Equivariant Seidel maps and a flat connection on equivariant symplectic cohomology

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Abstract

We construct shift operators on equivariant symplectic cohomology which generalise the shift operators on equivariant quantum cohomology in algebraic geometry. That is, given a Hamiltonian action of the torus T, we assign to a cocharacter of T an endomorphism of $(S^1 \times T)$-equivariant Floer cohomology based on the equivariant Floer Seidel map. We prove the shift operator commutes with a connection. This connection is a multivariate version of Seidel's q-connection on S^1-equivariant Floer cohomology and generalises the Dubrovin connection on equivariant quantum cohomology. We prove that the connection is flat, which was conjectured by Seidel. As an application, we compute these algebraic structures for a few specific examples and provide a method to compute them for toric manifolds using the moment polytope.

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Chapter 1

Introduction

I have written two academic papers which present the results from my DPhil. The first is *"An intertwining relation for equivariant Seidel maps"* [LJ20] and the second is *"Shift operators and flat connections on equivariant symplectic cohomology"* [LJ21]. These are copied verbatim in Chapter 2 and Chapter 3 respectively. Each paper has its own introduction that explains the context of the paper and its contribution to the literature.

In this introduction, we will present our results together with some background more informally.

1.1 Equivariant symplectic cohomology

We are interested in symplectic manifolds (M, ω) which are *convex*. This condition means M is the union of a compact symplectic manifold with boundary and a symplectic manifold ([1, *1*) , d($R\alpha$)), where ($, \alpha$) is a closed contact manifold and R is the coordinate of [1, *1*). The subset [1, *1*) is the *convex end* of M. The convexity assumption means that, outside of a compact set, M is well-behaved enough that many constructions which are defined for closed symplectic manifolds may be defined on M after a few modifications. One intriguing aspect of convex symplectic manifolds is the interaction between the Reeb dynamics of ($, \alpha$) and the symplectic structure of (M, ω).

Liouville manifolds are a subfamily of convex symplectic manifolds for which ω is exact on all of M. We are also interested in convex symplectic manifolds which are not exact, such as the negative line bundle O_{P^1}(1) over P^1 of degree 1.

The *symplectic cohomology* SH (M) of a convex symplectic manifold is an extremely powerful and commonly-used invariant [Sei08]. By definition, it is the direct limit of Floer cohomology FH (M, λ) as the *slope* λ tends to infinity. In turn, the *(Hamiltonian) Floer cohomology* FH (M, λ) is the cohomology of the Floer cochain complex, which is generated by the Hamiltonian orbits of an S^1-dependent Hamiltonian function $H : S^1$ M ! R.

These *Hamiltonian orbits* are maps $x : S^1 \to M$ which flow along the *Hamiltonian vector field* $X_{H,t}$, which is the S^1-dependent vector field uniquely determined by $\omega(\cdot, X_{H,t}) = dH_t$. We assume the Hamiltonian is *linear of slope* λ, which means that $H = \lambda R +$ constant holds outside of a compact set (independently of S^1).

While symplectic cohomology is constructed using Hamiltonian dynamics, it captures other information about M as well. For example, there is a canonical map $c^* : H^*(M) \to SH^*(M)$, and if this map is not an isomorphism, then Σ has a closed Reeb orbit. This can be shown by choosing a Hamiltonian H which is an (S^1-independent) Morse function on $M \setminus \{\text{convex end}\}$, while on the convex end it is a quadratic function of R until it reaches the desired gradient λ. Such a Hamiltonian is *admissible*. The Hamiltonian orbits of an admissible Hamiltonian are the constant orbits at the critical points of H on $M \setminus \{\text{convex end}\}$ and orbits $x(t) = (R_p, \gamma(tp))$ where γ is a closed Reeb orbit of period $p \in (0, \lambda)$ and $\frac{dH}{dR} = p$ at R_p. The subcomplex generated by the critical points is equivalent to the Morse cochain complex of the Morse function on $M \setminus \{\text{convex end}\}$, and hence it recovers $H^*(M)$ in cohomology. If there are no Reeb orbits, then the constant orbits at the critical points are the only Hamiltonian orbits, and hence $c^* : H^*(M) \to SH^*(M)$ is an isomorphism of cochain complexes.

We are mostly concerned with a variant of symplectic cohomology called *S^1-equivariant symplectic cohomology* $SH^*_{S^1}(M)$, which is the direct limit of *S^1-equivariant Floer cohomology* $FH^*_{S^1}(M, \lambda)$ as $\lambda \to \infty$. This invariant incorporates the natural S^1-action on loops $x : S^1 \to M$ given by

$$(\theta \cdot x)(t) = x(t - \theta) \tag{1.1.1}$$

for $\theta \in S^1$. Viterbo introduced S^1-equivariant symplectic cohomology [Vit96, Section 5] and Seidel gave another construction [Sei08, Section 8b], though the two constructions are isomorphic [BO17, Proposition 2.5]. Algebraically, $SH^*_{S^1}(M)$ is a module over the ring $H^*(BS^1) = \mathbb{Z}[\mathbf{u}]$, where $\mathbf{u}$ has degree 2.

Floer cohomology is inspired by the Morse cohomology of the contractible loop space $\mathcal{L}M = \{\text{contractible } x : S^1 \to M\}$, which is an infinite-dimensional manifold. Similarly, S^1-equivariant Floer cohomology is inspired by the S^1-equivariant Morse cohomology of $\mathcal{L}M$. The localisation theorem for S^1-equivariant (Morse) cohomology [AB84, Theorem 3.5] states that, for a compact manifold X with an S^1-action ρ, the inclusion of the fixed point set $X^{S^1} \hookrightarrow X$ induces an isomorphism

$$H^*_{S^1}(X) \otimes_{\mathbb{Z}[\mathbf{u}]} \mathbb{Q}[\mathbf{u}^{-1}] \cong H^*(X^{S^1}) \otimes_{\mathbb{Z}} \mathbb{Q}[\mathbf{u}^{-1}]. \tag{1.1.2}$$

The fixed point set of $\mathcal{L}M$ equipped with the action (1.1.1) is the set of constant maps, which is naturally isomorphic to M. Zhao showed that S^1-equivariant Floer cohomology has a localisation theorem that corresponds to the inclusion $M \hookrightarrow LM$ [Zha19]. This theorem says there are isomorphisms

$$SH_{S^1}(M) \otimes_{\mathbb{Z}[\mathbf{u}]} \mathbb{Q}[\mathbf{u}^{-1}] \cong FH_{S^1}(M,\lambda) \otimes_{\mathbb{Z}[\mathbf{u}]} \mathbb{Q}[\mathbf{u}^{-1}] \cong H^*(M) \otimes_{\mathbb{Z}} \mathbb{Q}[\mathbf{u}^{-1}]. \tag{1.1.3}$$

She proves this using an (S^1-equivariant) admissible Hamiltonian. Under the localisation operation (that is tensoring with $\mathbb{Q}[\mathbf{u}^{-1}]$), the contributions of the constant orbits at the critical points are preserved while those of the nonconstant orbits are lost.

We can see this in the example of $M = \mathbb{C}^*$ [Zha19, Section 8.3]. For an admissible Hamiltonian on $\mathbb{C}^*$, there are precisely two constant orbits (which correspond to the two generators of $H^*(\mathbb{C}^*)$) and there is[1] one nonconstant orbit for each $k \in \mathbb{Z} \setminus 0$ given by $x_k(t) = R_k e^{2\pi i k t}$ which rotates k times around the origin. The symplectic cohomology is given by

$$SH^{\text{even}}_{S^1}(\mathbb{C}^*) = \mathbb{Z}[\mathbf{u}] \oplus \bigoplus_{k \in \mathbb{Z}\setminus 0} \frac{\mathbb{Z}}{k\mathbb{Z}}[\mathbf{u}] \qquad SH^{\text{odd}}_{S^1}(\mathbb{C}^*) = \mathbb{Z}[\mathbf{u}][+1], \tag{1.1.4}$$

where the copies of $\mathbb{Z}[\mathbf{u}]$ arise from the constant orbits and the k-th summand arises from the orbit x_k. The $[+1]$ means we increase the degree of each element by 1. As is clear, tensoring by $\mathbb{Q}[\mathbf{u}^{-1}]$ kills the contributions of the nonconstant orbits, leaving $H^*(\mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}[\mathbf{u}^{-1}]$.

For an admissible Hamiltonian, we can construct a filtration on the S^1-equivariant Floer cochain complex with respect to which the constant orbits at the critical points form a subcomplex. The resulting quotient complex is only generated by the nonconstant orbits. The direct limit of the resulting cohomology is the *positive S^1-equivariant symplectic cohomology* $SH^{*,+}_{S^1}(M)$. Since the nonconstant orbits correspond to the Reeb orbits on the contact manifold Σ, this invariant is most useful when looking at Reeb dynamics. Bourgeois and Oancea showed that a version of $SH^{*,+}_{S^1}(M)$ sometimes recovers a known contact invariant called *linearized contact homology* [BO17]. Gutt uses a version of $SH^{*,+}_{S^1}(M)$ to distinguish different contact structures on the same manifold [Gut17]. In [MR18], Mclean and Ritter look at convex symplectic manifolds for which $SH_{S^1}(M) = 0$ vanishes, and use a long exact sequence to find an isomorphism $SH^{*,+}_{S^1}(M) = H^{*+1}(M) \otimes \mathbb{F}$ for a given $\mathbb{Z}[\mathbf{u}]$-module $\mathbb{F}$. Through their analysis of the Reeb dynamics, they are able to deduce the Betti numbers of M.

1.1.1 Algebraic structures

S^1-equivariant symplectic cohomology has many algebraic structures which are parallels of analogous structures in S^1-equivariant cohomology. There is, of course, the $\mathbb{Z}[\mathbf{u}]$-module

[1] Technically, the admissible Hamiltonian has the nonconstant orbits for those k with $|k|$ appropriately bounded by the slope λ, however this restriction is lost as $\lambda \to \infty$.

structure which arises from $H^*(BS^1) = \mathbb{Z}[\mathbf{u}]$. There are, in fact, two ways to realise this module structure, either as an endomorphism of BS^1 or as Morse cup product (see Section 2.4.4.1 vs Section 2.4.4.2). We also have the localisation theorem (1.1.3). In addition, there is a long exact sequence

$$\to SH^*_{S^1}(M) \xrightarrow{\mathbf{u}} SH^{*+2}_{S^1}(M) \xrightarrow{\mathbf{u}\mapsto 0} SH^{*+2}(M) \to SH^{*+1}_{S^1}(M) \to \tag{1.1.5}$$

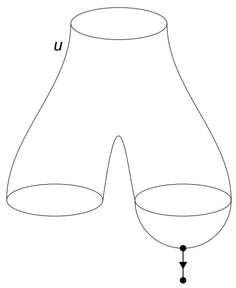
called the *Gysin exact sequence*. This sequence yields a *string bracket* on S^1-equivariant symplectic cohomology [BO13, page 366].

Similarly, $SH^*_{S^1}(M)$ inherits many algebraic structures from symplectic cohomology. There is a module homomorphism $c^*: QH^*_{S^1}(M, \mathrm{Id}_M) \to SH^*_{S^1}(M)$ from the S^1-equivariant quantum cohomology for the trivial S^1-action on M. The filtration arising from admissible Hamiltonians, together with the resulting positive S^1-equivariant symplectic cohomology, also come from symplectic cohomology.

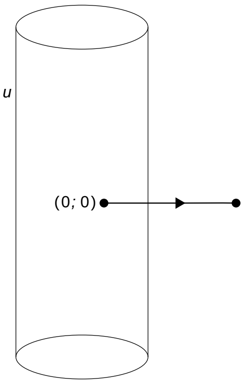
S^1-equivariant symplectic cohomology does not, however, inherit the *pair-of-pants product* from symplectic cohomology. This is because there is no way to simultaneously rotate the three boundary circles of the pair-of-pants configuration. This is in contrast to a cylinder whose two boundary circles can be simultaneously rotated – simply rotate the entire cylinder. This S^1-action on cylinders is how we define the S^1-equivariant Floer differential, which counts cylinders $u : \mathbb{R} \times S^1 \to M$ that satisfy the Floer equation. The symplectic cohomology $SH^*(M)$ is a graded-commutative associative unital algebra with respect to the pair-of-pants product, and this extends to a full TQFT structure on $SH^*(M)$ [Rit13; Sei08, Section (8a)] by using other maps like the pair-of-pants product. Additionally, capping with the PSS map $QH^*(M) \to SH^*(M)$ yields an action of quantum cohomology on symplectic cohomology as in Figure 1.1, which is a map $: QH^*(M) \otimes SH^*(M) \to SH^*(M)$. However, rotating the cylinder moves the point of intersection with the cycle in quantum cohomology, so like the pair-of-pants product, this *quantum action* does not extend to $SH^*_{S^1}(M)$.

In the construction of S^1-equivariant Floer cohomology, the Floer equation depends on points in the space $ES^1 = S^\infty$, so we might also look to use ES^1 to parameterise the point of intersection. For this, we would need an S^1-equivariant map $ES^1 \to S^1$, however none exists: the space ES^1 is contractible, so $\pi_1(ES^1) = 0$, but this is incompatible with S^1-equivariance which implies that the composition $S^1 \xrightarrow{e\cdot} ES^1 \to S^1$ is an isomorphism $S^1 \to S^1$ for some $e \in ES^1$.

We can define an S^1-equivariant map $ES^1 \to S^1$ on a *proper* subset of $W \times ES^1$, and it turns out this is enough to define a map on the S^1-equivariant Floer cochain complex. While this map resembles an S^1-equivariant quantum action, it is not a chain map so it does not induce a map on S^1-equivariant Floer cohomology. The standard way to show that a



(a) Pair-of-pants product with PSS map



(b) Quantum action $\alpha *$ on symplectic cohomology

Figure 1.1: A homotopy argument shows that the pair-of-pants product together with a PSS map (a) can be simplified to a quantum action (b) which counts Floer solutions $u : \mathbb{R} \times S^1 \to M$ which intersect with a locally-finite cycle representing α at $u(0,0)$. Rotating the cylinder by $\theta \in S^1$ would move this intersection point to $u(0,\theta)$.

map f is a chain map in Floer theory is using a homotopy argument between $f \circ d$ and $d \circ f$. For our map, however, the homotopy argument has an additional *error term* which arises as the point in ES^1 leaves the proper subset $W \subset ES^1$.

We can write the S^1-equivariant Floer cochain complex as a free Λ-module generated by a chosen basis, where the Novikov ring Λ is a ring with a formal parameter q. The operation which differentiates with respect to the Novikov variable q and multiplies by $\mathbf{u}$ is well-defined on the cochain complex, however it also fails to be a chain map. The error term is the same error term as for the S^1-equivariant quantum action, but with the opposite sign. Therefore the sum of the S^1-equivariant quantum action and this differentiation operation is a well-defined chain map.

Seidel introduced this concept and motivated it as a way to produce a well-defined differentiation operation on S^1-equivariant Floer cohomology [Sei18]. He called the resulting algebraic structure the *q-connection* because it is built out of differentiating with respect to the Novikov variable q and has similar algebraic properties to a connection on a vector bundle. In Seidel's original definition, he only differentiated by the special class $[\omega] \in H^2(M)$. In our second paper (Chapter 3), we define a 'connection' in which we differentiate by any class $\alpha \in H^2(M)$. (It should be clarified that this work is not related to the other connection in [Sei18] in which Seidel differentiates with respect to $\mathbf{u}$.)

Theorem 1.1.1. *For every $\alpha \in H^2(M;\mathbb{Z})$, there is a $\mathbb{Z}[\mathbf{u}]$-module endomorphism*

$$\nabla_\alpha : FH^*_{S^1}(M,\lambda) \to FH^{*+2}_{S^1}(M,\lambda) \tag{1.1.6}$$

which satisfies the Leibniz rule

$$\nabla(fx) = f\nabla(x) + \mathbf{u}\left(\frac{d}{d\alpha}f\right)x \tag{1.1.7}$$

and makes the following diagram commute.

$$\begin{array}{ccc} FH_{S^1}(M,\lambda) & \xrightarrow{\nabla} & FH_{S^1}^{+2}(M,\lambda) \\ \big\downarrow{\scriptstyle \mathbf{u}\mapsto 0} & & \big\downarrow{\scriptstyle \mathbf{u}\mapsto 0} \\ FH(M,\lambda) & \longrightarrow & FH^{+2}(M,\lambda) \end{array} \tag{1.1.8}$$

These maps commute with continuation maps, and hence induce maps on S^1-equivariant symplectic cohomology which satisfy (1.1.7).

With differentiation by multiple degree-2 classes comes a "genuinely new question" [Sei18, Section 2a] of whether these operations commute. We answer this question positively.

Theorem 1.1.2. *The connection ∇ is* flat, *which means ∇ and ∇ commute for any $\alpha, \beta \in H^2(M)$.*

The connection ∇ is inspired by a Dubrovin connection ∇^{Dub} on $QH_{S^1}(M, \mathrm{Id}_M)$ which is given by the sum of a differentiation operation $\mathbf{u}\frac{d}{d}$ with the quantum product by α. In fact, the two connections ∇ and ∇^{Dub} are preserved by the PSS maps which identify $QH_{S^1}(M, \mathrm{Id}_M)$ with $FH_{S^1}(M,\lambda)$ for small $\lambda > 0$.

Corollary 1.1.3. *The map $c : QH_{S^1}(M, \mathrm{Id}_M) \to SH_{S^1}(M)$ satisfies $\nabla^{\mathrm{Dub}} = \nabla \, c$.*

1.2 Seidel maps

Let M be a closed symplectic manifold and let σ be a Hamiltonian S^1-action on M. The S^1-action induces an automorphism of the loop space $\sigma \in \mathrm{Aut}\, LM$ which is given by $(\sigma x)(t) = \sigma_t(x(t))$. Seidel showed that this automorphism takes a Hamiltonian orbit x of H to a Hamiltonian orbit σx of a *pullback Hamiltonian* σH [Sei97]. The map $x \mapsto \sigma x$ induces an isomorphism on Floer cohomology

$$FS(\sigma) : FH(M;H) \to FH^{+j}(M;\sigma H) \tag{1.2.1}$$

which we call the *Floer Seidel map*. In fact, $FS(\sigma)$ is an isomorphism of the underlying Floer cochain complexes.

Recall that for closed symplectic manifolds, the PSS map PSS : $QH(M) \to FH(M)$ is an isomorphism between quantum cohomology and Floer cohomology. Seidel constructed a map on quantum cohomology which agrees with $\mathrm{PSS}^{-1} \, FS(\sigma) \, \mathrm{PSS}$. This map

$$QS(\sigma) : QH(M) \to QH^{+j}(M), \tag{1.2.2}$$

which we call the *quantum Seidel map*, counts pseudoholomorphic sections of a *clutching bundle* with fibre M and base S^2. Since the composition $\mathrm{PSS}^{-1} \circ FS(\sigma) \circ \mathrm{PSS}$ is a composition of isomorphisms, the quantum Seidel map $QS(\sigma)$ is also an isomorphism. It intertwines the quantum product, so it satisfies $QS(\sigma)(\alpha * x) = \alpha * QS(\sigma)(x)$.

These maps are computable in a number of cases. On $\mathbb{P}^2$, quantum cohomology is the ring $\mathbb{Z}[q^{\pm 1}][x]/(x^3 - q)$, where the Novikov variable q has degree 6 and x is the generator of $H^2(\mathbb{P}^2)$. The quantum Seidel map for the S^1-action which acts on the first coordinate is simply multiplication by x.

Ritter extended Seidel's constructions to convex symplectic manifolds [Rit14; Rit16]. We impose a condition which ensures that σ is compatible with the convex end. Namely, the Hamiltonian K of the S^1-action σ is linear of slope κ. With this assumption, the construction for $FS(\sigma)$ goes through completely, giving an isomorphism

$$FS(\sigma) : FH^*(M, \lambda; H) \to FH^{*+|\sigma|}(M, \lambda - \kappa; \sigma^* H). \tag{1.2.3}$$

Notice that the pullback Hamiltonian $\sigma^* H$ has slope $\lambda - \kappa$. As for closed manifolds, the quantum Seidel map $QS(\sigma)$ is equal to the composition

$$QS(\sigma) = \mathrm{PSS}^{-1} \circ \varphi \circ FS(\sigma) \circ \mathrm{PSS}, \tag{1.2.4}$$

but an additional continuation map φ is required to change the slope from $\lambda - \kappa$ back to λ. Continuation maps can only increase the slope, so $QS(\sigma)$ is only defined for $\kappa \geq 0$. Since continuation maps are not necessarily isomorphisms, the quantum Seidel map is not necessarily an isomorphism for convex manifolds. Ritter also gives a direct construction for $QS(\sigma)$ which counts sections of the clutching bundle.

We can compute the quantum Seidel map for a number of convex manifolds. The quantum cohomology of $\mathcal{O}_{\mathbb{P}^1}(-1)$ is the ring $\mathbb{Z}[q^{\pm 1}][x]/(x^2 + qx)$, where the Novikov variable q is in degree 2 and x is the Poincaré dual of a fibre, so x generates $H^2(\mathcal{O}_{\mathbb{P}^1}(-1))$. The quantum Seidel map for the S^1-action which rotates the fibres of the bundle about the zero section is multiplication by x. Clearly this map is not an isomorphism, for $x + q$ lies in the kernel. The quantum Seidel map can be used to compute symplectic cohomology [Rit14], which results in $SH^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathbb{Z}[q^{\pm 1}][x]/(x + q)$. We quotient by the kernel of the quantum Seidel map.

1.2.1 Equivariant Seidel maps

Maulik and Okounkov defined the first version of equivariant quantum Seidel maps as part of their work on quiver varieties [MO19, Section 8]. This is a map

$$QS_{S^1}(\sigma) : QH_{S^1}^*(M, \mathrm{Id}_M) \to QH_{S^1}^{*+|\sigma|}(M, \sigma^{-1}). \tag{1.2.5}$$

Note the codomain is equipped with the S^1-action $\sigma \circ \mathrm{Id}_M = \sigma^{-1}$. Unlike equivariant Floer cohomology, there is a well-defined ring structure on S^1-equivariant quantum cohomology given by the equivariant quantum product $\star$. This raises the natural question of whether $QS_{S^1}(\sigma)$ intertwines $\star$ like its nonequivariant counterpart. The answer is no, and instead we have the *intertwining relation* due to Maulik and Okounkov which states

$$QS_{S^1}(\sigma)(\alpha \star x) - \alpha \star QS_{S^1}(\sigma)(x) = \mathbf{u} WQS_{S^1}(\sigma, \alpha)(x), \tag{1.2.6}$$

where $WQS_{S^1}(\sigma, \alpha)$ is a weighted version of the S^1-equivariant quantum Seidel map which counts sections according to the number of intersections it has with a locally-finite class representing α.

In the algebraic geometry literature, the equivariant quantum Seidel map was used to compute the equivariant quantum product on the Springer resolution [BMO11]. Later, Iritani developed a version of the equivariant quantum Seidel map which applies to equivariant big quantum cohomology [Iri17]. He showed that this version also satisfies the intertwining relation. He used the construction to recover Givental's mirror theorem, which describes the equivariant genus-zero Gromov-Witten invariants of a toric variety.

Our goal was to extend the definition of $QS_{S^1}(\sigma)$ to an S^1-equivariant Floer Seidel map, and establish a similar intertwining relation for this new map. The technical machinery available in S^1-equivariant Floer cohomology is different to the algebrogeometric machinery that Maulik and Okounkov use to define their S^1-equivariant quantum Seidel map. First, their definition of $QS_{S^1}(\sigma)$ uses *virtual fundamental classes* to count the sections, and second, their proof of the intertwining relation uses a technique called *virtual localisation*. Neither of these tools exist in the Floer theory setting. In our first paper [LJ20], we give a new definition of $QS_{S^1}(\sigma)$ which uses a perturbation of an almost complex structure to ensure that sections have well-defined counts. We give a new proof of the intertwining relation which is Morse-theoretic. We define a homotopy, and the boundary of its 1-dimensional moduli spaces recover the intertwining relation on cohomology.

The challenges in defining this homotopy are remarkably similar to those faced when defining $\alpha \star$ in S^1-equivariant Floer cohomology. (In fact this homotopy inspired our later definition of $\alpha \star$ since this work was performed first chronologically.) The clutching bundle is $E(\sigma) \to S^2$, and the S^1-action rotates the base of the clutching bundle, which is the sphere S^2. The desired homotopy would count sections $u : S^2 \to E(\sigma)$ of the clutching bundle which intersect a locally-finite cycle representing α at some specified point $z_0 \in S^2$. Of course, this point z_0 can depend on a point in $ES^1 = S^\infty$, but it must vary S^1-equivariantly. Thus, when z_0 is not one of the two fixed points of the S^1-action on S^2, we are again seeking an S^1-equivariant map $ES^1 \to S^1$. No global map exists, but it is sufficient to define the map on a *proper* subset $W \subset ES^1$. The term on the right-hand side of (1.2.6) arises as the point in ES^1 leaves W.

Theorem 1.2.1. *Our S^1-equivariant quantum Seidel map $QS_{S^1}(\sigma)$ satisfies the intertwining relation* (1.2.6)*, where $\mathbf{u}$ acts geometrically.*

We also define a S^1-equivariant Floer Seidel map $FS_{S^1}(\sigma)$ in our first paper. It is an isomorphism

$$FS_{S^1}(\sigma) : FH_{S^1}(M, \mathrm{Id}_M; H) \to FH_{S^1}^{*+|\sigma|}(M, \sigma^{-1}; \sigma^* H). \tag{1.2.7}$$

Like (1.2.5), the codomain has an additional action on M, whereas the S^1-action for the domain is simply the rotation action on loops. This map is an isomorphism on the cochain complex. It also satisfies a similar compatibility result to (1.2.4) with the S^1-equivariant quantum Seidel map.

The Dubrovin connection on S^1-equivariant quantum cohomology is given by $\nabla^{\mathrm{Dub}} = \mathbf{u}\frac{d}{d\alpha} + \alpha \star$. As in [MO19, Section 1.4], an alternative way to phrase the intertwining relation is

$$\nabla^{\mathrm{Dub}} \circ QS_{S^1}(\sigma) = QS_{S^1}(\sigma) \circ \nabla^{\mathrm{Dub}}. \tag{1.2.8}$$

This combines the intertwining relation (1.2.6) with

$$\frac{d}{d\alpha} QS_{S^1}(\sigma)(x) - QS_{S^1}(\sigma)\left(\frac{d}{d\alpha}(x)\right) = WQS_{S^1}(\sigma, \alpha)(x). \tag{1.2.9}$$

In our second paper [LJ21], we establish the Floer-theoretic analogue of this result. The proof is heavily inspired by our earlier proof of the intertwining relation for $QS_{S^1}(\sigma)$.

Theorem 1.2.2. *∇ and $FS_{S^1}(\sigma)$ commute on S^1-equivariant Floer cohomology.*[2]

1.2.2 Shift operators

Thus far, we have considered only a single S^1-action on M. Let us now take a Hamiltonian torus action $\rho : T \times M \to M$. A *cocharacter* is a group homomorphism $\sigma : S^1 \to T$, and associated to the cocharacter σ is the S^1-action $\rho \circ \sigma : S^1 \times M \to M$. We have an induced T-action on the loop space $\mathcal{L}M$, as well as the S^1-action which rotates loops as in (1.1.1). To keep the different actions separate, define $\widehat{T} = S^1_0 \times T$ (we denote this copy of S^1 by S^1_0 to avoid confusion with the other copies of S^1 in our work).

We get $\widehat{T}$-equivariant versions of our earlier invariants. That is, we have $\widehat{T}$-equivariant quantum cohomology $QH_{\widehat{T}}(M, \rho)$ (here, S^1_0 acts trivially on M) and $\widehat{T}$-equivariant Floer cohomology $FH_{\widehat{T}}(M, \rho, \lambda)$ (here, S^1_0 acts by rotating the domain of loops, as described above).

[2] The domain and codomain of the map $FS_{S^1}(\sigma)$ have different S^1-actions on $\mathcal{L}M$, so technically there is a different connection ∇ for each such S^1-action. Theorem 1.2.3 avoids this slight-of-hand.

Each of these modules has a flat connection ∇ just like their S^1-equivariant analogues, and these connections are compatible with each other under $\widehat{T}$-equivariant PSS maps.

Associated to any[3] cocharacter σ is a $\widehat{T}$-equivariant quantum Seidel map $QS_{\widehat{T}}(\sigma)$ (for which S^1_0 rotates the base S^2 of the clutching bundle $E(\sigma)$) and a $\widehat{T}$-equivariant Floer Seidel map $FS_{\widehat{T}}(\sigma)$. The codomain of each of these two maps has a different *pullback $\widehat{T}$-action*, as per the S^1-equivariant constructions. However for $\widehat{T}$, the pullback $\widehat{T}$-action is isomorphic to the original $\widehat{T}$-action, up to an isomorphism $\widehat{\sigma} \in \mathrm{Aut}(\widehat{T})$. Associated to this isomorphism $\widehat{\sigma} : \widehat{T} \to \widehat{T}$ is an isomorphism of $\widehat{T}$-equivariant cohomology denoted $(B\widehat{\sigma})^*$. We define versions of $(B\widehat{\sigma})^*$ for $\widehat{T}$-equivariant quantum cohomology and $\widehat{T}$-equivariant Floer cohomology.

The shift operators on $\widehat{T}$-equivariant quantum cohomology and Floer cohomology are the maps

$$\mathbb{S}_{\widetilde{\sigma}} = (B\widehat{\sigma})^* \circ QS_{\widehat{T}}(\sigma) : QH^*_{\widehat{T}}(M,\rho) \to QH^{*+|\widetilde{\sigma}|}_{\widehat{T}}(M,\rho) \tag{1.2.10}$$

and

$$\mathbb{S}_{\widetilde{\sigma}} = (B\widehat{\sigma})^* \circ \varphi \circ FS_{\widehat{T}}(\sigma) : FH^*_{\widehat{T}}(M,\rho,\lambda) \to FH^{*+|\widetilde{\sigma}|}_{\widehat{T}}(M,\rho,\lambda) \tag{1.2.11}$$

respectively. The map $(B\widehat{\sigma})^*$ undoes the change in $\widehat{T}$-action and the continuation map φ in (1.2.11) undoes the change in slope, so both shift operators are graded endomorphisms. From $(B\widehat{\sigma})^*$, they inherit the relation

$$\mathbb{S}_{\widetilde{\sigma}}(fx) = (B\widehat{\sigma})^*(f)\, \mathbb{S}_{\widetilde{\sigma}}(x) \tag{1.2.12}$$

for $f \in H^*(B\widehat{T})$. Note that (1.2.12) implies that the shift operators are not $H^*(B\widehat{T})$-module endomorphisms. The shift operators are compatible under the PSS map. Our main theorem states that the shift operators commute with the connection.

Theorem 1.2.3. *The connection ∇_{ξ} and the shift operator $\mathbb{S}_{\widetilde{\sigma}}$ commute on $\widehat{T}$-equivariant Floer cohomology.*

The combined algebraic structure induced by the connection together with the shift operator is called a *difference-differential connection*. In Theorem 1.1.2, we showed that the maps ∇_{ξ} and $\nabla_{\xi'}$ commute. In Theorem 1.2.3, we showed that the maps ∇_{ξ} and $\mathbb{S}_{\widetilde{\sigma}}$ commute. It is also true for elementary reasons that the maps $\mathbb{S}_{\widetilde{\sigma}}$ and $\mathbb{S}_{\widetilde{\sigma}'}$ commute. These three commutativity statements together mean that the difference-differential connection is *flat*. Under the limit $\lambda \to \infty$, this yields the following theorem.

Theorem 1.2.4. *There is a flat difference-differential connection on $\widehat{T}$-equivariant symplectic cohomology.*

[3] More precisely, any cocharacter which induces an S^1-action whose Hamiltonian has nonnegative slope.

This theorem represents an important enhancement of the algebraic structures which are available on equivariant symplectic cohomology.

1.2.3 Toric manifolds

We can compute our algebraic structures for *toric manifolds*. These are manifolds M for which $\rho : T \times M \to M$ is an effective torus action with $\dim T = \frac{1}{2} \dim M$. Toric manifolds are determined by a moment polytope, and combinatorial analysis of the polytope recovers many important invariants of the corresponding toric manifold. The Batyrev presentation of quantum cohomology is an important example, and this can be derived using quantum Seidel maps [MT06, Section 5]. This result was extended from closed toric manifolds to various toric line bundles by Ritter [Rit16]. Using a special almost complex structure, we are able to compute that $QS(\sigma_D)(1) = \bar{D}$ for divisors D, where the action $\rho \circ \sigma_D$ rotates positively around D and $\bar{D}$ is the Poincaré dual of D. As such, the map $QS(\sigma_D)$ is simply quantum multiplication by $\bar{D}$.

We are able to deduce $S_{\sigma_D}(1) = \bar{D}$ in our work as well. Of course, our shift operator respects the connection, and not quantum multiplication. However we are able to show that the connection is a sufficiently rich algebraic structure to determine S_{σ_D}, and we provide an algorithm to compute it.

Theorem 1.2.5. *We can compute the shift operator on the $\widehat{T}$-equivariant quantum cohomology of toric manifolds by combining $S_{\sigma_D}(1) = \bar{D}$ with (1.2.12) and the fact it commutes with the connection ∇.*

Let's take $\mathbb{P}^2$ as an example. Its three divisors are D_0, D_1, D_2, where D_i is the zero locus of the i-th coordinate. Let $x_i = \bar{D}_i$. The $\widehat{T}$-equivariant quantum cohomology is the ring $\mathbb{Z}[q^{-1}][x_0, x_1, x_2][\mathbf{u}]/(x_0 x_1 x_2 - q)$, where the Novikov variable q has degree 6. Let σ be the cocharacter which induces the S^1-action which rotates positively about D_1. We have $S_\sigma(1) = x_1$, and we can deduce S_σ on other elements using the connection, following our algorithm. This yields, for example,

$$S_\sigma(x_0) = x_0 x_1 \tag{1.2.13}$$

$$S_\sigma(x_1) = x_1(x_1 - \mathbf{u}) \tag{1.2.14}$$

$$S_\sigma(x_2) = x_1 x_2. \tag{1.2.15}$$

We work through this example fully in Section 3.5.4.

For convex toric manifolds, we conjecture that the determinant of the shift operator captures $\widehat{T}$-equivariant information about the Reeb dynamics (Conjecture 3.5.12). If true, this result would provide a precise relationship between the Reeb dynamics and the pseudoholomorphic sections of a clutching bundle.

Chapter 2

An intertwining relation for equivariant Seidel maps

2.1 Introduction

For us, *equivariant* will always mean S^1-equivariant.

The Seidel maps on the quantum and Floer cohomology of a closed symplectic manifold M are two maps associated to a Hamiltonian S^1-action σ on M [Sei97]. They are compatible with each other via the PSS isomorphisms, which are maps that identify quantum cohomology and Floer cohomology [Sei97, Theorem 8.2]. McDuff and Tolman used Seidel maps to recover the Batyrev presentation of the quantum cohomology of closed toric manifolds [MT06]. Ritter extended the definition of Seidel maps to (non-closed) convex symplectic manifolds [Rit14]. He used the Seidel maps to determine the quantum cohomology and the symplectic cohomology of convex toric manifolds [Rit16].

Let ρ be a Hamiltonian S^1-action on a closed or convex symplectic manifold M. The equivariant quantum cohomology $EQH(M)$ has three important compatible algebraic structures: it is a ring equipped with the equivariant quantum product ; it is a module over the Novikov ring ; and it has a geometric $\mathbb{Z}[\mathbf{u}]$-module structure denoted $\smile$. In this paper, we introduce an equivariant quantum Seidel map corresponding to an additional Hamiltonian S^1-action σ on M which commutes with ρ. We could, for example, set $\rho = \mathrm{Id}_M$, or we could let ρ and σ be two S^1-actions which are part of a Hamiltonian torus action on M.

Theorem 2.1.1. *There is an equivariant quantum Seidel map*

$$EQS_{\tilde{\sigma}} : EQH(M) \to EQH^{+2I(\tilde{\sigma})}(M) \tag{2.1.1}$$

which is a $\mathbb{Z}[\mathbf{u}]$*-module homorphism. The codomain of* (2.1.1) *is the equivariant quantum cohomology of* M *with the* pullback action

$$\sigma^* \rho = \sigma^{-1}\rho. \tag{2.1.2}$$

For closed symplectic manifolds M with a Hamiltonian S^1-action σ, Kirwan showed there is a (non-canonical) $\mathbb{Q}[\mathbf{u}]$-module isomorphism

$$EH^*(M;\mathbb{Q}) = H^*(M;\mathbb{Q}) \otimes \mathbb{Z}[\mathbf{u}] \tag{2.1.3}$$

using Morse inequalities [Kir84, Proposition 5.8]. Her proof extends to our convex setting since we assume σ is *linear at infinity* (see Section 2.3.3). The equivariant quantum Seidel map is an isomorphism for closed manifolds by (2.7.3), and hence it is a *canonical* isomorphism. The equivariant quantum Seidel map is not an isomorphism for convex manifolds in general.

Corollary 2.1.2 (Canonical isomorphism for closed manifolds). *When M is a closed symplectic manifold, the equivariant quantum Seidel map is a canonical $\widehat{\ }\mathbb{Z}[\mathbf{u}]$-module isomorphism*

$$EQH^*(M) \to EQH_{\mathrm{Id}}^{*+2I(\widetilde{\sigma})}(M) = \left(QH^*(M)\,\widehat{\ }\,\mathbb{Z}[\mathbf{u}]\right)^{*+2I(\widetilde{\sigma})}. \tag{2.1.4}$$

(The grading shift $+2I(\widetilde{\sigma})$ is not canonical since it depends on a choice of lift $\widetilde{\sigma}$ as in Section 2.3.3.)

Our main theorem describes the relationship between $EQS_{\widetilde{\sigma}}$ and equivariant quantum multiplication. Unlike for the (non-equivariant) quantum Seidel map, this relationship is not simply that the map commutes with the quantum product. We can nonetheless exploit the relationship to derive the equivariant quantum product, which we demonstrate for a few examples in Sections 2.2.3 and 2.8.

Theorem 2.1.3 (Intertwining relation). *The equation*

$$EQS_{\widetilde{\sigma}}(x * \alpha^+) - EQS_{\widetilde{\sigma}}(x) * \alpha^- = \mathbf{u} \smile EQS_{\widetilde{\sigma};\alpha}(x) \tag{2.1.5}$$

holds for all $x \in EQH^*(M)$. Here, $\alpha^+ \in EH^*(M)$ and $\alpha^- \in EH^*(M)$ are two equivariant cohomology classes which are related via the clutching bundle, and

$$EQS_{\widetilde{\sigma};\alpha} : EQH^*(M) \to EQH^{*+2I(\widetilde{\sigma})+|\alpha|-2}(M) \tag{2.1.6}$$

is a map defined in Section 2.7.4.1.

Maulik and Okounkov gave the first definition of equivariant quantum Seidel maps as part of their work on quiver varieties [MO19, Section 8]. They also proved that the maps satisfied the intertwining relation [MO19, Proposition 8.1]. Braverman, Maulik and Okounkov used equivariant quantum Seidel maps to derive the equivariant quantum product of the Springer resolution [BMO11]. Iritani recovered Givental's mirror theorem using a new version of the equivariant quantum Seidel map which applied to big equivariant quantum cohomology [Iri17]. Givental's mirror theorem describes the equivariant genus-zero Gromov-Witten invariants of a toric variety.

Maulik and Okounkov's work on the equivariant quantum Seidel map applies to smooth quasi-projective varieties X with a holomorphic symplectic structure which are equipped with the action of a reductive group G (such X are (real) symplectic manifolds with $c_1 = 0$). Iritani's work applies to smooth toric varieties X that admit a projective morphism to an affine variety and for which the action of the algebraic torus G on X satisfies some positivity condition. The definitions of the equivariant quantum Seidel map in this algebrogeometric context use *equivariant virtual fundamental classes* to count stable maps. The proofs of the intertwining relation in this context make heavy use of an algebrogeometric technique called *virtual localisation*, which reduces counting stable maps to counting only the G-fixed stable maps.

In contrast, our results apply to closed or convex symplectic manifolds M which satisfy a monotonicity condition. We use a Borel model for equivariant quantum cohomology, which combines the Morse cohomology of our manifold M with the Morse cohomology of the classifying space of S^1. This model is preferable in our context because it readily extends to equivariant Floer theory. In our Borel model, we perturb the data on M using the classifying space of S^1 to ensure that the moduli spaces are smooth manifolds. We therefore avoid using virtual fundamental classes to count stable maps, and instead just count the 0-dimensional moduli spaces. The G-fixed stable maps used by virtual localisation are not pseudoholomorphic curves for the perturbed data, however, so virtual localisation will not work in our context. To remedy this, we provide a new proof of the intertwining relation Theorem 2.1.3 which has a Morse-theoretic flavour: we construct an explicit 1-dimensional moduli space whose boundary gives the relation.

We also introduce an equivariant Floer Seidel map, which is an isomorphism on equivariant Floer cohomology. We use a Borel model for equivariant Floer cohomology, which combines Morse theory on the classifying space of S^1 with Floer theory on M. The S^1-action $\rho : S^1 \times M \to M$ induces an S^1-action on Hamiltonian orbits $x : S^1 \to M$, given by

$$(\theta \cdot x)(t) = \rho_\theta (x(t - \theta)), \tag{2.1.7}$$

and it is with respect to this action that we define equivariant Floer cohomology. The equivariant Floer Seidel map is the identity map on the classifying space of S^1 and it maps the Hamiltonian orbit $x : S^1 \to M$ to the Hamiltonian orbit $(\sigma \cdot x)(t) = \sigma_t^{-1}(x(t))$. Much like the equivariant quantum Seidel map (2.1.1), the codomain of the equivariant Floer

Seidel map is the equivariant Floer cohomology not for the action (2.1.7), but instead for the corresponding action induced by the pullback action $\sigma^* \rho$.

The equivariant Floer Seidel map commutes with continuation maps, which means it induces a map on equivariant symplectic cohomology. Recall that the equivariant symplectic cohomology of M is the direct limit of equivariant Floer cohomology as the slope of the Hamiltonian function increases (see Section 2.4.3).

Theorem 2.1.4. *There is an equivariant Floer Seidel map on equivariant symplectic cohomology*

$$EFS_{\tilde{\sigma}} : ESH^*(M) \to ESH^{*+2I(\tilde{\sigma})}(M) \tag{2.1.8}$$

which is a $\widehat{\ }\,\mathbb{Z}[\mathbf{u}]$*-module isomorphism. The diagram*

$$\begin{array}{ccc} EQH^*(M) & \xrightarrow{EQS_{\tilde{\sigma}}} & EQH^{*+2I(\tilde{\sigma})}(M) \\ \downarrow & & \downarrow \\ ESH^*(M) & \xrightarrow{EFS_{\tilde{\sigma}}} & ESH^{*+2I(\tilde{\sigma})}(M) \end{array} \tag{2.1.9}$$

commutes, where the vertical arrows denote equivariant c^* maps.

Equivariant symplectic cohomology has attracted attention in recent years because, while similar to (non-equivariant) symplectic cohomology, it possesses a number of different properties. Chiefly, it distinguishes Hamiltonian orbits with different stabilizer groups more readily than does its non-equivariant counterpart. This is useful because these different orbits have different geometric significance (for certain choices of Hamiltonians). For example, the constant orbits (with stabilizer group S^1) capture the topology of M. Zhao restricted to the constant orbits by localising the ring $\mathbb{Z}[\mathbf{u}]$, and obtained the isomorphism

$$ESH^*(M) \otimes_{\mathbb{Z}[\mathbf{u}]} \mathbb{Q}[\mathbf{u}, \mathbf{u}^{-1}] \cong H^*(M) \otimes_{\mathbb{Z}} \mathbb{Q}[\mathbf{u}, \mathbf{u}^{-1}] \tag{2.1.10}$$

for completions of Liouville domains [Zha19, Theorem 1.1]. This is unlike (non-equivariant) symplectic cohomology, which vanishes for subcritical Stein manifolds (the vanishing follows from [Cie02, Theorem 1.1, part 1] and $SH^*(\mathbb{C}^N) = 0$ [Oan04, Section 3]).

On the other hand, the nonconstant orbits (with finite stablizer groups) capture the Reeb dynamics. We can restrict to the nonconstant orbits by looking at the positive part of $ESH^*(M)$. Bourgeois and Oancea showed that the positive part of equivariant symplectic homology is isomorphic to linearized contact homology [BO17] while Gutt used it to distinguish nonisomorphic contact structures on spheres [Gut17, Theorem 1.4]. In fact, McLean and Ritter used $ESH^*(M)$ to classify orbits with different finite stablizer groups in their new proof of the McKay correspondence [MR18].

One downside of equivariant symplectic cohomology is that it lacks an interesting algebraic structure. (Non-equivariant) symplectic cohomology has the pair-of-pants product, which equips it with a graded-commutative, associative and unital ring structure. This ring structure can in fact be upgraded to an entire TQFT structure on symplectic cohomology [Rit13; Sei08, Section (8a)]. In contrast, equivariant symplectic cohomology only has the $\mathbb{Z}[\mathbf{u}]$-module structure, which arises as the cohomology of the classifying space of S^1.

Seidel described a new algebraic structure on equivariant Floer cohomology, which he called the *q-connection* [Sei18]. This structure enhances the module structure, but is less rich than a full ring structure. Seidel's q-connection is based upon the *quantum connection*, which is a map $\nabla_\alpha : EQH^*(M) \to EQH^{*+2}(M)$ that combines multiplication by α with a differentiation-like operation applied to the Novikov variable. Throughout the algebro-geometric literature on equivariant quantum Seidel maps, the intertwining relation is often written as

$$\nabla_\alpha \circ S = S \circ \nabla_\alpha , \tag{2.1.11}$$

where S is a variant of the equivariant quantum Seidel map [MO19; BMO11; Iri17]. In our upcoming paper [LJ21], we will show that a variant of our equivariant Floer Seidel map commutes with the q-connection, just like (2.1.11). This represents a further enhancement of the available algebraic structures on equivariant symplectic cohomology. The result is proved with a Floer theory version of our new proof of the intertwining relation Theorem 2.1.3.

2.1.1 Outline

We give an overview of the background material for our work in the next section (Section 2.2), as well as giving more information about and intuition for our constructions and results. Section 2.2.3 contains an overview of our example calculations.

In Section 2.3, we introduce the Floer Seidel maps in detail, clarifying our assumptions and conventions for symplectic cohomology. We introduce equivariant Floer theory and its associated module structures in Section 2.4 before defining the equivariant Floer Seidel map in Section 2.5.

In Section 2.6, we introduce the quantum Seidel map (Section 2.6.2) and equivariant quantum cohomology (Section 2.6.3). We define the equivariant quantum Seidel map in Section 2.7 and prove the intertwining relation in Section 2.7.4.

Section 2.8 contains the details of our three example calculations.

2.2 Overview

2.2.1 Background

2.2.1.1 Seidel maps on closed symplectic manifolds

Let M be a closed monotone symplectic manifold and let σ be a Hamiltonian circle action on M. In [Sei97], Seidel defined a pair of automorphisms associated to σ, one on Floer cohomology and one on quantum cohomology. To distinguish between these maps, we call the former the *Floer Seidel map* and the latter the *quantum Seidel map*.

Definition 2.2.1 (Floer Seidel map). Recall that the Floer cochain complex $FC^*(M;H)$ associated to the time-dependent Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ is freely-generated by the 1-periodic Hamiltonian orbits of H over the Novikov ring Λ. Given a Hamiltonian orbit $x : S^1 \to M$ of H, the pullback orbit $\sigma^* x$ given by

$$(\sigma^* x)(t) = \sigma_t^{-1}(x(t)) \tag{2.2.1}$$

is a Hamiltonian orbit of the pullback Hamiltonian $\sigma^* H$. Moreover, the assignment $x \mapsto \sigma^* x$ is a bijection between the orbits of H and the orbits of $\sigma^* H$. This assignment can be upgraded to an isomorphism of Floer cochain complexes

$$FS_{\tilde{\sigma}} : FC^*(M;H) \to FC^{*+2I(\tilde{\sigma})}(M;\sigma^* H) \tag{2.2.2}$$

by using a *lift* $\tilde{\sigma}$ of the circle action σ to keep track of the information recorded by the Novikov ring. The quantity $I(\tilde{\sigma})$ is a Maslov index associated to $\tilde{\sigma}$ (see Section 2.3.3 for details).

The *Floer Seidel map* $FS_{\tilde{\sigma}}$ is the map induced on Floer cohomology by (2.2.2). It satisfies $FS_{\tilde{\sigma}}(a * b) = a * FS_{\tilde{\sigma}}(b)$, where $*$ denotes the pair-of-pants product.

Definition 2.2.2 (Quantum Seidel map). Define a *clutching bundle* E over S^2 with fibre M as follows. The sphere is the union of its northern hemisphere $\mathbb{D}^-$ and its southern hemisphere $\mathbb{D}^+$. Each hemisphere is isomorphic to a closed unit disc, and the two hemispheres are glued along the equator $S^1 = \partial\mathbb{D}^- = \partial\mathbb{D}^+$ to get the sphere. The clutching bundle is the union of the trivial bundles $\mathbb{D}^- \times M$ and $\mathbb{D}^+ \times M$, glued along the equator by the relation

$$\partial\mathbb{D}^- \times M \ni (t,m) \leftrightarrow (t,\sigma_t(m)) \in \partial\mathbb{D}^+ \times M. \tag{2.2.3}$$

Thus we twist the bundle by σ when passing from the northern hemisphere to the southern hemisphere.

The *quantum Seidel map* is an isomorphism $QS_{\tilde{\sigma}} : QH^*(M) \to QH^{*+2I(\tilde{\sigma})}(M)$ which counts pseudoholomorphic sections of this clutching bundle. It satisfies

$$QS_{\tilde{\sigma}}(a * b) = a * QS_{\tilde{\sigma}}(b), \tag{2.2.4}$$

where $*$ denotes the quantum product. It follows that the quantum Seidel map is given by quantum product multiplication by the invertible element $QS_{\widetilde{\sigma}}(1)$. This element is the *Seidel element of* $\widetilde{\sigma}$.

Using the PSS isomorphism to identify $QH^*(M)$ with Floer cohomology, the Floer Seidel map and the quantum Seidel map are identified. Roughly, each hemisphere in the clutching bundle corresponds to a PSS map, and the twisting along the equator corresponds to the Floer Seidel map.

More generally, Seidel maps may be defined for loops in the Hamiltonian symplectomorphism group $\mathrm{Ham}(M)$ based at the identity Id_M. (Technically, we must use a cover $\widetilde{\mathrm{Ham}}(M)$, though we omit details here.) The Seidel maps are homotopy invariants, so that the assignment

$$\begin{aligned} \pi_1(\widetilde{\mathrm{Ham}}(M), \mathrm{Id}_M) &\to QH^*(M) \\ \widetilde{\sigma} &\mapsto QS_{\widetilde{\sigma}}(1) \end{aligned} \tag{2.2.5}$$

is a group homomorphism. The map (2.2.5) is the *Seidel representation*.

Here is a selection of results whose proofs use Seidel maps. It is by no means complete.

- An algorithmic and combinatorial computation of quantum cohomology from the moment polytope of a toric symplectic manifold [MT06, Section 5].
- An isomorphism $QH^*(M) = QH^*((\mathbb{P}^1)^n)$ whenever M admits a semifree circle action with nonempty isolated fixed point set [Gon06].
- Computation of the Gromov width and Hofer-Zehnder capacity in terms of the values of the Hamiltonian of σ on fixed components, under the assumption that σ is semi-free (and its maximal fixed locus is a point) [HS17].

2.2.1.2 Convex manifolds

The symplectic manifold (M, ω) is *convex* (sometimes *conical* in the literature) if it is symplectomorphic to $\Sigma \times [1, \infty)$ with symplectic form $\omega = \mathrm{d}(R\alpha)$ away from a relatively compact subset $M_0 \subset M$. Here, Σ is a closed contact manifold with contact form α and R is the coordinate of $[1, \infty)$.

The cotangent bundle of a closed manifold is convex, and more generally so is the completion of a Liouville domain. In addition, there are many examples of convex symplectic manifolds whose symplectic forms are not globally exact, such as the total space of the line bundle $\mathcal{O}_{\mathbb{P}^N}(-1)$.

In the region $\Sigma \times [1, \infty)$, the *convex end*, the symplectic form is exact and it grows infinitely big as $R \to \infty$. Moreover the formula $\omega = \mathrm{d}(R\alpha) = \mathrm{d}R \wedge \alpha + R\mathrm{d}\alpha$ decomposes the tangent space into the two-dimensional subspace $\mathbb{R}\partial_R + \mathbb{R}X$ and the contact distribution $\ker \alpha$. (Here, X is the Reeb vector field, which is characterised by $\alpha(X) = 1$ and $d\alpha(X, \cdot) = 0$.) These properties of ω constrain the Hamiltonian dynamics of a Hamiltonian function H to a compact region of the form $M_0 \cup (\quad [1, R_0])$ so long as $H = \lambda R + \mu$ is a linear function of R outside this region (some further conditions are also required, see Section 2.3.2).

Once the Hamiltonian dynamics are restricted to a compact region, the machinery of Floer cohomology applies, so we can define Floer cohomology for convex symplectic manifolds (which satisfy an appropriate monotonicity assumption) and Hamiltonian functions which are linear outside a compact region. Floer cohomology depends on the slope λ of the Hamiltonian, unlike for closed manifolds whose Floer cohomology is entirely independent of the Hamiltonian. *Symplectic cohomology* $SH(M)$ is the direct limit of Floer cohomology as the slope λ tends to infinity. The PSS maps are isomorphisms between the quantum cohomology $QH(M)$ and the Floer cohomology of a Hamiltonian with sufficiently small (positive) slope.

2.2.1.3 Seidel maps on convex symplectic manifolds

In [Rit14], Ritter extended the construction of Seidel maps to Hamiltonian circle actions σ on convex symplectic manifolds, so long as the Hamiltonian K of σ is linear, with nonnegative slope, outside a compact region. The Floer Seidel map is an isomorphism between $FH(M; H)$ and $FH^{+2I(\tilde{\sigma})}(M; \sigma H)$ as for closed manifolds, however the pullback Hamiltonian σH may have a different slope to H. The limit of the Floer Seidel maps as the slope λ of H tends to infinity induces an automorphism $FS_{\tilde{\sigma}}$ of symplectic cohomology $SH(M)$.

The quantum Seidel map on a convex symplectic manifold is not necessarily an isomorphism, but instead is merely a module homomorphism. Ritter computed the quantum Seidel maps on the total spaces of the line bundles $\mathcal{O}_{\mathbb{P}^N}(\ \ 1)$. His calculations demonstrate that, even in elementary examples, the quantum Seidel map may fail to be injective and surjective.

The fact that the Floer Seidel map is an isomorphism even though the quantum Seidel map may fail to be injective and surjective is explained by the following commutative diagram. The non-bijectivity of the quantum Seidel map corresponds to the non-bijectivity of the continuation map.

$$
\begin{array}{ccc}
QH^*(M) & \xrightarrow{\quad QS_{\tilde{\sigma}} \quad} & QH^{*+2I(\tilde{\sigma})}(M) \\
\text{PSS map} \downarrow \cong & & \cong \uparrow \text{PSS map} \\
FH^*(M; H^{\text{small}}) & & FH^{*+2I(\tilde{\sigma})}(M; H^{\text{small}}) \\
& \searrow^{\cong}_{FS_{\tilde{\sigma}}} \qquad \nearrow_{\text{continuation map}} & \\
& FH^{*+2I(\tilde{\sigma})}(M; \sigma^* H^{\text{small}}) &
\end{array}
\tag{2.2.6}
$$

The Hamiltonian H^{small} has small (positive) slope, but the Hamiltonian $\sigma^* H^{\text{small}}$ typically has negative slope.

When the slope of K is positive, the quantum Seidel map may be used to compute symplectic cohomology [Rit14, Theorem 22]. This approach explicitly computes symplectic cohomology for various line bundles of the form $O_{\mathbb{P}^N}(-k)$ [Rit14, Theorem 5] and provides an algorithm to compute the symplectic cohomology of a Fano toric negative line bundle using its moment polytope [Rit16, Theorem 1.5].

2.2.1.4 Equivariant Floer cohomology

Recall that Floer cohomology is inspired by the Morse cohomology of the loop space $\mathcal{L}M = \{\text{contractible } x : S^1 \to M\}$. The loop space $\mathcal{L}M$ has a canonical circle action which rotates the loops. Equivariant Floer cohomology is analogously inspired by the equivariant Morse cohomology of $\mathcal{L}M$ with this rotation action.

Viterbo introduced the first version of equivariant Floer cohomology [Vit96, Section 5], and later Seidel introduced a second version [Sei08, Section 8b]. Bourgeois and Oancea showed these different versions are equivalent [BO17, Proposition 2.5]. We use Seidel's approach, since its analysis is far simpler. For more information on the history, see [BO17, Section 2]. Our conventions differ from those in the literature in three important ways: we use a different relation for the Borel homotopy quotient (Remark 2.4.3), we use a geometric module structure (Section 2.4.4.2) and we incorporate actions on the manifold M.

Denote by S^∞ the limit of the inclusions $S^1 \hookrightarrow S^3 \hookrightarrow S^5 \hookrightarrow \cdots$, where S^{2k-1} is thought of as the unit sphere in $\mathbb{C}^k$. It is a contractible space with a free circle action. Given any space X with a circle action, its *Borel homotopy quotient*, denoted $S^\infty \times_{S^1} X$, is the quotient of $S^\infty \times X$ by the relation $(w, \theta \cdot x) \sim (\theta^{-1} \cdot w, x)$ for all $\theta \in S^1$. *Equivariant cohomology* is the cohomology of $S^\infty \times_{S^1} X$.

Informally, the equivariant Morse cohomology of the Borel homotopy quotient may be obtained by doing Morse theory on S^∞ and on X, and quotienting the moduli spaces by the induced relation. Similarly, the equivariant Floer cohomology of M is obtained by doing

Morse theory on S^∞ and Floer theory on M, and quotienting the moduli spaces by the induced relation.

The *equivariant Floer cochain complex* is generated by $\sim$-equivalence classes $[(w,x)]$, where w is a critical point in S^∞ and x is a 1-periodic Hamiltonian orbit of a time-dependent Hamiltonian H^{eq}_w. The function $H^{\mathrm{eq}} : S^\infty \times S^1 \times M \to \mathbb{R}$ must satisfy the identity

$$H^{\mathrm{eq}}_{\theta w,t}(m) = H^{\mathrm{eq}}_{w,t+\theta}(m) \tag{2.2.7}$$

in order that the relation $\sim$ make sense on the pairs $(w,x) \in S^\infty \times \mathcal{L}M$. Such $\sim$-equivalence classes are *equivariant Hamiltonian orbits*.

Similarly, consider $\sim$-equivalence classes $[(v,u)]$ where $v : \mathbb{R} \to S^\infty$ is a Morse trajectory and $u : \mathbb{R} \times S^1 \to M$ is a Floer solution of the s-dependent Hamiltonian $H^{\mathrm{eq}}_{v(s)}$. The differential on the equivariant Floer cochain complex counts these equivalence classes modulo the free $\mathbb{R}$-action of translation.

The cohomology of S^∞/S^1 is $\mathbb{Z}[\mathbf{u}]$, where $\mathbf{u}$ is a formal variable in degree 2. For certain choices of Floer data, the equivariant Floer cochain complex[1] is $FC^*(M) \widehat{\otimes} \mathbb{Z}[\mathbf{u}]$ with differential $d^{\mathrm{eq}} = d + o(\mathbf{u})$. Here $FC^*(M)$ is the (non-equivariant) Floer cochain complex and d is the (non-equivariant) Floer differential.

The resulting *equivariant Floer cohomology* $EFH^*(M; H^{\mathrm{eq}})$ is a graded module over the Novikov ring Λ. Moreover, it has a $\mathbb{Z}[\mathbf{u}]$-module structure coming from the description of the cochain complex above. In fact, it has another $\mathbb{Z}[\mathbf{u}]$-module structure given by an equivariant cup product type construction, which we call the *geometric module structure* and denote $\smile$.

Much like in the non-equivariant setup in Section 2.2.1.2, $EFH^*(M; H^{\mathrm{eq}})$ only depends on the slope of H^{eq}, and the limit of $EFH^*(M; H^{\mathrm{eq}})$ as the slopes increase to infinity is the *equivariant symplectic cohomology* $ESH^*(M)$. Equivariant PSS maps identify $EFH^*(M; H^{\mathrm{eq}})$ and $EQH^*(M)$ when H^{eq} has sufficiently small (positive) slope. Here, $EQH^*(M)$ is the equivariant quantum cohomology of M for the trivial identity circle action on M.

For us, we have a Hamiltonian action σ acting on M. This means that the loop space $\mathcal{L}(M)$ has another canonical circle action given by

$$\theta \cdot (\, t \mapsto x(t)\,) = (\, t \mapsto \sigma_\theta(x(t-\theta))\,). \tag{2.2.8}$$

This action combines the rotation action on the domain of the loops with the action σ on the target space M. All of the equivariant constructions above generalise to the new action,

[1] We may need infinitely-many powers of $\mathbf{u}$ so we must use some kind of completed tensor product. Some authors use $\mathbb{Z}[\![\mathbf{u}]\!]$, however we use a slightly smaller cochain complex (2.4.14). With our approach, $EFH^*(M)$ is graded in the conventional sense.

and we denote these versions with a subscript σ. For example, $ESH^*_\sigma(M)$ is the equivariant symplectic cohomology corresponding to (2.2.8).

2.2.2 Equivariant Seidel maps

In this paper, we define new variants of the Floer Seidel map and the quantum Seidel map on equivariant Floer cohomology and equivariant quantum cohomology respectively. We prove a number of initial properties of these maps, which for the most part are just like the non-equivariant maps. The main exception is that the equivariant quantum Seidel map and the equivariant quantum product do not commute (Theorem 2.1.3).

The extension of the Floer Seidel map to the equivariant setting is nontrivial because the equivariant Floer Seidel map pulls back the action (2.2.8). A similar phenomenon occurs with the equivariant quantum Seidel map. No new analysis is required, however, since our constructions only use an S^1-parameterised version of the analysis used to define the Seidel maps in [Rit14].

2.2.2.1 Definitions

Let M be a convex[2] symplectic manifold which is either monotone or whose first Chern class vanishes. Let σ and ρ be two commuting Hamiltonian circle actions on M whose Hamiltonian functions are linear outside a compact subset of M. Assume the Hamiltonian K_σ of σ has nonnegative slope.

Recall the cochain complex for equivariant Floer cohomology $EFH^*(M; H^{\text{eq}})$ is generated by equivariant Hamiltonian orbits, which are certain equivalence classes of pairs $(w, x) \in S^\infty \times \mathcal{L}M$ under the equivalence relation $(\theta w, x(t)) \sim (w, \rho^{-\theta}(x(t-\theta)))$.

In order for the map $[(w, x)] \mapsto [(w, \sigma^* x)]$ to be well-defined, the equivalence class $[(w, \sigma^* x)]$ must be considered with the relation $\sim_{\sigma^*\rho}$, which corresponds to the pullback circle action $\sigma^*\rho = \sigma^{-1}\rho$. Once we account for the change in the action, the definition of the Floer Seidel map extends naturally.

Definition 2.2.3 (Equivariant Floer Seidel map). The *equivariant Floer Seidel map* is the map

$$EFS_{\tilde\sigma} : EFH^*(M; H^{\text{eq}}) \to EFH^{*+2I(\tilde\sigma)}(M; \sigma^* H^{\text{eq}}) \tag{2.2.9}$$

given by $[(w, x)] \mapsto [(w, \sigma^* x)]$ on equivariant Hamiltonian orbits. It is a -module isomorphism on the cochain complex which is compatible with algebraic $\mathbb{Z}[\mathbf{u}]$-module structure.

[2] Our construction works for closed manifolds as well, but we only discuss the convex case here to simplify the discussion. We treat both cases in the rest of the paper (see Remark 2.3.3).

On cohomology, it is compatible with the geometric Z[u]-module structure and with continuation maps. Under the limit as the slope of the equivariant Hamiltonians tends to infinity, the maps induce a well-defined isomorphism

$$EFS_{\tilde{}} : ESH\ (M)\ \to\ ESH\ ^{+2I(\tilde{})}(M). \tag{2.2.10}$$

We discuss how the equivariant Floer Seidel map is compatible with filtrations and the Gysin sequence in Section 2.5.2.

For the quantum Seidel map, we put an action on the clutching bundle which lifts the natural rotation action of the sphere and which restricts to the action ρ on the fibre above the south pole. The "twisting by σ" across the equator forces the action on the fibre above the north pole to be $\sigma\ \rho$. With this action on the clutching bundle, the quantum Seidel map extends naturally to the equivariant setup.

