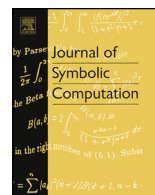




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Hypergeometric-type sequences

Bertrand Teguia Tabuguia^{a,b}^a Department of Computer Science, University of Oxford, UK^b Max Planck Institute for Software Systems, Saarbrücken, Germany

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ABSTRACT

We introduce hypergeometric-type sequences. They are linear combinations of interlaced hypergeometric sequences (of arbitrary interlacements). We prove that they form a subring of the ring of holonomic sequences. An interesting family of sequences in this class are those defined by trigonometric functions with linear arguments in the index and π , such as Chebyshev polynomials, $\left(\sin^2(n\pi/4) \cdot \cos(n\pi/6)\right)_n$, and compositions like $(\sin(\cos(n\pi/3)\pi))_n$.

We describe an algorithm that computes a hypergeometric-type normal form of a given holonomic n th term whenever it exists. Our implementation enables us to generate several identities for terms defined via trigonometric functions.

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1. Introduction

The connection between summation and linear difference equations with polynomial coefficients dates back to Fasenmyer (Fasenmyer, 1945, 1949; Koepf, 2014). These equations, called holonomic or P-recursive (or P-finite), easily lead to closed forms of the corresponding sums when the equations are made of two non-zero terms. The resulting equations have the form

$$P(n) a_{n+m} = Q(n) a_n, \quad (1)$$

where P and Q are polynomials. When $m = 1$, the corresponding solution is a hypergeometric term. For $m > 1$, we say that the solution is m -fold hypergeometric.

E-mail address: bertrand.teguia@cs.ox.ac.uk.

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Zeilberger (1990b) and Chyzak (2000) generalized the algorithmic approach to finding holonomic recurrence equations for sums. Zeilberger proposed an efficient algorithm for dealing with sums whose summands are hypergeometric terms (Zeilberger, 1990a). It resulted in several automatic proofs of special functions and combinatorial identities in synergy with his collaboration with Wilf (Wilf and Zeilberger, 1990) for the WZ method, which cleverly uses Gosper's algorithm (Gosper, 1978). The recipe for systematic finding of hypergeometric identities could not be ready before the availability of an algorithm for finding all hypergeometric term solutions of P-recursive equations. For instance, for a summation problem like

$$s_n := \sum_{k=-\infty}^{\infty} (-1)^k \binom{n}{k} \binom{d \cdot k}{n}, \quad d \in \mathbb{N},$$

Zeilberger's original algorithm finds a recurrence equation of order d while it can be proven that $s_n = (-d)^n$ – a hypergeometric term.

In 1993, Petkovšek completed the recipe by proposing algorithm *Hyper* (Petkovšek, 1992). Thanks to that, the computer algebra community benefited from the book $A = B$ (Petkovšek et al., 1996), which gathers all fundamental results, at least for that moment. The present paper does not intend to investigate symbolic summation. For further results in this direction, we recommend the following non-exhaustive list of references (Paule, 1995; Koepf, 1995; Abramov and Petkovšek, 2001; Koutschan, 2013; Chyzak et al., 2009; Chyzak, 2014; Chen et al., 2018), (Kauers, 2023, Chapter 5).

Petkovšek's algorithm did not only serve combinatorial identities but also formal power series. Koepf establishes a connection with his work (Koepf, 1992), where he also highlighted the need for an algorithm for finding m -fold hypergeometric term solutions of holonomic recurrences. Petkovšek and Salvy (1993) proposed a way to adapt *Hyper* to this context. Abramov further investigated the study of such solutions (Abramov, 2000). Ryabenko gave the first concrete implementation in the Maple computer algebra system (CAS) (Ryabenko, 2000).

Mark van Hoeij improved Petkovšek's algorithm to a much more efficient version (van Hoeij, 1999; Cluzeau and van Hoeij, 2006). A theoretical algorithm for its generalization to the m -fold case was first proposed in Horn et al. (2012). We here recall results from the author's Ph.D. thesis (Teguia Tabuguia, 2020) (see also Teguia Tabuguia and Koepf (2022b)) as a refreshment before introducing hypergeometric-type sequences.

Definition 1 (*Proper hypergeometric-type power series* (Teguia Tabuguia and Koepf, 2021a)). For an expansion around zero, a series $S(z)$ is said to be of “proper” hypergeometric type if it can be written as

$$S(z) := \sum_{j=1}^J S_j(z), \quad S_j(z) := \sum_{i=0}^{m_j-1} \sum_{n=0}^{\infty} a_i(m_j n + i) z^{m_j n + i}, \quad (2)$$

where $m_j, J \in \mathbb{N}, m_j \neq 0$, and $a_i(n)$ is a linear combination of m_j -fold hypergeometric terms. Thus, a proper hypergeometric-type power series is a linear combination of formal power series whose coefficients are m -fold hypergeometric terms. A proper hypergeometric-type function is a function that can be expanded as a proper hypergeometric-type power series.

The word “proper” in Definition 1 is used to lighten the definition in Teguia Tabuguia and Koepf (2021b,a) by neglecting Laurent-Puiseux series. Note that in contrast with the definition given in the original papers (Teguia Tabuguia, 2020; Teguia Tabuguia and Koepf, 2022b), here we highlight that the coefficients a_i 's are not necessarily built from the same hypergeometric terms, and this is in perfect agreement with the scope of the formal power series algorithm proposed there.

Recently, Koepf and the author designed *mfoldHyper*, an algorithm that extends the algorithms by Petkovšek and van Hoeij to find all m -fold hypergeometric term solutions of P-recursive equations. It has the advantage of offering a better efficiency than the algorithm from Petkovšek and Salvy (1993); Ryabenko (2000) (see also Teguia Tabuguia (2021)). Algorithm *mfoldHyper* helped to

design a complete algorithm to convert a univariate holonomic function into a hypergeometric-type power series (Teguia Tabuguia, 2020; Teguia Tabuguia and Koepf, 2022b). The resulting algorithm is available from Maple 2022 as `convert/FormalPowerSeries`, and `mfoldHyper` as `LREtools:-mhypergeomsols`, all from the FPS package at Koepf and Teguia Tabuguia (2022). The Maxima version of the package is in the process of being integrated into Maxima.

We observed that the concept of hypergeometric type can be adapted to sequences. In this regard, a similar development as that of formal power series would enable a compact definition of closed forms for terms that are usually expressed in cases depending on some properties satisfied by the index. Roughly speaking, a hypergeometric-type sequence is a sequence whose general term (n th term) is a linear combination of m -fold hypergeometric terms. We will give a formal definition in the next section. Several examples can be generated with trigonometric functions. In this case, the property satisfied by the index refers to its remainder with respect to some non-negative integer.

Throughout this paper, we will always assume that n is an integer, usually non-negative, i.e., $n \in \mathbb{N} := \{0, 1, 2, \dots\}$. It may sound intriguing to notice that there seems to be no CAS that uses a symbolic computation algorithm to find normal forms free of unevaluated trigonometric functions. Some examples are $\sin^2(n\pi/4)$, $\sin(\cos(n\pi/3)\pi)$, etc. Of course, one could always eliminate trigonometric functions using simplifications with Euler's formulas (see, for instance, `convert/-exp` in Maple or `TrigToExp` in Mathematica); however, this would not define normal forms as the simplifications used after conversion into exponentials may be tailored to the given trigonometric expressions. Because Maple seems to be the only CAS implementing `mfoldHyper` or its analog from Ryabenko (2000), and such a simplification of trigonometric sequences is not available in Maple, it is reasonable to see our idea as a novel method. It sets a symbolic approach for finding closed forms of a more general class of sequences, for which we also develop a theoretical framework.

Our approach to hypergeometric-type sequences resembles that of hypergeometric-type power series. We consider three main steps: given a term h_n ,

1. find a P-recursive equation satisfied by h_n ;
2. find a basis of all m -fold hypergeometric term solutions of that equation (using `mfoldHyper`, for instance);
3. use initial values from h_n to deduce a hypergeometric-type normal form for h_n .

The three steps would be successful if h_n is the term of a hypergeometric-type sequence (or simply a hypergeometric-type term).

In the next section, we define hypergeometric-type sequences, study their structure, and state some properties, like their link to hypergeometric-type functions. After discussing canonical and normal forms of hypergeometric-type terms, Section 3 details the three steps of our algorithmic method. In Section 4, we present our current Maple implementation from the package `HyperTypeSeq`. We recommend using Maple versions between 2019 and 2021 because of some misbehavior of the package with the recent releases. The package is accessible via Github at Teguia Tabuguia. Below are two simple formulas automatically computed using our implementation.

```
> with(HyperTypeSeq):
> HTS(sin(n*Pi/4)^2,n)
```

$$\frac{1}{2} - \frac{(-1)^{\frac{n}{2}} \chi_{\{\text{mod}p(n,2)=0\}}}{2} \quad (3)$$

The sequence [A212579](#) from the OEIS (Sloane et al., 2003) satisfies the recurrence equation

```
> RE:= a(n) = a(n-1)+2*a(n-2)-a(n-3)-2*a(n-4)-a(n-5)+2*a(n-6)+a(n-7)-a(n-8)
```

$$RE := a(n) = a(n-1) + 2a(n-2) - a(n-3) - 2a(n-4) - a(n-5) + 2a(n-6) + a(n-7) - a(n-8).$$

Using the first thirteen initial values, our algorithm finds the closed form:

```
> REToHTS(RE, a(n), [0, 1, 8, 31, 80, 171, 308, 509, 780, 1137, 1584,
2143, 2812])
```

$$\frac{4}{9} + \frac{31}{12}n - 3n^2 + \frac{67}{36}n^3 - \frac{1}{4}n\chi_{\{modp(n,2)=0\}} - \frac{4}{9}\chi_{\{modp(n,3)=0\}} - \frac{8}{9}\chi_{\{modp(n,3)=1\}}. \quad (4)$$

In these outputs, $\chi_{\{modp(n,m)=j\}}$ denotes the indicator function for the set of integers with non-negative remainder $j \in \{0, \dots, m-1\}$ in their division by m ; and $\chi_{\{modp(n,1)=0\}} = 1$.

