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On trees and dual rotund norms☆

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Abstract

We give a necessary and sufficient condition for the existence of an equivalent dual rotund norm on $\mathscr{C}_0(\Upsilon)^* \equiv \ell_1(\Upsilon)$, where Υ is a tree. The condition is expressed succinctly, in terms of the embeddability of Υ into a particular totally ordered set, and compares very well with the analogous situation for local uniform rotundity. This resolves an open problem from Haydon's work in Asplund spaces, trees and renorming theory.

Keywords: Trees; Norms; Renorming theory; Strictly convex; Rotund; Locally uniformly rotund; Asplund spaces; Embeddability

1. Introduction

Renorming theory is concerned with determining the extent to which a given Banach space may be endowed with a new, equivalent norm, in possession of different, usually superior, geometrical properties than the original. It is common for authors to concentrate on the spaces $\mathscr{C}(K)$ of continuous real-valued functions defined on K, where K is compact and Hausdorff. Aside from the intrinsic interest and structural benefits that such spaces offer, results in this context can be significant to the general theory because the Banach spaces $\mathscr{C}(K)$ form a universal subclass; recall that any Banach space X may be embedded isometrically by evaluation into $\mathscr{C}(B_{X^*})$, where the dual ball B_{X^*} is taken in its w^*-topology. To see how this fact may be exploited in renorming theory,

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we refer the reader to [7,2]. In addition, we recommend [1] for an excellent account of the general theory up to 1993.

We recall two well-known geometrical properties of norms. The norm $||\cdot||$ on a Banach space X is *strictly convex*, *rotund*, or simply R, if $x = y$ whenever $||x|| = ||y|| = \frac{1}{2}||x+y||$. We call the norm $||\cdot||$ *locally uniformly rotund*, or LUR, if $\lim ||x - x_n|| = 0$ whenever $x \in X$ and the sequence $(x_n) \subseteq X$ satisfy $\lim ||x_n|| = ||x||$ and $\lim ||x + x_n|| = 2||x||$. Evidently, LUR norms are also R.

When L is locally compact and Hausdorff, $\mathscr{C}_0(L)$ denotes the space of continuous real-valued functions on L vanishing at infinity. In [4], Haydon provides equivalent conditions for the existence of many types of norm on $\mathscr{C}_0(\Upsilon)$, where Υ is a tree. A partial order $(\Upsilon, \preccurlyeq)$ is called a *tree* if, for every $t \in \Upsilon$, the set of strict predecessors $\{s \in \Upsilon \mid s \prec t\}$ is well-ordered. In this way, trees naturally generalise ordinal numbers. Given $t \in \Upsilon$, the height $\mathrm{ht}(t, \Upsilon)$ of t in Υ is the order type of its strict predecessors. For convenience, we introduce elements 0 and ∞, not in Υ, with the property that $0 \prec t \prec \infty$ for all $t \in \Upsilon$. Given $s \in \Upsilon \cup \{0\}$ and $t \in \Upsilon$ with $s \prec t$, we define the *interval* $(s, t] = \{\xi \in \Upsilon \mid s \prec \xi \preccurlyeq t\}$. In addition, if $t \in \Upsilon$, we define the *wedge* $[t, \infty) = \{u \in \Upsilon \mid t \preccurlyeq u\}$. The standard locally compact *order topology* on Υ takes as its basis the intervals $(s, t]$, where s and t range over the values above. With respect to this topology, the intervals are compact. We call totally ordered subsets of trees *chains*. The tree Υ is called *Hausdorff* if every non-empty chain in Υ has at most one minimal upper bound. It is clear that Υ satisfies this property if and only if the order topology is Hausdorff in the ordinary sense. For this reason, when considering $\mathscr{C}_0(\Upsilon)$, the tree Υ is always Hausdorff. It is evident from the definitions that Υ in its interval topology is scattered. Indeed, if $E \subseteq \Upsilon$ and $t \in E$ then the least element of $E \cap (0, t]$ is relatively isolated in E. It follows that $\mathscr{C}_0(\Upsilon)$ is an *Asplund space* and that the dual space of measures $\mathscr{C}_0(\Upsilon)^*$ identifies in the natural way with $\ell_1(\Upsilon)$. We say that a subset $A \subseteq \Upsilon$ is an *antichain* if no two distinct elements of A are comparable in the tree order. Given $t \in \Upsilon$, the antichain of *immediate successors* of t is given by t^+, and Υ^+ denotes the union $\bigcup_{t\in\Upsilon} t^+$. Finally, any subset Υ_0 of Υ is regarded as a *subtree*, with respect to the inherited order, though it is worth noting that the subspace topology of Υ_0, while always finer than that induced by the new order, is often strictly so. For a comprehensive survey of the theory of trees, we refer the reader to [10].

As demonstrated in the aforementioned survey, trees enjoy a close relationship with total orders. Given partial orders P and Q, the map $\rho : P \longrightarrow Q$ is called *increasing* if $\rho(x) \preccurlyeq \rho(y)$ whenever $x \preccurlyeq y$, and *strictly increasing* if $\rho(x) \prec \rho(y)$ whenever $x \prec y$. We say that P is *Q-embeddable*, written $P \preccurlyeq Q$, if there exists a strictly increasing map $\rho : P \longrightarrow Q$. It is a standard result in the theory of trees that $\Upsilon \preccurlyeq \mathbb{Q}$ if and only Υ is *special*, which is to say that Υ is a countable union of antichains. Another established fact is that if $\rho : \Upsilon \longrightarrow \mathbb{R}$ is strictly increasing, then Υ^+ is special. Indeed, given $u \in t^+$, we can select $\sigma(u) \in (\rho(t), \rho(u)) \cap \mathbb{Q}$, whence $\Upsilon^+ \preccurlyeq \mathbb{Q}$. As well as increasing maps, we will make use of decreasing and strictly decreasing functions, particularly real-valued ones; we say that $\rho : P \longrightarrow Q$ is *decreasing* if $\rho(x) \succcurlyeq \rho(y)$ whenever $x \preccurlyeq y$, and strictly so if $\rho(x) \succ \rho(y)$ whenever $x \prec y$.

Special trees, and to a lesser extent $\mathbb{R}$-embeddable trees, possess many good traits. The equivalent conditions in [4] are phrased in terms of increasing real-valued functions on Υ with additional combinatorial properties. In the case of dual R norms, the following sufficient condition is presented.

Theorem 1 (*Haydon [4]*). *Suppose that on the tree Υ there is an increasing function $\rho : \Upsilon \longrightarrow \mathbb{R}$ which is constant on no increasing sequence in Υ. Then there is an equivalent norm on $\mathscr{C}_0(\Upsilon)$ with rotund dual norm.*

As trees go, those satisfying the hypothesis of Theorem 1 are relatively well-behaved. By contrast, the known necessary conditions for the existence of such a norm are far less demanding. We say that a subset $G \subseteq \Upsilon$ is a *final part* if $u \in G$ whenever $t \in G$ and $t \preccurlyeq u$, and that final part G is *dense* if for any $t \in \Upsilon$, there exists $u \in G$ such that $t \preccurlyeq u$. A tree is called *Baire* if, whenever (G_n) is a countable sequence of dense final parts, the intersection is again dense. We deduce from [3,8] that if Υ contains a Baire subtree without maximal elements, then there is no norm on $\mathscr{C}_0(\Upsilon)$ with R dual norm.

Before stating our main result Theorem 6, we set it in context by combining known results to establish an equivalent condition for the existence of a dual LUR norm. Recall that a locally compact space L is called *σ-discrete* if it is a countable union of discrete subsets.

Theorem 2 (*Raja [6]*). *The space $\mathscr{C}_0(L)^*$ admits an equivalent dual* LUR *norm if and only if L is σ-discrete.*

We shall call a map $\pi : P \longrightarrow P$ on a partial order P *regressive* if $\pi(x) \prec x$ whenever x is not a minimal element. The theorem of Todorčević below amounts to a pressing down lemma for trees.

