mathbb{R})$ at $t = 0$. $\square$

Next, we consider the commutator between the operators $\mathcal{I}$ and $\varolessthan$, a result reminiscent of [CC18, Prop. 2.7].

Lemma B.7.2. *Let $T > 0$, $\eta, \eta_1, \eta_2 \in [0, 1)$ be such that $\eta = \eta_1 + \eta_2$, $\kappa > 0$, $\alpha \in (-\infty, (1 \wedge 2\kappa))$, $\beta \in \mathbb{R}$ and $m \in (0, 2)$. For every $u \in \mathscr{L}^{\kappa}_{\eta_1;T}\mathcal{C}^{\alpha}(\mathbb{T}^d; \mathbb{R})$ and $v \in C_{\eta_2;T}\mathcal{C}^{\beta}(\mathbb{T}^d; \mathbb{R})$, it holds that $\mathcal{I}[u \varolessthan v] - u \varolessthan \mathcal{I}[v] \in C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d; \mathbb{R})$ and*

$$\|\mathcal{I}[u \varolessthan v] - u \varolessthan \mathcal{I}[v]\|_{C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}} \lesssim_T \|u\|_{\mathscr{L}^{\kappa}_{\eta_1;T}\mathcal{C}^{\alpha}} \|v\|_{C_{\eta_2;T}\mathcal{C}^{\beta}}. \tag{B.7.1}$$

Proof. Let $u \in \mathscr{L}^{\kappa}_{\eta_1;T}\mathcal{C}^{\alpha}(\mathbb{T}^d;\mathbb{R})$ and $v \in C_{\eta_2;T}\mathcal{C}^{\beta}(\mathbb{T}^d;\mathbb{R})$, we first prove the bound (B.7.1) and then the regularity $\mathcal{I}[u \varolessthan v] - u \varolessthan \mathcal{I}[v] \in C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d;\mathbb{R})$. Let $t \in [0,T]$, by definition

$$\begin{aligned}
&\mathcal{I}[u \varolessthan v]_t - u_t \varolessthan \mathcal{I}[v]_t \\
&= \int_0^t P_{t-s}(u(s) \varolessthan v(s)) - u(t) \varolessthan P_{t-s}v(s)\,\mathrm{d}s \\
&= \int_0^t P_{t-s}(u(s) \varolessthan v(s)) - u(s) \varolessthan P_{t-s}v(s)\,\mathrm{d}s + \int_0^t (u(s)-u(t)) \varolessthan P_{t-s}v(s)\,\mathrm{d}s.
\end{aligned}$$

To bound the first summand, we apply [Per13, Lem. 5.3.20] and use that $\alpha < 1$ to estimate

$$\|P_{t-s}(u(s) \varolessthan v(s)) - u(s) \varolessthan P_{t-s}v(s)\|_{\mathcal{C}^{\alpha+\beta+m}} \lesssim |t-s|^{-m/2}(1\wedge s)^{-\eta}\|u\|_{C_{\eta_1;s}\mathcal{C}^{\alpha}}\|v\|_{C_{\eta_2;s}\mathcal{C}^{\beta}}.$$

Taking the supremum, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}(1\wedge t)^{\eta}\left\|\int_0^t P_{t-s}(u(s) \varolessthan v(s)) - u(s) \varolessthan P_{t-s}v(s)\,\mathrm{d}s\right\|_{\mathcal{C}^{\alpha+\beta+m}} \\
&\lesssim \Big(\sup_{t\in[0,T]}(1\wedge t)^{\eta}\int_0^t |t-s|^{-m/2}(1\wedge s)^{-\eta}\,\mathrm{d}s\Big)\|u\|_{C_{\eta_1;T}\mathcal{C}^{\alpha}}\|v\|_{C_{\eta_2;T}\mathcal{C}^{\beta}}.
\end{aligned}$$

We can use a case distinction over $t \in [0,1]$ and $t \in (1,\infty)$; and the assumptions $\eta < 1$, $m < 2$, to bound

$$\sup_{t\in[0,T]}(1\wedge t)^{\eta}\int_0^t |t-s|^{-m/2}(1\wedge s)^{-\eta}\,\mathrm{d}s \lesssim_T 1.$$

To bound the second summand, we apply Lemma B.5.1 and use that $\alpha - 2\kappa < 0$ to estimate

$$\begin{aligned}
&\|(u(s)-u(t)) \varolessthan P_{t-s}v(s)\|_{\mathcal{C}^{\alpha+\beta+m}} \\
&\lesssim \|u(s)-u(t)\|_{\mathcal{C}^{\alpha-2\kappa}}\|P_{t-s}v(s)\|_{\mathcal{C}^{\beta+2\kappa+m}} \\
&\lesssim |t-s|^{\kappa}(1\vee|t-s|^{-\kappa-m/2})(1\wedge s)^{-\eta}\|u\|_{C^{\kappa}_{\eta_1;t}\mathcal{C}^{\alpha-2\kappa}}\|v\|_{C_{\eta_2;s}\mathcal{C}^{\beta}}.
\end{aligned}$$

Taking the supremum, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}(1\wedge t)^{\eta}\left\|\int_0^t (u(s)-u(t)) \varolessthan P_{t-s}v(s)\,\mathrm{d}s\right\|_{\mathcal{C}^{\alpha+\beta+m}} \\
&\lesssim \Big(\sup_{t\in[0,T]}(1\wedge t)^{\eta}\int_0^t |t-s|^{\kappa}(1\vee|t-s|^{-\kappa-m/2})(1\wedge s)^{-\eta}\,\mathrm{d}s\Big)\|u\|_{C^{\kappa}_{\eta_1;T}\mathcal{C}^{\alpha-2\kappa}}\|v\|_{C_{\eta_2;T}\mathcal{C}^{\beta}}.
\end{aligned}$$

We can use a case distinction over $t \in [0,1]$, $t \in (1,2]$, and $t \in (2,\infty)$; and the assumptions $\eta < 1$, $m < 2$, to bound

$$\sup_{t\in[0,T]} (1 \wedge t)^\eta \int_0^t |t-s|^\kappa (1 \vee |t-s|^{-\kappa-m/2})(1\wedge s)^{-\eta}\, \mathrm{d}s \lesssim_T 1.$$

It follows that

$$\|\mathcal{I}[u \otimes v] - u \otimes \mathcal{I}[v]\|_{C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}} \lesssim_T \|u\|_{\mathscr{L}^\kappa_{\eta_1;T}\mathcal{C}^\alpha} \|v\|_{C_{\eta_2;T}\mathcal{C}^\beta},$$

which yields (B.7.1).

To establish $\mathcal{I}[u \otimes v] - u \otimes \mathcal{I}[v] \in C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d;\mathbb{R})$, we need to show that $(0,T] \ni t \mapsto (1\wedge t)^\eta(\mathcal{I}[u \otimes v]_t - u_t \otimes \mathcal{I}[v]_t)$ is continuous and that $\lim_{t\to 0}(1 \wedge t)^\eta(\mathcal{I}[u \otimes v]_t - u_t \otimes \mathcal{I}[v]_t)$ exists in $\mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d;\mathbb{R})$. The continuity in $(0,T]$ follows by the completeness of the continuous functions under the supremum norm and by taking limits along a smooth approximating sequence of v. To show the continuity at 0, we can use the same bounds as in the derivation of (B.7.1) to control for every $t \in (0, 1\wedge T]$,

$$(1\wedge t)^\eta \|\mathcal{I}[u \otimes v]_t - u_t \otimes \mathcal{I}[v]_t\|_{\mathcal{C}^{\alpha+\beta+m}} \lesssim t^{1-m/2},$$

which implies $\lim_{t\to 0}(1\wedge t)^\eta(\mathcal{I}[u \otimes v]_t - u_t \otimes \mathcal{I}[v]_t) = 0 \in \mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d;\mathbb{R})$. Therefore, $\mathcal{I}[u \otimes v] - u \otimes \mathcal{I}[v] \in C_{\eta;T}\mathcal{C}^{\alpha+\beta+m}(\mathbb{T}^d;\mathbb{R})$, which yields the claim. □

Finally, we present a commutator result between the operators $\otimes$ and $\odot$.

Lemma B.7.3. *Let $T > 0$, $\eta, \eta_1, \eta_2, \eta_3 \geq 0$ be such that $\eta = \eta_1 + \eta_2 + \eta_3$, $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$. We define*

$$\mathscr{C}(f,g,h) = (f \otimes g) \odot h - f(g \odot h), \qquad \textit{where}$$
$$(f,g,h) \in C_{\eta_1;T}C^\infty(\mathbb{T}^d;\mathbb{R}) \times C_{\eta_2;T}C^\infty(\mathbb{T}^d;\mathbb{R}) \times C_{\eta_3;T}C^\infty(\mathbb{T}^d;\mathbb{R}).$$

Then $\mathscr{C}$ extends to a bounded, trilinear operator

$$\mathscr{C}\colon C_{\eta_1;T}\mathcal{C}^\alpha(\mathbb{T}^d;\mathbb{R}) \times C_{\eta_2;T}\mathcal{C}^\beta(\mathbb{T}^d;\mathbb{R}) \times C_{\eta_3;T}\mathcal{C}^\gamma(\mathbb{T}^d;\mathbb{R}) \to C_{\eta;T}\mathcal{C}^{\alpha+\beta+\gamma}(\mathbb{T}^d;\mathbb{R}).$$

Proof. The result is a direct consequence of [GIP15, Lem. 2.4]. □

B.8 Blow-Up Spaces

In this section we introduce a space of continuous functions that may blow up in finite time. We follow the construction of [Cha+22, Sec. 1.5.1].

Definition B.8.1. *Let $(F, \|\cdot\|_F)$ be a normed vector space and let $\skull$ be a cemetery state. We define $F^{\skull} := F \cup \{\skull\}$ and equip it with the topology generated by the open balls of F and the sets of the form $\{u : \|u\|_F > R\} \cup \{\skull\}$ with $R > 0$.*

Next we define the space F_T^{sol} of continuous functions on $[0, T]$ with values in $F^{\skull}$ that cannot return from the cemetery state $\skull$.

Definition B.8.2. *We define the set*

$$F_T^{\mathrm{sol}} := \Big\{ f \in C([0,T]; F^{\skull}) : f|_{(T_F^{\skull}[f], T]} \equiv \skull \Big\},$$
$$\text{where} \quad T_F^{\skull}[f] := T \wedge \inf\{t \in [0,T] : f(t) = \skull\}.$$

We equip F_T^{sol} with a suitable metric.

Lemma B.8.3. *There exists a metric D_T^F on F_T^{sol} such that $D_T^F(f_n, f) \to 0$ as $n \to \infty$ if and only if for every $L > 0$,*

$$\lim_{n\to\infty} \sup_{t\in[0,T_L]} \|f_n(t) - f(t)\|_F = 0,$$

where

$$T_L := T \wedge L \wedge \inf\{t \in [0,T] : \|f_n(t)\|_F > L \text{ or } \|f(t)\|_F > L\}.$$

Proof. The existence of such a metric follows by [Cha+22, Lem. 1.2]. □

Finally, we equip F_T^{sol} with a weight at 0 to allow for irregular initial data.

Definition B.8.4. *We denote $u_{\eta\text{-wt}}(t) := (1\wedge t)^\eta u(t)$ for each $\eta \geq 0$, $u\colon (0,T] \to F^{\skull}$ and $t \in (0, T]$. We define*

$$F_{\eta;T}^{\mathrm{sol}} := \Big\{ u\colon (0,T] \to F^{\skull} : u_{\eta\text{-wt}}(0) := \lim_{t\to 0}(1\wedge t)^\eta u(t) \in F^{\skull},\ u_{\eta\text{-wt}} \in F_T^{\mathrm{sol}} \Big\}$$

and equip $F_{\eta;T}^{\mathrm{sol}}$ with the metric $D_{\eta;T}^F$ given by

$$D_{\eta;T}^F(u,v) := D_T^F(u_{\eta\text{-wt}}, v_{\eta\text{-wt}}), \qquad u, v \in F_{\eta;T}^{\mathrm{sol}}.$$

Let F be a separable, normed vector space, in the next lemma we show that F_T^{sol} is also separable.

Lemma B.8.5. *Let $T > 0$ and F be a separable, normed vector space. Then F_T^{sol} is separable.*

Proof. The space F_T^{sol} can be identified with a subspace of the metric space F^{sol} defined in [Cha+22, Sec. 1.5.1]. As discussed in [Cha+22, Sec. 1.5.1], F^{sol} is a separable metric space, hence F_T^{sol} is a separable subspace. □

In the next lemma we show that the map $T_F^{\skull} \colon F_T^{\mathrm{sol}} \to [0,T]$ is lower semi-continuous.

Lemma B.8.6. *The map $T_F^{\skull} \colon F_T^{\mathrm{sol}} \to [0,T]$ is lower semicontinuous.*

Proof. We define for every $L > 0$ and $f \in F_T^{\mathrm{sol}}$ the exceedance time $T_F^L[f] \coloneqq T \wedge L \wedge \inf\{t \in [0,T] : \|f(t)\|_F \geq L\}$. Let $S \in [0,T]$, we first show that the sublevel set

$$(T_F^L)^{-1}([0,S]) \coloneqq \{f \in F_T^{\mathrm{sol}} : T_F^L[f] \leq S\}$$

is closed in F_T^{sol}, which is equivalent to the lower semicontinuity of T_F^L.

The case $S \geq T \wedge L$ is trivial, since $(T_F^L)^{-1}([0, T \wedge L]) = F_T^{\mathrm{sol}}$, hence assume $S < T \wedge L$. Let $(f_n)_{n\in\mathbb{N}}$ be a converging sequence in $(T_F^L)^{-1}([0,S])$ with limit $f \in F_T^{\mathrm{sol}}$. Our aim is to show that $f \in (T_F^L)^{-1}([0,S])$, so let's assume for a contradiction that $T_F^L[f] > S$. The sequence of exceedance times $(T_F^L[f_n])_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0,S]$, hence by the compactness of $[0,S]$ we can extract a subsequence that converges (modulo re-labelling) to a limit $\lim_{n\to\infty} T_F^L[f_n] = S^* \in [0,S]$. Using the continuity of the norm, we obtain

$$\begin{aligned}\|f(S^*)\|_F &= \lim_{n\to\infty} \|f(T_F^L[f_n])\|_F \\ &\geq \liminf_{n\to\infty} \|f_n(T_F^L[f_n])\|_F - \limsup_{n\to\infty} \|f(T_F^L[f_n]) - f_n(T_F^L[f_n])\|_F = L,\end{aligned}$$

where we used $f_n \in (T_F^L)^{-1}([0,S])$ and $S < T \wedge L$ to deduce $\|f_n(T_F^L[f_n])\|_F = L$ and the definition of the metric D_T^F on F_T^{sol} (cf. Lemma B.8.3) to ensure that the second term vanishes.

Therefore, $\|f(S^*)\|_F \geq L$ and by the definition of $T_F^L[f]$, we obtain $T_F^L[f] \leq S^* \leq S$. It follows that the limit $f \in F_T^{\mathrm{sol}}$ satisfies $T_F^L[f] \leq S$, hence $f \in (T_F^L)^{-1}([0,S])$, which proves the lower semicontinuity of $T_F^L \colon F_T^{\mathrm{sol}} \to [0,T]$.

To deduce the lower semicontinuity of the blow-up time $T_F^{\text{☠}} \colon F_T^{\mathrm{sol}} \to [0,T]$, we use that $T_F^{\text{☠}}[f] = \sup_{L>0} T_F^L[f]$ and the general fact that lower semicontinuity is preserved under taking suprema. □

B.9 Space-Time Hölder Distributions

In this section we define the space of periodic space-time Hölder distributions (Definition B.9.2; see also Definition B.9.4 for the analogous notion on the torus), which we use in Section C.1 to quantify the regularity of space-time white noise (Theorem C.1.2).

Denote by $\mathfrak{s} = (\mathfrak{s}_i)_{i=0}^d = (2,1,\ldots,1)$ the parabolic scaling of $\mathbb{R} \times \mathbb{R}^d$ with norm $|\mathfrak{s}| = \sum_{i=0}^d \mathfrak{s}_i = 2 + d$.

Definition B.9.1 ([Hai14, (2.10) & (2.13)]). *Let $z = (t,x) \in \mathbb{R} \times \mathbb{R}^d$ and $z' = (t',x') \in \mathbb{R} \times \mathbb{R}^d$, we define the parabolic distance*

$$d_{\mathfrak{s}}(z,z') \coloneqq |t-t'|^{1/\mathfrak{s}_0} + \sum_{i=1}^d |x_i - x_i'|^{1/\mathfrak{s}_i}$$

and denote the parabolic unit ball by $B_{\mathfrak{s}}(0,1) \coloneqq \{z \in \mathbb{R} \times \mathbb{R}^d : |z|_{\mathfrak{s}} < 1\}$, where

$$|z|_{\mathfrak{s}} \coloneqq |t|^{1/\mathfrak{s}_0} + \sum_{i=1}^d |x_i|^{1/\mathfrak{s}_i}$$

(by a slight abuse of notation, since $|\cdot|_{\mathfrak{s}}$ is not homogeneous, hence fails to be a norm.) We define the scaling of a function by $\lambda \in (0,1]$ around the point z via

$$\mathcal{S}_{\mathfrak{s},z}^\lambda f(s,y) \coloneqq \lambda^{-|\mathfrak{s}|} f(\lambda^{-\mathfrak{s}_0}(s-t), \lambda^{-\mathfrak{s}_1}(y_1-x_1), \ldots, \lambda^{-\mathfrak{s}_d}(y_d-x_d)), \qquad f \colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}.$$

We can now define the space $\mathcal{C}_{\mathfrak{s},\mathrm{per}}^\alpha([0,T]\times\mathbb{R}^d)$ of periodic distributions of Hölder regularity $\alpha < 0$.

Definition B.9.2. *Let $\alpha < 0$ and $r_\alpha \coloneqq \lceil -\alpha \rceil$, we define for each compact $\mathfrak{K} \subset \mathbb{R} \times \mathbb{R}^d$ the seminorm*

$$\|u\|_{\mathcal{C}^\alpha_\mathfrak{s}(\mathfrak{K})} \coloneqq \sup_{\lambda \in (0,1]} \sup_{\eta \in \mathcal{B}^{r_\alpha}_\mathfrak{s}} \sup_{z \in \mathfrak{K}} \frac{|\langle u, \mathcal{S}^\lambda_{\mathfrak{s},z} \eta \rangle|}{\lambda^\alpha}, \qquad u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d).$$

where

$$\mathcal{B}^{r_\alpha}_\mathfrak{s} \coloneqq \{\eta \in C^{r_\alpha}(\mathbb{R} \times \mathbb{R}^d) : \operatorname{supp}(\eta) \subset B_\mathfrak{s}(0,1),\ \|\eta\|_{C^{r_\alpha}(\mathbb{R} \times \mathbb{R}^d)} \leq 1\},$$

and

$$\|\eta\|_{C^{r_\alpha}(\mathbb{R} \times \mathbb{R}^d)} \coloneqq \sup_{\substack{\nu \in \mathbb{N}_0^{1+d} \\ |\nu| \leq r_\alpha}} \sup_{z \in \mathbb{R} \times \mathbb{R}^d} |\partial^\nu \eta(z)|.$$

We define $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d) \subset \mathcal{S}'_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^d)$ as the completion of $C^\infty([0,T] \times [0,1]^d)$ under the norm $\| \cdot \|_{\mathcal{C}^\alpha_\mathfrak{s}([0,T] \times [0,1]^d)}$, where we extend each element of $C^\infty([0,T] \times [0,1]^d)$ to $\mathbb{R} \times \mathbb{R}^d$ periodically in space and by 0 in time.