2. Structure and properties

We consider sequences in $\mathbb{K}^{\mathbb{N}}$, where \mathbb{K} is a field of characteristic zero. In general, \mathbb{K} is a number field or a field such that the field extension \mathbb{K}/\mathbb{Q} has a finite transcendence degree. We will use $n \in \mathbb{N}$ as the index variable. All our results may certainly extend to negative indices $-n \in \mathbb{Z}$, but we restrict ourselves to \mathbb{N} to fix the starting index at 0 or $n_0 \in \mathbb{N}$ for factorial-like sequences. As it turns out, this is enough to introduce all the necessary concepts. When referring to an arbitrary sequence in $\mathbb{K}^{\mathbb{N}}$, we will denote that sequence with parentheses as $(s)_n \in \mathbb{K}^{\mathbb{N}}$ or simply (s) . The n th term of a sequence $(s)_n$, also called its general term, is s_n (without the parentheses) or $s(n)$. We usually use the former notation; we use the latter when the subscript is already occupied. For instance, the term $T_1(n)$ is the n th term of the sequence $(T_1)_n$.

2.1. Interlacements: m -fold indicator sequences

Before diving into the concept of interlacement, let us recall a notion commonly used in set theory that will prove useful in the sequel. Let \mathcal{A} be a set, and $A \subset \mathcal{A}$. The indicator function of A , denoted χ_A , is defined as

$$\begin{aligned} \chi_A: \mathcal{A} &\longrightarrow \{0, 1\} \\ a &\mapsto \chi_A(a) := \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (5)$$

Definition 2 (m -fold indicator sequence). A sequence $(s)_n$ is said to be m -fold indicator if there exists a positive integer m_0 such that $(s)_n$ is the indicator function of the non-negative integers in a coset of $\mathbb{Z}/m_0\mathbb{Z}$. In this case, $(s)_n$ is called m_0 -fold indicator sequence, and m_0 is its characteristic.

Note that the definition naturally extends to the whole set of integers \mathbb{Z} but then requires a certain care for the starting index. Without a specific value, the terminology m -fold indicator sequence refers to any such sequence. In that case, m may also be used as the corresponding characteristic without ambiguity.

Example 1.

- $\chi_{\{2n+1, n \in \mathbb{N}\}}$: the indicator function of the set $\{2n+1, n \in \mathbb{N}\}$ of odd natural numbers is a 2-fold indicator sequence.
- $\chi_{\{3n+2, n \in \mathbb{N}\}}$ is an m -fold indicator sequence of characteristic 3. ■

The following proposition shows that the characteristic of an m -fold indicator sequence is unique.

Proposition 1. Let m be a positive integer such that the sequence $(s)_n$ is m -fold indicator. Then m is unique.

Proof. Denote by $[j]_m$, $j \in \{0, \dots, m-1\}$, the non-negative integers of the coset in $\mathbb{Z}/m\mathbb{Z}$ of integers with remainder $j \geq 0$ in their division by m . Let m_1 and m_2 be two distinct positive integers such that the sequence $(s)_n$ is m_1 -fold indicator and m_2 -fold indicator. To obtain a contradiction, we only have to find an index n at which $s_n = 0$ and $s_n = 1$. Without loss of generality, we can assume that $m_1 < m_2$. Suppose that $(s)_n$ is the indicator sequence of $[j_1]_{m_1}$ and $[j_2]_{m_2}$, $j_1 \in \{0, \dots, m_1-1\}$, $j_2 \in \{0, \dots, m_2-1\}$. If $j_1 \neq j_2$ then we are done since $s_{j_1} = 1$ as an m_1 -fold indicator sequence but $s_{j_1} = 0$ as an m_2 -fold indicator sequence. If $j_1 = j_2$, then $j_1 + m_1 \in [j_1]_{m_1}$, but $j_1 + m_1 \notin [j_2]_{m_2}$. Hence, taking n as $j_1 + m_1$ yields a contradiction. Therefore we must have $m_1 = m_2$. \square

A direct consequence of Proposition 1 is that we can count the number m -fold indicator sequences of characteristic m . The following corollary is also used as a definition to uniquely identify m -fold indicator sequences and fix a notation that we will use in the rest of the paper.

Corollary 1. *There are exactly m m -fold indicator sequences of characteristic m . For any $j \in \{0, \dots, m-1\}$, the m -fold indicator sequence of characteristic m and remainder j , denoted $(\chi_{\{j \bmod m\}})_n$, is defined by the general term*

$$\chi_{\{j \bmod m\}}(n) = \chi_{\{n \equiv j \bmod m\}} = \begin{cases} 1 & \text{if } n \equiv j \bmod m \\ 0 & \text{otherwise} \end{cases}, \quad n \in \mathbb{N}, \quad (6)$$

where $n \equiv j \bmod m$ means that j is the non-negative remainder of n in its division by m .

We mention that m -fold indicator sequences are intrinsically discussed in Abramov (2000), but the attention there is on solving differential equations rather than studying the object itself. Here, we present definitions and properties to capture the mathematical essence of the concept of interlacement, which we will use later to define hypergeometric-type sequences. Our notation is closer to that used in Kauers (2023, Section 2.2, page 108), where $\chi_{\{n \equiv j \bmod m\}}$ is written as $\delta_{n \bmod m, j}$. However, there, m -fold indicator sequences are only considered when they have the same characteristic, in which case, their sums and products are straightforward to deduce.

We will now see what happens when we add and multiply m -fold indicator sequences of arbitrary characteristics and remainders. Since 0 is not a divisor of any integer and that 1 divides them all, we conventionally note $(\chi_{\{0 \bmod 0\}})_n := (\chi_{\{\bmod 0\}})_n$ the zero sequence $(0, 0, \dots)$, and $(\chi_{\{0 \bmod 1\}})_n := (\chi_{\{\bmod 1\}})_n$ the one sequence $(1, 1, \dots)$. These conventions make the next statements of this subsection more precise.

Proposition 2 (Sum of m -fold indicator sequences). *The sum of two non-zero m -fold indicator sequences is not an m -fold indicator sequence.*

Proof. If there are indices where the two m -fold indicator sequences take the same value 1, then their sum takes the value 2 at those indices and thus cannot be an m -fold indicator sequence. We could stop here as this first case necessarily happens for two m -fold indicator sequences. On the other hand, assuming that there are no indices where the two m -fold indicator sequences take the value 1 and that it is m -fold indicator would imply that their sum is an m -fold indicator sequence with two characteristics, which is impossible according to Proposition 1. \square

For products of m -fold indicator sequences, let us first look at some illustrative examples.

Example 2 (Product of m -fold indicator sequences, part I). Let examine the product $(\chi_{\{1 \bmod 4\}})_n \cdot (\chi_{\{1 \bmod 6\}})_n$. Table 1 presents some of the first indices where the terms of both sequences are 1, with their coincidences colored in red.

From this table one observes that the term $\chi_{\{n \equiv 1 \bmod 4\}} \cdot \chi_{\{n \equiv 1 \bmod 6\}}$ is apparently 1 periodically. The period corresponds to the arithmetic progression $12n + 1 = \text{lcm}(4, 6) \cdot n + 1$, where lcm stands for least common multiple. \blacksquare

Table 1

Indices where $\chi_{\{n \equiv 1 \pmod{4}\}} = 1$ and $\chi_{\{n \equiv 1 \pmod{6}\}} = 1$ for $n \leq 37$. The red color is for the coincidences. (For interpretation of the colors in the table(s), the reader is referred to the web version of this article.)

$n \equiv 1 \pmod{4}$	1	5	9	13	17	21	25	29	33	37
$n \equiv 1 \pmod{6}$	1		7	13		19	25		31	37

Table 2

Indices where $\chi_{\{n \equiv 1 \pmod{4}\}} = 1$ and $\chi_{\{n \equiv 2 \pmod{6}\}} = 1$ for $n \leq 38$. No coincidence occurs.

$n \equiv 1 \pmod{4}$	1	5	9	13	17	21	25	29	33	37
$n \equiv 2 \pmod{6}$		2	8		14	20		26	32	38

Table 3

Indices where $\chi_{\{n \equiv 3 \pmod{4}\}} = 1$ and $\chi_{\{n \equiv 2 \pmod{5}\}} = 1$ for $n \leq 67$. Four coincidences: $n = 7, n = 27, n = 47$, and $n = 67$.

