Transcendence of Numbers Related to Episturmian Words

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Ut in omnibus glorificetur Deus!

Abstract

This thesis relates combinatorial properties of sequences to arithmetic properties of the numbers that they represent. A guiding principle is that numbers whose b-expansion has low subword complexity are either rational or transcendental. This heuristic was confirmed by Ferenczi and Maduit, who proved that for $b > 1$ an integer, numbers whose b-expansion is a Sturmian sequence or an Arnoux-Rauzy sequence are transcendental. (These can be considered as the simplest non-ultimately-periodic sequences in terms of their subword complexity.) Subsequent work of Adamczewski and Bugeaud extended this result by proving that all numbers whose b-expansion has linear subword complexity are rational or transcendental, again for b an integer. The latter authors obtained related results to the case of b an algebraic base under certain combinatorial properties of the sequence, which depend on b.

The main contribution of this thesis is providing a transcendence result which applies to arbitrary algebraic bases. We introduce a new combinatorial condition on sequences and prove a transcendence result for numbers of the form $\alpha := \sum_{n=1}^{\infty} u_n \beta^{-n}$ where β is any algebraic number such that $|\beta| > 1$ and $\boldsymbol{u} = u_1 u_2 \cdots$ is a sequence of algebraic numbers satisfying the above-mentioned criterion. In particular we prove that all Episturmian (a generalisation of Arnoux-Rauzy words) words satisfy this criterion.

Keywords: transcendental number, algebraic base, Subspace Theorem, Sturmian word, Episturmian word, Tribonacci word

Contents

Notations

- $\mathbb{N}$ denotes the set of all positive natural numbers.
- $\mathbb{N}_0$ denotes the set of all natural numbers including 0.
- $[\![a, b]\!]$ denotes the set $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$.
- $[a, b]$ denotes the set $\{i \in \mathbb{R} \mid a \leq i \leq b\}$.
- $|w|$ is the length of the word w.
- $\mathsf{Alp}(w)$ denotes the set of letters occurring in the (potentially infinite) word w.
- $\mathsf{Ult}(w)$ denotes the set of letters occurring infinitely many times in the infinite word w.

Introduction

It is a familiar fact that the decimal expansion of a rational number is ultimately periodic. This remains true even if we write the rational number in any integer base b. A natural follow-up question arises: what does the b-expansion of an irrational number look like? Intuition may suggest that they behave "randomly" in some sense, but it turns out that there are many irrational numbers whose integer base expansion is fairly regular. One such example is the following one proposed by Liouville in 1844 [**21**]:

$$\sum_{j=1}^{\infty} \frac{1}{10^{j!}} = 0.110001000000000000000001000000..$$

In fact, this number is supposedly the first number in the history to be proved transcendental. Recall that a number is transcendental if it is not algebraic, *i.e.*, there is no polynomial with rational coefficients such that the number is a root of that polynomial. Familiar examples of algebraic numbers include $\sqrt{2}$ and the golden ratio $\frac{\sqrt{5}-1}{2}$. Transcendental numbers include the aforementioned number of Liouville as well as the numbers π, e.

The transcendence of the above number follows from the fact that it can be very well approximated by rational numbers, which is not possible in the case of algebraic irrational numbers. This is well known since Roth proved his celebrated theorem for which he was awarded Fields medal [**27**]. It states that for an irrational algebraic number ξ and any positive real ϵ, there are only a finite number of rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

This bound is tight since every irrational number can be approximated by infinitely many irrational numbers in the case the exponent is equal to two.[1] In this thesis, we employ a generalised version of this theorem, called Subspace Theorem which we survey in Section 2.2.

[1] by Dirichlet Approximation Theorem.

These results suggest that exactly all algebraic irrational numbers behave "randomly". This is captured by the conjecture of Borel [**11**] stating that all algebraic irrational numbers are normal – for the base b, each block consisting of letters from $\{1, \ldots, b-1\}$ of length n occurs in the b-expansion of that number with frequency $1/b^n$. For instance, one can find the word 3005727 in the decimal expansion of a normal number infinitely often with the frequency $1/10^7$. This conjecture is far from reach since we do not know whether ordinary constants such as π, e or $\sqrt{2}$ are normal [**9**].

Despite this, partial results have been obtained, particularly in the area of words with linear subword complexity. Given an infinite sequence $\boldsymbol{u}$, let $\mathsf{sc}(n)$ denote the number of subwords of $\boldsymbol{u}$ of length n. Notice that b-expansion of a normal number has $\mathsf{sc}(n) = b^n$ for every n and a word is ultimately periodic if and only if there is $n \in \mathbb{N}$ such that $\mathsf{sc}(n) = n$ [**23**].

A significant breakthrough was achieved by Ferenczi and Mauduit in [**16**] where they prove that every number whose b-expansion satisfies $\mathsf{sc}(n) = n + 1$ is transcendental. Words satisfying $\mathsf{sc}(n) = n + 1$ are called Sturmian and they are of great importance in combinatorics, dynamical systems, geometry, *etc.* This result was strengthened by Adamczewski and Bugead by proving that given a b-expansion of some number ξ, if

$$\liminf_{n \to \infty} \frac{\mathsf{sc}(n)}{n} < +\infty$$

then ξ is transcendental [**4**].

In both works, the authors used a combinatorial criterion similar to the following one:

Theorem A. *For a given number ξ whose b-expansion is an infinite sequence $\boldsymbol{w}$, if there are infinitely many words u_n, v_n, v'_n and a positive real number m such that*

1. *$u_n v_n v'_n$ is a prefix of $\boldsymbol{w}$,*
2. *v'_n is a prefix of v_n and $|v'_n| \geq m \cdot |v_n|$*
3. *$|v_n|$ is strictly increasing,*
4. *$|u_n|/|v_n|$ is bounded by a constant,*

then the number ξ is transcendental.

The basic idea is that if we can find infinitely long words such that they repeat at least in some linear part, then we can construct good rational approximations which implies, using the Subspace Theorem, that the related number is transcendental.

From a different perspective, Hartmanis and Stearns conjectured that no algebraic irrational number can be generated by a Turing machine in linear time [**18**]. This question is important since it relates to questions such as time complexity of matrix multiplication [**12**]. So far, we know that every automatic number is either rational or transcendental [**4**] and the same holds for numbers generated by deterministic pushdown automaton [**1**].

In the previous discussion, we assumed that the base is an integer. However, if we allow ourselves to use algebraic numbers, we return to the initial question at hand. Before we survey the known results, let us explain how a number can be written in a non-integer base. Let us have a base β, a set of digits $\mathcal{H}$ – these can be integers $1, \dots, \lfloor|\beta|\rfloor$, or they can be arbitrary complex number – and an infinite word over this alphabet $\boldsymbol{u} = u_1 u_2 \dots \in \mathcal{H}^\omega$. Then, we easily interpret this word as a number in base β as

$$\xi = \sum_{j=1}^{\infty} u_j \beta^{-j}.$$

This series always converges if $\mathcal{H}$ is finite.

First of all, many properties that were true in the case of an integer base hold no more. For instance, a single number can have multiple representations in the same base, *e.g.*, $(0.011)_\Phi = (0.1)_\Phi$ where Φ is the golden ratio. Furthermore, an ultimately periodic word does not necessarily denote a rational number but lies in $\mathbb{Q}(\beta)$ which is an algebraic number field generated by β. Surprisingly, the other way around is not always true, *i.e.*, there are non-ultimately periodic words whose interpretation in base β lie in $\mathbb{Q}(\beta)$. Both directions hold exactly in the case of a Pisot[2] number [**30**].

The above facts render both Hartmanis-Stearns conjecture and conjecture about normal numbers false in the case of an arbitrary algebraic base. However, this does not mean that studying numbers in algebraic bases is of no use. So-called β-expansions have connections to ergodic theory, theoretical computer science, tilings, *etc.* (see for instance [**10**]).

Addressing transcendence of numbers in algebraic bases, Adamczewski and Bugeaud [**2**] proved that if the base β satisfies the following inequality, then the word $\boldsymbol{u}$ inter-

[2]Pisot numbers are all algebraic integers β such that $|\beta_i| < 1$ for all its conjugates β_i except β.

preted in base β is either transcendental or lies in $\mathbb{Q}(\beta)$:

$$\mathsf{dio}(\boldsymbol{u}) > \frac{M(\beta)}{\log|\beta|}.$$

Here, $M(\beta)$ is the Mahler measure of β and $\mathsf{dio}(\boldsymbol{u})$ is the supremum of satisfactory m in the criterion from Theorem A. Another result from [**3**] states that every word satisfying the criterion from Theorem A is transcendental if interpreted in a base which is a Pisot number. A brief survey of known results is provided in Section 2.4. Unfortunately, both results are restrictive in terms of admissible bases.

The main contribution of this thesis can be summarized as follows: We provide a new transcendence result that works with any algebraic base $\beta, |\beta| > 1$ and an infinite sequence satisfying some combinatorial criterion. Sequences which satisfy this criterion are called *echoic*[3], indicating that they are similar to a periodic word with some minor errors (like an echo of a sound). In section 2.5, we prove the following:

Theorem 2.15. *Let $\boldsymbol{u} = u_1u_2\cdots \in \mathcal{H}^\omega$ be an infinite echoic sequence over a finite set of numbers $\mathcal{H}$ and β an algebraic base $|\beta| > 1$. Then, the number $\alpha := \sum_{j=1}^{\infty} u_j\beta^{-j}$ is transcendental or belongs to $\mathbb{Q}(\beta, \mathcal{H})$.*

This approach was first proposed in [**22**] defining *stuttering sequences* leading to a result concerning Sturmian words. However, the definition of a stuttering sequence allows only a constant number of mismatches, which makes the applicability limited.

Our approach is much stronger since we prove that not only all Sturmian words are echoic, but also Arnoux-Rauzy words are and their generalisation – Episturmian words (Theorem 3.15).

Episturmian words are of great importance because of their combinatorial properties which are similar to Sturmian words. See more in the section 1.6. The standard example of an episturmian word that is not Sturmian is the Tribonacci word:

$$0102010010201010201001020102010010201010201001020100\cdots$$

This thesis is split into three chapters: The first chapter defines all necessary concepts related to infinite words and word differences. In the second chapter, we survey the evolution of the Subspace Theorem, show how it was used in the previous cases and finally prove our main result which is providing a new combinatorial criterion in Section 2.5. The third chapter applies the theorem for various classes of infinite sequences. First, explaining the ideas on Sturmian words and the Tribonacci word, and then fitting the combinatorial criterion to Episturmian words in Section 3.3.

[3]See Definition 2.14.

Chapter 1

Stringology

This chapter formally defines all concepts necessary for working with finite and infinite words. We provide a brand new concept – word difference – which ought to simplify our reasoning about "subtracting" words. In the third section we define morphic words, *e.g.* Fibonacci word, Tribonacci word, Thue-Morse word, and show some of their basic properties. Finally, the fourth section defines episturmian words which are our main concern in this thesis.

1.1 Preliminaries

An *alphabet* Σ is a finite set of letters. A *word* is a finite or infinite one-way sequence of letters from Σ. By ε, we denote the empty word.

Let Σ be an alphabet. Given a word $w = w_1 w_2 \dots w_n \in \Sigma^*$ where $w_1, \dots, w_n \in \Sigma$, we denote the *reversal* of w as $\tilde{w} = w_n w_{n-1} \cdots w_1$.

We index words from 1, *i.e.*, for a given (potentially infinite) word $\boldsymbol{u} = u_1 u_2 \dots \in \Sigma^* \cup \Sigma^\omega$, the index of u_1 is 1, the index of u_2 is 2 *etc.*

Let Σ be an alphabet and $\boldsymbol{w} \in \Sigma^\omega$ an infinite word. By $\mathsf{fac}_n(\boldsymbol{w})$, we denote the set of all *factors* of the word $\boldsymbol{w}$ of length n. That is $\mathsf{fac}_n(\boldsymbol{w}) = \{u \mid |u| = n, \boldsymbol{w} = vu\boldsymbol{v}$ for some $v \in \Sigma^*, \boldsymbol{v} \in \Sigma^\omega\}$. By $\mathsf{fac}(\boldsymbol{w})$, we denote the set of all factors of the word $\boldsymbol{w}$ of any length, *i.e.*, $\mathsf{fac}(\boldsymbol{w}) = \bigcup_{n \in \mathbb{N}_0} \mathsf{fac}_n(\boldsymbol{w})$.

For an infinite word $\boldsymbol{w} \in \Sigma^\omega$, we define *subword complexity* $\mathsf{sc} : \mathbb{N} \to \mathbb{N}$, where $\mathsf{sc}(n)$ is the size of the set of all factors of length n in $\boldsymbol{w}$, *i.e.*, $\mathsf{sc}(n) = |\mathsf{fac}_n(\boldsymbol{w})|$.

An infinite word $\boldsymbol{w} \in \Sigma^\omega$ is periodic if there exist $v \in \Sigma^*$ such that $\boldsymbol{w} = v^\omega$ and it is ultimately periodic if there exists $u, v \in \Sigma^*$ such that $\boldsymbol{w} = uv^\omega$.

For a finite word, we define an operation of *circular shift* $T : \Sigma^* \to \Sigma^*$ defined as $T(aw) = wa$ where $w \in \Sigma^*, a \in \Sigma$. This is naturally extended to infinite words as $T(a\boldsymbol{u}) = \boldsymbol{u}$ where $\boldsymbol{u} \in \Sigma^\omega, a \in \Sigma$.

Let Σ be an alphabet and $w \in \Sigma^* \cup \Sigma^\omega$ a potentially infinite word. For every letter $a \in \Sigma$, we define *letter return time* $\mathsf{lrt}_a : \Sigma^* \to \mathbb{N} \cup \{\infty\}$ of a letter a in a word w as the largest distance between two occurrences of the letter a in w, *i.e.*, $\mathsf{lrt}_a(w)$ is the smallest number $K \in \mathbb{N}$ such that $a \in \mathsf{Alp}(v)$ for every $v \in \mathsf{fac}_K(w)$. If $\Sigma' \subseteq \Sigma$ then $\mathsf{lrt}_{\Sigma'}(w) = \max\{\mathsf{lrt}_a(w) \mid a \in \Sigma'\}$.

1.2 Word Difference

For an alphabet Σ, given two equal-length words $u = u_1 u_2 \dots u_k \in \Sigma^*$ and $v = v_1 v_2 \dots v_k \in \Sigma^*$ with $k \in \mathbb{N}$, we define word difference $u \ominus v$ to be the word $w = w_1 w_2 \dots w_k$ over alphabet $\Sigma^\ominus := \Sigma \times \Sigma$ where $w_i = (u_i, v_i)$ for each $i \in 1, \dots, k$. We sometimes denote (a, b) as $\frac{a}{b}$ and (a, a) as $\bullet$ for every $a, b \in \Sigma$. In the case of a difference between words over numbers, letter difference becomes digit-wise subtraction.

Hence, if we have $u = 001039$ and $v = 021094$, we denote $u \ominus v$ either of following ways:

$$(0,0)(0,2)(1,1)(0,0)(3,9)(9,4)$$

$$\bullet \tfrac{0}{2} \bullet \bullet \tfrac{3}{9} \tfrac{9}{4}$$

$$0(-2)00(-6)5$$

Additionally we define negation of a word $w = \frac{u_1}{v_1} \cdots \frac{u_k}{v_k} \in (\Sigma^\ominus)^*$ as $\ominus : (\Sigma^\ominus)^* \to (\Sigma^\ominus)^*$ where $\ominus w = \frac{v_1}{u_1} \cdots \frac{v_k}{u_k}$. Hence, for every $v, w \in \Sigma^*$ such that $|v| = |w|$,

$$(v \ominus w) = \ominus(w \ominus v).$$

Next, we define operator $\diamond : (\Sigma^\ominus)^* \to (\Sigma^\ominus)^*$ which trims any leading and tailing $\bullet$, *i.e.*, for a word w, $w^\diamond$ is a word neither starting nor ending with "$\bullet$" such that $w = \bullet^k w^\diamond \bullet^l$ for some $k, l \in \mathbb{N}$. In particular, if $w \in \bullet^*$ then $w^\diamond = \varepsilon$.

1.3 Laurent Series and Polynomials Related to Words

Let $\mathcal{H}$ be a set of numbers. With a finite word $u = u_1 u_2 \cdots u_n$ where $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathcal{H}$, we associate the polynomial

$$\sum_{j=1}^{n} u_j X^{n-j}.$$

With an infinite word $\boldsymbol{u} = u_1 u_2 \cdots \in \mathcal{H}^\omega$, we associate the Laurent series

$$\sum_{j=1}^{\infty} u_j X^{-j}.$$

If $w \in \Sigma^* \cup \Sigma^\omega$ is a (potentially infinite) word difference $w = \frac{u_1}{v_1}\frac{u_2}{v_2}\cdots$ where $u_1, u_2, \ldots, v_1, v_2, \ldots \in \Sigma$, we take $u_j - v_j$ as the number for every $j \in \mathbb{N}$.

For a (potentially infinite) word $w \in \Sigma^* (\in \Sigma^\omega)$, and its related polynomial (Laurent series) $p(X)$, we denote $w_\beta := p(\beta)$ where $\beta \in \mathbb{C}$.

1.4 Morphic Words

A *morphism* is a map $\phi : \Sigma \to \Sigma^*$ which is naturally extended to all words $\phi : \Sigma^* \cup \Sigma^\omega \to \Sigma^* \cup \Sigma^\omega$ such that $\phi(uv) = \phi(u)\phi(v)$ for any $u \in \Sigma^*, v \in \Sigma^* \cup \Sigma^\omega$. A famous example is the following morphism over an alphabet $\{0, 1\}$

$$\begin{aligned} \phi \quad 0 &\mapsto 01 \\ 1 &\mapsto 0. \end{aligned}$$

Applying this morphism multiple times (ϕ^1, $\phi^2, \ldots$) on the value 0, we notice that every word is a prefix of the following word.

$$\begin{aligned} \phi(0) &= 01 \\ \phi^2(0) &= 010 \\ \phi^3(0) &= 01001 \\ \phi^4(0) &= 01001010 \\ &\cdots \end{aligned}$$

Indeed, the above morphism has a fixed point which is the following word, known as the *Fibonacci word*

$$\lim_{n\to\infty} \phi^n(0) = 0100101001001010010100100101001001010100\ldots$$

Notice also that the lengths $|\phi^0(0)| = 1, |\phi^1(0)| = 2, |\phi^2(0)| = 3, |\phi^3(0)| = 5, \ldots$ form the Fibonacci sequence. Furthermore, the ratio $\frac{\#_0(u)}{|u|}$ is equal to the golden ratio in the limit, where u is a prefix of the Fibonacci word and $\#_0(u)$ is the number of occurrences of 0 in u.

A morphism has a fixed point on a letter a if and only if it is *prolongable* on the letter a, which happens if there is a non-empty word u such that $\phi(a) = au$ and $\phi^n(u)$ is non-empty for all $n \in \mathbb{N}$. Then, the fixed point is

$$a\phi(a)\phi^2(a)\phi^3(a)\phi^4(a)\cdots.$$

Similarly, the *Tribonacci word* over an alphabet $\{0, 1, 2\}$, is generated by

$$\begin{aligned} \phi \quad 0 &\mapsto 01 \\ 1 &\mapsto 02 \\ 2 &\mapsto 0 \end{aligned}$$

and has the fixed point:

$$0102010010201010201001020102010010201010201001020100\ldots$$

If there exists $k \in \mathbb{N}$ such that for every $a \in \Sigma$, there exists $u \in \Sigma^k$ and $\psi(a) = u$, then we say the morphism ϕ is *k-uniform*. Cobham proved that an infinite word is generated by a finite automaton over an alphabet of size k if and only if it is an image by coding[1] of a word that is generated by a k-uniform morphism [**13**]. An emblematic example of a k-uniform morphism is the Thue-Morse word generated by:

$$\begin{aligned} \phi \quad 0 &\mapsto 01 \\ 1 &\mapsto 10 \end{aligned}$$

which induces the infinite word

$$0110100110010110100101100110100110010110011010010 11\ldots$$

1.5 Sturmian Words

Sturmian words are probably the most researched class of infinite binary words. They arise naturally in multiple areas such as number theory, symbolic dynamics, theoretical physics, or theoretical computer science, and they share various combinatorial, geometrical, and algebraic properties. See [**6**] for more.

