A Two-Asset Jump Diffusion Model with Correlation



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Chapter 1

Introduction

Here we concern ourselves with the pricing of options on financial stocks. The market for options is huge, for example in January 2007 on average 10 million options contracts were traded per day in the United States. Therefore need for an accurate option pricing model is obvious as, bearing in mind the volumes being traded, any slight inaccuracies in the price could cause great losses to an investor.

In practice the models presented by Fischer Black and Myron Scholes [2] in their seminal paper published in 1973 are most widely used by practitioners in the markets and in recent years most research into derivative pricing has concentrated on refining these models. The popularity of the models is mainly due to the fact that the majority of the required inputs are observable variables; the only unobservable variable required is the volatility of the underlying stock. The model assumes that the price of the underlying asset follows a geometric Brownian motion (GBM),

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{1.1}$$

Under this assumption the stock price follows a log-normal distribution between any two points in time. Black and Scholes then used the no arbitrage principle (NAP) to derive a unique price for the option. On top of the assumption of a GBM modelling the evolution of the underlying stock price, the Black-Scholes model also rests on the following assumptions about the market:

1. there are no transaction costs or taxes;
2. trading takes place continuously in time,
3. borrowing and short-selling are allowed,
4. borrowing and lending rates are equal,
5. the short-term interest rate is known and is constant,
6. the stock does not pay dividends,

Further research has refined the original Black-Scholes model and shown that the method holds when the interest rate is assumed to be non-constant or even stochastic; when a stock pays dividends; when restrictions are made on the use of short sales and when the option is of the American type, i.e. it can be exercised any time before or at the expiry date.

As mentioned above, mispricing of an option can lead to huge financial losses and empirical studies of the prices admitted by the Black-Scholes model show that using this model this is the case, see Black [3]. The critical assumptions in the Black-Scholes derivation is that trading takes place continuously in time and that the stock price has a continuous sample path with probability one.

Realistically, continuous trading is not possible, however rendering the Black-Scholes model invalid because of this would be an over-reaction as the continuous trading solution is a valid asymptotic approximation to the discrete trading solution, provided the stock price dynamics have continuous sample paths (which of course they are not, but are close to).

The Black-Scholes arbitrage portfolio under continuous trading has zero-risk due to continuous hedging. However under discrete trading conditions we introduce some risk since the market moves between trades. The portfolio risk has the order of the trading interval length and thus the risk will tend to zero as the interval between trades tends to zero. Therefore provided the time interval between trades is not too large, the error between the Black-Scholes price and the realistic discrete trading price will not differ by much.

The Black-Scholes model is not valid though, even if we trade in the continuous limit, if the stock price dynamics do not have a continuous sample path. The Black-Scholes formula is valid if the stock price can only change by a small amount over a small interval of time. Again empirical studies show that this is not the case and a more sophisticated model of the underlying stock price is required. Market returns are generally leptokurtic meaning the market distribution has heavier tails than a normal distribution. The model should permit large random fluctuations such as crashes or upsurges. The market distribution is generally negatively skewed since downward outliers are usually larger than upward outliers.

Robert Merton [16] proposed a solution to this problem. He suggested adding another term to the evolution of the underlying stock price given by equation (1.1), this new term would model jumps in the asset price. The evolution for the underlying stock becomes,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + dJ, \tag{1.2}$$

where dJ represents a jump term.

1.1 Modelling Jumps in the Underlying Stock Price

Using equation (1.2), we will have two types of changes affecting the overall change in the stock price. As with the original Black-Scholes model we have fluctuations in the price due to general economic factors such as supply and demand, changes

in economic outlook etc. These factors cause small movements in the price and are modelled by a geometric Brownian motion with a constant drift term. The jump term models the arrival of important information into the market that will have an abnormal effect on the price. This information could be industry specific or even firm specific. By its very nature important information only arrives at discrete points in time and will be modelled by a jump.

The causes of the jumps in the stock price can be put in three separate categories.

1. **Firm specific jumps** - These jumps only affect individual firms. They may be caused by news entering the market about an individual firm's profit report or management news etc.

2. **Industry/sector specific jumps** - These jumps are caused by news entering the market that may only affect a specific industry, for example a national holiday in which weather is particularly bad may affect the stocks reliant on the British tourism industry.

3. **Market specific jumps** - These jumps affect every company in the market. They may be caused by news affecting the general market such as interest rates, credit spreads or oil prices. Not all firms are affected in the same way however, for example some firms may jumps up, others down and the jumps may be of varying magnitude.

When modelling an option dependent on the one individual stock, differentiation between causes of jumps is not necessary since we only need to know when the stock jumps and not what caused the jump or whether other stocks were affected by the same information. However when modelling two stocks the different causes of jumps is of interest to us and jumps may occur independently in each stock or they may be common to both.

What properties should these jumps have? Here we assume that each jump in the stock price is independent of the others. This assumption is not necessarily realistic but we make the assumption for modelling purposes. Our jump term will take the form of a random variable J that can take positive or negative values and determines the jump magnitude, multiplied by an integer valued process N_t that initiates a jump. N_t must have the following properties,

- $N_t - N_s$ is independent of N_s.
- $N_0 = 0, N_t \in \mathbb{N}^+$,
- $N_s \leq N_t$ if $s < t$.

We also require that in a given short space of time δt, the likelihood of a jump is roughly proportional to the length of δt. The proportionality constant is denoted by λ and is called the jump intensity. If δt is taken to be small, the probability of two jumps in the interval is negligible. Therefore another property our counting process

must satisfy is,

$$N_{t+\delta t} = \begin{cases} N_t & \text{with probability } 1 - \lambda\delta t - o(\delta t) \\ N_t + 1 & \text{with probability } \lambda\delta t + o(\delta t) \\ N_t + k & \text{with probability } o(\delta t). \end{cases} .$$

A process satisfying the four properties above is called a Poisson process with intensity λ. It has the Poisson probability density function,

$$\mathbb{P}(N_t = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t} \qquad j = 0, 1, 2, \ldots$$

Proof. Define $p_n(t) \equiv \mathbb{P}(N_t = n)$. Consider $p_n(t + \delta t)$, i.e. $\mathbb{P}(N_{t+\delta t} = n)$. There are $n + 1$ ways in which $N_{t+\delta t}$ can equal n:

1. $N_t = n$ and no jumps occur in δt,
2. $N_t = n - k$ and k jumps occur in δt, $\qquad k = 1, 2, \ldots, n$.

These events are disjoint therefore

$$\begin{aligned} p_n(t + \delta t) &= \sum_{0 \le m \le n} \mathbb{P}(N_t = m)\mathbb{P}(N_{t+\delta t} = n | N_t = m) \\ &= \sum_{0 \le m \le n} \mathbb{P}(N_t = m)\mathbb{P}(n - m \text{ arrivals in } \delta t) \\ &= \mathbb{P}(N_t = n)(1 - \lambda\delta t) + \mathbb{P}(N_t = n - 1)(\lambda\delta t) + o(\delta t) \\ &= p_n(t)(1 - \lambda\delta t) + p_{n-1}(t)(\lambda\delta t) + o(\delta t). \end{aligned}$$

This final equation holds for $n \neq 0$. For $n = 0$ we find

$$p_0(t + \delta t) = p_0(t)(1 - \lambda\delta t) + o(\delta t).$$

We expand $p_n(t + \delta_t)$ about t using Taylor series,

$$p_n(t + \delta t) = p_n(t) + \delta t \frac{dp_n(t)}{dt} + (\delta t)^2 \frac{d^2 p_n(t)}{dt^2} + \ldots$$

Therefore our two equations for $p_n(t)$ and $p_0(t)$ become

$$p_n(t) + \delta t \frac{dp_n(t)}{dt} + \cdots = p_n(t)(1 - \lambda\delta t) + p_{n-1}(t)(\lambda\delta t) + o(\delta t)$$

and

$$p_0(t) + \delta t \frac{dp_0(t)}{dt} + \cdots = p_0(t)(1 - \lambda\delta t) + o(\delta t).$$

Therefore upon cancelling the $p_n(t)$ and $p_0(t)$ from both sides respectively, dividing through by δt and letting $\delta t \downarrow 0$ we get

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t) \qquad n = 1, 2, \ldots \tag{1.3}$$

and

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t). \tag{1.4}$$

We now have a system of equations with boundary conditions $p_n(0) = \delta_{n0}$ where δ_{ij} is the Kronecker Delta. We solve this system by induction.

Equation (1.4) with $p_0(t) = 1$ is solved to give

$$p_0(t) = e^{-\lambda t}. \tag{1.5}$$

We substitute equation (1.5) into equation (1.3) with $n = 1$ and $p_1(t) = 0$ and solve to get

$$p_1(t) = \lambda t e^{-\lambda t}.$$

Now suppose that

$$p_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$$

and look to solve equation (1.3). We have

$$\frac{dp_n(t)}{dt} = \lambda^n t^{n-1} e^{-\lambda t} - \lambda p_n(t)$$

with $p_n(0) = 0$. This is solved by

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

as required.

□

Now we must consider how the jump variable J will be distributed. Suppose that information enters the market causing an instantaneous jump in the asset price after which the price has moved from S_t to $J_t S_t$. Here J_t is the absolute magnitude of the jump. Therefore the relative price change will be given by

$$\frac{dS_t}{S_t} = \frac{J_t S_t - S_t}{S_t} = J_t - 1.$$

The simplest option would be to choose all jumps to be equal to some constant $J_t = J^c$ for all t, however this would be highly unrealistic. It would not be ridiculous to assume that the larger the magnitude of the jump, the less probable it would be. Merton [16] suggests modelling the jump variables as non-negative log-normal random variables in order to provide a more realistic jump term. A random variable X has a log-normal distribution with mean μ and variance σ if $\ln(X)$ is normally distributed with mean μ and variance σ. Equivalently X has a log-normal distribution if $X = \exp(Y)$ where Y has a normal distribution with mean μ and variance σ. If

$$X \sim N(\alpha, \delta) \text{ then } X \sim \text{log-normal}(e^{\alpha + \frac{1}{2}\delta^2}, e^{2\alpha + \delta^2}(e^{\delta^2} - 1)).$$

The jump variable J is distributed as follows,

$$j_t = \ln(J_t) \sim \text{ IID } N(\alpha, \delta^2). \tag{1.6}$$

The relative jump size is $(J_t - 1)$ where J_t is log-normally distributed with mean α and variance δ^2, therefore

$$\begin{aligned} &\mathbb{E}[(J_t - 1)] = e^{\alpha + \frac{1}{2}\delta^2} - 1 \equiv K \\ &\mathbb{E}\big[\big((J_t - 1) - \mathbb{E}[J_t - 1]\big)^2\big] = e^{2\alpha + \delta^2}(e^{\delta^2} - 1). \end{aligned} \tag{1.7}$$

1.2 The Merton Model

Using the last section, the evolution of an asset under a jump diffusion model is

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (J_t - 1)dN_t.$$

Here μ and σ are constants. W_t is a Wiener process with respect to the market probability measure $\mathbb{P}$. N_t is a Poisson process (with constant intensity λ) with respect to the market probability measure also, the Poisson process is assumed to be independent of the Wiener process. J_t is the random jump variable that is assumed to follow the log-normal distribution. We also assume that J is independent of the other two random processes in the evolution, namely the Wiener process and Poisson process. The variable J is also independent through time,

$$\text{Cov}(J_s, J_t) = 0 \quad \text{for} \quad s \neq t.$$

The model evolves as follows:

$$\frac{dS_t}{S_t} = \begin{cases} \mu dt + \sigma dW_t & \text{if no Poisson event occurs} \\ \mu dt + \sigma dW_t + (J_t - 1) & \text{if a Poisson event occurs} \end{cases}.$$

Therefore if a jump is initiated and the log-normal random variable takes the value 1.1, the stock price will jump up by 10%. Likewise a drawing of 0.9 for J_t will cause a 10% fall in the stock price.

We now look to solve equation (1.2) and find an expression for the log prices $\ln(S_t)$. For this we need to use Ito's formula for jump-diffusion processes as given by Cont and Tankov [7], we will introduce this shortly.

1.2.1 Notation

Due to the instantaneous nature of jumps in the price of an asset, we will have a discontinuity in the price when a jump occurs. To distinguish between the prices either side of the discontinuity we introduce the following notation. If at time t a jump occurs, denoted by ΔS_t then

$$S_{t-} = \lim_{r \uparrow t} S_r \quad \text{and} \quad S_t = S_{t-} + \Delta S_t.$$

Theorem 1.2.1. (Ito's Formula for Jump-Diffusions) *Suppose X_t is a jump diffusion process with evolution given by*

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s + \sum_{i=1}^{N_t} \Delta X_i,$$

where a_t is the drift term, b_t is the volatility term and ΔX_i corresponds to jump i in the stock price. Then Cont and Tankov [7] state that for a function $f(X_t, t)$,

$$\begin{aligned} df(X_t,t) = &\frac{\partial f(X_t,t)}{\partial t}dt + a_t\frac{\partial f(X_t,t)}{\partial x}dt + \frac{b_t^2}{2}\frac{\partial^2 f(X_t,t)}{\partial x^2}dt \\ &+ b_t\frac{\partial f(X_t,t)}{\partial x}dW_t + [f(X_{t-} + \Delta X_t) - f(X_{t-})]. \end{aligned} \tag{1.8}$$

Using this theorem for $f(.) = \ln(.)$ and S_t described by the stochastic differential equation (1.2) we get

$$\begin{aligned} d\ln S_t &= \frac{\partial \ln S_t}{\partial t}dt + \mu S_t\frac{\partial \ln S_t}{\partial S_t}dt + \sigma S_t\frac{\partial \ln S_t}{\partial \ln S_t}dW_t + [\ln j_t S_t - \ln S_t] \\ &= \mu S_t\frac{1}{S_t}dt + \frac{\sigma^2 S_t^2}{2}\left(-\frac{1}{S_t^2}\right)dt + \sigma S_t\frac{1}{S_t}dW_t + [\ln J_t + \ln S_t - \ln S_t] \\ &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t + \ln J_t. \end{aligned}$$

This is solved to give

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} \ln J_i. \tag{1.9}$$

This is clearly the same as it would be in the Black Scholes geometric Brownian motion case except for the sum of log-normal jumps. We must be careful when adding a term of this type to the price process. The term

$$Q_t = \sum_{i=1}^{N_t} \ln J_i$$

is called a compound Poisson process and adding a term of this form to a geometric Brownian motion, as we have done above, will affect the drift of the asset. We see using a moment generating function argument (Appendix 1) that

$$\mathbb{E}\left[\sum_{i=1}^{N_t} \ln J_i\right] = \lambda t K$$

where K is defined in equation (1.7). Therefore the addition of a compound Poisson process will increase the mean of the asset by $\lambda t K$.

We introduce the compensated Poisson process

$$Q_t - \lambda t K.$$

This process is a martingale.

Proof.

$$\mathbb{E}[Q_t - K\lambda t|\mathcal{F}_s] = \mathbb{E}[Q_t - Q_s|\mathcal{F}_s] + Q_s - K\lambda t.$$

Due to the memorylessness of exponential random variables (see [18]) we have

$$\mathbb{E}[Q_t - Q_s|\mathcal{F}_s] = \mathbb{E}[Q_t - Q_s] = K\lambda t - K\lambda s = K\lambda(t-s).$$

Therefore

$$\mathbb{E}[Q_t - K\lambda t|\mathcal{F}_s] = K\lambda(t-s) + Q_s - K\lambda t = Q_s - K\lambda s.$$

□

Since the compensated Poisson process is martingale we can added it to the price process without affecting the drift of the asset μ. Hence equations (1.2) and (1.9) must become

$$\frac{dS_t}{S_t} = (\mu - \lambda K)dt + \sigma dW_t + (J_t - 1)dN_t \tag{1.10}$$

and

$$\ln S_t = \ln S_0 + \left(\mu - \lambda K - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} \ln J_i. \tag{1.11}$$

Upon taking exponentials of equation (1.11) we get the solution for S_t,

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} j_i\right\}. \tag{1.12}$$

where the variable $j = \ln J$ is normally distributed as stated in equation (1.6). Equivalently we can write

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \prod_{i=1}^{N_t} J_i. \tag{1.13}$$

Notice that if all the jumps that occur in the interval $(0, t)$ are zero then $\sum_{i=1}^{N_t} j_i = 0$ or $\prod_{i=1}^{N_t} J_i = 1$ as we would expect.