$n \equiv 3 \pmod{4}$	3	7	11	15	19	23	27	31	35	39
$n \equiv 2 \pmod{5}$	2	7		12	17	22	27		32	37
$n \equiv 3 \pmod{4}$			43	47	51	55	59	63	67	
$n \equiv 2 \pmod{5}$		42		47		52	57	62	67	

It turns out that the observation in Example 2 hides a general fact partly established by the following lemma.

Lemma 1. Let $m_1, m_2 \in \mathbb{N} \setminus \{0\}$, and $j \in \{0, 1, \dots, \min\{m_1, m_2\} - 1\}$. Then

$$\chi_{\{n \equiv j \pmod{m_1}\}} \cdot \chi_{\{n \equiv j \pmod{m_2}\}} = \chi_{\{n \equiv j \pmod{\text{lcm}(m_1, m_2)}\}}. \quad (7)$$

In other words, the product of two m -fold indicator sequences of the same remainder is an m -fold indicator sequence of this same remainder.

Proof. $(\chi_{\{j \pmod{m_1}\}})_n$ is the indicator sequence of $[j]_{m_1}$ and $(\chi_{\{j \pmod{m_2}\}})_n$ is the one of $[j]_{m_2}$ (this notation was introduced in Proposition 1). Thus for all non-negative integers n , $\chi_{\{n \equiv j \pmod{m_1}\}} \cdot \chi_{\{n \equiv j \pmod{m_2}\}}$ is 1 if and only if $n \equiv j \pmod{m_1}$ and $n \equiv j \pmod{m_2}$. This implies that there exist non-negative integers k_1 and k_2 , such that $n = m_1 \cdot k_1 + j = m_2 \cdot k_2 + j$. So $m_1 \cdot k_1 = m_2 \cdot k_2$. Thus $m_1 \mid m_2 \cdot k_1$ (m_1 divides $m_2 \cdot k_1$) and $m_2 \mid m_1 \cdot k_1$. Let $\mu = \text{lcm}(m_1, m_2)$. Then $\mu \mid m_1 \cdot k_1$ and $\mu \mid m_2 \cdot k_2$ since both m_1 and m_2 divide $m_1 \cdot k_1$ and $m_2 \cdot k_2$. Therefore $n \equiv j \pmod{\mu}$ and we conclude that $\chi_{\{n \equiv j \pmod{m_1}\}} \cdot \chi_{\{n \equiv j \pmod{m_2}\}} = \chi_{\{n \equiv j \pmod{\mu}\}}$ by the uniqueness property (see Proposition 1). \square

Thus, we are now certain that $\chi_{\{n \equiv 1 \pmod{4}\}} \cdot \chi_{\{n \equiv 1 \pmod{6}\}} = \chi_{\{n \equiv 1 \pmod{12}\}}$ for all $n \in \mathbb{N}$. Let us consider the case of different remainders and generalize Lemma 1.

Example 3 (Product of m -fold indicator sequences, part II). We consider the product $(\chi_{\{1 \pmod{4}\}})_n \cdot (\chi_{\{2 \pmod{6}\}})_n$. Table 2 presents some of the first indices where the terms of both sequences are 1, showing no coincidence for $n \leq 38$.

We claim that $\chi_{\{n \equiv 1 \pmod{4}\}} \cdot \chi_{\{n \equiv 2 \pmod{6}\}} = 0 = \chi_{\{n \equiv 0 \pmod{0}\}}$ for all $n \in \mathbb{N}$. \blacksquare

Example 4 (Product of m -fold indicator sequences, part III). We want to find $(\chi_{\{3 \pmod{4}\}})_n \cdot (\chi_{\{2 \pmod{5}\}})_n$. We consider the terms of indices $n \leq 67$ (see Table 3).

Claim: $\chi_{\{n \equiv 3 \pmod{4}\}} \cdot \chi_{\{n \equiv 2 \pmod{5}\}} = \chi_{\{n \equiv 7 \pmod{\text{lcm}(4,5)}\}} = \chi_{\{n \equiv 7 \pmod{20}\}}$ for all $n \in \mathbb{N}$. \blacksquare

Together with Lemma 1, the following statement plays a crucial role in proving one of our main results. They establish that the set of m -fold indicator sequences is multiplicatively closed.

Lemma 2 (Product of m -fold indicator sequences). *The product of two distinct m -fold indicator sequences of distinct remainders is an m -fold indicator sequence.*

Proof. Let $\chi_{\{j_1 \bmod m_1\}}$ and $\chi_{\{j_2 \bmod m_2\}}$ be two m -fold indicator sequences such that $j_1 \neq j_2$. Let $\mu = \text{lcm}(m_1, m_2)$ and

$$\mathcal{N} := \left\{ j \in \mathbb{N} : j < \mu \text{ and there exist } j_1, j_2 \in \mathbb{N}, j_1 < m_1, j_2 < m_2, \begin{cases} j \equiv j_1 \bmod m_1, \\ j \equiv j_2 \bmod m_2. \end{cases} \right\}. \quad (8)$$

We consider two cases: $\mathcal{N} = \emptyset$ and $\mathcal{N} \neq \emptyset$.

Case 1: $\mathcal{N} = \emptyset$. This means that for indices less than μ , there is no coincidence of 1 between $\chi_{\{n \equiv j_1 \bmod m_1\}}$ and $\chi_{\{n \equiv j_2 \bmod m_2\}}$. We show that when this happens, the coincidence will not occur, and thus, the corresponding product of m -fold indicator sequences is the zero sequence. Let $n \in \mathbb{N}$ such that $n \equiv j_1 \bmod m_1 \equiv j_2 \bmod m_2$. So there exist $k_1, k_2 \in \mathbb{N}$, such that $m_1 \cdot k_1 + j_1 = m_2 \cdot k_2 + j_2$. Since $\mathcal{N} = \emptyset$, there exist $k_3, j_3 \in \mathbb{N}$, $k_3 > 0$ and $j_3 < \mu$ such that $n = \mu \cdot k_3 + j_3$. Thus we have

$$m_1 \cdot k_1 + j_1 = m_2 \cdot k_2 + j_2 = \mu \cdot k_3 + j_3.$$

We obtain a contradiction since this implies $j_3 \in \mathcal{N}$. Therefore if $\mathcal{N} = \emptyset$ then $\chi_{\{n \equiv j_1 \bmod m_1\}} \cdot \chi_{\{n \equiv j_2 \bmod m_2\}} = 0$ for all $n \in \mathbb{N}$.

Case 2: $\mathcal{N} \neq \emptyset$. To prove that the product is an m -fold indicator sequence, we only need to show that $|\mathcal{N}| = 1$, i.e., \mathcal{N} has only one element. We proceed by contradiction. As a subset of \mathbb{N} , \mathcal{N} has a least element. Let j_0 be that element. Then, following the reasoning of the first case, we can find integers k_1, k_2 such that

$$j_0 = m_1 \cdot k_1 + j_1 = m_2 \cdot k_2 + j_2.$$

Let $j'_0 \in \mathcal{N}$, $j'_0 \neq j_0$. Then $j'_0 > j_0$ and we can find k'_1 and k'_2 such that

$$j'_0 = m_1 \cdot k'_1 + j'_1 = m_2 \cdot k'_2 + j'_2.$$

Thus

$$\begin{aligned} j'_0 - j_0 &= m_1(k'_1 - k_1) + j'_1 - j_1 \\ &= m_2(k'_2 - k_2) + j'_2 - j_2. \end{aligned}$$

Hence $j'_0 - j_0 \in \mathcal{N}$. In fact, for every $l \in \mathbb{N}$, if $j'_0 - l \cdot j_0 \geq 0$, then $j'_0 - l \cdot j_0 \in \mathcal{N}$. By Euclidean division, let us write $j'_0 = j_0 \cdot q + r$, $0 \leq r < j_0$. Then $r = j'_0 - j_0 \cdot q \in \mathcal{N}$, contradicting the fact that j_0 is the smallest element in \mathcal{N} . Therefore if $\mathcal{N} \neq \emptyset$ then

$$\chi_{\{n \equiv j_1 \bmod m_1\}} \cdot \chi_{\{n \equiv j_2 \bmod m_2\}} = \chi_{\{n \equiv j_0 \bmod \mu\}}$$

for all $n \in \mathbb{N}$, where j_0 is the unique element of \mathcal{N} . \square

One can also prove Lemma 2 using the *Chinese Remainder Theorem*. The proof of Lemma 2 is constructive and gives an algorithmic way to find products of m -fold indicator sequences. Remark that they form a multiplicative group where the one sequence $\chi_{\{0 \bmod 1\}}$ is the unit element.

Example 5.