Lemma B.9.3. *For every $\alpha < 0$ it follows that the normed vector space $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)$ is complete and separable.*

Proof. Completeness is clear since $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)$ is defined as a completion. To prove separability it suffices to show the existence of a countable subset of $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)$ which can approximate every $u \in C^\infty([0,T] \times [0,1]^d)$ (extended to $\mathbb{R} \times \mathbb{R}^d$ periodically in space and by 0 in time) under the norm $\| \cdot \|_{\mathcal{C}^\alpha_\mathfrak{s}([0,T] \times [0,1]^d)}$.

An application of the Stone–Weierstrass theorem shows that for each $\varepsilon > 0$ there exists a polynomial $p \colon [0,T] \times [0,1]^d \to \mathbb{R}$ such that $\|u - p\|_{L^\infty([0,T] \times [0,1]^d)} \leq \varepsilon$. We extend this polynomial to $\mathbb{R} \times \mathbb{R}^d$ periodically in space and by 0 in time; in particular it still holds that $\|u - p\|_{L^\infty(\overline{\mathfrak{K}}_1)} \leq \varepsilon$, where $\mathfrak{K} = [0,T] \times [0,1]^d$ and $\overline{\mathfrak{K}}_1$ denotes the 1-fattening of $\mathfrak{K}$, that is, the closure of $\mathfrak{K} + B_\mathfrak{s}(0,1)$. We can now control the $\mathcal{C}^\alpha_\mathfrak{s}(\mathfrak{K})$-norm of $u - p$ by its $L^\infty(\overline{\mathfrak{K}}_1)$-norm, since for every $\lambda \in (0,1]$, $\eta \in \mathcal{B}^{r_\alpha}_\mathfrak{s}$ and $z \in \mathfrak{K}$,

$$|\langle u - p, \mathcal{S}^\lambda_{\mathfrak{s},z} \eta \rangle| \leq \|u - p\|_{L^\infty(\overline{\mathfrak{K}}_1)} \|\eta\|_{L^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})} \lesssim \|u - p\|_{L^\infty(\overline{\mathfrak{K}}_1)} \leq \varepsilon,$$

which implies

$$\|u - p\|_{\mathcal{C}^\alpha_\mathfrak{s}(\mathfrak{K})} = \sup_{\lambda \in (0,1]} \sup_{\eta \in \mathcal{B}^{r_\alpha}_\mathfrak{s}} \sup_{z \in \mathfrak{K}} \frac{|\langle u - p, \mathcal{S}^\lambda_{\mathfrak{s},z} \eta \rangle|}{\lambda^\alpha} \lesssim \varepsilon,$$

showing that the (suitably extended) polynomials are dense in $C^\infty([0,T] \times [0,1]^d)$ under the norm $\|\cdot\|_{\mathcal{C}^\alpha_{\mathfrak{s}}(\mathfrak{K})}$. A similar argument then implies that the (suitably extended) polynomials with rational coefficients are dense as well, which yields the separability of $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)$. $\square$

We define the space $\mathcal{C}^\alpha_{\mathfrak{s}}([0,T] \times \mathbb{T}^d)$ as those distributions in $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$, whose periodization lies in $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)$.

Definition B.9.4. *Let $\Gamma\colon \mathcal{S}'(\mathbb{R} \times \mathbb{T}^d) \to \mathcal{S}'_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^d)$ be a linear homeomorphism (which exists by Lemma B.9.6 below.) We define for every $\alpha < 0$,*

$$\mathcal{C}^\alpha_{\mathfrak{s}}([0,T] \times \mathbb{T}^d) := \{u \in \mathcal{S}'(\mathbb{R} \times \mathbb{T}^d) : \Gamma(u) \in \mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T] \times \mathbb{R}^d)\}$$

and equip $\mathcal{C}^\alpha_{\mathfrak{s}}([0,T] \times \mathbb{T}^d)$ with the natural norm that turns Γ into a linear isometric isomorphism.

Lemma B.9.5. *For every $\alpha < 0$ it follows that $\mathcal{C}^\alpha_{\mathfrak{s}}([0,T] \times \mathbb{T}^d)$ is a separable Banach space.*

Proof. Completeness and separability are preserved under isometric isomorphisms; hence the claim follows by Lemma B.9.3. $\square$

The following lemma is classical, here we follow the presentation contained in the unpublished notes [vZui22].

Lemma B.9.6 ([vZui22, Lem. 31.3])**.** *Let $\Psi\colon \mathcal{S}(\mathbb{R}^d) \to C^\infty_{\mathrm{per}}(\mathbb{R}^d)$ be given by*

$$\Psi(\phi) := \sum_{z \in \mathbb{Z}^d} \mathcal{T}_z \phi, \qquad \mathcal{T}_z \phi(x) := \phi(x - z)$$

and $\gamma\colon C^\infty_{\mathrm{per}}(\mathbb{R}^d) \to C^\infty(\mathbb{T}^d)$ be the linear homeomorphism

$$\gamma(\phi)([x]) := \phi(x), \qquad [x] := x + \mathbb{Z}^d \in \mathbb{T}^d.$$

Then $\Gamma\colon \mathcal{S}'(\mathbb{T}^d) \to \mathcal{S}'_{\mathrm{per}}(\mathbb{R}^d)$ given by

$$\langle \Gamma(u), \phi \rangle = \langle u, \gamma(\Psi(\phi)) \rangle$$

is a linear homeomorphism with inverse

$$\Gamma^{-1} \colon \mathcal{S}'_{\mathrm{per}}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{T}^d), \qquad \langle \Gamma^{-1}(u), \phi \rangle = \langle u, \chi \gamma^{-1}(\phi) \rangle,$$

where $\chi \in \mathcal{S}(\mathbb{R}^d)$ *is such that* $(\mathcal{T}_z \chi)_{z \in \mathbb{Z}^d}$ *forms a partition of unity (which exists by [vZui22, Lem. 29.3].) The above extends to* $\mathcal{S}'_{\mathrm{per}}(\mathbb{R} \times \mathbb{R}^d)$ *and* $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^d)$ mutatis mutandis*.*

C
Space-Time White Noise

Contents

In this appendix we show the existence and regularity of space-time white noise (Theorem C.1.2), construct an abstract Wiener space (Theorem C.2.1) and consider iterated Itô integrals (Section C.3).

C.1 Existence and Regularity

We first define space-time white noise as a family of Gaussian random variables.

Definition C.1.1 ([Hai14, Sec. 10.1]). *We define a space-time white noise on $[0,T]\times\mathbb{T}^2$ to be a family of real-valued, centred Gaussian random variables $\{\xi(\phi) : \phi \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R})\}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $\mathbb{E}[\xi(\phi)\xi(\psi)] = \langle \phi,\psi\rangle_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R})}$ for every $\phi,\psi \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R})$.*

By an application of Kolmogorov's extension theorem, there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a family as in Definition C.1.1. Indeed, following [Jan97, Ex. 1.27], we may take Ω to be the algebraic dual to $L^2([0,T]\times\mathbb{T}^2)$, define for all $h\in L^2([0,T]\times\mathbb{T}^2)$ the functions $\xi_h\colon\Omega\to\mathbb{R}$, $\omega\mapsto\xi_h(\omega):=\omega(h)$, and set $\mathcal{F}=\sigma(\xi_h : h\in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}))$. Then there exists a unique probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ such that each ξ_h is centred Gaussian with variance $\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R})}$.

Next we show that there exists a modification of this family that is a random distribution in $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^2;\mathbb{R})$ and that this modification lies in the space $\mathcal{C}^\alpha_{\mathfrak{s}}([0,T]\times\mathbb{T}^2;\mathbb{R})$ (see Definition B.9.4).

Theorem C.1.2. *A space-time white noise on $[0,T]\times\mathbb{T}^2$ as defined in Definition C.1.1 exists as a random distribution in $\mathcal{S}'(\mathbb{R}\times\mathbb{T}^2;\mathbb{R})$ and is almost surely an element of $\mathcal{C}^\alpha_{\mathfrak{s}}([0,T]\times\mathbb{T}^2;\mathbb{R})$ for each $\alpha<-2$.*

Proof. Following [Hai14, below Rem. 10.1], let us consider the periodically-extended noise

$$\mathcal{S}(\mathbb{R}\times\mathbb{R}^2;\mathbb{R})\ni\phi\mapsto\Gamma(\xi\mathbb{1}_{[0,T]})(\phi),$$

where Γ denotes the homeomorphism constructed in Lemma B.9.6. For each $\alpha<-2$ it follows by [Hai14, Lem. 10.2] that $\Gamma(\xi\mathbb{1}_{[0,T]})$ is almost surely a random variable in $\mathcal{C}^\alpha_{\mathfrak{s},\mathrm{per}}([0,T]\times\mathbb{R}^2;\mathbb{R})$. In particular, by Definition B.9.4 we obtain almost surely $\xi\mathbb{1}_{[0,T]}\in\mathcal{C}^\alpha_{\mathfrak{s}}([0,T]\times\mathbb{T}^2;\mathbb{R})$. $\square$

Given two independent space-time white noises ξ^1 and ξ^2 as in Definition C.1.1 on the same probability space, we can define a vector-valued space-time white noise $\boldsymbol{\xi}=(\xi^1,\xi^2)$ by

$$\boldsymbol{\xi}(\phi):=(\xi^1(\phi),\xi^2(\phi)),\qquad\phi\in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}).\tag{C.1.1}$$

With this definition it is clear how to pass from scalar-valued to vector-valued noise. In the remainder of this appendix, for the sake of notation, we only consider scalar-valued noise and omit the target space from our definitions.

C.2 Abstract Wiener Space

In the following we do not distinguish between $L^2([0,T]\times\mathbb{T}^2)$ and the functions in $L^2(\mathbb{R}\times\mathbb{T}^2)$ that are zero on the complement of $[0,T]$.

Theorem C.2.1. *For every $\alpha < -2$ it follows that the law of the space-time white noise on $[0,T]\times\mathbb{T}^2$ can be realized on the abstract Wiener space $(B,\mathcal{H}_\mu,\mu)$ with separable Banach space $B \coloneqq \mathcal{C}^\alpha_{\mathfrak{s}}([0,T]\times\mathbb{T}^2)$, Cameron–Martin space $\mathcal{H}_\mu = L^2([0,T]\times\mathbb{T}^2)$ and Gaussian probability measure $\mu = \mathrm{Law}(\xi)$.*

Proof. It follows by Theorem C.1.2 that μ is a Borel probability measure on B. Hence to prove that μ is a Gaussian probability measure in the sense of [Led96, Sec. 4], we need to show that the law of each continuous linear functional on B is Gaussian.

It follows by the Cauchy–Schwarz inequality and the assumption $\alpha < -2$, that $L^2([0,T]\times\mathbb{T}^2) \subset \mathcal{C}^\alpha_{\mathfrak{s}}([0,T]\times\mathbb{T}^2) = B$, which implies by standard arguments $B' \subset L^2([0,T]\times\mathbb{T}^2)$. Consequently we obtain by Definition C.1.1 that for each $l \in B'$, $l_\#\mu$ is a centred Gaussian measure on $\mathbb{R}$, which implies that μ is a Borel probability measure on B.

It remains to identify the Cameron–Martin space $\mathcal{H}_\mu$. To do so, we first consider the reproducing kernel Hilbert space $\mathcal{R}_\mu$, i.e. the closure of B' in $L^2(B,\mu)$, which is isomorphic to $\mathcal{H}_\mu$. Using a duality argument, we obtain $C^\infty_{\mathrm{c}}((0,T)\times\mathbb{T}^2) \subset B'$ (where we extend each element of $C^\infty_{\mathrm{c}}((0,T)\times\mathbb{T}^2)$ by 0 to $\mathbb{R}\times\mathbb{T}^2$). The closure of B' in $L^2(B,\mu)$ coincides with the closure of B' in $L^2([0,T]\times\mathbb{T}^2)$ since for all $l \in B' \subset L^2([0,T]\times\mathbb{T}^2)$ it holds that $\|l\|_{L^2(B,\mu)} = \|l\|_{L^2([0,T]\times\mathbb{T}^2)}$. Hence, using that $C^\infty_{\mathrm{c}}((0,T)\times\mathbb{T}^2) \subset B'$, we obtain $\mathcal{R}_\mu = L^2([0,T]\times\mathbb{T}^2)$.

By [Led96, Sec. 4] there exists a linear isometric isomorphism

$$\iota\colon \mathcal{R}_\mu \to \mathcal{H}_\mu, \qquad \iota(l) \coloneqq \int_B x l(x)\mu(\mathrm{d}x),$$

so that it suffices to identify each element $\iota(l) \in \mathcal{H}_\mu$ with an element of $L^2([0,T]\times\mathbb{T}^2)$. Let $\phi \in \mathcal{S}(\mathbb{R}\times\mathbb{T}^2)$, we obtain

$$\langle \iota(l), \phi\rangle = \int_B \phi(x) l(x)\mu(\mathrm{d}x) = \langle l, \phi\rangle_{L^2([0,T]\times\mathbb{T}^2)},$$

which shows that $\iota(l)$ coincides with the tempered distribution

$$u_l \in \mathcal{S}'(\mathbb{R}\times\mathbb{T}^2), \qquad \mathcal{S}(\mathbb{R}\times\mathbb{T}^2) \ni \phi \mapsto u_l(\phi) = \int_{[0,T]\times\mathbb{T}^2} l(z)\phi(z)\,\mathrm{d}z.$$

Using that the map $L^2(\mathbb{R}\times\mathbb{T}^2) \ni l \mapsto u_l \in \mathcal{S}'(\mathbb{R}\times\mathbb{T}^2)$ is an embedding, we can identify $\iota(l)$ and l, which yields $\mathcal{H}_\mu = L^2([0,T]\times\mathbb{T}^2)$. $\square$

C.3 Iterated Itô Integrals

In this section we define the notion of an iterated Itô integral (Definition C.3.2), relate it to our space-time white noise (Lemma C.3.1), introduce Itô's isometry and Nelson's estimate (Lemma C.3.3 & C.3.8), show that each integral yields an element of a homogenous Wiener chaos (Lemma C.3.9) and characterize its homogeneous part (Lemma C.3.10). We further define an iterated Itô integral with mollification and heterogeneity (Definition C.3.4), discuss in Lemma C.3.5 how those can be controlled by passing to real space and then introduce an interpolation argument to show that mollified iterated Itô integrals converge to their un-mollified counterparts (Lemma C.3.6). See also [MWX17; GP17] for different instances of the same arguments.

Lemma C.3.1. *Let $\boldsymbol{\xi}$ be a vector-valued space-time white noise on $[0,T]\times\mathbb{T}^2$ as in Section C.1. Then there exists a family of complex-valued Brownian motions $(W^j(\,\cdot\,,\omega))_{\omega\in\mathbb{Z}^2, j=1,2}$ (on the same underlying probability space) that satisfy $\overline{W^j(\,\cdot\,,\omega)} = W^j(\,\cdot\,,-\omega)$ but are otherwise independent.*

Proof. Let $\boldsymbol{\xi}$ be a vector-valued space-time white noise on $[0,T]\times\mathbb{T}^2$ as in Section C.1. Let $t\in[0,T]$, $\omega\in\mathbb{Z}^2$, $j=1,2$ and define

$$B^j(t,\omega) \coloneqq \boldsymbol{\xi}(\mathbb{1}_{[0,t]}(\,\cdot\,)\otimes\cos(2\pi\langle\omega,\,\cdot\,\rangle)\otimes\delta_{j,\,\cdot\,}),$$
$$\tilde{B}^j(t,\omega) \coloneqq \boldsymbol{\xi}(\mathbb{1}_{[0,t]}(\,\cdot\,)\otimes -\sin(2\pi\langle\omega,\,\cdot\,\rangle)\otimes\delta_{j,\,\cdot\,}),$$

which yields a family of Brownian motions such that $\mathbb{E}[(B^j(t,\omega))^2] = t/2$, $\mathbb{E}[(\tilde{B}^j(t,\omega))^2] = t/2$, $B^j(t,\omega) = B^j(t,-\omega)$, $\tilde{B}^j(t,\omega) = -\tilde{B}^j(t,-\omega)$ and which are otherwise independent. Using this family, we can define the complex Brownian motion $W^j(t,\omega) \coloneqq B^j(t,\omega) + \mathrm{i}\tilde{B}^j(t,\omega)$. $\square$

Let $n \in \mathbb{N}$, $D \subset (0,\infty)^n$ and $\phi \in L^2(D \times \mathbb{T}^{2n}; \mathbb{C}^{2n})$. We define the spatial Fourier transform of ϕ by

$$\begin{aligned}&\widehat{\phi}(u_1, \omega_1, j_1, \ldots, u_n, \omega_n, j_n)\\ &\coloneqq \int_{(\mathbb{T}^2)^n} \mathrm{e}^{-2\pi \mathrm{i}(\langle \omega_1, x_1\rangle + \ldots + \langle \omega_n, x_n\rangle)} \phi(u_1, x_1, j_1, \ldots, u_n, x_n, j_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n.\end{aligned}$$

We can now define iterated Itô integrals.

Definition C.3.2 (Iterated Itô integral). *Let $n \in \mathbb{N}$ and*

$$(0,\infty)^n_> \coloneqq \{(u_1, \ldots, u_n) \in (0,\infty)^n : u_1 > u_2 > \ldots > u_n\}.$$

We define the iterated Itô integral acting on $\phi \in L^2((0,\infty)^n_> \times \mathbb{T}^{2n}; \mathbb{C}^{2n})$ by

$$\begin{aligned}I^n(\phi) \coloneqq \sum_{\omega_1, \ldots, \omega_n \in \mathbb{Z}^2} \sum_{j_1, \ldots, j_n = 1}^{2} \int_0^\infty \mathrm{d}W^{j_1}(u_1, \omega_1) \ldots \int_0^{u_{n-1}} \mathrm{d}W^{j_n}(u_n, \omega_n)&\\ \widehat{\phi}(u_1, -\omega_1, j_1, \ldots, u_n, -\omega_n, j_n).&\end{aligned}$$

For $n = 0$, it is conventional to set $I^0 \coloneqq \mathrm{id}_{\mathbb{C}}$.

We can identify second moments of iterated Itô integrals by a version of Itô's isometry.

Lemma C.3.3 (Itô's isometry). *Let $n \in \mathbb{N}$ and $\phi \in L^2((0,\infty)^n_> \times \mathbb{T}^{2n}; \mathbb{C}^{2n})$, then*

$$\begin{aligned}\mathbb{E}[|I^n(\phi)|^2] &= \sum_{\omega_1, \ldots, \omega_n \in \mathbb{Z}^2} \sum_{j_1, \ldots, j_n = 1}^{2} \int_0^\infty \mathrm{d}u_1 \ldots \int_0^{u_{n-1}} \mathrm{d}u_n |\widehat{\phi}(u_1, -\omega_1, j_1, \ldots, u_n, -\omega_n, j_n)|^2\\ &= \|\phi\|^2_{L^2((0,\infty)^n_> \times \mathbb{T}^{2n} \times \{1,2\}^n; \mathbb{C})}.\end{aligned}$$

Proof. The claim follows by Itô's isometry, applied inductively to each stochastic integral. □

Let $T > 0$ be a time horizon. Next we define iterated Itô integrals with mollification (1.2.6)–(1.2.7) and heterogeneity $\sigma\colon [0,T] \times \mathbb{T}^2 \to \mathbb{R}$ (Definition C.3.4) and show how one may control the influence of the heterogeneity by passing to real space (Lemma C.3.5).

