

A Correlation-sensitive Calibration of a Stochastic Volatility LIBOR Market Model



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1 Introduction

The LIBOR market model (LMM) framework for modelling forward interest rates, first introduced by Brace et al. [1] and Musiela and Rutkowski [2], is a well-established tool for the pricing and risk management of interest rate derivatives. In its lognormal form, it is easy and cheap (in terms of computational effort) to calibrate to market-implied swaption volatilities across the entire swaption grid, and was the standard for pricing complex products before smiles appeared in the interest rate markets. The incorporation of monotonically decreasing smiles to the LMM is fairly straightforward and can be done without too drastic a compensation made in the ease and speed of calibration, but to recover full implied volatility smiles is mathematically and computationally challenging.

A lognormal LMM produces the same implied Black volatilities for swaptions of the same expiry but of different strikes, yet we observe that European swaptions of different maturities exhibit different volatility smiles. Forays into extensions of the LMM to account for the full volatility smile can be broadly classified into three main categories: allowing for a local volatility function, incorporating stochastic volatility, and imposing jump-diffusion dynamics. Arguably the most successful of these three has been stochastic volatility extensions of the LMM, in which each forward rate follows a process with a deterministic local volatility function with a stochastic volatility perturbation applied to all LIBOR rates. Work done so far in this strand can be found in papers by Andersen and Andreasen in [3], Joshi and Rebonato in [4], to name a few.

In order to obtain correct price and hedge ratios, one needs to accurately specify and calibrate the underlying volatility and correlation structure of an LMM. Extracting information about the forward rate volatilities can be done using market prices of caps/floors and swaptions with relative ease, but implying the correlations in this manner leads to some difficulties as swaptions, which are usually the main calibration instrument, depend only mildly on the forward rate correlations [5]. Since the prices of vanilla options only depend of the terminal distribution of the underlying, it is still acceptable to use a model separately calibrated for each option expiry. However, this is not the case for exotic instruments as their prices typically depend on the full dynamics of the model. To overcome this problem, one can consider adding other derivatives apart from caps, floors, or swaptions to the calibration. The instruments that we are interest in adding to the calibration are constant maturity swap (CMS) spread caps and floors, which are far more correlation sensitive and have become more liquid in recent years.

While the pricing of CMS spread options via Monte Carlo simulations in a LMM is conceptually straightforward, it is computationally expensive thus making it unattractive to be included in a calibration procedure. In a recent paper, Lutz and Kiesel [6] introduce an efficient semi-closed formula for the pricing of CMS spread options in a Heston-type stochastic volatility LMM. Using this formula, we seek to include CMS spread options in the calibration process of this forward rate model, following closely the approach taken by Piterbarg [7] and Lutz [8]. In addition, we consider

several parametric forms of the correlation structure and evaluate how well each of them manages to capture correlation information from the market.

We organise this paper as follows. Section 2 introduces the basic instruments in our interest rate market and standardises notation used throughout the rest of the paper. We describe the stochastic volatility LMM we are working in in section 3 and also state pricing formulae for our calibration instruments. Section 4 discusses the correlation forms that we consider, and we detail the calibration process as well as present our results in Section 5.

2 The Term Structure of Interest Rates

The *term structure* of interest rates refers to the relationship between interest rates and their time to maturity. In this section, we will use zero coupon bonds as the basic building blocks of the interest rate market leading into our discussion of the market instruments we will use in later sections. The material in this section can be found in several other textbook treatments such as [9], [10], and [11], but we adapt these accounts to standardise notation and make our discussion self-consistent.

2.1 Bonds, Forward Rates, and Swaps

Zero-coupon bonds are the main traded objects in the fixed income markets and are the basic idealised vehicles for traded debt. We will denote the price at time t of a zero-coupon bond paying one unit of currency at maturity $T \geq t$ by $P(t, T)$, also called a T -bond. Clearly, barring any default risk, $P(t, t) = 1$ for all t . We further assume that $P(t, T)$ is differentiable with respect to T and that the price process $P(t, T)$ is adapted and strictly positive for $t \in [0, T]$.

A no-arbitrage argument allows us to define simply-compounded risk-free interest rates, known as *forward rates*, over future periods in terms of the values of zero-coupon bonds. At time t , we sell one T_1 -bond for $P(t, T_1)$ and buy $\frac{P(t, T_1)}{P(t, T_2)}$ T_2 -bonds. We would then pay 1 at T_1 and receive $\frac{P(t, T_1)}{P(t, T_2)}$ at T_2 . This allows us to define the forward rate for the period $[T_1, T_2]$ as seen at time t as

$$L(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right). \quad (1)$$

We will frequently use the notation $L_i(t) = L(t, T_i, T_{i+1})$. Given a discrete set of times $t = T_0 < T_1 < \dots < T_i$, we can express the price of a T_i -bond at time $t < T_1$ in terms of forward rates using (1),

$$P_i(t) \equiv P(t, T_i) = \prod_{k=0}^i \frac{1}{1 + (T_{k+1} - T_k)L_k(t)}. \quad (2)$$

An *interest rate swap* is an agreement between two parties to exchange cash flows at fixed payment dates $T_{\alpha+1}, T_{\alpha+2}, \dots, T_{\beta}$. Typically, one of the cash flows is based on a floating interest rate, and we have an associated set of dates known as *fixing dates*, $T_{\alpha}, T_{\alpha+1}, \dots, T_{\beta-1}$, where the floating cash flows for the payment dates T_i are determined by accruing $L_{i-1}(T_{i-1})$ over the period

$\Delta_i = T_i - T_{i-1}$. The fixing and payment dates are collectively known as the *tenor structure*. The holder of a *payer (receiver) swap* makes fixed (floating) payments in exchange for floating (fixed) cash flows, where the floating payments typically depend on LIBOR rates. If the fixed payments are made at a predetermined rate K , we can discount the cash flows at each payment date T_i to recover the value of the payer swap at time $t \leq T_\alpha$ to be

$$\sum_{i=\alpha+1}^{\beta} P(t, T_i) \Delta_i (L_{i-1}(T_{i-1}) - K). \quad (3)$$

The receiver swap is simply a payer swap with reversed cash flows and can be priced similarly. The *swap rate* $S_{\alpha\beta}(t)$ is the value of K that makes the swap value (3) at time $t \leq T_\alpha$ equal to zero. Solving this equation gives

$$S_{\alpha\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \Delta_i P(t, T_i)}. \quad (4)$$

2.2 Interest Rate Derivatives

The relationships that we introduced in the previous section hold true regardless of the dynamics of the term structure as they were based on static no-arbitrage arguments. In this section, we introduce instruments that we will be using to calibrate our forward rate model and whose valuation is dependent on how one chooses to model the term structure dynamics. We prespecify a tenor structure $0 = T_0 < T_1 < \dots < T_n$ with intervals $\Delta_i = T_i - T_{i-1}, i = 1, \dots, n$, for convenience.

Swaptions

Options on swaps, known as *swaptions*, are options to enter into a swap agreement. Together, caps, floors, and swaptions make up the bulk of derivatives traded in the interest rate market. A European *payer swaption* is an option to enter into a payer swap at the swaption strike K on a date T_α which usually coincides with the first fixing date of the swap. Receiver swaptions are defined in a completely analogous manner. By using the relation (4), one can rewrite the time- t value of a payer swap (3) as

$$(S_{\alpha\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} \Delta_i P(t, T_i). \quad (5)$$

Thus, whether or not to exercise the swaption on expiry is immediately clear – the swaption is exercised if $S_{\alpha\beta}(T_\alpha) > K$. At expiry T_α , the payoff of the payer swaption is clearly

$$(S_{\alpha\beta}(t) - K)^+ \sum_{i=\alpha+1}^{\beta} \Delta_i P(T_\alpha, T_i). \quad (6)$$

Observe that the *annuity* term $\sum_{i=\alpha+1}^{\beta} \Delta_i P(t, T_i)$ is simply a sum of zero-coupon bonds which are traded assets in the market. Thus, we can use it as a numéraire and we see that the swap rate (4)

must be a martingale under the associated measure $\mathbb{Q}^{\alpha\beta}$ since its numerator is just a portfolio of traded assets in the market. Taking the dynamics of $S_{\alpha\beta}(t)$ to be lognormal under $\mathbb{Q}^{\alpha\beta}$,

$$dS_{\alpha\beta}(t) = \sigma S_{\alpha\beta}(t) dW^{\mathbb{Q}^{\alpha\beta}}, \quad (7)$$

one gets Black's formula for a European payer swaption by taking the expectation of (6) under $\mathbb{Q}^{\alpha\beta}$. Market prices of swaptions are quoted in terms of these implied Black volatilities.

CMS Spread Options

CMS spread options are options written on the difference between two CMS rates and they allow investors to express their views on the future shape of the yield curve. Given that CMS spread options depend on the co-movement of swap rates, their prices are highly sensitive to correlations between forward rates in the market.