- $(\chi_{\{1 \bmod 4\}})_n \cdot (\chi_{\{2 \bmod 6\}})_n = (\chi_{\{1 \bmod 0\}})_n = 0$.
- $(\chi_{\{3 \bmod 4\}})_n \cdot (\chi_{\{2 \bmod 5\}})_n = (\chi_{\{7 \bmod 20\}})_n$.
- $(\chi_{\{1 \bmod 2\}})_n \cdot (\chi_{\{1 \bmod 3\}})_n = (\chi_{\{1 \bmod 6\}})_n = 0$. \blacksquare

2.2. Definition and structure

We are now ready to define hypergeometric-type sequences. The idea is to encompass every possible linear combination of interlaced hypergeometric terms we can think of.

Definition 3 (*Hypergeometric-type sequence*). A sequence $(s)_n$ is said to be of hypergeometric type if there exist finitely many m -fold indicator sequences $(\chi_{\{j_1 \bmod m_1\}})_n, \dots, (\chi_{\{j_l \bmod m_l\}})_n$ such that its general term s_n writes

$$s_n = H_1(\sigma_1(n)) \cdot \chi_{\{n \equiv j_1 \bmod m_1\}} + H_2(\sigma_2(n)) \cdot \chi_{\{n \equiv j_2 \bmod m_2\}} + \dots + H_l(\sigma_l(n)) \cdot \chi_{\{n \equiv j_l \bmod m_l\}}, \quad (9)$$

where $\sigma_i: \mathbb{N} \rightarrow \mathbb{Q}$ is such that $\sigma_i(m_i \cdot n + j_i) \in \mathbb{N}$, and $H_i(n)$ is a \mathbb{K} -linear combination of hypergeometric terms, $i = 1, \dots, l$. We call the H_i 's the coefficients of s_n (or $(s)_n$).

Remark 1.

- In Definition 3, we used $H_i(\sigma(n))$ instead of $H_{i\sigma(n)}$ to ease the notation and avoid confusion with indices.
- The specification of “finitely many” is mainly considered for algorithmic computation, though it seems unfeasible to envision arithmetic operations when the sum in (9) is infinite.
- When one of the m -fold indicator terms in (9) is $\chi_{\{0 \bmod 1\}}$, the corresponding summand is replaced by its coefficient.

Let (\mathcal{H}_T) denote the set of hypergeometric-type sequences, and \mathcal{H}_T be the set of their general terms. Unless otherwise stated, we assume that if $(s)_n \in (\mathcal{H}_T)$ then $s_n \in \mathcal{H}_T$. Every hypergeometric sequence is of hypergeometric type; in particular, every polynomial and rational sequence is of hypergeometric type.

Example 6 (*An example from the OEIS (Sloane et al., 2003)*). The general term, say a_n , of the sequence [A307717](#) counts the number of palindromic squares, n^2 , of length n (in the decimal basis) such that n is also palindromic. Its explicit formula (see for instance Kauers and Koutschan, 2022; Teguia Tabuguia and Koepf, 2022a) is given by

$$a_n := \begin{cases} 0 & \text{if } n \equiv 0 \bmod 2 \\ \frac{195+203n-15n^2+n^3}{192} & \text{if } n \equiv 1 \bmod 4 \\ \frac{501+107n-9n^2+n^3}{384} & \text{if } n \equiv 3 \bmod 4 \end{cases}. \quad (10)$$

The sequence $(a)_n$ is, of course, of hypergeometric type since its general term can be written as

$$a_n = \frac{195 + 203n - 15n^2 + n^3}{192} \chi_{\{n \equiv 1 \bmod 4\}} + \frac{501 + 107n - 9n^2 + n^3}{384} \chi_{\{n \equiv 3 \bmod 4\}}. \quad \blacksquare \quad (11)$$

Example 7. Let us consider two hypergeometric terms h_n and g_n . By using subsequences of $(h)_n$ and $(g)_n$, we can construct several hypergeometric-type sequences. For instance, the two general terms

$$u_n := h_{(n-1)/4} \chi_{\{n \equiv 1 \bmod 4\}} + h_{(n-2)/5} \chi_{\{n \equiv 2 \bmod 5\}}, \quad (12)$$

$$v_n := g_{(n-2)/6} \chi_{\{n \equiv 2 \bmod 6\}} + g_{(n-3)/4} \chi_{\{n \equiv 3 \bmod 4\}}, \quad (13)$$

are of hypergeometric type. Moreover, by definition, their sum

$$u_n + v_n = h_{(n-1)/4} \chi_{\{n \equiv 1 \bmod 4\}} + g_{(n-3)/4} \chi_{\{n \equiv 3 \bmod 4\}} + h_{(n-2)/5} \chi_{\{n \equiv 2 \bmod 5\}} + g_{(n-2)/6} \chi_{\{n \equiv 2 \bmod 6\}}, \quad (14)$$

is also of hypergeometric type. For their product, using Lemma 2, one can easily show that

$$u_n v_n = h_{(n-2)/5} g_{(n-3)/4} \chi_{\{n \equiv 7 \pmod{20}\}}. \quad (15)$$

Thus their product is also of hypergeometric type since the product of hypergeometric terms is a hypergeometric term. ■

The previous example illustrates a general fact concerning the closure properties of hypergeometric-type sequences. We establish it in the following theorem.

Theorem 1. *The set (\mathcal{H}_T) of hypergeometric-type sequences is a ring.*

Proof. Let $(s)_n, (s')_n \in (\mathcal{H}_T)$, and denote by \mathfrak{M} and \mathfrak{M}' the sets of m -fold indicator terms occurring in s_n and s'_n , respectively. So we have

$$\begin{aligned} s_n &= \sum_{\chi_i \in \mathfrak{M}} H_i(\sigma_i(n)) \chi_i(n), \\ s'_n &= \sum_{\chi'_i \in \mathfrak{M}'} H'_i(\sigma'_i(n)) \chi'_i(n). \end{aligned}$$

Then, we can write

$$\begin{aligned} s_n + s'_n &= \sum_{\chi_i \in \mathfrak{M} \cap \mathfrak{M}'} (H_i(\sigma_i(n)) + H'_i(\sigma'_i(n))) \chi_i(n) + \sum_{\chi_i \in \mathfrak{M} \setminus \mathfrak{M}'} H_i(\sigma_i(n)) \chi_i(n) \\ &\quad + \sum_{\chi'_i \in \mathfrak{M}' \setminus \mathfrak{M}} H'_i(\sigma'_i(n)) \chi'_i(n) \in \mathcal{H}_T. \end{aligned} \quad (16)$$

Hence $(s + s')_n \in (\mathcal{H}_T)$.

By distributivity of the multiplication with respect to addition, the product $s_n \cdot s'_n$ yields a sum of terms of the form

$$H_i(\sigma_i(n)) \cdot H'_j(\sigma'_j(n)) \cdot \chi_i(n) \cdot \chi'_j(n), \quad \chi_i \in \mathfrak{M}, \chi'_j \in \mathfrak{M}'. \quad (17)$$

Since the product of hypergeometric terms is a hypergeometric term, and that from Lemma 1 and Lemma 2 we know that $\chi_i(n) \cdot \chi'_j(n)$ is an m -fold indicator term, say $\chi''_{i,j}$, we deduce that (17) can be written as $H''_{i,j}(\sigma''_{i,j}(n)) \chi''_{i,j}(n)$, where $H''_{i,j}$ is a linear combination of hypergeometric terms. Hence $(s \cdot s')_n \in (\mathcal{H}_T)$.

In conclusion (\mathcal{H}_T) is a subring of $\mathbb{K}^{\mathbb{N}}$. □

2.3. Generating functions

With their LLL-based technique of guessing, Kauers and Koutschan were able to find a 6th-order holonomic recurrence equation of degree 9 for the OEIS sequence [A307717](#) of Example 6 from its first 70 terms. We have seen that this sequence is of hypergeometric type. We want to prove that such an equation always exists for any hypergeometric-type sequence.

Proposition 3. *Every hypergeometric-type sequence is P-recursive.*

Proof. We give sufficient arguments that show how to construct a holonomic recurrence equation satisfied by a given hypergeometric-type term. A basic example is given in Section 3.2. Let $(s)_n \in (\mathcal{H}_T)$ such that

$$s_n := \sum_{i=1}^l H_i(\sigma_i(n)) \chi_{\{n \equiv j_i \pmod{m_i}\}}.$$

Then for all $n \in \mathbb{N}$, $1 \leq i \leq l$, $s_{m_i n + j_i} = H_i(\sigma_i(m_i n + j_i)) + \epsilon_i(n)$. We neglect $\epsilon_i(n)$ and look for a recurrence equation for $u_i(m_i n + j_i) := H_i(\sigma_i(m_i n + j_i))$. Let l_i be the number of m_i -fold hypergeometric terms in $u_i(m_i n + j_i)$. Since $u_i(m_i n + j_i)$ is a linear combination of m_i -fold hypergeometric terms, a recurrence equation of order at most $m_i \cdot l_i$ can be computed (see, e.g., Teguia Tabuguia, 2021, Section 2). This yields a recurrence equation for the index $m_i n + j_i$. To obtain a recurrence in n , one substitutes n by $(n - j_i)/m_i$ in the equation. Let us denote by r_i the order of the resulting holonomic equation.

Notice that for all $1 \leq i \leq l$, any m_j -fold hypergeometric terms in $\epsilon_i(n)$, $j \neq i$, are considered in one of the $u_k(n)$, $k = 1, \dots, l$. Therefore the span of all m -fold hypergeometric terms in s_n is fully covered by the solution space of the system defined by the l constructed holonomic recurrence equations.

