

# BOUNDED LENGTH INTERVALS CONTAINING TWO PRIMES AND AN ALMOST-PRIME

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**ABSTRACT.** Goldston, Pintz and Yıldırım have shown that if the primes have ‘level of distribution’  $\theta$  for some  $\theta > 1/2$  then there exists a constant  $C(\theta)$ , such that there are infinitely many integers  $n$  for which the interval  $[n, n + C(\theta)]$  contains two primes. We show under the same assumption that for any integer  $k \geq 1$  there exists constants  $D(\theta, k)$  and  $r(\theta, k)$ , such that there are infinitely many integers  $n$  for which the interval  $[n, n + D(\theta, k)]$  contains two primes and  $k$  almost-primes, with all of the almost-primes having at most  $r(\theta, k)$  prime factors. If  $\theta$  can be taken as large as 0.99, and provided that numbers with 2, 3, or 4 prime factors also have level of distribution 0.99, we show that there are infinitely many integers  $n$  such that the interval  $[n, n + 90]$  contains 2 primes and an almost-prime with at most 4 prime factors. This extends a result of Pintz [9].

## 1. INTRODUCTION

We are interested in trying to understand how small gaps between primes can be. If we let  $p_n$  denote the  $n^{\text{th}}$  prime, it is conjectured that

$$(1.1) \quad \liminf_n (p_{n+1} - p_n) = 2.$$

This is the famous twin prime conjecture. Unfortunately we appear unable to prove any results of this strength. The best unconditional result is due to Goldston, Pintz and Yıldırım [5] which states that

$$(1.2) \quad \liminf_n \frac{p_{n+1} - p_n}{\sqrt{\log p_n (\log \log p_n)^2}} < \infty.$$

Therefore we do not know that  $\liminf p_{n+1} - p_n$  is finite.

The method of [5] relies heavily on results about primes in arithmetic progressions. We say that the primes have ‘level of distribution’  $\theta$  if for any constant  $A$  there is a constant  $C = C(A)$  such that

$$(1.3) \quad \sum_{q \leq x^\theta (\log x)^{-C}} \max_{\substack{a \\ (a, q) = 1}} \left| \sum_{\substack{p \equiv a \pmod{q} \\ p \leq x}} 1 - \frac{\text{Li}(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

The Bombieri-Vinogradov theorem states that the primes have level of distribution  $1/2$ , and this is a major ingredient in the proof of the Goldston-Pintz-Yıldırım result.

If we could improve the Bombieri-Vinogradov theorem to show that the primes have level of distribution  $\theta$  for some constant  $\theta > 1/2$ , then it would follow from [4][Theorem 1] that there is a constant  $D = D(\theta)$  such that

$$(1.4) \quad \liminf_n (p_{n+1} - p_n) < D,$$

and so there would be infinitely many bounded gaps between primes. It is believed that such improvements to the Bombieri-Vinogradov theorem are true, and Elliott and Halberstam [1] conjectured the following much stronger result.

**Conjecture** (Elliott-Halberstam Conjecture). *For any fixed  $\epsilon > 0$ , the primes have level of distribution  $1 - \epsilon$ .*

Friedlander and Granville [2] have shown that the primes do not have level of distribution 1, and so the Elliott-Halberstam conjecture represents the strongest possible result of this type.

Under the Elliott-Halberstam conjecture the Goldston-Pintz-Yıldırım method gives [4] that

$$(1.5) \quad \liminf_n (p_{n+1} - p_n) \leq 16.$$

If we consider the length of 3 or more consecutive primes, however, we are unable to prove as strong results, even under the full strength of the Elliott-Halberstam conjecture. In particular we are unable to prove that there are infinitely many intervals of bounded length that contain at least 3 primes. The Goldston-Pintz-Yıldırım methods can still be used, but even with the Elliott-Halberstam conjecture we are only able to prove that

$$(1.6) \quad \liminf_n \frac{p_{n+2} - p_n}{\log p_n} = 0.$$

This should be contrasted with the following conjecture.

**Conjecture** (Prime  $k$ -tuples conjecture). *Let  $\mathcal{L} = \{L_1, \dots, L_k\}$  be a set of integer linear functions whose product has no fixed prime divisor. Then there are infinitely many  $n$  for which all of  $L_1(n), L_2(n), \dots, L_k(n)$  are simultaneously prime.*

By ‘no fixed prime divisor’ above we mean that for every prime  $p$  there is an integer  $n_p$  such that  $L_i(n_p)$  is coprime to  $p$  for all  $1 \leq i \leq k$ . We call such a set of linear functions *admissible*.