As with interest rate caps and floors, one can decompose a CMS spread cap or floor into a sum of CMS spread caplet or floorlet payoffs of the form

$$(wS_{\alpha\beta}(T_n) - wS_{\alpha\beta'}(T_n) - wK)^+, \quad w = \pm 1, \quad (8)$$

where $K \in \mathbb{R}$ is the strike and $S_{\alpha\beta}$ and $S_{\alpha\beta'}$ are swap rates with the same first fixing date T_α but different maturities. Payment of such a payoff commonly takes place at the first payment date of the underlying swap tenors, $T_{\alpha+1}$, so we can price a CMS spread cap or floor by summing discounted payoffs of the form

$$P(0, T_{\alpha+1}) \mathbb{E}^{\mathbb{Q}^{T_{\alpha+1}}} [(wS_{\alpha\beta}(T_n) - wS_{\alpha\beta'}(T_n) - wK)^+]. \quad (9)$$

Even in the simplest case where we assume that $S_{\alpha\beta}$ and $S_{\alpha\beta'}$ are lognormally distributed, finding an analytical formula for (9) is difficult because the distribution of a sum of lognormal random variables is unknown.

3 Model Dynamics

3.1 General Modelling Framework

We use a standard mathematical setting much like in [9] and [10] to model the dynamics of our forward rates. Here, we outline the general modelling framework. Let $W(t) = (W_1(t), \dots, W_K(t))$ be a K -dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)$ the natural filtration of $W(t)$. Suppose that we have N traded assets $S_1, \dots, S_N$ whose evolutions are described by the processes

$$dS_i(t) = \mu_i(t)dt + \sum_{j=1}^K \sigma_{ij}(t) dW_j(t), \quad i = 1, \dots, N, \quad (10)$$

where the μ_i 's and σ_{ij} 's satisfy certain regularity conditions. If we define

$$|\sigma_i(t)| = \sqrt{\sum_{j=1}^K \sigma_{ij}^2(t)}, \quad (11)$$

(10) can be rewritten as

$$dS_i(t) = \mu_i(t)dt + |\sigma_i(t)| \sum_{j=1}^K \frac{\sigma_{ij}(t)}{|\sigma_i(t)|} dW_j(t). \quad (12)$$

Let $b_{ij}(t) = \frac{\sigma_{ij}(t)}{|\sigma_i(t)|}$, then (12) can be expressed in a more compact form as

$$dS_i(t) = \mu_i(t)dt + |\sigma_i(t)| \sum_{j=1}^K b_{ij} dW_j(t). \quad (13)$$

Writing $dX_i(t) = \sum_{j=1}^K b_{ij}(t) dW_j(t)$, we observe that

$$X_i(t) = \sum_{j=1}^K \int_0^t b_{ij}(u) dW_j(u) \quad (14)$$

is a $\mathbb{P}$ -local martingale and

$$\begin{aligned} \langle X_i \rangle(t) &= \sum_{j=1}^K \int_0^t b_{ij}^2(u) dW_j(u) \\ &= \sum_{j=1}^K \int_0^t \frac{\sigma_{ij}^2(u)}{|\sigma_i(u)|^2} dW_j(u) \\ &= t. \end{aligned}$$

By the Levy characterisation of Brownian motion, each $X_i(t)$ is a $\mathbb{P}$ -Brownian motion for $i = 1, \dots, N$. Thus, we can specify the dynamics of $S_1, \dots, S_N$ as

$$dS_i(t) = \mu_i(t)dt + |\sigma_i(t)| dX_i(t), \quad (15)$$

where the $X_i(t)$ are 1-dimensional $\mathbb{P}$ -Brownian motions with instantaneous correlations

$$dX_i(t) dX_j(t) = \sum_{l=1}^K b_{il} b_{jl} dt \equiv \rho_{ij}(t) dt. \quad (16)$$

Let us denote by b the N by K matrix of elements b_{ij} . Then, it is a simple calculation to show that the correlations between the drivers of each asset can be expressed as a matrix

$$\rho = bb^T.$$

3.2 A Heston-type Stochastic Volatility Forward Rate Model

Let $0 = T_0 < T_1 < \dots < T_n$ be our tenor structure with associated year fractions $\Delta_i = T_i - T_{i-1}$, $i = 1, \dots, n$. From (1), we have that

$$P(t, T_{i+1})L_i(t) = \frac{P(t, T_i) - P(t, T_{i+1})}{\Delta_{i+1}}. \quad (17)$$

Since $P(t, T_{i+1})L_i(t)$ is a sum of traded assets, one can easily see that $L_i(t)$ is a martingale under the measure $\mathbb{Q}^{T_{i+1}}$, which is the measure associated with taking $P(t, T_{i+1})$ as numéraire. This is also known as the T_{i+1} -forward measure. The quantities that are modelled under the LMM class of models are the forward rates

$$dL_i(t) = \sum_{k=1}^d \sigma_{ik}(t) dW_k^{T_{i+1}}(t), \quad i = 1, \dots, n-1, \quad (18)$$

where $W^{T_{i+1}}(t) = (W_1^{T_{i+1}}(t), \dots, W_d^{T_{i+1}}(t))$ is a standard d -dimensional Brownian motion in the measure $\mathbb{Q}^{T_{i+1}}$. Under our general modelling framework (15), we can rewrite this as

$$dL_i(t) = |\sigma_i(t)| dZ^{T_{i+1}}(t), \quad i = 1, \dots, n-1, \quad (19)$$

where $|\sigma_i(t)|$ are given as in (11) and $Z^{T_{i+1}}(t)$ are standard Brownian motions in their respective measures. Note that only the forward rate $L_i(t)$ is a martingale in the $\mathbb{Q}^{T_{i+1}}$ measure. To establish dynamics in other probability measures, the following result, which is a simple application of Girsanov's theorem, from stochastic calculus is needed:

$$dW^{T_i} = dW^{T_{i+1}} - \frac{\Delta_{i+1}\sigma_i(t)}{1 + \Delta_{i+1}L_i(t)} dt, \quad (20)$$

where $\sigma_i(t) = (\sigma_{i1}(t), \dots, \sigma_{id}(t))$. To fully specify the forward rate model, one would then have to decide on functional forms of the $\sigma_i(t)$ for each forward rate being modelled. Equivalently, in the general modelling framework (15), one would specify the instantaneous volatilities $|\sigma_i(t)|$ and correlations ρ_{ij} between the drivers of each rate.

Being able to recover all volatility smiles across the entire swaption grid is an essential characteristic of any model being used to price exotic interest rate derivatives. The basic lognormal forward rate model fails in this regard as it is unable to capture volatility smiles in the market. While a model can be separately calibrated for vanilla options of different expiries due to their dependence only on the terminal distribution of the underlying, the value of structured interest rate derivatives typically depend on swaption volatilities with different maturities. Thus, the accurate value of such instruments calls for a model that should be able to match market volatilities for all strike and expiry combinations. One of the first models capable of capturing volatility smiles across the entire swaption grid is an extension of the model from [12] presented in [7].

Assume a generic measure $\mathbb{P}$. Let the stochastic variance process V follow a Cox-Ingersoll-Ross (CIR) process defined by

$$\begin{aligned} dV(t) &= \kappa(1 - V(t))dt + \xi\sqrt{V(t)}dZ(t), \\ V(0) &= 1, \end{aligned} \tag{21}$$

with κ and ξ positive constants and Z a $\mathbb{P}$ -Brownian motion. Then, the dynamics of the spanning forward rates are given by

$$\begin{aligned} dL_i(t) &= \mu_i(t)dt + (\beta_i(t)L_i(t) + (1 - \beta_i(t))L_i(0))|\sigma_i(t)|\sqrt{V(t)}dW_i(t), \\ dW_i(t)dW_j(t) &= \rho_{ij}(t), \quad i = 1, \dots, n-1 \end{aligned} \tag{22}$$

where each $W_i(t)$ are $\mathbb{P}$ -Brownian motions independent of $V(t)$. $|\sigma_i(t)|$ are the exogenously specified instantaneous volatility functions and $\rho_{ij}(t)$ are the correlations between the drivers of each forward rate. $\mu_i(t)$ are numéraire-specific drift functions that guarantee lack of arbitrage within the model. The value of each of the input parameters in (22) have different effects on the implied Black volatility smile. $|\sigma_i(t)|$ is a scaling factor and determines the overall level of the smile. $\beta_i(t)$ controls the slope of the volatility smile at the money, and ξ affects the curvature of the smile. The parameter κ is the speed of mean reversion and affects the speed at which the smile flattens out at time to expiry increases. Note that the stochastic volatility process V is the same for all forward rates and only the skews $\beta_i(t)$ and instantaneous volatilities $|\sigma_i(t)|$ are allowed to vary across different forward rates. Extending the model further with rate-specific volatility smile curvatures is certainly possible, but [7] points out that this is unnecessary since a single stochastic volatility driver captures well the curvature of volatility smiles for different swaptions.

3.3 Approximate Swap Rate Dynamics

Since swaps are just an exchange of payments involving forward rates, one can choose swap rates to be the quantities being modelled and price products in a forward swap framework. Going back to (4), we see that the obvious choice of numéraire are the annuities $\sum_{i=\alpha+1}^{\beta} \Delta_i P(t, T_i)$ and one can posit, as in the lognormal forward model, that forward swap rates follow a lognormal diffusion process under their respective annuity measures. Note, however, that these two model classes are not equivalent as lognormal forward rates (under their respective measures) do not give rise to lognormal swap rates under the associated annuity measure. A natural question to ask is then what kind of dynamics do we get for forward swap rates coming from a forward rate model. In the basic lognormal forward model, it can be shown that swap rates follow approximate lognormal dynamics.

A CMS spread option depends on more than one swap rate, so we shall have to turn to a forward swap model to model the simultaneous evolution of swap rates under a single probability measure.