Sturmian words are exactly those words with subword complexity $\mathsf{sc}(n) = n + 1$ for every $n \in \mathbb{N}$. Since an infinite word is ultimately periodic if and only if $\mathsf{sc}(n) = n$ for some $n \in \mathbb{N}$ [**23**], one can view Sturmian words as aperiodic words with the lowest subword complexity. Notice that this definition implies that Sturmian words are over a binary alphabet. We use $\Sigma = \{0, 1\}$. There are many other definitions such as that Sturmian words are aperiodic and balanced[2], they are formed as the cutting sequence

[1] A coding of a word is a 1-uniform morphism, *i.e.*, morphism $\phi : \Sigma \to \Sigma$.

[2] A word is balanced if the number of 0's in any two factors of the same length differ by at most 1.

of a line with some irrational slope with the unit grid [**14**], and they also arise as the coding of a rotation on a unit circle [**23**].

Let $\boldsymbol{u}$ be a Sturmian word. From the definition, one has that for each $n \in \mathbb{N}$ there exist n words $w \in \mathsf{fac}_n(\boldsymbol{u})$ of length n such that exactly one of $w0 \in \mathsf{fac}_{n+1}(\boldsymbol{u})$ or $w1 \in \mathsf{fac}_{n+1}(\boldsymbol{u})$ holds, and there is one special word $v \in \mathsf{fac}_n(\boldsymbol{u})$ such that $v0, v1 \in \mathsf{fac}_{n+1}(\boldsymbol{u})$. Necessarily, the same holds for appending a letter to the left and actually, the left special word is the reversal of the right special word. In the next section, we show how this is used to define Arnoux-Rauzy words and Episturmian words.

A Sturmian word is *standard* if all its left special factors are prefixes of the word. It is known that for each Sturmian word $\boldsymbol{u}$, there is exactly one standard Sturmian word $\boldsymbol{v}$ such that $\mathsf{fac}(\boldsymbol{u}) = \mathsf{fac}(\boldsymbol{v})$. This word is also called the *characteristic word* of $\boldsymbol{u}$.

Non-standard Sturmian words are somehow a shifted version of their respective characteristic word. This is best described using the following equivalent definition of Sturmian words: Let $s \in [0,1]$ be an irrational number and $i \in [0,1]$. They are referred to as the *slope* and *intercept*, respectively. We define an infinite sequence $\boldsymbol{w} = w_1 w_2 \ldots$ where

$$w_n = \lfloor ns + i \rfloor - \lfloor (n-1)s + i \rfloor.$$

In this case, the Sturmian word is standard if and only if $i = 0$. All words with the same slope have the same set of factors.

For a standard Sturmian word $\boldsymbol{u}$ there exist an infinite sequence $\langle w_n \rangle_{n=0}^{\infty}$ of prefixes of $\boldsymbol{u}$ satisfying

$$w_{n+1} = w_n^{s_n} w_{n-1}$$

for every $n \in \mathbb{N}$ where $s_1, s_2, \ldots \in \mathbb{N}$ and $\{w_0, w_1\} = \{0, 1\}$. Furthermore, $[0; s_1 + 1, s_2, s_3, \ldots]$ is the continued fraction expansion of the slope s of $\boldsymbol{u}$. [**5**].

There is also a connection between Sturmian words and morphism. Let

$$\psi_0 : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array} \qquad\qquad \psi_1 : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 10 \end{array}$$

Given a standard Sturmian word $\boldsymbol{u}$ with the sequence $s_1, s_2, \ldots$ and $w_0 = 0, w_1 = 1$, let $\Delta = 0^{s_1} 1^{s_2} 0^{s_3} \cdots = d_1 d_2 \cdots$ be called a *directive sequence* where $d_1, d_2, \ldots, \in \{0, 1\}$. Also, $s_1, s_2, \ldots$ are called the *partial quotients* of Δ. Then, the word $\boldsymbol{u}$ arises as the following limit:

$$\boldsymbol{u} = \lim_{n \to \infty} \psi_{d_1} \cdots \psi_{d_n}(d_{n+1})$$

1.6 Arnoux-Rauzy and Episturmian Words

The property of having exactly one special left factor and one special right factor for every $n \in \mathbb{N}$ can be generalised to any alphabet. Infinite words satisfying this property are called *Arnoux-Rauzy words*. They were extensively studied for their geometrical realizations such as *Rauzy fractals* [**7**], interval exchanges [**8**], and S-adic dynamical systems [**26**].

Arnoux-Rauzy words posses the property that they are closed under factor reversal, *i.e.*, if u is a factor, then its reversal is also a factor. Thus, one can generalise Arnoux-Rauzy words to infinite words that are closed under reversal and have at most one special left factor for each length. These words, introduced by Droubay, Justin and Pirillo [**15**], are called *episturmian* and share similar properties with Sturmian words. See the survey of Glen and Justin [**17**] for an overview of known results.

Formally, let Σ be an alphabet. An infinite word $\boldsymbol{u}$ is *episturmian* if

1. $\tilde{u} \in \mathsf{fac}(\boldsymbol{u})$ for every factor $u \in \mathsf{fac}(\boldsymbol{u})$,
2. for every $v \in \mathsf{fac}(\boldsymbol{u})$ of length $n \in \mathbb{N}$ except one word u, there is exactly one letter $a \in \Sigma$ such that $va \in \mathsf{fac}_{n+1}(\boldsymbol{u})$; the word u is called the *right special factor*.[3]

An episturmian word that has all left special factors as its prefix is called standard. Standard episturmian is shortened to *epistandard*. For every episturmian word $\boldsymbol{u}$, there exist a unique epistandard word $\boldsymbol{v}$ such that $\mathsf{fac}(\boldsymbol{u}) = \mathsf{fac}(\boldsymbol{v})$.

There is another viewpoint on episturmian words using morphisms and *spins*. Let Σ be an alphabet and $\bar{\Sigma} = \{\bar{a} \mid a \in \Sigma\}$ be the spinned version of Σ. For every $a, b \in \Sigma$, we define

$$\psi_a(b) = \begin{cases} a \text{ if } b = a \\ ab \text{ otherwise} \end{cases} \qquad \psi_{\bar{a}}(b) = \begin{cases} a \text{ if } b = a \\ ba \text{ otherwise} \end{cases}$$

We denote $\breve{\Sigma} = \Sigma \cup \bar{\Sigma}$. A letter $\breve{a} \in \breve{\Sigma}$ has spin 1 if $\breve{a} = \bar{a}$ and spin 0 if $\breve{a} = a$. For a letter $a \in \Sigma$, $\breve{a}$ is the spinned version of a.

Theorem 1.1 ([**20**], Thm. 3.10)**.** *An infinite word $\boldsymbol{u} \in \Sigma^\omega$ is episturmian if and only if there exists an infinite spinned word $\breve{\Delta} = \breve{d}_1\breve{d}_2 \cdots \in \breve{\Sigma}^\omega$ and an infinite sequence $\langle \boldsymbol{u}^{(n)} \rangle_{n=0}^{\infty}$ of infinite recurrent*[4] *words such that*

$$\boldsymbol{u}^{(0)} = \boldsymbol{u} \quad \text{and} \quad \boldsymbol{u}^{(n-1)} = \psi_{\breve{d}_n}(\boldsymbol{u}^{(n)})$$

[3] Since $\boldsymbol{u}$ is closed under reversal, there is also a unique left special factor which is the reversal of the right special word.

[4] A word w is recurrent if every factor $u \in \mathsf{fac}(w)$ occurs infinitely often in w.

for every $n \in \mathbb{N}$.

It is a known fact that if all spins of $\breve{\Delta}$ are 0, the word $\boldsymbol{u}$ is epistandard.

Chapter 2

Number Theory and Transcendence

In this chapter, we explore results concerning transcendence of numbers related to infinite sequences. The first section lays down number-theoretic preliminaries including embeddings, norms and places. The second section surveys the evolution of the Subspace Theorem from Liouville's inequality up to Schlickewei's p-adic version of the Subspace Theorem for number fields. In the third section, we reproduce the proof of transcendence of numbers related to Sturmian sequences and to sequences with linear subword complexity, for an integer base b. The fourth section describes difficulties in trying to generalise the former results to algebraic bases and outlines known results in this area. We end the chapter proving theorem 2.15 which is the main contribution of this thesis.

2.1 Preliminaries

We denote the set of polynomials with coefficients in a set $\mathcal{H} \subseteq \mathbb{C}$ as $\mathcal{H}[X] = \{a_d X^n + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \mid a_d, \ldots, a_0 \in \mathcal{H}, d \in \mathbb{N}_0\}$. For a comprehensive introduction to number theory, we refer the reader to [**19**].

Algebraic numbers, as mentioned above, are exactly those complex numbers γ for which there exists a polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(\gamma) = 0$. With every algebraic number γ we associate the minimal polynomial of γ over $\mathbb{Q}$ – $m(X) \in \mathbb{Q}[X]$ – which is the polynomial with the least degree such that it is irreducible, it has leading coefficient 1, and $m(\gamma) = 0$. A polynomial $f(X) \in \mathbb{Q}[X]$ is irreducible if it is divisible only by 1 and by itself, *i.e.*, there are no polynomials $g(X), h(X) \in \mathbb{Q}[X]$ such that $f(X) = g(X)h(X)$ and neither $g(X)$ nor $h(X)$ are constant. For every algebraic number γ, there is a unique minimal polynomial and we denote $deg(\gamma)$ the degree of that polynomial. Throughout, we work only with minimal polynomials over $\mathbb{Q}$.

A number field generated by γ over $\mathbb{Q}$ is $\mathbb{Q}(\gamma)$, by which we mean the smallest field extension containing element γ and all rational numbers. In other words, the set obtained by applying addition, subtraction, multiplication and division to the set of rational numbers and γ. It is a known fact that for algebraic γ,

$$\mathbb{Q}(\gamma) \cong \frac{\mathbb{Q}[X]}{(m(X))}$$

where $(m(X))$ is the ideal, generated by the minimal polynomial of γ. Hence, every element of $\mathbb{Q}(\gamma)$ can be written as a polynomial in γ with coefficients in $\mathbb{Q}$ with degree smaller than $deg(\gamma)$.

For a given algebraic number γ and its minimal polynomial $m(X)$ we call *conjugates of* γ all roots of $m(X)$. Since all roots of an irreducible polynomial are distinct, there are exactly $deg(\gamma)$ conjugates of γ. Conjugates can be grouped into real ones (having zero imaginary part) and complex ones which are paired, *i.e.*, for $a, b \in \mathbb{R}$, if $a + bi$ is a conjugate, then also $a - bi$ is a conjugate.

Given a field extension of algebraic numbers K, if the dimension $[K : \mathbb{Q}]$ is finite, we say that K is a number field. Moreover, for a number field K, there is a primitive element $\gamma \in K$ such that $\mathbb{Q}(\gamma) = K$. Since γ is the generator of K, every element of K can be written as $\sum_{i=0}^{deg(\gamma)-1} a_i \gamma^i$ for some $a_0, \ldots, a_{d-1} \in \mathbb{Q}$.

For example, a number field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ which is a number field generated by $\sqrt{2}$ and $\sqrt{3}$ is also generated by $\sqrt{2} + \sqrt{3}$.

In order to use the Subspace Theorem, we need to define several concepts. Namely embeddings, prime ideals, places, and Weil height.

Firstly, we define embeddings of a number field. Given a number field K, an embedding is an injective homomorphism $K \hookrightarrow \mathbb{C}$. If $\gamma_1, \ldots, \gamma_d$ are exactly all the roots of a minimal polynomial $m(X) \in \mathbb{Q}[X]$ related to the number field $K \cong \frac{\mathbb{Q}[X]}{(m(X))}$, then for any embedding $K \hookrightarrow \mathbb{C}$ is isomorphic to

$$\begin{aligned} \sigma_i : \mathbb{Q}(\gamma) &\hookrightarrow \mathbb{C} \\ \sum_{j=0}^{d-1} a_j \gamma^j &\mapsto \sum_{j=0}^{d-1} a_j \gamma_i^j \end{aligned}$$

where $i \in 1, \ldots, d$, $\gamma \in \gamma_1, \ldots, \gamma_d$ and $a_0, \ldots, a_{d-1} \in \mathbb{Q}$. Note that the set of embeddings is the same independently of the choice of γ. Also if $\gamma_i = \bar{\gamma}_j$, then $\sigma_i \cong \bar{\sigma}_j$. Hence, we have r real embeddings and s pairs of complex embeddings where r is the number of real roots of $m(X)$ and $2s$ is the number of complex roots of $m(X)$. Obviously, one of the embeddings (the one associated with $\gamma_i = \gamma$) is the identity map $x \mapsto x$. As we will see later, this induces the usual absolute value.

By $\mathcal{O}_K$, we mean the set of all numbers from K which are roots of some monic polynomial with integer coefficients; that is, the algebraic integers.

Before we use embeddings to define infinite places, let us define p-adic absolute values. By p-adic absolute value, we express the "occurrence" of the number p in a given rational number a. Formally, given a rational number $a = \frac{b}{d}$, $b, d \in \mathbb{Z}, d \neq 0$, we can write it uniquely as $a = \frac{b'}{d'}p^c$ where $b', d', c \in \mathbb{Z}, b' \nmid p, d' \nmid p$ for any prime p. Then, for p-adic absolute value, we have $|a|_p = p^{-c}$. For instance, $|14|_7 = \frac{1}{7}$, $|\frac{3}{40}|_2 = 8$, $|\frac{4}{9}|_5 = 1$.

The product formula states that for any rational x,

$$|x| \prod_{p \text{ prime}} |x|_p = 1.$$

This can be generalised into any number field but we need the concept of absolute values and places. For a number field K, an absolute value $|\cdot| : K \hookrightarrow \mathbb{R}_0^+$ is a map satisfying for every $x, y \in K$:

- $|x| = 0$ if and only if $x = 0$
- $|x||y| = |xy|$
- $|x + y| \leq |x| + |y|$.

Absolute values can be split into three groups: trivial, archimedean and non-archimedean. There is one special trivial absolute value that has $|a| = 1$ for all $a \neq 0$ and $|0| = 0$. Non-archimedean are those for which $|x + y| \leq \max(|x|, |y|)$. Archimedean are all other absolute values. An example of an archimedean absolute value is the standard absolute value and an example of a non-archimedean absolute value is any p-adic absolute value.

Two absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if there is $c \in \mathbb{R}^+$ such that $|a|_1 = |a|_2^c$ for all $a \in K$. An equivalence class over this equivalence relation is called a place. We do not include the trivial absolute value as a place. Places corresponding to non-archimedean absolute values are called finite, and places corresponding to archimedean absolute values are called infinite.

Ostrowski's Lemma states that for rational numbers, there are only two types of places: the place containing the usual absolute value and a place for every p-adic absolute value.

For a number field K, an absolute value from a finite place is equivalent to one of the form

$$|x|_{\mathfrak{p}} = N(\mathfrak{p})^{-ord_{\mathfrak{p}}(x)}$$

where $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$, $ord_{\mathfrak{p}}(x)$ is defined for every $x \in \mathcal{O}_K$ as the largest m such that $x \in \mathfrak{p}^m$, and $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$. The definition is extended multiplicatively, *i.e.*, $ord_{\mathfrak{p}}(x \cdot y) = ord_{\mathfrak{p}}(x) + ord_{\mathfrak{p}}(y)$.

Regarding infinite places for a number field K, an absolute value $|\cdot|_v$ from an infinite place is equivalent to an absolute value

$$|x|_v = |\sigma(x)|$$

where $\sigma : K \hookrightarrow \mathbb{C}$ is some embedding on K and $|\cdot|$ is the usual absolute value.

Let us denote $\mathcal{M}(K)$ the set of all places of a number field K. Since we have the freedom of choosing which absolute value is the representative of a place, we select them such that the product formula holds. The product formula states: for any number field K and any $x \in K$:

$$\prod_{v \in \mathcal{M}(K)} |x|_v = 1.$$

For every $x \in K$ and $v \in \mathcal{M}(K)$, we define representative absolute value $|x|_v$ as:

- $|x|_v := |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to a real embedding $\sigma : K \hookrightarrow \mathbb{R}$,
- $|x|_v := |\sigma(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to a conjugate pair of complex embeddings $\sigma : K \hookrightarrow \mathbb{C}$,
- $|x|_v := |N(\mathfrak{p})|^{-ord_{\mathfrak{p}}(x)/[K:\mathbb{Q}]}$ if v corresponds to a finite place related to a prime ideal $\mathfrak{p}$.

For a number field K and for any $m \in \mathbb{N}$, the Weil height $H : K^m \to K$ of a tuple $x = (x_1, x_2, \ldots, x_m)$ is

$$H(x) = \prod_{v \in \mathcal{M}(K)} \max(|x_1|_v, \ldots, |x_m|_v, 1).$$

The value $H(x)$ is independent of the choice of the number field K.

Let $p(X) \in \mathbb{Q}[X]$ and $b \in \mathbb{Q}$ be the leading coefficient of $p(X)$. The Mahler measure of $p(X)$ is defined as

$$M(p) := |b| \prod_{i=1}^{deg(p)} \max(1, |\beta_i|)$$

where $\beta_1, \ldots, \beta_{deg(p)} \in \mathbb{C}$ are all roots of $p(X)$.

Proposition 2.1. *Let K be a number field and $D \in \mathbb{R}$. For every polynomial $p(X) = a_tX^t + \cdots + a_1X + a_0$ with rational coefficients $|a_i| \leq D$ for every $i \in 0, \ldots, t$, and an algebraic number $\beta \in K$,*

$$H(p(\beta)) \leq t^{\deg(\beta)} \cdot C(\beta, D)^t$$

where $C(\beta, D)$ is a constant dependent only on β and D

Proof. Let M_0 be all finite places of $\mathcal{M}(K)$ and M_1 all infinite places of $\mathcal{M}(K)$.

$$\begin{aligned}
H(p(\beta)) &= \prod_{v \in \mathcal{M}(K)} \max(|p(\beta)|_v, 1) \\
&\leq \prod_{v \in M_0} \max(|a_t\beta^t|_v, \ldots, |a_1\beta^1|_v, |a_0|, 1) \\
&\quad \prod_{v \in M_1} (t+1) \cdot c \cdot \max(|\beta|_v, 1)^t \\
&\leq \prod_{v \in M_0} \max(|a_t|_v, \ldots, |a_0|_v) \cdot \max(|\beta|_v, 1)^t \\
&\quad \prod_{v \in M_1} (t+1) \cdot c \cdot \max(|\beta|_v, 1)^t \\
&\leq H(\beta)^t \cdot H(a_t, \ldots, a_0) \cdot ((t+1)c)^{deg(\beta)}
\end{aligned}$$

where $c \in \mathbb{R}$ is an upper bound $c := \max\{|a_i|_v \mid i \in 0, \ldots, t; v \in \mathcal{M}(K)\}$. The first inequality follows from the fact that $|x + y|_v \leq \max(|x|_v, |y|_v)$ for all finite v and $|a_i\beta^i|_v \leq c|\beta^t|_v$ for all $i \in 1, \ldots, t$. The last inequality follows from the definition of H and the fact that $|M_1| \leq deg(\beta)$. □

2.2 The Subspace Theorem

Among various tools for showing transcendence, the Subspace Theorem has proved to be very useful especially for infinite words with some repetition pattern. In this section we survey the history of its versions and reproduce some transcendence proofs along the way.