We note that  $\{n, n+2, n+6\}$  is an admissible set of linear functions, and so the prime  $k$ -tuples conjecture predicts that  $\liminf_n (p_{n+2} - p_n) \leq 6$  (it is easy to verify that one cannot have  $p_{n+2} - p_n < 6$  for  $n > 2$ ). More generally, for any constant  $k > 0$  the conjecture predicts that  $\liminf (p_{n+k} - p_n) < \infty$ , and so there are infinitely many intervals of bounded size containing at least  $k$  primes.

At the moment the prime  $k$ -tuples conjecture appears beyond the techniques currently available to us. As an approximation to the conjecture, it is common to look for *almost-prime* numbers instead of primes, where almost-prime indicates that the number has only a ‘few’ prime factors.

Graham, Goldston, Pintz and Yıldırım [3] have shown that given an integer  $k$ , there are infinitely many intervals of bounded length (depending on  $k$ ) containing at least  $k$  integers each with exactly two prime factors. It is a classical result of Halberstam and Richert [6] that there are infinitely many intervals of bounded length (depending on  $k$ ) which contain

a prime and at least  $k$  numbers each with at most  $r$  prime factors for  $r$  sufficiently large (depending on  $k$ ). Pintz [9] has shown that if the primes have level of distribution  $\theta > 1/2$  then for  $k$  sufficiently large (depending on  $\theta$ ) an admissible  $k$ -tuple contains two primes and all other components have a bounded number of prime factors (depending only on  $k$  and  $\theta$ ).

We extend this work of Pintz by investigating, under the assumption that the primes have level of distribution  $\theta > 1/2$ , whether there are infinitely many intervals of bounded length (depending on  $k$ ) containing 2 primes and  $k$  numbers each with at most  $r$  prime factors, for some explicitly given  $r$ .

## 2. INITIAL HYPOTHESES

We will work with an assumption either on the distribution of primes in arithmetic progressions of level  $\theta$ , or a stronger assumption on numbers with exactly  $r$  prime factors each of which is of a given size.

Given constants  $0 \leq \eta_i \leq \delta_i \leq 1$  for  $1 \leq i \leq r$  we define

$$(2.1) \quad \beta_{r,\eta,\delta}(n) = \begin{cases} 1, & n = p_1 p_2 \dots p_r \text{ with } n^{\eta_i} \leq p_i \leq n^{\delta_i} \text{ for } 1 \leq i \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

We put

$$(2.2) \quad \Delta(x; q, a) = \sum_{\substack{x < p \leq 2x \\ p \equiv a \pmod{q}}} 1 - \frac{1}{\phi(q)} \sum_{x < p \leq 2x} 1,$$

$$(2.3) \quad \Delta_{r,\eta,\delta}(x; q, a) = \sum_{\substack{x < p \leq 2x \\ p \equiv a \pmod{q}}} \beta_{r,\eta,\delta}(n) - \frac{1}{\phi(q)} \sum_{x < p \leq 2x} \beta_{r,\eta,\delta}(n),$$

$$(2.4) \quad \Delta^*(x; q) = \max_{y \leq x} \max_{\substack{a \\ (a,q)=1}} |\Delta(y; q, a)|,$$

$$(2.5) \quad \Delta_{r,\eta,\delta}^*(x; q) = \max_{y \leq x} \max_{\substack{a \\ (a,q)=1}} |\Delta_{r,\eta,\delta}(y; q, a)|.$$

We can now state the two hypotheses that we will consider, the Bombieri-Vinogradov hypothesis of level  $\theta$ ,  $BV(\theta)$ , and the generalised Bombieri-Vinogradov hypothesis of level  $\theta$  for  $E_r$  numbers,  $GBV(\theta, r)$ .

**Hypothesis  $BV(\theta)$ .** For every constant  $A > 0$  and integer  $h > 0$  there is a constant  $C = C(A, h)$  such that if  $Q \leq x^\theta (\log x)^{-C}$  then we have

$$\sum_{q \leq Q} \mu^2(q) h^{\omega(q)} \Delta^*(x; q) \ll_A x (\log x)^{-A}.$$

**Hypothesis  $GBV(\theta, r)$ .** For every constant  $A > 0$  and integer  $h > 0$  there is a constant  $C = C(A, h)$  such that if  $Q \leq x^\theta (\log x)^{-C}$  then uniformly for  $0 \leq \eta_i \leq \delta_i \leq 1$  ( $1 \leq i \leq r$ ) we have

$$\sum_{q \leq Q} \mu^2(q) h^{\omega(q)} \Delta_{r,\eta,\delta}^*(x; q) \ll_A x (\log x)^{-A}.$$

We note that by standard arguments in sieve methods (see, for example, [6][Lemma 3.5]) Hypothesis  $BV(\theta)$  follows from the primes having level of distribution  $\theta$ .

