

Manifold Tensor Categories



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Introduction

Group theory is the study of symmetries: The set of symmetries of an object is equipped with a unital associative multiplication given by composition of symmetries, and every symmetry admits an inverse. The symmetries of an object equipped with smooth structure assemble into a group with smooth structure: a *Lie group*. Tensor category theory is the study of *quantum symmetries*, a generalisation of ordinary symmetries inspired by quantum physics [Web20]. In this thesis, we examine tensor categories with smooth structure – the analogue of Lie groups in the world of quantum symmetries.

A tensor category is a linear¹ category \mathbf{C} equipped with a monoidal product functor

$$\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C},$$

a \otimes -unit $\mathbb{1} \in \mathbf{C}$, and morphisms enforcing associativity and unitality. *Braided* tensor categories are further equipped with a natural isomorphism $x \otimes y \xrightarrow{\cong} y \otimes x$ for all $x, y \in \mathbf{C}$. When this braiding satisfies a certain non-degeneracy condition [Tur92], it is called a *modular* tensor category. In this introduction, we further assume every object in a tensor category is *dualisable*. Dualisability is a generalisation of invertibility and equivalent to finite-dimensionality in the tensor category of vector spaces. We review these definitions in detail in Section 1.1.1. A textbook account of the theory of tensor categories is given in [EGNO16].

The term quantum symmetry is generally used rather loosely, but it has a precise meaning in Algebraic Quantum Field Theory. Let A be an algebra (such as the algebra of operators of a quantum field theory²). A classical symmetry of A is given by an invertible algebra homomorphism $A \xrightarrow{\cong} A$. These classical symmetries assemble into the group of automorphisms $\text{Aut}(A)$. A *quantum* symmetry of A is an A - A -bimodule X . These quantum symmetries assemble into the tensor category ${}_A\mathbf{Mod}_A$ of A - A -bimodules. Its tensor product – the composition of quantum symmetries of A – is given by the usual tensor product of A - A -bimodules. This really is a generalisation of the notion of a symmetry: the *unit* symmetry of A is given by A considered as a bimodule ${}_AA_A$ over itself, and an automorphism $\phi \in \text{Aut}(A)$ appears in ${}_A\mathbf{Mod}_A$ as ${}_AA_{A^\phi}$, the module A with canonical left action and ϕ -twisted right action.

Given a tensor category \mathbf{C} and an algebra A , we may define an action of \mathbf{C} on A by giving a functor of tensor categories

$$\mathbf{C} \rightarrow {}_A\mathbf{Mod}_A.$$

¹We work over \mathbb{C} throughout, but a large body of work is devoted to the study of tensor categories over more general basefields.

²Depending on context, this algebra is usually considered as a C^* -algebra or a von Neumann algebra.

This generalises the notion of a group action $G \rightarrow \text{Aut}(A)$. Just as classical symmetries of objects are actions of groups, so quantum symmetries are actions of tensor categories.

Quantum symmetries of operator algebras have been studied intensely, for example in [MS92; EK⁺98; ASS17]. Many authors work in the setting of *quantum groups* [Dri86; CP95; Str07; Kas12], which are related to braided tensor categories by Tannaka duality.³ For more general examples of actions of tensor categories, see the talks at [PPTW⁺21], in particular [Yam21; Hen21; Evi21; Jon21].

Tensor categories also naturally appear in the functorial approach to quantum field theory. In this picture, one views a quantum field theory as a symmetric monoidal functor

$$\mathbf{Bord} \rightarrow \mathcal{D}$$

from a bordism category to a category \mathcal{D} . If \mathbf{Bord} is just viewed as a category of ordinary manifolds, this is the definition of a *topological quantum field theory* (TQFT) [Seg88; Ati88]. This setup may be generalised to allow for manifolds with geometric structures, see e.g. [ST11]. A *quantum field theory with defects* is a functor as above, except the domain \mathbf{Bord} is replaced by a bordism n -category of decorated manifolds [Lur08, Sec 4]. A particular model for 2-dimensional topological defect quantum field theories is spelled out in [Car16].

In certain quantum field theories, the structure of defects is captured by tensor categories (and their higher categorical analogues). This has been made precise for topological defects in chiral conformal field theories [FFRS02]. The study of topological defects in full conformal field theory largely reduces to the study of algebras in tensor categories [FFRS07; FFRS10; FSY21]. Closely related is the picture of *topological defect lines* introduced in [Cha⁺19; TW19]: they naturally assemble into tensor categories. For a treatment from the point of view of lattice QFT, see [AFM16; AFM20].

A tensor category with the property that every object splits as a finite direct sum of irreducible objects $\{X_i\}$ is called a (finitely) *semisimple* tensor category. Then $\{X_i\}$ is called the *set of simple objects* of the tensor category. A semisimple tensor category is *finite* if its set of simple objects is finite. A *fusion category* is a finite semisimple tensor category whose monoidal unit is a simple object.

One may directly build TQFTs from fusion categories. The first example of such a construction was given by Reshetikhin and Turaev [RT91]. They construct a TQFT from a modular tensor category (a braided fusion category satisfying a non-degeneracy condition). In particular, they recover the Chern-Simons theory [Wit89] associated to a semisimple Lie group G and positive-definite level $k \in H^4(\mathcal{B}G, \mathbb{Z})$ when fed the category $\mathbf{Rep}^k LG$ of positive-energy representations of the corresponding loop group at level k . Notable is also the Turaev-Viro construction [TV92; BW96], which yields a TQFT from a fusion category equipped with a *spherical* structure. If one works exclusively with framed manifolds, no spherical structure is needed [DSPS20].

³Quantum groups are certain Hopf algebras equipped with extra structure. The tensor category associated to a quantum group H is its category of representations $\mathbf{Rep}H$.

Manifold Tensor Categories

The first result of this thesis is a definition which makes precise the notion of a tensor category with a manifold of simple objects. We also construct a number of basic examples.

A lot of attention has been focused on the study of fusion categories. The condition of having finitely many simple objects is baked into the TQFT setup (at least if one works sufficiently locally, see [BDSPV15, App A] for a precise statement). In the study of Conformal Field Theory, one often restricts attention to rational CFTs precisely to stay in the finite-dimensional realm.

We examine instead what happens when the set of simple objects is not necessarily finite, but equipped with a smooth structure. We call the resulting notion a *manifold tensor category* when the objects form a manifold (Definition 2.4.0.2). Immediate examples of such tensor categories are provided by representation theory: the complex representations of the group \mathbb{Z} form a tensor category with a manifold \mathbb{C}^\times of simple objects. We go further and introduce tensor categories with an orbifold of simple objects (*orbifold tensor categories*), with examples provided by representations of virtually abelian groups such as $D_\infty = \mathbb{Z}/2 \rtimes \mathbb{Z}$ (see Sections 2.1 and 2.4).

Manifold tensor categories also appear in the study of CFTs at central charge $c = 1$: It is shown in [TW21; CL21] that the quantum symmetries of such CFTs assemble into topological spaces (the authors work without the smooth structure present on these spaces).

A particular class of manifold tensor categories was already introduced in the study of Chern-Simons theory [FHLT10]: Freed, Hopkins, Lurie and Teleman introduce the *categorified group ring* $\mathbf{Vec}^\tau[T]$ associated to a compact torus T and a cohomology class $\tau \in H^3(\mathcal{B}T, \mathbb{C}^\times)$.⁴ We build the analogous categorified group ring $\mathbf{Vec}^\omega[G]$ associated to a general Lie group G and 3-cocycle $\omega \in H^3(\mathcal{B}G, \mathbb{C}^\times)$. This is a manifold tensor category whose simple objects form the manifold underlying G , and whose tensor product is given by the group multiplication. The 3-cocycle ω encodes the associator.

Gerbes and Twisted Sheaves

Defining orbifold tensor categories requires us to introduce a linear category whose simple objects ‘form an orbifold’. Motivated by the fact that the category of skyscraper sheaves (Definition 1.2.1.6) on a manifold M is naturally equivalent to the category of M -graded vector spaces

$$\mathbf{Sky}(M) \simeq \mathbf{Vec}^{\oplus M},$$

we work with skyscraper sheaves over orbifolds.

We retain the smooth structure present on the underlying orbifold \mathcal{M} by keeping track of skyscraper sheaves in families. To this end, we upgrade $\mathbf{Sky}(M)$ to a stack $\mathbf{Sky}_{\mathcal{M}}$ over the site of smooth manifolds.

However, this setup is not flexible enough for our purposes yet. Consider the categorified group ring $\mathbf{Vec}^{\omega \neq 0}[\mathrm{SU}(2)]$. Its simple objects form S^3 , the underlying manifold

⁴The authors of [FHLT10] choose to work with the group $H^4(\mathcal{B}T, \mathbb{Z})$ instead. There is a canonical isomorphism $H^3(\mathcal{B}T, \mathbb{C}^\times) \xrightarrow{\simeq} H^4(\mathcal{B}T, \mathbb{Z})$ coming from the exponential exact sequence $\mathbb{Z} \hookrightarrow \mathbb{C} \twoheadrightarrow \mathbb{C}^\times$.

of $\mathrm{SU}(2)$, but it does not admit a linear equivalence to the stack \mathbf{Sky}_{S^3} : There is a global S^3 -family of skyscraper sheaves on S^3 , whose value at $x \in S^3$ is the skyscraper sheaf \mathbb{C}_x . No such family exists for $\mathbf{Vec}^{\omega \neq 0}[\mathrm{SU}(2)]$: A 3-cocycle $\omega \in H^3(\mathcal{B}\mathrm{SU}(2), \mathbb{C}^\times)$ transgresses to a 2-cocycle $\theta \in H^2(S^3, \mathbb{C}^\times) \simeq H^3(S^3, \mathbb{Z}) \simeq \mathbb{Z}$. A global S^3 -family of objects in $\mathbf{Vec}^\omega[\mathrm{SU}(2)]$ induces a trivialisation of θ , but θ is cohomologically non-trivial if ω is.

More geometrically, the choice of associator $\omega \in H^3(\mathcal{B}\mathrm{SU}(2), \mathbb{C}^\times)$ induces (the equivalence class of) a \mathbb{C}^\times -gerbe (Definition 2.2.4.2) on S^3 . A \mathbb{C}^\times -gerbe \mathcal{G} over an orbifold \mathcal{M} allows a twisting of the notion of a sheaf on \mathcal{M} . We introduce the stack of twisted skyscraper sheaves over an orbifold in Definition 2.2.6.9. The linear stack underlying $\mathbf{Vec}^\omega[\mathrm{SU}(2)]$ is equivalent to the stack $\mathbf{Sky}_{S^3}^{\mathcal{G}}$ of \mathcal{G} -twisted skyscraper sheaves over S^3 , where \mathcal{G} is the \mathbb{C}^\times -gerbe induced by ω .

A large part of this thesis is devoted to the study of twisted sheaves. In Appendix A, we generalise the pullback and pushforward functors from ordinary sheaves to twisted sheaves over orbifolds,⁵ and show that they are adjoint. In Section 2.3, we prove a number of results about the stack of twisted skyscraper sheaves over an orbifold, establishing in particular a local image factorisation for morphisms (Proposition 2.3.2.21) and a local normal form for objects (Corollary 2.3.2.16). In Section 2.5, we examine how the smooth structure of the stack of twisted skyscraper sheaves constrains possible monoidal structures.

Interpolated Fusion Categories

We provide a general method to construct new manifold tensor categories from a categorified group ring $\mathbf{Vec}^\omega[G]$ and a fusion category \mathbf{D} , and give a number of examples.

In Section 2.4.3, we study group algebras in orbifold tensor categories. We show (Proposition 2.4.3.6) that modules over such an algebra A in an orbifold tensor category $\underline{\mathbf{C}}$ assemble into a linear category $\mathbf{Mod}_A(\underline{\mathbf{C}})$ whose simple objects form an orbifold. A lift of A to the Drinfel'd centre [Müg03; Str07] $\mathcal{Z}\underline{\mathbf{C}}$ of $\underline{\mathbf{C}}$ imbues $\mathbf{Mod}_A(\underline{\mathbf{C}})$ with a monoidal structure, making it an orbifold tensor category (Theorem 2.4.4.2). This is a smooth version of the statement for ordinary tensor categories [Par95]. We use this fact to provide a construction of new families of manifold tensor categories in Section 2.4.5. This construction takes as input a fusion category \mathbf{D} with a braided group algebra $\mathbf{Vec}^\beta[H] \hookrightarrow \mathcal{Z}\mathbf{D}$ in its Drinfel'd centre, together with a categorified group ring $\mathbf{Vec}^\omega[G]$, chosen such that its Drinfel'd centre contains the oppositely braided group algebra $\mathbf{Vec}^{-\beta}[H] \hookrightarrow \mathcal{Z}\mathbf{Vec}^\omega[G]$. From this data, we construct a manifold tensor category

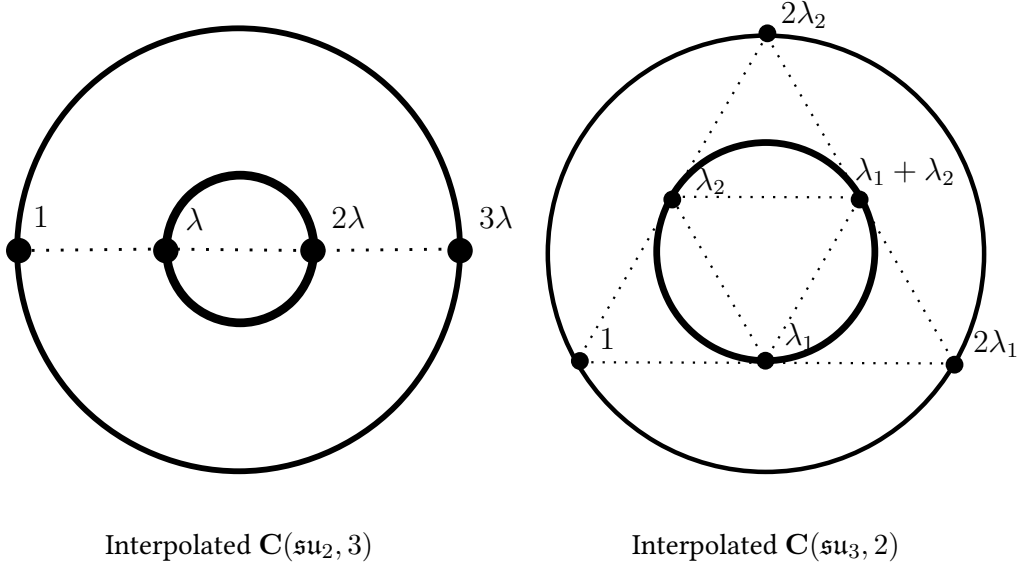
$$\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vec}[H]} \mathbf{D},$$

whose manifold of simple objects is given by the quotient $(\mathrm{ob} \mathbf{D} \times G)/H$.

There is a canonical monoidal morphism $\mathbf{D} \rightarrow \mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vec}[H]} \mathbf{D}$. If G is connected, simple objects in the same orbit of $H \curvearrowright \mathrm{ob} \mathbf{D}$ land in the same connected component of $(\mathrm{ob} \mathbf{D} \times G)/H$. We call these manifold tensor categories *interpolated fusion categories*: the manifold *interpolates* between simple objects of \mathbf{D} which are in the same H -orbit.

⁵In fact, we work in the slightly greater generality of étale stacks.

We give explicit examples of these constructions in Sections 2.4.6 and 2.4.7, where we interpolate Tambara-Yamagami categories and quantum group categories. We reproduce below Figure 2.3, which depicts the simple objects for two interpolated quantum group categories — see Sections 2.1 and 2.4.6 for more details.



Interpolated Tambara-Yamagami categories were independently discovered from a physics perspective in [TW19; TW21].

Smooth Dualisability

We show that pointwise duals in certain orbifold tensor categories automatically assemble into smooth families of duals.

Recall that a Lie group is a manifold G , equipped with a choice of unit $e \in G$, a smooth multiplication map $G \times G \rightarrow G$, such that each element $g \in G$ admits an inverse. The map $(-)^{-1} : G \rightarrow G$ sending an element to its inverse is automatically smooth (we recall this as Lemma 2.6.0.1). We not only categorify this statement, but in fact generalise the condition of invertibility to dualisability.

We introduce the notion of *orbifusion categories*: orbifold tensor categories all of whose simple objects are pointwise dualisable (and whose monoidal unit $\mathbb{1}$ is a simple object). We further assume the orbifold of simple objects is *effective* (see Section 2.2.2).

Theorem (Theorem 2.6.0.7, Corollary 2.6.0.8). *An orbifusion category $\underline{\mathbf{C}}$ admits a smooth dualising functor $\underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}}^{\text{op}}$.*

The Centre of the String 2-Group

Chapter 3 is concerned with the computation of the invertible part of $\mathcal{L}\text{Vec}^\omega[G]$. The construction of interpolated fusion categories in Chapter 2 requires us to find group algebras in the Drinfel'd centre $\mathcal{L}\text{Vec}^\omega[G]$ of $\text{Vec}^\omega[G]$. The following Lemma compels us to focus on the maximal sub-2-group of $\text{Vec}^\omega[G]$.

Lemma (Lemma 2.4.5.1).

$$(\mathcal{L}\mathbf{Vec}^\omega[G])^\times \simeq \mathcal{L}(\mathbf{Vec}^\omega[G]^\times)$$

The smooth structure of $\mathbf{Vec}^\omega[G]$ equips its maximal sub-2-group

$$G_\omega := (\mathbf{Vec}^\omega[G])^\times,$$

with the structure of a smooth 2-group. It is known as the String 2-group [SP11] associated to the Lie group G and 3-cocycle $\omega \in H^3(\mathcal{B}G, \mathbb{C}^\times)$. We compute $\mathcal{L}G_\omega$ for G compact and connected in Chapter 3. The result is interesting in its own right:

Theorem (Theorem 3.6.3.3). *Let G be a semisimple compact connected Lie group and $k \in H^4(\mathcal{B}G, \mathbb{Z}) \simeq H^3(\mathcal{B}G, \mathbb{C}^\times)$.⁶ Then there is a braided equivalence*

$$\mathcal{L}G_k = (\mathbf{Rep}^k LG)^\times$$

between its Drinfel'd centre and the category of level k positive energy representations of the loop group $LG = C^\infty(S^1, G)$.

The category $\mathbf{Rep}^k LG$ is the modular tensor category corresponding (under the Reshetikhin-Turaev construction) to Chern-Simons theory. The Drinfel'd centre of the String 2-group G_k therefore recovers the invertible defects of the Chern-Simons theory associated to G and k .

Structure of this Thesis

We review background material in Chapter 1: In Section 1.1, we discuss monoidal structures, monoid/algebra objects and modules in the context of monoidal categories and bicategories. In Section 1.2, we give a brief reminder on sheaves and stacks. Chapter 1 may be safely skipped by readers familiar with these notions.

Chapters 2 and 3 may be read independently. We use the results of Chapter 3 to construct interpolated fusion categories in Section 2.4. Apart from that, the two chapters are logically independent, and may be viewed as standalone papers. We have chosen to give both of them their own introduction.

⁶The cohomology class is subject to the further conditions of being positive-definite and not pulling back to level 2 along a map $E_8 \rightarrow G$ – see Section 3.6.3 for details.

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Chapter 1

Background

In this section, we review helpful background material for the remainder of the thesis. This mostly serves to introduce notation: we do not prove any new statements, and will omit details when convenient. We provide suitable references instead.

Chapters 2 and 3 take place in the world of (2-)monoids in monoidal bicategories, which we review in Section 1.1. The specific monoidal bicategories we work in will be categories of stacks over \mathbf{Man} , the site of smooth manifolds. Sheaves and stacks are reviewed in Section 1.2.

1.1 Monoidal Structures

A unital (1-)monoid is a set equipped with a unit element and an associative multiplication. This is the beginning of a hierarchy: the notion of a unital n -monoid makes sense in a unital $(n + 1)$ -monoid. A unital 1-monoid is called commutative if its multiplication commutes with the swap map in the category of sets. This notion makes sense in a monoidal category equipped with a braiding (a braided unital 2-monoid). More generally, there are n levels of “commutativity” in the world of n -monoids, and the notion of a k -commutative n -monoid makes sense inside a k -commutative $(n + 1)$ -monoid. See [BD98] for a precise statement, and [BS10] for a discussion of n -categories with varying levels of commutativity.

In this Section, we recall some of the above theory in the context of *tensor categories* (linear 1-monoids), and in the context of *monoidal bicategories* (2-monoids).

1.1.1 Tensor Categories

In this section, we work towards the definition of a fusion category. We begin by introducing the necessary notions from linear category theory before we go on to introduce tensor structures and duals. We provide examples along the way. A textbook account of the theory of tensor categories may be found in [EGNO16].

We denote by \mathbf{Vec}^∞ the category of vector spaces over \mathbb{C} , and by $\mathbf{Vec} \subset \mathbf{Vec}^\infty$ the full subcategory on finite-dimensional vector spaces. We use the term *linear category* to refer to a category enriched over \mathbf{Vec}^∞ .

Let \mathbf{C} be a linear category.

Definition 1.1.1.1. A *zero object* in \mathbf{C} is an object $0 \in \mathbf{C}$ such that every object admits a unique map to and from 0 .

The zero object induces a morphism $0 : X \rightarrow 0 \rightarrow Y$ between any pair of objects. These morphisms are called *zero morphisms*.

Definition 1.1.1.2. A *summand* of an object $X \in \mathbf{C}$ is an object $Y \in \mathbf{C}$ with a pair of maps

$$X \xrightarrow{p} Y \xleftarrow{i} X$$

satisfying $p \circ i = \text{id}_Y$.

Definition 1.1.1.3. A *direct sum* of two objects $X, Y \in \mathbf{C}$ is an object Z equipped with four maps

$$X \begin{array}{c} \xleftarrow{p_X} \\ \xrightarrow{i_X} \end{array} Z \begin{array}{c} \xleftarrow{p_Y} \\ \xrightarrow{i_Y} \end{array} Y$$

exhibiting X and Y as summands of Z which are further orthogonal and exhaust Z :

$$p_Y \circ i_X = 0 = p_X \circ i_Y \qquad i_X \circ p_X + i_Y \circ p_Y = \text{id}_Z.$$

Direct sums are unique up to unique isomorphism, so it makes sense to speak of “the” direct sum, denoted by $X \oplus Y$. Direct sums are simultaneously limits and colimits. Their universal property may be written entirely as a set of equations. This makes them an example of an *absolute limit*: a limit preserved by any functor.

Definition 1.1.1.4. An object $X \in \mathbf{C}$ is *simple*, if every monomorphism $Y \hookrightarrow X$ is either an isomorphism or a zero morphism.

The inclusion of a summand is always a monomorphism, so this implies summands of a simple morphism are isomorphic to it.

EXAMPLE 1.1.1.5. A vector space $V \in \mathbf{Vec}^\infty$ is simple iff it is 1-dimensional.

EXAMPLE 1.1.1.6. Let X be a set. We denote by $\mathbf{Vec}[X]$ the linear category of X -graded vector spaces. Its objects are finite direct sums of objects of the form V_x , where $V \in \mathbf{Vec}$ is a finite-dimensional vector space and $x \in X$. The morphism spaces in $\mathbf{Vec}[X]$ are generated under direct sum by $\text{Hom}(V_x, W_x) = \text{Hom}_{\mathbf{Vec}}(V, W)$ and $\text{Hom}(V_x, W_{x'}) = 0$ when $x \neq x'$. The simple objects of $\mathbf{Vec}[X]$ are those of the form V_x where V is 1-dimensional.

EXAMPLE 1.1.1.7. Let $\mathbf{Rep}(G)$ be the category of finite-dimensional complex representations of a group G . The simple objects of $\mathbf{Rep}(G)$ are the irreducible representations.

The classical Schur’s Lemma from representation theory admits a generalisation to linear categories:

Lemma 1.1.1.8 (Schur’s Lemma). *For any pair of simple objects $X, Y \in \mathbf{C}$ the morphism space $\text{Hom}(X, Y)$ is either 0 or 1-dimensional (in which case $X \cong Y$).*

In particular, the endomorphism ring of a simple object $X \in \mathbf{C}$ may be identified with the complex numbers $\text{End}(X) = \mathbb{C}$.

Definition 1.1.1.9. A linear category \mathbf{C} is *semisimple*, if every object is a finite direct sum of simple objects, and every such finite direct sum exists.

This condition implies, in particular, that all idempotents

$$e : X \rightarrow X \text{ s.t. } e^2 = e$$

admit a splitting: a direct summand $Y \subset X$ whose structure maps

$$X \xrightarrow{p} Y \xleftarrow{i} X$$

compose to $i \circ p = e$.

The category \mathbf{Vec} is semisimple, and so is $\mathbf{Vec}[X]$. Semisimplicity of $\mathbf{Rep}(G)$ depends on the group: when G is finite, $\mathbf{Rep}(G)$ is automatically semisimple.

Definition 1.1.1.10. A *tensor category* is a linear category \mathbf{C} , equipped with a choice of *unit object* $\mathbb{1} \in \mathbf{C}$, a *tensor product* functor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, an *associator* natural isomorphism

$$\alpha : - \otimes (- \otimes -) \Rightarrow - \otimes (- \otimes -),$$

and *unitors* natural isomorphisms

$$\begin{aligned} l : \mathbb{1} \otimes - &\Rightarrow - \\ r : - \otimes \mathbb{1} &\Rightarrow -. \end{aligned}$$

These data are subject to the *triangle equations*

$$r_X \otimes \text{id}_Y = (\text{id}_X \otimes l_Y) \circ \alpha_{X, \mathbb{1}, Y}$$

for all $X, Y \in \mathbf{C}$ and the *pentagon equations*

$$(\text{id}_W \otimes \alpha_{X, Y, Z}) \circ \alpha_{W, X \otimes Y, Z} \circ (\alpha_{W, X, Y} \otimes \text{id}_Z) = \alpha_{W, X, Y \otimes Z} \circ \alpha_{W \otimes X, Y, Z},$$

for all $W, X, Y, Z \in \mathbf{C}$.

As explained in [EGNO16, Rmk 2.2.9], the unitors present in Definition 1.1.1.10 may be suppressed: the data $(\mathbf{C}, \mathbb{1}, \otimes, \alpha, l, r)$ is equivalent to the data $(\mathbf{C}, \mathbb{1}, \otimes, \alpha)$, together with an explicit choice of isomorphism $\mathbb{1} \otimes \mathbb{1} \rightarrow \mathbb{1}$. We will usually suppress unitors in this thesis.

Definition 1.1.1.11. A *tensor functor* of tensor categories is a linear functor $F : \mathbf{C} \rightarrow \mathbf{D}$, equipped with natural transformations $\mu : (F-) \otimes (F-) \xrightarrow{\cong} F(- \otimes -)$ and $\varepsilon : F\mathbb{1}_{\mathbf{C}} \rightarrow \mathbb{1}_{\mathbf{D}}$ satisfying associativity and unitality conditions.