If the dynamics of our forward rates are given by (22), Piterbarg [7] shows that forward swap rates approximately follow the same dynamics.

Let $S_{\alpha\beta}(t)$ be the swap rate for a swap with first fixing date T_α and last payment date T_β . The swap measure $\mathbb{Q}^{\alpha\beta}$ is the measure obtained by taking $\sum_{i=\alpha+1}^\beta \Delta_i P(t, T_i)$ as the numéraire, under which $S_{\alpha\beta}(t)$ is a martingale. We denote by $W^{\alpha\beta}$ a d -dimensional standard Brownian motion under $\mathbb{Q}^{\alpha\beta}$. In our stochastic volatility forward rate model (22), the swap rate dynamics can be approximately written as

$$dS_{\alpha\beta}(t) = (\beta_{\alpha\beta}(t)S_{\alpha\beta}(t) + (1 - \beta_{\alpha\beta}(t))S_{\alpha\beta}(0))\sqrt{V(t)}\sigma_{\alpha\beta}(t) \cdot dW^{\alpha\beta}(t), \quad (23)$$

where

$$\sigma_{\alpha\beta}(t) = \sum_{i=n}^{m-1} q_i^{\alpha\beta} \sigma_i(t), \quad (24)$$

$$\beta_{\alpha\beta}(t) = \sum_{i=n}^{m-1} p_i^{\alpha\beta} \beta_i(t), \quad (25)$$

and

$$q_i^{\alpha\beta} = \frac{L_i(0)}{S_{\alpha\beta}(0)} \frac{\partial S_{\alpha\beta}(0)}{\partial L_i(0)}, \quad (26)$$

$$p_i^{\alpha\beta} = \frac{\sigma_i(t)\sigma_{\alpha\beta}(t)'}{|\sigma_{\alpha\beta}(t)|^2}. \quad (27)$$

The approximate SDE for the swap rates $S_{\alpha\beta}$ has the same form as the exact SDE for the forward rates.

3.4 Effective Dynamics

If we take $\sigma_{\alpha\beta}(t)$ and $\beta_{\alpha\beta}(t)$ to be constant in (23), the model reduces to the well-known displaced-diffusion stochastic volatility model [12] for which efficient numerical methods for pricing European options exist. Piterbarg's approach [7] to implementing a successful calibration for the model (23) is to come up with formulae that relate the time-dependent parameters $\sigma_{\alpha\beta}(t)$ and $\beta_{\alpha\beta}(t)$ to effective constant ones $\bar{\sigma}_{\alpha\beta}$ and $\bar{\beta}_{\alpha\beta}$. Market-implied parameters $\bar{\sigma}_{\alpha\beta}$ and $\bar{\beta}_{\alpha\beta}$ are typically maintained by vanilla options trading desks, and by way of these 'averaging' formulae, the calibration can be set up without the need for costly direct/inverse option valuations during the non-linear optimisation process. We state the results of [7] where a full derivation of these formulae can be found.

Using our general modelling framework, we can write the dynamics of each swap or forward rate as

$$\begin{aligned} dS(t) &= \sigma(t)(\beta(t)S(t) + (1 - \beta(t))S(0))\sqrt{V(t)}dW(t) \\ dV(t) &= \kappa(1 - V(t))dt + \xi\sqrt{V(t)}dZ(t), \quad V(0) = 1, \\ dW(t)dZ(t) &= 0, \end{aligned} \quad (28)$$

under the appropriate measure where $W(t)$ and $Z(t)$ are standard Brownian motions. Over the time horizon $[0, T]$, the effective skew $\bar{\beta}$ for the equation

$$dS(t) = \sigma(t)(\bar{\beta}S(t) + (1 - \bar{\beta})S(0))\sqrt{V(t)}dW(t) \quad (29)$$

is given by

$$\bar{\beta} = \int_0^T \beta(t)w(t)dt, \quad (30)$$

where the weights $w(\cdot)$ are given by

$$w(t) = \frac{v^2(t)\sigma^2(t)}{\int_0^T v^2(t)\sigma^2(t)dt},$$

$$v^2(t) = V_0^2 \int_0^t \sigma^2(s)ds + V_0\xi^2 e^{-\kappa t} \int_0^t \sigma^2(s) \frac{e^{\kappa s} - e^{-\kappa s}}{2\kappa} ds.$$

Given the effective skew $\bar{\beta}$ for (30), the effective volatility $\bar{\sigma}$ of the process is given by the solution to

$$\varphi_0 \left(-\frac{g'' \left(V_0 \int_0^{T_n} \sigma^2(t)dt \right)}{g' \left(V_0 \int_0^{T_n} \sigma^2(t)dt \right)} \bar{\sigma}^2 \right) = \varphi \left(-\frac{g'' \left(V_0 \int_0^{T_n} \sigma^2(t)dt \right)}{g' \left(V_0 \int_0^{T_n} \sigma^2(t)dt \right)} \right), \quad (31)$$

$$\varphi(\mu) = \mathbb{E} \exp(-\mu \bar{V}(T)),$$

$$\varphi_0(\mu) = \mathbb{E} \exp(-\mu \int_0^T V(t)dt),$$

where

$$\bar{V}(T) = \int_0^T \sigma^2(t)V(t)dt,$$

$$g(x) = \frac{S(0)}{\bar{\beta}} (2\Phi(\bar{\beta}\sqrt{x}/2) - 1),$$

and Φ is the cumulative distribution function for a standard normal random variable. Finally, we have the effective dynamics of each swap and forward rate over the time horizon $[0, T]$ given by

$$dS(t) = \bar{\sigma}(\bar{\beta}S(t) + (1 - \bar{\beta})S(0))\sqrt{V(t)}dW(t), \quad (32)$$

$$dV(t) = \kappa(1 - V(t))dt + \xi\sqrt{V(t)}dZ(t), \quad V(0) = 1$$

$$dW(t)dZ(t) = 0.$$

3.5 Pricing Approaches

A successful calibration procedure depends on fast European pricing implementations in the model. While we have managed to reduce (28) to a constant parameter one in (32), Monte Carlo valuation is still too costly for calibration purposes. This makes it necessary to use computationally inexpensive yet numerically accurate approximations. In this section, we present approximate pricing formulae for European payer swaptions and CMS spread options in the model (32).

Swaptions

We state a result from [11] that gives us a formula for evaluating the price of an European call option on an underlying asset whose dynamics, under the appropriate measure, are given by (32). Let $S^* = \bar{\beta}S + (1 - \bar{\beta})L$ and $K^* = \bar{\beta}K + (1 - \bar{\beta})L$. Then, the time-zero price of a call option with strike K and expiry T in the model (32) is given by

$$c_{SV}(0, S, T, K) = \frac{1}{\bar{\beta}} c_{\text{Black}}(0, S^*, T, K^*, \bar{\sigma}\bar{\beta}) - \frac{K^*}{2\pi\bar{\beta}} \int_{-\infty}^{\infty} \frac{e^{(1/2+iw)\ln(S^*/K^*)q(1/2+iw)}}{w^2 + 1/4} dw, \quad (33)$$

where $c_{\text{Black}}(0, S^*, T, K^*, \bar{\sigma}\bar{\beta})$ is the Black formula for spot S^* , strike K^* , expiry T and volatility $\bar{\sigma}\bar{\beta}$. The function $q(\cdot)$ is defined as

$$q(u) = \Psi_V \left(\frac{1}{2} (\bar{\sigma}\bar{\beta})^2 u(u-1), u; T \right) - \exp \left(\frac{1}{2} \bar{\sigma}^2 \bar{\beta}^2 T u(u-1) \right), \quad (34)$$

where

$$\Psi_V(v, u; T) = \frac{2\kappa}{\xi^2} \left(\ln \left(\frac{2\gamma}{(\kappa + \gamma) - e^{-\gamma T}(\kappa - \gamma)} \right) + (\kappa - \gamma)T \right) + \frac{2v(1 - e^{-\gamma T})}{(\kappa + \gamma)(1 - e^{-\gamma T}) + 2\gamma e^{-\gamma T}},$$

$$\gamma = \sqrt{\kappa^2 - 2\xi^2 v}.$$

A discussion of how to efficiently evaluate this expression numerically can be found in [11].