Finally, using the addition closure property of P-recursive sequences, one can compute a holonomic equation of order at most $r_1 + \dots + r_l$ (see Salvy and Zimmermann GFUN, 1994; Mallinger, 1996; Koutschan, 2014; Kauers et al., 2015) satisfied by s_n . Hence $(s)_n$ is P-recursive. \square

From Proposition 3, we can say that the generating functions of hypergeometric-type sequences are D-finite functions (Kauers, 2023). However, we can be more specific. From Definition 1, we can establish a natural link between hypergeometric-type sequences and proper hypergeometric-type power series (Teguia Tabuguia, 2020; Teguia Tabuguia and Koepf, 2022b).

Example 8. Let us consider the proper hypergeometric-type power series $\cos(z) + \sin(z)$. We have

$$\cos(z) + \sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} z^{2n+1} \quad (18)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^{n/2}}{n!} \chi_{\{n \equiv 0 \pmod{2}\}} + \frac{(-1)^{(n-1)/2}}{n!} \chi_{\{n \equiv 1 \pmod{2}\}} \right) z^n. \quad (19)$$

Thus, the general coefficient of the power series of $\cos(z) + \sin(z)$ is a hypergeometric-type term. \blacksquare

Proposition 4. *There is a one-to-one correspondence between proper hypergeometric-type power series and hypergeometric-type sequences.*

Proof. This is established by the following equality

$$\sum_{i=0}^{m_j-1} \sum_{n=0}^{\infty} a_i(m_j n + i) z^{m_j n + i} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{m_j-1} a_i(n) \chi_{\{n \equiv i \pmod{m_j}\}} \right) z^n, \quad (20)$$

which naturally links Definition 1 and Definition 3. \square

In other words, the generating functions of hypergeometric-type sequences are hypergeometric-type functions. Thanks to the correspondence in Proposition 4, we can relate hypergeometric-type sequences to $\mathbb{K}(x)$ -linear combinations of some special functions such as Bessel functions, Airy functions, and trigonometric functions.

3. Algorithmic approach

Our motivation to study hypergeometric-type sequences came from the purpose of this section. Hypergeometric-type terms may be present in the sciences as trigonometric or elliptic functions with discrete arguments. A connection between hypergeometric terms and elliptic curves can be found in

Saito et al. (2013, Section 1.3). Our purpose is to bring those representations into “standard” forms. Recent results in Dreyfus (2022) show that the only meromorphic differentially algebraic functions (Ait El Manssour et al., 2023; Teguia, 2023) that are P-recursive are made with periodic functions and exponentials. It is thus reasonable to think of an algorithmic approach to convert the related discrete functions into hypergeometric-type normal forms.

In this section, it is essential to understand the difference between an object $(1, -1, 1, \dots) \in (\mathcal{H}_T)$ and its term $(-1)^n \in \mathcal{H}_T$. The former represents the sequence $((-1)^n)_n$ as a mathematical object, and the latter as its ‘closed’ mathematical writing. We recommend Geddes et al. (1992, Chapter 3) to remove any ambiguity in this formalism.

3.1. Canonical and normal forms

The definition of normal and canonical forms for hypergeometric-type sequences entails assessing computability in the ring (\mathcal{H}_T) . Indeed, an element of (\mathcal{H}_T) may have many equivalent representations in \mathcal{H}_T .

Example 9 (Distinct representations of same hypergeometric-type terms, part I). The sequence $(\sin^2(n\pi/4))$ is of hypergeometric type and has the two following representations:

$$\sin^2\left(\frac{n\pi}{4}\right) = \frac{1}{2} \left(1 - (-1)^{\frac{n}{2}} \chi_{\{n \equiv 0 \pmod{2}\}}\right) \quad (21)$$

$$= \frac{1}{2} \left(1 - \frac{I^n + (-I)^n}{2}\right), \quad (22)$$

where $I = \sqrt{-1}$ is the imaginary unit. ■

The idea of a canonical form is to have a unique and “simple” representation of a mathematical object in a certain class, here \mathcal{H}_T . Observe that the representation (22) of $s_n := \sin^2(n\pi/4)$ requires to work on $\mathbb{Q}(I)$, whereas (21) is a formula over \mathbb{Q} (no extension field). Thus, (21) and (22) are valid in $\mathbb{Q}(I)$, making (22) less appropriate as a canonical form of s_n . Hence writing formulas over the minimal field extension reduces the possible representations of a hypergeometric-type term. Unfortunately, uniqueness remains an issue even in base fields. We give two examples below.

Example 10 (Distinct representations of same hypergeometric-type terms, part II).

1. The general coefficient of $\cosh(z)$ has the two formulas:

$$\frac{1}{n!} \chi_{\{n \equiv 0 \pmod{2}\}} = \frac{1 + (-1)^n}{2 \cdot n!}. \quad (23)$$

2. The following was observed from different formulas of the general term of A212579:

$$\frac{31}{3} - \chi_{\{n \equiv 0 \pmod{2}\}} = \frac{1}{2} \left(\frac{59}{3} - (-1)^n \right). \quad \blacksquare \quad (24)$$

While for compactness reasons, one might prefer the left-hand side for the first item in Example 10, both sides seem to have a relatively similar compactness for the second item. We may also choose to define a canonical form by eliminating all alternations in the formula. For instance, $(-1)^n$ can be written as $\chi_{\{n \equiv 0 \pmod{2}\}} - \chi_{\{n \equiv 1 \pmod{2}\}}$, the latter form being seen as canonical. In this view, our canonical forms would be the left-hand sides in Example 10. For Example 9, (21) may be further simplified since $(-1)^{n/2} \chi_{\{n \equiv 0 \pmod{2}\}}$ alternates between $-\chi_{\{n \equiv 0 \pmod{2}\}}$ and $\chi_{\{n \equiv 0 \pmod{2}\}}$. So, a hypergeometric-type canonical form of $\sin^2(n\pi/4)$ would contain 4-fold indicator terms, which sounds reasonable with the 4 occurring in its expression. However, as the second item in Example 10 shows, further simplifications need to be done to reduce the number of m -fold indicator terms after

substituting alternating elements. The fact that $\chi_{\{n \equiv 0 \pmod{2}\}}$ survived allows us to think that there might be another way to write the formula with $\chi_{\{n \equiv 1 \pmod{2}\}}$. Therefore, uniqueness may not still be guaranteed.

Why do we care about canonical form at all? The main reason is that it completely solves (theoretically) the zero-equivalence problem in \mathcal{H}_T . However, our motivation is to bring expressions not written as elements of \mathcal{H}_T into easily recognizable hypergeometric-type terms whenever possible. The above discussion presents the difficulty of defining a canonical form in \mathcal{H}_T and forces us to reduce ourselves to normal forms. We have already introduced them, but we give the definition below for formal reference.

Definition 4 (A normal form in \mathcal{H}_T). Any representation of a hypergeometric-type term as in (9) is a normal form.

Having stated our normal form, we need to give an algorithm for the normalization. Note that this still solves the zero equivalence problem by the unique representation of the zero sequence. The remaining part of the paper describes the algorithm sustaining this fact.

3.2. Finding holonomic recurrence equations

This subsection addresses finding a holonomic recurrence equation satisfied by a given hypergeometric-type “expression”. It means that the given formula is not necessarily in the form of (9). The algorithm behind this conversion is the first step of our general algorithm toward finding hypergeometric-type normal forms. We mention that there is no fundamental result in this part of the paper because there are many existing software to compute univariate holonomic (differential and difference) equations. Some references are Salvy and Zimmermann GFUN (1994); Mallinger (1996); Koutschan (2014); Kauers et al. (2015); Koepf and Teguia Tabuguia (2022). However, unlike the differential case for which software packages are easily accessible for any computer algebra system, general-purpose algorithms for finding recurrences from holonomic expressions do not seem available in the difference case. For instance, the well-known GFUN package (Salvy and Zimmermann GFUN, 1994) misses such an implementation. That is one reason why we decided to include this subsection. We adapt the `HOLONOMICDE` algorithm (available within Maple 2022 as `DEtools:-FindODE`) of Koepf and Teguia Tabuguia (2022) to the case of recurrences. For details on the original algorithm, see the explanation from Koepf (1992). Given a term s_n , the aim is to find $C_1, \dots, C_N \in \mathbb{Q}(n)$ such that

$$s_{n+N} + C_N \cdot s_{n+N-1} + \dots + C_1 \cdot s_n = 0.$$

We consider an efficient variant of this method as proposed in Teguia Tabuguia (2020, Section 4.1.2), Teguia Tabuguia and Koepf (2022b, Section 2). The idea is to write s_n and its N first shifts in the same basis and solve the linear system that expresses their linear dependency over $\mathbb{K}(n)$. We explain how the algorithm works in the following example.

Example 11. Let $s_n := n! + \frac{1}{n!}$.