EXAMPLE 1.1.1.12. The categories \mathbf{Vec} and \mathbf{Vec}^∞ both become tensor categories when equipped with the usual tensor product of vector spaces. The unitors and associators are the canonical identity morphisms.¹

EXAMPLE 1.1.1.13. Recall the linear category $\mathbf{Vec}[X]$ from Example 1.1.1.6. Equip the set X with a unital, associative multiplication \cdot . Then $\mathbf{Vec}[X]$ inherits a tensor structure given by the convolution product

$$V_x \otimes W_{x'} = (V \otimes W)_{x \cdot x'},$$

with associators and unitors induced by those on \mathbf{Vec} .

EXAMPLE 1.1.1.14. One may ask what other solutions to the pentagon equation there are for $\mathbf{Vec}[X]$ equipped with the convolution product. It is straightforward to check that the pentagon equation boils down to the equation of a 3-cocycle in the monoid cohomology of X with coefficients in \mathbb{C}^\times . This is the cohomology of the classifying space $\mathcal{B}X$ of X . Given such a 3-cocycle $\omega \in Z^3(\mathcal{B}X, \mathbb{C}^\times)$, we denote the resulting tensor category by $\mathbf{Vec}^\omega[X]$. It has the same structure as $\mathbf{Vec}[X]$, with the exception that the associator has components additively extended from

$$\alpha_{V_x, V'_y, V''_z} = \omega(x, y, z) \cdot \alpha_{V, V', V''}^{\mathbf{Vec}}.$$

It is straightforward to check that a tensor functor $\mathbf{Vec}^\omega[X] \rightarrow \mathbf{Vec}^{\omega'}[X]$ whose underlying linear functor is the identity $\mathbf{Vec}[X] \rightarrow \mathbf{Vec}[X]$ is equivalent to the data of a 2-cocycle $\mu \in Z^2(\mathcal{B}X, \mathbb{C}^\times)$ whose boundary is $d\mu = \omega - \omega'$. It is in this sense that tensor structures on $\mathbf{Vec}[X]$ are classified by the cohomology group $H^3(\mathcal{B}X, \mathbb{C}^\times)$.

EXAMPLE 1.1.1.15. Let M be a manifold. Then the linear category $\mathbf{Vect}(M)$ of vector bundles over M admits a tensor structure given by the (pointwise) tensor product of vector bundles.

EXAMPLE 1.1.1.16. The category $\mathbf{Rep}(G)$ of representations of a group G is a tensor category under the pointwise tensor product of representations.

Definition 1.1.1.17. Let $X \in \mathbf{C}$ be an object in a linear category. A *left dual* for X is an object X^\vee , such that there exist evaluation and coevaluation morphisms

$$\mathrm{ev} : X^\vee \otimes X \rightarrow \mathbb{1} \qquad \mathrm{coev} : \mathbb{1} \rightarrow X \otimes X^\vee,$$

satisfying the *snake equations*: The composites

$$\begin{aligned} X &\xrightarrow{\mathrm{coev} \otimes \mathrm{id}_X} (X \otimes X^\vee) \otimes X \xrightarrow{\alpha_{X, X^\vee, X}} X \otimes (X^\vee \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}} X \\ X^\vee &\xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{coev}} X^\vee \otimes (X \otimes X^\vee) \xrightarrow{\alpha_{X^\vee, X, X^\vee}^{-1}} (X^\vee \otimes X) \otimes X^\vee \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_{X^\vee}} X^\vee \end{aligned}$$

must be equal to the respective identity morphisms. In that case, X is called a *right dual* of X^\vee . An object that admits both a right and left dual is called *dualisable*. A tensor category \mathbf{C} is *autonomous* if every object $X \in \mathbf{C}$ is dualisable.

¹In fact, this monoidal structure on \mathbf{Vec}^∞ (considered as an ordinary category) is necessary to even define the notion of \mathbf{Vec}^∞ -enriched category. It is straightforward to check that this monoidal structure descends to the \mathbf{Vec}^∞ -enriched context.

Definition 1.1.1.18. An *invertible* object in a monoidal category is an object X , such that there exists an object X^{-1} with isomorphisms $X^{-1} \otimes X \cong \mathbb{1} \cong X \otimes X^{-1}$.

We call X^{-1} the inverse of X . It is unique up to isomorphism. It is straightforward to check that it may be equipped with duality data which identify it as both the left and right dual of X . Being invertible is a property, and the invertible objects of a tensor category form a subset of its dualisable objects.

EXAMPLE 1.1.1.19. In \mathbf{Vec}^∞ , duality data reduces to the usual notion of duality data familiar from linear algebra. The dualisable objects in \mathbf{Vec}^∞ are precisely the finite-dimensional vector spaces, making $\mathbf{Vec} \subset \mathbf{Vec}^\infty$ the maximal autonomous subcategory of \mathbf{Vec}^∞ . A vector space is invertible precisely when it is 1-dimensional.

EXAMPLE 1.1.1.20. The dual of a finite-dimensional G -representation $(V, \rho) \in \mathbf{Rep}(G)$ is provided by its contragredient representation $(V^\vee, \rho^\vee : g \mapsto \rho(g^{-1})^\vee)$.

EXAMPLE 1.1.1.21. Consider the tensor category $\mathbf{Vec}^\omega[X]$ of X -graded vector spaces with twisted associator (Example 1.1.1.14). An object of the form V_x admits a left dual precisely if x admits a left inverse x^l . A dual is then given by $V_{x^l}^\vee$. Let $X^\sim \subset X$ be the maximal subgroup of X , consisting of those elements $x \in X$ that admit a left and right inverse. Then $\mathbf{Vec}^\omega[X^\sim] \subset \mathbf{Vec}[X]$ is the maximal autonomous subcategory. An object $V_x \in \mathbf{Vec}^\omega[X]$ is invertible when $x \in X^\sim$ is invertible and V is 1-dimensional. All invertible objects of $\mathbf{Vec}^\omega[X]$ are of this form.

Definition 1.1.1.22. A *fusion category* is a semisimple autonomous tensor category, with finitely many isomorphism classes of simple objects, and simple monoidal unit $\mathbb{1}$.

EXAMPLE 1.1.1.23. The category $\mathbf{Rep}(G)$ of representations of a group G is fusion iff it is semisimple and there are only finitely many irreducible representations up to isomorphism. This is always the case if G is finite.

EXAMPLE 1.1.1.24. As explained in Example 1.1.1.21, the semisimple category $\mathbf{Vec}^\omega[X]$ is autonomous iff X is a group. Its isomorphism classes of simple objects are in correspondence with elements of X . Let H be a finite group and $\omega \in Z^3(\mathcal{B}H, \mathbb{C}^\times)$ a 3-cocycle. Then $\mathbf{Vec}^\omega[H]$ is a fusion category.

As explained in Example 1.1.1.14, tensor structures on $\mathbf{Vec}[H]$ which lift the product on G are classified up to equivalence by $H^3(\mathcal{B}H, \mathbb{C}^\times)$. The category $\mathbf{Vec}^\omega[H]$ has the special property that all of its simple objects are invertible. A fusion category with this property is called *pointed*. Every pointed fusion category is of the form $\mathbf{Vec}^\omega[H]$.

EXAMPLE 1.1.1.25. The smallest example of a non-pointed fusion category is given by the *Fibonacci categories* \mathbf{Fib}^\pm . They have two simple objects $\mathbb{1}$ and X , with tensor product given by

$$X \otimes X = \mathbb{1} \oplus X.$$

There are two non-equivalent solutions to the pentagon equation [Ost03b; MS89], both featuring the golden ratio ϕ .

Tambara and Yamagami obtained in [TY98] a complete classifications of fusion categories with a single non-invertible object. These are the Tambara-Yamagami categories.

EXAMPLE 1.1.1.26. The *Tambara-Yamagami category* associated to a finite abelian group A , a bicharacter $\chi : A \otimes A \rightarrow \mathbb{C}^\times$, and a choice of square root $\tau = \pm\sqrt{|A|^{-1}}$ has set of simple objects $A \cup \{m\}$. Let $a, b, c \in A$. The tensor product in $\mathbf{TY}(A, \chi, \tau)$ is given by

$$a \otimes b = a + b \qquad a \otimes m = m \otimes a = m \qquad m \otimes m = \bigoplus_{a \in A} a$$

and the non-trivial components of the associator take the values

$$\begin{aligned} \alpha_{a,m,b} &= \chi(a, b) \text{id}_m : m \rightarrow m \\ \alpha_{m,a,m} &= \bigoplus_{b,c \in A} \chi(a, b) \delta_{b,c} \text{id}_b : \bigoplus_{b \in A} b \rightarrow \bigoplus_{c \in A} c \\ \alpha_{m,m,m} &= \bigoplus_{b,c \in A} \tau \chi(b, c)^{-1} \text{id}_m : \bigoplus_{b \in A} m \rightarrow \bigoplus_{c \in A} m. \end{aligned}$$

1.1.2 Braided Tensor Categories and the Drinfel'd Centre

Let $X \in \mathbf{C}$ be an object in a tensor category.

Definition 1.1.2.1. A *half-braiding* on X is a natural isomorphism

$$\gamma : X \otimes - \xrightarrow{\cong} - \otimes X$$

satisfying, for all $Y, Z \in \mathbf{C}$, the hexagon equation

$$\alpha_{Y,Z,X} \circ \gamma_{Y \otimes Z} \circ \alpha_{X,Y,Z} = (\text{id}_Y \otimes \gamma_Z) \circ \alpha_{Y,X,Z} \circ (\gamma_Y \otimes \text{id}_Z).$$

The hexagon equation says that γ is multiplicative (ie. compatible with the tensor product), and has the consequence that there is a unique natural morphism $(X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$.

Definition 1.1.2.2. A *braiding* on a tensor category \mathbf{C} is a natural isomorphism

$$\beta : - \otimes \cdot \xrightarrow{\cong} \cdot \otimes -,$$

such that for all $X \in \mathbf{C}$, $\beta_{X,-} : X \otimes - \xrightarrow{\cong} - \otimes X$ is a half-braiding for $X \in \mathbf{C}$ and $\beta_{-,X} : - \otimes X \xrightarrow{\cong} X \otimes -$ is a half-braiding for X in \mathbf{C}^{op} , the category equipped with the opposite monoidal structure. A tensor category equipped with a braiding is called a *braided tensor category*.

A braiding is *symmetric* if $\beta_{b,a} \circ \beta_{a,b} = \text{id}_{a \otimes b}$ for all $a, b \in \mathbf{C}$. A tensor category equipped with a symmetric braiding is called a *symmetric tensor category*.

A braided functor of braided categories is a tensor functor whose structure natural transformations are compatible with the braiding. Ie. it is a property for a tensor functor to be braided.

EXAMPLE 1.1.2.3. The usual swap isomorphism of vector spaces $V \otimes W \rightarrow W \otimes V$ equips \mathbf{Vec} with a braiding. This braiding pulls back to a braiding on $\mathbf{Rep}(G)$ via the forgetful functor $\mathbf{Rep}(G) \rightarrow \mathbf{Vec}$.

EXAMPLE 1.1.2.4. The category of (twisted) G -graded vector spaces cannot admit a braiding if G is nonabelian, since the existence of a braiding on a category \mathbf{C} implies $X \otimes Y \cong Y \otimes X$ for all $X, Y \in \mathbf{C}$. Now assume $G = H$ is an abelian group. Braided structures on $\mathbf{Vec}[H]$ that lift the group structure on H turn out to be classified by their associated quadratic form

$$\begin{aligned} q : H &\rightarrow \mathbb{C}^\times \\ h &\mapsto \beta_{\mathbb{C}_h, \mathbb{C}_h}. \end{aligned}$$

This fact is proved in [EM54]. We discuss it in more detail in Chapter 3.

EXAMPLE 1.1.2.5. There are 4 non-equivalent braided tensor structures on the linear category $\mathbf{Vec}[\mathbb{Z}/2]$ that lift the group structure on $\mathbb{Z}/2$. We write $\mathbf{Vec}[\mathbb{Z}/2]$ to denote the standard braided tensor structure, given by the trivial associator and trivial braiding inherited from \mathbf{Vec} . Perhaps better-known is the category \mathbf{sVec} of super vector spaces: it also has trivial associator, but the self-braiding of the simple object \mathbb{C}_{-1} is $\beta_{-1, -1} = -1$.

The remaining two braided tensor structures are the semion \mathbf{Semi} and anti-semion $\overline{\mathbf{Semi}}$. Both of them have twisted associator, with $\alpha_{-1, -1, -1} = -1$. In \mathbf{Semi} , we have $\beta_{-1, -1} = \mathbf{i}$, and in $\overline{\mathbf{Semi}}$, $\beta_{-1, -1} = -\mathbf{i}$.

EXAMPLE 1.1.2.6. Let G be a compact simple connected Lie group, and $k \in Z^4(\mathcal{B}G, \mathbb{Z})$ a \mathbb{Z} -valued 4-cocycle. The loop group $LG = C^\infty(S^1, G)$ has an associated central extension

$$\mathbb{C}^\times \rightarrow \widetilde{LG}^k \rightarrow LG.$$

The representations of this central extension are extensively studied in [PS85]. There is a particularly interesting class of representations, cut out by the condition of having *positive energy*. The subcategory $\mathbf{Rep}^k(LG)$ of such representations is a braided tensor category. These categories play an important role in the study of Conformal and Topological Field Theories, in particular that of Chern-Simons Theory. When G is simply-connected, the category $\mathbf{Rep}^k(LG)$ is braided equivalent to the Quantum group category $\mathbf{C}(\mathfrak{g}, k)$ associated to its Lie algebra \mathfrak{g} and the level k . We describe these in more detail in Chapter 2.

The notion of the centre of a monoid categorifies to the setting of tensor categories.

Definition 1.1.2.7. The *Drinfel'd centre* $\mathcal{Z}\mathbf{C}$ of a tensor category \mathbf{C} has objects pairs (X, γ) where $X \in \mathbf{C}$ and γ is a half-braiding for X .

The centre comes equipped with a tensor functor $\mathcal{Z}\mathbf{C} \rightarrow \mathbf{C}$ which forgets the half-braiding. It also carries a braiding β , with components

$$\beta_{(X, \gamma), (Y, \gamma')} = \gamma_Y.$$

EXAMPLE 1.1.2.8. The Drinfel'd centres $\mathcal{Z}\mathbf{Vec}^\omega[G]$ of pointed fusion categories are computed explicitly in [CGR00; GM18]. Let $G = H$ be abelian and ω trivial. Then the simple objects of $\mathcal{Z}\mathbf{Vec}[H]$ correspond to pairs (h, χ) of an element $h \in H$ and a character $\chi \in \widehat{H}$. The simple object corresponding to (h, χ) is $\mathbb{C}_h \in \mathbf{Vec}[H]$, equipped with the half-braiding $\gamma_{\mathbb{C}_{h'}} = \chi(h')$. The resulting braiding on the centre is again given by pairing characters with the group elements.

The primary objective of Chapter 3 is the computation of the analogue of this centre in the context of smooth 2-groups.

1.1.3 Algebras and Modules

We recall how one may define algebras and modules internal to (braided) tensor categories.

Definition 1.1.3.1. An *algebra* in a tensor category $(\mathbf{C}, \mathbb{1}, \otimes)$ is an object $A \in \mathbf{C}$, together with a unit $\eta : \mathbb{1} \rightarrow A$ and multiplication $\mu : A \otimes A \rightarrow A$, satisfying unitality

$$\mu \circ (\eta \otimes \text{id}_A) = l_A \qquad \mu \circ (\text{id}_A \otimes \eta) = r_A$$

and associativity

$$\mu \circ (\mu \otimes \text{id}_A) = \mu \circ (\text{id}_A \otimes \mu).$$

Definition 1.1.3.2. A *commutative algebra* in a braided tensor category $(\mathbf{C}, \mathbb{1}, \otimes, \beta)$ is an algebra $(A, \mu) \in (\mathbf{C}, \mathbb{1}, \otimes)$, such that

$$\mu \circ \beta_{A,A} = \mu : A \otimes A \rightarrow A.$$

Definition 1.1.3.3. A (right) *module* over an algebra object $A \in \mathbf{C}$ is an object $M \in \mathbf{C}$, together with an action $a : M \otimes A \rightarrow M$, compatible with the multiplication: The two induced morphisms $(M \otimes A) \otimes A \rightarrow M$ given by

$$a \circ (\text{id}_A \otimes \mu_A) = a \circ (a \otimes \text{id}_A) \circ \alpha_{\mathbf{C}}$$

agree.

Right modules and morphisms between them (required to commute with the action maps) assemble into a category $\mathbf{Mod}_A(\mathbf{C})$. Left modules are defined exactly analogously.

When A is commutative, a right module structure on $M \in \mathbf{C}$ induces a left-module structure by precomposing with the braiding. This can be done in different ways. In particular, $a : M \otimes A \rightarrow M$ may be precomposed with either of $\beta_{A,M}, \beta_{M,A}^{-1} : A \otimes M \rightarrow M \otimes A$.

Definition 1.1.3.4. A right module $M \in \mathbf{Mod}_A(\mathbf{C})$ over a commutative algebra A is *local* if the two induced left-module structures agree.

In a symmetric tensor category, $\beta_{A,M} = \beta_{M,A}^{-1}$, and all modules over commutative algebras are automatically local.

1.1.4 Symmetric Monoidal Bicategories

In this section, we briefly recall bicategories, equipped with varying amounts of monoidal structure. The history of the definition of symmetric monoidal bicategories is long: A detailed review is given in [SP09, Sec 2.1]. Given the work of [GPS95] on tricategories, it is possible to define a monoidal bicategory in a single line.

Definition 1.1.4.1. A *monoidal bicategory* is a tricategory with a single object.

Of course, this definition is only helpful if one is already familiar with tricategories. We use Definition 1.1.4.1 as an excuse to reduce the amount of details we spell out below. We try to be precise about the data required, but will only give an idea of the coherence laws this data is required to satisfy. The interested reader is invited to consult the excellent [Lei98] for precise definitions of bicategories and morphisms between them, and [Sta16] for a detailed account of the definitions of monoidal, braided monoidal, sylleptic monoidal and symmetric monoidal bicategories.

Definition 1.1.4.2. A bicategory \mathcal{B} is given by the following data:

- a collection $\text{ob } \mathcal{B}$ of objects
- a category $\text{Hom}_{\mathcal{B}}(a, b)$ of morphisms for each pair $a, b \in \text{ob } \mathcal{B}$
- composition functors $\circ_{a,b,c} : \text{Hom}_{\mathcal{B}}(b, c) \times \text{Hom}_{\mathcal{B}}(a, b) \rightarrow \text{Hom}_{\mathcal{B}}(a, c)$
- identities $\text{id}_a \in \text{Hom}_{\mathcal{B}}(a, a)$.
- associator and unitor natural transformations

The associators identify the two different ways of composing three morphisms, and unitors trivialise composing with identity morphism. This data is subject to generalisations of the pentagon and triangle equations recalled in Section 1.1.1.

The objects of $\text{Hom}_{\mathcal{B}}(a, b)$ in the definition above are called 1-morphisms (or *1-cells*), and the morphisms between them in $\text{Hom}_{\mathcal{B}}(a, b)$ are the 2-morphisms (or *2-cells*) of \mathcal{B} .

A bicategory is called a *2-category* if its associator and unitor natural transformations are trivial. This means composition is associative and unital on the nose.

EXAMPLE 1.1.4.3. Every 1-category \mathbf{C} defines a bicategory whose objects and 1-morphisms are those of \mathbf{C} , and all of whose 2-morphisms, associators and unitors are trivial.

EXAMPLE 1.1.4.4. Let \mathbf{V} be a monoidal category. Then \mathbf{V} -enriched categories assemble into a bicategory \mathbf{VCat} , whose 1-cells are \mathbf{V} -functors and 2-cells are \mathbf{V} -natural transformations.

Definition 1.1.4.5. A functor of bicategories $\mathcal{A} \rightarrow \mathcal{B}$ is given by

- a map on objects $F : \text{ob } \mathcal{A} \rightarrow \text{ob } \mathcal{B}$
- functors on morphism categories $F_{a,b} : \text{Hom}_{\mathcal{A}}(a, b) \rightarrow \text{Hom}_{\mathcal{B}}(F(a), F(b))$
- compositor and identitor natural isomorphisms.

The compositor natural transformations implement compatibility of the functor with the composition maps in \mathcal{A} and \mathcal{B} . The identitors identify the image of identity 1-morphisms in \mathcal{A} with identity 1-morphisms in \mathcal{B} . The coherences this data has to satisfy are compatibility conditions of the compositors and identitors with the associator 2-cells and unitor 2-cells of \mathcal{A} and \mathcal{B} .

Definition 1.1.4.6. A transformation of functors of bicategories $t : F \rightarrow G$ between $F, G : \mathcal{A} \rightarrow \mathcal{B}$ consists of the data of

- a 1-cell $t(A) : F(A) \rightarrow G(A)$ for each object $A \in \mathcal{A}$
- natural transformations implementing lax naturality of $t(A)$

The natural transformations are 2-cells $t(f) : G(f) \circ t(A) \rightarrow t(B) \circ F(f)$ for each 1-cell $f : A \rightarrow A'$ in \mathcal{A} . The coherences again consist of compatibility conditions with compositors and unitors.

Definition 1.1.4.7. A *modification* of two transformations $m : s \rightarrow t$ consists of

- a 2-cell $m(A) : s(A) \rightarrow t(A)$ for each $A \in \mathcal{A}$,

subject to compatibility with the 2-cells $s(f), t(f)$ for all 1-cells f in \mathcal{A} .

We now turn to the definitions of monoidal structures on bicategories. As is to be expected, the coherence conditions involved grow bigger and more numerous. We will not attempt to describe them – an excellent geometric exposition is provided in [Sta16, Ch 4]. For a complete definition of monoidal bicategories, braided monoidal bicategories, sylleptic monoidal bicategories and symmetric monoidal bicategories all in one, see [SP09, Def 2.3].

Definition 1.1.4.8. A *monoidal bicategory* is given by a bicategory \mathcal{B} equipped with

1. a unit object $1 \in \mathcal{B}$
2. a product functor²

$$\boxtimes : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$$

3. invertible associator and unitor transformations

$$\begin{aligned} (- \boxtimes -) \boxtimes - &\xrightarrow{\cong} - \boxtimes (- \boxtimes -) \\ - \boxtimes 1 &\xrightarrow{\cong} \text{Id}_{\mathcal{B}} \\ 1 \boxtimes - &\xrightarrow{\cong} \text{Id}_{\mathcal{B}} \end{aligned}$$

4. invertible modifications filling the pentagon and triangle coherences (these take the form of equations in Definition 1.1.1.10), as well as an invertible modification implementing compatibility of the left and right unitor

The coherence theorems in [GPS95] allow us to work with less structure. Every monoidal bicategory is equivalent to a *monoidal 2-category*. That is a monoidal bicategory whose underlying bicategory is a 2-category. We follow [McC00] and work in this generality.

Definition 1.1.4.9. A *braided monoidal bicategory* is a monoidal bicategory equipped with

1. an invertible braiding transformation $b : \boxtimes \xrightarrow{\cong} \boxtimes \circ \text{swap}$, where $\text{swap} : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$ denotes the usual swap functor

²The product $\mathcal{A} \times \mathcal{B}$ of two bicategories is given by taking products of all data.

2. invertible modifications filling the two hexagon coherences (which are equations in Definition 1.1.2.2).

Definition 1.1.4.10. A *syllleptic monoidal bicategory* is a braided monoidal bicategory equipped with invertible modifications $\sigma_{x,y} : \text{Id}_{x \otimes y} \rightarrow b_{y,x} \otimes b_{x,y}$ for all $x, y \in \mathcal{B}$.

In a syllleptic monoidal bicategory, there are two modifications $b \xrightarrow{\cong} b \circ b \circ b$ corresponding to the two bracketings of the right hand side. (We neglect the compositor structure and identify both bracketings with $b \circ b \circ b$.)

Definition 1.1.4.11. A *symmetric monoidal bicategory* is a syllleptic monoidal bicategory in which the two canonical modifications $b \xrightarrow{\cong} b \circ b \circ b$ agree.

1.1.5 Monoids in Monoidal Bicategories

The notion of a monoid internal to a monoidal bicategory was introduced in [SD97] in the context Gray monoids. The slightly weaker framework of monoidal 2-categories is chosen in [McC00]. Definitions in the most general context of monoidal bicategories may be extracted from the definitions of 2-groups and braided 2-groups in [SP11, Sec 3] by removing the invertibility condition. All three of these contexts are equivalent by the coherence theorems of [GPS95].

NOTATION 1.1.5.1. We denote by *monoid* the notion more commonly referred to by the name *pseudomonoid* in the literature.

The bicategory with one object, one 1-morphism and one 2-morphism is a symmetric monoidal bicategory in a canonical way. By forgetting structure, it is also canonically a syllleptic, braided or simply a monoidal bicategory. Denote this bicategory by $\mathbf{1}$. As explained in [McC00, Sec 2-4], one may define

- monoids in a monoidal bicategory
- braided monoids in a braided monoidal bicategory
- symmetric monoids in a syllleptic monoidal bicategory

as monoidal/braided monoidal/syllleptic monoidal morphisms

$$\mathbf{1} \rightarrow \mathcal{B}.$$

Here, \mathcal{B} is monoidal/braided monoidal/syllleptic monoidal, and $\mathbf{1}$ is equipped with its canonical monoidal/braided monoidal/syllleptic monoidal structure. We flesh out the resulting structure below.

Definition 1.1.5.2. A *monoid* in a monoidal bicategory $(\mathcal{B}, \boxtimes, 1)$ is an object $C \in \mathcal{B}$, equipped with

1. a unit $\mathbb{1} : 1 \rightarrow C$,
2. a tensor morphism $\otimes : C \boxtimes C \rightarrow C$,

3. invertible associator and unitor 2-cells

$$\begin{aligned}\alpha &: (- \otimes -) \otimes - \xrightarrow{\cong} - \otimes (- \otimes -) \\ l &: (\mathbb{1} \otimes -) \xrightarrow{\cong} \text{Id}_{\mathbf{C}} \\ r &: (- \otimes \mathbb{1}) \xrightarrow{\cong} \text{Id}_{\mathbf{C}},\end{aligned}$$

satisfying the pentagon equation and triangle equation from Definition 1.1.1.10 (phrased internal to \mathcal{B}).

Definition 1.1.5.3. A *braided monoid* in a braided monoidal bicategory $(\mathcal{B}, \boxtimes, 1, b)$ is a monoid $(\mathbf{C}, \otimes, \mathbb{1})$ in \mathcal{B} , equipped with an invertible braiding 2-cell

$$\beta : \otimes \xrightarrow{\cong} \otimes \circ b.$$

satisfying the hexagon equation from Definition 1.1.2.2 (phrased internal to \mathcal{B}).

The 2-cell β above induces a 2-cell

$$\boxtimes \xrightarrow{\beta} \boxtimes \circ b \xrightarrow{\beta \circ b} \boxtimes \circ b \circ b.$$

If \mathcal{B} is equipped with a sylleptic monoidal structure, the syllepsis 2-cell $\sigma : \text{Id} \rightarrow b \circ b$ induces another 2-morphism $\boxtimes \circ \sigma : \boxtimes \rightarrow \boxtimes \circ b \circ b$.