CMS Spread Options

Lutz and Kiesel in [6] come up with a semi-closed formula for pricing CMS spread options in (32). We state their result here and direct the reader to [6] for a detailed discussion of the derivation. Let $S_{\alpha\beta_1} \equiv S_1$ and $S_{\alpha\beta_2} \equiv S_2$ be two swap rates on which a CMS spread option is written. In our swap rate model, S_1 and S_2 are assumed to have dynamics of the form

$$dS_i(t) = \mu_i(t)dt + (\beta_i(t)S_i(t) + (1 - \beta_i(t))S_i(0)) \sqrt{V(t)}\sigma_i(t) \cdot dW_i^{\mathbb{Q}^{T_{\alpha+1}}}(t), \quad i = 1, 2 \quad (35)$$

under the measure $\mathbb{Q}^{T_{\alpha+1}}$. To get effective dynamics like what we have in (32), Lutz and Kiesel make use of Piterbarg's effective skew and volatility formulae to get $\bar{\beta}_i$ and $\bar{\sigma}_i$ for each swap rate. The instantaneous correlation between the drivers of the two swap rates are approximated to be the average correlation over the life of the option

$$\bar{\rho} = \frac{\int_0^{T_\alpha} \sigma_1(t)\sigma_2(t)' dt}{\sqrt{\int_0^{T_\alpha} |\sigma_1(t)|^2 dt} \sqrt{\int_0^{T_\alpha} |\sigma_2(t)|^2 dt}}. \quad (36)$$

Finally, effective drift terms are approximated by

$$\bar{\mu}_i = -\frac{P(0, T_{\alpha+1})}{\sum_{j=\alpha+1}^{\beta_i} \Delta_j P(0, T_j)} \nabla F_i(L(0)) D\bar{\Sigma} D\nabla G_i(L(0))', \quad (37)$$

where

$$\begin{aligned} D &= \text{diag}(L_j(0)), & j &= 1, \dots, n \\ (\bar{\Sigma})_{jk} &= \frac{1}{T_\alpha} \int_0^{T_\alpha} \hat{\sigma}_j(t) \hat{\sigma}_k(t)' dt, & j, k &= 1, \dots, n \end{aligned}$$

with $\hat{\sigma}_j(t)$ denoting the instantaneous volatility for the forward rate $L_j(t)$. F_i and G_i are functions on $L(t) = (L_1(t), \dots, L_{\alpha-1}(t))$ such that

$$\begin{aligned} F_i(L(t)) &= S_i(t), \\ G_i(L(t)) &= \frac{1}{P(t, T_{\alpha+1})} \sum_{j=\alpha+1}^{\beta_i} \Delta_j P(t, T_j). \end{aligned}$$

With these approximations, we can write the effective dynamics of (35) as

$$\begin{aligned} dS_i(t) &= \bar{\mu}_i dt + (\bar{\beta}_i S_i(t) + (1 - \bar{\beta}_i) S_i(0)) \sqrt{V(t)} \bar{\sigma}_i dU_i(t), & i &= 1, 2 \\ dU_1(t) dU_2(t) &= \bar{\rho}, \end{aligned} \quad (38)$$

with U_1 and U_2 being standard Brownian motions under $\mathbb{Q}^{T_{\alpha+1}}$. Then, the time-zero price of a CMS spread option ($w = \pm 1$) is given by

$$\begin{aligned} P(0, T_n + \delta) \mathbb{E}^{\mathbb{Q}^{T_n + \delta}} [(w S_1(T_n) - w S_2(T_n) - w K)^+] \\ \approx P(0, T_n + \delta) \int_0^\infty \int_{-\infty}^\infty g(u, v) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du f(v) dv, \end{aligned} \quad (39)$$

where

$$\begin{aligned} g(u, v) &= \begin{cases} \left(\frac{S_1(0)}{\bar{\beta}_1} \exp(h(u, v)) - \hat{K}(u, v) \right) \mathbf{1}_{\{w=1\}} & , \hat{K}(u, v) \leq 0; \\ w \frac{S_1(0)}{\bar{\beta}_1} \exp(h(u, v)) \Phi \left(\frac{\ln \left(\frac{S_1(0)}{\bar{\beta}_1 \hat{K}(u, v)} \right) + h(u, v) + \frac{1}{2} \bar{\beta}_1^2 \bar{\sigma}_1^2 (1 - \bar{\rho}^2) v}{\bar{\beta}_1 \bar{\sigma}_1 \sqrt{v(1 - \bar{\rho}^2)}} \right) & , \hat{K}(u, v) > 0, \\ -w \hat{K}(u, v) \Phi \left(\frac{\ln \left(\frac{S_1(0)}{\bar{\beta}_1 \hat{K}(u, v)} \right) + h(u, v) - \frac{1}{2} \bar{\beta}_1^2 \bar{\sigma}_1^2 (1 - \bar{\rho}^2) v}{\bar{\beta}_1 \bar{\sigma}_1 \sqrt{v(1 - \bar{\rho}^2)}} \right) & \end{cases} \\ \tilde{K} &= K + \frac{1 - \bar{\beta}_1}{\bar{\beta}_1} S_1(0) - \frac{1 - \bar{\beta}_2}{\bar{\beta}_2} S_2(0), \\ \hat{K}(u, v) &= \tilde{K} + \frac{S_2(0)}{\bar{\beta}_2} \exp \left(\left(\frac{\bar{\beta}_2 \bar{\mu}_2}{S_2(0)} - \frac{\bar{\beta}_2^2 \bar{\sigma}_2^2}{2} \right) v + \bar{\beta}_2 \bar{\sigma}_2 \sqrt{v} u \right) \\ h(u, v) &= \left(\frac{\bar{\beta}_1 \bar{\mu}_1}{S_1(0)} - \frac{\bar{\beta}_1^2 \bar{\sigma}_1^2}{2} \bar{\rho}^2 \right) v + \bar{\rho} \bar{\beta}_1 \bar{\sigma}_1 \sqrt{v} u, \end{aligned}$$

and $f(v)$ is the density of the integrated variance $\int_0^T V(t) dt$, which is evaluated using the numerical method developed in [6].

As a test of our implementation and the accuracy of these numerical approximations, we evolved the forward rates $L_i(t)$ using a stylised parameter setting much like the one in [6]. We use the yield curve as given in [6] and parameterise instantaneous forward rate volatilities with

$$\sigma_i(t) = [a + b(T_i - t)] \exp(-c(T_i - t)) + d, \quad 1 \leq i \leq n,$$

where $a = 0.04$, $b = 0.32$, $c = 1.1$, $d = 0.17$. The correlation between forward rate drivers were specified by the form

$$\rho_{ij}(t) = \exp(-|T_i - T_j|\nu \exp(-\eta \min(T_i - t, T_j - t))), \quad 1 \leq i \leq n,$$

where $\nu = 0.11$, $\eta = 0.22$. Skew functions were given by the linear function

$$\beta_i(t) = \left(1 - \frac{T_i - t}{T_n}\right) 0.4 + \frac{T_i - t}{T_n} 0.9, \quad 1 \leq i \leq n,$$

and we set the mean reversion of variance $\kappa = 15\%$ and volatility of variance $\xi = 130\%$. The forward rates were evolved under the respective terminal measures related to each instrument using an Euler-Maruyama discretisation. An absorbing boundary condition was imposed for the simulation of the stochastic variance process, which was done by taking 100 steps between each time step of the forward rates.

Tables 1 and 2 compare our simulation results against values calculated using the approximation formulae. In general, we see that our approximations are in close agreement with the Monte Carlo prices and are good enough for calibration purposes.

Strike	2 × 5Y Payer			10 × 5Y Payer		
	4.00%	7.00%	10.0%	4.00%	7.00%	10.0%
MC Price	183	1.9	0.37	415	108	30
Std. Dev.	1	0.2	0.06	3	2	2
Approx.	189	2.2	0.40	417	110	29
Abs. Error	7	0.3	0.03	2	2	1

Table 1: Monte Carlo vs approximate swaption formula (33) prices in basis points. 100,000 paths, 2 steps per year.

Strike	5Y Maturity			10Y Maturity		
	-0.30%	0.10%	0.20%	-0.50%	-0.10%	0.40%
MC Price	69.2	43.6	37.2	70.5	49.2	37.9
Std. Dev.	0.3	0.2	0.2	0.6	0.6	0.4
Approx.	70.1	43.4	37.1	71.1	49.5	37.1
Abs. Error	0.9	0.2	0.1	0.6	0.3	0.8

Table 2: Monte Carlo vs approximate 10Y-2Y CMS spread caplet (39) prices in basis points. 100,000 paths, 2 steps per year.

4 Volatility and Correlation Specifications

Despite the name, the LMM is not a model in the true sense of the word; rather, it is a set of no-arbitrage conditions among forward rates. To get an actual model, we still have to specify the instantaneous volatilities and correlations, and in our displaced diffusion model (22), skew functions of the forward rates. We examine plausible choices of these functions in this section.

4.1 An Instantaneous Volatility Parametric Form

We choose an instantaneous volatility parameterisation given by

$$|\sigma_i(t)| = [a + b(T_i - t)]\exp(-c(T_i - t)) + d, \quad 1 \leq i \leq n. \quad (40)$$

Since the focus of this paper is on the correlation structure of the forward rates, we shall only consider this volatility parameterisation. The specification of instantaneous forward rate volatilities is an interesting subject in its own right and we direct the reader to [13] for a more involved discussion. This particular functional form was chosen because it satisfies several important criteria, namely:

- it's parameter values can be associated to observable economic properties,
- it is square-integrable,
- and it is time-homogenous.

[13] expands upon these criteria and provide other reasons why (40) is a good choice for the volatility function. Figure 1 shows some possible shapes of the volatility function. In an excited market (figure 1(b)), it is the forward rates closest to expiry that experience the most uncertainty, which explains a monotonically declining volatility function. For normal market conditions, (figure 1(a)), it is the actions of monetary authorities in the middle term that are most uncertain, giving rise to a humped volatility function. A possible way to allow for (40) to deviate from time-homogeneity and get better market fits is to introduce a forward-rate specific scaling factor ψ_i , giving us

$$|\sigma_i(t)| = \psi_i[a + b(T_i - t)]\exp(-c(T_i - t)) + d, \quad 1 \leq i \leq n. \quad (41)$$

We choose not to go down this route here, but point the reader to [4] for a discussion of the merits of this approach.