Theorem 3 (*Todorčević [9]*). *If Υ is a non-special tree and $\pi : \Upsilon \longrightarrow \Upsilon$ is regressive, then there exists $s \in \Upsilon$ such that the pre-image $\pi^{-1}(s)$ is non-special.*

Now we can give our equivalent condition for dual LUR norms. It is attributed to Raja as the author's contribution is only minor. Note that in all that follows, the w^*-topology of $\ell_1(\Upsilon)$ is always taken with respect to $\mathscr{C}_0(\Upsilon)$.

Theorem 4 (*Raja*). *If Υ is a tree, $\ell_1(\Upsilon)$ admits an equivalent dual LUR norm if and only if Υ is special.*

Proof. Given Theorem 2, it is enough to prove that Υ is special if and only if it is σ-discrete. The direct implication is clear as any antichain is evidently discrete. To prove the converse, we let (D_n) be a countable family of discrete, disjoint subsets that cover Υ. Given a non-minimal element $t \in \Upsilon$, let n be the unique integer such that $t \in D_n$. As D_n is discrete, we can find minimal $\pi(t) \prec t$, subject to the condition $(\pi(t), t] \cap D_n = \{t\}$. We claim that the resulting regressive map on Υ has no non-special pre-image. Indeed, if $s \in \Upsilon$ and $m \in \mathbb{N}$, then by inspection, $\pi^{-1}(s) \cap D_m$ is an antichain. The conclusion now follows from Theorem 3. $\square$

Equally, we may say that $\ell_1(\Upsilon)$ admits an equivalent dual LUR norm if and only if $\Upsilon \preccurlyeq \mathbb{Q}$. With respect to trees, the existence of dual R norms can be similarly characterised in terms of embeddings into a particular total order.

Definition 5. Let Y be the set of all strictly increasing, continuous, transfinite sequences $x = (x_\alpha)_{\alpha \leqslant \beta}$ of real numbers, where $0 \leqslant \beta < \omega_1$. Order Y by declaring that $x < y$ if and only if either y strictly extends x, or there is some ordinal α such that $x_\xi = y_\xi$ for $\xi < \alpha$ and $y_\alpha < x_\alpha$.

Observe that Y is not ordered in the usual lexicographic way. To give an indication of size, we remark that Y contains the lexicographic product $\mathbb{R}^\beta$, for any countable ordinal β. This will be proved later as a consequence of Proposition 20. The following theorem is the main result of this paper.

Theorem 6. *If Υ is a tree, $\ell_1(\Upsilon)$ admits an equivalent dual* R *norm if and only if $\Upsilon \preccurlyeq Y$.*

2. Embeddings, plateaux and partition trees

In this section, we prove the direct implication of Theorem 6, and go some way to proving its converse. Along the way, we introduce several tools that will be used in the next section. From now on, all trees are assumed to be Hausdorff. In addition, ω and ω_1 will denote the first infinite and uncountable ordinals, respectively.

Proposition 7. *If Υ is a tree and $\ell_1(\Upsilon)$ admits an equivalent dual* R *norm then $\Upsilon \preccurlyeq Y$.*

Proof. The proof comes in two parts. First of all, we construct a real-valued function σ defined on 'closed' intervals $[s, t] = \{\xi \in \Upsilon \mid s \preccurlyeq \xi \preccurlyeq t\}$, where $s, t \in \Upsilon$. Of course, the interval $[s, t]$ is uniquely defined by the pair (s, t); we choose interval notation simply because if $s \prec t$, then $(s, t]$ is a convenient way of expressing the interval $[s', t]$, where s' is the unique element of $s^+ \cap (0, t]$. Once defined, we use σ in the second part of the proof to establish a strictly increasing map $\rho : \Upsilon \longrightarrow Y$.

We proceed to the definition of σ. Let $||\cdot||$ be an equivalent dual R norm on $\ell_1(\Upsilon)$, and assume that $||x|| < 1$ whenever $||x||_1 = 1$. Recall that given a compact subset K of a locally compact space L, the probability measures on K form a w^*-compact and convex set in $\mathscr{C}_0(L)^*$. Given $s \preccurlyeq t$, define the w^*-compact convex set

$$\Sigma[s, t] = \{x \in \ell_1(\Upsilon) \mid x \text{ is positive}, ||x||_1 = 1, \text{supp } x \subseteq [s, t]\}$$

and the corresponding infimum $\sigma[s, t] = \inf\{||x|| \mid x \in \Sigma(s, t)\}$. We verify that σ satisfies the following three properties:

(1) $\sigma[\cdot, t]$ is increasing on $(0, t]$ and $\sigma[s, \cdot]$ is decreasing on the wedge $[s, \infty)$;
(2) if $s_n \nearrow s$ and $s \preccurlyeq t$ then $\sigma[s_n, t] \nearrow \sigma[s, t]$;
(3) if $r \preccurlyeq s \prec t$, then it is not true that $\sigma[r, s] = \sigma[r, t] = \sigma(s, t]$.

Property (1) of σ follows immediately by the definition. Remembering that dual norms are w^*-lower semicontinuous, by the compactness of $\Sigma[s,t]$, there exists $x^{[s,t]} \in \Sigma[s,t]$ such that $||x^{[s,t]}|| = \sigma[s,t]$. Moreover, by the rotundity of $||\cdot||$ and convexity of $\Sigma[s,t]$, this element is unique. Property (3) follows. To demonstrate that the second property holds, we must show that $\sigma[s,t] \leqslant \sup_n \sigma[s_n,t]$. By the compactness of $\Sigma[s,t]$, there exists an element $x \in \Sigma[s,t]$ which lies in the w^*-closure of the $x^{[s_n,t]}$. Then, given the w^*-lower semicontinuity of $||\cdot||$, we have

$$\sigma[s,t] \leqslant ||x|| \leqslant \sup_n ||x^{[s_n,t]}|| = \sup_n \sigma[s_n,t].$$

Now we turn to the second part of the proof. Given $t \in \Upsilon$, we define a strictly increasing, continuous, transfinite sequence $(t_\alpha)_{\alpha \leqslant \beta} \subseteq (0,t]$ by recursion. To begin, let t_0 be minimal in $(0,t]$. Suppose now that t_α has been constructed. If $t_\alpha = t$ then terminate the process by setting $\beta = \alpha$. Otherwise, observe that by property (1) of σ, $\sigma(t_\alpha,\cdot]$ is decreasing on $(t_\alpha,t]$. Let $t_{\alpha+1} \in (t_\alpha,t]$ be minimal, subject to the requirement that $\sigma(t_\alpha,\cdot]$ is constant on $[t_{\alpha+1},t]$. If α is a limit ordinal, and $t_\xi \prec t$ exists for all $\xi < \alpha$, then set $t = \sup_{\xi<\alpha} t_\xi$. This completes the recursion. The process stops eventually as (t_α) is strictly increasing.

To define ρ, set $\rho(t)_0 = 0$ and $\rho(t)_\alpha = \sup_{\xi<\alpha} \sigma(t_\xi,t]$ if $0 < \alpha \leqslant \beta$. By inspection, the sequence $\rho(t) = (\rho(t)_\alpha)_{\alpha \leqslant \beta}$ is increasing and continuous. In order to prove that $\rho(t) \in Y$, we will argue that $\rho(t)_\alpha < \rho(t)_{\alpha+1}$ for each countable $\alpha < \beta$. Provided $\beta < \omega_1$, this argument will demonstrate that $\rho(t)$ is strictly increasing on its domain and thus an element of Y. Fortunately, the same argument rules out the possibility that $\beta \geqslant \omega_1$ because in this case $(\rho(t)_\alpha)_{\alpha<\omega_1}$ would be a strictly increasing, uncountable sequence of real numbers.