## 3. STATEMENT OF RESULTS

**Theorem 3.1.** *Let  $k \geq 1$  be an integer. Let  $1/2 < \theta < 0.99$ . Assume Hypothesis  $BV(\theta)$  holds. Let*

$$r = \frac{240k^2}{(2\theta - 1)^3}.$$

*Then there are infinitely many integers  $n$  such that the interval  $[n, n + C(k, \theta)]$  contains two primes and  $k$  integers, each with at most  $r$  prime factors.*

**Theorem 3.2.** *Let  $\theta \geq 0.99$ , and assume Hypothesis  $GBV(\theta, r)$  holds for  $1 \leq r \leq 4$ . Then there exist infinitely many integers  $n$  such that the interval  $[n, n + 90]$  contains two primes and one other integer with at most 4 prime factors.*

We remark that the methods used to prove these results also give the unconditional existence of infinitely many short intervals  $[x, x + \epsilon \log x]$  containing two primes and an  $k$  integers each with at most  $O(k^2 \epsilon^{-3})$  prime factors, analogous to the original result of Goldston Pintz and Yıldırım [5].

## 4. PROOF OF THEOREM 3.1

We consider two finite disjoint sets of integer linear functions  $\mathcal{L}_1^{(1)} = \{L_1^{(1)}, \dots, L_{k_1}^{(1)}\}$  and  $\mathcal{L}_2^{(1)} = \{L_{k_1+1}^{(1)}, \dots, L_{k_1+k_2}^{(1)}\}$ , whose union  $\mathcal{L}^{(1)} = \mathcal{L}_1^{(1)} \cup \mathcal{L}_2^{(1)}$  is admissible. (We recall that a such set is admissible if for every prime  $p$  there is an integer  $n_p$  such that every function evaluated at  $n_p$  is coprime to  $p$ ).

We wish to show that there are infinitely many  $n$  for which two of the functions from  $\mathcal{L}_1^{(1)}$  take prime values at  $n$ , and all of the functions from  $\mathcal{L}^{(2)}$  take almost-prime values at  $n$ .

Since we are only interested in showing there are infinitely many such  $n$ , we adopt a normalisation of our linear functions, as done originally by Heath-Brown [7] which simplifies our argument. By considering  $L_i(n) = L_i^{(1)}(An + B)$  for suitable constants  $A$  and  $B$  we may assume that the functions  $L_i$  satisfy the following conditions.

- (1) The functions  $L_i(n) = a_i n + b_i$  ( $1 \leq i \leq k_1 + k_2$ ) are distinct with  $a_i > 0$ .
- (2) Each of the coefficients  $a_i$  is composed of the same primes none of which divides the  $b_j$ .
- (3) If  $i \neq j$ , then any prime factor of  $a_i b_j - a_j b_i$  divides each of the  $a_l$ .

We let  $\mathcal{L}_1 = \{L_1, \dots, L_{k_1}\}$  and  $\mathcal{L}_2 = \{L_{k_1+1}, \dots, L_{k_1+k_2}\}$ .

We now consider the sum

$$(4.1) \quad S(N; \mathcal{L}_1, \mathcal{L}_2) = \sum_{N \leq n \leq 2N} \left( \sum_{L \in \mathcal{L}_1} \chi_1(L(n)) + \sum_{L \in \mathcal{L}_2} \chi_r(L(n)) - k_2 - 1 \right) \left( \sum_{\substack{d | \prod_{i=1}^{k_1} L_i(n) \\ d \leq R}} \lambda_d \right)^2,$$

where

$$(4.2) \quad \chi_r(n) = \begin{cases} 1, & n \text{ has at most } r \text{ prime factors} \\ 0, & \text{otherwise,} \end{cases}$$

$$(4.3) \quad R = N^{\theta/2}(\log N)^{-C},$$

$$(4.4) \quad \Pi(n) = \prod_{L \in \mathcal{L}_1 \cup \mathcal{L}_2} L(n),$$

and the  $\lambda_d$  are real numbers which we declare later.  $C > 0$  is a constant chosen sufficiently large so we can use the estimates of hypotheses  $BV(\theta)$  or  $GBV(\theta, r)$ .

If we can show that  $S > 0$  then we know there must be at least one  $n \in [N, 2N]$  for which the terms in parentheses give a positive contribution to  $S$ . The second term in our expression for  $S$  is a square, and so is always non-negative. We see that the first term in parentheses is positive only when there are at least two primes and  $k_2$  numbers each with at most  $r$  prime factors amongst the  $L_i(n)$  ( $1 \leq i \leq k_1 + k_2$ ). If we choose all our original functions to be of the form  $L_i^{(1)}(n) = n + h_i$  (with  $h_i \geq 0$ ) then all these integers then lie in an interval  $[m, m + H]$ , where  $H = \max_i h_i$ .