4.2 Correlation Parameterisations

Before we go on to consider specific correlation structures, we first outline requirements that our correlation matrix must satisfy. These requirements come from both mathematical and financial underpinnings. From a mathematical standpoint, we want a valid correlation matrix ρ to satisfy the following requirements:

- ρ must be real and symmetric,
- $\rho_{ii} = 1$ for $i = 1, \dots, n$,
- ρ must be positive semi-definite.

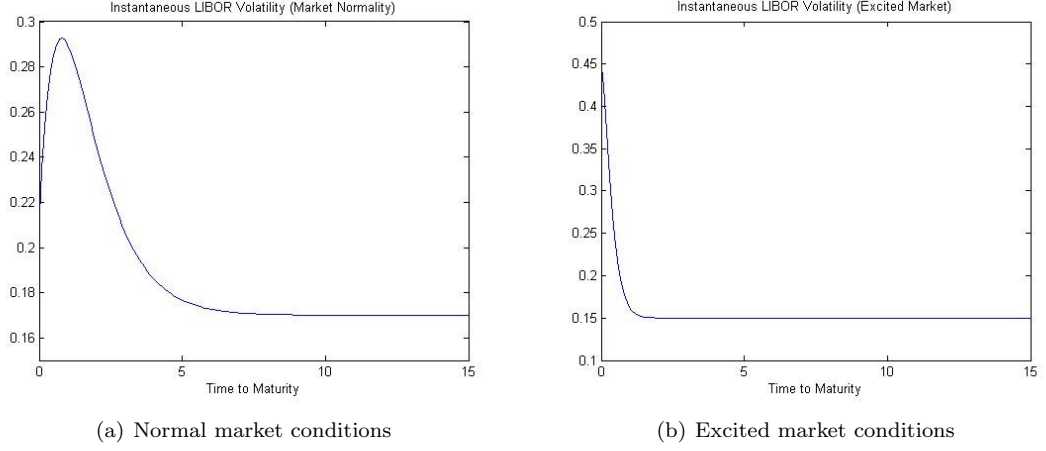


Figure 1: Instantaneous LIBOR volatilities under different market conditions

The positive semi-definiteness of ρ ensures that it has a Cholesky decomposition $\rho = bb^T$ as required in our general modelling framework (15). Aside from these mathematical requirements, we would also like our correlation matrix to describe certain empirical financial facts about the market:

- $i \mapsto \rho_{ij}$, $i \geq j$ is decreasing,
- $i \mapsto \rho_{i+p,i}$ is increasing for fixed $p \in \{1, \dots, n-1\}$.

The first requirement forces our correlation matrix to capture the fact that forward rates become more decorrelated the further apart they are. It has also been observed that given two pairs of forward rates that are the same ‘distance’ apart, the pair on the longer end of the curve should be more correlated than the pair on the shorter end. The second requirement ensures our correlation matrix to describe this.

The Simple Exponential Correlation

A simple correlation form is given by

$$\rho_{ij} = \exp(-\beta|T_i - T_j|), \quad \beta > 0, \quad i, j = 1, \dots, n. \quad (42)$$

For any valid value of β , the simple exponential parametric form always produces an admissible correlation matrix (in a mathematical sense). Figure 2 shows possible shapes of the simple exponential correlation matrix. While it has the desirable feature that forward rates decorrelate more the further apart they are, the correlation between forward rates is not dependent on the current time t , which is empirically a poor approximation. Yet, as [14] points out, this functional form remains popular in practice because of the computational advantage it has in evaluating the covariance elements

$$C_{ij}(k) = \int_{T_k}^{T_k+1} \sigma_i(u) \sigma_j(u) \rho_{ij}(u) du. \quad (43)$$

As we can see from (43), having a constant-time correlation matrix $\rho_{ij}(t) = \rho_{ij}$ significantly reduces the computational effort in evaluating these quantities if a judicious choice of the instantaneous volatility function has been made. Whether or not the lessened computational effort makes up for this unrealistic feature of the correlation structure thus depends on the instrument we are pricing. According to [5], swaption prices show an “almost total lack of dependence on the shape of the correlation function,” and their dependence on the specific correlation structure is modest as long as the average correlations among the forward rates are the same. The tradeoff between the accuracy of the correlation specification and computational effort would be acceptable in this case. On the other hand, prices of derivatives such as CMS spread options are much more correlation sensitive, so we seek further parametric forms to describe the correlation structure of our forward rates.

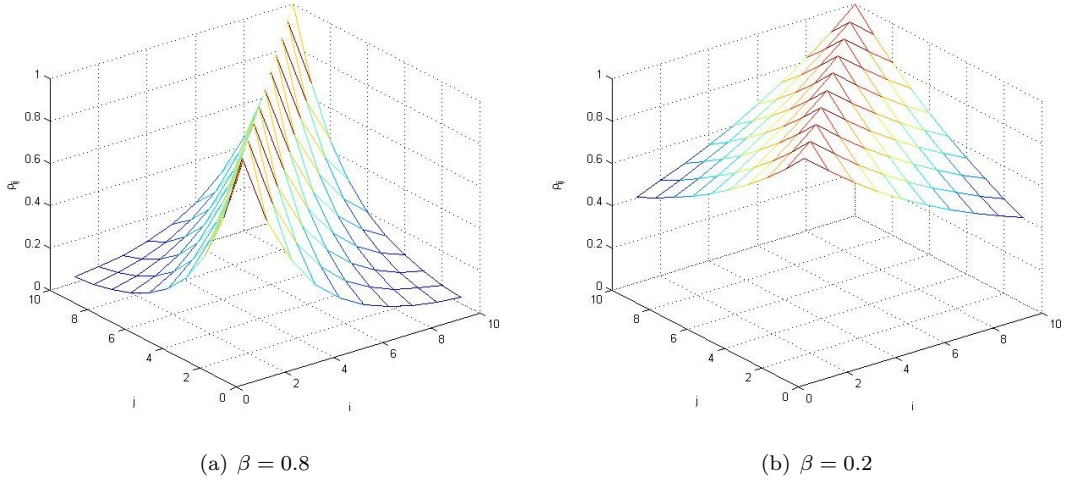


Figure 2: Possible shapes of the simple exponential correlation matrix (42)

Doust Correlation Function

There are many ways in which one can improve on the simple exponential form, one of which is the Doust parameterisation introduced in [14].

$$\begin{aligned}
 \hat{\rho}_{ij} &= \prod_{k=1}^{j-1} a_k & , i > j; \\
 \hat{\rho}_{ij} &= \rho_{ji} & , i < j; \\
 \hat{\rho}_{ii} &= 1 & , i, j = 1, \dots, n; \\
 a_k &\in [-1, 1] & , k = 1, \dots, n-1
 \end{aligned} \tag{44}$$

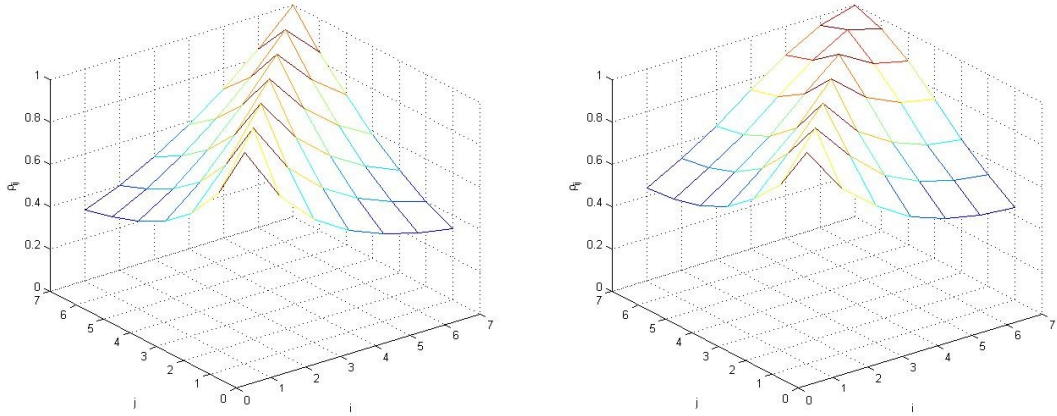
Now, we have $n - 1$ independent parameters that describe the correlation matrix, greatly increasing the degrees of freedom in our correlation parameterisation. While this might give us better fits to market-implied correlations, the Doust functional form is still overparameterised and is likely to lead

to noisy correlation matrices with unintuitive entries. [14] outlines a number of possible ways that we can systematically choose the a_i 's, but we shall stick with the original specification for comparison's sake. Figure 3 shows us two possible shapes of the Doust correlation matrix. In particular, we see that in figure 3(b), the Doust parameterisation manages to capture the convexity that was lacking in the simple exponential form – equidistant pairs are more correlated at the longer end of the curve than at the shorter end.

In the Doust correlation function, the correlation between forward rates goes to zero as the distance between fixing dates increase. A simple extension that ensures the correlation between forward rates goes asymptotically to some finite level ρ_∞ can be made at very little cost by imposing an extension to the original Doust correlation function $\hat{\rho}_{ij}$:

$$\rho_{ij} = \rho_\infty + (1 - \rho_\infty)\hat{\rho}_{ij}, \quad \rho_\infty \in [0, 1). \quad (45)$$

In fact, one can make this extension to any valid correlation function $\hat{\rho}_{ij}$ as long as $\rho_\infty \in [0, 1)$ (see [8] for a proof).