Given countable $\alpha < \beta$, note that by property (1) of σ, we have $\rho(t)_{\alpha+1} = \sigma(t_\alpha,t]$. If $\alpha = 0$ then the inequality clearly holds, as σ is never zero. If $\alpha = \xi+1$ for some ξ, then we have $\rho(t)_\alpha = \sigma(t_\xi,t]$, and by the definition of t_α, it follows that $\sigma(t_\xi,t] = \sigma(t_\xi,t_\alpha]$. Now by property (3) of σ, $\rho(t)_\alpha = \sigma(t_\xi,t] < \sigma(t_\alpha,t] = \rho(t)_{\alpha+1}$. Finally, if α is a limit ordinal, we claim that if $s \in [t_\alpha,t]$ then $\sigma[t_\alpha,s] = \rho(t)_\alpha$. Assuming this, we have $\sigma[t_\alpha,t_\alpha] = \sigma[t_\alpha,t]$, thus again by property (3) of σ, $\rho(t)_\alpha = \sigma[t_\alpha,t] < \sigma(t_\alpha,t] = \rho(t)_{\alpha+1}$. To prove our claim, let $s \in [t_\alpha,t]$, and remembering that α is countable, select an increasing, countable sequence of ordinals (ξ_n) with supremum α. We note that $\rho(t)_\alpha = \sup_n \sigma(t_{\xi_n},t]$ by property (1) of σ, and that $\sigma(t_{\xi_n},s] = \sigma(t_{\xi_n},t]$ because $\sigma(t_{\xi_n},\cdot]$ is constant on $[t_{\xi_n+1},t]$, and $s \in [t_\alpha,t] \subseteq [t_{\xi_n+1},t]$. As $\sup_n \sigma(t_{\xi_n},s] = \sigma[t_\alpha,s]$ by property (2) of σ, the claim follows.

We have established that $\rho(t) \in Y$. From now on, fix β_t in such a way that $\rho(t)$ is defined on $[0,\beta_t]$. To complete the proof, we must check that if $s \prec t$, then $\rho(s) < \rho(t)$. Given such s and t, let γ be the least ordinal such that $\rho(s)_\gamma$ and $\rho(t)_\gamma$ are not both defined and equal. By continuity, and the fact that $\rho(s)_0 = 0 = \rho(t)_0$, there exists an ordinal β such that $\gamma = \beta+1$. It follows that $\rho(s)_\alpha = \rho(t)_\alpha$ for all $\alpha \leqslant \beta$. Using transfinite induction, we prove that $s_\beta = t_\beta$. Certainly $s_0 = t_0$. If $\alpha < \beta$ and $s_\alpha = r = t_\alpha$, then $\sigma(r,s] = \rho(s)_{\alpha+1} = \rho(t)_{\alpha+1} = \sigma(r,t]$, meaning $\sigma(r,\cdot]$ is constant on $[s,t]$. By minimality, $s_{\alpha+1} = t_{\alpha+1}$. Limit stages of the induction follow immediately by continuity.

Now let $s_\beta = r = t_\beta$. As $r \preccurlyeq s \prec t$, it follows that $\rho(t)_\gamma = \rho(t)_{\beta+1}$ is defined, and in particular, $\beta < \beta_t$. Therefore, if $\beta = \beta_s$ then we are done, as $\rho(t)$ strictly extends $\rho(s)$ in this case. Otherwise, $\beta < \beta_s$, which means that $\rho(s)_\gamma$ is defined and equals $\sigma(r, s]$. Now, as $\rho(s)_\gamma \neq \rho(t)_\gamma$, we have

$$\rho(t)_\gamma = \sigma(r, t] \ < \ \sigma(r, s] = \rho(s)_\gamma$$

by property (1) of σ. Consequently, $\rho(s) < \rho(t)$ as required. □

Now we turn to the converse implication of Theorem 6. The following concept is of central importance.

Definition 8. A subset $W \subseteq \Upsilon$ is called a plateau if W has a least element 0_W and $W = \bigcup_{t\in W}[0_W, t]$. A partition $\mathscr{P}$ of Υ consisting solely of plateaux is called a plateau partition.

There are two basic observations to make at this stage. First of all, given a plateau V, the set $V\backslash\{0_V\} = \bigcup_{t\in V\backslash\{0_V\}}(0_V, t]$ is open in Υ. It follows that if $\mathscr{P}$ is a plateau partition, the set of least elements $H = \{0_V \mid V \in \mathscr{P}\}$ is a closed subtree of Υ. Our next result shows how increasing functions naturally induce plateau partitions.

Proposition 9. *Let $\rho : \Upsilon \longrightarrow \Sigma$ be an increasing function into a total order Σ. Then there exists a natural plateau partition $\mathscr{P}$ of Υ associated with ρ. Moreover, the restriction of ρ to the set of least elements $H = \{0_V \mid V \in \mathscr{P}\}$ is strictly increasing.*

Proof. Define an equivalence relation $\sim$ on Υ by declaring that $t \sim u$ if there exists $s \preccurlyeq t, u$ such that $\rho(t) = \rho(s) = \rho(u)$. To confirm transitivity of this relation, let $t \sim u$ and $u \sim v$. Then there exists $s \preccurlyeq t, u$ and $r \preccurlyeq u, v$ such that $\rho(t) = \rho(s) = \rho(u) = \rho(r) = \rho(v)$. Since $(0, u]$ is totally ordered, s and r are comparable. It follows that $t \sim v$. Let $\mathscr{P}$ be the corresponding partition. We must ensure each $W \in \mathscr{P}$ is a plateau. If $t \in W$, let s be the least element of $W \cap (0, t]$. If $v \in W$, there exists $r \in W \cap (0, s] \cap (0, v]$, hence $s = r \preccurlyeq v$. Therefore, s is the least element of W. Finally, if $u \in [s, v]$ then $u \in W$, because ρ is increasing. The second assertion of the proposition follows easily. If $0_V, 0_{V'} \in H$ and $0_V \prec 0_{V'}$ then necessarily $\rho(0_V) < \rho(0_{V'})$, as $0_{V'} \notin V$. □

We can also produce plateaux by taking intersections.

Proposition 10. *Let Υ be a tree and $\mathfrak{F}$ a family of plateaux with non-empty intersection W. Then W is a plateau, and $0_W = \sup_{V\in\mathfrak{F}} 0_V$.*

Proof. The least elements $(0_V)_{V\in\mathfrak{F}}$ form a chain with an upper bound. Indeed, if $u \in W$ then $0_V \preccurlyeq u$ for each $V \in \mathfrak{F}$. If we let $t = \sup_{V\in\mathfrak{F}} 0_V$, then t is the least element of W because $t \in [0_V, u] \subseteq V$ whenever $u \in W$ and $V \in \mathfrak{F}$. Moreover, if $u \in W$, then $[t, u] \subseteq V$ for all $V \in \mathfrak{F}$. This proves that W is a plateau. □

We spend the rest of this section developing a useful construction that serves as an intermediate step in the converse implication of Theorem 6. Elements of this construction appear implicitly in the proof of Proposition 7.

Definition 11. Given a tree Υ, let $(\mathscr{P}_\beta)_{\beta<\omega_1}$ be a sequence of plateau partitions satisfying:

(1) if $\alpha < \beta$ and $V \in \mathscr{P}_\alpha$, $W \in \mathscr{P}_\beta$, then either $W \subseteq V$ or $V \cap W$ is empty;
(2) if β is a limit ordinal and $W \in \mathscr{P}_\beta$, then

$$W = \bigcap\{V \mid V \in \mathscr{P}_\alpha, \alpha < \beta, W \subseteq V\};$$

(3) if $t \in \Upsilon$, there exists $\beta < \omega_1$, depending on t, such that $\{t\} \in \mathscr{P}_\beta$.

We call such a sequence admissible.

Let $(\mathscr{P}_\beta)_{\beta<\omega_1}$ be an admissible sequence. Property (1) means that if $\alpha < \beta$ then $\mathscr{P}_\beta$ is a refinement of $\mathscr{P}_\alpha$. The second property implies some continuity at limits, and property (3) is an exhaustion condition.