Definition C.3.4. *Let $n \in \mathbb{N}$, $T > 0$ and*

$$(0,T)^n_> \coloneqq \{(u_1, \ldots, u_n) \in (0,T)^n : u_1 > u_2 > \ldots > u_n\}$$

For every $\sigma \in C_T L^\infty(\mathbb{T}^2)$, $\delta \geq 0$ and φ as in (1.2.6) *we define the heterogeneous iterated Itô integral acting on $\phi \in L^2((0,T)^n_> \times \mathbb{T}^{2n}; \mathbb{C}^{2n})$ (with mollification if $\delta > 0$) by*

$$\begin{aligned} I^n_{\delta;\sigma}(\phi) \coloneqq \sum_{\omega_1,\dots,\omega_n \in \mathbb{Z}^2} \sum_{j_1,\dots,j_n=1}^{2} \sum_{m_1,\dots,m_n \in \mathbb{Z}^2} &\varphi(\delta m_1) \dots \varphi(\delta m_n) \\ \times \int_0^T \mathrm{d}W^{j_1}(u_1, m_1) \int_0^{u_1} \mathrm{d}W^{j_2}(u_2, m_2) \dots \int_0^{u_{n-1}} \mathrm{d}W^{j_n}(u_n, m_n) \\ \times \widehat{\sigma}(u_1, \omega_1 - m_1) \dots \widehat{\sigma}(u_n, \omega_n - m_n) \widehat{\phi}(u_1, -\omega_1, j_1, \dots, u_n, -\omega_n, j_n). \end{aligned}$$

Next we show how one may control the heterogeneity.

Lemma C.3.5. *Let $n \in \mathbb{N}$, $T > 0$, $\sigma \in C_T L^\infty(\mathbb{T}^2)$, $\delta \geq 0$ and φ be as in* (1.2.6)*. If $\delta = 0$, then*

$$\mathbb{E}[|I^n_{0;\sigma}(\phi)|^2] \leq \|\sigma\|^{2n}_{C_T L^\infty} \mathbb{E}[|I^n(\phi)|^2]. \tag{C.3.1}$$

If $\delta > 0$, then

$$\mathbb{E}[|I^n_{\delta;\sigma}(\phi)|^2] \leq \|\varphi\|^{2n}_{L^\infty} \|\sigma\|^{2n}_{C_T L^\infty} \mathbb{E}[|I^n(\phi)|^2]. \tag{C.3.2}$$

Proof. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$, $\boldsymbol{m} = (m_1, \dots, m_n)$, $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{j} = (j_1, \dots, j_n)$. We represent

$$\begin{aligned} &\widehat{\sigma}(u_1, \omega_1 - m_1) \dots \widehat{\sigma}(u_n, \omega_n - m_n) \widehat{\phi}(u_1, -\omega_1, j_1, \dots, u_n, -\omega_n, j_n) \\ &= \widehat{\sigma^{\otimes n}}(\boldsymbol{u}, \boldsymbol{\omega} - \boldsymbol{m}) \widehat{\phi}(\boldsymbol{u}, -\boldsymbol{\omega}, \boldsymbol{j}). \end{aligned}$$

Using that the Fourier transform turns products of $L^2(\mathbb{T}^{2n})$-functions into convolutions, we obtain

$$\sum_{\boldsymbol{\omega} \in (\mathbb{Z}^2)^{\times n}} \widehat{\sigma^{\otimes n}}(\boldsymbol{u}, \boldsymbol{\omega} - \boldsymbol{m}) \widehat{\phi}(\boldsymbol{u}, -\boldsymbol{\omega}, \boldsymbol{j}) = \mathscr{F}(\sigma^{\otimes n} \phi)(\boldsymbol{u}, -\boldsymbol{m}, \boldsymbol{j})$$

and consequently, $I^n_{\delta;\sigma}(\phi) = I^n(\psi_\delta^{\otimes n} * (\sigma^{\otimes n} \phi))$ for $\delta > 0$ and $I^n_{0;\sigma}(\phi) = I^n(\sigma^{\otimes n} \phi)$ for $\delta = 0$. To establish (C.3.2) for $\delta > 0$, we apply Itô's isometry (Lemma C.3.3), Parseval's theorem and use the boundedness of φ and σ to control

$$\begin{aligned}
&\mathbb{E}[|I^n_{\delta;\sigma}(\phi)|^2] \\
&= \sum_{\boldsymbol{j}\in\{1,2\}^{\times n}} \sum_{\boldsymbol{m}\in(\mathbb{Z}^2)^{\times n}} \int_0^T \mathrm{d}u_1 \dots \int_0^{u_{n-1}} \mathrm{d}u_n |\varphi^{\otimes n}(\delta\boldsymbol{m})|^2 |\mathscr{F}(\sigma^{\otimes n}\phi)(\boldsymbol{u},-\boldsymbol{m},\boldsymbol{j})|^2 \\
&\leq \|\varphi\|^{2n}_{L^\infty} \sum_{\boldsymbol{j}\in\{1,2\}^{\times n}} \sum_{\boldsymbol{m}\in(\mathbb{Z}^2)^{\times n}} \int_0^T \mathrm{d}u_1 \dots \int_0^{u_{n-1}} \mathrm{d}u_n |\mathscr{F}(\sigma^{\otimes n}\phi)(\boldsymbol{u},-\boldsymbol{m},\boldsymbol{j})|^2 \\
&= \|\varphi\|^{2n}_{L^\infty} \sum_{\boldsymbol{j}\in\{1,2\}^{\times n}} \int_0^T \mathrm{d}u_1 \dots \int_0^{u_{n-1}} \mathrm{d}u_n \int_{(\mathbb{T}^2)^n} \mathrm{d}\boldsymbol{x} |\sigma^{\otimes n}(\boldsymbol{u},\boldsymbol{x})|^2 |\phi(\boldsymbol{u},\boldsymbol{x},\boldsymbol{j})|^2 \\
&\leq \|\varphi\|^{2n}_{L^\infty} \|\sigma\|^{2n}_{C_T L^\infty} \sum_{\boldsymbol{j}\in\{1,2\}^{\times n}} \int_0^T \mathrm{d}u_1 \dots \int_0^{u_{n-1}} \mathrm{d}u_n \int_{(\mathbb{T}^2)^n} \mathrm{d}\boldsymbol{x} |\phi(\boldsymbol{u},\boldsymbol{x},\boldsymbol{j})|^2 \\
&= \|\varphi\|^{2n}_{L^\infty} \|\sigma\|^{2n}_{C_T L^\infty} \mathbb{E}[|I^n(\phi)|^2],
\end{aligned}$$

which yields (C.3.2). The derivation of (C.3.1) for $\delta = 0$ is analogous. □

Next we consider the second moment of the difference $I^n_{0;\sigma}(\phi) - I^n_{\delta;\sigma}(\phi)$, where we combine the idea of Lemma C.3.5 with an interpolation argument to extract a small power of δ. The resulting bound (C.3.3) is reminiscent of [GP17, Lem. 8.7] in that we also need to pass to a space of lower regularity.

Lemma C.3.6. *Let $n \in \mathbb{N}$, $T > 0$, $\sigma \in C_T\mathcal{H}^1(\mathbb{T}^2)$, $\delta \geq 0$ and φ be as in* (1.2.6). *Then for every $0 < \vartheta' < \vartheta \leq 1$ and $\phi \in L^2((0,T)^n_>; \mathcal{H}^\vartheta(\mathbb{T}^{2n}; \mathbb{C}^{2n}))$,*

$$\mathbb{E}[|I^n_{0;\sigma}(\phi) - I^n_{\delta;\sigma}(\phi)|^2] \lesssim \delta^{2\vartheta'} \|1-\varphi\|^2_{C^1} (1 + \|\varphi\|^{n-1}_{L^\infty})^2 \|\sigma\|^{2n}_{C_T\mathcal{H}^1} \mathbb{E}[|I^n((1-\Delta)^{\vartheta/2}\phi)|^2]. \tag{C.3.3}$$

Proof. We use induction and interpolation to bound for every $\vartheta' \in (0,1)$,

$$|1 - \varphi^{\otimes n}(\delta\boldsymbol{m})| \leq \delta^{\vartheta'} \|1-\varphi\|_{C^1} (1 + \|\varphi\|^{n-1}_{L^\infty}) n (1 + |\boldsymbol{m}|^2)^{\vartheta'/2},$$

which combined with Itô's isometry and the definition of $I^n_{0;\sigma}(\phi) - I^n_{\delta;\sigma}(\phi)$ yields

$$\begin{aligned}
&\mathbb{E}[|I^n_{0;\sigma}(\phi) - I^n_{\delta;\sigma}(\phi)|^2] \\
&\lesssim \delta^{2\vartheta'} \|1-\varphi\|^2_{C^1} (1 + \|\varphi\|^{n-1}_{L^\infty})^2 \\
&\quad \times \sum_{\boldsymbol{j}\in\{1,2\}^{\times n}} \sum_{\boldsymbol{m}\in(\mathbb{Z}^2)^{\times n}} \int_0^T \mathrm{d}u_1 \dots \int_0^{u_{n-1}} \mathrm{d}u_n (1 + |2\pi\boldsymbol{m}|^2)^{\vartheta'} |\mathscr{F}(\sigma^{\otimes n}\phi)(\boldsymbol{u},-\boldsymbol{m},\boldsymbol{j})|^2.
\end{aligned} \tag{C.3.4}$$

Let $0 < \vartheta' < \vartheta \le 1$, to control the integrand in (C.3.4), we use standard product estimates in Bessel potential spaces to obtain

$$\sum_{\boldsymbol{m}\in(\mathbb{Z}^2)^{\times n}} (1+|2\pi\boldsymbol{m}|^2)^{\vartheta'}|\mathscr{F}(\sigma^{\otimes n}\phi)(\boldsymbol{u},-\boldsymbol{m},\boldsymbol{j})|^2 = \|(\sigma^{\otimes n}\phi)(\boldsymbol{u},\,\cdot\,,\boldsymbol{j})\|^2_{\mathcal{H}^{\vartheta'}(\mathbb{T}^{2n})}$$
$$\lesssim \|\sigma\|^{2n}_{C_T\mathcal{H}^1(\mathbb{T}^2)}\|\phi(\boldsymbol{u},\,\cdot\,,\boldsymbol{j})\|^2_{\mathcal{H}^{\vartheta}(\mathbb{T}^{2n})}. \tag{C.3.5}$$

Plugging (C.3.5) into (C.3.4), we arrive at

$$\mathbb{E}[|I^n_{0;\sigma}(\phi) - I^n_{\delta;\sigma}(\phi)|^2] \lesssim \delta^{2\vartheta'}\|1-\varphi\|^2_{C^1}(1+\|\varphi\|^{n-1}_{L^\infty})^2\|\sigma\|^{2n}_{C_T\mathcal{H}^1}\mathbb{E}[|I^n((1-\Delta)^{\vartheta/2}\phi)|^2],$$

which yields (C.3.3). □

Remark C.3.7. In (C.3.3), to extract an exponent ϑ' of δ^2, we had to assume that $\sigma \in C_T\mathcal{H}^1(\mathbb{T}^2)$ and $\phi \in L^2((0,T)^n_>;\mathcal{H}^\vartheta(\mathbb{T}^{2n};\mathbb{C}^{2n}))$ for some $0 < \vartheta' < \vartheta \le 1$; in other words, we had to assume that σ is regular and also had to incur a small loss in the regularity of ϕ. Both assumptions can be lifted if one considers iterated Itô integrals based on the noise $\psi_\delta * (\sigma\boldsymbol{\xi})$ rather than $\sigma(\psi_\delta * \boldsymbol{\xi})$, where one can use that $\psi_\delta * (\sigma\boldsymbol{\xi})$ may be interpreted as a mollified, vector-valued space-time white noise on an L^2-space with measure $\sigma(u,x)^2\,\mathrm{d}u\,\mathrm{d}x$.

The following result, Nelson's estimate, allows us to bound pth-moments of iterated Itô integrals by their second moments.

Lemma C.3.8 (Nelson's estimate)**.** *Let $n \in \mathbb{N}$ and $p \in [2,\infty)$. Then there exists a $C = C(n,p) > 0$ such that for each $\phi \in L^2((0,\infty)^n_> \times \mathbb{T}^{2n};\mathbb{C}^{2n})$,*

$$\mathbb{E}[|I^n(\phi)|^p]^{1/p} \le C\mathbb{E}[|I^n(\phi)|^2]^{1/2}.$$

Proof. For a proof, see [Nua06; MWX17]. □

Let $\Sigma(1,\dots,n)$ denote the permutation group over $\{1,\dots,n\}$. We call $h \in L^2([0,T]^n \times \mathbb{T}^{2n};\mathbb{R}^{2n})$ symmetric, if

$$h(z_1,\dots,z_n) = \mathrm{Sym}(h)(z_1,\dots,z_n) \coloneqq \frac{1}{n!}\sum_{\varsigma\in\Sigma(1,\dots,n)} h(z_{\varsigma(1)},\dots,z_{\varsigma(n)}),$$

where for all $k \in \{1,\dots,n\}$ we denote $u_k \in [0,T]$, $x_k \in \mathbb{T}^2$, $j_k \in \{1,2\}$ and $z_k = (u_k, x_k, j_k)$. We denote the space of symmetric, square-integrable functions by $f \in L^2_{\mathrm{sym}}([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})$.

Next we show that each iterated Itô integrals of a symmetric integrand is an element of a homogeneous Wiener chaos as defined in (G.0.1).

Lemma C.3.9. *For each $n \in \mathbb{N}$ and $f \in L^2_{\mathrm{sym}}([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})$, it holds that $I^n(f) \in \mathcal{H}^{(n)}(\Omega, \mathbb{P}; \mathbb{R})$, where the homogeneous Wiener chaos $\mathcal{H}^{(n)}(\Omega, \mathbb{P}; \mathbb{R})$ is defined as in* (G.0.1).

Proof. Let $f \in L^2_{\mathrm{sym}}([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})$, it is now a consequence of [Nua06, below Thm. 1.1.2] that $I^n(f) \in \mathcal{H}^{(n)}(\Omega, \mathbb{P}; \mathbb{R})$. □

Using that each iterated Itô integral has zero mean, we obtain the following result (cf. [HW15, Sec. 4]).

Lemma C.3.10. *Let $n \in \mathbb{N}$, $f \in L^2_{\mathrm{sym}}([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})$ and $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$. It then follows that $n!\mathbb{E}[I^n(f)(\,\cdot\, + h)] = \langle f, h^{\otimes n}\rangle_{L^2([0,T]^n \times \mathbb{T}^{2n}; \mathbb{R}^{2n})}$.*

Sketch of Proof. For every $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ we can apply Lemma C.3.1 to identify the shifted vector-valued space-time white noise $\boldsymbol{\xi} + h$ with a complex Brownian motion plus drift, that is, for every $t \in [0,T]$, $\omega \in \mathbb{Z}^2$ and $j = 1, 2$,

$$(\boldsymbol{\xi} + h)(\mathbb{1}_{[0,t](\,\cdot\,)} \otimes \mathrm{e}^{-2\pi \mathrm{i}\langle \omega, \,\cdot\, \rangle} \otimes \delta_{j,\,\cdot\,}) = W^j(t,\omega) + \int_0^t \widehat{h}(s,\omega,j)\,\mathrm{d}s.$$

In the case of iterated Itô integrals of order 1 it follows that for every integrand $f \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$,

$$\begin{aligned}
\mathbb{E}[I^1(f)(\,\cdot\, + h)] &= \mathbb{E}\Big[\sum_{\omega_1 \in \mathbb{Z}^2} \sum_{j_1=1}^{2} \int_0^\infty \widehat{f}(u_1, -\omega_1, j_1)(\mathrm{d}W^{j_1}(u_1,\omega_1) + \widehat{h}(u_1,\omega_1,j_1)\,\mathrm{d}u_1)\Big] \\
&= \sum_{\omega_1 \in \mathbb{Z}^2} \sum_{j_1=1}^{2} \int_0^\infty \widehat{f}(u_1, -\omega_1, j_1)\widehat{h}(u_1,\omega_1,j_1)\,\mathrm{d}u_1 \\
&= \langle f, h\rangle_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)},
\end{aligned}$$

where in the second equality we used that Itô integrals have zero mean and in the third equality applied Parseval's theorem.

The case of general $n \in \mathbb{N}$ follows by linearity, where we use the symmetry of f to recover the integral over $[0,T]^n \times \mathbb{T}^{2n} \times \{1,2\}^n$ modulo a factor of $n!$. For the full proof, see [Klo17, Prop. 1.33]. $\square$

D
Shape-Coefficient Estimates

In this appendix we bound our shape coefficients in terms of $|t-s|^\gamma$ and $|\omega_k|^\beta$ for some $\gamma \in [0,1]$ and $\beta \in \mathbb{R}$. We first consider the shape coefficient for $\mathbf{Y}$ (cf. Definition 2.2.10).

Lemma D.0.1. *Let $s,t \geq 0$, $\gamma \in [0,1]$ and $\omega_1, \omega_2 \in 2\pi\mathbb{Z}^2 \setminus \{0\}$ be such that $\omega_1 + \omega_2 \neq 0$.*

1. *In the case $(\omega_1 \perp \omega_2)$, we obtain*

$$\mathsf{D}_{s,t}\mathbf{Y}(\omega_1,\omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-4+2\gamma}.$$

2. *In the case $\neg(\omega_1 \perp \omega_2)$, we obtain*

$$\begin{aligned}\mathsf{D}_{s,t}\mathbf{Y}(\omega_1,\omega_2) \lesssim{}& |t-s|^\gamma |\omega_1|^{-4+2\gamma}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2} \\ &+ |t-s|^\gamma |\omega_1|^{-4}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2+2\gamma}.\end{aligned}$$

The implied constants are uniform in ω_1, ω_2.

Proof. To bound our shape coefficients, we first derive explicit expressions, a step that was inspired by [TW18, Proof of Thm. 2.1]. To do so, we evaluate the exponential integrals in (2.2.8) and compute $\mathsf{D}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$ through (2.2.9). We can then further decompose

$$\mathsf{D}_{s,t}\mathbf{Y}(\omega_1,\omega_2) = \mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2) + \mathsf{DE}_{s,t}\mathbf{Y}(\omega_1,\omega_2),$$

into the *leading term* $\mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$ and the *error term* $\mathsf{DE}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$. The error term is generated by the zero initial condition of the noise, i.e. the remaining restriction $u_3, u_3' \geq 0$ in (2.2.8).

Assume $(\omega_1 \perp \omega_2)$, then

$$\begin{aligned}&\mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2)\\ &= \frac{1}{2^2}|\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-4}\Big(1-\mathrm{e}^{-|t-s||\omega_1+\omega_2|^2}-|t-s||\omega_1+\omega_2|^2\mathrm{e}^{-|t-s||\omega_1+\omega_2|^2}\Big)\end{aligned}$$

and

$$\begin{aligned}&\mathsf{DE}_{s,t}\mathbf{Y}(\omega_1,\omega_2)\\ &= \frac{1}{2^3}|\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-4}\Big(2t|\omega_1+\omega_2|^2(\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}-\mathrm{e}^{-2t|\omega_1+\omega_2|^2})\\ &\qquad\qquad + 2s|\omega_1+\omega_2|^2(\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}-\mathrm{e}^{-2s|\omega_1+\omega_2|^2})\\ &\qquad\qquad - (\mathrm{e}^{-t|\omega_1+\omega_2|^2}-\mathrm{e}^{-s|\omega_1+\omega_2|^2})^2\Big).\end{aligned}$$

We first consider $\mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$ and estimate by (B.5.1) for every $\gamma \in [0,1]$,

$$\mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2) \lesssim |t-s|^{\gamma}|\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-4+2\gamma}.$$

Next we estimate the error term $\mathsf{DE}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$. By symmetry it is enough to consider $s \leq t$, for which

$$\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}-\mathrm{e}^{-2s|\omega_1+\omega_2|^2} \leq 0.$$

We can then proceed to bound the remaining non-negative term of $\mathsf{DE}_{s,t}\mathbf{Y}(\omega_1,\omega_2)$ by (B.5.1) and (B.5.2), giving the result in the case $(\omega_1 \perp \omega_2)$.