We note here that an alternative method for adjusting the Merton model to account for the predictable parts of the jumps involves incorporating the expectation of the jump term into the Wiener process. By definition, a Wiener process satisfies $W_0 = 0$ and has normally distributed increments with zero mean,

$$W_t - W_s \sim N(0, t-s) \Longrightarrow W_t \sim N(0, t).$$

In order to incorporate the predictable expectation of the jump term, the mean of our Wiener process would have to become $-\frac{K\lambda}{\sigma}$.

1.3 Option Pricing under the Merton Model

Merton [16] derived a pricing formula for a European call option on the asset S under the Merton jump diffusion model. He used a delta-hedging argument similar to that used by Black and Scholes in the derivation of the call option pricing formula under geometric Brownian motion. If no jump occurs in the asset price then the only risk in the asset evolution comes from the Wiener process W_t. The portfolio that is long one call option and short Δ units of the asset S is perfectly hedged of this risk so the portfolio grows at the risk-free rate r. However if a jump does occur the portfolio is exposed to the jump risk as the hedge has not eliminated the risk associated with the Poisson process N_t. Merton makes the assumption that the risk associated with the jumps in the asset price is diversifiable since the jumps in the individual asset price are uncorrelated with the market as a whole, in other words the risk is unsystematic. If this is the case then the Capital Asset Pricing Model (CAPM) says the jump terms offer no risk premium and the asset still grows at the risk free rate. Merton derived a partial differential equation, which was solved by an infinite series. If we denote the t-price of the call on asset S with strike K under the Merton jump diffusion model by $V_M(S_t, K, r, T)$ where r is the risk-free rate and T is the option maturity, then

$$V_M(S_t, K, r, T) = \sum_{i=0}^{\infty} e^{-\overline{\lambda}(T-t)} \frac{(\overline{\lambda}(T-t))^i}{i!} V_{\text{call}}(S_t, r_i, \sigma_i, T), \tag{1.14}$$

where

$$\begin{aligned}
\overline{\lambda} &= \lambda e^{\alpha + \frac{\delta^2}{2}} \\
r_i &= r - \lambda(e^{\alpha + \frac{\delta^2}{2}} - 1) + i\frac{(\alpha + \frac{\delta^2}{2})}{(T-t)} \\
\sigma_i &= \sqrt{\sigma^2 + i\frac{\delta^2}{T-t}}
\end{aligned}$$

and the normally distributed jump variables j have mean α and variance δ^2. Here V_{call} is the Black Scholes call price. This pricing formula is not a closed formula as it involves an infinite sum that needs to be truncated in order to calculate a price. However the sum converges rapidly so accurate prices can be achieved with a relatively small number of terms. Figure 1.1 shows the prices admitted by the Merton jump diffusion model for two different sets of jump parameters compared with Black Scholes prices.

The prices admitted by the Merton model for these sets of jump parameters are nearly always greater than the Black Scholes prices and the difference between the prices increases as the jump parameters increase in magnitude. For options deep in-the-money or deep out-the-money the difference between the prices is smaller than the options that are at-the-money. We will discuss this in further detail in Section 3.3.

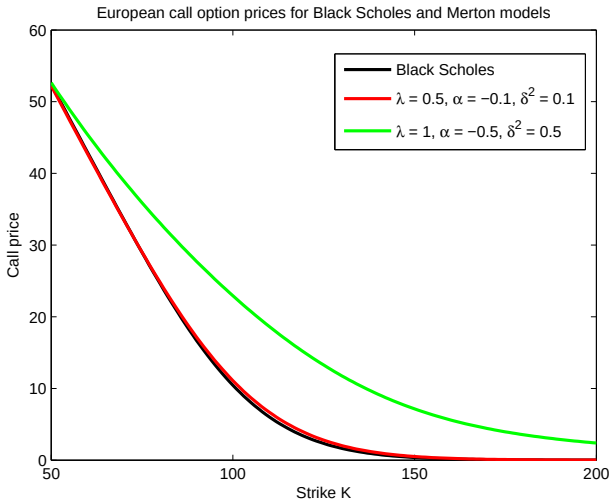


Figure 1.1: Black Scholes prices and Merton prices for a European call option.

1.4 The Structure of this Project

Under the assumption that the underlying stock price evolves according to a geometric Brownian motion we can compute explicit pricing formulas for calls and puts and also a wide range of exotic options too, for example see Haug [11]. However under a jump diffusion model, explicit pricing formulas are harder to come by. Merton [16] derived formulas for calls and puts under the jump diffusion model using a risk-hedging argument and these formulas have subsequently also been derived using equivalent martingale measures. Few explicit formulas for options under jump diffusion exist due to the increased complexity that jumps cause.

A further complication occurs when the exotic option to be priced has a payoff that is a function of two underlying assets rather than one. This is due to the correlation that may or may not exist between the two assets. In the geometric Brownian motion case the only correlation between two assets comes via the Wiener processes. In the jump diffusion environment however, it is possible and more realistic to have correlated jump terms. This is the main focus of this project. Previous work by Ike Dike [8] focused on spread options, which are exotic options with a payoff that is a function of typically two energy commodities. The underlying assets were modelled using jump diffusion processes in which the jump terms were perfectly correlated and the focus was on the numerical solution of the pricing equations.

In this project we first we propose a model that allows us to realistically imitate correlation between the jumps in two assets. We do not focus on a particular industry or commodity as a model of this type could be applied to any option on two correlated assets that exhibit discontinuous price processes. We then derive a pricing formula for an exchange option on two assets when the assets are modelled using the jump diffusion model with correlation.

We then price exchange options and another exotic option called a max-call under

the jump diffusion with correlation model using Monte Carlo methods. The price of an exchange option found using Monte Carlo methods can then be compared with the price admitted by the pricing formula already derived. We also investigate implied volatility and implied correlation of jump diffusion prices.

Chapter 2

Correlated Jumps

Suppose we choose to model two correlated assets $S^{(1)}$ and $S^{(2)}$ using a jump diffusion model. Both assets are modelled by equation (1.10),

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (\mu_i - \lambda_i K_i)dt + \sigma_i dW_t^{(i)} + (J^{(i)} - 1)dN_t^{(i)} \quad i = 1, 2. \tag{2.1}$$

Here μ_i is the constant mean of the asset i, σ_i is the constant volatility, $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener processes with correlation ρ_{12} and volatility matrix $\{\sigma_{ij}\}_{i,j=1}^2$. The variables $J^{(i)}$ are the log-normally distributed jump variables. The Poisson processes $N_t^{(1)}$ and $N_t^{(2)}$ have constant intensities denoted by λ_1 and λ_2 respectively.

The two jump terms are to be partially correlated meaning that if one asset jumps then the other will jump with a probability p. We construct $N_t^{(1)}$ and $N_t^{(2)}$ in such a manner to achieve this. We construct them using three independent Poisson processes denoted by $n_t^{(1)}$, $n_t^{(2)}$ and $n_t^{(3)}$. These independent Poisson processes $n_t^{(i)}$ have intensity denoted by λ_{i*} and the discrete probability density function

$$\mathbb{P}(n_t^{(i)} = j) = \frac{(\lambda_{i*}t)^j}{j!}e^{-\lambda_{i*}t} \tag{2.2}$$

derived in Section 1.1. We then construct $N_t^{(1)}$ and $N_t^{(2)}$ as follows,

$$N_t^{(1)} = n_t^{(1)} + n_t^{(3)} \tag{2.3}$$

$$N_t^{(2)} = n_t^{(2)} + n_t^{(3)}. \tag{2.4}$$

Changing the construction of the Poisson process $N_t^{(1)}$ in this way does not change it's distribution. This is seen using characteristic functions.

For a random variable Y the characteristic function of Y is defined as

$$\varphi_Y(\omega) = \mathbb{E}(e^{i\omega Y}).$$

If Y has probability density function f_Y, the characteristic function is given by

$$\mathbb{E}(e^{i\omega Y}) = \begin{cases} \int_{-\infty}^{\infty} e^{i\omega x} f_Y(x)dx & \text{if } Y \text{ is continuous} \\ \sum_{-\infty}^{\infty} e^{i\omega x} f_Y(x) & \text{if } Y \text{ is discrete.} \end{cases} \tag{2.5}$$

We know $n_t^{(i)}$ has a Poisson distribution with intensity λ_{i*} and discrete probability density function given by equation (2.2), therefore it has a characteristic function equal to

$$\phi_{n^{(i)}}(\theta) = \sum_{i=0}^{\infty} e^{i\theta x} \frac{(\lambda_{i*}t)^x e^{-\lambda_{i*}t}}{x!} = e^{-\lambda_{i*}t} \sum_{i=0}^{\infty} \frac{(\lambda_{i*}te^{i\theta})^x}{x!} = e^{\lambda_{i*}t(e^{i\theta}-1)}.$$

We define $N_t^{(i)} = n_t^{(i)} + n_t^{(3)}$ and using the fact that the characterstic function of the sum of two independent random variables is simply the product of the characteristic functions of the individual random variables we get,

$$\begin{aligned}\phi_{N^{(i)}}(\theta) &= \phi_{n^{(i)}}(\theta) \times \phi_{n^{(3)}}(\theta) \\ &= e^{\lambda_{i*}t(e^{i\theta}-1)} \times e^{\lambda_{3*}t(e^{i\theta}-1)} \\ &= e^{(\lambda_{i*}+\lambda_{3*})t(e^{i\theta}-1)}\end{aligned}$$

It follows that $N_t^{(i)} \sim \text{Poiss}(\lambda_{i*} + \lambda_{3*})$.

The Poisson processes $N_t^{(1)}$ and $N_t^{(2)}$ are capable of producing independent jumps through $n_t^{(1)}$ and $n_t^{(2)}$ and simultaneous jumps through $n_t^{(3)}$. Using work by M'Kendrick [17], a change of variables and integrating out $n_t^{(3)}$ results in the joint probability density function for $N_t^{(1)}$ and $N_t^{(2)}$,

$$\mathbb{P}(N_t^{(1)} = i, N_t^{(2)} = j) = \sum_{k=0}^{\min(i,j)} e^{-(\lambda_{1^*}+\lambda_{2^*}+\lambda_{3^*})} \frac{\lambda_{1^*}^{i-k}\lambda_{2^*}^{j-k}\lambda_{3^*}^{k}}{(i-k)!(j-k)!k!}. \tag{2.6}$$

Both assets are modelled by

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (\mu_i - \lambda_i K_i - \lambda_3 K_3)dt + \sigma_i dW_t^{(i)} + (J_t^{(i)} - 1)dn_t^{(i)} + (J_t^{(3)} - 1)dn_t^{(3)} \tag{2.7}$$

for $i = 1, 2$. Here $K_i = \exp\left(\alpha_i + \frac{1}{2}\delta_i^2\right)$ for $i = 1, 2, 3$. Using the same methods as in Section 1.2 this is solved for $S^{(i)}$ to give,

$$S_t^{(i)} = S_0^{(i)} \exp\left\{(\mu_i - \lambda_i K_i - \lambda_3 K_3)t + \sigma_i W_t^{(i)} + \sum_{k=0}^{n_t^{(i)}} j_k^{(i)} + \sum_{l=0}^{n_t^{(3)}} j_l^{(3)}\right\}. \tag{2.8}$$

Here we have chosen that there be three jump variables, one for each independent Poisson process. Figure 2.1 shows an example of a simulation of this model. We see that asset 1 exhibits an independent jump near time $t = 0.51$ whereas both assets exhibit common jumps near time $t = 0.22$ and $t = 0.65$. Alternatively we could have chosen not to differentiate between jumps initiated by either Poisson process in each asset. This is a simple case of the model outlined in equation (2.7) in which $J_t^{(i)} = J_t^{(3)}$. We will use this model when we derive a pricing formula for an exchange option in Section 4.

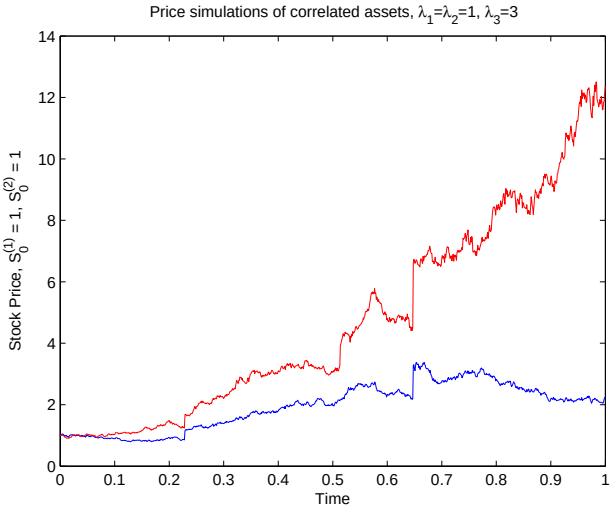


Figure 2.1: Simulation of asset prices; $\lambda_1 = \lambda_2 = 1, \lambda_3 = 3$.

2.1 Covariance of two assets

We have that the two assets are modelled by equation (2.1) and therefore

$$\mathbb{E}\left[\frac{dS_t^{(i)}}{S_t^{(i)}}\right] = \mu_i dt.$$

The variance of asset i is found by

$$\begin{aligned}\mathrm{Var}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}\right) &= \mathbb{E}\left[\left(\frac{dS_t^{(1)}}{S_t^{(1)}} - \mathbb{E}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}\right)\right)^2\right] \\ &= \mathbb{E}\left[\left(-(\lambda_i K_i + \lambda_3 K_3)dt + \sigma_1 dW_t^{(i)} + (J_t^{(i)} - 1)dn_t^{(1)} + (J_t^{(3)} - 1)dn_t^{(3)}\right)^2\right]\end{aligned}$$

Using the following rules

$$\begin{aligned}(dt)^2 &= 0, \\ dt dW_t^{(i)} &= 0 \qquad \text{for } i = 1, 2, \\ (dW_t^{(i)})^2 &= \sigma_i^2 dt \qquad \text{for } i = 1, 2, \\ dW_t^{(1)} dW_t^{(2)} &= \rho_{12} dt,\end{aligned}$$

this simplifies to

$$\mathrm{Var}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}\right) = (\sigma_1^2 + \lambda_i K_i^2 + \lambda_3 K_3^2)dt.$$

The covariance is found as follows.

$$\begin{aligned}
\mathrm{Cov}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}, \frac{dS_t^{(2)}}{S_t^{(2)}}\right) &= \mathbb{E}\left[\left(\frac{dS_t^{(1)}}{S_t^{(1)}} - \mathbb{E}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}\right)\right)\left(\frac{dS_t^{(2)}}{S_t^{(2)}} - \mathbb{E}\left(\frac{dS_t^{(2)}}{S_t^{(2)}}\right)\right)\right] \\
&= \mathbb{E}\Big[\big(-(\lambda_1 K_3 + \lambda_3 K_3)dt + \sigma_1 dW_t^{(1)} + (J_t^{(1)} - 1)dn_t^{(1)} + (J_t^{(3)} - 1)dn_t^{(3)}\big) \\
&\qquad \times \big(-(\lambda_2 K_2 + \lambda_3 K_3)dt + \sigma_2 dW_t^{(2)} + (J_t^{(2)} - 1)dn_t^{(2)} + (J_t^{(3)} - 1)dn_t^{(3)}\big)\Big]
\end{aligned}$$

Again using the rules outlined above, this simplifies to

$$\begin{aligned}
\mathrm{Cov}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}, \frac{dS_t^{(2)}}{S_t^{(2)}}\right) &= \mathbb{E}[\sigma_1\sigma_2\rho_{12} + (J_t^{(3)} - 1)(J_t^{(3)} - 1)dn_t^{(3)}] \\
&= (\sigma_1\sigma_2\rho_{12} + K_3^2\lambda_3)dt
\end{aligned}$$

The correlation of the two assets is defined as

$$\begin{aligned}
\mathrm{Corr}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}, \frac{dS_t^{(2)}}{S_t^{(2)}}\right) &= \frac{\mathrm{Cov}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}, \frac{dS_t^{(2)}}{S_t^{(2)}}\right)}{\sqrt{\mathrm{Var}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}\right)}\sqrt{\mathrm{Var}\left(\frac{dS_t^{(2)}}{S_t^{(2)}}\right)}} \\
&= \frac{\sigma_1\sigma_2\rho_{12} + K_3^2\lambda_3}{\sqrt{\sigma_1^2 + K_1^2\lambda_1 + K_3^2\lambda_3}\sqrt{\sigma_2^2 + K_2^2\lambda_2 + K_3^2\lambda_3}}. \qquad (2.9)
\end{aligned}$$

It is clear to see that increasing the correlation ρ_{12}, between the two Wiener processes will increase the correlation between the two assets. Increasing the size of λ_3 compared with λ_1 and λ_2, and increasing K_3 compared with K_3 and K_2 will also increase correlation between the assets.