We can use an admissible sequence of partitions to define a new tree that will form the basis for the construction of a dual R norm. The set $T = \bigcup_{\beta<\omega_1}\{\beta\} \times \mathscr{P}_\beta$ can be endowed with a partial order by declaring that $(\alpha, V) \preccurlyeq (\beta, W)$ if and only if $\alpha \leqslant \beta$ and $W \subseteq V$. Conditions (1) and (2) of admissibility ensure that $(T, \preccurlyeq)$ is a Hausdorff tree. However, T is not the tree we want because it is too big. This is where the third condition of admissibility can play a role. Observe that if $\{t\} \in \mathscr{P}_\alpha$, then the wedge of elements in T greater than $(\alpha, \{t\})$ is the totally ordered set $\{(\beta, \{t\}) \mid \alpha \leqslant \beta < \omega_1\}$. The tree T can be pruned of these wedges without losing any information about the sequence $(\mathscr{P}_\beta)$, and this is exactly what we do.

Definition 12. Let $(\mathscr{P}_\beta)_{\beta<\omega_1}$ be admissible, and let T be as above. Then the subtree

$$\Upsilon(\mathscr{P}) = \{(\beta, V) \in T \mid U \text{ is not a singleton whenever } (\alpha, U) \prec (\beta, V)\}$$

of T is called the partition tree of Υ with respect to $(\mathscr{P}_\beta)_{\beta<\omega_1}$.

In this way, (β, V) is a maximal element of $\Upsilon(\mathscr{P})$ if and only if V is a singleton. Partition trees exist in other contexts, see e.g. [10, Section 3]. This section ends with the following proposition, which is our promised intermediate step in the conclusion of Theorem 6.

Proposition 13. *Let Υ be a tree. If $\Upsilon \preccurlyeq Y$ then there exists an admissible sequence of partitions $(\mathscr{P}_\beta)_{\beta<\omega_1}$ that yields a partition tree $\Upsilon(\mathscr{P})$, and a strictly increasing function $\pi : \Upsilon(\mathscr{P}) \longrightarrow [0, 1]$. Moreover*:

(1) $\mathscr{P}_0 = \{[r, \infty) \mid r \in \Upsilon$ *is minimal*$\}$;

(2) *for any non-maximal* $(\beta, V) \in \Upsilon(\mathscr{P})$, *the map*

$$0_W \longmapsto \pi(\beta+1, W)$$

is strictly decreasing on the subtree of least elements

$$H_{(\beta,V)} = \{0_W \mid (\beta+1, W) \in (\beta, V)^+\}.$$

Proof. Let us suppose that $\rho : \Upsilon \longrightarrow Y$ is a strictly increasing map. Note that as the intervals $\mathbb{R}$ and $(0, 1)$ are isomorphic, we can consider elements of Y as sequences in $(0, 1)$. Then, if we append 0 to the beginning of each such sequence, we obtain a strictly increasing map from Y into the set $Y_0 = \{x = (x_\alpha)_{\alpha \leqslant \beta} \in Y \mid x \subseteq [0, 1) \text{ and } x_0 = 0\}$. As a result, and for convenience, we assume that ρ takes values in Y_0.

Before we define the partitions $\mathscr{P}_\beta$, we make a simple remark about the order on Y. If $x \leqslant y \leqslant z$ and $x_\alpha = z_\alpha$ for all $\alpha \leqslant \beta$, then y_α is defined and equal to x_α for all $\alpha \leqslant \beta$. This fact can be verified by transfinite induction, which we leave to the reader. To define $\mathscr{P}_\beta$, we consider an equivalence relation only marginally more complicated than the one defined in Proposition 9. Let $s \sim_\beta t$ if and only if $s = t$, or there exists $r \preccurlyeq s, t$ such that the quantities $\rho(r)_\alpha$, $\rho(s)_\alpha$ and $\rho(t)_\alpha$ are defined and equal for all $\alpha \leqslant \beta$. It is clear that $\sim_\beta$ is an equivalence relation. We let $\mathscr{P}_\beta$ be its corresponding partition, and claim $(\mathscr{P}_\beta)_{\beta<\omega_1}$ is an admissible sequence of plateau partitions. Our first duty is to prove that every element of $\mathscr{P}_\beta$ is indeed a plateau. Suppose that $t \in V \in \mathscr{P}_\beta$, and that V is not a singleton. As V is not a singleton, $\rho(u)_\alpha$ must be defined whenever $u \in V$ and $\alpha \leqslant \beta$. We claim that the least element s of $V \cap (0, t]$ is also the least element of V. If $u \in V$ then as $t \sim_\beta u$, it follows that there is some $r \preccurlyeq t, u$ such that $r \in V$. By minimality of s, we have $s \preccurlyeq r \preccurlyeq u$, therefore V has a least element 0_V. Now let $t \preccurlyeq u \preccurlyeq v$, with $t, v \in V$. By our observation above about the order of Y, we have $u \in V$. Consequently, V is a plateau.

We turn to the task of ensuring that $(\mathscr{P}_\beta)_{\beta<\omega_1}$ is admissible. Let $\rho(t)$ be defined on $[0, \beta_t]$. Then property (3) of Definition 11 follows straightaway, as $\{t\} \in \mathscr{P}_\beta$ whenever $\beta_t < \beta$. Now we verify property (1). Let $\alpha \leqslant \beta$, $V \in \mathscr{P}_\alpha$, $W \in \mathscr{P}_\beta$, and suppose $t \in V \cap W$. If W is a singleton then we are done. If W is not a singleton then $\rho(u)_\xi$ is defined and equal to $\rho(t)_\xi$ whenever $u \in W$ and $\xi \leqslant \beta$. Consequently $W \subseteq V$, giving us property (1) of admissibility. To confirm property (2) of admissibility, let β be a limit, $W \in \mathscr{P}_\beta$ and suppose $V_\alpha \in \mathscr{P}_\alpha$ satisfies $W \subseteq V_\alpha$ for all $\alpha < \beta$. We claim that $W = \bigcap_{\alpha<\beta} V_\alpha$. If V_α is a singleton for some α then there is nothing to prove. If we suppose otherwise, observe that if $u \in \bigcap_{\alpha<\beta} V_\alpha$, then $\rho(u)_\xi$ is defined for all $\xi \leqslant \alpha$ and $\alpha < \beta$, whence $\rho(u)_\beta$ is also defined by continuity. Now set $s = \sup 0_{V_\alpha} \preccurlyeq 0_W$. As each V_α is a plateau and $0_{V_\alpha} \preccurlyeq s \preccurlyeq 0_W$, it follows that $s \in V_\alpha$ for each α. Therefore, $\rho(s)_\alpha = \rho(0_W)_\alpha$ for all $\alpha < \beta$, and by continuity, $\rho(s)_\beta = \rho(0_W)_\beta$. It follows that $s \in W$, so $s = 0_W$. Now, if $u \in \bigcap_{\alpha<\beta} V_\alpha$, we have $0_W = s \preccurlyeq u$ and $\rho(u)_\alpha = \rho(s)_\alpha = \rho(0_W)_\alpha$ for all $\alpha \leqslant \beta$, again by the continuity of elements of Y. This proves that $u \in W$ as required.

Let $\Upsilon(\mathscr{P})$ be the associated partition tree. We prove that it satisfies the properties listed in the statement of this proposition. Firstly, by the definition of Y_0, $\rho(t)_0 = 0$ for all $t \in \Upsilon$, thus we certainly have property (1). Now we give our definition of π. If $(\beta, V) \in \mathscr{P}$ then it is possible that $\rho(0_V)_\beta$ is not defined. In this case $\beta > 0$. By the definition of $\Upsilon(\mathscr{P})$, if $(\alpha, U) \prec (\beta, V)$ then U is not a singleton, whence $\rho(0_V)_\alpha$ is defined whenever $\alpha < \beta$. Therefore, by continuity, it follows that β is not a limit ordinal. With this in mind, we define $\pi : \Upsilon(\mathscr{P}) \longrightarrow [0, 1]$ by

$$\pi(\beta, V) = \begin{cases} \rho(0_V)_\beta & \text{if } \rho(0_V)_\beta \text{ is defined,} \\ 1 & \text{otherwise.} \end{cases}$$

We prove that π is strictly increasing and satisfies property (2). If $(\alpha, U) \prec (\beta, V)$ then (α, U) is not maximal, whence U is not a singleton, $\rho(0_U)_\alpha$ is defined and $\pi(\alpha, U) < 1$. If $\rho(0_V)_\beta$ is not defined then we are done. Otherwise, note that $0_V \in U$, thus $\rho(0_V)_\alpha = \rho(0_U)_\alpha$, meaning $\pi(\beta, V) = \rho(0_V)_\beta > \rho(0_V)_\alpha = \rho(0_U)_\alpha = \pi(\alpha, U)$. Consequently, π is strictly increasing.