1. $N = 0$: since $\frac{n!}{1/n!} = n!^2 \notin \mathbb{Q}(n)$, we consider the basis (e_1, e_2) , where $e_1 = n!$ and $e_2 = 1/n!$. Thus

$$s_n = e_1 + e_2,$$

and at this stage, the matrix H of the components is $H = [1, 1]$. The rows of H are the components of s_{n+N} in the $(N+1)$ st basis.

2. $N = 1$:

$$s_{n+1} = (n+1)! + \frac{1}{(n+1)!} = (n+1) \cdot e_1 + \frac{1}{(n+1)} \cdot e_2.$$

$$H = \begin{bmatrix} 1 & 1 \\ n+1 & \frac{1}{n+1} \end{bmatrix}.$$

Since s_{n+1} and s_n are written in the same basis, we try to solve the system

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot C = \begin{bmatrix} -(n+1) \\ -\frac{1}{n+1} \end{bmatrix}, \quad C \in \mathbb{Q}(n).$$

The right-hand side is the negative transpose of the last row of H , and the matrix of the left-hand side is the transpose of the first N rows of H . The obtained system has no solution, so we move on to the next iteration.

3. $N = 2$:

$$s_{n+2} = (n+2)(n+1) \cdot e_1 + \frac{1}{(n+2)(n+1)} \cdot e_2.$$

$$H = \begin{bmatrix} 1 & 1 \\ n+1 & \frac{1}{n+1} \\ (n+2)(n+1) & \frac{1}{(n+2)(n+1)} \end{bmatrix}.$$

We solve the system

$$\begin{bmatrix} 1 & n+1 \\ 1 & \frac{1}{n+1} \end{bmatrix} \cdot C = \begin{bmatrix} -(n+2)(n+1) \\ -\frac{1}{(n+2)(n+1)} \end{bmatrix}, \quad C \in \mathbb{Q}(n)^2,$$

and get a unique solution

$$C = \begin{bmatrix} C_1 := \frac{(n+3)(n+1)^2}{n(n+2)^2} \\ C_2 := -\frac{(n^2+3n+1)(n^2+3n+3)}{n(n+2)^2} \end{bmatrix}.$$

Thus s_n satisfies the equation

$$a_{n+2} + C_2 a_{n+1} + C_1 a_n = 0.$$

After clearing denominators, we get the holonomic recurrence equation

$$(n+3)(n+1)^2 a_n - (n^2+3n+1)(n^2+3n+3) a_{n+1} + n(n+2)^2 a_{n+2} = 0, \quad (25)$$

satisfied by s_n . ■

Example 12. For $\sin^2(n\pi/4)$ the algorithm leads to the recurrence

$$-a_n + a_{n+1} - a_{n+2} + a_{n+3} = 0. \quad (26)$$

Of course, the algorithm works in this case because the expansion formulas of trigonometric functions are used. ■

The above-outlined algorithm cannot apply to hypergeometric-type terms written in normal forms. For those terms we use the construction highlighted in the proof of Proposition 3. We give one basic example to illustrate how it works.

Example 13. Let $s_n = \left(\frac{1}{3^{\frac{n}{2}}} + (-5)^{\frac{n}{2}}\right) \chi_{\{n \equiv 0 \pmod{2}\}} + 2^{\frac{n}{3}} \chi_{\{n \equiv 0 \pmod{3}\}}$.

We consider $u_1(2n) = \frac{1}{3^n} + (-5)^n$ and $u_2(3n) = 2^n$. To find a recurrence for $u_1(2n)$, we use the addition algorithm with 2-shifts and find

$$5u_1(2n) - 14u_1(2n+2) - 3u_1(2n+4) = 0.$$

Hence the equation for $u_1(n)$:

$$5u_1(n) - 14u_1(n+2) - 3u_1(n+4) = 0.$$

Similarly, $u_2(n)$ satisfies

$$2u_2(n) - u_2(n+3) = 0.$$

Finally, using the addition closure property for holonomic sequences we get the equation

$$10a_n - 28a_{n+2} - 5a_{n+3} - 6a_{n+4} + 14a_{n+5} + 3a_{n+7} = 0, \quad (27)$$

satisfied by s_n . ■

We will denote by $\text{HolonomicRE}(s_n, a(n), d)$ the algorithm that applies the algorithm in the proof of Proposition 3 if s_n is already in normal form, i.e., $s_n \in \mathcal{H}_T$, and the algorithm outlined in Example 11 otherwise. The output is either a holonomic recurrence equation of order at most $d \in \mathbb{N}$ in the indeterminate $a(n)$, or FAIL when such an equation is not found. We can omit d for hypergeometric-type terms since the recurrence is obtained by construction and not by search.

We mention that $\sin^2(z\pi/4)$ also satisfies a holonomic differential equation. The work in Dreyfus (2022) suggests that we can use trigonometric functions to generate terms that satisfy holonomic recurrence equations. The remaining steps of our algorithm help to verify whether these terms are of hypergeometric type or not.

3.3. Finding normal forms

Let $(s)_n \in (\mathcal{H}_T)$ such that the given expression s_n is not an element of \mathcal{H}_T . We want to find a representation of s_n in \mathcal{H}_T . Suppose that s_n is a solution of the following d th-order recurrence equation:

$$P_d(n)a_{n+d} + P_{d-1}(n)a_{n+d-1} + \dots + P_0(n)a_n = 0, \quad (28)$$

with polynomial coefficients $P_d, P_{d-1}, \dots, P_0 \in \mathbb{K}[x]$, $P_d P_0 \neq 0$. Using `mfoldHyper` (Teguia Tabuguia and Koepf, 2022b, Section 3), we can compute a basis of m -fold hypergeometric term solutions of (28). This may be written as

$$\mathfrak{B} := \{\{m_i, \mathfrak{B}_i\}, i = 1, \dots, N\} := \left\{ \left\{ m_i, \{h_{i,1}(m_i n), \dots, h_{i,l_i}(m_i n)\} \right\}, i = 1, \dots, N \right\}, \quad (29)$$

$m_i \in \mathbb{N} \setminus \{0\}$. For each $h_{i,j} \in \mathfrak{B}_i$, there are $m_i - 1$ other solutions, namely $h_{i,j}(m_i n + k_j)$, $k_j = 1, \dots, m_i - 1$. These other solutions can also be generated by `mfoldHyper` at the user's request. Note that the reason why the basis (29) is written in this form is because the primary purpose of `mfoldHyper` is to find general coefficients of formal power series. We recall that `mfoldHyper` is an extension of the algorithms by Petkovšek (Hyper) and van Hoeij (Petkovšek, 1992; van Hoeij, 1999; Teguia Tabuguia, 2021). Thanks to the correspondence of Proposition 4, the output of `mfoldHyper` can be easily used to find a hypergeometric-type representation of s_n .

To obtain a hypergeometric-type formula for s_n , we look for constant coefficients $c_{i,j,k_j} \in \mathbb{K}$, $i = 1, \dots, N$, $j = 1, \dots, l_i$, $k_j = 0, \dots, m_i - 1$, such that

$$s_n = \sum_{0 \leq k_j \leq m_i - 1, 1 \leq j \leq l_i, 1 \leq i \leq N} c_{i,j,k_j} h_{i,j,k_j}(n) \chi_{\{n \equiv k_j \pmod{m_i}\}}. \quad (30)$$

We evaluate both sides of (30) to obtain a Cramer system for the unknown c_{i,j,k_j} 's. Its solutions lead to a hypergeometric-type representation of s_n . We mention that the resulting system can have many solutions because some sub-bases or mixing of elements from the basis in (29) may span the same vector space over different field extensions. Nevertheless, we need to select one of them to get the normal form we want. Let us give some examples.

Example 14 ($s_n := \sin^2\left(\frac{n\pi}{4}\right)$). As presented in Example 12, s_n satisfies the recurrence equation:

$$-a_n + a_{n+1} - a_{n+2} + a_{n+3} = 0.$$

Algorithm `mfoldHyper` finds the following basis of solutions over \mathbb{Q} :

$$\{\{1, \{1\}\}, \{2, \{(-1)^n\}\}\}. \quad (31)$$

More solutions can be found if one enables computations over field extensions. This is avoided as much as possible to have the chance to obtain a normal form over the base field. We write

$$s_n = c_1 + c_2(-1)^{\frac{n}{2}} \chi_{\{n \equiv 0 \pmod{2}\}} + c_3(-1)^{\frac{n-1}{2}} \chi_{\{n \equiv 1 \pmod{2}\}},$$

and use the first terms s_0, s_1, s_2 to obtain the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + c_3 = \frac{1}{2} \\ c_1 - c_2 = 1 \end{cases}.$$

The system has a unique solution which leads to the following normal form for s_n :

$$s_n = \sin^2\left(\frac{n\pi}{4}\right) = \frac{1}{2} - \frac{(-1)^{\frac{n}{2}}}{2} \chi_{\{n \equiv 0 \pmod{2}\}} \quad \blacksquare \quad (32)$$