Figure 2.2 shows how the correlation between the assets varies when K_1 and K_2 are kept fixed and K_3 is varied between 0 and 2. For each plot we set $\sigma_1 = \sigma_2 = 0.5$, $\rho_{12} = 0.5$ and $\lambda_i = 1$ for $i = 1, 2, 3$.

Figure 2.3 shows how the correlation varies when λ_1 and λ_2 are kept fixed and λ_3 varies between 0 and 10. σ_1, σ_2 and ρ are as above and we set $K_i = 1$ for $i = 1, 2, 3$.

2.2 Characteristic Functions

We find the log-price process for asset $S^{(i)}$ by dividing equation (1.12) by $S_0^{(i)}$ and taking the natural logarithm. We denote the log-price process of asset $S^{(i)}$ by $X^{(i)}$,

$$X_t^{(i)} \equiv \ln\left(\frac{S_t^{(i)}}{S_0^{(i)}}\right) = \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3\right)t + \sigma_i W_t + \sum_{j=0}^{n_t^{(i)}} j_j^{(i)} + \sum_{k=0}^{n_t^{(3)}} j_k^{(3)},$$

where $j^{(i)}$ and $j^{(3)}$ are normally distributed as in equation (1.6). If the compound Poisson processes were removed the stock would follow a geometric Brownian motion

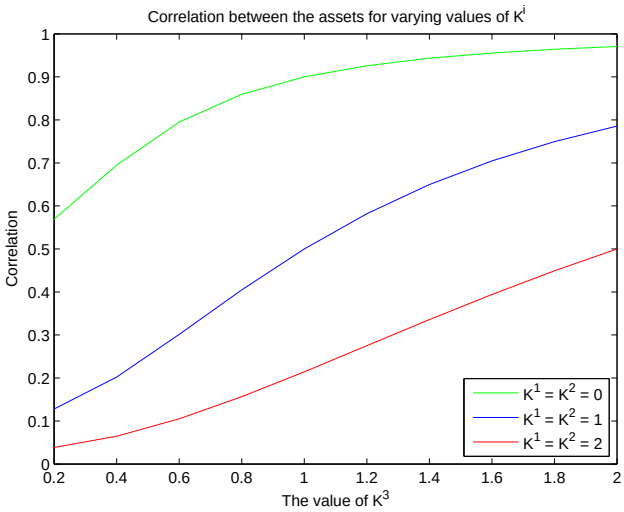


Figure 2.2: Correlation between the assets for varying values of K_3

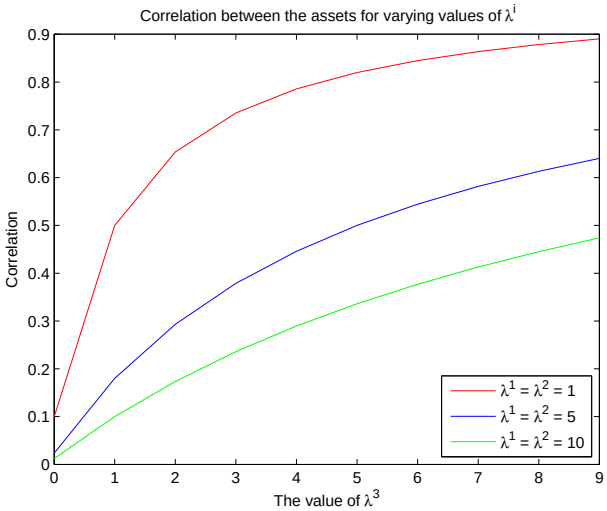


Figure 2.3: Correlation between the assets for varying values of λ_3

and as in the Black Scholes case, $X_t^{(i)}$ would be normally distributed. However the compound Poisson processes make the log-return process non-normal. We will use the law of total probability to derive an expression for the probability density function of the variable $X_t^{(i)}$. We will then calculate the characteristic function and deduce the moments of $X_t^{(i)}$.

A special case of the law of total probability for discrete random variables says that given n mutually exclusive events, $B_1, \ldots, B_n$, whose probabilities sum to one, we have

$$\mathbb{P}(A) = \sum_{i=0}^{n} \mathbb{P}(A|B_i).$$

Using the law of total probability and denoting the probability density function on $X_t^{(i)}$ by $f_{X_t^{(i)}}$ we have

$$f_{X_t^{(i)}} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(n_t^{(i)} = j)\mathbb{P}(n_t^{(3)} = k)\mathbb{P}(X_t^{(i)}|n_t^{(i)} = j, n_t^{(3)} = k),$$

where $\mathbb{P}(X_t^{(i)}|n_t^{(i)} = j, n_t^{(3)} = k)$ is the probability density function of $X_t^{(i)}$ conditional on the compound Poisson processes being equal to j and k respectively.

The probability density functions for the Poisson processes were introduced in Section 1.1. As mentioned above the probability density function of $X_t^{(i)}$ is not normal. However when we condition on the Poisson processes being equal j and k respectively, the probability density function of $X_t^{(i)}$ is normal. If the jump terms above are removed, $X_t^{(i)}$ becomes the familiar Black-Scholes log-return which is normally distributed. The jump variables j are also normally distributed. Therefore when conditioned on the Poisson processes $X_t^{(i)}$ is simply a sum of independent normally distributed variables, which itself is normal. We have that

$$j^{(i)} \sim N(\alpha_i, \delta_i^2) \quad \text{for } i = 1, 2, 3. \tag{2.10}$$

It follows that if we condition on $n_t^{(i)} = j$ and $n_t^{(3)} = k$,

$$X_t^{(i)} \sim N\left(\left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3\right)t + j\alpha_i + k\alpha_3, \sigma_i^2 t + j\delta_i^2 + k\delta_3^2\right).$$

If we let $\tilde{\mu} = (\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3)t + j\alpha_i + k\alpha_3$ and $\tilde{\sigma}^2 = \sigma_i^2 t + j\delta_i^2 + k\delta_3^2$ then

$$\mathbb{P}(X_t|n_t^i = j, n_t^3 = k) = \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left\{\frac{-(x-\tilde{\mu})^2}{2\tilde{\sigma}^2}\right\}.$$

Using the definition of a characteristic function in equation (2.5) and the definitions of $\tilde{\mu}$ and $\tilde{\sigma}^2$ from above the characteristic function of $X_t^{(i)}$ is

$$\begin{aligned}
\varphi_{X_t^{(i)}}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega x} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!}\frac{e^{-\lambda_3 t}(\lambda_3 t)^k}{k!}\frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left\{\frac{-(x-\tilde{\mu})^2}{2\tilde{\sigma}^2}\right\} \\
&= \int_{-\infty}^{\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!}\frac{e^{-\lambda_3 t}(\lambda_3 t)^k}{k!}\frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left\{\frac{-(x-\tilde{\mu})^2}{2\tilde{\sigma}^2} + i\omega x\right\}
\end{aligned}$$

We now complete the square within the exponential brackets to get

$$\varphi_{X_t^{(i)}}(\omega) = \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!} \frac{e^{-\lambda_3 t}(\lambda_3 t)^k}{k!} \exp\left\{ i\tilde{\mu}\omega - \frac{\omega^2 \tilde{\sigma}^2}{2} \right\}$$
$$\times \frac{1}{\tilde{\sigma}\sqrt{2\pi}} \exp\left\{ -\frac{x - (\tilde{\mu} + i\omega\tilde{\sigma^2})^2}{2\tilde{\sigma^2}} \right\} dx$$

We see that the terms on the second line are the only terms containing x, therefore they are the only terms that remain under the integral. They constitute the probability density function of a normally distributed variable with mean $\tilde{\mu} + i\omega\tilde{\sigma^2}$ and variance $\tilde{\sigma^2}$. Thus when integrated over the whole real line this term is equal to one. We now substitute $\tilde{\mu}$ and $\tilde{\sigma}^2$ back in and rearrange,

$$\varphi_{X_t^{(i)}}(\omega) =$$
$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!} \frac{e^{-\lambda_3 t}(\lambda_3 t)^k}{k!} \exp\left\{ i\omega \left[\left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3 \right) t + j\alpha_i + k\alpha_3 \right] \right.$$
$$\left. - \frac{\omega^2 \sigma_i^2 t + j\delta_i^2 + k\delta_3^2}{2} \right\} =$$
$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!} \frac{e^{-\lambda_3 t}(\lambda_3 t)^k}{k!} \exp\left\{ i\omega \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3 \right) t - \omega^2 \sigma_i^2 t \right\}$$
$$\times \exp\{ i\omega\alpha_i - \omega^2\delta_i^2 \}^j \exp\{ i\omega\alpha_3 - \omega^2\delta_3^2 \}^k =$$
$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t e^{\{i\omega\alpha_i - \omega^2\delta_i^2\}})^j}{j!} \frac{e^{-\lambda_3 t}(\lambda_3 t e^{\{i\omega\alpha_3 - \omega^2\delta_3^2\}})^k}{k!}$$
$$\times \exp\left\{ i\omega \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3 \right) t - \omega^2 \sigma_i^2 t \right\}.$$

We separate the double series into the product of two single infinite sums and therefore we get

$$\varphi_{X_t^{(i)}}(\omega) = \exp\left\{ \lambda_i t (e^{\{i\omega\alpha_i - \omega^2\delta_i^2\}} - 1) \right\} \exp\left\{ \lambda_3 t (e^{\{i\omega\alpha_3 - \omega^2\delta_3^2\}} - 1) \right\}$$
$$\times \exp\left\{ i\omega \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3 \right) t - \omega^2 \sigma_i^2 t \right\}. \tag{2.11}$$

The characteristic function of a Poisson distributed random variable $P \sim \text{Poiss}(\lambda)$ is

$$\varphi_P(\theta) = \exp\{ \lambda(e^{i\theta} - 1) \}$$

and the characteristic function of a normally distributed random variable $Q \sim N(\mu, \sigma^2)$ is

$$\varphi_Q(\theta) = \exp\left\{ \mu i\theta - \frac{\sigma^2\theta^2}{s} \right\}.$$

It is clear that the characterstic function of $X_t^{(i)}$ given in equation (2.11) is the product of the characteristic functions of two Poisson variables and the characteristic function of a normal variable, this result makes sense intuitively.

2.3 Moments

Now we have the characteristic function for the random variable $X_t^{(i)}$ we are interested in using it to analyse certain properties of the distribution of $X_t^{(i)}$. If we take the natural logarithm of $\varphi_{X_t^{(i)}}$ and expand as a Taylor series we get, [19],

$$g(\omega) = \ln \varphi_{X_t^{(i)}}(\omega) = \kappa_1(i\omega) + \kappa_2\frac{(i\omega)^2}{2} + \cdots + \kappa_r\frac{(i\omega)^r}{r!} + \ldots$$

where κ_i is the ith cumulant of the distribution of $X_t^{(i)}$. We can then find the mean, variance, skewness and excess kurtosis of $X_t^{(i)}$ using,

- $\mathbb{E}[X_t^{(i)}] = \kappa_1$,
- $\text{Var}[X_t^{(i)}] = \kappa_2$,
- $\text{Skewness}[X_t^{(i)}] = \frac{\kappa_3}{(\kappa_2)^{\frac{3}{2}}}$,
- $\text{Excess Kurtosis}[X_t^{(i)}] = \frac{\kappa_4}{\kappa_2^2}$.

The nth cumulant is found using the following:

$$\kappa_n = i^{-n}\frac{d^{(n)}g}{d\omega^{(n)}}(0) \tag{2.12}$$

where the superscript n denotes the nth derivative with respect to ω. Using equation (2.12) we find

- $\mathbb{E}[X_t^{(i)}] = \left(\mu_i - \frac{\sigma_i^2}{2} - \lambda_i K_i - \lambda_3 K_3\right)t + \lambda_i t\alpha_i + \lambda_3 t\alpha_3$,
- $\text{Var}[X_t^{(i)}] = \sigma_i t + \lambda_i t(\alpha_i^2 + \delta_i^2) + \lambda_3 t(\alpha_3^2 + \delta_3^2)$
- $\text{Skewness}[X_t^{(i)}] = \lambda_i t(3\delta_i^2\alpha_i + \alpha_i^2) + \lambda_3 t(3\delta_3^2\alpha_3 + \alpha_3^2)$
- $\text{Excess Kurtosis}[X_t^{(i)}] = \lambda_i t(3\delta_i^2 + 6\delta_i^2\alpha_i^2 + \alpha_i^2) + \lambda_3 t(3\delta_3^2 + 6\delta_3^2\alpha_3^2 + \alpha_3^2)$.

The evolution of an asset under a jump diffusion model is equal to a Black Scholes geometric Brownian motion when both λ_i and λ_3 are zero and/or when the expected jumps are exactly zero for all t, i.e when μ_i, μ_3, δ_i and δ_3 are zero. Trivially, it is clear from the expression of the variance of $X^{(i)}$ under a jump diffusion that if the jumps terms are exactly zero for all t, the variance of the asset under modelled using a jump diffusion is equal to that of a geometric Brownian motion. As the mean and variance of the jump variables increase, so does the variance of the asset, as one would expect.

The real interest though is in the skewness and kurtosis. The whole aim of this model is to incorporate skewness and leptokurticity of returns, characteristics of market returns that the Black Scholes model fails to reproduce. Provided that

$$\alpha_j \neq 0 \quad \text{and} \quad \delta_j^2 \neq -\frac{\alpha_j}{3} \quad \text{for } j = i, 3,$$

then the returns will be positively or negatively skewed. Whether the returns are positively or negatively skewed will depend on the relative sizes of the jump parameters.

The returns will display excess kurtosis provided the jump parameters are not exactly zero. This means the log-returns should display heavy tails that get thicker as the jump parameters increase.