Next assume $\neg(\omega_1 \perp \omega_2)$. We obtain the expressions

$$\begin{aligned}&\mathsf{DL}_{s,t}\mathbf{Y}(\omega_1,\omega_2)\\ &= \frac{1}{2^2}|\omega_1|^{-2}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2}\\ &\quad\times\Big(\frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}(\mathrm{e}^{-|t-s|(|\omega_1|^2+|\omega_2|^2)}-\mathrm{e}^{-|t-s||\omega_1+\omega_2|^2})\\ &\qquad+\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}(2-\mathrm{e}^{-|t-s||\omega_1+\omega_2|^2}-\mathrm{e}^{-|t-s|(|\omega_1|^2+|\omega_2|^2)})\Big)\end{aligned} \tag{D.0.1}$$

and

$$
\begin{aligned}
&\mathsf{DE}_{s,t}\Upsilon(\omega_1,\omega_2)\\
&= \frac{1}{2^2}|\omega_1|^{-2}|\omega_2|^{-2}\\
&\quad\times\Bigg(\frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}\\
&\qquad\quad\times (2(\mathrm{e}^{t(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2t|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2})\\
&\qquad\qquad\quad+2(\mathrm{e}^{s(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2s|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}))\\
&\qquad\qquad-\frac{1}{|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}(\mathrm{e}^{-t|\omega_1+\omega_2|^2}-\mathrm{e}^{-s|\omega_1+\omega_2|^2})^2\Bigg).
\end{aligned}
\tag{D.0.2}
$$

The first term in (D.0.1),

$$
\frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}(\mathrm{e}^{-|t-s|(|\omega_1|^2+|\omega_2|^2)}-\mathrm{e}^{-|t-s||\omega_1+\omega_2|^2})
$$

is non-positive so that by (B.5.1),

$$
\mathsf{DL}_{s,t}\Upsilon(\omega_1,\omega_2)\lesssim |t-s|^{\gamma}|\omega_1|^{-4+2\gamma}|\omega_2|^{-2}|\omega_1+\omega_2|^{-2}.
$$

In (D.0.2), we bound the second, non-positive term by 0. In the first term, we distinguish cases to fix the sign of the prefactor $(|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2)^{-1}$. Assume first $|\omega_1|^2+|\omega_2|^2<|\omega_1+\omega_2|^2$ and by symmetry $s\le t$. We obtain by (B.5.1) and (B.5.2),

$$
\begin{aligned}
&\frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}\\
&\quad\times\Big((\mathrm{e}^{t(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2t|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2})\\
&\qquad\quad+(\mathrm{e}^{s(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2s|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2})\Big)\\
&\le\frac{1}{|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}\\
&\quad\times\mathrm{e}^{-(t+s)(|\omega_1|^2+|\omega_2|^2)}(1-\mathrm{e}^{-t(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)})(1-\mathrm{e}^{-(t-s)|\omega_1+\omega_2|^2})\\
&\lesssim|t-s|^{\gamma}\frac{|\omega_1+\omega_2|^{2\gamma}}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2}\frac{t}{t+s}\\
&\lesssim|t-s|^{\gamma}|\omega_1+\omega_2|^{-2+2\gamma}|\omega_1|^{-2}.
\end{aligned}
\tag{D.0.3}
$$

If instead $|\omega_1+\omega_2|^2 < |\omega_1|^2+|\omega_2|^2$, $s \le t$, then

$$\begin{aligned}
&\frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2} \\
&\quad\times\Big((\mathrm{e}^{t(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2t|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}) \\
&\qquad\quad+(\mathrm{e}^{s(|\omega_1+\omega_2|^2-|\omega_1|^2-|\omega_2|^2)}-1)(\mathrm{e}^{-2s|\omega_1+\omega_2|^2}-\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2})\Big) \\
&\le \frac{1}{|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2}\frac{1}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2} \qquad \text{(D.0.4)}\\
&\quad\times(1-\mathrm{e}^{-t(|\omega_1|^2+|\omega_2|^2-|\omega_1+\omega_2|^2)})\mathrm{e}^{-(t+s)|\omega_1+\omega_2|^2}(1-\mathrm{e}^{-(t-s)|\omega_1+\omega_2|^2}) \\
&\lesssim |t-s|^\gamma|\omega_1+\omega_2|^{-2}\frac{|\omega_1+\omega_2|^{2\gamma}}{|\omega_1|^2+|\omega_2|^2+|\omega_1+\omega_2|^2}\frac{t}{t+s} \\
&\le |t-s|^\gamma|\omega_1+\omega_2|^{-2}|\omega_1|^{-2+2\gamma}.
\end{aligned}$$

By combining (D.0.2) with (D.0.3) and (D.0.4), we arrive at

$$\mathsf{DE}_{s,t}\mathsf{Y}(\omega_1,\omega_2) \lesssim |t-s|^\gamma|\omega_1|^{-2}|\omega_2|^{-2}(|\omega_1+\omega_2|^{-2+2\gamma}|\omega_1|^{-2}+|\omega_1+\omega_2|^{-2}|\omega_1|^{-2+2\gamma}).$$

This yields the claim. $\square$

Next we bound the shape coefficient $\mathsf{A}\blacktriangledown$, which appears in , ,  and  (cf. Definition 2.2.16)

Lemma D.0.2. *Let $s,t \ge 0$, $k=1,2$, $\gamma \in [0,1]$ and $C \ge 1$. Then uniformly in $\omega_1,\omega_2,\omega_3 \in \mathbb{Z}^2\setminus\{0\}$ such that $C^{-1}|\omega_1| \le |\omega_2| \le C|\omega_1|$, it holds that*

$$\mathsf{A}^k_{s,t}\blacktriangledown(\omega_1,\omega_2,\omega_3) \lesssim |t-s|^\gamma|\omega_2|^{2\gamma}|\omega_3|^{-1}.$$

Proof. The claim follows by the triangle inequality and repeated applications of (B.5.1). $\square$

The following lemma controls the shape coefficient $\mathsf{A}\boldsymbol{\vee}$, which appears in ,  and ,  (cf. Definition 2.2.14).

Lemma D.0.3. *Let $s,t \ge 0$, $k,k'=1,2$, $\gamma\in[0,1]$ and $C,C' \ge 1$. Then uniformly in $\omega_1,\omega_1',\omega_2 \in \mathbb{Z}^2\setminus\{0\}$ such that $C^{-1}|\omega_1| \le |\omega_2| \le C|\omega_1|$ and $(C')^{-1}|\omega_1'| \le |\omega_2| \le C'|\omega_1'|$, it holds that*

$$\mathsf{A}^{k,k'}_{s,t}\boldsymbol{\vee}(\omega_1,\omega_1',\omega_2) \lesssim |t-s|^\gamma|\omega_1|^{-1+\gamma}|\omega_1'|^{-1+\gamma}.$$

Proof. The claim follows by the triangle inequality and repeated applications of (B.5.1). $\square$

E
Summation Estimates

Contents

E.1 Basic Estimates

We prove a number of summation and discrete convolution estimates that are central to our bounds.

Recall that the Fourier multiplier of the elliptic operator $\partial_j\Phi$, denoted by $G^j(\omega) = 2\pi \mathrm{i}\omega^j|2\pi\mathrm{i}\omega|^{-2}\mathbb{1}_{\omega\neq 0}$ for all $\omega \in \mathbb{Z}^2$ and $j = 1, 2$, frequently appears in the diagrams considered in Subsection 2.2.2. The following lemma shows that $|G^j(\omega + \omega_1) - G^j(\omega_1)|$ decays like $|\omega + \omega_1|^{-2}$ for every $\omega_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}$, which is one order better than $|G^j(\omega + \omega_1)| \lesssim |\omega + \omega_1|^{-1}$.

Lemma E.1.1. *Let $j = 1, 2$, then uniformly in $\omega, \omega_1 \in \mathbb{Z}^2$ such that $\omega_1, \omega + \omega_1 \neq 0$. Then it holds that*

$$|G^j(\omega + \omega_1) - G^j(\omega_1)| \lesssim |\omega||\omega + \omega_1|^{-2}(1 + |\omega||\omega_1|^{-1}).$$

Proof. We compute

$$G^j(\omega+\omega_1)-G^j(\omega_1)=\frac{\mathrm{i}}{2\pi}\Big(\frac{\omega^j+\omega_1^j}{|\omega+\omega_1|^2}-\frac{\omega_1^j}{|\omega_1|^2}\Big),$$

which can be bounded in absolute value by

$$\begin{aligned}\left|\frac{\omega^j+\omega_1^j}{|\omega+\omega_1|^2}-\frac{\omega_1^j}{|\omega_1|^2}\right| &= \frac{|\omega^j|\omega_1|^2-\omega_1^j|\omega|^2-\omega_1^j 2\langle\omega,\omega_1\rangle|}{|\omega+\omega_1|^2|\omega_1|^2}\\ &\lesssim |\omega||\omega+\omega_1|^{-2}+|\omega|^2|\omega_1|^{-1}|\omega+\omega_1|^{-2}.\end{aligned}$$

This yields the claim. □

We apply the following summation estimates repeatedly to establish the regularities of our diagrams.

Lemma E.1.2. *It holds that uniformly in $\delta>0$,*

$$\sum_{\substack{k\in\mathbb{Z}^2\\ |k|\le\delta^{-1}}} 1\lesssim (1\vee\delta^{-2}) \tag{E.1.1}$$

and

$$\sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\}\\ |k|\le\delta^{-1}}} |k|^{-2}\lesssim (1\vee\log(\delta^{-1})). \tag{E.1.2}$$

What is more, for each $\alpha>2$,

$$\sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^{-\alpha}\lesssim_\alpha 1. \tag{E.1.3}$$

Proof. The proof follows by treating the left-hand sides in (E.1.1)–(E.1.3) as Riemann sums and by bounding the associated Riemann integrals. □

We make repeated use of the following convolution estimate to construct non-linear objects.

Lemma E.1.3 ([ZZ15, Lem. 3.10], [MWX17, Lem. 5 & Lem. 6])**.** *Let $\alpha,\beta\in\mathbb{R}$ be such that $\alpha+\beta>2$. We have uniformly in $\omega\in\mathbb{Z}^2$,*

$$\sum_{\substack{\omega_1\in\mathbb{Z}^2\setminus\{0,\omega\}\\ \omega_1\sim(\omega-\omega_1)}} |\omega_1|^{-\alpha}|\omega-\omega_1|^{-\beta}\lesssim_{\alpha+\beta}(1\vee|\omega|)^{-\alpha-\beta+2}.$$

If in addition $\alpha,\beta<2$, then we have uniformly in $\omega\in\mathbb{Z}^2$,

$$\sum_{\omega_1\in\mathbb{Z}^2\setminus\{0,\omega\}} |\omega_1|^{-\alpha}|\omega-\omega_1|^{-\beta}\lesssim_{\alpha,\beta,\alpha+\beta}(1\vee|\omega|)^{-\alpha-\beta+2}.$$

The next convolution result is useful for estimating correlated frequencies $\omega \neq \omega'$.

Lemma E.1.4. *Let $\alpha, \beta, \gamma \in (0,2)$ be such that $\alpha + \gamma > 2$ and $\beta + \gamma > 2$. Then for all $\omega, \omega' \in \mathbb{Z}^2 \setminus \{0\}$ such that $\omega \neq \omega'$ it uniformly holds that*

$$\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega'\}} |\omega - \omega_1|^{-\alpha} |\omega' - \omega_1|^{-\beta} |\omega_1|^{-\gamma} \lesssim |\omega - \omega'|^{-\beta} |\omega|^{-\alpha-\gamma+2} + |\omega - \omega'|^{-\alpha} |\omega'|^{-\beta-\gamma+2}.$$

Proof. The proof follows by two applications of Lemma E.1.3, one in the case $|\omega - \omega_1| \leq |\omega - \omega'|/2$ and the other in the complement. □

To derive finer estimates, it is useful to introduce a discrete paraproduct analogue to extend Lemma E.1.3, see (1.2.5) for the relevant notation.

Lemma E.1.5. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha > 2$ and $\beta \geq 0$. We have uniformly in $\omega \in \mathbb{Z}^2 \setminus \{0\}$,*

$$\sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega\} \\ \omega_1 \precsim (\omega - \omega_1)}} |\omega_1|^{-\alpha} |\omega - \omega_1|^{-\beta} \lesssim_\alpha |\omega|^{-\beta}.$$

Proof. The proof is immediate by (E.1.3) and the bound induced by (1.2.5). □

E.2 Double-Sum Estimates

When we take the Fourier transform of the noise $\sigma\boldsymbol{\xi}$, it generates convolutions of $\widehat{\sigma}(t, \omega - m_1)$ against $\mathrm{d}W^{j_1}(t, m_1)$ in $m_1 \in \mathbb{Z}^2$. We also generate convolutions in $\omega_k \in \mathbb{Z}^2$ by constructing non-linear objects such as $\nabla \cdot \mathcal{I}[\,\uparrow \nabla \Phi_{\uparrow}]$. Those steps lead to double sums over ω_k and m_k that do not factorize. In this section, we estimate those sums.

We apply the following estimate in Subsection 2.2.4 to construct .

Lemma E.2.1. *Let $\gamma \in [0, 1/2)$ and $\varepsilon \in (0, 1 - 2\gamma)$. Then uniformly in $\omega, \omega_1 \in \mathbb{Z}^2$, it holds that*

$$\begin{aligned} &\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0,\omega,\omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2} (1 + |\omega| |\omega_4|^{-1}) |\omega_4|^{2\gamma} |\omega - \omega_1 - \omega_4|^{-1} \\ &\quad \times \sum_{m_2 \in \mathbb{Z}^2} (1 + |\omega - \omega_1 - \omega_4 - m_2|^2)^{-1} (1 + |\omega_4 + m_2|^2)^{-1} \\ &\lesssim (1 \vee |\omega - \omega_1|)^{-2+\varepsilon} (1 \vee |\omega|)^{-1+2\gamma+\varepsilon}. \end{aligned}$$

Proof. We decompose the sum over $m_2 \in \mathbb{Z}^2$ into the regions $m_2 = \omega - \omega_1 - \omega_4$, $m_2 = -\omega_4$ and $m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}$. We only give the bound that involves the sum over $m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}$, as the other ones are simpler. We estimate

$$\begin{aligned}
&\sum_{m_2 \in \mathbb{Z}^2 \setminus \{\omega-\omega_1-\omega_4, -\omega_4\}} (1 + |\omega - \omega_1 - \omega_4 - m_2|^2)^{-1} (1 + |\omega_4 + m_2|^2)^{-1} \\
&\leq \sum_{m_2 \in \mathbb{Z}^2 \setminus \{\omega-\omega_1-\omega_4, -\omega_4\}} |\omega - \omega_1 - \omega_4 - m_2|^{-2} |\omega_4 + m_2|^{-2}.
\end{aligned}$$

Let $\varepsilon \in (0,1)$, we can then apply Lemma E.1.3 to bound

$$\begin{aligned}
&\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2} (1 + |\omega||\omega_4|^{-1}) |\omega_4|^{2\gamma} |\omega - \omega_1 - \omega_4|^{-1} \\
&\quad \times \sum_{m_2 \in \mathbb{Z}^2 \setminus \{\omega-\omega_1-\omega_4, -\omega_4\}} |\omega - \omega_1 - \omega_4 - m_2|^{-2} |\omega_4 + m_2|^{-2} \\
&\lesssim (1 \vee |\omega - \omega_1|)^{-2+2\varepsilon} \sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2} (1 + |\omega||\omega_4|^{-1}) |\omega_4|^{2\gamma} |\omega - \omega_1 - \omega_4|^{-1}.
\end{aligned}$$

Let $p \in (1, \infty)$, $q = p/(p-1)$ and $\delta > 0$. We obtain by Hölder's inequality,

$$\begin{aligned}
&\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2} (1 + |\omega||\omega_4|^{-1}) |\omega_4|^{2\gamma} |\omega - \omega_1 - \omega_4|^{-1} \\
&\leq \Big(\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2p} |\omega_4|^{\delta p} (1 + |\omega||\omega_4|^{-1})^p \Big)^{1/p} \\
&\quad \times \Big(\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\}} |\omega_4|^{-\delta q} |\omega_4|^{2\gamma q} |\omega - \omega_1 - \omega_4|^{-q} \Big)^{1/q}.
\end{aligned}$$

We assume $\gamma \in [0, 1/2)$, $2 < p$ and $1 - 2/p + 2\gamma < \delta < 2 - 2/p$. It follows that

$$p(2-\delta) > 2, \qquad q(\delta - 2\gamma) < 2, \qquad q < 2, \qquad q(\delta - 2\gamma + 1) > 2.$$

and consequently by Lemma E.1.3,

$$\begin{aligned}
&\Big(\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2p} |\omega_4|^{\delta p} (1 + |\omega||\omega_4|^{-1})^p \Big)^{1/p} \\
&\quad \times \Big(\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\}} |\omega_4|^{-\delta q} |\omega_4|^{2\gamma q} |\omega - \omega_1 - \omega_4|^{-q} \Big)^{1/q} \\
&\lesssim (1 \vee |\omega|)^{-2+\delta+2/p} (1 \vee |\omega - \omega_1|)^{-\delta + 2\gamma - 1 + 2/q}.
\end{aligned}$$

Let $\varepsilon \in (0, 1-2\gamma)$, we can now set $\delta = 1 - 2/p + 2\gamma + \varepsilon$ to conclude

$$\sum_{\substack{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega-\omega_1\} \\ (\omega-\omega_4) \sim \omega_4}} |\omega - \omega_4|^{-2}(1 + |\omega||\omega_4|^{-1})|\omega_4|^{2\gamma}|\omega - \omega_1 - \omega_4|^{-1}$$
$$\lesssim (1 \vee |\omega|)^{-1+2\gamma+\varepsilon}(1 \vee |\omega - \omega_1|)^{-\varepsilon},$$

which yields the claim. $\square$

We apply the following estimate in Subsection 2.2.5 to construct $^{\delta}$, $^{\delta}$ and $^{\delta}$. We use the restriction $|m_1| \leq \delta^{-1}$ induced by the cut-off $\varphi(\delta m_1)$ to establish a bound in terms of $\log(\delta^{-1})$.