Finally, we ensure that property (2) holds. Let $(\beta, V) \in \Upsilon(\mathscr{P})$ be non-maximal, and suppose that $0_W, 0_{W'} \in H_{(\beta,V)}$ satisfy $0_W \prec 0_{W'}$. We know that $\rho(0_V)_\alpha = \rho(0_W)_\alpha = \rho(0_{W'})_\alpha$ for all $\alpha \leqslant \beta$. We also know that $\rho(0_V) \leqslant \rho(0_W) < \rho(0_{W'})$. There are two cases to consider: either $\rho(0_W)_{\beta+1}$ is defined or it is not. If it is defined then so must $\rho(0_{W'})_{\beta+1}$, else $\rho(0_W)$ would be a strict extension of $\rho(0_{W'})$. Moreover, $\rho(0_W)_{\beta+1} \geqslant \rho(0_{W'})_{\beta+1}$, remembering the order of Y. However, we cannot have equality, lest $0_{W'}$ be in W. Consequently,

$$\pi(\beta + 1, W) = \rho(0_W)_{\beta+1} \; > \; \rho(0_{W'})_{\beta+1} = \pi(\beta + 1, W').$$

Instead, if $\rho(0_W)_{\beta+1}$ is not defined, then $\rho(0_V)_{\beta+1}$ is not defined either; otherwise, $\rho(0_V)$ would strictly extend $\rho(0_W)$. Therefore, $\rho(0_W) = \rho(0_V)$, $0_W = 0_V$, and $\pi(\beta + 1, W) = 1$. Now $\rho(0_{W'})_{\beta+1}$ must exist, because if it did not then we would have $0_{W'} = 0_V$ by the same argument. As a result

$$\pi(\beta + 1, W) = 1 \; > \; \rho(0_{W'})_{\beta+1} = \pi(\beta + 1, W').$$

This completes the proof. □

3. A dual rotund norm

Throughout this section, we only consider partition trees $\Upsilon(\mathscr{P})$ of the form described in Proposition 13, together with the corresponding admissible partitions $(\mathscr{P}_\beta)_{\beta<\omega_1}$. They are used to supply the basic building blocks of our dual rotund norm, and the means to sum these elements legitimately. We approach the question of legitimate summation by first associating with each $t \in \Upsilon$ two chains in $\Upsilon(\mathscr{P})$.

Recall that given $t \in \Upsilon$ and $\alpha < \omega_1$, the plateau V^t_α is the unique element of $\mathscr{P}_\alpha$ containing t. By property (3) of admissibility, there exists $\beta < \omega_1$ such that $V^t_\beta = \{t\}$.

Definition 14. If $\varphi = \varphi_t < \omega_1$ is minimal, subject to the condition $V^t_\varphi = \{t\}$, then we call $(\beta, V^t_\beta)_{\beta \leqslant \varphi}$ the primary sequence of t.

The primary sequence of t is a maximal chain, or a *branch*, in $\Upsilon(\mathscr{P})$. Our second chain in $\Upsilon(\mathscr{P})$ associated with t takes a little longer to describe. Note that if W is a plateau then so is $\overline{W}$, with least element 0_W. If $\beta < \omega_1$ then it is possible that there exists $W \in \mathscr{P}_\beta$ such that $t \in \overline{W}\backslash W$. The *secondary sequence* of t will contain all W for which this holds. Before giving the official definition, we take a look at how these sets W behave.

Proposition 15. *Let $\alpha \leqslant \beta$ and suppose $v \in \overline{U}\backslash U \cap \overline{V}\backslash V$ for some $U \in \mathscr{P}_\alpha$ and $V \in \mathscr{P}_\beta$. Then $V \cap U$ is non-empty, meaning $V \subseteq U$, and if $\alpha = \beta$ then $V = U$.*

Proof. As $v \in \overline{U}\backslash U$, there exists $s \in U \cap (0, v]$. Necessarily $s \prec v$, and similarly, there exists $t \in V \cap (s, v]$ with $t \prec v$. Again, there exists $u \in U \cap (t, v]$. Hence $s \prec t \prec u$ and $s, u \in U$, which forces $t \in U$. $\square$

Firstly, Proposition 15 tells us that if $W \in \mathscr{P}_\beta$ satisfies $t \in \overline{W}\backslash W$, then W is unique. We denote it by W^t_β. Secondly, if $\alpha \leqslant \beta$ and W^t_α, W^t_β both exist, then $W^t_\beta \subseteq W^t_\alpha$. Because W^t_β cannot be a singleton if it exists, (β, W^t_β) lies in $\Upsilon(\mathscr{P})$. Therefore, the elements (β, W^t_β) form a chain in $\Upsilon(\mathscr{P})$.

Now we find the ordinals β for which W^t_β exists. Observe that if $0_{V^t_\beta} \prec t$ then W^t_β cannot exist. Indeed, in this case, the only element of $\mathscr{P}_\beta$ which intersects the open set $(0_{V^t_\beta}, t]$ non-trivially is V^t_β itself, and by definition, $t \in V^t_\beta$. Hence if W^t_β exists then $t = 0_{V^t_\beta}$. Let $\theta = \theta_t < \omega_1$ be minimal, subject to the condition $t = 0_{V^t_\theta}$. It follows that if $\beta < \theta$ then W^t_β does not exist. Now let $\psi = \psi_t \geqslant \theta$ be minimal such that W^t_ψ does not exist. We show that if $\theta \leqslant \alpha \leqslant \beta$ and W^t_β exists then so does W^t_α. Indeed, given such W^t_β, choose $V \in \mathscr{P}_\alpha$ such that $0_{W^t_\beta} \in V$. Then $t \in \overline{V}$, since $W^t_\beta \subseteq V$. We claim $t \notin V$. Indeed, $0_V \preccurlyeq 0_{W^t_\beta} \prec t$, but since $\alpha \geqslant \theta$, it follows that $t \in V$ would force $t = 0_V$. Therefore, $W^t_\alpha = V$ exists. To summarise, W^t_β exists if and only if $\theta \leqslant \beta < \psi$. It is perfectly possible that for a given t, $\psi = \theta$ and thus W^t_β never exists.

Definition 16. The chain $(\beta, W^t_\beta)_{\theta \leqslant \beta < \psi}$ is called the secondary sequence of t.

The dual rotund norm will be built largely from terms dominated by non-zero multiples of $\|x\restriction_{\overline{V}}\|_1$, where $V \in \mathscr{P}_\beta$. Primary and secondary sequences will enable us to sum enough of them. For each $(\beta, V) \in \Upsilon(\mathscr{P})$ and successor $(\beta + 1, W) \in (\beta, V)^+$, fix $\delta_{(\beta+1,W)} = \pi(\beta + 1, W) - \pi(\beta, V) > 0$, where π is our map from Proposition 13.

Observe that if these elements $\delta_{(\beta+1,W)}$ are summed along a chain in $\Upsilon(\mathscr{P})$ then the total will not exceed unity, given the upper bound on π.