First we verify the skewness of the log-returns by plotting the probability densities of the log-returns for different choices of α_i and α_3. We take $\mu_i = 0.05, \sigma_i = 0.2$ and plot the log-returns over the interval $(0, 0.25)$ for the asset with starting price $S_0^{(i)} = 50$. We fix $\lambda_i = \lambda_3 = 0.5$ and $\delta_i = \delta_3 = 0.1$ and observe the cases when α_i and α_3 are both equal to -0.5, 0 and 0.5. Figure 2.4 plots the log-return densities for the three choices of jump means. We see that when the jump means are negative the

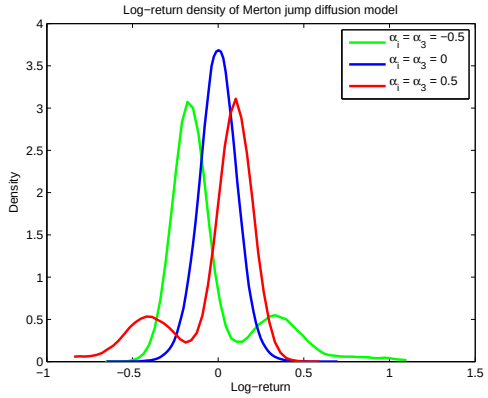


Figure 2.4: Log-return densities for various choices of α_i and α_3.

log-return densities are negatively skewed, when the the jump means are positive the log-returns are positively skewed and when the jump means are zero the log-return density is not skewed. This agrees fully with the analysis above and is confirmed by the summary statistics in Table 2.1. Also for non-negative jump means the log-returns are bi-model.

We now set the jump means to zero so that the skewness remains zero and keep the other parameters as in Figure 2.4 apart from the jump intensities λ_i and λ_3. If we increase these jump intensities we should see heavier tails in the log-return distribution. This is evident in Figure 2.5. As the jump intensities increase, the tails in the log-return density clearly get thicker. This is because an increase in jump intensity causes a greater number of jumps in the asset price. It is these jumps that cause the outlier returns. Extra outliers give the distribution heavier tails. This is

Jump means	mean(X_t)	var(X_t)	skewness(X_t)	excess kurtosis(X_t)
$\alpha_i = \alpha_3 =$ -0.5	-0.0237	0.0758	-1.7237	3.5601
$\alpha_i = \alpha_3 = 0$	6.0747e-04	0.0125	-0.0055	0.5080
$\alpha_i = \alpha_3 = 0.5$	-0.0422	0.0747	1.7086	3.5427

Table 2.1: Statistics of log-return densities for different jump means.

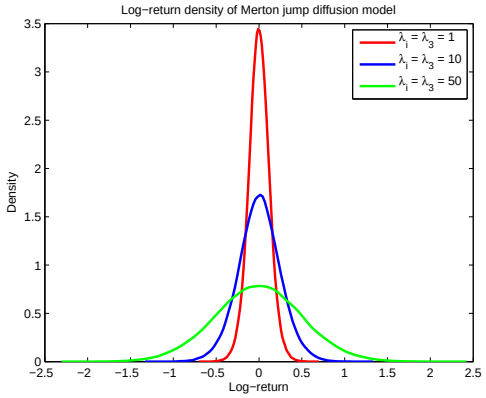


Figure 2.5: Log-return densities for various choices of λ_i and λ_3.

further ratified by a QQ-plot of the log returns when the jump means are set to zero in Figure 2.6. If the log-returns were normal the QQ-plot would be linear along the red line.

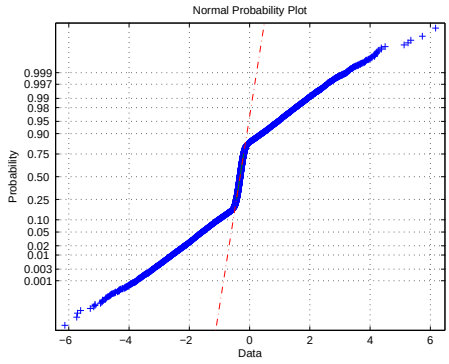


Figure 2.6: QQ-plot of log-returns when jump means are zero.

Chapter 3

Monte Carlo Pricing

3.1 Equivalent Martingale Measures and Market Incompleteness

The only probability measure we have mentioned so far is the market measure or physical measure denoted by $\mathbb{P}$. This is the measure actually observed in the market under which assets with more risk generally display a greater expected return. We are now interested in the equivalent martingale measure.

Two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on a sample space Ω with event spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively are equivalent if $\mathcal{F}_1 = \mathcal{F}_2$ and for all events ω in $\mathcal{F}_1$ we have that

$$\mathbb{P}_1(\omega) = 0 \iff \mathbb{P}_2(\omega) = 0.$$

Under a martingale measure all assets have an expected return equal to the risk-free rate irrespective of the amount of risk associated with the asset under the market measure $\mathbb{P}$. Therefore for an asset S_t we have

$$\mathbb{E}^{\mathrm{MM}}[S_t] = S_0 e^{rt}$$

if interest rates are constant where $\mathbb{E}^{\mathrm{MM}}$ is the expectation under the martingale measure.

In the Black Scholes model the asset evolution follows a geometric Brownian motion, which consists of a drift term and Brownian motion term as in equation (1.1). The only source of risk in this model comes from the Brownian motion term. We define

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$$

where r is the risk-free rate and the term $\frac{\mu - r}{\sigma}$ is the market price of risk. By Girsanov's theorem there exists a measure $\mathbb{Q}$ under which $\tilde{W}_t$ is a Brownian motion and under this measure the asset price evolution becomes

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,d\tilde{W}_t. \tag{3.1}$$

It is clear that the expected return of the asset is now equal to the risk-free rate and $\mathbb{Q}$ is the unique equivalent martingale measure.

When we introduce jump terms into the evolution of the asset as we have done in the previous section, the ideas above no longer hold. We can now also change the probability measure by changing the jump intensities λ_i and λ_3. Therefore for a space of paths of the price of the asset $S_t^{(i)}$, λ_i and λ_3 can be whatever we choose and we can then simply adjust the drift of the Brownian motion to make the new measure a martingale measure. We have an infinite number of choices for the jump intensities therefore we have an infinite number of equivalent martingale measures. This is known as market incompleteness.

In order to combat this problem we follow the example set by Merton and introduce the idea of systematic and unsystematic risk. Systematic risk is the risk inherent to the entire market. In Section 1.1 we discussed the possible types of jumps that could occur in an asset price. Jumps that are firm-specific or industry-specific are associated with unsystematic risk. We assume that the only cause of jumps in the asset price is firm-specific or industry-specific information entering the market. If the Capital Asset Pricing Model (CAPM) holds then this means the jump term is zero-beta and thus offers no risk premium. It follows that if we then make a measure change by adjusting the drift of the Brownian motion and apply Girsanov's theorem, the jump diffusion evolution is a martingale under the risk-neutral measure $\mathbb{Q}$.

Shreve [18] discusses a jump diffusion model under which a change of measure for the Wiener process and the jump intensities in the compound Poisson process results in a complete market model, however this model requires that the jump variables J be finitely discrete distributed. In our model we have assumed that the jump variables are log-normally distributed and are therefore clearly not finitely distributed.

3.2 Implementing Monte Carlo Methods

With many exotic derivatives, closed pricing formulas are practically impossible to derive. As the number of stocks on which the payoff is dependent increases, the complexity increases. Therefore we must use alternative methods for pricing. The basic idea behind Monte Carlo simulation is that running many simulations of the payoff under the risk-neutral measure $\mathbb{Q}$ and taking an average will give a good indication of the derivative price once discounted by the risk-free rate. Suppose an option on two assets has a payoff function $H(S_T^{(1)}, S_T^{(2)})$. If interest rates are constant, the price at time t under the equivalent martingale measure $\mathbb{Q}$ is given by

$$V(S_t^{(1)}, S_t^{(2)}, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H(S_T^{(1)}, S_T^{(2)})], \tag{3.2}$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation under this measure. For example a call option on asset S with strike K has payoff function $\Psi = \max(S - K, 0)$, therefore the t-price of the call option can be found by

$$\mathrm{V}_{\text{call}}(S, K, t) = e^{-r(T-t)}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\Psi\left(S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\hat{\mathbb{Q}}}\right\}\right)\right], \tag{3.3}$$

where $\hat{\mathbb{Q}}$ is the risk-neutral measure.

In order to price an option on two assets under the jump diffusion model we must first simulate the evolution of assets 1 and 2 under the risk-neutral measure. As discussed in the previous section we assume the jump terms offer no risk premium and we make a change of measure so the adjusted Brownian motion term becomes a martingale under the risk-neutral measure.

$$S_t^{(i)} = S_0^{(i)} \exp\left\{ (r - \lambda_i K_i - \lambda_3 K_3)t + \sigma_i d\tilde{W}_t^{(i)} + \sum_{j=1}^{n_t^{(i)}} j_j^{(i)} + \sum_{k=1}^{n_t^{(3)}} j_k^{(3)} \right\}, \tag{3.4}$$

where $\tilde{W}^{(i)}$ is a Wiener process under $\mathbb{Q}$. We aim to simulate the evolution of $X_t^{(i)} = \log\left(\frac{S_t^{(i)}}{S_0^{(i)}}\right)$ as we can then take exponentials and multiply by $S_0^{(i)}$ to get a sample path for the asset. In order to simulate equation (3.4) over the interval $(0, T)$, we discretize time into intervals of length Δt and over each interval $(t, t + \Delta t)$ follow the steps,

1. generate $Z \sim N(0, 1)$,
2. generate $N^i \sim \text{Poisson}(\lambda_i \Delta t)$ and $N^3 \sim \text{Poisson}(\lambda_3 \Delta t)$; if $N^i = N^3 = 0$ set $M^i = M^3 = 0$ and go to step 5,
3. generate $j_1^i, \ldots, j_{N^i}^i$ and $j_1^3, \ldots, j_{N^3}^3$ where $j_k^i \sim N(\alpha_i, \delta_i)$ for $k = 1, \ldots, N^i$ and $j_l^3 \sim N(\alpha_3, \delta_3)$ for $l = 1, \ldots, N^3$,
4. set $M^i = j_1^i + \cdots + j_{N^i}^i$ and $M^3 = j_1^3 + \cdots + j_{N^3}^3$,
5. set

$$X_{t+\Delta t}^i = X_t^i + (r - \frac{1}{2}\sigma_i^2 - \lambda_i K_i - \lambda_3 K_3)\Delta t + \sigma\sqrt{\Delta t} Z + M^i + M^3.$$

Figure 2.1 in Section 2 is a plot of the sample paths of two correlated assets modelled under jump diffusions using this simulation method.

When simulating the payoff of a European derivative, we are only interested in the value of the underlying assets at time T as this is the only time at which the derivative can be exercised. Therefore computing a value for $S_t^{(i)}$ at many points over the interval $(0, T)$ is not necessary. We take Δt to equal the whole interval $(0, T)$ and the steps outlined above can be repeated.

We compute n sample paths and use the value of the assets at time T to calculate the expected derivative payoff for each sample,

$$P_i = H(S_T^{(1)}, S_T^{(2)}) \quad \text{for} \quad 1, \ldots, n.$$

The estimate of the expected payoff $\tilde{P}$ is found by

$$\tilde{P} = \frac{1}{n} \sum_{i=1}^{n} P_i.$$

We then discount the expected payoff by dividing by the price of zero-coupon bond with price one at t and maturity at T. Thus

$$H(S_t^{(1)}, S_t^{(2)}, t) = e^{-rT}\tilde{P}.$$

As one would expect, the standard error of the estimated derivative price decreases as the number of sample paths increases. In fact the Monte Carlo price is guaranteed to converge to the actual price at the number of simulations tends to infinity. The standard error of the estimate is given by

$$\text{Error} = \sqrt{\left(\frac{1}{(1-m)^2}\sum_{i=1}^{m}(\tilde{P} - P_i)^2\right)}$$

This decreases proportionally with the square root of the number of simulations. If we would like to halve the error in the estimate we must run four times as many simulations. This is highlighted when we use Monte Carlo techniques to price a European call option using a varying number of samples. Figure 3.1 plots the Monte Carlo European call price for an increasing number of simulations against the Black Scholes price. The error between the Monte Carlo price and the Black Scholes price

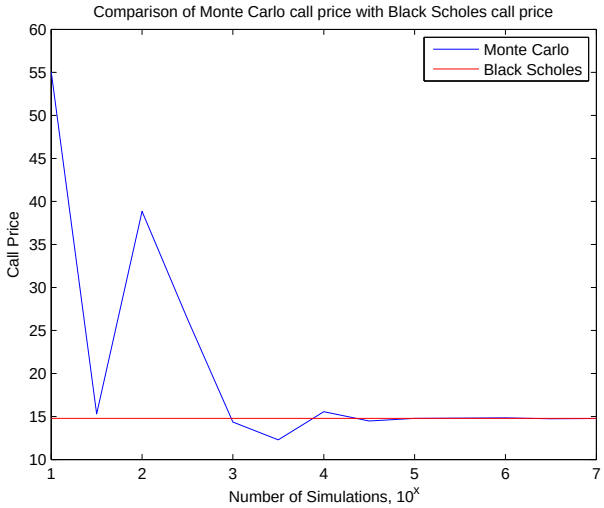


Figure 3.1: Monte Carlo and Black Scholes prices of a European call

is extremely small around 10^6 simulations. Therefore for the calculations we will run here we will use 10^6 simulations to ensure satisfactory accuracy. Obviously this number of simulations is by no means small and computing time can become and issue. Since Matlab is a vector based program the shortest computation time is achieved when the simulations are run simultaneously in vector format. However, this requires storing a huge amount of data as each simulation requires a number of random variables to be sampled and stored. For sample sizes greater than 10^7 we run into memory problems. A way to combat this is to run the simulations in

a loop with each loop computing one payoff and adding it to a sum that can be divided by the number of samples in the sum for the estimated payoff. This way we only need to store the data for one sample path at a time and we sidestep any memory problems. However this method is extremely time consuming in Matlab. Since Figure 3.1 suggests sample sizes of the order 10^6 suffice we will use the vector technique here.

3.3 Pricing Options on One Asset

Under the Black Scholes model the only source of volatility in the underlying asset price comes from the driving Wiener process. The standard deviation of the asset under a Black Scholes GBM is

$$\mathrm{sd}_{\mathrm{BS}}(S_t) = \sigma_{BS}\sqrt{t}, \tag{3.5}$$

where σ_{BS} is the standard deviation of the driving Wiener process. If we denote the standard deviation of the Wiener process in the jump diffusion model by $\mathrm{sd}_{\mathrm{jd}}$ then in light of Section 2.3, the standard deviation of the asset under a jump diffusion model is

$$\mathrm{sd}_{\mathrm{jd}}(S_t) = \sqrt{\big(\sigma_{\mathrm{jd}}^2 + \lambda_i(\alpha_i^2 + \delta_i^2) + \lambda_3(\alpha_3^2 + \delta_3^2)\big)t}.$$

Since the jump intensities λ_i and λ_3 are always positive it is clear that if we set $\sigma_{BS} = \sigma_{\mathrm{jd}}$ then the volatility of jump diffusion model will always be greater or equal to that of the Black Scholes model. The price of a European call option increases as volatility increases therefore we would expect the jump diffusion model to return European call option prices greater than those from the Black Scholes model. This is illustrated in Figures 3.2 and 3.3. We set $S_0 = 100$, $\sigma_{\mathrm{jd}} = \sigma_{BS} = 0.2$ and vary the jump diffusion parameters to compare the prices admitted by both models over a range of strikes. The different sets of jump diffusion parameters used are highlighted in Table 3.1.

	λ_1	λ_2	α_1	α_2	δ_1	δ_2
Set 1	0.5	0.5	-0.05	-0.05	0.05	0.05
Set 2	2	2	-0.1	-0.1	0.1	0.1
Set 3	1	1	-0.3	-0.3	0.3	0.3
Set 4	2	2	-0.3	-0.3	0.3	0.3

Table 3.1: The jump parameters used in each set for Figure 3.2, 3.3 and 3.6

We see that when the jump parameters are small as in set 1, there is very little difference between the Black Scholes price and the jump diffusion price for options that are deep in the money or deep out of the money, however the difference increases as we get closer to being at-the-money. When the jump parameters are large the difference between the two prices is an again largest when the options are near at-the-money, however as the jump parameters increase in magnitude the difference

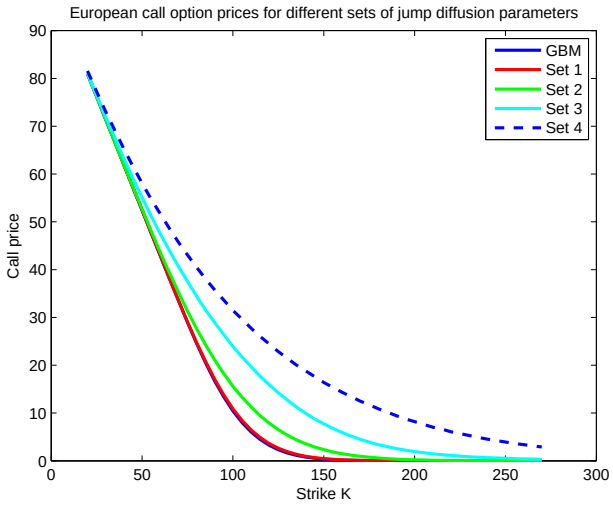


Figure 3.2: Black Scholes price and jump diffusion price for different sets of jump diffusion parameters for $30 \leq K \leq 270$.