Definition 2.2.4 (Equivariant quantum Seidel map). The *equivariant quantum Seidel map* is the map

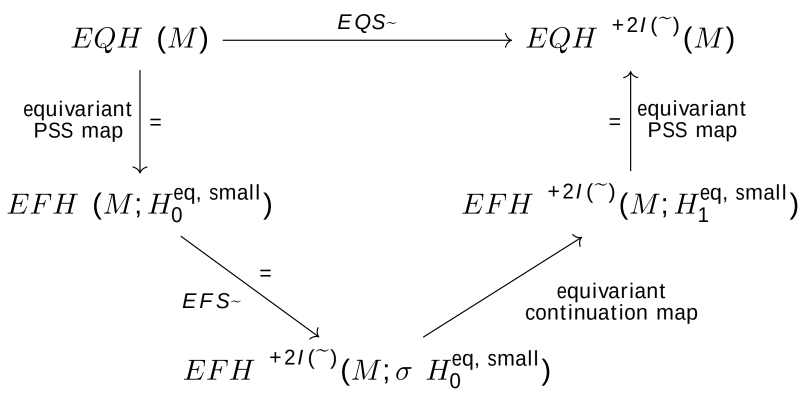
$$EQS_{\tilde{}} : EQH\ (M)\ \to\ EQH\ ^{+2I(\tilde{})}(M) \tag{2.2.11}$$

which counts equivalence classes $[(w, u)]$, where $w \in S^1$ and u is a pseudoholomorphic section of the clutching bundle. It is a -module homomorphism which is compatible with the algebraic and geometric Z[u]-module structures.

2.2.2.2 Properties

Equivariant PSS maps identify equivariant quantum cohomology with the equivariant Floer cohomology of an equivariant Hamiltonian of sufficiently small (positive) slope.

Proposition 2.2.5 (Compatibility with PSS maps). An analogous commutative diagram to (2.2.6) holds for the equivariant maps:



(2.2.12)

Here, $H_0^{\text{eq, small}}$ satisfies an equivariance condition analogous to (2.2.7) which corresponds to the action ρ. In contrast, $\sigma\ H_0^{\text{eq, small}}$ and $H_1^{\text{eq, small}}$ satisfy the equivariance condition which corresponds to the action $\sigma \ \rho$. Both $H_0^{\text{eq, small}}$ and $H_1^{\text{eq, small}}$ have small positive slope, but $\sigma \ H_0^{\text{eq, small}}$ has negative slope in general.

This proposition implies (2.1.9).

The maps for different actions compose exactly like the non-equivariant case as follows.

Proposition 2.2.6 (Composition of multiple actions). Let ρ, σ_1 and σ_2 be commuting Hamiltonian circle actions whose Hamiltonians are linear outside a compact subset of M. Suppose the Hamiltonians of σ_1 and σ_2 have nonnegative slope. The following diagrams commute. (We have omitted grading, the Hamiltonians and M from the notation.)

$$
\begin{array}{ccc}
EQH \xrightarrow{EQS_{\tilde\sigma_1}} EQH_{\sigma_1} \xrightarrow{EQS_{\tilde\sigma_2}} EQH_{(\sigma_2 \sigma_1)}, & EQH \xrightarrow{EQS_{\tilde\sigma_2 \tilde\sigma_1}} EQH_{(\sigma_2 \sigma_1)} \\
EFH \xrightarrow{EFS_{\tilde\sigma_1}} EFH_{\sigma_1} \xrightarrow{EFS_{\tilde\sigma_2}} EFH_{(\sigma_2 \sigma_1)}, & EFH \xrightarrow{EFS_{\tilde\sigma_2 \tilde\sigma_1}} EFH_{(\sigma_2 \sigma_1)}
\end{array}
\tag{2.2.13}
$$

The quantum Seidel map intertwines the quantum product. More precisely, given $\alpha \in QH^*(M)$, the operations $QS_{\tilde\sigma}$ and α commute, where α is the operation given by right-multiplication by α. Thus the relation $[QS_{\tilde\sigma}, \ \alpha] = 0$ holds.

The first complication for extending this relation to the equivariant setup is that the domain and codomain of $EQS_{\tilde\sigma}$ are different. We solve this by using the clutching bundle to relate the domain and codomain. Let $\alpha \in EH^*(E)$ be an equivariant cohomology class of the clutching bundle E with respect to the action we put on it. The restriction of α to the fibre over the south pole is a class $\alpha^+ \in EH^*(M)$, while the restriction to the fibre over the north pole is a class $\alpha^- \in EH^*(M)$.

Remark 2.2.7 (Finding α given α^+). A Mayer-Vietoris argument applied to the the two hemispheres of the clutching bundle yields the exact sequence

$$
\begin{aligned}
EH^*(E) &\to EH^*(M) \oplus EH^*(M) \to H^*(M) \\
\alpha &\mapsto (\alpha^+, \alpha^-).
\end{aligned}
\tag{2.2.14}
$$

Thus, given $\alpha^+ \in EH^*(M)$, we determine the class α by finding a class α^- whose restriction to non-equivariant cohomology is the same as that of α^+. This is readily possible if the map $EH^*(M) \to H^*(M)$ is surjective. This surjectivity holds if $\sigma \ \rho$ is the trivial action, and it holds for any action over rational coefficients by (2.1.3).

Having established how to compare classes in the domain and codomain of $EQS_{\tilde\sigma}$, we can ask whether the analogue of the intertwining relation $[QS_{\tilde\sigma}, \ \alpha] = 0$ holds in the equivariant setup. From the non-equivariant relation, we can immediately deduce any failure to commute is $o(\mathbf{u})$. The following theorem characterises the failure precisely.

Theorem 2.2.8 (Intertwining relation, Theorem 2.1.3). *The equation*

$$EQS_{\tilde{\sigma}}(x \star \alpha^+) - EQS_{\tilde{\sigma}}(x) \star \alpha^- = \mathbf{u} \smile EQS_{\tilde{\sigma},\alpha}(x) \tag{2.2.15}$$

holds for all $\alpha \in EH^*(E)$ *and* $x \in EQH^*(M)$*, where*

$$EQS_{\tilde{\sigma},\alpha} : EQH^*(M) \to EQH^{*+2I(\tilde{\sigma})+|\alpha|-2}(M) \tag{2.2.16}$$

is a map defined in Section 2.7.4.1.

In Theorem 2.1.3, the product $\star$ is the equivariant quantum product for the action ρ, and the symbol $\smile$ denotes the action of the geometric $\mathbb{Z}[\mathbf{u}]$-module structure. The map $EQS_{\tilde{\sigma},\alpha}$ counts (equivariant) pseudoholomorphic sections of the clutching bundle which are weighted by the (equivariant) class α. This weighting is easiest to understand when $\alpha \in EH^2(E)$ is a degree 2 class. In this case, the map $EQS_{\tilde{\sigma},\alpha}$ counts exactly the same pseudoholomorphic sections u as the map $EQS_{\tilde{\sigma}}$, but with the weight $\alpha(u_*([S^2]))$. For the full definition, see Section 2.7.4.1.

In the realm of algebraic geometry, Maulik and Okounkov proved an analogous[3] intertwining relation for quiver varieties [MO19, Proposition 8.2.1], and our formula resembles theirs when α has degree 2. They prove the relation using *virtual localization* [Hor03, Chapter 27]. This technique converts counting sections into counting only the sections which are invariant under the S^1-action on the clutching bundle. Any invariant section must be a constant section at a fixed point of the action σ on M, however invariant sections are allowed to bubble over the poles. The result is a decomposition of EQS into three maps: $B^- \star F \star B^+$. The map B^- counts bubbles over the north pole and B^+ counts bubbles over the south pole, both appropriately modified according to the virtual localization. The map F corresponds to the constant section at a fixed point of the action σ.

The intertwining relation is then proven for each of the three maps. For B^- and B^+, it is a consequence of standard relations corresponding to gravitational descendant invariants (namely, the divisor equation and the topological recursion relation [Hor03, Chapter 26]). For F, the intertwining relation is a topological result which relates α^- and α^+ when both classes are restricted to the fixed locus of σ on M.

In contrast, our proof has a Floer-theoretic flavour: we construct an explicit 1-dimensional moduli space whose boundary gives (2.2.15) on cohomology.

To motivate our proof, consider first the following proof of the intertwining of the non-equivariant quantum Seidel map. Define a 1-dimensional moduli space of pseudoholomorphic

[3] The sections of the bundle must intersect the input $x \in EQH^*(M)$ over the south pole. In Maulik and Okounkov's conventions, the sections intersect x over the north pole, and this is why their intertwining relation has different signs to ours.

sections which intersect a fixed Poincaré dual α^- of α along a fixed line of longitude $L \subset S^2$. The boundary of this moduli space occurs when the intersection point is at either pole. When it is at the south pole, we recover the term $QS(x * \alpha)$, while at the north pole we get the term $-QS(x) * \alpha$. Summing these boundary components gives the equation $QS(x * \alpha) - QS(x) * \alpha = 0$ as desired.

In the equivariant case, we must allow the line of longitude to vary with $w \in S^\infty$. This is because the intersection condition must be preserved by the equivalence relation $\sim$. We fix an equivariant assignment of lines of longitude $w \mapsto L_w$ for an invariant dense open set of $w \in S^\infty$. Note that a global assignment is not possible: the set of lines of longitude is isomorphic to S^1, however there are no equivariant maps $S^\infty \to S^1$.

For the equivariant 1-moduli space, we ask that the equivariant section $[(w, u)]$ satisfies $u(z) \in \alpha^-$ for some $z \in L_w$. As per the non-equivariant case, the two poles yield the two terms on the left-hand side of (2.2.15). The remaining term in (2.2.15) comes from a limit in which w exits the dense open set.

The computation behind Example 2.2.10 verifies that the right-hand side of (2.2.15) is nonzero even in straightforward cases.

2.2.3 Examples

We compute the equivariant quantum Seidel map for the following three spaces: the complex plane, complex projective space and the total space of the tautological line bundle $\mathcal{O}_{\mathbb{P}^N}(-1)$. Through these examples, we demonstrate how the map may be used to compute equivariant quantum cohomology and equivariant symplectic cohomology.

For the complex plane, we deduce the equivariant quantum Seidel map from the equivariant Floer complex which was computed in [Zha19, Section 8.1]. In both other examples, we find the map by directly computing some coefficients and deducing the rest by repeated application of the intertwining relation (2.1.5).

We use the parameterisation $S^1 = \mathbb{R}/\mathbb{Z}$ throughout.

Example 2.2.9 (Complex plane). The complex plane has a Hamiltonian circle action $\sigma_t(z) = e^{2\pi i t} z$. The origin $0_{\mathbb{C}} \in \mathbb{C}$ is the unique fixed point of σ. Thus $\mathbb{C}$ is equivariantly contractible and its symplectic form is globally exact, so for any nonnegative r, we have $EQH^*_r(\mathbb{C}) = \mathbb{Z}[\mathbf{u}]$. The equivariant quantum Seidel map is

$$\begin{aligned} EQS_\sigma : EQH^*_r(\mathbb{C}) &\to EQH^{*+2}_{(r+1)}(\mathbb{C}) \\ 1 &\mapsto (r+1)\mathbf{u}. \end{aligned} \tag{2.2.17}$$

Example 2.2.10 (Projective space). The complex projective space $\mathbb{P}^n$ with its Fubini-Study symplectic form has a Hamiltonian action σ given by

$$\theta \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : e^{2\pi i\theta} z_1 : \cdots : e^{2\pi i n\theta} z_n], \tag{2.2.18}$$

for any $(z_0, z_1, \cdots, z_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$. The σ-invariant Morse function $f_{\mathbb{P}^n}([z_0 : \cdots : z_n]) = \sum_{k=0}^{n} k|z_k|^2$ has critical points $e_0, \dots, e_n$, where e_k has Morse index $2k$. For any nonnegative integer r, we have

$$EQH^{-*}_{r}(\mathbb{P}^n) = \mathbb{Z}[q^{-1}] \,\hat{\otimes}\, \mathbb{Z}[\mathbf{u}]\langle e_0, \dots, e_n\rangle, \tag{2.2.19}$$

where the Novikov variable q is a formal variable of degree $2(n+1)$. The equivariant quantum Seidel map is the map

$$EQS_{\tilde{\sigma}} : EQH^{-*}_{r}(\mathbb{P}^n) \to EQH^{-*+2n}_{(r+1)}(\mathbb{P}^n) \tag{2.2.20}$$

given by

$$\begin{cases} e_0 \mapsto \sum_{l=0}^{n} (r+1)^{n-l}\mathbf{u}^{n-l} e_l, \\ e_k \mapsto \sum_{l=0}^{k-1} (r+1)^{k-1-l} q\mathbf{u}^{k-1-l} e_l, \quad k = 1, \dots, n. \end{cases} \tag{2.2.21}$$

Computing the quantum Seidel map and using (2.2.4) is one way to compute the (non-equivariant) quantum product on $\mathbb{P}^n$ [MT06, Section 5]. In our computations behind Example 2.2.10, we demonstrate this extends to the equivariant case: we use the equivariant quantum Seidel map and Theorem 2.1.3 to derive the equivariant quantum product on $\mathbb{P}^n$. Equivariant quantum Seidel maps have already been used to compute the equivariant quantum product on $\mathbb{P}^n$ [Iri17, Section 4.4], though our direct method is new. The product on $\mathbb{P}^n$ was previously known; it had been derived using equivariant quantum Littlewood–Richardson coefficients [Mih06].

Example 2.2.11 (Tautological line bundle). The total space $O_{\mathbb{P}^n}(-1)$ of the tautological line bundle over projective space $\mathbb{P}^n$ is a monotone convex symplectic manifold [Rit14, Section 7]. The fibres are symplectic submanifolds, and the circle action σ which rotates the fibres is a linear Hamiltonian circle action. The action σ fixes the image of the zero section Z, and, like Example 2.2.9, $O_{\mathbb{P}^n}(-1)$ equivariantly contracts onto Z with the trivial circle action. We use the same Morse function on $Z = \mathbb{P}^n$ as for Example 2.2.10, so we have

$$EQH^{-*}_{r}(O_{\mathbb{P}^n}(-1)) = \mathbb{Z}[q^{-1}] \,\hat{\otimes}\, \mathbb{Z}[\mathbf{u}]\langle e_0, \dots, e_n\rangle, \tag{2.2.22}$$

where the Novikov variable q now has degree $2n$ (so $\Lambda = \mathbb{Z}[q^{-1}]$ is the Novikov ring). The equivariant quantum Seidel map is the map

$$EQ\tilde{S} : EQH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1)) \to EQH^{*+2}_{\star_{(r+1)}}(\mathcal{O}_{\mathbb{P}^n}(-1)) \tag{2.2.23}$$

given by

$$e_k \mapsto \begin{cases} e_{k+1} + (r+1)\mathbf{u}e_k & k < n, \\ qe_1 + (r+1)\mathbf{u}e_n - (r+1)\mathbf{u}qe_0 & k = n. \end{cases} \tag{2.2.24}$$

Unlike the non-equivariant quantum Seidel map on $\mathcal{O}_{\mathbb{P}^n}(-1)$, this is an injective map.

From (2.2.24), we derive the equivariant quantum product as given by

$$e_1 \star_r e_k = \begin{cases} e_{k+1} & k < n, \\ -qe_1 + r\mathbf{u}qe_0 & k = n. \end{cases} \tag{2.2.25}$$

This product has a term which is not detected by either quantum cohomology or equivariant cohomology: it exists only in equivariant quantum cohomology.

We can use (2.2.24) to find the equivariant symplectic cohomology of $\mathcal{O}_{\mathbb{P}^n}(-1)$ using an argument of Ritter [Rit14, Theorem 22]. We deduce that the equivariant symplectic cohomology $ESH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1))$ is a $\Lambda\hat{\otimes}\mathbb{Z}[\mathbf{u}]$-module which is not finitely generated and which satisfies

$$EQH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1)) \subset ESH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1)) \subset \left(EQH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1))\right)_{\mathbb{Z}[\mathbf{u}]\setminus\{0\}}, \tag{2.2.26}$$

where the left inclusion is the equivariant c^* map and the module on the right is the localisation by $\mathbb{Z}[\mathbf{u}] \setminus \{0\}$. If we perform this localisation to all three modules, we find that the localised equivariant symplectic cohomology is isomorphic to the localised equivariant quantum cohomology, which is equivalent to a version of Zhao's localisation theorem (2.1.10).

We express $ESH^*_{\star_r}(\mathcal{O}_{\mathbb{P}^n}(-1))$ as a $\Lambda\hat{\otimes}\mathbb{Z}[\mathbf{u}]$-submodule of the localised equivariant quantum cohomology with an explicit set of generators. The generators are defined by a recurrence relation induced by (2.2.24).

2.3 Floer theory

In Section 2.3.1, we clarify the assumptions we place on our symplectic manifold. We proceed in Section 2.3.2 by defining Floer cohomology and symplectic cohomology using the same conventions as [Sei97; Rit14]. A more complete explanation of the construction of Floer cohomology may be found in [Sal97]. Next, we introduce the Hamiltonian circle actions which yield the Seidel map in Section 2.3.3, and define the Floer Seidel map in Section 2.3.4.

2.3.1 Symplectic manifolds

Let M be a $2n$-dimensional smooth manifold with a symplectic form ω. For convenience, we assume throughout that M is nonempty and connected. There are two additional conditions that we will impose on M. The first establishes a relationship between the cohomology class of ω and the first Chern class, and the second controls the behaviour of the symplectic form when the manifold is open.

Definition 2.3.1. Denote by $c_1 \in H^2(M)$ the first Chern class of the symplectic vector bundle (TM, ω). The symplectic manifold M is *nonnegatively monotone* if either:

- there is $\lambda \geq 0$ such that $\omega(A) = \lambda c_1(A)$ for all $A \in \pi_2(M)$; or
- $c_1(A) = 0$ for all $A \in \pi_2(M)$.

By an analogous argument to the proof of Lemma 1.1 in [HS95], a symplectic manifold is nonnegatively monotone if and only if the implication

$$c_1(A) < 0 \implies \omega(A) \geq 0 \tag{2.3.1}$$

holds for all $A \in \pi_2(M)$. Such a manifold has the property that, for any compatible almost complex structure, all pseudoholomorphic curves have nonnegative first Chern number.

Definition 2.3.2. A *convex symplectic manifold* is a symplectic manifold M that is equipped with a closed $(2n-1)$-dimensional manifold Σ, a contact form α on Σ and a map

$$\psi : \Sigma \times [1, \infty) \to M \tag{2.3.2}$$

such that ψ is a diffeomorphism onto its image, the set $M \setminus \psi(\Sigma \times [1, \infty))$ is relatively compact[4] and

$$\psi^* \omega = d(R\alpha) \tag{2.3.3}$$

holds on $\Sigma \times [1, \infty)$. Here, $R \in [1, \infty)$ is the *radial coordinate*[5] and the image of ψ is the *convex end* of M.

Remark 2.3.3. We emphasise three features of this definition.

- The manifold Σ, the contact form α and the diffeomorphism ψ are all part of the data of a convex symplectic manifold. Consequently, we can use the coordinates provided by ψ without worrying about whether our constructions are independent of this choice (c.f. Remark 2.3.10).

[4] A subset of a topological space is *relatively compact* if its closure is compact.
[5] The set $\{R \leq R_0\}$ is defined to be $M \setminus \psi(\Sigma \times (R_0, \infty))$ and is compact for all $R_0 \geq 1$.

- A closed symplectic manifold is convex since we allow the manifold ∂M to be empty.[6] With this convention, we are able to prove all of our results for closed manifolds and (non-trivially) convex symplectic manifolds simultaneously.

- The symplectic form ω does not have to be exact on all of M, since (2.3.3) applies only on the convex end of M. Indeed, symplectic forms on closed manifolds are never globally exact.

Henceforth, M will be a nonnegatively monotone convex symplectic manifold.

Finally, we assume that $\mathbb{R} \setminus \mathcal{R}$ is unbounded so that symplectic cohomology is a meaningful direct limit (see Definition 2.3.9). Here $\mathcal{R}$ is the set of Reeb periods as defined in (2.3.6).

Remark 2.3.4 (Orientations). This paper uses integral coefficients, and thus we require orientations on all moduli spaces. For this purpose, let $\mathfrak{o}$ be a coherent orientation, defined as in [Rit13, Appendix B]. The proof of all orientation signs in this paper is omitted.

2.3.2 Floer cohomology

2.3.2.1 Hamiltonian dynamics

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

A *(time-dependent[7]) Hamiltonian (function)* H is a smooth function $S^1 \times M \to \mathbb{R}$. Its *(time-dependent) Hamiltonian vector field* X_H is the unique S^1-family of vector fields defined by $\omega(\cdot, X_{H_t}) = \mathrm{d}H_t$. The *Hamiltonian flow* φ_H is the flow along the vector field X_H, so φ_H^t satisfies

$$\partial_t \left(\varphi_H^t(m)\right) = X_{H_t}\left(\varphi_H^t(m)\right). \tag{2.3.4}$$

A *Hamiltonian orbit* is a loop $x : S^1 \to M$ which satisfies $\partial_t x(t) = X_{H_t}(x(t))$ for all $t \in S^1$. It is *nondegenerate* if the linear map $D\varphi_H^1 : T_{x(0)}M \to T_{x(1)}M$ has no eigenvalue equal to 1. Denote by $\mathcal{P}(H)$ the set of Hamiltonian orbits of H.

The Hamiltonian H is *linear of slope* λ if the identity $H_t(\psi(y, R)) = \lambda R + \mu$ holds at infinity[8] for some constant μ. The vector field of such a Hamiltonian is a multiple of the

[6] We adopt the convention that the *empty manifold* is a disconnected closed oriented manifold of every dimension. This means that the dimension of a manifold X is not well-defined; the statement $\dim X = k$ is to be interpreted as 'X is k-dimensional'.

[7] Throughout, a *time-dependent* object is a smooth S^1-family of objects, and a B*-dependent* object is a smooth B-family of objects for any manifold B.

[8] In the context of convex manifolds, a condition holds *at infinity* if there exists $R_0 \geq 1$ such that the condition holds on $\psi(\partial M \times [R_0, \infty))$. Notice that if finitely-many conditions each hold individually at infinity, then their conjunction holds at infinity (i.e., they all hold in a common region at infinity). All statements hold tautologically at infinity on a closed symplectic manifold (see Remark 2.3.3).

Reeb vector field[9] X , so, at infinity, we have

$$X_{H_t} = \lambda X \quad 0 \in T \quad T[1, \infty) = TM. \tag{2.3.5}$$

As such, any Hamiltonian orbit in this region corresponds to a λ-periodic flow along the Reeb vector field.

A *Reeb period* is a nonzero number $\lambda \in \mathbb{R} \setminus \{0\}$ such that there exists a point $y \in$ such that the flow of the Reeb vector field from y is λ-periodic. If is empty, set $\mathcal{R} = \emptyset$, and otherwise set

$$\mathcal{R} = \{\text{Reeb periods}\} \cup \{0\}. \tag{2.3.6}$$

Notice $k\lambda \in \mathcal{R}$ for any $\lambda \in \mathcal{R}$ and any $k \in \mathbb{Z}$.

Thus, if H is linear of slope λ with $\lambda \notin \mathcal{R}$, then all Hamiltonian orbits of H lie in a compact region of M. If moreover all Hamiltonian orbits are nondegenerate, then $\mathcal{P}(H)$ is finite.

2.3.2.2 Almost complex structures

Let m be a point in the convex end of M. The ω-compatible almost complex structure J is *convex at the point* m if $dR \circ J = R\alpha$ holds at m. This means that the almost complex structure respects the direct sum decomposition

$$T_m M = (\mathbb{R}\partial_R \oplus \mathbb{R}X) \oplus \ker \alpha \tag{2.3.7}$$

and satisfies $J(R\partial_R) = X$ at m.

Let B be a manifold and let $\mathbf{J} = (\mathbf{J}_b)_{b \in B}$ be a smooth family of ω-compatible almost complex structures. The family $\mathbf{J}$ is *convex* if the almost complex structure $\mathbf{J}_b$ is convex in a common region at infinity for all $b \in B$.

Definition 2.3.5. A choice of *Floer data* is a pair $(H, \mathbf{J})$, where H is a linear time-dependent Hamiltonian with slope not in $\mathcal{R}$ and $\mathbf{J}$ is a convex time-dependent ω-compatible almost complex structure.

2.3.2.3 Pseudoholomorphic spheres

Let j be the standard almost complex structure on the sphere $\mathbb{P}^1$. Let

$$= \frac{\pi_2(M)}{\ker c_1 \cap \ker \omega}, \tag{2.3.8}$$

[9] The *Reeb vector field* X associated to is the vector field on uniquely defined by $\iota_X(d) = 0$ and $\iota_X = 1$.

where $c_1, \omega : \pi_2(M) \to \mathbb{Z}$ implitly use the Hurewitz homomorphism $\pi_2(M) \to H_2(M)$ A curve $u : \mathbb{P}^1 \to M$ is *J-holomorphic* if $(Du) \circ j = J \circ (Du)$ and u *represents the class $A \in$* if $[u] = A$.

Let B be a manifold and $\mathbf{J}$ be a convex B-dependent family of ω-compatible almost complex structures. The moduli space $\mathcal{M}(A, \mathbf{J})$ is the space of pairs (b, u), where u is a simple[10] $\mathbf{J}_b$-holomorphic curve representing $A \in$.

Denote by $V_k(\mathbf{J})$ the set of pairs $(b, m) \in B \times M$ such that m lies in the image of a nonconstant $\mathbf{J}_b$-holomorphic sphere u with $c_1(u) \leq k$.

2.3.2.4 Floer solutions

Denote by $\mathcal{L}M$ the space of contractible smooth maps $S^1 \to M$. Define a cover of this space $\widetilde{\mathcal{L}M}$ as the space of pairs (x, u) of loops $x \in \mathcal{L}M$ and smooth maps $u : D \to M$ satisfying $x(t) = u(e^{2\pi i t})$, considered up to the equivalence

$$(x, u) \sim (x, u') \iff \int_D u^* \widetilde{c_1} = \int_D u'^* \widetilde{c_1} \text{ and } \int_D u^* \omega = \int_D u'^* \omega, \tag{2.3.9}$$

where $\widetilde{c_1}$ is a differential 2-form representing the first Chern class c_1. The deck transformation group of $\widetilde{\mathcal{L}M}$ is , which acts on $\widetilde{\mathcal{L}M}$ by 'adding $A \in$ to the filling u'. This action is described explicitly in [HS95, Section 5]. Let

$$\widetilde{\mathcal{P}}(H) = \left\{ (x, u) \in \widetilde{\mathcal{L}M} \;:\; x \in \mathcal{P}(H) \right\}. \tag{2.3.10}$$

Let $(H, \mathbf{J})$ be a choice of Floer data. The *action functional* associated to H is the map $\mathcal{A}_H : \widetilde{\mathcal{L}M} \to \mathbb{R}$ given by

$$\mathcal{A}_H(x, u) = -\int_D u^* \omega + \int_{t=0}^{1} H_t(x(t))\, \mathrm{d}t. \tag{2.3.11}$$

The set of critical points of $\mathcal{A}_H$ equals $\widetilde{\mathcal{P}}(H)$. A smooth map $u : \mathbb{R} \times S^1 \to M$ is a *Floer solution* if it satisfies

$$\partial_s u + \mathbf{J}_t(\partial_t u - X_{H;t}) = 0 \tag{2.3.12}$$

for all $(s, t) \in \mathbb{R} \times S^1$. The left side of (2.3.12) is abbreviated by $\overline{\partial}_{H;\mathbf{J}}(u)$. The *energy* $E(u)$ of a map u is

$$E(u) = \int_{\mathbb{R} \times S^1} \|\partial_s u\|^2_{\mathbf{J}_t}\, \mathrm{d}s \wedge \mathrm{d}t, \tag{2.3.13}$$

---

[10] The holomorphic curve $u : \mathbb{P}^1 \to M$ is *multiply-covered* if it is a composition of a holomorphic branched covering map $\mathbb{P}^1 \to \mathbb{P}^1$ with degree strictly greater than 1 and a second holomorphic curve. The curve u is *simple* otherwise.

where $\|\cdot\|_J$ is the norm associated to the J-invariant metric $\omega(\cdot, J\cdot)$. Suppose the Hamiltonian orbits of H are nondegenerate. If the energy of the Floer solution u is finite, then there exist two Hamiltonian orbits $x_\pm$ such that

$$\lim_{s\to\pm\infty} u(s,t) = x_\pm(t) \qquad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0, \tag{2.3.14}$$

where the limits denote uniform convergence[11] in t, and moreover

$$E(u) = A_H(x_-, u_-) - A_H(x^+, u_- \# u) \tag{2.3.15}$$

for any filling u_- of x_-.

By a maximum principle [Rit13, Appendix D], there is a compact region of M such that all Floer solutions lie completely within the region.

Let $\widetilde{x}_- = (x_-, u_-) \in \widetilde{P}(H)$. Denote by $M(\widetilde{x}_-, \widetilde{x}^+)$ the moduli space of Floer solutions u which satisfy (2.3.14) and $u^+ = u_- \# u$.

2.3.2.5 Moduli space of Floer trajectories

In order to ensure the moduli space $M(\widetilde{x}_-, \widetilde{x}^+)$ is a smooth finite-dimensional manifold and has other desired behaviour, we impose a number of *regularity conditions* on the Floer data. For *regular* Floer data, the Hamiltonian orbits are nondegenerate, the moduli space $M(A; \mathbf{J})$ is a smooth canonically-oriented[12] manifold of dimension[13] $2n + 2c_1(A) + 1$ and the moduli space $M(\widetilde{x}_-, \widetilde{x}^+)$ of Floer solutions is a smooth oriented manifold of dimension

$$\dim M(\widetilde{x}_-, \widetilde{x}^+) = \mu(\widetilde{x}_-) - \mu(\widetilde{x}^+). \tag{2.3.16}$$

Here, we denote by $\mu(\widetilde{x})$ the Conley-Zehnder index for $\widetilde{x} \in \widetilde{P}(H)$. In addition, we have $(t, x(t)) \notin V_1(\mathbf{J})$ and $(t, u(s,t)) \notin V_0(\mathbf{J})$ for all Hamiltonian orbits x and Floer solutions $u \in M(\widetilde{x}_-, \widetilde{x}^+)$ when $\dim M(\widetilde{x}_-, \widetilde{x}^+) \leq 2$.

Remark 2.3.6 (Regularity). In this paper, we will not always list all of the regularity conditions, however we will mention those that are more uncommon.[14] The conditions are

[11] The uniform convergence is a priori with respect to the t-dependent metric $\|\cdot\|_{J_t}$, but the property holds for any t-dependent Riemannian metric.

[12] In order to orient all other moduli spaces in this paper, we have had to choose orientation data, such as coherent orientations (or orientations of the unstable manifolds in the case of Morse theory). The orientation of $M(A; \mathbf{J})$ is intrinsic, however it does rely on an orientation of the parameter space S^1, which we fix.

[13] The 1 in this formula corresponds to $\dim S^1$, and will be $\dim B$ for B-dependent almost complex structures.

[14] In order to find that moduli spaces are manifolds of a given dimension, the standard method is to find a Fredholm operator whose kernel describes the tangent space of the moduli space. The corresponding regularity condition is that the operator is onto, so that some version of the implicit function theorem may be applied. A version of the Sard-Smale theorem shows this to be generic [FHS95]. The nondegeneracy of the orbits follows by [AD14, Remark 5.4.8], and the avoidence of bubbling here is due to [HS95; Sei97].

always motivated by improving the behaviour of the moduli spaces. Here, we have used them to ensure the moduli spaces of Floer solutions have the structure of manifolds without bubbling in dimensions 1 and 2. In our notation, regularity conditions will always be satisfied by generic[15] data, and hence will exist.

Assume $(H, \mathbf{J})$ is regular. When $\widetilde{x}^- \neq \widetilde{x}^+$, the moduli space $\mathcal{M}(\widetilde{x}^-, \widetilde{x}^+)$ admits a smooth free $\mathbb{R}$-action given by s-translation. The quotient $\widetilde{\mathcal{M}}(\widetilde{x}^-, \widetilde{x}^+)$ is a smooth manifold. The 0-dimensional moduli spaces $\widetilde{\mathcal{M}}(\widetilde{x}^-, \widetilde{x}^+)$ are all compact by the so-called compactification argument (for example, see [Sal97, Section 3.1]).

The 1-dimensional moduli spaces $\widetilde{\mathcal{M}}(\widetilde{x}^-, \widetilde{x}^+)$ may be given the structure of an oriented compact 1-manifold with boundary by attaching the endpoints

$$\bigcup_{\substack{\widetilde{x}^0 \in \widetilde{\mathcal{P}}(H) \\ \mu(x^-) = \mu(x^0)+1}} \widetilde{\mathcal{M}}(\widetilde{x}^-, \widetilde{x}^0) \times \widetilde{\mathcal{M}}(\widetilde{x}^0, \widetilde{x}^+) \tag{2.3.17}$$

using the so-called gluing maps (for example, see [Sal97, Section 3.3]). Any element of (2.3.17), and more generally any chain of Floer trajectories, is a *broken* Floer trajectory.

2.3.2.6 Floer cochain complex

Let $(H, \mathbf{J})$ be a regular choice of Floer data.

Let q be a formal variable. A *formal power series with coefficients in* $\mathbb{Z}$ *and exponents in* Γ is a formal sum $\sum_{A \in \Gamma} \alpha_A q^A$. The monomial q^A is given the grading $2c_1(A) \in \mathbb{Z}$. Set Λ^k to be the group of formal power series which are supported by only monomials of grading k and satisfy the condition that the set

$$\{A \in \Gamma : \omega(A) \leq c, \alpha_A \neq 0\}$$

is finite for all $c \in \mathbb{R}$. The *Novikov ring* Λ is the $\mathbb{Z}$-graded ring $\bigoplus_{k \in \mathbb{Z}} \Lambda^k$.

The *degree-k Floer cochains* are the formal sums

$$\sum_{\substack{\widetilde{x} \in \widetilde{\mathcal{P}}(H) \\ \mu(\widetilde{x}) = k}} \alpha_{\widetilde{x}} \widetilde{x}$$

with integer coefficients such that the set $\left\{\widetilde{x} \in \widetilde{\mathcal{P}}(H) : \omega(\widetilde{x}) \leq c, \alpha_{\widetilde{x}} \neq 0\right\}$ is finite for all $c \in \mathbb{R}$. Denote by $FC^k(M; H, \mathbf{J})$ the set of degree-k Floer cochains. The *Floer cochain*

[15] In a topological space, a subset is *generic* or *of second category* if it is a countable intersection of open dense subsets. A condition on (Floer) data is *generic* if the set of data satisfying the condition forms a generic subset of all data.

complex $FC(M;H,\mathbf{J})$ *associated to* $\mathbf{J}$ *and* H is the finitely-generated free Z-graded -module $_{k \in \mathbb{Z}} FC^k(M;H,\mathbf{J})$. The -module structure is induced by the inverse of the -action on $\widetilde{P}(H)$, giving $q^A \ [x,u] = [x,(\ A)\#u]$.

The Floer cochain differential $d : FC(M;H,\mathbf{J}) \to FC^{+1}(M;H,\mathbf{J})$ is the degree-1 -module endomorphism given by[16]

$$d(\widetilde{x}^+) = \sum_{\substack{\widetilde{x} \in \widetilde{P}(H) \\ (\widetilde{x}) \ (\widetilde{x}^+)=1}} \ \sum_{[u] \in \widetilde{M}(\widetilde{x} \ ;\widetilde{x}^+)} o([u])\,\widetilde{x} \ , \tag{2.3.18}$$

where $o([u]) \in \{ 1\}$ denotes the orientation[17] of the point $[u] \in \widetilde{M}(\widetilde{x} \ ,\widetilde{x}^+)$ induced by the coherent orientation o.

Lemma 2.3.7. *The differential* d *satisfies* $d^2 = 0$.

The *Floer cohomology* $FH(M;H,\mathbf{J})$ *of* M *with Floer data* $(H,\mathbf{J})$ is the cohomology of $(FC(M;H,\mathbf{J}),d)$. It is a Z-graded -module.

Remark 2.3.8. Floer cohomology only depends on the slope of the Hamiltonian in the Floer data, and is otherwise independent of the choice of Floer data. This follows from the fact that, given two choices of Floer data whose Hamiltonians have the same slope, any monotone homotopy between them induces a continuation map which is an isomorphism (continuation maps are defined in Section 2.3.2.7). Floer cohomology is also independent of the coherent orientation o. For closed manifolds M, there is a -module isomorphism $FH(M;H,\mathbf{J}) = \quad H(M)$ (the PSS maps of (2.6.9) are isomorphisms).

2.3.2.7 Continuation maps

Given any two regular choices of Floer data $(H \ ,\mathbf{J} \ )$ and $(H^+,\mathbf{J}^+)$, a *homotopy* between them is a pair $(H_{s;t},\mathbf{J}_{s;t})$, where $H_{s;t}$ is a R S^1-dependent Hamiltonian and $\mathbf{J}_{s;t}$ is a convex R S^1-dependent (ω-compatible) almost complex structure, such that both are s-dependent only on a compact region of R S^1 and, respectively, equal H and $\mathbf{J}$ for $s \quad 0$. The homotopy is *monotone* if, at infinity, $H_{s;t} = h_s(R)$ and $\partial_s h_s^0(R) \quad 0$. A monotone homotopy exists only if H have slopes λ that satisfy $\lambda \quad \lambda^+$.

Given any monotone homotopy[18] $(H_{s;t},\mathbf{J}_{s;t})$ which is suitably regular, the *continuation map* $\varphi : FH(M;H^+,\mathbf{J}^+) \to FH(M;H \ ,\mathbf{J} \ )$ is defined by

$$\varphi(\widetilde{x}^+) = \sum_{\substack{\widetilde{x} \in \widetilde{P}(H \ ) \\ (\widetilde{x} \ ) = \ (\widetilde{x}^+)}} \ \sum_{u \in M(\widetilde{x} \ ;\widetilde{x}^+)} o(u)\,\widetilde{x} \ , \tag{2.3.19}$$

[16] A map of the form (2.3.18) *counts* the moduli spaces $\widetilde{M}(\widetilde{x} \ ;\widetilde{x}^+)$.

[17] Recall an orientation of a 0-dimensional manifold is a choice of 1 for each point of the manifold.

[18] The space of such regular monotone homotopies is nonempty whenever H have slopes that satisfy $^+$.

where $\mathcal{M}(\widetilde{x}^-, \widetilde{x}^+)$ is the moduli space of solutions to a parameterised version of the Floer equation (2.3.12) for the homotopy. Continuation maps are independent of the choice of homotopy. Moreover, the composition of two continuation maps is itself a continuation map.

Definition 2.3.9. The *symplectic cohomology* $SH^*(M)$ *of* M is the direct limit

$$\lim_{\rightarrow} FH^*(M; H, \mathbf{J}), \tag{2.3.20}$$

where the limit is over all choices of regular Floer data $(H, \mathbf{J})$ ordered by slope, and the maps between the Floer cohomologies are the continuation maps.

Remark 2.3.10. The symplectic cohomology $SH^*(M)$ a priori depends on the parameterisation of the convex end of M. Perturbations of this parameterisation do not affect $SH^*(M)$ (see [BR20, Theorem 1.9]). With our convention in Remark 2.3.3, symplectic cohomology is isomorphic to Floer cohomology for closed manifolds.

2.3.3 Hamiltonian circle actions

A *Hamiltonian circle action* on M is a smooth circle action $\sigma : S^1 \times M \to M$ which flows along the Hamiltonian vector field of some Hamiltonian function $K : M \to \mathbb{R}$. Such a circle action automatically preserves the symplectic structure. The action is *linear* if K is linear.

The *vector field* X_σ of a circle action σ is the vector field along which the action flows; it is given by $X_\sigma = \partial_t \sigma_t|_{t=0}$. Thus for our *Hamiltonian* circle action σ, the vector field X_σ equals the Hamiltonian vector field X_K of K.

Lemma 2.3.11. *If* M *has a linear Hamiltonian circle action* σ *of nonzero slope* κ*, then the positive Reeb periods form a discrete subset of* $(0, \infty)$*.*

Proof. Without loss of generality, suppose $\kappa > 0$ by using the action σ_t^{-1} if necessary. By (2.3.5), the action σ restricts to an S^1-action on Σ with vector field κX_Σ, where X_Σ is the Reeb vector field on (Σ, α). For any $y \in \Sigma$, the stablizer subgroup corresponding to $S^1 \cdot y$ is $\mathbb{Z}/l_y$ for some $l_y \in \mathbb{Z}^{>0}$. By [Aud04, Proposition 1.2.4], the set $\{l_y : y \in \Sigma\}$ is finite. Thus, the positive Reeb periods form a subset of

$$\frac{\kappa}{(\max l_y)!} \mathbb{Z}^{>0},$$

which is a discrete set as required. □

Remark 2.3.12. A contact manifold is *Besse* if every point is on a closed Reeb orbit (see [CGM20]). The above proof shows that is Besse for any convex symplectic manifold which admits a linear Hamiltonian circle action of nonzero slope.

A Hamiltonian circle action σ induces a diffeomorphism on the free loop space of M which is given by

$$(\sigma(x))(t) = \sigma_t \circ x(t) \tag{2.3.21}$$

for any loop $x : S^1 \to M$.

Lemma 2.3.13. *Let σ be a linear Hamiltonian circle action on a convex symplectic manifold M. The action σ has a fixed point, and hence the map* (2.3.21) *takes contractible loops to contractible loops.*

Proof. Let K have slope κ. If $\kappa \geq 0$, then the function $K : M \to \mathbb{R}$ has a minumum, so that it has at least one critical point. If $\kappa < 0$, then K has a maximum and hence a critical point. Critical points of K are fixed points of σ, so σ has a fixed point.

Let $m_0 \in M$ be a fixed point of σ. Let $x : S^1 \to M$ be any contractible loop. Let $u : D \to M$ be a smooth filling of x such that $u(z) = m_0$ for all $|z| < \frac{1}{2}$. Such a filling exists because M is connected. Define $\sigma \circ u : D \to M$ by

$$(\sigma \circ u)(re^{2\pi i t}) = \sigma_t \circ u(re^{2\pi i t}) \tag{2.3.22}$$

for all $t \in S^1$ and $r \in [0, 1]$. Since $u(z)$ is constantly a fixed point of σ for z in a neighbourhood of $0 \in D$, the map $\sigma \circ u$ is well-defined. The map $\sigma \circ u$ is a filling of $\sigma \circ x$. □

Remark 2.3.14. For convex manifolds, the linearity hypothesis is vital. The Hamiltonian action on T^*S^1 induced by rotation of S^1 is not linear, and the induced action on loops does not preserve contractibility. Seidel proved a more general result for closed manifolds which applies to loops in $\mathrm{Ham}(M)$ based at Id_M [Sei97, Lemma 2.2]. His proof used the Arnold conjecture for closed manifolds. For convex manifolds, Ritter observed (in a Technical Remark [Rit16, page 14]) that if the map (2.3.21) did not preserve contractibility, then symplectic cohomology vanished as did the quantum Seidel map of Section 2.6.2.2, which renders this case uninteresting from the point of view of the Seidel map, so it was discarded.

Lemma 2.3.13 means (2.3.21) restricts to a map $\sigma : \mathcal{L}M \to \mathcal{L}M$. This map may be lifted to the cover $\widetilde{\mathcal{L}M}$ by the argument of [Sei97, Lemma 2.4]. Denote the choice of a lift of σ by $\widetilde{\sigma}$.

Definition 2.3.15 (Maslov index of $\tilde{\sigma}$). Given a lift $\tilde{\sigma}$, and a point $(x, u) \in \widetilde{LM}$, let $\tilde{\sigma}(x, u) = (\sigma x, v)$. Let $\tau_x : (x^* TM, x^* \omega) \to (\mathbb{R}^{2n}, \Omega)$ be the restriction of a trivialisation of (TM, ω) on u and let $\tau_{\sigma x}$ be the restriction of a trivialisation on v. Here, Ω is the standard symplectic bilinear form on $\mathbb{R}^{2n}$. Define the loop of symplectic matrices $l(t)$ by

$$l(t) = \tau_{\sigma x}(t) D\sigma_t(x(t)) \tau_x(t)^{-1}. \tag{2.3.23}$$

The Maslov index $I(\tilde{\sigma})$ associated to this loop does not depend on the choice of the point (x, u) or on the choice of trivialisations, but it does depend on the choice of lift $\tilde{\sigma}$ of σ.

2.3.4 Floer Seidel map

Let $\tilde{\sigma}$ be a lift of a linear Hamiltonian circle action of slope κ. In [Sei97], Seidel defined a natural automorphism on Floer cohomology associated to $\tilde{\sigma}$ for closed symplectic manifolds, which was extended to convex symplectic manifolds in [Rit14].

Let $(H, \mathbf{J})$ be a regular choice of Floer data. The *pullback*[19] $\sigma^* \mathbf{J}$ of $\mathbf{J}$ by σ is

$$(\sigma^* \mathbf{J})_t = (D\sigma_t)^{-1} \mathbf{J}_t D\sigma_t \tag{2.3.24}$$

and the *pullback* $\sigma^* H$ of H by σ is given by

$$(\sigma^* H)_t(m) = H_t(\sigma_t(m)) - K(\sigma_t(m)) \tag{2.3.25}$$

for all $m \in M$. By the elementary calculations in [Pol01, Section 1.4], the Hamiltonian flows satisfy $\varphi^t_{\sigma^* H} = \sigma_t^{-1} \varphi^t_H$. The *pullback Floer data* $(\sigma^* H, \sigma^* \mathbf{J})$ is a regular choice of Floer data, with $\sigma^* H$ of slope $\lambda - \kappa$ if H is of slope λ.

The map (2.3.21) induces isomorphisms between the moduli spaces of Floer solutions of the pullback Floer data $(\sigma^* H, \sigma^* \mathbf{J})$ and of $(H, \mathbf{J})$. This isomorphism is orientation-preserving with respect to the orientations induced by o.

These pullback constructions yield a degree-$2I(\tilde{\sigma})$ Λ-module isomorphism $FS_{\tilde{\sigma}}$ on Floer cohomology. It is induced by the chain map

$$FS_{\tilde{\sigma}} : FC^*(M; \mathbf{J}, H) \to FC^{*+2I(\tilde{\sigma})}(M; \sigma^* \mathbf{J}, \sigma^* H) \tag{2.3.26}$$

is given by the Λ-linear extension of $\tilde{x} \mapsto \tilde{\sigma}^* \tilde{x}$, where $\tilde{\sigma}^* \tilde{x}$ is the preimage of $\tilde{x}$ under the map $\tilde{\sigma}$. By pulling back regular monotone homotopies, it is possible to establish that $FS_{\tilde{\sigma}}$ commutes with continuation maps, so that

$$\begin{array}{ccc} FH^*(M; \mathbf{J}^+, H^+) & \xrightarrow{FS_{\tilde{\sigma}}} & FH^{*+2I(\tilde{\sigma})}(M; \sigma^* \mathbf{J}^+, \sigma^* H^+) \\ \downarrow \varphi & & \downarrow \tilde{\varphi} \\ FH^*(M; \mathbf{J}^-, H^-) & \xrightarrow[FS_{\tilde{\sigma}}]{} & FH^{*+2I(\tilde{\sigma})}(M; \sigma^* \mathbf{J}^-, \sigma^* H^-) \end{array} \tag{2.3.27}$$

[19] This definition conforms to the conventions for pullbacks of tensor fields.

commutes. Hence $FS_{\widetilde{\sigma}}$ induces a degree-$2I(\widetilde{\sigma})$ automorphism of symplectic cohomology.

Seidel showed that $FS_{\widetilde{\sigma}}$ respects the module structure induced by the pair-of-pants product for closed manifolds [Sei97, Proposition 6.3], and this was extended to convex manifolds by Ritter [Rit14, Theorem 23]. The pair-of-pants product does not extend to the equivariant setup, so there is no analogue for this result.

2.4 Equivariant Floer theory

In this section, we define equivariant Floer cohomology. It should be considered in analogy to the Borel homotopy-quotient model for the equivariant cohomology of a topological space (see Section 2.4.2). We outline in Section 2.4.1 our conventions for the universal bundle of S^1, and proceed to explain our definition in Section 2.4.3.

In the literature, the S^1-action used in the definition of S^1-equivariant Floer cohomology is the action which rotates the domains of loops. In this paper, we incorporate an additional S^1-action on M into the definition. Using the trivial action on M in our definition recovers the usual definition, except we use a non-standard and more geometric construction for the $\mathbb{Z}[\mathbf{u}]$-module action (see Section 2.4.4.2). For completeness, we give the standard construction of the $\mathbb{Z}[\mathbf{u}]$-module action in Section 2.4.4.1, though we do not use it in this paper.

Our definition is based on the standard definition of the Borel homotopy-quotient rather than the variant which is common in the equivariant symplectic cohomology literature (see Remark 2.4.3).

Aside from these three key differences, our construction strongly resembles those already in the literature. In the following remark, we compare our conventions to those in other papers, however all these remaining differences are cosmetic.

Remark 2.4.1 (Conventions). Our definition is close to those of [BO17, Sections 2.2–2.3] and [Gut18, Section 2.3], except we use cohomological conventions and a Novikov ring. Our conventions are almost identical to Zhao's *periodic symplectic cohomology* in [Zha19; Zha16] and Seidel's definition in [Sei18], except that we use a direct sum convention for cohomology rather than a direct product (and we use a Novikov ring). Importantly, we do not use $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]/(\mathbf{u}\mathbb{Z}[\mathbf{u}])$ in our coefficient ring unlike [Sei08, Section 8b] and [MR18, Appendix B]; indeed this is not possible with our module operation.

2.4.1 Infinite sphere

The group S^1 acts freely on the odd-dimensional sphere S^{2k-1}, which we consider as the subset of $\mathbb{C}^k$ of norm 1, by multiplication

$$\theta \cdot w = e^{2\pi i \theta} w. \tag{2.4.1}$$

With this action, the manifold S^{2k-1} is a principal S^1-bundle over $\mathbb{CP}^{k-1}$. These spheres are equipped with canonical inclusion maps $i_k : S^{2k-1} \hookrightarrow S^{2k+1}$. Denote by S^∞ the direct limit of the odd-dimensional spheres under these inclusion maps.

Remark 2.4.2. While the topological space S^∞ is an CW complex, it is not compact or finite-dimensional, and it is not first countable[20] so it is not metrisable. As such, we do not want to directly work with S^∞ in our moduli spaces. Instead, we use the notation of S^∞ to simplify otherwise cumbersome statements which involve the limit of spheres. For example, by a *smooth* map $f : S^\infty \to \mathbb{R}$, we mean a sequence of smooth maps $f_k : S^{2k-1} \to \mathbb{R}$, which are compatible with the inclusions in the sense that $f_{k+1} \circ i_k = f_k$ and $\lim f_k = f$. In this way, the space S^∞ is a principal S^1-bundle, with projection map $\pi : S^\infty \to \mathbb{CP}^\infty$.

Define the smooth function $F : S^\infty \to \mathbb{R}$ by $(w_0, \ldots) \mapsto \sum_k k|w_k|^2$. The function F descends to a Morse-Smale function on $\mathbb{CP}^\infty$ whose unique critical point of index $2k$ is the standard basis vector $c_k = [0 : \cdots : 0 : 1 : 0 : \cdots]$ for all $k \geq 0$. Recall that the cohomology of $\mathbb{CP}^\infty$ is isomorphic to $\mathbb{Z}[\mathbf{u}]$, with $\mathbf{u}$ a formal variable of degree 2, where the critical point c_k corresponds to $\mathbf{u}^k$.

The space S^∞ is equipped with the round metric. For each k, fix an identification $S^1 \leftrightarrow c_k \subseteq S^\infty$. Extend this identification to the unstable and stable manifolds to get an equivariant map

$$\tau_k : W^u(c_k) \cup W^s(c_k) \to S^1, \tag{2.4.2}$$

defined by where the negative gradient flowline of F converges to along c_k.

The right-shift map $\mathbb{C}^k \to \mathbb{C}^{k+1}$ given by $(w_0, \ldots, w_{k-1}) \mapsto (0, w_0, \ldots, w_{k-1})$ induces an injective smooth map $U : S^\infty \to S^\infty$. The gradient vector field of F is U-invariant.

[20] Recall a topological space X is *first countable* if, for every point $x \in X$, there is a sequence $(U_k)_{k \in \mathbb{N}}$ of open subsets of X which contain x such that, for every open subset $U \subseteq X$ with $x \in U$, there is an inclusion $U_k \subseteq U$ for some $k \in \mathbb{N}$. All metric spaces are first countable.

To show S^∞ is not first countable, we show $\mathbb{R}^\infty = S^\infty \setminus \{pt\}$ is not first countable. Let $U_k \subseteq \mathbb{R}^\infty$ be any sequence of open subsets containing $0 \in \mathbb{R}^\infty$. Choose real numbers $\varepsilon_i^{(k)} \in (0, \infty)$ which satisfy $\left(-\varepsilon_1^{(k)}, \varepsilon_1^{(k)}\right) \times \cdots \times \left(-\varepsilon_k^{(k)}, \varepsilon_k^{(k)}\right) \subseteq U_k \cap \mathbb{R}^k$. The subset $\left(-\frac{1}{2}\varepsilon_1^{(1)}, \frac{1}{2}\varepsilon_1^{(1)}\right) \times \left(-\frac{1}{2}\varepsilon_2^{(2)}, \frac{1}{2}\varepsilon_2^{(2)}\right) \times \cdots \subseteq \mathbb{R}^\infty$ does not contain any of the open sets U_k.

2.4.2 Equivariant cohomology

Let X be a topological space with a continuous circle action $\rho : S^1 \times X \to X$. The *Borel homotopy quotient*, denoted $S^\infty \times_{S^1} X$, is the quotient of the product $S^\infty \times X$ by the relation $(\theta \cdot w, x) \sim (w, \theta \cdot x)$. Equivalently, it is the quotient of the product $S^\infty \times X$ by the free circle action

$$\theta \cdot (w, x) = (\theta^{-1} \cdot w, \theta \cdot x). \tag{2.4.3}$$

The *equivariant cohomology* of the pair (X, ρ) is $EH^*(X) = H^*(S^\infty \times_{S^1} X)$. The projection $S^\infty \times_{S^1} X \to S^\infty / S^1 = \mathbb{CP}^\infty$ induces a map $\mathbb{Z}[\mathbf{u}] = H^*(\mathbb{CP}^\infty) \to EH^*(X)$. Together with the cup product, this map gives equivariant cohomology the structure of a unital, associative and graded-commutative $\mathbb{Z}[\mathbf{u}]$-algebra.

Remark 2.4.3 (Diagonal action in literature). The literature for equivariant symplectic cohomology uses an alternative convention for the Borel homotopy quotient whereby the free diagonal action on $S^\infty \times X$ is used. The automorphism of S^∞ given by complex conjugation takes this diagonal action back to the standard action (2.4.3), however it is *not orientation-preserving* on the quotient $\mathbb{CP}^\infty$, so this different convention results in a different $\mathbb{Z}[\mathbf{u}]$-module structure. To correct for this, the transformation $\mathbf{u} \mapsto -\mathbf{u}$ must be used when changing convention.

Let (X, ρ_X) and (Y, ρ_Y) be two topological spaces with circle actions as above. The continuous map $f : X \to Y$ is *equivariant* if it intertwines the two actions, that is the identity $\rho_{Y,\theta}(f(x)) = f(\rho_{X,\theta}(x))$ holds. Such an equivariant map induces a well-defined map on the Borel homotopy quotients, and hence induces a natural *pullback map* $f^* : EH^*_{\rho_Y}(Y) \to EH^*_{\rho_X}(X)$.

2.4.3 Equivariant Floer cohomology

2.4.3.1 Equivariant Hamiltonian orbits

Let ρ be a symplectic circle action on M, and assume that the action flows along the Hamiltonian vector field of some linear Hamiltonian K at infinity. Such actions are *linear at infinity*. Notice that ρ preserves the radial coordinate R at infinity. The linear Hamiltonian circle actions of Section 2.3.3 satisfy this assumption by definition; the difference is that we do not assume the action is Hamiltonian on the entire manifold here.

The loop space LM, being a space of maps $S^1 \to M$, naturally inherits[21] two circle actions, one from the rotation action on the domain S^1 and the other from the circle action

[21] Let the group G act on sets X and Y on the left. Let $\mathrm{Map}(X, Y)$ denote the space of maps from X to Y. The action on Y induces an action on $\mathrm{Map}(X, Y)$ by post-composition. The action on $\mathrm{Map}(X, Y)$ given by $g \cdot f = f \circ g$ is a right action, instead of a left action. The action given by $g \cdot f = f \circ g^{-1}$ is still a left action, so this is the action *naturally inherited* by $\mathrm{Map}(X, Y)$. Here, $g \in G$ and $f \in \mathrm{Map}(X, Y)$.

on the codomain M. We combine these to get the circle action on LM which is given, for all $t, \theta \in S^1$ and all $x \in LM$, by

$$\theta \cdot (t \mapsto x(t)) = (t \mapsto \rho_\theta(x(t - \theta))). \tag{2.4.4}$$

An *equivariant Hamiltonian* is a smooth function $H^{\text{eq}} : S^\infty \times (S^1 \times M) \to \mathbb{R}$ which satisfies

$$H^{\text{eq}}_{w,t}(m) = H^{\text{eq}}_{\theta^{-1} w, t+\theta}(\rho_\theta(m)) \tag{2.4.5}$$

for all $t, \theta \in S^1$, all $w \in S^\infty$ and all $m \in M$. Notice that a function satisfying (2.4.5) is equivariant, in the sense of Section 2.4.2, with respect to the natural action on the domain, according to our convention in (2.4.3), and the trivial action on the codomain $\mathbb{R}$.

The equivariant Hamiltonian H^{eq} is *linear of slope* λ if there is R_0 such that the equation $H^{\text{eq}}_{w,t}(\psi(y, R)) = \lambda R$ holds when $R \geq R_0$.

An *equivariant Hamiltonian orbit* is an equivalence class $[w, x] \in S^\infty \times_{S^1} LM$ such that w is a critical point of $F : S^\infty \to \mathbb{R}$ and $x(t)$ is a Hamiltonian orbit of $H^{\text{eq}}_{w,t}(\cdot)$. The equivariance of the Hamiltonian H^{eq} guarantees this definition is independent of the choice of representative (w, x).

The action (2.4.4) on LM lifts canonically to an action on $\widetilde{LM}$. The action functional $\mathcal{A}_{H^{\text{eq}}} : S^\infty \times \widetilde{LM} \to \mathbb{R}$ given by

$$\mathcal{A}_{H^{\text{eq}}}(w, (x, u)) = -\int_D u^* \omega + \int_{t=0}^{1} H^{\text{eq}}_{w,t}(x(t))\, \mathrm{d}t \tag{2.4.6}$$

is invariant under the action combining the lift of (2.4.4) and (2.4.3), much like the equivariant Morse functions of Section 2.6.3.1.

2.4.3.2 Equivariant Floer data

An *equivariant*[22] *almost-complex structure* $\mathbf{J}^{\text{eq}}$ is a $S^\infty \times S^1$-family of almost-complex structures $\mathbf{J}^{\text{eq}}_{w,t}$ which makes the diagram

$$\begin{array}{ccc} T_m M & \xrightarrow{\mathbf{J}^{\text{eq}}_{w,t}} & T_m M \\ \downarrow D\rho_\theta & & \downarrow D\rho_\theta \\ T_{\rho_\theta(m)} M & \xrightarrow{\mathbf{J}^{\text{eq}}_{\theta^{-1} w, t+\theta}} & T_{\rho_\theta(m)} M \end{array} \tag{2.4.7}$$

commute for all $m \in M$.

[22] An almost-complex structure is a section of the bundle $\text{Aut}(TM) \to M$. Let $p : S^\infty \times S^1 \times M \to M$ be the natural projection map. An equivariant almost-complex structure is a map $S^\infty \times S^1 \times M \to p^* \text{Aut}(TM)$ which is equivariant in the usual sense.

Definition 2.4.4. The equivariant Hamiltonian H^{eq} *extends the sequence of (non-equivariant) Hamiltonians* H^k if

$$H^{\mathrm{eq}}_{w;t}(m) = H^k_{t+\tau_k(w)}(\rho_{-\tau_k(w)}(m)) \tag{2.4.8}$$

in a neighbourhood of c_k in $W^u(c_k) \cup W^s(c_k)$ for all $k \geq 0$. Likewise, the equivariant almost complex structure $\mathbf{J}^{\mathrm{eq}}$ *extends the sequence of almost complex structures* $\mathbf{J}^k$ if

$$\mathbf{J}^{\mathrm{eq}}_{t;w} = D\rho^{-1}_{-\tau_k(w)} \circ \mathbf{J}^k_{t+\tau_k(w)} \circ D\rho_{-\tau_k(w)} \tag{2.4.9}$$

in a neighbourhood of c_k in $W^u(c_k) \cup W^s(c_k)$ for all $k \geq 0$.

This condition constrains the variation of the data near the critical points c_k and thus allows us to appeal to continuation map methods later in Section 2.4.3.3. For such equivariant data, the equivariant Hamiltonian orbits are equivalence classes $[w, x]$ where $w \in c_k$ satisfies $\tau_k(w) = 0$ and $x \in P(H^k)$. We use the shorthand (c_k, x) for such equivariant Hamiltonian orbits.

Definition 2.4.5. A choice of *equivariant Floer data* is a pair $(H^{\mathrm{eq}}, \mathbf{J}^{\mathrm{eq}})$ consisting of a linear equivariant Hamiltonian function H^{eq} and a convex equivariant ω-compatible almost complex structure $\mathbf{J}^{\mathrm{eq}}$ which together extend a sequence of Floer data.

Remark 2.4.6. Typically, one chooses the equivariant Floer data so that it is the same at the critical points, however we relax this requirement here, allowing any sequence of Floer data. We need (2.4.8) and (2.4.9) to hold for a sequence of non-equivariant data $(H^k, \mathbf{J}^k)_{k \geq 0}$ in order to apply the standard continuation map techniques that guarantee the desired moduli space behaviour.

Proposition 2.4.7 (Existence of data). Let $(H^k, \mathbf{J}^k)$ be any sequence of Floer data whose Hamiltonians are all linear of the same slope λ and whose *at infinity conditions*[23] are all satisfied in a common region at infinity. There is equivariant Floer data which extends this sequence, and moreover the space of such data is contractible.

Proof. We can construct an equivariant Hamiltonian by an analogous argument to [BO17, Example 2.4]. First, we fix an invariant time-independent Hamiltonian H_∞ on M which is linear at infinity of slope λ. To do this, simply set $H_\infty = 0$ on $\{R < R_0\}$ and $H_\infty(\psi(y, R)) = h(R)$ on $\{R \geq R_0\}$ for an appropriate function h. Next, use a cut-off function near each c_k to interpolate between each H^k (appropriately interpreted via (2.4.8)) and the fixed H_∞. The result is an equivariant Hamiltonian of slope λ which extends the sequence as desired. This shows the desired existence. The space of all such equivariant Hamiltonians is convex and hence contractible.

[23] That is, the linearity of the Hamiltonians and the convexity of the almost complex structures.

For the equivariant almost complex structure, consider the symplectic vector bundle

$$
\begin{array}{c}
E = S^{\infty} \times_{S^1} TM/S^1 \\
\downarrow \\
B = S^{\infty} \times_{S^1} M/S^1.
\end{array}
\tag{2.4.10}
$$

The space of (compatible) almost complex structures on E is nonempty and contractible [MS98, Proposition 2.63]. The proof of this constructs a retraction r from the space of inner products on E (which is convex and hence contractible) to the space of compatible almost complex structures on E. By restricting this map r to an appropriate subspace of inner products, we will get a retraction to the space of convex almost complex structures which extend the given sequence $\mathbf{J}^k$. It is sufficient for the subspace to be nonempty and contractible to complete the proof.

To guarantee convexity, we restrict to inner products g which satisfy

$$
\begin{cases}
g(\partial_R, R\partial_R) = 1 \\
g(\partial_R, X) = 0 \\
g(X, X) = R \\
\ker(\alpha) \text{ is } g\text{-orthogonal to } \partial_R \text{ and } X
\end{cases}
\tag{2.4.11}
$$

at infinity. To ensure the almost complex structures will extend $\mathbf{J}^k$, we further restrict to inner products which take appropriate fixed values over neighbourhoods of $c_k \times_{S^1} M/S^1$ for each k. The space of inner products which satisfy these two conditions remains convex and nonempty, as desired. □

2.4.3.3 Equivariant Floer solutions

The map $(v, u) : \mathbb{R} \to S^{\infty} \times LM$ *satisfies the equivariant Floer equation* if v is a negative gradient flowline of $F : S^{\infty} \to \mathbb{R}$ and u satisfies the equation

$$
\partial_s u + \mathbf{J}^{\mathrm{eq}}_{v(s);t}\left(\partial_t u - X_{H_{v(s);t}(\,)}\right) = 0. \tag{2.4.12}
$$

An *equivariant Floer solution* is an equivalence class $[v, u] \in C^{\infty}(\mathbb{R}; S^{\infty} \times LM)/S^1$. The equivariant Floer solution $[v, u]$ *converges to equivariant Hamiltonian orbits* (c_k, x) if the limits $\lim_{s \to \infty} v(s) = w$ and (2.3.14) hold for some choices of representatives $[w, x]$ of the critical points.