Definition 1.1.5.4. A *symmetric monoid* in a sylleptic monoidal bicategory $(\mathcal{B}, \boxtimes, 1, b, \sigma)$ is a braided monoid $(\mathbf{C}, \otimes, \mathbb{1}, \beta)$ in \mathcal{B} , such that the two canonical 2-cells $\boxtimes \rightarrow \boxtimes \circ b \circ b$ agree.

EXAMPLE 1.1.5.5. The archetypical example is provided by the bicategory $\mathbf{V}\text{-Cat}$ of \mathbf{V} -enriched 1-categories. (Here, \mathbf{V} is some symmetric monoidal 1-category.) $\mathbf{V}\text{-Cat}$ is symmetric monoidal under Kelly's product of enriched categories [KK82]. Monoids therein are \mathbf{V} -enriched monoidal categories, braided monoids are \mathbf{V} -enriched braided monoidal categories, and symmetric monoids are \mathbf{V} -enriched symmetric monoidal categories. In particular, for $\mathbf{V} = \mathbf{Vec}^\infty$, we recover Definition 1.1.1.10 of a tensor category.

In analogy with the lower dimensional situation, we may ask for modules over monoid objects. Let $(\mathbf{C}, \otimes, \mathbb{1})$ be a monoid in a monoidal bicategory $(\mathcal{B}, \boxtimes, 1)$.

Definition 1.1.5.6. A (right) *module* over \mathbf{C} is an object $\mathbf{M} \in \mathcal{B}$, equipped with

- an action map

$$\odot : \mathbf{M} \boxtimes \mathbf{C} \rightarrow \mathbf{M}$$

- an invertible compatibility 2-cell

$$s : (- \odot -) \odot - \xrightarrow{\cong} - \odot (- \otimes -),$$

satisfying a coherence relating s and the associator 2-cell of \mathbf{C} , as well as a unitality condition.

EXAMPLE 1.1.5.7. In the monoidal bicategory $\mathbf{V}\text{-Cat}$, modules over monoids recover the usual notion of module categories over monoidal categories. See [Ost03a] for a discussion of the case $\mathbf{V} = \mathbf{Vec}^\infty$.

1.2 Sheaves and Stacks

Associated to a subset of a manifold³ $U \subset M$ is its ring of functions $C^\infty(U)$. There is a restriction map $C^\infty(U) \rightarrow C^\infty(V)$ associated to any inclusion $V \hookrightarrow U$. Given a cover $\coprod U_i \rightarrow U$ and functions $f_i \in C^\infty(U_i)$ which are compatible on overlaps $U_i \cap U_j$, there is a unique function $f \in C^\infty(U)$ that restricts to the f_i . In other words, the assignment

$$U \mapsto C^\infty(U)$$

satisfies the *descent* or *gluing* condition. Such an assignment is called a *sheaf*.

Stepping up the categorical ladder, we may instead consider the assignment

$$U \mapsto \mathbf{Vect}(U),$$

sending a subset $U \subset M$ to the category of vector bundles over it. There are restriction functors $\mathbf{Vect}(U) \rightarrow \mathbf{Vect}(V)$ associated to $V \hookrightarrow U$. This assignment also satisfies a descent condition over a cover $\coprod U_i \rightarrow U$: given vector bundles $P_i \in \mathbf{Vect}(U_i)$ and isomorphisms $\phi_{ij} : P_i|_{U_{ij}} \xrightarrow{\cong} P_j|_{U_{ij}}$ which are compatible on triple overlaps U_{ijk} , there is a vector bundle $P \in \mathbf{Vect}(U)$, unique up to unique isomorphism, that glues this data. This makes \mathbf{Vect} an example of a *2-sheaf* or *stack*.

Below, we briefly recall some notions from the theory of sheaves and stacks relevant to this thesis. Standard references for this topic are [SGA4; Ten75; KS05; MM12; Moe02; Joh02; Gir66]. We recommend [Vis04; Met03] as thorough introductions.

1.2.1 Sheaves

Recall that a *site* is a category \mathbf{C} equipped with a *Grothendieck topology*: for each object $U \in \mathbf{C}$, a collection of covers $\{f_i : U_i \rightarrow U\}_{i \in I}$. We assume that all fibre products $U_{ij \dots k} := U_i \times_U U_j \times_U \dots \times_U U_k$ exist for each cover. We use suggestive indexing for the projections $p_{ij \dots k} : U_{i'j' \dots k'} \rightarrow U_{ij \dots k}$ when the domain is clear from context. Subcategories and overcategories of a site inherit Grothendieck topologies obtained by restricting along the forgetful functors. A full subcategory \mathbf{D} of a site \mathbf{C} is *dense* if each object in \mathbf{C} admits a cover by objects in \mathbf{D} . Recall that we use \mathbf{Vec}^∞ to denote the category of vector spaces over \mathbb{C} (possibly infinite-dimensional), and $\mathbf{Vec} \subset \mathbf{Vec}^\infty$ to denote the full subcategory of finite-dimensional vector spaces.

EXAMPLE 1.2.1.1. We equip the category \mathbf{Man} of smooth, paracompact manifolds with the structure of a site by specifying covers to be surjective submersions. It has a dense subsite \mathbf{CartSp} , the full subcategory on Cartesian spaces \mathbb{R}^n .

Definition 1.2.1.2. A *sheaf* over a site \mathbf{C} (valued in vector spaces) is a functor $\mathcal{F} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Vec}^\infty$ that satisfies *descent*: For any cover $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$, the map

$$\mathcal{F}(U) \rightarrow \mathcal{F}(\mathcal{U}) := \lim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}) \right)$$

is an isomorphism. We denote the category of sheaves by $\mathbf{Shv}(\mathbf{C}) \subset \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Vec}^\infty)$.

³We will always talk about C^∞ -smooth paracompact manifolds in this thesis.

An element $s \in \mathcal{F}(U)$ is called a *section of \mathcal{F} over U* . Given a cover $\mathcal{U} = \{f_i : U_i \rightarrow U\}$, we can restrict s to the cover via the *restriction maps* $f_i^* := \mathcal{F}(f_i)$. The resulting sections $f_i^*s \in \mathcal{F}(U_i)$ will agree upon further restriction to the overlaps U_{ij} . The descent condition requires that given sections $s_i \in \mathcal{F}(U_i)$ which restrict to equal sections in $\mathcal{F}(U_{ij})$, there is a *unique* section over U which restricts to the s_i . We refer to a compatible choice of sections over a cover as a *descent datum*. The descent condition can now be stated succinctly: *Descent data for a sheaf can be glued uniquely*.

Lemma 1.2.1.3 (Comparison Lemma). *The restriction functor $\mathbf{Shv}(\mathbf{C}) \rightarrow \mathbf{Shv}(\mathbf{D})$ induced by the inclusion of a dense subsite $\mathbf{D} \hookrightarrow \mathbf{C}$ is an equivalence of categories.*

EXAMPLE 1.2.1.4. A sheaf over \mathbf{Man} is completely determined by its restriction to \mathbf{CartSp} .

The category $\mathbf{Shv}(\mathbf{C})$ directly inherits the additive structure of \mathbf{Vec}^∞ . The inclusion of the full subcategory $\iota : \mathbf{Shv}(\mathbf{C}) \hookrightarrow \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Vec}^\infty)$ admits a left adjoint, the *sheafification functor*

$$(-)^\# : \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Vec}^\infty) \rightarrow \mathbf{Shv}(\mathbf{C}).$$

The counit of this adjunction is a natural isomorphism $(\iota-)^{\#} \xrightarrow{\cong} \text{Id}$, which allows us to transport the monoidal structure on \mathbf{Vec}^∞ across to $\mathbf{Shv}(\mathbf{C})$. For $\mathcal{F}, \mathcal{G} \in \mathbf{Shv}(\mathbf{C})$,

$$\mathcal{F} \otimes \mathcal{G} := (\iota\mathcal{F} \otimes \iota\mathcal{G})^\#.$$

The functors above are additive and so $\mathbf{Shv}(\mathbf{C})$ receives the structure of a symmetric tensor category.

Sheaves on a topological space X are sheaves over the site $\mathbf{Op}(X)$ of open sets of X . By abuse of notation, we denote the associated sheaf category by $\mathbf{Shv}(X)$. A map $f : X \rightarrow Y$ of topological spaces induces an adjoint pair of tensor functors

$$f^{-1} : \mathbf{Shv}(Y) \rightleftarrows \mathbf{Shv}(X) : f_*.$$

The *pushforward* is given by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)),$$

and the *pullback* $f^{-1}\mathcal{F}$ can be computed as the sheafification of the functor

$$\text{pre}f^{-1}\mathcal{F} : U \mapsto \text{colim}_{V \supset f(U)} \mathcal{F}(V).$$

The *stalk* of a sheaf $\mathcal{F} \in \mathbf{Shv}(M)$ at a point $x : * \rightarrow M$ is the pullback $x^{-1}\mathcal{F} \in \mathbf{Shv}(*) = \mathbf{Vec}^\infty$. The *support* of \mathcal{F} is the subset of points of M with nonzero stalk.

A *ringed site* is a site together with an algebra object in its category of sheaves, the *structure ring* or *structure sheaf* \mathcal{O} . When working with a ringed site $(\mathbf{C}, \mathcal{O})$, we will use $\mathbf{Shv}_{\mathcal{O}}(\mathbf{C})$ to refer to the category of \mathcal{O} -modules.

Let M be a manifold. The sheaf of smooth functions $C_M^\infty : \mathbf{Op}(M)^{\text{op}} \rightarrow \mathbf{Vec}^\infty$ equips $\mathbf{Op}(M)$ with the structure of a ringed site. A smooth map $f : M \rightarrow N$ of manifolds induces a morphism of structure sheaves $f^\# : f^{-1}C_N^\infty \rightarrow C_M^\infty$, and the adjoint functor pair can be upgraded to an adjunction of C^∞ -modules. The pushforward of sheaves automatically descends to a functor of module categories $f_* : \mathbf{Shv}_{C_M^\infty}(M) \rightarrow \mathbf{Shv}_{C_N^\infty}(N)$ (and we indeed denote it by the same symbol). Its left adjoint is given by

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}C_N^\infty} C_M^\infty.$$

Definition 1.2.1.5. A *vector bundle* on M is a C^∞ -module in $\mathcal{F} \in \mathbf{Shv}_{C^\infty}(M)$, for which there exists an open cover $\{f_i : U_i \rightarrow M\}$ over which \mathcal{F} is free of finite rank: for all i ,

$$f_i^* \mathcal{F} \cong (C_{U_i}^\infty)^{\oplus r_i}$$

for some $r_i \in \mathbb{N}$.

Definition 1.2.1.6. A *skyscraper sheaf* on a manifold M is a C^∞ -module with support a finite subset of M , such that pullbacks over any point are finite-dimensional.

The *category of skyscraper sheaves* on M will play a central role in the definition of manifold categories (Definition 2.4.0.3). The category of skyscraper sheaves on M is equivalent to a direct sum of copies of \mathbf{Vec} , indexed by the underlying set of M . It is in this sense that we view skyscraper sheaves on M as *vector spaces graded by M* . Using the language of stacks, we will also keep track of the smooth structure of M .

1.2.2 Stacks

Let \mathbf{C} be a 1-category, and \mathbf{Cat} the 2-category of categories. We will write $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Cat}$ to denote the maximally weak notion of functor between them: a morphism of underlying bicategories.

Definition 1.2.2.1. A *stack* over a site \mathbf{C} is a functor $\mathcal{F} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Cat}$ satisfying *descent*: The natural functor

$$\mathcal{F}(U) \rightarrow \mathcal{F}(\mathcal{U}) := \lim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}) \rightrightarrows \prod_{i,j,k} \mathcal{F}(U_{ijk}) \right)$$

is an equivalence for each cover $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$ of U .

The 2-limit $\mathcal{F}(\mathcal{U})$ associated to a cover $\mathcal{U} = \{U_i \rightarrow U\}$ can be computed explicitly as the *category of descent data*. It has

- objects: collections of sections $\{X_i \in \mathcal{F}(U_i)\}$, equipped with isomorphisms on overlaps $\{\phi_{ij} : p_i^* X_i \xrightarrow{\cong} p_j^* X_j \in \mathcal{F}(U_{ij})\}$. The isomorphisms are required to be compatible on triple intersections:

$$p_{ij}^* \phi_{ij} \circ p_{jk}^* \phi_{jk} = p_{ik}^* \phi_{ik}$$

in $\mathcal{F}(U_{ijk})$ for all triples i, j, k .

- morphisms $\{\{X_i\}, \{\phi_{ij}\}\} \rightarrow \{\{X'_i\}, \{\phi'_{ij}\}\}$: collections $\{f_i : X_i \rightarrow X'_i\}$, satisfying

$$\phi'_{ij} \circ p_i^* f_i = p_j^* f_j \circ \phi_{ij}$$

in $\mathcal{F}(U_{ij})$ for all pairs i, j .

REMARK 1.2.2.2. The notion of descent datum is visibly a categorification of that for sheaves: Given a cover $\mathcal{U} = \{U_i\}$ of U , and sections $\{X_i \in \mathcal{F}(U_i)\}$ of a stack, we can no longer ask them to restrict to *equal* sections over the overlaps U_{ij} . Rather, a descent datum now further contains isomorphisms of these restricted sections in $\mathcal{F}(U_{ij})$. These isomorphisms are then required to satisfy an equation over triple overlaps.

The bicategory \mathbf{St}/\mathbf{C} of stacks over \mathbf{C} is the full subcategory of $\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Cat})$ on those functors that satisfy descent. The data of a stack is a category $\mathcal{F}(U)$ for each $U \in \mathbf{C}$, a *pullback functor* $f^* := \mathcal{F}(f)$ for each morphism $f \in \mathbf{C}$, as well as invertible natural transformations $\eta_{f,g} : f^* \circ g^* \rightarrow (g \circ f)^*$, $\epsilon_U : \text{id}_U^* \rightarrow \text{Id}_{\mathcal{F}(U)}$ which we leave implicit. This data is required to satisfy appropriate coherence conditions.

Proposition 1.2.2.3. [Gir66] *The inclusion $\mathbf{St}/\mathbf{C} \hookrightarrow \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Cat})$ admits a left 2-adjoint, the stackification functor*

$$(-)^{\#} : \mathbf{St}/\mathbf{C} \rightarrow \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Cat}).$$

The universal property of the adjoint implies there is a morphism $\mathcal{F} \rightarrow \mathcal{F}^{\#}$.

Lemma 1.2.2.4. [SP, Tag 02ZN] *For any section of the stackification $X \in \mathcal{F}^{\#}(U)$, there exists a cover $\mathcal{U} = \{f_i : U_i \rightarrow U\}$ such that all $f_i^* X \in \mathcal{F}^{\#}(U_i)$ are in the essential image of $\mathcal{F}(U_i) \rightarrow \mathcal{F}^{\#}(U_i)$.*

In other words, each section of $\mathcal{F}^{\#}$ still locally decomposes into sections of \mathcal{F} .

EXAMPLE 1.2.2.5. The *classifying stack* $\mathcal{B}G = [*/G]$ of principal G -bundles is a stack on \mathbf{Man} . It assigns to a manifold U the category of G -principal bundles over U . $\mathcal{B}G$ can be obtained as the stackification of the prestack which assigns to U its category of trivial G -principal bundles. In this example, Lemma 1.2.2.4 amounts to the statement that any principal G -bundle is locally trivial.

EXAMPLE 1.2.2.6. The *representable stack* \underline{X} associated to an object $X \in \mathbf{C}$ assigns to $Y \in \mathbf{C}$ the discrete category $\underline{X}(Y) := \text{Hom}(Y, X)$.

The Yoneda Lemma familiar from presheaves carries over to this context.

Lemma 1.2.2.7 (2-Yoneda Lemma). *Let $X \in \mathbf{C}$, $\mathcal{Y} \in \mathbf{St}/\mathbf{C}$, then*

$$\text{Hom}_{\mathbf{St}/\mathbf{C}}(\underline{X}, \mathcal{Y}) \simeq \mathcal{Y}(X).$$

We will hence use the notations $X : U \rightarrow \mathcal{F}$ and $X \in \mathcal{F}(U)$ interchangeably to refer to a section of \mathcal{F} over U .

Chapter 2

Manifold and Orbifold Tensor Categories

In this chapter, we introduce manifold and orbifold tensor categories, formalising the idea of a fusion category with a manifold or orbifold of simple objects. The interplay of smooth and monoidal structures is examined, culminating in the categorification of a theorem of Lie theory: Pointwise dualisability implies dualisability in families. We work over \mathbb{C} throughout.

2.1 Introduction

A fusion category is a linear autonomous monoidal semisimple category with finitely many simple objects (see Section 1.1.1). Up to equivalence, the data of a fusion category consists of a finite set of simple objects $\{X_i\}$ together with the data of a monoidal structure, often encoded as a pair of fusion relations

$$X_i \cdot X_j = \sum_k N_{ij}^k X_k$$

and associators

$$\alpha_{i,j,k} : (X_i \otimes X_j) \otimes X_k \xrightarrow{\cong} X_i \otimes (X_j \otimes X_k).$$

The theory of fusion categories has links to Conformal Field Theory [BDH09; HP17], Topological Quantum Field Theory [KJB10; DSPS20; Hen17b], and subfactor theory of von Neumann algebras [FRS08; CMS11; GS12; JMS14]. A detailed account of the theory is given in [EGNO16].

One may build a tensor category $\text{Vec}^\omega[G]$ associated to any group G and group cocycle $\omega \in Z^3(\mathcal{B}G, \mathbb{C}^\times)$: The category has simple objects $\{\mathbb{C}_g\}$, and monoidal structure linearly extended from

$$\begin{aligned} \mathbb{C}_g \otimes \mathbb{C}_h &= \mathbb{C}_{gh} \\ \alpha_{g,h,k} &= \omega(g, h, k) \in \text{End}(\mathbb{C}_{ghk}) = \mathbb{C}. \end{aligned}$$

The pentagon equation is satisfied precisely because ω is a cocycle. Every object has left and right duals, in particular $(\mathbb{C}_g)^\vee = \mathbb{C}_{g^{-1}}$. We refer to $\mathbf{Vec}^\omega[G]$ as the categorified ω -twisted group ring of G . If G is finite, $\mathbf{Vec}^\omega[G]$ is a fusion category.

If G is a Lie group, $\mathbf{Vec}^\omega[G]$ inherits the smoothness of G . We encode this smooth structure by considering $\mathbf{Vec}^\omega[G]$ as a stack over the site \mathbf{Man} of manifolds. The String 2-group G_ω of Chapter 3 is the maximal smooth sub-2-group¹

$$G_\omega = \mathbf{Vec}^\omega[G]^\times.$$

These categories have a genuine manifold of simple objects: the Lie group G . For $G = T$ a torus, such tensor categories were already considered in [FHLT10], where they played a central role in the study of toral Chern-Simons theory.

The tensor categories introduced above are all group-like: every simple object $X \in \mathbf{C}$ has an inverse Y such that $X \otimes Y \cong Y \otimes X \cong \mathbb{1}$, the unit object of the category. Many interesting fusion categories are not of this kind, for example the representation categories $\mathbf{Rep}(H)$ of finite (non-abelian) groups, the quantum group categories $\mathbf{C}(\mathfrak{g}, k)$ (Example 1.1.2.6) and the Tambara-Yamagami categories $\mathbf{TY}(A, \chi, \tau)$ (Example 1.1.1.26).

Manifold tensor categories are a generalisation of fusion categories that make precise the notion of a *tensor category with a manifold of simple objects*. They should not be viewed as exotic objects. Indeed, already the finite-dimensional (complex) representations of the group \mathbb{Z} form such a category: its simple objects assemble into the manifold \mathbb{C}^\times . The unitary representations form a tensor subcategory with manifold of simple objects S^1 .

We further allow ourselves to talk about *orbifolds of simple objects*. Natural examples may again be found in representation theory. Let Γ be a virtually abelian group, ie. a group fitting into a short exact sequence

$$\mathbb{Z}^n \rightarrow \Gamma \rightarrow H$$

where H is a finite group. The category of (unitary) representations of Γ is an orbifold tensor category. The simple objects of $\mathbf{Rep}^u(\Gamma)$ form the moduli space of points of the orbifold $[T^n/H]$, where $T^n = \widehat{\mathbb{Z}^n}$ is the n -torus of representations of \mathbb{Z}^n . The H -action is induced by the conjugation action of H on Γ . See Figure 2.1 for a depiction of the moduli space of simple objects in a few examples.

The space of simple objects in Figures 2.1b and 2.1c does not itself form an orbifold. In fact, the moduli space of simple objects in these two examples is not even Hausdorff. It is the *moduli space of simple skyscraper sheaves* on the orbifolds $[S^1/\Sigma_2]$ and $[(S^1 \times S^1)/\Sigma_3]$, respectively. Orbifold tensor categories (Definition 2.4.0.2) are defined such that their simple objects form this moduli space. This is what we mean by a tensor category with an orbifold of simple objects.

Any fusion category is an example of a manifold tensor category whose manifold of simple objects is a finite set (Section 2.4.1). The product of two orbifold tensor categories is again an orbifold tensor category, and so a further source of examples are simple products of the form $\mathbf{C} \times \mathbf{Vec}^\omega[G]$, where \mathbf{C} is a fusion category. These product examples are hardly

¹Technically, we treat G_ω as a central extension of G by $[*/U(1)]$. The maximal smooth sub-2-group of $\mathbf{Vec}^\omega[G]$ recovers the corresponding extension of G by $[*/\mathbb{C}^\times]$.

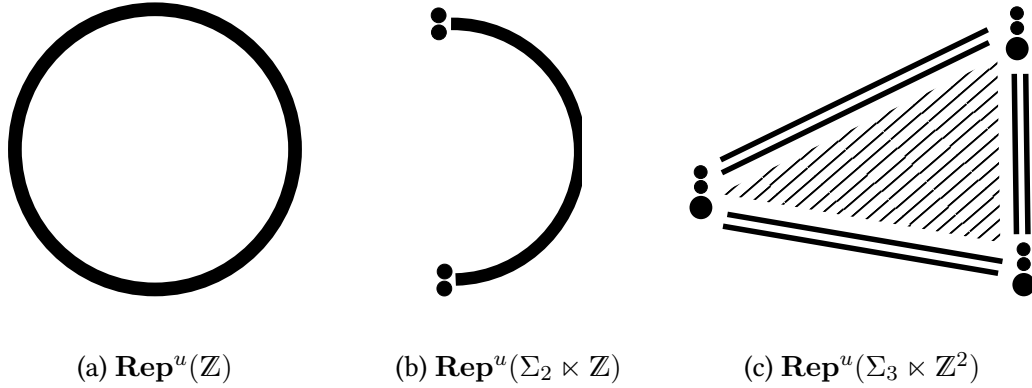


Figure 2.1: Orbifold tensor categories arising in the representation theory of virtually abelian groups.

interesting by themselves, but we leverage them to build further orbifold tensor categories in Sections 2.4.6 and 2.4.7. Recall that there is a Tambara-Yamagami category $\mathbf{TY}(H, \chi, \tau)$ associated to a finite abelian group H , a symmetric non-degenerate bicharacter $\chi : H \otimes H \rightarrow \mathbb{C}^\times$, and a square root τ of $1/|H|$ in \mathbb{C}^\times (see Example 1.1.1.26). It has set of simple objects $H \coprod \{m\}$, and fusion rules ($a, b \in H$):

$$a \otimes b \cong ab \qquad a \otimes m \cong m \cong m \otimes a \qquad m \otimes m \cong \bigoplus_{a \in H} a.$$

The expression for the associator involves χ and τ . We build families of *interpolated Tambara-Yamagami categories* whose manifolds of simples are of the form $G \coprod G/H$, for a Lie group G and central subgroup $H \subset Z(G)$. These categories contain a Tambara-Yamagami category $\mathbf{TY}(H, \chi, \tau)$ as a sub-manifold tensor category (for suitable χ and τ). The fusion rules are direct extensions of the above ($g, g' \in G$):

$$g \otimes g' \cong gg' \qquad g \otimes g'H \cong gg'H \cong gH \otimes g' \qquad gH \otimes g'H \cong \bigoplus_{h \in H} gg'h.$$

We show an example of the resulting manifold of simple objects in Figure 2.2.

In a similar vein, we combine the quantum group categories $\mathbf{C}(\mathfrak{g}, k)$ with smooth categorified group rings $\mathbf{Vec}^\omega[G]$.² We build examples with underlying manifolds of the form $G \coprod^n$ and $(G \coprod^n) \coprod G/H$, for compact connected Lie groups G and central subgroups $H \subset Z(G)$. Figure 2.3 shows two examples of interpolated quantum group categories. An interpolated fusion category $\underline{\mathbf{C}}$ built from $\mathbf{Vec}^\omega[G]$ and a fusion category \mathbf{D} comes with a monoidal functor

$$\mathbf{Vec}^\omega[G] \boxtimes \mathbf{D} \rightarrow \underline{\mathbf{C}},$$

and the monoidal structure on $\underline{\mathbf{C}}$ may be completely read off from this functor.

The construction of the examples above requires a specific choice of associator ω on $\mathbf{Vec}^\omega[G]$. The underlying smooth linear category of $\mathbf{Vec}[G]$ is the moduli stack of

²Recall from Example 1.1.2.6 that $\mathbf{C}(\mathfrak{g}, k)$ is equivalent to $\mathbf{Rep}^k LG$ for the simply-connected Lie group G integrating \mathfrak{g} . In Chapter 2 we denote these categories by $\mathbf{C}(\mathfrak{g}, k)$, considering them as examples of fusion categories. In Chapter 3, we prefer to make the connection to loop groups and denote them by $\mathbf{Rep}^k LG$.