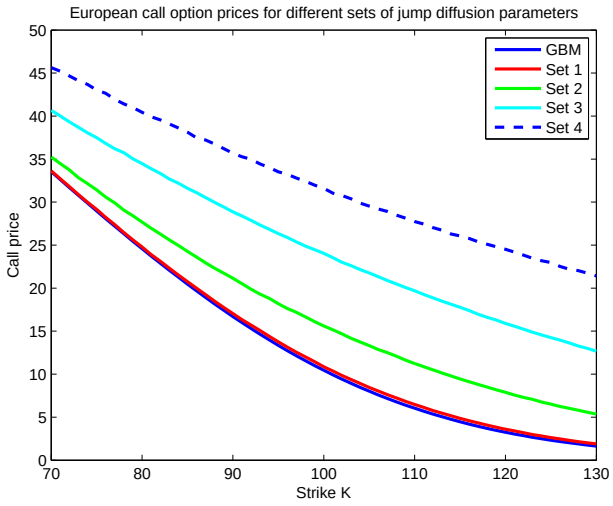


Figure 3.3: Black Scholes price and jump diffusion price for different sets of jump diffusion parameters for $70 \leq K \leq 130$.

between the options deep in-the-money and deep out-of-the-money is also large, as we would expect.

Now we choose a set of jump diffusion parameters and calculate the standard deviation of the asset under this model using equation (3.5). We will then set the Black Scholes standard deviation equal to this jump diffusion standard deviation, i.e.

$$\sigma_{BS}\sqrt{t} = \sqrt{(\sigma_{\text{jd}}^2 + \lambda_i(\alpha_i^2 + \delta_i^2) + \lambda_3(\alpha_3^2 + \delta_3^2))t}.$$

If we choose the standard deviation of the two models to be the same then we should be to see the effect of the skewness and leptokurticity on the option prices since the mean and variance of the assets under both models should match but the skewness and excess kurtosis will not.

If we take the jump parameters as in set 2 outlined in Table 3.1 then

$$\text{sd}_{\text{jd}}(S_t) = \sqrt{(\sigma_{\text{jd}}^2 + \lambda_i(\alpha_i^2 + \delta_i^2) + \lambda_3(\alpha_3^2 + \delta_3^2))t} = 0.34641.$$

Figure 3.4 plots the jump diffusion prices for the parameters of set 2 and the Black Scholes prices for $\text{sd}_{\text{BS}}(S_t) = 0.34641$ over a range of strikes. The prices match closely

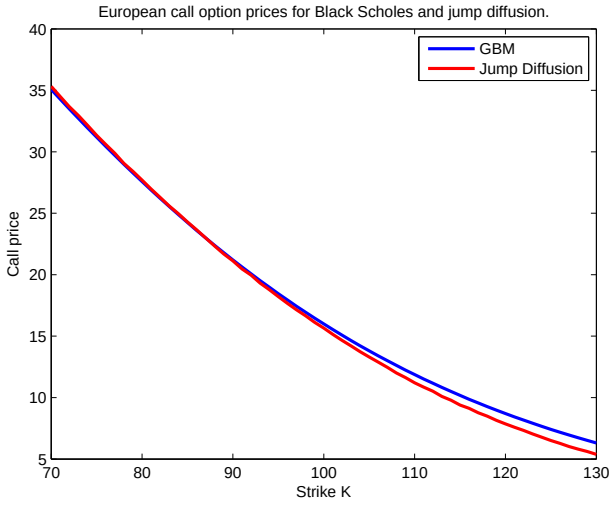


Figure 3.4: Black Scholes price and jump diffusion price when $\text{sd}_{\text{BS}}(S_t) = \text{sd}_{\text{jd}}(S_t)$.

for at-the-money options and deep out-the-money options but as the strike passes the stock starting price of 100 the the difference between the two prices increases and the jump diffusion model slightly under-estimates the price compared with the Black Scholes price. One possible explanation for this is that under geometric Brownian motion an option that is deep in-the-money has less risk of retreating back to being out-the money whereas adding the jump terms adds risk and the chance of the price falling is therefore greater.

3.4 Implied Volatility

The Black Scholes pricing equation for a European call on an asset S requires a number of input parameters to compute a price,

$$V_{\text{call}}(S_0, K, r, T, \sigma_{\text{BS}}) = \text{price}.$$

These being the current value of the stock, the strike of the option, the risk-free interest rate, the time until expiry of the option and the stock volatility. The first four input parameters are either observable or are predetermined conditions of the contract. The final input however, the stock volatility, is unobservable. This means the value of the option depends on an estimate of the constant volatility of the underlying stock over the interval $(0, T)$. The option price is a monotonically increasing function of the volatility, as illustrated in Figure 3.5. Therefore the higher the volatility of

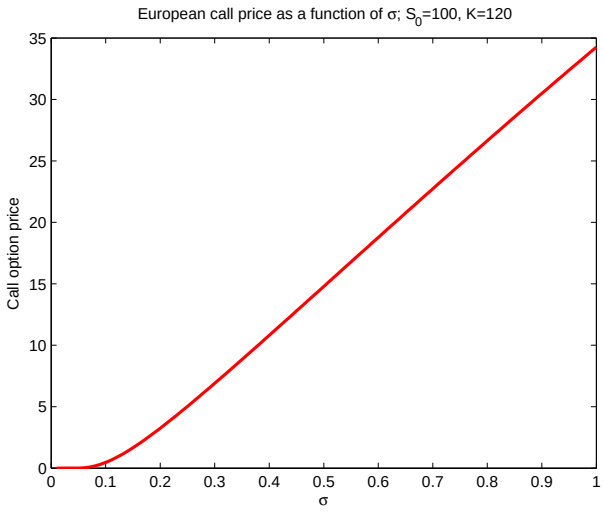


Figure 3.5: European call price as a function of σ.

the underlying stock, the higher the option price. It also means that for every option price there exists a unique volatility that must realise that price when other inputs are kept constant. This is the implied volatility of that price. In general, options on the same asset but with different strikes and expiration dates have different implied volatilities. For example, the plots of the implied volatility of market call option prices against the option strike often resemble a smile meaning deep in-the-money or deep out-of-the-money options have higher implied volatilities than at-the-money options. This contradicts the assumption made in the Black-Scholes model that the underlier has constant volatility. Calculating the implied volatility for the market price of a call option over a range or strikes or maturity dates gives an insight into the market expectations of the asset volatility and is therefore of interest to an investor.

We can compute call prices under the jump diffusion model using Monte Carlo methods for a range of strikes. We can then use these prices to find the Black Scholes

implied volatility. The Black Scholes formula is difficult to invert in a closed form so we have to use a numerical method. If $V_{\text{call}}(.)$ is the Black Scholes price as function of volatility and C_{jd} is the call price from the jump diffusion model, we need to find σ_{imp} such that

$$V_{\text{call}}(\sigma_{\text{imp}}) - C_{\text{jd}} = 0$$

using a root finding function. Matlab actually has an inbuilt function that computes the implied volatility given the required inputs. Figure 3.6 plots the implied volatilities of jump diffusion call prices for different sets of jump diffusion parameters. The sets of jump diffusion parameters used are outlined in Table 3.2.

	λ_1	λ_2	λ_3	α_1	α_2	α_3	δ_1	δ_2	δ_3
Set 1	0.5	0.5	1	-0.05	-0.05	-0.05	0.05	0.05	0.05
Set 2	1	1	1	-0.1	-0.1	-0.1	0.1	0.1	0.1
Set 3	1	1	1	-0.2	-0.2	-0.3	0.05	0.05	0.05
Set 4	0.5	0.5	1	-0.05	-0.05	-0.05	0.2	0.2	0.3

Table 3.2: The jump parameters used in each set for Figure 3.6

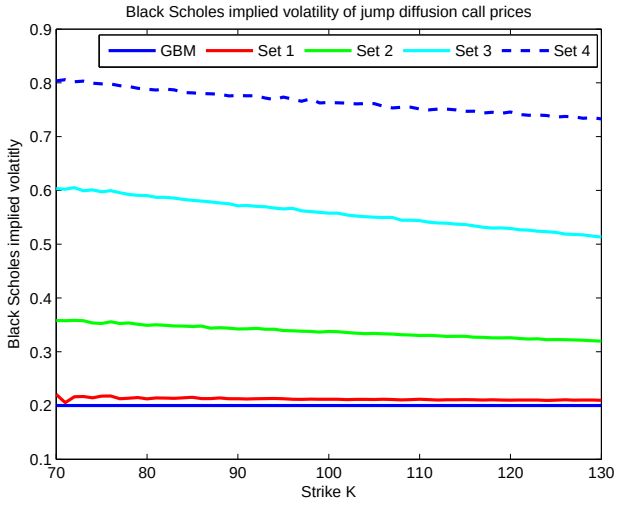


Figure 3.6: Implied volatilities of jump diffusion call prices for different sets of jump parameters; $70 \leq K \leq 130$.

As expected the prices given by the model with the largest jump parameters display the highest implied volatility. The implied volatility of the Black Scholes model is perfectly constant as we would hope but the jump diffusion prices display slightly greater implied volatilities for smaller strike prices.

Chapter 4

Exotic Options

4.1 Exchange Options

Consider two assets $S^{(1)}$ and $S^{(2)}$, which have prices at time t denoted by $S_t^{(1)}$ and $S_t^{(2)}$ respectively. We assume no dividends are paid on the stock and all returns are from capital gains only. An exchange option gives the holder the right but not the obligation to exchange a quantity of asset $S^{(1)}$ for a quantity of asset $S^{(2)}$. Here we will discuss a European exchange option, which can only be exercised at the expiry date T in the future. Exchange options can occur in many different contexts. For example in a buyout or takeover, holders of shares in the target company may be given the option of exchanging their holdings for a quantity of shares in the acquiring company. Another example is in currency markets, an option to swap a quantity of one foreign currency for a predetermined volume of another is an exchange option.

Given the definition above, the holder would only exercise the option if at time T the value of stock $S^{(2)}$ was greater than $S^{(1)}$. Therefore the payoff is given by

$$E(S_T^{(1)}, S_T^{(2)}) = \max(Q_2 S_T^{(2)} - Q_1 S_T^{(1)}, 0), \tag{4.1}$$

where Q_i is the quantity of asset i. If we assume that fractions of stocks can be traded, without loss of generality we can define the option as the option to swap one unit of asset $S^{(1)}$ for $\frac{Q_2}{Q_1}$ units of asset $S^{(2)}$. Here we will assume that $\frac{Q_2}{Q_1} = 1$. Upon exercising the option the holder could then instantaneously sell the more expensive asset $S^{(2)}$ in the market to receive a profit or hold it in his/her portfolio having acquired it for cheaper than the market price. Notice that the definition of a European exchange option on the assets $S^{(1)}$ and $S^{(2)}$ with expiry T is also the definition of a European call option on asset $S^{(2)}$ with strike price $S_T^{(1)}$ and expiry T.

Margrabe [14] derived a pricing formula for an exchange option when the underlying assets are modelled using a geometric Brownian motion. If we denote the t-price of a European exchange option by $V_{\mathrm{E}}(S^{(1)}, S^{(2)}, \rho, t)$ then

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, \rho, t) = S^{(1)} N(d_1) - S^{(2)} N(d_2) \tag{4.2}$$

where $N(.)$ is the cumulative standard normal distribution and

$$d_1 = \frac{\ln\left(\frac{S^{(1)}}{S^{(2)}}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}.$$

It is clear from the formula above that the correlation between the two assets also has an effect on the option price.

4.2 Pricing Formula for an Exchange Option

We will aim to derive a pricing equation for the exchange option under the jump diffusion model for correlated assets. We make the assumption that the jumps in asset i are drawn from the log normal random variable $J^{(i)}$ irrespective of whether the jump was initiated by the independent Poisson process $n_t^{(i)}$ or the common Poisson process $n_t^{(3)}$. Therefore both assets are modelled by

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \big(\mu_i - (\lambda_i + \lambda_3)K_i\big)dt + \sigma_i dW_t^{(i)} + (J^{(i)} - 1)(dn_t^{(i)} + dn_t^{(3)}) \quad \text{for } i = 1, 2. \tag{4.3}$$

Notice that

$$E(S_T^{(1)}, S_T^{(2)}) = \max(S_T^{(2)} - S_T^{(1)}, 0) = S_T^{(1)} \max\left(\frac{S_T^{(2)}}{S_T^{(1)}} - 1, 0\right)$$

so if we let $\xi = \frac{S^{(2)}}{S^{(1)}}$ this becomes

$$S_T^{(1)} \max(\xi_T - 1, 0) = S_T^{(1)}\Psi(\xi_T) = \phi(\xi_T). \tag{4.4}$$

We choose asset $S^{(1)}$ to be the numeraire. Since cash is not explicitly involved in this problem, we can measure the value of the assets and the option relative to the asset $S^{(1)}$ rather than an arbitrary monetary denomination and thus reduce the number of variables in the problem. This is what we have achieved above. ξ is the value of $S^{(2)}$ relative to $S^{(1)}$ and Φ is the value of $E(S_T^{(1)}, S_T^{(2)})$ relative to $S^{(1)}$. Obviously the value of $S^{(1)}$ relative to $S^{(1)}$ is always unity therefore we remove a variable from the problem. If there was a cash term in the payoff function as well as the two stocks, for example a strike price K, then the change of numeraire would not decrease the number of variables as the fixed term K would become $\frac{K}{S^{(1)}}$, which is also now a variable. The payoff defined in equation (4.4) is a European call option on the asset ξ with a strike of 1. Merton [16] derived an expression for the price of a European call option under a jump diffusion model, therefore if we can derive an expression for the evolution of ξ, we can use similar techniques to derive an explicit formula for an exchange option.