(a) $a_1 = 0.6788, a_2 = 0.7025, a_3 = 0.7353, a_4 = 0.7650$, (b) $a_1 = 0.7249, a_2 = 0.7697, a_3 = 0.7745, a_4 = 0.8294$,
 $a_5 = 0.7869, a_6 = 0.8048$ $a_5 = 0.8842, a_6 = 0.9440$

Figure 3: Possible shapes of the Doust correlation matrix (44). $\rho_\infty = 0.2$

Lutz 5-Parametric Form

In the Doust parameterisation, whether or not the resulting correlation matrix possesses the financial characteristics that we want depends on how we choose the parameters a_i . Schoenmakers and Coffey [15] present an appealing approach to specifying correlation parametric forms where the mathematical and financial requirements are satisfied by construction. However, the resulting parameterisations only have a small number of parameters (typically two or three) and are not as flexible as, for example, (44). With such a small number of parameters, we cannot hope to capture detailed information about specific parts of the correlation surface. In [8], Lutz comes up with an

alternate way of characterising matrices from the Schoenmakers-Coffey family, resulting in a much richer parameterisation with which we might hope to capture correlation information implied by CMS spread options. The Lutz 5-parametric form is given by

$$\rho_{ij} = \rho_{\infty} + (1 - \rho_{\infty}) \left[\exp(-\beta((i-1)^{\alpha} + (j-1)^{\alpha})) + \frac{\vartheta_{ij}}{\sqrt{\vartheta_{ii}\vartheta_{jj}}} \sqrt{(1 - \exp(-2\beta(i-1)^{\alpha}))(1 - \exp(-2\beta(j-1)^{\alpha}))} \right], \quad i, j = 1, \dots, n; \quad (46)$$

$$\vartheta_{ij} = \begin{cases} 1 & , \min(i, j) = 1; \\ \min(i-1, j-1) & , \min(i, j) > 1, \quad \xi_i \xi_j = 1; \\ \frac{(\xi_i \xi_j)^{\min(i-1, j-1)} - 1}{1 - 1/(\xi_i \xi_j)} & , \min(i, j) > 1, \quad \xi_i \xi_j \neq 1; \end{cases}$$

$$\xi_i = \exp \left(-\frac{1}{i-1} \left(\frac{i-2}{n-2} \gamma + \frac{n-i}{n-2} \delta \right) \right), \quad \alpha, \beta > 0, \gamma, \delta \in \mathbb{R}, \rho_{\infty} \in [0, 1).$$

Note that unlike the simple specification (42), i and j in the Lutz 5-parametric form represent the time to maturity indices, assuming the forward rates have a roughly equidistant tenor grid. Figure 4 shows us some possible shapes of the correlation matrix resulting from the Lutz 5-parametric form. Intuitively, α and β control the correlation for shorter maturities while γ and δ act on the longer dates.

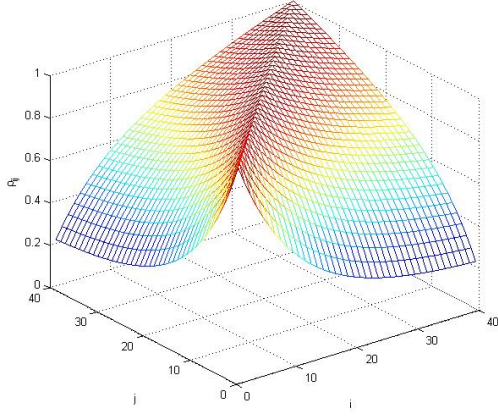
The correlation functions that have been presented all lead to positive definite full-rank correlation matrices. While these parsimonious parametric forms significantly reduces the number of parameters compared to specifying the $n(n-1)/2$ independent entries individually, reduced-rank correlations still have the advantage of being more computationally efficient and are discussed further in [16]. An alternative approach to calibrating correlations to the market is to exogenously specify the correlation matrix (for example, by fitting to historical correlations or expressing a future view) and only use the volatility parameters for calibration purposes.

5 Calibration

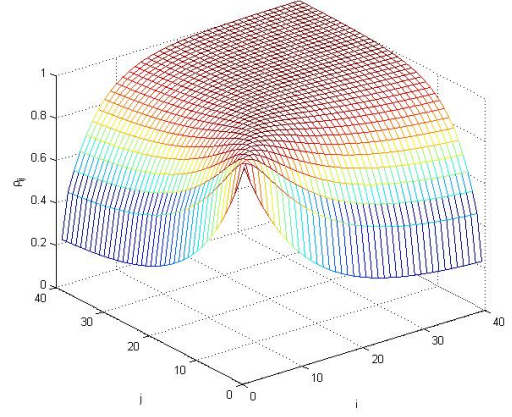
5.1 Market Data

We calibrate 6 month forward Euribor rates over a 30 year period. The market data we used for calibration was provided by Dr. Jörg Kienitz of Deutsche Postbank. This consisted of

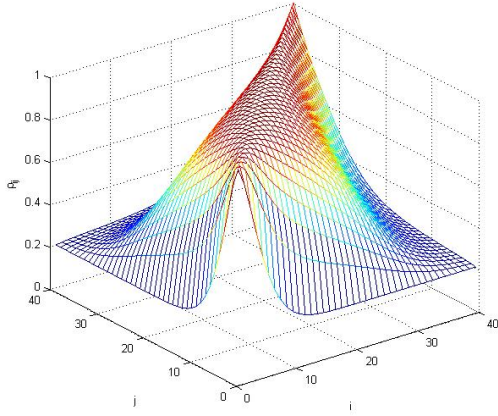
- zero-coupon discount rates for various dates in a 30 year time horizon,
- implied Black volatilities of EUR swaptions with expiries 1, 2, 5, and 10 years on swaps with maturities 2, 5, 10, 20, and 30 years at strikes -1.5%, -1.0%, -0.5%, -0.25%, +0%, +0.25%, +0.5%, +1.0%, and +1.5% relative to the at-the-money strike.



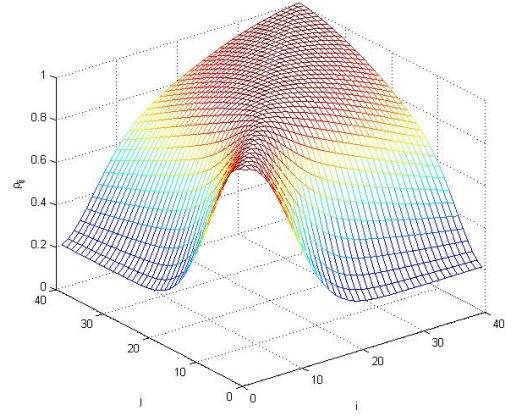
(a) $\alpha=1.0, \beta=0.1, \gamma=0.0, \delta=0.0$



(b) $\alpha=1.0, \beta=0.1, \gamma=10.0, \delta=4.5$



(c) $\alpha=1.5, \beta=0.1, \gamma=-3.4, \delta=8.3$



(d) $\alpha=2.2, \beta=0.006, \gamma=0.95, \delta=3.6$

Figure 4: Possible shapes of the Lutz 5-parametric correlation matrix (46). $\rho_\infty = 0.2$

- and prices of EUR 10Y-2Y CMS spread options with maturities 1, 2, 3, 4, 5, 7, 10, 15, and 20 years at strikes -0.25%, -0.1%, +0%, +0.25%, +0.5%, +0.75%, +1.0%, and +1.5% relative to the at-the-money strike.

The CMS spread option prices were stripped to give prices of individual CMS spread caplets and floorlets. We interpolated the discount rate data using a cubic spline method to give the 6-month forward rate curve shown in figure 5.1. As we can see, the forward rate curve is relatively steep and corresponds to an excited market environment. Note that we will only be considering forward rates with maturities up to 30 years and so will only be calibrating to a subset of the market prices provided.

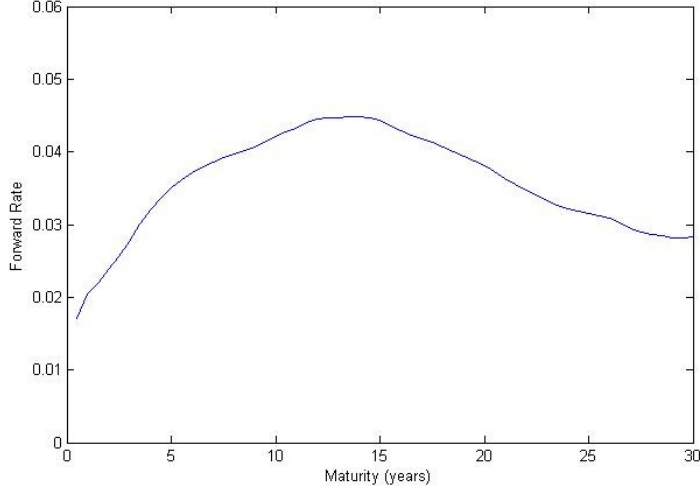


Figure 5: Initial six month forward Euribor curve

5.2 Calibration Procedure

We follow the calibration procedure as set out in [7] and [8]. For the reader's convenience, we state again the model that we will be calibrating. The underlying forward and swap rates of swaptions and CMS spread options are assumed to follow

$$\begin{aligned} dS(t) &= \bar{\sigma}(\bar{\beta}S(t) + (1 - \bar{\beta})S(0))\sqrt{V(t)}dW(t) \\ dV(t) &= \kappa(1 - V(t))dt + \xi\sqrt{V(t)}dZ(t), \quad V(0) = 1 \\ dW(t)dZ(t) &= 0. \end{aligned}$$