Thus it is sufficient to show that  $S > 0$  for any large  $N$  to prove Theorem 3.1. We can get such a bound by following a method similar to Goldston, Pintz and Yıldırım [5], which we refer to as the GPY method.

To simplify notation we put

$$(4.5) \quad \Lambda^2(n) = \left( \sum_{\substack{d \mid \Pi(n) \\ d \leq R}} \lambda_d \right)^2.$$

To avoid confusion we mention that  $\Lambda^2(n)$  is unrelated to the von-Mangoldt function.

We expect to be able to show that  $S > 0$  for suitably large  $k_1$  and  $r$  (depending on  $k_2$ ) when the primes have level of distribution  $\theta > 1/2$ . This is because the original GPY method shows that for sufficiently large size of  $k_1$  (depending on  $k_2$  and  $\epsilon$ ) we can choose the  $\lambda_d$  to give

$$(4.6) \quad \sum_{N \leq n \leq 2N} \sum_{L \in \mathcal{L}_1} \chi_1(L(n)) \Lambda^2(n) \geq (2\theta - \epsilon) \sum_{N \leq n \leq 2N} \Lambda^2(n).$$

Moreover, since  $\Lambda^2(n)$  is small when  $\Pi(n)$  has many prime factors, we expect for sufficiently large  $r$  (depending on  $k_2$  and  $\epsilon$ ) that

$$(4.7) \quad \sum_{N \leq n \leq 2N} \sum_{L \in \mathcal{L}_2} (1 - \chi_r(L(n))) \Lambda^2(n) \leq \epsilon \sum_{N \leq n \leq 2N} \Lambda^2(n).$$

And so provided that  $\theta > 1/2 + \epsilon$  we expect that

$$(4.8) \quad S \gg_{\epsilon} \sum_{N \leq n \leq 2N} \Lambda^2(n) > 0.$$

Although the method of Graham, Goldston, Pintz and Yıldırım allows one to estimate similar sums involving numbers with a fixed number of prime factors, these results rely on level-of-distribution results for such numbers, which we are not assuming in Theorem 3.1.

Instead we proceed by noting that any integer which is at most  $2N$  and has more than  $r$  prime factors must have a prime factor of size at most  $(2N)^{1/(r+1)}$ . Thus for  $n \leq 2N$

$$(4.9) \quad \chi_r(n) \geq 1 - \sum_{\substack{p|n \\ p \leq (2N)^{1/(r+1)}}} 1.$$

Substituting this into our expression for  $S$  we have

$$(4.10) \quad \begin{aligned} S &\geq \sum_{N \leq n \leq 2N} \left( \sum_{L \in \mathcal{L}_1} \chi_1(L(n)) - 1 - \sum_{L \in \mathcal{L}_2} \sum_{\substack{p|L(n) \\ p \leq (2N)^{1/(r+1)}}} 1 \right) \Lambda^2(n) \\ &= \sum_{L \in \mathcal{L}_1} Q_1(L) - Q_2 - \sum_{L \in \mathcal{L}_2} Q_3(L), \end{aligned}$$

where

$$(4.11) \quad Q_1(L) = \sum_{N \leq n \leq 2N} \chi_1(L(n)) \Lambda^2(n),$$

$$(4.12) \quad Q_2 = \sum_{N \leq n \leq 2N} \Lambda^2(n),$$

$$(4.13) \quad Q_3(L) = \sum_{N \leq n \leq 2N} \sum_{\substack{p|L(n) \\ p \leq (2N)^{1/(r+1)}}} \Lambda^2(n).$$

The choice of good values for  $\lambda_d$  and the corresponding evaluation of  $Q_1, Q_2, Q_3$  already exists in the literature. We quote from [8][Proposition 4.1] taking  $W_0(t) = 1$  and [3][Theorem 7 and Theorem 9]. We note that [3][Theorem 9] does not require that  $E_2$ -numbers have level of distribution  $\theta$ , so Hypothesis  $BV(\theta)$  is sufficient for the statement to hold. These results give for a fixed polynomial  $P$ , for  $k = k_1 + k_2 = \#(\mathcal{L}_1 \cup \mathcal{L}_2)$ , for  $L \in \mathcal{L}$  and for sufficiently large  $C$  that we can choose the  $\lambda_d$  such that

$$(4.14) \quad Q_1(L) \sim \frac{\mathfrak{S}(\mathcal{L})N(\log R)^{k+1}}{(k-2)! \log N} \int_0^1 \tilde{P}(1-t)^2 t^{k-2} dt,$$