From Liouville to Ridout

The hunt for transcendental numbers was initiated by Liouville proving the following inequality:

Theorem 2.2 (Liouville 1844 [**21**]). *For each algebraic number* ξ *of a degree* $deg(\xi) \geq 2$*, there exists a number* c_ξ *such that*

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c_\xi}{q^{deg(\xi)}}$$

for every rational number p/q *with* $q \geq 1$*.*

As a direct consequence, if there is a number such that for every $d \in \mathbb{N}$, there exists p/q such that $|\xi - p/q| < 1/q^d$, then this number is necessarily transcendental[1].

The number Liouville constructed is the following:

$$\xi = \sum_{j=1}^{\infty} \frac{1}{10^{j!}} = 0.110001000000000000000010000\cdots$$

His proof worked as follows: For every $n \in \mathbb{N}$, we define an approximation of ξ as

$$\xi_n = \sum_{j=1}^{n} \frac{1}{10^{j!}} = \frac{P_n}{10^{n!}}$$

where $P_n \in \mathbb{N}$ is a number smaller than $10^{n!}$ such that

$$\xi - \xi_n = \xi - \frac{P_n}{10^{n!}} = \sum_{j=n+1}^{\infty} \frac{1}{10^{j!}} < \frac{2}{10^{(n+1)!}} = \frac{2}{(10^{n!})^{(n+1)}}.$$

Applying the contrapositive of Liouville's inequality, ξ is transcendental.

Following from these observations, one can be interested in the concept of a, so-called, *irrationality measure*. For a given number ξ, let R be a set of positive real numbers η such that there are only finitely many rational numbers p/q for which

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\eta}.$$

Then, the irrationality measure of ξ, denoted as $\eta(\xi)$ is the infimum of R. Observe that multiplying the right-hand side by some positive real constant does not affect the (in)finiteness of the set of suitable rational numbers p/q.

Liouville's inequality shows that if $\eta(\xi) = \infty$, then ξ is transcendental. Dirichlet's Approximation Theorem gives us that $\eta(\xi) \geq 2$ for every irrational number and $\eta(\xi) = 1$ for every rational number.

A major improvement in this area was provided by Roth:

[1]These numbers are referred to as Liouville's numbers.

Theorem 2.3 (Roth 1955 [**27**]). *Let ξ be an algebraic number. Then, for every $\epsilon > 0$ there are only finitely many rational numbers p/q such that*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

Roth's theorem implies that $\eta(\xi) = 2$ for any algebraic irrational number ξ. Using Roth's theorem, we are able to prove transcendence of the following number:

$$\chi = \sum_{j=1}^{\infty} \frac{1}{10^{3^j}}.$$

For every n, we define

$$\chi_n = \sum_{j=1}^{n} \frac{1}{10^{3^j}} = \frac{P_n}{10^{3^n}}$$

for suitable $P_n \in \mathbb{Z}$ such that

$$\chi - \chi_n = \chi - \frac{P_n}{10^{3^n}} = \sum_{j=n+1}^{\infty} \frac{1}{10^{3^n}} < \frac{2}{10^{3^{n+1}}} = \frac{2}{(10^{3^n})^3}.$$

Intuitively, Roth's theorem says that when a number is "well approximable" by rational numbers (with an exponent greater than 2), it is transcendental.

However, even with Roth's Theorem, it is unclear whether numbers with irrationality measure equal to 2 are transcendental or algebraic. For instance,

$$\vartheta = \sum_{j=1}^{\infty} \frac{1}{10^{2^j}}.$$

In this case, a p-adic extension of Roth's Theorem turns out to be useful.

The p-adic version of Roth's Theorem was introduced in 1957 by Ridout. The basic idea is to strengthen the theorem in the case any of the integers p, q which we use to approximate number ξ is made up of only infinitely many primes. Formally,

Theorem 2.4 (Ridout 1957 [**25**]). *Given an algebraic number ξ, a positive real number ϵ, and a finite set of primes S, there is only a finite number of rational numbers p/q such that*

$$\left(\prod_{v\in S} |p|_v \cdot |q|_v\right) \cdot \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

If we set $S = \{2, 5\}$ and we choose to approximate ϑ by $\sum_{j=1}^{n} \frac{1}{10^{2^j}} = \frac{P_n}{10^{2^n}}$, then

$$|P_n|_2 \cdot |10^{2^n}|_2 \cdot |P_n|_5 \cdot |10^{2^n}|_5 \cdot \left|\vartheta - \frac{P_n}{10^{2^n}}\right| < \frac{1}{10^{2^n}} \cdot \frac{2}{10^{2^{n+1}}} = \frac{2}{(10^{2^n})^3}.$$

Schlickewei's Subspace Theorem

We present a major generalisation of p-adic Roth's Theorem by Schlickewei and then his full generalisation to all number fields. We refer to the former as the basic Subspace Theorem and to the latter as the Subspace Theorem.

Theorem 2.5 (Schlickewei 1976 [**28**]). *Let S be a finite set of primes containing ∞ and let $m \in \mathbb{N}$. For every $p \in S$ let $L_{1,p}, L_{2,p}, \ldots, L_{m,p}$ be linearly independent linear forms in m variables with algebraic coefficients. Then, for any positive real ϵ the solutions $\boldsymbol{x} \in \mathbb{Z}^m$ to the inequality*

$$\prod_{p\in S}\prod_{i=1}^{m} |L_{i,p}|_p \leq (max(|x_1|, |x_2|, \ldots, |x_m|))^{-\epsilon} = H(\boldsymbol{x})^{\epsilon}$$

are contained in finitely many proper linear subspaces of $\mathbb{Q}^m$.

By linear form in m variables with algebraic coefficient, we mean $L(\boldsymbol{x}) = c_1x_1 + \ldots, c_mx_m$ where $c_1, \ldots, c_m$ are all algebraic. Linearly independent means that there are no algebraic $a_1, \ldots, a_m$ such that $a_1L_1 + \cdots + a_mL_m = 0$ and $(a_1, \ldots, a_m)$ is non-zero. Also, by $|\cdot|_\infty$, we mean the usual absolute value $|\cdot|$ and by $|\cdot|_p$, we mean p-adic absolute value on $\mathbb{Q}$.

Now, we are ready to state the full version of the Subspace Theorem which we use in this thesis. We work in a number field K. By $\mathcal{O}_S$, we mean all S-integers which is the set of all $\gamma \in K$ such that $|\gamma|_v \leq 1$ for all finite places $v \notin S$, where S is a set of places.

Theorem 2.6 (Schlickewei 1977 [**29**]). *Let $S \subseteq \mathcal{M}(K)$ be a finite set of places containing all infinite places, and let $m \geq 2$. For every $v \in S$, let $L_{1,v}, \ldots, L_{m,v}$ be linearly independent linear forms in m variables with algebraic coefficients. Then, for every positive real ϵ, the solutions $\boldsymbol{x} \in \mathcal{O}_S^m$ to the inequality*

$$\prod_{v\in S}\prod_{i=1}^{m} |L_{i,v}|_v \leq H(\boldsymbol{x})^{-\epsilon}$$

are contained in finitely many proper linear subspaces of K^m.

2.3 Transcendence of Numbers in an Integer Base

In the previous section, we stated the main tools for proving transcendence of sequences with some repetitive structure. This section surveys particular theorems about transcendence of numbers related to sequences in an integer base.

In the pioneering work by Ferenczi and Mauduit [**16**], they proved that any Sturmian word interpreted in any integer base is a transcendental number. The idea is similar to the one we used in the previous section. We are looking for some regular patterns that we use to construct "good enough" rational approximations. In this and all of the following proofs, the structure of the proof is as follows: 1) we prove our class of sequences (*e.g.*, Sturmian words) satisfy some combinatorial criterion, 2) we show that the combinatorial criterion is sufficient to construct "good enough" rational approximations and hence to prove transcendence.

Ferenczi and Mauduit defined a combinatorial criterion that is sufficient to prove transcendence for an integer base. Later, Adamczewski, Bugeaud and Luca provided similar but stronger criterion [**4**]. Here, we state the latter one and prove the transcendence result related to it.

Definition 2.7. Let $\boldsymbol{u} = u_1u_2\cdots \in \Sigma^\omega$ be an infinite sequence over a finite alphabet Σ. If there is real positive $\delta > 1$ and infinitely many words $v_n, w_n \in \Sigma^*$ such that

(i) $v_n w_n^\delta$ is a prefix of $\boldsymbol{u}$

(ii) $\langle |w_n| \rangle_{n=1}^\infty$ is strictly increasing

(iii) $\langle |v_n|/|w_n| \rangle_{n=1}^\infty$ is bounded from above

then $\boldsymbol{u}$ is called *stammering*[2].

If w is a word and $c \in \mathbb{R}^+$, by w^c we mean $w^{\lfloor c \rfloor}v$ where v is the largest prefix of w such that $|v| \leq (c - \lfloor c \rfloor)|w|$.

Theorem 2.8. *Let $b \in \mathbb{N}$, $b \geq 2$ and $\boldsymbol{u} = u_1u_2\cdots$ be an infinite non-eventually periodic stammering sequence over a finite alphabet $\{1, \ldots, b-1\}$. Then, the number $\alpha := \sum_{j=1}^\infty u_j b^{-j}$ is transcendental.*

Proof. There is a simple proof of this using the basic Subspace Theorem. Let $r_n := |v_n|$, and $s_n := |w_n|$ for all $n \in \mathbb{N}$. We define α_n to be a rational number approximating α such that its b-expansion $\boldsymbol{v}_n$ satisfies: first r_n digits of $\boldsymbol{v}_n$ are identical to the first r_n digits of $\boldsymbol{u}$ and the next s_n digits of $\boldsymbol{v}_n$ – which we set to be the same as the next s_n digits of $\boldsymbol{u}$ – repeat to infinity. Using (i) from 2.7, we know that $\boldsymbol{v}_n$ and $\boldsymbol{u}$ start to differ at index $r_n + s_n\delta$ which we exploit to prove transcendence of α.

To illustrate, for every $n \in \mathbb{N}$, we have

$$\boldsymbol{v}_n = 0.u_1u_2\cdots u_{r_n}u_{r_n+1}\cdots u_{r_n+s_n}u_{r_n+1}u_{r_n+2}\cdots u_{r_n+s_n}u_{r_n+1}\cdots$$

[2]This name was used in [**4**].

and $\alpha_n = \sum_{j=1}^{\infty} v_j b^{-j}$ where v_j is the jth digit of $\boldsymbol{v}_n$. This is also equal to

$$\alpha_n = \frac{P_n}{b^{r_n}(b^{s_n} - 1)}$$

where $P_n := \sum_{j=0}^{r_n+s_n-1} u_{r_n+s_n-j} b^j$.

As we said above, for every $n \in \mathbb{N}$, (i) of 2.7 gives us that

$$\alpha - \alpha_n < \frac{1}{b^{r_n+s_n\delta}}. \tag{2.1}$$

Now, we are ready to use the basic Subspace Theorem. We set S to be the set of prime divisors of b along with the usual absolute value ∞. Set $m = 3$ and linear forms as follows: For every $v \in S - \{\infty\}$, let:

$$L_{1,v} = x_1, L_{2,v} = x_2, L_{3,v} = x_3.$$

For usual absolute value, we have

$$L_{1,\infty} = \alpha x_1 - \alpha x_2 - x_3, L_{2,\infty} = x_2, L_{3,\infty} = x_3.$$

Obviously, linear forms $L_{1,v}, L_{2,v}, L_{3,v}$ are linearly independent for every $v \in S$. Also, their coefficients are all algebraic since we assume α is algebraic. This will lead us to a contradiction.

Next, we try to prove that

$$\prod_{v \in S} \prod_{i \in 1,2,3} |L_{i,v}(\boldsymbol{a}_n)|_v < H(\boldsymbol{a}_n)^{-\epsilon}$$

for some positive real ϵ, where $\boldsymbol{a}_n = (b^{r_n+s_n}, b^{r_n}, P_n)$.

Since S contains all prime divisors of b, we can apply the product formula to get

$$\prod_{v \in S} |L_{1,v}(\boldsymbol{a}_n)| \leq 1$$

and similarly

$$\prod_{v \in S} |L_{2,v}(\boldsymbol{a}_n)| \leq 1.$$

For the third variable, we have

$$|L_{3,\infty}(\boldsymbol{a}_n)| = |b^{r_n+s_n}\alpha - b^{r_n}\alpha - P_n| = |\alpha(b^{r_n}(b^{s_n} - 1)) - P_n| < \frac{1}{b^{(\delta-1)s_n}}$$

where the last equality is obtained using inequality (2.1). Furthermore, for every $v \in S - \{\infty\}$, we have $|L_{3,v}(\boldsymbol{a}_n)| \leq 1$ which altogether gives

$$\prod_{v \in S \cup \{\infty\}} \prod_{i \in 1,2,3} |L_{i,v}|_v < \frac{1}{b^{(\delta-1)s_n}}$$

From the definition of H and since $\boldsymbol{a}_n$ consists only of integers, we have that $H(\boldsymbol{a}_n) \leq |b|^{r_n+s_n}$ for all n. Using the property (iii) of 2.7, there exists $D > 0$ such that $r_n/s_n < D$ for all n. Hence, $H(\boldsymbol{a}_n) \leq |b|^{(D+1)s_n}$ This gives us

$$\prod_{v \in S \cup \{\infty\}} \prod_{i \in 1,2,3} |L_{i,v}|_v < \frac{1}{b^{(\delta-1)s_n}} \leq H(\boldsymbol{a}_n)^{-\epsilon}$$

where $\epsilon = (\delta - 1)/(D + 1)$.

Applying the basic Subspace Theorem, we are given finitely many non-zero linear forms with rational coefficients such that for every n, $\boldsymbol{a}_n$ vanishes on at least one of them. Therefore, there is some linear form L for which there are infinitely many $\boldsymbol{a}_n$ which vanish on L. Let $L(\boldsymbol{x}) = z_1x_1 + z_2x_2 + z_3x_3$, $z_1, z_2, z_3 \in \mathbb{Q}$. For every $\boldsymbol{a}_n$, we divide $L(\boldsymbol{a}_n)$ by $b^{r_n+s_n}$ and using property (ii) of 2.7, we look at the value in limit.

$$\lim_{n\to\infty} L(\boldsymbol{a}_n)b^{-(r_n+s_n)} = z_1 + z_2 \lim_{n\to\infty} \frac{1}{b^{s_n}} + z_3 \lim_{n\to\infty} \frac{P_n}{b^{r_n+s_n}} = z_1 + z_3\alpha = 0. \qquad (2.2)$$

However, the last equality is possible only if α is equal to $\frac{-z_1}{z_3}$ which is a rational number. But since our base b is an integer and every rational number in an integer base is ultimately periodic, α would need to have ultimately periodic form which is not our case. Therefore, α is transcendental. □

Sturmian and Arnoux-Rauzy words

Once we have established transcendence of numbers related to stammering sequences, we can apply this knowledge to some known classes of sequences.

The first application of this result was done by Ferenczi and Maduit in [**16**] where they proved that all Sturmian words are stammering sequences. We indicate the proof for standard Sturmian words since for non-standard Sturmian words we just need to apply a suitable shift.

Firstly, let us recall that for a standard Sturmian word $\boldsymbol{u} = u_1u_2 \cdots \in \Sigma$ over a binary alphabet Σ, there exists a sequence of words $w_0, w_1, \ldots \in \Sigma^*$ and a sequence of numbers $s_1, s_2, \ldots \in \mathbb{N}$ such that

$$w_{n+1} = w_n^{s_n} w_{n-1}$$

for every $n \in \mathbb{N}$, w_n is a prefix of $\boldsymbol{u}$ for every $n \in \mathbb{N}$, and $\{w_0, w_1\} = \{0, 1\}$.

Applying the recursive formula twice, $w_nw_{n-1}w_n$ is a prefix of w_{n+2} which is a prefix of $\boldsymbol{u}$, for all $n \geq 2$. Since $|w_n|/|w_nw_{n-1}| \geq \frac{1}{2}$ we can set $u_n := \varepsilon, v_n := w_nw_{n-1}$ and $\delta := \frac{1}{2}$ which fits the definition of a stammering sequence.

A similar argument can be also made for Arnoux-Rauzy words proving that they are stammering.

Sequences with linear subword complexity

In [**3**], Adamczewski, Bugeaud and Luca proved that if the b-expansion of a number η has linear subword complexity then it is transcendental; $b \in \mathbb{N}, b \geq 2$. A sequence has linear subword complexity if there exists a constant $C \in \mathbb{N}$ such that the subword complexity $\mathsf{sc}(n) \leq Cn$ for all $n \in \mathbb{N}$.

They showed that for every $n \in \mathbb{N}$, if we take the prefix of $\boldsymbol{u}$ of length $(C+1)n$, then there is a subword of length n which occurs twice in the prefix. Precisely, there are words $U_n, V_n, W_n, X_n, Y_n, \in \Sigma^*$ such that

$$U_n V_n X_n Z_n = U_n Y_n V_n Z_n$$

and $|U_n V_n X_n Z_n| = (C+1)n$, $|V_n| = n$, $|Y_n| > 0$. We are trying to find u_n, v_n satisfying the properties (i) and (iii) of 2.7 for all n and some constant δ (condition (ii) is satisfied from the fact that we can do this for any $n \in \mathbb{N}$).

In [**3**], they split the cases as follows:

1. If $|X_n| \geq |V_n|$, then there is a word A_n such that $U_n V_n X_n Z_n = U_n V_n A_n V_n Z_n$. Hence, we can set $u_n := U_n$ and $v_n := V_n A_n$.

2. If $|V_n| > |X_n| \geq |V_n|/3$, then there are words A_n, B_n such that $U_n V_n X_n Z_n = U_n V_n^{1/3} A_n V_n^{1/3} A_n B_n$. Then, $u_n := U_n$ and $v_n := V_n^{1/3} A_n$.

3. If $|V_n|/3 > |X_n|$, then $|V_n|$ starts with $|X_n|$ repeating multiple times which we use to find suitable u_n and v_n.

In all cases, we can prove that $|u_n||v_n|^{1+1/C}$ is a prefix of $\boldsymbol{u}$ and $|u_n|/|v_n| < C/4$ for all $n \in \mathbb{N}$. Therefore, we can fit it into the above argument to get a transcendence result.

This is a groundbreaking result that implies transcendence of all irrational automatic numbers since their subword complexity is linear.

2.4 Transcendence of Numbers in an Algebraic Base

After obtaining the above results, an immediate question is raised regarding transcendence of sequences in non-integer bases. We work in an algebraic base β. Let $\mathcal{H}$ be a finite set of integers and $\boldsymbol{u} = u_1 u_2 \cdots \in \mathcal{H}^\omega$. Then, $\boldsymbol{u}$ is associated with the number

$$\alpha := \sum_{j=1}^{\infty} u_j \beta^{-j}.$$

Before stating results for an algebraic base, let us explain why it is not obvious to generalise results from the previous section to any base. First of all, we realise that we cannot use the basic Subspace Theorem since it does not allow us to have non-integer values for a_n. Therefore, we use the full version of the Subspace Theorem. There, we work over a number field K related to the base β. In order for us to use the product formula, we choose the set of finite places to contain all divisors of β. The Subspace Theorem also forces us to use all infinite places. We define all linear forms as previously and we get to evaluate the linear forms on a_n which we define identically. We obtain $\prod_{v\in S}|L_{i,v}(a_n)| = 1$ for $i \in 1, 2$ using the generalised product formula. But once we try to bound $|L_{3,v}|_v$ for some non-standard absolute value v, we notice that it might be greater than 1.