For regular equivariant Floer data, the moduli space of equivariant Floer solutions $M((c_k, \widetilde{x}), (c_{k^+}, \widetilde{x}^+))$ is a smooth oriented manifold of dimension $2k - 2k^+ + \mu(\widetilde{x}) - \mu(\widetilde{x})$. The moduli space is empty unless $2k - 2k^+ \geq 0$. It admits a smooth $\mathbb{R}$-action via s-translation that is free if the equivariant Hamiltonian orbits are distinct. (The case when $k = k^+$ is canonically identical to the non-equivariant case using the maps τ_{k^+}.)

The dimension-0 moduli spaces are compact and the dimension-1 moduli spaces admit a compactification via gluing broken trajectories as per Section 2.3.2.5. The phenomenon of bubbling is avoided using regularity conditions.[24]

Remark 2.4.8 (Weak$^+$ monotonicity). A symplectic manifold satisfies *weak$^+$ monotonicity* if the implication

$$2 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0 \tag{2.4.13}$$

holds for all $A \in \pi_2(M)$. Weak$^+$ monotonicity is insufficient to exclude bubbling by standard arguments for the equivariant definitions, even though it is sufficient for the non-equivariant definitions. For regular time-dependent almost complex structures $\mathbf{J}$, the moduli space of $\mathbf{J}$-holomorphic spheres of class A has dimension $2n + 2c_1(A) + 1$. Therefore the moduli space is empty when $c_1(A) \ll 0$ is large and negative, because the dimension is negative. In the equivariant setup, however, the moduli space of $\mathbf{J}^{\mathrm{eq}}$-holomorphic spheres has 'dimension' $2n + 2c_1(A) + 1 + \dim(S^\infty/S^1)$. Since this 'dimension' is positive for any $c_1(A)$, the same argument does not apply. Instead, we require nonnegative monotonicity which prohibits any $\mathbf{J}^{\mathrm{eq}}$-holomorphic spheres with negative first Chern class via (2.3.1).

2.4.3.4 Equivariant Floer cochain complex

Let $(H^{\mathrm{eq}}, \mathbf{J}^{\mathrm{eq}})$ be a regular choice of equivariant Floer data. The *degree-l equivariant Floer cochains* are the formal sums

$$\sum_{k \geq 0} \sum_{\substack{\tilde{x} \in \widetilde{P}(H^k) \\ \mu(\tilde{x}) = l - 2k}} \alpha_{k;\tilde{x}} (c_k, \tilde{x}) \tag{2.4.14}$$

with integer coefficients such that the set

$$\left\{ \tilde{x} \in \widetilde{P}(H^k) \ : \ \omega(\tilde{x}) \leq c, \ \alpha_{k;\tilde{x}} \neq 0 \right\}$$

is finite for all $c \in \mathbb{R}$ and $k \geq 0$. Denote by $EFC^l(M; H^{\mathrm{eq}})$ the $\mathbb{Z}$-module of such degree-l equivariant Floer cochains. The *equivariant Floer cochain complex* is the $\mathbb{Z}$-graded -module $\bigoplus_{k \in \mathbb{Z}} EFC^k(M; H^{\mathrm{eq}})$.

The *equivariant Floer cochain differential* is the -module degree-1 endomorphism $d : EFC^*(M; H^{\mathrm{eq}}) \to EFC^{*+1}(M; H^{\mathrm{eq}})$ given by

$$d(c_{k^+}, \tilde{x}^+) = \sum_{\substack{k^- \geq k^+ \\ \tilde{x}^- \in \widetilde{P}(H^{k^-}) \\ 2k^- - 2k^+ + \mu(\tilde{x}^-) - \mu(\tilde{x}^+) = 1}} \sum_{[v;u] \in \widetilde{M}((c_{k^-};\tilde{x}^-))} \mathrm{o}([v,u]) \, (c_{k^-}, \tilde{x}^-). \tag{2.4.15}$$

[24] Explicitly, the regularity conditions ensure $((w;t);x(t)) \notin V_1(\mathbf{J}^{\mathrm{eq}})$ for all equivariant Hamiltonian orbits $(w;x)$ and $((v(s);t);u(s;t)) \notin V_0(\mathbf{J}^{\mathrm{eq}})$ for all equivariant Floer solutions $(v;u)$ occuring in moduli spaces of dimension 1. These conditions are the equivariant analogues of those used by [HS95].

The differential indeed satisfies $d^2 = 0$, and the *equivariant Floer cohomology*, denoted $EFH^*(M; H^{\text{eq}})$, is the cohomology of $(EFC^*(M; H^{\text{eq}}), d)$. It is a $\mathbb{Z}$-graded Λ-module.

Standard homotopy techniques ensure $EFH^*(M; H^{\text{eq}})$ is dependent only on the slope of the Hamiltonian. Moreover, equivariant monotone homotopies induce equivariant continuation maps. The *equivariant symplectic cohomology* $ESH^*(M)$ is the direct limit of the resulting system, just as in the non-equivariant case.

2.4.4 Module structures

2.4.4.1 Algebraic module structure

This section describes the $\mathbb{Z}[\mathbf{u}]$-module action used in the literature.

Recall the right-shift operator on S^∞ denoted U. Assume that the maps τ_k are U-compatible and that the equivariant Hamiltonian H^{eq} and the equivariant almost-complex structure $\mathbf{J}^{\text{eq}}$ are U-invariant. These assumptions yield isomorphisms

$$M((c_{k^-}, \widetilde{x}^-), (c_{k^+}, \widetilde{x}^+)) = M((c_{k^-+r}, \widetilde{x}^-), (c_{k^++r}, \widetilde{x}^+))$$

for all $r \geq 0$. With respect to the $\mathbb{Z}[\mathbf{u}]$-module structure on $EFC^*(M; H^{\text{eq}})$ given by $\mathbf{u} \cdot (c_k, \widetilde{x}) = (c_{k+1}, \widetilde{x})$, the differential d is a $\Lambda[\mathbf{u}]$-module endomorphism. Hence under this assumption, the equivariant Floer cohomology is a $\Lambda[\mathbf{u}]$-module.

2.4.4.2 Geometric module structure

The Morse cup product counts 'Y'-shaped flowlines. We adapt this product to get a $\mathbb{Z}[\mathbf{u}]$-module structure on equivariant Floer cohomology which reflects the geometric behaviour of equivariant flowlines in S^∞. The additional conditions placed on data for this construction are generic, in contrast to the invariance assumptions of Section 2.4.4.1 which are not generic.

Take an S^1-invariant s-dependent perturbation[25] of the function $F : S^\infty \to \mathbb{R}$. This is a smooth function $[0, \infty) \to C^\infty(S^\infty)$, $s \mapsto F_s$.

Given two equivariant Hamiltonian orbits $(c_{k^-}, \widetilde{x}^-)$ and $(c_{k^+}, \widetilde{x}^+)$, consider the moduli space of S^1-equivalence classes of triples (v, u, v'), where $[v, u]$ is an equivariant Floer solution converging to the two equivariant Hamiltonian orbits and $v' : [0, \infty) \to S^\infty$ is a negative gradient flowline of F_s which converges to a point on c_k and satisfies $v'(0) = v(0)$. We have drawn this configuration in Figure 2.1.

[25] Throughout, any s-dependent perturbation is s-dependent only on a bounded interval.

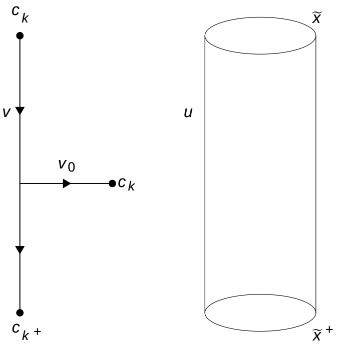


Figure 2.1: The geometric module action uses a Morse 'Y'-shaped flowline on S^1.

For a regular[26] choice of the perturbed Morse function F_S, this moduli space is an oriented smooth manifold of dimension

$$2k \quad 2k^+ \quad 2k + \mu(\widetilde{x}\ ) \quad \mu(\widetilde{x}\ ). \tag{2.4.17}$$

The 0-dimensional moduli spaces are compact and the 1-dimensional moduli spaces admit a compactification by broken solutions. The map

$$\mathbf{u}^k : EFC\ (M; H^{\mathrm{eq}}) \ ! \ EFC\ ^{+2k}(M; H^{\mathrm{eq}}) \tag{2.4.18}$$

which counts the 0-dimensional moduli spaces commutes with the equivariant differential. Moreover, a standard argument yields a chain homotopy between $\mathbf{u}^k \quad \mathbf{u}^{k^0}$ and $\mathbf{u}^{k+k^0}$. This gives equivariant Floer cohomology the structure of a Z[$\mathbf{u}$]-module. To distinguish this module structure from the one in Section 2.4.4.1, we denote this new action by $\mathbf{u} \smile$.

Remark 2.4.9 (Comparison of Z[$\mathbf{u}$]-module structures). Suppose the equivariant Floer data satisfies the conditions for the module structures in both Section 2.4.4.1 and this section. Let us compare the algebraic product $\mathbf{u}^k \quad (c_l, \widetilde{x}) = (c_{l+k}, \widetilde{x})$ with the geometric product $\mathbf{u}^k \smile (c_l, \widetilde{x})$. Suppose $[(v, u, v^0)]$ is a solution to the geometric product $\mathbf{u}^k \smile (c_l, \widetilde{x})$ with end point $(c_k\ , \widetilde{x}\ )$. The 'Y'-shaped flowline $[v, v^0]$ is a solution to the cup product on S^1/S^1, though it may not be isolated. Therefore we have $k \qquad k + l$ since there are no solutions to the cup product otherwise. Moreover for $k \quad = k + l$, the 'Y'-shaped flowline

[26] In order to avoid bubbling, we moreover assume that the map

$$M(A; \mathbf{J}^{\mathrm{eq}})_F \quad S^1) = S^1 \quad \mathrm{PSL}\ \mathrm{P}^1 \quad M\ ((c_k\ ; \widetilde{x}\ ); (c_{k^+}; \widetilde{x}^+))\ !\ (F \quad S^1\ M) \quad (F \quad S^1\ M) \tag{2.4.16}$$

given by $([(w; t; u_{\mathrm{P}^1}); p]; (v; u))$ 7! $(w; u_{\mathrm{P}^1}(p); v(0); u(0; t))$ is transversal to the diagonal for all A 2 . Here, F denotes the intersection $W^u(\mathbf{u}^k\ ) \setminus W^s(\mathbf{u}^{k^+}) \quad S^1$. The domain of the map has dimension $2k \quad 2k^+ + 2n + 2c_1(A) \quad 3 + (2k \quad + \quad (\widetilde{x}\ ) \quad 2k^+ \quad (\widetilde{x}^+))$. When the moduli space of equivariant Floer solutions has dimension 1 or 2 and $c_1(A) = 0$, the map (2.4.16) is transversal to the diagonal only if the intersection of the image with the diagonal is empty. This recovers one of the conditions for regular equivariant data. Data will generically satisfy this condition by an argument analogous to the proof of [HS95, Theorem 3.2].

is isolated, so u is a continuation map between $\tilde{x}$ and $\tilde{x}$ (and this continuation map is an isomorphism because the Hamiltonian has not changed). As such, we can informally say that the two products agree on the c_{k+l} term. That said, the geometric product may have other terms on the $c_{k'}$ terms with $k' > k + l$ unlike the algebraic product. The author has not determined whether the two products are chain homotopic, however anticipates that the moduli spaces in [Sei18, Sections 3 and 5] can be extended to get a chain homotopy.

2.5 Equivariant Floer Seidel map

In this section, we extend the definition of the Floer Seidel map of Section 2.3.4 to the equivariant setup introduced in Section 2.4.

Let $\tilde{\sigma}$ be a lift of a linear Hamiltonian circle action and let ρ be a symplectic circle action which is linear at infinity, as per Section 2.4.3.1. Assume that the σ and ρ commute.

2.5.1 Equivariant Floer Seidel map definition

Let $(H^{\text{eq}}, \mathbf{J}^{\text{eq}})$ be a regular choice of equivariant Floer data for the action ρ. The *pull-back equivariant Floer data* $(\sigma^* H^{\text{eq}}, \sigma^* \mathbf{J}^{\text{eq}})$ are given by the same formulae as in the non-equivariant case (equations (2.3.24) and (2.3.25)), so we have

$$(\sigma^* \mathbf{J}^{\text{eq}})_{w,t} = (D\sigma_t)^{-1} \mathbf{J}^{\text{eq}}_{w,t} D\sigma_t \tag{2.5.1}$$

and

$$(\sigma^* H^{\text{eq}})_{w,t}(m) = H^{\text{eq}}_{w,t}(\sigma_t(m)) - K_\sigma(\sigma_t(m)) \tag{2.5.2}$$

for all $w \in S^1$ and $m \in M$.

The pullback equivariant Floer data are regular equivariant Floer data for the pullback action $\sigma^* \rho$. We show this for the pullback Hamiltonian as follows.

Proof of equivariance of (2.5.2). Since σ and ρ commute, the Lie bracket of their vector fields vanishes. The function $\omega(X_\sigma, X_\rho) : M \to \mathbb{R}$ has Hamiltonian vector field equal to this bracket $[X_\sigma, X_\rho]$ [Sil01, Proposition 18.3], and is therefore constant. The value of the constant is 0 because σ has a fixed point by Lemma 2.3.11. This yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(K_\sigma(\rho_\theta(m))) = (\mathrm{d}K_\sigma)_{\rho_\theta(m)}((X_\rho)_{\rho_\theta(m)}) = \omega(X_\sigma, X_\rho)|_{\rho_\theta(m)} = 0, \tag{2.5.3}$$

from which we deduce that K_σ is constant along ρ. We use this in line (2.5.4b) to deduce the desired equivariance condition:

$$
\begin{aligned}
(\sigma\ H^{\text{eq}})\ {}_{1\ w;t+}\ \big((\sigma\ \rho)\ (m)\big) & \\
= H^{\text{eq}}{}_{1\ w;t+}\ \big(\sigma_{t+}\ ((\sigma\ \rho)\ (m))\big)\quad K\ \big(\sigma_{t+}\ ((\sigma\ \rho)\ (m))\big) & \qquad (2.5.4a)\\
= H^{\text{eq}}{}_{1\ w;t+}\ \big(\rho\ (\sigma_t(m))\big)\quad K\ \big(\rho\ (\sigma_t(m))\big) & \qquad (2.5.4b)\\
= H^{\text{eq}}_{w;t}(\sigma_t(m))\quad K\ \big(\sigma_t(m)\big) & \qquad (2.5.4c)\\
= (\sigma\ H^{\text{eq}})_{w;t}(m). & \qquad (2.5.4d)
\end{aligned}
$$

□

If H^{eq} has slope λ and the Hamiltonian K of σ has slope κ, then the pullback $\sigma\ H^{\text{eq}}$ has slope $\lambda\ \ \kappa$.

Recall that, by definition, $\widetilde{\sigma}$ is a lift of the automorphism $\sigma : LM \to LM$, given by $(\sigma(x))(t) = \sigma_t\ \ x(t)$, to an automorphism of $\widetilde{LM}$. We have a commutative diagram of automorphisms on $S^1\ \ LM$

$$
\begin{array}{ccc}
(w, t \mapsto x(t)) & \xrightarrow{\ \text{Id}_{S^1}\ \ \sim\ } & (w, t \mapsto \sigma_t(x(t))) \\
\big\downarrow \text{for} & & \big\downarrow \text{for} \\
(\theta\ ^1\ w, t \mapsto (\sigma\ \rho)\ (x(t\ \ \theta))) & \xrightarrow{\ \text{Id}_{S^1}\ \ \sim\ } & (\theta\ ^1\ w, t \mapsto \rho\ (\sigma_t\ \ (x(t\ \ \theta))))
\end{array}
\qquad (2.5.5)
$$

which lifts to $S^1\ \ \widetilde{LM}$. Just like the non-equivariant case, the map $(\text{Id}_{S^1}\ \ \widetilde{\sigma})\ ^1$ takes equivariant Hamiltonian orbits of H^{eq} to equivariant orbits of $\sigma\ H^{\text{eq}}$, and induces similar isomorphisms on the moduli spaces of equivariant Floer solutions. As such, we get an isomorphism of cochain complexes

$$
\begin{aligned}
EFS_{\sim} : EFC\ (M; \mathbf{J}^{\text{eq}}, H^{\text{eq}}) &\to EFC^{\ +2I(\sim)}(M; \sigma\ \mathbf{J}^{\text{eq}}, \sigma\ H^{\text{eq}}) \\
(c_k, \widetilde{x}) &\mapsto (c_k, \widetilde{\sigma}\ \widetilde{x}).
\end{aligned}
\qquad (2.5.6)
$$

This *equivariant Floer Seidel map* preserves both the geometric and algebraic module structures on cohomology because it induces isomorphisms between the relevant moduli spaces.

Remark 2.5.1. The diagram (2.5.5) commutes if and only if σ is indeed an action, and fails to commute if σ is merely a based loop in Ham(M, ω). This diagram is the reason that our equivariant construction requires this stronger assumption, unlike the non-equivariant case.

As per the non-equivariant case, we can pullback regular equivariant monotone homotopies. Thus for any equivariant continuation map φ we get the following commutative diagram.

$$
\begin{array}{ccc}
EFC^*(M; \mathbf{J}^{\mathrm{eq};+}, H^{\mathrm{eq};+}) & \xrightarrow{EFS_{\tilde\sigma}} & EFC^{*+2I(\tilde\sigma)}(M; \sigma^* \mathbf{J}^{\mathrm{eq};+}, \sigma^* H^{\mathrm{eq};+}) \\
\varphi \big\downarrow & & \big\downarrow \varphi \\
EFC^*(M; \mathbf{J}^{\mathrm{eq};-}, H^{\mathrm{eq};-}) & \xrightarrow[EFS_{\tilde\sigma}]{} & EFC^{*+2I(\tilde\sigma)}(M; \sigma^* \mathbf{J}^{\mathrm{eq};-}, \sigma^* H^{\mathrm{eq};-})
\end{array}
\tag{2.5.7}
$$

When σ is linear of strictly positive slope κ, this offers a way to compute equivariant symplectic cohomology (this construction is the equivariant version of [Rit14, Theorem 22]). Let $\varepsilon > 0$ be smaller than any positive Reeb period. For each $r \in \mathbb{Z}_{\geq 0}$, choose equivariant Floer data $(H_r^{\mathrm{eq}}, \mathbf{J}_r^{\mathrm{eq}})$ of slope ε for the action $\sigma^{-r}\rho$. The pullback data $((\sigma^{-r})^* H_r^{\mathrm{eq}}, (\sigma^{-r})^* \mathbf{J}_r^{\mathrm{eq}})$ is equivariant for the action ρ and has slope $\varepsilon + r\kappa$. Consider the following commutative diagram, where φ_r are the continuation maps between the relevant Floer data. We have omitted M, degrees and the almost complex structure from the notation for clarity.

$$
\begin{array}{ccccccc}
EFH^*(H_0^{\mathrm{eq}}) & \xrightarrow{\varphi_0} & EFH^*((\sigma^{-1})^* H_1^{\mathrm{eq}}) & \xrightarrow{(\sigma^{-1})^*\varphi_1} & EFH^*((\sigma^{-2})^* H_2^{\mathrm{eq}}) & \xrightarrow{(\sigma^{-2})^*\varphi_2} & \cdots \\
 & & \cong \big\downarrow EFS_{\tilde\sigma} & & \cong \big\downarrow EFS_{\tilde\sigma} & & \\
 & \dashrightarrow & EFH^{*-I_1}(H_1^{\mathrm{eq}}) & \xrightarrow{\varphi_1} & EFH^{*-I_1}((\sigma^{-1})^* H_2^{\mathrm{eq}}) & \xrightarrow{(\sigma^{-1})^*\varphi_2} & \cdots \\
 & & & & \cong \big\downarrow EFS_{\tilde\sigma} & & \\
 & & & \dashrightarrow & EFH^{*-I_2}(H_2^{\mathrm{eq}}) & \xrightarrow{\varphi_2} & \cdots \\
 & & & & & \dashrightarrow & \cdots
\end{array}
\tag{2.5.8}
$$

The direct limit of the top line of (2.5.8) is, by definition, the equivariant symplectic cohomology of M for the action ρ. The diagram shows this is isomorphic to the direct limit of the dashed maps $EFS_{\tilde\sigma} \circ \varphi_r$. In Section 2.7.3, we find another way to characterise these dashed maps using equivariant quantum cohomology. This diagram is crucial for our calculations in Section 2.8.1 and Section 2.8.3.

2.5.2 Equivariant Floer Seidel map properties

The equivariant Floer Seidel map is a very natural operation, and as such it preserves many of the structures associated to equivariant Floer cohomology. It is a module map for both the geometric and algebraic module structures, and is compatible with continuation maps as described above. As such, after taking a direct limit, we get an isomorphism of equivariant symplectic cohomologies $EFS_{\tilde\sigma} : ESH^*(M) \to ESH^{*+2I(\tilde\sigma)}(M)$. In the remainder of this section, we describe two further structures on equivariant symplectic cohomology which are compatible with the equivariant Floer Seidel map.

2.5.2.1 Gysin sequence

Analogously to [BO17], there is a long exact sequence on equivariant symplectic cohomology

$$\to ESH^{*}(M) \xrightarrow{\mathbf{u}} ESH^{*+2}(M) \to SH^{*+2}(M) \to ESH^{*+1}(M) \to \cdots . \qquad (2.5.9)$$

This is an immediate algebraic consequence of our definitions because there is a short exact sequence of cochain complexes, where $\mathbf{u}$ denotes the algebraic $\mathbb{Z}[\mathbf{u}]$-module operation. The second map $ESH^{*}(M) \to SH^{*}(M)$ is the map induced by $(c_0, \widetilde{x}) \mapsto \widetilde{x}$ and $(c_k, \widetilde{x}) \mapsto 0$ for $k > 0$ on equivariant Floer cohomology. This long exact sequence is the *Gysin exact sequence* of equivariant symplectic cohomology.

The equivariant Floer Seidel maps are isomorphisms on the cochain complexes which are compatible with the maps in (2.5.9), and therefore fit into the following commutative diagram.

$$\begin{array}{ccccccc}
\longrightarrow ESH^{*}(M) & \xrightarrow{\mathbf{u}} & ESH^{*+2}(M) & \longrightarrow & SH^{*+2}(M) & \longrightarrow \\
\downarrow EFS_{\tilde{}} & & \downarrow EFS_{\tilde{}} & & \downarrow FS_{\tilde{}} & \\
\to ESH^{*+2I(\tilde{})}(M) & \xrightarrow{\mathbf{u}} & ESH^{*+2+2I(\tilde{})}(M) & \to & SH^{*+2+2I(\tilde{})}(M) & \to
\end{array} \qquad (2.5.10)$$

2.5.2.2 Filtration and positive equivariant symplectic cohomology

For exact symplectic manifolds, the Floer cochain complexes may be equipped with an action filtration which distinguishes between different orbits [Vit99]. Roughly, the orbits of a Hamiltonian with small[27] positive slope correspond, via the PSS map of Section 2.6.2.3, to the quantum cohomology of M. Conversely, orbits which occur only for Hamiltonians with larger slopes correspond to Reeb orbits on , with the period of the Reeb orbit corresponding to the slope of the Hamiltonian at the radius of the orbit, as per (2.3.5).

In our setup, however, the action functional (2.3.11) fails to provide a filtration for two reasons. First, since we have not assumed M is exact, the value of $A_H(x)$ depends on the lift of x to $\widetilde{LM}$. Second, the action functional may not decrease along equivariant Floer trajectories that are not constant in S^{∞} (see [BO17, page 3867]). These issues may be resolved by choosing special equivariant Floer data and adding a cut-off function to the action filtration, an approach taken in [MR18, Appendix D]. We briefly outline this construction.

Fix two numbers[28] $1 < R_0 < R_1 < \infty$. Our equivariant Hamiltonian H^{eq} will have positive small slope at $R = R_0$, be quadratic and increasing on $R_0 < R < R_1$, and it will

[27] The slope of a Hamiltonian is *small* if it is smaller than any positive Reeb period, i.e. it satisfies $< \min(R \cap (0, \infty))$.

[28] Choose R_0 large enough so that is linear on $\{R > R_0\}$.

be linear on $R > R_1$; in particular, it will only depend on R in the region $R > R_0$, modulo a small perturbation we will ignore. We write $H^{\mathrm{eq}}_{w,t}(\psi(y, R)) = h(R)$ to emphasize this. Fix a *cut-off function* $\beta : [1, \infty) \to \mathbb{R}$; this is a smooth increasing function which is 0 on $[1, R_0)$ and has increasing gradient on the open interval (R_0, R_1). Define $f_h : \mathbb{R} \to \mathbb{R}$ to be

$$f_h(R) = \int_0^R \beta'(r)h'(r)\,\mathrm{d}r \tag{2.5.11}$$

and define the function $F_h : \mathcal{L}M \to \mathbb{R}$ by

$$F_h(x) = -\int_{S^1} x^*(\beta(R)\alpha) + \int_{t=0}^{1} f_h(R \circ x)\,\mathrm{d}t. \tag{2.5.12}$$

Notice how (2.5.12) resembles the action functional (2.3.11) in the exact setup if we were to change β to $\mathrm{Id}_{\mathbb{R}}$ (of course this choice doesn't satisfy the requirements for β). The function F_h decreases along any equivariant Floer trajectory. Therefore the inclusion

$$EFC^*(M; H^{\mathrm{eq}}; F_h \geq 0) \hookrightarrow EFC^*(M; H^{\mathrm{eq}}), \tag{2.5.13}$$

of the subcomplex generated by the orbits which satisfy $F_h \geq 0$ is a chain map.

The cohomology of the quotient cochain complex of (2.5.13) is the *positive equivariant Floer cohomology*, denoted $EFH^{*,+}(M; H^{\mathrm{eq}})$. This construction is compatible with continuation maps, so we can take a direct limit of $EFH^{*,+}(M; H^{\mathrm{eq}})$ under continuation maps with the slopes increasing. This direct limit is the *positive equivariant symplectic cohomology* of M, and is written $ESH^{*,+}(M)$. The equivariant PSS maps of Section 2.7.3 give an isomorphism between the cohomology of the subcomplex with equivariant quantum cohomology $EFH^*(M; H^{\mathrm{eq}}; F_h \geq 0) = EQH^*(M)$. Associated to the short exact sequence induced by the inclusion (2.5.13), there is a long exact sequence

$$\cdots \to EQH^*(M) \to ESH^*(M) \to ESH^{*,+}(M) \to EQH^{*+1}(M) \to \cdots. \tag{2.5.14}$$

We have the following compatibility result between the filtration and the equivariant Floer Seidel map.

Theorem 2.5.2. *For any equivariant Hamiltonian orbit (c_k, x), the filtration satisfies*

$$F_{\sigma^* h}(\sigma \cdot x) = F_h(x). \tag{2.5.15}$$

Proof. By definition, we have $(\sigma^* h)(R) = h(R) - \kappa R$, so that $f_{\sigma^* h}(R) = f_h(R) - \kappa\beta(R)$, where κ is the slope of σ. Without loss of generality, let x occur in the region $\{R_0 < R < R_1\}$

since otherwise both sides of (2.5.15) are 0. Suppose that x has period l. The period of $\sigma \; x$ is $l \quad \kappa$. We have

$$
\begin{aligned}
F \;\; h(\sigma \; x) &= \int_{S^1} (\sigma \; x) \; (\beta(R)\alpha) + \int_{t=0}^{1} f \;\; h(R \;\; (\sigma \; x))\,\mathrm{d}t \\
&= \beta(R \;\; (\sigma \; x))\,(l \quad \kappa) + \int_{t=0}^{1} f \;\; h(R \;\; (\sigma \; x))\,\mathrm{d}t \\
&= \beta(R \;\; x)\,(l \quad \kappa) + \int_{t=0}^{1} f \;\; h(R \;\; x)\,\mathrm{d}t \\
&= \beta(R \;\; x)\,(l \quad \kappa) + \int_{t=0}^{1} f_h(R \;\; x) \quad \kappa\beta(R \;\; x)\,\mathrm{d}t \\
&= l\beta(R \;\; x) + \int_{t=0}^{1} f_h(R \;\; x)\,\mathrm{d}t \\
&= \int_{S^1} x \; (\beta(R)\alpha) + \int_{t=0}^{1} f_h(R \;\; x)\,\mathrm{d}t \\
&= F_h(x).
\end{aligned}
$$

□

Unfortunately, the pullback Hamiltonian $\sigma \; H^{\mathrm{eq}}$ does not satisfy the condition that its slope at R_0 is small; instead its slope at R_0 will be $\varepsilon \quad \kappa$ if H^{eq} had slope ε at R_0. To rectify the situation for positive equivariant symplectic cohomology, we have to consider the subcomplex generated by orbits with $F \;\; h \quad f \;\; h(R_0^{\prime})$, where $R_0^{\prime}$ is the radius at which $h^{\prime} \quad \kappa = 0$. The *positive equivariant Floer Seidel map* is the composition of the equivariant Floer Seidel map with the map induced on the quotient complexes corresponding to the inclusions of these subcomplexes. More explicitly, abusing notation, we have the following diagram.

$$
\begin{array}{ccc}
ESH_{\;;+} & = & ESH\;(F_h < 0) \\
\Big\downarrow EFS_{\tilde{\ };+} & & \Big\downarrow EFS_{\tilde{\ }} \\
 & & ESH^{\;+2I(\tilde{\ })}(F \;\; h < 0) \\
 & & \Big\downarrow \\
ESH^{\;+2I(\tilde{\ })}_{\;;+} & = & ESH^{\;+2I(\tilde{\ })}(F \;\; h < f \;\; h(R_0^{\prime}))
\end{array}
\tag{2.5.16}
$$

This gives us the following morphism of long exact sequences.

$$
\begin{array}{ccccccccc}
\longrightarrow EQH & \longrightarrow & ESH & \longrightarrow & ESH_{\;;+} & \longrightarrow & EQH^{\;+1} & \longrightarrow \\
\Big\downarrow EQS_{\tilde{\ }} & & = \Big\downarrow EFS_{\tilde{\ }} & & \Big\downarrow EFS_{\tilde{\ };+} & & \Big\downarrow EQS_{\tilde{\ }} & \\
\to EQH^{\;+2I(\tilde{\ })} & \to & ESH^{\;+2I(\tilde{\ })} & \to & ESH^{\;+2I(\tilde{\ })}_{\;;+} & \to & EQH^{\;+2I(\tilde{\ })+1} & \to
\end{array}
\tag{2.5.17}
$$

The five lemma applied to the diagram (2.5.17) implies that $EFS_{\tilde{\ };+}$ is an isomorphism if and only if $EQS_{\tilde{\ }}$ is an isomorphism.

2.6 Quantum theory

2.6.1 Quantum cohomology

2.6.1.1 Morse cohomology

Fix a Riemannian metric on M. Let $f : M \to \mathbb{R}$ be a Morse-Smale function which increases in the radial coordinate direction at infinity. Thus the inequality $\partial_R(f(\psi(y, R))) > 0$ holds at infinity. Denote by $\mathrm{Crit}(f)$ the finite set of critical points of f. The *Morse index* $\mathrm{ind}(x)$ of a critical point x is the dimension of the maximal subspace of the tangent space at x on which the Hessian of f is negative definite. The *Morse cohomology of* M is the cohomology of the cochain complex freely generated by $\mathrm{Crit}(f)$ whose differential counts negative gradient trajectories between critical points. It is isomorphic to the (singular) cohomology of M. Through the choice of an orientation of each unstable manifold, the count of trajectories is signed, so that Morse cohomology is a $\mathbb{Z}$-graded Abelian group.

2.6.1.2 Quantum product

Let J be a regular convex ω-compatible almost complex structure. Fix distinct points $p^-, p_1^+, p_2^+ \in \mathbb{P}^1$. The quantum product counts quadruples $(u, \gamma^-, \gamma_1^+, \gamma_2^+)$, where $u : \mathbb{P}^1 \to M$ is a simple (or constant) J-holomorphic sphere and $\gamma^- : (-\infty, 0] \to M$ and $\gamma_i^+ : [0, \infty) \to M$ are negative gradient flowlines, with the intersection conditions $\gamma_i^+(0) = u(p_i^+)$ and $\gamma^-(0) = u(p^-)$. Denote by $\mathcal{M}(x^-, x_1^+, x_2^+; A)$ the space of such quadruples where u represents $A \in \Gamma$ and the limits $\gamma_i^+(s) \to x_i^+$ and $\gamma^-(s) \to x^-$ hold as $s \to \pm\infty$.

With a regular choice of three s-dependent perturbations f_1^+, f_2^+, f^- of the Morse-Smale function f, the moduli spaces will all be smooth oriented manifolds with

$$\dim \mathcal{M}(x^-, x_1^+, x_2^+; A) = \mathrm{ind}(x^-) - \mathrm{ind}(x_1^+) - \mathrm{ind}(x_2^+) + 2c_1(A). \tag{2.6.1}$$

Via standard compactification and gluing arguments,[29] the 0-dimensional moduli space is compact and the 1-dimensional moduli space may be compactified to a manifold whose boundary is made up of the broken trajectories (the sphere will not bubble by regularity). As such, the map $C^*(M; f; \Lambda)^{\otimes 2} \to C^*(M; f; \Lambda)$ given by

$$x_1^+ * x_2^+ = \sum_{\substack{A \in \Gamma \\ x^- \in \mathrm{Crit}(f) \\ \dim \mathcal{M}(x^-; x_1^+; x_2^+; A) = 0}} \; \sum_{(u; \gamma^\pm) \in \mathcal{M}(x^-; x_1^+; x_2^+; A)} o(u, \gamma^\pm)\, q^A x^- \tag{2.6.2}$$

is a chain map, and hence it induces a product structure on the Morse cohomology of M with coefficients in the Novikov ring Λ. This product is unital, skew-commutative and associative,

[29] In the region at infinity where J is convex, any J-holomorphic sphere cannot achieve a maximal value of R by a maximum principle. As such, all the holomorphic spheres lie in a compact region and standard compactification results apply.

and induces the structure of a $\mathbb{Z}$-graded Λ-algebra on $H^*(M; f; \Lambda)$. This is the *quantum cohomology* $QH^*(M)$ of the manifold M, and the product is the *quantum product*.

Remark 2.6.1. The quantum cohomology a priori depends on the Riemannian metric, the Morse-Smale function f and its three perturbations, the chosen orientations of the unstable manifolds and the almost-complex structure. However the dependence on all of this data may be removed up to canonical isomorphism via standard homotopy arguments. Moreover, quantum cohomology is independent of the parameterisation of the convex end because this information is used only to constrain all flowlines and spheres to a compact region.

2.6.2 Quantum Seidel map

Let $\widetilde{\sigma}$ be a lifted linear Hamiltonian circle action on M with nonnegative slope.

2.6.2.1 Clutching construction

In this section, we define a symplectic M-bundle E over the sphere associated to the action σ.

Base space The sphere S^2 is the union of its upper hemisphere $\mathbb{D}^-$ and lower hemisphere $\mathbb{D}^+$. Each hemisphere is a copy of the closed unit disc in the complex plane. The equator of the sphere is the circle $S^1 = \mathbb{R}/\mathbb{Z}$. We identify the boundaries of the hemispheres with the equator via

$$t \in S^1 \leftrightarrow e^{2\pi it} \in \partial\mathbb{D}^- \leftrightarrow e^{-2\pi it} \in \partial\mathbb{D}^+. \tag{2.6.3}$$

The poles of the sphere are the points $z_\pm = 0 \in \mathbb{D}^\pm$. The complement of the poles in the sphere is isomorphic to a cylinder via the map

$$\mathbb{R} \times S^1 \ni (s,t) \mapsto \begin{cases} e^{2\pi(s+it)} \in \mathbb{D}^- & \text{if } s \leq 0, \\ e^{-2\pi(s+it)} \in \mathbb{D}^+ & \text{if } s \geq 0. \end{cases} \tag{2.6.4}$$

Remark 2.6.2. Our notation is opposite to that of Seidel [Sei97] and Ritter [Rit14], so our $\mathbb{D}^\pm$ correspond to their $D^\mp$.

Total space The smooth manifold E is the union of the manifolds $\mathbb{D}^\pm \times M$ glued along the boundaries via

$$\partial\mathbb{D}^- \times M \ni (e^{2\pi it}, m) \leftrightarrow (e^{-2\pi it}, \sigma_t m) \in \partial\mathbb{D}^+ \times M. \tag{2.6.5}$$

The projection map $\pi : E \to S^2$ is the union of the projection maps $\mathbb{D}^\pm \times M \to \mathbb{D}^\pm$. Let $\iota_\pm : M \to \{z_\pm\} \times M \subset E$ be the fibre inclusion maps over the poles.

Symplectic bilinear form Denote by $T^{\mathrm{vert}}E$ the kernel of $D\pi$. With $\pi_M : \mathbb{D}^{\pm} \times M \to M$ the projection map, the vector space $T^{\mathrm{vert}}_{(w,m)}E$ is equipped with a symplectic bilinear form $\Omega_{(w,m)} = (\pi_M)^*_{(w,m)}\omega_m$. Since the circle action σ is symplectic, these symplectic bilinear forms agree along the equator, so that $T^{\mathrm{vert}}E \to E$ is a symplectic vector bundle with symplectic bilinear form Ω.

Global 2-form There is a closed 2-form $\widehat{\Omega}$ on E which restricts to Ω on $T^{\mathrm{vert}}E$. The construction of $\widehat{\Omega}$ for convex symplectic manifolds, due to Ritter [Rit14, Section 5], uses a special[30] pair of Hamiltonians $H^{E,\pm} : \mathbb{D}^{\pm} \times M \to \mathbb{R}$ to modify the fibrewise symplectic form ω so that it becomes a well-defined closed 2-form.

Almost complex structures The sphere has an (almost) complex structure j given by $\pm i$ on $\mathbb{D}^{\pm}$. Denote by $\mathcal{J}(E)$ the space of almost complex structures $\widehat{\mathbf{J}}$ on E which satisfy the following properties:

- $D\pi$ is $(\widehat{\mathbf{J}}, j)$-holomorphic.
- $\widehat{\mathbf{J}}|_{T^{\mathrm{vert}}E}$ is a convex Ω-compatible almost complex structure on $T^{\mathrm{vert}}E$.
- At infinity, $\widehat{\mathbf{J}}$ has the form

$$\widehat{\mathbf{J}}_{(z,m)} = \begin{pmatrix} j & 0 \\ \mathrm{d}s \otimes X_{H^{E,\pm}_z} - \mathrm{d}t \otimes \mathbf{J}_z X_{H^{E,\pm}_z} & \mathbf{J}_z \end{pmatrix} \tag{2.6.6}$$

with respect to the decomposition $T_{(z,m)}E = T_z\mathbb{D}^{\pm} \oplus T_mM$ and the coordinates (s,t) on the sphere from (2.6.4), denoting by $\mathbf{J}_z$ the fibrewise restriction $\widehat{\mathbf{J}}|_{T^{\mathrm{vert}}E}$ on each hemisphere.

Given any $\widehat{\mathbf{J}} \in \mathcal{J}(E)$, the 2-form $\widehat{\Omega} + c\pi^*\omega_{S^2}$ is symplectic and $\widehat{\mathbf{J}}$ is $(\widehat{\Omega} + c\pi^*\omega_{S^2})$-compatible for large enough $c > 0$. Here, we denote by ω_{S^2} the standard symplectic form on S^2.

Remark 2.6.3. The motivation for (2.6.6) is that any $\widehat{\mathbf{J}}$-holomorphic section is locally a Floer solution for $(H^E, \mathbf{J})$ whenever (2.6.6) applies, and hence a maximum principle forbids any (non-fixed) $\widehat{\mathbf{J}}$-holomorphic sections outside a compact region.

[30] The functions $H^{E,\pm} : \mathbb{D}^{\pm} \times M \to \mathbb{R}$ must vanish near the poles, be independent of the s-coordinate near the equator, and glue according to $H^{E,+}_t = -H^{E,-}_{-t}$. The gluing condition ensures the Hamiltonian vector field in $T^{\mathrm{vert}}E$ is well-defined along the equator. Moreover, $H^{E,\pm}$ must both be monotone, by which we mean that in a region at infinity, the functions are dependent only on the radial coordinate R and the s-coordinate of (2.6.4), and satisfy $\partial_s H^{E,\pm} \leq 0$. We assume λ is linear of nonnegative slope precisely so that such Hamiltonian functions exist.

Sections Two sections $s_1, s_2 : S^2 \to E$ are *-equivalent* if the conditions $\widehat{\ }(s_1) = \widehat{\ }(s_2)$ and $c_1(T^{\text{vert}}E)(s_1) = c_1(T^{\text{vert}}E)(s_2)$ hold, where $c_1(T^{\text{vert}}E)$ is the first Chern class of the symplectic vector bundle $T^{\text{vert}}E \to E$. The property of -equivalence is independent of the choice of global 2-form $\widehat{\ }$ which restricts to . Moreover, the group acts freely and transitively on -equivalence classes of sections.

Given any lift $\widetilde{\sigma}$ and any $\widetilde{x} \in \widetilde{LM}$, we can produce a section by setting[31] $z \mapsto (z, \widetilde{x}(z))$ on D and $z \mapsto (z, \widetilde{\sigma}(\widetilde{x})(\overline{z}))$ on D$^+$. The -equivalence class of this section is independent of the choice of $\widetilde{x}$. We denote it by $S_{\widetilde{\sigma}}$. It satisfies $I(\widetilde{\sigma}) = c_1(T^{\text{vert}}E)(S_{\widetilde{\sigma}})$. Every -equivalence class is $S_{\widetilde{\sigma}} + A$ for a unique $A \in$.

Fixed sections For every fixed point $m \in M$ of the circle action σ, there is a constant section $s_m : z \mapsto (z, m)$. The section s_m is the *fixed section at* m. For any fixed section s_m, we have that $c_1(T^{\text{vert}}E)(s_m)$ equals the sum of the weights of the action around m [MT06, Lemma 2.2].

Remark 2.6.4 (Minimal fixed sections). A *minimal* fixed section is a fixed section s_m at a point m in the minimal locus of the Hamiltonian K, i.e. $K(m) = \min(K)$. Minimal fixed sections are $(j, \widehat{\mathbf{J}})$-holomorphic for a restricted class of almost complex structures $\widehat{\mathbf{J}}$ (see [MT06, Definition 2.3] and the preceding text). In this setting, we moreover have $\widehat{\ }(u) > \widehat{\ }(s_m)$ for any minimal fixed section s_m and any $(j, \widehat{\mathbf{J}})$-holomorphic section u which is not a minimal fixed section [MT06, Lemma 3.1].

2.6.2.2 Quantum Seidel map definition

The objects of focus are $(j, \widehat{\mathbf{J}})$-holomorphic sections of E, for a suitably regular $\widehat{\mathbf{J}} \in \mathcal{J}(E)$ which we use throughout this section. The moduli space $\mathcal{M}(S)$ of $(j, \widehat{\mathbf{J}})$-holomorphic sections which are in -equivalence class S is a smooth manifold of dimension $2n + 2c_1(T^{\text{vert}}E)(S)$. A sequence of such sections $u_r \in \mathcal{M}(S)$ with $\widehat{\ }(u_r)$ bounded will have a subsequence which converges to a section with bubbles in the fibres.

For critical points $x \in \text{Crit}(f)$, denote by $\mathcal{M}(x, x^+; S)$ the moduli space of triples (γ, γ^+, u) where $\gamma : (\infty, 0] \to M$ and $\gamma^+ : [0, \infty) \to M$ are negative gradient trajectories of f and $u \in \mathcal{M}(S)$ is a section which satisfies $u(z) = \gamma(0)$ at the poles. Here, the functions $f : \mathbb{R} \times M \to \mathbb{R}$ are s-dependent perturbations of f, chosen such that the data $(f, \widehat{\mathbf{J}})$ satisfies some regularity conditions. These regularity conditions ensure that $\mathcal{M}(x, x^+; S)$ is a smooth manifold of dimension $\text{ind}(x) \quad \text{ind}(x^+) + 2c_1(T^{\text{vert}}E)(S)$.

[31] More precisely, by $\widetilde{x}(z)$, we mean $u(z)$ for a choice of filling u of x, and similarly for $\widetilde{\ }\widetilde{x}(\overline{z})$. The resulting -equivalence class is independent of the choice.

Define a degree-$2I(\tilde\sigma)$ chain map $QS_{\tilde\sigma} : C^*(M;f;\Lambda) \to C^{*+2I(\tilde\sigma)}(M;f;\Lambda)$ by

$$x^+ \mapsto \sum_{\substack{x^- \in \mathrm{Crit}(f) \\ A \in \Gamma \\ \dim \mathcal{M}(x^-,x^+;\tilde S + A) = 0}} \; \sum_{(\gamma,u) \in \mathcal{M}(x^-,x^+;\tilde S + A)} o((\gamma, u))\, q^A x^- . \tag{2.6.7}$$

The regularity conditions we impose ensure that QS is a chain map so it induces a map on quantum cohomology. Moreover, QS intertwines quantum multiplication in $QH^*(M)$, giving

$$QS_{\tilde\sigma}(x \star y) = x \star QS_{\tilde\sigma}(y) \tag{2.6.8}$$

for all $x, y \in H^*(M;f;\Lambda)$.

Remark 2.6.5. There are two more-or-less equivalent methods to proving this intertwining relation (2.6.8). One approach is to prove the intertwining of the Floer Seidel map with the pair-of-pants product [Sei97, Proposition 6.3], and apply the ring isomorphisms PSS from (2.6.9) to deduce the desired result. This is the approach taken by Seidel. This method will not extend to the equivariant setup for two reasons: the pair-of-pants product has no equivariant version and the homotopy Seidel constructs to prove the intertwining with the pair-of-pants product involves reparameterising the path $\sigma : S^1 \to \mathrm{Ham}(M)$, which cannot be done while maintaining (2.4.4) (see Remark 2.5.1). The second approach directly constructs a chain homotopy either side of (2.6.8). While a standard argument, I believe it has not appeared in the literature. It is this second approach we extend in Section 2.7.4. The non-equivariant argument may be derived from the equivariant argument by removing the flowlines in S^∞.

2.6.2.3 Gluing construction

The PSS isomorphism is a ring isomorphism between the Floer cohomology and the quantum cohomology of a weakly monotone closed symplectic manifold constructed in [PSS96]. As a map, the PSS isomorphism counts *spiked discs*. These are maps from the disc to M which near the boundary act like Floer solutions and which near the centre act like a pseudoholomorphic sphere, together with half-flowlines[32] between a critical point and the centre of the disc, the *spikes*. To extend the definition of these maps to convex symplectic manifolds, we make the following definition.

Definition 2.6.6. The (time-dependent) Hamiltonian $H^0 : S^1 \to C^\infty(M)$ has *slope zero* if H^0 is C^2-bounded and, at infinity, $H^0_t(\psi(y,R)) = h(R)$ for a smooth function $h : (R_0, \infty) \to \mathbb{R}$ which satisfies $0 < h' < T_{\min}$, $h'' < 0$ and $h' \to 0$, where $T_{\min} \in (0, \infty]$ is the minimal Reeb period.

[32] i.e. a flowline with domain either $[0;\infty)$ or $(-\infty;0]$.

While such a Hamiltonian is not linear, it still satisfies (2.3.5) and a maximum principle at infinity. Thus we can define Floer cohomology for a regular choice of Floer data $(H^0, \mathbf{J})$. The PSS construction yields a pair of ring isomorphisms between Floer cohomology and quantum cohomology

$$\mathrm{PSS}^- : FH^*(M; H^0) \to QH^*(M), \qquad \mathrm{PSS}^+ : QH^*(M) \to FH^*(M; H^0) \tag{2.6.9}$$

which are mutual inverses [Rit14, Theorem 37].

Seidel's gluing argument [Sei97, Section 8] proves the following diagram is commutative.

$$\begin{array}{ccc} QH^*(M) & \xrightarrow{\quad QS_{\tilde\sigma}\quad} & QH^{*+2I(\tilde\sigma)}(M) \\ \mathrm{PSS}^+ \downarrow \cong & & \cong \uparrow \mathrm{PSS}^- \\ FH^*(M; H^0) & & FH^{*+2I(\tilde\sigma)}(M; H^0) \\ & \searrow^{\cong}_{FS_{\tilde\sigma}} \qquad \nearrow_{\text{continuation map}} & \\ & FH^{*+2I(\tilde\sigma)}(M; \sigma^* H^0) & \end{array} \tag{2.6.10}$$

In [Rit14], Ritter used a non-equivariant version of (2.5.8) together with (2.6.10) to show that if σ has positive slope, then the direct limit of the direct system

$$QH^*(M) \xrightarrow{QS_{\tilde\sigma}} QH^{*+2I(\tilde\sigma)}(M) \xrightarrow{QS_{\tilde\sigma}} QH^{*+4I(\tilde\sigma)}(M) \xrightarrow{QS_{\tilde\sigma}} \cdots \tag{2.6.11}$$

is isomorphic to symplectic cohomology. This offers a method to calculate symplectic cohomology because the quantum Seidel map is quantum multiplication by the element $QS_{\tilde\sigma}(1)$.

2.6.3 Equivariant Quantum cohomology

2.6.3.1 Equivariant Morse cohomology

Let ρ be a smooth circle action on M, and fix a ρ-invariant Riemannian metric on M. An *equivariant Morse function* is a function $f^{eq} : S^\infty \times M \to \mathbb{R}$ which is invariant under the free action (2.4.3), and which extends Morse functions $f^k : M \to \mathbb{R}$ analogously to Definition 2.4.4. We assume that, at infinity, the function $f^{eq}_w(\cdot)$ is increasing in the radial coordinate direction for all $w \in S^\infty$.

Analogously to the equivariant Floer cohomology construction of Section 2.4, an *equivariant critical point* is an equivalence class $[w, x] \in S^\infty \times_{S^1} M$ such that w is a critical point of F and x is a critical point of $f^{eq}_w(\cdot)$. The index of such a critical point is $\mathrm{ind}(w; F) + \mathrm{ind}(x; f^{eq})$. We use the notation $(c_k, x) \in \mathrm{Crit}(f^{eq})$ for equivariant critical points and $|c_k, x|$ for their indices.

For a suitably regular equivariant Morse function, the moduli spaces of *equivariant negative gradient trajectories* are smooth oriented manifolds. *Equivariant (Morse) cohomology*

$EH(M)$ is the cohomology of the cochain complex generated over Z by the equivariant critical points whose differential counts the equivariant negative gradient trajectories. As in Section 2.6.1.1, an orientation of the unstable manifolds must be chosen in order that the count is signed. We omit the details. The Z[$\mathbf{u}$]-module actions of Section 2.4.4.1 and Section 2.4.4.2 may be defined on equivariant Morse cohomology, and we opt for the geometric action $\mathbf{u} \smile$ which counts 'Y'-shaped graphs.

2.6.3.2 Equivariant quantum product

Let ρ be a symplectic circle action which is linear at infinity. We use a 'Y'-shaped flowline in S^1 for the equivariant quantum product so that it resembles a deformed equivariant cup product. In order to construct the moduli space, we need three s-dependent perturbations F, F_1^+, F_2^+ of the standard function on S^1 and three s-dependent perturbations $f^{\mathrm{eq};}, f_1^{\mathrm{eq};+}, f_2^{\mathrm{eq};+}$ of the equivariant Morse function on M. We also need a regular convex S^1-dependent ω-compatible almost complex structure J^{eq}, which is equivariant in the sense that $J^{\mathrm{eq}}_w = (D\sigma)^{1} J^{\mathrm{eq}}{}_{1\,w} D\sigma$ for all θ 2 S^1 and w 2 S^1.

We consider septuples $(v, v_1^+, v_2^+, \gamma, \gamma_1^+, \gamma_2^+, u)$, where v are negative gradient flowlines of F satisfying $v_1^+(0) = v_2^+(0) = v(0)$ and γ are negative gradient flowlines of $(f^{\mathrm{eq};})_v$ ( ) which intersect at 0 with a simple (or constant) $J^{\mathrm{eq}}_{v(0)}$-holomorphic sphere u at the points p 2 P[1] (see Figure 2.2). When the data is sufficiently regular, the moduli space of (S^1-equivalence classes of) such septuples with limits (c_k, x) 2 Crit(f^{eq}) and with u representing A 2 is a smooth oriented manifold of dimension

$$\left|c_k, x\right| + 2c_1(A) \quad \left|c_{k_1^+}, x_1^+\right| \quad \left|c_{k_2^+}, x_2^+\right|. \tag{2.6.12}$$

The *equivariant quantum product* counts the 0-dimensional moduli spaces.

The *equivariant quantum cohomology* $EQH(M)$ is the cohomology of the cochain complex

$$EQC^l(M) = \prod_{k=0}^{1} \bigoplus_{(c_k;x)2\mathrm{Crit}(f^{\mathrm{eq}})} \quad {}^{l\ jc_k;xj} h(c_k, x)i \tag{2.6.13}$$

with the equivariant Morse differential. It is a graded [$\mathbf{u}$]-module for both the algebraic and the geometric $\mathbf{u}$-actions. The product is a unital, graded-commutative, associative product on $EQH(M)$ compatible with the -module structure and the geometric[33] Z[$\mathbf{u}$]-module structure. All of these properties follow from standard homotopy proofs, as does the independence from all the data chosen.

Via the algebraic Z[$\mathbf{u}$]-module structure, the equivariant quantum cochain complex (2.6.13) is the graded completed tensor product $QC(M) \,\widehat{}\,$ Z[$\mathbf{u}$]. By the *graded completed*

[33] It is difficult to see any relationship between the algebraic Z[$\mathbf{u}$]-module structure and the equivariant product, hence our decision to use the geometric module structure in this paper.

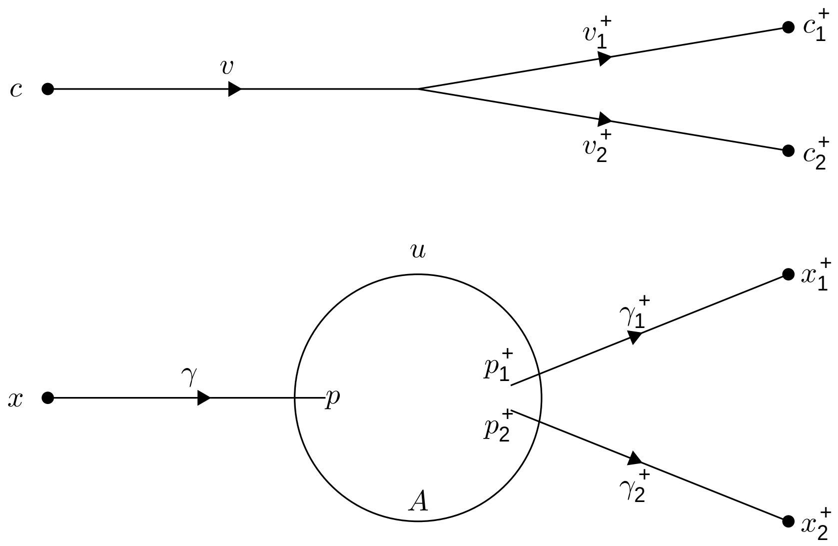


Figure 2.2: The equivariant quantum product counts equivalence classes of septuples $(v^-, v_1^+, v_2^+, \gamma^-, \gamma_1^+, \gamma_2^+, u)$. The 'Y'-shaped graph above maps to S^1 while the configuration below the graph maps to M.

tensor product $A \hat{\otimes} \mathbb{Z}[\mathbf{u}]$, where A is a graded $\mathbb{Z}$-module, we mean the graded $\mathbb{Z}$-module whose grading-l subgroup is

$$(A \hat{\otimes} \mathbb{Z}[\mathbf{u}])^l = \prod_{k=0}^{\infty} A^{l-2k} \otimes_{\mathbb{Z}} \mathbb{Z}\,\mathbf{u}^k. \tag{2.6.14}$$

This is a variant of the completed tensor product used by Zhao [Zha19, Section 2], but which is a graded module in the conventional sense.

2.7 Equivariant quantum Seidel map

Let $\widetilde{\sigma}$ be a lift of a linear Hamiltonian circle action of nonnegative slope and ρ a symplectic circle action linear at infinity. Assume σ and ρ commute.

2.7.1 Clutching bundle action

The sphere S^2 has a natural rotation action given by $\theta \cdot (s,t) = (s, t+\theta)$ away from the poles, using the parameterisation (2.6.4).

The clutching bundle E from Section 2.6.2.1 admits a smooth circle action given by

$$\begin{cases} \mathbb{D}^- \times M \ni (z,m) \mapsto (e^{2\pi i\theta} z, (\sigma_\theta \circ \rho_\theta)(m)) \in \mathbb{D}^- \times M \\ \mathbb{D}^+ \times M \ni (z,m) \mapsto (e^{-2\pi i\theta} z, \rho_\theta(m)) \in \mathbb{D}^+ \times M \end{cases}, \tag{2.7.1}$$

which glues correctly along the equator. We denote this action by ρ_E. The projection map is equivariant with respect to the action ρ_E on E and to the rotation action on S^2. The

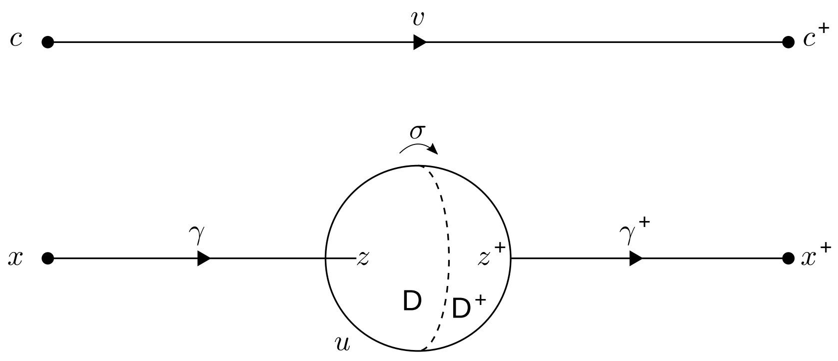


Figure 2.3: The equivariant quantum Seidel map counts equivalence classes of quadruples $(v, \gamma^-, \gamma^+, u)$. The map u is a section of the clutching bundle which twists the fibres by σ when passing from the upper hemisphere D^- to the lower hemisphere D^+. The flowlines $\gamma^{\pm}$ map to the manifold M, which is identified with the fibres of the clutching bundle over the poles $z^{\pm}$.

poles $z^{\pm}$ are fixed points of the rotation of S^2, so ρ_E restricts to a circle action in each of the fibres $E_{z^{\pm}}$: the action on E_{z^+} is the action ρ and the action on E_{z^-} is the action $\sigma^{-1} \rho$. Here, we have identified these fibres $E_{z^{\pm}}$ with M via the inclusion maps $\iota^{\pm}$.

2.7.2 Equivariant quantum Seidel map definition

The equivariant quantum Seidel map is a version of the quantum Seidel map QS which is equivariant with respect to the circle action ρ_E of Section 2.7.1. Since the action ρ_E restricts to different actions on the two fibres over the poles, the equivariant quantum Seidel map will map between the equivariant quantum cohomology for these two different circle actions.

Let $f^{eq;\pm}$ be equivariant Morse-Smale functions for the circle actions in the corresponding to the fibre $E_{z^{\pm}}$, so $f^{eq;+}$ is equivariant with respect to the action ρ and $f^{eq;-}$ is equivariant with respect to the action $\sigma^{-1} \rho$.

We require s-dependent perturbations $f_s^{eq;\pm}$ of the equivariant Morse data and an S^∞-dependent almost complex structure $\widehat{\mathbf{J}}^{eq}$ which is equivariant in the sense of Section 2.6.3.2 with respect to ρ_E. This almost complex structure should have similar properties to the non-equivariant $\widehat{\mathbf{J}}$, so that $D\pi$ is holomorphic and the fibrewise restriction $\widehat{\mathbf{J}}^{eq}|_{T^{vert}E}$ is a ω-compatible almost complex structure for all $w \in S^\infty$, and $\widehat{\mathbf{J}}^{eq}|_{T^{vert}E}$ is convex and $\widehat{\mathbf{J}}^{eq}$ has the form (2.6.6) at all points in a region at infinity.

The regularity conditions will guarantee the moduli spaces below are smooth manifolds which in dimensions 0 and 1 compactify without bubbling.

The equivariant quantum Seidel map counts (S^1-equivalence classes of) quadruples $(v, \gamma^-, \gamma^+, u)$, where v is a flowline in S^∞, the curves $\gamma^+ : [0, \infty) \to M$ and $\gamma^- : (-\infty, 0] \to$

M are equivariant $f^{\text{eq};}_{S}$ $(v(s),\ )$-flowlines and u is a $\widehat{\mathbf{J}}^{\text{eq}}_{v(0)}$-holomorphic section satisfying $u(z\ ) = \gamma\ (0)$ (see Figure 2.3). It is a degree-$2I(\widetilde{\sigma})$ -module homomorphism $EQS_{\widetilde{}}$: $EQH\ (M)\ !\ EQH^{\ +2I(\widetilde{\ })}(M)$. A standard homotopy argument shows that $EQS_{\widetilde{}}$ commutes with the geometric Z[$\mathbf{u}$]-module structure.

2.7.3 Equivariant gluing

The results of Section 2.6.2.3 extend to the equivariant setup. The equivariant PSS maps are the [$\mathbf{u}$]-module isomorphisms

$$\begin{aligned} E\text{PSS}\ \ &: EFH\ (M; H^{\text{eq};0})\ !\ EQH\ (M) \\ E\text{PSS}^{+} &: EQH\ (M)\ !\ EFH\ (M; H^{\text{eq};0}) \end{aligned} \tag{2.7.2}$$

which count equivariant spiked discs. Here, $H^{\text{eq};0}$ is an equivariant Hamiltonian of slope zero with respect to the action ρ. The equivariant version of (2.6.10) is the following commutative diagram.

$$\begin{array}{ccccc} EQH\ (M) & & \xrightarrow{EQS_{\widetilde{}}} & & EQH^{\ +2I(\widetilde{\ })}(M) \\ {\scriptstyle EPSS^{+}}\downarrow{\scriptstyle =} & & & & {\scriptstyle =}\uparrow{\scriptstyle EPSS} \\ EFH\ (M; H^{\text{eq};0}) & & & & EFH^{\ +2I(\widetilde{\ })}(M; H^{\text{eq};0}) \\ & \searrow^{=}_{EFS_{\widetilde{}}} & & \nearrow_{\text{continuation map}} & \\ & & EFH^{\ +2I(\widetilde{\ })}(M; \sigma\ H^{\text{eq};0}) & & \end{array} \tag{2.7.3}$$

If σ has positive slope, then Ritter's argument (2.5.8) combined with (2.7.3) implies that the direct limit of the direct system

$$EQH\ (M)\ \overset{EQS_{\widetilde{}}}{!}\ EQH^{\ +2I(\widetilde{\ })}(M)\ \overset{EQS_{\widetilde{}}}{!}\ EQH^{\ +4I(\widetilde{\ })}_{(\ 2)}(M)\ \overset{EQS_{\widetilde{}}}{!} \tag{2.7.4}$$

is isomorphic to equivariant symplectic cohomology $ESH\ (M)$.

2.7.4 Intertwining relation

The intertwining result (2.6.8) does not hold in our equivariant setup. Instead, we have the formula

$$EQS_{\widetilde{}}(x\ \ ((\iota^{+})\ \ ))\quad EQS_{\widetilde{}}(x)\quad ((\iota\ )\ \ ) = \mathbf{u} \smile EQS_{\widetilde{};}\ (x). \tag{2.7.5}$$

In this equation, $2\ EH_{\ E}(E)$ is an equivariant cohomology class of the clutching bundle, and the map $EQS_{\widetilde{};}$ is the -weighted equivariant quantum Seidel map defined in the following section. The proof of (2.7.5) takes up the rest of this section.