Lemma 17. *If* $x \in \ell_1(\Upsilon)$ *then*

$$\sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\|x\restriction_{\overline{W}}\|_1 \leqslant 2\|x\|_1$$

and

$$\sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\|x\restriction_{W}\|_1 \leqslant \|x\|_1.$$

Proof. We prove the first inequality, but leave the second as it is similar. For $G \subseteq \Upsilon$, set $\varepsilon_{(t,G)} = 1$ if $t \in G$, and $\varepsilon_{(t,G)} = 0$ otherwise. We have

$$\sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\|x\restriction_{\overline{W}}\|_1 = \sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)} \sum_{t\in\Upsilon} \varepsilon_{(t,\overline{W})}\,|x_t|$$
$$= \sum_{t\in\Upsilon} |x_t| \sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\,\varepsilon_{(t,\overline{W})}.$$

Fix t for a moment. Given $(\beta+1, W)$, if $\varepsilon_{(t,\overline{W})} = 1$ then either $W = V^t_{\beta+1}$, or $W = W^t_{\beta+1}$, should it exist. There are no other possibilities. Therefore,

$$\sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\,\varepsilon_{(t,\overline{W})} = \sum_{\beta+1\,\leqslant\,\varphi} \delta_{(\beta+1,V^t_{\beta+1})} + \sum_{\theta\,\leqslant\,\beta+1<\psi} \delta_{(\beta+1,W^t_{\beta+1})},$$

which will not exceed 2, given that we are summing along the primary and secondary sequences of t, which are both chains in $\Upsilon(\mathscr{P})$. □

Now we are ready to define the building blocks of the norm. Fix $(\beta, V) \in \Upsilon(\mathscr{P})$. Since V is a plateau, $A = \overline{V}\backslash V$ is an antichain. Indeed, if $s, u \in \overline{V}$ and $s \prec u$ then there exists $t \in (s, u] \cap V$, forcing $s \in [0_V, t] \subseteq V$. Being an antichain, A is closed and discrete. It follows that $\mathscr{C}_0(A) = c_0(A)$, and from [1, Theorem II.7.4], $\ell_1(A)$ admits an equivalent dual LUR norm $\|\cdot\|_A$ that we can choose to satisfy $\|\cdot\|_A \leqslant \|\cdot\|_1$. Given a positive integer n, we define

$$|||x|||_n^2 = \inf\{n^{-1}\|y\|_A^2 + \|x-y\|_1^2 \mid y \in \ell_1(A)\}$$

and by mimicking the proof of [1, Theorem II.2.1], we see that $|||\cdot|||_n$ is an equivalent dual norm on $\ell_1(\overline{V}\backslash\{0_V\}) \equiv \mathscr{C}_0(\overline{V}\backslash\{0_V\})^*$ satisfying $|||\cdot|||_n \leqslant \|\cdot\|_1$. The next lemma exposes our motivation for introducing such frequently used 'inf-convolution norms'; variants of the proof are common in the literature, but we include it anyway for the sake of completeness.

Lemma 18. *If* $x, z \in \ell_1(\overline{V}\backslash\{0_V\})$ *satisfy*

$$2|||x|||_n^2 + 2|||z|||_n^2 - |||x+z|||_n^2 = 0$$

for all n, *then* $x\restriction_A = z\restriction_A$.

Proof. By compactness and the w^*-lower semicontinuity of $||\cdot||_A$, there exists $x^n \in \ell_1(A)$ such that $|||x|||_n = n^{-1}||x^n||_A^2 + ||x - x^n||_1^2$. Corresponding elements z^n may be chosen for z. A standard convexity argument [1, Fact II.2.3] yields

$$2||x^n||_A^2 + 2||z^n||_A^2 - ||x^n + z^n||_A^2 = 0,$$

whence $x^n = z^n$, as $||\cdot||_A$ is rotund. To finish, it suffices to show that $\lim_n ||x\restriction_A - x^n||_1 = 0$, while acknowledging that the corresponding expression will also hold for z. Indeed,

$$\begin{aligned} ||x\restriction_A - x^n||_1 &= ||x - x^n||_1 - ||x\restriction_{V\backslash\{0_V\}}||_1 \\ &\leqslant |||x|||_n - ||x\restriction_{V\backslash\{0_V\}}||_1 \\ &\leqslant [n^{-1}||x\restriction_A||_A^2 + ||x\restriction_{V\backslash\{0_V\}}||_1^2]^{\frac{1}{2}} - ||x\restriction_{V\backslash\{0_V\}}||_1, \end{aligned}$$

which tends to 0 as $n \to \infty$. □

As (β, V) will vary in that which follows, we relabel $|||\cdot|||_n$ to $|||\cdot|||_{(\beta,V,n)}$. Observe that the assignment $x \mapsto x\restriction_{\overline{V}\backslash\{0_V\}}$ from $\ell_1(\Upsilon)$ to $\ell_1(\overline{V}\backslash\{0_V\})$ is a dual map, as $\overline{V}\backslash\{0_V\}$ is open in Υ. Therefore, $x \mapsto |||x\restriction_{\overline{V}\backslash\{0_V\}}|||_{(\beta,V,n)}$ defines a w^*-lower semicontinuous seminorm on $\ell_1(\Upsilon)$.

We have need of a further class of w^*-lower semicontinuous seminorms. If Υ is a tree, $\rho : \Upsilon \longrightarrow [0, 1]$ a decreasing map, and $G \subseteq \Upsilon$ an open set, then $\rho \cdot \mathbf{1}_G$ is lower semicontinuous. Consequently, the map

$$x \longmapsto F(\rho, G, x) = \sum_{t\in G} \rho(t)|x_t|$$

is w^*-lower semicontinuous on $\ell_1(\Upsilon)$. Moreover, if (G_n) is any decreasing sequence of sets with intersection G, then we can apply the Monotone Convergence Theorem to the measure $F(\rho, \cdot, x)$ to conclude that $\lim_n F(\rho, G_n, x) = F(\rho, G, x)$. Armed with this final class of functions, we can finish the proof of Theorem 6.

Proposition 19. *Suppose that* Υ *admits a partition tree of the type given in Proposition* 13. *Then* $\ell_1(\Upsilon)$ *admits a dual* R *norm.*

Proof. Recall the subtrees $H_{(\beta,V)}$ from Proposition 13. Let $H^+_{(\beta,V)}$ be the set of immediate successors of $H_{(\beta,V)}$; in other words, the set of all $v \in H_{(\beta,V)}$ such that $t \prec v$ for some $t \in H_{(\beta,V)}$, and moreover, $t \prec u \prec v$ for no $u \in H_{(\beta,V)}$. When $(\beta, V) \in \Upsilon(\mathscr{P})$ is not maximal, define a function $\mu_{(\beta,V)} : V \longrightarrow (0, 1]$ by setting $\mu_{(\beta,V)}(t) = \delta_{(\beta+1,W)}$ whenever $t \in W$ and $(\beta + 1, W) \in (\beta, V)^+$. As $0_W \mapsto \pi(\beta + 1, W)$ is strictly decreasing on $H_{(\beta,V)}$, the map $\mu_{(\beta,V)}$ is decreasing on V. Moreover, as $H_{(\beta,V)} \preccurlyeq \mathbb{R}$, we may decompose the successors $H^+_{(\beta,V)}$ into a countable family of antichains $(E_{(\beta,V,n)})$. If $0_W \in E_{(\beta,V,n)}$, note that

$$[0_W, \infty) \cap V = \bigcup\{W' \mid 0_{W'} \in [0_W, \infty) \cap H_{(\beta,V)}\}.$$

Let $F_{(\beta,V,0_W)}(\cdot) = F(\mu_{(\beta,V)}, (0_W, \infty) \cap V, \cdot)$, where $(0_W, \infty) = [0_W, \infty)\backslash\{0_W\}$. Observe the identity

$$F_{(\beta,V,0_W)}(x) + \delta_{(\beta+1,W)}|x_{0_W}| = \sum_{0_{W'}\in[0_W,\infty)\cap H_{(\beta,V)}} \delta_{(\beta+1,W')}\|x\restriction_{W'}\|_1.$$