Example 15 ($s_n := \sin\left(\frac{\pi}{6}\cos(n\pi)\right)\sin\left(\frac{n\pi}{4}\right)$). The given term satisfies the recurrence equation:

$$a_n + \sqrt{2}a_{n+1} + a_{n+2} = 0. \quad (33)$$

This equation does not have m -fold hypergeometric term solutions over \mathbb{Q} , not even over $\mathbb{Q}(\sqrt{2})$. Enabling extension fields allows `mfoldHyper` to find the basis of solutions

$$\left\{\left\{1, \left\{\left(\text{RootOf}\left(1 + \sqrt{2}X + X^2\right)\right)^n\right\}\right\}\right\}, \quad (34)$$

where $(\text{RootOf}(P(X)))^n$ is a compact notation of α^n , for all $\alpha, P(\alpha) = 0$. Thus we have two hypergeometric terms over $\mathbb{Q}(\sqrt{2}, I)$. The algorithm can proceed as in Example 14 and find a hypergeometric-type representation of s_n . However, as this is not what our implementation will do (Section 4), we want to present a technique that our implementation does to avoid field extensions. This may also justify why we cannot always find normal forms over base fields. The point is, as designed, the algorithm of Section 3.2 tries to compute a holonomic recurrence equation of the smallest order. However, the least-order recurrence equation may not contain term solutions over the base field of the given hypergeometric-type sequence. Thus, it might be relevant to look for other recurrence equations. To do so, we ask the algorithm to search for a recurrence equation between 2-shifts of s_n , i.e., $s_n, s_{n+2}, s_{n+4}, \dots$. We obtain the two-term recurrence equation

$$a_n + a_{n+4} = 0. \quad (35)$$

Hence, the basis of term solutions over \mathbb{Q} :

$$\{\{4, \{(-1)^n\}\}\}. \quad (36)$$

At this stage, we are sure to obtain a normal form in the corresponding base field since any algebraic number in the formula will come from the evaluation of s_n . For the ansatz

$$s_n = c_0(-1)^{\frac{n}{4}} \chi_{\{n \equiv 0 \pmod{4}\}} + c_1(-1)^{\frac{n-1}{4}} \chi_{\{n \equiv 1 \pmod{4}\}} + c_2(-1)^{\frac{n-2}{4}} \chi_{\{n \equiv 2 \pmod{4}\}} + c_3(-1)^{\frac{n-3}{4}} \chi_{\{n \equiv 3 \pmod{4}\}},$$

we get the linear system:

$$\begin{cases} c_0 = 0 \\ c_1 = -\frac{\sqrt{2}}{4} \\ c_2 = \frac{1}{2} \\ c_3 = -\frac{\sqrt{2}}{4} \end{cases}.$$

Therefore

$$\begin{aligned} s_n &= \sin\left(\frac{\pi}{6} \cos(n\pi)\right) \sin\left(\frac{n\pi}{4}\right) \\ &= -\frac{(-1)^{\frac{n-1}{4}} \sqrt{2}}{2} \chi_{\{n \equiv 1 \pmod{4}\}} + \frac{1}{2} (-1)^{\frac{n-2}{4}} \chi_{\{n \equiv 2 \pmod{4}\}} - \frac{(-1)^{\frac{n-3}{4}} \sqrt{2}}{4} \chi_{\{n \equiv 3 \pmod{4}\}} \quad \blacksquare \end{aligned}$$

Let us present all the steps of our algorithmic approach to detecting hypergeometric-type terms by writing them in normal forms.

Algorithm 1 Finding hypergeometric-type formulas

Input: A general term s_n of a sequence $(s_n) \in \mathbb{K}^{\mathbb{N}}$, and a positive integer d . If $s_n \in \mathcal{H}_T$, then d may be computed as: the sum of (number of hypergeometric term in each coefficient) \times (the corresponding characteristic).

Output: Either

- FAIL, meaning that “no holonomic recurrence equation of order at most d was found”;
- a holonomic recurrence equation of order at most d with enough initial values to identify (s_n) uniquely: this means that “ $(s_n) \notin (\mathcal{H}_T)$ ”;
- a hypergeometric-type normal form, meaning that “ $(s_n) \in (\mathcal{H}_T)$ ”.
 1. Apply `HolonomicRE`($s_n, a(n), d$) (Section 3.2) and call the result RE .
 2. If $RE = \text{FAIL}$ then stop and return it. //comment: d may be small.
 3. RE is a holonomic recurrence equation of order $r \leq d$. Use `mfoldHyper` to compute a basis of m -fold hypergeometric term solutions of RE over \mathbb{K} and denote it \mathfrak{B} .
 4. If \mathfrak{B} is empty then stop and return RE together with $a_0 = s_0, \dots, a_{r-1} = s_{r-1}$.
 5. \mathfrak{B} is not empty and has the form

$$\mathfrak{B} := \left\{ \{m_i, \{h_{i,1}(m_i n), \dots, h_{i,l_i}(m_i n)\}\}, i = 1, \dots, N \right\},$$

as in (29). Let

$$u_n := \sum_{0 \leq k_j \leq m_i - 1, 1 \leq j \leq l_i, 1 \leq i \leq N} c_{i,j,k_j} h_{i,j,k_j}(n) \chi_{\{n \equiv k_j \pmod{m_i}\}},$$

as in (30), with the unknown constants $c_{i,j,k_j} \in \mathbb{K}$, $i = 1, \dots, N$, $j = 1, \dots, l_i$, $k_j = 0, \dots, m_i - 1$.

6. Let p be the number of constant c_{i,j,k_j} . $p = \sum_{i=1}^N m_i \cdot l_i$
 7. Let E_0 be a finite set of non-negative integers that are not roots of the leading and the trailing polynomial coefficients of RE , such that E_0 evaluates $u_n = s_n$ to a linear system of rank at least p .
 8. Solve the linear system $u_j = s_j$, $j \in E_0$, and let S be the set of solutions.
 9. If S is empty then stop and return RE together with $a_0 = s_0, \dots, a_{r-1} = s_{r-1}$.
 10. Return the substitution of a solution in S into u_n .
-

Remark 2.

- Note that we omitted the steps where we try to avoid field extensions to simplify the algorithm. The idea is to use `mfoldHyper` over \mathbb{Q} in step 3 with a few more recurrences satisfied by s_n , and see if it leads to a non-empty S at step 9.
- The reason for avoiding roots of the leading and the trailing coefficients is a singularity issue. See the discussion in Kauers (2023, Section 2.2).
- The set E_0 in step 7 can be chosen as $\{n_0, \dots, n_0 + p\}$, where n_0 is a non-negative integer strictly greater than the maximum integer root of the leading and the trailing polynomial coefficients. However, finding n_0 that way may not be the best approach when symbolic values occur in the equation. One could look for such integer intervals by evaluation at consecutive indices starting

from 0. The latter approach would be inappropriate in today's computer only if the recurrence equation has thousands or billions of m -fold hypergeometric term solutions.

Theorem 2. *Algorithm 1 is correct.*

Theorem 2 is deduced from the previous paragraphs and Remark 2. Algorithm 1 is a transformation for finding normal forms of hypergeometric-type terms for which holonomic recurrence equations are found in its first step. The zero sequence may be returned as 0 or, most likely, as the zeroth-order holonomic recurrence equation. Thus, we can identify distinct hypergeometric-type terms.

4. Implementation

We implemented Algorithm 1 with Maple as a command in the package `HyperTypeSeq` (Teguia Tabuguia). The package currently contains three commands: `HolonomicRE`, `REtoHTS`, and `HTS`.

1. `HolonomicRE` adapts `HolonomicDE` from FPS (Koeppf and Teguia Tabuguia, 2022) to search for a holonomic recurrence equation from an expression and a given bound. The syntax is

`HolonomicRE(expr, a(n), maxreorder = d, reshift = t),`

where `maxreorder` and `reshift` are optional with default values 10 and 1, respectively. `expr` is a term in n , and `a` is the name of the unknown for the equation. `maxreorder` is the maximum order of the holonomic recurrence equation sought, and `reshift` is the minimal possible shift of `a(n)` in the recurrence equation sought. The current version of `HolonomicRE` still misses an implementation for finding recurrence equations from hypergeometric-type terms containing m -fold indicator terms.

2. `REtoHTS` applies Algorithm 1 from step 3. The syntax is

`REtoHTS(RE, a(n), P).`

`RE` is the holonomic recurrence equation and `a(n)` is the unknown term in it. `P` is a procedure for computing values of the sequence at any index. `P` can also be a list of initial values; however, the list must contain the values of the evaluations of `expr` starting from 0.

With finding holonomic recurrence equations for sequences in enumerative combinatorics, `REtoHTS` may be useful for finding new formulas.

3. `HTS` implements Algorithm 1 with the syntax

`HTS(expr, n),`

with self-explanatory arguments from the previous commands. The argument `maxreorder` is also optional for `HTS`.

For our implementation $\chi_{\{n \equiv j \pmod m\}} = \chi_{\{m \mid \text{modp}(n, m) = j\}}$. We can now present more sophisticated conversions of trigonometric expressions into hypergeometric-type terms. We encountered an implementation issue with Maple 2022 and Maple 2023; we could not obtain some formulas that Maple 2019 and Maple 2021 found within seconds with our code. Maple 2022 and Maple 2023 keep running. Simple checking on our implementation tells us that the problem comes from the linear system solver `SolveTools:-Linear`. We will see one of the expressions that led to this misbehavior. Thus, note that all the formulas in the examples below are obtained within seconds ($\leq 4s$) with Maple 2021.