Implying Market Parameters

Using the pricing formula (33) for European swaptions, we extract a market-implied effective volatility $\bar{\sigma}_x^*$ and effective skew $\bar{\beta}_x^*$ for each forward swap rate S_x by minimising the function

$$\sum_j \left(\frac{1}{\bar{\beta}_x} c_{\text{Black}}(0, S_x^*, T, K_j^*, \bar{\sigma}_x \bar{\beta}_x) - c_{\text{market}}(T, K_j) \right)^2, \quad (47)$$

where we are summing across all swaption strikes. Ideally, we would also have liked to imply the stochastic volatility parameters κ^* and ξ^* from the market, but we found that these were very unstable in our optimisation as these parameters only had a minimal effect on swaption prices through the pricing formula (33). If instruments more sensitive to the values of the stochastic volatility parameters were to be included in the calibration, one would then do a global optimisation across all underlyings S_i to imply market values of these parameters. For purposes of this discussion, we set the mean reversion of variance $\kappa = 15\%$ and volatility of variance $\xi = 130\%$. These values for the parameters for the CIR stochastic volatility process generate rather convex volatility smiles, in

accordance with what is observed in the markets. Further, the relatively large value of κ is in line with the steep forward rate curve that we see with the current market data.

Optimisation

Before we go on to discuss how we might include CMS spread options in our calibration procedure, we first describe the calibration process if we used swaptions alone. Suppose that we have selected a correlation structure for calibration. As mentioned in section 4, we choose (40) to be our instantaneous volatility function. We follow [6] and [7] by parameterising the $\beta_i(t)$ not by time and forward rate indices, but by the time and time to maturity and denote them by $B(t, \tau)$. The connection between the two is given by

$$\beta_i(t) = B(t, T_i - t). \quad (48)$$

A set of knot points (t_j, τ_k) are then chosen on which $B(t, \tau)$ is calibrated, with all other intermediate values computed using bilinear interpolation. The advantage of this is that we are able to significantly reduce the number of parameters, as opposed to say a piecewise constant parameterisation of each $\beta_i(t)$, thus avoiding overfitting and we get more stable calibration results. Parameterising the skew functions in this manner, the time-homogeneity in the skew structure can then easily be checked by plotting $B(t, \tau)$ against τ . If the model skew structure were time-homogeneous, the plots of $B(t, \tau)$ against τ would coincide for each value of t .

With parameterisations of $\sigma_i(t)$ and $\beta_i(t)$ fully specified, the aim is to obtain model parameters for them by optimising with respect to the market-implied $\bar{\sigma}_x^*$ and $\bar{\beta}_x^*$ for each swap rate. The $\sigma_i(t)$ and $\beta_i(t)$ can be related to swap rate volatility and skew functions through (24) and (25). With these time-dependent parameters, we are then able to calculate model-implied effective volatilities and skews $\bar{\sigma}_x$ and $\bar{\beta}_x$, respectively, for each swap rate. So, we see that each model-implied effective volatility and skew is a function of the following parameters

- a , b , c , and d , in our volatility parameterisation (40),
- parameters of the chosen correlation structure,
- and the knot points of our skew parameterisation $B(t, \tau)$.

Thus, calibrating the model comes down to minimising the objective function

$$\sum_x \alpha_x (\bar{\sigma}_x(\hat{v}, \hat{\rho}, \hat{B}) - \bar{\sigma}_x^*)^2 + \sum_x \alpha_x (\bar{\beta}_x(\hat{v}, \hat{\rho}, \hat{B}) - \bar{\beta}_x^*)^2 \quad (49)$$

across the swap rates S_x , where $(\hat{v}, \hat{\rho}, \hat{B})$ is the triple of volatility, correlation, and skew parameters, respectively, and the α_x denote exogenously specified weights. Additional terms such as

$$\sum_{i,j} (B(t_i, \tau_j) - B(t_{i-1}, \tau_j))^2 \quad (50)$$

can be added to penalise deviation from a time-homogenous term structure, but we leave this for the interested reader to pursue.

Instead of combining volatility and skew calibrations in the same optimisation, [7] suggests splitting the minimisation of (49) into the two subproblems of minimising

$$\sum_x \alpha_x (\bar{\sigma}_x(\hat{v}, \hat{\rho}, \hat{B}) - \bar{\sigma}_x^*)^2 \quad \text{and} \quad \sum_x \alpha_x (\bar{\beta}_x(\hat{v}, \hat{\rho}, \hat{B}) - \bar{\beta}_x^*)^2 \quad (51)$$

dequentially to make the calibration faster and more stable. This is justified by the fact that the dependence of the swap rate volailities $\sigma_x(t)$ depend only very mildly on the forward rate skews $\beta_i(t)$ because, as Piterbarg [7] points out, the value of the local volatility function at the money $S_x = S_x(0)$ is independent of the skew parameter $\bar{\beta}_x$.

The calibration algorithm can thus be summarised as follows:

1. Imply market parameters $\bar{\beta}_x^*$ and $\bar{\sigma}_x^*$ for each swap rate by minimising the objective function (47). If required, optimise for market values of κ^* and ξ^* across all calibration instruments.
2. Set skew parameters B_{ij} to some initial value ([6] and [7] both chose this to be the average of all market skews $\bar{\beta}_i^*$). Calibrate the instantaneous volatility functions to $\bar{\sigma}_x^*$ using the Piterbarg effective volatility formula (31).
3. With the model volatilities calculated in the previous step, calibrate the model skews $\beta_i(t)$ to the market-implied $\bar{\beta}_x^*$ using the Piterbarg effective skew formula (30).
4. With the updated model skews $\beta_i(t)$, recalibrate the $\sigma_i(t)$ to the $\bar{\sigma}_x^*$ by repeating the second step.

Steps 3 and 4 can be repeated until stable results are obtained, but both [6] and [7] report that one cycle is often enough to obtain a good fit.

Extracting Correlations

Given initial swap rates $S(0) = (S_1(0), S_2(0))$ with market-implied effective skews and volatilities $\bar{\beta}^* = (\bar{\beta}_1^*, \bar{\beta}_2^*)$ and $\bar{\sigma}^* = (\bar{\sigma}_1^*, \bar{\sigma}_2^*)$, respectively, and stochastic volatility parameters ξ^* and κ^* , Lutz [8] solves

$$CMS_{spread}(K, T; S(0), \bar{\mu}, \bar{\sigma}^*, \bar{\beta}^*, \xi^*, \kappa^*, \bar{\rho}_{imp}) = \widehat{CMS}_{spread}(K, T) \quad (52)$$

for $\bar{\rho}_{imp}$ to extract market-implied swap rate correlations, where $\widehat{CMS}_{spread}(K, T)$ denotes the market price of a CMS spread option with strike K and maturity T , and CMS_{spread} is our CMS spread option pricing formula (39). To calculate the approximation $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2)$ using (37), the instantaneous forward rate volatilities are required in the computation of the average covariance matrix

$$(\bar{\Sigma})_{jk} = \frac{1}{T_\alpha} \int_0^{T_\alpha} \hat{\sigma}_j(t) \hat{\sigma}_k(t)' dt, \quad j, k = 1, \dots, n. \quad (53)$$

As we have yet to calibrate the model parameters, a proxy has to be used so that we can solve for $\bar{\rho}_{imp}$. Since we are excluding caplets from our calibration, Lutz's suggestion of using market-implied caplet volatilities is not possible here. Thus, we propose to use market-implied effective volatilities of the shortest-tenor swap rates

$$(\bar{\Sigma})_{jk} = \bar{\sigma}_j^* \bar{\sigma}_k^* \rho_{jk}, \quad j, k = 1, \dots, n, \quad (54)$$

where $(\rho)_{jk}$ is some simple average correlation matrix such as $(\rho)_{jk} = 0.5(1 + \exp(-0.15|T_k - T_j|))$.

With the market-implied swap rate correlations $\bar{\rho}_{imp}$, we are ready to include CMS spread options in our calibration. Lutz [8] points out that CMS spread option prices calculated using (39) are only mildly sensitive to the swap rate skews $\bar{\beta}$, and that the main determinants of their value are the effective swap rate volatilities $\bar{\sigma}$ and the effective swap rate correlation $\bar{\rho}$ calculated in the model with (36). Thus, he proposes only including CMS spread options in steps 1, 3, and 4 of the calibration algorithm above. Thus, a detailed calibration algorithm with the inclusion of CMS spread options is:

1. Imply market parameters $\bar{\beta}_x^*$ and $\bar{\sigma}_x^*$ for each swap rate by minimising the objective function (47). If required, optimise for market values of κ^* and ξ^* across all calibration instruments.
2. Back out market-implied swap rate correlations by solving (52) and using an approximation for $(\bar{\Sigma})_{jk}$.
3. Set skew parameters B_{ij} to some initial value. Calibrate the instantaneous volatility functions to $\bar{\sigma}^*$ and $\bar{\rho}_{imp}$. Effective swap rate correlation in the model is calculated using (36).
4. With the model volatilities calculated in the previous step, calibrate the model skews $\beta_i(t)$ to the market-implied $\bar{\beta}_x^*$ using the Piterbarg effective skew formula (30).
5. With the updated model skews $\beta_i(t)$ and volatilities $\sigma_i(t)$, recalibrate the $\sigma_i(t)$ to the $\bar{\sigma}_x^*$ and $\bar{\rho}_{imp}$ by repeating the second step.