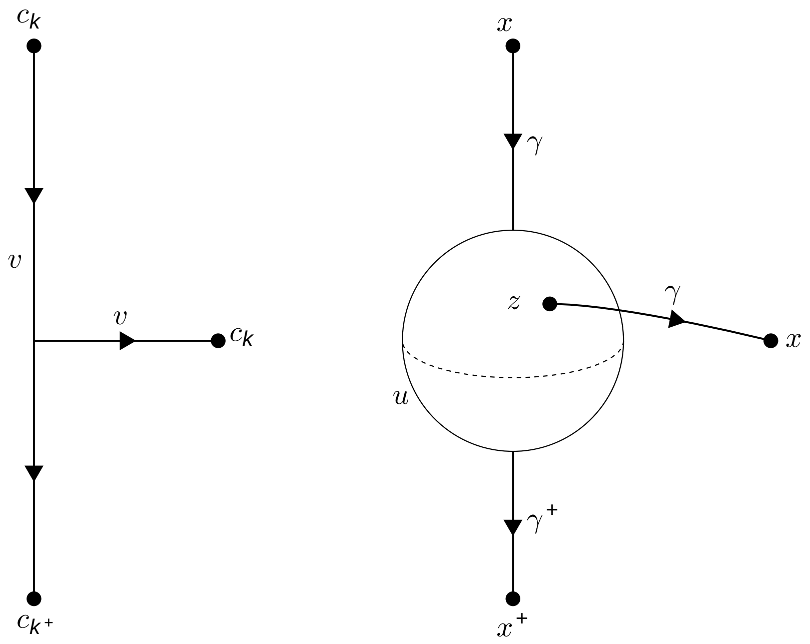


Figure 2.4: The weighted equivariant quantum Seidel map counts equivalence classes of septuples $(v, \gamma^+, \gamma \ , u; v \ , \gamma \ , z \ )$. The section u of the clutching bundle intersects the flowline γ over the point $z \in S^2$.

2.7.4.1 Weighted equivariant quantum Seidel map

Fix an invariant Riemannian metric on the clutching bundle E for the action ρ_E, and let f_E^{eq} be an equivariant Morse-Smale function for this action.

Take a regular choice of the data $f_S^{eq;}$ and $\widehat{\mathbf{J}}^{eq}$ from Section 2.7.2. We use an s-dependent perturbation $f_{E;s}^{eq;}$ of f_E^{eq} on $[0, \infty)$. We will consider septuples $(v, \gamma^+, \gamma \ , u; v \ , \gamma \ , z \ )$ where $(v, \gamma^+, \gamma \ , u)$ is a quadruple from Section 2.7.2, $z \in S^2$ is a point in the sphere and $(v \ , \gamma \ ) : [0, \infty) \to S^1 \ E$ is an equivariant $f_{E;s}^{eq;}$ flowline satisfying $v \ (0) = v(0)$ and $\gamma \ (0) = u(z \ )$ (see Figure 2.4). Given any equivariant critical point $= (c_k \ , x \ )$ of f_E^{eq}, together with equivariant critical points $x \ = (c_k \ , x \ ) \in \mathrm{Crit}(f^{eq})$ and a class $A \in$, denote by $M(x \ , x^+, A; \ )$ the moduli space of S^1-equivalence classes of septuples as above with the obvious limits and with u of class $\tilde{S} + A$. For regular perturbations, these moduli spaces are smooth oriented manifolds of dimension

$$\dim M(x \ , x^+, A; \ ) = |x \ | \quad |x^+| \quad | \ | + 2c_1(A) + 2. \tag{2.7.6}$$

The $+2$ comes from the 2-dimensional freedom of the point $z \in S^2$. Moreover, we can assume $z \notin \{z \ \}$ for moduli spaces of dimension 0 and 1 by imposing further regularity conditions on $f_{E;s}^{eq;}$.

Counting this moduli space yields a map

$$EQS_{\tilde{\ };} : QC(M) \to QC^{+2I(\tilde{\ })+|\ |-2}(M). \tag{2.7.7}$$

Our definition may be immediately extended linearly to any $\in EC_E(E)$. The 1-dimensional moduli spaces are compactified by any of the flowlines breaking since there is no bubbling by regularity. This yields the equation

$$d\,EQS_{\tilde{\ };}(x) = EQS_{\tilde{\ };}\,d(x) + (-1)^{|x|}EQS_{\tilde{\ };d}(x). \tag{2.7.8}$$

Thus for closed Morse cochains , the map $EQS_{\tilde{\ };}$ is a chain map.

Remark 2.7.1 (Interpretation). Let $k = 0$ and $|x| = 2$. For any quadruple (v, γ^+, γ, u) from Section 2.7.2, the flowline v generically flows to the minimum $c_0 = c_k$. Therefore, the count of flowlines γ with $\gamma(0) = u(z)$ for some $z \in S^2$ and $\gamma(\infty) = x$ recovers the number $[x](u[S^2])$. This is the evaluation of the degree-2 cohomology class $[x]$ on the degree-2 homology class $(u[S^2])$. As such, the map $EQS_{\tilde{\ };}$ is a weighted version of $EQS_{\tilde{\ }}$ under which any section u has weight $[x](u[S^2])$.

2.7.4.2 1-dimensional moduli space

Fix the line of longitude $L = \mathbb{R}_{>0} \cap D \subset S^2$, which does not include the poles. One way to derive the commutativity of the quantum product and the quantum Seidel map $QS_{\tilde{\ }}$ is to take a non-equivariant version of the moduli space from Section 2.7.4.1 in which $z \in L$. The 1-dimensional moduli space is compactified by breaking one of the flowlines, or allowing a bubble over the pole when $z \to z$. If x is closed, this yields a chain homotopy between $(\iota) x \ QS_{\tilde{\ }}()$ and $QS_{\tilde{\ }}((\iota^+) x\ )$.

The intersection condition $z \in L$ does not transform correctly under the S^1-action on the equivariant moduli space, however. To rectify this, take a further s-dependent perturbation F^0_s of the Morse function F on S^∞. We consider octuples $(v, \gamma^+, \gamma, u; v, \gamma, z; v^0)$ which extend the septuples in the construction of Section 2.7.4.1 (see Figure 2.5). The map $v^0 : [0, \infty) \to S^\infty$ is a F^0_s-flowline which satisfies the intersection $v^0(0) = v(0)$ and the limit $v^0(\infty) \in c_0$. We moreover impose $z \in \tau_0(v^0(\infty))^{-1} L$, where the action on S^2 is the rotation action defined in Section 2.7.1. Notice that this condition is preserved for all $(v^0(\infty), z) \in c_0 \times S^2$ by the natural circle action on $c_0 \times S^2$ because $c_0 \subset S^\infty$ has the inverse action in accordance with (2.4.3). The effect of this construction is that the 2-dimensional freedom of the point z has been reduced to a 1-dimensional freedom 'along L'; the second dimension of freedom has been absorbed into the S^1-action.

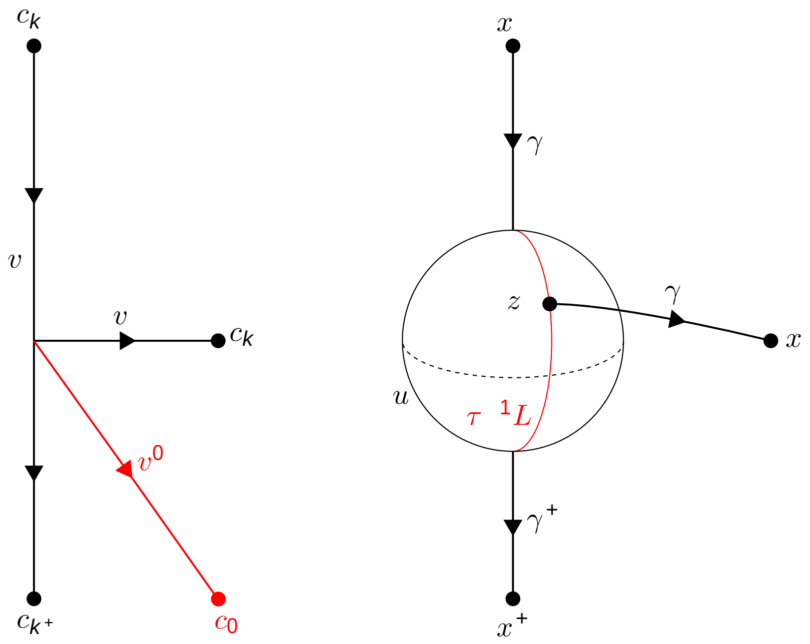


Figure 2.5: The map K counts equivalence classes of tuples like the weighted equivariant quantum Seidel map in Figure 2.4, however with an additional flowline v^0 in S^{1}. Moreover, we restrict to tuples which satisfy $z\ \ 2\ \tau\ ^1L$, where $\tau = \tau_0(v^0(1))$ is determined by the additional flowline in S^{1}.

Remark 2.7.2 (Regularity). We impose regularity conditions on the data so that the moduli spaces of the above octuples are smooth manifolds, as well as the moduli spaces of the above octuples without the condition $z\ \ 2\ \tau_0(v^0(1))\ ^1\ \ L$ and with $v^0(1)\ 2\ c_k$ for any k. Moreover, we use regularity conditions to avoid unnecessary intersections over the poles by asking that the projection $[v(0), z\ \ ] : M\ \ !\ S^{1}\ \ {}_{S^1} S^2$ intersects $CP^{1}\ \ \ f z\ \ g$ transversally for all the above moduli spaces. We impose further regularity conditions to ensure that we control the behaviour of configurations with bubbles just as in Seidel's argument [Sei97, Section 7].

We quotient by the free S^1-action to get the moduli space of S^1-equivalence classes of above octuples with the obvious constraints, which we denote by $M\ \ (x\ \ , x^+, A;\ \ )$. It is a smooth oriented manifold of dimension

$$\dim M\ \ (x\ \ , x^+, A;\ \ ) = |x\ \ |\ \ \ |x^+|\ \ \ j\ \ j + 2c_1(A) + 1. \tag{2.7.9}$$

Denote by K the map $EQC\ \ (M)\ \ !\ \ EQC\ \ ^{+2I(\tilde{\ })+j\ \ j\ \ 1}(M)$ which counts these moduli spaces.

2.7.4.3 Boundary of the moduli space

By compactification and gluing arguments, the 1-dimensional moduli spaces $M\ \ (x\ \ , x^+, A;\ \ )$ will have a boundary composed of broken flowlines and bubbled spheres. The sum of these boundary components will be zero. Subject to further homotopies, this sum yields the desired relation (2.7.5). We list the various components of the boundary below, and detail how they contribute to the sum.

$(vj_{(\ \ 1\ ;0]}, \gamma\ \ )$ breaking We get a contribution $d\ K\ \ (x^+)$.

$(vj_{[0;\ 1\ )}, \gamma^+)$ breaking We get a contribution $K\ \ (dx^+)$.

$(v\ \ , \gamma\ \ )$ breaking We get a contribution $(\ \ 1)^{jx^+j} K_d\ \ (x^+)$. If is a closed cochain, then this contribution vanishes.

v^0 breaking As a consequence of the regularity conditions, the only possible breaking of the flowline v^0 will be to the critical point c_1 2 Crit(F). Consider such a breaking, but before we have taken the quotient by the S^1-action. The broken flowline consists of a flowline v^0 whose limit is $v^0(\ 1\ ) = y$ 2 c_1 and a second unparameterised F-flowline from y to τ 2 $S^1 = c_0$, the identification $S^1 = c_0$ given by the map τ_0. In such a configuration, the point z cannot be either of the poles because of the regularity conditions. As a consequence, the point z uniquely determines τ 2 S^1 via z 2 $\tau\ \ ^1\ L$. It may be explicitly shown that there is a unique F-flowline whose limits are any specified points of c_0 and c_1. It follows that the flowline from y to τ may be omitted without loss of generality.

A standard homotopy argument will separate the flowlines v^0 and v in S^1 (see Figure 2.6). Therefore when v^0 breaks, we get a contribution of $\mathbf{u} \smile EQS_{\tilde{};}\ (x^+)$ to the sum, up to chain homotopy.

$z\ \ !\ z\ \ $ with bubbling Due to the regularity conditions, the only possible bubbling configuration is a single bubble in the fibre over one of the two poles z with the flowline to x starting within the fibre. In this configuration, we get z 2 fz g, so the condition z 2 $\tau_0(v^0(\ 1\ ))\ \ ^1\ L$ is automatically satisfied (for any value of $v^0(\ 1\ )$). As such, the flowline v^0 may be omitted without losing any information because it no longer constrains the point z .

We will treat the case of a bubble over the pole z^+. If the bubble is of class B 2 , then the section u is of class $\tilde{S} + A\ \ \ B$. We use a standard homotopy argument to insert a broken flowline between the section and the bubble, and also to extend the flowline

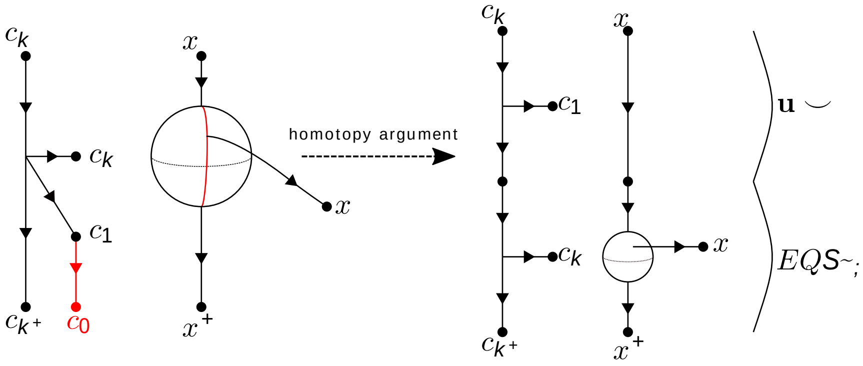


Figure 2.6: When the flowline v^0 breaks, we can use a homotopy argument to split the moduli space into multiplication by $\mathbf{u}$ and the weighted equivariant Seidel map. The flowline from c_1 to c_0 contains redundant information so we omit it.

to so it breaks into a half-flowline to (ι^+) and an equivariant functorial flowline[34] from there to (see Figure 2.7). The result is a contribution of $EQS\tilde{}(x^+ \quad ((\iota^+) \quad))$ to the sum, up to chain homotopy.

Analogously, a bubble over the pole z yields a contribution

$$EQS\tilde{}(x^+) \qquad ((\iota \ ) \quad) \tag{2.7.10}$$

to the sum, up to chain homotopy.

Summing these contributions, we get the equation

$$\mathbf{u} \smile EQS\tilde{}_{;}\ (x) + EQS\tilde{}(x \quad ((\iota^+) \quad)) \quad EQS\tilde{}(x) \quad ((\iota \ ) \quad) = 0 \tag{2.7.11}$$

for x 2 $EQH\ (M)$, which rearranges to give (2.7.5) as desired.

2.8 Examples

2.8.1 Complex plane

The complex plane C is an exact open symplectic manifold whose symplectic form is given by ω = dx ^ dy at points z = x + iy 2 C . The contact form α = πdt on S^1 and the isomorphism ψ : S^1 [1, 1) ! C given by (t, R) 7! $\sqrt{R}e^{2\ it}$ gives C the structure of a

[34]Our construction of the pullback maps is a variant of [RV14, Section 1.3], in which we allow s-dependent perturbations of the Morse data instead of perturbing the function. Let M be manifolds with Morse-Smale functions f : M ! R (and metrics and orientation data). Given any ' : M ! M^+, a *functorial flowline* is a pair of half-flowlines (: (1 ; 0] ! M ; $^+$: [0; 1) ! M^+) of s-dependent perturbations of the functions f which satisfy '((0)) = $^+$(0). When ' is an immersion or a submersion, it follows from standard arguments that the moduli spaces of functorial flowlines between critical points x is a smooth manifold of dimension ind(x ; f) ind(x^+; f^+). The map which counts these moduli spaces is a chain map and is denoted ' . The equivariant pullback maps are defined analogously.

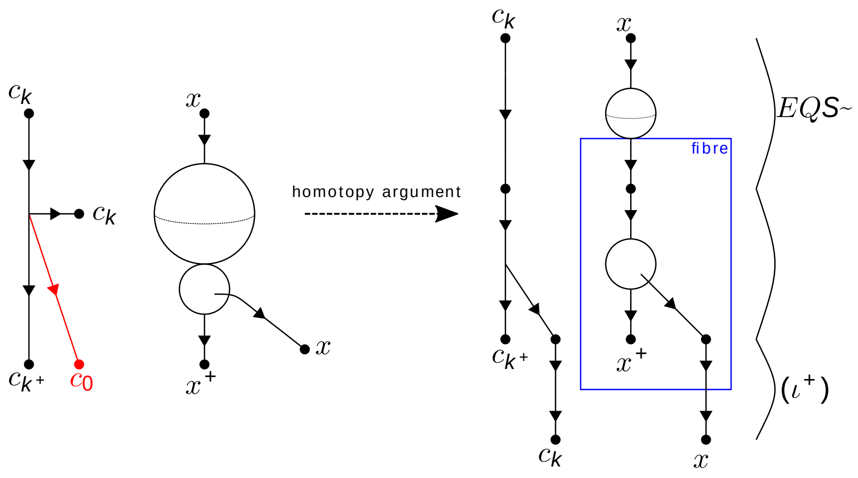


Figure 2.7: When the section bubbles over the south pole, we can use a homotopy argument to get an equivariant quantum product in the fibre above the south pole followed by the map $EQS_{\widetilde{\sigma}}$. The flowline to c_0 is redundant, so we remove it.

convex symplectic manifold. The set of Reeb periods is $R = \pi\mathbb{Z}$. The smooth circle action $\sigma : S^1 \times \mathbb{C} \to \mathbb{C}$ given by $\theta \cdot z = e^{2\pi i\theta} z$ has Hamiltonian $K_\sigma(z) = \pi|z|^2$ and is linear of slope π. The action has a unique fixed point $0_{\mathbb{C}} \in \mathbb{C}$. The lift of σ to $\widetilde{L\mathbb{C}}$ is unique because $\mathbb{C}$ is contractible. This lift $\widetilde{\sigma}$ fixes the point $(S^1 \mapsto 0_{\mathbb{C}}, D \mapsto 0_{\mathbb{C}})$ and has Maslov index $I(\widetilde{\sigma}) = 2$.

Let $H_{\text{linear}} : \mathbb{C} \to \mathbb{R}$ be the autonomous linear Hamiltonian $z \mapsto \lambda|z|^2$. If $\lambda \notin R$, then the unique Hamiltonian orbit of H_{linear} is the constant loop at $0_{\mathbb{C}}$ and has Conley-Zehnder index $-2\lfloor \frac{\lambda}{\pi} \rfloor$ [Oan04, Section 3.2]. Therefore, for $\lambda \notin R$, the Floer cochain complex is

$$FC^*(\mathbb{C}; H_{\text{linear}}) = \begin{cases} \mathbb{Z} & \text{if } * = -2\lfloor \frac{\lambda}{\pi} \rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8.1}$$

The symplectic cohomology of $\mathbb{C}$ thus vanishes. In contrast, the equivariant cohomology $ESH^*_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C})$ is isomorphic to $\mathbb{Q}[\mathbf{u}, \mathbf{u}^{-1}]$ as a $\mathbb{Z}[\mathbf{u}]$-module for the geometric and the algebraic module structures. We derive this isomorphism in the course of our discussion below (see (2.8.7)). With the equivariant Seidel map, which is an isomorphism, we get further $\mathbb{Z}[\mathbf{u}]$-module isomorphisms $ESH^*_{\sigma^r}(\mathbb{C}) \cong \mathbb{Q}[\mathbf{u}, \mathbf{u}^{-1}]$ for $r \in \mathbb{Z}$.

Give $\mathbb{C}$ the standard Riemannian metric, which is σ-invariant. The Hamiltonian function K_σ is Morse-Smale with respect to this metric and has exactly one critical point: the fixed point[35] $0_{\mathbb{C}}$ of Morse index 0. Since K_σ is invariant, the function $K^{\text{eq}}_\sigma : S^\infty \times \mathbb{C} \to \mathbb{R}$ given by $(w, z) \mapsto K_\sigma(z)$ is a regular equivariant Morse function on $\mathbb{C}$. Thus the Morse cohomology of $\mathbb{C}$ is $H^*(\mathbb{C}; K_\sigma) = \mathbb{Z}\langle 0_{\mathbb{C}} \rangle$ and the equivariant Morse cohomology is $EH^*_{\sigma^r}(\mathbb{C}; K^{\text{eq}}_\sigma) = \mathbb{Z}[\mathbf{u}]\langle 0_{\mathbb{C}} \rangle$ for any $r \in \mathbb{Z}$. Since the Novikov ring is $\Lambda = \mathbb{Z}$, the quantum cohomology and

[35] Give $W^u(0_{\mathbb{C}}) = \{0_{\mathbb{C}}\}$ the orientation $+1$.

the equivariant quantum cohomology are simply the Morse cohomology and the equivariant Morse cohomology respectively.

We compute the equivariant quantum Seidel map for the lifted action $\widetilde{\sigma}$. We can do this for the underlying actions $\rho = (\sigma^{r}) \times \mathrm{Id}_{\mathbb{C}} = \sigma^{r}$, where r is nonnegative and $\mathrm{Id}_{\mathbb{C}}$ is the identity action on $\mathbb{C}$.

Theorem 2.8.1. *Let $r \geq 0$. The equivariant quantum Seidel map*

$$EQS_{\tilde{\sigma}} : EH^{*}_{-r}(\mathbb{C}; K^{eq}) \to EH^{*+2}_{-(r+1)}(\mathbb{C}; K^{eq}) \tag{2.8.2}$$

is the $\mathbb{Z}[\mathbf{u}]$*-linear extension of* $0_{\mathbb{C}} \mapsto (r+1)\mathbf{u}\, 0_{\mathbb{C}}$.

Rather than finding the equivariant quantum Seidel map directly, we will appeal to (2.7.3) and opt to find the continuation map instead. We will use the sequence of Hamiltonians defined by Zhao in [Zha19, Section 8.1], which we outline below.

For each nonnegative integer $s \in \mathbb{Z}_{\geq 0}$, let $H^{s\pi+1}_{\text{quadratic}} : S^1 \times \mathbb{C} \to \mathbb{R}$ be a time-dependent Hamiltonian function such that:

- On $|z|^2 < 1$, the function is negative, achieves its minimum at $0_{\mathbb{C}}$ and is Morse with exactly one critical point at $0_{\mathbb{C}}$;
- On $1 < |z|^2 < s\pi + 2$, the function $H^{s\pi+1}_{\text{quadratic}}$ equals $\frac{1}{2}(|z|^2 - 1)^2$ plus a small time-dependent perturbation around $|z|^2 = j\pi + 1$ for $j = 1, \dots, s$; and
- On $|z|^2 > s\pi + 2$, the function $H^{s\pi+1}_{\text{quadratic}}$ is linear[36] of slope $s\pi + 1$.

The Hamiltonian orbits of $H^{s\pi+1}_{\text{quadratic}}$ occur only at $0_{\mathbb{C}}$ and at $R = |z|^2 = j\pi + 1$. Modulo the perturbations, the slope of $H^{s\pi+1}_{\text{quadratic}}$ when $R = j\pi + 1$ is

$$\frac{\mathrm{d}\left(H^{s\pi+1}_{\text{quadratic}}\right)}{\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}R}\frac{1}{2}(|z|^2 - 1)^2 = R - 1 = j\pi. \tag{2.8.3}$$

Thus the slope at $R = j\pi + 1$ is the Reeb period $j\pi \in \mathcal{R}$. If we hadn't perturbed $H^{s\pi+1}_{\text{quadratic}}$ in this region, we would get a S^1-family of Hamiltonian orbits corresponding to the Reeb orbit of period $j\pi$, as per (2.3.5). Instead, as a result of Zhao's perturbation, we get two Hamiltonian orbits, which we denote by x_{2j-1} and x_{2j}. Denote by x_0 the constant Hamiltonian orbit at $0_{\mathbb{C}}$. The orbit x_l has degree $-l$ for all $0 \leq l \leq 2s$.

With this choice of Hamiltonian, the equivariant Floer cochain complex (with respect to the trivial action $\mathrm{Id}_{\mathbb{C}}$) is given by

$$EFC_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{s\pi+1}_{\text{quadratic}}) = \bigoplus_{k \geq 0} \bigoplus_{j=0}^{2s} \mathbb{Z}\langle (c_k, x_j) \rangle. \tag{2.8.4}$$

[36] We slightly change $H^{s\pi+1}_{\text{quadratic}}$ near $|z|^2 = s\pi + 2$ so that the function is smooth.

Lemma 2.8.2 ([Zha19, Section 8.1]). *There exist choices of all remaining data such that the differential of* (2.8.4) *is given by*[37]

$$d(c_k, x_{2j-1}) = (c_k, x_{2j-2}) - j(c_{k+1}, x_{2j}), \qquad d(c_k, x_{2j}) = 0 \tag{2.8.5}$$

and the continuation map $\kappa_s : EFC_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{s+1}_{\text{quadratic}}) \to EFC_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{(s+1)+1}_{\text{quadratic}})$ *is the inclusion map on the cochain complex.*

Using this explicit cochain complex, we deduce that the inclusion[38] map $\mathbb{Z}[\mathbf{u}]\langle x_{2s}\rangle = \bigoplus_k \mathbb{Z}\langle (c_k, x_{2s})\rangle \hookrightarrow EFC_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{s+1}_{\text{quadratic}})$ induces an isomorphism on cohomology. Moreover, with respect to this isomorphism, the continuation map κ_s is the map $x_{2s} \mapsto (s+1)\mathbf{u} \cdot x_{2(s+1)}$. Explicitly, we have

$$\begin{array}{ccc} EFH_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{s+1}_{\text{quadratic}}) & \xrightarrow{\kappa_s} & EFH_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{(s+1)+1}_{\text{quadratic}}) \\ \cong\uparrow & & \cong\uparrow \\ \mathbb{Z}[\mathbf{u}]\langle x_{2s}\rangle & \xrightarrow[x_{2s} \mapsto (s+1)\mathbf{u} \cdot x_{2(s+1)}]{} & \mathbb{Z}[\mathbf{u}]\langle x_{2(s+1)}\rangle \end{array} \tag{2.8.6}$$

so that the map κ_s is really multiplication by $(s+1)\mathbf{u}$.

To compute the equivariant symplectic cohomology of $\mathbb{C}$, it is enough to consider the direct limits of the continuation maps κ_s because the slopes of $H^{s+1}_{\text{quadratic}}$ are arbitrarily large. The map κ_s is multiplication by $(s+1)\mathbf{u}$, so it contributes to “allowing division” by $(s+1)\mathbf{u}$ in the direct limit. Thus we get the isomorphism

$$ESH_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}) = \mathbb{Q}[\mathbf{u}, \mathbf{u}^{-1}]. \tag{2.8.7}$$

Proof of Theorem 2.8.1. Fix $r \geq 0$. Consider the following diagram, which is a combination of (2.5.7) and (2.7.3). The top square commutes on the cochain complexes whereas the bottom square commutes on cohomology.

$$\begin{array}{ccc} EFC_{\sigma_0^{2r}}(\mathbb{C}; H^{r+1}_{\text{quadratic}}) & \xrightarrow{\kappa_r} & EFC_{\sigma_0^{2r}}(\mathbb{C}; H^{(r+1)+1}_{\text{quadratic}}) \\ \cong\downarrow EFS_{\sim r+1} & & \cong\downarrow EFS_{\sim r+1} \\ EFC^{+2}_{\sigma_{(r+1)}}(\mathbb{C}; (\sigma^{r+1})^* H^{r+1}_{\text{quadratic}}) & \xrightarrow{\kappa'_{(r+1)}} & EFC^{+2}_{\sigma_{(r+1)}}(\mathbb{C}; (\sigma^{r+1})^* H^{(r+1)+1}_{\text{quadratic}}) \\ \cong\uparrow EFS_{\sim} & & \\ EFC_{\sigma_r}(\mathbb{C}; (\sigma^r)^* H^{r+1}_{\text{quadratic}}) & & \downarrow EPSS^{-}_{\sigma_{(r+1)}} \\ \uparrow EPSS^{+}_{\sigma_r} & & \\ EQC_{\sigma_r}(\mathbb{C}) & \xrightarrow{EQS_{\sim}} & EQC^{+2}_{\sigma_{(r+1)}}(\mathbb{C}) \end{array} \tag{2.8.8}$$

[37] Zhao derived the equation $d(c_k, x_{2j-1}) = (c_k, x_{2j-2}) + j(c_{k+1}, x_{2j})$, which has different signs to (2.8.5). Her result changes to (2.8.5) once we apply the rule $\mathbf{u} \mapsto -\mathbf{u}$ to account for our different conventions, as in Remark 2.4.3.

[38] The equation $\mathbf{u} \wedge (c_k, x_{2s}) = (c_{k+1}, x_{2s})$ holds in $EFC_{\mathrm{Id}_{\mathbb{C}}}(\mathbb{C}; H^{s+1}_{\text{quadratic}})$ for degree reasons, so the inclusion is a $\mathbb{Z}[\mathbf{u}]$-module map for the geometric module structure.

Here, we use the subscript on $EPSS$ to record the underlying circle action so we can distinguish the different maps. Moreover, we use the identity $(\sigma^r)^* \mathrm{Id}_{\mathbb{C}} = \sigma'^{\,r}$ to simplify notation.

An inspection of Zhao's explicit perturbation finds that there is[39] $\varepsilon \in \{\pm 1\}$ such that $EPSS^+_{\sigma'^s}(0_{\mathbb{C}}) = \varepsilon\,(\sigma^s)^* x_{2s}$ for all $s \geq 0$ because the asymptotic behaviour of a spiked disc is the same for all these maps (see [Rit13, Appendix B] for a characterisation of the coherent orientation).

Thus, in cohomology, the element $0_{\mathbb{C}}$ is mapped in (2.8.8) as below.

$$
\begin{array}{ccc}
\varepsilon\, x_{2r} & \longmapsto & \varepsilon\,(r+1)\mathbf{u}\, x_{2(r+1)} \\
\uparrow & & \downarrow \\
\varepsilon\,(\sigma^{r+1})^* x_{2r} & \longmapsto & \varepsilon\,(r+1)\mathbf{u}\,(\sigma^{r+1})^* x_{2(r+1)} \\
\uparrow & & \downarrow \\
\varepsilon\,(\sigma^{r})^* x_{2r} & & \\
\uparrow & & \downarrow \\
0_{\mathbb{C}} & \longmapsto & \varepsilon^2\,(r+1)\mathbf{u}\, 0_{\mathbb{C}}
\end{array}
\tag{2.8.9}
$$

Thus we have $0_{\mathbb{C}} \mapsto (r+1)\mathbf{u}\, 0_{\mathbb{C}}$ as desired. □

This result generalises to $\mathbb{C}^n$ and the action $\theta \cdot (z_1, \ldots, z_n) = (e^{2\pi i\theta} z_1, \ldots, e^{2\pi i\theta} z_n)$. This action σ also has exactly one fixed point, $0_{\mathbb{C}^n}$. There exist analogous data to describe the equivariant cohomology with $0_{\mathbb{C}^n}$ the unique minimal critical point. Since $\mathbb{C}^n$ is exact and equivariantly contractible, the equivariant quantum product is trivial.

Theorem 2.8.3. *Let $r \geq 0$. The equivariant quantum Seidel map $EQS_{\tilde{\sigma}^r} : EH^*_{r}(\mathbb{C}^n) \to EH^{*+2n}_{(r+1)}(\mathbb{C}^n)$ is the $\mathbb{Z}[\mathbf{u}]$-linear extension of the assignment $0_{\mathbb{C}^n} \mapsto ((r+1)\mathbf{u})^n\, 0_{\mathbb{C}^n}$.*

Proof. A similar approach to the calculation for $\mathbb{C}$, using the spectral sequences from [MR18, Corollary 7.2] to deduce the differential, yields the desired formula, however only up to sign. Instead, we will derive the $\mathbb{C}^n$ case directly from Theorem 2.8.1.

By Remark 2.6.4, the only holomorphic section of the clutching bundle is the minimal fixed section at $0_{\mathbb{C}^n}$. It follows that $EQS_{\tilde{\sigma}}(0_{\mathbb{C}^n})$ may be characterised by intersecting equivariant flowlines with the fixed point $0_{\mathbb{C}^n}$. Thus $EQS_{\tilde{\sigma}}(0_{\mathbb{C}^n}) = A^{0_{\mathbb{C}^n}}_{(r+1)}(0_{\mathbb{C}^n})$, where we define the map A below. The theorem immediately follows from (2.8.11), which allows us to express $A^{0_{\mathbb{C}^n}}_{(r+1)}(0_{\mathbb{C}^n})$ as the n-th power of $A^{0_{\mathbb{C}}}_{(r+1)}(0_{\mathbb{C}})$, and the computation in $\mathbb{C}$ from Theorem 2.8.1.

[39] This ε will depend on the choice of coherent orientation that was made in Lemma 2.8.2.

Let X be a manifold with a smooth circle action φ and fixed point $p \in X$. Equip X with an equivariant Morse function. The map $A^p_\varphi : EH^*_\varphi(X) \to EH^{*+\dim(X)}_\varphi(X)$ counts equivariant s-dependent negative gradient trajectories that intersect p at $s = 0$. Notice that, since p is a fixed point, this intersection condition is independent of the representative of the equivariant trajectory.

Given two such manifolds X and Y with circle actions φ_X and φ_Y and fixed points p_X and p_Y respectively, consider the following diagram.

$$\begin{array}{ccc} S^1 \times_{S^1} (X \times Y) & \xrightarrow{\pi_X} & S^1 \times_{S^1} X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow \\ S^1 \times_{S^1} Y & \longrightarrow & S^1/S^1 \end{array} \tag{2.8.10}$$

Standard homotopies yield the equation

$$A^{(p_X,p_Y)}_{(\varphi_X,\varphi_Y)}(\pi_X^* \smile \pi_Y^*) = (\pi_X^* A^{p_X}_{\varphi_X}) \smile (\pi_Y^* A^{p_Y}_{\varphi_Y}). \tag{2.8.11}$$

□

2.8.2 Projective space

The complex projective space $\mathbb{P}^n$ is a closed monotone Kähler manifold with the Fubini-Study symplectic form ω_{FS} and the Fubini-Study metric. Its Novikov ring is $\Lambda = \mathbb{Z}[q, q^{-1}]$, where q is a formal variable of degree $2(n+1)$. The standard Morse function on $\mathbb{P}^n$ is the function $f_{\mathbb{P}^n}([z_0 : \cdots : z_n]) = \sum_{k=0}^{n} k|z_k|^2$. The critical points of $f_{\mathbb{P}^n}$ are the unit vectors[40] e_k, each with Morse index $\mathrm{ind}(e_k; f_{\mathbb{P}^n}) = 2k$. The Hamiltonian circle action σ given by $\theta \cdot [z_0 : z_1 : \cdots : z_n] = [z_0 : e^{2\pi i\theta} z_1 : \cdots : e^{2\pi i n\theta} z_n]$ preserves the metric and the function $f_{\mathbb{P}^n}$. As per Section 2.8.1, this means we can form a canonical equivariant Morse function $f^{eq}_{\mathbb{P}^n}$ from $f_{\mathbb{P}^n}$.

We use the Morse functions $f_{\mathbb{P}^n}$ and $f^{eq}_{\mathbb{P}_n}$ to describe a basis for the various cohomologies below. In (2.8.12), we give module isomorphisms[41] to each of the cohomologies, and describe the corresponding products[42] (which are determined by the given information). The global

[40] Give $W^u(e_k)$ the orientation that comes naturally from the complex structure.

[41] As for $\mathbb{C}$, we have $\mathbf{u} \wedge (c_l; e_k) = (c_{l+1}; e_k)$, so we use the shorthand $\mathbf{u}^l e_k$ for the equivariant critical point $(c_l; e_k)$.

[42] For projective space, there is a $\mathbb{Z}[\mathbf{u}]$-module isomorphism $EH^*_\sigma(\mathbb{P}^n) = \mathbb{Z}[\mathbf{u}] \otimes H^*(\mathbb{P}^n)$, which is a version of (2.1.3) with integral coefficients. The decomposition is not natural, however. As we see in (2.8.12), even the equivariant cup product does not respect this decomposition.

minimum e_0 is the unit for all products. In the following, r is any nonnegative integer.

$$\begin{aligned}
H^*(\mathbb{P}^n) &= \mathbb{Z}\langle e_k\rangle_{k=0}^n, & e_1 \smile e_k &= \begin{cases} e_{k+1} & 0 < k < n, \\ 0 & k = n. \end{cases} \\
EH^*_r(\mathbb{P}^n) &= \mathbb{Z}[\mathbf{u}]\langle e_k\rangle_{k=0}^n, & e_1 \smile_r e_k &= \begin{cases} e_{k+1} - r\mathbf{u}e_k & 0 < k < n, \\ -r\mathbf{u}e_n & k = n. \end{cases} \\
QH^*(\mathbb{P}^n) &= \Lambda\langle e_k\rangle_{k=0}^n, & e_1 * e_k &= \begin{cases} e_{k+1} & 0 < k < n, \\ qe_0 & k = n. \end{cases} \\
EQH^*_r(\mathbb{P}^n) &= \hat{\Lambda}\mathbb{Z}[\mathbf{u}]\langle e_k\rangle_{k=0}^n, & e_1 *_r e_k &= \begin{cases} e_{k+1} - r\mathbf{u}e_k & 0 < k < n, \\ qe_0 - r\mathbf{u}e_n & k = n. \end{cases}
\end{aligned} \tag{2.8.12}$$

Let $\tilde{\sigma}$ be the unique lift of σ to $\widetilde{L\mathbb{P}^n}$ which fixes the point $(S^1 \mapsto e_0, D \mapsto e_0)$. It has Maslov index $2n$. For $r \geq 0$, the equivariant quantum Seidel map $EQS_{\tilde{\sigma}} : EQH^*_r(\mathbb{P}^n) \to EQH^{*+2n}_{(r+1)}(\mathbb{P}^n)$ is given by

$$\begin{cases} e_0 \mapsto \displaystyle\sum_{l=0}^{n} (r+1)^{n-l}\mathbf{u}^{n-l} e_l \\ e_k \mapsto \displaystyle\sum_{l=0}^{k-1} (r+1)^{k-1-l} q\mathbf{u}^{k-1-l} e_l. \end{cases} \tag{2.8.13}$$

Proof. Our method of determining the equivariant coefficients in (2.8.12) and (2.8.13) is to find certain coefficients directly and to deduce the remaining coefficients by repeated application of the intertwining formula (2.7.5). We demonstrate our method for $\mathbb{P}^2$.

The non-equivariant products and non-equivariant quantum Seidel maps are known for $\mathbb{P}^2$ (for example, by [MT06, (5.13)]), hence we can write the equivariant quantum products, using unknown integer coefficients, as

$$e_1 *_r e_1 = e_2 + \alpha_r \mathbf{u} e_1 + \beta_r \mathbf{u}^2 e_0 \tag{2.8.14}$$

$$e_1 *_r e_2 = qe_0 + \gamma_r \mathbf{u} e_2 + \delta_r \mathbf{u}^2 e_1 + \varepsilon_r \mathbf{u}^3 e_0 \tag{2.8.15}$$

and the equivariant quantum Seidel map $EQS_{\tilde{\sigma}} : EQH^*_r(\mathbb{P}^2) \to EQH^{*+4}_{(r+1)}(\mathbb{P}^2)$ as

$$\begin{cases} e_0 \mapsto e_2 + A_r\mathbf{u}e_1 + B_r\mathbf{u}^2 e_0 \\ e_1 \mapsto qe_0 + C_r\mathbf{u}e_2 + D_r\mathbf{u}^2 e_1 + E_r\mathbf{u}^3 e_0 \\ e_2 \mapsto qe_1 + F_r q\mathbf{u}e_0 + G_r\mathbf{u}^2 e_2 + H_r\mathbf{u}^3 e_1 + I_r\mathbf{u}^4 e_0. \end{cases} \tag{2.8.16}$$

By Remark 2.6.4, the only holomorphic section which contributes a q^0 term in (2.8.16) is the minimal fixed section at e_0. By using only small perturbations of the invariant Morse function $f_{\mathbb{P}^2}$, we immediately deduce $\beta_r, \delta_r, \varepsilon_r = 0$ and $C_r, D_r, E_r, G_r, H_r, I_r = 0$ (hence fading these terms above) because otherwise the function would increase along its negative gradient trajectories. Moreover, the coefficient B_r is a local count of equivariant trajectories that intersect the fixed point e_0, and is therefore $(r+1)^2$ since this is the same count as in Theorem 2.8.3. Finally, since the Borel space $S^\infty \times_{S^1} \mathbb{P}^2$ decomposes as $\mathbb{CP}^\infty \times \mathbb{P}^2$ in the $r = 0$ case, we have $\alpha_0, \gamma_0 = 0$.

We apply the intertwining relation (2.7.4) with[43] $\alpha = e_1$ and $x = e_k$. By Remark 2.7.1 and an algebraic topology calculation, the α-weighted equivariant quantum Seidel map counts the fixed section at e_0 with weight 0 and any section corresponding to q^1 with weight 1. With $x = e_0$, we get

$$\begin{aligned}
0 &= EQS_{\sim}(e_0 \star_r e_1) - EQS_{\sim}(e_0) \star_{(r+1)} e_1 - \mathbf{u} EQS_{\sim;e_1}(e_0) \\
&= EQS_{\sim}(e_1) - (e_2 + A_r \mathbf{u} e_1 + (r+1)^2 \mathbf{u}^2 e_0) \star_{(r+1)} e_1 - 0 \\
&= q e_0 - (q e_0 + \gamma_{r+1} \mathbf{u} e_2) - A_r \mathbf{u}(e_2 + \alpha_{r+1} \mathbf{u} e_1) - (r+1)^2 \mathbf{u}^2 e_1 \\
&= -(\gamma_{r+1} + A_r)\mathbf{u} e_2 - (A_r \alpha_{r+1} + (r+1)^2)\mathbf{u}^2 e_1,
\end{aligned} \tag{2.8.17}$$

with $x = e_1$, we get

$$\begin{aligned}
0 &= EQS_{\sim}(e_1 \star_r e_1) - EQS_{\sim}(e_1) \star_{(r+1)} e_1 - \mathbf{u} EQS_{\sim;e_1}(e_1) \\
&= EQS_{\sim}(e_2 + \alpha_r \mathbf{u} e_1) - q e_0 \star_{(r+1)} e_1 - \mathbf{u}(q e_0) \\
&= (q e_1 + F_r q \mathbf{u} e_0) + \alpha_r \mathbf{u}(q e_0) - q e_1 - q \mathbf{u} e_0 \\
&= (F_r + \alpha_r - 1) q \mathbf{u} e_0,
\end{aligned} \tag{2.8.18}$$

and with $x = e_2$, we get

$$\begin{aligned}
0 &= EQS_{\sim}(e_2 \star_r e_1) - EQS_{\sim}(e_2) \star_{(r+1)} e_1 - \mathbf{u} EQS_{\sim;e_1}(e_2) \\
&= EQS_{\sim}(q e_0 + \gamma_r \mathbf{u} e_2) - (q e_1 + F_r q \mathbf{u} e_0) \star_{(r+1)} e_1 - \mathbf{u}(q e_1 + F_r q \mathbf{u} e_0) \\
&= q(e_2 + A_r \mathbf{u} e_1 + (r+1)^2 \mathbf{u}^2 e_0) + \gamma_r \mathbf{u}(q e_1 + F_r \mathbf{u} q e_0) \\
&\qquad - q(e_2 + \alpha_{r+1} \mathbf{u} e_1) - F_r q \mathbf{u} e_1 - q \mathbf{u} e_1 - F_r q \mathbf{u}^2 e_0 \\
&= (A_r + \gamma_r - \alpha_{r+1} - F_r - 1) q \mathbf{u} e_1 + ((r+1)^2 + \gamma_r F_r - F_r) q \mathbf{u}^2 e_0.
\end{aligned} \tag{2.8.19}$$

The coefficients of (2.8.17), (2.8.18) and (2.8.19) yield the following silmultaneous equations.

$$\gamma_{r+1} = -A_r \tag{2.8.20}$$

$$A_r \alpha_{r+1} = -(r+1)^2 \tag{2.8.21}$$

$$F_r = -\alpha_r + 1 \tag{2.8.22}$$

$$\alpha_{r+1} - A_r = \gamma_r - 1 - F_r \tag{2.8.23}$$

$$(\gamma_r - 1) F_r = -(r+1)^2 \tag{2.8.24}$$

[43] There is a degree-2 equivariant cohomology class $\boldsymbol{\alpha}$ in the clutching bundle which restricts to e_1 at both poles.

By induction, set $\alpha_r, \gamma_r = -r$. Either (2.8.22) or (2.8.24) yields $F_r = r+1$. The unique solution to (2.8.21) and (2.8.23) is $\alpha_{r+1} = -(r+1)$ and $A_r = r+1$. Finally, (2.8.20) yields $\gamma_{r+1} = -(r+1)$. This proves the induction hypothesis $\alpha_{r+1}, \gamma_{r+1} = -(r+1)$ and hence completes the proof. □

Remark 2.8.4 (Inverse action). The inverse circle action σ^{-1} is another Hamiltonian circle action for which we can compute the equivariant quantum Seidel map. We have $EQS_{\tilde\sigma} EQS_{\tilde\sigma^{-1}} = EQS_{\tilde\sigma\tilde\sigma^{-1}} = \mathrm{Id}$, so we can deduce the map $EQS_{\tilde\sigma^{-1}}$ by inverting (2.8.13). Thus the map $EQS_{\tilde\sigma^{-1}} : EQH^{*}_{-(r+1)}(\mathbb{P}^n) \to EQH^{*+2n}_{-r}(\mathbb{P}^n)$ is given by

$$\begin{cases} e_0 \mapsto q^{-1} e_1, & \\ e_k \mapsto -(r+1)\mathbf{u} q^{-1} e_k + q^{-1} e_{k+1}, & 1 \leq k < n \\ e_n \mapsto -(r+1)\mathbf{u} q^{-1} e_n + e_0. & \end{cases} \tag{2.8.25}$$

The assignment $e_0 \mapsto q^{-1} e_1$ may be deduced directly via minimal fixed sections using Remark 2.6.4. Here, e_1 is the Poincaré dual of the subset $\{z_0 = 0\}$, which is the minimal locus of $-K$. No sections other than these minimal fixed sections can contribute to $EQS_{\tilde\sigma^{-1}}(e_0)$ for degree reasons.

2.8.3 Tautological line bundle on projective space

The total space $\mathcal{O}_{\mathbb{P}^n}(-1)$ of the tautological line bundle over projective space $\mathbb{P}^n$ is one of the negative line bundles studied by Ritter [Rit14]. The elements of $\mathcal{O}_{\mathbb{P}^n}(-1)$ are of the form $([\mathbf{z}], \mathbf{v}) = ([z_0 : \ldots : z_n], (v_0, \ldots, v_n)) \in \mathbb{P}^n \times \mathbb{C}^{n+1}$, where $\mathbf{z}$ and $\mathbf{v}$ are linearly dependent. Denote by Z the image of the zero section, giving $Z = \{\mathbf{v} = 0\} = \mathbb{P}^n$.

There is a symplectic form ω on $\mathcal{O}_{\mathbb{P}^n}(-1)$ such that $(\mathcal{O}_{\mathbb{P}^n}(-1), \omega)$ is a monotone convex symplectic manifold whose fibres and whose submanifold Z are symplectic submanifolds [Rit14, Section 7]. Its Novikov ring is $\Lambda = \mathbb{Z}[q, q^{-1}]$, where q is a formal variable of degree $2n$.

Let σ be the linear Hamiltonian circle action given by $\theta \cdot ([\mathbf{z}], \mathbf{v}) = ([\mathbf{z}], e^{2\pi i \theta} \mathbf{v})$. It rotates the fibres and its fixed point set is Z. Set $\tilde\sigma$ to be the unique lift of σ which fixes the points $(S^1 \mapsto ([\mathbf{z}], 0), D \mapsto ([\mathbf{z}], 0))$. The Maslov index of $\tilde\sigma$ is 2.

The invariant Morse function given by $f(([\mathbf{z}], \mathbf{v})) = \sum_{k=0}^{n} k|z_k|^2 + |v_k|^2$ has critical points at each of the unit vectors in Z. Denote the k-th such critical point[44] by e_k. It has Morse index $\mathrm{ind}(e_k; f) = 2k$. Give $\mathcal{O}_{\mathbb{P}^n}(-1)$ the metric which is the restriction of the standard metric on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, so that we recover the Fubini-Study metric along the zero section.

[44] Orient $W^u(e_k)$ according to the natural complex structure.

Thus the Morse cohomology of $\mathcal{O}_{\mathbb{P}^n}(-1)$ is

$$H^*(\mathcal{O}_{\mathbb{P}^n}(-1)) = \mathbb{Z}\langle e_k\rangle_{k=0}^{n}, \qquad e_1 \smile e_k = \begin{cases} e_{k+1} & k < n, \\ 0 & k = n. \end{cases} \tag{2.8.26}$$

Since $\mathcal{O}_{\mathbb{P}^n}(-1)$ equivariantly retracts onto Z (with the trivial circle action), its equivariant cohomology $EH^*_r(\mathcal{O}_{\mathbb{P}^n}(-1))$ is ring isomorphic to $\mathbb{Z}[\mathbf{u}] \otimes_{\mathbb{Z}} H^*(\mathcal{O}_{\mathbb{P}^n}(-1))$.

The quantum cohomology is

$$QH^*(\mathcal{O}_{\mathbb{P}^n}(-1)) = \Lambda\langle e_k\rangle_{k=0}^{n}, \qquad e_1 * e_k = \begin{cases} e_{k+1} & k < n, \\ -qe_1 & k = n. \end{cases} \tag{2.8.27}$$

The quantum product is deduced via (2.6.8) from the the quantum Seidel map which Ritter computed [Rit14, Lemma 60]:

$$QS_{\sim}(e_k) = \begin{cases} -e_{k+1} & k < n, \\ qe_1 & k = n. \end{cases} \tag{2.8.28}$$

The equivariant quantum cohomology is $\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]\langle e_k\rangle_{k=0}^{n}$. The equivariant quantum product and equivariant quantum Seidel maps are given by the following theorems.

Theorem 2.8.5. *For $r \geq 0$, the equivariant quantum product on $EQH^*_r(\mathcal{O}_{\mathbb{P}^n}(-1))$ is given by*

$$e_1 \star_r e_k = \begin{cases} e_{k+1} & k < n, \\ -qe_1 + r\mathbf{u}qe_0 & k = n. \end{cases} \tag{2.8.29}$$

Theorem 2.8.6. *For $r \geq 0$, the equivariant quantum Seidel map*

$$EQS_{\sim} : EQH^*_r(\mathcal{O}_{\mathbb{P}^n}(-1)) \to EQH^{*+2}_{(r+1)}(\mathcal{O}_{\mathbb{P}^n}(-1)) \tag{2.8.30}$$

is given by

$$e_k \mapsto \begin{cases} -e_{k+1} + (r+1)\mathbf{u}e_k & k < n, \\ qe_1 + (r+1)\mathbf{u}e_n - (r+1)\mathbf{u}qe_0 & k = n. \end{cases} \tag{2.8.31}$$

Proof of Theorem 2.8.5 and Theorem 2.8.6. We use exactly the same method as that we used in Section 2.8.2. Write the equivariant quantum products and equivariant quantum Seidel maps using variables for the unknown coefficients. By using only a small perturbation of the Morse function, deduce that all coefficients are zero apart from those which occur above. The coefficients of the $\mathbf{u}e_k$ terms and the $\mathbf{u}e_n$ term in (2.8.31) are all $(r+1)$ because the fibre locally resembles $\mathbb{C}$. Apply the intertwining formula (2.7.5) to the elements e_{n-1} and e_n to deduce the two remaining coefficients. □

2.8.3.1 Deducing equivariant symplectic cohomology

The symplectic cohomology is the limit of the quantum Seidel map (2.8.28). We have $QS_{\tilde{}}(e_n + qe_0) = 0$, and $QS_{\tilde{}}$ is an isomorphism after quotienting by $e_n + qe_0$. Thus we get Λ-algebra and Λ-module isomorphisms

$$SH^*(\mathcal{O}_{\mathbb{P}^n}(-1)) \underset{\text{alg.}}{\cong} \frac{\mathbb{Z}[q, q^{-1}]\langle e_k\rangle_{k=0}^n}{(e_n + qe_0)} \underset{\text{mod.}}{\cong} \bigoplus_{k=0}^{n-1} \mathbb{Z}[q, q^{-1}]\langle e_k\rangle. \tag{2.8.32}$$

In particular, in every even degree it has one copy of $\mathbb{Z}$.

The equivariant quantum Seidel map $EQS_{\tilde{}}$ from (2.8.30) is injective, and its determinant is $\det EQS_{\tilde{}} = (r+1)^{n+1}\mathbf{u}^{n+1}$. To find the direct limit of the maps $EQS_{\tilde{}}$, we must use a different strategy.

We localise the ring $\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]$ so the determinants are invertible. We denote this localisation by $\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]_{\text{local}}$. The degree-$l$ elements in $\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]_{\text{local}}$ are given as per the graded completed tensor product (2.6.14), except finitely-many negative powers of $\mathbf{u}$ are permitted, and we tensor with $\mathbb{Q}$. Denote by $EQH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))_{\text{local}}$ the tensor product $EQH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \left(\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]_{\text{local}}\right)$.

Consider the following commutative diagram.

$$\begin{array}{ccccc} EQH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1)) & \xrightarrow{EQS_{\tilde{}}} & EQH^{*+2}_1(\mathcal{O}_{\mathbb{P}^n}(-1)) & \xrightarrow{EQS_{\tilde{}}} & EQH^{*+2}_2(\mathcal{O}_{\mathbb{P}^n}(-1)) \\ \downarrow & \swarrow_{EQS_{\tilde{}}^{-1}} & & & \\ EQH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))_{\text{local}} & & \xleftarrow{EQS_{\tilde{}}^{-1} \circ EQS_{\tilde{}}^{-1}} & & \end{array} \tag{2.8.33}$$

The direct limit of the maps $EQS_{\tilde{}}$ is the image of the (injective) dashed maps above. This gives

$$ESH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1)) = \bigcup_{p=0}^{\infty} \left\{ \text{image}\left((EQS_{\tilde{}}^{-1})^p : EQH^{*+2p}_p \to EQH^*_{0;\,\text{local}} \right) \right\}. \tag{2.8.34}$$

We immediately deduce that $ESH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))$ is not a finitely-generated $\Lambda \hat{\otimes} \mathbb{Z}[\mathbf{u}]$-module because this is a strictly increasing chain of submodules. Moreover, we can follow the element e_n under the maps $EQS_{\tilde{}}$ to get

$$\begin{aligned} e_n \mapsto qe_1 + o(\mathbf{u}) \mapsto qe_2 + o(\mathbf{u}) \mapsto \cdots \mapsto qe_n + o(\mathbf{u}) \\ \mapsto q^2 e_1 + o(\mathbf{u}) \mapsto \cdots, \end{aligned} \tag{2.8.35}$$

which implies that e_n is not divisible by $\mathbf{u}$ in the direct limit (for none of the images are divisible by $\mathbf{u}$). Therefore $ESH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))$ is a proper submodule of the localisation.

2.8.3.2 Finding generators

Recall the *adjugate* (or *adjoint*) of a nonsingular matrix A is the unique matrix A^* such that $A^*A = AA^* = \det(A)\mathrm{Id}$, so that if the inverse of A exists, it is $A^{-1} = \frac{1}{\det(A)}A^*$. Therefore, to find the image of the inverses $(EQS_{\sim}^{-1})^p$, we find the image of the adjugates (which is a submodule of $EQH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))$ without localisation), and then divide these elements by the determinants.

We have $\det(EQS_{\sim}^p) = \prod_{r=0}^{p-1}((r+1)^{n+1}\mathbf{u}^{n+1})$. We characterise the image of the adjugate below.

Using (2.8.31), we have $EQS_{\sim}(e_n + qe_0) = (r+1)\mathbf{u}e_n$. Thus, with respect to the ordered basis $\langle e_n + qe_0, e_1, e_2, \ldots, e_{n-1}, e_n\rangle$, the map $EQS_{\sim}$ is given by the following matrix.

$$A_r = \begin{pmatrix} 0 & 0 & 0 & & 0 & (r+1)\mathbf{u} \\ 0 & (r+1)\mathbf{u} & 0 & & 0 & q \\ 0 & 1 & (r+1)\mathbf{u} & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & (r+1)\mathbf{u} & 0 \\ -(r+1)\mathbf{u} & 0 & 0 & & 1 & 2(r+1)\mathbf{u} \end{pmatrix} \tag{2.8.36}$$

Note that if we permute the first and last column, the matrix (2.8.36) becomes lower triangular. Set $x_k^p = A_0^* \cdots A_{p-1}^* e_k$ for $k = 1, \ldots, n$, and set $x_0^p = A_0^* \cdots A_{p-1}^*(e_n + qe_0)$. Thus the image of $(EQS_{\sim})^p$ above is the span of $\{x_0^p, \ldots, x_n^p\}$. Using (2.8.36), we can set up a recursive formula for the x_k^p, which starts like

$$\begin{cases} x_n^{p+1} = (p+1)^n\mathbf{u}^n x_0^p \\ x_{n-1}^{p+1} = (p+1)^n\mathbf{u}^n x_{n-1}^p + (p+1)^{n-1}\mathbf{u}^{n-1}x_0^p \\ \ldots \end{cases} \tag{2.8.37}$$

Example 2.8.7 ($n = 1$). For $\mathcal{O}_{\mathbb{P}^1}(-1)$, the system (2.8.37) becomes

$$\begin{cases} x_1^{p+1} = (p+1)\mathbf{u}x_0^p \\ x_0^{p+1} = (q + 2(p+1)\mathbf{u})x_0^p - (p+1)\mathbf{u}x_1^p \end{cases} \tag{2.8.38}$$

and has solution $x_0^p = q^{p-1}((q + p(p+1)\mathbf{u})e_0 - \mathbf{u}e_1) + o(\mathbf{u}^2)$. Since $x_1^{p+1} = (p+1)\mathbf{u}x_0^p$, one copy of $(p+1)\mathbf{u}$ cancels upon division by the determinant. This yields the presentation

$$ESH^*_0(\mathcal{O}_{\mathbb{P}^1}(-1)) = \widehat{\mathbb{Z}[\mathbf{u}]}\left\langle \frac{x_0^p}{1^2 \cdots p^2(p+1)\mathbf{u}^{2p+1}} \right\rangle_{p=1}^{\infty}. \tag{2.8.39}$$

Remark 2.8.8 (Possible nicer presentation). The author was unable to establish whether there is an element $X \in ESH^*_0(\mathcal{O}_{\mathbb{P}^n}(-1))$ which is divisible by every power of $\mathbf{u}$, or indeed by every determinant. If so, this would produce an isomorphism

$$ESH^*_0(\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \widehat{\mathbb{Z}[\mathbf{u}]} \oplus \widehat{\mathbb{Z}[\mathbf{u}]}_{\mathrm{local}} \tag{2.8.40}$$

in the $n = 1$ case, and similar isomorphisms for $n > 1$.

Chapter 3

Shift operators and flat connections on equivariant symplectic cohomology

3.1 Introduction

3.1.1 Flat connections on S^1-equivariant symplectic cohomology

The S^1-equivariant symplectic cohomology $SH_{S^1}(M)$ of a convex symplectic manifold M is a $\mathbb{Z}[\mathbf{u}]$-module invariant introduced by Viterbo [Vit96, Section 5], and later developed by Seidel [Sei08, Section 8b] and Bourgeois and Oancea [BO17] among others. It builds on symplectic cohomology by incorporating the natural S^1-action on the loop space $\mathcal{L}M = \{\text{contractible } x : S^1 \to M\}$ given by $(\theta \cdot x)(t) = x(t - \theta)$ for $\theta \in S^1$. This S^1-action readily distinguishes constant and nonconstant Hamiltonian orbits by their stabilizer groups. The constant orbits capture topological information about M via a localisation theorem [Zha19, Theorem 1.1], while the nonconstant orbits give rise to the *positive S^1-equivariant symplectic cohomology* $SH^{*,+}_{S^1}(M)$ of M, which is an effective invariant for distinguishing contact structures [BO17; Gut17, Theorem 1.4]. McLean and Ritter used S^1-equivariant symplectic cohomology to deduce the cohomology of a crepant resolution of an isolated singularity by analysing the nonconstant orbits, establishing a new proof of the McKay correspondance [MR18].

The S^1-action breaks the definition of the pair-of-pants product. Equipped with this product, symplectic cohomology is a graded-commutative, associative and unital algebra, and in fact has a full TQFT structure [Rit13; Sei08, Section (8a)]. Without the pair-of-pants product, $SH_{S^1}(M)$ has only its module structure, a poor offering compared to the assortment of algebraic structures available on symplectic cohomology.

To remedy this, Seidel defined the *q-connection*, which is an additive endomorphism

$$\nabla_q : FH_{S^1}(M, \lambda) \to FH^{*+2}_{S^1}(M, \lambda) \tag{3.1.1}$$

of the S^1-equivariant Floer cohomology of a Hamiltonian function with slope λ [Sei18, Section (2a)]. The endomorphism satisfies the Leibniz rule

$$\nabla_q(fx) = f\nabla_q(x) + \mathbf{u}(\partial_q f)x \tag{3.1.2}$$

for $f \in \Lambda[\![\mathbf{u}]\!]$ and $x \in FH^*_{S^1}(M,\lambda)$. Under Seidel's assumptions, the Novikov ring Λ is a ring of formal power series in q and $\partial_q : \Lambda[\![\mathbf{u}]\!] \to \Lambda[\![\mathbf{u}]\!]$ is the operation which differentiates with respect to q.

Seidel's q-connection is an example of a *differential connection*, an algebraic structure that abstracts the algebraic properties of a connection on a vector bundle (Section 3.2.1). The existence of a map satisfying (3.1.2) indicates that the $\Lambda[\![\mathbf{u}]\!]$-module $FH^*_{S^1}(M,\lambda)$ respects the differentiation operation $\mathbf{u}\partial_q : \Lambda[\![\mathbf{u}]\!] \to \Lambda[\![\mathbf{u}]\!]$ in the natural way. As such, the map ∇_q upgrades the underlying $\Lambda[\![\mathbf{u}]\!]$-module structure on $FH^*_{S^1}(M,\lambda)$. Differential connections are useful when working with maps which preserve the connection. For example, when we compute shift operators in Section 3.5, the shift operators preserve the connection and we can determine each shift operator by its value on a single element. Compare this to how a linear map is determined by its values on a basis.

The operation $\mathbf{u}\partial_q$ on the S^1-equivariant Floer cochain complex is not a cochain map, so a correction term is required when defining ∇_q. We can understand this correction term by reducing to non-equivariant cohomology. We have the following commutative diagram.

$$\begin{array}{ccc} FH^*_{S^1}(M,\lambda) & \xrightarrow{\nabla_q} & FH^{*+2}_{S^1}(M,\lambda) \\ \big\downarrow{\scriptstyle \mathbf{u}\mapsto 0} & & \big\downarrow{\scriptstyle \mathbf{u}\mapsto 0} \\ FH^*(M,\lambda) & \xrightarrow{q^{-1}[\omega]*} & FH^{*+2}(M,\lambda) \end{array} \tag{3.1.3}$$

The quantum action $[\omega]*$ by the cohomology class $[\omega] \in H^2(M;\mathbb{R})$ counts Floer solutions $u : \mathbb{R} \times S^1 \to M$ which intersect a locally-finite cycle representing $[\omega]$ at $u(0,0)$. The map $q^{-1}[\omega]*$ is morally similar to performing ∂_q, but 'on the Floer solutions': for the operation ∂_q, we first multiply by the exponent of q (which records the symplectic energy $[\omega](A)$ of classes $A \in H_2(M)$), and then we divide by q.

In this paper, we define a generalisation of Seidel's q-connection for any degree-2 cohomology class.

Theorem 3.1.1. *For every $\alpha \in H^2(M;\mathbb{Z})$, there is a $\mathbb{Z}[\mathbf{u}]$-module endomorphism*

$$\nabla_\alpha : FH^*_{S^1}(M,\lambda) \to FH^{*+2}_{S^1}(M,\lambda) \tag{3.1.4}$$

which satisfies the Leibniz rule

$$\nabla_\alpha(fx) = f\nabla_\alpha(x) + \mathbf{u}\left(\frac{d}{d\alpha}f\right)x \tag{3.1.5}$$

and makes the following diagram commute.

$$\begin{array}{ccc} FH^*_{S^1}(M,\lambda) & \xrightarrow{\nabla_\alpha} & FH^{*+2}_{S^1}(M,\lambda) \\ \downarrow{\scriptstyle \mathbf{u}\mapsto 0} & & \downarrow{\scriptstyle \mathbf{u}\mapsto 0} \\ FH^*(M,\lambda) & \xrightarrow{\quad} & FH^{*+2}(M,\lambda) \end{array} \tag{3.1.6}$$

These maps commute with continuation maps, and hence they induce $\mathbb{Z}[\mathbf{u}]$*-module endomorphisms*

$$\nabla_\alpha : SH^*_{S^1}(M) \to SH^{*+2}_{S^1}(M) \tag{3.1.7}$$

on S^1*-equivariant symplectic cohomology which satisfy* (3.1.5).