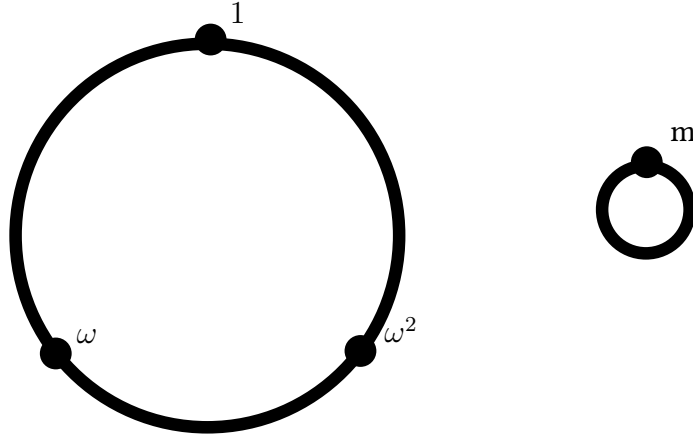
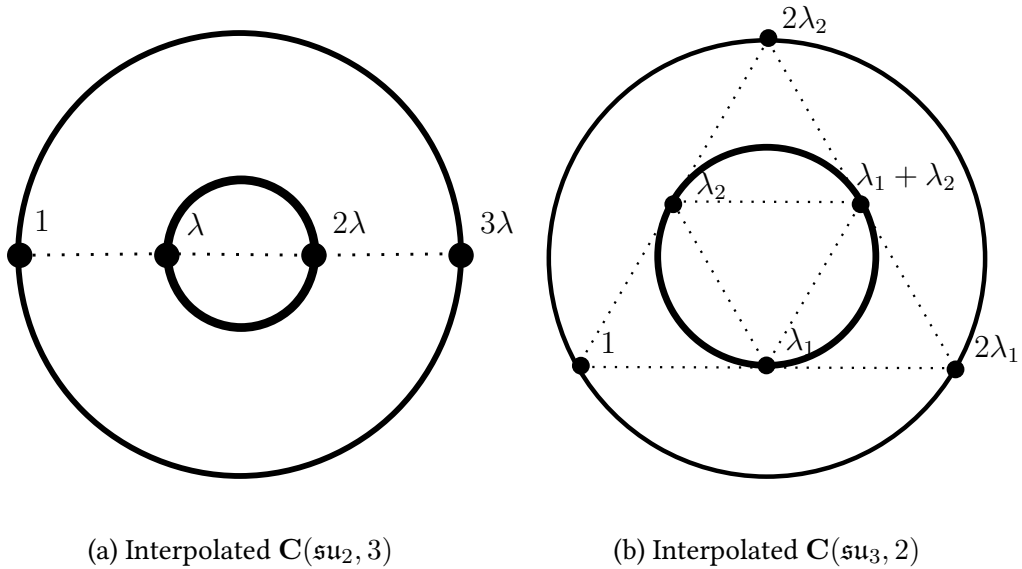


Figure 2.2: Interpolated Tambara-Yamagami category for the subgroup $\mathbb{Z}/3 \subset S^1$. The simples of $\mathbf{TY}(\mathbb{Z}/3, \chi, \tau)$ are marked. The size of the right circle indicates that multiplication by elements in the left circle is given by the $\times 3$ map. All objects on the right circle are non-invertible.



(a) Interpolated $\mathbf{C}(\mathfrak{su}_2, 3)$

(b) Interpolated $\mathbf{C}(\mathfrak{su}_3, 2)$

Figure 2.3: Simple objects of interpolated quantum group categories associated to subgroups of S^1 . Objects of $\mathbf{C}(\mathfrak{g}, k)$ are marked. The dotted lines are a mere visual aid: they emphasise the shape of the Weyl alcove of the quantum group category. The action of the unit component on the other circle is the usual left action of S^1 on itself in both categories depicted.

skyscraper sheaves on the manifold G . Not all associators on $\omega \in H^3(\mathcal{B}G, \mathbb{C}^\times)$ can be realised as part of a monoidal structure on this stack. In fact, the transgression homomorphism

$$H^3(\mathcal{B}G, \mathbb{C}^\times) \rightarrow H^2(G, \mathbb{C}^\times)$$

sends an associator ω to the equivalence class of a *gerbe* \mathcal{G}_ω over the manifold G .³ The associator ω gives rise to a monoidal structure on the moduli stack of \mathcal{G}_ω -twisted skyscraper sheaves over G . We discuss gerbes and twisted sheaves in Section 2.2.4, and in more detail in Appendix A.

Besides introducing orbifold tensor categories and constructing examples, we study their general structure. Most of our results are proved under the assumption that the underlying orbifold is effective (a notion we recall in Section 2.2.2). We show that if the monoidal unit is supported at a point with trivial stabiliser, then every point of the orbifold of simple objects has trivial stabiliser. In other words, manifold tensor categories are precisely those effective orbifold tensor categories whose monoidal unit has trivial stabiliser (Corollary 2.5.1.5). The main theorem of this chapter, however, is a categorification of the following statement in classical Lie theory (recalled as Lemma 2.6.0.1):

Lemma. *A smooth monoid all of whose elements are invertible is a Lie group: the map $x \mapsto x^{-1}$ assigning inverses is automatically smooth.*

Making use of the Inverse Function Theorem, we show that this statement admits the following categorification (Theorem 2.6.0.7):

Theorem (Smooth Dualisability). *In an orbifold tensor category with simple monoidal unit, pointwise dualisability implies dualisability of all smooth families of objects.*

We begin by introducing twisted sheaves on orbifolds and smooth tensor stacks in Section 2.2. In particular, we show that a gerbe over an orbifold point with stabiliser Γ is essentially the data of a 2-cocycle $\theta \in Z(\mathcal{B}\Gamma, \mathbb{C}^\times)$, and that twisted skyscraper sheaves over this orbifold point correspond to θ -twisted representations of Γ . In Section 2.3, we introduce the notion of a linear category with an orbifold of simple objects (*orbisimple categories*). We study the structure of such categories, and show that they may be thought of as linearisations of gerbes. We give the definition of orbifold tensor categories in Section 2.4. The remainder of that section is devoted to the construction of explicit examples, in particular the families of interpolated fusion categories. To this end, we study central group actions on orbifold tensor categories. Section 2.5 is concerned with structural results on orbifold tensor categories: we study the interplay between the smooth and the monoidal structure. In Section 2.6, we prove the Smooth Dualisability Theorem.

2.2 Twisted Sheaves on Orbifolds

We will work over \mathbb{C} throughout. The category of \mathbb{C} -vector spaces will be denoted by \mathbf{Vec}^∞ , the subcategory of finite-dimensional vector spaces by $\mathbf{Vec} \subset \mathbf{Vec}^\infty$. We write

³The transgression map on cohomology arises from the universal G -bundle over $\mathcal{B}G$. It was lifted to the context of gerbes in [Car⁺05].

$\mathbf{Vec}^\infty\text{-Cat}$ to denote the bicategory of \mathbf{Vec}^∞ -enriched categories, and $\mathbf{VCat} \subset \mathbf{Vec}^\infty\text{-Cat}$ to denote the full subcategory on those categories that are complete under direct sums.

A reminder on sheaves and stacks is given in Section 1.2. We denote by \mathbf{Man} the site of smooth (\mathbb{R} -)manifolds, equipped with the surjective submersion Grothendieck topology, and the structure ring C^∞ of smooth \mathbb{C} -valued functions. Given a cover $Y \rightarrow M$ in \mathbf{Man} , we denote the associated iterated fibre products by $Y^{[n]} := Y \times_M Y \times_M \cdots Y$. We use $\mathbf{Shv}_{C^\infty}(M)$ to denote the category of C^∞ -modules over a manifold M , and freely refer to such C^∞ -modules simply as sheaves. A particular class of C^∞ -modules is provided by vector bundles $E \rightarrow M$, which we identify with their sheaves of sections.

2.2.1 Twisted Representations

Recall that a finite-dimensional \mathbb{C} -linear representation of a discrete group Γ is a vector space $V \in \mathbf{Vec}$, equipped with a homomorphism $\rho : \Gamma \rightarrow \text{End}(V)$. This may be rephrased by saying a \mathbb{C} -linear representation is a morphism $\underline{\Gamma} \rightarrow \mathbf{Vec}$, where $\underline{\Gamma}$ denotes the category with a single object whose endomorphisms form the group Γ and \mathbf{Vec} denotes the category of \mathbb{C} -modules. The endomorphisms assigned to group elements $g, h, gh \in \Gamma$ satisfy the condition

$$\rho(g)\rho(h) = \rho(gh).$$

In a *projective representation*, this equation only holds up to a scalar:

$$\rho(g)\rho(h) = \theta(g, h)\rho(gh).$$

In the above, $\theta : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ is a map of sets, and \mathbb{C}^\times acts via the usual \mathbb{C} -module structure on $V \in \mathbf{Vec}$. Requiring associativity of ρ yields

$$\theta(g, h)\theta(gh, k) = \theta(h, k)\theta(g, hk)$$

for every triple $g, h, k \in \Gamma$. This is the equation of a 2-cocycle $\theta \in Z^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$. We call a representation of the above form a *θ -twisted representation* of Γ . Twisted representations for a specific 2-cocycle assemble into a category $\mathbf{Rep}^\theta(\Gamma)$.

EXAMPLE 2.2.1.1. Denote two chosen generators of the Klein four group $\mathbb{V} = \mathbb{Z}/2 \times \mathbb{Z}/2$ by x and p . We pick a 2-cocycle $\theta : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}^\times$ representing the generator of $H^2(\mathcal{B}\mathbb{V}, \mathbb{C}^\times) \simeq \mathbb{Z}/2$, setting $\theta(a, b) = 1$ except

$$\theta(p, x) = \theta(xp, xp) = \theta(p, xp) = \theta(xp, x) = -1.$$

A 2-dimensional θ -twisted representation π of \mathbb{V} is furnished by the Pauli matrices

$$x \mapsto \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad p \mapsto \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy the necessary and sufficient relations $\sigma_x^2 = \sigma_z^2 = \text{Id}$ and $\sigma_x\sigma_z = -\sigma_z\sigma_x$.

The tensor product of a θ -twisted representation with a θ' -twisted representation is a $\theta \cdot \theta'$ -twisted representation, where \cdot denotes the group operation in $Z^2(\Gamma, \mathbb{C}^\times)$. A trivialisation $\theta'/\theta = d\kappa$ is equivalently a 1-dimensional θ/θ' -projective representation

$$\kappa(x)\kappa(y) = \frac{\theta(x, y)}{\theta'(x, y)}\kappa(xy).$$

Tensoring with κ constitutes an explicit equivalence

$$\mathbf{Rep}^{\theta'}(\Gamma) \xrightarrow{\cong} \mathbf{Rep}^\theta(\Gamma).$$

In particular, a trivialisation $\theta = d\kappa$ gives rise to an equivalence $\mathbf{Rep}(\Gamma) \xrightarrow{\cong} \mathbf{Rep}^\theta(\Gamma)$ between θ -twisted representations and untwisted representations. The discussion above has the following corollary.

Corollary 2.2.1.2. *The category $\mathbf{Rep}^\theta(\Gamma)$ contains a 1-dimensional representation if and only if θ is trivialisable.*

REMARK 2.2.1.3. A 2-cocycle θ also defines a central extension⁴

$$\mathbb{C}^\times \rightarrow \tilde{\Gamma} \rightarrow \Gamma,$$

where $\tilde{\Gamma}$ is the set $\mathbb{C}^\times \times \Gamma$ equipped with a multiplication

$$(z, \gamma) \cdot_\theta (z', \gamma') = (z \cdot z' \cdot \theta(\gamma, \gamma'), \gamma \cdot \gamma').$$

Note that the extension built from θ comes with a distinguished section of sets $\Gamma \rightarrow \tilde{\Gamma}$ sending $\gamma \mapsto (1, \gamma)$. The central extension is classified up to equivalence by the cohomology class $[\theta] \in H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$. Every 1-chain κ such that $d\kappa = \theta/\theta'$ defines an explicit isomorphism of the corresponding extensions. The θ -twisted representations of Γ in fact correspond to representations of $\tilde{\Gamma}$ where the central $\mathbb{C}^\times \hookrightarrow \tilde{\Gamma}$ acts by the identity character. We will not need this correspondence in this thesis.

To discuss the restriction-induction adjunction, it is useful to think about twisted representations using the language of rings and modules.

Definition 2.2.1.4. The θ -twisted group ring of Γ is the (non-commutative) algebra $\mathbb{C}^\theta[\Gamma]$ with underlying vector space the \mathbb{C} -span of the set Γ , and multiplication defined by

$$\underline{\gamma_1} \cdot \underline{\gamma_2} = \theta(\gamma_1, \gamma_2)\underline{\gamma_1\gamma_2}.$$

The θ -twisted representations of Γ defined above are precisely the modules over this group ring

$$\mathbf{Rep}^\theta(\Gamma) = \mathbf{Mod}_{\mathbb{C}^\theta[\Gamma]}.$$

The θ -twisted group ring plays the role of the *regular representation* in ordinary representation theory. Using twisted character theory (as developed in [Che15]), one may prove that the θ -twisted regular representation decomposes in the usual way. Denote by $\mathbf{Irr}^\theta(G)$ a set of representatives for the θ -twisted irreducible representations of G .

⁴We made the assumption that Γ is discrete. As a result all \mathbb{C}^\times -extensions of Γ are split as topological spaces, and we may avoid dealing with Segal-Mitchison cocycles (see Definition 3.3.2.1).

Proposition 2.2.1.5 ([Che15, Prop 2.3]). *The twisted group ring decomposes as a direct sum of irreducible representations and contains each of them with multiplicity their dimension:*

$$\mathbb{C}^\theta[\Gamma] \cong \bigoplus_{V \in \text{Irr}^\theta(G)} V^{\oplus \dim V}.$$

A homomorphism of groups $f : H \rightarrow \Gamma$ induces a pullback 2-cocycle $f^*\theta$ for H , and a homomorphism

$$f : \mathbb{C}^{f^*\theta}[H] \rightarrow \mathbb{C}^\theta[\Gamma]$$

(we abuse notation to denote it by the same letter).

Recall the *restriction* functor $f^* : \mathbf{Mod}_S \rightarrow \mathbf{Mod}_R$ associated to a map of rings $f : R \rightarrow S$ is given by $f^* = \text{Hom}_S({}_S S_R, -) = {}_R S_S \otimes_S -$, and thus has both a left and a right adjoint, supplied by the \otimes -Hom-adjunction:

$$(f_! \dashv f^* \dashv f_*) : \mathbf{Mod}_R \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} \mathbf{Mod}_S.$$

The left adjoint $f_! = {}_S S_R \otimes -$ is called *extension of scalars*. When S and R are twisted group rings, this operation corresponds to *induction* of twisted representations. The right adjoint $f_* = \text{Hom}_R({}_R S_S, -)$ is called *coextension of scalars* and specialises to *coinduction* for group rings.

Proposition 2.2.1.6. *If $f : H \hookrightarrow \Gamma$ is an inclusion of finite groups, induction is equivalent to coinduction:*

$$f_! \simeq f_* : \mathbf{Mod}_{\mathbb{C}^{f^*\theta}[H]} \rightarrow \mathbf{Mod}_{\mathbb{C}^\theta[\Gamma]}.$$

The proof below is adapted to the θ -twisted case from [Yua12; Mic20].

Proof. Let V be a left $\mathbb{C}^{f^*\theta}[H]$ -module. The isomorphism between the induction and coinduction functor is furnished by the twisted Nakayama isomorphism

$$\begin{aligned} \text{Nak} : \text{Hom}_{\mathbb{C}^{f^*\theta}[H]}(\mathbb{C}^\theta[\Gamma], V) &\rightarrow \mathbb{C}^\theta[\Gamma] \otimes_{\mathbb{C}^{f^*\theta}[H]} V \\ \phi &\mapsto \frac{1}{|H|} \sum_{x \in \Gamma} \frac{1}{\theta(x^{-1}, x)} x^{-1} \otimes \phi(\underline{x}), \end{aligned}$$

which is manifestly natural in V . Its inverse is given by

$$\begin{aligned} \text{Nak}^{-1} : \mathbb{C}^\theta[\Gamma] \otimes_{\mathbb{C}^{f^*\theta}[H]} V &\rightarrow \text{Hom}_{\mathbb{C}^{f^*\theta}[H]}(\mathbb{C}^\theta[\Gamma], V) \\ \underline{x} \otimes v &\mapsto \theta(x^{-1}, x) \sum_{h \in H} \frac{1}{\theta(h, x^{-1})} h x^{-1*} \otimes h.v, \end{aligned}$$

where $\underline{x}^* \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^\theta[G], \mathbb{C})$ denotes the function that sends $\underline{x} \mapsto 1$ and all other generators to 0.

It is tedious but straightforward to check that these morphisms are indeed well-defined and compatible with the $\mathbb{C}^\theta[\Gamma]$ -module structure. We show why the map Nak^{-1} is well-defined as an example. In $\mathbb{C}^\theta[\Gamma] \otimes_{\mathbb{C}^{f^*\theta}[H]} V$,

$$\underline{x} \otimes v = \underline{x} h^{-1} h \otimes v = \frac{1}{\theta(x h^{-1}, h)} x h^{-1} \otimes h.v$$

for all $h \in H$. Under Nak^{-1} , this is sent to

$$\begin{aligned} \frac{1}{\theta(xh^{-1}, h)} xh^{-1} \otimes h.v &\mapsto \frac{\theta(hx^{-1}, xh^{-1})}{\theta(xh^{-1}, h)} \sum_{h' \in H} \frac{1}{\theta(h', hx^{-1})} h'hx^{-1*} \otimes h'.h.v \\ &= \sum_{h' \in H} \frac{\theta(hx^{-1}, xh^{-1})\theta(h', h)}{\theta(xh^{-1}, h)\theta(h', hx^{-1})} h'hx^{-1*} \otimes (h'h).v \\ &= \sum_{h'' \in H} \frac{\theta(hx^{-1}, xh^{-1})\theta(h''h^{-1}, h)}{\theta(xh^{-1}, h)\theta(h''h^{-1}, hx^{-1})} h''x^{-1*} \otimes h''.v \end{aligned}$$

The cocycle conditions for the triples $(y, 1, z)$ and (y, y^{-1}, y) imply that $\theta(y, 1) = \theta(1, z)$ and $\theta(y, y^{-1}) = \theta(y^{-1}, y)$ for all $y, z \in \Gamma$. Together with these facts, the cocycle conditions for the triples (xh^{-1}, h, x^{-1}) and $(h''h^{-1}, h, x^{-1})$ complete the above verification:

$$\begin{aligned} \sum_{h'' \in H} \frac{\theta(hx^{-1}, xh^{-1})\theta(h''h^{-1}, h)}{\theta(xh^{-1}, h)\theta(h''h^{-1}, hx^{-1})} h''x^{-1*} \otimes h''.v &= \sum_{h'' \in H} \frac{\theta(x^{-1}, x)}{\theta(h'', x^{-1})} h''x^{-1*} \otimes h''.v \\ &= \text{Nak}^{-1}(\underline{x} \otimes v). \quad \square \end{aligned}$$

Definition 2.2.1.7 ([Lau06]). An *ambidextrous adjunction* is a triple of adjunctions

$$F \dashv G \dashv F' : \mathbf{D} \begin{array}{c} \xrightarrow{\quad} \\ \xleftrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathbf{C}$$

equipped with an isomorphism of functors $F \simeq F'$.

The endofunctor $T = F'G \simeq FG \in \text{End}(\mathbf{C})$ is simultaneously a monad because of the adjunction $G \dashv F'$ and a comonad because $F \dashv G$. In other words, it is simultaneously an algebra (T, μ, η) and a coalgebra (T, δ, ε) in $\text{End}(\mathbf{C})$. The multiplication μ and comultiplication δ satisfy the Frobenius axiom

$$(\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \delta) = \delta \circ \mu = (\mathbb{1} \otimes \mu) \circ (\delta \otimes \mathbb{1})$$

by the interchange law in bicategories (see [Lau06]). This structure is referred to as a *Frobenius monad* (first studied in [Str04a]).

By Proposition 2.2.1.6, we obtain an ambidextrous adjunction

$$\begin{array}{ccc} & f^* & \\ & \curvearrowleft & \\ \mathbf{Mod}_{\mathbf{C}^{f^*\theta[H]}} & \perp \& \top & \mathbf{Mod}_{\mathbf{C}^\theta[\Gamma]} \\ & \curvearrowright & \\ & f_* & \end{array}$$

Denote the associated monad on $\mathbf{Mod}_{\mathbf{C}^\theta[\Gamma]}$ by $(T, \mu, \eta, \delta, \varepsilon)$. For definiteness, we set

$$T(V) = \mathbf{C}^\theta[\Gamma] \otimes_{\mathbf{C}^{f^*\theta[H]}} \mathbf{C}^\theta[\Gamma] \otimes_{\mathbf{C}^\theta[\Gamma]} V.$$

Then the counit $\varepsilon : T \rightarrow \text{Id}$ is given by evaluation

$$\begin{aligned} \varepsilon : T(V) &\rightarrow V \\ \underline{x} \otimes \underline{x}' \otimes v &\mapsto \underline{x}.x'.v = \theta(x, x')\underline{x}x'.v \end{aligned}$$

while the unit $\eta : \text{Id} \rightarrow T$ uses the Nakayama isomorphism given in the proof of Proposition 2.2.1.6

$$\begin{aligned} \eta_V : V &\rightarrow T(V) \\ v &\mapsto \frac{1}{|H|} \sum_{x \in \Gamma} \frac{1}{\theta(x^{-1}, x)} x^{-1} \otimes x \otimes v. \end{aligned}$$

The composite $\varepsilon \circ \eta$, an endomorphism of the identity functor, amounts to multiplication by a scalar:

$$\begin{aligned} \varepsilon_V \circ \eta_V : V &\rightarrow V \\ v &\mapsto \frac{1}{|H|} \sum_{x \in \Gamma} v = \frac{|\Gamma|}{|H|} v. \end{aligned}$$

This way, the natural transformation

$$e := \frac{|H|}{|\Gamma|} \eta \circ \varepsilon : T \rightarrow T$$

defines an idempotent on T , split by

$$T \begin{array}{c} \xleftarrow{\frac{|H|}{|\Gamma|} \eta} \\ \xrightarrow{\varepsilon} \text{Id} \end{array}$$

This identifies the identity functor as a direct summand of T , and in particular $V \in \mathbf{Mod}_{\mathbb{C}^\theta[\Gamma]}$ as a direct summand of $T(V)$.

Corollary 2.2.1.8. *The category $\mathbf{Mod}_{\mathbb{C}^\theta[\Gamma]}$ is a Karoubi completion of the essential image of $\mathbf{Mod}_{\mathbb{C}^{f^*\theta}[H]}$ under the induction functor. In other words, every $\mathbb{C}^\theta[\Gamma]$ -module is a direct summand of a module induced up from the subgroup H .*

Proof. It suffices to show that every $V \in \mathbf{Mod}_{\mathbb{C}^\theta[\Gamma]}$ is a summand of a module of the form f_*W , where $W \in \mathbf{Mod}_{\mathbb{C}^{f^*\theta}[H]}$. Setting $W = f^*V$, V is a summand of $f_*W = T(V)$ by the discussion above. \square

In the special case where $H = * \hookrightarrow \Gamma$ is the inclusion of the trivial subgroup, the restriction functor takes the underlying vector space of a θ -twisted Γ -representation. The induction functor sends $V \in \mathbf{Vec}$ to the representation

$$f_*V = \mathbb{C}^\theta[\Gamma] \otimes V.$$

2.2.2 Orbifolds as Groupoids

Just as a manifold is locally modelled on \mathbb{R}^n , an orbifold is locally modelled on \mathbb{R}^n/Γ , where Γ is a finite group acting on \mathbb{R}^n . As argued in [MP97; Moe02; Ler10], orbifolds are best thought of as stacks presented by Lie groupoids.

A Lie groupoid G_\bullet is a pair of manifolds G_1, G_0 with two surjective submersions $s, t : G_1 \rightrightarrows G_0$ and smooth maps

$$\circ : G_1 \times_{G_0} G_1 \rightarrow G_1 \quad \text{id}_- : G_0 \rightarrow G_1 \quad (-)^{-1} : G_1 \rightarrow G_1,$$

giving it the structure of a groupoid internal to \mathbf{Man} . The stabiliser group G_x of a point $x \in G_0$ is the Lie group of arrows $x \rightarrow x$.

The Lie groupoid G_\bullet is called *proper* if the map

$$G_1 \xrightarrow{(s,t)} G_0 \times G_0$$

is proper. It is *étale* if the maps s, t are local homeomorphisms $G_1 \rightarrow G_0$. In a proper Lie groupoid, all stabiliser groups are compact. In an étale Lie groupoid, they are discrete. In a proper étale Lie groupoid, all stabiliser groups are finite.

EXAMPLE 2.2.2.1. The Lie groupoid $M \rightrightarrows M$ associated to a manifold M is a proper étale Lie groupoid.

EXAMPLE 2.2.2.2. The translation groupoid $[M/\Gamma]$ associated to a (right) action $M \curvearrowright \Gamma$ is given by

$$M \times \Gamma \xrightarrow{\pi_M} M.$$

It is proper iff the group action is proper in the traditional sense. It is étale iff Γ is discrete.

EXAMPLE 2.2.2.3. The classifying stack $[\ast/\Gamma]$ is the special case where M is a point. It is proper iff Γ is compact and étale iff Γ is discrete.

A smooth functor of Lie groupoids $G_\bullet \rightarrow H_\bullet$ is a pair of smooth maps $G_0 \rightarrow H_0, G_1 \rightarrow H_1$ compatible with composition and units. Denote by

$$\text{Hom}(-, G_\bullet) : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Cat}$$

the 2-presheaf that sends a manifold M to the category of smooth functors $(M \rightrightarrows M) \rightarrow (G_1 \rightrightarrows G_0)$. The stack associated to a Lie groupoid is given by the stackification of this 2-presheaf, $\text{Hom}(-, G_\bullet)^\#$. We call it the stack represented by G_\bullet .

Recall that a *differentiable stack* is a stack which may be represented by a Lie groupoid.

Definition 2.2.2.4. A differentiable stack is an *orbifold* if it is representable by a proper étale Lie groupoid.

When G_\bullet is an orbifold, the stabiliser group G_x of $x \in G_0$ acts on a small neighbourhood of x [Car11, Prop III.5.2]. Recall a group action is called *effective* if the only element acting trivially is the identity element.

Definition 2.2.2.5. An orbifold is *effective* if all stabilisers G_x act effectively on a small neighbourhood of x .

Effectivity can be checked locally. Every orbifold is locally of the form $[M/\Gamma]$. Effectivity of the orbifold is then equivalent to the effectivity of the group action $M \curvearrowright \Gamma$ for each such local patch. In particular, effectivity implies that a generic point of the orbifold has trivial stabiliser.

EXAMPLE 2.2.2.6. The stack $[*/H]$ is an orbifold iff the group H is finite. If H is nontrivial, $[*/H]$ is *not* an effective orbifold.

Let $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$ denote the functor sending a category to its set of isomorphism classes. The points of an orbifold \mathcal{M} form a category $\mathcal{M}(*)$. Its set of isomorphism classes $|\mathcal{M}| := \pi_0 \mathcal{M}(*)$ comes equipped with the following data: Each functor $f : M \rightarrow \mathcal{M}$ induces a map of sets $|f| : M \rightarrow |\mathcal{M}|$, and f is determined by $|f|$ up to isomorphism. The collection of maps to $|\mathcal{M}|$ obtained this way assemble into a sheaf of sets

$$\pi_0 \circ \mathcal{M} : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Cat} \rightarrow \mathbf{Set},$$

whose value on the point is $|\mathcal{M}|$. In fact, the sheaf satisfies the condition of being *concrete* [BH11]:

Definition 2.2.2.7. A sheaf $\mathcal{F} : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Set}$ is *concrete* if for all $U \in \mathbf{Man}$, the map

$$\begin{aligned} \mathcal{F}(U) &\rightarrow [\text{Hom}(*, U), \mathcal{F}(*)] \\ p &\mapsto (u \mapsto u^*p) \end{aligned}$$

is injective.