Lutz [8] notes that this method only works well if the calibrated model-implied swap rate parameters are a close match to the market-implied ones. However, this is generally the case so this method will suffice for our purposes. See [8] for further discussion about this issue.

5.3 Calibration Results

Tables 3 and 4 show the market-implied effective volatilities $\bar{\sigma}^*$ and skews $\bar{\beta}^*$ for each forward swap rate. With these parameters, we use the correlation extraction method discussed in the previous section to get an initial estimate for the correlation between 10 year and 2 year swap rates. These are presented in table 5. Despite our best efforts in setting up the problem similarly to Lutz [8] for easy comparison, we found that calibration under such a scenario to be tremendously difficult in our

implementation. For example, unlike the 30+ seconds that Lutz takes to calibrate his models using swaption volatilities and implied correlations, running the first volatility optimisation (step 2 in the implied correlation algorithm) took upwards of 1 hour to complete even in the simplest exponential correlation scenario, and even then, the model parameters made no sense. Whether this is due to our implementation in a high-level package like Matlab (Lutz uses C++) or because of our different market setting is still unknown at this point. Nevertheless, we present our calibration results for 1 year swaptions with 2, 5, and 10 year maturities and the 10Y-2Y CMS spread option expiring in 1 year. Note that by modelling on the much smaller subset of 6 month forward rates with expiries up to 21.5 years, the price of our CMS spread options carries information about all the forward rates in our model.

Knot points were defined for times $t_i = 0, 2, 3, 4, 5, 6, 7, 11$ years and at times-to-maturity $\tau_j = 0, 2, 3, 4, 5, 6, 7, 11$ years. Excluding correlation parameters, we have 68 parameters in our model so far, 4 from the volatility function (40) and 64 from the knot points of our grid parameterisation.

We plot calibrated time-dependent model parameters in figure 6. From figures 6(c) and 6(d), it is clear that our model is not time-homogeneous as the skews are dependent on calendar time. Despite the problems in calibration described earlier, we find that our model fits well to the market-implied parameters and that the volatility and skew plots for both correlation parameterisations are not entirely outrageous. In fact, the root-mean-squared error was 0.02% for both the volatility and correlation optimisations of both parameterisations, though such a low error is expected because of the number of instruments we are calibrating to. Note, however, the marked difference in the correlation structures because of the inclusion of an asymptotic correlation parameter in the Doust parameterisation. Looking at the parameterised instantaneous volatility curves figures ?? and ??, we find that the market is experiencing very near-term excitement, much in line with the steep 6 month forward rate that we have. All this leads us to believe in the correctness of our approach, but we still have to look into the unreasonably long calculations as we include longer-dated forward rates.

Expiry	Maturity				
	2Y	5Y	10Y	20Y	30Y
1Y	34.89%	28.73%	23.99%	22.07%	23.59%
2Y	30.95%	26.54%	23.14%	21.58%	23.60%
5Y	23.08%	20.89%	19.95%	19.68%	21.85%
10Y	17.31%	16.76%	17.37%	17.88%	18.80%

Table 3: Effective forward swap volatilities $\bar{\sigma}^*$ extracted from swaption smiles. $\kappa = 15\%$, $\xi = 130\%$

Expiry	Maturity				
	2Y	5Y	10Y	20Y	30Y
1Y	43.32%	15.61%	-12.21%	-51.39%	-67.48%
2Y	45.41%	13.12%	-4.25%	-22.32%	-28.00%
5Y	24.98%	12.33%	-2.85%	-5.77%	-0.67%
10Y	11.55%	5.41%	-2.26%	-0.37%	3.31%

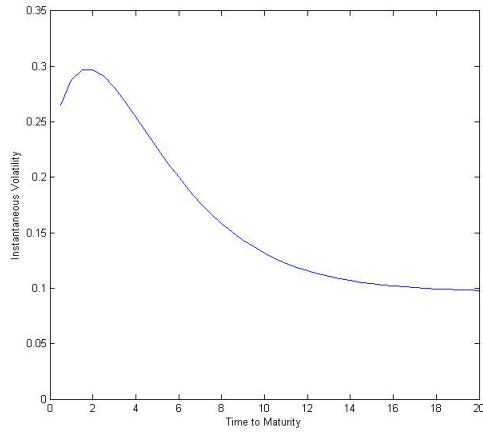
Table 4: Effective forward swap skews $\bar{\beta}^*$ extracted from swaption smiles. $\kappa = 15\%$, $\xi = 130\%$

10Y-2Y Swap Rate Correlation				
Fixing date	1Y	2Y	5Y	10Y
Correlation	0.6422	0.8582	0.9166	0.8894

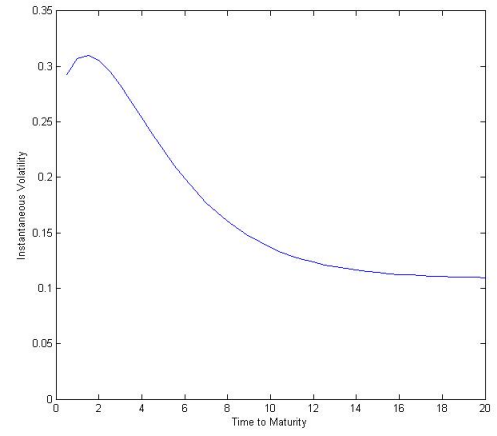
Table 5: Initial approximation of $\bar{\rho}$

6 Concluding Remarks

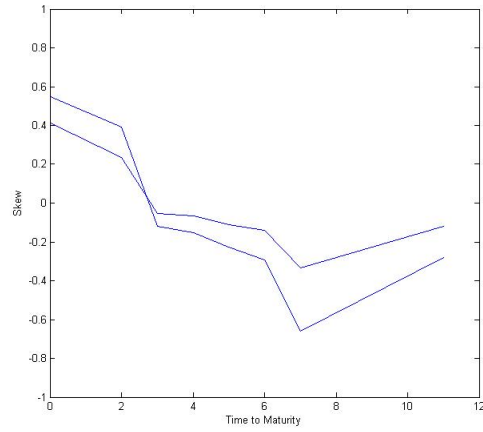
In this paper, we have presented and calibrated a Heston-type stochastic volatility displaced-diffusion LMM with different correlation parameterisations. While we have not been able to calibrate the model on the entire swaption grid, we find that in the shorter term, this model provides a realistic description of the market with a relatively good fit to market parameters. Further, we were able to include CMS spread options in our calibration by using a pricing formula developed by Lutz and Kiesel [6] for this model, allowing us to extract detailed information about different parts of the correlation surface that swaptions and caplets are not able to provide.



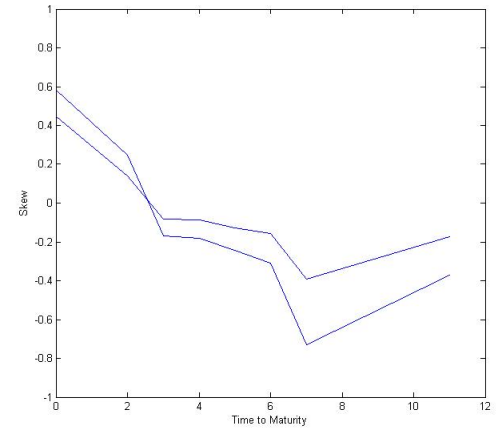
(a) Instantaneous Volatility, Simple Exponential



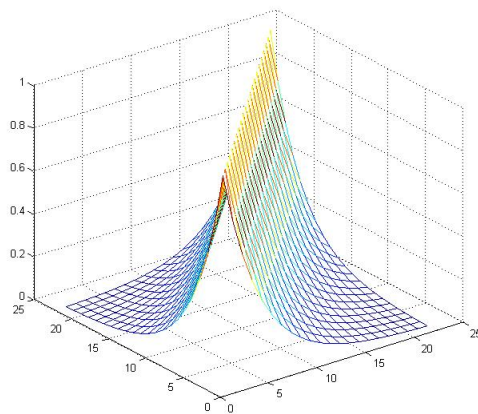
(b) Instantaneous Volatility, Doust



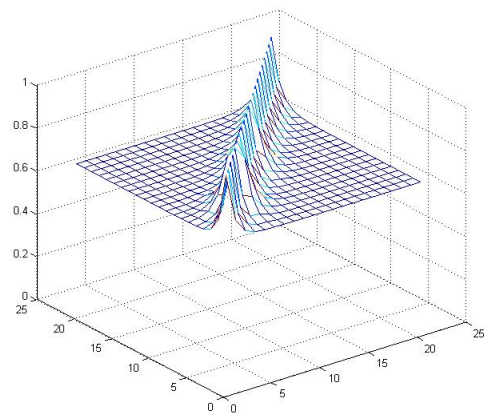
(c) Instantaneous Skew, Simple Exponential. $t=0$ on top, $t=2$ below



(d) Instantaneous Skew, Doust. $t=0$ on top, $t=2$ below



(e) Correlation Surface, Simple Exponential



(f) Correlation Surface, Doust

Figure 6: Model-parameterised skews, volatilities, and correlations of the simple exponential and Doust correlation parameterisations

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