Our setup differs from Seidel's construction in two important ways. First, instead of Seidel's formal $\mathbb{Z}[\mathbf{u}]$-module structure, our $\mathbb{Z}[\mathbf{u}]$-module structure arises from a Morse cup product construction in the classifying space $BS^1 = \mathbb{CP}^\infty$, as in our previous paper [LJ20, Section 4.2.2]. This uses the identification $\mathbb{Z}[\mathbf{u}] = H^*(BS^1)$. Second, our Novikov ring records classes $A \in H_2(M;\mathbb{Z})$, unlike Seidel's Novikov ring which records only the symplectic energy $[\omega](A)$. The operation $\frac{d}{d\alpha}$ is given by

$$\frac{d}{d\alpha}(q^A) = \alpha(A)q^A. \tag{3.1.8}$$

Notice this operation does not change the exponent of q, unlike normal differentiation.

For any $\alpha, \beta \in H^2(M)$, the two operations $\frac{d}{d\alpha}$ and $\frac{d}{d\beta}$ commute. We show that the corresponding maps ∇_α and ∇_β also commute. This condition is called *flatness*, continuing the analogy with connections on a vector bundle (see Example 3.2.3). The result was anticipated by Seidel [Sei18, Section (2a)], but is new.

Theorem 3.1.2 (Flatness). *The maps* ∇_α *and* ∇_β *commute for any* $\alpha, \beta \in H^2(M)$.

Like Seidel's q-connection, our connection ∇_α can be viewed as a Floer-theoretic analogue of a Dubrovin connection on S^1-equivariant quantum cohomology $QH^*_{S^1}(M)$ which differentiates with respect to the Novikov variable. Here, S^1 acts trivially on M. The connection is given by

$$\begin{aligned} \nabla_\alpha &: QH^*_{S^1}(M) \to QH^{*+2}_{S^1}(M) \\ \nabla_\alpha(fx) &= \mathbf{u}\left(\frac{d}{d\alpha}f\right)x + \alpha \star (fx) \end{aligned} \tag{3.1.9}$$

for $f \in \Lambda$ and $x \in H^*_{S^1}(M)$, using the isomorphism $QH^*_{S^1}(M) = \Lambda \otimes H^*_{S^1}(M)$. For small slopes, our connection (3.1.4) agrees with the connection (3.1.9) under the PSS isomorphism. S^1-equivariant quantum cohomology has many different connections, and all of these connections are flat by a general argument [Dub92]. The flatness corresponds to important geometric properties of the S^1-equivariant quantum product—namely its graded-commutativity and associativity.

Like the connection (3.1.9), our connection on $FH^{*}_{S^1}(M,\lambda)$ is the sum of a formal differentiation operation $\mathbf{u}\frac{d}{d}$ with an operation α . The S^1-equivariant quantum action α counts Floer solutions $u : \mathbb{R} \times S^1 \to M$ which intersect with α at $u(0,t_0)$, where the value of $t_0 \in S^1$ is determined by an equivariant construction on the classifying space BS^1. An additional correction term $\mathbf{u}w^B$ is required because $FC^{*}_{S^1}(M,\lambda)$ does not have a canonical basis unlike quantum cohomology. Roughly speaking, w^B acts like a geometric version of the $\frac{d}{d}$ operator on the choices of cappings of the contractible Hamiltonian orbits that generate the cochain complex. The two terms in (3.1.9) are individually chain maps, whereas in the Floer case it is only the sum $\mathbf{u}\frac{d}{d} + \alpha \quad \mathbf{u}w^B$ which is a chain map on $FC^{*}_{S^1}(M,\lambda)$. The operation α corresponds to the correction term in Seidel's q-connection.

3.1.2 Flat connections on $\widehat{T}$-equivariant symplectic cohomology

In this paper, we work with a Hamiltonian action ρ of the k-dimensional torus T on our convex symplectic manifold M (see Section 3.3.1 and Section 3.3.6 for our full assumptions). In Section 3.5, we work with toric manifolds, for which ρ is an effective Hamiltonian T-action with $k = \frac{1}{2}\dim M$, but our constructions here apply for any k. In particular, the case $k = 0$ was described in Section 3.1.1.

The torus T acts pointwise on the loops in $\mathcal{L}M$ via ρ. Combined with the S^1-action from Section 3.1.1, this yields a natural $S^1 \times T$-action on $\mathcal{L}M$. Denote a copy of S^1 by S^1_0, set $\widehat{T} = S^1_0 \times T$ and denote by $\widehat{\rho}$ the $\widehat{T}$-action on M determined by $\widehat{\rho}j_T = \rho$ and $\widehat{\rho}j_{S^1_0} = \mathrm{Id}_M$. We define $\widehat{T}$-equivariant Floer cohomology $FH^{*}_{\widehat{T}}(M,\widehat{\rho},\lambda)$ similarly to S^1-equivariant Floer cohomology, combining Morse theory on the classifying space $B\widehat{T}$ with Floer theory on M. The $\widehat{T}$-equivariant symplectic cohomology of M is the direct limit of $FH^{*}_{\widehat{T}}(M,\widehat{\rho},\lambda)$ as $\lambda \to \infty$. These are both $H^{*}(B\widehat{T})$-modules. Associated to the projection $\widehat{T} \to S^1_0$ is a special element $\widehat{y}_0 \in H^{*}(B\widehat{T})$, which has similar properties to $\mathbf{u}$ in Section 3.1.1.

Theorem 3.1.3 (Flat connection). *Let $\mu \in M$ be a fixed point of ρ. For every $\alpha \in H^2(M;\mathbb{Z})$, there is a $H^{*}(B\widehat{T})$-module endomorphism*

$$\nabla : FH^{*}_{\widehat{T}}(M,\widehat{\rho},\lambda) \to FH^{*+2}_{\widehat{T}}(M,\widehat{\rho},\lambda) \tag{3.1.10}$$

which satisfies the Leibniz rule

$$\nabla(fx) = f\nabla(x) + \widehat{y}_0\left(\frac{d}{d\alpha}f\right)x \tag{3.1.11}$$

and makes the following diagram commute.

$$\begin{array}{ccc} FH^{*}_{\widehat{T}}(M,\widehat{\rho},\lambda) & \xrightarrow{\nabla} & FH^{*+2}_{\widehat{T}}(M,\widehat{\rho},\lambda) \\ \downarrow & & \downarrow \\ FH^{*}(M,\lambda) & \longrightarrow & FH^{*+2}(M,\lambda) \end{array} \tag{3.1.12}$$

In (3.1.12)*, the vertical maps are induced by the restriction map* $H(B\widehat{T}) \to H(\mathrm{pt})$*. The maps* ∇ *commute with continuation maps, so they induce maps on* $\widehat{T}$*-equivariant symplectic cohomology which satisfy* (3.1.11)*. Moreover, the maps* ∇ *and* ∇ *commute for any* $\alpha, \beta \in H^2(M)$*, so* ∇ *is flat.*

The connection is given by the sum

$$\nabla = \hat{y}_0 \frac{d}{d\alpha} + \alpha \quad \hat{y}_0 w^B. \tag{3.1.13}$$

The fixed point μ is used to lift the class α to a class $\alpha \in H_{\widehat{T}}(M, \rho)$. Note that Theorem 3.1.1 and Theorem 3.1.2 are the $k = 0$ case of Theorem 3.1.3, with μ any point of M.

3.1.3 Equivariant Seidel maps and shift operators

A *cocharacter* of the torus T is a group homomorphism $\sigma : S^1 \to T$. The composition $\rho \ \sigma$ is a Hamiltonian S^1-action on M. This S^1-action naturally induces an automorphism of the loop space $LM = \{x : S^1 \to M\}$ given by $((\rho \ \sigma)(x))(t) = \rho_{(t)}(x(t))$. Seidel defined an isomorphism

$$FS(\sigma, \mu) : FH(M; H) \to FH^{+j\,;\,j}(M; \sigma H) \tag{3.1.14}$$

which corresponds to the pullback by this automorphism $(\rho \ \sigma) : LM \to LM$ [Sei97]. We call this isomorphism the *Floer Seidel map*. Seidel's construction applies to Hamiltonian S^1-actions on closed symplectic manifolds. Ritter extended the definition to *linear* Hamiltonian S^1-actions on convex symplectic manifolds [Rit14], and further generalised the linearity condition in [Rit16]. The Floer Seidel map is an isomorphism of the underlying Floer cochain complexes, it preserves continuation maps, and it intertwines the pair-of-pants product. Moreover, the Floer Seidel map satisfies

$$FS(\sigma, \mu) \quad FS(\sigma', \mu) = FS(\sigma + \sigma', \mu) \tag{3.1.15}$$

for any two cocharacters σ, σ'.

We introduced the S^1*-equivariant Floer Seidel map* $FS_{S^1}(\sigma, \mu)$ in our previous work [LJ20]. This map combines the identity map on BS^1 with the Floer Seidel map on M. Like the non-equivariant Floer Seidel map, $FS_{S^1}(\sigma, \mu)$ is an isomorphism of the underlying equivariant Floer cochain complexes, it preserves continuation maps, and it satisfies (3.1.15). The S^1-action on M changes under the S^1-equivariant Floer Seidel map, in a similar way to how the Hamiltonian H changes to the pullback Hamiltonian $\sigma \ H$.

The S^1-equivariant construction readily extends to the $\widehat{T}$-equivariant setup. We assume the S^1-action $\rho \circ \sigma : S^1 \times M \to M$ satisfies a linearity condition for all cocharacters $\sigma \in \operatorname{Cochar} T$. We define the *$\widehat{T}$-equivariant Floer Seidel map*

$$FS_{\widehat{T}}(\sigma,\mu) : FH^*_{\widehat{T}}(M,\widehat{\rho},\lambda;H) \to FH^{*+|\mu|}_{\widehat{T}}(M,\sigma^*\widehat{\rho},\lambda-\kappa;\sigma^*H) \tag{3.1.16}$$

analogously to the S^1-equivariant Floer Seidel map. The $\widehat{T}$-action $\sigma^*\widehat{\rho}$ is determined by $(\sigma^*\widehat{\rho})|_T = \rho$ and $(\sigma^*\widehat{\rho})|_{S^1_0} = (\rho\circ\sigma)^{-1}$. Let $\widehat{\sigma} : \widehat{T} \to \widehat{T}$ be the automorphism $(a,\mathbf{t}) \mapsto (a,\mathbf{t}+\sigma(a))$, so that $\sigma^*\widehat{\rho}$ is given by[1] $\widehat{\rho}\circ\widehat{\sigma}^{-1}$. By the functorial properties of equivariant cohomology, associated to $\widehat{\sigma}$ is an automorphism $(B\widehat{\sigma})^* : H^*(B\widehat{T}) \to H^*(B\widehat{T})$ and an isomorphism

$$(B\widehat{\sigma})^* : H^*_{\widehat{T}}(M,\sigma^*\widehat{\rho}) \to H^*_{\widehat{T}}(M,\widehat{\rho}). \tag{3.1.17}$$

The map (3.1.17) is not an $H^*(B\widehat{T})$-module homomorphism and instead satisfies

$$(B\widehat{\sigma})^*(fx) = (B\widehat{\sigma})^*(f)\,(B\widehat{\sigma})^*(x) \tag{3.1.18}$$

for $f \in H^*(B\widehat{T})$ and $x \in H^*_{\widehat{T}}(M,\sigma^*\widehat{\rho})$. There is a similar map $(B\widehat{\sigma})^*$ on $\widehat{T}$-equivariant Floer cohomology which satisfies (3.1.18), and it 'undoes' the change of $\widehat{T}$-action in (3.1.16).

If the slope (function) $\kappa_\sigma \geq 0$ of the linear (admissible[2]) action $\rho\circ\sigma$ is nonnegative, then the slope of the pullback Hamiltonian σ^*H, given by $\lambda-\kappa_\sigma$, is smaller than the slope λ of H. Denote by $\operatorname{Cochar}_{\geq 0} T$ the set of cocharacters σ for which $\rho\circ\sigma$ satisfies this nonnegativity condition. It is a commutative monoid inside the lattice of cocharacters $\operatorname{Cochar} T$. For $\sigma \in \operatorname{Cochar}_{\geq 0} T$, the continuation map φ_{κ_σ} is well-defined and 'undoes' the change of slope.

Putting this together, the composition

$$\mathsf{S}_\sigma = \varphi_{\kappa_\sigma}\circ(B\widehat{\sigma})^*\circ FS_{\widehat{T}}(\sigma,\mu) : FH^*_{\widehat{T}}(M,\widehat{\rho},\lambda) \to FH^{*+|\mu|}_{\widehat{T}}(M,\widehat{\rho},\lambda) \tag{3.1.19}$$

is an endomorphism of $\widehat{T}$-equivariant Floer cohomology for all $\sigma \in \operatorname{Cochar}_{\geq 0} T$. The maps $FS_{\widehat{T}}(\sigma,\mu)$ and φ_{κ_σ} are $H^*(B\widehat{T})$-module homomorphisms, but $(B\widehat{\sigma})^*$ satisfies (3.1.18). Therefore S_σ is not a $H^*(B\widehat{T})$-module homomorphism, and instead satisfies

$$\mathsf{S}_\sigma(fx) = (B\widehat{\sigma})^*(f)\,\mathsf{S}_\sigma(x) \tag{3.1.20}$$

for $f \in H^*(B\widehat{T})$.

[1] The map $\operatorname{Cochar} T \to \operatorname{Aut}(\widehat{T})$ given by $\sigma \mapsto \widehat{\sigma}$ is a $(\operatorname{Cochar} T)$-action on $\widehat{T}$. The induced $(\operatorname{Cochar} T)$-action on maps $\widehat{\rho} : \widehat{T} \to \operatorname{Aut}(M)$ naturally precomposes with the *inverse* of the $(\operatorname{Cochar} T)$-action on $\widehat{T}$.

[2] See Section 3.3.6 for our definition of admissibility.

Theorem 3.1.4 (Flat shift operator). *For every cocharacter σ in the commutative monoid* Cochar $^0 T$*, there is a -module endomorphism*

$$\mathrm{S} : FH_{\widehat{T}}(M, \widehat{\rho}, \lambda) \to FH_{\widehat{T}}^{+j \,;\, j}(M, \widehat{\rho}, \lambda) \tag{3.1.21}$$

which satisfies (3.1.20) *and makes the following diagram commute.*

$$\begin{array}{ccc} FH_{\widehat{T}}(M, \widehat{\rho}, \lambda) & \xrightarrow{\mathrm{S}} & FH_{\widehat{T}}^{+j \,;\, j}(M, \widehat{\rho}, \lambda) \\ \downarrow & & \downarrow \\ FH(M, \lambda) & \xrightarrow{\quad FS(\,;\,)} & FH^{+j \,;\, j}(M, \lambda) \end{array} \tag{3.1.22}$$

In (3.1.22)*, the vertical maps are induced by the restriction map $H(B\widehat{T}) \to H(\mathrm{pt})$ and the bottom map is the composition of a (non-equivariant) continuation map with* (3.1.14)*. Moreover, we have* $\mathrm{S}\,\mathrm{S}_{0} = \mathrm{S}_{+\,0}$ *for any two cocharacters $\sigma, \sigma' \in$* Cochar $^0 T$*.*

The map S is called a *shift operator*. The automorphism $\widehat{\sigma} : \widehat{T} \to \widehat{T}$ induces a 'shift' in the lattice of characters of $\widehat{T}$. This 'shift' is represented in the algebra $H(B\widehat{T})$ by the isomorphism $(B\widehat{\sigma})$. The operator S captures this 'shift' in the $H(B\widehat{T})$-module $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ because it satisfies (3.1.20).

The map (3.1.21) commutes with continuation maps, and hence induces a map S on $\widehat{T}$-equivariant symplectic cohomology $SH_{\widehat{T}}(M, \widehat{\rho})$ that satisfies (3.1.20). The continuation map in (3.1.19) is absorbed by the continuation maps in the direct limit which defines $SH_{\widehat{T}}(M, \widehat{\rho})$. This has two consequences for the shift operators on $SH_{\widehat{T}}(M, \widehat{\rho})$: they are isomorphisms and they are well-defined even for cocharacters σ without the nonnegativity condition. The shift operator on $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ does not have these two properties in general.

Corollary 3.1.5 (Shift operator representation). *For every cocharacter $\sigma \in$* Cochar T*, we have a shift operator* $\mathrm{S} : SH_{\widehat{T}}(M, \widehat{\rho}) \to SH_{\widehat{T}}^{+j \,;\, j}(M, \widehat{\rho})$ *which is a -module automorphism satisfying* (3.1.20)*. This yields a representation*

$$\mathrm{S} : \mathrm{Cochar}\, T \to \mathrm{Aut}\,(SH_{\widehat{T}}(M, \widehat{\rho})) \tag{3.1.23}$$

of the commutative group of cocharacters of T.

We have two collections of endomorphisms of $\widehat{T}$-equivariant Floer cohomology which naturally augment the module structures: the connection ∇ augments the -module structure and the shift operator S augments the $H(B\widehat{T})$-module structure. The two module structures commute, inducing a $H(B\widehat{T})$-module structure on $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$. Our main theorem states that the augmentations of these module structures also commute.

Theorem 3.1.6 (Flatness). *The maps* S *and* r *commute on* $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ *for any* α 2 $H^2(M)$ *and any* σ 2 Cochar ^0T. *This also holds on* $SH_{\widehat{T}}(M, \widehat{\rho})$ *for any* σ 2 Cochar T.

A key step in our proof of Theorem 3.1.6 is a Morse homotopy argument which describes how $FS_{\widehat{T}}(\sigma, \mu)$ and α intertwine as operations. This argument builds on our proof of the analogous *intertwining relation* in equivariant quantum cohomology, Theorem 3.1.7, which describes how the equivariant quantum Seidel map and the equivariant quantum product α intertwine [LJ20, Section 7.4].

3.1.4 $\widehat{T}$-equivariant quantum cohomology

The quantum Seidel map $QS(\sigma, \mu) : QH(M)$! $QH^{+j\,;\,j}(M)$ was developed by Seidel [Sei97]. It counts pseudoholomorphic sections of a clutching bundle over P^1 with fibre M. It is compatible with the Floer Seidel map under PSS isomorphisms for closed M [Sei97, Section 8]. The analogous compatibility result for convex M involves a continuation map to undo the change in slope [Rit14, page 1039], and as such $QS(\sigma, \mu)$ is not an isomorphism in general. The quantum Seidel map intertwines the quantum product and satisfies

$$QS(\sigma, \mu) \quad QS(\sigma', \mu) = QS(\sigma + \sigma', \mu) \tag{3.1.24}$$

for any two cocharacters σ, σ'.

Equivariant quantum Seidel maps were initially developed by Maulik and Okounkov in their study of quiver varieties [MO19, Section 8]. In the construction of the $\widehat{T}$-equivariant quantum Seidel map $QS_{\widehat{T}}(\sigma, \mu)$, the group S^1_0 acts by rotating P^1 while T acts fibrewise on the clutching bundle. With a technique called *virtual localisation*, they proved the *intertwining relation*, Theorem 3.1.7, which describes how the equivariant quantum Seidel map interacts with the equivariant quantum product. They interpreted this relation as the flatness of a difference-differential connection [MO19, Section 1.4]. Braverman, Maulik and Okounkov used equivariant quantum Seidel maps to derive the equivariant quantum product on the Springer resolution [BMO11]. Iritani used equivariant quantum Seidel maps to describe shift operators on the big equivariant quantum cohomology of toric varieties and proved flatness in this setting [Iri17].

Theorem 3.1.7 (Intertwining relation). *The equation*

$$QS_{\widehat{T}}(\sigma, \mu)(x \quad \alpha^+) \quad QS_{\widehat{T}}(\sigma, \mu)(x) \quad \alpha \quad = \mathbf{u}\, WQS_{\widehat{T}}(\sigma, \mu, \alpha)(x) \tag{3.1.25}$$

holds for all x 2 $QH_{\widehat{T}}(M, \widehat{\rho})$. *Here,* α^+ 2 $H^2_{\widehat{T}}(M, \widehat{\rho})$ *and* α 2 $H^2_{\widehat{T}}(M, \sigma \quad \widehat{\rho})$ *are two equivariant cohomology classes that are related via the clutching bundle and*

$$WQS_{\widehat{T}}(\sigma, \mu, \alpha) : QH_{\widehat{T}}(M, \widehat{\rho}) \; ! \; QH_{\widehat{T}}^{+j\,;\,j}(M, \sigma \quad \widehat{\rho}) \tag{3.1.26}$$

is a weighted version of the $\widehat{T}$-equivariant quantum Seidel map defined in (3.3.40).

The algebrogeometric technique of virtual localisation does not work in equivariant Floer cohomology. In [LJ20], we gave a new Morse-theoretic proof of the intertwining relation which does not use virtual localisation. This proof gives the relation as the boundary of a 1-dimensional moduli space which is made up of a configuration on $ES^1 = S^{\infty}$ and a configuration on M which depends on the configuration on ES^1. We wanted the configuration on M to depend on an element of S^1, but there is no globally-defined equivariant map $ES^1 \to S^1$. A key insight in our proof was that a map $ES^1 \to S^1$ defined on an open dense S^1-invariant *proper* subset of ES^1 was sufficient. This approach gives rise to an additional boundary component in which the configuration on ES^1 leaves the proper subset. We use this insight in many of the moduli spaces in this paper (with $E\widehat{T}$ replacing ES^1). For example, in our construction of the operation α on $FH_{\widehat{T}}(M,\widehat{\rho},\lambda)$, the intersection of α and the Floer solution $u : \mathbb{R} \times S^1 \to M$ occurs at $u(0,t_0)$ for an element $t_0 \in S^1$ determined by the configuration in $E\widehat{T}$.

The $\widehat{T}$-equivariant quantum cohomology $QH_{\widehat{T}}(M,\widehat{\rho})$ of M is equipped with a connection ∇ and a shift operator $\mathbb{S}$, given by the composition $\mathbb{S} = (B\widehat{\sigma}) \ QS_{\widehat{T}}(\sigma,\mu)$. The flatness result that ∇ and $\mathbb{S}$ commute on $QH_{\widehat{T}}(M,\widehat{\rho})$ is equivalent to the intertwining relation (see the proof of Theorem 3.3.6). We prove the intertwining relation in $\widehat{T}$-equivariant quantum cohomology using a $\widehat{T}$-equivariant version of our Morse-theoretic proof. Under $\widehat{T}$-equivariant PSS isomorphisms, which identify $QH_{\widehat{T}}(M,\widehat{\rho})$ with $FH_{\widehat{T}}(M,\widehat{\rho},\lambda)$ for small slopes $\lambda > 0$, the connections and shift operators on these different modules are identified.

Seidel conjectured that his q-connection corresponds to the equivariant quantum connection under the equivariant Floer Seidel map and the PSS isomorphism [Sei18, Remark 5.6]. We confirm the analogous result holds in our setup. Let σ be a cocharacter which induces a S^1-action of strictly positive slope $\kappa > 0$. The $\widehat{T}$-equivariant Floer Seidel map is an isomorphism

$$FS_{\widehat{T}}(\sigma,\mu) : FH_{\widehat{T}}(M,\widehat{\rho},\kappa + \varepsilon) \to FH_{\widehat{T}}^{+|\ |}(M,\sigma \ \widehat{\rho},\varepsilon) \tag{3.1.27}$$

which preserves the connection ∇. For small $\varepsilon > 0$, the right-hand side is isomorphic to $QH_{\widehat{T}}(M,\sigma \ \widehat{\rho})$ via the PSS isomorphism, and the PSS isomorphism preserves the connection also. Therefore, moving to the S^1-equivariant setup, the isomorphism

$$\mathrm{PSS} \ FS_{S^1}(\sigma,\mu) : FH_{S^1}(M,\kappa + \varepsilon) \to QH_{S^1}^{+|\ |}(M,\rho \ (\ \sigma)) \tag{3.1.28}$$

preserves the connection. The S^1-action on S^1-equivariant Floer cohomology rotates loops with no pointwise action on M, meanwhile the S^1-action on S^1-equivariant quantum cohomology is[3] $\rho \ (\ \sigma)$.

[3] For the Borel quotient in the construction of equivariant cohomology, we use the antidiagonal action

3.1.5 Toric manifolds

A *toric manifold* M is a symplectic manifold equipped with an effective Hamiltonian T-action with $\dim T = \frac{1}{2}\dim M$. It is determined up to equivariant symplectomorphism by its *moment polytope*, a convex polytope in $\mathbb{R}^{\dim T}$. The combinatorial data describing the moment polytope can be used to compute many invariants of the corresponding toric manifold, including its quantum cohomology [MT06, Proposition 5.2] and its T-equivariant cohomology [Cox11, Theorem 12.4.14]. We combine these two results to find a presentation for the $\widehat{T}$-equivariant quantum cohomology (Theorem 3.5.6). We use the combinatorial data to compute the connection ∇ and the shift operator $\mathbb{S}$ in this presentation (Section 3.5.3).

Theorem 3.1.8. *For toric manifolds, the shift operator* $\mathbb{S}$ *on* $QH_{\widehat{T}}(M, \widehat{\rho})$ *is determined by its value at 1 via the analogues to* (3.1.20) *and* Theorem 3.1.6. *Both of these conditions may be expressed combinatorially using the moment polytope. Moreover, we provide a simple combinatorial recipe to compute* $\mathbb{S}(1)$ *from the polytope data.*

Ritter showed that the moment polytope of a convex toric manifold M gives rise to a neat presentation for $SH^*(M)$ [Rit16, Theorem 1.5(2)]. The $\widehat{T}$-equivariant version of Ritter's argument does not produce a presentation for $\widehat{T}$-equivariant symplectic cohomology, but we can nonetheless use it to deduce properties of $SH_{\widehat{T}}(M, \widehat{\rho})$. In particular, for $M = \mathcal{O}_{\mathbb{P}^1}(-1)$, we deduce that there are elements $d_i \in H^2(B\widehat{T})$ for $i \in \mathbb{Z}_{>0}$ such that every element of $SH_{\widehat{T}}(M, \widehat{\rho})$ can be written in the form

$$\frac{x}{d_1 \cdots d_r} \tag{3.1.29}$$

for $x \in QH_{\widehat{T}}(M, \widehat{\rho})$ and $r > 0$. The elements d_i arise as the determinants of the shift operators, and they are determined by the Reeb dynamics of the sphere bundle $S\mathcal{O}_{\mathbb{P}^1}(-1)$. While it is beyond the scope of this paper to verify, we conjecture that this relationship between the determinants of the shift operators and the Reeb dynamics holds in general (Conjecture 3.5.12).

3.1.6 Notation

In [LJ20], we only looked at S^1-equivariant constructions, denoting S^1-equivariant invariants with an E (EQS, EQH^*, etc.). In this paper, we consider S^1-equivariant, T-equivariant and $\widehat{T}$-equivariant constructions, so we specify the group in this paper's notation (QS_{S^1}, QH_T, etc.).

(3.3.8) whereas Seidel used the diagonal action. The minus sign in ( ) is not present in Seidel's conjecture, but it arises merely as a consequence of this convention.

The Novikov ring in this paper records classes $A \in H_2(M)$ (definition in Section 3.3.2), whereas the Novikov ring in our previous paper recorded only $c_1(TM, \omega)(A)$ and $[\omega](A)$ [LJ20, Section 3.2.6]. We use a fixed point μ to construct a 'reference section' of the clutching bundle, replacing the *lifted* S^1*-actions* $\widetilde{\sigma}$ from our previous paper. For the q-connection, Seidel assumed $c_1(TM, \omega) = 0$ and his Novikov ring recorded only $[\omega](A)$ [Sei18, Section 2a].

We introduce the manifold M in Section 3.3.1, the torus T and the action ρ in Section 3.3.6, and the extended torus $\widehat{T}$ in Section 3.3.11.

3.1.7 Outline

Section 3.2 We introduce connections and shift operators as algebraic structures. We discuss how the terminology is inspired by bundles.

Section 3.3 We list the assumptions on M and ρ. We construct the connection and shift operator on $\widehat{T}$-equivariant quantum cohomology. We prove flatness in Theorem 3.3.6.

Section 3.4 We construct $\widehat{T}$-equivariant Floer cohomology together with its connection and shift operator. We prove flatness in Theorem 3.4.10.

Section 3.5 We compute the $\widehat{T}$-equivariant quantum cohomology of a toric manifold, together with its connection and shift operator. We demonstrate the methods for $\mathbb{P}^2$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Section 3.6 We postpone a few topological proofs to the appendix.

3.2 Connections as an algebraic structure

3.2.1 Differential connections

This viewpoint on connections is taken from [Kos86, Chapter 1]. It abstracts the definition of connections on vector bundles (see Example 3.2.3).

Let k be an integral domain. Let A be a unital, commutative and associative algebra over k. A *derivation* or *vector field* on A is a k-linear map $X : A \to A$ which satisfies the Leibniz rule

$$X(ab) = (Xa)b + a(Xb). \tag{3.2.1}$$

Denote by $\mathcal{X}(A)$ the set of all derivations on A. It has the structure of an A-module with post-multiplication and the structure of a k-Lie algebra with the commutator $[X, Y] = XY - YX$.

Let P be a unitary[4] A-module. A *(differential) connection on P* or a *derivation law in P* is an A-module map $\nabla : \mathfrak{X}(A) \to \mathrm{Hom}_k(P, P)$ which satisfies the Leibniz rule

$$\nabla_X(ap) = (Xa)p + a\nabla_X(p). \tag{3.2.2}$$

The *curvature* of ∇ is the k-linear map $R^\nabla : \mathfrak{X}(A) \times \mathfrak{X}(A) \to \mathrm{Hom}_A(P, P)$ given by

$$R^\nabla_{X,Y} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}. \tag{3.2.3}$$

A connection is *flat* if its curvature vanishes.

Example 3.2.1 (Canonical connections). The A-module A has a *canonical connection* given by $\nabla_X(a) = Xa$. The free A-module P with specified basis $\{p_i : i \in I\}$ has a *canonical connection* given by $\nabla_X(\sum_i a_i p_i) = \sum_i (Xa_i)p_i$.

Example 3.2.2 (Polynomial rings). Let $A = k[t]$. The standard differentiation operation $\partial_t : t^n \mapsto nt^{n-1}$ generates $\mathfrak{X}(k[t])$.

Example 3.2.3 (Differentiable manifolds). Let $E \to B$ be a smooth vector bundle over a smooth manifold B. Let $k = \mathbb{R}$, let $A = C^\infty(B)$ be the smooth $\mathbb{R}$-valued functions on B, and let $P = C^\infty(B, E)$ be the smooth sections of the vector bundle E. The *vector fields on A* are precisely the smooth vector fields on B, so $\mathfrak{X}(A) = C^\infty(B, TB)$. Any connection on $E \to B$, in the conventional sense, is a *connection in P*.

Example 3.2.4 (Non-existence). Let $A = k[t]$ and $P = {}^{k[t]}/_{(t)}$. The module P has no connection, since (3.2.2) yields

$$0 + (t) = \nabla_{\partial_t}(t \cdot (1 + (t))) = (\partial_t t)(1 + (t)) + t\nabla_{\partial_t}((1 + (t))) = 1 + (t) \neq 0 + (t) \tag{3.2.4}$$

for any connection ∇.

Lemma 3.2.5 (Flatness sufficient on a generating set). *Suppose the set of vector fields $\{X_\alpha\}_{\alpha \in I}$ generates $\mathfrak{X}(A)$ as an A-module. If $R^\nabla_{X_\alpha, X_\beta} = 0$ for all $\alpha, \beta \in I$, then the connection ∇ is flat.*

Proof. This follows immediately from the Leibniz rules. □

Definition 3.2.6 (Partial connection). In some cases, it is undesirable to define a connection ∇_X for all $X \in \mathfrak{X}(A)$. Instead, we might define the connection $\nabla^{\mathcal{X}}_X$ for all X in a given k-submodule $\mathcal{X} \subset \mathfrak{X}(A)$. The connection $\nabla^{\mathcal{X}}$ must still be A-linear, whenever this condition makes sense. That is, for all $a \in A$ and $X \in \mathcal{X}$, the relation $\nabla^{\mathcal{X}}_{aX} = a\nabla^{\mathcal{X}}_X$ holds whenever $aX \in \mathcal{X}$. Equivalently, $\nabla^{\mathcal{X}}$ is the restriction of an A-module map $A \otimes \mathcal{X} \to \mathrm{Hom}_k(P, P)$ which satisfies (3.2.2). The *curvature* of $\nabla^{\mathcal{X}}$, defined exactly as in (3.2.3), is well-defined if $\mathcal{X}$ is a k-Lie subalgebra of $\mathfrak{X}(A)$.

[4] In a *unitary* A-module P, the relation $1_A \cdot p = p$ holds for all $p \in P$.

3.2.2 Difference connections

Let S be any set. We will call the elements of S the *fundamental directions*.

A *shift operator* on A is an assignment to each fundamental direction $\sigma \in S$ an algebra homomorphism $E_\sigma : A \to A$ such that E_σ and E_τ commute for any $\sigma, \tau \in S$. The corresponding *difference operator* is the map $X^E : S \to \mathrm{Hom}_k(A, A)$ which assigns to each fundamental direction σ the k-linear map $X^E_\sigma = E_\sigma - \mathrm{Id}_A$. It satisfies a *deformed Leibniz rule*

$$X^E_\sigma(ab) = (X^E_\sigma a)b + (E_\sigma a)(X^E_\sigma b). \tag{3.2.5}$$

Note this rule is symmetric in a and b because A is commutative.

A *difference connection* or *discrete connection* on P is a map $\nabla : S \to \mathrm{Hom}_k(P, P)$ which satisfies the *deformed Leibniz rule*

$$\nabla_\sigma(ap) = (X^E_\sigma a)p + (E_\sigma a)\nabla_\sigma(p). \tag{3.2.6}$$

The *curvature* $R^\nabla : S \times S \to \mathrm{Hom}_k(P, P)$ is given by $R^\nabla_{\sigma,\tau} = \nabla_\sigma \nabla_\tau - \nabla_\tau \nabla_\sigma$, and ∇ is *flat* if the curvature vanishes. Unlike the curvature of a differential connection, the codomain of R^∇ is not the A-module endomorphisms of P. Instead, we have the relation

$$R^\nabla_{\sigma,\tau}(ap) = (E_\sigma E_\tau a)\, R^\nabla_{\sigma,\tau}(p) \tag{3.2.7}$$

for all fundamental directions σ and τ.

An equivalent formulation of difference connections is via shift operators. A *shift operator* Σ on P is an assignment to each fundamental direction $\sigma \in S$ a k-linear map $\Sigma_\sigma : P \to P$ which satisfies $\Sigma_\sigma(ap) = (E_\sigma a)\,\Sigma_\sigma(p)$. The map $\nabla^\Sigma_\sigma = \Sigma_\sigma - \mathrm{Id}_P$ is a difference connection. The flatness of ∇^Σ is equivalent to the commutativity of the maps Σ_σ and $\Sigma_{\sigma'}$ for any $\sigma, \sigma' \in S$.

Flat shift operators may also be defined on commutative monoids[5] M. A *shift operator* on A for M is a monoid map $E : M \to \mathrm{AlgHom}_k(A, A)$. A *flat shift operator* Σ on P for M is an assignment to each $\sigma \in M$ a k-linear map $\Sigma_\sigma : P \to P$ which satisfies $\Sigma_\sigma(ap) = (E_\sigma a)\,\Sigma_\sigma(p)$ and $\Sigma_{\sigma\sigma'} = \Sigma_\sigma \Sigma_{\sigma'}$ for all $\sigma, \sigma' \in M$. A flat shift operator Σ defined for a set S may be extended to the *commutative monoid* $\bigoplus_{\sigma \in S} \mathbb{Z}_{\geq 0}\langle\sigma\rangle$ generated by S. Explicitly, the shift operator assigned to the product $\sigma_1 \cdots \sigma_m$ is $\Sigma_{\sigma_1} \cdots \Sigma_{\sigma_m}$.

Example 3.2.7 (Lattice bundles). Let S be a finite set of linearly independent vectors in $\mathbb{R}^n$ and let $L = \bigoplus_{\sigma \in S} \mathbb{Z}\sigma$ be the lattice generated by S. Let V be any k-module. Set $A = \mathrm{Map}(L, k)$, the k-algebra of all k-valued functions on L, and $P = \mathrm{Map}(L, V)$. Define

[5] A *commutative monoid* M is a set with a commutative associative binary operation which has a two-sided identity element. It does not have an inverse operation, unlike a group. $(\mathbb{Z}_{\geq 0}, +, 0)$ is a commutative monoid.

$E_\sigma(a)(l) = a(l+\sigma)$, so the function $E_\sigma(a)$ is the composition of a with a shift in L by σ. Similarly, define $\mathsf{S}_\sigma(p)(l) = p(l+\sigma)$. The resulting difference connection is flat.

Remark 3.2.8 (Comparison of Example 3.2.3 and Example 3.2.7). Take the setup of Example 3.2.7. The directional derivative of $a : \mathbb{R}^n \to \mathbb{R}$ at 0 in the direction σ is the limit of $\frac{1}{t}(a(t\sigma) - a(0))$ as $t \to 0$. The difference operator X^E_σ should be considered as pre-limit version of the directional derivative, where without loss of generality we have scaled so that $t = 1$.

3.2.3 Difference-differential connections

Let k, A, P and S be as before. Let $\mathcal{X}$ be a k-module of vector fields and let E be a shift operator on A with respect to S. A *difference-differential connection* on P is a map $\nabla : S \sqcup \mathcal{X} \to \mathrm{Hom}_k(P,P)$ which is a difference connection on S and a partial differential connection on $\mathcal{X}$.

Suppose $\mathcal{X}$ is a k-Lie algebra and $[X, X^E_\sigma] = 0$ for all $X \in \mathcal{X}$ and all $\sigma \in S$. The *curvature* of ∇ is the map $R^\nabla : (S \sqcup \mathcal{X})^2 \to \mathrm{Hom}_k(P,P)$ given by (3.2.3), interpreting σ as X^E_σ in the formula where appropriate. Since shift operators in different fundamental directions commute, the commutator $[X^E_\sigma, X^E_\tau]$ is zero, so that R^∇ coincides with the curvature of a difference connection when restricted to $S \times S$. The connection ∇ is *flat* if R^∇ vanishes.

Difference-differential connections may be described using shift operators. In this context, a *difference-differential connection* is a pair (S, ∇) consisting of a shift operator S and a differential connection ∇.

Lemma 3.2.9 (Flatness sufficient on a generating set). *Suppose the shift operators on A are monomorphisms and that the set of vector fields $\mathcal{B} = \{X_i\}_{i \in I} \subseteq \mathcal{X}$ generates $A \otimes \mathcal{X}$ as an A-module. If $R^\nabla = 0$ on $(S \sqcup \mathcal{B})^2$, then the connection ∇ is flat.*

Proof. Lemma 3.2.5 shows that R^∇ vanishes on $\mathcal{X}^2$. By the anti-symmetry and additivity of R^∇, it is sufficient to show $R^\nabla_{aX_i,\sigma} = 0$ for any $a \in A$ such that $aX_i \in \mathcal{X}$. We have

$$0 = a[X_i, X^E_\sigma] - [aX_i, X^E_\sigma] = X^E_\sigma(a)\,(E_\sigma X_i) = E_\sigma((X^E_\sigma(a))X_i), \tag{3.2.8}$$

from which we deduce $(X^E_\sigma(a))X_i = 0$ holds because E_σ is injective. Hence we derive

$$\begin{aligned} R^\nabla_{aX_i,\sigma} &= \nabla_{aX_i}\nabla_\sigma - \nabla_\sigma(a\nabla_{X_i}) \\ &= \nabla_{(E_\sigma a)X_i}\nabla_\sigma - X^E_\sigma(a)\nabla_{X_i} - (E_\sigma a)\nabla_\sigma\nabla_{X_i} \\ &= (E_\sigma a)[\nabla_{X_i}, \nabla_\sigma] - \nabla_{X^E_\sigma(a)X_i} \\ &= 0. \end{aligned} \tag{3.2.9}$$

□

Remark 3.2.10 (Grading). Suppose A and P are $\mathbb{Z}$-graded. The A-module of derivations is naturally graded by the degree of the (homogeneous) derivations. A *degree-d* differential connection is thus a differential connection which is degree-d as an A-module map $X(A) \to \mathrm{Hom}_k(P, P)$. Note that shift operators on A are degree-0 maps since they preserve 1, and hence difference operators are degree-0 also. A difference connection is *degree-d_S* if ∇_σ is a degree-$d_S(\sigma)$ map for all $\sigma \in S$, where $d_S : S \to \mathbb{Z}$ is a given map. These definitions may be combined to get a *degree-(d_S, d)* difference-differential connection.

3.3 Equivariant quantum cohomology

3.3.1 Assumptions on symplectic manifold

Let (M, ω) be a $2n$-dimensional symplectic manifold. Assume M is nonempty and connected for convenience.

We constrain the possible bubbling configurations of pseudoholomorphic spheres by imposing that M is *nonnegatively monotone*. This means that the implication

$$c_1(TM, \omega)(A) < 0 \implies \omega(A) \geq 0 \tag{3.3.1}$$

holds for all $A \in \pi_2(M)$. It is sufficient that M is monotone or that $c_1(TM, \omega)$ is zero.

A *convex structure* on M is a pair (ψ, Σ), where Σ is a closed contact manifold with contact form α and $\psi : [1, \infty) \times \Sigma \to M$ is a diffeomorphism onto its image, such that $M \setminus \psi([1, \infty) \times \Sigma)$ is a relatively compact subset of M and the equation

$$\psi^* \omega = \mathrm{d}(R\alpha) \tag{3.3.2}$$

holds on $[1, \infty) \times \Sigma$. The *convex end* (the image of ψ) is captured by the notation $\{R \geq 1\}$ using the *radial coordinate* $R \in [1, \infty)$. Similarly, its complement is given by the notation $\{R < 1\}$.

Equation (3.3.2) states that the symplectic form ω is exact on the convex end, however this does not imply that ω is exact on all of M. We allow $\Sigma = \emptyset$ so that closed symplectic manifolds have a convex structure and are included in our analysis.

The Reeb periods of the contact manifold (Σ, α) are the periods of the closed paths along the Reeb vector field X_Σ. (X_Σ is determined by $\mathrm{d}\alpha(X_\Sigma, \cdot) = 0$ and $\alpha(X_\Sigma) = 1$.) Denote by $\mathcal{R}$ the set of all Reeb periods (including 0 if Σ is nonempty). We assume that M has a convex structure (ψ, Σ) for which $\mathbb{R} \setminus \mathcal{R}$ is unbounded. This unboundedness assumption ensures symplectic cohomology can be defined.

Finally, to facilitate lifting cohomology classes to equivariant cohomology, we further assume that M is *simply connected*. Our work involves a torus T of dimension k acting on M (the action is introduced in Section 3.3.6). For results with a trivial T-action (or indeed with $k = 0$), we do not need to make the assumption that M is simply connected. In this case, the lifting is straightforward, and our results hold as stated without the simply connectedness assumption.

Henceforth, M will be a nonnegatively monotone and simply connected symplectic manifold with a convex structure $(\psi, \ )$ for which $\mathbb{R} \setminus R$ is unbounded.

3.3.2 Novikov ring

The Novikov ring is an algebraic tool for recording the homology class of pseudoholomorphic spheres. We use the following variation.

A degree-l element of the Novikov ring is a formal sum of monomials q^A with exponents $A \in H_2(M)$ and coefficients in $\mathbb{Z}$, subject to the grading requirement that $2c_1(TM, \omega)(A) = l$ and the symplectic energy requirement that, for any $c \in \mathbb{R}$, there are only finitely-many supported A with $\omega(A)$ c. We use sum notation, even though elements can be infinite sums. The *Novikov ring* is the direct sum of its homogeneous parts.

3.3.3 Morse cohomology

Fix a Riemannian metric on M. Let $f : M \to \mathbb{R}$ be a Morse-Smale function which increases in the radial direction, so that $\partial_R f(\psi(y, R)) > 0$ holds at infinity. Such Morse-Smale functions are *convex*. Let $\mathrm{Crit}(f, M)$ denote the set of critical points of f. The *Morse index* $|x|$ of a critical point $x \in \mathrm{Crit}(f, M)$ is the dimension of the maximal subspace of the tangent space at x on which the Hessian of f is negative definite.

The *Morse cochain complex* is $\mathbb{Z}\langle \mathrm{Crit}(f, M)\rangle$, and the *Morse differential* counts the negative gradient trajectories between critical points. Explicitly, given two critical points $x^\pm \in \mathrm{Crit}(f, M)$, we consider the moduli space

$$\mathcal{M}(x^-, x^+) = \left\{ \gamma : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \partial_s(\gamma(s)) = (-\nabla f)_{\gamma(s)} \\ \gamma(\pm\infty) = x^\pm \end{array} \right\}. \tag{3.3.3}$$

It has dimension $|x^-| - |x^+|$ and an $\mathbb{R}$-action given by s-translation. The differential is given by

$$dx^+ = \sum_{\substack{x^- \in \mathrm{Crit}(f; M) \\ |x^-| = |x^+| + 1}} \#\left(\frac{\mathcal{M}(x^-, x^+)}{\mathbb{R}}\right) x^-, \tag{3.3.4}$$

where $\#(\cdot)$ denotes the signed count of a closed 0-dimensional manifold. We denote the *Morse cohomology* of M by $H^*(M; f)$ to emphasise the fact that Morse cohomology recovers ordinary cohomology and does not depend on f.

We suppress orientation information from our notation. A choice of orientation for each unstable manifold $W^u(x)$ is nonetheless required to ensure the moduli spaces of trajectories may be consistently oriented.

For convenience, a *flowline* is a map $\gamma : \mathbb{R} \to M$ satisfying

$$\partial_s(\gamma(s)) = (-\nabla f)_{\gamma(s)} \tag{3.3.5}$$

with finite energy $\int \|\partial_s \gamma\|^2$. The finite energy requirement ensures that a flowline converges to critical points of f as $s \to \pm\infty$. A map $\gamma^- : (-\infty, 0] \to \mathbb{R}$ satisfying (3.3.5) with finite energy is a *half$^-$ flowline* and a map $\gamma^+ : [0, \infty) \to M$ satisfying (3.3.5) with finite energy is a *half$^+$ flowline*.

3.3.4 Quantum product

Let J be a regular ω-compatible almost complex structure on M, so that the moduli space $\mathcal{M}(A; J)$ of J-holomorphic maps $u : \mathbb{P}^1 \to M$ with $u_*[\mathbb{P}^1] = A$ has dimension $2n + 2c_1(TM, \omega)(A)$, for any $A \in H_2(M)$. We moreover assume that J is *convex*, which means that $-dR \circ J = R\alpha$ holds at infinity, using the coordinates provided by ψ. This compatibility with the convex structure allows us to prove compactness results for the moduli spaces $\mathcal{M}(A; J)$ using the maximum principle [Rit13, Appendix D].

The Morse cup product counts 'Y'-shaped graphs, and the (Morse) quantum product is a deformation of the Morse cup product. We define it as follows.

Let $p_0^-, p_1^+, p_2^+ \in \mathbb{P}^1$ be fixed distinct points. A *deformed 'Y'-shaped graph* is a quadruple $(\gamma_0^-, \gamma_1^+, \gamma_2^+, u)$ where $\gamma_i^\pm$ are half$^\pm$ flowlines and $u : \mathbb{P}^1 \to M$ is a J-holomorphic map that satisfies $u(p_i^\pm) = \gamma_i^\pm(0)$. Given the critical points $x_0^-, x_1^+, x_2^+ \in \mathrm{Crit}(f, M)$ and the class $A \in H_2(M)$, the moduli space $\mathcal{M}(x_0^-, x_1^+, x_2^+, A)$ of deformed 'Y'-shaped graphs that satisfy $\gamma_i^\pm(\pm\infty) = x_i^\pm$ and $u \in \mathcal{M}(A; J)$ is a smooth manifold of dimension

$$|x_0^-| - |x_1^+| - |x_2^+| + 2c_1(TM, \omega)(A). \tag{3.3.6}$$

Technically, we must use three s-dependent perturbations of the Morse function, one for each half flowline, to guarantee the moduli space is a smooth manifold. Such s-dependent perturbations may only depend on $s \in \mathbb{R}$ in a compact interval. To keep our notation simple, we refer to flowlines that use a Morse function perturbed in this way as *perturbed* flowlines (though we keep the perturbations themselves implicit).

Quantum cohomology $QH^*(M; f)$ is the cohomology of $\Lambda \otimes \mathbb{Z}\langle \mathrm{Crit}(f, M)\rangle$ with the Morse differential (3.3.4), so the Λ-module isomorphism $QH^*(M; f) \cong \Lambda \otimes H^*(M; f)$ holds. Quantum cohomology is equipped with the *quantum product* $*$, which is given by

$$x_1^+ * x_2^+ = \sum_{\substack{A \in H_2(M) \\ x_0 \in \mathrm{Crit}(f;M) \\ (3.3.6)=0}} \# M(x_0, x_1^+, x_2^+, A)\, q^A x_0 . \tag{3.3.7}$$

The quantum product is graded-commutative, associative and unital. These facts are proved by standard homotopy arguments.

3.3.5 Equivariant cohomology: the general case

Let G be a closed connected smooth Abelian Lie group (G is always a torus in this paper). Let $\rho : G \times X \to X$ be a continuous G-action on a topological space X. G-equivariant cohomology is a functorial invariant of the pair (X, ρ). It is constructed as follows.

Let EG be a contractible space with a free G-action. The *classifying space* of G is the quotient $BG = EG/G$ and the *universal bundle* of G is the bundle $EG \to BG$. The cohomology $H^*(BG)$ of the classifying space depends only on G. The *Borel homotopy quotient* $EG \times_G X$ is the quotient of $EG \times X$ by the relation

$$(e, g \cdot x) \sim (g \cdot e, x). \tag{3.3.8}$$

Equivalently, the Borel homotopy quotient can be obtained by taking the quotient of $EG \times X$ by the antidiagonal action $g \cdot (e, x) = (g^{-1} \cdot e, g \cdot x)$. The *$G$-equivariant cohomology* of (X, ρ) is the cohomology ring $H^*(EG \times_G X)$, and it is denoted $H_G^*(X, \rho)$. The projection map $EG \times_G X \to BG$ induces a $H^*(BG)$-algebra structure on $H_G^*(X, \rho)$.

G-equivariant cohomology respects G-equivariant continuous maps between topological spaces with G-actions. Moreover, it is functorial with respect to group homomorphisms: let $\varphi : G \to H$ be a Lie group homomorpism and let $E\varphi : EG \to EH$ be a continuous map[6] which satisfies

$$(E\varphi)(g \cdot e) = \varphi(g) \cdot (E\varphi)(e). \tag{3.3.9}$$

The induced map on the classifying spaces, $B\varphi : BG \to BH$, is well-defined and continuous, so it induces a map $(B\varphi)^* : H^*(BH) \to H^*(BG)$ on cohomology. Suppose that $f : X \to Y$ is a continuous map which satisfies

$$f(g \cdot x) = \varphi(g) \cdot f(x) \tag{3.3.10}$$

[6] A map satisfying (3.3.9) may be constructed as follows. Let $(EG)'$ and EH be arbitrary. Set $EG = (EG)' \times EH$, with the action $g \cdot (e'_G, e_H) = (g \cdot e'_G, \varphi(g) \cdot e_H)$. The projection map $EG \to EH$ satisfies (3.3.9).

with respect to a G-action ρ_X on X and an H-action ρ_Y on Y. The map $(E\varphi, f) : EG \times X \to EH \times Y$ descends to the Borel homotopy quotients, and hence induces a map

$$(E\varphi, f)^* : H^*_H(Y, \rho_Y) \to H^*_G(X, \rho_X). \tag{3.3.11}$$

When f is the identity map on $X = Y$, the map (3.3.11) is denoted $(B\varphi)^*$, and when φ is the identity map on $G = H$, the map (3.3.11) is denoted f^*.

Associated to the fibre bundle $EG \times_G X \to BG$ with fibre X, there is a sequence

$$H^*(BG) \to H^*_G(X, \rho) \to H^*(X). \tag{3.3.12}$$

The second map is induced by fibre inclusion, and it is well-defined on cohomology because G and EG are path-connected.

3.3.6 The torus

Let T denote the k-dimensional (real) torus, with $0 \leq k \leq n$. We always use the parameterisation $S^1 = \mathbb{R}/\mathbb{Z}$ for the circle. The *characters* of the torus are the homomorphisms $\chi : T \to S^1$, and the *cocharacters* are the homomorphisms $\sigma : S^1 \to T$. We denote the spaces of characters and cocharacters by Char T and Cochar T respectively. Each space is a lattice isomorphic to $\mathbb{Z}^k$. There is a natural pairing Cochar $T \times$ Char $T \to \mathbb{Z}$ which records the multiplicity of the composition $\sigma \circ \chi : S^1 \to S^1$; we denote it by (σ, χ).

A *basis* for T is a k-tuple of cocharacters $\tau = (\tau_1, \ldots, \tau_k)$ such that $\tau : (S^1)^k \to T$ is an isomorphism.

Let $\rho : T \times M \to M$ be a Hamiltonian T-action on M, such that $\rho \circ \sigma$ is an admissible Hamiltonian S^1-action for all $\sigma \in$ Cochar T. The S^1-action $\rho \circ \sigma$ is *admissible* if its Hamiltonian $K_\sigma : M \to \mathbb{R}$ satisfies

$$K_\sigma(\psi(R, y)) = \kappa_\sigma(y)R + \text{constant} \tag{3.3.13}$$

at infinity for a Reeb flow-invariant function $\kappa_\sigma : \Sigma \to \mathbb{R}$. Note that, unlike for the admissible Hamiltonians in Section 3.4.1, we have $\kappa_\sigma \notin L$ because the flow of X_{K_σ} is an S^1-action, so it has 1-periodic orbits at infinity. The admissibility assumption ensures the radial coordinate R is preserved by ρ at infinity.

We assume the action ρ has a fixed point $\mu \in M$. The existence of a fixed point is guaranteed for $k \geq 1$ by [LJ20, Lemma 3.13 and Remark 3.14]. The toric manifolds in Section 3.5 have fixed points.

3.3.7 Classifying space for the torus

A standard choice of ES^1 is the infinite sphere S^∞, which is defined to be the limit of the odd-dimensional spheres $S^1 \hookrightarrow S^3 \hookrightarrow \cdots$ using embeddings as the unit spheres $S^{2r-1} \subset \mathbb{C}^r$. The standard rotation action on complex space restricts to the spheres, and induces a S^1-action on S^∞. The corresponding quotient S^∞/S^1 is the infinite complex projective space $\mathbb{CP}^\infty$. It follows that $H^*(BS^1) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[y]$ is a polynomial ring with y in degree 2.

Given a character $\chi : T \to S^1$, we get a map $(B\chi)^* : H^*(BS^1) \to H^*(BT)$. Define $[\chi] = (B\chi)^*(y) \in H^*(BT)$. The map $\chi \mapsto [\chi]$ naturally extends to the symmetric algebra[7] $\mathrm{Sym}(\mathrm{Char}\, T)$ and is an isomorphism of rings $\mathrm{Sym}(\mathrm{Char}\, T) \cong H^*(BT)$.

More concretely, we can use a basis τ for T to construct a universal bundle explicitly. Write T as a product $(S^1)^k$, using the basis τ. Set $ET = (S^\infty)^k$, and equip it with the natural action coming from this product decomposition. If τ_i^* is the character dual to τ_i in the basis τ, and we set $y_i = [\tau_i^*]$, then we can write $H^*(BT) = \mathbb{Z}[y_1, y_2, \ldots, y_k]$. The cohomology ring does not depend on the choice of τ (Remark 3.3.2).

Proposition 3.3.1. The sequence (3.3.12) is a short exact sequence in degree 2 for (M, ρ).

Proof. If the action ρ is trivial, or $k = 0$ holds, the result is immediate. Otherwise, we use the assumption that M is simply connected. Consider the following commutative diagram. The top row is the homotopy long exact sequence for the fibre bundle $M \to ET \times_T M \to BT$, the bottom row is the homology version of (3.3.12), and the vertical maps are the Hurewicz homomorphisms.

$$\begin{array}{ccccccccc} \pi_3(BT) & \longrightarrow & \pi_2(M) & \longrightarrow & \pi_2(ET \times_T M) & \longrightarrow & \pi_2(BT) & \longrightarrow & \pi_1(M) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & H_2(M) & \longrightarrow & H_2(ET \times_T M) & \longrightarrow & H_2(BT) & & \end{array} \tag{3.3.14}$$

The group $\pi_3(BT) = 0$ vanishes (this may be derived using the homotopy long exact sequence for the fibre bundle $S^1 \to S^\infty \to \mathbb{CP}^\infty$). The group $\pi_1(M) = 0$ vanishes since M is simply connected. Therefore the top row of (3.3.14) is a short exact sequence. Since $\pi_1(M) = 0$ vanishes, the vertical Hurewicz homomorphisms are isomorphisms by the Hurewicz theorem. The proposition follows by applying the universal coefficient theorem to the short exact sequence in homology (the exactness of the sequence is preserved because $H_2(BT)$ is free). □

[7] The *symmetric algebra* $\mathrm{Sym}(V)$ associated to the abelian group V is $\mathbb{Z} \oplus V \oplus \mathrm{Sym}^2(V) \oplus \cdots$, where $\mathrm{Sym}^2(V) = V \otimes V / v \otimes v' - v' \otimes v$ corresponds to commutative products of two elements of V. Essentially, it is a way to write down a polynomial ring without choosing a basis for V.

3.3.8 Equivariant Morse theory

Let us start by defining Morse data for the standard universal bundle for S^1, namely $S^\infty \to \mathbb{CP}^\infty$. Elements of S^∞ are vectors $(e_0, e_1, \ldots) \in \mathbb{C}^\infty$ with unit norm (and only finitely-many nonzero entries). The function $g_{S^1} : S^\infty \to \mathbb{R}$ given by $g_{S^1}(e) = \sum_{r \geq 0} r|e_r|^2$ descends to a Morse function on $\mathbb{CP}^\infty$. It has exactly one critical point in each even degree, in such a way that the degree-$2r$ critical point is the r-th standard basis vector. We use the round metric on S^∞. It immediately follows that $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[y]$, wherein y^r corresponds to the degree-$2r$ critical point.

For the torus T with basis τ, we use the product decomposition to define Morse data for $(S^\infty)^k \to (\mathbb{CP}^\infty)^k$. Give $(S^\infty)^k$ the product round metric and set the function $g : (S^\infty)^k \to \mathbb{R}$ to be the sum of the functions g_{S^1} on each copy of S^∞. The function g descends to a Morse function on $(\mathbb{CP}^\infty)^k$.

The critical points in $(\mathbb{CP}^\infty)^k$ are in bijection with $(\mathbb{Z}_{\geq 0})^k$, where the vector $c = (c_1, \ldots, c_k) \in (\mathbb{Z}_{\geq 0})^k$ corresponds to the critical point comprising of the c_i-th unit vector in the i-th copy of $\mathbb{CP}^\infty$. We will use this notation to describe $\mathrm{Crit}(g, (\mathbb{CP}^\infty)^k)$, and the notation y^c for the corresponding class in $H^*((\mathbb{CP}^\infty)^k)$. For each critical point $c \in \mathrm{Crit}(g, (\mathbb{CP}^\infty)^k)$, there is a T-orbit of points in $(S^\infty)^k$, all of which are critical points of $g : (S^\infty)^k \to \mathbb{R}$. Such critical points $\varepsilon \in \mathrm{Crit}(g, (S^\infty)^k)$ are identified by the notation $[\varepsilon] = c$.

From now on, the notation ET will mean $(S^\infty)^k$, taken with respect to the basis τ, and BT will mean $(\mathbb{CP}^\infty)^k$.

Fix a ρ-invariant Riemannian metric on M. A *T-equivariant Morse function* on M is a function $f^{\mathrm{eq}} : ET \times M \to \mathbb{R}$ which is T-invariant, so $f^{\mathrm{eq}}(\mathbf{t}^{-1} e, \rho_{\mathbf{t}}(m)) = f^{\mathrm{eq}}(e, m)$ holds for all $\mathbf{t} \in T$; which is convex for all $e \in ET$ (as defined in Section 3.3.3); and which satisfies the following condition about critical points. For every critical point $\varepsilon \in \mathrm{Crit}(g, ET)$, the function $f^{\mathrm{eq}}(\varepsilon, \cdot)$ is a convex Morse-Smale function on M, and moreover $f^{\mathrm{eq}}(\varepsilon, \cdot) = f^{\mathrm{eq}}(e, \cdot)$ for all e in a neighbourhood of ε in $W^u(\varepsilon) \cup W^s(\varepsilon)$.

A *T-equivariant critical point* of f^{eq} is a T-equivalence class of points $(\varepsilon, x) \in ET \times M$ where $\varepsilon \in \mathrm{Crit}(g, ET)$ and $x \in \mathrm{Crit}(f^{\mathrm{eq}}(\varepsilon, \cdot), M)$ are critical points. Such T-equivariant points are denoted $[\varepsilon, x]$, and the set of T-equivariant points is denoted $\mathrm{Crit}^{\mathrm{eq}}(f^{\mathrm{eq}}, M)$. The *degree* $|\varepsilon, x|$ of $[\varepsilon, x]$ is simply the sum of the degrees of ε and x.

A *T-equivariant flowline* of f^{eq} is a T-equivalence class of pairs (v, γ) such that $v : \mathbb{R} \to ET$ is a flowline of g and $\gamma : \mathbb{R} \to M$ is a flowline of $f^{\mathrm{eq}}(v(s), \cdot)$. Explicitly, γ satisfies $\partial_s(\gamma(s)) = -(\nabla f^{\mathrm{eq}}(v(s), \cdot))_{\gamma(s)}$ and has finite energy. The flowline γ depends on s only on a compact interval of $\mathbb{R}$ because $f^{\mathrm{eq}}(v(-\infty), \cdot) = f^{\mathrm{eq}}(v(s), \cdot)$ holds for all sufficiently large $-s \gg 0$. As such, the flowline γ behaves analytically like a continuation map. The torus T acts on pairs (v, γ) by simultaneously acting on the codomains of the two maps, though it acts on ET by the inverse of the usual action. This mimics the natural T-action on maps $(v,\gamma):\mathbb{R}\to ET\times M$ induced by the antidiagonal action on $ET\times M$. The space of T-equivariant flowlines from $[\varepsilon^-,x^-]$ to $[\varepsilon^+,x^+]$ has dimension $|\varepsilon^-,x^-|-|\varepsilon^+,x^+|$.

Exactly like the non-equivariant Morse cohomology definition in Section 3.3.3, the *T-equivariant Morse cochain complex* is $\mathbb{Z}\langle \mathrm{Crit}^{\mathrm{eq}}(f^{\mathrm{eq}},M)\rangle$ and the *T-equivariant Morse differential* counts $\mathbb{R}$-equivalence classes of T-equivariant flowlines (see Figure 3.1a). We denote the resulting *T-equivariant Morse cohomology* by $H^*_T(M;f^{\mathrm{eq}})$.

The *(geometric*[8]*) $H^*(BT)$-module structure* is given by counting 'Y'-shaped graphs in ET paired with flowlines in M (see Figure 3.1b). Such a configuration is a T-equivalence class of triples (v,v_0,γ) where $[v,\gamma]$ is a T-equivariant flowline, and v_0 is a perturbed half$^+$ flowline of g satisfying $v_0(0)=v(0)$. For a critical point $c\in\mathrm{Crit}(g,BT)$ and two T-equivariant critical points $[\varepsilon^-,x^-]\in\mathrm{Crit}^{\mathrm{eq}}(f^{\mathrm{eq}},M)$, let $\mathcal{M}([\varepsilon^-,x^-];c)$ be the moduli space of such $[(v,v_0,\gamma)]$ which satisfy $[v(-\infty),\gamma(-\infty)]=[\varepsilon^-,x^-]$ and $[v_0(+\infty)]=c$. The module structure is given by

$$y^c * [\varepsilon^+,x^+] = \sum_{\substack{[\varepsilon^-,x^-]\\ \dim\mathcal{M}=0}} \#\mathcal{M}([\varepsilon^-,x^-];c)\,[\varepsilon^-,x^-]. \tag{3.3.15}$$

Remark 3.3.2 (Dependence on the basis τ). Our construction of ET uses the basis τ of T, but T-equivariant cohomology does not depend on τ. While there are many ways to show independence of τ, not all approaches also allow us to show that the T-equivariant quantum product (3.3.17) is independent of τ. Here we describe a Morse-theoretic construction which uses carefully-chosen Morse data to simplify the moduli spaces. In this remark, we are not concerned with independence of the Morse data on ET because this can be demonstrated using standard homotopy arguments.