$$(4.15) \quad Q_2 \sim \frac{\mathfrak{S}(\mathcal{L})N(\log R)^k}{(k-1)!} \int_0^1 P(1-t)^2 t^{k-1} dt,$$

$$(4.16) \quad Q_3(L) \sim \frac{\mathfrak{S}(\mathcal{L})N(\log R)^k}{(k-1)!} I,$$

where

$$(4.17) \quad \mathfrak{S}(\mathcal{L}) \text{ is a positive constant depending only on } \mathcal{L},$$

$$(4.18) \quad \tilde{P}(x) = \int_0^x P(t) dt,$$

$$(4.19) \quad \begin{aligned} I &= \int_0^\delta \frac{1}{y} \int_{1-y}^1 P(1-t)^2 t^{k-1} dt dy \\ &\quad + \int_0^\delta \frac{1}{y} \int_0^{1-y} (P(1-t) - P(1-t-y))^2 t^{k-1} dt dy, \end{aligned}$$

$$(4.20) \quad \delta = \frac{2}{\theta(r+1)}.$$

Here the asymptotic for  $Q_3$  is valid only for  $R^2(2N)^{1/(r+1)} \leq N(\log N)^{-C}$ , and so we introduce the condition

$$(4.21) \quad r + 1 > \frac{1}{1 - \theta}$$

to ensure that this is satisfied for  $N$  sufficiently large. All the other asymptotics are valid without further conditions.

We choose the polynomial

$$(4.22) \quad P(x) = x^l,$$

where  $l \geq 0$  is an integer to be declared later. To ease notation we let

$$(4.23) \quad C(\mathcal{L}) = \frac{\mathfrak{S}(\mathcal{L})N(\log R)^k(2l)!}{(k + 2l)!}.$$

We note that for  $a, b \in \mathbb{N}$

$$(4.24) \quad \int_0^1 x^a(1-x)^b dx = \frac{a!b!}{(a+b+1)!}.$$

Thus we see that

$$(4.25) \quad \sum_{L \in \mathcal{L}_1} Q_1(L) \sim \theta \left( \frac{2l+1}{l+1} \right) \left( \frac{k_1}{k+2l+1} \right) C(\mathcal{L}),$$

$$(4.26) \quad Q_2 \sim C(\mathcal{L}).$$

We follow a similar approach to Graham, Goldston, Pintz and Yıldırım [3] to estimate  $Q_3$ . We let

$$(4.27) \quad I = \int_0^\delta \frac{F(y)}{y} dy,$$

where

$$(4.28) \quad \begin{aligned} F(y) &= \int_{1-y}^1 P(1-t)^2 t^{k-1} dt + \int_0^{1-y} (P(1-t) - P(1-t-y))^2 t^{k-1} dt \\ &= \int_0^1 P(1-t)^2 t^{k-1} dt + \int_0^{1-y} P(1-t-y)^2 t^{k-1} dt \\ &\quad - 2 \int_0^{1-y} P(1-t)P(1-t-y) t^{k-1} dt. \end{aligned}$$

We recall that  $P(x) = x^l$ , and note that

$$(4.29) \quad P(1-t) = (1-t)^l = (1-t-y)^l + \sum_{j=1}^l \binom{l}{j} y^j (1-t-y)^{l-j}.$$

Thus

$$\begin{aligned}
 F(y) &= \int_0^1 (1-t)^{2l} t^{k-1} dt + \int_0^{1-y} (1-t-y)^{2l} t^{k-1} dt \\
 &\quad - 2 \int_0^{1-y} (1-t-y)^{2l} t^{k-1} dt - 2 \sum_{j=1}^l \binom{l}{j} y^j \int_0^{1-y} (1-t-y)^{2l-j} t^{k-1} dt \\
 &\leq \int_0^1 (1-t)^{2l} t^{k-1} dt - \int_0^{1-y} (1-t-y)^{2l} t^{k-1} dt \\
 (4.30) \quad &= \frac{(k-1)!(2l)!}{(k+2l)!} (1 - (1-y)^{k+2l}).
 \end{aligned}$$

Substituting this into (4.27) gives

$$(4.31) \quad I = \int_0^\delta \frac{F(y)}{y} dy \leq \frac{(k-1)!(2l)!}{(k+2l)!} \int_0^\delta \frac{1 - (1-y)^{k+2l}}{y} dy$$

$$(4.32) \quad = \frac{(k-1)!(2l)!}{(k+2l)!} \sum_{j=0}^{k+2l-1} \int_0^\delta (1-y)^j dy$$

$$(4.33) \quad \leq \frac{(k-1)!(2l)!}{(k+2l)!} \delta(k+2l).$$

Therefore from (4.16), (4.23) and (4.33), for any  $L \in \mathcal{L}_2$  we have that

$$(4.34) \quad Q_3(L) \leq C(\mathcal{L}) (\delta(k+2l) + o(1)).$$

Substituting (4.25), (4.26) and (4.34) into (4.10) we see

$$(4.35) \quad S \geq C(\mathcal{L}) \left( \theta \left( \frac{2l+1}{l+1} \right) \left( \frac{k_1}{k+2l+1} \right) - 1 - k_2 \delta(k+2l) + o(1) \right).$$