This condition says that sections over $U \in \mathbf{Man}$ are ‘allowed families of sections over $* \in \mathbf{Man}$ ’: every section is determined by its restrictions over each point $* \rightarrow U$. It was shown in [BH11] that this data is equivalent to that of a *diffeological space* in the sense of [Sou80; Sou84; IZ13]. The sheaf $\pi_0 \circ \mathcal{M}$ equips the set $(\pi_0 \circ \mathcal{M})(*) = |\mathcal{M}|$ with a *diffeology*. The value of the sheaf on $U \in \mathbf{Man}$ is the set of maps of underlying sets $U \rightarrow |\mathcal{M}|$ that admit a lift to a map of orbifolds. This gives $|\mathcal{M}|$ a topology: the *D-topology* [IZ13] is the finest topology for which all these maps are continuous. For an orbifold represented by a Lie groupoid $G_\bullet = G_1 \rightrightarrows G_0$, this topology is the topology of the quotient space $|G_\bullet| = G_0/G_1$.⁵ This is true more generally for differentiable stacks [WW14].

Theorem 2.2.2.8 ([Wat17, Thm B]). *The diffeological space $|\mathcal{M}|$ encodes the orbifold \mathcal{M} up to equivalence.*

2.2.3 Sheaves over Orbifolds

We recall the relationship between an atlas and a representing groupoid for a stack. Then we introduce a number of sites associated to an étale stack, most notably the small site associated to an atlas.

Definition 2.2.3.1. A map of smooth stacks $F : \mathcal{X} \rightarrow \mathcal{Y}$ is *representable* if for every map $M \rightarrow \mathcal{Y}$ with domain a manifold, the pullback $M \times_{\mathcal{Y}} \mathcal{X}$ is representable by a manifold. It is *representably submersive* (we also call this a *cover*) if the maps $M \times_{\mathcal{Y}} \mathcal{X} \rightarrow M$ are further surjective submersions. It is *representably étale* if the maps $M \times_{\mathcal{Y}} \mathcal{X} \rightarrow M$ are étale maps (ie. local diffeomorphisms).

⁵The quotient space is obtained by quotienting out the underlying topological space of G_0 by the equivalence relation induced by G_1 .

The fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ of two maps of stacks $f : \mathcal{X} \rightarrow \mathcal{Z} \leftarrow \mathcal{Y} : g$ is computed object-wise. It assigns to $U \in \mathbf{Man}$ the category with:

- objects: triples $(X \in \mathcal{X}(U), Y \in \mathcal{Y}(U), \phi : f(X) \xrightarrow{\cong} g(Y) \in \mathcal{Z}(U))$
- morphisms $(X, Y, \phi) \rightarrow (X', Y', \phi')$: tuples $(\alpha : X \rightarrow X', \beta : Y \rightarrow Y')$ such that $\phi' \circ f(\alpha) = g(\beta) \circ \phi$.

Definition 2.2.3.2. An *atlas* of a differentiable stack \mathcal{X} is a cover $X_0 \rightarrow \mathcal{X}$ of \mathcal{X} by a manifold X_0 . If the cover is representably étale, we call it an *étale atlas*.

Proposition 2.2.3.3 ([Ler10, Prop 4.31]). *Given a stack \mathcal{X} with an atlas $X_0 \rightarrow \mathcal{X}$, it is represented by the Lie groupoid*

$$X_0 \times_{\mathcal{X}} X_0 \rightrightarrows X_0.$$

Corollary 2.2.3.4. *A stack is differentiable iff it admits an atlas. It is étale iff it admits an étale atlas.*

Proof. The Lie groupoid constructed in Proposition 2.2.3.3 has source and target morphisms given by the pullbacks of the atlas projection along itself. A representably étale map $X_0 \rightarrow \mathcal{X}$ pulls back to étale source and target maps $X_0 \times_{\mathcal{X}} X_0 \rightarrow X_0$, thus giving the presented stack the structure of an étale stack. This covers the ‘if’ direction of the statement.

If \mathcal{X} is represented by $X_1 \rightrightarrows X_0$, there is a natural map $\pi : X_0 \rightarrow \mathcal{X}$ induced by the identity map on objects. It is straightforward to check that X_1 represents $X_0 \times_{\mathcal{X}} X_0$. The map $X_0 \rightarrow \mathcal{X}$ is representably étale if and only if the pullback maps $X_0 \times_{\mathcal{X}} X_0$ are: as $X_0 \rightarrow \mathcal{X}$ is a representable submersion, it suffices to check that the pullback atlas over any point $* \rightarrow \mathcal{X}$ is étale (ie. discrete). Any point may be factored as $* \rightarrow X_0 \rightarrow \mathcal{X}$, and the resulting atlas $* \times_{\mathcal{X}} X_0 \rightarrow *$ is pulled back from the étale map $X_0 \times_{\mathcal{X}} X_0 \rightarrow X_0$. It is thus étale itself, which shows $X_0 \times_{\mathcal{X}} X_0$ is representably étale. \square

EXAMPLE 2.2.3.5. The morphism $\xi : * \rightarrow [*/G]$ is representable. For any morphism $f : M \rightarrow [*/G]$, the pullback $M \times_{[*/G]} *$ is the total space of the G -bundle classified by f . The map ξ is further *representably étale* if G is discrete. More generally, morphisms $[*/H] \rightarrow [*/G]$ are in bijection with homomorphisms $H \rightarrow G$. They are representable iff the associated homomorphism is injective, and representably étale iff the quotient G/H is discrete.

Definition 2.2.3.6. The *big site* of a stack \mathcal{X} has underlying category \mathbf{Man}/\mathcal{X} , whose objects are maps $f : M \rightarrow \mathcal{X}$, where $M \in \mathbf{Man}$. A morphism $(f : M \rightarrow \mathcal{X}) \rightarrow (g : N \rightarrow \mathcal{X})$ is a pair $(h : M \rightarrow N, \phi : g \circ h \Rightarrow f)$ making

$$\begin{array}{ccc}
 M & \xrightarrow{h} & N \\
 & \searrow f & \swarrow g \\
 & & \mathcal{M}
 \end{array}
 \quad \phi : g \circ h \Rightarrow f$$

commute. The structure as a ringed site is inherited via the forgetful functor $\mathbf{Man}/\mathcal{X} \rightarrow \mathbf{Man}$. We retain the notation C^∞ for the structure sheaf.

Sheaves over the big site of \mathcal{X} are to be thought of as families of sheaves over \mathbf{Man} *parameterised by \mathcal{X}* . In particular, sheaves over the big site of the point $* \in \mathbf{Man}$ are sheaves over the site $\mathbf{Man}/* \simeq \mathbf{Man}$. This is markedly different from the category of sheaves over the site $\mathbf{Op}(*)$.

For an étale stack \mathcal{M} , one can define a site which recovers the behaviour of the site $\mathbf{Op}(X)$ of open subsets of a space X . (See in particular [Car10].) We discuss other versions of this site in Appendix A.

Definition 2.2.3.7. The *small site* (also called the *étale site*) of an étale stack \mathcal{M} is the full subsite $\mathrm{Et}(\mathcal{M}) \subset \mathbf{Man}/\mathcal{X}$ on objects $(f : M \rightarrow \mathcal{M})$ where f is representably étale.

Definition 2.2.3.8. A *sheaf/stack* on an étale stack is a sheaf/stack on its small site.

It is not hard to check that for a manifold M , the site $\mathbf{Op}(M)$ is dense in $\mathrm{Et}(M)$, validating the above definition. We will use $\mathbf{Shv}_{C^\infty}(\mathcal{M})$ to denote the category of C^∞ -modules over the small site of \mathcal{M} , and refer to them simply as sheaves over \mathcal{M} .

EXAMPLE 2.2.3.9. The small site of $[*/\Gamma]$ has a dense subsite with a single object $*$ whose endomorphisms form the group Γ . The structure sheaf C^∞ assigns \mathbb{C} with its usual ring structure to $*$. The category of sheaves of C^∞ -modules is the category

$$\mathbf{Shv}_{C^\infty}([*/\Gamma]) = \mathbf{Rep}(\Gamma)$$

of \mathbb{C} -linear representations of Γ .

Let \mathcal{F} be a sheaf over an orbifold \mathcal{M} . For each point $x : * \rightarrow \mathcal{M}$, one may compute the pullback $x^*\mathcal{F} \in \mathbf{Shv}_{C^\infty}(*) = \mathbf{Vec}$. The points for which this pullback is non-empty form a full subcategory $\mathbf{Supp} \mathcal{F} \subset \mathcal{M}(*)$.

Definition 2.2.3.10. The *support* of a sheaf over an orbifold is the diffeological subspace of the quotient space given by

$$\mathrm{supp} \mathcal{F} := \pi_0 \mathbf{Supp} \mathcal{F} \subset \pi_0 \mathcal{M}(*) = |\mathcal{M}|.$$

The support plays well with pushforwards and pullbacks.

Lemma 2.2.3.11. Let $\mathcal{F} \in \mathbf{Shv}_{C^\infty}(\mathcal{M})$, $\mathcal{F}' \in \mathbf{Shv}_{C^\infty}(\mathcal{N})$, be sheaves of C^∞ -modules over étale stacks and $f : \mathcal{M} \rightarrow \mathcal{N}$ a map between the stacks. Then

$$\mathrm{supp} f_* \mathcal{F} = f(\mathrm{supp} \mathcal{F}) \subset |\mathcal{N}| \quad \mathrm{supp} f^* \mathcal{F}' = f^{-1}(\mathrm{supp} \mathcal{F}') \subset |\mathcal{M}|$$

Proof. This may be checked pointwise. The stalk of the pushforward/pullback sheaf is non-empty precisely when the condition in the statement is satisfied. \square

EXAMPLE 2.2.3.12. Let E be a vector bundle over a manifold M and $f : M \rightarrow \mathcal{N}$ a map to an étale stack. Then the support of $f_* E$ is $\mathrm{im} f \subset |\mathcal{N}|$.

2.2.4 Gerbes and Twisted Sheaves

We will follow [SP11] and give a modern definition of gerbes. We introduce \mathcal{G} -twisted sheaves (associated to a \mathbb{C}^\times -gerbe \mathcal{G}) in Definition 2.2.4.11.

Let G be a group object in \mathbf{DiffSt} (see [SP09, Def 41] – we recall this in Definition 3.3.1.2), and \mathcal{X} a differentiable stack. Denote by $\mathbf{DiffSt}/\mathcal{X}$ the overcategory of \mathcal{X} . The object $\mathcal{X} \times G$ is then a group object in $\mathbf{DiffSt}/\mathcal{X}$.

Definition 2.2.4.1 ([SP11, Def 70]). A *principal G -bundle over \mathcal{X}* is a stack $\mathcal{G} \in \mathbf{DiffSt}/\mathcal{X}$, equipped with a right action of $\mathcal{X} \times G$, locally equivalent to the trivial principal G -bundle: There exists a cover $f : \mathcal{Y} \rightarrow \mathcal{X}$ such that the pullback $f^*\mathcal{G} = \mathcal{Y} \times_{\mathcal{X}} \mathcal{G} \in \mathbf{DiffSt}/\mathcal{Y}$ is G -equivariantly equivalent to $\mathcal{Y} \times G$ with the standard $\mathcal{Y} \times G$ -action.

A principal G -bundle is also called a G -torsor.

Definition 2.2.4.2. A \mathbb{C}^\times -gerbe over \mathcal{X} is a principal $[\mathbb{C}^\times]$ -bundle over \mathcal{X} .

We will exclusively work with \mathbb{C}^\times -gerbes, and just refer to them as gerbes. The *trivial gerbe* over a stack \mathcal{X} is the gerbe

$$\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times [\mathbb{C}^\times]$$

equipped with the projection morphism $\mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{X}$. When the base is clear from context, we omit the subscript and simply write \mathcal{I} to denote the trivial gerbe.

We may pull back a gerbe $\mathcal{G} \rightarrow \mathcal{X}$ along a map $\mathcal{Y} \rightarrow \mathcal{X}$ of stacks by forming the fibre product

$$\begin{array}{ccc} f^*\mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X}. \end{array}$$

The trivial gerbe $\mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{X}$ pulls back to the trivial gerbe $\mathcal{I}_{\mathcal{Y}} \rightarrow \mathcal{Y}$ for any map $\mathcal{Y} \rightarrow \mathcal{X}$.

Definition 2.2.4.3. A *trivialisation* of a gerbe $\mathcal{G} \rightarrow \mathcal{X}$ is an isomorphism $\mathcal{I}_{\mathcal{X}} \xrightarrow{\cong} \mathcal{G}$ of gerbes over \mathcal{X} . We call a gerbe *trivialisable* if it admits a trivialisation.

Every trivialisation gives rise to a section $\mathcal{X} \rightarrow \mathcal{G}$ by composing $\mathcal{I}_{\mathcal{X}} \xrightarrow{\cong} \mathcal{G}$ with the canonical section $\mathcal{X} \rightarrow \mathcal{I}_{\mathcal{X}}$ of the trivial gerbe. The $[\mathbb{C}^\times]$ -torsor structure makes this assignment an equivalence between trivialisations and sections.

By definition, every differentiable stack admits a cover by a manifold, at atlas. Every cover $\mathcal{Y} \rightarrow \mathcal{X}$ is refined by such an atlas. This means we may always replace \mathcal{Y} in Definition 2.2.4.1 by a manifold. The local triviality of gerbes allows for a description of gerbes over differentiable stacks via \mathbb{C}^\times -extensions of Lie groupoids.

Definition 2.2.4.4. A *central \mathbb{C}^\times -extension* of a Lie groupoid X_\bullet is a smooth functor of Lie groupoids

$$\begin{array}{ccc} P & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{\text{id}} & X_0 \end{array}$$

together with a \mathbb{C}^\times -action on P that makes $P \rightarrow X_1$ a principal \mathbb{C}^\times -bundle. The action is required to commute with composition of morphisms in $P \rightrightarrows X_0$.

EXAMPLE 2.2.4.5. Central Lie groupoid \mathbb{C}^\times -extensions of $[*/G] = G \rightrightarrows *$ are in bijection with central Lie group \mathbb{C}^\times -extensions of G .

There is a dictionary between \mathbb{C}^\times -gerbes over differentiable stacks and central \mathbb{C}^\times -extensions of Lie groupoids.

Lemma 2.2.4.6 ([LGSX09; BX11]). *Let $X_\bullet = X_1 \rightrightarrows X_0$ be a Lie groupoid presenting a stack \mathcal{X} . There is a one-to-one correspondence between equivalence classes of \mathbb{C}^\times -gerbes \mathcal{G} over \mathcal{X} equipped with a trivialisation over X_0 , and isomorphism classes of central \mathbb{C}^\times -extensions $\mathcal{G}_\bullet \rightarrow X_\bullet$.*

Given a presentation of a gerbe $\mathcal{G} \rightarrow \mathcal{X}$ by a \mathbb{C}^\times -extension $P \rightarrow X_1 \rightrightarrows X_0$, we may extract a 2-cocycle $\theta \in Z^2(\mathcal{X}, \mathbb{C}^\times)$ [FHRZ08, App A]: If we choose X_0 fine enough, we may assume X_1 is a disjoint union of contractible manifolds, and the \mathbb{C}^\times -bundle $P \rightarrow X_1$ is equivalent to $X_1 \times \mathbb{C}^\times \rightarrow X_1$. The class of the \mathbb{C}^\times -bundle in $H^1(X_1, \mathbb{C}^\times)$ becomes a class in the cohomology of \mathcal{X} (see [Beh04] for a nice review of cohomology theories for stacks). The class $[\theta] \in H^2(\mathcal{X}, \mathbb{C}^\times)$ of this 2-cocycle only depends on the equivalence class of the gerbe $\mathcal{G} \rightarrow \mathcal{X}$. The converse is also true:

Theorem 2.2.4.7 ([Gir66],[LGSX09, Thm 3.13]). *\mathbb{C}^\times -gerbes over \mathcal{X} are classified up to equivalence by their associated class in $H^2(\mathcal{X}, \mathbb{C}^\times)$.*

In particular, every gerbe over a stack \mathcal{X} with trivial $H^2(\mathcal{X}, \mathbb{C}^\times)$ admits a trivialisation.

Corollary 2.2.4.8. *Let U be a contractible Euclidean space with a Γ -action (we assume Γ is finite). Equivalence classes of \mathbb{C}^\times -gerbes over the stack $[U/\Gamma]$ are in bijection with $H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$.*

Proof. By assumption, U is contractible, so $H^2(U, \mathbb{C}^\times) = 0$. By Theorem 2.2.4.7, every gerbe over $[U/\Gamma]$ admits a trivialisation over the atlas $U \rightarrow [U/\Gamma]$. Lemma 2.2.4.6 then identifies equivalence classes of gerbes over $[U/\Gamma]$ with equivalence classes of central \mathbb{C}^\times -extensions of the Lie groupoid $U \times \Gamma \rightrightarrows U$. Up to equivalence, every \mathbb{C}^\times -bundle P over $U \times \Gamma$ is trivial, so the only extra data is the composition on the morphism space P of the extension. It is easily checked that this encodes above each point $u \in U$ a central extension of Γ by \mathbb{C}^\times . This central extension may vary smoothly across U , but as Γ is finite, so is the group $H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$ of equivalence classes of central extensions. The induced morphism $U \rightarrow H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$ is thus constant. This establishes a map from \mathbb{C}^\times -gerbes to $H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$. Groupoid extensions which induce equivalent central extensions of Γ by \mathbb{C}^\times are equivalent. \square

REMARK 2.2.4.9. More conceptually, the above statement is a consequence of the fact that the stack $[U/\Gamma]$ has the homotopy type of $\mathcal{B}\Gamma$ (in the ∞ -topos of smooth ∞ -stacks), hence $H^2([U/\Gamma], \mathbb{C}^\times) = H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$.

Every orbifold may be covered by suborbifolds of the form $[U/\Gamma]$, where U is an open ball in some Cartesian space \mathbb{R}^n , equipped with a right action by a finite group Γ . Pick a point $x_0 \in U$ with maximal stabiliser $\text{Aut}(x_0) = \Gamma$. Any gerbe \mathcal{G} over the orbifold pulls back to a gerbe $[\ast/\tilde{\Gamma}] \rightarrow [\ast/\Gamma]$. Let $\mathbb{C}^\times \hookrightarrow \tilde{\Gamma} \twoheadrightarrow \Gamma$ be some central extension representing this gerbe according to Corollary 2.2.4.8. The class of this central extension in $H^2(\mathcal{B}\Gamma, \mathbb{C}^\times)$ represents this gerbe up to equivalence. We obtain a useful local normal form for \mathbb{C}^\times -gerbes over these quotient suborbifolds. We record the outcome of this discussion in the following lemma:

Lemma 2.2.4.10. *Let $[U/\Gamma]$ be a quotient suborbifold of an orbifold \mathcal{M} equipped with a gerbe \mathcal{G} . Assume further U is contractible, and contains a Γ -fixed point over which \mathcal{G} is classified by the central extension*

$$\mathbb{C}^\times \hookrightarrow \tilde{\Gamma} \twoheadrightarrow \Gamma.$$

Then the restriction of \mathcal{G} to $[U/\Gamma]$ admits a presentation by the constant \mathbb{C}^\times -extension

$$\begin{array}{ccc} U \times \tilde{\Gamma} & \xrightarrow{\text{id} \times \pi} & U \times \Gamma \\ \Downarrow & & \Downarrow \\ U & \xrightarrow{\text{id}} & U \end{array}$$

of Lie groupoids.

A \mathbb{C}^\times -gerbe \mathcal{G} over an étale stack \mathcal{M} allows a twisting of the notion of sheaf over said stack. The following material is introduced in more detail in Appendix A. We begin by describing the notion of a twisted sheaf for a trivialisable gerbe. If \mathcal{G} is trivialisable over \mathcal{M} , the category of \mathcal{G} -twisted sheaves over it has objects pairs $(\tau : \mathcal{I} \xrightarrow{\simeq} \mathcal{G}, \mathcal{F} \in \text{Shv}_{\mathbb{C}^\infty}(\mathcal{M}))$, which we write as

$$\tau \cdot \mathcal{F}.$$

The morphisms $\tau \cdot \mathcal{F} \rightarrow \tau' \cdot \mathcal{F}'$ are given by morphisms of sheaves “twisted by the trivialisations”: the composite of trivialisations $\tau^{-1}\tau'$ is an endomorphism of \mathcal{I} , and thus defines a \mathbb{C}^\times -bundle over \mathcal{M} . We denote the action of the associated complex line bundle $\mathcal{L}_{\tau, \tau'}$ on a sheaf \mathcal{F}' by

$$\tau^{-1}\tau' \cdot \mathcal{F}' := \mathcal{L}_{\tau, \tau'} \otimes_{\mathbb{C}^\infty} \mathcal{F}'.$$

A morphism $\phi : \tau \cdot \mathcal{F} \rightarrow \tau' \cdot \mathcal{F}'$ is a morphism of sheaves

$$\bar{\phi} : \mathcal{F} \rightarrow \tau^{-1}\tau' \cdot \mathcal{F}'.$$

The composite

$$\tau \cdot \mathcal{F} \xrightarrow{\phi} \tau' \cdot \mathcal{F}' \xrightarrow{\phi'} \tau'' \cdot \mathcal{F}''$$

is given by the morphisms of \mathbb{C}^∞ -modules

$$\mathcal{F} \xrightarrow{\bar{\phi}} \tau^{-1}\tau' \cdot \mathcal{F}' \xrightarrow{\tau^{-1}\tau' \cdot \bar{\phi}'} \tau^{-1}\tau'\tau'^{-1}\tau'' \cdot \mathcal{F}'' \simeq \tau^{-1}\tau'' \cdot \mathcal{F}''.$$

We denote the resulting category by $\mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$. A choice of trivialisation τ induces a functor

$$\begin{aligned} \tau \cdot - : \mathbf{Shv}_{C^\infty}(\mathcal{M}) &\rightarrow \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \\ \mathcal{F} &\mapsto \tau \cdot \mathcal{F}, \end{aligned}$$

and this functor is an equivalence (see Lemma A.3.0.5 and below). This way, \mathcal{G} -twisted sheaves for a trivialisable gerbe are (non-canonically) identified with usual sheaves of C^∞ -modules.

A map $f : \mathcal{N} \rightarrow \mathcal{M}$ of étale stacks induces a pullback functor

$$\begin{aligned} f^* : \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) &\rightarrow \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \\ \tau \cdot \mathcal{F} &\mapsto f^*\tau \cdot f^*\mathcal{F}, \end{aligned}$$

which makes the square

$$\begin{array}{ccc} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) & \xrightarrow{f^*} & \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \\ \tau \cdot - \downarrow & & \downarrow f^*\tau \cdot - \\ \mathbf{Shv}_{C^\infty}(\mathcal{M}) & \xrightarrow{f^*} & \mathbf{Shv}_{C^\infty}(\mathcal{N}) \end{array}$$

commute for any choice of trivialisation τ .

Now we drop the assumption that \mathcal{G} is trivialisable. The étale site of \mathcal{M} has a dense subsite $\text{Et}^{\mathcal{G}}(\mathcal{M}) \subset \text{Et}(\mathcal{M})$ on objects $u : U \rightarrow \mathcal{M}$ where the pullback gerbe $u^*\mathcal{G}$ admits a trivialisation. The assignment

$$\begin{aligned} \mathbf{Shv}^{\mathcal{G}} : \text{Et}^{\mathcal{G}}(\mathcal{M})^{\text{op}} &\rightarrow \mathbf{VCat} \subset \mathbf{Vec}^\infty\text{-Cat} \\ (u : U \rightarrow \mathcal{M}) &\mapsto \mathbf{Shv}^{u^*\mathcal{G}}(U) \\ ((f, \phi) : v \rightarrow u) &\mapsto (f^* : \mathbf{Shv}^{u^*\mathcal{G}}(U) \rightarrow \mathbf{Shv}^{v^*\mathcal{G}}(V)) \end{aligned}$$

extends to a \mathbf{VCat} -valued stack over \mathcal{M} (Lemma A.3.0.7).

Definition 2.2.4.11. A \mathcal{G} -twisted sheaf over the étale stack \mathcal{M} is a global section of $\mathbf{Shv}^{\mathcal{G}}$.

Given any cover $\pi : Y \twoheadrightarrow \mathcal{M}$ over which \mathcal{G} is trivialisable, the category of \mathcal{G} -twisted sheaves over \mathcal{M} is equivalent to the category of descent data of $\mathbf{Shv}^{\mathcal{G}}$ over Y . A \mathcal{G} -twisted sheaf is represented by a twisted sheaf $\tau \cdot \mathcal{F} \in \mathbf{Shv}^{\pi^*\mathcal{G}}(Y)$, equipped with a descent isomorphism

$$\phi : p_1^*(\tau \cdot \mathcal{F}) \xrightarrow{\sim} p_2^*(\tau \cdot \mathcal{F})$$

in $\mathbf{Shv}^{\pi^*\mathcal{G}}(Y^{[2]})$, subject to the condition that

$$\begin{array}{ccc} & p_2^*(\tau \cdot \mathcal{F}) & \\ p_{12}^*\phi \nearrow & & \searrow p_{23}^*\phi \\ p_1^*(\tau \cdot \mathcal{F}) & \xrightarrow{p_{13}^*\phi} & p_3^*(\tau \cdot \mathcal{F}) \end{array}$$

commutes.

Given a specific trivialisation $\tau : \mathcal{I} \xrightarrow{\simeq} \pi^*\mathcal{G}$ over Y with associated line bundle $\mathcal{L} = \mathcal{L}_{p_1^*\tau, p_2^*\tau}$ over $Y^{[2]}$, we may use the equivalence $\tau \cdot - : \mathbf{Shv}_{\mathbf{C}^\infty}(Y) \xrightarrow{\simeq} \mathbf{Shv}^{\pi^*\mathcal{G}}(Y)$ to replace the datum $\tau \cdot \mathcal{F}$ above with a sheaf of C^∞ -modules over Y . This identifies the category of descent data with the category of pairs $(\mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y), \bar{\phi} : p_1^*\mathcal{F} \xrightarrow{\simeq} \mathcal{L} \otimes p_2^*\mathcal{F})$, satisfying the cocycle condition

$$\begin{array}{ccc} p_{12}^*\mathcal{L} \otimes p_2^*\mathcal{F} & \xrightarrow{p_{12}^*\mathcal{L} \otimes p_{23}^*\bar{\phi}} & p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \otimes p_3^*\mathcal{F} \\ p_{12}^*\bar{\phi} \uparrow & & \wr \\ p_1^*\mathcal{F} & \xrightarrow{p_{13}^*\bar{\phi}} & p_{13}^*\mathcal{L} \otimes p_3^*\mathcal{F}, \end{array}$$

where the natural isomorphism $p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \simeq p_{13}^*\mathcal{L}$ is part of the data of the gerbe.⁶

EXAMPLE 2.2.4.12. Consider the gerbe \mathcal{G} over $[*/\Gamma]$ corresponding to a group extension $\mathbb{C}^\times \rightarrow \tilde{\Gamma} \rightarrow \Gamma$. The gerbe admits a trivialisation over the atlas $Y = * \rightarrow [*/\Gamma]$, with associated line bundle over $Y^{[2]} = * \times_{*/\Gamma} * = \Gamma$ given by $\mathcal{L} = \tilde{\Gamma} \rightarrow \Gamma$. We may pick a trivialisation $\tilde{\Gamma} \cong \mathbb{C}^\times \times \Gamma$. Then the isomorphism of line bundles $p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \xrightarrow{\simeq} p_{13}^*\mathcal{L}$ over $Y^{[3]} = \Gamma \times \Gamma$ is the data of a 2-cocycle $\theta : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$. By Lemma 2.2.4.10, all gerbes over $[*/\Gamma]$ are of this form (up to equivalence).