Lemma E.2.2. *Let $\varepsilon \in (0, 1/2)$ and $\delta > 0$. Then uniformly in $\omega \in \mathbb{Z}^2 \setminus \{0\}$ it holds that*

$$\sum_{\substack{m_1 \in \mathbb{Z}^2 \\ |m_1| \leq \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}(1 + |\omega||\omega - \omega_1|^{-1})$$
$$\lesssim |\omega|^{-2+3\varepsilon}(1 \vee \log(\delta^{-1})). \tag{E.2.1}$$

Proof. To bound (E.2.1) it suffices to estimate the two parts

$$\sum_{\substack{m_1 \in \mathbb{Z}^2 \\ |m_1| \leq \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2} \tag{E.2.2}$$

and

$$\sum_{\substack{m_1 \in \mathbb{Z}^2 \\ |m_1| \leq \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}|\omega - \omega_1|^{-1}. \tag{E.2.3}$$

Let us consider (E.2.2). We decompose the sum over m_1 into the regions $m_1 = 0$, $m_1 = -\omega$ and $m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}$. The sum over $m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}$ is given by

$$\sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\} \\ |m_1| \leq \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}.$$

The sum over $\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}$ can then be further decomposed into the regions $\omega_1 = m_1$, $\omega_1 = \omega + m_1$ and $\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}$. We only give the bound that involves the sums over $m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}$ and $\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}$. Using that $\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}$, we may estimate

$$\begin{aligned}
&\sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\}} (1 + |\omega_1 - m_1|^2)^{-1} (1 + |\omega - \omega_1 + m_1|^2)^{-1} |\omega_1|^{-2} \\
&\le \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2}.
\end{aligned}$$

Introducing the dyadic partition of unity $(\varrho_q)_{q \in \mathbb{N}_{-1}}$ (cf. (1.2.4) & (1.2.5)), we decompose this sum into

$$\begin{aligned}
&\sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2} \\
&= \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ \omega_1 \lesssim (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2} && \text{(E.2.4)} \\
&\quad + \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ \omega_1 \gtrsim (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2} && \text{(E.2.5)} \\
&\quad + \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ \omega_1 \sim (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2}. && \text{(E.2.6)}
\end{aligned}$$

Assume $\varepsilon < 2/3$, the terms (E.2.4) and (E.2.6) can be estimated by two applications of Lemma E.1.3,

$$\begin{aligned}
&\sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ |\omega_1| \le 64/9 |\omega - \omega_1 + m_1|}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2} \\
&\lesssim \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} |\omega + m_1|^{-2} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ |\omega_1| \le 64/9 |\omega - \omega_1 + m_1|}} |\omega_1 - m_1|^{-2} |\omega_1|^{-2} \\
&\lesssim \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} |\omega + m_1|^{-2} |m_1|^{-2+2\varepsilon} \lesssim |\omega|^{-2+3\varepsilon}.
\end{aligned}$$

The second term (E.2.5) can be estimated by Lemma E.1.3 and (E.1.2),

$$\begin{aligned}
&\sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ \omega_1 \gtrsim (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2} \\
&\lesssim \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} |\omega + m_1|^{-2} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0,\omega,m_1,\omega+m_1\} \\ \omega_1 \gtrsim (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} \\
&\lesssim |\omega|^{-2+2\varepsilon} \sum_{\substack{m_1 \in \mathbb{Z}^2 \setminus \{0,-\omega\} \\ |m_1| \le \delta^{-1}}} |\omega + m_1|^{-2} \lesssim |\omega|^{-2+3\varepsilon} (1 \vee \log(\delta^{-1})).
\end{aligned}$$

Decomposing (E.2.2) as discussed and bounding the resulting terms yields

$$\sum_{\substack{m_1\in\mathbb{Z}^2\\|m_1|\le\delta^{-1}}}\sum_{\omega_1\in\mathbb{Z}^2\setminus\{0,\omega\}}(1+|\omega_1-m_1|^2)^{-1}(1+|\omega-\omega_1+m_1|^2)^{-1}|\omega_1|^{-2}\lesssim|\omega|^{-2+3\varepsilon}(1\vee\log(\delta^{-1})).$$

Assume $\varepsilon\in(0,1/2)$, the bound on (E.2.3) then follows in a similar manner. $\square$

F

Positivity of Diffusion-Advection Equations

Let $u \in C_1^2((0,T] \times \mathbb{T}^d) \cap C([0,T] \times \mathbb{T}^d)$ be the classical solution (if it exists) to the general diffusion-advection equation

$$\begin{cases} (\partial_t - \Delta)u = \nabla \cdot (bu), & \text{in } (0,T] \times \mathbb{T}^d, \\ u|_{t=0} = u_0, & \text{on } \mathbb{T}^d, \end{cases} \tag{F.0.1}$$

with positive initial data $u_0 \in C(\mathbb{T}^d)$ and advection $b \in C([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ satisfying $\nabla \cdot b \in C([0,T] \times \mathbb{T}^d; \mathbb{R})$. In the next proposition we show that u stays bounded away from zero, which is used to prove Lemma 3.2.4 (where existence of a solution is guaranteed.)

Proposition F.0.1. *Let $u_0 \in C(\mathbb{T}^d)$ be such that $u_0 > 0$ and assume $b \in C([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ satisfies $\nabla \cdot b \in C([0,T] \times \mathbb{T}^d; \mathbb{R})$. Let $u \in C_1^2((0,T] \times \mathbb{T}^d) \cap C([0,T] \times \mathbb{T}^d)$ be any classical solution to* (F.0.1). *Then there exists a constant $c > 0$ such that $u(t,x) \geq c$ for all $(t,x) \in [0,T] \times \mathbb{T}^d$.*

Proof. Using the product rule and passing between the torus and the full space, we find that $u \in C_1^2((0,T] \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ is a periodic solution to

$$\begin{cases} \partial_t u = \Delta u - uV + b \cdot \nabla u, & \text{in } (0,T] \times \mathbb{R}^d, \\ u|_{t=0} = u_0, & \text{on } \mathbb{R}^d, \end{cases}$$

where we denote $V \coloneqq -\nabla \cdot b$ and abuse notation to identify functions on $\mathbb{T}^d$ with periodic functions on $\mathbb{R}^d$. Using the assumption $\nabla \cdot b \in C([0,T] \times \mathbb{T}^d)$, it follows that $V \in C([0,T] \times \mathbb{R}^d)$ and $\|V\|_{C([0,T]\times\mathbb{R}^d)} < \infty$.

Let $t \in [0,T]$ and for every $v\colon [0,t] \to \mathbb{R}$ define the time reversal $\tilde{v}(s) \coloneqq v(t-s)$. It follows by [RW00, Chpt. V, Thm. 23.5 & Thm. 20.1] that for every $x \in \mathbb{R}^d$ there exists a weak solution $((Y_s)_{s\geq 0}, (\mathcal{F}_s)_{s\geq 0})$ to the SDE,

$$\begin{cases} \mathrm{d}Y_s = \tilde{b}(s, Y_s)\mathbb{1}_{s\leq t}\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}B_s, \\ \;Y_0 = x, \end{cases}$$

where we used $b \in C([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ to ensure that $(s, y) \mapsto \tilde{b}(s, y)\mathbb{1}_{s\leq t}$ is a previsible path functional in the sense of [RW00, Chpt. V, Def. 8.3] and bounded in a neighbourhood of time t. By an application of [EK86, Chpt. 8, Thm. 1.7 & Chpt. 4, Cor. 4.3], this solution is unique in law.

We denote $A_s \coloneqq \int_0^s \tilde{V}(r, Y_r)\,\mathrm{d}r$ and $X_s \coloneqq \mathrm{e}^{-A_s}\tilde{u}(s, Y_s)$, where $\tilde{u}(s) \coloneqq u(t-s)$ solves

$$\begin{cases} -\partial_s\tilde{u} = \Delta\tilde{u} - \tilde{u}\tilde{V} + \tilde{b}\cdot\nabla\tilde{u}, & \text{in } [0,t) \times \mathbb{R}^d, \\ \;\tilde{u}|_{s=t} = u_0, & \text{on } \mathbb{R}^d. \end{cases}$$

It then follows by Itô's formula (cf. [KS91, Chpt. 3, Thm. 3.6]) that for every $s < t$,

$$\begin{aligned} \mathrm{d}X_s &= -\mathrm{e}^{-A_s}\tilde{u}(s,Y_s)\tilde{V}(s,Y_s)\,\mathrm{d}s + \mathrm{e}^{-A_s}\partial_s\tilde{u}(s,Y_s)\,\mathrm{d}s + \mathrm{e}^{-A_s}\nabla\tilde{u}(s,Y_s)\cdot\tilde{b}(s,y_s)\,\mathrm{d}s \\ &\qquad + \sqrt{2}\mathrm{e}^{-A_s}\nabla\tilde{u}(s,Y_s)\cdot\mathrm{d}B_s + \mathrm{e}^{-A_s}\Delta\tilde{u}(s,Y_s)\,\mathrm{d}s \\ &= \sqrt{2}\mathrm{e}^{-A_s}\nabla\tilde{u}(s,Y_s)\cdot\mathrm{d}B_s, \end{aligned}$$

which implies that $(X_s)_{s\in[0,t)}$ is a continuous local martingale up to time t.

Let $(T_k)_{k\in\mathbb{N}}$ be a localizing sequence, i.e. each T_k is a stopping time, $T_k < T_{k+1}$, $t = \sup_{k\in\mathbb{N}} T_k$ and $(X_{s\wedge T_k})_{s\geq 0}$ is a martingale for every $k \in \mathbb{N}$. It follows that $X_{s\wedge T_k} \to X_{s\wedge t}$ almost surely as $k \to \infty$. Furthermore,

$$\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s|\Big] \leq \|u\|_{C([0,t]\times\mathbb{T}^d)}\mathrm{e}^{t\|V\|_{C([0,t]\times\mathbb{T}^d)}} < \infty$$

and each $|X_{s\wedge T_k}|$ is dominated by $\sup_{s\in[0,t]}|X_s|$, which is integrable by the above. Therefore we obtain by the dominated convergence theorem for all $0 \leq r < s < \infty$,

$$\mathbb{E}[X_{s\wedge t} \mid \mathcal{F}_r] \leftarrow \mathbb{E}[X_{s\wedge T_k} \mid \mathcal{F}_r] = X_{r\wedge T_k} \to X_{r\wedge t},$$

so that $(X_{s\wedge t})_{s\geq 0}$ is a martingale.

We can then apply the martingale property to deduce

$$u(t,x)=\tilde{u}(0,x)=\mathbb{E}[X_0]=\mathbb{E}[X_t]=\mathbb{E}[\mathrm{e}^{-A_t}u(0,Y_t)]=\mathbb{E}\Big[\mathrm{e}^{-\int_0^t \tilde{V}(r,Y_r)\,\mathrm{d}r}u_0(Y_t)\Big].$$

Using that $\mathbb{T}^d$ is compact, it follows by the extreme value theorem that there exists a constant $c_0>0$ such that $u_0>c_0$. Therefore for all $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u(t,x)=\mathbb{E}\Big[\mathrm{e}^{-\int_0^t \tilde{V}(r,Y_r)\,\mathrm{d}r}u_0(Y_t)\Big]>c_0\mathrm{e}^{-t\|V\|_{C([0,t]\times\mathbb{T}^d)}}\geq c_0\mathrm{e}^{-T\|V\|_{C([0,T]\times\mathbb{T}^d)}}>0,$$

which yields the claim. $\square$

G
Generalized Freidlin–Wentzell Theory

In this appendix we generalize Freidlin and Wentzell's theory [FW12] to (direct sums of) random variables taking values in Banach space-valued inhomogeneous Wiener chaoses (Theorem G.0.4).

A similar program has already been carried out in [HW15, Sec. 3] and the main observation of the present appendix is that the results of [HW15, Sec. 3] carry over to situations in which the notion of convergence [HW15, (3.9)] is replaced by (G.0.4) (where one retains powers of the noise intensity for lower-order Wiener chaoses), if one assumes that the limit has no lower-order contributions (G.0.5). It is this change that allows us to apply Theorem G.0.4 directly to the (canonical) enhancement constructed in Theorem 3.4.1, rather than going through a perturbation argument as in [HW15, Thm. 4.7], which does not generically apply for us as we wish to treat cases where the white-noise induced renormalisation is strictly a distribution rather than a smooth function (or as in [HW15] a constant.)

Let $(B, \mathcal{H}_\mu, \mu)$ be an abstract Wiener space where B denotes a real, separable Banach space and μ a centred Gaussian probability measure on B with Cameron–Martin space $(\mathcal{H}_\mu, \|\cdot\|_{\mathcal{H}_\mu})$. Let $\mathcal{R}_\mu$ be the associated reproducing kernel Hilbert space, i.e. the completion of the dual space B' in $L^2(B, \mu)$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{R}_\mu$ such that $e_k \in B'$ for every $k \in \mathbb{N}$, which exists by an application of the Gram–Schmidt procedure.

We define the Hermite polynomials $H_n(x)$, for every $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, through the series expansion

$$\lambda \mapsto \exp\Big(\lambda x - \frac{1}{2}\lambda^2\Big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} H_n(x)$$

and the generalized Hermite polynomials $H_\nu(\boldsymbol{x})$, for every multi-index $\nu \in \mathbb{N}_0^{\mathbb{N}}$ with finitely-many non-zero entries and $\boldsymbol{x} \in \mathbb{R}^{\mathbb{N}}$, by

$$H_\nu(\boldsymbol{x}) \coloneqq \prod_{i=1}^{\infty} H_{\nu_i}(x_i),$$

which we extend to Banach space-valued arguments $\xi \in B$ via

$$H_\nu(\xi) \coloneqq H_\nu((\langle \xi, e_i \rangle)_{i\in\mathbb{N}}).$$

Let E be a real, separable Banach space, we denote by $L^2(B, \mu; E)$ the space of functions $\Psi\colon B \to E$ that are Bochner measurable and square integrable under μ. We can now define the E-valued homogeneous Wiener chaos of degree $k \in \mathbb{N}$, denoted by $\mathcal{H}^{(k)}(B, \mu; E)$, as in [HW15, (3.3)]:

$$\mathcal{H}^{(k)}(B, \mu; E) \coloneqq \Big\{\Psi \in L^2(B, \mu; E) : \int_B \Psi(\xi) H_\nu(\xi) \mu(\mathrm{d}\xi) = 0 \text{ for all } |\nu| \neq k\Big\}, \tag{G.0.1}$$

where the absolute value of the multi-index ν is given by $|\nu| \coloneqq \sum_{i=1}^{\infty} \nu_i$. We further set $\mathcal{H}^{(0)}(B, \mu; E) \coloneqq E$.

Definition G.0.1. *Let $\mathcal{W}$ be a finite index set. For each $\tau \in \mathcal{W}$, let K_τ be a natural number, E_τ be a real, separable Banach space and Ψ_τ be an element of the E_τ-valued inhomogeneous Wiener chaos of order K_τ, i.e.*

$$\Psi_\tau \coloneqq \sum_{k=0}^{K_\tau} \Psi_{\tau,k}, \qquad \Psi_{\tau,k} \in \mathcal{H}^{(k)}(B, \mu; E_\tau).$$

Denote $\mathbf{E} = \bigoplus_{\tau\in\mathcal{W}} E_\tau$ and let $\boldsymbol{\Psi}$ be given by

$$\boldsymbol{\Psi} \coloneqq \bigoplus_{\tau\in\mathcal{W}} \Psi_\tau = \bigoplus_{\tau\in\mathcal{W}} \sum_{k=0}^{K_\tau} \Psi_{\tau,k}, \qquad \Psi_{\tau,k} \in \mathcal{H}^{(k)}(B, \mu; E_\tau). \tag{G.0.2}$$

Given $\varepsilon > 0$ *and a random variable* $\mathbf{\Psi}$ *of the form* (G.0.2)*, we define the rescaling*

$$\mathbf{\Psi}^{(\varepsilon)} \coloneqq \bigoplus_{\tau \in \mathcal{W}} \varepsilon^{\frac{K_\tau}{2}} \Psi_\tau.$$

Definition G.0.2. *Let* $\mathbf{\Psi}$ *be a random variable of the form* (G.0.2)*,* $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ *and* $(\mathbf{\Psi}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon > 0}$ *be a sequence such that*

$$\mathbf{\Psi}^{(\varepsilon)}_{\delta(\varepsilon)} = \bigoplus_{\tau \in \mathcal{W}} \varepsilon^{\frac{K_\tau}{2}} \sum_{k=0}^{K_\tau} \Psi_{\delta(\varepsilon);\tau,k}, \qquad \Psi_{\delta(\varepsilon);\tau,k} \in \mathcal{H}^{(k)}(B, \mu; E_\tau). \tag{G.0.3}$$

We say that $\mathbf{\Psi}^{(\varepsilon)}_{\delta(\varepsilon)}$ *converges to* $\mathbf{\Psi}$ *as* $\varepsilon \to 0$ *if for each* $\tau \in \mathcal{W}$ *and* $k \leq K_\tau$*,*

$$\lim_{\varepsilon \to 0} \varepsilon^{K_\tau - k} \int_B \|\Psi_{\delta(\varepsilon);\tau,k}(\xi) - \Psi_{\tau,k}(\xi)\|^2_{E_\tau} \mu(\mathrm{d}\xi) = 0. \tag{G.0.4}$$

Definition G.0.3. *Let* $\mathbf{\Psi}$ *be a random variable of the form* (G.0.2)*. We define the homogeneous part* $\mathbf{\Psi}_{\mathrm{hom}} \colon \mathcal{H}_\mu \to \mathbf{E}$ *of* $\mathbf{\Psi}$ *by*

$$\mathbf{\Psi}_{\mathrm{hom}} \coloneqq \bigoplus_{\tau \in \mathcal{W}} (\Psi_{\tau, K_\tau})_{\mathrm{hom}},$$

where for each $\tau \in \mathcal{W}$ *and* $h \in \mathcal{H}_\mu$*,*

$$(\Psi_{\tau,K_\tau})_{\mathrm{hom}}(h) \coloneqq \int_B \Psi_{\tau,K_\tau}(\xi + h) \mu(\mathrm{d}\xi).$$

The homogeneous part $(\Psi_{\tau,K_\tau})_{\mathrm{hom}}(h)$ is well-defined by the Cameron–Martin theorem [Led96, (4.11)] and [HW15, (3.8)].

We can now state a large deviation principle for random variables of the form (G.0.3) that converge in the sense of (G.0.4) to a limit that does not have any contributions from lower-order Wiener chaoses.

Theorem G.0.4. *Let* $(\mathbf{\Psi}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon > 0}$ *be a family of random variables of the form* (G.0.3) *converging in the sense of* (G.0.4) *to a random variable* $\mathbf{\Psi}$ *such that*

$$\mathbf{\Psi} = \bigoplus_{\tau \in \mathcal{W}} \Psi_{\tau,K_\tau}, \qquad \Psi_{\tau,K_\tau} \in \mathcal{H}^{(K_\tau)}(B, \mu; E_\tau). \tag{G.0.5}$$

It then follows that $(\mathbf{\Psi}^{(\varepsilon)}_{\delta(\varepsilon)})_{\varepsilon > 0}$ *satisfies a large deviation principle in* $\mathbf{E}$ *with rate* ε *and good rate function* $\mathscr{I}_{\mathbf{\Psi}}$ *given by*

$$\mathscr{I}_{\mathbf{\Psi}}(\boldsymbol{s}) \coloneqq \inf\Big\{ \frac{1}{2} \|h\|^2_{\mathcal{H}_\mu} : h \in \mathcal{H}_\mu \text{ with } \mathbf{\Psi}_{\mathrm{hom}}(h) = \boldsymbol{s} \Big\}.$$

Proof. The key observation is that the proof of [HW15, Lem. 3.7] carries over if we assume (G.0.4) and (G.0.5) instead of [HW15, (3.9)]. One can then adapt the strategy of [HW15, Thm. 3.5] to our setting. $\square$

H

The Enhancement Driven by a Cameron–Martin Element

In this appendix we analyse the regularity of the mild solution $\mathbf{I}^h = \nabla \cdot \mathcal{I}[\sqrt{\rho_{\mathrm{det}}} h]$ driven by a Cameron–Martin element $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ (Lemmas H.0.1 & H.0.2), which we then use to construct the enhancement $\mathbb{X}^h = (\mathbf{I}^h, \mathbf{Y}^h, \mathbf{\Psi}^h, \mathbf{V}^h)$ (cf. (H.0.3)) appearing in Theorem 3.4.7 (Lemma H.0.3).