We let

$$(4.36) \quad k+2l+1 = \lceil C_1(2\theta-1)^{-2} \rceil, \quad l+1 = \lceil C_2(2\theta-1)^{-1} \rceil$$

for some  $C_1, C_2$ . This gives

$$\begin{aligned}
 \frac{S}{C(\mathcal{L})} &\geq \theta \left( 2 - \frac{2\theta-1}{C_2} \right) \left( 1 - \frac{2C_2(2\theta-1)}{C_1} - \frac{(k_2+1)(2\theta-1)^2}{C_1} \right) - 1 \\
 &\quad - k_2 \delta C_1(2\theta-1)^{-2} + o(1) \\
 &\geq (2\theta-1) \left( 1 - \frac{\theta}{C_2} - \frac{4\theta C_2}{C_1} - \frac{2k_2(2\theta-1)\theta}{C_1} + \frac{(1+k_2)\theta(2\theta-1)^2}{C_1 C_2} \right) \\
 (4.37) \quad &\quad - k_2 \delta C_1(2\theta-1)^{-2} + o(1).
 \end{aligned}$$

We let

$$(4.38) \quad C_1 = 40k_2, \quad C_2 = 3.$$

We see from (4.36) that this choice of  $C_1$  and  $C_2$  corresponds to positive integer values for  $k_1$  and  $l$  for any choice of  $0.5 < \theta \leq 0.99$  or  $k_2$ , and so is a valid choice.

We notice that with this choice the second expression in parentheses in the last line of (4.37) is

$$(4.39) \quad \left( 1 - \left( \frac{40\theta + 6\theta(2\theta-1) - \theta(2\theta-1)^2}{120} \right) - \left( \frac{36\theta - \theta(2\theta-1)^2}{120k_2} \right) \right).$$



Since  $k_2$  is a positive integer and  $1/2 < \theta \leq 1$ , (4.39) is smallest when  $k_2 = 1$ . This gives

$$\begin{aligned} \frac{S}{C(\mathcal{L})} &\geq (2\theta - 1) \left( \frac{30 - 17\theta - 5\theta^2 + 2\theta^3}{30} \right) - 40k_2^2(2\theta - 1)^{-2}\delta + o(1) \\ (4.40) \quad &\geq (2\theta - 1) \left( \frac{31 - 21\theta}{30} \right) - 40k_2^2(2\theta - 1)^{-2}\delta + o(1). \end{aligned}$$

Thus  $S > 0$  for large  $N$  if  $\delta$  is chosen such that

$$(4.41) \quad \delta < \frac{(2\theta - 1)^3(31 - 21\theta)}{1200k_2^2}.$$

We recall  $\delta = 2/\theta(r + 1)$ , so  $S$  is positive provided  $r$  is chosen larger than

$$(4.42) \quad \frac{2400k_2^2}{(2\theta - 1)^3\theta(31 - 21\theta)} - 1 < \frac{240k_2^2}{(2\theta - 1)^3}.$$

We note that if  $r = 240k_2^2/(2\theta - 1)^3$  then for  $\theta < 0.99$  the condition (4.21) is satisfied. This completes the proof of Theorem 3.1.

We remark that by choosing  $L(n) = n + h$  with  $h < H$  for  $L \in \mathcal{L}_2$  and  $L(n) = n + h$  with  $h > H$  for  $L \in \mathcal{L}_1$  we can ensure that of the  $k_2 + 2$  almost-primes we find, the largest two are primes.

## 5. PROOF OF THEOREM 3.2

We can get better quantitative results for the number of prime factors involved in our almost-prime if we assume a fixed level of distribution result for almost-primes and for primes, and then follow the work of [3].