To illustrate this, imagine we have $\beta = \frac{60}{47}$ and we try to evaluate it on p-adic absolute value for $p = 47$. Then, $|\beta|_{47} = 47$ which is much greater than β. Therefore, it is not at all clear whether these absolute values are small enough so that the overall product is sufficiently small. We give a short survey on how this problem was avoided.

Pisot and Salem numbers

A number β is *Pisot* if it algebraic integer, $|\beta| > 1$ and $|\beta_i| < 1$ for all its conjugates β_i except β. A number β is *Salem* number if it algebraic integer, $|\beta| > 1$ and $|\beta| \leq 1$ for all its conjugates β_i except β and there is at least one conjugate β_j such that $|\beta_j| = 1$.

Some partial results were obtained in [**3**] by Adamczewski and Bugeaud restricting the base to be a Pisot or a Salem number. We restate their results here

Theorem 2.9. *Let β be a Pisot or a Salem number and $\boldsymbol{u} = u_1u_2\cdots \in \mathcal{H}^\omega$ a stammering sequence. Then the number $\alpha := \sum_{j=1}^{\infty} u_j|\beta|^{-j}$ is either transcendental or belongs to $\mathbb{Q}(\beta)$.*

We continue as we started at the beginning of this section. Since β is an algebraic integer (since it is Pisot or Salem) and all coefficients are rational integers, then $|P_n|_v \leq 1$ for any finite place v. In the case of infinite places, one can use the properties of absolute values to obtain

$$
\begin{aligned}
|P_n|_v &\leq \sum_{i=0}^{r_n+s_n} |c_i\beta^i|_v \\
&\leq \sum_{i=0}^{r_n+s_n} D && \text{since}|\beta_j| \leq 1 \text{ for all conjugates } \beta_j \text{ of } \beta \\
&\leq (r_n+s_n)D && \text{definition of } D
\end{aligned}
$$

for some integers $c_1, \ldots, c_{r_n+s_n}$ where D is the upper bound $D := \max(|c|_v \mid c \in \mathbb{Z}, |c| \leq \mathcal{H}-\mathcal{H}, v \in \mathcal{M}(\mathbb{Q}(\beta)))$ where $\mathcal{H}-\mathcal{H} = \{a-b \mid a, b \in \mathcal{H}\}$. Putting it all together, we get

$$\prod_{v\in S}\prod_{i\in 1,2,3} |L_{i,v}|_v \leq ((r_n+s_n)D)^{deg(\beta)}|\beta|^{-(\delta-1)s_n}.$$

Using the Proposition 2.1, we can find positive real ϵ such that $H(a_n)^\epsilon \leq \beta^{(\delta-1)s_n/2}$. Since $\delta - 1 > 0$, then for big enough $n \in \mathbb{N}$ we have

$$\prod_{v\in S}\prod_{i\in 1,2,3} |L_{i,v}|_v \leq H(a_n)^{-\epsilon}.$$

Hence, we can apply the Subspace Theorem which gives us that all points $a_n = (\beta^{r_n+s_n}, \beta^{r_n}, P_n(\beta))$ lie in finitely many proper linear subspaces of $\mathbb{Q}(\beta)^3$. Hence, there is a non-zero linear form $L(x) = z_1x_1 + z_2x_2 + z_3x_3$ such that $L(a_n) = 0$ for infinitely many n. Taking n to infinity and dividing every equation by $\beta^{r_n+s_n}$, we get

$$\lim_{n\to\infty}\frac{L(a_n)}{\beta^{r_n+s_n}} = z_1 + z_3\alpha = 0$$

which implies that α belongs to $\mathbb{Q}(\beta)$ or it is transcendental if the initial assumption is false.

Theorem 2.10. *Let β be a Pisot number and $\boldsymbol{u} = u_1u_2\cdots \in \mathcal{H}^\omega$ a stammering sequence. Then the number $\alpha := \sum_{j=1}^{\infty} u_j|\beta|^{-j}$ is transcendental.*

The only difference to the previous proof is that in the very last part when we are given $z_1, z_3 \in \mathbb{Q}(\beta)$ such that $z_1 + z_3\alpha = 0$ we can conclude a contradiction since every Pisot number belonging to $\mathbb{Q}(\beta)$ is ultimately periodic but $\boldsymbol{u}$ is not ultimately periodic.

Diophantine Exponent

Another approach proposed in [**2**] make use of, so-called, *diophantine exponent.* Let us recall the properties of Definition 2.7. Obviously, the smaller the δ in the criterion, the shorter the repetition and therefore the weaker the ability to exploit this repetition. In the case of an integer base, it is sufficient to have any positive real $\delta > 1$. However, [**2**] showed that for an algebraic base β, if the δ is big enough, we can prove a transcendence result.

Definition 2.11. Let $\mathsf{dio}(\boldsymbol{u})$ be the supremum of δ such that $\boldsymbol{u}$ is a stammering sequence for this δ.

The scope of $\mathsf{dio}(\boldsymbol{u})$ is $1 \leq \mathsf{dio}(\boldsymbol{u}) \leq +\infty$. The concept of diophantine exponent is not only helpful as the unifying interface for previously stated theorems but also allows us to prove interesting things about sequences in an algebraic base β.

In [**2**], Adamczewski and Bugeaud proved the following:

Theorem 2.12. *Let β be an algebraic number $|\beta| > 1$ and $\boldsymbol{u} = u_1 u_2 \cdots$ an infinite sequence of bounded rational integers. If*

$$\mathsf{dio}(\boldsymbol{u}) > \frac{\log M(\beta)}{\log |\beta|},$$

then the real number

$$\alpha := \sum_{j=1}^{\infty} u_j \beta^{-j}$$

lies in $\mathbb{Q}(\beta)$ or is transcendental.

By $M(\beta)$ we mean the Mahler measure of β. This theorem initiates researching the diophantine exponent for various infinite sequences because, for example, if an infinite sequence has an infinite diophantine exponent then the result holds for any $|\beta| > 1$.

For an episturmian word directed by $\Delta = d_1^{a_1} d_2^{a_2} \cdots \in \Sigma^\omega$ where $a_1, a_2, \ldots, \in \mathbb{N}$, Peltomäki proved that the sequence has $\mathsf{dio}(\boldsymbol{u}) = \infty$ if and only if the sequence $a_1, a_2, \ldots$ is unbounded [**24**]. Therefore, with the theorem above, there are still plenty of episturmian sequences $\boldsymbol{u}$ and bases β such that it is unclear whether the word related to $\boldsymbol{u}$ in base β is transcendental, lies in $\mathbb{Q}(\beta)$ or elsewhere.

Finitely many discrepancies

A completely new approach was proposed in [**22**] by Luca, Ouaknine, and Worrell. They noticed that given a Sturmian word $\boldsymbol{u}$, if we define periodic words $\boldsymbol{v}_n$ as in the section 2.3 then $\boldsymbol{u}$ and $\boldsymbol{v}_n$ differ in constantly many indices every $|v_n|$ where v_n is the period of $\boldsymbol{v}_n$. In other words, a Sturmian word has an infinite diophantine exponent if the repetition is allowed constantly many mismatches. Since our main contribution in this thesis is a generalisation of this idea, let us explore this in detail.

First, we define, the combinatorial criterion: An infinite sequence $\boldsymbol{u} = u_1 u_2 \cdots \in \Sigma^\omega$ over a finite alphabet Σ is called *stuttering* if for every $m \in \mathbb{N}$ there exists $d \in \mathbb{N}$, an increasing sequence of integer $\langle r_n \rangle_{n=1}^{\infty}$, and sequence of integers $\langle s_n \rangle_{n=1}^{\infty}$ such that

$$u_{r_n+1} u_{r_n+2} \cdots u_{r_n+ms_n} \text{ and } u_{r_n+s_n+1} u_{r_n+s_n+2} \cdots u_{r_n+(m+1)s_n}$$

differ in at most d indices and r_n/s_n is bounded by a constant.

The reason why this is similar to $\mathsf{dio}(\boldsymbol{u}) = \infty$ is because m can be made arbitrarily big. That is why we can prove a transcendence result for arbitrary algebraic base β, $|\beta| > 1$.

To illustrate why there is only a constant number of discrepancies, let us take the Fibonacci word and compare it with itself shifted by 5:

$$\begin{aligned} \boldsymbol{u} =& 01001010010010100101001001010010010100101001001010010 \cdots \\ T^5(\boldsymbol{u}) =& 01001001010010100100101001001010010100100101001010010 \cdots \\ \boldsymbol{u} \ominus T^5(\boldsymbol{u}) =& 000000\tfrac{1}{0}\tfrac{0}{1}00000000000\tfrac{1}{0}\tfrac{0}{1}000000\tfrac{1}{0}\tfrac{0}{1}00000000000\tfrac{1}{0}\tfrac{0}{1}00000000000 \cdots \end{aligned}$$

We notice that there are at most two mismatches every 8 indices. In general, if we shift the Fibonacci word $\boldsymbol{u}$ by some Fibonacci number F_n and compare it with $\boldsymbol{u}$, then in every consecutive F_{n+1} indices, there are at most two mismatches. We explain this behaviour in the next chapter.

Now, we show that we fit the Subspace Theorem for our purposes. We introduce d new variables which correspond to d mismatches. Thus, we choose the number of linear forms m to be $3+d$, S to be as usual and linear forms to be defined as $L_{i,v} = x_i$ for every $i \in 1, \ldots, 3+d$, $v \in S$ except $(i, v) = (1, v_0)$ where v_0 is the usual absolute value. In that case, we set $L_{1,v_0} = x_1 - x_2 - \cdots - x_{3+d}$. Let

$$\boldsymbol{a}_n = (\beta^{r_n+s_n}, \beta^{r_n}, P_n, \beta^{-t_{n,1}}, \ldots, \beta^{-t_{n,d}})$$

where P_n is defined as usual and $t_{n,1}, \ldots, t_{n,d} \in \mathbb{N}$ are indices of mismatches between

$$u_{r_n+1} \cdots u_{r_n+ms_n} \ominus u_{r_n+s_n+1} \cdots u_{r_n+(m+1)s_n}$$

in increasing order. Observe that using the criterion from the definition of stuttering sequences, we have

$$|L_{1,v_0}(\boldsymbol{a}_n)| = |\beta^{r_n+s_n}\alpha - \beta^{r_n}\alpha - P_n(\beta) - \sum_{i=1}^{d} \beta^{-t_{n,i}}| < \beta^{-r_n+ms_n}. \tag{2.3}$$

This is great since we can make the right hand side arbitrarily small in terms of multiples of s_n. However, we are concerned about the absolute values of the new linear forms in the overall product

$$\prod_{v\in\mathcal{M}(K)} \prod_{i\in 1,\ldots,3+k} |L_{i,v}(\boldsymbol{a}_n)|_v.$$

Fortunately, we can use the product formula to get

$$\prod_{v\in S} |\beta|^{-i_{n,i}} = 1$$

for any $n \in \mathbb{N}$ and any $i \in 1,\ldots,d$. Therefore, the additional d variables do not contribute to the overall product and since we can choose m to be arbitrarily big, we obtain the following:

Theorem 2.13. *Let β be an algebraic number $|\beta| > 1$ and $\boldsymbol{u} = u_1u_2\cdots \in \mathcal{H}$ a stuttering sequence over a bounded set of integers $\mathcal{H}$. Then, the number $\alpha := \sum_{j=1}^{\infty} u_j\beta^{-j}$ is transcendental or belongs to $\mathbb{Q}(\beta)$.*

In [**22**], they extended this result in two ways. First of all, they allowed the set of digits to be arbitrary algebraic numbers. This changes the proof only slightly since we change the number field from $\mathbb{Q}(\beta)$ to $\mathbb{Q}(\beta,\mathcal{H})$.

The other extension they provide is that they prove that α cannot lie in $\mathbb{Q}(\beta,\mathcal{H})$ under some other restrictive conditions applicable to Sturmian words. The basic idea is that if α belongs to $\mathbb{Q}(\beta,\mathcal{H})$, then the value of $\sum_{i=1}^{d}\beta^{-t_{n,i}}$ would need to be very small which is impossible for infinitely many $n \in \mathbb{N}$.

2.5 Echoic Sequences

In this section, we define echoic sequences which is a generalisation of stuttering sequences and prove Theorem 2.15 which is one of the main contributions of this thesis. An application of this result to episturmian words is presented in the next chapter.

Let us recall that a finite word $w \in \mathcal{H}^*$ is associated with a polynomial and an infinite word $\boldsymbol{w} \in \mathcal{H}^\omega$ is associated with a Laurent series. By w_β or $\boldsymbol{w}_\beta$, we mean $p(\beta)$ where $p(X)$ is the related polynomial or Laurent series, respectively.

Definition 2.14. Let Σ be a finite alphabet. An infinite sequence $\boldsymbol{u} = u_1 u_2 \cdots \in \Sigma^\omega$ is said to be *echoic* if for every $m \in \mathbb{N}$ and positive real ρ, there exist sequences $\langle r_n \rangle_{n=0}^\infty$ and $\langle s_n \rangle_{n=0}^\infty$ of positive integers and an integer $d \in \mathbb{N}$ such that:

(i) s_n is strictly increasing,

(ii) $r_n / s_n \leq C$ for some constant $C \in \mathbb{N}$,

(iii) For all $n \in \mathbb{N}$, there exist $x_{n,1}, \ldots, x_{n,d}, y_{n,1}, \ldots, y_{n,d} \in \mathbb{N}$ such that the set of indices in which equal-length words

$$u_1 \cdots u_{r_n + m s_n} \text{ and } u_{s_n+1} \cdots u_{r_n + (m+1) s_n}$$

differ is a subset of $[\![1, r_n]\!] \cup \bigcup_{i=1}^d [\![x_{n,i}, y_{n,i}]\!]$ where $1 \leq x_{n,i} \leq y_{n,i} \leq r_n + m s_n$ for all $i \in 1, \ldots, d$, and

$$\frac{\sum_{i=1}^d y_{n,i} - x_{n,i}}{m s_n} < \rho.$$

Intuitively, a prefix of an echoic sequence $\boldsymbol{u}$ is "similar" to $U_n V_n^m$ where $U_n V_n$ is a prefix of $\boldsymbol{u}$ such that $|U_n| = r_n$ and $|V_n| = s_n$. By similar, we mean that there are at most $\rho \cdot m s_n$ mismatches grouped in d intervals.

We can assume that $y_{n,i} \leq x_{n,i+1}$ for all $i = 1, \ldots, d-1$. For every $n \in \mathbb{N}$, let us denote

$$\boldsymbol{v}^{(n)} = T^{s_n}(u) = v_1^{(n)} v_2^{(n)} \cdots$$

where $\boldsymbol{v}^{(n)} \in \Sigma^\omega, v_1^{(n)}, v_2^{(n)}, \ldots, \in \Sigma$.

For a given m, ρ, we set

$$\delta_{m,\rho,n,i}(X) := \sum_{j=x_{n,i}}^{y_{n,i}} (v_j^{(n)} - u_j) X^{-j}$$

for all $n \in \mathbb{N}$ and $i \in 1, \ldots, d$. We omit writing m and ρ if they are clear from the context.

Theorem 2.15. *Let $\boldsymbol{u} = u_1 u_2 \cdots \in \mathcal{H}^\omega$ be an echoic sequence over a finite set of algebraic numbers $\mathcal{H}$. For an algebraic number β with $|\beta| > 1$, the number $\alpha := (\boldsymbol{u})_\beta = \sum_{j=0}^\infty u_j \beta^{-j}$ is transcendental or belongs to $\mathbb{Q}(\beta, \mathcal{H})$.*

Proof. We suppose that α is algebraic and we prove that it belongs to $\mathbb{Q}(\beta, \mathcal{H})$. Let $K := \mathbb{Q}(\beta, \mathcal{H})$ be the field generated by β and numbers from $\mathcal{H}$. Let S be the set of places of K comprising of all infinite places and all finite places coresponding to prime divisors of β and prime divisors of $\mathcal{H}$. Let $v_0 \in S$ denote the infinite place corresponding to the identity embedding $K \hookrightarrow \mathbb{C}$.

Let $c \in \mathbb{N}$ such that $c \geq |\sum_{j=1}^{\infty} a_j \beta^{-j}|$ for any $a_1, a_2, \ldots \in \mathcal{H}$. This always exist since $\mathcal{H}$ is finite. Applying the property (iii) of 2.14 (for values of m and ρ to be determined later), we obtain a number $d \in \mathbb{N}$, sequences $\langle r_n \rangle_{n=0}^{\infty}$, $\langle s_n \rangle_{n=0}^{\infty}$ and for each $n \in \mathbb{N}$ a set of intervals $[\![x_{n,1}, y_{n,1}]\!], \ldots, [\![x_{n,d}, y_{n,d}]\!]$ such that:

$$|\beta^{r_n+s_n}\alpha - \beta^{r_n}\alpha - P_n(\beta) - \sum_{i=1}^{d} \delta_{n,i}(\beta)| < c|\beta|^{-ms_n}, \tag{2.4}$$

where $P_n(\beta) := \sum_{j=1}^{s_n} u_j \beta^{r_n+s_n-j} + \sum_{j=s_n+1}^{r_n+s_n} (u_j - u_{j-s_n}) \beta^{r_n+s_n-j}$. To visualise, let us look at the difference $\beta^{r_n+s_n}\alpha - \beta^{r_n}\alpha = \beta^{r_n+s_n}(\alpha - \alpha\beta^{-s_n})$:

$$\begin{array}{rl} \alpha = & (u_1 u_2 \cdots u_{s_n} u_{s_n+1} \cdots u_{s_n+r_n} \cdots u_{s_n+x_1} \cdots u_{s_n+y_1} \cdots u_{s_n+x_d} \cdots u_{s_n+y_d} \cdots)_\beta \\ \alpha\beta^{-s_n} = & (0 \quad \cdots \quad 0 \quad u_1 \quad \cdots u_{r_n} \quad \cdots u_{x_1} \quad \cdots u_{y_1} \quad \cdots u_{x_d} \quad \cdots u_{y_d} \quad \cdots)_\beta \\ \alpha - \alpha\beta^{-s_n} = & \qquad P_n(\beta) \qquad + \quad \delta_{n,1}(\beta) \quad + \cdots + \quad \delta_{n,d}(\beta) \quad \cdots \end{array}$$

We set linear forms $L_{i,v}$ for every $i \in 1, \ldots, 3+d$ and $v \in S$ as

$$L_{i,v}(x_1, \ldots, x_{3+d}) := x_i, \text{ for all } (i,v) \neq (1, v_0) \tag{2.5}$$

$$L_{1,v_0}(x_1, \ldots, x_{3+d}) := \alpha x_1 - \alpha x_2 - x_3 - \sum_{i=1}^{d} x_i. \tag{2.6}$$

It is easy to see that $L_{1,v}, L_{2,v}, \ldots, L_{3+d,v}$ are linearly independent for every $v \in S$.[3] Let

$$\boldsymbol{a}_n := (\beta^{r_n+s_n}, \beta^{r_n}, P_n(\beta), \delta_{n,1}(\beta), \ldots, \delta_{n,d}(\beta)).$$

Let us define $M \geq 2$ to be an upper bound of the set $\{|a|_v : v \in S, a \in \mathcal{H} \cup (\mathcal{H} - \mathcal{H}) \cup \{\beta, \beta^{-1}\}\}$ where $\mathcal{H} - \mathcal{H} = \{a - b \mid a, b \in \mathcal{H}\}$. Since S contains all finite places corresponding to divisors of β, we can apply the product formula to obtain

$$|\beta^{r_n+s_n}|^{1/deg(\beta)} \cdot \prod_{v \in S/v_0} |L_{1,v}(\boldsymbol{a}_n)|_v = 1 \qquad \text{and} \qquad \prod_{v \in S} |L_{2,v}(\boldsymbol{a}_n)|_v = 1$$

Hence, $\prod_{v \in S/v_0} |L_{1,v}(\boldsymbol{a}_n)|_v \leq 1$ and $|L_{1,v_0}(\boldsymbol{a}_n)|_{v_0} \leq c|\beta|^{-ms_n/deg(\beta)}$.