The category of \mathcal{G} -twisted sheaves over $[*/\Gamma]$ is then given by a vector space V (the C^∞ -module over $*$), equipped with an isomorphism of pullback sheaves over Γ . This isomorphism is the data of a \mathbb{C} -linear isomorphism $\phi_\gamma : V \rightarrow V$ for all $\gamma \in \Gamma$. The cocycle condition is twisted by θ , and precisely recovers the definition of a θ -twisted representation.

2.2.5 The Twisted Equivariantisation Adjunction

Let $\mathbf{C} \in \mathbf{VCat}$ be a \mathbf{Vec}^∞ -enriched category complete under direct sums, equipped with a group action $\Gamma \rightarrow \text{End}(\mathbf{C})$. The *equivariantisation* of $\Gamma \curvearrowright \mathbf{C}$ is a category \mathbf{C}^Γ , whose objects are objects $X \in \mathbf{C}$ equipped with Γ -equivariance data. When Γ is finite, the obvious forgetful functor $u : \mathbf{C}^\Gamma \rightarrow \mathbf{C}$ admits a simultaneous left and right adjoint, the induction functor I . It sends an object $X \in \mathbf{C}$ to the orbit of X under the Γ -action $X \mapsto \bigoplus_{\gamma \in \Gamma} \gamma \cdot X$, equipped with the obvious equivariance data.

An example of this in the \mathbf{Vec}^∞ -enriched context is the category of sheaves of C^∞ -modules over a quotient orbifold $[M/\Gamma]$:

$$\mathbf{Shv}_{\mathbf{C}^\infty}([M/\Gamma]) \simeq \mathbf{Shv}_{\mathbf{C}^\infty}(M)^\Gamma.$$

The data of the ambidextrous adjunction between u and I induces a Frobenius monad on $\mathbf{Shv}_{\mathbf{C}^\infty}([M/\Gamma])$, and in particular exhibits every object of $\mathbf{Shv}_{\mathbf{C}^\infty}([M/\Gamma])$ as a direct summand of an object of the form $I(X)$.⁷ In this section, we show that these results also hold for twisted sheaves in the presence of a non-trivial gerbe over $[M/\Gamma]$. The statement below is Proposition A.4.0.4 in Appendix A.

⁶In the triangular cocycle condition, this equivalence is hidden in the composition operation between morphisms of twisted sheaves.

⁷Ambidextrous adjunctions and the associated Frobenius monad are discussed more in Section 2.2.1.

Proposition 2.2.5.1. *The functor $f^* : \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \rightarrow \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N})$ induced by a map $f : \mathcal{N} \rightarrow \mathcal{M}$ of étale stacks admits a right adjoint*

$$f_* : \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \rightarrow \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}).$$

We call f_* the *pushforward of twisted sheaves*. Recall that a trivialisation of \mathcal{G} identifies the pullback of twisted sheaves with the pullback of C^∞ -modules. By uniqueness of adjoints, the same is true for the pushforward: a trivialisation of \mathcal{G} (over the target of the map f) identifies the pushforward of twisted sheaves along f with the ordinary pushforward.

EXAMPLE 2.2.5.2. Let $\mathcal{G} \rightarrow [*/\Gamma]$ be the gerbe corresponding to the 2-cocycle $\theta : \Gamma \times \Gamma \rightarrow \mathbb{C}^\times$ as in Example 2.2.4.12. A homomorphism of groups $H \hookrightarrow \Gamma$ gives rise to a map $h : [*/H] \rightarrow [*/\Gamma]$. The associated adjunction $h^* : \mathbf{Rep}^{\alpha|_H}(H) \rightleftarrows \mathbf{Rep}^\alpha(\Gamma) : h_*$ is the restriction-induction adjunction of twisted representations given in Section 2.2.1.

We now generalise Example 2.2.5.2 to the case of a quotient orbifold $[U/\Gamma]$, where U is contractible. Up to equivalence, any gerbe over $[U/\Gamma]$ is of the form $[U/\tilde{\Gamma}] \rightarrow [U/\Gamma]$, where $\tilde{\Gamma} = \Gamma \times \mathbb{C}^\times$ with multiplication twisted by a 2-cocycle θ . This is the local normal form for gerbes established in Lemma 2.2.4.10. We pick a gerbe \mathcal{G} over $[U/\Gamma]$ and assume it is of this form.

We view a \mathcal{G} -twisted sheaf over $[U/\Gamma]$ as a sheaf \mathcal{F} over U , together with a descent isomorphism $\phi : p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$ over $U \times_{[U/\Gamma]} U = U \times \Gamma$, satisfying the cocycle condition. In the case at hand, this is a set of isomorphisms $\{\phi_\gamma := \phi|_{U \times \gamma} : \gamma^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}\}_{\gamma \in \Gamma}$, such that

$$\phi_{\gamma'} \circ \phi_\gamma = \theta(\gamma', \gamma) \cdot \phi_{\gamma' \cdot \gamma}$$

(cf. Example 2.2.4.12). Using this description, the pullback associated to the cover $q : U \rightarrow [U/\Gamma]$ is given by

$$\begin{aligned} q^* : \mathbf{Shv}^{\mathcal{G}}([U/\Gamma]) &\rightarrow \mathbf{Shv}_{\mathbb{C}^\infty}(U) \\ (\mathcal{F}, \phi) &\mapsto \mathcal{F}. \end{aligned}$$

Proposition 2.2.5.3. *The functor q^* has a simultaneous left and right adjoint, the induction functor*

$$\begin{aligned} I : \mathbf{Shv}_{\mathbb{C}^\infty}(U) &\rightarrow \mathbf{Shv}^{\mathcal{G}}([U/\Gamma]) \\ \mathcal{F} &\mapsto \left(\bigoplus_{\gamma \in \Gamma} \gamma^* \mathcal{F}, \phi \right), \end{aligned}$$

where the isomorphism ϕ over $U \times \Gamma$ has components

$$\phi_{\gamma_1} : \bigoplus_{\gamma \in \Gamma} \gamma_1^* \gamma^* \mathcal{F} \xrightarrow{\sim} \bigoplus_{\gamma \in \Gamma} (\gamma_1 \cdot \gamma)^* \mathcal{F} \xrightarrow{\oplus \theta(\gamma_1, \gamma)} \bigoplus_{\gamma \in \Gamma} (\gamma_1 \cdot \gamma)^* \mathcal{F}$$

given by the 2-cocycle θ .⁸

⁸Note that because Γ acts on U from the right, the pullback morphisms compose as $\gamma_1^* \gamma^* \simeq (\gamma_1 \cdot \gamma)^*$.

Note that the functor I is indeed well-defined: the condition $\phi_{\gamma'} \circ \phi_\gamma = \theta(\gamma', \gamma) \cdot \phi_{\gamma' \cdot \gamma}$ follows from the 2-cocycle condition satisfied by θ .

Proof. We show the adjunction $q^* \dashv I$. The adjunction $I \dashv q^*$ is proved the same way. We do this by exhibiting a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathcal{G}}([U/\Gamma])}((\mathcal{F}, \phi), I\mathcal{F}') \simeq \mathrm{Hom}_{\mathbf{Shv}_{\mathbf{C}^\infty}(U)}(\mathcal{F}, \mathcal{F}').$$

A map $(\mathcal{F}, \phi) \rightarrow I\mathcal{F}'$ is a θ -twisted equivariant map $\beta : \mathcal{F} \rightarrow \bigoplus_\gamma \gamma^* \mathcal{F}'$. The natural equivalence of Hom-spaces is given by

$$\left(\beta : \mathcal{F} \rightarrow \bigoplus_\gamma \gamma^* \mathcal{F}' \right) \mapsto (\beta_e : \mathcal{F} \rightarrow \mathcal{F}').$$

Its inverse is the map

$$(\beta_e : \mathcal{F} \rightarrow \mathcal{F}') \mapsto (\beta' : \mathcal{F} \rightarrow \bigoplus_\gamma \gamma^* \mathcal{F}'),$$

with component $\beta'_\gamma : \gamma^* \mathcal{F} \rightarrow \mathcal{F}'$ given by

$$\beta'_\gamma = (\gamma^* \beta_e) \circ \phi_\gamma.$$

□

By the uniqueness of adjoints, there is a natural isomorphism of functors

$$I \simeq q_*.$$

Corollary 2.2.5.4. *Every twisted sheaf on $[U/\Gamma]$ is a summand of a twisted sheaf in the image of the functor*

$$q_* : \mathbf{Shv}_{\mathbf{C}^\infty}(U) \rightarrow \mathbf{Shv}^{\mathcal{G}}([U/\Gamma]).$$

Proof. The following computation takes place in $\mathrm{End}(\mathbf{Shv}^{\mathcal{G}}([U/\Gamma]))$. Examining the adjunction data given in the proof of Proposition 2.2.5.3, the counit and unit compose to the identity natural transformation

$$1 : \mathrm{Id}_{\mathbf{Shv}^{\mathcal{G}}([U/\Gamma])} \xrightarrow{\eta} q_* q^* \xrightarrow{\varepsilon} \mathrm{Id}_{\mathbf{Shv}^{\mathcal{G}}([U/\Gamma])},$$

hence the composite

$$e : q_* q^* \xrightarrow{\varepsilon} \mathrm{Id}_{\mathbf{Shv}^{\mathcal{G}}([U/\Gamma])} \xrightarrow{\eta} q_* q^*$$

is an idempotent, split by ε and η . They identify the identity functor as a summand of $q_* q^*$. In particular, this means every object V is a summand of $q_* q^* V$ which is manifestly in the image of q_* . □

2.2.6 The Stack of Twisted Skyscraper Sheaves

Let $\mathcal{G} \rightarrow \mathcal{M}$ be an étale stack equipped with a gerbe \mathcal{G} . We make the two definitions below in analogy with the usual support of a sheaf (Definition 2.2.3.10), and untwisted skyscraper sheaves (Definition 1.2.1.6).

Definition 2.2.6.1. The *support* of a twisted sheaf $\mathcal{F} \in \mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ is the subset

$$\mathrm{supp} \mathcal{F} := \{[x] \in |\mathcal{M}| \text{ s.t. } x^* \mathcal{F} \neq 0\} \subset |\mathcal{M}|.$$

Definition 2.2.6.2. A \mathcal{G} -*twisted skyscraper sheaf* over \mathcal{M} is a \mathcal{G} -twisted sheaf over \mathcal{M} with finite support, such that the pullback along any point is finite-dimensional as a C^∞ -module.

EXAMPLE 2.2.6.3. Every \mathcal{G} -twisted skyscraper sheaf is a direct sum of \mathcal{G} -twisted sheaves whose support is (the equivalence class of) a single point $x \rightarrow \mathcal{M}$. As spelled out in Example 2.2.4.12, such sheaves correspond (non-canonically!) to twisted representations of the stabiliser group Γ_x .

Let $\mathcal{G} \rightarrow M$ be a gerbe over a manifold. A choice of trivialisation τ_x of $x^* \mathcal{G}$ for each point $x : * \hookrightarrow M$ induces an equivalence of categories between the category of \mathcal{G} -twisted skyscraper sheaves and untwisted skyscraper sheaves over M . This equivalence is not canonical: different choices of trivialisations lead to different equivalences. When \mathcal{G} is non-trivial, it does not admit a global trivialisation, and the τ_x cannot be chosen to vary smoothly with $x \in M$.

The difference between twisted and untwisted skyscraper sheaves is detected by families of such sheaves. We directly describe the *stack of families of \mathcal{G} -twisted sheaves over \mathcal{M}* . It assigns

$$\begin{aligned} \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}} &: \mathbf{Man}^{\mathrm{op}} \rightarrow \mathbf{VCat} \\ U &\mapsto \mathbf{Shv}^{p_{\mathcal{M}}^* \mathcal{G}}(\mathcal{M} \times U), \end{aligned}$$

where $p_{\mathcal{M}} : \mathcal{M} \times U \rightarrow \mathcal{M}$ is the usual projection. A morphism $f : V \rightarrow U \in \mathbf{Man}$ is sent to the pullback morphism

$$(\mathrm{id}_{\mathcal{M}} \times f)^* : \mathbf{Shv}^{p_{\mathcal{M}}^* \mathcal{G}}(\mathcal{M} \times U) \rightarrow \mathbf{Shv}^{p_{\mathcal{M}}^* \mathcal{G}}(\mathcal{M} \times V).$$

This assignment uses the isomorphism

$$((\mathrm{id}_{\mathcal{M}} \times f) \circ p_{\mathcal{M}})^* \mathcal{G} \xrightarrow{\cong} p_{\mathcal{M}}^* \mathcal{G}$$

and the associated canonical equivalence

$$\mathbf{Shv}^{((\mathrm{id}_{\mathcal{M}} \times f) \circ p_{\mathcal{M}})^* \mathcal{G}}(\mathcal{M} \times V) \xrightarrow{\cong} \mathbf{Shv}^{p_{\mathcal{M}}^* \mathcal{G}}(\mathcal{M} \times V).$$

NOTATION 2.2.6.4. To avoid clutter, we drop the pullbacks along $p_{\mathcal{M}}$ in the superscript from now on.

The category assigned to $* \in \mathbf{Man}$ is the category $\mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ of \mathcal{G} -twisted sheaves over \mathcal{M} . The subcategory $\mathbf{Sky}^{\mathcal{G}}(\mathcal{M}) \subset \mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ of twisted skyscraper sheaves is cut out by the condition that the twisted sheaves have finite support.

A twisted skyscraper sheaf may be thought of as a finite set of points $x : * \rightarrow \mathcal{M}$, equipped with finite-dimensional \mathcal{G}_x -twisted representations of their stabiliser groups Γ_x in \mathcal{M} . Varying such a skyscraper sheaf smoothly should amount to smoothly varying the points and representations involved. The remainder of this section is dedicated to making this notion precise.

Let $U \in \mathbf{Man}$ be a parameterising manifold and $\pi : Y \twoheadrightarrow \mathcal{M}$ a cover of \mathcal{M} , equipped with a trivialisation $\tau : \mathcal{I} \xrightarrow{\simeq} \pi^*\mathcal{G}$ of the pullback gerbe. A map $f : U \rightarrow Y$ defines a graph

$$\mathrm{Gr}(f) := U \xrightarrow{\Delta} U \times U \xrightarrow{f \times \mathrm{id}_U} Y \times U.$$

Pushforward along $\mathrm{Gr}(f)$ is a functor

$$\mathrm{Gr}(f)_* : \mathbf{Shv}_{\mathbf{C}^\infty}(U) \rightarrow \mathbf{Shv}_{\mathbf{C}^\infty}(Y \times U) \xrightarrow{\tau} \mathbf{Shv}^{\pi^*\mathcal{G}}(Y \times U).$$

A U -family of skyscraper sheaves over the point is a U -family of vector spaces (ie. a vector bundle over U).

Definition 2.2.6.5. A sheaf $\mathcal{S} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y \times U)$ is a *basic* U -family of skyscraper sheaves if it decomposes as a finite direct sum of pushforwards of vector bundles on U via the graph of a smooth function:

$$\mathcal{S} \cong \bigoplus_{i=0}^r \mathrm{Gr}(f_i)_* E_i$$

for $f_i : U \rightarrow Y$, $E_i \in \mathbf{Vect}(U) \subset \mathbf{Shv}_{\mathbf{C}^\infty}(U)$.

Let $Y \twoheadrightarrow \mathcal{M}$ be a cover of \mathcal{M} and $p : \tilde{U} \rightarrow U$ a cover of the parameterising family U . The map $\pi \times p : Y \times \tilde{U} \twoheadrightarrow \mathcal{M} \times U$ and the trivialisation τ of the gerbe induce an adjunction

$$\mathbf{Shv}_{\mathbf{C}^\infty}(Y \times \tilde{U}) \rightleftarrows \mathbf{Shv}^{\mathcal{G}}(\mathcal{M} \times U).$$

We define U -families of twisted skyscraper sheaves over an orbifold \mathcal{M} in a way that is manifestly local on \mathcal{M} and on U :

Definition 2.2.6.6. A twisted sheaf $\mathcal{S} \in \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}(U)$ is a U -family of \mathcal{G} -twisted skyscraper sheaves on \mathcal{M} if there exists a cover $Y \twoheadrightarrow \mathcal{M}$ with a trivialisation of the pullback gerbe, and a cover $\tilde{U} \twoheadrightarrow U$, such that the pullback of \mathcal{S} to $Y \times \tilde{U}$ is a basic \tilde{U} -family.

We denote the subcategory of U -families of skyscraper sheaves by

$$\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U) \subset \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}(U).$$

EXAMPLE 2.2.6.7. Over the point $* \in \mathbf{Man}$ (equipped with the trivial gerbe \mathcal{I}), the condition of being a basic U -family of skyscraper sheaves in $\mathbf{Shv}_{\mathbf{C}^\infty}(* \times U) = \mathbf{Shv}_{\mathbf{C}^\infty}(U)$ is equivalent to that of being a vector bundle over U . The stack of skyscraper sheaves over the point is thus

$$\mathbf{Sky}_* \simeq \mathbf{Vect},$$

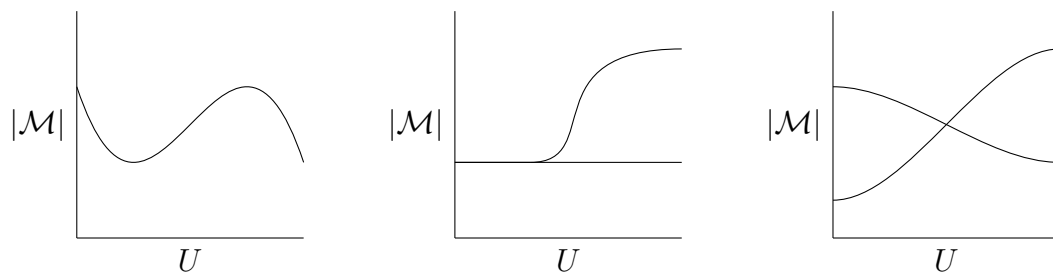
the stack of vector bundles over \mathbf{Man} . The functor $\mathbf{Shv}_{\mathcal{C}^\infty}$ takes coproducts $\coprod_i (\mathcal{G}_i \rightarrow M_i)$ of manifolds equipped with gerbes to products of categories, and hence so does \mathbf{Sky} . In particular, for a finite set $S \in \mathbf{Man}$,

$$\mathbf{Sky}_S \simeq \mathbf{Vect}^{\oplus S},$$

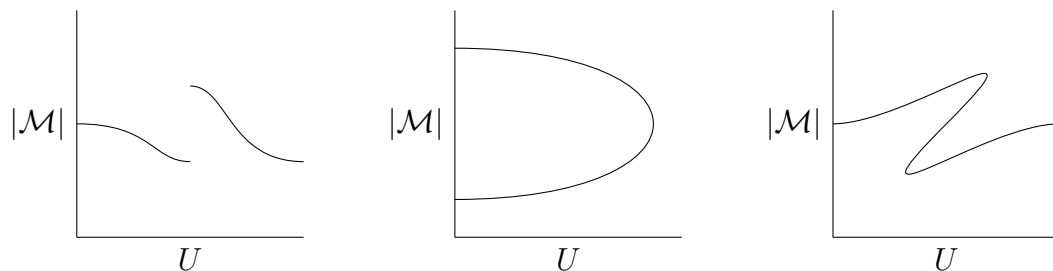
the direct sum of S copies of \mathbf{Vect} .

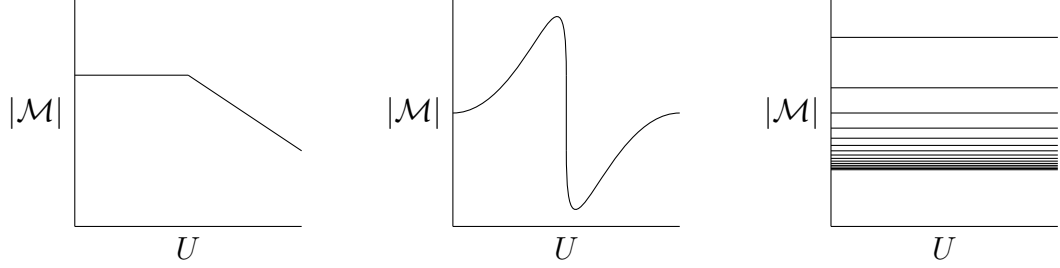
The restriction of a U -family of \mathcal{G} -twisted skyscraper sheaves on \mathcal{M} along any point $* \rightarrow U$ is a \mathcal{G} -twisted skyscraper sheaf on \mathcal{M} . Up to isomorphism, this is a finite set of points $x_i : * \rightarrow \mathcal{M}$, together with finite-dimensional θ_i -twisted representations of the stabilisers Γ_{x_i} , where θ_i denotes a 2-cocycle representing the restriction of the gerbe \mathcal{G} to $[*/\Gamma_{x_i}]$. The definition of U -families in particular implies that the total rank (the sum of the dimensions of the representations involved) is locally constant across U .

EXAMPLE 2.2.6.8. Some consequences of the definition may already be seen by looking at the support of a U -family of skyscraper sheaves. As an object $\mathcal{S} \in \mathbf{Shv}^{\mathcal{G}}(U \times \mathcal{M})$, the support of a U -family is a subset of $U \times |\mathcal{M}|$. The support of a pushforward sheaf $\mathrm{Gr}(f)_* E$ is given by the graph of $|f| : U \rightarrow \mathcal{M}$. This means the support of a family of twisted skyscraper sheaves must be locally (in the parameterising family U) given by the union of finitely many such graphs. The reader may like to think of them as graphs of smooth, multivalued functions $U \rightarrow |\mathcal{M}|$. The following subsets can arise as the support of \mathcal{S}



while the subsets below may not.





Definition 2.2.6.9. The stack of \mathcal{G} -twisted skyscraper sheaves is the full substack

$$\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}} \subset \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$$

which assigns to $U \in \mathbf{Man}$ the category of U -families of \mathcal{G} -twisted skyscraper sheaves on \mathcal{M} .

Lemma 2.2.6.10. *The condition above indeed cuts out a substack.*

Proof. The condition of being a U -family is manifestly local in U , so it is preserved under gluing. We prove that the condition of being a family of skyscraper sheaves is preserved by the pullback morphisms in $\mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$. Let $\mathcal{S} \in \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U) \subset \mathbf{Shv}^{\mathcal{G}}(\mathcal{M} \times U)$. By Definition 2.2.6.6, there exists a product cover

$$\pi \times p : Y \times \tilde{U} \rightarrow \mathcal{M} \times U$$

with a trivialisation τ of the pullback gerbe, such that $(\pi \times p)^* \mathcal{S}$ decomposes as a finite direct sum

$$(\pi \times p)^* \mathcal{S} \cong \bigoplus_{i=0}^r \tau \cdot \mathrm{Gr}(f_i)_* E_i,$$

where $f_i : \tilde{U} \rightarrow Y$ are smooth maps and $E_i \rightarrow \tilde{U}$ vector bundles over \tilde{U} .

Consider the pullback morphism $(\mathrm{id}_{\mathcal{M}} \times g)^*$ associated to a map $g : V \rightarrow U$. The product cover $\pi \times p$ pulls back to a product cover $\pi \times g^* p : Y \times \tilde{V} \rightarrow \mathcal{M} \times V$:

$$\begin{array}{ccc} Y \times \tilde{V} & \xrightarrow{\mathrm{id}_Y \times \tilde{g}} & Y \times \tilde{U} \\ \pi \times g^* p \downarrow & \lrcorner & \downarrow \pi \times p \\ \mathcal{M} \times V & \xrightarrow{\mathrm{id}_{\mathcal{M}} \times g} & \mathcal{M} \times U. \end{array}$$

Pullbacks compose, so

$$(\pi \times g^* p)^* (\mathrm{id}_{\mathcal{M}} \times g)^* \mathcal{S} \simeq (\mathrm{id}_Y \times \tilde{g})^* (\pi \times p)^* \mathcal{S},$$

which reduces the proof to showing that

$$(\mathrm{id}_Y \times \tilde{g})^* (\tau \cdot \mathrm{Gr}(f_i)_* E_i)$$

is a basic \tilde{V} -family of skyscraper sheaves. This follows from base change (Lemma A.4.0.2) along the pullback diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{g}} & \tilde{U} \\ \text{Gr}(f_i \circ \tilde{g}) \downarrow & \lrcorner & \downarrow \text{Gr}(f_i) \\ Y \times \tilde{V} & \xrightarrow{\text{id}_Y \times \tilde{g}} & Y \times \tilde{U}. \end{array}$$

It yields an isomorphism

$$(\text{id}_Y \times \tilde{g})^* \text{Gr}(f_i)_* E_i \simeq \text{Gr}(f_i \circ \tilde{g})_* \tilde{g}^* E_i.$$

The right hand side is the pushforward of the vector bundle $\tilde{g}^* E_i \rightarrow V$ along the graph of the smooth map $f_i \circ \tilde{g} : V \rightarrow Y$, and induces a basic V -family of skyscraper sheaves

$$((\text{id}_Y \times \tilde{g})^* \tau) \cdot \text{Gr}(f_i \circ \tilde{g})_* \tilde{g}^* E_i. \quad \square$$

2.2.7 Linear Stacks

Recall we write \mathbf{VCat} to denote the bicategory of \oplus -complete \mathbf{Vec}^∞ -enriched categories. It is symmetric monoidal under the tensor product of enriched \oplus -complete categories introduced by Kelly in [KK82, Ch 6.5] and studied in more detail in [Fra13]. Let $\mathbf{A}, \mathbf{B} \in \mathbf{VCat}$, then their tensor product is obtained by first taking the ordinary product of enriched categories, and then completing freely under direct sums. The resulting category $\mathbf{A} \boxtimes \mathbf{B}$ has objects formal finite direct sums $\bigoplus_i a_i \boxtimes b_i$ (where $a_i \in \mathbf{A}, b_i \in \mathbf{B}$) and the Hom spaces are extended linearly from

$$\text{Hom}(a \boxtimes b, a' \boxtimes b') := \text{Hom}_{\mathbf{A}}(a, a') \otimes \text{Hom}_{\mathbf{B}}(b, b').$$

This tensor product extends the Deligne tensor product used in the fusion category literature [EGNO16, Ch 1.11].