Example 16 (Some expressions of hypergeometric type).

1.

```
> with(HyperTypeSeq):
> HTS(sin(Pi*cos(n*Pi)/6)*cos(n*Pi/4), n)
```

$$\frac{(-1)^{\frac{n}{4}} \chi_{\{modp(n,4)=0\}}}{2} - \frac{\sqrt{2} (-1)^{\frac{n}{4}-\frac{1}{4}} \chi_{\{modp(n,4)=1\}}}{4} + \frac{\sqrt{2} (-1)^{\frac{n}{4}-\frac{3}{4}} \chi_{\{modp(n,4)=3\}}}{4} \quad (37)$$

2. > HTS (sin (cos (n*Pi/3) *Pi) , n)

$$(-1)^{\frac{n}{3}-\frac{1}{3}} \chi_{\{modp(n,3)=1\}} - (-1)^{\frac{n}{3}-\frac{2}{3}} \chi_{\{modp(n,3)=2\}} \quad (38)$$

3. > HTS (tan (n*Pi/4) , n)

$$\chi_{\{modp(n,4)=1\}} + \left(\lim_{n \rightarrow 2} \tan \left(\frac{n\pi}{4} \right) \right) \chi_{\{modp(n,4)=2\}} - \chi_{\{modp(n,4)=3\}} \quad (39)$$

4. > HTS (tan (n*Pi/3) , n)

$$\sqrt{3} \chi_{\{modp(n,3)=1\}} - \sqrt{3} \chi_{\{modp(n,3)=2\}} \quad (40)$$

5. Chebyshev polynomials:

> HTS (cos (n*arccos (x)) , n)

$$\frac{(x - \sqrt{x^2 - 1})^n}{2} + \frac{(x + \sqrt{x^2 - 1})^n}{2} \quad (41)$$

6. > HTS (sin (n*Pi/6) *cos (n*Pi/3) -sin (n*Pi/2) , n)

$$-\frac{I\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)^n}{4} + \frac{I\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)^n}{4} - \frac{(-1)^{\frac{n}{2}-\frac{1}{2}} \chi_{\{modp(n,2)=1\}}}{2} \quad (42)$$

7. > HTS (sin (n*Pi/4) ^2*cos (n*Pi/6) ^2 , n)

$$\begin{aligned} & \frac{1}{4} - \frac{(-1)^n}{8} + \frac{\left(\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^n}{8} + \frac{\left(\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^n}{8} - \frac{I(-1)^{\frac{n}{2}-\frac{1}{2}} \chi_{\{modp(n,2)=1\}}}{8} \\ & - \frac{3(-1)^{\frac{n}{3}} \chi_{\{modp(n,3)=0\}}}{8} - \frac{3I(-1)^{\frac{n}{6}-\frac{1}{2}} \chi_{\{modp(n,6)=3\}}}{8} \end{aligned} \quad (43)$$

The following formula could not be obtained with Maple 2023 and Maple 2022. That is the reason why we used Maple 2021 for all the examples in Example 16.

8. > HTS (sin (n*Pi/4) ^2*cos (n*Pi/6) ^4 , n, maxreorder=12)

$$\begin{aligned} & \frac{9}{32} + \left(\left(\frac{3}{32} + \frac{I\sqrt{3}}{8} \right) (-1)^{\frac{n}{2}} + \frac{\left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^{\frac{n}{2}}}{4} \right) \chi_{\{modp(n,2)=0\}} \\ & - \frac{9\chi_{\{modp(n,3)=0\}}}{32} + \frac{I\sqrt{3} (-1)^{\frac{n}{3}-\frac{2}{3}} \chi_{\{modp(n,3)=2\}}}{4} + \frac{\left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}\right)^{\frac{n}{4}} \chi_{\{modp(n,4)=0\}}}{2} \\ & + \left(-\frac{27}{32} - \frac{I\sqrt{3}}{8} \right) (-1)^{\frac{n}{6}} \chi_{\{modp(n,6)=0\}} - \frac{\sqrt{3} (-1)^{\frac{n}{6}-\frac{5}{6}} \chi_{\{modp(n,6)=5\}}}{4} \blacksquare \end{aligned} \quad (44)$$

Example 17 (Collatz sequence beginning at 21, OEIS:[A033481](#)). The general term s_n of the sequence is defined by the recursion

$$s_0 = 21, s_{n+1} = \begin{cases} \frac{s_n}{2} & \text{if } s_n \equiv 0 \pmod{2} \\ 3s_n + 1 & \text{if } s_n \equiv 1 \pmod{2} \end{cases} \quad \text{for all } n \geq 1. \quad (45)$$

The generating function of $(s)_n$ is given by

$$f(z) := \frac{-7z^7 - 14z^6 - 28z^5 - 56z^4 - 5z^3 + 32z^2 + 64z + 21}{(1-z)(z^2+z+1)}.$$

This function is a non-proper hypergeometric-type function as FPS (Koepef and Teguia Tabuguia, 2022) finds the power series formula

$$f(z) := 7z^4 + 14z^3 + 28z^2 + 63z + 19 + \sum_{n=0}^{\infty} 4z^n + \sum_{n=0}^{\infty} -2z^{3n} + \sum_{n=0}^{\infty} -3z^{3n+1}.$$

Thus $(s)_n \notin (\mathcal{H}_T)$. However, if we remove the polynomial part from the expansion, i.e., we consider

$$g(z) := f(z) - (7z^4 + 14z^3 + 28z^2 + 63z + 19),$$

then the resulting sequence of coefficients is of hypergeometric type. The formula can be deduced either with FPS or its 'child' HTS. As we removed the polynomial part in the expansion of $f(z)$, the new sequence is $(u)_n = (s)_n - (19, 63, 28, 14, 7, 0, 0, \dots)$. Using FPS: -FindRE we find the following recurrence equation satisfied by u_n :

```
> RE:=FPS:-FindRE(f-(7*z^4 + 14*z^3 + 28*z^2 + 63*z + 19),z,u(n))
```

$$\begin{aligned} RE := & (-n+1)u(n) + (4n-12)u(n-4) + (n-1)u(n-3) + (2n+2)u(n-2) \\ & + (-4n+12)u(n-1) + (-2n-2)u(n+1) = 0. \end{aligned} \quad (46)$$

Hence the formula

```
> REtoHTS(RE,u(n),[2, 1, 4, 2, 1, 4])
```

$$4 - 2\chi_{\{modp(n,3)=0\}} - 3\chi_{\{modp(n,3)=1\}}. \quad (47)$$

The main point in this example is that formulas of solutions to holonomic recurrence equations can be found with enough initial values. We usually prefer to supply a procedure instead of a list of values, as this will allow the code to use as many values as necessary. For this example, the syntax would be:

```
> U:=proc(n) U(n):=subs([n=n-1,u=U],solve(RE,u(n+1))) end proc:
U(0):=2:U(1):=1:U(2):=4:U(3):=2:U(4):=1:U(5):=4:
> REtoHTS(RE,u(n),U):
```

We hid the output as it is precisely (47). ■

5. Conclusion

In conclusion, this article introduced hypergeometric-type sequences with a formalism of interlacement described by m -fold indicator sequences. We showed that these sequences are generated by proper hypergeometric-type series. It may be possible to generalize the study to include proper Laurent-Puiseux series of hypergeometric type. For Puiseux series, the corresponding interlacements may be viewed as α -fold indicator sequences for some $\alpha \in \mathbb{Q}$.

We proved that \mathcal{H}_T is a ring and presented an algorithm to decide whether a given holonomic term is of hypergeometric type or not. The latter comes as a complement of the algorithms by Petkovšek (1992); van Hoeij (1999) to detect when a given holonomic term can be written as a linear combination of interlaced hypergeometric terms.

It is worth mentioning that C-finite sequences (Zeilberger, 1990b), also called LRS (linear recurrence sequence) (Ouaknine and Worrell, 2012), form a subclass of hypergeometric-type sequences. The inclusion is immediate from their writing as exponential polynomials (Chonev et al., 2023). Ouaknine and Worrell showed that one can decide if any C-finite sequence of order 5 or less is positive (Ouaknine and Worrell, 2014). Could the same conclusion hold for hypergeometric-type sequences that satisfy holonomic recurrence equations of order at most 5? The target is, of course, a particular case (see Proposition 3) of the general class of holonomic sequences for which the positivity problem is only partially studied (Kauers and Pillwein, 2010; Pillwein, 2013; Ibrahim and Salvy, 2024).

We end with an observation concerning the generating functions of m -fold indicator sequences. It is easy to see that

$$f_{m,j}(z) := \frac{z^j}{1-z^m} = \sum_{n=0}^{\infty} \chi_{\{n \equiv j \pmod{m}\}} z^n, \quad m, j \in \mathbb{N}, j < m. \quad (48)$$

As m -fold indicator sequences may be regarded as a basis of a free module, it sounds interesting to study the structure of proper hypergeometric-type functions and see their relation to the $f_{m,j}$'s.

CRediT authorship contribution statement

Bertrand Teguia Tabuguia: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

A reference to the accompanying software is provided in the manuscript

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