Since the wedges $[0_W, \infty)$ are disjoint for $0_W \in E_{(\beta,V,n)}$, it follows that

$$\begin{aligned}\sum_{0_W\in E_{(\beta,V,n)}} F_{(\beta,V,0_W)}(x) &\leqslant \sum_{0_{W'}\in H_{(\beta,V)}} \delta_{(\beta+1,W')}\|x\restriction_{W'}\|_1\\ &= \sum_{(\beta+1,W')\in(\beta,V)^+} \delta_{(\beta+1,W')}\|x\restriction_{W'}\|_1\end{aligned}$$

which gives

$$\sum_{(\beta,V)\in\Upsilon(\mathscr{P})}\ \sum_{0_W\in E_{(\beta,V,n)}} F_{(\beta,V,0_W)}(x) \leqslant \sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta_{(\beta+1,W)}\|x\restriction_W\|_1. \tag{1}$$

Now we are ready to define an equivalent dual norm that we claim is rotund. Let Υ° be the isolated points of Υ. Since every $\{t\} \subseteq \Upsilon^\circ$ is open in Υ, it follows that $x \mapsto \|x\restriction_{\Upsilon^\circ}\|_2^2$ is w^*-lower semicontinuous on $\ell_1(\Upsilon)$, where $\|\cdot\|_2$ is the standard 2-norm. Let

$$\begin{aligned}|||x|||^2 = \|x\|_1^2 + \|x\restriction_{\Upsilon^\circ}\|_2^2 &+ \sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta^2_{(\beta+1,W)}\|x\restriction_{W\backslash\{0_W\}}\|_1^2\\ &+ \sum_{n\geqslant 1} 2^{-n} \sum_{(\beta+1,W)\in\Upsilon(\mathscr{P})^+} \delta^2_{(\beta+1,W)}|||x\restriction_{\overline{W}\backslash\{0_W\}}|||^2_{(\beta+1,W,n)}\\ &+ \sum_{n\geqslant 1} 2^{-n} \sum_{(\beta,V)\in\Upsilon(\mathscr{P})}\ \sum_{0_W\in E_{(\beta,V,n)}} F_{(\beta,V,0_W)}(x)^2.\end{aligned}$$

By Lemma 17, Eq. (1) above, and the fact that $|||\cdot|||_{(\beta,V,n)} \leqslant |||\cdot|||_1$, we obtain $|||x||| \leqslant \sqrt{8}||x||_1$. As all the summands are w^*-lower semicontinuous, $|||\cdot|||$ is a dual norm. It remains to show that $|||\cdot|||$ is rotund. Suppose $x, z \in \ell_1(\Upsilon)$ satisfy $|||x||| = |||z||| = \frac{1}{2}|||x+z|||$. We must prove that $x_t = z_t$ for all $t \in \Upsilon$. By the second term in the definition of $|||\cdot|||$, together with a standard convexity argument, it follows that $x\restriction_{\Upsilon^\circ} = z\restriction_{\Upsilon^\circ}$.

The remaining elements of Υ, the limits, are divided into three types. Each is treated separately. Take a limit t and recall from our treatment of secondary sequences that θ is minimal, subject to the condition $t = 0_{V^t_\theta}$. By property (1) of Proposition 13 and that fact that as t is a limit, it is not minimal in Υ, we have $\theta > 0$. We say the limit t is of *type I* if $\theta = \beta + 1$ is a successor ordinal. In this case, let $V = V^t_\beta$. By minimality of θ, $0_V \prec t$. We deduce that (β, V) is not a singleton and is not maximal in $\Upsilon(\mathscr{P})$. Moreover, we observe that $t \in H_{(\beta,V)}$, since $\theta = \beta + 1$. Furthermore, we see that t is not the least element of the subtree $H_{(\beta,V)}$, again because $0_V \prec t$. It follows that either $t \in H^+_{(\beta,V)}$ or t is a limit of $H_{(\beta,V)}$. If $t \in H^+_{(\beta,V)}$, then we say t is of *type Ia*. Otherwise, we say t is of *type Ib*. Finally, if θ is not a successor but a limit ordinal, then t is of *type II*.

Let t be of type Ia. Suppose $t = 0_{W'} \in H^+_{(\beta,V)}$ and 0_W is the immediate predecessor of t in $H_{(\beta,V)}$. Because t is a limit in Υ, it must be that $t \in \overline{W}\backslash W$. Indeed, $[0_W, t]\backslash\{t\} \subseteq W$, lest 0_W not be the immediate predecessor of t in $H_{(\beta,V)}$. Consequently, we can use the fourth term in the definition of $|||\cdot|||$, together with convexity arguments and Lemma 18, to conclude that $x_t = z_t$.

Now suppose t is of type Ib. Let $0_{W_m} \in E_{(\beta,V,n_m)}$ be a sequence in $H^+_{(\beta,V)}$ converging to t. By applying convexity arguments to the fifth term in the definition of $|||\cdot|||$, we have

$$F_{(\beta,V,0_{W_m})}(x) = F_{(\beta,V,0_{W_m})}(z) = \tfrac{1}{2}F_{(\beta,V,0_{W_m})}(x+z)$$

and as $\bigcap_n [0_{W_m}, \infty) \cap V = [t, \infty) \cap V$, we take intersections to obtain

$$F(\mu, [t,\infty) \cap V, x) = F(\mu, [t,\infty) \cap V, z) = \tfrac{1}{2}F(\mu, [t,\infty) \cap V, x+z), \tag{2}$$

where $\mu = \mu_{(\beta,V)}$. These equalities hold for all type Ib elements in $H_{(\beta,V)}$. Let

$$P = \{v \in H_{(\beta,V)} \mid v \text{ is of type Ib, and minimal subject to } t \prec v\}$$

and

$$Q = \{u \in H^+_{(\beta,V)} \mid t \prec u, \text{ and } v \prec u \text{ for no } v \in P\}.$$

It follows that

$$[t, \infty) \cap H_{(\beta, V)} = \{t\} \cup Q \cup \bigcup_{v \in P} ([v, \infty) \cap H_{(\beta, V)})$$

and

$$\begin{aligned} F(\mu, [t, \infty) \cap V, x) = {} & \delta_{(\beta+1, W)} |x_t| + \delta_{(\beta+1, W)} \| x\restriction_{W \setminus \{t\}} \|_1 \\ & + \sum_{0_{W'} \in Q} \delta_{(\beta+1, W')} |x_{0_{W'}}| \\ & + \sum_{0_{W'} \in Q} \delta_{(\beta+1, W')} \| x\restriction_{W' \setminus \{0_{W'}\}} \|_1 \\ & + \sum_{v \in P} F(\mu, [v, \infty) \cap V, x) \end{aligned} \tag{3}$$

with corresponding identities for z and $\frac{1}{2}(x+z)$, which we call sister expressions. By convexity arguments and the third term in the definition of $|||\cdot|||$, we see that

$$\| x\restriction_{U \setminus \{0_U\}} \|_1 = \| z\restriction_{U \setminus \{0_U\}} \|_1 = \tfrac{1}{2} \| (x+z)\restriction_{U \setminus \{0_U\}} \|_1, \tag{4}$$

whenever $(\alpha + 1, U) \in \Upsilon(\mathscr{P})^+$. In particular, (4) holds if $\alpha = \beta$ and $0_U \in Q \cup \{t\}$. As each $0_{W'} \in Q$ is either an isolated point of Υ or of type Ia, we know already that $x_{0_{W'}} = z_{0_{W'}}$. Therefore, if we apply (2) and (4) to Eq. (3) and its sister expressions, we obtain $|x_t| = |z_t| = \frac{1}{2}|x_t + z_t|$, whence $x_t = z_t$.