Let $t \in [0,T]$ and $\omega \in \mathbb{Z}^2$, we can represent the Fourier transform of $\mathbf{I}^h$ by

$$\widehat{\mathbf{I}^h}(t,\omega) = \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} \int_0^t H^j_{t-u}(\omega) \widehat{\sigma}(u, \omega - m) \widehat{h}(u,m,j) \, \mathrm{d}u. \tag{H.0.1}$$

The next lemma establishes the regularity of $\mathbf{I}^h$ as a (Hölder-)continuous function in time.

Lemma H.0.1. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 \geq 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$, $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ and $\kappa \in [0, 1/2]$ it holds that*

$$\| \mathbf{I}^h \|_{C^\kappa_T \mathcal{H}^{-2\kappa}} \lesssim \| \sqrt{\rho_{\mathrm{det}}} \|_{C([0,T] \times \mathbb{T}^2)} \| h \|_{L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)}$$

and in particular $\mathbf{I}^h \in C_T L^2(\mathbb{T}^2)$.

Proof. An application of Lemma 3.2.5 yields $\sqrt{\rho_{\mathrm{det}}} \in C([0,T]\times\mathbb{T}^2)$; in particular $\sqrt{\rho_{\mathrm{det}}} h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ and upon redefining h, we may assume that $\Psi^h = \nabla\cdot\mathcal{I}[h]$.

Let $\omega \in \mathbb{Z}^2$ and $s,t \in [0,T]$ with $s<t$. We bound by Jensen's inequality and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&|\widehat{\Psi^h}(t,\omega) - \widehat{\Psi^h}(s,\omega)|^2 \\
&\lesssim \sum_{j=1}^{2}\Big|\int_0^s (H^j_{t-u}(\omega) - H^j_{s-u}(\omega))\widehat{h}(u,\omega,j)\,\mathrm{d}u\Big|^2 + \Big|\int_s^t H^j_{t-u}(\omega)\widehat{h}(u,\omega,j)\,\mathrm{d}u\Big|^2 \\
&\le \sum_{j=1}^{2}\Big(\int_0^s |H^j_{t-u}(\omega) - H^j_{s-u}(\omega)|^2\,\mathrm{d}u\Big)\Big(\int_0^s |\widehat{h}(u,\omega,j)|^2\,\mathrm{d}u\Big) \\
&\qquad + \Big(\int_s^t |H^j_{t-u}(\omega)|^2\,\mathrm{d}u\Big)\Big(\int_s^t |\widehat{h}(u,\omega,j)|^2\,\mathrm{d}u\Big).
\end{aligned}
$$

It follows by interpolation that for all $\kappa \in [0,1/2]$,

$$
|\widehat{\Psi^h}(t,\omega) - \widehat{\Psi^h}(s,\omega)|^2 \lesssim |t-s|^{2\kappa}|\omega|^{4\kappa}\|\widehat{h}(\omega)\|^2_{L^2([0,T];\mathbb{R}^2)}, \tag{H.0.2}
$$

which allows us to estimate by (1.2.1),

$$
\|\Psi^h_t - \Psi^h_s\|^2_{\mathcal{H}^{-2\kappa}} = \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^{-2\kappa}|\widehat{\Psi^h}(t,\omega) - \widehat{\Psi^h}(s,\omega)|^2 \lesssim |t-s|^{2\kappa}\|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}.
$$

By the definition of the $C^\kappa_T\mathcal{H}^{-2\kappa}(\mathbb{T}^2)$-norm, we obtain the bound

$$
\|\Psi^h\|_{C^\kappa_T\mathcal{H}^{-2\kappa}} \lesssim \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}.
$$

To establish $\Psi^h \in C_T L^2(\mathbb{T}^2)$, we approximate and use the completeness of $C_T L^2(\mathbb{T}^2)$ under the supremum-norm. Let $(h_\delta)_{\delta>0}$ be a sequence such that $h_\delta \in L^2([0,T];\mathcal{H}^\gamma(\mathbb{T}^2;\mathbb{R}^2))$ for every $\delta>0$ and $\gamma \ge 0$, and such that $h_\delta \to h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ as $\delta\to 0$. An application of (H.0.2) yields for all $s,t\in[0,T]$ with $s<t$,

$$
\begin{aligned}
\|\Psi^{h_\delta}_t - \Psi^{h_\delta}_s\|^2_{\mathcal{H}^{\gamma-2\kappa}} &\lesssim |t-s|^{2\kappa}\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^{\gamma}\|\widehat{h_\delta}(\omega)\|^2_{L^2([0,T];\mathbb{R}^2)} \\
&= |t-s|^{2\kappa}\|h_\delta\|^2_{L^2([0,T];\mathcal{H}^\gamma(\mathbb{T}^2;\mathbb{R}^2))},
\end{aligned}
$$

hence $\|\Psi^{h_\delta}\|_{C^\kappa_T\mathcal{H}^{\gamma-2\kappa}} < \infty$, which, upon choosing $\gamma \ge 2\kappa$, implies $\Psi^{h_\delta} \in C_T L^2(\mathbb{T}^2)$. Another application of (H.0.2) allows us to deduce

$$
\|\Psi^h - \Psi^{h_\delta}\|^2_{C_T L^2} \lesssim \|h-h_\delta\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} \to 0 \quad \text{as } \delta\to 0
$$

and using that $C_T L^2(\mathbb{T}^2)$ is complete we obtain $\mathbf{I}^h \in C_T L^2(\mathbb{T}^2)$, which yields the claim. $\square$

The next lemma establishes the regularity of $\mathbf{I}^h$ as a square-integrable function in time.

Lemma H.0.2. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 \geq 0$ and $\overline{\rho_0} = 1$, $\rho_{\det}$ be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$ and $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$ it holds that*

$$\|\mathbf{I}^h\|_{L_T^2 \mathcal{H}^1} \lesssim \|\sqrt{\rho_{\det}}\|_{C([0,T]\times\mathbb{T}^2)} \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}.$$

and in particular $\mathbf{I}^h \in L_T^2 \mathcal{H}^1(\mathbb{T}^2)$.

Proof. As in the proof of Lemma H.0.1, we may assume that $\mathbf{I}^h = \nabla \cdot \mathcal{I}[h]$. Using (H.0.1), we decompose

$$\|\mathbf{I}^h\|_{L_T^2\mathcal{H}^1}^2 = \int_0^T \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} (1+|2\pi\omega|^2) \Big| \sum_{j=1}^2 2\pi \mathrm{i} \omega^j \int_0^t \mathrm{e}^{-|t-u||2\pi\omega|^2} \widehat{h}(u,\omega,j) \,\mathrm{d}u \Big|^2 \,\mathrm{d}t.$$

We then apply the Cauchy–Schwarz inequality followed by Young's convolution inequality to estimate

$$\begin{aligned}
\|\mathbf{I}^h\|_{L_T^2\mathcal{H}^1}^2 &\leq \int_0^T \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} (1+|2\pi\omega|^2)|2\pi\omega|^2 \sum_{j=1}^2 \Big| \int_0^t \mathrm{e}^{-|t-u||2\pi\omega|^2} \widehat{h}(u,\omega,j)\,\mathrm{d}u \Big|^2 \,\mathrm{d}t \\
&\leq \sum_{j=1}^2 \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} (1+|2\pi\omega|^2)|2\pi\omega|^2 \Big(\int_0^T \mathrm{e}^{-t|2\pi\omega|^2} \,\mathrm{d}t \Big)^2 \Big(\int_0^T |\widehat{h}(t,\omega,j)|^2 \,\mathrm{d}t \Big) \\
&\lesssim \sum_{j=1}^2 \sum_{\omega\in\mathbb{Z}^2} \int_0^T |\widehat{h}(t,\omega,j)|^2 \,\mathrm{d}t = \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}^2,
\end{aligned}$$

which yields the claim. $\square$

Let $\mathbb{X}^h := (\mathbf{I}^h, \mathbf{Y}^h, \mathbf{\Psi}^h, \mathbf{V}^h)$ be the enhancement driven by the Cameron–Martin element $h \in L^2([0,T] \times \mathbb{T}^2; \mathbb{R}^2)$, i.e.

$$\mathbf{I}^h := \nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}} h], \quad \mathbf{Y}^h := \nabla \cdot \mathcal{I}[\mathbf{I}^h \nabla \Phi_{\mathbf{I}^h}],$$

$$\mathbf{\Psi}^h := \mathbf{Y}^h \odot \nabla\Phi_{\mathbf{I}^h} + \nabla\Phi_{\mathbf{Y}^h} \odot \mathbf{I}^h, \quad \mathbf{V}^h := \nabla\mathcal{I}[\mathbf{I}^h] \odot \nabla\Phi_{\mathbf{I}^h} + \nabla^2\mathcal{I}[\Phi_{\mathbf{I}^h}] \odot \mathbf{I}^h. \tag{H.0.3}$$

We can construct $\mathbb{X}^h$ by combining Lemma H.0.1, Schauder's estimate (Lemma B.5.2) and Bony's estimate.

Lemma H.0.3. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 \geq 0$ and $\overline{\rho_0} = 1$, ρ_{det} be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$, $h \in L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$, $\alpha < -2$ and $\kappa \in (0,1/2)$, it holds that $\mathbb{X}^h = (\mathbf{I}^h, \mathbf{Y}^h, \mathbf{\Psi}^h, \mathbf{V}^h) \in \mathcal{X}_T^{\alpha,\kappa}$ is well-defined as a distribution and*

$$\|\mathbb{X}^h\|_{\mathcal{X}_T^{\alpha,\kappa}} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|_{C_T L^\infty} \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} (1 \vee \|\sqrt{\rho_{\mathrm{det}}}\|^2_{C_T L^\infty} \|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}).$$

Proof. Let $\vartheta \in (0,1/2)$ and $\kappa \in (0,1/2)$ be such that $3\vartheta < 1-2\kappa$. We bound the $\mathscr{L}_T^\kappa \mathcal{C}^{-1-\vartheta}(\mathbb{T}^2)$-norm of $\mathbf{I}^h$ by the Besov embedding [GIP15, Lem. A.2] and Lemma H.0.1,

$$\|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-1-\vartheta}} \lesssim \|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{-\vartheta}} \lesssim \|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{0}} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|_{C([0,T]\times\mathbb{T}^2)} \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}.$$

We bound the $\mathscr{L}_T^\kappa \mathcal{C}^{-2\vartheta}(\mathbb{T}^2)$-norm of $\mathbf{Y}^h$ by Schauder's estimate (Lemma B.5.2), the Besov embedding and Bony's estimate,

$$\|\mathbf{Y}^h\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-2\vartheta}} \lesssim_T \|\mathbf{I}^h \nabla\Phi_{\mathbf{I}^h}\|_{C_T \mathcal{C}^{-1-2\vartheta}} \lesssim \|\mathbf{I}^h \nabla\Phi_{\mathbf{I}^h}\|_{C_T \mathcal{B}_{2,2}^{-2\vartheta}} \lesssim \|\mathbf{I}^h\|_{C_T \mathcal{B}_{2,2}^{-\vartheta}} \|\nabla\Phi_{\mathbf{I}^h}\|_{C_T \mathcal{B}_{2,2}^{1-\vartheta}}.$$

We bound the $\mathscr{L}_T^\kappa \mathcal{C}^{-3\vartheta}(\mathbb{T}^2)$-norm of $\mathbf{\Psi}^h$ by the Besov embedding and Bony's resonant product estimate,

$$\begin{aligned}
\|\mathbf{\Psi}^h\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-3\vartheta}} &\lesssim \|\mathbf{\Psi}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{1-3\vartheta}} \\
&\lesssim \|\mathbf{Y}^h\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-2\vartheta}} \|\nabla\Phi_{\mathbf{I}^h}\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{1-\vartheta}} + \|\nabla\Phi_{\mathbf{Y}^h}\|_{\mathscr{L}_T^\kappa \mathcal{C}^{1-2\vartheta}} \|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{-\vartheta}}.
\end{aligned}$$

We bound the $\mathscr{L}_T^\kappa \mathcal{C}^{-2\vartheta}(\mathbb{T}^2)$-norm of $\mathbf{V}^h$ by the Besov embedding, Bony's resonant product estimate and Schauder's estimate,

$$\begin{aligned}
\|\mathbf{V}^h\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-2\vartheta}} &\lesssim \|\mathbf{V}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{1-2\vartheta}} \\
&\lesssim \|\nabla\mathcal{I}[\mathbf{I}^h]\|_{\mathscr{L}_T^\kappa \mathcal{C}^{-\vartheta}} \|\nabla\Phi_{\mathbf{I}^h}\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{1-\vartheta}} + \|\nabla^2\mathcal{I}[\Phi_{\mathbf{I}^h}]\|_{\mathscr{L}_T^\kappa \mathcal{C}^{1-\vartheta}} \|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{-\vartheta}} \\
&\lesssim_T \|\mathbf{I}^h\|_{C_T \mathcal{C}^{-1-\vartheta}} \|\mathbf{I}^h\|_{\mathscr{L}_T^\kappa \mathcal{B}_{2,2}^{-\vartheta}}.
\end{aligned}$$

We can now choose $\alpha = -2 - \vartheta$ to obtain

$$\|\mathbb{X}^h\|_{\mathcal{X}_T^{\alpha,\kappa}} \lesssim \|\sqrt{\rho_{\mathrm{det}}}\|_{C_T L^\infty} \|h\|_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)} \big(1 \vee \|\sqrt{\rho_{\mathrm{det}}}\|^2_{C_T L^\infty} \|h\|^2_{L^2([0,T]\times\mathbb{T}^2;\mathbb{R}^2)}\big).$$

To show that $\mathbb{X}^h \in \boldsymbol{\mathcal{X}}_T^{\alpha,\kappa}$ it suffices to construct a sequence of smooth approximations to $\mathbf{I}^h$, see e.g. the proof of Lemma H.0.1. $\square$

I

Regular Solutions to the Stochastic Heat Equation

In this appendix we analyse the regularity of the mild solution $\boldsymbol{\uparrow}^\delta = \nabla \cdot \mathcal{I}[\sqrt{\rho_{\text{det}}}\boldsymbol{\xi}^\delta]$, where $\delta > 0$ is the correlation length and $\boldsymbol{\xi}^\delta = \psi_\delta * \boldsymbol{\xi}$ denotes the mollification (in space) of a vector-valued space-time white noise $\boldsymbol{\xi}$ against a mollifier ψ_δ defined as in (1.2.7). Our goal is to establish that $\boldsymbol{\uparrow}^\delta \in C_T L^2(\mathbb{T}^2) \cap L_T^2 \mathcal{H}^1(\mathbb{T}^2)$, if the initial data ρ_0 of ρ_{det} is assumed to be continuous, positive and of unit mass (Proposition I.0.4).

To show Proposition I.0.4, we first need to prove generic bounds on $\boldsymbol{\uparrow}^\delta$ that depend on $\sqrt{\rho_{\text{det}}}$ only through its existence and regularity (cf. Lemmas I.0.1, I.0.2 and I.0.3). Therefore, to keep the notation concise, we denote $\sigma = \sqrt{\rho_{\text{det}}}$ and treat σ as a generic function of a given regularity until further notice.

Let $\gamma \geq 0$, $\gamma' \in [-1, \gamma)$, $\kappa \in [0, 1/2)$ and $\eta \in [0, 1/2)$ be such that $2\eta < \gamma - \gamma'$ and $2\eta < 1 - 2\kappa$. Below we establish $\boldsymbol{\uparrow}^\delta \in \mathscr{L}_T^\kappa \mathcal{H}^{\gamma'}(\mathbb{T}^2)$ if $\sigma \in C_{\eta;T}\mathcal{H}^\gamma(\mathbb{T}^2)$ (Lemma I.0.1), $\boldsymbol{\uparrow}^\delta \in L_T^2 \mathcal{H}^\gamma(\mathbb{T}^2)$ if $\sigma \in L_T^2 \mathcal{H}^\gamma(\mathbb{T}^2)$ (Lemma I.0.2) and $\boldsymbol{\uparrow}^\delta \in L_T^\infty \mathcal{H}^{\gamma'}(\mathbb{T}^2)$ if $\sigma \in C_T \mathcal{H}^{(\gamma' \vee 0)}(\mathbb{T}^2) \cap L_T^2 \mathcal{H}^{\gamma'+1}(\mathbb{T}^2)$ (Lemma I.0.3), each almost surely. We then use all of these results to establish Proposition I.0.4.

The Fourier transform of a vector-valued space-time white noise is given by a family complex-valued Brownian motions $W^j(t, m) \coloneqq \boldsymbol{\xi}(\mathbb{1}_{[0,t]}(\,\cdot\,) \otimes \mathrm{e}^{-2\pi \mathrm{i}\langle m, \cdot \rangle} \otimes \delta_{j, \cdot})$, where $t \geq 0$, $m \in \mathbb{Z}^2$ and $j = 1, 2$ (see Lemma C.3.1). Therefore, we can represent the Fourier transform of Ψ^{δ} by

$$\widehat{\Psi^{\delta}}(t,\omega) = \sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\int_0^t H^j_{t-u}(\omega)\widehat{\sigma}(u,\omega-m)\varphi(\delta m)\,\mathrm{d}W^j(u,m), \qquad t\in[0,T],\ \omega\in\mathbb{Z}^2, \tag{I.0.1}$$

see also (2.2.5). Hence taking the Fourier inverse, we can deduce the real-space representation

$$\Psi^{\delta}(t,x) = \sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)\int_0^t \partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})(x)\,\mathrm{d}W^j(u,m), \qquad x\in\mathbb{T}^2. \tag{I.0.2}$$

The next lemma establishes the regularity of Ψ^{δ} in an interpolation space (see Section B.3).

Lemma I.0.1. *Let $T>0$, $\gamma\geq 0$, $\gamma'\in[-1,\gamma)$, $\kappa\in[0,1/2)$ and $\eta\in[0,1/2)$ be such that $2\eta<\gamma-\gamma'$ and $2\eta<1-2\kappa$. Then for all $\sigma\in C_{\eta;T}\mathcal{H}^{\gamma}(\mathbb{T}^2)$, $\delta>0$ and $p\in[1,\infty)$ it holds that*

$$\mathbb{E}[\|\Psi^{\delta}\|^p_{\mathscr{L}^{\kappa}_T\mathcal{H}^{\gamma'}}]^{1/p} \lesssim_{p,T} (1+\delta^{-\gamma-1})\|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma}} \tag{I.0.3}$$

and $\Psi^{\delta}\in\mathscr{L}^{\kappa}_T\mathcal{H}^{\gamma'}(\mathbb{T}^2)$ almost surely.

Proof. Using the real-space representation (I.0.2), we first establish for $\gamma\geq 0$, $\gamma'\in[-1,\gamma)$ and $\eta\in[0,1/2)$ satisfying $2\eta\leq\gamma-\gamma'$, that

$$\sup_{t\in[0,T]}\mathbb{E}[\|\Psi^{\delta}_t\|^2_{\mathcal{H}^{\gamma'}}]^{1/2} \lesssim_T (1+\delta^{-\gamma-1})\|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma}} \tag{I.0.4}$$

and in particular $\Psi^{\delta}_t\in\mathcal{H}^{\gamma'}(\mathbb{T}^2)$ almost surely for each $t\in[0,T]$.