Let τ and τ' be two bases of T. Let E and E' denote the corresponding constructions of ET (both spaces are $(S^\infty)^k$, but they have different T-actions). Let g and g' denote the corresponding Morse functions. The product $E\times E'$ with the product T-action satisfies the criteria for ET: it is a contractible space with a free T-action. It is sufficient to show E and $E\times E'$ yield isomorphic constructions of T-equivariant cohomology. Set $\pi_E : E\times E'\to E$. The induced projection $(E\times E')/T\to E/T$ is a fibration with contractible fibre E'. The function $g+g':(E\times E')/T\to\mathbb{R}$ is Morse-Bott with critical submanifolds isomorphic to T. Pick a perfect Morse function on each critical submanifold. Given a critical point $\varepsilon\in E$ of g, denote by $\min(\varepsilon)\in(g')^{-1}(0)\subset E'$ the unique point such that $[\varepsilon,\min(\varepsilon)]$ is the minimum on the critical submanifold $((T\cdot\varepsilon)\times(g')^{-1}(0))/T\subset(E\times E')/T$.

[8] An alternative *algebraic module structure* is given by shifting coordinates in BT (see [LJ20, Section 4.4.1]).

Fix a T-equivariant Morse function $f^{\text{eq}}_E : E \times M \to \mathbb{R}$ from the construction of T-equivariant cohomology which uses E. We can use the functions f^{eq}_E and $g + g'$, together with the Morse functions on the critical submanifolds, to define a Morse-Bott construction of T-equivariant cohomology which uses $E \times E'$. Consider the map $\pi_E : \text{Crit}(E \times_T M) \to \text{Crit}((E \times E') \times_T M)$ given by $\pi_E([(\varepsilon, x)]) = [((\varepsilon, \min(\varepsilon)), x)]$. The map π_E induces a map on the Morse cochain complex, and we prove that π_E is a chain map by computing the differential $d([((\varepsilon^+, \min(\varepsilon^+)), x^+)])$. This differential counts isolated equivariant flowlines, and these are either flowlines that flow from $[((\varepsilon^-, \min(\varepsilon^-)), x^-)]$, which correspond to flowlines on $E \times_T M$, or flowlines that are constant in $E \times M$, which correspond to the flowlines of the perfect Morse function on $T \cdot \min(\varepsilon^-)$. These latter flowlines cancel with each other because the Morse differential of a perfect Morse function on T is zero. Additionally, $\pi_E : H^*(E \times_T M) \to H^*((E \times E') \times_T M)$ is an isomorphism. It is easiest to prove this by appealing to topological properties of pullback maps, using the functorial flowline construction below to assert π_E is indeed a pullback map, however it is also possible to prove this directly by computing the differential on $(E \times E')/T$. In any case, we have explicitly constructed a pullback map $\pi_E : H^*(E \times_T M) \to H^*((E \times E') \times_T M)$ and found that it is an isomorphism, demonstrating the required independence.

In general, pullback maps like π_E count functorial flowlines,[9] however in our setup the only isolated functorial flowlines for π_E are constant in E and M, and flow from the minimum in E', so π_E is given by the above formula. Since the T-equivariant Morse function does not depend on E', the two half flowlines on M join to create a single flowline on M. It is this property that allows us to deduce that the T-equivariant quantum product is independent of the basis τ.

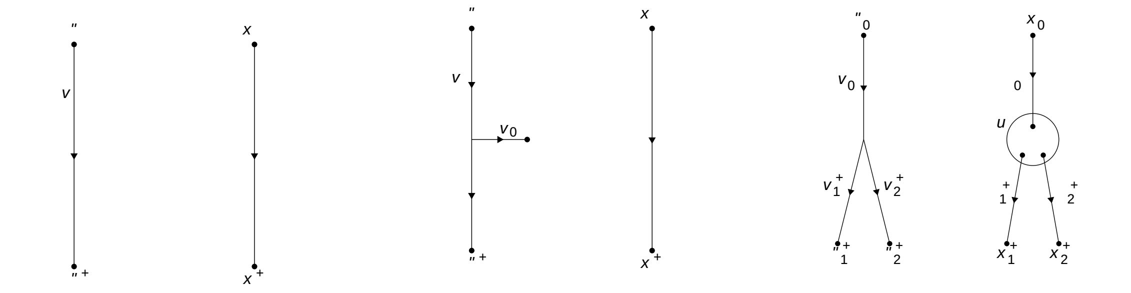
3.3.9 Equivariant almost complex structures

A *T-equivariant almost complex structure* J^{eq} is a choice of almost complex structure J^{eq}_e for each $e \in ET$ such that the diagram

$$\begin{array}{ccc} T_m M & \xrightarrow{J^{\text{eq}}_{e;m}} & T_m M \\ \big\downarrow D\rho_{\mathbf{t}} & & \big\downarrow D\rho_{\mathbf{t}} \\ T_{\mathbf{t}\cdot m} M & \xrightarrow{J^{\text{eq}}_{\mathbf{t}^{-1}\cdot e;\mathbf{t}\cdot m}} & T_{\mathbf{t}\cdot m} M \end{array} \tag{3.3.16}$$

commutes. This condition implies that the space of pairs (e, u) of elements $e \in ET$ and J^{eq}_e-holomorphic maps $u : \mathbb{P}^1 \to M$ inherits a natural T-action given by $\mathbf{t} \cdot (e, u) = (\mathbf{t}^{-1} \cdot e, \rho_{\mathbf{t}} \circ u)$. We will assume that our T-equivariant almost complex structures are everywhere

[9] For the map $\varphi : M^- \to M^+$, a *functorial flowline* is a pair of perturbed half flowlines $\gamma_\pm$ in M which satisfy $\varphi(\gamma_-(0)) = \gamma_+(0)$. See [LJ20, Footnote 34] or [RV14, Section 1.3].



(a) Equivariant differential

(b) Geometric $H^*(BT)$-module structure

(c) Equivariant quantum product

Figure 3.1: Our diagrams for equivariant configurations have two components. The left component describes the part that is mapped to ET while the right component describes the part that is mapped to M. In general, our Morse flowlines flow downwards on the page, in the direction that the Morse function decreases. As such, the 'Y'-shaped graphs are upside-down when compared to the letter 'Y'.

ω-compatible, regular and convex. Here, *convex* means there exists $R_0 \in [1, \infty)$ such that $dR \circ J^{eq}_{e;m} = R\alpha$ holds for all $e \in ET$ and all $m \in \{R \geq R_0\}$.

The regularity ensures that the space $M(A; J^{eq}, K)$ of pairs (e, u) with $e \in K$ and $u_*[\mathbb{P}^1] = A$ is a manifold of dimension $2n + 2c_1(TM, \omega)(A) + \dim K$ for the T-invariant compact subsets $K = (S^{2r-1})^k \subset ET$.

3.3.10 Equivariant quantum cohomology

Analogously to non-equivariant quantum cohomology, the *T-equivariant quantum cohomology* $QH^*_{T;\Lambda}(M, \rho; f^{eq})$ of M is the cohomology of $\Lambda \hat{\otimes} \mathbb{Z}\langle \mathrm{Crit}^{eq}(f^{eq}, M)\rangle$ with the T-equivariant Morse differential. Unlike the non-equivariant case, the set $\mathrm{Crit}^{eq}(f^{eq}, M)$ is typically infinite, so we use the graded completed tensor product[10] $\hat{\otimes}$ in place of the standard tensor product $\otimes$. The Λ-module $QH^*_{T;\Lambda}(M, \rho; f^{eq})$ is immediately a $\Lambda \hat{\otimes} H^*(BT)$-module by combining the formal Novikov ring action with the geometric $H^*(BT)$-module structure.

The T-equivariant quantum product $*$ gives $QH^*_{T;\Lambda}(M, \rho; f^{eq})$ the structure of a $\Lambda \hat{\otimes} H^*(BT)$-algebra. It is defined by counting 'Y'-shaped graphs in ET paired with deformed 'Y'-shaped graphs in M (see Figure 3.1c). Explicitly, a *T-equivariant deformed 'Y'-shaped graph* is a T-equivalence class of septuples $(v_0^-, v_1^+, v_2^+, \gamma_0^-, \gamma_1^+, \gamma_2^+, u)$ where $[v_i^\pm, \gamma_i^\pm]$ are perturbed T-equivariant half flowlines and $u : \mathbb{P}^1 \to M$ is a $J^{eq}_{v_0^-(0)}$-holomorphic map subject to the

[10] The *graded completed tensor product* of two $\mathbb{Z}$-graded $\mathbb{Z}$-modules A and B is $A \hat{\otimes} B = \bigoplus_{l \in \mathbb{Z}} (A \hat{\otimes} B)^l$, where the degree-$l$ summand is $(A \hat{\otimes} B)^l = \prod_{l' \in \mathbb{Z}} A^{l'} \otimes B^{l-l'}$. For us, any degree-$l$ element in the $\mathbb{Z}$-module $\Lambda \hat{\otimes} \mathbb{Z}\langle \mathrm{Crit}^{eq}(f^{eq}; M)\rangle$ has the property that, given $c \in \mathbb{R}$ and $l', l'' \in \mathbb{Z}$, only finitely-many of the terms $q^A \otimes [\varepsilon; x]$ with $|q^A| = l'$, $\omega(A) \leq c$ and $|[\varepsilon]| \leq l''$ are supported. This is an equivariant extension of the corresponding condition which defines the Novikov ring (see Section 3.3.2).

conditions $v_0(0) = v_1^+(0) = v_2^+(0)$ and $u(p_i) = \gamma_i(0)$. Let $\mathcal{M}([\varepsilon_i, x_i], A)$ be the moduli space of such T-equivariant deformed 'Y'-shaped graphs with the obvious critical point conditions and $[v_0(0), u] \in \mathcal{M}^{\mathrm{eq}}(A; J^{\mathrm{eq}})$. The *$T$-equivariant quantum product* is given by

$$[\varepsilon_1^+, x_1^+] \ast [\varepsilon_2^+, x_2^+] = \sum_{\substack{A \in H_2(M) \\ [\varepsilon_0, x_0] \in \mathrm{Crit}^{\mathrm{eq}}(f^{\mathrm{eq}}; M) \\ \dim \mathcal{M} = 0}} \# \mathcal{M}([\varepsilon_i, x_i], A)\, q^A [\varepsilon_0, x_0]. \tag{3.3.17}$$

Like the non-equivariant quantum product, the T-equivariant quantum product is unital, graded-commutative and associative by standard homotopy arguments. Moreover, it is independent of the choice of T-equivariant Morse function f^{eq} and the choice of T-equivariant almost complex structure J^{eq}.

3.3.11 The extended torus

Denote a copy of the circle S^1 by S^1_0. Set $\widehat{T} = S^1_0 \times T$. The character $\widehat{T} \to S^1$ which is a projection to S^1_0 is denoted $\widehat{\chi}_0$, and the corresponding variable in $H^*(B\widehat{T})$ is denoted $\widehat{y}_0$. We will use the notation $\widehat{\ }$ to denote actions, bases, (co)characters, etc. which correspond to $\widehat{T}$.

We are interested in the $\widehat{T}$-equivariant quantum cohomology $QH^*_{\widehat{T},\widehat{\rho}}(M, \widehat{\rho})$. Here, the $\widehat{T}$-action $\widehat{\rho}$ on M induced by ρ is uniquely determined by $\widehat{\rho}|_T = \rho$ and $\widehat{\rho}|_{S^1_0} = \mathrm{Id}_M$. We use bases $\widehat{\tau}$ of $\widehat{T}$ which are given by

$$\begin{cases} \widehat{\tau}_0(s) = (s, \tau_0(s)), \\ \widehat{\tau}_i(s) = (0, \tau_i(s)), \end{cases} \tag{3.3.18}$$

where τ is a basis of T and $\tau_0 : S^1 \to T$ is a further cocharacter of T. Such bases satisfy $(\widehat{\tau}_i, \widehat{\chi}_0) = \delta_{i,0}$, where δ is the Kronecker delta. Certain constructions use the basis $\widehat{\tau}$, but the cohomological invariants are independent of the choice by Remark 3.3.2.

3.3.12 Lifting to equivariant quantum cohomology

Let $\mu \in M$ be a fixed point. The inclusion map $\{\mu\} \to M$ is $\widehat{T}$-equivariant, so there is an induced map $\mu^* : H^*_{\widehat{T}}(M) \to H^*_{\widehat{T}}(\{\mu\}) = H^*(B\widehat{T})$ as in (3.3.11). The map μ^* induces a splitting of the short exact sequence

$$0 \longrightarrow H^2(B\widehat{T}) \longrightarrow H^2_{\widehat{T}}(M, \widehat{\rho}) \longrightarrow H^2(M) \longrightarrow 0. \tag{3.3.19}$$

The short exact sequence (3.3.19) is the degree-2 part of (3.3.12) for $(M, \widehat{\rho})$, and it is exact by Proposition 3.3.1. The corresponding dashed map is

$$H^2(M) \to H^2_{\widehat{T}}(M, \widehat{\rho}), \quad \alpha \mapsto \widehat{\alpha} \quad \text{for } \alpha \in H^2(M). \tag{3.3.20}$$

3.3.13 Differential connection

There is a differential connection on $QH_{\widehat{T}}(M,\widehat{\rho})$, in the sense of Definition 3.2.6. To define this connection, we must describe the data (k, A, X, P, ∇), as in Section 3.2.1.

The integral domain k is $H(BS^1_0) = \mathbb{Z}[\widehat{y}_0]$. The k-algebra A is $\widehat{H}(B\widehat{T})$. The A-module P is $QH_{\widehat{T}}(M,\widehat{\rho})$, which we will abbreviate by Q for convenience.

Associated to a cohomology class $\alpha \in H^2(M)$, we have a derivation $\widehat{y}_0\frac{d}{d\alpha}$ on $\widehat{H}(B\widehat{T})$ given by

$$\left(\widehat{y}_0\frac{d}{d\alpha}\right)(q^A y^c) = \alpha(A) q^A \widehat{y}_0 y^c. \tag{3.3.21}$$

Note that the operation $\frac{d}{d\alpha}$ does not change the exponent of q, so it actually behaves more like the derivation $t\frac{d}{dt}$ than pure differentiation $\frac{d}{dt}$ in a polynomial ring $k[t]$ (see Example 3.2.2). Nonetheless, the map (3.3.21) satisfies the Leibniz rule (3.2.1). The space of derivations is

$$X = [\widehat{y}_0] \otimes H^2(M), \tag{3.3.22}$$

where $a \otimes \alpha \in X$ corresponds to the derivation $a\,\widehat{y}_0\frac{d}{d\alpha}$ for $a \in [\widehat{y}_0]$.

Let $\mu \in M$ be a fixed point. Given $\alpha \in H^2(M)$, the map

$$\nabla_\alpha = \widehat{y}_0\frac{d}{d\alpha} + \alpha : Q \to Q \tag{3.3.23}$$

given by

$$\nabla_\alpha(q^A[\varepsilon, x]) = \alpha(A)\widehat{y}_0\, q^A[\varepsilon, x] + \alpha(q^A[\varepsilon, x]) \tag{3.3.24}$$

satisfies the Leibniz rule (3.2.2). It naturally extends to a connection

$$\nabla : X \to \mathrm{Hom}_{\mathbb{Z}[\widehat{y}_0]}(Q, Q). \tag{3.3.25}$$

Theorem 3.3.3. *The connection ∇ is flat.*

Proof. By Lemma 3.2.5, it is sufficient to check that the curvature (3.2.3) vanishes for any two classes $\alpha, \beta \in H^2(M)$. This further reduces to verifying $[\nabla_\alpha, \nabla_\beta] = 0$ because the derivations $\widehat{y}_0\frac{d}{d\alpha}$ and $\widehat{y}_0\frac{d}{d\beta}$ commute.

Let us write out the terms of $\nabla_\alpha\nabla_\beta(q^A[\varepsilon, x])$:

$$\begin{aligned}
\nabla_\alpha\nabla_\beta(q^A[\varepsilon, x]) &= \nabla_\alpha\left(\beta(A)\, q^A\widehat{y}_0[\varepsilon, x] + \beta(q^A[\varepsilon, x])\right) \\
&= \alpha(A)\beta(A)\, q^A\widehat{y}_0^2[\varepsilon, x] && (3.3.26)\\
&\quad + \beta(A)\,\alpha\, q^A\widehat{y}_0[\varepsilon, x] && (3.3.27)\\
&\quad + \left(\widehat{y}_0\frac{d}{d\alpha}\right)(\beta\, q^A[\varepsilon, x]) && (3.3.28)\\
&\quad + \alpha(\beta\, q^A[\varepsilon, x]). && (3.3.29)
\end{aligned}$$

The first term (3.3.26) is clearly symmetric in α and β. The last term (3.3.29) is also symmetric in α and β because the $\widehat{T}$-equivariant quantum product is graded-commutative and each of α and β has even degree.

We will show that the remaining two terms (3.3.27) and (3.3.28) cancel with their counterparts in r r . It is these four terms together that cancel (they do not pair o).

Analogously to how the $\widehat{T}$-equivariant quantum product : Q ^{2} ! Q counts $\widehat{T}$-equivariant deformed 'Y'-shaped graphs, the map : Q ^{3} ! Q counts $\widehat{T}$*-equivariant deformed ' '-shaped graphs* (see Figure 3.2a). These graphs are $\widehat{T}$-equivalence classes of 10-tuples $(v_0, v_1^+, v_2^+, v_3^+, \gamma_0, \gamma_1^+, \gamma_2^+, \gamma_3^+, u, p_3^+)$ where $[v_i, \gamma_i]$ are $\widehat{T}$-equivariant perturbed half flowlines, u : P^1 ! M is a $J^{eq}_{v_0(0)}$-holomorphic map and p_3^+ is a point in P^1 n $\{p_0, p_1^+, p_2^+\}$, subject to the conditions $v_0(0) = v_1^+(0) = v_2^+(0) = v_3^+(0)$ and $u(p_i) = \gamma_i(0)$.

is graded-commutative. We show this by permuting the $\widehat{T}$-equivariant half$^+$ flowlines. Suppose the 10-tuple $(v_0, v_1^+, v_2^+, v_3^+, \gamma_0, \gamma_1^+, \gamma_2^+, \gamma_3^+, u, p_3^+)$ is counted by (a,b,c). Let φ : P^1 ! P^1 be the unique automorphism which fixes p_0 and p_1^+ and satisfies $\varphi(p_2^+) = p_3^+$. The 10-tuple $(v_0, v_1^+, v_3^+, v_2^+, \gamma_0, \gamma_1^+, \gamma_3^+, \gamma_2^+, u$ φ, φ $^1(p_2^+))$ is counted by the map (a,c,b). Here, we have switched $[v_2^+, \gamma_2^+]$ and $[v_3^+, \gamma_3^+]$ and modified P^1 to accommodate the switch. Implicitly, we are appealing to the fact that is independent of the perturbations of the $\widehat{T}$-equivariant Morse functions. Altogether, this yields (a,b,c) = (1)^{jbjjcj} (a,c,b). The same argument demonstrates (a,b,c) = (1)^{jajjbj} (b,a,c).

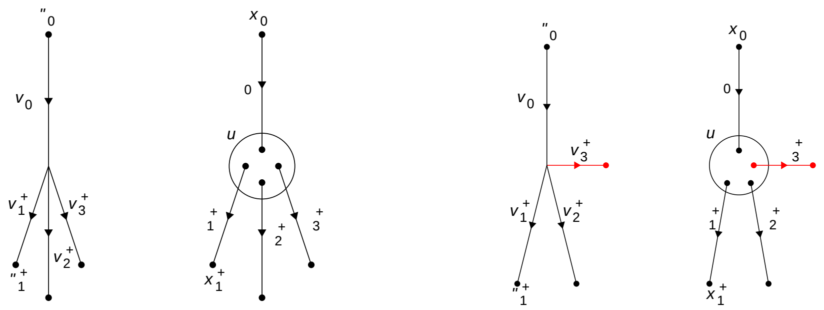
In the counts for $(q^A[\varepsilon,x],\alpha\ ,\beta\ )$, the 2-dimensional freedom of the point p_3^+ is matched by the 2-dimensional intersection condition $[v_3^+, \gamma_3^+]$(*1*) = β . As such, these are the same counts as for $(q^A[\varepsilon,x])$ α , except where the $\widehat{T}$-equivariant deformed 'Y'-shaped graphs have weights $\beta(u$ [P^1]) (see Figure 3.2b). This gives the relation

$$(q^A[\varepsilon,x],\alpha\ ,\beta\ ) = \left(\widehat{y}_0\frac{d}{d\beta}\right)(\alpha\quad q^A[\varepsilon,x])\quad\alpha\quad\left(\left(\widehat{y}_0\frac{d}{d\beta}\right)(q^A[\varepsilon,x])\right). \tag{3.3.30}$$

This relation, together with the same relation with α and β switched and the graded-commutativity statement $(q^A[\varepsilon,x],\alpha\ ,\beta\ )$ = $(q^A[\varepsilon,x],\beta\ ,\alpha\ )$, completes the proof of flatness. □

3.3.14 Clutching bundle

Fix a cocharacter σ *2* Cochar T for which the action ρ σ is admissible for a nonnegative slope function. We denote the set of such ρ*-nonnegative cocharacters* by Cochar $^0 T$. The *clutching bundle* $E(\sigma)$ is a symplectic M-bundle over the sphere S^2 which is associated to σ. We outline the construction here, but further details for closed M are found in [Sei97] and for convex M in [Rit14].



(a) $\widehat{T}$-equivariant deformed ‘ ’-shaped graph

(b) β -weighted equivariant product by α

Figure 3.2: The $\widehat{T}$-equivariant deformed ‘ ’-shaped graphs that counts are the same configurations that the β -weighted equivariant product by α counts. Only the interpretation of the flowline $[v_3^+, \gamma_3^+]$ changes between the two maps.

Write the sphere S^2 as a union of its two hemispheres $\mathbb{D}^-$ and $\mathbb{D}^+$. Each hemisphere is a closed disc, and they are glued along the equator. Use coordinates for $\mathbb{D}^\pm$ from the closed unit disc in $\mathbb{C}$ and for the equator $S^1 = \mathbb{R}/\mathbb{Z}$, so the gluing identifications are

$$\theta \in S^1 \leftrightarrow e^{2\pi i\theta} \in \partial\mathbb{D}^- \leftrightarrow e^{-2\pi i\theta} \in \partial\mathbb{D}^+. \tag{3.3.31}$$

The bundle $E(\sigma)$ is the union of the trivial bundles $\mathbb{D}^\pm \times M$ over the hemispheres, glued over the equator via

$$\partial\mathbb{D}^- \times M \ni (e^{2\pi i\theta}, m) \leftrightarrow (e^{-2\pi i\theta}, \rho_{\sigma(\theta)}(m)) \in \partial\mathbb{D}^+ \times M. \tag{3.3.32}$$

Denote the projection map by $\pi : E(\sigma) \to S^2$. The vertical tangent bundle $T^{\mathrm{vert}}E(\sigma) = \ker D\pi$ is naturally equipped with the symplectic bilinear form $\Omega_{z;m} = \omega_m$ for all $z \in \mathbb{D}^\pm$. The action $\rho \circ \sigma$ is symplectic, so Ω is well-defined.

The circle S^1_0 acts on S^2 by rotation. The fixed points of the S^1_0-action are the poles $z^\pm = 0 \in \mathbb{D}^\pm$. The complement of the poles $S^2 \setminus \{z^\pm\}$ is isomorphic to an open cylinder via

$$\mathbb{R} \times S^1 \ni (s, \theta) \mapsto \begin{cases} e^{2\pi(s+i\theta)} \in \mathbb{D}^- & \text{if } s \leq 0 \\ e^{-2\pi(s+i\theta)} \in \mathbb{D}^+ & \text{if } s \geq 0, \end{cases} \tag{3.3.33}$$

and the circle S^1_0 acts via $a \cdot (s, \theta) = (s, \theta + a)$. This rotation action on the sphere S^2 lifts to a $\widehat{T}$-action on $E(\sigma)$. The element $(a, \mathbf{t}) \in \widehat{T} = S^1_0 \times T$ acts via

$$\begin{array}{llllll} \mathbb{D}^- \times M & \ni & (z, m) & \mapsto & (e^{2\pi ia}z, \rho_{\mathbf{t}\sigma(a)}(m)) & \in \mathbb{D}^- \times M \\ \mathbb{D}^+ \times M & \ni & (z, m) & \mapsto & (e^{-2\pi ia}z, \rho_{\mathbf{t}}(m)) & \in \mathbb{D}^+ \times M \end{array} \tag{3.3.34}$$

The fibres above the poles $z^\pm$ are $\widehat{T}$-invariant under this action. Identifying each fibre with M using the local trivialisation $\mathbb{D}^\pm \times M$, the $\widehat{T}$-action over the south pole z^+ is $\widehat{\rho}$ and the

$\widehat{T}$-action over the north pole z is $\sigma \quad \widehat{\rho} = \widehat{\rho} \quad \widehat{\sigma}^{\ 1}$, where $\widehat{\sigma}$ is the automorphism

$$\widehat{\sigma} : \widehat{T} \to \widehat{T}, \quad (a, \mathbf{t}) \mapsto (a, \mathbf{t} + \sigma(a)). \tag{3.3.35}$$

3.3.15 Sections of the clutching bundle

We are concerned with pseudoholomorphic sections of $\pi : E(\sigma) \to S^2$. With this in mind, let j be the complex structure on S^2 induced by (3.3.33). Let $\mathbf{J}$ be an almost complex structure on $E(\sigma)$ for which π is a $(\mathbf{J}, j)$-holomorphic map, which restricts to a convex -compatible almost complex structure on $T^{\text{vert}}E(\sigma)$ and which satisfies a Floer-type convexity condition outside of a compact subset (see [Rit14, Definition 26]). Such an almost complex structure is *admissible*. For a regular admissible $\mathbf{J}$, the space of $(j, \mathbf{J})$-holomorphic sections $u : S^2 \to E(\sigma)$ with $u \ [S^2] = A \in H_2(E(\sigma))$ is a manifold of dimension $2n + 2c_1(T^{\text{vert}}E(\sigma), \ )(A)$.

Let $\mu \in M$ be a fixed point. There is a *fixed section* $u \ : S^2 \to E(\sigma)$ associated to μ given by $u \ (z) = (z, \mu)$ for $z \in S^2$. Set $|\sigma, \mu| = \ 2c_1(T^{\text{vert}}E(\sigma), \ )(u \ [S^2])$. This quantity $|\sigma, \mu|$ is equal to twice the sum of the weights of $\rho \quad \sigma$ around μ [MT06, Lemma 2.2]. The section u splits the short exact sequence

$$0 \longrightarrow H_2(M) \xrightarrow{\ z\ } H_2(E(\sigma)) \longrightarrow H_2(S^2) = \mathbb{Z} \longrightarrow 0. \tag{3.3.36}$$

from Lemma 3.6.1. In (3.3.36), the map $z \ : M \to E(\sigma)$ is the inclusion map of the fibre above the north pole z . Given a class $A \in H_2(M)$, set $A \ = z \ (A) + u \ [S^2]$. Note $2c_1(T^{\text{vert}}E(\sigma), \ )(A \ ) = 2c_1(TM, \omega)(A) \quad |\sigma, \mu|$. The assignment $A \mapsto A$ is a bijection between classes $A \in H_2(M)$, which are recorded by the Novikov ring, and those classes in $H_2(E(\sigma))$ which can represent sections of $E(\sigma)$.

A *spiked section* is a triple $(\gamma \ , u, \gamma^+)$ of perturbed half flowlines γ in M together with a $(j, \mathbf{J})$-holomorphic section u which satisfies $u(z \ ) = \gamma \ (0)$. Let $\mathcal{M}(x \ , x^+, A \ )$ denote the space of spiked sections $(\gamma \ , u, \gamma^+)$ which satisfy $\gamma \ (\ 1) = x$ and $u \ [S^2] = A$. The *quantum Seidel map*

$$QS(\sigma, \mu) : QH \ (M; f) \to QH^{\ +|\ ;\ |}(M; f) \tag{3.3.37}$$

counts spiked sections; it is given by

$$QS(\sigma, \mu)\ (x^+) = \sum_{\substack{x \ ;A \\ \dim \mathcal{M} = 0}} \# \mathcal{M}(x \ , x^+, A \ )\ q^A x \ . \tag{3.3.38}$$

The quantum Seidel map commutes with the quantum product.

3.3.16 The equivariant quantum Seidel map

Analogously to Section 3.3.9, a $\widehat{T}$*-equivariant admissible almost complex structure* $\mathbf{J}^{eq}$ is a choice of admissible almost complex structure on $E(\sigma)$ for each $e \in E\widehat{T}$ which makes an analogous diagram to (3.3.16) commute. The convexity of $\mathbf{J}^{eq}|_{T^{vert}E(\ )}$ and the Floer-type convexity condition must hold in a common region at infinity for all $e \in E\widehat{T}$.

The $\widehat{T}$*-equivariant quantum Seidel map*

$$QS_{\widehat{T}}(\sigma,\mu) : QH_{\widehat{T}}(M,\widehat{\rho}; f^{eq}_{+}) \to QH_{\widehat{T}}^{+|\ |}(M,\sigma\ \widehat{\rho}; f^{eq}) \tag{3.3.39}$$

counts $\widehat{T}$-equivariant spiked sections, analogously to (3.3.37). These $\widehat{T}$*-equivariant spiked sections* are $\widehat{T}$-equivalence classes of quadruples $(v,\gamma\ ,u,\gamma^{+})$ where v is a flowline in $E\widehat{T}$, γ are perturbed half flowlines of $f^{eq}(v(s),\ )$ and u is a $(j,\mathbf{J}^{eq}_{v(0)})$-holomorphic section which satisfies $u(z\ ) = \gamma\ (0)$. See Figure 3.3a. The two $\widehat{T}$-equivariant Morse functions $f^{eq} : E\widehat{T}\ M \to \mathbb{R}$ satisfy different relations because the domain and codomain have different $\widehat{T}$-actions.

A homotopy argument shows that $QS_{\widehat{T}}(\sigma,\mu)$ commutes with the geometric $H\ (B\widehat{T})$-action.

3.3.17 Intertwining relation

Let $\alpha \in H^2_{\widehat{T}}(E(\sigma))$ be a $\widehat{T}$-equivariant degree-2 coclass for the $\widehat{T}$-action (3.3.34) on the total space $E(\sigma)$. Denote by α the restrictions of α to the fibres above the poles z . This gives two elements $\alpha\ \in H^2_{\widehat{T}}(M,\sigma\ \widehat{\rho})$ and $\alpha^{+} \in H^2_{\widehat{T}}(M,\widehat{\rho})$.

The α*-weighted* $\widehat{T}$*-equivariant quantum Seidel map*

$$WQS_{\widehat{T}}(\sigma,\mu,\alpha) : QH_{\widehat{T}}(M,\widehat{\rho}; f^{eq}_{+}) \to QH_{\widehat{T}}^{+|\ |}(M,\sigma\ \widehat{\rho}; f^{eq}) \tag{3.3.40}$$

counts $\widehat{T}$-equivariant spiked sections $[v,\gamma\ ,u,\gamma^{+}]$, but weighted by $\alpha([v(0),u([S^2])])$. We construct $WQS_{\widehat{T}}(\sigma,\mu,\alpha)$ in the proof of Theorem 3.3.4 using a $\widehat{T}$-equivariant half^{+} flowline to α to record the weight.

Theorem 3.3.4 (Intertwining relation). *The relation*

$$QS_{\widehat{T}}(\sigma,\mu)(x\ \ \alpha^{+}) \quad QS_{\widehat{T}}(\sigma,\mu)(x)\ \ \alpha\ = \widehat{y}_0\ WQS_{\widehat{T}}(\sigma,\mu,\alpha)(x) \tag{3.3.41}$$

holds for all $x \in QH_{\widehat{T}}(M,\widehat{\rho}; f^{eq}_{+})$.

Theorem 3.3.4 is the $\widehat{T}$-equivariant version of the S^1-equivariant intertwining relation that we showed in [LJ20], and the proof is almost identical. The new proofs in Section 3.4 use similar ideas.

Sketch proof. Fix a $\widehat{T}$-equivariant identification $\min(g) \to \widehat{T}$, where $\min(g)$ is the minimal locus of $g : E\widehat{T} \to \mathbb{R}$. Composing with the character $\widehat{\chi}_0 : \widehat{T} \to S^1_0$, this gives a S^1_0-equivariant map $\arg : \min(g) \to S^1_0$.

Likewise, define the S^1_0-equivariant map $\arg : S^2 \setminus \{z_\pm\} \to S^1_0$ via $(s, \theta) \mapsto \theta$, using the cylindrical coordinates from (3.3.33).

The homotopy which gives (3.3.41) on cohomology is given by counting $\widehat{T}$-equivalence classes of octuples $(v, \gamma^-, u, \gamma^+, v^{\min}, v_0, \gamma_0, z_0)$ where $[v, \gamma^-, u, \gamma^+]$ is a $\widehat{T}$-equivariant spiked section, $v^{\min}$ is a perturbed half$^+$ flowline in $E\widehat{T}$, $[v_0, \gamma_0]$ is a $\widehat{T}$-equivariant half$^+$ flowline that converges to α and $z_0 \in S^2 \setminus \{z_\pm\}$ is a point which together satisfy the conditions $v(0) = v^{\min}(0) = v_0(0)$, $\gamma_0(0) = u(z_0)$ and $\arg(z_0) + \arg(v^{\min}(+\infty)) = 0$. See Figure 3.3b. The last condition leaves a 1-dimensional freedom in the point z_0 along the line of longitude between the poles and with fixed argument.

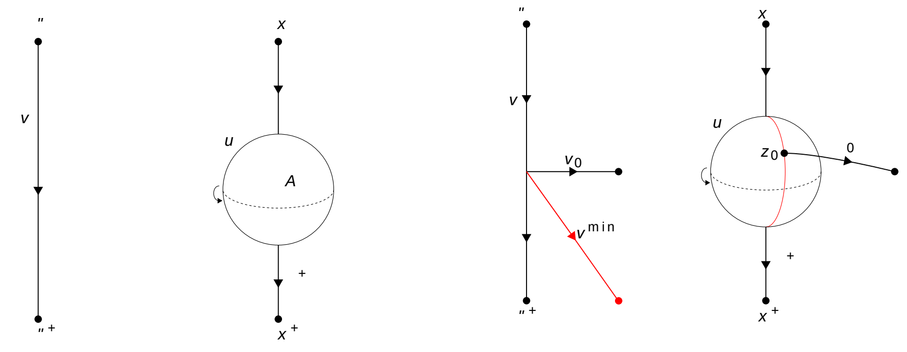
The map $WQS_{\widehat{T}}(\sigma, \mu, \alpha)$ is given by the same count, but with $v^{\min}$ omitted. Each section u is counted as many times as there are intersections of $u(S^2)$ and $W^s(\alpha)$. The 2-dimensional freedom of $z_0 \in S^2 \setminus \{z_\pm\}$ is cancelled by the 2-dimensional intersection condition with α.

The two terms on the left-hand side of (3.3.41) arise as the point z_0 converges to one of the poles $z_\pm$. The term on the right-hand side of (3.3.41) arises as $v^{\min}$ breaks. The half$^+$ flowline $v^{\min}$ breaks into a pair of flowlines, the first of which is a half$^+$ flowline from $v(0)$ to a critical point ε, where the critical point $c = [\varepsilon]$ corresponds to $\widehat{y}_0$ (the critical point c exists and is unique because of our choice of basis satisfying (3.3.18)). The second flowline is from ε to a point $\varepsilon^{\min} \in \min(g)$ which satisfies $\arg(\varepsilon^{\min}) = \arg(z_0)$. It turns out that the point $\varepsilon^{\min}$ and the entire second flowline are uniquely determined by ε and z_0, so we discard this second flowline from the moduli space without losing any information. Using a homotopy, we decouple the flowline to c from the rest of the configuration, and the resulting moduli spaces yield the maps $\widehat{y}_0$ and $WQS_{\widehat{T}}(\sigma, \mu, \alpha)$ respectively (more details in [LJ20]). □

The fact there is no global S^1_0-equivariant function $E\widehat{T} \to S^1_0$ is why the right-hand side of (3.3.41) is nonzero. The best we can do is define such a function on $W^s(\min(g))$, which is a dense subset. In the proof, the function $W^s(\min(g)) \to S^1_0$ arises as the composition of flowing along the flowline $v^{\min}$ with the map $\arg : \min(g) \to S^1_0$.

3.3.18 Pullback group isomorphisms

The group isomorphism $\widehat{\sigma} : \widehat{T} \to \widehat{T}$ associated to a cocharacter $\sigma : S^1 \to T$ is given by $(a, \mathbf{t}) \mapsto (a, \mathbf{t} + \sigma(a))$. The induced map $(B\widehat{\sigma})^* : H^*(B\widehat{T}) \to H^*(B\widehat{T})$ is given by $[\widehat{\chi}] \mapsto [\widehat{\chi} \circ \widehat{\sigma}]$ for characters $\widehat{\chi} : \widehat{T} \to S^1$. Note that we have $\widehat{\chi}_0 = \widehat{\chi}_0 \circ \widehat{\sigma}$, giving $(B\widehat{\sigma})^*(\widehat{y}_0) = \widehat{y}_0$.



(a) $\widehat{T}$-equivariant quantum Seidel map

(b) Homotopy from proof of intertwining relation

Figure 3.3: An additional flowline $v^{\min}$ in $E\widehat{T}$ is used to describe a path between the poles. It is from this path that the half$^+$ flowline γ_0 flows.

In this paper, we are using an explicit classifying bundle $(S^\infty)^{k+1} \to (\mathbb{CP}^\infty)^{k+1}$ for $E\widehat{T} \to B\widehat{T}$ which is associated to a basis $\widehat{\tau}$. Denote by ϱ the coordinate-wise action of $(S^1)^{k+1}$ on $(S^\infty)^{k+1}$. The action of $\widehat{T}$ on $(S^\infty)^{k+1}$ is $\varrho \circ \widehat{\tau}^{-1}$, where $\widehat{\tau} : (S^1)^{k+1} \to \widehat{T}$ is the isomorphism associated to the basis $\widehat{\tau}$. The identity map $\mathrm{Id} : (S^\infty)^{k+1} \to (S^\infty)^{k+1}$ satisfies

$$\mathrm{Id}\big(\varrho(\widehat{\tau}^{-1}(\widehat{\mathbf{t}}), \widehat{e})\big) = \varrho\big((\widehat{\sigma} \circ \widehat{\tau})^{-1}(\widehat{\sigma}(\widehat{\mathbf{t}})), \mathrm{Id}(\widehat{e})\big) \tag{3.3.42}$$

for all $\widehat{e} \in (S^\infty)^{k+1}$ and all $\widehat{\mathbf{t}} \in \widehat{T}$. This relation (3.3.42) is the relation (3.3.9) applied to our setting. Therefore the identity map $\mathrm{Id} : (S^\infty)^{k+1} \to (S^\infty)^{k+1}$ is $E\widehat{\sigma}$, where the domain has associated basis $\widehat{\tau}$ and the codomain has the associated basis $\widehat{\sigma} \circ \widehat{\tau}$.

This explicit construction of $E\widehat{\sigma}$ as the identity map using different bases extends to the Borel homotopy quotients. The identity map on the Borel homotopy quotients induces the isomorphism

$$(B\widehat{\sigma})^* : H^*_{\widehat{T};\widehat{\sigma}\circ\widehat{\tau}}(M, \widehat{\rho} \circ \widehat{\sigma}^{-1}; f^{\mathrm{eq}}) \to H^*_{\widehat{T};\widehat{\tau}}(M, \widehat{\rho}; f^{\mathrm{eq}}). \tag{3.3.43}$$

Denote the basis $\widehat{\sigma} \circ \widehat{\tau}$ by $\sigma \cdot \widehat{\tau}$.

The map (3.3.43) is compatible with the $\widehat{T}$-equivariant quantum product $*$ because it is essentially the identity map. Thus $(B\widehat{\sigma})^*(x * y) = (B\widehat{\sigma})^*(x) * (B\widehat{\sigma})^*(y)$ holds.

3.3.19 Shift operator

The *shift operator* on quantum cohomology is the composition

$$\mathsf{S}_\sigma = (B\widehat{\sigma})^* \circ QS_{\widehat{T}}(\sigma, \mu) : QH^*_{\widehat{T};\widehat{\tau}}(M, \widehat{\rho}) \to QH^{*+I(\sigma,\mu)}_{\widehat{T};\sigma\cdot\widehat{\tau}}(M, \widehat{\rho}). \tag{3.3.44}$$

The shift operator is defined on the commutative monoid $\mathrm{Cochar}^{\geq 0}\, T$ of ρ-nonnegative cocharacters $\sigma : S^1 \to T$.

Theorem 3.3.5. *The shift operator* S_σ *is flat (and hence well-defined on* Cochar${}^0 T$*).*

Proof. We must show $S_\sigma S_{\sigma'} = S_{\sigma+\sigma'}$ for any two $\sigma, \sigma' \in$ Cochar${}^0 T$. This statement can be proved directly by gluing sections exactly as in the non-equivariant case [MS04, Theorem 11.4.3]. On the other hand, it is much easier to show $S_\sigma S_{\sigma'} = S_{\sigma+\sigma'}$ directly for the shift operators S_σ on $\widehat{T}$-equivariant Floer cohomology. We can deduce the quantum statement from the Floer statement, Theorem 3.4.10, using $\widehat{T}$-equivariant PSS isomorphisms (see Proposition 3.4.9). This second approach is analogous to that taken in [Sei97]. □

Theorem 3.3.6. *The difference-differential connection* $(S_\sigma, \nabla_\alpha)$ *is flat.*

Proof. The differential connection ∇_α is flat by Theorem 3.3.3. The shift operator S_σ is flat by Theorem 3.3.5.

It remains to show $[\nabla_\alpha, S_\sigma] = 0$ for any $\alpha \in H^2(M)$ and any $\sigma \in$ Cochar${}^0 T$. By Lemma 3.6.2, there is a class $\beta \in H^2_{\widehat{T}}(E(\sigma))$ which satisfies $\beta^+ = \alpha$, $(B\widehat{\sigma})^*(\beta^-) = \alpha$ and $\beta(A_\sigma) = \alpha(A)$ for $A \in H_2(M)$. The first two conditions imply that the intertwining relation Theorem 3.3.4 applied to β, post-composed with $(B\widehat{\sigma})^*$, gives

$$S_\sigma(x \star \alpha) - S_\sigma(x) \star \alpha = (B\widehat{\sigma})^* \widehat{y}_0 \, WQS_{\widehat{T}}(\sigma, \mu, \beta)(x) \tag{3.3.45}$$

for $x \in QH_{\widehat{T};\widehat{\rho}}(M, \widehat{\rho})$. The third condition $\beta(A_\sigma) = \alpha(A)$ gives

$$\left(\widehat{y}_0 \frac{d}{d\alpha}\right) S_\sigma(x) - S_\sigma\left(\left(\widehat{y}_0 \frac{d}{d\alpha}\right) x\right) = (B\widehat{\sigma})^* \widehat{y}_0 \, WQS_{\widehat{T}}(\sigma, \mu, \beta)(x), \tag{3.3.46}$$

analogously to (3.3.30). The difference (3.3.46) − (3.3.45) is $[\nabla_\alpha, S_\sigma](x) = 0$. □

3.4 Equivariant Floer cohomology

The symplectic manifold (M, ω) has the same assumptions as Section 3.3.1.

3.4.1 Hamiltonian orbits

The time-dependent Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ is *linear of slope* $\lambda \in \mathbb{R} \setminus \mathcal{R}$ if the equation $H_t(\psi(R, y)) = \lambda R$ + constant holds outside a compact set. The (1-periodic Hamiltonian) orbits $x : S^1 \to M$ of such a Hamiltonian H are the solutions of $\partial_t x(t) = X_{H;t}$, where the Hamiltonian vector field $X_{H;t}$ is determined by $\omega(\cdot, X_{H;t}) = dH_t(\cdot)$. At infinity, the Hamiltonian vector field $X_{H;t} = \lambda X_{\mathcal{R}}$ is a multiple of the Reeb vector field, but since $\lambda \in \mathbb{R} \setminus \mathcal{R}$ is not a Reeb period, there are no 1-periodic Hamiltonian orbits in this region. It follows that a regular linear Hamiltonian of slope λ has only finitely-many orbits.

The linearity assumption on Hamiltonian torus actions is very restrictive, so we instead use a generalised linearity condition introduced by Ritter [Rit16, Appendix C]. This condition replaces the constant λ with a function $\lambda : \Sigma \to \mathbb{R}$ which is Reeb flow-invariant. For such a function, set X_λ to be the vector field on Σ determined by

$$\begin{aligned} \alpha(X_\lambda) &= \lambda \\ d\alpha(\cdot, X_\lambda) &= d\lambda(\cdot). \end{aligned} \tag{3.4.1}$$

Let L be the set of Reeb flow-invariant functions $\lambda : \Sigma \to \mathbb{R}$ for which the vector field X_λ has no closed orbits of period 1.

The time-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$ is *admissible* if

$$H_t(\psi(R, y)) = \lambda(y)R + \text{constant} \tag{3.4.2}$$

holds at infinity for some $\lambda \in L$. The function λ is the *slope* (or *slope function*) of H. The Hamiltonian vector field of H_t equals the vector field X_λ at infinity, and hence there are no Hamiltonian orbits outside of a compact set by assumption. Therefore, a regular admissible Hamiltonian has only finitely-many orbits as desired. Denote the set of orbits of H by $P(H)$.

Lemma 3.4.1. *For $\lambda \in \mathbb{R} \setminus R$, the constant function $\lambda(y) \equiv \lambda$ lies in L. In particular, linear Hamiltonians are admissible.*

Proof. For the constant function, we have $X_\lambda = \lambda X_\alpha$, and this vector field has no 1-periodic orbits by the definition of R. □

Lemma 3.4.2. *Positive Reeb flow-invariant functions $\lambda : \Sigma \to \mathbb{R}$ generically lie in L.*

Proof. The periodic orbits of a generic contact form are non-degenerate [Bou03a, Lemma 2] (more details in [Gut10, Section 3.4], see also [Bou03b, Lemma 2.3]). In the proof, we perturb a given contact form α by rescaling it by a positive Reeb flow-invariant function $f : \Sigma \to \mathbb{R}$. For generic such functions f, the periodic orbits of $f\alpha$ are non-degenerate. A direct calculation shows that the vector field X_λ equals the Reeb vector field $X_{\frac{1}{\lambda}\alpha}$ of the rescaled contact form $\frac{1}{\lambda}\alpha$. Therefore, the periodic orbits of X_λ are non-degenerate for generic λ. In particular, the periods of these orbits are discrete, so generically there will be no orbits with period 1. □

Lemma 3.4.2 immediately extends to negative functions by reversing the parameterisation of the orbits. The author expects that the positivity/negativity assumption is not necessary, perhaps by a modification of the argument that the orbits of a generic contact form are non-degenerate. It is, however, beyond the scope of this work to prove this.

3.4.2 Fillings

A *filling* of an orbit $x : S^1 \to M$ is a continuous map $f : D \to M$ which satisfies $f(e^{2 \, it}) = x(t)$, where $D = \{|z| \;\; 1\} \;\; \mathbb{C}$ is a 2-disc. Two fillings f, f' of the same orbit x combine to give a map $(f \sqcup_x f') : S^2 = D \sqcup_{S^1} D \to M$ (orient $D \sqcup_{S^1} D$ with the standard orientation on the first copy of D and the opposite orientation on the second copy of D). These fillings are *equivalent* if $(f \sqcup_x f')_* [S^2] = 0 \in H_2(M)$ holds. If $(f \sqcup_x f')_* [S^2] = A \in H_2(M)$ holds, then we write $[f] = A \# [f']$ for the corresponding equivalence classes. This coincides with the connected sum of A with f' (see [HS95]). We denote an orbit x together with an equivalence class of fillings by $\widetilde{x}$. Denote the set of such pairs by $\widetilde{P}(H)$.

The Conley-Zehnder index $|\widetilde{x}|$ of $\widetilde{x}$ is well-defined, for a filling of x induces a trivialisation of $x^* TM$. We use the convention for the Conley-Zehnder index for which the Conley-Zehnder index of a critical point of a small Hamiltonian recovers the Morse index, as in [Rit14]. For this convention, we have $|A \# \widetilde{x}| = |\widetilde{x}| \;\; 2c_1(TM, \omega)(A)$.

For each orbit $x_i \in P(H)$, fix a filling $\widetilde{x}_i \in \widetilde{P}(H)$ for the orbit. The *Floer cochain complex* $FC^*(M, \lambda; H)$ is the graded $\;$-module $\langle \widetilde{x}_i \rangle$. It is independent of the choices of fillings because the alternative filling $\widetilde{x}_i' = (\;\; A) \# \widetilde{x}_i$ corresponds to the element $q^A \widetilde{x}_i$. Equivalently, we have $FC^*(M, \lambda; H) = \;\; \mathbb{Z}\langle \widetilde{P}(H) \rangle / \;\;$, where the relation $\;$ is generated by

$$q^A \widetilde{x} \quad (\;\; A) \# \widetilde{x}. \tag{3.4.3}$$

Remark 3.4.3 (Cohomological conventions for fillings). Poincaré duality can be established in Morse theory by reversing the flowlines [AD14, Section 4.3]. The *homology Morse differential* for the Morse function f is $d(x^- ; f) = x^+ + \cdots$, and it counts (R-equivalence classes of) flowlines $\gamma(s)$ from the critical point x^- to the critical point x^+. The *cohomology Morse differential* for f, in contrast, is $d(x^+; f) = x^- + \cdots$. If we reverse the flowline $\gamma(s)$, we get the flowline $\gamma(-s)$ of the Morse function $-f$ which goes from x^+ to x^-. The cohomology Morse differential for $-f$ is $d(x^-; -f) = x^+ + \cdots$, so it coincides with $d(x^-; f)$. Therefore the Morse homology of f is isomorphic to the Morse cohomology of $-f$, as desired.

Similarly, in Floer theory, we reverse the Floer solutions to recover Poincaré duality. If $u(s, t)$ is a Floer solution for H_t, then $u(-s, -t)$ is a Floer solution for $-H_{-t}$. The homological convention $q^A \widetilde{x} \;\; A \# \widetilde{x}$ for fillings was described in [HS95, Section 5]. Under Poincaré duality, the filling $f : D \to M$ for $x(t)$ is mapped to the filling $\overline{f} : D \to M$ for $x(-t)$, where $\overline{f}$ denotes the precomposition of f with the complex conjugate map. Therefore, the filling $A \# f$ is mapped to $(-A) \# \overline{f}$ under Poincaré duality. With our Novikov ring $\;$ from Section 3.3.2, the relation (3.4.3) is the correct convention for the $\;$-ring action on

Floer cohomology. This remark corrects the convention in [Rit14, Section 2.7] and an earlier draft of [LJ20].

3.4.3 Floer solutions

A *Floer datum* is a pair (H, J) consisting of a time-dependent admissible Hamiltonian H and a time-dependent convex ω-compatible almost complex structure J. A *Floer solution* is a smooth map $u : \mathbb{R} \times S^1 \to M$ which satisfies

$$\partial_s u + J_t(\partial_t u - X_{H,t}) = 0 \tag{3.4.4}$$

for all $(s, t) \in \mathbb{R} \times S^1$ and whose energy

$$E(u) = \int_{\mathbb{R} \times S^1} \|\partial_s u\|^2_{\omega(\cdot, J_t \cdot)} \, ds \, dt \tag{3.4.5}$$

is finite. Here, the norm is taken with respect to the time-dependent metric $\omega(\cdot, J_t \cdot)$.

For a regular Floer datum (H, J), we consider the moduli space $\mathcal{M}(\tilde{x}^-, \tilde{x}^+)$ of Floer solutions u which satisfy the uniform limits $u(s, \cdot) \to x^{\pm}(\cdot)$ and $\partial_s u(s, \cdot) \to 0$ as $s \to \pm\infty$ and the condition $\tilde{x}^+ = u \# \tilde{x}^-$ on the fillings.[11] This moduli space is a smooth manifold of dimension $|\tilde{x}^-| - |\tilde{x}^+|$ and has a free $\mathbb{R}$-action given by s-translation (except for constant u). Moreover, the Floer solutions are restricted to a compact subset of M by a No Escape Lemma [Rit16, Theorem 9.2]. The moduli space may be compactified by broken Floer solutions with bubbling via standard compactification and gluing theorems, and bubbling for $\dim \mathcal{M}(\tilde{x}^-, \tilde{x}^+) \leq 2$ is avoided by regularity (see [Sal97]).

The *Floer differential* counts Floer solutions mod s-translation. It is given by

$$d\tilde{x}^+ = \sum_{\substack{\tilde{x}^- \in \tilde{\mathcal{P}}(H) \\ |\tilde{x}^-| - |\tilde{x}^+| = 1}} \# \left(\frac{\mathcal{M}(\tilde{x}^-, \tilde{x}^+)}{\mathbb{R}} \right) \tilde{x}^- . \tag{3.4.6}$$

The orientation of $\mathcal{M}(\tilde{x}^-, \tilde{x}^+)$ comes from a choice of *coherent orientation*, which shall be implicit throughout (see [Rit13, Appendix B] for details).

The *Floer cohomology* $FH^*(M, \lambda; H, J)$ is the cohomology of $(FC^*(M, \lambda; H), d)$.

3.4.4 Continuation maps

Given two Floer data (H^-, J^-) and (H^+, J^+), a *homotopy* between them is an $\mathbb{R}$-dependent family of Floer data (H_s, J_s) which depends on $s \in \mathbb{R}$ only on a bounded interval and equals $(H^{\pm}, J^{\pm})$ as $s \to \pm\infty$. The homotopy is *monotone* if the slope of H_s is nonincreasing as s

[11] Here, the notation $u \# \tilde{x}^-$ denotes the cylinder $u : \mathbb{R} \times S^1 \to M$ and the closed disc $\tilde{x}^- : D \to M$ glued along their common boundary $x^- : S^1 \to M$, which is a filling of the orbit x^+.

increases. Notice this means the slope functions $\lambda^\pm$ of $H^\pm$ satisfy $\lambda^- \le \lambda^+$. Denote by δ the difference $\lambda^- - \lambda^+ \le 0$ of the slope functions $\lambda^\pm$ of $H^\pm$.

A *continuation solution* is a smooth map $u : \mathbb{R} \times S^1 \to M$ which satisfies

$$\partial_s u + J_{s,t}(\partial_t u - X_{H_{s,t}}) = 0 \tag{3.4.7}$$

and whose energy

$$E(u) = \int_{\mathbb{R}\times S^1} \|\partial_s u\|^2_{\omega(\cdot, J_{s,t}\cdot)} \, \mathrm{d}s \, \mathrm{d}t \tag{3.4.8}$$

is finite. That is, a continuation solution is a Floer solution, but with s-dependent Floer data. The moduli space $\mathcal{M}_s(\tilde{x}^-, \tilde{x}^+)$ of continuation solutions between $\tilde{x}^\pm \in \widetilde{\mathcal{P}}(H^\pm)$ is defined as for Floer solutions. For regular Floer data and a regular monotone homotopy, the moduli spaces $\mathcal{M}_s(\tilde{x}^-, \tilde{x}^+)$ are smooth manifolds of dimension $|\tilde{x}^-| - |\tilde{x}^+|$. The map which counts continuation solutions is a chain map. The *continuation map* between the Floer data is the induced map on Floer cohomology

$$\varphi : FH^*(M, \lambda^+; H^+, J^+) \to FH^*(M, \lambda^-; H^-, J^-) \tag{3.4.9}$$

and it is independent of the choice of monotone homotopy. The composition of two continuation maps is itself a continuation map. Moreover, Floer cohomology only depends on the slope of the Hamiltonian, and is otherwise independent of the choice of Floer datum.

The *symplectic cohomology* $SH^*(M)$ of M is the direct limit of the direct system $FH^*(M, \lambda)$ for $\lambda \in L$. The maps of the direct system are the continuation maps, and the underlying poset is $(L, \le)$. Note that the constant functions in Lemma 3.4.1 form a cofinal subset of L, so our definition of $SH^*(M)$ agrees with the standard definition of $SH^*(M)$ which uses only linear Hamiltonians.

3.4.5 Equivariant Floer cohomology

Let T and $\rho : T \times M \to M$ be as in Section 3.3.6. The extended torus $\widehat{T} = S^1_0 \times T$ acts on the (contractible) loop space $\mathcal{L}M = \{\text{contractible } x : S^1 \to M\}$ via

$$((a, \mathbf{t}) \cdot x)(t) = \widehat{\rho}_{(a,\mathbf{t})}(x(t - a)) = \rho_{\mathbf{t}}(x(t - a)). \tag{3.4.10}$$

Recall that S^1_0 acts trivially under $\widehat{\rho}$ as in Section 3.3.11. Our model of $\widehat{T}$-equivariant Floer cohomology is a Borel-style model which captures the action (3.4.10) on the Hamiltonian orbits.

Fix a basis $\widehat{\tau}$ for $\widehat{T}$. Let $E\widehat{T}$ be the space $(S^\infty)^{k+1}$ defined using the basis and let $g : E\widehat{T} \to \mathbb{R}$ be the corresponding Morse function, as in Section 3.3.8. A *$\widehat{T}$-equivariant Hamiltonian function* is a function $H^{\mathrm{eq}} : E\widehat{T} \times S^1 \times M \to \mathbb{R}$ which is $\widehat{T}$-invariant, so it satisfies $H^{\mathrm{eq}}_{e,t}(m) = H^{\mathrm{eq}}_{(a,t)^{-1}e,a+t}(\widehat{\rho}_{(a,t)}(m))$, and which satisfies the following condition on the critical points in $E\widehat{T}$. For every critical point $\varepsilon \in \mathrm{Crit}(g, E\widehat{T})$, the function $H^{\mathrm{eq}}_{\varepsilon,\cdot}(\cdot)$ is a regular Hamiltonian on $S^1 \times M$, and moreover $H^{\mathrm{eq}}_{\varepsilon,\cdot}(\cdot) = H^{\mathrm{eq}}_{e,\cdot}(\cdot)$ for all e in a neighbourhood of ε in $W^u(\varepsilon) \cup W^s(\varepsilon)$. Moreover, we ask that H^{eq} is *admissible* for some ρ-invariant slope function $\lambda \in \mathcal{L}$, which means that there is $R_0 > 1$ such that $H^{\mathrm{eq}}_{\cdot}(\psi(R, y)) = \lambda R + \text{constant}$ for all $R \geq R_0$.

A $\widehat{T}$*-equivariant Hamiltonian orbit* is a $\widehat{T}$-equivalence class $[\varepsilon, x] \in E\widehat{T} \times_{\widehat{T}} \mathcal{L}M$ with $\varepsilon \in \mathrm{Crit}(g, E\widehat{T})$ and $x \in \mathcal{P}(H^{\mathrm{eq}}_{\varepsilon,\cdot})$. The set of such equivariant orbits is denoted $\mathcal{P}^{\mathrm{eq}}(H^{\mathrm{eq}})$. We can attach equivalence classes of fillings as per Section 3.4.2. The corresponding set is denoted $\widetilde{\mathcal{P}}^{\mathrm{eq}}(H^{\mathrm{eq}})$ and the index of $[\varepsilon, \widetilde{x}] \in \widetilde{\mathcal{P}}^{\mathrm{eq}}(H^{\mathrm{eq}})$ is $|\varepsilon, \widetilde{x}| = |\varepsilon| + |\widetilde{x}|$.

For each equivariant orbit $[\varepsilon, x] \in \mathcal{P}^{\mathrm{eq}}(H^{\mathrm{eq}})$, fix a filling $\widetilde{x}_B$ of x. Let $\mathcal{B} \subset \widetilde{\mathcal{P}}^{\mathrm{eq}}(H^{\mathrm{eq}})$ be the set of such equivariant orbits with fillings. Such a set is a *basis of fillings*. The $\widehat{T}$*-equivariant Floer cochain complex* $FC_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\mathrm{eq}})$ is the $\mathbb{Z}$-graded Λ-module with degree-l summand

$$FC^l_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\mathrm{eq}}) = \prod_{[\varepsilon, \widetilde{x}_B] \in \mathcal{B}} \left(\Lambda\langle[\varepsilon, \widetilde{x}_B]\rangle\right)^l. \tag{3.4.11}$$

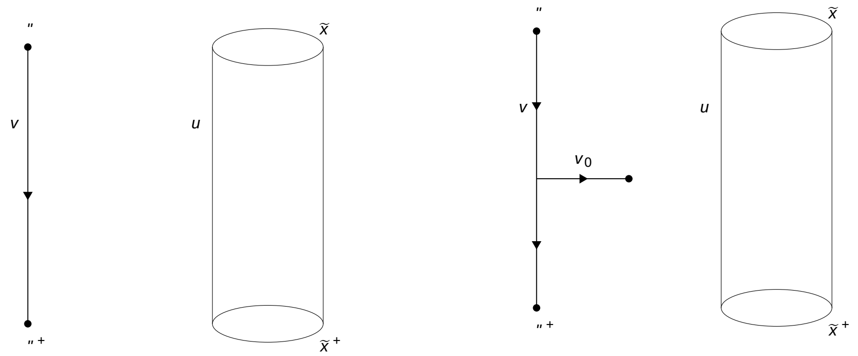
Let us unpack this definition. First, this is independent of the choices in $\mathcal{B}$ just as in the non-equivariant definition: the equivariant orbit $[\varepsilon, (-A)\#\widetilde{x}_B] \in \widetilde{\mathcal{P}}^{\mathrm{eq}}(H^{\mathrm{eq}})$ corresponds to the element $q^A[\varepsilon, \widetilde{x}_B]$. Second, for each critical point $c \in \mathrm{Crit}(g, B\widehat{T})$, there are finitely-many equivariant orbits $[\varepsilon, x] \in \mathcal{P}^{\mathrm{eq}}(H^{\mathrm{eq}})$ with $[\varepsilon] = c$. Thus the part of the complex (3.4.11) made up of such orbits is isomorphic to $FC^{l-|c|}(M, \lambda; H^{\mathrm{eq}}_{\varepsilon,\cdot})$ for any ε with $[\varepsilon] = c$.

A $\widehat{T}$*-equivariant time-dependent almost complex structure* J^{eq} is a choice of almost complex structure $J^{\mathrm{eq}}_{e,t}$ for all $\widehat{\mathbf{t}} \in E\widehat{T}$ and all $t \in S^1$ such that the diagram

$$\begin{array}{ccc} T_m M & \xrightarrow{\;J^{\mathrm{eq}}_{e,t}\;} & T_m M \\ \big\downarrow{\scriptstyle D\widehat{\rho}_{(a,t)}} & & \big\downarrow{\scriptstyle D\widehat{\rho}_{(a,t)}} \\ T_{\widehat{\rho}_{(a,t)}(m)} M & \xrightarrow{\;J^{\mathrm{eq}}_{(a,t)^{-1}e,t+a}\;} & T_{\widehat{\rho}_{(a,t)}(m)} M \end{array} \tag{3.4.12}$$

commutes, and which satisfies the following condition on the critical points in $E\widehat{T}$. For every critical point $\varepsilon \in \mathrm{Crit}(g, E\widehat{T})$, the time-dependent almost complex structure $J^{\mathrm{eq}}_{\varepsilon,\cdot}$ is regular, and moreover $J^{\mathrm{eq}}_{\varepsilon,\cdot} = J^{\mathrm{eq}}_{e,\cdot}$ for all e in a neighbourhood of ε in $W^u(\varepsilon) \cup W^s(\varepsilon)$. A $\widehat{T}$*-equivariant Floer datum* is a pair $(H^{\mathrm{eq}}, J^{\mathrm{eq}})$ comprising an admissible $\widehat{T}$-equivariant time-dependent Hamiltonian function of slope λ and a convex ω-compatible $\widehat{T}$-equivariant time-dependent almost complex structure J^{eq}.

A $\widehat{T}$*-equivariant Floer solution* is a $\widehat{T}$-equivalence class of pairs (v, u) where $v : \mathbb{R} \to E\widehat{T}$ is a flowline of g and $u : \mathbb{R} \times S^1 \to M$ is a continuation solution for the s-dependent Floer



(a) Equivariant differential

(b) Geometric $H^*(B\widehat{T})$-module structure

Figure 3.4: The equivariant Floer differential and geometric module action acutely resemble those of equivariant Morse cohomology (Figure 3.1). The cylinders on the right-hand side of the diagrams denote a Floer cylinder $\mathbb{R}\times S^1 \to M$.

data $(H^{\mathrm{eq}}_{v(s),\cdot}, J^{\mathrm{eq}}_{v(s),\cdot})$. See Figure 3.4a. Notice how the condition on the $\widehat{T}$-equivariant Floer datum near the critical points in $E\widehat{T}$ means this s-dependent Floer data depends on s only on a bounded interval. The $\widehat{T}$-action on such pairs is given by $(a,\mathbf{t})\cdot(v,u) = ((a,\mathbf{t})^{-1}\cdot v, u')$ with $u'(s,t) = \widehat{\rho}_{(a,\mathbf{t})}(u(s,t-a))$.