Finally, we suppose that t is of type II. Let $\beta_n < \theta$ define a sequence of ordinals that increases up to θ. Set $V_n = V^t_{\beta_n}$ and $V = V^t_\theta = \bigcap_n V_n$. Since $\beta_n < \theta$ for each n, it follows that $0_{V_n} \prec t$ and therefore $V = \bigcap_n V_n \setminus \{0_{V_n}\}$. By the third term in the definition of the norm, we have

$$\| x\restriction_{V_n \setminus \{0_{V_n}\}} \|_1 = \| z\restriction_{V_n \setminus \{0_{V_n}\}} \|_1 = \tfrac{1}{2} \| (x+z)\restriction_{V_n \setminus \{0_{V_n}\}} \|_1,$$

whence

$$\| x\restriction_V \|_1 = \| z\restriction_V \|_1 = \tfrac{1}{2} \| (x+z)\restriction_V \|_1 \tag{5}$$

by taking intersections. Now we have

$$\| x\restriction_V \|_1 = |x_t| + \sum_{0_W \in H_{(\beta, V)} \setminus \{t\}} |x_{0_W}| + \sum_{0_W \in H_{(\beta, V)}} \| x\restriction_{W \setminus \{0_W\}} \|_1 \tag{6}$$

again with sister expressions for z and $\frac{1}{2}(x+z)$. As isolated points and type I limits have been treated already, we know that $x_{0_W} = z_{0_W}$ whenever $0_W \in H_{(\beta,V)} \backslash \{t\}$. Consequently, by applying (4) and (5) to Eq. (6) and its sisters, it follows that $|x_t| = |z_t| = \frac{1}{2}|x_t + z_t|$, whence $x_t = z_t$. This completes the proof. □

4. Examples

In the final section, we see how Theorem 1 emerges as a corollary of Theorem 6, and give examples of trees Υ with the property that $\ell_1(\Upsilon)$ admits an equivalent dual R norm, but not by virtue of Haydon's condition. All the results of this section follow from the next proposition, which exposes a universality property of Y. Although it can be sharpened, it is sufficient for our needs. Recall that by Cantor's theorem, $\mathbb{Q}^n \preccurlyeq \mathbb{Q}$ whenever $n < \omega$.

Proposition 20. *If $\beta < \omega_1$ then $Y^\beta \preccurlyeq Y$.*

Proof. It is enough to prove that $Y_0^\beta \preccurlyeq Y_0$, where Y_0 is the order defined in the proof of Proposition 13. Moreover, we will regard elements of Y_0 as compact, well-ordered subsets of $[0, 1)$, with least element 0. In the subset interpretation, elements $x, y \in Y_0$ have a maximal common initial segment u. It follows that $x < y$ if and only if either $u = x$ and $y\backslash u$ is non-empty, or if both $x\backslash u$ and $y\backslash u$ are non-empty, and the least element of the former strictly exceeds that of the latter.

Now take $\beta < \omega_1$, and fix some $x = (x_\alpha)_{\alpha \leqslant \beta} \in Y_0$. If $y = (y^\alpha)_{\alpha<\beta} \in Y_0^\beta$, define

$$\pi(y) = \bigcup_{\alpha<\beta} \{x_\alpha + (x_{\alpha+1} - x_\alpha)t \mid t \in y^\alpha\} \cup \{x_\beta\}.$$

We leave to the reader the routine tasks of proving that π is well-defined and strictly increasing. □

As $\omega \preccurlyeq \mathbb{R} \preccurlyeq Y$, we see immediately from Proposition 20 that $\mathbb{R}\times\omega \preccurlyeq Y$, where $\mathbb{R}\times\omega$ is ordered lexicographically. Theorem 1 is now a direct consequence of Theorem 6 and the next proposition.

Proposition 21. *If Υ admits an increasing function $\rho : \Upsilon \longrightarrow \mathbb{R}$ that is constant on no infinite chain in Υ, then $\Upsilon \preccurlyeq \mathbb{R} \times \omega$.*

Proof. Let $\mathscr{P}$ be the plateau partition of Υ with respect to ρ, furnished by Proposition 9. As there are no infinite chains on which ρ is constant, given any $V \in \mathscr{P}$ and $t \in V$, the height $\mathrm{ht}(t, V)$ of t with respect to V is finite. Define the map $\mu : \Upsilon \longrightarrow \mathbb{R} \times \omega$ by $\mu(t) = (\rho(t), \mathrm{ht}(t, V_t))$, where V_t is the unique element of $\mathscr{P}$ containing t. We prove that μ is strictly increasing. Let $s \prec t$. If $\rho(s) < \rho(t)$ then we are done. Otherwise, $V_s = V = V_t$ and $\mathrm{ht}(s, V) < \mathrm{ht}(t, V)$. □

Our examples use the quintessential operation of tree construction, first introduced by Kurepa. Given a partial order P, define

$$\sigma P = \{A \subseteq P \mid A \text{ is well-ordered}\}$$

and order it by declaring that $A \preccurlyeq B$ if and only if A is an initial segment of B. With respect to this order, σP is a Hausdorff tree. The next theorem hints at why the σ-operation is so useful.

Theorem 22 (*Kurepa [5]*). *If P is a partial order then $\sigma P \not\preccurlyeq P$.*

Despite Theorem 22, the following elementary observation demonstrates that in some cases, σP is almost P-embeddable.

Proposition 23. *Suppose that Σ is a complete total order, in the sense that every subset $A \subseteq \Sigma$ has a least upper bound, denoted by* $\sup A$. *Then $\sigma\Sigma \preccurlyeq \Sigma \times \{0, 1\}$, where $\Sigma \times \{0, 1\}$ is ordered lexicographically.*

Proof. Observe that $A \mapsto \sup A$ defines an increasing map on $\sigma\Sigma$ that takes values in Σ. Moreover, if $A \prec B$ and $\sup B = \sup A$, then $\sup A \notin A$ and $B = A \cup \{\sup A\}$. Indeed, if $x \in B \setminus A$ then $\sup A \leqslant x \leqslant \sup B = \sup A$. Therefore, the assignment $\rho(A) = (\sup A, i)$, where $i = 0$ if A has no greatest element, and $i = 1$ otherwise, defines a strictly increasing map taking values in $\Sigma \times \{0, 1\}$. □

Now consider the family $\sigma(\mathbb{R}^\beta)$, where $2 \leqslant \beta < \omega_1$ and $\mathbb{R}^\beta$ is ordered lexicographically. Proposition 20 confirms the assertion, made in Section 1, that $\mathbb{R}^\beta \preccurlyeq Y$ whenever $\beta < \omega_1$. By Theorem 22, $\sigma(\mathbb{R}^\beta) \not\preccurlyeq \mathbb{R}^\beta$. In particular, $\sigma(\mathbb{R}^\beta) \not\preccurlyeq \mathbb{R} \times \omega$ if $\beta \geqslant 2$, thus by Proposition 21, it is not possible to apply Theorem 1 to $\sigma(\mathbb{R}^\beta)$. On the other hand, we claim that $\sigma(\mathbb{R}^\beta) \preccurlyeq \mathbb{R}^{\beta+1} \preccurlyeq Y$, and therefore $\ell_1(\sigma(\mathbb{R}^\beta))$ admits an equivalent dual R norm by Theorem 6. To prove the claim, note that $[0, 1]^\beta$ is a complete order, and that elements of $\sigma(\mathbb{R}^\beta)$ may be considered as subsets of $(0, 1)^\beta$. Hence, we can apply Proposition 23 to conclude that $\sigma(\mathbb{R}^\beta) \preccurlyeq [0, 1]^\beta \times \{0, 1\} \preccurlyeq \mathbb{R}^{\beta+1}$ as required.

We finish by observing that this family of trees forms a strictly increasing hierarchy with respect to the quasi-order $\preccurlyeq$. Indeed, if $\alpha < \beta$ then $\sigma(\mathbb{R}^\alpha) \preccurlyeq \mathbb{R}^\beta$ and $\sigma(\mathbb{R}^\beta) \not\preccurlyeq \mathbb{R}^\beta$, thus $\sigma(\mathbb{R}^\beta) \not\preccurlyeq \sigma(\mathbb{R}^\alpha)$. In particular, these trees are mutually non-isomorphic as partial orders.

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