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{T}^2)$. Using the definition of $\mathcal{H}^{\gamma'}(\mathbb{T}^2)$ and Parseval's identity, we obtain for every $t\in[0,T]$,

$$\begin{aligned}\|\Psi^{\delta}_t\|^2_{\mathcal{H}^{\gamma'}} &= \|(1-\Delta)^{\gamma'/2}\Psi^{\delta}_t\|^2_{L^2}\\ &= \sum_{k=1}^{\infty}\left|\left\langle e_k, \sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)\int_0^t(1-\Delta)^{\gamma'/2}\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\,\mathrm{d}W^j(u,m)\right\rangle_{L^2}\right|^2.\end{aligned}$$

An application of the stochastic Fubini theorem [Ver12] combined with Itô's isometry yields

$$\begin{aligned}\mathbb{E}[\|\Psi^{\delta}_{t}\|^2_{\mathcal{H}^{\gamma'}}] &= \sum_{k=1}^{\infty}\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_0^t |\langle e_k, (1-\Delta)^{\gamma'/2}\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\rangle_{L^2}|^2\,\mathrm{d}u \\ &= \sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_0^t \|(1-\Delta)^{\gamma'/2}\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{L^2}\,\mathrm{d}u.\end{aligned} \tag{I.0.5}$$

We now distinguish the cases $\gamma' \in [-1, \gamma-1)$ and $\gamma' \in [\gamma-1, \gamma)$. First let us assume that $\gamma' \in [\gamma-1,\gamma)$. We can bound the integrand by an application of Schauder's estimate (Lemma B.5.1),

$$\begin{aligned}\|(1-\Delta)^{\gamma'/2}\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|_{L^2} &\lesssim \|P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|_{\mathcal{H}^{\gamma'+1}} \\ &\lesssim (1 \vee |t-u|^{-\frac{1+\gamma'-\gamma}{2}})\|\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle}\|_{\mathcal{H}^{\gamma}},\end{aligned}$$

where we used that $\gamma \leq \gamma'+1$. We bound by a change of variables and Jensen's inequality uniformly in $u \in [0,T]$,

$$\begin{aligned}&\|\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle}\|^2_{\mathcal{H}^{\gamma}} \\ &= \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^{\gamma}|\widehat{\sigma}(u,\omega-m)|^2 = \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi(\omega+m)|^2)^{\gamma}|\widehat{\sigma}(u,\omega)|^2 \\ &\lesssim \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2+|2\pi m|^2)^{\gamma}|\widehat{\sigma}(u,\omega)|^2 \lesssim \|\sigma(u,\,\cdot\,)\|^2_{\mathcal{H}^{\gamma}} + (1+|2\pi m|^2)^{\gamma}\|\sigma(u,\,\cdot\,)\|^2_{L^2} \\ &\lesssim (1+|2\pi m|^2)^{\gamma}\|\sigma(u,\,\cdot\,)\|^2_{\mathcal{H}^{\gamma}},\end{aligned} \tag{I.0.6}$$

where the second inequality follows for $\gamma \in [0,1]$ by the sub-additivity of $x \mapsto x^{\gamma}$ and for $\gamma > 1$ by another application of Jensen's inequality.

Consequently if $\gamma' \in [\gamma-1,\gamma)$ we may estimate uniformly in $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}[\|\Psi^{\delta}_{t}\|^2_{\mathcal{H}^{\gamma'}}] &\lesssim \|\sigma\|^2_{C_{\eta;T}\mathcal{H}^{\gamma}}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2(1+|2\pi m|^2)^{\gamma}\int_0^t (1\vee|t-u|^{-\frac{1+\gamma'-\gamma}{2}})^2(1\wedge u)^{-2\eta}\,\mathrm{d}u \\ &\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^{\gamma}},\end{aligned} \tag{I.0.7}$$

where we used $\gamma' < \gamma$ and $2\eta < 1$ to integrate the singularity; $2\eta \leq \gamma-\gamma'$ to obtain a bound that is uniform in $t \in [0,T]$; and $\mathrm{supp}(\varphi) \subset B(0,1)$ to estimate the sum over $m \in \mathbb{Z}^2$.

For $\gamma' \in [-1, \gamma - 1)$, we estimate instead

$$\mathbb{E}[\|\upharpoonleft_t^\delta\|_{\mathcal{H}^{\gamma'}}^2] \lesssim_T (1+\delta^{-1})^{2(\gamma'+2)} \|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma'+1}}^2 \lesssim (1+\delta^{-1})^{2(\gamma+1)} \|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma}}^2,$$

where in the first inequality we used that $\gamma' \in [\gamma', \gamma'+1)$ and (I.0.7) with $\gamma = \gamma' + 1 \geq 0$.

To summarise, it follows for all $\gamma \geq 0$, $\gamma' \in [-1, \gamma)$ and $\eta \in [0, 1/2)$ satisfying $2\eta \leq \gamma - \gamma'$, that

$$\sup_{t \in [0,T]} \mathbb{E}[\|\upharpoonleft_t^\delta\|_{C_T\mathcal{H}^{\gamma'}}^2] \lesssim_T (1+\delta^{-1})^{2(\gamma+1)} \|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma}}^2.$$

which yields (I.0.4) and that $\upharpoonleft_t^\delta \in \mathcal{H}^{\gamma'}(\mathbb{T}^2)$ almost surely for all $t \in [0, T]$.

Next by an application of Kolmogorov's continuity criterion we establish (I.0.3) and the parabolic regularity $\upharpoonleft^\delta \in \mathscr{L}_T^\kappa \mathcal{H}^{\gamma'}(\mathbb{T}^2)$ for arbitrary $\gamma \geq 0$, $\gamma' \in [-1, \gamma)$, $\kappa \in [0, 1/2)$ and $\eta \in [0, 1/2)$ satisfying $2\eta < \gamma - \gamma'$ and $2\eta < 1 - 2\kappa$.

Let $\kappa' \in (\kappa, 1/2]$ be such that $2\eta \leq 1 - 2\kappa'$. Let $s < t \in [0, T]$; we distinguish the cases $|t-s| \leq 1$ and $|t-s| > 1$ and first assume $|t-s| \leq 1$. Using the representation

$$\upharpoonleft_t^\delta = P_{t-s}\upharpoonleft_s^\delta + \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} \varphi(\delta m) \int_s^t \partial_j P_{t-u}(\sigma(u, \cdot\,) \mathrm{e}^{2\pi \mathrm{i}\langle m, \cdot\, \rangle}) \, \mathrm{d}W^j(u, m)$$

combined with the fact that increments of $(W^j(\,\cdot\,, m))_{m \in \mathbb{Z}^2, j \in \{1,2\}}$ over $[s, t]$ are independent of $\upharpoonleft_s^\delta$, we obtain

$$\begin{aligned}\mathbb{E}[\|\upharpoonleft_t^\delta - \upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'-2\kappa'}}^2] &= \mathbb{E}[\|P_{t-s}\upharpoonleft_s^\delta - \upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'-2\kappa'}}^2] \\ &\quad + \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \int_s^t \|\partial_j P_{t-u}(\sigma(u, \cdot\,) \mathrm{e}^{2\pi \mathrm{i}\langle m, \cdot\, \rangle})\|_{\mathcal{H}^{\gamma'-2\kappa'}}^2 \, \mathrm{d}u.\end{aligned} \tag{I.0.8}$$

We bound the first summand in (I.0.8) by Schauder's estimate (Lemma B.5.1),

$$\|P_{t-s}\upharpoonleft_s^\delta - \upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'-2\kappa'}} \lesssim |t-s|^{\kappa'} \|\upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'}},$$

where we used that $\gamma' - 2\kappa' \leq \gamma' \leq \gamma' - 2\kappa' + 2$. Therefore by (I.0.4), we obtain uniformly in $s < t \in [0, T]$,

$$\mathbb{E}[\|P_{t-s}\upharpoonleft_s^\delta - \upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'-2\kappa'}}^2] \lesssim |t-s|^{2\kappa'} \mathbb{E}[\|\upharpoonleft_s^\delta\|_{\mathcal{H}^{\gamma'}}^2] \lesssim_T |t-s|^{2\kappa'} (1+\delta^{-\gamma-1})^2 \|\sigma\|_{C_{\eta;T}\mathcal{H}^{\gamma}}^2. \tag{I.0.9}$$

To control the second summand in (I.0.8), we further distinguish the cases $2\kappa' \leq 1+\gamma'-\gamma$ and $2\kappa' > 1+\gamma'-\gamma$. Assume $2\kappa' \leq 1+\gamma'-\gamma$, we estimate the integrand as in (I.0.4) and obtain the singularity

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_s^t \|\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}\,\mathrm{d}u\\ &\lesssim (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^{\gamma}}\int_s^t (1\vee|t-u|^{-\frac{1+\gamma'-\gamma-2\kappa'}{2}})^2(1\wedge u)^{-2\eta}\,\mathrm{d}u.\end{aligned}$$

Using that $|t-s|\leq 1$, we may bound

$$\begin{aligned}\int_s^t (1\vee|t-u|^{-\frac{1+\gamma'-\gamma-2\kappa'}{2}})^2(1\wedge u)^{-2\eta}\,\mathrm{d}u &= \int_s^t |t-u|^{-(1+\gamma'-\gamma-2\kappa')}(1\wedge u)^{-2\eta}\,\mathrm{d}u\\ &\lesssim_T |t-s|^{2\kappa'},\end{aligned}$$

where the singularity is integrable by $\gamma' < \gamma$ and $2\eta < 1$; and the bound is uniform since $2\eta \leq \gamma-\gamma'$. Therefore if $2\kappa' \leq 1+\gamma'-\gamma$, we can control the second summand in (I.0.8) by

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_s^t \|\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}\,\mathrm{d}u\\ &\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^{\gamma}}|t-s|^{2\kappa'}.\end{aligned}$$

Now assume $2\kappa' > 1+\gamma'-\gamma$; we may skip the application of Schauder's estimate, that is,

$$\|P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|_{\mathcal{H}^{\gamma'-2\kappa'+1}} \lesssim \|P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|_{\mathcal{H}^{\gamma}} \lesssim \|\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle}\|_{\mathcal{H}^{\gamma}},$$

to control the second summand in (I.0.8) by the singularity

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_s^t \|\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}\,\mathrm{d}u\\ &\lesssim (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^{\gamma}}\int_s^t (1\wedge u)^{-2\eta}\,\mathrm{d}u.\end{aligned}$$

We may bound

$$\int_s^t (1\wedge u)^{-2\eta}\,\mathrm{d}u \lesssim_T |t-s|^{2\kappa'},$$

where the singularity is integrable by $2\eta < 1$; and the bound is uniform by $2\eta \leq 1-2\kappa'$. Therefore if $2\kappa' > 1+\gamma'-\gamma$, we can control the second summand in (I.0.8) by

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_s^t\|\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}\,\mathrm{d}u\\ &\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{2\kappa'}.\end{aligned}$$

To summarize, for all $\kappa'\in(\kappa,1/2]$ such that $2\eta\le 1-2\kappa'$, we can bound the second summand in (I.0.8) by

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_s^t\|\partial_j P_{t-u}(\sigma(u,\,\cdot\,)\mathrm{e}^{2\pi\mathrm{i}\langle m,\,\cdot\,\rangle})\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}\,\mathrm{d}u\\ &\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{2\kappa'}.\end{aligned}\tag{I.0.10}$$

Hence by (I.0.8), (I.0.9) and (I.0.10), we obtain uniformly in $|t-s|\le 1$ the bound

$$\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{2\kappa'}.$$

For $|t-s|>1$, we instead estimate by (I.0.4),

$$\begin{aligned}\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]&\lesssim\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]+\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_s\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]\\ &\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{2\kappa'},\end{aligned}$$

which yields for all $s,t\in[0,T]$ the bound

$$\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]\lesssim_T (1+\delta^{-1})^{2(\gamma+1)}\|\sigma\|^2_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{2\kappa'}.$$

Let $p\in[1,\infty)$, to control the $L^p(\mathbb{P})$-norm, we use Nelson's estimate [FV07, Lem. 2] and Hölder's inequality to deduce

$$\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^p_{\mathcal{H}^{\gamma'-2\kappa'}}]^{1/p}\lesssim_p\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^2_{\mathcal{H}^{\gamma'-2\kappa'}}]^{1/2},$$

which, together with the $L^2(\mathbb{P})$-bound above, yields

$$\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta_t-\rotatebox[origin=c]{180}{!}^\delta_s\|^p_{\mathcal{H}^{\gamma'-2\kappa'}}]^{1/p}\lesssim_{p,T}(1+\delta^{-\gamma-1})\|\sigma\|_{C_{\eta;T}\mathcal{H}^\gamma}|t-s|^{\kappa'}.$$

An application of Kolmogorov's continuity criterion [FV10, Thm. A.10] then implies that there exists a modification (which we do not relabel) such that for all $p>1/\kappa'$ and $\kappa\in[0,\kappa'-1/p)$,

$$\mathbb{E}[\|\rotatebox[origin=c]{180}{!}^\delta\|^p_{C^\kappa_T\mathcal{H}^{\gamma'-2\kappa'}}]^{1/p}\lesssim_{p,T}(1+\delta^{-\gamma-1})\|\sigma\|_{C_{\eta;T}\mathcal{H}^\gamma}.\tag{I.0.11}$$

Another application of Hölder's inequality shows that this bound continuous to hold for every $\kappa \in [0, \kappa')$ and $p \in [1, \infty)$.

The next step is to generalise the result to arbitrary exponents $\kappa \in [0, 1/2)$ and to avoid the loss of $2(\kappa' - \kappa)$ space regularity in (I.0.11). Let $\gamma \geq 0$, $\gamma' \in [-1, \gamma)$, $\kappa \in [0, 1/2)$ and $\eta \in [0, 1/2)$ be such that $2\eta < \gamma - \gamma'$ and $2\eta < 1 - 2\kappa$. We can find $\kappa' \in (\kappa, 1/2]$ such that $\gamma' + 2(\kappa' - \kappa) \in [-1, \gamma)$, $2\eta \leq \gamma - (\gamma' + 2(\kappa' - \kappa))$ and $2\eta \leq 1 - 2\kappa'$. An application of (I.0.11) with these inflated exponents yields

$$\mathbb{E}[\|\text{¡}^{\delta}\|^p_{C^\kappa_T \mathcal{H}^{\gamma' - 2\kappa}}]^{1/p} = \mathbb{E}[\|\text{¡}^{\delta}\|^p_{C^\kappa_T \mathcal{H}^{\gamma' + 2(\kappa' - \kappa) - 2\kappa'}}]^{1/p} \lesssim_{p,T} (1 + \delta^{-\gamma - 1}) \|\sigma\|_{C_{\eta;T} \mathcal{H}^\gamma},$$

which proves (I.0.3). Further, an application of Lemma B.3.3 shows $\text{¡}^{\delta} \in \mathscr{L}^\kappa_T \mathcal{H}^{\gamma'}(\mathbb{T}^2)$ almost surely. □

The next lemma establishes the regularity of ¡^{δ} as a square-integrable function in time.

Lemma I.0.2. *Let $T > 0$, $\gamma \geq 0$ and $\sigma \in L^2_T \mathcal{H}^\gamma(\mathbb{T}^2)$. Then for all $\delta > 0$ and $p \in [1, \infty)$ it holds that*

$$\mathbb{E}[\|\text{¡}^{\delta}\|^p_{L^2_T \mathcal{H}^\gamma}]^{1/p} \lesssim_p (1 + \delta^{-\gamma - 1}) \|\sigma\|_{L^2_T \mathcal{H}^\gamma}.$$

and in particular $\text{¡}^{\delta} \in L^2_T \mathcal{H}^\gamma(\mathbb{T}^2)$ almost surely.

Proof. We apply Itô's isometry (I.0.5), Parseval's theorem and Tonelli's theorem to obtain

$$\begin{aligned}
&\mathbb{E}[\|\text{¡}^{\delta}\|^2_{L^2_T \mathcal{H}^\gamma}] \\
&= \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \int_0^T \int_0^t \|(1 - \Delta)^{\gamma/2} \partial_j P_{t-u}(\sigma(u, \cdot\,) \mathrm{e}^{2\pi \mathrm{i} \langle m, \cdot \rangle})\|^2_{L^2} \,\mathrm{d}u \,\mathrm{d}t \\
&= \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \sum_{\omega \in \mathbb{Z}^2} (1 + |2\pi\omega|^2)^\gamma |2\pi\omega|^2 \int_0^T \int_0^t \mathrm{e}^{-2|t-u||2\pi\omega|^2} |\widehat{\sigma}(u, \omega - m)|^2 \,\mathrm{d}u \,\mathrm{d}t \\
&= \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \sum_{\omega \in \mathbb{Z}^2} (1 + |2\pi\omega|^2)^\gamma |2\pi\omega|^2 \int_0^T |\widehat{\sigma}(u, \omega - m)|^2 \int_u^T \mathrm{e}^{-2|t-u||2\pi\omega|^2} \,\mathrm{d}t \,\mathrm{d}u \\
&= \frac{1}{2} \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \sum_{\omega \in \mathbb{Z}^2} (1 + |2\pi\omega|^2)^\gamma \int_0^T |\widehat{\sigma}(u, \omega - m)|^2 (1 - \mathrm{e}^{-2|T-u||2\pi\omega|^2}) \,\mathrm{d}u,
\end{aligned}$$

which we can bound by

$$\mathbb{E}[\| \mathbf{\uparrow}^{\delta} \|_{L_T^2 \mathcal{H}^\gamma}^2] \lesssim \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 \int_0^T \sum_{\omega \in \mathbb{Z}^2} (1 + |2\pi\omega|^2)^\gamma |\widehat{\sigma}(u, \omega - m)|^2 \, \mathrm{d}u.$$

Using that $\gamma \geq 0$, we bound the integrand as in (I.0.6) uniformly in $u \in [0,T]$,

$$\sum_{\omega \in \mathbb{Z}^2} (1 + |2\pi\omega|^2)^\gamma |\widehat{\sigma}(u, \omega - m)|^2 \lesssim (1 + |2\pi m|^2)^\gamma \|\sigma(u, \,\cdot\,)\|_{\mathcal{H}^\gamma}^2,$$

which yields

$$\begin{aligned} \mathbb{E}[\| \mathbf{\uparrow}^{\delta} \|_{L_T^2 \mathcal{H}^\gamma}^2] &\lesssim \sum_{m \in \mathbb{Z}^2} \varphi(\delta m)^2 (1 + |2\pi m|^2)^\gamma \int_0^T \|\sigma(u, \,\cdot\,)\|_{\mathcal{H}^\gamma}^2 \, \mathrm{d}u \\ &\lesssim (1 + \delta^{-1})^{2(\gamma+1)} \|\sigma\|_{L_T^2 \mathcal{H}^\gamma}^2, \end{aligned}$$

where in the last inequality we applied (E.1.1) to control the sum over $m \in \mathbb{Z}^2$. The claim for arbitrary $p \in [1, \infty)$ then follows by Nelson's estimate [FV07, Lem. 2] and Hölder's inequality. □

The next lemma establishes a uniform bound on $\mathbf{\uparrow}^{\delta}$ without trading space regularity for time regularity (cf. Lemma I.0.1).