In order to reduce the pricing formula for the European exchange option to a European call option we look to price an exchange option using the expected value of the discounted cash-flows under the risk-neutral measure. Using the equations for

the evolution of $S^{(1)}$ and $S^{(2)}$ under $\mathbb{Q}$ we derive an expression for ξ_t under the risk neutral measure $\mathbb{Q}$. Then we must find the Radon Nikodym derivative $\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}$ so that under the measure $\mathbb{Q}$ the numeraire $S^{(1)}$ is of the form

$$S_T^{(1)} = \S 0_t e^{r(T-t)} \frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}\bigg|_T$$

and ξ_t is a martingale under $\tilde{\mathbb{Q}}$. Using equation (3.2) the t-price of the exchange option using discounted cash-flows is

$$\begin{aligned} V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max(S_T^{(2)} - S_T^{(1)}, 0)] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T^{(1)} \max(\xi_T - 1, 0)] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_0^{(1)} e^{r(T-t)} \frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} \max(\xi_T - 1, 0)\right] \\ &= e^{-r(T-t)} S_0^{(1)} e^{r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max\left((\xi_T - 1)\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}}, 0\right)\right] \\ &= S_0^{(1)}\mathbb{E}^{\tilde{\mathbb{Q}}}[\max(\xi_T - 1, 0)]. \end{aligned} \tag{4.5}$$

Our first task is to find an expression for

$$\log(\xi) = \log\left(\frac{S^{(2)}}{S^{(1)}}\right)$$

under the risk-neutral measure. This is achieved by finding the evolutions of $\log(S^{(1)})$ and $\log(S^{(2)})$ and using the property of the log function that says

$$\log(\xi) = \log\left(\frac{S^{(2)}}{S^{(1)}}\right) = \log(S^{(2)}) - \log(S^{(1)}). \tag{4.6}$$

4.3 Evolution of $\log(\xi)$

$S^{(i)}$ is defined as in equation (4.3). We discussed how to change into the risk-neutral measure in Section 3.1. We do this by adjusting the drift of the Wiener process to incorporate the excess drift over the risk-free rate and leave the jump terms untouched as they are unsystematic. Then

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (r - (\lambda_i + \lambda_3)K_i)dt + \sigma_i d\widetilde{W^{(i)}}_t + (J^{(i)} - 1)(dn_t^{(i)} + dn_t^{(3)}) \quad \text{for } i = 1, 2 \tag{4.7}$$

where $\widetilde{W^{(i)}}_t$ is a Wiener process under the martingale measure $\mathbb{Q}$. Since the compensated Poisson process is a martingale, it is clear that the discounted stock price is martingale under $\mathbb{Q}$.

We now look to find an expression for $d\log(S^{(i)})$ We calculate the derivatives of the logarithm function and apply Ito's lemma for jump diffusions as introduced in Theorem 1.2.1,

$$\begin{aligned}
d\log(S^{(i)}) =& \frac{S^{(i)}}{S^{(i)}}\left[(r-(\lambda_i+\lambda_3)K_i)dt+\sigma_i\widetilde{dW_t^{(i)}}\right] \\
&-\frac{(S^{(i)})^2}{2(S^{(i)})^2}\left[\left(r-(\lambda_i+\lambda_3)K_i)dt+\sigma_i\widetilde{dW_t^{(i)}}\right)^2\right] \\
&+\log(S_{t-}^{(i)}+\Delta S_t^{(i)})-\log(S_t^{(i)}) \\
=&\left(r-(\lambda_i+\lambda_3)K_i-\frac{\sigma_i^2}{2}\right)dt-\sigma_i\widetilde{dW_t^{(i)}} \\
&+\log\left(S_{t-}^{(i)}\times J_t^{(i)}(dn_t^{(i)}+dn_t^{(3)})\right)-\log(S_{t-}^{(i)}) \\
=&\left(r-(\lambda_i+\lambda_3)K_i-\frac{\sigma_i^2}{2}\right)dt+\sigma_i\widetilde{dW_t^{(i)}}+\log\left(J_t^{(i)}(dn_t^{(i)}+dn_t^{(3)})\right). \quad (4.8)
\end{aligned}$$

Using equation (4.6) we have that

$$\begin{aligned}
d\log(\xi) =& d\log(S^{(2)})-d\log(S^{(1)}) \\
=&\left(r-(\lambda_2+\lambda_3)K_2-\frac{\sigma_2^2}{2}\right)dt+\sigma_2\widetilde{dW_t^{(2)}} \\
&+\log S_{t-}^{(2)}+\log(j_t^{(2)}(dn_t^{(2)}+dn_t^{(3)}))-\log(S_{t-}^{(i)}) \\
&-\left(r-(\lambda_1+\lambda_3)K_1-\frac{\sigma_1^2}{2}\right)dt-\sigma_1\widetilde{dW_t^{(1)}} \\
&-\log S_{t-}^{(2)}-\log(j_t^{(1)}(dn_t^{(1)}+dn_t^{(3)}))+\log(S_{t-}^{(i)})\Big] \\
=&\left((\lambda_1+\lambda_3)K_1-(\lambda_2+\lambda_3)K_2+\frac{1}{2}(\sigma_1^2-\sigma_2^2)\right)dt-\sigma_1\widetilde{dW^{(1)}}_t+\sigma_2\widetilde{dW^{(2)}}_t \\
&+\log\left(J^{(2)}(dn_t^{(2)}+dn_t^{(3)})\right)-\log\left(J^{(1)}(dn_t^{(1)}+dn_t^{(3)})\right). \quad (4.9)
\end{aligned}$$

4.4 Change of Measure

Using equation (4.8) the process $\log(S^{(1)})$ evolves under $\mathbb{Q}$ as follows,

$$S_t^1=S_0^1\exp\left\{\left(r-(\lambda_1K_3+\lambda_3K_3)-\frac{\sigma_1^2}{2}\right)t+\sigma_1\widetilde{W_t^{(1)}}+\sum_{k=1}^{n_t^{(i)}+n_t^{(3)}}j_k^{(i)}\right\}$$

where $j^{(i)}$ is normally distributed with mean α_i and variance δ^2. When discounted by the risk-free rate this process is not a martingale under $\mathbb{Q}$ due to the $-\frac{\sigma_1^2}{2}t$ term. We look to make a change of measure from $\mathbb{Q}$ to $\tilde{\mathbb{Q}}$ so that when discounted by the risk-free rate $\log(S^{(1)})$ is a martingale under $\tilde{\mathbb{Q}}$.

Since $\widetilde{W^{(1)}}_t \sim N(0, \sqrt{t})$ we have

$$M \equiv -\frac{\sigma_1^2}{2}t + \widetilde{W^{(1)}}_t \sim N\left(\frac{\sigma_1^2}{2}, \sigma_1\sqrt{t}\right)$$

and by equation (1.7) we have that

$$\mathbb{E}^{\mathbb{Q}}[\exp(M)] = 1.$$

We define the new probability measure $\tilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ with Radon Nikodym derivative

$$\frac{d\mathbb{Q}}{d\tilde{\mathbb{Q}}} = \exp\left\{-\frac{\sigma_1^2}{2}t + \sigma_1\widetilde{W_t^{(1)}}\right\}.$$

The jump terms are risk-free under both measures as we made the assumption that they are unsystematic so the Radon-Nikodym derivative does not involve these terms. Under $\mathbb{Q}$ we can write

$$S_T^{(1)} = S_0^{(1)}e^{rT}\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_T.$$

By Girsanov's theorem we have that the process

$$d\widehat{W^{(1)}} = d\widetilde{W^{(1)}} - \sigma_1 dt \tag{4.10}$$

is a Wiener process under the risk neutral probability space $(\Omega, \mathcal{A}, \mathcal{F}, \tilde{\mathbb{Q}})$. We write $d\widetilde{W^{(2)}}$ as

$$d\widetilde{W^{(2)}} = \rho_{12}d\widetilde{W^{(1)}} + \sqrt{1-\rho_{12}^2}dW'$$

where W' is independent of $\widetilde{W^{(1)}}$ under $\mathbb{Q}$. However under $\tilde{\mathbb{Q}}$, W' remains a Wiener process independent of $\widehat{W^{(1)}}$. Hence $d\widehat{W^{(2)}}$ is defined by

$$d\widehat{W^{(2)}} = \rho_{12}d\widehat{W^{(1)}} + \sqrt{1-\rho_{12}^2}dW' \tag{4.11}$$

is a Wiener process under $\tilde{\mathbb{Q}}$. Using equation (4.10) we can write

$$d\widehat{W^{(2)}} = d\widetilde{W^{(2)}} - \rho_{12}\sigma_1 dt. \tag{4.12}$$

Using equation (4.11) it is easy to see

$$\sigma_2 d\widehat{W^{(2)}} - \sigma_1 d\widehat{W^{(1)}} = (\sigma_2\rho_{12} - \sigma_1)d\widehat{W^{(1)}} + \sigma_2\sqrt{1-\rho_{12}^2}dW' \equiv dW^{\tilde{\mathbb{Q}}}$$

and we can easily verify that

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}}\big[dW^{\tilde{\mathbb{Q}}}\big] &= \mathbb{E}^{\tilde{\mathbb{Q}}}\big[(\sigma_2\rho_{12} - \sigma_1)d\widehat{W^{(1)}} + \sigma_1\sqrt{1-\rho_{12}^2}dW'\big] \\
&= (\sigma_2\rho_{12} - \sigma_1)\mathbb{E}^{\tilde{\mathbb{Q}}}\big[d\widehat{W^{(1)}}\big] + \sigma_1\sqrt{1-\rho_{12}^2}\mathbb{E}^{\tilde{\mathbb{Q}}}\big[dW'\big] = 0 \\
\mathrm{Var}\big[dW^{\tilde{\mathbb{Q}}}\big] &= (\sigma_2\rho_{12} - \sigma_1)^2\mathrm{Var}\big[d\widehat{W^{(1)}}\big] + \big(\sigma_1\sqrt{1-\rho_{12}^2}\big)^2\mathrm{Var}\big[dW'\big] \\
&= (\sigma_2^2 + \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2)dt \equiv \sigma^2 dt.
\end{aligned}$$

Therefore $W^{\tilde{\mathbb{Q}}}$ is a Wiener process under $\tilde{\mathbb{Q}}$. Using equation (4.10) and (4.12) we have

$$\sigma_2 d\widetilde{W^{(2)}} - \sigma_1 d\widetilde{W^{(1)}} = \sigma_2 d\widehat{W^{(2)}} - \sigma_1 d\widehat{W^{(1)}} + \rho_{12}\sigma_1\sigma_2 dt - \sigma_1^2.$$

It follows from equation (4.9) that the evolution of $\log(\xi)$ under $\tilde{\mathbb{Q}}$ becomes

$$\begin{aligned} d\log(\xi) = &\left((\lambda_1+\lambda_3)K_1 - (\lambda_2+\lambda_3)K_2 + \frac{1}{2}\sigma^2\right)dt + \sigma dW^{\tilde{\mathbb{Q}}} \\ &+ \log\left(j^{(2)}(dn_t^{(2)} + dn_t^{(3)})\right) - \log\left(j^{(1)}(dn_t^{(1)} + dn_t^{(3)})\right). \end{aligned} \tag{4.13}$$

Solving this stochastic differential equation for ξ_t gives,

$$\xi_t = \xi_0 \exp\left\{\left((\lambda_1+\lambda_3)K_3 - (\lambda_2+\lambda_3)K_2 - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\tilde{\mathbb{Q}}} - \sum_{k=0}^{N_t^{(1)}} j_k^{(1)} + \sum_{l=0}^{N_t^{(2)}} j_l^{(2)}\right\}, \tag{4.14}$$

where $N^{(1)} = n^{(1)} + n^{(3)}$ and $N^{(2)} = n^{(2)} + n^{(3)}$. Here $W^{\tilde{\mathbb{Q}}}$ is a geometric Brownian motion under $(\Omega, \mathcal{F}, \tilde{\mathbb{Q}})$ and since the compensated jump terms are martingales, ξ_t is a martingale under $\tilde{\mathbb{Q}}$ when discounted by the risk-free rate e^{-rt}.

If the payoff for the European exchange option is $E(S_T^{(1)}, S_T^{(2)})$ then using equation (4.5) the price at time t is

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = S^{(1)}\mathbb{E}^{\tilde{\mathbb{Q}}}[\max(\xi_T - 1, 0)] = S^{(1)}\mathbb{E}^{\tilde{\mathbb{Q}}}[\Psi(\xi_T)].$$

In order to save space we let $(\lambda_1+\lambda_3)K_3 - (\lambda_2+\lambda_3)K_2 - \frac{1}{2}\sigma^2 = \widetilde{\lambda K}$ and substituting in the expression for ξ gives

$$\begin{aligned} &V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = \\ &\quad S^{(1)}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\Psi\left(\xi_t \exp\left\{\underbrace{\widetilde{\lambda K}t + \sigma W_t^{\tilde{\mathbb{Q}}} - \sum_{k=0}^{n_t^{(1)}+n_t^{(3)}} j_k^{(1)} + \sum_{l=0}^{n_t^{(2)}+n_t^{(3)}} j_l^{(2)}}_{\dagger}\right\}\right)\right]. \end{aligned} \tag{4.15}$$

In Section 2.2 we said that if we conditioned on the Poisson processes $n^{(1)}, n^{(2)}$ and $n^{(3)}$, the term labelled † in equation (4.15) is normally distributed. This is because the jump variables $j^{(1)}$ and $j^{(2)}$ are normally distributed after the log transformation. The term labelled † is therefore a sum of independent normal random variables.

In Section 2 we introduced a joint probability density function for two Poisson processes constructed from three independent Poisson processes in the following way,

$$N_t^{(1)} = n_t^{(1)} + n_t^{(3)} \tag{4.16}$$

$$N_t^{(2)} = n_t^{(2)} + n_t^{(3)}. \tag{4.17}$$

The joint probability density function was defined as

$$P_{ij} \equiv \mathbb{P}(N_t^{(1)} = i, N_t^{(2)} = j) = \sum_{k=0}^{\min(i,j)} e^{-(\lambda_{1^*}+\lambda_{2^*}+\lambda_{3^*})} \frac{\lambda_{1^*}^{i-k}\lambda_{2^*}^{j-k}\lambda_{3^*}^{k}}{(i-k)!(j-k)!k!}.$$

Therefore using the law of total probability as we did in Section 2.2 we have

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij} S^{(1)} \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\Psi\left(\xi_t \exp\left\{\widetilde{\lambda K}t + \sigma W_t^{\tilde{\mathbb{Q}}} - \sum_{k=0}^{i} j_k^{(3)} + \sum_{l=0}^{j} j_l^{(2)}\right\}\right)\right]. \tag{4.18}$$

Now it is clear that when we have conditioned on the Poisson processes, if the jump variables are distributed as in equation (2.10) then the term labelled † is normally distributed,

$$\dagger = \widetilde{\lambda K}t + \sigma W_t^{\tilde{\mathbb{Q}}} - \sum_{k=0}^{i} j_k^{(1)} + \sum_{l=0}^{j} j_l^{(2)} \sim N\left(\widetilde{\lambda K}t - i\alpha_1 + j\alpha_2, \sigma^2 t + i\delta_1 + j\delta_2\right).$$

We rearrange this without changing how it is distributed,

$$\widetilde{\lambda K}t - i\alpha_1 + j\alpha_2 + \sqrt{\frac{\sigma^2 t + i\delta_1 + j\delta_2}{t}} W_t^{\tilde{\mathbb{Q}}} \sim N\left(\widetilde{\lambda K}t - \alpha_1 + j\alpha_2, \sigma^2 t + i\delta_1 + j\delta_2\right).$$

Since a normal distribution is uniquely determined by its mean and variance we can write

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij} S^{(1)} \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\Psi\left(\xi_t \exp\left\{\widetilde{\lambda K}t - i\alpha_1 + j\alpha_2 + \sqrt{\frac{\sigma^2 t + i\delta_1 + j\delta_2}{t}} W_t^{\tilde{\mathbb{Q}}}\right\}\right)\right].$$

We have now removed the sum of jump variables and incorporated it into the Wiener process. The expectation term now looks like the Black-Scholes call pricing formula in equation (3.3) only with redefined parameters. After manipulation of the terms inside the brackets we get

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij} S^{(1)} \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\Psi\left(\xi_t \exp\left\{\left(r_{ij} - \frac{1}{2}\sigma_{ij}^2\right)t + \sigma_{ij} W_t^{\tilde{\mathbb{Q}}}\right\}\right)\right],$$

where

$$\begin{aligned}
K_i &= \exp\left\{\alpha_i + \frac{\delta_i^2}{2}\right\} - 1 \qquad \text{for } i = 1, 2, \\
r_{ij} &= (\lambda_1 + \lambda_3)K_1 - (\lambda_2 + \lambda_3)K_2 - \frac{i}{t}\left(\alpha_1 - \frac{\delta_1^2}{2}\right) + \frac{j}{t}\left(\alpha_2 + \frac{\delta_2^2}{2}\right), \\
\sigma_{ij} &= \sigma^2 + \frac{i\delta_1^2 + j\delta_2^2}{t}, \\
\sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}.
\end{aligned}$$

Using the Black-Scholes formula for a call option [2] we have,

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij}\left[S^{(2)} N(d_1^{(i,j)}) - S^{(1)} N(d_2^{(i,j)})\right],$$

where

$$d_1^{(i,j)} = \frac{\ln\left(\frac{S^{(2)}}{S^{(1)}}\right) + \left(r_{ij} + \frac{\sigma_{ij}^2}{2}\right)(T-t)}{\sigma_{ij}\sqrt{T-t}}, \qquad d_2^{(i,j)} = d_1^{(i,j)} - \sigma_{ij}\sqrt{T_t}.$$

This formula is not a closed pricing formula as it involves an infinite double sum. However, we can truncate after a relatively small number of terms as the sum rapidly converges.