We consider the same sum  $S$ , but now we assume that  $\mathcal{L}_2 = \{L_0\}$  and we take  $r = 4$ . Thus  $k_2 = 1$  and  $k = k_1 + 1$ .

$$\begin{aligned} S &= S(N; \mathcal{L}_1, \{h_0\}) = \sum_{N \leq n \leq 2N} \left( \sum_{L \in \mathcal{L}_1} \chi_1(L(n)) + \chi_4(L_0(n)) - 2 \right) \left( \sum_{\substack{d | \Pi(n) \\ d \leq R}} \lambda_d \right)^2 \\ (5.1) \quad &= \sum_{L \in \mathcal{L}_1} Q_1(L) + Q'_1(L_0) - Q_2, \end{aligned}$$

where  $Q_1(L)$ ,  $Q_2$  are as before and

$$(5.2) \quad Q'_1 = \sum_{N \leq n \leq 2N} \chi_4(L_0(n)) \left( \sum_{\substack{d | \Pi(n) \\ d \leq R}} \lambda_d \right)^2$$

$$(5.3) \quad R = N^{0.99/2}(\log N)^C.$$

As before,  $C$  is a suitably large positive constant.

We split the contribution to  $Q'_1$  depending on whether  $L_0(n)$  has exactly 1, 2, 3 or 4 prime factors. Thus

$$(5.4) \quad Q'_1 = Q_1(L_0) + Q'_{12} + Q'_{13} + Q'_{14},$$

where

$$(5.5) \quad Q'_{1j} = \sum_{N \leq n \leq 2N} \beta_j(L_0(n)) \left( \sum_{\substack{d|11(n) \\ d \leq R}} \lambda_d \right)^2,$$

and

$$(5.6) \quad \beta_j(n) = \begin{cases} 1, & n \text{ has exactly } j \text{ prime factors} \\ 0, & \text{otherwise.} \end{cases}$$

For technical reasons we find it harder to deal with terms arising when  $L(n)$  has a prime factor less than  $N^\epsilon$  or no prime factor greater than  $N^{1/2}$ . Thus we obtain a lower bound for  $Q'_{1j}$  by replacing  $\beta_j(L_0(n))$  with  $\beta'_j(L_0(n))$ , where

$$(5.7) \quad \beta'_j(n) = \begin{cases} 1, & n = p_1 p_2 \dots p_j \text{ with } n^\epsilon < p_1 < \dots < p_j \text{ and } n^{0.505} < p_j \\ 0, & \text{otherwise.} \end{cases}$$

We can then obtain these asymptotic lower bounds. By following an equivalent argument to [10] and [8][Proposition 4.2] but using Hypothesis GBV(0.99,  $j$ ) to bound the error terms we have

$$(5.8) \quad Q'_{1j} \geq (1 + o(1)) \frac{\mathfrak{S}(\mathcal{L})(\log R)^{k+1}}{(k-2)! \log N} J_j,$$

where

$$(5.9) \quad J_r = \int_{(x_1, \dots, x_{r-1}) \in \mathcal{A}_r} \frac{I_1(Bx_1, \dots, Bx_{r-1})}{\left(\prod_{i=1}^{r-1} x_i\right) \left(1 - \sum_{i=1}^{r-1} x_i\right)} dx_1 \dots dx_{r-1},$$

$$(5.10) \quad I_1 = \int_0^1 \left( \sum_{J \subset \{1, \dots, r-1\}} (-1)^{|J|} \tilde{P}^+(1-t - \sum_{i \in J} x_i) \right)^2 t^{k-2} dt,$$

$$(5.11) \quad \tilde{P}^+(x) = \begin{cases} \int_0^x P(t) dt, & x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$(5.12) \quad B = \frac{2}{0.99},$$

$$(5.13) \quad \mathcal{A}_r = \left\{ x \in [0, 1]^{r-1} : \epsilon < x_1 < \dots < x_{r-1}, \sum_{i=1}^{r-1} x_i < B^{-1} \right\}.$$

As before, by Hypothesis BV(0.99) we also have for any  $L \in \mathcal{L}$

$$(5.14) \quad Q_1(L) \sim \frac{\mathfrak{S}(\mathcal{L})(\log R)^{k+1}}{(k-2)! \log N} \int_0^1 \tilde{P}(1-t)^2 t^{k-2} dt,$$

$$(5.15) \quad Q_2 \sim \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!} \int_0^1 P(1-t)^2 t^{k-1} dt.$$

Thus we have that

$$(5.16) \quad S \geq \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!} \left( \frac{0.99(k-1)}{2} (kJ_1 + J_2 + J_3 + J_4) - 2I_0 + o(1) \right),$$

where  $J_r$  is given above and

$$(5.17) \quad I_0 = \int_0^1 P(1-t)^2 t^{k-1} dt.$$

Therefore given a polynomial  $P$  we can get an asymptotic lower bound for  $S$  by explicitly calculating the integrals  $I_0, J_1, J_2, J_3$  and  $J_4$ .