Lemma I.0.3. *Let $T > 0$, $\gamma \in [-1, \infty)$ and $\sigma \in C_T \mathcal{H}^{\gamma \vee 0}(\mathbb{T}^2) \cap L_T^2 \mathcal{H}^{\gamma+1}(\mathbb{T}^2)$. Then for all $\delta > 0$ and $p \in [1, \infty)$ it holds that*

$$\mathbb{E}[\| \mathbf{\uparrow}^{\delta} \|_{L_T^\infty \mathcal{H}^\gamma}^p]^{1/p} \lesssim_p (1 + \delta^{-\gamma-2}) \max\{\|\sigma\|_{C_T \mathcal{H}^{\gamma \vee 0}}, \|\sigma\|_{L_T^2 \mathcal{H}^{\gamma+1}}\}.$$

Proof. For all $\omega \in \mathbb{Z}^2$ we obtain by (I.0.1) the stochastic differential equation of the Fourier transform $(\widehat{\mathbf{\uparrow}^{\delta}}(t, \omega))_{t \in [0,T]}$,

$$\mathrm{d}\widehat{\mathbf{\uparrow}^{\delta}}(t, \omega) = -|2\pi\omega|^2 \widehat{\mathbf{\uparrow}^{\delta}}(t, \omega) \, \mathrm{d}t + \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} 2\pi \mathrm{i} \omega^j \widehat{\sigma}(t, \omega - m) \varphi(\delta m) \, \mathrm{d}W^j(t, m),$$

whose (complex) martingale part we denote by

$$\widehat{M^\delta}(t, \omega) \coloneqq \sum_{j=1}^{2} \sum_{m \in \mathbb{Z}^2} \int_0^t 2\pi \mathrm{i} \omega^j \widehat{\sigma}(u, \omega - m) \varphi(\delta m) \, \mathrm{d}W^j(u, m).$$

The (sesquilinear) covariation of $\widehat{M^\delta}(t, \omega)$ is given by

$$\mathrm{d}[\widehat{M^\delta}(\,\cdot\,, \omega), \widehat{M^\delta}(\,\cdot\,, \omega')]_t = \langle 2\pi\omega, 2\pi\omega' \rangle \sum_{m \in \mathbb{Z}^2} \widehat{\sigma}(t, \omega - m) \overline{\widehat{\sigma}(t, \omega' - m)} \varphi(\delta m)^2 \, \mathrm{d}t$$

and we denote the complex quadratic variation by $[\,\cdot\,]_t \coloneqq [\,\cdot\,,\,\cdot\,]_t$. It then follows by an application of the complex Itô formula, that

$$
\begin{aligned}
&\mathrm{d}|\widehat{\upharpoonleft^\delta}(t,\omega)|^2\\
&= \widehat{\upharpoonleft^\delta}(t,-\omega)\,\mathrm{d}\widehat{\upharpoonleft^\delta}(t,\omega) + \widehat{\upharpoonleft^\delta}(t,\omega)\,\mathrm{d}\widehat{\upharpoonleft^\delta}(t,-\omega) + \mathrm{d}[\widehat{\upharpoonleft^\delta}(\,\cdot\,,\omega)]_t\\
&= -2|\widehat{\upharpoonleft^\delta}(t,\omega)|^2|2\pi\omega|^2\,\mathrm{d}t + \widehat{\upharpoonleft^\delta}(t,-\omega)\,\mathrm{d}\widehat{M^\delta}(t,\omega) + \widehat{\upharpoonleft^\delta}(t,\omega)\,\mathrm{d}\widehat{M^\delta}(t,-\omega) + \mathrm{d}[\widehat{M^\delta}(\,\cdot\,,\omega)]_t,
\end{aligned}
$$

which, combined with (1.2.1), yields

$$
\begin{aligned}
\|\upharpoonleft^\delta_t\|^2_{\mathcal{H}^\gamma} = -2\int_0^t \|\nabla \upharpoonleft^\delta_u\|^2_{\mathcal{H}^\gamma}\,\mathrm{d}u + 2\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma \int_0^t \widehat{\upharpoonleft^\delta}(u,-\omega)\,\mathrm{d}\widehat{M^\delta}(u,\omega)\\
+ \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma[\widehat{M^\delta}(\,\cdot\,,\omega)]_t.
\end{aligned}
\tag{I.0.12}
$$

Let $p\in[1,\infty)$, taking the supremum of (I.0.12) we can deduce the uniform bound

$$
\begin{aligned}
\|\upharpoonleft^\delta\|^p_{L^\infty_T\mathcal{H}^\gamma} \le 2^{p/2}\sup_{t\in[0,T]}\Big|\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma\int_0^t \widehat{\upharpoonleft^\delta}(u,-\omega)\,\mathrm{d}\widehat{M^\delta}(u,\omega)\Big|^{p/2}\\
+\Big(\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma[\widehat{M^\delta}(\,\cdot\,,\omega)]_T\Big)^{p/2}.
\end{aligned}
\tag{I.0.13}
$$

To control the first term in (I.0.13), we use the (complex) Burkholder–Davis–Gundy inequality [FV10, Thm. 14.6], several applications of the Cauchy–Schwarz inequality and (I.0.6) to deduce for every $p\in[2,\infty)$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma\int_0^t \widehat{\upharpoonleft^\delta}(u,-\omega)\,\mathrm{d}\widehat{M^\delta}(u,\omega)\Big|^{p/2}\Big]\\
&\lesssim_p \mathbb{E}\Big[\Big[\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma\int_0^\cdot \widehat{\upharpoonleft^\delta}(u,-\omega)\,\mathrm{d}\widehat{M^\delta}(u,\omega)\Big]_T^{p/4}\Big]\\
&\le \|\sigma\|^{p/2}_{C_T\mathcal{H}^{\gamma\vee 0}}\mathbb{E}[\|\upharpoonleft^\delta\|^{p/2}_{L^2_T\mathcal{H}^{\gamma+1}}](1+\delta^{-1})^{((\gamma\vee 0)+1)p/2}.
\end{aligned}
\tag{I.0.14}
$$

To control the second term in (I.0.13), we estimate by (I.0.6) using that $\gamma+1\ge 0$,

$$
\begin{aligned}
&\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma[\widehat{M^\delta}(\,\cdot\,,\omega)]_T\\
&= \sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma|2\pi\omega|^2\sum_{m\in\mathbb{Z}^2}\int_0^T|\widehat{\sigma}(u,\omega-m)|^2\varphi(\delta m)^2\,\mathrm{d}u\\
&= \sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2\int_0^T\sum_{\omega\in\mathbb{Z}^2}(1+|2\pi\omega|^2)^\gamma|2\pi\omega|^2|\widehat{\sigma}(u,\omega-m)|^2\,\mathrm{d}u\\
&\lesssim \sum_{m\in\mathbb{Z}^2}\varphi(\delta m)^2(1+|2\pi m|^2)^{\gamma+1}\int_0^T\|\sigma_u\|^2_{\mathcal{H}^{\gamma+1}}\,\mathrm{d}u\\
&= (1+\delta^{-1})^{2(\gamma+2)}\|\sigma\|^2_{L^2_T\mathcal{H}^{\gamma+1}}.
\end{aligned}
\tag{I.0.15}
$$

Therefore it follows by (I.0.13), (I.0.14), (I.0.15) and Lemma I.0.2 that for every $p \in [2, \infty)$,

$$\begin{aligned}\mathbb{E}[\|\mathbf{!}^\delta\|^p_{L^\infty_T \mathcal{H}^\gamma}] &\lesssim_p (1+\delta^{-1})^{((\gamma\vee 0)+1)p/2}\|\sigma\|^{p/2}_{C_T\mathcal{H}^{\gamma\vee 0}}\mathbb{E}[\|\mathbf{!}^\delta\|^{p/2}_{L^2_T\mathcal{H}^{\gamma+1}}] \\ &\quad + (1+\delta^{-1})^{(\gamma+2)p}\|\sigma\|^p_{L^2_T\mathcal{H}^{\gamma+1}} \\ &\lesssim (1+\delta^{-1})^{(\gamma+2)p}\max\{\|\sigma\|_{C_T\mathcal{H}^{\gamma\vee 0}}, \|\sigma\|_{L^2_T\mathcal{H}^{\gamma+1}}\}^p.\end{aligned}$$

To generalize the bound to $p \in [1, \infty)$, it suffices to apply Hölder's inequality, which yields the claim. □

In the next proposition we specialize $\sigma = \sqrt{\rho_{\det}}$, where $\rho_{\det}$ is a solution to the deterministic Keller–Segel equation (3.2.1) with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$ (see Lemma 3.2.3). A combination of Lemmas I.0.1, I.0.2 and I.0.3 with results of Section 3.2 yields the existence and regularity of $\mathbf{!}^\delta = \nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}}\boldsymbol{\xi}^\delta]$ in $C_T L^2(\mathbb{T}^2) \cap L^2_T\mathcal{H}^1(\mathbb{T}^2)$ if ρ_0 is assumed to be continuous, positive and of unit mass.

Proposition I.0.4. *Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$, $\rho_{\det}$ be the weak solution to* (3.2.1) *with initial data ρ_0 and chemotactic sensitivity $\chi \in \mathbb{R}$, and T^* be its maximal time of existence (see Lemma 3.2.3). Then for all $T < T^*$ and $\delta > 0$ it holds that $\mathbf{!}^\delta := \nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}}\boldsymbol{\xi}^\delta] \in C_T L^2(\mathbb{T}^2) \cap L^2_T\mathcal{H}^1(\mathbb{T}^2)$ almost surely and we can bound for every $p \in [1, \infty)$ and $\gamma \in [-1, 0]$,*

$$\mathbb{E}\Big[\|\mathbf{!}^\delta\|^p_{C_T\mathcal{H}^\gamma \cap L^2_T\mathcal{H}^{\gamma+1}}\Big]^{1/p} \lesssim_p (1+\delta^{-\gamma-2})\|\sqrt{\rho_{\det}}\|_{C_T L^2 \cap L^2_T\mathcal{H}^1}. \tag{I.0.16}$$

Proof. Let $\rho_0 \in C(\mathbb{T}^2)$ be such that $\rho_0 > 0$ and $\overline{\rho_0} = 1$. An application of Lemma 3.2.5 yields $\sqrt{\rho_{\det}} \in L^2_T\mathcal{H}^1(\mathbb{T}^2)$, which combined with Lemma I.0.2 allows us to control for every $\gamma \in [-1, 0]$,

$$\mathbb{E}[\|\mathbf{!}^\delta\|^p_{L^2_T\mathcal{H}^{\gamma+1}}]^{1/p} \lesssim_p (1+\delta^{-\gamma-2})\|\sqrt{\rho_{\det}}\|_{L^2_T\mathcal{H}^{\gamma+1}} \lesssim (1+\delta^{-\gamma-2})\|\sqrt{\rho_{\det}}\|_{L^2_T\mathcal{H}^1} \tag{I.0.17}$$

and deduce $\mathbf{!}^\delta \in L^2_T\mathcal{H}^{\gamma+1}(\mathbb{T}^2)$ almost surely.

To establish $\mathbf{!}^\delta \in C_T L^2(\mathbb{T}^2)$ almost surely, we use Lemma I.0.3 and an approximation argument. By the density of the smooth functions in $C(\mathbb{T}^2)$, we can find a sequence $(\rho_0^n)_{n\in\mathbb{N}}$ such that $\rho_0^n \in C^\infty(\mathbb{T}^2)$, $\rho_0^n > 0$, $\overline{\rho_0^n} = 1$ for every $n \in \mathbb{N}$

and $\rho_0^n \to \rho_0$ in $C(\mathbb{T}^2)$ as $n \to \infty$. Denote by $(\rho_{\det}^n)_{n\in\mathbb{N}}$ the corresponding sequence of solutions to (3.2.1) with initial data $(\rho_0^n)_{n\in\mathbb{N}}$ and chemotactic sensitivity $\chi \in \mathbb{R}$. It follows by (the proof of) Theorem 3.2.1 that $\rho_{\det}^n \to \rho_{\det}$ in $C_T L^2(\mathbb{T}^2) \cap L_T^2\mathcal{H}^1(\mathbb{T}^2)$ and an application of [BCD11a, Cor. 2.91] combined with a cut-off argument as in Lemma B.6.1 yields $\sqrt{\rho_{\det}^n} \to \sqrt{\rho_{\det}} \in C_T L^2(\mathbb{T}^2) \cap L_T^2\mathcal{H}^1(\mathbb{T}^2)$. Using that $\rho_0^n \in C^\infty(\mathbb{T}^2) \subset \mathcal{H}^\vartheta(\mathbb{T}^2)$ for every $\vartheta > 0$, we can apply Lemma 3.2.5 to deduce $\sqrt{\rho_{\det}^n} \in C_{\eta;T}\mathcal{H}^{\vartheta+2\eta}(\mathbb{T}^2)$ for each $\eta \in (0,1/2)$. Hence, Lemma I.0.1 yields $\nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}^n}\boldsymbol{\xi}^\delta] \in C_T L^2(\mathbb{T}^2)$ almost surely (where we use that $2\eta < \gamma + 2\eta$) and Lemma I.0.3 yields $\nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}^n}\boldsymbol{\xi}^\delta] \to \nabla \cdot \mathcal{I}[\sqrt{\rho_{\det}}\boldsymbol{\xi}^\delta] = \boldsymbol{\uparrow}^\delta$ in $L_T^\infty L^2(\mathbb{T}^2)$ as $n \to \infty$ almost surely along a subsequence (which we do not relabel). It follows that $\boldsymbol{\uparrow}^\delta \in C_T L^2(\mathbb{T}^2)$ almost surely by the completeness of $C_T L^2(\mathbb{T}^2)$ under the uniform norm.

Finally, another application of Lemma I.0.3 yields

$$
\begin{aligned}
\mathbb{E}[\|\boldsymbol{\uparrow}^\delta\|_{C_T\mathcal{H}^\gamma}^p]^{1/p} &\lesssim_p (1+\delta^{-\gamma-2}) \max\{\|\sqrt{\rho_{\det}}\|_{C_T L^2}, \|\sqrt{\rho_{\det}}\|_{L_T^2\mathcal{H}^{\gamma+1}}\} \\
&\lesssim (1+\delta^{-\gamma-2}) \max\{\|\sqrt{\rho_{\det}}\|_{C_T L^2}, \|\sqrt{\rho_{\det}}\|_{L_T^2\mathcal{H}^{1}}\}
\end{aligned}
$$

which combined with (I.0.17) proves (I.0.16). $\square$

J Glossaries

J.1 Function and Distribution Spaces

Space	Description	Reference
$C^\infty(\mathbb{T}^2)$	The smooth functions on $\mathbb{T}^2$.	Sec. 1.2
$\mathcal{S}'(\mathbb{T}^2)$	The distributions on $\mathbb{T}^2$.	Sec. 1.2
$\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2)$	The completion of $C^\infty(\mathbb{T}^2)$ under the norm $\lVert\cdot\rVert_{\mathcal{B}^\alpha_{p,q}}$.	Def. B.2.1
$\mathcal{C}^\alpha(\mathbb{T}^2)$	The Hölder–Besov space $\mathcal{C}^\alpha(\mathbb{T}^2) = \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^2)$.	Def. B.2.1
$\mathcal{H}^\alpha(\mathbb{T}^2)$	The Sobolev space $\mathcal{H}^\alpha(\mathbb{T}^2) = \mathcal{B}^\alpha_{2,2}(\mathbb{T}^2)$.	Sec. 1.2
$C_T E$	The continuous functions $f\colon [0,T] \to E$.	Sec. 1.2
$C^\kappa_T E$	The closure of smooth functions in the κ-Hölder norm.	Sec. 1.2
$C_{\eta;T} E$	The continuous functions $f\colon (0,T] \to E$ with blow-up of at most $t^{-\eta}$.	Sec. 1.2
$C^\kappa_{\eta;T} E$	The κ-Hölder functions $f\colon (0,T] \to E$ with blow-up of at most $t^{-\eta}$.	Sec. 1.2
$\mathscr{L}^\kappa_{\eta;T}\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2)$	The weighted interpolation space $\mathscr{L}^\kappa_{\eta;T}\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2) = C^\kappa_{\eta;T}\mathcal{B}^{\alpha-2\kappa}_{p,q}(\mathbb{T}^2) \cap C_{\eta;T}\mathcal{B}^\alpha_{p,q}(\mathbb{T}^2)$.	Def. B.3.1
$\mathcal{X}^{\alpha,\kappa}_T$	The space of enhanced rough noises.	Def. 2.2.1
$\mathscr{D}_T$	The space of paracontrolled distributions.	Def. 2.3.4
$F^{\skull}$	The normed vector space F with cemetery state $\skull$.	Def. B.8.1
$F^{\mathrm{sol}}_{\eta;T}$	The continuous $f\colon (0,T] \to F^{\skull}$ that do not resurrect, with weight η at 0.	Def. B.8.4

J.2 Noise Objects and Feynman Diagrams

Diagram	Description	Reference
$\boldsymbol{\xi}$	The vector-valued space-time white-noise $\boldsymbol{\xi} = (\xi^1, \xi^2)$.	App. C
	$= \nabla \cdot \mathcal{I}[\sigma \boldsymbol{\xi}]$, where $\sigma \in C_T\mathcal{H}^2(\mathbb{T}^2)$ in Chapter 2	
	and $\sigma = \sqrt{\rho_{\det}}$ in Chapters 1 & 3.	(2.2.2)
	$= \mathbb{E}[\nabla \cdot \mathcal{I}[\ \nabla\Phi\ ]]$.	Subsec. 2.2.2
	$= \nabla \cdot \mathcal{I}[\ \nabla\Phi\ ] -\ =\ $.	Subsec. 2.2.2
	$=\ \odot \nabla\Phi\ + \nabla\Phi\ \odot\ $	
	$=\ +\ +\ +\ +\ $.	Subsec. 2.2.2
	$=\ +\ $.	Subsec. 2.2.2
	$= \nabla\mathcal{I}[\ ] \odot \nabla\Phi\ + \nabla^2\mathcal{I}[\Phi\ ] \odot\ =\ +\ +\ $.	Subsec. 2.2.2
	$=\ +\ $.	Subsec. 2.2.2
$\mathbb{X}$	$= (\ ,\ ,\ ,\ )$ the renormalised enhancement.	Thm. 2.2.2
φ	$\in C^\infty(\mathbb{R}^2)$ of compact support, $\mathrm{supp}(\varphi) \subset B(0,1)$,	
	even and such that $\varphi(0) = 1$.	(1.2.6)
ψ_δ	$= \sum_{\omega \in \mathbb{Z}^2} \mathrm{e}^{2\pi \mathrm{i}\langle \omega, x\rangle} \varphi(\delta\omega)$, where $\delta > 0$.	(1.2.7)
$^\delta$	$= \nabla \cdot \mathcal{I}[\sigma(\psi_\delta * \boldsymbol{\xi})]$.	(2.2.5)
$\mathbb{X}^\delta$	$= (\ ^\delta,\ ^\delta,\ ^\delta,\ ^\delta)$ the prelimiting	
	renormormalised enhancement.	Thm. 2.2.2
$^\delta_{\mathrm{can}}$	$= \nabla \cdot \mathcal{I}[\ ^\delta \nabla\Phi_{\ ^\delta}] =\ ^\delta +\ ^\delta$.	Subsec. 2.2.2
$^\delta_{\mathrm{can}}$	$=\ ^\delta_{\mathrm{can}} \odot \nabla\Phi_{\ ^\delta} + \nabla\Phi_{\ ^\delta_{\mathrm{can}}} \odot\ ^\delta =\ ^\delta +\ ^\delta +\ ^\delta$.	Subsec. 2.2.2
$\mathbb{X}^\delta_{\mathrm{can}}$	$= (\ ^\delta,\ ^\delta_{\mathrm{can}},\ ^\delta_{\mathrm{can}},\ ^\delta)$ the canonical enhancement.	Thm. 2.2.2
$\mathbb{X}^{(\varepsilon)}_\delta$	$= (\varepsilon^{1/2}\ ^\delta, \varepsilon\ ^\delta_{\mathrm{can}}, \varepsilon^{3/2}\ ^\delta_{\mathrm{can}}, \varepsilon\ ^\delta)$ the rescaled	
	canonical enhancement.	Thm. 3.4.1
	(C.3.1) applied to .	Subsec. 2.2.3
	(C.3.1) applied to .	Subsec. 2.2.4

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