The $\widehat{T}$-equivariant Floer solution $[v,u]$ *converges* to the $\widehat{T}$-equivariant orbits $[\varepsilon^\pm,\widetilde{x}^\pm]$ as $s\to\pm\infty$ if $v\in \mathcal{M}(\varepsilon^-,\varepsilon^+)$ and $u\in\mathcal{M}_s(\widetilde{x}^-,\widetilde{x}^+)$ hold for some choices of representatives. For a regular $\widehat{T}$-equivariant Floer datum, the moduli space $\mathcal{M}([\varepsilon^-,\widetilde{x}^-],[\varepsilon^+,\widetilde{x}^+])$ of such $\widehat{T}$-equivariant Floer solutions is a smooth manifold of dimension $|\varepsilon^-,\widetilde{x}^-|-|\varepsilon^+,\widetilde{x}^+|$. The *$\widehat{T}$-equivariant Floer differential* counts equivariant Floer solutions mod s-translation. It is given by

$$d[\varepsilon^+,\widetilde{x}^+] = \sum_{\substack{[\varepsilon^-,\widetilde{x}^-]\in\widetilde{P}^{\mathrm{eq}}(H^{\mathrm{eq}})\\ |\varepsilon^-,\widetilde{x}^-|-|\varepsilon^+,\widetilde{x}^+|=1}} \#\left(\frac{\mathcal{M}([\varepsilon^-,\widetilde{x}^-],[\varepsilon^+,\widetilde{x}^+])}{\mathbb{R}}\right)[\varepsilon^-,\widetilde{x}^-]. \qquad (3.4.13)$$

The *$\widehat{T}$-equivariant Floer cohomology* $FH^*_{\widehat{T},\widehat{\tau}}(M,\widehat{\rho},\lambda;H^{\mathrm{eq}},J^{\mathrm{eq}})$ is the cohomology of the $\widehat{T}$-equivariant Floer cochain complex. It is independent of the $\widehat{T}$-equivariant Floer datum by standard homotopy arguments and it is independent of $\widehat{\tau}$ by Remark 3.3.2.

As with the equivariant Morse cohomology in Section 3.3.8, $\widehat{T}$-equivariant Floer cohomology has a (geometric) $H^*(B\widehat{T})$-module structure. It is given by counting $\widehat{T}$-equivariant Floer solutions together with a perturbed half-flowline in $E\widehat{T}$ (see Figure 3.4b). Combined with the formal u-multiplication action on the Floer cochain complex, this yields a $\widehat{H}^*(B\widehat{T})$-module structure on $\widehat{T}$-equivariant Floer cohomology.

As with (non-equivariant) Floer cohomology, we can construct $\widehat{T}$-equivariant continuation maps corresponding to $\widehat{T}$-equivariant monotone homotopies between different $\widehat{T}$-equivariant Floer data. These $\widehat{T}$-equivariant continuation maps have the same properties: they compose and they are isomorphisms if the slopes are the same. Therefore $\widehat{T}$-equivariant Floer cohomology depends only on the slope of the $\widehat{T}$-equivariant Hamiltonian.

$\widehat{T}$-equivariant symplectic cohomology $SH_{\widehat{T}}(M, \widehat{\rho})$ is the direct limit of the direct system $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ for $\lambda \in L$. As in the non-equivariant setup, Lemma 3.4.1 shows that the constant slope functions $\lambda \in \mathbb{R} \setminus R$ form a cofinal subset of L.

3.4.6 Differentiation

Fix a regular $\widehat{T}$-equivariant Floer datum $(H^{\mathrm{eq}}, J^{\mathrm{eq}})$ and a basis of fillings B. For $\alpha \in H^2(M)$, the differentiation map

$$\frac{d}{d\alpha} : FC_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\mathrm{eq}}, J^{\mathrm{eq}}) \to FC_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\mathrm{eq}}, J^{\mathrm{eq}}) \tag{3.4.14}$$

is given by $q^A[\varepsilon, \widetilde{x}] \mapsto \alpha(A) q^A[\varepsilon, \widetilde{x}]$ for all $[\varepsilon, \widetilde{x}] \in B$. This map depends on the choice of basis B. The differenatiation map $\frac{d}{d\alpha}$ is *not* a chain map. Instead, we have

$$\frac{d}{d\alpha} \circ d - d \circ \frac{d}{d\alpha} = d^B, \tag{3.4.15}$$

where d^B is the α-weighted differential (3.4.17) for the basis B.

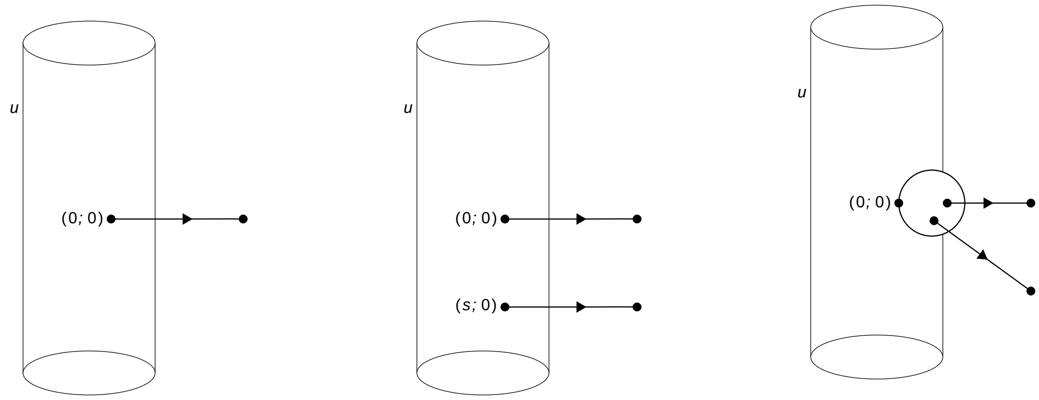
An alternative expression for the Floer differential using the filling basis B is

$$d[\varepsilon^+, \widetilde{x}_B^+] = \sum_{\substack{A \in H_2(M) \\ [\varepsilon^-, \widetilde{x}_B^-] \in B \\ |\varepsilon^-; (-A)\#\widetilde{x}_B^-| - |\varepsilon^+; \widetilde{x}_B^+| = 1}} \; \sum_{\mathbb{R}\cdot[v;u] \in \frac{\mathcal{M}([\varepsilon^-; (-A)\#\widetilde{x}_B^-]; [\varepsilon^+; \widetilde{x}_B^+])}{\mathbb{R}}} \#(\mathbb{R}\cdot[v, u])\, q^A[\varepsilon^-, \widetilde{x}_B^-]. \tag{3.4.16}$$

In this expression, the count operation $\#(\mathbb{R}\cdot[v, u])$ simply captures the sign associated to this point of the quotient moduli space. Note that $u\#((-A)\#\widetilde{x}_B^-) = \widetilde{x}_B^+$ holds, so A is determined by the Floer solution u and the fillings $\widetilde{x}_B$. The *α-weighted differential* d^B for B is given by

$$d^B[\varepsilon^+, \widetilde{x}_B^+] = \sum_{\substack{A \in H_2(M) \\ [\varepsilon^-, \widetilde{x}_B^-] \in B \\ |\varepsilon^-; (-A)\#\widetilde{x}_B^-| - |\varepsilon^+; \widetilde{x}_B^+| = 1}} \; \sum_{\mathbb{R}\cdot[v;u] \in \frac{\mathcal{M}([\varepsilon^-; (-A)\#\widetilde{x}_B^-]; [\varepsilon^+; \widetilde{x}_B^+])}{\mathbb{R}}} \#(\mathbb{R}\cdot[v, u])\, \alpha(A) q^A[\varepsilon^-, \widetilde{x}_B^-]. \tag{3.4.17}$$

The α-weighted differential d^B has an additional *weight* $\alpha(A)$, but otherwise resembles (3.4.16).



(a) Quantum action on Floer cohomology

(b) Homotopy with two outgoing half$^+$ flowlines

(c) Bubbled configuration as the half$^+$ flowlines come together

Figure 3.5: To show the quantum action on Floer cohomology (a) satisfies $\alpha * (\beta * \tilde{x}) = (\alpha * \beta) * \tilde{x}$, construct a homotopy with two half$^+$ flowlines as in (b). The limit $s \to \infty$ gives the left-hand side $\alpha * (\beta * \tilde{x})$ as the Floer solution breaks, while the limit $s \to 0^+$ gives the bubbled configuration (c) which is homotopic to the right-hand side $(\alpha * \beta) * \tilde{x}$.

3.4.7 Equivariant quantum action

The quantum cohomology action on Floer cohomology is the map $: QH^*(M) \otimes FH^*(M, \lambda) \to FH^*(M, \lambda)$ which counts Floer solutions u together with a perturbed half$^+$ flowline γ_0 from $u(0,0) = \gamma_0(0)$ (see Figure 3.5a). The map satisfies $\alpha * (\beta * \tilde{x}) = (\alpha * \beta) * \tilde{x}$, where the *quantum* product $(\alpha * \beta)$ arises as the Floer solution may bubble when the two perturbed half$^+$ flowlines come together (see Figure 3.5c).

To extend this action to the $\widehat{T}$-equivariant setting, we must change the $u(0,0) = \gamma_0(0)$ condition so that it respects the rotation action (3.4.10). Our strategy is just like the homotopy we constructed to prove the intertwining relation, so we use an additional perturbed half$^+$ flowline in $E\widehat{T}$ together with the argument function $\text{arg} : \text{min}(g) \to S^1_0$ to control the input t_0 in the new condition $u(0, t_0) = \gamma_0(0)$.

Explicitly, the new map

$$: H^2_{\widehat{T}}(M, \widehat{\rho}) \otimes FC^*_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\text{eq}}, J^{\text{eq}}) \to FC^*_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{\text{eq}}, J^{\text{eq}}) \qquad (3.4.18)$$

counts $\widehat{T}$-equivalence classes of sextuples $(v, u, v_0, \gamma_0, v^{\text{min}}, t_0)$ where $[v, u]$ is a $\widehat{T}$-equivariant Floer solution, $[v_0, \gamma_0]$ is a $\widehat{T}$-equivariant perturbed half$^+$ flowline, v^{min} is a perturbed half$^+$ flowline in $E\widehat{T}$ and $t_0 \in S^1$ is an element of the circle which together satisfy $v(0) = v_0(0) = v^{\text{min}}(0)$, $u(0, t_0) = \gamma_0(0)$ and $\text{arg}(v^{\text{min}}(+\infty)) + t_0 = 0$. See Figure 3.6a. We are interested in applying the action to the classes $\alpha \in H^2_{\widehat{T}}(M, \widehat{\rho})$.

Using the same Morse data as for the flowline $[v_0, \gamma_0]$ in the construction of (3.4.18), let $f : D \to M$ be a filling for $[\varepsilon, \widetilde{x}]$ for which $(\varepsilon, f(D)) \cap W^s_{\text{pert}}(\alpha)$ is a transverse intersection. Here, the *(perturbed) stable manifold* $W^s_{\text{pert}}(\alpha)$ of $\widehat{T}$-equivariant critical point α is the set of points $(e, m) \in E\widehat{T} \times M$ for which there is a perturbed half$^+$ flowline $[v_0, \gamma_0]$ to α with $(v_0, \gamma_0)(0) = (e, m)$. By regularity, no intersections occur on the Hamiltonian orbit $[\varepsilon, x]$ itself. Set

$$w([\varepsilon, \widetilde{x}]) = \#((\varepsilon, f(D)) \cap W^s_{\text{pert}}(\alpha))\, [\varepsilon, \widetilde{x}], \tag{3.4.19}$$

a degree-0 map on the $\widehat{T}$-equivariant Floer cochain complex. Similarly, let w^B be the -linear map given by

$$w^B(q^A[\varepsilon, \widetilde{x}_B]) = \#((\varepsilon, f_B(D)) \cap W^s_{\text{pert}}(\alpha))\, q^A[\varepsilon, \widetilde{x}_B], \tag{3.4.20}$$

where f_B is a filling for $[\varepsilon, \widetilde{x}_B]$. The convention (3.4.3) yields

$$\frac{d}{d\alpha} w^B = w. \tag{3.4.21}$$

Both w and w^B are independent of the actual fillings f and f_B used, but they do depend on the choice of Morse data for $W^s_{\text{pert}}(\alpha)$).

Let the (basis-free) α*-weighted differential* d be given by

$$d[\varepsilon^+, \widetilde{x}^+] = \sum_{\substack{[\varepsilon^-, \widetilde{x}^-] \in \widetilde{P}^{\text{eq}}(H^{\text{eq}}) \\ |\varepsilon^-, \widetilde{x}^-| - |\varepsilon^+, \widetilde{x}^+| = 1 \\ \mathbb{R}[(v,u)] \in \frac{M([\varepsilon^-, \widetilde{x}^-]; [\varepsilon^+, \widetilde{x}^+])}{\mathbb{R}}}} \#((v, u)(\mathbb{R} \times S^1) \cap W^s_{\text{pert}}(\alpha))\, [\varepsilon^-, \widetilde{x}^-]. \tag{3.4.22}$$

This map counts Floer solutions weighted by their intersections with $W^s_{\text{pert}}(\alpha)$, and, unlike d^B, it does not incorporate any intersections with the fillings. Combining the definitions yields

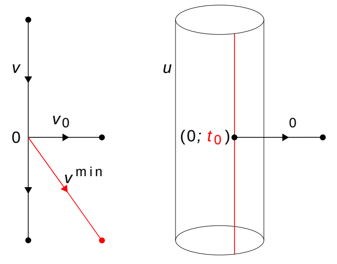
$$d = d^B w^B d + d w^B. \tag{3.4.23}$$

Proposition 3.4.4. For the class $\alpha \in H^2_{\widehat{T}}(M, \widehat{\rho})$, we have

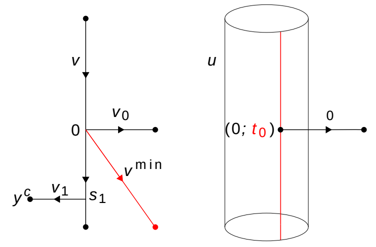
$$d(\alpha) \alpha d() = \widehat{y}_0 d. \tag{3.4.24}$$

Proof. The proof is analogous to the proof of the intertwining relation Theorem 3.3.4. The 1-dimensional moduli space of $\widehat{T}$-equivalence classes $[v, u, v_0, \gamma_0, v^{\min}, t_0]$ has a boundary given by breaking either the $\widehat{T}$-equivariant Floer solution $[v, u]$, breaking the $\widehat{T}$-equivariant perturbed half$^+$ flowline to α or breaking the perturbed half$^+$ flowline $v^{\min}$.

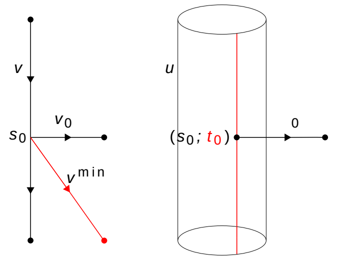
The terms on the left-hand side of (3.4.24) come from the $\widehat{T}$-equivariant Floer solution breaking, and the term on the right-hand side comes from the perturbed half$^+$ flowline $v^{\min}$ breaking. When the perturbed half$^+$ flowline $v^{\min}$ breaks, it breaks into a perturbed half$^+$ flowline to $c \in \text{Crit}(g, B\widehat{T})$ and a flowline (considered modulo s-translation) between



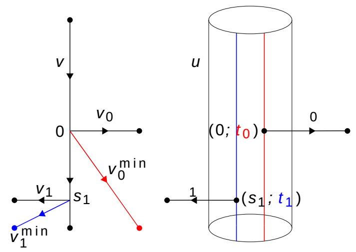
(a) Equivariant action α on equivariant Floer cohomology



(b) Homotopy for geometric module structure and connection



(c) Homotopy for continuation maps and connection



(d) Homotopy for flatness

Figure 3.6: The connection comprises a formal differentiation operation and the equivariant action α in (a). The action is modified into three different homotopies which are used to show the connection is compatible with the geometric H $(B\widehat{T})$-module structure (b), compatible with continuation maps (c) and flat (d).

c and min(g). Only the case $y^c = \widehat{y}_0$ has a 0-dimensional moduli space by regularity. The flowline between c and min(g) is uniquely determined by the rest of the configuration, for its endpoint in min(g) must satisfy arg( ) + t_0 = 0. Therefore the moduli space is unchanged if we omit the flowline between c and min(g) (we omit the argument relation too, so now t_0 2 S^1 is arbitrary). The perturbed half$^+$ flowline to c contributes the multiplication by $\widehat{y}_0$. The $\widehat{T}$-equivariant half$^+$ flowline to α means (R-equivalence classes of) $\widehat{T}$-equivariant Floer solutions $[v, u]$ are counted as many times as they intersect the (perturbed) stable manifold of α . □

3.4.8 Floer connection

The *Floer connection* r ;B on $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{eq}, J^{eq})$ is the collection of maps

$$r^{;B} : FH_{\widehat{T}}(M, \widehat{\rho}, \lambda; H^{eq}, J^{eq}) \; ! \; FH_{\widehat{T}}^{+2}(M, \widehat{\rho}, \lambda; H^{eq}, J^{eq})$$
$$r^{;B}(\,) = \widehat{y}_0 \frac{d}{d\alpha}(\,) + \alpha \quad (\,) \quad \widehat{y}_0 w^B(\,) \tag{3.4.25}$$

for each $\alpha \in H^2(M)$. The map $r^{;B}$ is a chain map because the error terms in (3.4.15) and (3.4.24) cancel with each other. The Leibniz rule is satisfied for multiplication with because the differentiation operation $\widehat{y}_0 \frac{d}{d}$ satisfies the Leibniz rule on the cochain complex while the other two maps are -linear on the cochain complex. The difference $\frac{d}{d} \quad w^B$ is independent of B by (3.4.21), and hence $r^{;B}$ is independent of B. This independence can also be shown using a homotopy construction based on continuation maps (Proposition 3.4.6).

Proposition 3.4.5. $r^{;B}$ is a $H(B\widehat{T})$-linear map.

Proof. We construct another homotopy in the intertwining relation style. Fix $y^c \in H(B\widehat{T})$ and $\alpha \in H^2(M)$. We have

$$(\quad w \quad)(y^c \quad) \quad y^c(\quad w \quad)(\,) = y^c(\,) \tag{3.4.26}$$

where y^c denotes the α-weighted geometric action of y^c, defined analogously to (3.4.22). The homotopy counts $\widehat{T}$-equivalence classes of septuples $(v, u, v_0, \gamma_0, v^{\min}, t_0, s_1, v_1)$ where $(v, u, v_0, \gamma_0, v^{\min}, t_0)$ is as for (3.4.18) and v_1 is a perturbed half$^+$ flowline in $E\widehat{T}$ which satisfies $v(s_1) = v_1(0)$ and $s_1 \in \mathbb{R} \setminus \{0\}$. See Figure 3.6b. The components of the 1-dimensional moduli space boundary which survive to cohomology are the limits $s_1 \to \infty$ and the breaking of the perturbed half$^+$ flowline $v^{\min}$. The left and right limits $s_1 \to 0$ cancel with each other. This yields

$$y^c(\alpha \quad (\,)) \quad \alpha \quad (y^c \quad) \quad \widehat{y}_0 y^c(\,). \tag{3.4.27}$$

For simplicity, we have omitted the terms which arise as the $\widehat{T}$-equivariant Floer solution breaks. These are the terms $d \quad h$ and $h \quad d$, where h denotes the homotopy which counts the $\widehat{T}$-equivalence classes of sextuples described above. The symbol denotes that each side is homotopic to the other. This is a slight abuse of notation because the two sides are not chain maps, however when combined with (3.4.26) in (3.4.28), both sides are indeed chain maps.

While the derivation of (3.4.27) is mostly like the proof of Proposition 3.4.4, there is an additional step of performing an s-translation when interpreting the breaking of $v^{\min}$ as $\widehat{y}_0 y^c(\,)$. More precisely, precomposing $[v, u]$ with the translation $s \mapsto s + s_1$ changes the intersection condition $v(s_1) = v_1(0)$ to $v(0) = v_1(0)$. After the translation, the intersection between $[v, u]$ and $[v_0, \gamma_0]$ occurs at an unconstrained point $(\quad s_1, t_0) \in \mathbb{R} \quad S^1$. Therefore, we recover an α-weighted count of the configurations which determine the 'geometric action of y^c' map.

Together, equations (3.4.26) and (3.4.27) yield

$$[r^{;B}, y^c](\,) = \widehat{y}_0 ((\quad w \quad)(y^c \quad) \quad y^c(\quad w \quad)(\,)) \quad (y^c(\alpha \quad (\,)) \quad \alpha \quad (y^c \quad)) \quad 0 \tag{3.4.28}$$

as desired. □

3.4.9 Connection and continuation maps

Let $(H^{\text{eq};} \ , J^{\text{eq};} \ )$ be two $\widehat{T}$-equivariant Floer data, and let B be filling bases for the data. Suppose the slope functions λ of $H^{\text{eq};}$ satisfy $\delta = \lambda \quad \lambda^+ \quad 0$, and let $(H_s^{\text{eq}}, J_s^{\text{eq}})$ be a regular $\widehat{T}$-equivariant monotone homotopy between the data. The corresponding $\widehat{T}$-equivariant continuation map φ is given for $[\varepsilon^+, \widetilde{x}^+_{B^+}] \in B^+$ by

$$\varphi\, [\varepsilon^+, \widetilde{x}^+_{B^+}] = \sum_{\substack{A \in H_2(M) \\ [\varepsilon \ , \widetilde{x}_B \ ] \in B \\ |\varepsilon \ ;(\ A)\#\widetilde{x}_B \ | \ |\varepsilon^+, \widetilde{x}^+_{B^+}| = 0}} \ \sum_{[v;u] \in M_s([\varepsilon \ ;(\ A)\#\widetilde{x}_B \ ];[\varepsilon^+, \widetilde{x}^+_{B^+}])} \#([v,u])\, q^A [\varepsilon \ , \widetilde{x}_B \ ]. \tag{3.4.29}$$

Proposition 3.4.6. The connections $r^{\ ;B}$ satisfy

$$r^{\ ;B} \quad \varphi \ = \varphi \quad r^{\ ;B^+}. \tag{3.4.30}$$

Proof. This is another variation of the intertwining relation proof. Fix data for the $\widehat{T}$-equivariant quantum actions $\ _B$. Choose an s-dependent homotopy between this Morse data. We have

$$(\ w \ )(\varphi \ ) \quad \varphi \ (\ w \ ) = \varphi \ , \tag{3.4.31}$$

where φ is the α-weighted continuation map which counts $\widehat{T}$-equivariant continuation solutions weighted by their intersections with s-dependent half$^+$ flowlines for the s-dependent homotopy. Thus φ is the s-dependent version of (3.4.22).

Define a map which counts $\widehat{T}$-equivalence classes of septuples $(v, u, v_0, \gamma_0, v^{\text{min}}, s_0, t_0)$ where $(v, u, v_0, \gamma_0, v^{\text{min}}, t_0)$ differs from (3.4.18) only in its intersection conditions: instead we have $v(s_0) = v_0(0) = v^{\text{min}}(0)$ and $u(s_0, t_0) = \gamma_0(0)$. See Figure 3.6c. Thus for $s_0 \quad 0$, the half$^+$ flowlines use the data corresponding to $\ _B$. The limits $s_0 \to \ 1$ and the breaking of the half$^+$ flowline v^{min} form the boundary of the 1-dimensional moduli space, and yield

$$\varphi \ (\alpha \quad _{B^+} \ (\ )) \quad \alpha \quad _B \ \varphi \ (\ ) \quad \widehat{y}_0 \varphi \ (\ ), \tag{3.4.32}$$

which together with (3.4.31) gives the desired compatibility. □

In particular, the connection r on $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ is well-defined and independent of the choice of $\widehat{T}$-equivariant Floer datum and filling basis. Moreover, the connection is well-defined in the direct limit as $\lambda \to \ 1$, implying the following corollary.

Corollary 3.4.7. *There is an induced connection r on $\widehat{T}$-equivariant symplectic cohomology $SH_{\widehat{T}}(M, \widehat{\rho})$.*

The proof of Proposition 3.4.6 may be adapted to prove that the connection r on $FH_{\widehat{T}}(M, \widehat{\rho}, 0)$ is isomorphic to the connection (3.3.25), and the isomorphisms are the $\widehat{T}$-equivariant PSS maps. Here, a Hamiltonian of slope 0 and the corresponding PSS maps are defined as in [Rit14, Theorem 37].

3.4.10 Flatness of Floer connection

The Floer connection is flat. The proof is a Floer-theoretic version of the proof of Theorem 3.3.3, and it is more involved because α $(\beta$ $)$ and β $(\alpha$ $)$ are not chain maps, let alone equal.

Theorem 3.4.8. *The connection r is flat.*

Proof. As per the proof of Theorem 3.3.3, we need only check the commutativity of r and r for classes α, β 2 $H^2(M)$. The maps (w) and (w) clearly commute. We have

$$(\ w \ )(\alpha \quad) \quad \alpha \quad (\ w \ )(\ ) = (\alpha \quad) \ (\ ), \tag{3.4.33}$$

where $(\alpha$ $)$ denotes the β-weighted $\widehat{T}$-equivariant quantum action of α , analogously to (3.4.22). An analogous equation holds for β and α. Below, we apply Proposition 3.4.5 to get from (3.4.34) to (3.4.35), and we apply (3.4.33) to get from (3.4.36) to (3.4.37). This gives us

$$[r \ , r \ ](\ ) = r \ (\widehat{y}_0(\ w \ )(\ ) + \beta \quad (\ )) \quad r \ (\widehat{y}_0(\ w \ )(\ ) + \alpha \quad (\ )) \tag{3.4.34}$$

$$= \widehat{y}_0 \, r \ ((\ w \ )(\ )) + r \ (\beta \quad (\ ))$$

$$\widehat{y}_0 \, r \ ((\ w \ )(\ )) \quad r \ (\alpha \quad (\ )) \tag{3.4.35}$$

$$= \widehat{y}_0^2(\ w \ )((\ w \ )(\ )) + \widehat{y}_0\alpha \quad ((\ w \ )(\ )) + \widehat{y}_0(\ w \ )(\beta \quad (\ ))$$

$$+ \alpha \quad (\beta \quad (\ ))$$

$$\widehat{y}_0^2(\ w \ )((\ w \ )(\ )) \quad \widehat{y}_0\beta \quad ((\ w \ )(\ )) \quad \widehat{y}_0(\ w \ )(\alpha \quad (\ ))$$

$$\beta \quad (\alpha \quad (\ )) \tag{3.4.36}$$

$$= \quad \widehat{y}_0(\alpha \quad)(\ ) + \widehat{y}_0(\beta \quad)(\ ) + \alpha \quad (\beta \quad (\ )) \quad \beta \quad (\alpha \quad (\ )). \tag{3.4.37}$$

We will find a homotopy which shows that (3.4.37) is chain homotopic to zero. The homotopy is drawn in Figure 3.6d. It counts $\widehat{T}$-equivalence classes of 11-tuples $(v, u, v_0, \gamma_0, v_0^{\text{min}}, t_0, v_1, \gamma_1, v_1^{\text{min}}, s_1, t_1)$, where $[v, u]$ is a $\widehat{T}$-equivariant Floer solution, $[v_i, \gamma_i]$ are $\widehat{T}$-equivariant perturbed half$^+$ flowlines, v_i^{min} are perturbed half$^+$ flowlines in $E\widehat{T}$, t_i 2 S^1 are elements of the circle and s_1 2 R n f0g is a real number. The 11-tuples satisfy the conditions

$$\begin{aligned} v(0) &= v_0(0) = v_0^{\text{min}}(0), \\ v(s_1) &= v_1(0) = v_1^{\text{min}}(0), \\ u(0, t_0) &= \gamma_0(0), \\ u(s_1, t_1) &= \gamma_1(0) \end{aligned} \qquad \begin{aligned} \arg(v_i^{\text{min}}(+1)) + t_i &= 0, \\ [v_0, \gamma_0](+1) &= \alpha \ , \\ [v_1, \gamma_1](+1) &= \beta \ . \end{aligned} \tag{3.4.38}$$

The boundary components of the 1-dimensional moduli space that contribute to (3.4.37) are the limits s_1 ! 1 and the breaking of each of the flowlines v_0^{min} and v_1^{min}. The limit $s_1 \to +\infty$ contributes the term $\alpha \quad (\beta \quad ( ))$ and the limit $s_1 \to \quad \infty$ contributes $\beta \quad (\alpha \quad ( ))$. When v_1^{min} breaks, we get $\widehat{y}_0(\beta \quad) \; ( )$ as in the proof of Proposition 3.4.4. Using s-translation as in the proof of Proposition 3.4.5, we recover the term $\widehat{y}_0(\alpha \quad) \; ( )$ when v_0^{min} breaks.

The boundary components for the left and right limits $s_1 \to 0$ cancel with each other. The breaking of $[v, u]$ contributes the $hd + dh$, where h is the chain homotopy map which counts isolated $\widehat{T}$-equivalence classes of such 11-tuples. Therefore (3.4.37) is chain homotopic to 0, and this completes the proof. □

3.4.11 Floer Seidel map

Let $\sigma \in \text{Cochar}^{\,0}\, T$ be a ρ-nonnegative cocharacter, that is a cocharacter for which $\rho \quad \sigma$ is an admissible S^1-action of nonnegative slope on M. The *pullback Hamiltonian* $\sigma \; H$ of the Hamiltonian H is given by

$$(\sigma \; H)_t(m) = H_t(\rho \;_{(t)}(m)) \quad K \; (\rho \;_{(t)}(m)), \tag{3.4.39}$$

where K is the Hamiltonian of the action $\rho \quad \sigma$ (without loss of generality, impose min $K \; =$ 0 to remove the freedom of choosing a constant). The assignment $x \mapsto \sigma \; x$, where $\sigma \; x$ is given by $t \mapsto \rho \;_{(\;\; t)} x(t)$, is a bijection $P(H) \to P(\sigma \; H)$.

Let κ be the slope function of K. If H is admissible for the ρ-invariant slope function $\lambda \in L$, then $\sigma \; H$ is admissible for $\lambda \quad \kappa \in L$. That X has no 1-periodic orbits immediately follows from the fact that $P(H) \to P(\sigma \; H)$ is a bijection.

The *pullback almost complex structure* $\sigma \; J$ is given by $(\sigma \; J)_t = (D\rho \;_{(t)})^{-1} \; J_t \;\; D\rho \;_{(t)}$. The assignment $u \mapsto \sigma \; u$, where $\sigma \; x$ is given by $(s, t) \mapsto \rho \;_{(\;\; t)} u(s, t)$, is a bijection between the Floer solutions of Floer datum (H, J) and those of $(\sigma \; H, \sigma \; J)$ [Sei97, Lemma 4.3].

Let $\mu \in M$ be a fixed point. Every orbit $\widetilde{x}$ has a choice of filling f for which $f(0) = \mu$. For such a filling f, the *pullback filling* $\sigma \; f$ is well-defined. Explicitly, this is given by $(\sigma \; f)(e^{2 \;\; (s+it)}) = \rho \;_{(\;\; t)} f(e^{2 \;\; (s+it)})$. Define $(\sigma, \mu) \; \widetilde{x}$ to be the Hamiltonian orbit $\sigma \; x$ with the equivalence class of fillings $[\sigma \; f]$, with f as above.

The *Floer Seidel map* is the map

$$FS(\sigma, \mu) : FH \;\; (M, \lambda; H, J) \to FH^{\;+j \;;\; j}(M, \lambda \quad \kappa \;; \sigma \; H, \sigma \; J) \tag{3.4.40}$$

given by $\widetilde{x} \mapsto (\sigma, \mu) \; \widetilde{x}$, and it is an isomorphism of cochain complexes. The Floer Seidel map commutes with continuation maps [Sei97, Corollary 4.8], and we can show this using s-dependent pullback constructions analogous to the above. As such, $FS(\sigma, \mu)$ is independent of the Floer datum.

3.4.12 Equivariant Floer Seidel map

In [LJ20], we defined an S^1-equivariant Floer Seidel map. The only modification required to upgrade the non-equivariant construction in Section 3.4.11 to the equivariant setup was to incorporate a pullback of the action on the contractible loop space LM. The same is true for the $\widehat{T}$-equivariant Floer Seidel map. The action on LM changes from

$$((a,\mathbf{t})\cdot x)(t) = \widehat{\rho}_{(a,\mathbf{t})}(x(t-a)) = \rho_{\mathbf{t}}(x(t-a)),$$

as in (3.4.10), to

$$((a,\mathbf{t})\cdot x)(t) = (\sigma^{*}\widehat{\rho})_{(a,\mathbf{t})}(x(t-a)) = \rho_{\mathbf{t}-\ell(a)}(x(t-a)). \tag{3.4.41}$$

This change in action from $\widehat{\rho}$ to $\sigma^{*}\widehat{\rho}$ is compatible with the pullback data, so the $\widehat{T}$-equivariant pullback Hamiltonian $\sigma^{*}H^{\mathrm{eq}}$, defined as in (3.4.39), is $\widehat{T}$-equivariant with respect to the action $\sigma^{*}\widehat{\rho}$; it satisfies

$$(\sigma^{*}H^{\mathrm{eq}})_{e,t}(m) = (\sigma^{*}H^{\mathrm{eq}})_{(a,\mathbf{t})^{-1}\cdot e,a+t}((\sigma^{*}\widehat{\rho})_{(a,\mathbf{t})}(m)). \tag{3.4.42}$$

The *$\widehat{T}$-equivariant Floer Seidel map* is the map

$$FS_{\widehat{T}}(\sigma,\mu) : FH^{*}_{\widehat{T}}(M,\widehat{\rho},\lambda;H^{\mathrm{eq}},J^{\mathrm{eq}}) \to FH^{*+|\tilde{\sigma}|}_{\widehat{T}}(M,\sigma^{*}\widehat{\rho},\lambda-\kappa;\sigma^{*}H^{\mathrm{eq}},\sigma^{*}J^{\mathrm{eq}}) \tag{3.4.43}$$

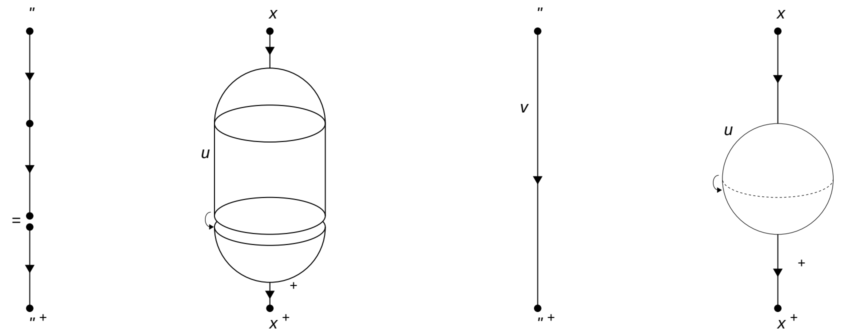
given by $[\varepsilon,\widetilde{x}] \mapsto [\varepsilon,(\sigma,\mu)^{*}\widetilde{x}]$. Like its non-equivariant counterpart, the $\widehat{T}$-equivariant Floer Seidel map is an isomorphism of the cochain complexes, and it commutes with $\widehat{T}$-equivariant continuation maps. It is compatible with the $\widehat{H}^{*}(B\widehat{T})$-module structure.

Proposition 3.4.9. There is a commutative diagram

$$\begin{array}{ccc}
QH^{*}_{\widehat{T}}(M,\widehat{\rho}) & \xrightarrow{QS_{\widehat{T}}(\sigma,\mu)} & QH^{*+|\tilde{\sigma}|}_{\widehat{T}}(M,\sigma^{*}\widehat{\rho}) \\
\downarrow{\scriptstyle \widehat{T}\text{-equivariant PSS map}\ \cong} & & \uparrow{\scriptstyle \cong\ \widehat{T}\text{-equivariant PSS map}} \\
FH^{*}_{\widehat{T}}(M,\widehat{\rho},0) & & FH^{*+|\tilde{\sigma}|}_{\widehat{T}}(M,\sigma^{*}\widehat{\rho},0) \\
\searrow{\scriptstyle FS_{\widehat{T}}(\sigma,\mu)\ \cong} & & \nearrow{\scriptstyle \widehat{T}\text{-equivariant continuation map}} \\
& FH^{*+|\tilde{\sigma}|}_{\widehat{T}}(M,\sigma^{*}\widehat{\rho},-\kappa) &
\end{array} \tag{3.4.44}$$

where slope 0 Hamiltonians and PSS maps are defined as in [Rit14, Theorem 37].

Proof. This is an equivariant version of [Sei97, Section 8], or more precisely its extension from closed manifolds to convex manifolds in [Rit14, Section 5.7]. The proof otherwise extends to our setup (see Figure 3.7). □



(a) Equivariant PSS maps together with Floer Seidel map and continuation map

(b) Equivariant quantum Seidel map from Figure 3.3a

Figure 3.7: Gluing together PSS maps, the Floer Seidel map and a continuation map (a) yields the quantum Seidel map (b).

3.4.13 Shift operator

As in Section 3.3.18, there is a pullback isomorphism

$$(B\widehat{\sigma})^{*} : FH^{*}_{\widehat{T},\widehat{\lambda}}(M, \sigma\,\widehat{\rho}, \lambda) \to FH^{*}_{\widehat{T},\,\widehat{\lambda}}(M, \widehat{\rho}, \lambda). \tag{3.4.45}$$

This map translates between different notation which ultimately describe the same moduli spaces. Recall $(B\widehat{\sigma})^{*}$ satisfies $(B\widehat{\sigma})^{*}([\widehat{\chi}]) = [\widehat{\chi} \circ \widehat{\sigma}](B\widehat{\sigma})^{*}()$ for characters $\widehat{\chi} : \widehat{T} \to S^{1}$.

The *shift operator* on $\widehat{T}$-equivariant Floer cohomology is the map

$$\mathrm{S} : FH^{*}_{\widehat{T},\widehat{\lambda}}(M, \widehat{\rho}, \lambda) \to FH^{*+j}_{\widehat{T},\,\widehat{\lambda}}(M, \widehat{\rho}, \lambda) \tag{3.4.46}$$

given by the composition

$$\mathrm{S} = (B\widehat{\sigma})^{*} \circ \varphi \circ FS_{\widehat{T}}(\sigma, \mu), \tag{3.4.47}$$

where φ is the continuation map which increases the slope by κ. Note that the three maps in (3.4.47) which make up S commute, so the order of these maps is unimportant.

Theorem 3.4.10. *The difference-differential connection* (S, ∇) *on $\widehat{T}$-equivariant Floer cohomology is flat.*

We prove Theorem 3.4.10 in Section 3.4.14.

This difference-differential connection on the $\widehat{T}$-equivariant Floer cohomology of slope 0 is isomorphic to the difference-differential connection on $\widehat{T}$-equivariant quantum cohomology. That the PSS isomorphism preserves the connection ∇ follows from an adaptation of Proposition 3.4.6, and that it preserves the shift operator S follows from Proposition 3.4.9.

The difference-differential connection also commutes with $\widehat{T}$-equivariant continuation maps. Therefore there is an induced difference-differential connection on $\widehat{T}$-equivariant symplectic cohomology. This difference-differential connection will be flat, since Theorem 3.4.10 holds for all slopes in the direct limit.

Remark 3.4.11 (Other cocharacters). Unlike the quantum Seidel map, the *Floer* Seidel map itself is well-defined for cocharacters σ which induce a S^1-action $\rho \circ \sigma$ which is admissible for a slope function κ which does not satisfy $\kappa \geq 0$ [Rit14, Remark, page 1046]. Similarly, the $\widehat{T}$-equivariant Floer Seidel map $FS_{\widehat{T}}(\sigma, \mu)$ is well-defined for such cocharacters and is an isomorphism of the cochain complexes. In (3.4.43), the slope λ in the domain changes to $\lambda - \kappa$. When $\kappa \geq 0$ is not satisfied, these slopes do not satisfy $\lambda - \kappa \leq \lambda$, and hence the change in slope cannot be undone with a continuation map. Therefore we cannot construct a shift operator on $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ corresponding to σ.

On the other hand, the composition $(B\widehat{\sigma}) \circ FS_{\widehat{T}}(\sigma, \mu)$ commutes with continuation maps, and hence induces a shift operator $\mathbb{S}$ on $\widehat{T}$-equivariant *symplectic* cohomology $SH_{\widehat{T}}(M, \widehat{\rho})$. This shift operator commutes with the connection ∇ by Proposition 3.4.12 and Proposition 3.4.16. Therefore $SH_{\widehat{T}}(M, \widehat{\rho})$ has a flat difference-differential connection defined on the group of all cocharacters, not just on the monoid $\mathrm{Cochar}^{\geq 0} T$.

3.4.14 Proof of flatness of connection

To show ∇ commutes with $\mathbb{S}$, we will show that ∇ commutes each of the three maps in the composition (3.4.47) separately.

Proposition 3.4.12. The map ∇ commutes with the $\widehat{T}$-equivariant Floer Seidel map $FS_{\widehat{T}}(\sigma, \mu)$.

Proof. We use the $\widehat{T}$-equivariant Floer data $(H^{\mathrm{eq}}, J^{\mathrm{eq}})$ and the filling basis B on the domain $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$. We use the pullback data $(\sigma^* H^{\mathrm{eq}}, \sigma^* J^{\mathrm{eq}})$ and the pullback filling basis $(\sigma, \mu)^* B$ on the codomain. With these choices, $FS_{\widehat{T}}(\sigma, \mu)$ is an isomorphism between the filling bases, and therefore preserves the differentiation operation $\frac{d}{d\alpha}$. The map $FS_{\widehat{T}}(\sigma, \mu)$ does not modify flowlines in $E\widehat{T}$, so it induces a natural isomorphism on the moduli spaces used to define the map $\widehat{y}_0$. Thus, with the data we have chosen, $FS_{\widehat{T}}(\sigma, \mu)$ commutes with $\widehat{y}_0 \frac{d}{d\alpha}$ on the cochain complexes, giving

$$FS_{\widehat{T}}(\sigma, \mu) \circ \left(\widehat{y}_0 \frac{d}{d\alpha}\right) = \left(\widehat{y}_0 \frac{d}{d\alpha}\right) \circ FS_{\widehat{T}}(\sigma, \mu). \tag{3.4.48}$$

The two other terms in ∇, namely α and w^B, both depend on the choice of $\widehat{T}$-equivariant Morse data on M. The $\widehat{T}$-action on M in the domain of $FS_{\widehat{T}}(\sigma, \mu)$ is $\widehat{\rho}$, while on the codomain it is $\sigma * \widehat{\rho}$. The fact that these two actions are different means that we cannot use the same $\widehat{T}$-equivariant Morse data in defining these terms of ∇. To emphasise the different underlying $\widehat{T}$-action, we will incorporate a subscript $\widehat{\rho}$ or $\sigma * \widehat{\rho}$ in our notation. Note that each of the classes $\alpha_{\widehat{\rho}} \in H^2_{\widehat{T}}(M, \widehat{\rho})$ and $\alpha_{\sigma * \widehat{\rho}} \in H^2_{\widehat{T}}(M, \sigma * \widehat{\rho})$ is independently defined using the split short exact sequence (3.3.19), however the relation $(B\widehat{\sigma})^* \alpha_{\sigma * \widehat{\rho}} = \alpha_{\widehat{\rho}}$ readily follows by a diagram chasing argument.

We will show

$$FS_{\widehat{T}}(\sigma, \mu) \circ \left(\alpha_{\widehat{\rho}} * \widehat{y}_0 w^B_{\cdot;\widehat{\rho}}\right) \simeq \left(\alpha_{\sigma * \widehat{\rho}} * \widehat{y}_0 w^{(\cdot;\cdot)B}_{\cdot;\sigma * \widehat{\rho}}\right) \circ FS_{\widehat{T}}(\sigma, \mu), \tag{3.4.49}$$

where $\simeq$ denotes a homotopy equivalence. This means there is a map K such that $dK + Kd$ is equal to the difference of the two sides of (3.4.49), and this makes sense even though neither side is a chain map. Combining (3.4.48) and (3.4.49) yields a chain homotopy between the chain maps $FS_{\widehat{T}}(\sigma, \mu) \circ \nabla$ and $\nabla \circ FS_{\widehat{T}}(\sigma, \mu)$, completing the proof.

To construct a homotopy between the two sides of (3.4.49), we will need to find a homotopy between $\alpha_{\widehat{\rho}}$ and $\alpha_{\sigma * \widehat{\rho}}$. These are $\widehat{T}$-equivariant cohomology classes for different actions, so we will use the clutching bundle $E(\sigma)$ from Section 3.3.14 to create the homotopy. By Lemma 3.6.2, there is a class $\beta \in H^2_{\widehat{T}}(E(\sigma))$ which satisfies

$$(I^+_{z^+})^* \beta = \alpha_{\widehat{\rho}}, \quad (I^-_{z^-})^* \beta = \alpha_{\sigma * \widehat{\rho}}, \quad \beta(A^-) = \alpha(A) \text{ for } A \in H_2(M). \tag{3.4.50}$$

Here, we have denoted the fibre inclusions over the poles $z^\pm \in S^2$ by $I^\pm_{z^\pm} : M \to E(\sigma)$. Thus, informally, the two classes $\alpha_{\widehat{\rho}}$ and $\alpha_{\sigma * \widehat{\rho}}$ coincide in the clutching bundle, and we have reduced the problem to finding a homotopy which moves the intersection with the class β from the fibre over one pole to the fibre over the other pole. We define this homotopy below, however note that we do require further homotopies in our proof of (3.4.49).

Definition 3.4.13 (The homotopy K). The map K^+, which we have drawn in Figure 3.8, counts $\widehat{T}$-equivalence classes of 7-tuples $(v, u, v^{\min}, v_0, \gamma_0, s_0, t_0)$, where $[v, u]$ is a $\widehat{T}$-equivariant Floer solution, $v^{\min}$ is a perturbed half$^+$ flowline in $E\widehat{T}$, $[v_0, \gamma_0]$ is a perturbed $\widehat{T}$-equivariant half$^+$ flowline in $E(\sigma)$ which converges to β at $+\infty$, $s_0 \in (0, \infty)$ is a positive real number and $t_0 \in S^1$ is an element of the circle. Together, they must satisfy $v(0) = v^{\min}(0) = v_0(0)$, $\arg(v^{\min}(+\infty)) + t_0 = 0$ and $I^+_{(s_0;t_0)}(u(0, t_0)) = \gamma_0(0)$. Here, $I^+_{(s_0;t_0)} : M \to E(\sigma)$ is the fibre inclusion map over the point $(s_0, t_0) \in S^2$, using coordinates from (3.3.33), which uses the trivialisation over the pole z^+ to identify the fibre with M. The map K^+ is an endomorphism of the $\widehat{T}$-equivariant Floer cochain complex for the data $(H^{\mathrm{eq}}, J^{\mathrm{eq}})$.

Similarly, we define a map K^- which counts similar $\widehat{T}$-equivalence classes of 7-tuples, but these instead satisfy $I^-_{(s_0;t_0)}(u(0, t_0)) = \gamma_0(0)$ and $s_0 \in (-\infty, 0)$. The map $I^-_{(s_0;t_0)} : M \to E(\sigma)$ is the fibre inclusion map using the trivialisation over the pole z^- to identify the fibres

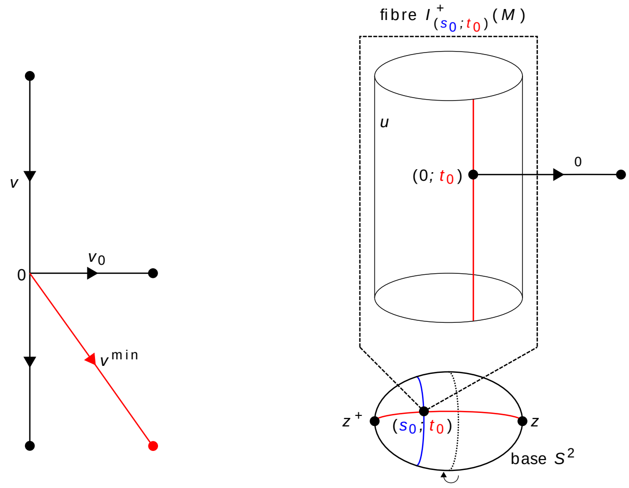


Figure 3.8: For the map K^+, we embed the Floer cylinder in the bundle $E(\sigma)$, and the half$^+$ flowline γ_0 to β lives in the bundle.

with M. The map K is an endomorphism of the $\widehat{T}$-equivariant Floer cochain complex for the pullback data $(\sigma\ H^{\text{eq}}, \sigma\ J^{\text{eq}})$.

Finally, define

$$K = (FS_{\widehat{T}}(\sigma, \mu) \quad K^+) + (K \qquad FS_{\widehat{T}}(\sigma, \mu)), \tag{3.4.51}$$

which is a map $K : FH_{\widehat{T}}(M, \widehat{\rho}, \lambda) \ ! \ FH_{\widehat{T}}^{+j \ ; \ j+1}(M, \sigma \quad \widehat{\rho}, \lambda \quad \kappa \ )$.

Two of the boundary components of the 1-dimensional moduli spaces for K will be the limits s_0 ! 1 . To characterise these boundary components, let $K_{\ 1}$ be the maps defined as per K in Definition 3.4.13, except with the intersection condition over the fibre at (s_0, t_0) replaced by the condition $I_z \ (u(0, t_0)) = \gamma_0(0)$. The moduli spaces for $K_{\ 1}$ do not record the parameter s_0 since this intersection condition does not depend on s_0.

Denote by $w^{B;+}$ the endomorphism of $FH_{\widehat{T}}(M, \widehat{\rho}, \lambda)$ which counts the intersections of $(\varepsilon, I^+_{z^+}(f(D)))$ with $W^s_{\text{pert}}(\beta)$ for the $\widehat{T}$-equivariant Hamiltonian orbit $[\varepsilon, \widetilde{x}]$ with filling f. This is the natural analogue of $w^B_{;\,\widehat{}}$ which uses β instead of α . Similarly define the endomorphism $w^{(\ ; \ ) \ B;}$ of $FH_{\widehat{T}}(M, \sigma \quad \widehat{\rho}, \lambda \quad \kappa \ )$, which is the analogue of $w^{(\ ; \ ) \ B}_{;\ \widehat{}}$ that uses β.

We now have the ingredients to prove (3.4.49), and therefore complete the proof. Lemma 3.4.14 states that we can indeed change to maps using β over the poles, implying that (3.4.49) is equivalent to

$$FS_{\widehat{T}}(\sigma, \mu) \quad \left(K^+_{+\ 1} \quad \widehat{y}_0 w^{B;+}\right) \quad \left(K_{\ 1} \quad \widehat{y}_0 w^{(\ ; \ ) \ B;}\right) \quad FS_{\widehat{T}}(\sigma, \mu). \tag{3.4.52}$$

We show (3.4.52) holds in Lemma 3.4.15 using the homotopy K. Therefore (3.4.49) holds as required. □

Lemma 3.4.14. *We have homotopy equivalences*

$$K^+_{+\infty} \quad \widehat{y}_0 w^{B;+} \quad \alpha_{\frown} \quad \widehat{y}_0 w^{B}_{;\frown} \tag{3.4.53}$$

$$K_{\infty} \quad \widehat{y}_0 w^{(\ ;\ )\,B;} \quad \alpha_{\ \frown} \quad \widehat{y}_0 w^{(\ ;\ )\,B}_{;\ \frown}. \tag{3.4.54}$$

Proof. We will show (3.4.53) holds; the same argument proves (3.4.54) as well. Denote by $f^{E(\ )} : E\widehat{T} \quad E(\sigma) \to \mathbb{R}$ the equivariant Morse function on $E(\sigma)$ and by $f^M : E\widehat{T} \quad M \to \mathbb{R}$ the equivariant Morse function on M for the action $\widehat{\rho}$. We have $\beta \in H^2_{\widehat{T}}(E(\sigma); f^{E(\ )})$ and $\alpha_{\frown} \in H^2_{\widehat{T}}(M, \widehat{\rho}; f^M)$. Let $f^{D^+} : D^+ \to \mathbb{R}$ be the function $f^{D^+}(z) = |z|^2$. Let $f^+ : E\widehat{T} \quad E(\sigma) \to \mathbb{R}$ be a Morse function which extends the sum $f^M + f^{D^+} : E\widehat{T} \quad \pi^{\ 1}(D^+) \to \mathbb{R}$. The Morse function f^+ has no flowlines which flow away from the fibre $I^+_{z^+}(M)$.

We will construct a homotopy which modifies the flowline $[v_0, \gamma_0]$ in the definition of $K^+_{+\infty}$ so that it uses f^+ instead of $f^{E(\ )}$. Since $[v_0, \gamma_0]$ is, in fact, an s-*dependent* flowline, we need s-dependent perturbations of the above Morse functions. The perturbation of $f^{E(\ )}$ is unrestricted, however for f^+ we must use a perturbation for which no flowline flows away from the fibre $I^+_{z^+}(M)$. For this, take a perturbation f^M_s of f^M, and impose $f^+_s = f^M_s + f^{D^+}$ on $\mathbb{R} \quad E\widehat{T} \quad \pi^{\ 1}(D^+)$.

Let $f^{+;\mathrm{hty}}_{\ ;s} : (0, \infty) \quad \mathbb{R} \quad E\widehat{T} \quad E(\sigma) \to \mathbb{R}$ be a function which also depends on a real parameter $\lambda \in (0, \infty)$ and which satifies $f^{+;\mathrm{hty}}_{\ ;s} = f^+_s$ on $s \quad \lambda \quad 1$ and $f^{+;\mathrm{hty}}_{\ ;s} = f^{E(\ )}_s$ on $s \quad \lambda$. Let h be the map which counts $\widehat{T}$-equivalence classes of tuples $(v, u, v^{\mathrm{min}}, v_0, \gamma_0, s_0, t_0, \lambda)$, where $\lambda \in (0, \infty)$ is a new parameter and $[v, u, v^{\mathrm{min}}, v_0, \gamma_0, s_0, t_0]$ is as per the definition of $K^+_{+\infty}$ in Definition 3.4.13, except that $[v_0, \gamma_0]$ is an s-dependent flowline for the Morse function $f^{+;\mathrm{hty}}_{\ ;s}$. We have drawn this configuration in Figure 3.9.

We will show the boundary of the 1-dimensional moduli spaces yields (3.4.53). The contributions from the equivariant Floer solution breaking and from the flowline $[v_0, \gamma_0]$ breaking produce the terms $dh + hd$. Under the limit $\lambda \to 0$, $[v_0, \gamma_0]$ is a flowline of $f^{E(\ )}_s$, so this boundary component yields the term $K^+_{+\infty}$. We will show that the limit $\lambda \to \infty$ yields the term $\alpha_{\frown}$ and that the breaking of v^{min} yields the remaining two terms in (3.4.53), up to a further homotopy equivalence.

Under the limit $\lambda \to \infty$, the flowline $[v_0, \gamma_0]$ breaks into a pair of flowlines: a half$^+$ flowline $[v_1, \gamma_1]$ from $[v(0), I^+_{z^+}(u(0, t_0))]$ to a $\widehat{T}$-equivariant critical point $[\varepsilon_1, x_1]$ and a further flowline $[v_2, \gamma_2]$ from $[\varepsilon_1, x_1]$ to β. The flowline $[v_1, \gamma_1]$ uses the Morse function f^+_s, so it lies completely within the fibre $I^+_{z^+}(M)$. Identifying this fibre with M, we see that $[v_1, \gamma_1]$ is a half$^+$ flowline of f^M_s. The other flowline $[v_2, \gamma_2]$ uses f^+ on $s \quad 1$ and uses $f^{E(\ )}$ on $s \quad 0$. To see this, consider a translation of the original flowline $[v_0, \gamma_0]$ by λ so that it is a flowline with domain $[\ \lambda, \infty)$ which uses f^+_s for $s \in [\ \lambda,\ 1]$ and $f^{E(\ )}_s$ for $s \in [0, \infty)$. Under the limit $\lambda \to \infty$, the flowline $[v_2, \gamma_2]$ is the component of the broken flowline according to these

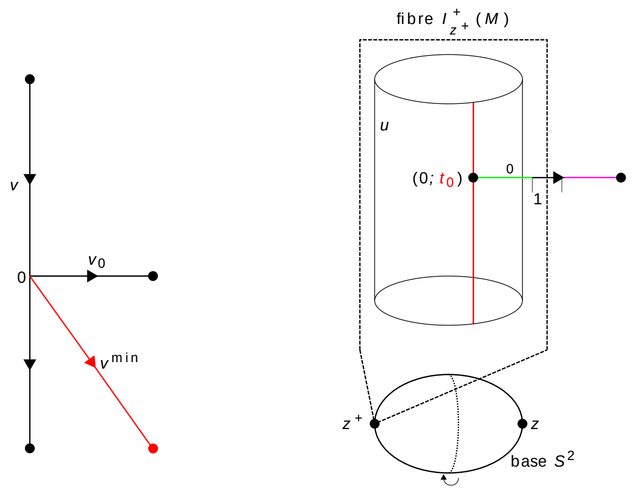


Figure 3.9: For the map h, we embed the Floer cylinder in the bundle $E(\sigma)$ in the fibre above z^+, and the half$^+$ flowline γ_0 to β lives in the bundle. The flowline γ_0 uses the data f_s^+ on $[0, \lambda - 1]$ and it uses the data $f_s^{E(\ )}$ on $[\lambda, \infty)$.

coordinates on $[v_0, \gamma_0]$. In particular, the standard Morse continuation map between the $\widehat{T}$-equivariant Morse functions $f^{E(\ )}$ and f^+ counts the flowlines $[v_2, \gamma_2]$ by definition. We restrict to those flowlines $[v_2, \gamma_2]$ which flow from the fibre $I^+_{z^+}(M)$ to get

$$\alpha_{\frown} = (I^+_{z^+})^* \beta = \sum_{\substack{[\varepsilon_1,x_1]\in \mathrm{Crit}^{eq}(f^+) \\ \text{with } x_1 \in I^+_{z^+}(M)}} \; \sum_{[v_2,\gamma_2]\in \mathcal{M}([\varepsilon_1,x_1];\ )} \#([v_2,\gamma_2])\,[\varepsilon_1, x_1]. \tag{3.4.55}$$

We can also derive (3.4.55) using functorial flowlines to model the pullback map $(I^+_{z^+})^*$ (see footnote 9). In summary, $[v_1, \gamma_1]$ is really a half$^+$ flowline in M which uses the Morse function f^+ and which flows to $\alpha_{\frown}$. Therefore, the $\lambda \to \infty$ limit exactly equals $\alpha_{\frown}$.

As per our previous proofs, the breaking of $v^{\min}$ is isolated exactly when it breaks with the intermediate critical point $\widehat{y}_0$, and moreover we can decouple the contribution of the flowline to $\widehat{y}_0$ from the rest of the configuration. Therefore the contribution from $v^{\min}$ breaking is homotopy equivalent to $\widehat{y}_0 \ \ C$. Here, C counts $\widehat{T}$-equivalence classes of tuples $(v, u, v_0, \gamma_0, t_0, \lambda)$ where $[v, u]$ is a $\widehat{T}$-equivariant Floer solution, $[v_0, \gamma_0]$ is a half$^+$ flowline to β, $t_0 \in S^1$ is an unconstrained point on the circle and $\lambda \in (0, \infty)$ is a positive number. The tuple satisfies the conditions $v(0) = v_0(0)$ and $I^+_{+\infty}(u(0, t_0)) = \gamma_0(0)$. The moduli space has dimension $|\varepsilon_{\ }, \widetilde{x}_{\ }| - |\varepsilon^+, \widetilde{x}^+| - |\beta| + 2$, where the $+2$ comes from the freedom in the parameters t_0 and λ. Since $|\beta| = 2$, these moduli spaces are isolated only when $[v, u]$ is constant in s. Therefore C counts the isolated flowlines $[v_0, \gamma_0]$ of $f^{+;\mathrm{hty}}_{;s}$ which satisfy $v_0(0) = \varepsilon$ and $\gamma_0(0) = x(t_0)$ for the given $\widehat{T}$-equivariant Hamiltonian orbit $[\varepsilon, \widetilde{x}]$, where t_0 and λ can vary.

We will show $C \quad = w^{B}_{;\frown} \quad w^{B;+}$. The maps $w^{B}_{;\frown}$ and $w^{B;+}$ count the intersections of the prescribed filling of $[\varepsilon, \widetilde{x}]$ with the perturbed stable manifolds of $\alpha_{\frown}$ and β respectively. These perturbed stable manifolds use the Morse functions f_s^M and $f_s^{E(\ )}$ respectively. Consider now the intersections of the filling with the perturbed stable manifold of β for the function $f^{+;\text{hty}}_{;s}$, for fixed λ 2 (0, 1). The limit as λ ! 1 recovers $w^{B}_{;\frown}$ (using (3.4.55) as above) and the limit λ ! 0 recovers $w^{B;+}$. As λ varies, the intersections will vary smoothly within the filling. At isolated values of λ, however, the intersections may enter or leave the filling. When this happens, we will have an intersection which lies on the boundary of the filling, and this boundary is the Hamiltonian orbit x. Thus, C exactly records the new and lost intersections, giving $C \quad = w^{B}_{;\frown} \quad w^{B;+}$ as desired. □

Lemma 3.4.15. *The homotopy equivalence* (3.4.52) *holds.*

Proof. We will show that the map $K = (FS_{\widehat{T}}(\sigma, \mu) \quad K^{+}) + (K \quad FS_{\widehat{T}}(\sigma, \mu))$ from Definition 3.4.13 is a homotopy[12] between the two sides of (3.4.52). To show this, we must consider each boundary component of the 1-dimensional moduli spaces of K. The limits as s_0 ! 1 recover the $K_{\ 1}$ terms. We will show that the boundary components corresponding to the limits s_0 ! 0 cancel with each other and that the breaking of the flowline v^{min} contributes the remaining terms $FS_{\widehat{T}}(\sigma, \mu) \quad \widehat{y}_0 w^{B;+}$ and $\widehat{y}_0 w^{(\ ;\ )\ B;} \quad FS_{\widehat{T}}(\sigma, \mu)$.

First, we will show that the limits s_0 ! 0 cancel with each other. Let K_0 denote the versions of the maps K with the parameter $s_0 = 0$ fixed, so that K_0 is the s_0 ! 0 boundary component of K. If $[v, u, v^{\text{min}}, v_0, \gamma_0, 0, t_0]$ is a $\widehat{T}$-equivalence class counted by K_0^{+}, then $[v, \sigma\ u, v^{\text{min}}, v_0, \gamma_0, 0, t_0]$ is counted by K_0. Indeed, the clutching bundle is essentially constructed so that this holds. This identification extends to an isomorphism between the moduli spaces for K_0^{+} and K_0. The isomorphism is sign-reversing because the limits s_0 ! 0 approach in opposite directions. Therefore the contributions of $FS_{\widehat{T}}(\sigma, \mu)$ K_0^{+} and $K_0 \quad FS_{\widehat{T}}(\sigma, \mu)$ cancel with each other, as desired.

Next, we consider the contribution of the boundary component in which the half^{+} flowline v^{min} breaks. As is standard in our intertwining-style proofs, only the breaking to $\widehat{y}_0$ is isolated and we can use a homotopy to decouple the flowline to $\widehat{y}_0$ from the rest of the configuration. Therefore, the contribution of this boundary component is chain homotopic to $FS_{\widehat{T}}(\sigma, \mu) \quad \widehat{y}_0 \quad K^{+}_{\widehat{y}_0} + \widehat{y}_0 \quad K_{\widehat{y}_0} \quad FS_{\widehat{T}}(\sigma, \mu)$, where the maps $K_{\widehat{y}_0}$ count $\widehat{T}$-equivalence classes of 6-tuples $(v, u, v_0, \gamma_0, s_0, t_0)$ just as for the maps K, but without the flowline v^{min}. The dimension of these moduli spaces is $j\varepsilon \ , \widetilde{x}\ j \quad j\varepsilon^{+}, \widetilde{x}^{+} j \quad j\beta j + 2$. The +2 comes from the freedom in the parameters s_0 and t_0. Recall $j\beta j = 2$. Thus the moduli spaces are isolated

[12] Technically, we add a further homotopy to K to achieve this, but K is the important part of the homotopy.

only when $|\varepsilon^-, \widetilde{x}^-| = |\varepsilon^+, \widetilde{x}^+|$ holds, and this equation holds exactly for the Floer solutions which are constant in s.