4.4.1 Implementing the Pricing Formula for an Exchange Option

We now implement the formula derived in the previous section in Matlab. Firstly this allows us check that the model agrees with the Margrabe model but also allows us to investigate the effect that varying the jump parameters and ultimately the correlation has on the exchange option price. In this model, since each asset only has one jump variable for two driving Poisson processes, we can only vary the correlation by varying the magnitude of λ_3 relative to λ_1 and λ_2. Using the same method as in Section 2 we have that the correlation between the two assets under this model is

$$\text{Corr}\left(\frac{dS_t^{(1)}}{S_t^{(1)}}, \frac{dS_t^{(2)}}{S_t^{(2)}}\right) = \frac{\sigma_1\sigma_2\rho_{12} + K_1K_2\lambda_3}{\sqrt{\sigma_1^2 + K_1^2(\lambda_1+\lambda_3)+}\sqrt{\sigma_2^2 + K_2^2(\lambda_2+\lambda_3)}}. \tag{4.19}$$

In each figure below we set $S_0^{(1)} = S_0^{(2)} = 100$, $\sigma_1 = 0.6$, $\sigma_2 = 0.4$ and $T = 1$.

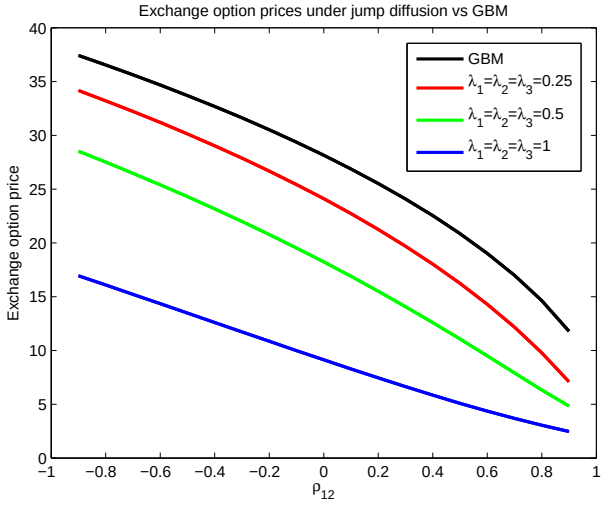


Figure 4.1: Exchange option prices when only the jump intensities are varied between 0 and 1.

We will begin with Figure 4.1 by plotting the exchange option prices for different jump intensities alongside the prices admitted by Margrabe's formula as in equation (4.2). The first thing to notice about the Margrabe prices is that the exchange

option prices are lowest when the two assets exhibit strong positive correlation. The exchange option prices increase monotonically as the correlation moves towards perfect negative correlation. This is because the option payoff is greater if the difference between the two asset prices is greater at the expiry. If the assets are positively correlated then as one moves in one direction the other is likely to move in that direction as well and the difference between the two prices will not increase. If they are negatively correlated then as one stock moves in one direction the other is more likely to move in the opposite direction hence increasing the difference between the prices.

We set $\lambda_1 = \lambda_2 = \lambda_3$ and vary the jump intensities together at the same rate. This allows us to observe the effect that an increased number of jumps in the underlying asset has on the price. Increasing the jump intensities causes the exchange option price to fall. Even a small value for the jump intensities causes a significant deviation from the Margrabe price.

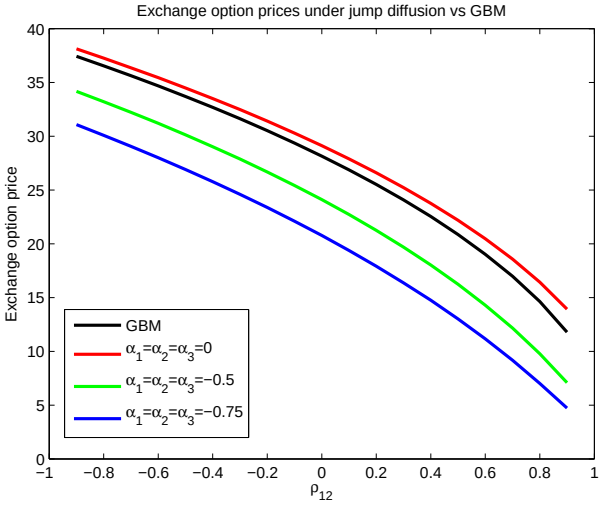


Figure 4.2: Exchange option prices when only the jump mean is varied between 0 and -0.75.

Figure 4.2 plots the exchange option prices for different values for the jump mean. We set $\alpha_1 = \alpha_2 = \alpha_3$ and vary the magnitude of these parameters. When the jump means are small the exchange option prices under the jump diffusion model match the Margrabe formula prices very closely although they are slightly higher for all values of ρ_{12}. As the jump means increase, the gap between the prices admitted by the two models increases but now the jump diffusion prices are significantly lower.

Finally Figure 4.3 plots the exchange option prices for different values of jump variance. Here the difference between the jump diffusion price and the Margrabe price increases as the jump variances increase but the jump diffusion prices are now greater than the Margrabe prices. One possible explanation for this is the added variance in the jump diffusion model. The exchange option is more expensive if there is an increased possibility that the asset prices may be further apart at the payoff. Adding

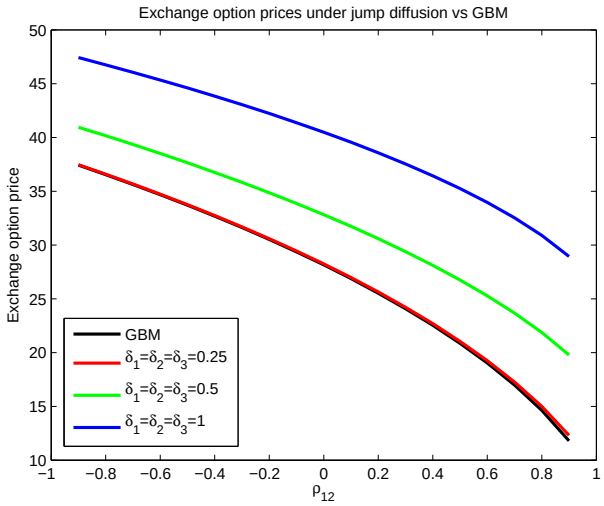


Figure 4.3: Exchange option prices when only the jump variance is varied between 0.25 and 1.

extra variance to the jumps increases the range in which the asset prices are likely to move and thus increases the probability of a greater payoff.

We now look to vary the correlation of the assets through varying the jump parameters. As mentioned at the beginning of this section, we can vary the correlation of the assets by changing the size of λ_3 relative to λ_1 and λ_2. The Margrabe price as a function of ρ_{12} in the previous figures suggests that an increase in positive correlation between the assets should decrease the price.

Figure 4.4 plots the exchange option prices for three different sets of jump intensities. The correlation of between the assets when $\rho_{12} = 0.5$ for the different cases is highlighted in Table 4.1. Since the correlation between the assets under jump diffusion varies linearly as we vary ρ_{12} we only need to compare correlations for one value of ρ_{12}.

	Jump intensities	Correlation
Set 1	$\lambda_1 = \lambda_2 = 0.5, \lambda_3 = 1$	0.524
Set 2	$\lambda_1 = \lambda_2 = 0.5, \lambda_3 = 3$	0.767
Set 3	$\lambda_1 = \lambda_2 = 1, \lambda_3 = 0.5$	0.262

Table 4.1: The correlation added by jumps in Figure 4.4

We see that an increase in positive correlation through the jump parameters does not cause a monotone decrease in the price. Although set 1 models a higher positive correlation than set 3, the prices for set 1 are extremely close to set 3 and are greater for certain values of ρ_{12}. This is due to the fact that we have increased λ_1 and λ_2 in order to decrease the correlation and this also affects the prices, as highlighted in Figure 4.1. This is also the case for set 2. We increased λ_3 by a significant amount in

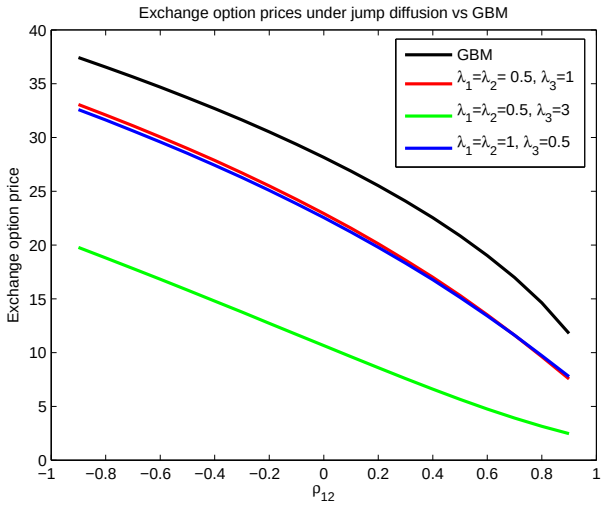


Figure 4.4: Exchange option prices when the correlation varies through jump intensities.

order to achieve a higher correlation, which alone should decrease the price but the increase in the jump intensity has decreased the price further.

It is evident that, unlike when using the Margrabe formula, there is not a straightforward relationship between the correlation in the jumps and the exchange option price. There are a number of ways we can vary the jump correlation through the jump parameters and each change in an individual jump parameter has a different effect on the price.

4.5 Pricing an Exchange Option using Monte Carlo

It is also possible to price an exchange option using the ideas outlined in Section 3. Since

$$V_{\mathrm{E}}(S^{(1)}, S^{(2)}, t) = S_0^{(1)} \mathbb{E}^{\tilde{\mathbb{Q}}}[\max(\xi_T - 1, 0)]$$

we can run a number simulations of ξ under the risk-neutral measure as given in equation (4.14). For each simulation we calculate the payoff and we can then take an average of these payoffs and discount using the risk-free rate. As the number of simulations increases the average of the payoffs should tend to the actual price. We can compare these prices with those admitted under a geometric Brownian motion model using Margrabe's formula. Figure 4.5 shows how the price of an exchange option varies with the correlation of the Wiener processes ρ_{12}, for a number of different jump parameters. Sets 1,3 and 4 all show similar correlation between the assets yet the prices are not all the same. Set 1 has the parameters smallest in magnitude and therefore the prices admitted by this model match those admitted by Margrabe's formula most closely. Sets 3 and 4 have the same jump intensities but the jump means and jump variances are reversed. Overall this swap has little effect on the

	λ_1	λ_2	λ_3	α_1	α_2	δ_1	δ_2	Corr when $\rho_{12} = 0.5$
Set 1	0.5	0.5	1	-0.05	-0.05	0.05	0.05	0.5150
Set 2	1	1	1	-0.1	-0.1	0.1	0.1	0.37545
Set 3	0.5	0.5	1	-0.2	-0.2	0.05	0.05	0.5066
Set 4	0.5	0.5	1	-0.05	-0.05	0.2	0.2	0.5159

Table 4.2: The jump parameters and the correlation between the assets for each set in Figure 4.5

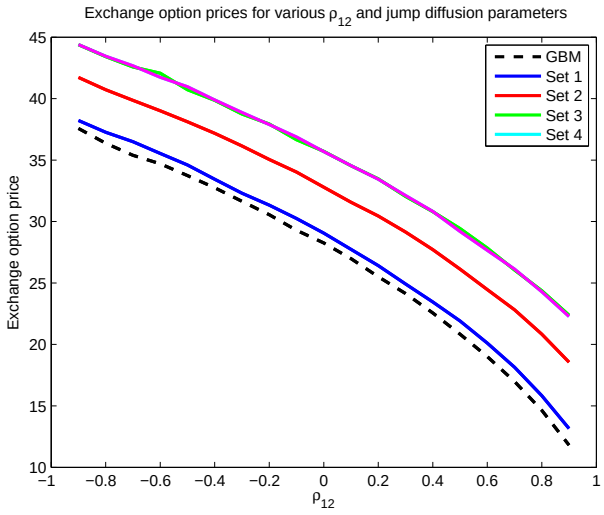


Figure 4.5: Jump diffusion exchange prices for varying ρ_{12} and jump diffusion parameters.

difference in the prices or the correlation. Set 2 shows the lowest positive correlation so we may expect an increase in the prices compared with other sets, but the jump intensities are the largest in this set, which should decrease the prices. We see a trade off and the prices fall between those of set 3 and 4 and the Margrabe prices. This again highlights to complicated relationship between the jump parameters and the exchange option prices.

The jump parameters used in the Monte Carlo pricing are relatively small compared to those used in the pricing with the formula derived in the previous section. The prices found using small parameters are generally higher than Margrabe's, whereas those found using larger parameters are generally lower.

4.6 Max-Call Options

A European max-call option is an exotic option whose payoff depends on two assets $S^{(1)}$ and $S^{(2)}$. The derivative gives the holder the right, but not obligation, to buy the highest valued stock between $S^{(1)}$ and $S^{(2)}$ for a predetermined strike price X at

time T in the future. The payoff is therefore

$$\max(\max(S_T^{(1)}, S_T^{(2)}) - X, 0). \tag{4.20}$$

Note that since

$$\max(S^{(1)}, S^{(2)}) = S^{(1)} + \max(S^{(2)} - S^{(1)}, 0)$$

and

$$\max(S^{(2)} - S^{(1)}, 0) \geq 0$$

it follows that

$$\begin{aligned}\max(\max(S_T^{(1)}, S_T^{(2)}) - X), 0) &= \max(S^{(1)} + \max(S_T^{(2)} - S_T^{(1)}) - X), 0) \\ &\geq \max(S^{(1)} - X, 0).\end{aligned}$$

Therefore the price of a max-call on the assets $S^{(1)}$ and $S^{(2)}$ will always be greater or equal to the price of a call on the individual asset $S^{(1)}$ or $S^{(2)}$ with the same strike.

A closed formula for the t-price of a max call option when the underlying asset follows a geometric Brownian motion was derived by Stulz [20]. The inputs for the formula are similar to those required for the Black-Scholes formula for calls and puts; we need the starting prices and volatilities of both stocks, the risk-free rate, the strike and the length of time until expiry of the option. However we also require the correlation between the two assets, denoted by ρ.

If we denote the price of the max call at time-0 by $V_{\max}(S^{(1)}, S^{(2)}, X, T, \rho)$ then

$$\begin{aligned}V_{\max}(S^{(1)}, S^{(2)}, X, T, \rho) &= S^{(1)} e^{-rT} M(y_1, d; \rho_1) \\ &\quad + S^2 e^{-rT} M(y_2, -d + \sigma\sqrt{T}; \rho_2) \\ &\quad - K e^{-rT} \times [1 - M(-y_1 + \sigma_1\sqrt{T}, -y_2 + \sigma_2\sqrt{T}; \rho)]\end{aligned} \tag{4.21}$$

where $M(a, b; \rho)$ is the bivariate cumulative normal distribution for two assets with correlation ρ. We define

$$\begin{aligned}d &= \frac{\ln\left(\frac{S^{(1)}}{S^{(2)}}\right) + \left(\frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \qquad y_i = \frac{\ln\left(\frac{S^{(1)}}{K}\right) + \left(\frac{\sigma_i^2}{2}\right) T}{\sigma_i\sqrt{T}} \\ \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \qquad \rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma} \qquad \rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma}.\end{aligned}$$

When implementing this pricing formula, we need to calculate the bivariate cumulative normal distribution. We use the method for computing this integral derived by Drezner [9].