[3] Notice that we could have chosen any $i \in 1, \ldots, 3+d$ to bear the special linear form L_{1,v_0} and linear independence would be maintained.

Furthermore,

$$\begin{aligned}
\prod_{v\in S}|L_{3,v}(\boldsymbol{a}_n)|_v &= \prod_{v\in S}|\sum_{j=0}^{r_n+s_n} u_j\beta^{r_n+s_n-j}|_v \\
&\leq \prod_{v\in S}\sum_{j=0}^{r_n}|u_j|_v|\beta|_v^{r_n+s_n-j} && \text{basic properties of absolute values} \\
&\leq \prod_{v\in S}\sum_{j=0}^{r_n+s_n} M^{j+1} && \text{definition of } M \\
&\leq \prod_{v\in S} M^{r_n+s_n+2} && \text{since } M\geq 2 \\
&\leq M^{|S|(r_n+s_n+2)}
\end{aligned}$$

Similarly, for each $i\in 1,\ldots,d$, we obtain

$$\begin{aligned}
\prod_{v\in S}|L_{i,v}(\boldsymbol{a}_n)|_v &= \prod_{v\in S}|\sum_{j=x_i}^{y_i}(u_{j+s_n}-u_j)\beta^{-j}|_v \\
&\leq M^{|S|(y_i-x_i+2)}
\end{aligned} \tag{2.7}$$

Altogether,

$$\begin{aligned}
\prod_{v\in S}\prod_{j=1}^{3+d}|L_{i,v}(\boldsymbol{a}_n)|_v &\leq c|\beta|^{-ms_n/deg(\beta)}\cdot M^{|S|(r_n+s_n+2)}\cdot\prod_{i=1}^{d}M^{|S|(y_i-x_i+2)} \\
&= c|\beta|^{-ms_r/deg(\beta)}\cdot|\beta|^{\log_{|\beta|}(M)|S|(r_n+s_n+2+\sum_{i=1}^{d}(y_i-x_i+2))} \\
&\leq c|\beta|^{-ms_n(\frac{1}{deg(\beta)}-\log_{|\beta|}(M)|S|(\frac{r_n+s_n+2d+2}{ms_n}+\rho))} && \text{(iii) of 2.14} \\
&\leq c|\beta|^{-ms_n(\frac{1}{deg(\beta)}-\log_{|\beta|}(M)|S|(\frac{C+1}{m}+\frac{2d+2}{ms_n}+\rho))} && \text{(ii) of 2.14}
\end{aligned}$$

which is smaller than $|\beta|^{-s_n}$ for sufficiently large m, sufficiently large n, and sufficiently small ρ. This is the point where we can select appropriate m and ρ. Now, the Weil height of $\boldsymbol{a}_n$ is

$$H(\boldsymbol{a}_n)^{\epsilon}=\left(\prod_{v\in M(K)}\max(|\beta^{r_n+s_n}|_v,|\beta^{r_n}|_v,|P_n(\beta)|_v,|\delta_{n,1}(\beta)|_v,\ldots,|\delta_{n,d}(\beta)|_v)\right)^{\epsilon}\leq|\beta|^{s_n}$$

for suitable $\epsilon\in\mathbb{R}^+$, using the Proposition 2.1. Thus,

$$\prod_{v\in S}\prod_{j=1}^{3+d}|L_{i,v}(b_n)|_v\leq|\beta|^{-s_n}\leq H(b_n)^{-\epsilon}$$

which means we can apply the Subspace theorem to obtain non-zero linear forms $L_1,\ldots,L_t$ for some $t\in\mathbb{N}$ with coefficients in K such that for every $\boldsymbol{a}_n$, there is

$i \in 1, \ldots, t$ such that $L_i(\boldsymbol{a}_n) = 0$. Hence, there is a linear form $L \in L_1, \ldots, L_t$ for which there are infinitely many n such that $L(\boldsymbol{a}_n) = 0$. That is, there exist $z_1, \ldots, z_{3+d}$ such that

$$z_1 \beta^{r_n + s_n} + z_2 \beta^{r_n} + z_3 P_n(\beta) + \sum_{i=1}^{d} z_{3+i} \delta_{n,i}(\beta) = 0$$

for infinitely many $n \in \mathbb{N}$.

Dividing by $\beta^{r_n+s_n}$ and letting n tend to infinity, we have

$$z_1 + z_3 \lim_{n \to \infty} \frac{P_n(\beta)}{\beta^{r_n+s_n}} = z_1 + z_3 \alpha = 0.$$

Hence, $\alpha \in \mathbb{Q}(\beta, \mathcal{H})$. □

2.6 Future Work

Our definition of echoic sequences can be generalised to capture sequences that do not have such a strong property. It can be generalised in two directions.

Firstly, one can make the number of repetitions a variable of the sequence. In other words, for any infinite word $\boldsymbol{u}$, one can define $\lambda(\boldsymbol{u})$[4] as follows:

Definition 2.16. Let Σ be a finite alphabet and $\boldsymbol{u} = u_1 u_2 \cdots \in \Sigma^\omega$ an infinite sequence. Then, $\lambda(\boldsymbol{u})$ is the supremum of $m \in \mathbb{R}$ satisfying:[5] for all positive real ρ there exist sequences of integers $\langle r_n \rangle_{n=0}^{\infty}$, $\langle s_n \rangle_{n=0}^{\infty}$, a number $d \in \mathbb{N}$, and for every $n \in \mathbb{N}, i \in 1, \ldots, d$ intervals $[\![x_{n,1}, y_{n,1}]\!], \ldots, [\![x_{n,d}, y_{n,d}]\!]$ such that they satisfy the property from Definition 2.14.

This actually generalises the notion of diophantine exponent since $\mathsf{dio}(\boldsymbol{u}) \leq \lambda(\boldsymbol{u})$ for any infinite word $\boldsymbol{u}$. This is due to the fact that $\mathsf{dio}(\boldsymbol{u})$ expresses a repetition without an error whereas $\lambda(\boldsymbol{u})$ allows discrepancies.

In another direction, one can make the "similarity ratio" ρ a variable, *i.e.*, for an infinite word $\boldsymbol{u}$, we define $\kappa(\boldsymbol{u})$ followingly:

Definition 2.17. Let Σ be a finite alphabet and $\boldsymbol{u} = u_1 u_2 \cdots \in \Sigma^\omega$ an infinite sequence. Then, $\kappa(\boldsymbol{u})$ is an infimum of $\epsilon \in [0, 1]$ satisfying: for all $m \in \mathbb{N}$ there exist sequences $\langle r_n \rangle_{n=0}^{\infty}$, $\langle s_n \rangle_{n=0}^{\infty}$, a number $d \in \mathbb{N}$, and for every $n \in \mathbb{N}, i \in 1, \ldots, d$ intervals $[\![x_{n,1}, y_{n,1}]\!], \ldots, [\![x_{n,d}, y_{n,d}]\!]$ such that they satisfy the property from the definition 2.14.

[4]The usage of λ here and κ below is just demonstrative and does not relate to any known concepts.

[5]Note that we have changed the scope of m to be positive real numbers instead of natural numbers. The only difference is that the property (iii) has to round ms_n to an integer.

This captures words which have infinite repetition but with some fixed error. However, the most general definition would be to make both the repetition m and the similarity ratio ρ parameters.

In the next chapter, we apply Theorem 2.15 to all episturmian words since we prove they are echoic. The next steps we suggest is to try to apply these results to automatic sequences or to sequences with linear subword complexity in general. Hopefully, one can identify the set of permissible algebraic bases β using the parameters m and ρ.

Chapter 3

Applications of the New Result

Before we proceed to prove that all episturmian words are echoic, let us demonstrate the main ideas on Sturmian words and the Tribonacci word.

Given an alphabet Σ, let us recall morphisms $\psi_a : \Sigma \to \Sigma^*$ and $\psi_{\bar{a}} : \Sigma \to \Sigma^*$ defined by $\psi_a(c) = ac$, $\psi_a(a) = a$, $\psi_{\breve{a}}(c) = ca$ and $\psi_{\breve{a}}(a) = a$ for every $a, c \in \Sigma, a \neq c$. Also if $\breve{\Delta} = \breve{d}_1\breve{d}_2 \cdots \in \breve{\Sigma}^\omega$ is a directive sequence of an episturmian word $\boldsymbol{u} = u_1u_2 \cdots \in \Sigma^\omega$, then we write $\psi_{\breve{n}} = \psi_{\breve{d}_1}\psi_{\breve{d}_1} \cdots \psi_{\breve{d}_n}$,$\psi_n = \psi_{d_1}\psi_{d_1} \cdots \psi_{d_n}$, and the following holds for all epistandard words, *i.e.*, for episturmian words where $\breve{\Delta} = \Delta$:

$$\boldsymbol{u} = \lim_{n\to\infty} \psi_n(d_{n+1}).$$

3.1 Sturmian Words

Even though the Sturmian case was covered in [**22**], we use Sturmian words to demonstrate our proof in detail. For simplicity, we assume only standard Sturmian words. We deal with all non-standard episturmian words in Section 3.3.

For a directive sequence $\Delta = 001001001\cdots = (001)^\omega$, we have

$$\begin{aligned}
\psi_\varepsilon(0) &= 0\\
\psi_1(0) = \psi_0(0) &= 0\\
\psi_2(1) = \psi_0\psi_0(1) &= 001\\
\psi_3(0) = \psi_0\psi_0\psi_1(0) &= 0010\\
\psi_4(0) = \psi_0\psi_0\psi_1\psi_0(0) &= 0010\\
\psi_5(1) = \psi_0\psi_0\psi_1\psi_0\psi_0(1) &= 00100010001\\
\psi_6(0) = \psi_0\psi_0\psi_1\psi_0\psi_0\psi_1(0) &= 001000100010010\\
\psi_7(0) = \psi_0\psi_0\psi_1\psi_0\psi_0\psi_1\psi_0(0) &= 001000100010010\\
\psi_8(1) = \psi_0\psi_0\psi_1\psi_0\psi_0\psi_1\psi_0\psi_0(1) &= 0010001000100100010001000100100010001\\
&\cdots
\end{aligned}$$

This induces the infinite sequence

$$\boldsymbol{u} = 00100010001001000100010001001000100010001001 0\cdots$$

Now, let us observe what happens if we shift this sequence by $|\psi_2(1)| = 3$. We get

$$\boldsymbol{u}^{(2)} = 000100010010001000100010010001000100010010001\cdots$$

If we take word difference $\boldsymbol{u} \ominus \boldsymbol{u}^{(2)}$, we get

$$\boldsymbol{u} \ominus \boldsymbol{u}^{(2)} = \bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\bullet\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\bullet\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\bullet\bullet\bullet\bullet\bullet\tfrac{1}{0}\tfrac{0}{1}\cdots$$

We see that most part of the word is matched except for some discrepancies which occur in pairs $\tfrac{1}{0}\tfrac{0}{1}$ and furthermore, the gaps between these groups are only of lengths 2 and 5.

The whole sequence $\boldsymbol{u}$ can be built up of two types of finite words: 001 and 0. Moreover, there are never two words of 0 next to each other. So in the case of the sequence $\boldsymbol{u}$, it can be grouped as follows:

$$\boldsymbol{u} = [001][0][001][0][001][001][0][001][0][001][0][001][001][0][001][0][001][0][001][001][0]\cdots$$

And similarly, since $\boldsymbol{u}^{(2)}$ is $\boldsymbol{u}$ shifted by $|001|$, it can be grouped as

$$\boldsymbol{u}^{(2)} = [0][001][0][001][001][0][001][0][001][0][001][001][0][001][0][001][0][001][001][0][001]\cdots$$

We realise that the word difference can be built up of only two types of word differences:

1. $([001][0]) \ominus ([0][001]) = \bullet\bullet\frac{1}{0}\frac{0}{1}$

2. $([001]) \ominus ([001]) = \bullet\bullet\bullet$

Actually, this happens with $\boldsymbol{u} \ominus T^{|\psi_n(d_{n+1})|}(\boldsymbol{u})$ for all $n \in \mathbb{N}$. In [**22**], they proved this from a different perspective – Sturmian word as the sequence induced by a rotation on a unit circle. Here, we give an argument based on the definition of Sturmian words using morphisms.

Let us assume we shift $\boldsymbol{u}$ by $|\psi_n(d_{n+1})|$ to get $\boldsymbol{u}^{(n)}$. If we take a new directive sequence $\Delta' = d_{n+1}d_{n+2}\cdots$ and its related infinite word $\boldsymbol{v}$, we notice that $\boldsymbol{u} = \psi_n(\boldsymbol{v})$ (from Theorem 1.1). Next, since the last morphism applied to $\boldsymbol{v}$ is $\psi_{d_{n+1}}$, every 2-factor of $\boldsymbol{v}$ contains letter d_{n+1} and $\boldsymbol{v}$ starts with the letter d_{n+1}. Let $a := d_{n+1}$ and b be the other letter. It is clear that one can factor $\boldsymbol{v}$ into two types of words: a and ab. Therefore, one can factor $\boldsymbol{u}$ into two types of words: $\psi_n(a)$ and $\psi_n(ab) = \psi_n(a)\psi_n(b)$. Since our shift is equal to $|\psi_n(a)|$, our shifted sequence $\boldsymbol{u}^{(n)}$ has $\psi_n(ba)$ where $\boldsymbol{u}$ has $\psi_n(ab)$ and $\psi_n(a)$ where $\boldsymbol{u}$ has $\psi_n(a)$. For instance, it may look like this:

$$\begin{aligned} \boldsymbol{u} &= \psi_n(a)\psi_n(ab)\psi_n(a)\psi_n(a)\psi_n(ab)\cdots \\ \boldsymbol{u}^{(n)} &= \psi_n(a)\psi_n(ba)\psi_n(a)\psi_n(a)\psi_n(ba)\cdots \end{aligned}$$

Therefore, the whole difference $\boldsymbol{u} \ominus \boldsymbol{u}^{(n)}$ can be built up of two types of word differences: 1) $\psi_n(a) \ominus \psi_n(a)$ and 2) $\psi_n(ab) \ominus \psi_n(ba)$. Comparison 1) is, obviously, a match of length $|\psi_n(a)|$.

Let us look closely at the difference $\psi_n(ab) \ominus \psi_n(ba)$. Without loss of generality, let us assume $d_n = a$. Then, $\psi_n(ab) = \psi_{n-1}(aab) = \psi_{n-1}(a)\psi_{n-1}(ab)$ and $\psi_{n-1}(aba) = \psi_{n-1}(a)\psi_{n-1}(ba)$. Using an induction, we obtain

$$\begin{aligned} \psi_n(ab) \ominus \psi_n(ba) &= (\psi_{n-1}(d_n) \ominus \psi_{n-1}(d_n))\psi_{n-1}(ab) \ominus \psi_{n-1}(ba) \\ &= (\psi_{n-1}(d_n) \ominus \psi_{n-1}(d_n))(\psi_{n-2}(d_{n-1}) \ominus \psi_{n-2}(d_{n-2}))\psi_{n-2}(ab) \ominus \psi_{n-2}(ba) \\ &\cdots \\ &= \bullet^{|\psi_{n-1}(d_n)|+|\psi_{n-2}(d_{n-1})|+\cdots+|\psi_1(d_2)|+|d_1|}\frac{a}{b}\frac{b}{a} \end{aligned}$$

This precisely explains, why there are exactly 5 matches before $\frac{0}{1}\frac{1}{0}$ in $\boldsymbol{u} \ominus \boldsymbol{u}^{(3)}$ since $|\psi_2(d_3)| + |\psi_1(d_2)| + |d_1| = |001| + |0| + |0| = 5$. Observe that $(001)(0)(0) = 00100 = \mathsf{Pali}(001)$ which generalises to $\psi_{n-1}(d_n)\psi_{n-2}(d_{n-1})\cdots\psi_1(d_2)d_1 = \mathsf{Pali}(d_1\cdots d_n)$. This is an inductive consequence of Justin's Formula we state in section 3.3.

Now, we know the reason why the difference $\boldsymbol{u} \ominus T^{|\psi_n(d_{n+1})|}(\boldsymbol{u})$ can be made up of two types of blocks where one of them consists of only matches and the other has only two discrepancies at the end.

We use this knowledge to prove that $\boldsymbol{u}$ is echoic. Let us have $m \in \mathbb{N}$ and $\rho > 0$. We set $r_n := 0, s_n := |\psi_n(d_{n+1})|$ for every $n \in \mathbb{N}$. From what we showed above, there is at most one pair of mismatches every $|\psi_n(d_{n+1})|$ letters in $\boldsymbol{u} \ominus T^{|\psi_n(d_{n+1})|}(\boldsymbol{u})$ which means that we can set $d := 2m$ and the intervals $[\![x, x+1]\!]$ according to the index of the mismatches. This ends the proof that standard Sturmian words are echoic.

Being an echoic sequence $\boldsymbol{u}$ over an alphabet $\{c, d\}$, Theorem 2.14 gives us that $\boldsymbol{u}_\beta$ is transcendental or belongs to $\mathbb{Q}(\beta, c, d)$ for any algebraic base $\beta, |\beta| > 1$.[1]

One can actually prove transcendence [**22**] which follows from the fact that for all $n \in \mathbb{N}$, the polynomial related to mismatches between $\psi_n(01) \ominus \psi_n(10)$ is equal to $\pm(c-d)(\beta - 1)$. The main idea of the proof is that these polynomials cannot vanish separately ($c \neq d$ and $|\beta| > 1$) and since the gaps between two such polynomials expand with n, they cannot vanish together, leading to a contradiction with the output of the Subspace Theorem.

However, for a different structure of mismatches, this might not be true. Let us imagine that the mismatches have a different form, *e.g.*, the related polynomial is $(b-a)\beta^3 + (c-b)\beta^2 + (b-c)\beta + (a-b)$ where a, b, c are three different algebraic numbers. This polynomial is related to a group of mismatches of the form $\frac{b}{a}\frac{c}{b}\frac{b}{c}\frac{a}{b}$. If we set $a = 1, b = 0, c = -\frac{3}{2}$, then the polynomial vanishes on $\beta = -2$.

This cannot happen with Sturmian words but it can happen with episturmian words. Let us have a word $\boldsymbol{u} = u_1u_2u_3\cdots \in \Sigma^\omega$ directed by $\Delta = bcbabababa\cdots = bc(ba)^\omega$. One can prove that the difference $\boldsymbol{u} \ominus T^{|\psi_n(d_{n+1})|}(\boldsymbol{u})$ consists only of the mismatches of the form $\frac{b}{a}\frac{c}{b}\frac{b}{c}\frac{a}{b}$. Consequently,

$$\alpha = \sum_{j=1}^{\infty} u_j\beta^{-j} = \frac{(bcb)_\beta}{(\beta)^3 - 1} = \frac{3}{-9} = \frac{(bcbbcba)_\beta}{(\beta)^7 - 1} = \frac{48 - 6 + 1}{129} = \cdots = -\frac{1}{3}.$$

Hence, even though we can prove that any number related to a Sturmian word is transcendental in any algebraic base for distinct digits, one needs to be very careful with the transcendence result for episturmian words.