Explicitly we have for  $r = 1$

$$(5.18) \quad J_1 = \int_0^1 \tilde{P}(1-t)^2 t^{k-2} dt.$$

Similarly for  $r = 2$  we have

$$(5.19) \quad J_2 = J_{21} + J_{22} + O(\epsilon),$$

where

$$(5.20) \quad J_{21} = \int_0^1 \frac{B}{x(B-x)} \int_0^{1-x} (\tilde{P}(1-t) - \tilde{P}(1-t-x))^2 t^{k-2} dt dx,$$

$$(5.21) \quad J_{22} = \int_0^1 \frac{B}{x(B-x)} \int_{1-x}^1 \tilde{P}(1-t)^2 t^{k-2} dt dx.$$

Similarly for  $r = 3$  we have

$$(5.22) \quad J_3 = J_{31} + J_{32} + J_{33} + J_{34} + O(\epsilon),$$

where

$$(5.23) \quad J_{31} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-x}^1 (\tilde{P}(1-t))^2 t^{k-2} dt dy dx,$$

$$(5.24) \quad J_{32} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-y}^{1-x} (\tilde{P}(1-t) - \tilde{P}(1-t-x))^2 t^{k-2} dt dy dx,$$

$$(5.25) \quad J_{33} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-x-y}^{1-y} (\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y))^2 t^{k-2} dt dy dx,$$

$$(5.26) \quad J_{34} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_0^{1-x-y} (\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) + \tilde{P}(1-t-x-y))^2 t^{k-2} dt dy dx.$$

Finally for  $r = 4$  we have

$$(5.27) \quad J_4 = J_{41} + J_{42} + J_{43} + J_{44} + J_{45} + J_{46} + J_{47} + J_{48} + J_{49} + J_{410} + J_{411} + O(\epsilon),$$

where

$$(5.28) \quad J_{41} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x}^1 (\tilde{P}(1-t))^2 t^{k-2} dt dz dy dx,$$

$$(5.29) \quad J_{42} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-y}^{1-x} (\tilde{P}(1-t) - \tilde{P}(1-t-x))^2 t^{k-2} dt dz dy dx,$$

$$(5.30) \quad J_{43} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y}^{1-y} (\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y))^2 t^{k-2} dt dz dy dx,$$

$$(5.30) \quad J_{44} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-z}^{1-y} (\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y))^2 t^{k-2} dt dz dy dx,$$

$$(5.31) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{45} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-z}^{1-x-y}$$

$$(5.32) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) + \tilde{P}(1-t-x-y) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{46} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y}^{1-z}$$

$$(5.33) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{47} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-z}^{1-z}$$

$$(5.34) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{48} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-z}^{1-x-y}$$

$$(5.35) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{49} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-y-z}^{1-x-z}$$

$$(5.36) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) + \tilde{P}(1-t-x-z) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{410} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y-z}^{1-y-z}$$

$$(5.37) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) + \tilde{P}(1-t-x-z) + \tilde{P}(1-t-y-z) \right)^2 t^{k-2} dt dz dy dz,$$

$$J_{411} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_0^{1-x-y-z}$$

$$(5.38) \quad \left( \tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) + \tilde{P}(1-t-x-z) + \tilde{P}(1-t-y-z) - \tilde{P}(1-t-x-y-z) \right)^2 t^{k-2} dt dz dy dz.$$

We choose  $k = 22$  and  $P(t) = 1 + 60t - 300t^2 + 3500t^3$  and find that

$$(5.39) \quad I_0 = \frac{121351}{59202} = 2.04978\dots,$$

$$(5.40) \quad J_1 = \frac{228380}{18027009} = 0.01266\dots,$$

$$(5.41) \quad J_2 \geq 0.041 + O(\epsilon),$$

$$(5.42) \quad J_3 \geq 0.048 + O(\epsilon),$$

$$(5.43) \quad J_4 \geq 0.028 + O(\epsilon).$$

Thus we have that

$$\begin{aligned}
 S &\geq \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!} \left( \frac{0.99(k-1)}{2} (kJ_1 + J_2 + J_3 + J_4) - 2I_0 + O(\epsilon) + o(1) \right) \\
 (5.44) \quad &\geq \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!} (0.013 + O(\epsilon) + o(1)).
 \end{aligned}$$

In particular, for  $N$  sufficiently large and  $\epsilon$  sufficiently small we have  $S > 0$ , and so there are infinitely many  $n$  for which an admissible 22-tuple attains at least two prime values and one value with at most 4 prime factors.

The set  $\{0, 6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 84, 86, 90\}$  is an admissible 22-tuple, and so the interval  $[n, n+90]$  infinitely often contains at least two primes and an integer with at most 4 prime factors.

We remark here that if we can take the level of distribution  $\theta = 1 - \delta$  for every  $\delta > 0$  then we can take  $k = 19$  instead of 22, which reduces the length of the interval to 80.

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