Let $[v,u] \in [\varepsilon, x]$ be a $\widehat{T}$-equivariant Floer solution which is constant in $s \in \mathbb{R}$. Let $C_{[v,u]}$ be the total count of the moduli spaces of $K^+_{\widehat{y}_0}$ which use the Floer solution $[v,u]$ and the moduli spaces of $K^-_{\widehat{y}_0}$ which use $[v, \sigma \cdot u]$. The map $FS_{\widehat{T}}(\sigma,\mu) \circ K^+_{\widehat{y}_0} + K^-_{\widehat{y}_0} \circ FS_{\widehat{T}}(\sigma,\mu)$ satisfies

$$[\varepsilon, \widetilde{x}_B] \mapsto C_{[v,u]} \cdot [\varepsilon, (\sigma,\mu) \cdot \widetilde{x}_B]. \tag{3.4.56}$$

Since $I^+_{(s_0,t_0)}(x(t_0)) = I^-_{(s_0,t_0)}(\sigma \cdot x(t_0))$ holds, the count $C_{[v,u]}$ is the same as the count of the points $(s_0,t_0) \in \mathbb{R} \times S^1$ for which $(\varepsilon, I^+_{(s_0,t_0)}(x(t_0))) \in W^s_{\text{pert}}(\beta)$ is satisfied. The map $(s,t) \mapsto I^+_{(s,t)}(x(t))$ describes a cylinder in $E(\sigma)$ which lifts the cylinder $S^2 \setminus \{z_\pm\}$, and $C_{[v,u]}$ is the number of intersections of this cylinder with $W^s_{\text{pert}}(\beta)$.

We use the prescribed fillings of $\widetilde{x}_B$ and $(\sigma,\mu) \cdot \widetilde{x}_B$ to cap either end of this cylinder. Recall that the expressions $w^{B,+}([\varepsilon, \widetilde{x}_B])$ and $w^{(\sigma,\mu)^*B,-}([\varepsilon, (\sigma,\mu) \cdot \widetilde{x}_B])$ capture the intersections of these fillings with $W^s_{\text{pert}}(\beta)$. The capped cylinder is contractible by the definition of the pullback filling basis, and hence the signed sum of its intersections with $W^s_{\text{pert}}(\beta)$ is 0. This yields

$$FS_{\widehat{T}}(\sigma,\mu) \circ K^+_{\widehat{y}_0} + K^-_{\widehat{y}_0} \circ FS_{\widehat{T}}(\sigma,\mu) = FS_{\widehat{T}}(\sigma,\mu) \circ w^{B,+} - w^{(\sigma,\mu)^*B,-} \circ FS_{\widehat{T}}(\sigma,\mu), \tag{3.4.57}$$

since the left-hand side is (3.4.56).

In summary, the contribution of the breaking of the half$^+$ flowline $v^{\min}$ is homotopic to $FS_{\widehat{T}}(\sigma,\mu) \circ \widehat{y}_0 \cdot K^+_{\widehat{y}_0} + \widehat{y}_0 \cdot K^-_{\widehat{y}_0} \circ FS_{\widehat{T}}(\sigma,\mu)$. This map is equal to $\widehat{y}_0 (FS_{\widehat{T}}(\sigma,\mu) \circ K^+_{\widehat{y}_0} + K^-_{\widehat{y}_0} \circ FS_{\widehat{T}}(\sigma,\mu))$ because $FS_{\widehat{T}}(\sigma,\mu)$ commutes with $\widehat{y}_0$. This final map recovers the required terms by (3.4.57). □

Proposition 3.4.16. The map ∇ commutes with the map $(B\widehat{\sigma})^*$.

Proof. The pullback isomorphism $(B\widehat{\sigma})^*$ does not modify the filling basis, so it preserves the differentiation operation $\frac{d}{d\alpha}$. Moreover, it exactly preserves the moduli spaces which are used by the maps $\widehat{y}_0$, w^B and α. Therefore the pullback isomorphism $(B\widehat{\sigma})^*$ commutes with the connection $\nabla = \widehat{y}_0 \frac{d}{d\alpha} + \alpha - \widehat{y}_0 w^B$ on the cochain complexes. Thus, we have

$$(B\widehat{\sigma})^* \circ \left(\widehat{y}_0 \frac{d}{d\alpha} + (\alpha\,\cdot) - \widehat{y}_0 w^B \right) = \left(\widehat{y}_0 \frac{d}{d\alpha} + (\alpha\,\cdot) - \widehat{y}_0 w^B \right) \circ (B\widehat{\sigma})^* \tag{3.4.58}$$

as desired. □

Proposition 3.4.17. The map ∇ commutes with the shift operator $\mathbb{S}$.

Proof. The shift operator is the composition $(B\widehat{\sigma}) \circ \varphi \circ FS_{\widehat{T}}(\sigma,\mu)$. We showed that the connection ∇ commutes with the map $(B\widehat{\sigma})$ in Proposition 3.4.16, with the continuation map φ in Proposition 3.4.6, and with the Floer Seidel map $FS_{\widehat{T}}(\sigma,\mu)$ in Proposition 3.4.12. Therefore the connection also commutes with the composition as desired. ☐

Proof of Theorem 3.4.10. The flatness theorem Theorem 3.4.10 comprises three separate claims. The first claim is that ∇_α commutes with ∇_β for two classes $\alpha, \beta \in H^2(M)$, which was shown in Theorem 3.4.8. The second claim is that ∇ commutes with $\mathbb{S}_\sigma$, which was shown in Proposition 3.4.17. The third and final claim is $\mathbb{S}_\sigma \mathbb{S}_{\sigma'} = \mathbb{S}_{\sigma+\sigma'}$ for two ρ-nonnegative cocharacters $\sigma, \sigma' \in \mathrm{Cochar}^{\geq 0} T$. This final claim follows readily from the fact that the maps which make up (3.4.47) commute with each other, and individually compose: we have $(B\widehat{\sigma})(B\widehat{\sigma'}) = (B(\sigma + \sigma'))$, $\varphi_\sigma \varphi_{\sigma'} = \varphi_{\sigma+\sigma'}$ and $FS_{\widehat{T}}(\sigma,\mu) FS_{\widehat{T}}(\sigma',\mu) = FS_{\widehat{T}}(\sigma + \sigma', \mu)$. ☐

3.5 Toric manifolds

While we discuss toric manifolds in Sections 3.5.1–3.5.3, the reader may wish to consult Section 3.5.4 for a running example (the projective plane $\mathbb{P}^2$).

3.5.1 Closed toric manifold

Let M be a closed monotone $2n$-dimensional symplectic manifold with an effective Hamiltonian action $\rho : T \times M \to M$ of the n-dimensional torus $T = (S^1)^n$. Such a pair (M, ρ) is a monotone *closed toric manifold*. It satisfies all of the hypotheses of Sections 3.3.1 and 3.3.6 because M is simply connected [Cox11, Theorem 12.1.10].

With the decomposition $T = (S^1)^n$, we identify the Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$ with $\mathbb{R}^n$. The lattice of cocharacters $\mathrm{Cochar}\, T$ is naturally isomorphic to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{t}$ under the map $\sigma \mapsto \partial \sigma|_{t=0} \in \mathfrak{t}$. Similarly, we identify $\mathfrak{t}^\vee$ with $\mathbb{R}^n$, such that the lattice of characters $\mathrm{Char}\, T$ is identified with $\mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{t}^\vee$.

The moment map $\mu_\rho : M \to \mathfrak{t}^\vee$ of ρ is uniquely determined up to an additive constant. The *moment polytope* Δ is the image of μ_ρ; it is a convex polytope in $\mathfrak{t}^\vee = \mathbb{R}^n$. The *facets* F_i of Δ are its codimension-1 faces. There exist uniquely-determined real numbers $\lambda_i \in \mathbb{R}$ such that the moment polytope is given by

$$\Delta = \{y \in \mathbb{R}^n : \langle e_i, y\rangle \geq \lambda_i \text{ for } i = 1, \ldots, N\}, \tag{3.5.1}$$

where $e_i \in \mathbb{Z}^n$ is the *primitive inward-pointing normal vector* to the facet F_i for each $i = 1, \ldots, N$. The moment polytope determines the pair (M, ρ) up to equivariant symplectomorphism by Delzant's theorem.

The faces of Δ are the nonempty subsets of the form $F_I = \{\langle y, e_i\rangle = \lambda_i \text{ for } i \in I\}$ for a subset of indices $I = \{i_1, \dots, i_p\} \subset \{1, \dots, N\}$. Associate to each face F_I the cone $C_I \subset \mathbb{R}^n = \mathfrak{t}$ given by the $\mathbb{R}_{\geq 0}$-span of the vectors $\{e_i : i \in I\}$. Thus we assign the ray $\mathbb{R}_{\geq 0} \cdot e_i$ to the facet F_i. The *(inward) normal fan* Σ is the fan comprising these cones. The fan is *complete*: the union of the cones in Σ is all of $\mathbb{R}^n$; and *smooth*: the generators $\{e_i : i \in I\}$ of each cone extend to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Let $\mathbb{Z}[x_1, \dots, x_N]$ be the polynomial ring with formal variables x_i in degree 2. The *linear relations* are the polynomials $\sum_i \langle e_i, y\rangle x_i$, for all $y \in \mathbb{Z}^n$ (or for all y in a basis of $\mathbb{Z}^n$). Let J_{lin} be the ideal generated by the linear relations. The *Stanley-Reisner ideal* is the ideal

$$J_{\text{SR}} = \langle x_{i_1} \cdots x_{i_p} \mid i_1, \dots, i_p \text{ are distinct and do not generate a cone of } \Sigma \rangle. \tag{3.5.2}$$

Set $D_i = \mu^{-1}(F_i)$, and denote by $D_i^{\vee} \in H^2(M)$ the Poincaré dual of D_i.

Theorem 3.5.1 (Cohomology presentation [Cox11, Theorem 12.4.4]). *The map $x_i \mapsto D_i^{\vee}$ induces a ring isomorphism*

$$\mathbb{Z}[x_1, \dots, x_N]/(J_{lin} + J_{SR}) \xrightarrow{\sim} H^*(M). \tag{3.5.3}$$

The map satisfies $\sum_i x_i \mapsto c_1(TM, \omega)$ and $-\sum \lambda_i x_i \mapsto [\omega]$.

Poincaré duality yields $H_2(M) = H^{2n-2}(M)$ because the cohomology ring has no torsion [Fra10, Proposition 1.5]. We can use this description of $H_2(M)$ together with the expressions for $c_1(TM, \omega)$ and $[\omega]$ from Theorem 3.5.1 to construct our Novikov ring Λ.

Let $\sigma_i \in \mathrm{Cochar}\, T$ be the cocharacter corresponding to the vector $e_i \in \mathbb{Z}^n \subset \mathfrak{t}$. The set D_i is the (minimal) fixed locus of $\rho \circ \sigma_i$. Given a vertex v of Δ, let $\mu_v = \mu^{-1}(v)$ be the corresponding fixed point of ρ in M. For any vertex v of the facet F_i, we have $QS(\sigma_i, \mu_v)(1) = D_i^{\vee}$ [MT06, Example 5.3].

Lemma 3.5.2. *Let e be an edge of Δ between two vertices $v_{\pm}$, and let $\gamma \in \mathbb{Z}^n$ be the vector from v_- to v_+. The Seidel map satisfies*

$$QS(\sigma_i, \mu_{v_-})(1) = q^{\langle e_i, \gamma\rangle [e]} QS(\sigma_i, \mu_{v_+})(1), \tag{3.5.4}$$

where $[e] \in H_2(M)$ is the class $[\mu^{-1} e]$.

Proof. It is sufficient to show $u_{v_+}^{\sigma}[S^2] = u_{v_-}^{\sigma}[S^2] + z(\langle e_i, \gamma\rangle [e])$ (this notation is from Section 3.3.15), and moreover it is sufficient to show this on $\mu^{-1}(e) = \mathbb{P}^1$. This can be either shown with a direct homotopy between the two fixed sections or deduced from previous calculations on $\mathbb{P}^1$ (for example [LJ20, Section 8.2]). □

Any edge e is the intersection of $n-1$ facets $e = F_{i_1} \cap \cdots \cap F_{i_{n-1}}$, so it is easy to describe the class $[e] = (x_{i_1} \cdots x_{i_{n-1}})^{\vee}$ using our presentation of $H_2(M)$. Repeated application of Lemma 3.5.2 along a path of edges from a given vertex v to a vertex lying on F_i yields

$$QS(\sigma_i, \mu_v)(1) = q^{d_i} D_{\bar{i}} \tag{3.5.5}$$

for a class $d_i \in H_2(M)$.

The set of distinct indices $I = \{i_1, \ldots, i_p\}$ is *primitive* if the cone C_I is not in Σ, but the cone $C_{I'}$ is in Σ for any proper subset $I' \subsetneq I$. Let I be primitive. The vector $v = e_{i_1} + \cdots + e_{i_p}$ lies in a cone C_J in Σ, for some subset of indices $J = \{j_1, \ldots, j_s\}$. Therefore there exist positive integers $c_1, \ldots, c_s \in \mathbb{Z}^{>0}$ which give

$$e_{i_1} + \cdots + e_{i_p} = c_1 e_{j_1} + \cdots + c_s e_{j_s}. \tag{3.5.6}$$

The indices $j_1, \ldots, j_s$ and integers $c_1, \ldots, c_s$ are unique up to permutation [Bat93, Definition 5.2], and moreover the sets I and J are disjoint.

Fix a vertex v of Δ. Set $z_i = q^{d_i} x_i \in \Lambda[x_1, \ldots, x_N]$, with d_i as in (3.5.5). The *quantum Stanley-Reisner ideal* is

$$J_{\text{qSR}} = \langle z_{i_1} \cdots z_{i_p} - z_{j_1}^{c_1} \cdots z_{j_s}^{c_s} \mid I = \{i_1, \ldots, i_p\} \text{ is primitive} \rangle. \tag{3.5.7}$$

Theorem 3.5.3 (Quantum cohomology presentation [MT06, Proposition 5.2]). *The map $x_i \mapsto D_{\bar{i}}$ induces a ring isomorphism*

$$\Lambda[x_1, \ldots, x_N]/(J_{lin} + J_{\text{qSR}}) \xrightarrow{\cong} QH^*(M). \tag{3.5.8}$$

Sketch proof. The map $x_i \mapsto D_{\bar{i}}$ satisfies $z_i \mapsto QS(\sigma_i, \mu_v)(1)$. The equality of vectors (3.5.6) means the corresponding cocharacters are equal, giving $QS(\sigma_{i_1} \cdots \sigma_{i_p}, \mu_v)(1) = QS(\sigma_{j_1}^{c_1} \cdots \sigma_{j_s}^{c_s}, \mu_v)(1)$. The left-hand side may be expanded as a composition of Seidel maps $QS(\sigma_{i_1}, \mu_v) \circ \cdots \circ QS(\sigma_{i_p}, \mu_v)(1)$ by (3.1.24), and then further rearranged to a product $QS(\sigma_{i_1}, \mu_v)(1) * \cdots * QS(\sigma_{i_p}, \mu_v)(1)$ because the quantum Seidel map is a module map with respect to quantum multiplication. The same procedure applies to the right-hand side. Therefore the generators of J_{qSR} vanish under $z_i \mapsto QS(\sigma_i, \mu_v)(1)$, so the map (3.5.8) is well-defined.

The relation $z_{i_1} \cdots z_{i_p} - z_{j_1}^{c_1} \cdots z_{j_s}^{c_s}$ for the primitive set I can be rearranged to

$$x_{i_1} \cdots x_{i_p} - q^{d_I} x_{j_1}^{c_1} \cdots x_{j_s}^{c_s} \tag{3.5.9}$$

with $\omega(d_I) > 0$ [Bat93]. Therefore the relations in J_{qSR} are exactly the relations in J_{SR}, except they have higher-order corrections. Here, a monomial is *higher-order* if its Λ-coefficient q^A satisfies $\omega(A) > 0$. We deduce the map (3.5.8) is both surjective and injective from Theorem 3.5.1 using induction, recursively incorporating the higher-order corrections [MT06, Lemma 5.1]. □

3.5.2 Presentation of equivariant quantum cohomology

Recall the canonical isomorphism $\mathrm{Sym}(\mathrm{Char}\, T) = H^*(BT)$ from Section 3.3.7. Therefore $H^2(BT)$ is isomorphic to the lattice $\mathbb{Z}^n \cong \mathfrak{t}$. Via this isomorphism, we define a map $H^2(BT) = \mathbb{Z}^n \to \mathbb{Z}[x_1, \ldots, x_N]$ by

$$y \mapsto -\sum_{i=1}^{N} \langle e_i, y \rangle x_i. \tag{3.5.10}$$

Associated to the T-invariant subset $D_i \subset M$, there is an *equivariant cohomology class* $[D_i]_T \in H^2_T(M, \rho)$ [Cox11, Proposition 12.4.13]. These are the T-equivariant analogues of the Poincaré duals $D_i^{\vee}$.

Theorem 3.5.4 (Equivariant cohomology presentation [Cox11, Theorem 12.4.14]). *The map $x_i \mapsto [D_i]_T$ induces a ring isomorphism*

$$\mathbb{Z}[x_1, \ldots, x_N]/J_{SR} \xrightarrow{\cong} H_T^*(M, \rho). \tag{3.5.11}$$

This extends to a $H^*(BT)$-algebra isomorphism under (3.5.10).

Remark 3.5.5 (Surprising minus sign). The minus sign in (3.5.10) is perhaps surprising. Consider the case $T = S^1$. We use the universal bundle $S^{\infty} \to \mathbb{C}P^{\infty}$ for S^1 and we have $H^*(\mathbb{C}P^{\infty}) = \mathbb{Z}[y]$ (see Section 3.3.7). Our S^1-bundle $S^{\infty} \to \mathbb{C}P^{\infty}$ is the unit sphere bundle of the complex line bundle $\mathcal{O}_{\mathbb{C}P^{\infty}}(+1) \to \mathbb{C}P^{\infty}$. The generator y is equal to the first Chern class $c_1(\mathcal{O}_{\mathbb{C}P^{\infty}}(+1))$.

Consider $\mathbb{C}$ with the standard S^1-action. The Borel homotopy quotient $S^{\infty} \times_{S^1} \mathbb{C} \to \mathbb{C}P^{\infty}$ is isomorphic to $\mathcal{O}_{\mathbb{C}P^{\infty}}(-1) \to \mathbb{C}P^{\infty}$, where the minus sign arises since the action on S^{∞} is reversed. The class $y = c_1(\mathcal{O}_{\mathbb{C}P^{\infty}}(+1))$ is equal to $-c_1(\mathcal{O}_{\mathbb{C}P^{\infty}}(-1))$, where $c_1(\mathcal{O}_{\mathbb{C}P^{\infty}}(-1)) = c_1(S^{\infty} \times_{S^1} \mathbb{C})$ plays the same role as the equivariant cohomology classes $[D_i]_T$.

Theorem 3.5.6 (Equivariant quantum cohomology presentation). *The map $x_i \mapsto [D_i]_T$ induces a ring isomorphism*

$$\widehat{\mathbb{Z}}[x_1, \ldots, x_N]/J_{qSR} \xrightarrow{\cong} QH_T^*(M, \rho). \tag{3.5.12}$$

This extends to a $H^*(BT)$-algebra isomorphism under (3.5.10).

Proof. This proof is the T-equivariant analogue of the proof of Theorem 3.5.3.

Exactly like the non-equivariant case, $QS_T(\sigma_i, \mu_v)(1)$ only counts the fixed sections of the clutching bundle $E(\sigma_i)$ at points in D_i. Therefore the map (3.5.12) satisfies $z_i \mapsto q^{d_i}[D_i]_T = QS_T(\sigma_i, \mu_v)(1)$ just like (3.5.5). The T-equivariant quantum Seidel map intertwines the T-equivariant quantum product (it is the $\widehat{T}$-equivariant quantum Seidel map which does not), hence the map (3.5.12) vanishes on J_{qSR}. Thus (3.5.12) is well-defined.

Let B_k be the ideal in $\mathbb{Z}[x_1,\dots,x_N]$ generated by the degree-k products of the polynomials in (3.5.10). Equivalently, B_k is generated by the image of $\mathrm{Sym}^k(\mathrm{Char}\,T)$. Notice $H^*_T(M,\rho) = \lim \mathbb{Z}[x_1,\dots,x_N]/(J_{\mathrm{SR}} + B_k)$ as $k \to \infty$ under the isomorphism (3.5.11). Quotienting by B_k acts like restricting $S^\infty \times_{S^1} M$ to $S^{2k-1} \times_{S^1} M$ in the $T = S^1$ case, and similarly in general.

Elements of $\widehat{\Lambda}\mathbb{Z}[x_1,\dots,x_N]$ are formal sums of monomials $q^A \otimes p$ with a finiteness property: there are only finitely many such monomials with $p \notin B_k$ and the energy $\omega(A)$ bounded above. For any given k, we can use the methods of higher-order corrections in q from Theorem 3.5.3 to show injectivity and surjectivity modulo B_k. Letting $k \to \infty$, we deduce the bijectivity of (3.5.12). □

3.5.3 Connection

First, we modify the presentation (3.5.12) to get a presentation for $QH^*_{\widehat{T}}(M,\widehat{\rho})$: the map $x_i \mapsto [D_i]_{\widehat{T}}$ and $\widehat{y}_0 \mapsto \widehat{y}_0$ induces a $H^*(B\widehat{T})$-module isomorphism

$$\widehat{\Lambda}\mathbb{Z}[x_1,\dots,x_N,\widehat{y}_0]/J_{\mathrm{qSR}} \xrightarrow{\cong} QH^*_{\widehat{T}}(M,\widehat{\rho}). \tag{3.5.13}$$

Since the copy of S^1_0 in $\widehat{T} = S^1_0 \times T$ acts trivially on M, we merely incorporate a new formal variable $\widehat{y}_0$ which generates $H^*(BS^1_0)$. We write $H^*(B\widehat{T}) = \mathrm{Sym}(\mathrm{Char}\,T) \otimes \mathbb{Z}[\widehat{y}_0]$.

Let v be a vertex of the polytope Δ, and let $N(v) = \{i : v \in F_i\}$ be the set of indices for the facets neighbouring v. The *star* of v is the union of the interiors of all faces containing v, denoted $\mathrm{Star}(v)$. The facets F_i with $i \in N(v)$ are exactly the facets which have nonempty intersection with $\mathrm{Star}(v)$. Therefore the restriction map $H^2_{\widehat{T}}(M,\widehat{\rho}) \to H^2_{\widehat{T}}(\mu^{-1}\mathrm{Star}(v),\widehat{\rho})$ satisfies $[D_i]_{\widehat{T}} \mapsto 0$ if and only if $i \notin N(v)$. The inclusion $\{\mu_v\} \to \mu^{-1}\mathrm{Star}(v)$ is an equivariant retraction, so $H^2_{\widehat{T}}(\mu^{-1}\mathrm{Star}(v),\widehat{\rho})$ is naturally isomorphic to $H^2_{\widehat{T}}(\{\mu_v\})$. This determines the map μ_v in (3.5.16).

Recall the set $\{e_i : i \in N(v)\}$ is a basis for $\mathbb{Z}^n \cong \mathfrak{t}$ since Δ is smooth. Let $\{y_i^\vee : i \in N(v)\}$ be the $\langle\cdot,\cdot\rangle$-dual basis to $\{e_i : i \in N(v)\}$. The linear relation corresponding to $y_i^\vee$ is of the form

$$x_i = -\sum_{j \notin N(v)} a_{ji} x_i \tag{3.5.14}$$

for each $i \in N(v)$. Therefore the map

$$\mathbb{Z}\langle x_i : i \notin N(v)\rangle \to \mathbb{Z}\langle x_1,\dots,x_N\rangle/(\text{linear relations}) \tag{3.5.15}$$

is an isomorphism. The right-hand side of (3.5.15) is the degree 2 summand of the presentation (3.5.3) because the elements of J_{SR} are at least degree 4. Putting this together, the short exact sequence (3.3.19) is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(B\widehat{T}) & \longrightarrow & H^2_{\widehat{T}}(M,\widehat{\rho}) & \longrightarrow & H^2(M) & \longrightarrow & 0 \\ && \Big\| && {=}\Big\uparrow && {=}\Big\uparrow && \\ 0 & \longrightarrow & H^2(B\widehat{T}) & \longrightarrow & \mathbb{Z}\langle x_i, \widehat{y}_0\rangle & \longrightarrow & \mathbb{Z}\langle x_i : i \notin N(v)\rangle & \longrightarrow & 0. \end{array} \tag{3.5.16}$$

v ; $_v(x_i) = 0 \;(\;)\; i \in N(v)$; $x_i \mapsto x_i$

It is easiest to define the differential connection for x_i with $i \notin N(v)$, for then the lifting map $\alpha \mapsto \alpha\ ^v$ is simply given by $x_i \mapsto x_i$. The differential connection is

$$r^v_i = \widehat{y}_0 \frac{d}{dx_i} + x_i \text{ for } i \notin N(v). \tag{3.5.17}$$

Remark 3.5.7 (Differentiation). The equivariant quantum cohomology $QH_{\widehat{T}}(M,\widehat{\rho})$ is isomorphic as a $\widehat{\ }H\ (B\widehat{T})$-module to $\widehat{\ }(H\ (B\widehat{T})\quad H\ (M))$. Therefore the elements of $QH_{\widehat{T}}(M,\widehat{\rho})$ may be written as formal sums of monomials $q^A p(y,\widehat{y}_0) r(x)$ where $p(y,\widehat{y}_0) \in H\ (B\widehat{T})$ is any monomial and $r(x) \in \mathbb{Z}[x_1,\dots,x_N]$ does not vanish under (3.5.3). The differentiation operation $\frac{d}{dx_i}$ is defined term-wise when elements are *written in this form*. We have $\frac{d}{dx_i}(q^A p(y,\widehat{y}_0) r(x)) = x_i(A)\ q^A p(y,\widehat{y}_0) r(x)$ for such monomials.

The shift operator $\mathbb{S}^v_{\ \ i} = \mathbb{S}^v_i$ satisfies

$$\mathbb{S}^v_i(1) = z_i = q^{d_i} x_i, \tag{3.5.18}$$

where $d_i \in H_2(M)$ is defined as in (3.5.5). We have

$$(B\widehat{\sigma}_i)\ (y) = y \quad \widehat{\sigma}_i = y + \langle e_i, y\rangle \widehat{y}_0, \tag{3.5.19}$$

where in the second equality we write out the character $y \quad \widehat{\sigma}_i$ in terms of the original character y and $\widehat{y}_0$.

Theorem 3.5.8. *We can compute $\mathbb{S}^v_i$ on the whole ring $QH_{\widehat{T}}(M,\widehat{\rho})$ with respect to the presentation (3.5.13) by combining (3.5.18) with the flatness equations $\mathbb{S}^v_i r^v_j = r^v_j \mathbb{S}^v_i$ for $j \notin N(v)$, where r^v_j is given by (3.5.17), and the relations $\mathbb{S}^v_i(yx) = (B\widehat{\sigma}_i)\ (y)\ \mathbb{S}^v_i(x)$ for $y \in \mathbb{Z}^n \quad t$, where $(B\widehat{\sigma}_i)\ (y)$ is given by (3.5.19).*

Moreover since the set of normal vectors $\{e_i\}_{i=1}^N$ spans $\mathbb{Z}^n$ = Cochar T, the shift operators $\mathbb{S}^v_i$ may be combined to compute $\mathbb{S}^v$ for any cocharacter σ.

Proof. We prove that $\mathbb{S}^v_i$ is determined by these ingredients on all monomials. Since $\mathbb{S}^v_i$ satisfies $\mathbb{S}^v_i(\widehat{y}_0 x) = \widehat{y}_0 \mathbb{S}^v_i(x)$, we need only consider monomials in the variables x_j. We use induction on the degree k of the monomial. The base case $k = 0$ is (3.5.18). For the induction step, take a monomial $x_j p$, where p is a monomial of degree k.

If $j \notin N(v)$ holds, we rearrange the flatness equation for $r_j^\vee$ to get

$$S_i^\vee(x_j p) = r_j^\vee S_i^\vee(p) - \hat{y}_0 S_i^\vee\left(\frac{d}{dx_j}p\right). \tag{3.5.20}$$

By the induction hypothesis, the right-hand side of (3.5.20) is known. Here, we are using the fact that $\frac{d}{dx_j}p$ is a sum of monomials of degree $\leq k$ with coefficients in the Novikov ring. This follows because, to put p into the form required by Remark 3.5.7, we apply the relations (3.5.10) and (3.5.9), both of which only decrease the degree of the monomials.

Conversely, if $j \in N(v)$ holds, then the linear relation (3.5.14) combined with (3.5.10) yields

$$S_i^\vee(x_j p) = \sum_{l \notin N(v)} -a_{lj} S_i^\vee(x_l p) - S_i^\vee(y_j^\vee p), \tag{3.5.21}$$

where $y_j^\vee$ is the dual to e_j The shift operator was computed above for the monomials $x_l p$ with $l \notin N(v)$, and we have

$$S_i^\vee(y_j^\vee p) = (y_j^\vee + \langle e_i, y_j^\vee\rangle \hat{y}_0)\, S_i^\vee(p) \tag{3.5.22}$$

using (3.5.19). Therefore the right-hand side of (3.5.21) is determined. □

3.5.4 Projective plane

An instructive example of a closed monotone toric manifold is the complex projective plane $\mathbb{CP}^2$. The 2-dimensional torus $T = (S^1)^2$ acts on $\mathbb{CP}^2$ via $(t_1, t_2) \cdot [w_0, w_1, w_2] = [w_0, e^{2\pi i t_1} w_1, e^{2\pi i t_2} w_2]$. The moment map $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$ given by

$$\mu([w_0, w_1, w_2]) = \frac{1}{\sum_i |w_i|^2}(|w_1|^2, |w_2|^2) \tag{3.5.23}$$

yields the moment polytope and fan as in Figure 3.10. The vertices $v_0 = (0,0)$, $v_1 = (1,0)$ and $v_2 = (0,1)$ correspond to the fixed points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$ respectively. The inward-pointing normal vectors are $e_0 = (-1,-1)$, $e_1 = (1,0)$ and $e_2 = (0,1)$, and the corresponding real numbers for (3.5.1) are $\lambda_0 = -1$, $\lambda_1 = 0$ and $\lambda_2 = 0$ respectively. The facets F_i correspond to the invariant subsets $D_i = \{w_i = 0\}$.

The linear relations are $x_1 - x_0$ and $x_2 - x_0$, and the Stanley-Reisner ideal is $J_{SR} = \langle x_0 x_1 x_2 \rangle$. The presentation of the cohomology ring $H^*(\mathbb{CP}^2)$ from Theorem 3.5.1 is

$$\mathbb{Z}[x_0, x_1, x_2]/\langle x_1 - x_0, x_2 - x_0, x_0 x_1 x_2 \rangle. \tag{3.5.24}$$

The map $x_i \mapsto x$ induces an isomorphism to $\mathbb{Z}[x]/x^3$, which is the standard presentation of $H^*(\mathbb{CP}^2)$. Using this presentation, we have $c_1(T\mathbb{CP}^2, \omega) = 3x$ and $[\omega] = x$. Therefore, the Novikov ring is $\Lambda = \mathbb{Z}[q]$, with q in degree 6, and $\frac{d}{dx} q^k = k q^k$ acts like $q \frac{d}{dq}$.

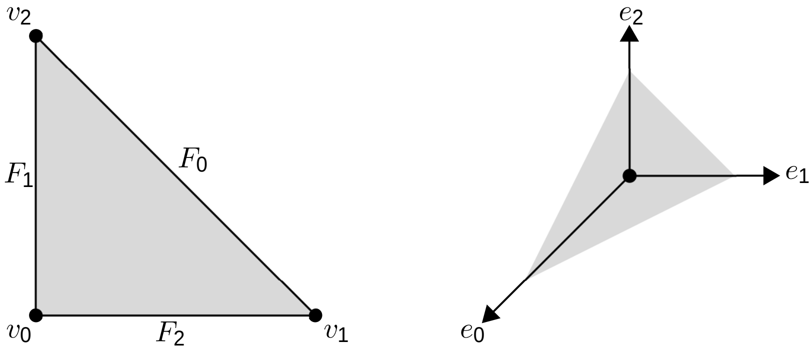


Figure 3.10: The moment polytope and the inner normal fan for $\mathbb{P}^2$.

Set $v = v_0$. This yields $z_1 = x_1$ and $z_2 = x_2$. To find z_0, we use the edge $e = F_2$ from v_0 to v_1 which has vector $\gamma = (1, 0)$. Lemma 3.5.2 yields

$$QS(\sigma_0, \mu_{v_0})(1) = q^{\langle e_0^*, [e]\rangle} QS(\sigma_0, \mu_{v_1})(1) = q^{-1} QS(\sigma_0, \mu_{v_1})(1), \tag{3.5.25}$$

giving $z_0 = q^{-1}x_0$. The only primitive set is $\{0, 1, 2\}$, for which (3.5.6) reads $e_0 + e_1 + e_2 = 0$. Thus the quantum Stanley-Reisner ideal is $J_{\mathrm{qSR}} = \langle z_0z_1z_2 - 1\rangle = \langle x_0x_1x_2 - q\rangle$. Theorem 3.5.3 yields the presentation

$$QH^*(\mathbb{P}^2) = \frac{[x_0, x_1, x_2]}{\langle x_1 - x_0, x_2 - x_0, x_0x_1x_2 - q\rangle}. \tag{3.5.26}$$

The map $x_i \mapsto x$ recovers the familiar presentation $[x]/(x^3 - q)$.

The lattice $\mathbb{Z}^2 \otimes t$ is generated by the vectors $y_1 = (1, 0)$ and $y_2 = (0, 1)$, so we write $H^*(BT) = \mathbb{Z}[y_1, y_2]$. Theorem 3.5.4 gives the presentation

$$H_T(\mathbb{P}^2) = \frac{\mathbb{Z}[x_0, x_1, x_2]}{\langle x_0x_1x_2\rangle} \tag{3.5.27}$$

for equivariant cohomology, with the $\mathbb{Z}[y_1, y_2]$-module structure arising from the map $\mathbb{Z}[y_1, y_2] \to \mathbb{Z}[x_0, x_1, x_2]$ given by

$$\begin{cases} y_1 \mapsto x_0 - x_1 \\ y_2 \mapsto x_0 - x_2, \end{cases} \tag{3.5.28}$$

as in (3.5.10). Via Theorem 3.5.6, the map (3.5.28) provides the module structure for the presentations of equivariant quantum cohomology

$$QH_T(\mathbb{P}^2) = \frac{\widehat{\mathbb{Z}}[x_0, x_1, x_2]}{\langle x_0x_1x_2 - q\rangle}, \qquad QH_{\widehat{T}}(\mathbb{P}^2) = \frac{\widehat{\mathbb{Z}}[x_0, x_1, x_2, \widehat{y}_0]}{\langle x_0x_1x_2 - q\rangle}. \tag{3.5.29}$$

Only the facet F_0 does not contain v_0, so the differential connection is

$$\nabla_0^{v_0} = \widehat{y}_0 q\frac{d}{dq} + x_0. \tag{3.5.30}$$

We demonstrate how to compute the shift operator $\mathsf{S}_1^{v_0}$. We have $\mathsf{S}_1^{v_0}(1) = x_1$. By applying the flatness equation to 1, we find $\mathsf{S}_1^{v_0}(x_0) = x_0 x_1$:

$$\begin{aligned} 0 &= \mathsf{S}_1^{v_0}\, r_0^{v_0}(1) \quad r_0^{v_0}\mathsf{S}_1^{v_0}(1) \\ &= \mathsf{S}_1^{v_0}(x_0) \quad r_0^{v_0}(x_1) \\ &= \mathsf{S}_1^{v_0}(x_0) \quad x_0 x_1. \end{aligned} \tag{3.5.31}$$

We use (3.5.28) to deduce the value of $\mathsf{S}_1^{v_0}$ for the remaining monomials x_1 and x_2:

$$\begin{aligned} \mathsf{S}_1^{v_0}(x_1) &= \mathsf{S}_1^{v_0}(x_0 \quad y_1) & \qquad \mathsf{S}_1^{v_0}(x_2) &= \mathsf{S}_1^{v_0}(x_0 \quad y_2) \\ &= x_0 x_1 \quad (y_1 + \widehat{y}_0)x_1 & &= x_0 x_1 \quad y_2 x_1 \\ &= x_1^2 \quad \widehat{y}_0 x_1 & &= x_1 x_2 \\ &= x_1(x_1 \quad \widehat{y}_0) & & \end{aligned} \tag{3.5.32}$$

We derive $\mathsf{S}_1^{v_0}$ for the monomials in degree 4 by repeating these steps: first apply the flatness equation to each of the monomials x_0, x_1 and x_2; and second use (3.5.28) to deduce $\mathsf{S}_1^{v_0}$ on the remaining monomials. This yields

$$\begin{aligned} \mathsf{S}_1^{v_0}(x_0^2) &= x_0^2 x_1 & \qquad \mathsf{S}_1^{v_0}(x_1^2) &= x_1(x_1 \quad \widehat{y}_0)^2 \\ \mathsf{S}_1^{v_0}(x_0 x_1) &= x_0 x_1(x_1 \quad \widehat{y}_0) & \qquad \mathsf{S}_1^{v_0}(x_1 x_2) &= x_1(x_1 \quad \widehat{y}_0)x_2 \\ \mathsf{S}_1^{v_0}(x_0 x_2) &= x_0 x_1 x_2 & \qquad \mathsf{S}_1^{v_0}(x_2^2) &= x_1 x_2^2. \end{aligned} \tag{3.5.33}$$

The pattern $\mathsf{S}_1^{v_0}(x_0^{k_0} x_1^{k_1} x_2^{k_2}) = x_0^{k_0} x_1 (x_1 \quad \widehat{y}_0)^{k_1} x_2^{k_2}$ holds for $k_0 + k_1 + k_2 \quad 2$ because the differentiation operation vanishes for such polynomials. It fails in general, however. For example, applying the flatness equation to $x_1 x_2$ yields

$$0 = \mathsf{S}_1^{v_0}\, r_0^{v_0}(x_1 x_2) \quad r_0^{v_0}\mathsf{S}_1^{v_0}(x_1 x_2) \tag{3.5.34}$$

$$= \mathsf{S}_1^{v_0}(x_0 x_1 x_2) \quad r_0^{v_0}(x_1(x_1 \quad \widehat{y}_0)x_2) \tag{3.5.35}$$

$$= \mathsf{S}_1^{v_0}(q) \quad \widehat{y}_0 q \frac{d}{dq}(x_1(x_1 \quad \widehat{y}_0)x_2) \quad x_0(x_1(x_1 \quad \widehat{y}_0)x_2) \tag{3.5.36}$$

$$= q x_1 \quad \widehat{y}_0 q \quad q(x_1 \quad \widehat{y}_0) \tag{3.5.37}$$

$$= 0. \tag{3.5.38}$$

The input to the differentiation operation in (3.5.36) must be rearranged to the format described in Remark 3.5.7,

$$\begin{aligned} x_1(x_1 \quad \widehat{y}_0)x_2 &= (x_0 \quad y_1)(x_1 \quad \widehat{y}_0)x_2 \\ &= q + \widehat{y}_0 x_0 x_2 + \widehat{y}_0 y_1 x_2 \quad y_1 x_1 x_2, \end{aligned} \tag{3.5.39}$$

before differentiating with respect to the variable q. A naive differentiation 'as is' would have missed the q term. The only monomials $r(x)$ 2 $\mathbb{Z}[x_0, x_1, x_2]$ which survive under (3.5.3) are the monomials $r(x) = x_0^{k_0} x_1^{k_1} x_2^{k_2}$ with $k_0 + k_1 + k_2 \quad 2$. Therefore we have completely described $\mathsf{S}_1^{v_0}$ since the input can be put in the format of Remark 3.5.7.

3.5.5 Toric negative line bundle

While we discuss toric negative line bundles in this section, the reader may wish to consult Section 3.5.6 for a running example (the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$).

Let (B, ρ_B) be a closed monotone toric manifold with symplectic form ω_B. Let λ_B denote the monotonicity constant[13] of B. We use a moment polytope $\Delta_B = \{\langle e_i^B, y\rangle \geq -\lambda_i^B\} \subset \mathbb{R}^n$ for B, and we denote the T^B-invariant subsets corresponding to its facets F_i^B by D_i^B.

The complex line bundle $\pi : E \to B$ is *negative* if $c_1(E) = -k[\omega_B]$ holds for some positive $k > 0$. There is a symplectic form ω_E on such E for which (E, ω_E) has a convex structure (ψ, SE) [Oan08; Rit14, Section 7]. Here, SE is the sphere bundle. Moreover, (E, ω_E) is monotone if and only if $k < \lambda_B$ holds, and in this case the monotonicity constant of E is $\lambda_E = \lambda_B - k$ [Rit16, Section 4.3]. We assume $0 < k < \lambda_B$.

We follow Ritter's procedure to construct a moment polytope for E [Rit16, Section 7]. The line bundle E is isomorphic to $\mathcal{O}(\sum_i m_i D_i^B)$ for integers $m_i \in \mathbb{Z}$. Ritter gives a construction to find these integers when ω_B is primitive[14] [Rit16, Sections 7.4-7.6]. The moment polytope Δ_E lies in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. With respect to this decomposition, set

$$\begin{aligned} e_i^E &= (e_i^B, -m_i) & \lambda_i^E &= \lambda_i^B & &\text{for } i = 1, \ldots, N \\ e_{N+1}^E &= (0, 1) & \lambda_{N+1}^E &= 0, \end{aligned} \tag{3.5.40}$$

and let Δ_E be the polytope defined by $\{\langle e_i^E, y\rangle \geq -\lambda_i^E\} \subset \mathbb{R}^{n+1}$. The inward normal fan of Δ_E is not complete because E is not closed. The first N facets of Δ_E correspond to $D_i^E = \pi^{-1} D_i^B$, while the last corresponds to the zero section $D_{N+1}^E = B \subset E$.

Denote by σ_i^E the cocharacter of $T^E = (S^1)^{n+1}$ corresponding to the vector e_i^E. The S^1-action $\rho^E \circ \sigma_i^E$ is admissible, and in fact $\sigma_i^E \in \operatorname{Cochar}_{\Delta_E}^0 T^E$ holds.

Theorem 3.5.9. *The presentations of cohomology, quantum cohomology, equivariant cohomology and equivariant quantum cohomology from Theorems 3.5.1, 3.5.3, 3.5.4 and 3.5.6 hold for E and Δ_E. Moreover, given a vertex v of Δ_E, the methods of Section 3.5.3 apply to compute $r_i^{\,v}$ for $i \notin N(v)$ and $S_i^{\,v}$ for all i.*

The only exception is that we can only compute $S_\sigma^{\,v}$ for those cocharacters $\sigma \in \operatorname{Cochar} T^E$ which are nonnegative sums of the cocharacters σ_i. Such σ lie in a submonoid of $\operatorname{Cochar}_{\Delta_E}^0 T^E$ in general. We cannot define the shift operator for all cocharacters because the shift operator is only defined for ρ^E-nonnegative cocharacters.

[13] The *monotonicity constant* of a monotone symplectic manifold (M, ω) is the positive number $\lambda > 0$ for which $c_1(TM) = \lambda[\omega]$ holds.
[14] The symplectic form ω on M is *primitive* if $[\omega] \in H^2(M; \mathbb{Z})$ is integral and is not a multiple of another class.

The cocharacter $\sigma = \sigma_{N+1}$ induces the fibre-wise rotation action about the zero section. The Hamiltonian of this action is linear of positive slope [Rit14, Section 7.6], so the direct limit of the sequence

$$QH^*(E) \xrightarrow{QS(\sigma;\mu_v)} QH^{*+|\mu_v|}(E) \xrightarrow{QS(\sigma;\mu_v)} QH^{*+2|\mu_v|}(E) \xrightarrow{QS(\sigma;\mu_v)} \cdots \tag{3.5.41}$$

is naturally isomorphic to the symplectic cohomology $SH^*(E)$ [Rit14, Theorem 22]. All the modules in this sequence are $QH^*(E)$ and the maps $QS(\sigma, \mu_v)$ are module maps given by multiplication. Therefore, the direct limit of the sequence may be characterised as

$$SH^*(E) = \frac{QH^*(E)}{\ker(QS(\sigma,\mu_v)^n)} \tag{3.5.42}$$

because the sequence $QS(\sigma,\mu_v)^p(QH^*(E))$ stabilizes by $p = n$ [Rit14, Theorem 1]. Combining (3.5.42) with (3.5.8) yields the following powerful result.

Theorem 3.5.10 (Symplectic cohomology presentation [Rit16, Theorem 1.5(2)]). *The map* $x_i \mapsto c^*(D_{\hat{T}})$ *induces a ring isomorphism*

$$[x_1^{\pm 1}, \ldots, x_{N+1}^{\pm 1}]/(J_{lin} + J_{qSR}) \xrightarrow{\cong} SH^*(E), \tag{3.5.43}$$

where $c^* : QH^*(E) \to SH^*(E)$ *is the canonical map.*

Similarly, the direct limit of the sequence

$$QH_{\hat{T}}^*(E,\hat{\rho}) \xrightarrow{QS_{\hat{T}}(\sigma;\mu_v)} QH_{\hat{T}}^{*+|\mu_v|}(E,\hat{\sigma}^*\hat{\rho}) \xrightarrow{QS_{\hat{T}}(\sigma;\mu_v)} \cdots, \tag{3.5.44}$$

is the equivariant symplectic cohomology $SH_{\hat{T}}^*(E,\hat{\rho})$ [LJ20, Equation (7.4)]. Since the map $(B\hat{\sigma})^*$ is an isomorphism, the sequence (3.5.44) is naturally isomorphic to

$$QH_{\hat{T}}^*(E,\hat{\rho}) \xrightarrow{S_{\mu_v}} QH_{\hat{T}}^{*+|\mu_v|}(E,\hat{\rho}) \xrightarrow{S_{\mu_v}} QH_{\hat{T}}^{*+2|\mu_v|}(E,\hat{\rho}) \xrightarrow{S_{\mu_v}} \cdots. \tag{3.5.45}$$

This second sequence preserves the difference-differential connection $(S_{\mu_v}, \nabla_{\mu_v})$ by Proposition 3.4.17. Unfortunately, both of sequences (3.5.44) and (3.5.45) do not satisfy the key properties of (3.5.41) which we used to derive (3.5.42). Specifically, the modules are different in (3.5.44), while the maps in (3.5.45) are not module maps (they are twisted by $(B\hat{\sigma})^*$). These sequences may nonetheless be used to understand $SH_{\hat{T}}^*(E,\hat{\rho})$, as in Section 3.5.6.

3.5.6 Tautological line bundle

The tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ is a negative line bundle with $k = 1$. The base $\mathbb{P}^1$ together with the S^1-action

$$t \cdot [w_0, w_1] = [w_0, e^{2\pi i t} w_1] \tag{3.5.46}$$

Figure 3.11: The moment polytope and the inner normal fan for $\mathbb{P}^1$.

admits the moment polytope from Figure 3.11, defined by $e_0 = -1$, $e_1 = 1$ and $\lambda_0 = -1$, $\lambda_1 = 0$.

The bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ is isomorphic to $\mathcal{O}(-D_1) \to \mathbb{P}^1$ [Rit16, Example, page 2021]. Therefore the total space $\mathcal{O}_{\mathbb{P}^1}(-1)$ admits the moment polytope from Figure 3.12, defined by its inward normal vectors $e_0 = (-1, 0)$, $e_1 = (1, 1)$ and $e_2 = (0, 1)$; and $\lambda_0 = -1$, $\lambda_1 = 0$ and $\lambda_2 = 0$. The unique primitive set is $\{0, 1\}$, for which (3.5.6) reads $e_0 + e_1 = 1e_2$, so the quantum Stanley-Reisner ideal is $J_{qSR} = \langle z_0 z_1 - z_2 \rangle$. The Novikov ring is the polynomial ring $\Lambda = \mathbb{Z}[q]$ with q in degree 2. Theorem 3.5.9 yields the presentation

$$QH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1)) = \frac{\widehat{\Lambda}\mathbb{Z}[x_0, x_1, x_2, \widehat{y}_0]}{\langle x_0 x_1 - q x_2 \rangle}, \tag{3.5.47}$$

where the $\widehat{\Lambda}\mathbb{Z}[\widehat{y}_0, y_1, y_2]$-module structure is induced by the map

$$\begin{cases} y_1 \mapsto x_0 - x_1 \\ y_2 \mapsto -x_1 - x_2. \end{cases} \tag{3.5.48}$$

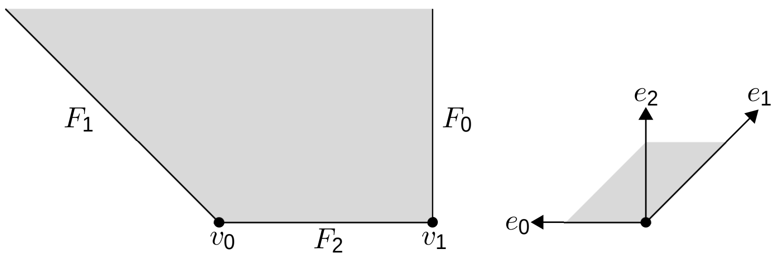


Figure 3.12: The moment polytope and the inner normal fan for $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Fix the vertex $v_0 = (0, 0)$ with $\mu_{v_0} = [1, 0]$. The differential connection is $\nabla_0^{v_0} = \widehat{y}_0 q \frac{d}{dq} + x_0$. Using the same approach as Section 3.5.4, we find that the shift operator $\mathbb{S}_2^{v_0}$ satisfies

$$\mathbb{S}_2^{v_0}(1) = x_2 \tag{3.5.49}$$

$$\mathbb{S}_2^{v_0}(x_0) = x_0 x_2 \tag{3.5.50}$$

$$\mathbb{S}_2^{v_0}(x_1) = x_1 x_2 \tag{3.5.51}$$

$$\mathbb{S}_2^{v_0}(x_2) = (x_2 - \widehat{y}_0) x_2. \tag{3.5.52}$$

Set $A = \widehat{\ }\mathbb{Z}[\widehat{y}_0, y_1, y_2]$. Consider the sequence (3.5.44). The modules in this sequence are $QH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1), (\widehat{\sigma}_2^r)^* \widehat{\rho})$ for $r \geq 0$. They are all naturally isomorphic to $A\langle 1, x_2 \rangle$, the free A-module with basis $\langle 1, x_2 \rangle$. The r-th map in the sequence,

$$QS_{\widehat{T}}(\sigma_2, \mu_{v_0}) : QH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1), (\widehat{\sigma}_2^{r-1})^* \widehat{\rho}) \to QH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1), (\widehat{\sigma}_2^r)^* \widehat{\rho}), \tag{3.5.53}$$

is equal to the composition $((B\widehat{\sigma}_2)^*)^{-r} \circ S_2^{v_0} \circ ((B\widehat{\sigma}_2)^*)^{r-1}$. By rearranging (3.5.52), we find that (3.5.53) is the A-module map given by

$$\begin{cases} 1 \mapsto x_2 \\ x_2 \mapsto (y_2 - r\widehat{y}_0)(r\widehat{y}_0 + y_1 - y_2) + (q + y_1 - 2y_2 + (2r-1)\widehat{y}_0)x_2. \end{cases} \tag{3.5.54}$$

with respect to the basis $\langle 1, x_2 \rangle$. Its determinant is

$$d_r = (y_2 - r\widehat{y}_0)(y_2 - y_1 - r\widehat{y}_0). \tag{3.5.55}$$

The maps (3.5.54) are isomorphisms over the fraction field Frac(A), and hence we have an isomorphism of A-modules

$$\mathrm{Frac}(A) \otimes_A SH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1)) = \mathrm{Frac}(A)\langle 1, x_2 \rangle. \tag{3.5.56}$$

The induced inclusion $SH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1)) \hookrightarrow \mathrm{Frac}(A)\langle 1, x_2 \rangle$ is injective, and thus every element of $SH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1))$ is of the form[15]

$$\frac{\alpha}{d_1 \cdots d_r} \tag{3.5.57}$$

for some $r \geq 0$ and some $\alpha \in A\langle 1, x_2 \rangle$. Note that $SH_{\widehat{T}}(\mathcal{O}_{\mathbb{P}^1}(-1))$ does not surject onto the set of elements of the form (3.5.57).

The non-equivariant quantum Seidel maps in the analogous sequence (3.5.41) are all the multiplication-by-x_2 map, giving

$$SH^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \ [x_0, x_1, x_2^{-1}]/(J_{\mathrm{lin}} + J_{\mathrm{SR}}). \tag{3.5.58}$$

The linear relations are $x_1 - x_0$ and $x_1 + x_2$, so (3.5.58) is equivalent to (3.5.43).

Remark 3.5.11 (Reeb orbits). The contact manifold $S\mathcal{O}_{\mathbb{P}^1}(-1)$ is isomorphic to S^3, which we write as the unit sphere in $\mathbb{C}^2$. The induced T-action on S^3 is given by $(\mathrm{t}_1, \mathrm{t}_2) \cdot (w_0, w_1) = (e^{2\pi i(\mathrm{t}_2 - \mathrm{t}_1)}w_0, e^{2\pi i \mathrm{t}_2}w_1)$, where the t_1 action comes from (3.5.46) and the t_2 action rotates the fibres. The flow along the Reeb vector field is given by $\theta \cdot (w_0, w_1) = (e^{2\pi i\theta} w_0, e^{2\pi i\theta} w_1)$. Let $\gamma_r : S^1 \to S^3$ be the Reeb orbit $\theta \mapsto (e^{2\pi ir\theta}, 0)$ for $r \in \mathbb{Z}_{>0}$, and consider the $\widehat{T}$-orbit

[15] Consider by analogy the sequence $\mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_3} \cdots$ for nonzero integers $d_r \in \mathbb{Z} \setminus 0$. This sequence is isomorphic to the sequence of inclusions $\mathbb{Z} \hookrightarrow \frac{1}{d_1}\mathbb{Z} \hookrightarrow \frac{1}{d_1 d_2}\mathbb{Z} \hookrightarrow \cdots$. The direct limit is naturally a $\mathbb{Z}$-submodule of $\mathbb{Q} = \mathrm{Frac}(\mathbb{Z})$ and every element of the direct limit is a quotient of an element $\in \mathbb{Z}$ by a product $d_1 d_2 \cdots d_r$.

$\widehat{T}\gamma_r$ in $LO_{\mathbb{P}^1}(-1)$. This $\widehat{T}$-orbit is isomorphic to S^1 because the Reeb orbits are determined by their value at 0. Therefore the map $\widehat{T} \to \widehat{T}\gamma_r$ is a map $\widehat{T} \to S^1$. As a character, this map $\widehat{T} \to S^1$ corresponds to $y_2 - y_1 - r\widehat{y}_0$. Repeating this with the Reeb orbits $\theta \mapsto (0, e^{2\pi i r\theta})$ yields $y_2 - r\widehat{y}_0$.

Conjecture 3.5.12 (Significance of the denominators). *In general, the denominators in* (3.5.57) *will be the product of the characters corresponding to those Reeb orbits whose $\widehat{T}$-orbit is isomorphic to S^1.*

3.6 Topological proofs

Each of these lemmas is clear if $\sigma = \mathrm{Id}_M$, so without loss of generality we assume that M is simply connected.

Lemma 3.6.1. *The sequence*

$$0 \longrightarrow H_2(M) \xrightarrow{\;z\;} H_2(E(\sigma)) \longrightarrow H_2(S^2) = \mathbb{Z} \longrightarrow 0. \tag{3.6.1}$$

is a short exact sequence.

Proof. The sphere S^2 is the union of $\mathbb{C}^+ = S^2 \setminus \{z^-\}$ and $\mathbb{C}^- = S^2 \setminus \{z^+\}$, and the intersection of these sets is the cylinder $\mathbb{R} \times S^1$. We use the Mayer-Vietoris sequence of the bundle $\pi : E(\sigma) \to S^2$ associated to this decomposition of the sphere. The relevant part of the Mayer-Vietoris long exact sequence is

$$\begin{aligned} &H_2(\pi^{-1}(\mathbb{R} \times S^1)) \to H_2(\pi^{-1}(\mathbb{C}^-)) \oplus H_2(\pi^{-1}(\mathbb{C}^+)) \to H_2(\pi^{-1}(S^2)) \\ &\quad \to H_1(\pi^{-1}(\mathbb{R} \times S^1)). \end{aligned} \tag{3.6.2}$$

We have a homeomorphism $\pi^{-1}(\mathbb{R} \times S^1) = M \times S^1 \times \mathbb{R}$ from the trivialisation of $E(\sigma)$ over the pole z^+. This homeomorphism yields $H_2(\pi^{-1}(\mathbb{R} \times S^1)) = H_2(M) \oplus H_1(M) \otimes H_1(S^1)$ and $H_1(\pi^{-1}(\mathbb{R} \times S^1)) = H_1(M) \oplus H_1(S^1)$ by the Kunneth theorem. Since M is simply connected, we have $H_1(M) = 0$, which gives $H_2(\pi^{-1}(\mathbb{R} \times S^1)) = H_2(M)$ and $H_1(\pi^{-1}(\mathbb{R} \times S^1)) = H_1(S^1)$. Moreover, the degree-2 isomorphism is independent of the choice of trivialising the bundle over the different poles $z^\pm$ (a chain homotopy between the maps $H_2(M) \to H_2(\pi^{-1}(\mathbb{R} \times S^1))$ is found by considering the inclusions $M \to E(\sigma)$ to the fibres at points $(s, 0) \in \mathbb{R} \times S^1 \subset S^2$).

Contracting each of $\mathbb{C}^\pm$ to the pole $z^\pm$ converts (3.6.2) into

$$\begin{aligned} H_2(M) &\longrightarrow H_2(M) \oplus H_2(M) \longrightarrow H_2(E(\sigma)) \longrightarrow H_1(S^1) \\ x &\longmapsto (x, x). \end{aligned} \tag{3.6.3}$$

Therefore, one of the copies of $H_2(M)$ cancels with the first term of the sequence.

Finally, we have a commutative diagram

$$\begin{array}{ccc} H_2(E(\sigma)) & \longrightarrow & H_1(S^1) \\ \downarrow & & \| \\ H_2(S^2) & \xrightarrow{=} & H_1(S^1) \end{array} \tag{3.6.4}$$

where the isomorphism $H_2(S^2) \to H_1(S^1)$ comes from the Mayer-Vietoris sequence for the sphere S^2. The lemma follows from combining (3.6.3) and (3.6.4). □

Lemma 3.6.2. *There is a class* $\beta \in H^2_{\widehat{T}}(E(\sigma))$ *which satisfies* $\beta^+ = \alpha$, $(B\widehat{\sigma})(\beta\ ) = \alpha$ *and* $\beta(A\ ) = \alpha(A)$ *for* $A \in H_2(M)$.

Proof. Write the sphere S^2 as the union of $\mathbb{C}^+ = S^2 \setminus \{z\ \}$ and $\mathbb{C}\ = S^2 \setminus \{z^+\}$, as in Lemma 3.6.1. We will apply the ($\widehat{T}$-equivariant cohomology) Mayer-Vietoris sequence to the bundle $\pi : E(\sigma) \to S^2$ using this decomposition of the sphere. The relevant part of this Mayer-Vietoris long exact sequence is

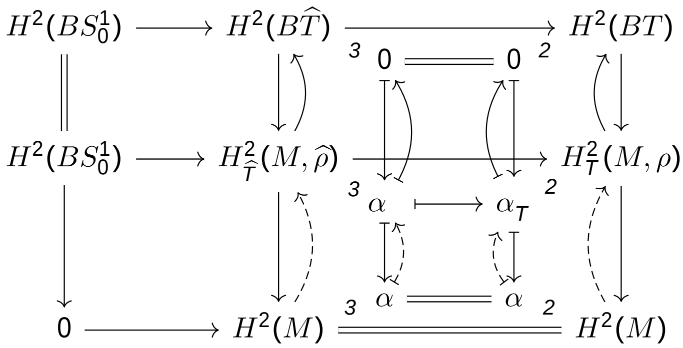
$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad H^1_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) \\ \hookrightarrow H^2_{\widehat{T}}(\pi^{\ 1}(S^2)) \to H^2_{\widehat{T}}(\pi^{\ 1}(\mathbb{C}\ )) \quad H^2_{\widehat{T}}(\pi^{\ 1}(\mathbb{C}^+)) \to H^2_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)). \end{array} \tag{3.6.5}$$

The second group in the sequence is $H^2_{\widehat{T}}(E(\sigma))$. We will show $H^1_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) = 0$, and then use the rest of the sequence to construct the element $\beta \in H^2_{\widehat{T}}(E(\sigma))$.

The inclusion $M \to \pi^{\ 1}(\mathbb{R}\ \ S^1)$ along the equator at $\theta = 0$ is independent of the choice of trivialisation between D $\ M$. As per (3.3.11), this inclusion yields a natural map $H_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) \to H_T(M)$, with the corresponding group map $T \to \widehat{T}$. The inclusion fits into a fibre bundle $M \to \pi^{\ 1}(\mathbb{R}\ \ S^1) \to \mathbb{R}\ \ S^1$ which pairs with the groups $T \to \widehat{T} \to S^1_0$. That is, T acts on M, $\widehat{T}$ acts on $\pi^{\ 1}(\mathbb{R}\ \ S^1)$ and S^1_0 acts on $\mathbb{R}\ \ S^1$, and the maps satisfy (3.3.10) with respect to these actions. The S^1_0-action on $\mathbb{R}\ \ S^1$ is free, which gives $H_{S^1_0}(\mathbb{R}\ \ S^1) = \mathbb{Z}$. Therefore the map $H_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) \to H_T(M)$ is a canonical isomorphism (because the Leray-Serre spectral sequence degenerates on the first page). But $H^1_T(M) = 0$ vanishes when M is simply connected ($\pi_1(ET\ \ _T M) = 0$ follows from the long exact sequence of homotopy groups of the Borel quotient $ET\ \ _T M$). Therefore $H^1_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) = 0$.

From the above argument, we also get a canonical isomorphism $H^2_{\widehat{T}}(\pi^{\ 1}(\mathbb{R}\ \ S^1)) \to H^2_T(M, \rho)$. As per the definition of $\alpha\ $ in Section 3.3.12, define $\alpha_T \in H^2_T(M, \rho)$ via the analogous split short exact sequence to (3.3.19).

Just as C is equivariantly contractible to 0, the manifold $\pi^{-1}(\mathbb{C}^+)$ will $\widehat{T}$-equivariantly retract onto the fibre $\pi^{-1}(z^+) = M$ with action $\widehat{\rho}$. We have the following commutative diagram (the splitting maps commute only in the obvious way).



(3.6.6)

The classes α, α and α_T are drawn in the diagram. The vertical short exact sequences are precisely the sequences which define α and α_T. From the diagram, we *deduce* $\alpha \mapsto \alpha_T$ on the second row.

Similarly, $\pi^{-1}(\mathbb{C})$ contracts onto the fibre $\pi^{-1}(z) = M$ with action $\sigma\ \widehat{\rho}$. The same diagram as (3.6.6) holds for $H^2_{\widehat{T}}(M, \sigma\ \widehat{\rho})$ giving $((B\widehat{\sigma}))^{-1}(\alpha) \mapsto \alpha_T$.

Thus the Mayer-Vietoris sequence (3.6.5) becomes

$$\begin{array}{ccccccc} 0 \longrightarrow H^2_{\widehat{T}}(E(\sigma)) \longrightarrow & H^2_{\widehat{T}}(M, \sigma\ \widehat{\rho}) & H^2_{\widehat{T}}(M, \widehat{\rho}) & \longrightarrow H^2_T(M, \rho) \\ & ((B\widehat{\sigma}))^{-1}(\alpha) & 0 & \longmapsto \alpha_T \\ & 0 & \alpha & \longmapsto \alpha_T . \end{array} \qquad (3.6.7)$$

The minus sign in α_T simply comes from our Mayer-Vietoris conventions. The desired class β is the preimage of $((B\widehat{\sigma}))^{-1}(\alpha)\ \ \alpha_T$. The classes β are, by definition, the restrictions of β to the fibres above the poles z, so $\beta^+ = \alpha$ and $(B\widehat{\sigma})(\beta) = \alpha$ are automatically satisfied.

The fibre inclusion $E(\sigma) \to E\widehat{T} \times_{\widehat{T}} E(\sigma)$ induces a map $H_2(E(\sigma)) \to H^{\widehat{T}}_2(E(\sigma))$. Let A_{eq} denote the image of $A = z(A) + u[S^2]$ under this map. The claim $\beta(A) = \alpha(A)$ precisely means $\beta(A_{\text{eq}}) = \alpha(A)$. To prove this claim, we consider the two terms of A separately. Diagram chasing using the fixed section $u : S^2 \to E(\sigma)$ with (3.6.5) yields $(u)\beta = 0 \in H^2_{\widehat{T}}(S^2)$, and hence $\beta((u[S^2])_{\text{eq}}) = 0$. Our definition of β readily yields $\beta \mapsto \alpha$ under $H^2_{\widehat{T}}(E(\sigma)) \to H^2_{\widehat{T}}(M) \to H^2(M)$, and hence we have $\beta(A_{\text{eq}}) = \beta((z(A))_{\text{eq}}) = \alpha(A)$ as required. $\square$

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