3.2 The Tribonacci Word

In this section, we explore the structure of mismatches in the case of the Tribonacci word and how it can help us to prove that it is an echoic sequence.

[1] Recall that $\boldsymbol{u}_\beta$ is the Laurent series related to the word $\boldsymbol{u}$ in base β.

The Tribonacci word is a fixed point of the following morphism:

$$\begin{aligned}\psi : & a \mapsto ab \\ & b \mapsto ac \\ & c \mapsto a\end{aligned}$$

which induces the word

$$\boldsymbol{u} = abacabaabacababacabaabacabacabaabacababacabaabacabaabacababacaba\cdots$$

Since it is also an episturmian word, it has a related directive sequence $\Delta = abcabcabc\cdots = (abc)^{\omega}$. Let us do the same as in the case of Sturmian words: shift the word by $|\psi_n(d_{n+1})|$ for various n to see how the structure of mismatches manifest. Let us shift by $|\psi_2(d_3)|$:

$$\boldsymbol{u}^{(2)} = abaabacababacabaabacabacabaabacababacabaabacabaabacababacabaab\cdots$$

$$\boldsymbol{u} \ominus \boldsymbol{u}^{(2)} = \bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}\bullet\bullet\bullet\frac{c}{b}\bullet\frac{b}{c}\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}\bullet\bullet\bullet\bullet\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}\bullet\bullet\bullet\frac{c}{b}\bullet\frac{b}{c}\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\cdots$$

At first glance, it looks like the structure of mismatches is far more complicated than in the Sturmian case. However, let us do the same analysis as in the previous section. There exists an infinite word $\boldsymbol{v} = v_1 v_2 \cdots \Sigma^{\omega}$ directed by $\Delta' = cabcab\cdots$ such that $\boldsymbol{u} = \psi_2(\boldsymbol{v})$. Hence, one can group $\boldsymbol{u}$ into three groups, respectively defined as $\psi_2(a) = aba, \psi_2(b) = ab, \psi_2(c) = abac$. Since every 2-factor of $\boldsymbol{v}$ contains letter $d_3 = c$ and $\boldsymbol{v}$ starts with c, then if we shift $\boldsymbol{u}$ by $|\psi_2(c)|$, we can group the whole difference $\boldsymbol{u} \ominus \boldsymbol{u}^{(3)}$ into three types of blocks:

- $\psi_2(c) \ominus \psi_2(c) = \bullet\bullet\bullet\bullet$
- $\psi_2(ca) \ominus \psi_2(ac) = \bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}$
- $\psi_2(cb) \ominus \psi_2(bc) = \bullet\bullet\bullet\frac{c}{b}\bullet\frac{b}{c}$

Let us continue to observe mismatches structure for another n:

$$\boldsymbol{u} \ominus \boldsymbol{u}_{(5)} = \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\frac{b}{c}\bullet\frac{c}{b}\bullet\frac{b}{a}\frac{a}{b}\bullet\frac{b}{c}\bullet\frac{c}{b}\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\frac{b}{a}\frac{a}{b}\frac{c}{a}\frac{a}{c}\frac{b}{a}\frac{a}{b}\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\cdots$$

Here, the three types of blocks are

- $\psi_4(b) \ominus \psi_4(b) = \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$
- $\psi_4(ba) \ominus \psi_4(ab) = \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\frac{b}{a}\frac{a}{b}\frac{c}{a}\frac{a}{c}\frac{b}{a}\frac{a}{b}$

- $\psi_4(bc) \ominus \psi_4(cb) = \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\frac{b}{c}\bullet\frac{c}{b}\bullet\frac{b}{a}\frac{a}{b}\bullet\frac{b}{c}\bullet\frac{c}{b}$

We start to notice that these mismatches are not of a constant shape as was the case of Sturmian words but they are somehow growing. Moreover, we observe that they have a recursive structure.

In order to explore this in detail, we use the same technique as in the Sturmian case – trying to understand $\psi_n(d_{n+1}d_{n+2})$ as a function of previous $\psi_{n-1}(d_n d_{n+1})$.

We need to focus only on these word differences for every $n \in \mathbb{N}$:

1. $U_n = \psi_n(d_{n+1}d_{n+2}) \ominus \psi_n(d_{n+2}d_{n+1})$,
2. $V_n = \psi_n(d_{n+1}d_{n+3}) \ominus \psi_n(d_{n+3}d_{n+1})$,
3. $W_n = \psi_n(d_{n+2}d_{n+3}) \ominus \psi_n(d_{n+3}d_{n+2})$.

We can view them as a table, where the column is one of these three types and each row is labelled by $n \in \mathbb{N}$.

n	$d_1 \cdots d_n$	U_n	V_n	W_n
0	ε	$\frac{a}{b}\frac{b}{a}$	$\frac{a}{c}\frac{c}{a}$	$\frac{b}{c}\frac{c}{b}$
1	a	$\bullet\frac{b}{c}\bullet\frac{c}{b}$	$\bullet\frac{b}{a}\frac{a}{b}$	$\bullet\frac{c}{a}\frac{a}{c}$
2	ab	$\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}$	$\bullet\bullet\bullet\frac{c}{b}\bullet\frac{b}{c}$	$\bullet\bullet\bullet\frac{a}{b}\frac{b}{a}$
3	abc	$\bullet^7\frac{a}{b}\frac{b}{a}\frac{a}{c}\frac{c}{a}\frac{a}{b}\frac{b}{a}$	$\bullet^7\frac{a}{c}\frac{b}{a}\frac{a}{b}\frac{c}{a}$	$\bullet^7\frac{b}{c}\bullet\frac{c}{b}$
4	$abca$	$\bullet^{14}\frac{b}{c}\bullet\frac{c}{b}\bullet\frac{b}{a}\frac{a}{b}\bullet\frac{b}{c}\bullet\frac{c}{b}$	$\bullet^{14}\frac{b}{a}\frac{a}{b}\frac{c}{a}\frac{a}{c}\frac{b}{a}\frac{a}{b}$	$\bullet^{14}\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}$
5	$abcab$	$\bullet^{27}\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}\bullet\bullet\bullet\frac{c}{b}\bullet\frac{b}{c}\bullet\bullet\bullet\frac{c}{a}\frac{a}{b}\frac{b}{a}\frac{a}{c}$	$\bullet^{27}\frac{c}{b}\bullet\frac{b}{c}\bullet\frac{a}{b}\frac{b}{a}\bullet\frac{c}{b}\bullet\frac{b}{c}$	$\bullet^{27}\frac{a}{b}\frac{b}{a}\frac{a}{c}\frac{c}{a}\frac{a}{b}\frac{b}{a}$
...	...	...	...	...

First of all, we observe that the ratio of mismatches compared to the overall length decreases. We make use of this to prove that for any ρ, we can find suitable intervals of mismatches.

Secondly, the length of the prefix of $\bullet$ is always $|\mathsf{Pali}(d_1 \cdots d_n)|$. This actually holds for any $\psi_w(ab) \ominus \psi_w(ba)$ $w \in \Sigma^*, a, b \in \Sigma$ (see Proposition 3.11).

Thirdly, we observe the following relationships between these words for all $n \geq 3$:

$$U_n = \bullet^{|\psi_n(d_{n+1})|}(U_{n-3}V_{n-3}U_{n-3}), \tag{3.1}$$

$$V_n = \bullet^{|\psi_n(d_{n+3})|}(\ominus U_{n-1}), \tag{3.2}$$

$$W_n = \bullet^{|\psi_n(d_{n+3})|}(\ominus V_{n-1}). \tag{3.3}$$

These formulas will eventually generalise to the Proposition 3.11. Without loss of generality, let us assume $n = 3k$ for some $k \in \mathbb{N}$. We derive the formula (3.2) for V_n:

$$
\begin{aligned}
V_n &= \psi_n(ac) \ominus \psi_n(ca) \\
&= \psi_{n-1}(cac) \ominus \psi_{n-1}(cca) \\
&= (\psi_n(c) \ominus \psi_n(c))(\psi_{n-1}(ac) \ominus \psi_{n-1}(ca)) \qquad \text{since } \psi_n(c) = \psi_{n-1}(c) \\
&= \bullet^{|\psi_n(c)|}(\ominus U_{n-1})
\end{aligned}
$$

The case of W_n is done similarly. Let us now focus on U_n:

$$
\begin{aligned}
U_n &= \psi_n(ab) \ominus \psi_n(ba) \\
&= \psi_{n-1}(cacb) \ominus \psi_{n-1}(cbca) \\
&= \psi_{n-2}(bcbabcb) \ominus \psi_{n-2}(bcbbcba) \\
&= \psi_{n-3}(abacabaabacab) \ominus \psi_{n-3}(abacababacaba) \\
&= \psi_n(a)\psi_{n-3}(ab)\psi_{n-3}(ac)\psi_{n-3}(ab) \ominus \psi_n(a)\psi_{n-3}(ba)\psi_{n-3}(ca)\psi_{n-3}(ba) \\
&= \bullet^{|\psi_n(a)|}(U_{n-3}V_{n-3}U_{n-3})
\end{aligned}
$$

Additionally, since $|\psi_{n-3}(abacaba)| \geq \psi_{n-3}(abacab)|$ we have

$$
|\psi_n(a)| \geq |U_{n-3}V_{n-3}U_{n-3}|. \tag{3.4}
$$

We combine formulas (3.1), (3.2), and (3.3) together to

$$
U_n = \bullet^{|\psi_n(d_{n+1})|} U_{n-3} \bullet^{|\psi_{n-3}(d_{n+2})|}(\ominus U_{n-4}) U_{n-3}. \tag{3.5}
$$

Using (3.4) we can bound the ratio of matches in the word U_n from below by

$$
\frac{|\psi_n(d_{n+1})|}{|\psi_n(d_{n+1})| + |U_{n-3}V_{n-3}U_{n-3}|} \geq \frac{1}{2}. \tag{3.6}
$$

Using an induction on ρ, we prove that for any $\rho \in \mathbb{R}^+$, we can find $d \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that all mismatches of U_n can be grouped into d intervals such that their overall length is smaller than $\rho \cdot |U_n|$. We say that (ρ, d, n_0) is a suitable triple.

We are given $\rho \in \mathbb{R}^+$ and using the inductive hypothesis on 2ρ, we are given $n_0 \in \mathbb{N}$ and $d \in \mathbb{N}$ such that $(2\rho, d, n_0)$ is a suitable triple. We show that $(\rho, 3d, n_0 + 4)$ is a suitable triple. For every $n \in \mathbb{N}, n \geq n_0 + 4$, using the formula (3.5) and the inductive hypothesis, we know that the number of mismatches in U_n is smaller than

$$
2\rho|U_{n-3}| + 2\rho|U_{n-4}| + 2\rho|U_{n-3}|
$$

and using the inequality (3.6), this is smaller than $\rho|U_n|$. Also, since all mismatches of either U_{n-3} or U_{n-4} can be grouped into d intervals, the overall number of intervals in U_n is $3d$.

The base case $\rho = 1$ is trivial. Therefore, we proved that the proportion of mismatches in U_n tends to zero with $n \to \infty$ and that we can effectively group all intervals. This proves that the Tribonacci word is an echoic sequence.

Since in the case of the Tribonacci word, the number of mismatches is not constant, we cannot apply [**22**] to obtain a pure transcendence result. However, we conjecture that the number $\boldsymbol{u}_\beta$ is transcendental, where $\boldsymbol{u}$ is the Tribonacci word and β is an algebraic base $|\beta| > 1$. Additionally, we conjecture that the same holds for any D-bonacci word.[2] Here, we provide an intuition for believing this:

For any $n \in \mathbb{N}_0$, $\delta_n(X)$ denotes the polynomial related to the word difference U_n. From (3.5), we have that

$$\delta_n(X) = (X^{a_1} + 1)\delta_{n-3}(X) + X^{a_2}\delta_{n-4}(X)$$

where $a_1, a_2 \in \mathbb{N}$. Let us assume there exists $n_0 \in \mathbb{N}$ such that $\delta_n(\beta) = 0$ for all $n \in \mathbb{N}$, $n > n_0$. If $\delta_{n_0}(\beta) \neq 0$, then we arrive at a contradiction since $\delta_{n_0+4}(\beta) = (\beta^{a_1} + 1)\delta_{n+1}(\beta) + \beta^{a_2}\delta_n(\beta)$ and $\delta_{n_0+4}(\beta) = \delta_{n+1}(\beta) = 0$. Thus, we have that $\delta_0(\beta) = (a - b)(\beta - 1) = 0$ which is impossible since $a \neq b$. Therefore, a sufficient step to prove transcendence of α is to prove that the assumption that α is algebraic yields that there is $n_0 \in \mathbb{N}$ such that $\delta_n(\beta) = 0$ for all $n \in \mathbb{N}$, $n > n_0$.

3.3 Episturmian Words

In this section, we prove that all episturmian words are echoic. In the previous sections, given an infinite (Sturmian or Tribonacci) word $\boldsymbol{u} = u_1 u_2 \cdots$, we simply said that the mismatches of the difference $u_1 \cdots u_{ms_n} \ominus u_{s_n+1} \cdots u_{(m+1)s_n}$ consist of at most m words of type $\psi_n(ab) \ominus \psi_n(ba)$ where $a, b \in \Sigma$, thus we can reduce the problem to the analysis of $\psi_n(ab) \ominus \psi_n(ba)$. In the case of episturmian words, there may be some block of mismatches as a prefix. Hence, we need to find an appropriate shift such that we can analyse the mismatches only as blocks of type $\psi_n(ab) \ominus \psi_n(ba)$ where $a, b \in \Sigma$. This is done in Lemma 3.5.

Before we analyse $\psi_n(ab) \ominus \psi_n(ba)$ for some $a, b \in \Sigma$, we notice that if the directive sequence Δ has unbounded $\mathsf{lrt}_\Sigma(\Delta)$[3], then our argument may not work. We solve this

[2] For $D \in \mathbb{N}$, the D-bonacci word is a fixed point of the morphism $0 \mapsto 01, 1 \mapsto 02, \ldots, D \mapsto 0$.

[3] Recall that for a (potentially infinite) word u, $\mathsf{lrt}(u)$ is the maximum of the letter return time in u for all letters from Σ.

issue by finding arbitrary long parts of the directive sequence Δ which "behave" like a sequence with letter return time bounded by a constant. This is proved in Lemma 3.6.

With this assumption, we analyse $\psi_n(ab) \ominus \psi_n(ba)$ where $a, b \in \Sigma$ and we prove that all mismatches can be grouped into constant number of intervals (based on ρ and some other constants) such that the ratio of overall length of these intervals compared to the overall length is smaller than ρ. This is proved in Lemma 3.14.

Definition 3.1. For every $w \in \Sigma$, let us define $\pi_w : \Sigma^* \times \Sigma^* \to (\Sigma^{\ominus})^*$ as

$$\pi_w(u, v) := \psi_w(u) \ominus \psi_w(v)$$

where $u, v \in \Sigma^*$.

We start by stating some known results.

Proposition 3.2 (Justin's Formula [**20**], p. 287.)**.** *For any $u, v \in \Sigma^*$, we have*

$$\mathsf{Pali}(uv) = \psi_u(\mathsf{Pali}(v))\mathsf{Pali}(u).$$

Proposition 3.3 ([**20**], Prop. 3.15.)**.** *Let Σ be an alphabet, $w \in \Sigma^*$ and $\breve{u} \in \breve{\Sigma}^*$. Then, there exist $n \in \mathbb{N}_0$ such that*

$$\psi_{\breve{u}}(w) = T^n(\psi_u(w)).$$

The number n is called the shifting factor *of $\breve{u}$.*

Proposition 3.4. *Let Σ be an alphabet and $u \in \Sigma^+, a \in \Sigma$. If $a \notin \mathsf{Alp}(u)$ we have $\psi_u(a) = \mathsf{Pali}(u)a$. If $a \in \mathsf{Alp}(u)$ we have $\psi_u(a)\mathsf{Pali}(v) = \mathsf{Pali}(u)$ where $v, w \in \Sigma^*$ such that $u = vaw$.*

Proof. The first part is clear from the fact that if $a \notin \mathsf{Alp}(u)$ then

$$\psi_u(a)\mathsf{Pali}(u) = \mathsf{Pali}(ua) = \mathsf{Pali}(u)a\mathsf{Pali}(u)$$

where the first equality is from Justin's formula and the second from the definition of Pali.

For second part,

$$\psi_{vaw}(a)\mathsf{Pali}(v) = \psi_{va}(\mathsf{Pali}(w)a)\mathsf{Pali}(v) = \psi_v(\mathsf{Pali}(aw))\mathsf{Pali}(v) = \mathsf{Pali}(u)$$

where the first equality is proved above, second and third equalities are applications of Justin's Formula. ☐

The following Lemma states that for a given word $\boldsymbol{u}$ there exists a small shift $r \in \mathbb{N}$ such that the first m terms of $T^r(\boldsymbol{u})$ consist only of the first m letters of the directive sequence.

Lemma 3.5. *Let Σ be an alphabet, $\boldsymbol{u} = u_1u_2u_3\cdots \in \Sigma^\omega$ be an episturmian word, and $\breve{\Delta} = \breve{d}_1\breve{d}_2\cdots \in \breve{\Sigma}^\omega$ a related directive sequence. Then, for every $m \in \mathbb{N}$, there exist $r \in \mathbb{N}, 1 \leq r \leq m+1$ such that $\mathsf{Alp}(u_{r+1}u_{r+2}\cdots u_{r+m}) \subseteq \mathsf{Alp}(d_1d_2\cdots d_m)$.*

Proof. Let $\Sigma' = \mathsf{Alp}(d_1d_2\cdots d_m)$. From the Theorem 1.1, we get an infinite sequence $\langle \boldsymbol{u}^{(n)}\rangle_{n=1}^{\infty}$ of infinite recurrent words such that $\boldsymbol{u}^{(0)} = \boldsymbol{u}$ and $\boldsymbol{u}^{(i)} = \psi_{\breve{d}_i}(\boldsymbol{u}^{(i+1)})$ for every $i \in \mathbb{N}_0$.

Let $\boldsymbol{u}^{(m)} = u_1^{(m)}u_2^{(m)}\cdots \in \Sigma^\omega$ and $j_1, j_2 \in \mathbb{N}, j_1 < j_2$ be the first two occurrences of a letter from $\Sigma - \Sigma'$ in $\boldsymbol{u}^{(m)}$, *i.e.*, $\mathsf{Alp}(u_1^{(m)}\cdots u_{j_1-1}^{(m)}), \mathsf{Alp}(u_{j_1+1}^{(m)}\cdots u_{j_2-1}^{(m)}) \subseteq \Sigma'$. and $u_{j_1}^{(m)}, u_{j_2}^{(m)} \notin \Sigma'$. If no such j_1, j_2 exist, then $\psi_{\breve{d}_1\cdots\breve{d}_m}(u_1^{(m)}\cdots u_{2m+1}^{(m)})$ is a prefix of $\boldsymbol{u}$, it has length at least $2m+1$, and it contains at most one letter from $\Sigma - \Sigma'$ (since there is at most one letter from $\Sigma - \Sigma'$ in $\boldsymbol{u}^{(m)}$). Hence, either $r = 1$ or $r = j_1$ would be suitable where j_1 is the occurrence of a letter from $\Sigma - \Sigma'$ in $\boldsymbol{u}$.