We will price a European max call option under the jump diffusion model using Monte Carlo methods. We should note here that we revert back to the original correlated asset model, as stated in equation (2.8). In this model each Poisson process has a jump variable associated with it. This gives us extra freedom to vary the correlation between the assets as now not only varying λ_3 relative to λ_1 and λ_2 affects the correlation, we can also vary K_3 compared with K_1 and K_2. This form of the

model does not allow us to derive an explicit pricing formula for an exchange option of the form derived in the previous section and it is similarly so for a max-call option, hence why we price using Monte Carlo methods here.

We will compare the jump diffusion price with the price admitted by Margrabe's formula from above. We will investigate the effect that varying the correlation has on the price. Figure 4.6 plots the max-call prices admitted by Margrabe's formula model above and the jump diffusion Monte Carlo price for four different sets of jump diffusion parameters. These sets are outlined in Table 4.3. The correlation column contains the level of correlation modelled by each set of parameters calculated using equation (2.9) when $\rho_{12} = 0.5$.

	λ_1	λ_2	λ_3	α_1	α_2	α_3	δ_1	δ_2	δ_3	Corr
Set 1	0.5	0.5	1	-0.05	-0.05	-0.05	0.05	0.05	0.05	0.6350
Set 2	1	1	1	-0.1	-0.1	-0.1	0.1	0.1	0.1	0.4955
Set 3	1	1	1	-0.2	-0.2	-0.3	0.05	0.05	0.05	0.4532
Set 4	0.5	0.5	1	-0.05	-0.05	-0.05	0.2	0.2	0.3	0.6461

Table 4.3: The jump parameters used in each set for Figure 4.6

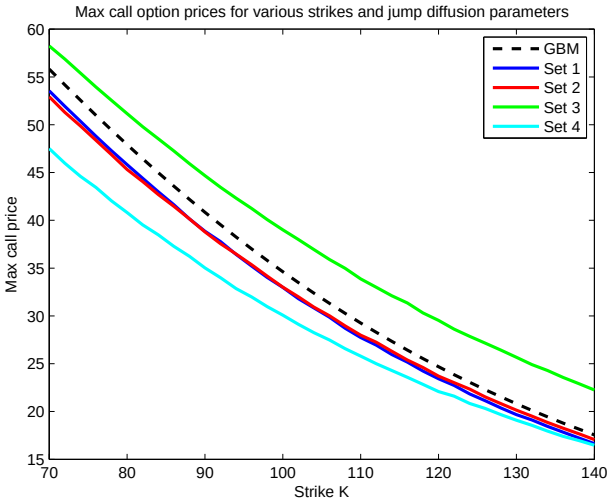


Figure 4.6: Max call prices comparing jump diffusion Monte Carlo and geometric Brownian motion

Set 1 and set 4 both model similar levels of correlation between the two assets although set 1 achieves this through smaller jump intensities, jump means and jump variances. The difference between the prices is greatest for out-the-money options and the prices tend to each other as the options move towards being in-the-money. Set 2 models a significantly lower correlation compared with set 1 yet the jump intensities are the same. The prices match closely across the range of strikes with set 2 slightly

under-pricing out-the-money options and over-pricing in-the-money options compared with set 1. Set 3 models a similar level of correlation to set 2 but with jumps that have means of higher magnitude and lower variance. The prices for set 3 are consistently higher than all the other models. As with the exchange options, it is clear that there is a complicated relationship between the different jump parameters and the max-call option price.

4.7 Implied Correlation

We can see from equation (4.21) that the max-call formula requires a number of input parameters. As with the Black Scholes formula for a call option on one asset many of these are either observable or conditions on the option contract, such as starting price of the two stocks, strike price, risk-free rate and time until maturity of the option. The final three inputs are the volatility of each asset and the correlation between the two assets. These three inputs are not freely observable in the market and must be estimated. Therefore a max-call price is dependent on estimates of these parameters. We saw in Section 3.4 that for a call option on one asset it was possible to calculate implied volatility for any call price due to the fact that a call price is monotonically increasing in volatility. We hope to be able to find a similar result for the implied correlation of a max-call price. The implied correlation of an max-call option price is the correlation that returns that price using Stulz's formula when all other inputs are kept constant. The implied correlation between two stocks is of interest to us. We assume that the correlation is constant over the interval (t, T) when we use the formula derived by Stulz to find a t-price of a max-call option. However in reality the correlation will vary over this interval, Finding the implied correlation of market max-call prices will give us an insight into the market's expectations of the correlation between the assets.

Stulz's formula in equation (4.21) is not invertible in practice therefore we must revert to a numerical method. If the max call price as a function of the correlation ρ is denoted by $V_{\max}(.)$ then we must find ρ_{imp} such that

$$V_{\max}(\rho_{\text{imp}}) - \text{price} = 0 \tag{4.22}$$

using a root finding function in Matlab, such as fsolve. Figure 4.7 shows the max call price under GBM as we vary the correlation between Wiener processes of the assets.

As we can see for the plots in Figure 4.7 the max-call price as a function of ρ_{12} is often humped. This means the implied correlation for a given price is not unique in general. If we used a numerical method to try and solve equation (4.22) then our results may not be accurate as the implied correlation for a given price found by the method may depend on the starting value of the search or the local nature of the prices as a function of ρ_{12}.

The max-call payoff surface is in Figure 4.8. If a call option on one asset is a 'bet' on whether the asset will be worth more than the strike at the maturity of the option, then a max-call option is a 'bet' on whether at least one of two assets will be greater than the strike at maturity of the option. If the assets have perfect positive

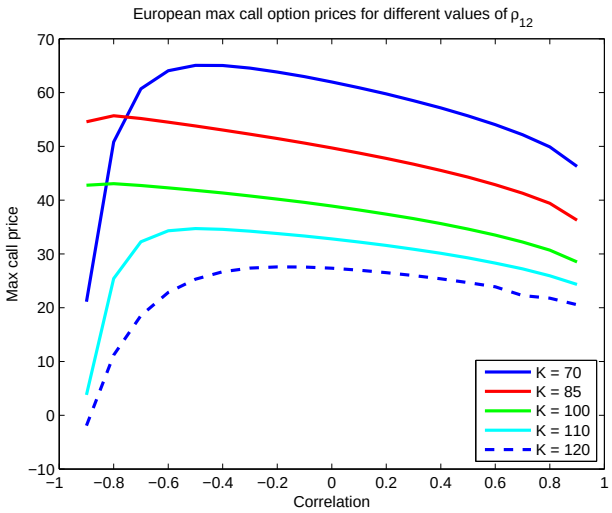


Figure 4.7: Max call prices for a range of values for ρ_{12} and strikes.

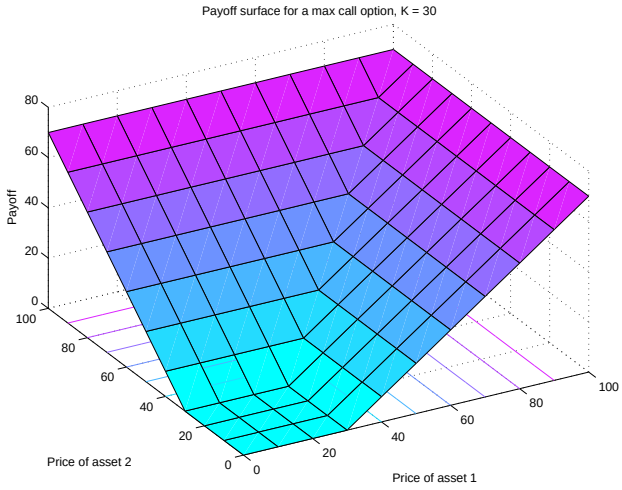


Figure 4.8: The max call option payoff surface.

correlation then they effectively become the same asset and the max-call reduces to a simple call option. If the assets show perfect negative correlation then if one asset price moves up, the other will move down meaning if the assets have the same starting price, at most one of the assets can be above the starting price at maturity of the option. This is effectively a simple call again on the asset that moves up. As the magnitude of the correlation decreases the likelihood of one of the assets being above the strike at the maturity increases. This explains the humped nature of the max-call price as a function of ρ. Therefore to find the implied correlation of two assets, we must find another option with a price that is monotonically increasing or decreasing with ρ_{12}.

As we saw in Section 4.5 the price of an exchange option is monotonically decreasing as a function of ρ_{12} over the interval $-1 < \rho_{12} < 1$. Therefore for each exchange option price there must be a unique correlation that admits that price when all other input parameters are kept constant. The payoff surface for an exchange option is in Figure 4.9.

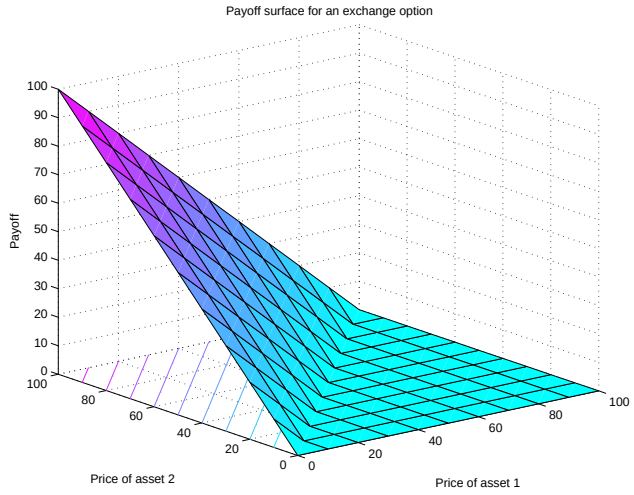


Figure 4.9: The exchange option payoff surface.

We can therefore calculate the implied correlation of the exchange option prices plotted in Figure 4.5. These were found using the jump parameters outlined in Table 4.2. Since the prices in Figure 4.5 were all greater than the Margrabe prices, the implied volatilities are all less positive or more negative than the implied volatility of the Margrabe prices. This is because the Margrabe prices monotonically decrease as correlation moves from perfect negative correlation to perfect positive correlation.

In order to find another option that may allow us to compute the implied correlation of two assets, we must find an option with a payoff surface similar to that of an exchange option.

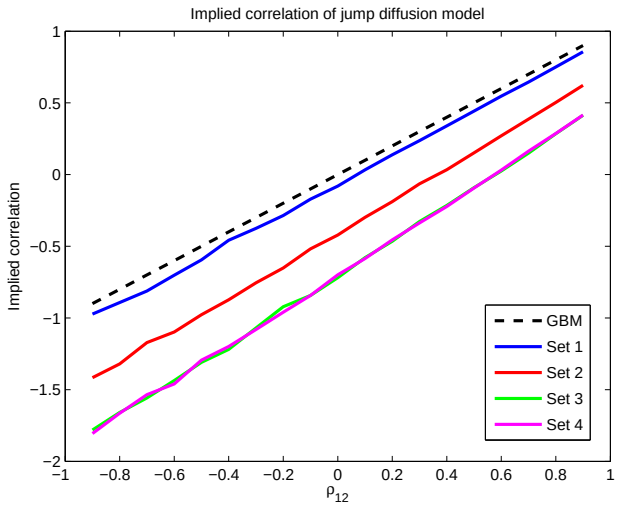


Figure 4.10: Implied correlation of exchange option jump diffusion prices for different jump parameters.

Chapter 5

Conclusion

In this project we have presented a jump diffusion model for two assets that allows us to model partial correlation between jumps in the asset prices. We used two forms of the model, one in which each driving Poisson process had a jump variable associated with it. This allows greater freedom to model correlation as we can vary both the jump intensity for the common jump and the jump mean and variance. However this model did not allow an explicit exchange option pricing formula using the joint probability density function for the two Poisson processes constructed from three independent Poisson processes. Therefore we introduced the second model in which each asset had only one jump variable driven by two independent Poisson processes. In this model we lost a degree of freedom as we could no longer isolate the common jumps to vary their mean and variance but it allowed us to derive a pricing formula for the exchange option. We also priced the exchange option using Monte Carlo methods on the more complicated jump diffusion model with correlation. Upon implementing the two methods we found the prices matched well.

Pricing of the exchange option requires inputting the same parameters as in the geometric Brownian motion case but we must also input either seven or nine extra jump parameters depending on which version of the model is used. These parameters would have to be estimated in practice before the model could be used to price any options and there may be a number of choices of parameters that could arguably model the assets equally well. This may be a downside to the model, as mentioned in the introduction, the Black-Scholes model is favoured by practitioners due the small number of input parameters needed to gain an accurate option price. We also found that there is a complicated relationship between the various jump parameters and the option prices. In the geometric Brownian motion model, an exchange option is monotonic in correlation but for a jump diffusion model this is far from the case.

We priced a max-call option using Monte Carlo methods with a view to finding the implied correlation of the two underlying assets by inverting Stulz's max-call option formula as a function of ρ_{12}. However, we found that the max-call option was not monotonic in ρ_{12} and a unique implied correlation could not be found. Therefore we had to revert to the exchange option in order to find the implied correlation of the two assets as this type of option is monotonic in ρ_{12}.

One final note on the comparisons between jump diffusion prices and those admit-

ted by models based on a geometric Brownian motion. It is clear that the addition of jumps to an asset evolution adds extra variance to the asset model. Therefore it is not wise to compare the models when the standard deviation of the geometric Brownian motion equals the standard deviation of the Wiener process in the jump diffusion model as the jump model will always have greater variance. We included GBM prices in most of our figures as a basis for comparison of the jump diffusion prices and our main interest was the correlation between the assets. Further analysis of exchange and max-call option prices in which we set the overall variance of the GBM models equal to that of the jump diffusion model may provide interesting results.

Appendix A

Expectation of a compound Poisson process

Define the random variable Z as a compound Poisson process,

$$Z_t = \sum_{i=0}^{N_t} J_i, \tag{A.1}$$

where N_t is a Poisson process and J are independent and identically distributed random variables.

For a discrete random variable X, the *probability generating function of* X is defined as

$$G_X(x) = \mathbb{E}\left[x^X\right].$$

The *moment generating function* of a random variable Y is defined as

$$M_Y(x) = \mathbb{E}\left[e^{xY}\right] \qquad \text{for } x \in \mathbb{R}.$$

The moment generating function of the variable Z defined in equation A.1 is

$$M_Z(x) = \mathbb{E}\left[e^{xZ}\right] = \mathbb{E}\left[e^{x\sum_{i=0}^{N} J}\right] \overset{*}{=} \mathbb{E}\left[\mathbb{E}\left[e^{x\sum_{i=0}^{N} J}\middle| N\right]\right] = \mathbb{E}\left[M_{\sum_{i=0}^{N} J}(x)\right], \tag{A.2}$$

where N is no longer a random variable. The step labelled by * uses the law of total expectation [1]. The moment generating function of a sum of independent random variables is equal to the product of the moment generating functions of each independent random variable. Equation (A.2) becomes

$$\mathbb{E}\left[M_{Y_1}(x) M_{Y_2}(x) \dots M_{Y_N}\right]. \tag{A.3}$$

The random variables Y are IID therefore their moment generating functions are all equal to $M_Y(x)$. It follows that

$$M_Z(x) = \mathbb{E}\left[M_Y(x)^N\right] = G_N\left(M_Y(x)\right).$$

The expectation of a random variable can be found by differentiating it's moment generating function once with respect to x and evaluating at $x = 0$ or differentiating it's probability generating function and evaluating at $x = 1$. The mean of Z_t is

$$M'_Z(x) = G'_N(M_Y(x))M'_Y(x).$$

Evaluated at $x = 0$ we have

$$M'_Z(0) = G'_N(M_Y(0))M'_Y(0) = G'_N(1)M'_Y(0) = \mathbb{E}[N]\mathbb{E}[Y].$$

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