Theorem 2.2.7.1 ([Fra13, Thm 27]). *Let \mathbf{A}, \mathbf{B} be two semisimple abelian categories with finite-dimensional Hom-spaces and objects of finite length. The product $\mathbf{A} \boxtimes \mathbf{B}$ is equivalent to their Deligne tensor product.*

In particular, $\mathbf{A} \boxtimes \mathbf{B}$ is again semisimple abelian with finite-dimensional Hom-spaces and objects of finite length.

The symmetric monoidal structure on \mathbf{VCat} induces a symmetric monoidal structure on the bicategory \mathbf{VSt} of \mathbf{VCat} -valued stacks over \mathbf{Man} . The product of $\mathcal{X}, \mathcal{Y} \in \mathbf{VSt}$ is computed by first forming the pointwise product

$$\mathcal{X} \boxtimes^0 \mathcal{Y} : U \mapsto \mathcal{X}(U) \boxtimes \mathcal{Y}(U),$$

and then stackifying to obtain

$$\mathcal{X} \boxtimes \mathcal{Y} := (\mathcal{X} \boxtimes^0 \mathcal{Y})^\#.$$

Every category $\mathbf{C} \in \mathbf{VCat}$ is canonically a module over \mathbf{Vec} : $\mathbf{C}^r \in \mathbf{Vec}$ acts by $X \mapsto X^{\oplus r}$, and morphisms act via the \mathbf{Vec} -module structure of \mathbf{Vec}^∞ .

We may pull back \mathbf{Vec} along $\mathbf{Man} \rightarrow *$ to obtain the locally constant stack $\mathbf{Vec}^{\text{const}}$, which assigns the category \mathbf{Vec} to any Cartesian space $U \in \mathbf{CartSp} \subset \mathbf{Man}$ and the identity functor to smooth maps between them. Any stack in $\mathbf{VSt}/\mathbf{Man}$ is canonically equipped with an action by $\mathbf{Vec}^{\text{const}}$.

EXAMPLE 2.2.7.2. For the stack $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ of \mathcal{G} -twisted skyscraper sheaves over an orbifold \mathcal{M} , the action by a vector space $V \in \mathbf{Vec}$ corresponds to tensoring twisted sheaves $\mathcal{F} \in \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U) \subset \mathbf{Shv}^{\mathcal{G}}(\mathcal{M} \times U)$ with the sheaf of sections $C^\infty(\mathcal{M} \times U, V)$ of the trivial V -bundle over $\mathcal{M} \times U$. We view this as tensoring the twisted skyscraper sheaf over each point $u \in U$ with the vector space V .

The \mathbf{Vec} -action in Example 2.2.7.2 admits an immediate extension: Instead of tensoring the skyscraper sheaf over each point $u \in U$ with the same vector space V , we may tensor the U -family with a vector bundle P over U . This equips the stack of skyscraper sheaves with an action by \mathbf{Vect} , the \mathbf{Vec} -enriched stack of vector bundles. The stack \mathbf{Vect} is symmetric monoidal via the usual tensor product of vector bundles.

Definition 2.2.7.3. A *linear stack* is a \mathbf{Vect} -module, ie. a stack $\mathcal{F} \in \mathbf{VSt}/\mathbf{Man}$ equipped with an action of the stack of vector bundles.

The *bicategory of linear stacks* \mathbf{LinSt} is the bicategory of \mathbf{Vect} -modules. A 1-morphism of \mathbf{Vect} -modules \mathcal{X}, \mathcal{Y} is a map

$$F : \mathcal{X} \rightarrow \mathcal{Y}$$

of underlying \mathbf{Vec} -enriched stacks, equipped with an invertible 2-cell s implementing compatibility with the \mathbf{Vect} -module structures $\odot_{\mathcal{X}}, \odot_{\mathcal{Y}}$. For each $U \in \mathbf{Man}$, $P \in \mathbf{Vect}(U)$, $X \in \mathcal{X}(U)$, this 2-cell is an equivalence

$$s_U(X, P) : F(X \odot_{\mathcal{X}} P) \xrightarrow{\simeq} F(X) \odot_{\mathcal{Y}} P.$$

A 2-morphism between two 1-morphisms $(F, s), (F', s') : \mathcal{X} \rightarrow \mathcal{Y}$ is a 2-morphism $a : F \rightarrow F'$, satisfying the compatibility condition

$$\begin{array}{ccc} F(X \odot_{\mathcal{X}} P) & \xrightarrow{s_U(X, P)} & F(X) \odot_{\mathcal{Y}} P \\ a(X \odot P) \downarrow & & \downarrow a(X) \odot P \\ F'(X \odot_{\mathcal{X}} P) & \xrightarrow{s'_U(X, P)} & F'(X) \odot_{\mathcal{Y}} P \end{array}$$

for all triples U, P, X as above.

Let $X, Y \in \mathcal{X}(U)$ be two sections of a linear stack \mathcal{X} over $U \in \mathbf{Man}$. The restriction functors upgrade $\text{Hom}_{\mathcal{X}(U)}(X, Y)$ to a presheaf

$$\begin{aligned} \underline{\text{Hom}}(X, Y) : (\mathbf{Man}/U)^{\text{op}} &\rightarrow \mathbf{Vec}^\infty \\ (f : V \rightarrow U) &\mapsto \text{Hom}(f^* X, f^* Y). \end{aligned}$$

The descent condition implies that this is actually a sheaf on \mathbf{Man}/U . The \mathbf{Vect} -module structure upgrades this sheaf to a C^∞ -module: We use the natural isomorphisms $Y \simeq Y \odot C^\infty$ and $C^\infty \simeq \underline{\mathbf{Hom}}_{\mathbf{Vect}}(C^\infty, C^\infty)$, to write down the action map

$$\underline{\mathbf{Hom}}(X, Y) \otimes C^\infty \simeq \underline{\mathbf{Hom}}(X, Y \odot C^\infty) \otimes \underline{\mathbf{Hom}}(C^\infty, C^\infty) \rightarrow \underline{\mathbf{Hom}}(X, Y).$$

Lemma 2.2.7.4. *The action map extends to an isomorphism of C^∞ -modules*

$$\underline{\mathbf{Hom}}(X, Y) \otimes_{C^\infty} P \simeq \underline{\mathbf{Hom}}(X, Y \odot P)$$

natural in $X, Y \in \mathcal{X}(U)$ and $P \in \mathbf{Vect}(U)$.

Proof. We use the natural isomorphism $P \simeq \underline{\mathbf{Hom}}_{\mathbf{Vect}}(C^\infty, P)$ to build the map of C^∞ -modules

$$\underline{\mathbf{Hom}}(X, Y \odot C^\infty) \otimes_{C^\infty} \underline{\mathbf{Hom}}_{\mathbf{Vect}}(C^\infty, P) \rightarrow \underline{\mathbf{Hom}}(X, Y \odot P).$$

It is manifestly natural in X, Y and P . That this map is an isomorphism may be checked locally. Hence, we may assume $P \simeq (C^\infty)^{\oplus r}$, and the two sides are isomorphic to

$$(\underline{\mathbf{Hom}}(X, Y))^{\oplus r} \simeq \underline{\mathbf{Hom}}(X, Y^{\oplus r}). \quad \square$$

The monoidal product of two linear stacks \mathcal{X}, \mathcal{Y} is computed by first computing the pointwise product $\mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y}(U)$: Its objects are formal direct sums $\bigoplus_i X_i \boxtimes Y_i$ for $X_i \in \mathcal{X}(U), Y_i \in \mathcal{Y}(U)$, and the morphism spaces are extended linearly from

$$\mathbf{Hom}(X \boxtimes Y, X' \boxtimes Y') := \mathbf{Hom}(X, X') \otimes_{C^\infty(U)} \mathbf{Hom}(Y, Y').$$

Definition 2.2.7.5. The product of linear stacks \mathcal{X}, \mathcal{Y} is the linear stack

$$\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y} := (\mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y})^\#.$$

Note there is a map $\mathcal{X} \boxtimes \mathcal{Y} \rightarrow \mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y}$ given by the identity on objects, and by the natural quotient map

$$\mathbf{Hom}(X, X') \otimes_{\mathbf{C}} \mathbf{Hom}(Y, Y') \rightarrow \mathbf{Hom}(X, X') \otimes_{C^\infty(U)} \mathbf{Hom}(Y, Y')$$

on \mathbf{Hom} -spaces over $U \in \mathbf{Man}$. Postcomposing with the natural map from a stack to its stackification yields the quotient map

$$\mathcal{X} \boxtimes \mathcal{Y} \rightarrow \mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}.$$

Lemma 2.2.7.6. *A map of \mathbf{Vect} -modules*

$$F : \mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y} \rightarrow \mathcal{Z}$$

is specified (up to equivalence) by a morphism of \mathbf{Vect} -modules

$$F^0 : \mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y} \rightarrow \mathcal{Z},$$

subject to the following condition: For every parameterising manifold $U \in \mathbf{Man}$ and object $x \boxtimes y \in \mathcal{X}(U) \boxtimes \mathcal{Y}(U)$, the map induced by F_U^0 on the endomorphisms

$$\mathbf{End}(x \boxtimes y) = \mathbf{End}(x) \otimes \mathbf{End}(y) \rightarrow \mathbf{End}(F_U^0(x \boxtimes y))$$

is C^∞ -bilinear, ie. F_U^0 sends both $c \cdot \text{id}_x \otimes \text{id}_y$ and $\text{id}_x \otimes c \cdot \text{id}_y$ to $c \cdot \text{id}_{F_U^0(x \boxtimes y)}$ for all smooth functions $c \in C^\infty(U)$.

Proof. The condition on F^0 ensures that the morphism $F^0 : \mathcal{X} \boxtimes^0 \mathcal{Y} \rightarrow \mathcal{Z}$ factors through the quotient map

$$\mathcal{X} \boxtimes^0 \mathcal{Y} \rightarrow \mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y}.$$

By the universal property of stackfication, the induced map $\mathcal{X} \boxtimes_{\mathbf{Vect}}^0 \mathcal{Y} \rightarrow \mathcal{Z}$ extends to a map $\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y} \rightarrow \mathcal{Z}$, unique up to an invertible 2-cell. \square

Lemma 2.2.7.7. *The product $\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}$ presents the colimit of the diagram*

$$\mathcal{X} \boxtimes \mathbf{Vect} \boxtimes \mathbf{Vect} \boxtimes \mathcal{Y} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{X} \boxtimes \mathbf{Vect} \boxtimes \mathcal{Y} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{X} \boxtimes \mathcal{Y}$$

coequalising the action of \mathbf{Vect} on \mathcal{X} and \mathcal{Y} .⁹

Proof. We show this by a direct computation. It is enough to check the statement on 2-presheaves: as a left adjoint, stackfication preserves colimits.

Denote the colimit of the diagram, evaluated at $U \in \mathbf{Man}$ by $C(U)$. By the universal property of the colimit, a functor $F : C(U) \rightarrow \mathbf{D}$ into a category $\mathbf{D} \in \mathbf{VCat}$ is the data of a functor $F : \mathcal{X} \boxtimes \mathcal{Y} \rightarrow \mathbf{D}$, equipped with an invertible 2-cell κ whose components

$$\kappa_U(P, X, Y) : F((X \odot P) \boxtimes Y) \rightarrow (X \boxtimes (Y \odot P)),$$

are natural in $P \in \mathbf{Vect}(U)$, $X \in \mathcal{X}(U)$, $Y \in \mathcal{Y}(U)$. This 2-cell is required to be multiplicative, in that

$$\begin{array}{ccc} F(((X \odot P) \odot Q) \boxtimes Y) & \xrightarrow{\kappa_U(X \odot P, Q, Y)} & F((X \odot P) \boxtimes (Y \odot Q)) \\ \simeq \downarrow & & \downarrow \kappa_U(X, P, Y \odot Q) \\ F((X \odot (P \otimes Q)) \boxtimes Y) & \xrightarrow{\kappa_U(X, P \otimes Q, Y)} & F(X \boxtimes (Y \odot (P \otimes Q))) \end{array}$$

commutes (the unlabeled morphism is part of the \mathbf{Vect} -module structure). Further, it is required to be unital: The triangle

$$\begin{array}{ccc} & F(X \boxtimes Y) & \\ & \swarrow \quad \searrow & \\ F((X \odot C^\infty) \boxtimes Y) & \xrightarrow{\kappa_U(X, C^\infty, Y)} & F(X \boxtimes (Y \odot C^\infty)) \end{array}$$

whose top sides are given by the unitors of the \mathbf{Vect} -action, commutes. A map $\underline{F} : C \rightarrow \underline{\mathbf{D}}$ from the colimit to a \mathbf{VCat} -valued stack $\underline{\mathbf{D}}$ carries this structure pointwise, and requires the 2-cells to be compatible under the pullback morphisms.

The unitality and naturality of the 2-cell κ imply that the induced functor $\underline{F}^0 : \mathcal{X} \boxtimes^0 \mathcal{Y} \rightarrow \underline{\mathbf{D}}$ satisfies the condition of Lemma 2.2.7.6. Hence, \underline{F} factors through a morphism $\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y} \rightarrow \underline{\mathbf{D}}$.

⁹The splitting maps in the simplicial diagram are given by the unit $C^\infty \in \mathbf{Vect}$. We suppress 2-cells.

The construction above provides a natural functor

$$\mathrm{Hom}(C, \underline{\mathbf{D}}) \rightarrow \mathrm{Hom}(\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}, \underline{\mathbf{D}}).$$

We now construct a natural inverse to this functor. Let $\underline{F}' : \mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y} \rightarrow \underline{\mathbf{D}}$ be a map of \mathbf{VCat} -valued stacks. We need to give the data of a map $\mathcal{X} \boxtimes \mathcal{Y} \rightarrow \underline{\mathbf{D}}$ and a 2-cell κ as above. The former is obtained by precomposing \underline{F}' with the quotient morphism $\mathcal{X} \boxtimes \mathcal{Y} \rightarrow \mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}$. The latter is induced by the natural isomorphism

$$(X \odot P) \boxtimes Y \simeq X \boxtimes (Y \odot P),$$

provided by Lemma 2.2.7.4:

$$\begin{aligned} \underline{\mathrm{Hom}}(A \boxtimes B, (X \odot P) \boxtimes Y) &= \underline{\mathrm{Hom}}(A, X \odot P) \otimes_{C^\infty} \underline{\mathrm{Hom}}(B, Y) \\ &\simeq \underline{\mathrm{Hom}}(A, X) \otimes_{C^\infty} P \otimes_{C^\infty} \underline{\mathrm{Hom}}(B, Y) \\ &\simeq \underline{\mathrm{Hom}}(A \boxtimes B, X \boxtimes (Y \odot P)). \end{aligned}$$

(This is natural in $A, X \in \mathcal{X}(U)$, $B, Y \in \mathcal{Y}(U)$ and $P \in \mathbf{Vect}(U)$.) The isomorphism above is one of C^∞ -modules, which implies compatibility with the pullback functors of the stack. The natural isomorphism is also compatible with the tensor product of vector bundles, and unital. As such, we obtain a map $C \rightarrow \underline{\mathbf{D}}$.

Thus far, we have constructed natural maps $\mathrm{Hom}(C, \underline{\mathbf{D}}) \rightleftarrows \mathrm{Hom}(\mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}, \underline{\mathbf{D}})$. By the co-Yoneda Lemma, these are really maps $C \rightleftarrows \mathcal{X} \boxtimes_{\mathbf{Vect}} \mathcal{Y}$. It is straightforward to check they are mutual inverses. \square

Given a category $\mathbf{D} \in \mathbf{VCat}$, there is an induced linear stack

$$\underline{\mathbf{D}} := \mathbf{D}^{\mathrm{const}} \boxtimes \mathbf{Vect}.$$

Proposition 2.2.7.8. *The assignment*

$$\mathbf{D} \mapsto \underline{\mathbf{D}}$$

extends to a fully faithful, monoidal embedding

$$\mathbf{VCat} \hookrightarrow \mathbf{LinSt}.$$

Proof. The map induced on Hom-spaces by the above assignment is

$$\begin{aligned} \mathrm{Hom}_{\mathbf{VCat}}(C, \mathbf{D}) &\rightarrow \mathrm{Hom}_{\mathbf{Mod}_{\mathbf{Vect}}}(\underline{C}, \underline{\mathbf{D}}) \\ F &\mapsto F \boxtimes \mathrm{Id}_{\mathbf{Vect}}. \end{aligned}$$

This is an equivalence of bicategories. We exhibit an inverse by sending a map of stacks $G : \underline{C} \rightarrow \underline{\mathbf{D}}$ to the functor

$$G_0 : C \xrightarrow{\simeq} C \boxtimes \mathbf{Vect} \xrightarrow{G^{(*)}} \mathbf{D} \boxtimes \mathbf{Vect} \xleftarrow{\simeq} \mathbf{D}.$$

Here, we make explicit use of the equivalence $\underline{\mathbf{C}}(*) \simeq \mathbf{C} \boxtimes \mathbf{Vect}$, and the unitor $\mathbf{C} \simeq \mathbf{C} \boxtimes \mathbf{Vect}$.¹⁰ The composite

$$F \mapsto F \boxtimes \mathrm{Id}_{\mathbf{Vect}} \mapsto (F \boxtimes \mathrm{Id}_{\mathbf{Vect}})_0 \simeq F$$

is the identity on the nose. In the other direction, $G \mapsto G_0 \mapsto G_0 \boxtimes \mathrm{Id}_{\mathbf{Vect}}$. There is a natural equivalence between G and $G_0 \boxtimes \mathrm{Id}_{\mathbf{Vect}}$, whose component at $c \boxtimes P \in \underline{\mathbf{C}}(U)$ is

$$G^0(c) \boxtimes P \simeq (G^0(c) \boxtimes C^\infty(U)) \odot P \simeq G(c \boxtimes C^\infty(U)) \odot P \simeq G(c \boxtimes P).$$

The outer two equivalences use the \mathbf{Vect} -module structure of G , and the middle equivalence uses compatibility with pullback along the map $U \rightarrow *$.

The monoidal structure of the 2-functor is constructed via the equivalence

$$(\mathbf{C}^{\mathrm{const}} \boxtimes \mathbf{Vect}) \boxtimes_{\mathbf{Vect}} (\mathbf{D}^{\mathrm{const}} \boxtimes \mathbf{Vect}) \simeq (\mathbf{C} \boxtimes \mathbf{D})^{\mathrm{const}} \boxtimes (\mathbf{Vect} \boxtimes_{\mathbf{Vect}} \mathbf{Vect}),$$

which is natural in \mathbf{C} and \mathbf{D} . By Lemma 2.2.7.7,

$$\mathbf{Vect} \boxtimes_{\mathbf{Vect}} \mathbf{Vect} \simeq \mathbf{Vect},$$

and we get the required 2-cell

$$\underline{\mathbf{C}} \boxtimes_{\mathbf{Vect}} \underline{\mathbf{D}} \xrightarrow{\simeq} \underline{\mathbf{C} \boxtimes \mathbf{D}}.$$

The remaining pieces of data for the monoidal 2-functor is the map of monoidal units

$$\mathbf{Vect} \simeq \mathbf{Vec}^{\mathrm{const}} \boxtimes \mathbf{Vect},$$

and 2-cells relating the associators and unitors (see [Hou13, Def 3.3]). The coherence conditions for a functor of monoidal bicategories are difficult to check, but in this case vastly simplified by the fact that the associators in \mathbf{VCat} and \mathbf{LinSt} can be chosen to be identities. \square

REMARK 2.2.7.9. Proposition 2.2.7.8 says that the functor $(-)^{\mathrm{const}} \boxtimes \mathbf{Vect}$ freely turns \mathbf{Vec} -modules into \mathbf{Vect} -modules.

2.3 Orbisimple Categories

A fusion category is a linear monoidal category \mathbf{C} whose underlying linear category admits an equivalence

$$\mathbf{C} \stackrel{\mathrm{lin.}}{\cong} \mathbf{Vec}^{\oplus S},$$

where S is a finite set. To define orbifold tensor categories, we would like to replace the right hand side with a symbol $\mathbf{Vec}^{\oplus \mathcal{M}}$, where \mathcal{M} is an orbifold. The corresponding object must keep track of the smooth structure of \mathcal{M} . Its role in this story is played by the stack $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ of Definition 2.2.6.9.

¹⁰The unitor is part of the canonical \mathbf{Vec} -module structure on objects of \mathbf{VCat} .

Definition 2.3.0.1. An *orbisimple category* is a linear stack over \mathbf{Man} equivalent to $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ for some orbifold \mathcal{M} and gerbe \mathcal{G} over \mathcal{M} . If \mathcal{M} can be represented by a manifold, we also call such a stack a *manisimple category*.

We use the adjective *compact* when \mathcal{M} can be chosen to be compact.

REMARK 2.3.0.2. Orbisimple and manisimple are shortenings of the more descriptive but unwieldy terms *orbifold/manifold semisimple*.

In this section, we establish a number of structural results about orbisimple categories. We begin with the study of the underlying linear semisimple category, the value of an orbisimple category on the point. The category of U -families of objects is no longer semisimple for non-trivial U , and not all morphisms $\mathcal{A} \rightarrow \mathcal{B}$ admit an epi-mono/image factorisation $\mathcal{A} \twoheadrightarrow \mathcal{C} \hookrightarrow \mathcal{B}$. The central result of this section is the fact that *locally*, such factorisations still exist (Proposition 2.3.2.21).

2.3.1 The Category of Points

Let \mathcal{M} be an orbifold, and \mathcal{G} a gerbe over it. We study the linear stack $\underline{\mathbf{C}} := \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$, and in particular $\mathbf{C} := \underline{\mathbf{C}}(*)$, the category of \mathcal{G} -twisted skyscraper sheaves on \mathcal{M} . By definition, any orbisimple category admits an equivalence to such a stack.

NOTATION 2.3.1.1. We denote by $\mathbf{C}^U := \underline{\mathbf{C}}(U)$ the category of U -families. An object $\mathcal{S} \in \mathbf{C}^U$ is also suggestively denoted by $\mathcal{S} : U \rightarrow \mathbf{C}$. Given a point $x : * \rightarrow U$, we denote the pullback by $\mathcal{S}(x) := x^*\mathcal{S}$. For the pullback along a map $V \rightarrow U$, we write $\mathcal{S}|_V$. We will use the same notation for morphisms of families and their pullbacks.

Let $x : * \rightarrow \mathcal{M}$ be a point of \mathcal{M} . We write $\Gamma_x := \text{End}(x)$ for its stabiliser group (which is finite by assumption). We denote the restriction of a gerbe $\mathcal{G} \rightarrow \mathcal{M}$ along x by $\mathcal{G}_x = x^*\mathcal{G}$. By Corollary 2.2.4.8, the gerbe is locally (around x) characterised by a class $[\theta_x] \in \mathbb{H}^2(\mathcal{B}\Gamma_x, \mathbb{C}^\times)$. We write θ_x to denote a representing 2-cocycle for this cohomology class.

Definition 2.3.1.2. We call $\mathbf{C}^{(x)} := (x^*\underline{\mathbf{C}})(*)$ the *category at x* .

The category at x includes into \mathbf{C} as the full subcategory on skyscraper sheaves whose support is concentrated at $[x]$. We abuse notation and denote by $\mathbf{C}^{(x)}$ its essential image under this inclusion. There are no morphisms between sheaves with disjoint support, so non-isomorphic points $x \not\cong y$ of \mathcal{M} yield disjoint full subcategories $\mathbf{C}^{(x)} \cap \mathbf{C}^{(y)} = \{0\}$. Meanwhile, an isomorphism $\phi : x \xrightarrow{\cong} x'$ induces an equivalence $\phi^* : x^* \xrightarrow{\cong} x'^*$, and by extension $\mathbf{C}^{(x)} \simeq \mathbf{C}^{(x')}$.

Lemma 2.3.1.3. *Let $x : * \rightarrow \mathcal{M}$ be a point of \mathcal{M} with stabiliser Γ_x , and a representing 2-cocycle θ_x for \mathcal{G}_x . Denote by \mathbb{C} the canonical line bundle over the point. Then*

1. $\mathbf{C}^{(x)} \cong \mathbf{Rep}^{\theta_x}(\Gamma_x)$ as linear categories, such that $x_*\mathbb{C} \cong \mathbb{C}^{\theta_x}[\Gamma_x]$
2. $\mathbf{C}^{(x)} \subset \mathbf{C}$ is the subcategory Karoubi-generated by $x_*\mathbb{C}$
3. Every simple object of \mathbf{C} is contained in $\mathbf{C}^{(x)}$ for some point $x : * \rightarrow \mathcal{M}$.

Proof. Property 1 is discussed in Example 2.2.4.12, and Property 2 is Corollary 2.2.1.8. A skyscraper sheaf with support at multiple non-isomorphic points splits into a direct sum of skyscraper sheaves with support at a single isomorphism class. Hence, a simple skyscraper sheaf must be contained in some $\mathbf{C}^{(x)}$, which establishes Property 3. \square

Denote by $\text{Irr } \mathbf{C}$ the set of isomorphism classes of simple objects of \mathbf{C} .

Corollary 2.3.1.4. *A choice of representing points $\{x : * \rightarrow \mathcal{M}\}$ and representing 2-cocycles θ_x for all $[x] \in |\mathcal{M}|$ yields an explicit bijection of sets*

$$\text{Irr } \mathbf{C} \xrightarrow{\cong} \coprod_{[x] \in |\mathcal{M}|} \text{Irr } \mathbf{C}^{(x)} = \coprod_{[x] \in |\mathcal{M}|} \text{Irr } \mathbf{Rep}^{\theta_x}(\Gamma_x),$$

and in fact,

$$\mathbf{C} \xrightarrow{\cong} \bigoplus_{[x] \in |\mathcal{M}|} \mathbf{Rep}^{\theta_x}(\Gamma_x)$$

as linear categories. In particular, \mathbf{C} is a direct sum of semisimple abelian categories, and hence semisimple and abelian itself.

A choice of representatives for $|\mathcal{M}|$ associates to a simple object $X \in \mathbf{C}$ a point x and an irreducible θ_x -twisted representation ρ of its stabiliser Γ_x . The dimension of the underlying vector space of ρ and $|\Gamma_x|$ are independent of the choices made.

Definition 2.3.1.5. With notation as above, the *dimension* of a simple object is

$$\dim X := \frac{\text{vdim } \rho}{|\Gamma_x|},$$

where vdim denotes the *vector space dimension* of the representation ρ . This is extended linearly to all objects in \mathbf{C} .