Let

$$v_1 := \psi_{\breve{d}_1\cdots\breve{d}_m}(u_{j_1}^{(m)}) \text{ and } v_2 := \psi_{\breve{d}_1\cdots\breve{d}_m}(u_{j_2}^{(m)})$$

Since $u_{j_1}^{(m)}, u_{j_2}^{(m)} \notin \Sigma'$, then using Proposition 3.4, we have

$$\psi_{d_1\cdots d_m}(u_{j_1}^{(m)}) = \mathsf{Pali}(d_1\cdots d_m)u_{j_1}^{(m)} \text{ and } \psi_{d_1\cdots d_m}(u_{j_2}^{(m)}) = \mathsf{Pali}(d_1\cdots d_m)u_{j_2}^{(m)}.$$

Using Proposition 3.3, we have

$$v_1 = w_2u_{j_1}^{(m)}w_1 \text{ and } v_2 = w_2u_{j_2}^{(m)}w_1$$

where $w_1, w_2 \in \Sigma'^*$ and $w_1w_2 = \mathsf{Pali}(d_1\cdots d_m)$. Altogether, $\boldsymbol{u}$ has a prefix

$$\psi_{\breve{d}_1\cdots\breve{d}_m}(u_1^{(m)}\cdots u_{j_1-1}^{(m)})w_2u_{j_1}^{(m)}w_1\psi_{\breve{d}_1\cdots\breve{d}_m}(u_{j_1+1}^{(m)}\cdots u_{j_2-1}^{(m)})w_2u_{j_2}^{(m)}w_1.$$

Notice that $\mathsf{Alp}(\psi_{\breve{d}_1\cdots\breve{d}_m}(u_1^{(m)}\cdots u_{j_1-1}^{(m)})), \mathsf{Alp}(\psi_{\breve{d}_1\cdots\breve{d}_m}(u_{j_1+1}^{(m)}\cdots u_{j_2-1}^{(m)})) \subseteq \Sigma'$. Thus, if $|\psi_{\breve{d}_1\cdots\breve{d}_m}(u_1^{(m)}\cdots u_{j_1-1}^{(m)})w_2| \geq m+1$, one sets $r = 1$, otherwise since $|w_1| + |w_2| = |\mathsf{Pali}(d_1\cdots d_m)| \geq m$, one can set $r = |\psi_{\breve{d}_1\cdots\breve{d}_m}(u_1^{(m)}\cdots u_{j_1-1}^{(m)})w_2u_{j_1}^{(m)}| \leq m+1$. □

The next Lemma establishes that in any directive sequence Δ there are arbitrarily long "local parts" $d_{p_k}\cdots d_{q_k}$ which have bounded $\mathsf{lrt}_{\Sigma'}(d_{p_k}\cdots d_{q_k})$ for some alphabet $\Sigma' \subseteq \Sigma$ and this bound is the same for all $k \in \mathbb{N}$.

Lemma 3.6. *Let $\Delta = d_1, d_2, \cdots$ be an infinite sequence over a finite alphabet Σ. There exist $K \in \mathbb{N}$, an alphabet $\Sigma_b \subseteq \Sigma$ and sequences $\langle p_k \rangle_{k=1}^{\infty}, \langle q_k \rangle_{k=1}^{\infty}$ such that for every $k \in \mathbb{N}$:*

(i) $p_k \leq q_k$ *for every* $k \in \mathbb{N}$,

(ii) $q_k - p_k = \omega(1)$, i.e., *the sequence* $\langle q_k - p_k \rangle_{k=1}^{\infty}$ *is unbounded,*

(iii) $\mathsf{Alp}(d_{p_k} \cdots d_{q_k}) \subseteq \Sigma_b$,

(iv) $\mathsf{lrt}_{\Sigma'}(d_{p_k} \cdots d_{q_k}) \leq K$, i.e., *every letter from* Σ_b *has letter return time bounded by* K *in* $d_{p_k} \cdots d_{q_k}$.

Proof. Let us have $\Sigma_b, \Sigma_f, \Sigma_u \subseteq \Sigma$, where Σ_b contains exactly the letters having bounded letter return time in Δ, Σ_f contains exactly the letters occurring only finitely many times in Δ, and Σ_u contains every other letter. We will manipulate these sets throughout the proof.

First, we drop a prefix of Δ containing letters from Σ_f, leaving a directive sequence consisting only of letters from Σ_b and Σ_u. Let K be an upper bound on the letter return time for all letters from Σ_b. If $\Sigma_u = \emptyset$, we are done since any subsequence of Δ has letter return time bounded by K and it consists only of letters from Σ_b.

If Σ_u is non-empty, let us pick a letter $a \in \Sigma_u$ and remove it from Σ_u. We split the whole infinite sequence Δ into subsequences based on the occurrence of a. Let $a_1, a_2, \ldots$ be all occurrences of a in Δ. We set $\langle p_k \rangle_{k=0}^{\infty}$ and $\langle q_k \rangle_{k=0}^{\infty}$ to be infinite sequences of integers such tha $p_k := a_k + 1$ and $q_k := a_{k+1} - 1$ for every $k \in \mathbb{N}$ except if $a_k + 1 = a_{k+1}$. Additionaly, we remove all p_i, q_i such that $q_i - p_i \leq q_{i-1} - p_{i-1}$. In other words, we make $q_k - p_k$ strictly increasing. We also know that the alphabet of these sequence is $\Sigma_b \cup \Sigma_u$, *i.e.*, $\mathsf{Alp}(d_{p_k} d_{p_k+1} \cdots d_{q_k}) \subseteq \Sigma_b \cup \Sigma_u$ for all $k \in \mathbb{N}$.

Next, we inductively remove all letters from Σ_u, either moving them to Σ_b, or removing them completely. We maintain that our sequences $\langle p_k \rangle_{k=0}^{\infty}$ and $\langle q_k \rangle_{k=0}^{\infty}$ satisfy properties (i), (ii), $\mathsf{Alp}(d_{p_k} \cdots d_{q_k}) \subseteq \Sigma_b \cup \Sigma_u$ for all $k \in \mathbb{N}$, and that there exist $K \in \mathbb{N}$ such that $\mathsf{lrt}_{\Sigma_b}(d_{p_k} \cdots d_{q_k}) \leq K$ for every $k \in \mathbb{N}$. Once we remove all letters from Σ_u, the proof is finished since $\mathsf{Alp}(d_{p_k} \cdots d_{q_k}) \subseteq \Sigma_b$ and there is a suitable bound K.

Let $b \in \Sigma_u$ and for each $k \in \mathbb{N}$ let $b_{1,k}, b_{2,k}, \ldots, b_{n_k,k}$ be the sequence of occurrences of the letter b in $d_{p_k} \cdots d_{q_k}$ where $n_k \in \mathbb{N}_0$. We split sequences $\langle p_k \rangle_{k=0}^{\infty}$ and $\langle q_k \rangle_{k=0}^{\infty}$ based on the occurrence of the letter b. In other words, let $\langle p'_k \rangle_{k=0}^{\infty}$ be the sequence of integers defined as

$$p_1, b_{1,1} - 1, b_{2,1} - 1, \ldots, b_{n_1,1} - 1, p_2, b_{1,2} - 1, \ldots, b_{n_2,2} - 1, p_3, \ldots$$

and similarly, we define $\langle q'_k \rangle_{k=0}^{\infty}$ as

$$b_{1,1}+1, b_{2,1}+1, \ldots, b_{n_1,1}+1, q_1, b_{1,2}+1, \ldots, b_{n_2,2}+1, q_2, \ldots$$

We remove all q'_k and p'_k such that $q'_k \leq p'_k$. Observe that $\mathsf{Alp}(d_{p'_k} \cdots d_{q'_k}) = \Sigma_b \cup \Sigma_u - \{b\}$ for all $k \in \mathbb{N}$.

We distinguish two cases:

1. There is a number $D \in \mathbb{N}$ such that $q'_k - p'_k \leq D$ for all $k \in \mathbb{N}$. Let K be the previous upper bound on the letter return time in $d_{p_k} \cdots d_{q_k}$ for every $k \in \mathbb{N}$. We move letter b to Σ_b and we set our new upper bound on the letter return time to be $\max(D, K)$. Since all invariants are maintained for $\langle p_k \rangle_{k=0}^{\infty}$ and $\langle q_k \rangle_{k=0}^{\infty}$ and $\max(D, K)$, we have proved the induction step.

2. There is no number $D \in \mathbb{N}$ such that $q'_k - p'_k \leq D$ for all $k \in \mathbb{N}$. We remove all p'_i, q'_i such that $q'_i - p'_i < q'_{i-1} - p'_{i-1}$. We remove letter b from Σ_u. Notice that $\langle p'_k \rangle_{k=0}^{\infty}$ and $\langle q'_k \rangle_{k=0}^{\infty}$ satisfy all invariants with the same bound K, which completes the induction step.

Therefore, we can remove all letters from Σ_u which ends the proof. □

Let us demostrate this process on the sequence $\Delta = d_1 d_2 \cdots \in \{a, b, c\}^{\omega}$ defined as follows:

$$d_i = \begin{cases} a \text{ if } i = 2^k \text{ for some } k \in \mathbb{N} \\ b \text{ if } i = 2^{2k} + j \text{ for some } k, j \in \mathbb{N}_0, 1 \leq j < 2^{2k} \\ c \text{ otherwise} \end{cases}$$

This induces

$$\Delta = aacabbbacccccccabbbbbbbbbbbbbbbacccccccccccccccc \cdots$$

We have $\Sigma_b = \emptyset$ and $\Sigma_u = \{a, b, c\}$. We split the sequence Δ into the following intervals using the letter a and we remove a from Σ_u:

$$\begin{aligned} &[\![d_{p_1}, d_{q_1}]\!] = c \\ &[\![d_{p_2}, d_{q_2}]\!] = bbb \\ &[\![d_{p_3}, d_{q_3}]\!] = ccccccc \\ &[\![d_{p_4}, d_{q_4}]\!] = bbbbbbbbbbbbbbb \\ &\ldots \end{aligned}$$

Since a was in Σ_u, the lengths of these sequences are unbounded. Let us choose the letter b as the next letter from Σ_u. We obtain new sequences $\langle p'_k \rangle_{k=0}^{\infty}$ and $\langle q'_k \rangle_{k=0}^{\infty}$ which look as follows:

$$\begin{aligned}
&[\![d_{p'_1}, d_{q'_1}]\!] = c\\
&[\![d_{p'_2}, d_{q'_2}]\!] = ccccccc\\
&[\![d_{p'_3}, d_{q'_3}]\!] = ccccccccccccccccccccccccccccccc\\
&\ldots
\end{aligned}$$

Since the lengths of these sequences are unbounded, we remove b from Σ_u and we proceed to letter c. However, we notice that the letter c occurs with a letter return time bounded in all these sequences. Since it is the only letter occurring in these sequences and their lengths are unbounded, we have found suitable "local parts" of the directive sequence.

Before we prove Lemma 3.14 which deals with the mismatches in $\pi_w(ab, ba)$, we need a few more supplementary propositions.

Proposition 3.7. *Given an alphabet Σ, $u, w \in \Sigma^*$, $a, b \in \Sigma$ such that $a, b \notin \mathsf{Alp}(w)$, we have*

$$\pi_{uaw}(ab, ba) = \pi_u(\mathsf{Pali}(aw)\mathsf{Pali}(aw)b, \mathsf{Pali}(aw)b\mathsf{Pali}(aw)).$$

We say $|aw|$ is the depth rate *of uaw.*

Proof.

$$\begin{aligned}
\pi_{uaw}(ab, ba) &\overset{(1)}{=} \pi_{ua}(\psi_w(ab), \psi_w(ba))\\
&\overset{(2)}{=} \pi_{ua}(\mathsf{Pali}(w)a\mathsf{Pali}(w)b, \mathsf{Pali}(w)b\mathsf{Pali}(w)a)\\
&\overset{(3)}{=} \pi_u(\mathsf{Pali}(aw)\mathsf{Pali}(aw)b \ominus \mathsf{Pali}(aw)b\mathsf{Pali}(aw)).
\end{aligned}$$

Part (1) holds since $\psi_{aw}(ab) \ominus \psi_{aw}(ba) = \psi_a\psi_w(ab) \ominus \psi_a\psi_w(ba) = \pi_a(\psi_w(ab), \psi_w(ba))$. Implication (2) is an application of Proposition 3.4. In implication (3) we simply use Theorem 3.2. □

Proposition 3.8. *Let Σ be an alphabet and $w \in \Sigma^*$. Then $|\mathsf{Pali}(w)| \leq 2^{|w|+1}$.*

Proof. From the definition of Pali, $\mathsf{Pali}(wx) = (\mathsf{Pali}(w)x)^{(+)}$ which is the longest if $x \notin \mathsf{Alp}(w)$ in which case $\mathsf{Pali}(wx) = \mathsf{Pali}(w)x\mathsf{Pali}(w)$ and therefore, $|\mathsf{Pali}(wx)| \leq 2|\mathsf{Pali}(w)| + 1$. Since $|\mathsf{Pali}(x)| = 1$ for every letter $x \in \Sigma$, we can bound the length of $\mathsf{Pali}(w)$ by $2^{|w|+1}$. □

Proposition 3.9. *Let Σ be an alphabet and $a, b \in \Sigma$ such that $a \neq b$. Then, for every $u \in \Sigma$, $\mathsf{Pali}(u)$ is a prefix of $\psi_u(ab)$.*

Proof. We prove it inductively on the length of u. The case $u \in \Sigma$ is trivial. Now, if $a \notin \mathsf{Alp}(u)$, we have that $\mathsf{Pali}(u)a = \psi_u(a)$. Otherwise, let $v, w \in \Sigma^*$ such that $u = vaw$ and $a \notin \mathsf{Alp}(w)$. Then, using Proposition 3.4, we have that $\mathsf{Pali}(u)$ is a prefix of $\psi_u(ab) = \psi_u(a)\psi_u(b)$ if $\mathsf{Pali}(v)$ is a prefix of $\psi_v(\psi_{bw}(a))$ which holds using the inductive hypothesis. □

Proposition 3.10. *Let Σ be an alphabet. Let $w, v \in \Sigma^*, a \in \Sigma$ such that $a \in \mathsf{Alp}(w)$. Then $\psi_w(a)$ is a prefix of $\psi_w(va)$.*

Proof. The case of $v = a$ is trivial. Let us assume $v \neq a$. Since $a \in \mathsf{Alp}(w)$, then using the Proposition 3.4, $\psi_w(a)$ is a prefix of $\mathsf{Pali}(w)$. Now, Proposition 3.9 gives us that $\mathsf{Pali}(w)$ is a prefix of $\psi_w(va)$ which ends the proof. □

Let us recall that for a word $w \in \Sigma$, $w^\diamond$ is the trimmed version of w removing all initial and trailing $\bullet$.

Proposition 3.11. *Given an alphabet Σ, let $u, w, v \in \Sigma^*$, $a, u_1, \ldots, u_n \in \Sigma$ where $u = u_1 \cdots u_n$ and $a \notin \mathsf{Alp}(v) \cup \mathsf{Alp}(u)$. Then:*

$$\pi_{wav}(au, ua) = \bullet^{|\psi_{wav}(a)|+c}(\pi_{wav}(au_1, u_1a))^\diamond \bullet^c (\pi_{wav}(au_2, u_2a))^\diamond \cdots \bullet^c (\pi_{wav}(au_n, u_na))^\diamond$$

where $c = |\mathsf{Pali}(w)|$. Furthermore, the longest common prefix of $\psi_{wav}(au)$ and $\psi_{wav}(ua)$ is $\mathsf{Pali}(w)$.

Proof. Let us first characterise $\pi_{wav}(au_i, u_ia)$ for any $i \in 1, \ldots, k$. We know that $u_i \neq a$.

If $u_i \in \mathsf{Alp}(v)$, let $v = v'u_iv''$ where $v', v'' \in \Sigma^*$ such that $u_i \notin \mathsf{Alp}(v'')$. Then,

$$\begin{aligned}
\psi_{wav}(a) &= \psi_{wav'}(\mathsf{Pali}(u_iv')a) && \text{Proposition 3.4} && (3.7)\\
&= \psi_{wav'}(\mathsf{Pali}(u_iv'))\psi_{wav'}(a) && && (3.8)\\
&= \psi_{wav'u_i}(\mathsf{Pali}(v')u_i)\psi_{wav'}(a) && \text{Proposition 3.4} && (3.9)\\
&= \psi_{wav}(u_i)\psi_{wav'}(a) && \text{Proposition 3.3,} && (3.10)
\end{aligned}$$

(A) $\psi_{wav'u_iv''}(a)$ is a prefix of $\psi_{wav'u_iv''}(u_ia)$ from Proposition 3.10,

(B) $\psi_{wav'}(a)$ is a prefix of $\psi_{wav'u_iv''}(a)$ from (3.10) and (A),

(C) $\mathsf{Pali}(wav')$ is a prefix of $\psi_{wav'}(\psi_{u_iv''}(a))$ from Proposition 3.9 and since $a \neq u_i$,

(D) $\psi_{wav'u_iv''}(a)$ is a prefix of $\mathsf{Pali}(wav'u_iv'')$ from Proposition 3.4,

(E) $\mathsf{Pali}(wav') = \psi_{wav'}(a)\mathsf{Pali}(w)$ from 3.4,

(F) $\mathsf{Pali}(wav')u_i$ is a prefix of $\psi_{wav'u_iv''}(a)$ from (D) and (E),

(G) $\mathsf{Pali}(w)$ is a prefix of $\psi_{wav'u_iv''}(u_i)$ from 3.9 and since $a \neq u_i$,

(H) $\psi_{wav'u_iv''}(u_i)$ is a prefix of $\mathsf{Pali}(wav'u_iv')$ from Proposition 3.4,

(I) $\mathsf{Pali}(w)a$ is a prefix of $\psi_{wav'u_iv''}(u_i)$ from (G) and (H),

Based on (I), let $U_i \in \Sigma^*$ such that $\psi_{wav'u_iv'}(u_i) = \mathsf{Pali}(w)aU_i$. Similary, based on (F), let $A_i \in \Sigma^*$ such that $\psi_{wav'u_iv'}(a) = \mathsf{Pali}(wav')u_iA_i$. Putting all together, we get

$$\begin{aligned}
\pi_{wav}(au_i, u_ia) &= \psi_{wav}(au_i) \ominus \psi_{wav}(u_ia) \\
&= \psi_{wav}(u_i)\psi_{wav'}(a)\psi_{wav}(u_i) \ominus \psi_{wav}(u_i)\psi_{wav}(a) && \text{from (3.10)} \\
&= \psi_{wav}(u_i)\psi_{wav'}(a)\psi_{wav}(u_i) \ominus \psi_{wav}(u_i)\mathsf{Pali}(wav')u_iA_i && \text{definition of } A_i \\
&= \psi_{wav}(u_i)\psi_{wav'}(a)\psi_{wav}(u_i) \ominus \psi_{wav}(u_i)\psi_{wav'}(a)\mathsf{Pali}(w)u_iA_i && \text{from (E)} \\
&= \bullet^{|\psi_{wav}(a)|}\mathsf{Pali}(w)aU_i \ominus \mathsf{Pali}(w)u_iA_i && \text{definition of } U_i \\
&= \bullet^{|\psi_{wav}(a)|+|\mathsf{Pali}(w)|}aU_i \ominus u_iA_i
\end{aligned}$$

Since the last letter of U_i is u_i and the last letter of A_i is a, we have completely described $\pi_{wav}(au_i, u_ia) = \bullet^{|\psi_{wav}(a)|+|\mathsf{Pali}(w)|}aU_i \ominus u_iA_i$.

Taking a closer look at the case $u_i \notin \mathsf{Alp}(v)$, we notice that the situation is practically symmetrical. If $u_i \notin \mathsf{Alp}(w)$, then one gets that $\mathsf{Pali}(w)u_i$ is a suffix of $\psi_{wav}(u_i)$ and $\mathsf{Pali}(w)a$ is a prefix of $\psi_{wav}(u_1)$. If $u_i \in \mathsf{Alp}(w)$, then the situation is symmetrical. We leave to the reader to check that both words $\psi_{wav}(au_i)$ and $\psi_{wav}(u_ia)$ have $\psi_{wav}(a)\mathsf{Pali}(w)$ as a prefix.

We finish the proof using an induction on the length of u. The base case was established above. Using Proposition 3.10, we know that $\psi_{wav}(a)$ is a prefix of $\psi_{wav}(u_ka)$. Hence, the prefix of $P_k := \pi_{wav}(au_1 \cdots u_k, u_1 \cdots u_ka)$ of length $|\psi_{wav}(au_1 \cdots u_{k-1})|$ is of the form $\pi_{wav}(au_1 \cdots u_{k-1}, u_1 \cdots u_{k-1}a)$ which is of the form as stated above, using the inductive hypothesis. Hence, we need to analyse the suffix of P_k of length $|\psi_{wav}(u_k)|$ which is identical to the suffix of $\pi_{wav}(au_k, u_ka)$ of length $|\psi_{wau_k}(u_k)|$ which is identical to $\bullet^{|\mathsf{Pali}(w)|}\pi_{wav}(au_k, u_ka)^{\diamond}$ as we have proved above. Thus, the induction is complete.

The additional statement is trivial since $\psi_{wav}(a)\mathsf{Pali}(w) = \mathsf{Pali}(u)$ from Proposition 3.4. □

Proposition 3.12. *Let Σ be an alphabet, $w \in \Sigma^*$, and $a, b, c \in \Sigma$ be three distinct letters. Then*

$$|\psi_w(a)| + |\psi_w(b)| > |\psi_w(c)|.$$

Proof. We prove it inductively on the length of w. The base case is trivial since $|a| + |b| > |c|$. Let $w = ud$ for some $u \in \Sigma^*, d \in \Sigma$ and we prove the induction step. First we assume that $d = a$ (without loss of generality, this the same as $d = b$). Then, since $\psi_w(a) = \psi_u(a)$, $\psi_w(b) = \psi_u(d)\psi_u(b)$, and $\psi_w(c) = \psi_u(d)\psi_u(c)$, we have

$$|\psi_w(a)| + |\psi_w(b)| = |\psi_u(a)| + |\psi_w(d)| + |\psi_u(b)| > |\psi_w(d)| + |\psi_u(c)| = |\psi_w(c)|$$

where the inequality is due to the inductive hypothesis. If $d = c$, then we do the same to obtain

$$|\psi_w(a)| + |\psi_w(b)| = |\psi_u(a)| + 2|\psi_w(d)| + |\psi_u(b)| > |\psi_w(c)|.$$

□

Proposition 3.13. *Let Σ be an alphabet with at least two letters, $u \in \Sigma^*$, and $a, b \in \Sigma$, such that $a \neq b$. Then, for any $k \in \mathbb{N}$,*

$$|\psi_{uab^k}(b)| \leq |\psi_{uab^k}(c)| \leq (k+2)|\psi_{uab^k}(b)|$$

for every $c \in \Sigma$.

Proof. The case of $c = b$ is trivial. Thus, we assume $c \neq b$. Firstly $|\psi_{uab^k}(b)| = |\psi_{ua}(b)| \leq |\psi_{uab^{k-1}}(b)| + |\psi_{uab^{k-1}}(c)| = |\psi_{uab^k}(c)|$. We also have $|\psi_{ua}(a)| \leq |\psi_{ua}(b)|$. Let $D := |\psi_{ua}(b)|$. Then, using Proposition 3.12, we have that $|\psi_{ua}(c)| \leq |\psi_{ua}(a)| + |\psi_{ua}(b)| \leq 2D$. Now, since $\psi_{uab^k}(c) = \psi_{ua}(b^k c)$, we have that $|\psi_{uab^k}(c)| = kD + |\psi_{ua}(c)| \leq (k+2)D$. □

Lemma 3.14. *Let $\rho \in \mathbb{R}^+$, $K \in \mathbb{N}$, Σ an alphabet, and $\Sigma' \subseteq \Sigma$ an alphabet. Let $D := 6^{K+1}$, $l := \lceil \log_{\frac{D-1}{D}}(\rho) \rceil$. For any words $u_0 \in \Sigma^*, u \in \Sigma'^*$ with $|u| \geq K(l+1)$ and $\mathsf{lrt}_{\Sigma'}(u) \leq K$, one has that for all $a, b \in \Sigma'$, all mismatches of $\pi_{u_0 u}(ab, ba)$ can be grouped into at most $2^{(K+2)l}$ intervals with the overall length smaller than $\rho \cdot |\pi_{u_0 u}(ab, ba)|$.*

Proof. Without loss of generality, let $u = v_1 a v_2 b v_3$ where $v_1, v_2, v_3 \in \Sigma^*$ such that $a, b \notin \mathsf{Alp}(v_3)$ and $a \notin \mathsf{Alp}(v_2)$. Since we have bounded letter return time, we know that $|v_2 b v_3| < K$. Using Proposition 3.7, we have

$$\pi_{u_0 u}(ab, ba) = \bullet^{|\psi_{u_0 v_1 a v_2}(\mathsf{Pali}(b v_3))|} \pi_{u_0 v_1 a v_2}(\mathsf{Pali}(b v_3) a, a \mathsf{Pali}(b v_3))$$

and using Proposition 3.11, we have

$$\pi_{u_0v_1av_2}(\mathsf{Pali}(bv_3)a, a\mathsf{Pali}(bv_3)) =$$
$$\bullet^{|\mathsf{Pali}(u_0v_1av_2)|}(\pi_{u_0v_1av_2}(w_1a, aw_1))^{\diamond}\bullet^{c}(\pi_{u_0v_1av_2}(w_2a, aw_2))^{\diamond}\cdots\bullet^{c}(\pi_{u_0v_1av_2}(w_ka, aw_k))^{\diamond}$$

where $c = |\mathsf{Pali}(u_0v_1)|, k \in \mathbb{N}$, $w_1w_2\cdots w_k = \mathsf{Pali}(bv_3)$, and $k \leq 2^{K+2}$ from $|v_3| < K$ and Proposition 3.8.

Let us call the application of both Propositions to some $\pi_w(cd, dc)^{\diamond}$, $w \in \Sigma^*, c, d \in \Sigma'$ a *recursion call*. We can apply a recursion call if $\mathsf{lrt}_{\Sigma'}(w) \leq K$. Notice that for every $i \in 1, \ldots, k$ we can apply a recursion call for $\pi_{u_0v_1av_2}(w_ia, aw_i)^{\diamond}$ again. Since $|u| \geq K(l+1)$, we can get to the recursion depth of at least l. Let us then apply the recursion call to the depth l which splits the whole word $\pi_{u_0u}(ab, ba)$ into $(2^{K+2})^l$ words of type $\pi_w(cd, dc)^{\diamond}$ for some $w \in \Sigma^*, c, d \in \Sigma$. All other parts of the word are matches. Hence, we can group all mismatches into at most $(2^{K+2})^l$ intervals.

Let us now focus on the proportion of mismatches compared to the overall length. We do it by bounding the number of matches from below. In each recursion call, we split a word of type $\pi_w(cd, dc)^{\diamond}$ into $k \in \mathbb{N}$ words $\pi_{w'}(f_ie_i, e_if_i)$ and $k-1$ words $\bullet^{\mathsf{Pali}(w'')}$ where $w, w', w'' \in \Sigma^*$, $c, d, e_i, f_i \in \Sigma'$ for every $i \in 1, \ldots, k$, and w' is a prefix of w'' satisfying $|w''|+K \geq |w'|$ (the last inequality is from the fact that last K letters of w' contain all letters from Σ' which follows from the bounded letter return time assumption). Now,

$$\begin{aligned}
|\pi_w(cd, dc)^{\diamond}| &\leq \sum_{i=1}^{k} |\psi_{w'}(e_if_i)| + (k-1)\cdot|\mathsf{Pali}(w'')| && \text{since } |\psi_w(a)| \leq 2\cdot|\mathsf{Pali}(w)| \\
&\leq k\cdot|\mathsf{Pali}(w')| + (k-1)\cdot|\mathsf{Pali}(w'')| && \text{using Proposition 3.4} \\
&\leq (3^K+1)k\cdot|\mathsf{Pali}(w'')| && \text{since } |\mathsf{Pali}(w')| \leq 3^K\cdot|\mathsf{Pali}(w'')| \\
&\leq 6^{K+1}\cdot|\mathsf{Pali}(w'')| && \text{since } k \leq 2^{K+1}
\end{aligned}$$

Therefore, by applying a recursion call to $\pi_w(cd, dc)^{\diamond}$, we are guaranteed that at least

$$\frac{(k-1)|\mathsf{Pali}(w'')|}{|\pi_w(cd, dc)^{\diamond}|} \geq \frac{1}{6^{K+1}}.$$

of the length of the word $\pi_w(cd, dc)^{\diamond}$ is to be a match. We use $D = 6^{K+1}$.

This bound assumes that for every $i \in 1, \ldots, d$ $\pi_{w'}(f_ie_i, e_if_i)$ consists only of mismatches. However, we can apply a recursion call again to get the same bound for each subpart individually. Therefore, since we have applied the recursion call to the depth l, we get that the length of matches compared to the overall length is bounded below by

$$\begin{aligned}
&\overbrace{\frac{1}{D}+\frac{D-1}{D}\left(\frac{1}{D}+\frac{D-1}{D}\left(\frac{1}{D}\cdots\right)\right)}^{l\text{-times}} \\
&=\frac{1}{D}\cdot\sum_{j=0}^{l}\frac{D-1}{D} \\
&=\frac{1}{D}\cdot\frac{1-\left(\frac{D-1}{D}\right)^{l}}{1-\frac{D-1}{D}} \\
&=1-\left(\frac{D-1}{D}\right)^{l}.
\end{aligned}$$

From the definition of l, the overall length of intervals is smaller than $\left(\frac{D-1}{D}\right)^{l}\leq\rho$. $\square$

Theorem 3.15. *Every episturmian word is an echoic sequence.*

Proof. Given an alphabet Σ, consider an episturmian word $\boldsymbol{u}$ with a directive sequence $\breve{\Delta}=\breve{d}_1\breve{d}_2\cdots$ and infinite sequence $\langle\boldsymbol{u}^{(n)}\rangle_{n=0}^{\infty}$ of infinite recurrent words satisfying $\boldsymbol{u}^{(0)}=\boldsymbol{u}$ and $\boldsymbol{u}^{(i)}=\psi_{\breve{d}_i}(\boldsymbol{u}^{(i+1)})$ for every $i\in\mathbb{N}_0$. We are given $m\in\mathbb{N}$ and $\rho\in(0,1]$ and we want to find suitable sequences $\langle r_n\rangle_{n=1}^{\infty}$ and $\langle s_n\rangle_{n=1}^{\infty}$, $d\in\mathbb{N}$ and intervals $x_{n,i},y_{n,i}$ for every $n\in\mathbb{N},i\in 1,\ldots,d$ satisfying the echoic conditions.

Using Lemma 3.6, we get sequences $\langle p_n\rangle_{n=1}^{\infty}$ and $\langle q_n\rangle_{n=1}^{\infty}$, alphabet Σ_b and a bound K satisfying the conditions from the Lemma. We set $D:=6^{K+1}$, $l:=\lceil\log_{\frac{D-1}{D}}(\rho)\rceil$, $d:=m2^{(K+2)l}$. Let $n_0\in\mathbb{N}$ such that $q_n-p_n>K(l+1)+m$ for every $n\in\mathbb{N},n>n_0$. For every $n\in\mathbb{N}$, $n>n_0$, we define

$$s_n:=|\psi_{q_n-m-1}(d_{fq_n-m})|.$$

Though the choice of this might seem mysterious, it will be clear from the usage of different Lemmas. Basically, we shift the whole sequence $\boldsymbol{u}$ by s_n which gives us that $\boldsymbol{u}\ominus T^{s_n}(\boldsymbol{u})$ can be seen as a sequence $w'w_{c_1}w_{c_2}\cdots$ where

$$w_c=\begin{cases}\pi_{\breve{d}_1\cdots\breve{d}_{q_n-m-1}}(d_{q_n-m}c,cd_{q_n-m}) \text{ if } c\neq d_{q_n-m}\\ \pi_{\breve{d}_1\cdots d_{q_n-m-1}}(d_{q_n-m},d_{q_n-m})=\bullet^{|\psi_n(d_{q_n-m})|} \text{ otherwise,}\end{cases}$$

w' is a prefix of $w_{u_1^{(q_n-m)}}$, and $c_1c_2c_3\cdots=T(\boldsymbol{u}^{(q_n-m)})$. This is a trivial observation we have done in the previous two sections. The only difference is the initial prefix w' which is due to the fact that if we apply $\psi_{\breve{w}}$ for some $\breve{w}\in\breve{\Sigma}^*$, we basically apply ψ_w and then apply some shifting factor (see Proposition 3.3).

Now, we want to reduce the problem to finding mismatches in individual w_c for some c but first, we need to ensure that the "local part" of the directive sequence behaves like a sequence with a bounded letter return time. We give Lemma 3.5 the sequence $\boldsymbol{u}^{(q_m-m)}$ and it provides us with r such that

$$\mathsf{Alp}(u_{r+1}^{(q_n-m)}\cdots u_{r+m}^{(q_n-m)}) \subseteq \mathsf{Alp}(\breve{d}_{q_n-m+1}\cdots \breve{d}_{q_n}) \subseteq \Sigma'.$$

The latter inclusion follows from the property (iii) of Lemma 3.6 we used to construct q_n. This implies that $c_r, c_{r+1}, \dots, c_{r+m} \in \Sigma'$ and since $d_{q_n-m-1-K(l+1)}, \dots, d_{q_n-m-1} \in \Sigma'$ again from the property (iii) of Lemma 3.6, we can apply Lemma 3.14 separately for every $w_{c_r}, \dots, w_{c_{r+m}}$ to obtain $m \cdot 2^{(K+2)l}$ intervals of mismatches with overall length smaller than $\rho \cdot |w_{c_r}\cdots w_{c_{r+m}}|$.

The one last thing we need to do is to define the shift r_n because $w_{c_r}\cdots w_{c_{r+m}}$ is not necessarily a prefix of $\boldsymbol{u} \ominus T^{s_n}(\boldsymbol{u})$. We set

$$r_n := |w' w_{c_1} \cdots w_{c_{r-1}}|$$

which solves this problem but we need to ensure that r_n/s_n is smaller than some constant. We have

$$\begin{aligned} r_n &\leq |\psi_{d_1\cdots d_{q_n-m-1}}(u_1^{(q_n-m)}\cdots u_{2r}^{(q_n-m)})| && \text{since } |\psi_{\breve{d}_{q_n-m}}(u_1^{(q_n-m)}c_1\cdots c_{r-1})| \leq 2r \\ &\leq K|\psi_{d_1\cdots d_{q_n-m-1}}(d_{q_n-m})| && \text{Proposition 3.13, bound } K \text{ and assuming } |\Sigma'| \geq 2 \\ &\leq K \cdot s_n. \end{aligned}$$

Notice that we have assumed that $|\Sigma'| \geq 2$ so we can apply the Proposition. However, if the contrary is true and $\Sigma' = \{a\}$ for some $a \in \Sigma$, then it means the partial quotients[4] of Δ are unbounded which means the diophantine exponent of $\boldsymbol{u}$ is infinite [**24**]. □

Combining Theorem 2.15 and Theorem 3.15, we obtain

Corollary 3.16. *Let $\boldsymbol{u} = u_1u_2\cdots \in \mathcal{H}$ be an episturmian word over an finite alphabet of algebraic numbers $\mathcal{H}$ and an algebraic number $\beta, |\beta| > 1$. Then, the number $\alpha := \sum_{j=1}^{\infty} u_j\beta^{-j}$ is transcendental or belongs to $\mathbb{Q}(\beta, \mathcal{H})$.*

Corollary 3.17. *Every infinite word $\boldsymbol{u}$ for which there exists an echoic word $\boldsymbol{v}$ and $k \in \mathbb{N}$ such that $\boldsymbol{u} = T^k(\boldsymbol{v})$ or $T^k(\boldsymbol{u}) = \boldsymbol{v}$, is echoic.*

[4]Partial quotients of a directive sequence $\Delta = d_1^{s_1}d_2^{s_2}\cdots$ are $s_1, s_2 \in \mathbb{N}$.

Proof. For a given $m \in \mathbb{N}$ and positive real ρ, let $\langle r'_n \rangle_{n=1}^{\infty}$ be the sequence obtained from echoic word $\boldsymbol{v}$ for any $m+1$ and ρ. If $\boldsymbol{u} = T^k(\boldsymbol{v})$, then for $\boldsymbol{u}$ we set $r_n := \min(r'_n - k, 0)$ and everything else as for $\boldsymbol{v}$ If $T^k(\boldsymbol{u}) = \boldsymbol{v}$, then for $\boldsymbol{u}$ we set $r_n := r'_n + k$, and everything else as for $\boldsymbol{v}$. It is easy to check that all properties are satisfied for big enough n. □

Conclusion

Ferenczi and Maduit initiated research focused on proving transcendence of aperiodic sequences with low subword complexity [**16**]. Major contributions were made by Adamczewski and Bugeaud proving that every number whose b-expansion has linear subword complexity is rational or transcendental [**3**]. However, both of these results assume the base to be an integer and all results which concern the base to be an algebraic number restrict the set to Pisot numbers or to numbers big enough in terms of diophantine exponent [**2**].

The main contribution of this thesis is providing a combinatorial criterion that allows us to prove a transcendence result for any algebraic base β, $|\beta| > 1$. In section 2.5, we defined the notion of echoic sequences which is a generalisation of stuttering sequences presented in [**22**]. We proved that for a finite set of algebraic numbers $\mathcal{H}$, algebraic number β, $|\beta| > 1$, and an echoic sequence $\boldsymbol{u} \in \mathcal{H}^\omega$, $\boldsymbol{u}_\beta$[5] is transcendental or belongs to $\mathbb{Q}(\beta, \mathcal{H})$ (Theorem 2.15).

For future work, we proposed parametrization of the definition of echoic sequences which generalises the concept of diophantine exponent. Since diophantine exponent is an established tool for proving transcendence of numbers related to various infinite sequences, this concept could prove very useful in future research. We propose the class of automatic sequences and the class of sequences with linear subword complexity to be the target of the subsequent research.

In the third chapter, we showed that any Episturmian word is an echoic sequence (Theorem 3.15) which implies that any Sturmian or Arnoux-Rauzy sequence is an echoic sequence and thus the Theorem 2.15 can be applied to it.

In the case of Sturmian words, one can strengthen this result by proving that the related number cannot lie in $\mathbb{Q}(\beta, \mathcal{H})$ implying transcendence [**22**]. We conjecture that the same can be obtained for any D-bonacci word, including the Tribonacci word. This conjecture cannot be extended to all episturmian words since there is

[5] $\boldsymbol{w}_\beta = \sum_{j=1}^{\infty} u_j \beta^{-j}$ where $\boldsymbol{u} = u_1 u_2 \cdots$

an episturmian word that is related to a rational number if interpreted in a negative integer base (see end of section 3.1).

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