EXAMPLE 2.3.1.6. Consider the orbifold $\mathcal{M} = [\mathbb{R}/(\mathbb{Z}/2)]$, equipped with the trivial gerbe. Its quotient space is $|\mathcal{M}| = \mathbb{R}_0^+$. The category $\mathbf{Sky}_{\mathcal{M}}(*)$ is generated under direct sum by a half-line of objects $\mathbf{C}_{x>0}$, and a pair of objects $\mathbf{C}_0^+, \mathbf{C}_0^- \in \mathbf{Rep}(\mathbb{Z}/2)$, corresponding to the trivial and sign representation of the stabiliser $\mathbb{Z}/2$ at $0 : * \rightarrow [\mathbb{R}/(\mathbb{Z}/2)]$. They assemble into the moduli space of simple objects as



and their dimensions are

$$\dim \mathbf{C}_{x>0} = 1 \qquad \dim \mathbf{C}_0^+ = \dim \mathbf{C}_0^- = \frac{1}{2}.$$

Definition 2.3.1.7. The *support* $\text{supp } X$ of a simple object $X \in \mathbf{C}$ is the isomorphism class $[x] \in |\mathcal{M}|$ such that X is in the essential image of $\mathbf{C}^{(x)}$. It is extended to \mathbf{C} by sending direct sums to unions of sets.

Ie. the support of an object $Z \in \mathbf{C}$ is the union of the supports of the simple objects it decomposes into.

REMARK 2.3.1.8. Each object $X \in \mathbf{C}$ is a summand of an object of the form

$$\overline{X} = \bigoplus_{i=0}^n x_{i,*} E_i,$$

where $n \in \mathbb{N}$ is a positive integer, the $x_i : * \rightarrow \mathcal{M}$ are points of the orbifold, and the E_i are vector spaces (= vector bundles over the point). The support of X is then a subset of the union of the points $[x_i] : * \rightarrow |\mathcal{M}|$.

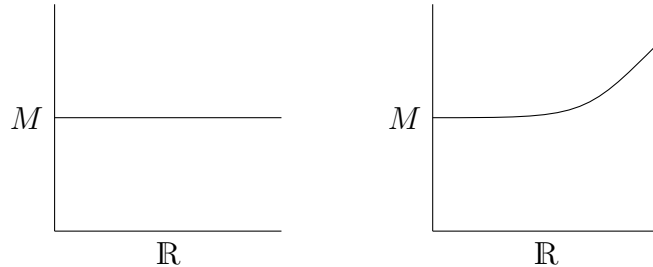
2.3.2 Morphisms of Families

This section is the most technical part of this chapter. We study morphism spaces between families of twisted skyscraper sheaves. In an abelian category \mathbf{D} , every morphism $F : A \rightarrow B$ admits an epi-mono factorisation

$$F : A \twoheadrightarrow \text{im}F \hookrightarrow B.$$

If \mathbf{D} is further semisimple, any morphism may be written as a matrix of maps of simple summands. In this case, $\text{im}F$ may be identified both as a summand of A and a summand of B . We established in Corollary 2.3.1.4 that the category \mathbf{C} is semisimple and abelian. This is not the case for \mathbf{C}^U unless U is discrete:

EXAMPLE 2.3.2.1. Consider two simple \mathbb{R} -families \mathcal{A}, \mathcal{B} of skyscraper sheaves over a manifold M . Assume further the support $\text{supp } \mathcal{A}(x) = \text{supp } \mathcal{B}(x)$ agrees for $x < 0 \in \mathbb{R}$ and is disjoint for $x > 0$ (see the example picture below).



Then there exists a morphism $F : \mathcal{A} \rightarrow \mathcal{B}$, non-zero on $x < 0$ and zero for $x \geq 0$. The families \mathcal{A} and \mathcal{B} don't share any summands, yet there exists a non-zero map between them. Hence $\mathbf{C}^{\mathbb{R}}$ is not semisimple. The map F also admits neither kernel nor cokernel, hence $\mathbf{C}^{\mathbb{R}}$ is not abelian.

However, the epi-mono factorisation via a summand still holds locally in \mathbf{C}^U : We show in Proposition 2.3.2.21 that for every morphism $F : \mathcal{A} \rightarrow \mathcal{B}$ and $x \in U$, there locally exists an isomorphism of direct summands of \mathcal{A} and \mathcal{B} which is equal to F at x . (The isomorphism does not, in general, equal F away from x – not even in a small neighbourhood.) This local factorisation result has an important implication, which we record as Corollary 2.3.2.22: The existence of a morphism $F : \mathcal{A} \rightarrow \mathcal{B}$ which is non-zero at $x \in U$ implies that \mathcal{A} and \mathcal{B} share a non-trivial summand in a neighbourhood of x . Our strategy to prove Proposition 2.3.2.21 will be to construct a local embedding of the morphism space $\text{Hom}_{\mathbf{C}^U}(\mathcal{A}, \mathcal{B})$ into a space of smooth matrix-valued functions.

Let $\mathcal{G} \rightarrow \mathcal{M}$ be an orbifold equipped with a gerbe. We may choose a cover of \mathcal{M} by a disjoint union $\coprod_i [M_i/\Gamma_i] \twoheadrightarrow \mathcal{M}$ of quotient orbifolds such that all M_i are contractible and all Γ_i are finite. Let $\iota : [M/\Gamma] \hookrightarrow \mathcal{M}$ be the inclusion of such a quotient suborbifold. It induces a pair of adjoint maps of stacks

$$\mathbf{Shv}_{[M/\Gamma]}^{\iota^* \mathcal{G}} \xleftarrow[\iota_*]{\iota^*} \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}},$$

which identifies $\mathbf{Shv}_{[M/\Gamma]}^{\iota^* \mathcal{G}}$ as the full substack of $\mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$ on (families of) sheaves whose support is concentrated in $[M/\Gamma] \subset \mathcal{M}$. Any family of skyscraper sheaves $\mathcal{S} \in \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$ locally decomposes into a direct sum of families that factor through the inclusion of disjoint suborbifolds of \mathcal{M} . Hence, we may study the local structure of $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ by studying $\mathbf{Sky}_{[M/\Gamma]}^{\iota^* \mathcal{G}}$.

Consider a quotient orbifold $[M/\Gamma]$ as above with gerbe \mathcal{G} , equipped with an atlas $M \twoheadrightarrow [M/\Gamma]$ and a trivialisation of the pullback gerbe over M . The twisted equivariant adjunction of Proposition 2.2.5.3 gives an adjoint pair of maps of stacks

$$\mathbf{Shv}_M^{\mathcal{I}} \xleftarrow[I=q_*]{u=q^*} \mathbf{Shv}_{[M/\Gamma]}^{\mathcal{G}}.$$

The components of these maps at $U \in \mathbf{Man}$ are given by the adjoint pair of functors

$$\mathbf{Shv}_{\mathbf{C}^\infty}(M \times U) \xleftarrow[I_U := (q \times \text{id}_U)_*]{u_U := (q \times \text{id}_U)^*} \mathbf{Shv}^{\mathcal{G}}([M/\Gamma] \times U).$$

Both I_U and u_U preserve the property of being a family of twisted skyscraper sheaves, hence this pair of maps restricts to an adjunction

$$\mathbf{Sky}_M^{\mathcal{I}} \xleftarrow[I]{u} \mathbf{Sky}_{[M/\Gamma]}^{\mathcal{G}}.$$

Every U -family $\mathcal{T} \in \mathbf{Sky}_M(U) \subset \mathbf{Shv}_{\mathbf{C}^\infty}(M \times U)$ locally decomposes as a direct sum of pushforwards of vector bundles: There exists a cover $Y \twoheadrightarrow U$ with a trivialisation of the pullback gerbe, a finite set of functions $f_i : Y \rightarrow M$, and vector bundles E_i over Y such that

$$\mathcal{T}|_Y \cong \bigoplus_i \text{Gr}(f_i)_* E_i.$$

The image of a family $\text{Gr}(f)_* E$ under the induction functor I associated to $M \twoheadrightarrow [M/\Gamma]$ is the family $\text{Gr}(q \circ f)_* E$. Any vector bundle E locally splits as a direct sum of line bundles, so any family in the image of I locally decomposes as the pushforward of a line bundle via the graph of a function $g : Y \rightarrow [M/\Gamma]$. Conversely, if a family is locally of this form, it is in the image of an induction functor.

Definition 2.3.2.2. A family $\mathcal{S} : U \rightarrow \mathbf{C}$ is a *basic mobile* if it is locally equivalent to a family of the form $\text{Gr}(f)_* C^\infty$, where f is a smooth map and C^∞ is the (sheaf of sections of) the trivial line bundle over U . A family is *mobile* if it is locally a direct sum of basic mobiles.

By the discussion above, mobile families are precisely the families which locally lie in the image of an induction functor.

EXAMPLE 2.3.2.3. Recall from Lemma 2.3.1.3 that $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(*)$ decomposes as a direct sum

$$\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(*) \cong \bigoplus_{[x] \in |\mathcal{M}|} \mathbf{Rep}^{\theta_x}(\Gamma_x).$$

The mobile objects (mobile $*$ -families) are precisely those objects which are isomorphic to direct sums of twisted group rings

$$\mathbf{C}^{\theta_x}[\Gamma_x] \in \mathbf{Rep}^{\theta_x}(\Gamma_x).$$

Definition 2.3.2.4. Let $s : U \rightarrow \mathcal{M}$ be a smooth map and τ a trivialisation of the pullback gerbe over U . The *basic mobile induced by s and τ* is the U -family

$$\mathrm{Ind}^{\tau}(s) := \mathrm{Gr}(s)_{*} \tau \cdot C^{\infty}.$$

Given a set of smooth maps $\{s_i : U \rightarrow \mathcal{M}\}$ and trivialisations τ_i , we denote the associated mobile by

$$\mathrm{Ind}^{\{\tau_i\}}\{s_i\} := \bigoplus_i \mathrm{Ind}^{\tau_i} s_i := \bigoplus_i \mathrm{Gr}(s_i)_{*} \tau_i \cdot C^{\infty},$$

and call it the *mobile induced by $\{s_i, \tau_i\}$* .

Mobile families play well with pullbacks. Given a map $f : V \rightarrow U$, there is a canonical isomorphism

$$f^* \mathrm{Ind}^{\tau}(s) \simeq \mathrm{Ind}^{f^* \tau}(s \circ f) \in \mathbf{C}^V.$$

Pullbacks of basic mobiles yield basic mobiles and thus the same is true for mobile families. Each family $\mathcal{A} : U \rightarrow \mathbf{C}$ is locally a summand of a mobile family $\overline{\mathcal{A}}$.

Lemma 2.3.2.5. *Let $U \in \mathbf{Man}$ be contractible and $f : U \rightarrow \mathcal{M}$ a smooth map. For any pair (τ, τ') of trivialisations of the pullback gerbe $f^* \mathcal{G}$, the families induced by (f, τ) and (f, τ') are abstractly isomorphic:*

$$\mathrm{Ind}^{\tau}(f) \upharpoonright_Y \cong \mathrm{Ind}^{\tau'}(f) \upharpoonright_Y.$$

Proof. The different choice of trivialisation amounts to tensoring with the line bundle $\mathcal{L}_{\tau, \tau'}$ (associated to $\tau^{-1} \tau'$), pushed forward from U via $\mathrm{Gr}(f)$. As U is contractible, this line bundle is trivial, and tensoring with it does not change the isomorphism type. \square

Every manifold admits a cover by contractible manifolds, hence we may locally ignore the chosen trivialisations when we are only working up to isomorphism.

NOTATION 2.3.2.6. Let $f : U \rightarrow \mathcal{M}$ be as in Lemma 2.3.2.5. We write $\mathrm{Ind}(f)$ to denote the isomorphism type of the family $\mathrm{Ind}^{\tau}(f)$.

Definition 2.3.2.7. A *mobile cover* of a family \mathcal{A} is a mobile family $\overline{\mathcal{A}}$, equipped with data $\mathcal{A} \hookrightarrow \overline{\mathcal{A}} \twoheadrightarrow \mathcal{A}$ identifying \mathcal{A} as a summand, such that the support of the two families agrees pointwise:

$$\mathrm{supp}(\overline{\mathcal{A}}) = \mathrm{supp}(\mathcal{A}) \subset |\mathcal{M}|.$$

Lemma 2.3.2.8. *Each $\mathcal{A} : U \rightarrow \mathbf{C}$ locally admits a mobile cover around any $x \in U$. Such a cover can be obtained by applying the equivariantisation monads Iu associated to a choice of orbifold charts around $\text{supp } \mathcal{A}(x)$.*

Proof. Pick pairwise non-isomorphic representatives $y_i : * \rightarrow \mathcal{M}$ for $\text{supp } \mathcal{A}(x) \subset |\mathcal{M}|$, and pick pairwise disjoint orbifold charts $M_i \rightarrow [M_i/\Gamma_i] \hookrightarrow \mathcal{M}$ around the points y_i . There exists a neighbourhood V of $x \in U$ such that $\mathcal{A}|_V$ splits into a direct sum of families with support concentrated in one of the orbifold charts.

Without loss of generality, we may assume that \mathcal{A} has support concentrated in a single quotient orbifold $[M/\Gamma]$. Let $I \dashv u$ be the twisted equivariantisation adjunction associated to the orbifold chart $M \rightarrow [M/\Gamma]$. The family $Iu\mathcal{A}$ is mobile because it is in the image of I . It is easy to check that its support pointwise agrees with that of \mathcal{A} . Lastly, Corollary 2.2.5.4 tells us that the unit and counit of the adjunction identify \mathcal{A} as a direct summand of $Iu\mathcal{A}$. \square

The monad Iu is trivial for an orbifold patch where the quotient group Γ is trivial. For an effective orbifold, this is the generic region.

Corollary 2.3.2.9. *Assume \mathcal{M} is an effective orbifold. Let $\mathcal{A} : U \rightarrow \mathbf{C}$ be such that $\text{supp } \mathcal{A}(x)$ is contained in the generic region of the orbifold. Then \mathcal{A} is mobile in a neighbourhood of x .*

Let $q : M \rightarrow \mathcal{M}$ be an étale map and τ a trivialisation of the pullback gerbe $q^*\mathcal{G}$. Let $\mathcal{A} : U \rightarrow \mathbf{C}$ be a U -family of \mathcal{G} -twisted skyscraper sheaves with support contained in the image of q .

Definition 2.3.2.10. *A set of support lifts for \mathcal{A} with respect to $(q : M \rightarrow \mathcal{M}, \tau : \mathcal{I} \xrightarrow{\cong} q^*\mathcal{G})$ is a collection $\{\tilde{s}_i : U \rightarrow M\}_{i \in I}$, such that the induced mobile family*

$$\text{Ind}^{\{\tilde{s}_i^* \tau\}}(q \circ \tilde{s}_i) = \bigoplus_{i \in I} \text{Ind}^{\tilde{s}_i^* \tau}(q \circ \tilde{s}_i)$$

is a mobile cover of \mathcal{A} .

Given that we can always find mobile covers locally (Lemma 2.3.2.8), we can always locally find support lifts:

Lemma 2.3.2.11. *For any family $\mathcal{A} : U \rightarrow \mathbf{C}$, and $x \in U$, there exists an open neighbourhood $V \ni x$, a chart $M \rightarrow \mathcal{M}$ and support lifts $\{\tilde{s}_i : V \rightarrow M\}$ for \mathcal{A} .*

There are families that indeed only admit these lifts locally.

EXAMPLE 2.3.2.12. Denote by S^1 the standard circle. There is a multivalued smooth map $S^1 \rightarrow S^1$, sending each point x to a pair of opposite points $\pm x/2$, rotating at half speed. There are S^1 -families of skyscraper sheaves on S^1 with this support function, but they do not admit a global support lift as there exists no map $S^1 \rightarrow S^1$ of degree 1/2.

Proposition 2.3.2.13. *Let $\mathcal{A}, \mathcal{B} : U \rightarrow \mathbf{C}$ be two U -families. There exists a cover $Y \rightarrow U$, and a faithful embedding*

$$\text{Hom}_{\mathbf{C}^Y}(\mathcal{A}|_Y, \mathcal{B}|_Y) \hookrightarrow C^\infty(Y, \text{Mat}_{m \times n}(\mathbf{C})) = \text{Mat}_{m \times n}(C_Y^\infty)$$

into a space of smooth functions valued in complex $(m \times n)$ -matrices for some $m, n \in \mathbb{N}$.

Proof. We prove the statement in a neighbourhood V of a point $x \in U$. By choosing V small enough, we may assume the family $\mathcal{A}|_V$ splits as a direct sum $\oplus_i \mathcal{A}_i$ of families whose supports are contained in pairwise disjoint suborbifolds $[M_i/\Gamma_i] \hookrightarrow \mathcal{M}$. By further restricting V , we may assume the same is true for \mathcal{B} . Families with disjoint support have no non-zero morphisms between, so it suffices to treat the case where \mathcal{A} and \mathcal{B} factor through the same quotient orbifold $[M/\Gamma] \hookrightarrow \mathcal{M}$.

Denote the induced gerbe over $[M/\Gamma]$ by \mathcal{G} , and equip $q : M \rightarrow [M/\Gamma]$ with a trivialisation of $q^*\mathcal{G}$. Now consider the twisted equivariantisation adjunction associated to $q : M \rightarrow [M/\Gamma]$. We may think of $\mathcal{A}, \mathcal{B} \in \mathbf{Sky}_{[M/\Gamma]}^{\mathcal{G}}(U) \subset \mathbf{Shv}^{\mathcal{G}}([M/\Gamma] \times U)$ as objects in $\mathbf{Sky}_M(U) \subset \mathbf{Shv}_{C^\infty}(M \times U)$, equipped with \mathcal{G} -twisted Γ -equivariance data. The forgetful functor u discards the equivariance data of an equivariant sheaf. It faithfully embeds

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathcal{G}}([M/\Gamma] \times U)}(\mathcal{A}, \mathcal{B}) \hookrightarrow \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(M \times V)}(u\mathcal{A}, u\mathcal{B})$$

as the Γ -equivariant morphisms.

By possibly restricting V further, we may assume $u\mathcal{A}$ and $u\mathcal{B}$ are of the form

$$u\mathcal{A} \cong \bigoplus_{i=0}^m \mathrm{Gr}(f_i)_* C^\infty \qquad u\mathcal{B} \cong \bigoplus_{j=0}^n \mathrm{Gr}(g_j)_* C^\infty,$$

where $\{f_i : V \rightarrow M\}$ and $\{g_j : V \rightarrow M\}$ are smooth functions. Using the isomorphism of the pullback-pushforward adjunction,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(M \times V)}(u\mathcal{A}, u\mathcal{B}) &\cong \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(M \times V)}\left(\bigoplus_{i=0}^m \mathrm{Gr}(f_i)_* C^\infty, \bigoplus_{j=0}^n \mathrm{Gr}(g_j)_* C^\infty\right) \\ &\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(V)}(\mathrm{Gr}(g_j)^* \mathrm{Gr}(f_i)_* C^\infty, C^\infty). \end{aligned}$$

We examine a single summand of the form

$$\mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(V)}(\mathrm{Gr}(g)^* \mathrm{Gr}(f)_* C^\infty, C^\infty).$$

By Lemma 2.2.3.11, we know the sheaf $\mathrm{Gr}(g)^* \mathrm{Gr}(f)_* C^\infty$ is zero on the complement of the subset $V_{f=g} \subset V$ where the functions are equal. If a morphism $\mathrm{Gr}(g)^* \mathrm{Gr}(f)_* C^\infty \rightarrow C^\infty$ is non-zero at a point $v \in V$, it must stay non-zero in a neighbourhood of v . Hence, such a morphism can only be non-zero on the interior $V_{f=g}^\circ$ of $V_{f=g}$. Restricted to $V_{f=g}^\circ$, however, $\mathrm{Gr}(g)^* \mathrm{Gr}(f)_* C^\infty = C^\infty$, so the Hom-space becomes

$$\mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(V_{f=g}^\circ)}(C^\infty, C^\infty) = C^\infty(V_{f=g}^\circ, \mathbb{C}).$$

This way, we obtain an embedding

$$\mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(M \times V)}(u\mathcal{A}, u\mathcal{B}) \hookrightarrow \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(V)}(C^\infty, C^\infty)^{\oplus m \times n} \simeq \mathrm{Mat}_{m \times n}(C_V^\infty).$$

The image of the embedding is the subspace of those matrix-valued functions which are \mathcal{G} -twisted Γ -equivariant¹¹ and whose (i, j) -component is zero outside the open set $V_{f_i=g_j}^\circ$. \square

¹¹Recall that the summands of $u\mathcal{A}$ and $u\mathcal{B}$ are acted upon by Γ . The identifications made above translate this into an action on the set of rows and the set of columns of the matrices.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism of U -families, and pick an embedding

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \hookrightarrow \mathrm{Mat}_{m \times n}(C^\infty)$$

around $x \in U$. This identifies $F(x)$ with a matrix. There is an open neighbourhood V of x where this matrix is in the image of the embedding. Denote by $\bar{F} : \mathcal{A}|_V \rightarrow \mathcal{B}|_V$ the morphism of V -families which is sent to the constant function at $F(x)$ under the above embedding.

NOTATION 2.3.2.14. Let $\mathcal{S} : U \rightarrow \mathbf{C}$ and $p \in \mathrm{End}(\mathcal{S})$ a split idempotent. We denote the summand splitting the idempotent by (\mathcal{S}, p) .

Lemma 2.3.2.15. *Let $\bar{\mathcal{A}} : U \rightarrow \mathbf{C}$ be a U -family, and $p \in \mathrm{End}(\bar{\mathcal{A}})$ an idempotent. Let \bar{p} be a map obtained as above by locally extending p_x for some point $x \in U$. Then \bar{p}_x is an idempotent. Let $\bar{\mathbf{C}}^U$ be a completion of \mathbf{C}^U where both p and \bar{p} are split. Then the summands of $\bar{\mathcal{A}}$ picked out by p and \bar{p} are isomorphic in a neighbourhood of x .*

Proof. Idempotency can be checked pointwise, thus \bar{p} is an idempotent. We will use the idempotents themselves to establish the isomorphism. An idempotent matrix has the property that all its eigenvalues are either 0 or 1. Eigenvalues vary smoothly in a smooth family of matrices, hence the rank of an idempotent matrix is locally constant. Now assume p and \bar{p} are split by summands $(\bar{\mathcal{A}}, p)$ and $(\bar{\mathcal{A}}, \bar{p})$, respectively. The maps p and \bar{p} now define morphisms $(\bar{\mathcal{A}}, p) \rightleftarrows (\bar{\mathcal{A}}, \bar{p})$. As $p(x) = \bar{p}(x)$, the maps are isomorphisms at x , ie. they have full rank. This is then also true in an open neighbourhood, which proves the desired isomorphism. \square

By Lemma 2.3.2.8, every family $\mathcal{A} : U \rightarrow \mathbf{C}$ is locally a summand of a family of the form $\bar{\mathcal{A}} = \bigoplus_i \mathrm{Gr}(f_i)_* E_i$. This summand is picked out by an idempotent $p \in \mathrm{End}(\bar{\mathcal{A}})$. We show that this idempotent may be diagonalised around x .

Corollary 2.3.2.16 (Local Normal Form). *A family $\mathcal{A} : U \rightarrow \mathbf{C}$ locally decomposes as*

$$\mathcal{A} \cong \bigoplus_i (\mathrm{Gr}(s_i)_* \tau_i \cdot E_i, p_i),$$

where the $s_i : U \rightarrow \mathcal{M}$ are smooth maps, the τ_i are trivialisations of the pullback gerbes, the E_i are vector bundles on U , and the p_i are idempotents on $\mathrm{Gr}(s_i)_* \tau_i \cdot E_i$. The set $\{s_i\}$ can be chosen to be obtained from any set of local support lifts $\{\tilde{s}_i : U \rightarrow M\}$ for \mathcal{A} to a chart $q : M \rightarrow \mathcal{M}$.

Proof. Near any point $x \in U$, any set of local support lifts corresponds to a mobile cover $\tilde{\mathcal{A}} \cong \bigoplus_i \mathrm{Gr}(s_i)_* E_i$ of \mathcal{A} , where $s_i = q \circ \tilde{s}_i$. Around x , \mathcal{A} is a summand of $\tilde{\mathcal{A}}$, and thus picked out by an idempotent p on \mathcal{A} . At x , we can decompose the projection as stated above (in such a way that it doesn't mix the summands). By Lemma 2.3.2.15, we can assume idempotents are locally constant, so the decomposition holds on an open neighbourhood of x . \square

Corollary 2.3.2.17. *The category \mathbf{C}^U of U -families is Karoubi complete.*

Proof. By definition $\mathbf{C}^U = \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U) \subset \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}(U)$ is cut out by the condition that $\mathcal{S} \in \mathbf{C}^U$ locally decomposes as a direct sum of sheaves of the form $\mathrm{Gr}(f)_{*}\tau \cdot E$ for some smooth map f , trivialisation τ of $f^*\mathcal{G}$ and vector bundle E . The category $\mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$ is manifestly Karoubi-complete, so it suffices to check this condition is preserved under the operation of taking a summand.

Let $\mathcal{S} : U \rightarrow \mathbf{C}$ and $p \in \mathrm{End}(\mathcal{S})$ an idempotent. This idempotent splits in $\mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$, and we denote the corresponding summand by (\mathcal{S}, p) . The sheaf \mathcal{S} locally pulls back to an object of the form $u\mathcal{S} = \bigoplus_i \mathrm{Gr}(f_i)_{*}\tau_i \cdot E_i$. The splitting of p pulls back to a sheaf $(u\mathcal{S}, up)$. We must show that this sheaf is again of the basic form above. By Corollary 2.3.2.16, we may assume the idempotent is diagonal in the direct-sum decomposition. It hence suffices to consider a single summand $\mathrm{Gr}(f)_{*}\tau \cdot E$, equipped with an idempotent p . As $\mathrm{Gr}(f)$ is an embedding, p is really an idempotent on E , which is split by the subbundle (E, p) . Hence $(\mathrm{Gr}(f)_{*}\tau \cdot E, p)$ is represented by the basic family $\mathrm{Gr}(f)_{*}\tau \cdot (E, p)$, which finishes the proof. \square

Recall the dimension of an object $X \in \mathbf{C}$ (Definition 2.3.1.5).

Lemma 2.3.2.18. *The dimension of an object in a family \mathcal{S} is locally constant.*

Proof. By the local form of families given above, it is enough to show this for a single family of the form $(\mathrm{Gr}(s)_{*}E, p)$, supported in a quotient suborbifold $[M/\Gamma]$ of \mathcal{M} . Factoring the pushforward through a cover $M \rightarrow [M/\Gamma]$, we compute the vector space dimension of $\mathrm{Gr}(s)_{*}E$ at $x \in U$ as $\mathrm{vdim} \mathrm{Gr}(s(x))_{*}E = \mathrm{vdim} I\mathrm{Gr}(\tilde{s}(x))_{*}E = |\Gamma_{s(x)}| \cdot \mathrm{vdim} E_x$. (The induction functor sums over the Γ -orbit of $\mathrm{Gr}(\tilde{s})_{*}$, so E_x gets pushed onto $\tilde{s}(x)$ with multiplicity its stabiliser group.) Now, $\mathrm{vdim} E_x$ is locally constant as E is a vector bundle, thus so is $\mathrm{dim} \mathrm{Gr}(s(x))_{*}E = \mathrm{vdim} \mathrm{Gr}(s(x))_{*}E / |\Gamma_{s(x)}| = \mathrm{vdim} E_x$. The endomorphism p is an idempotent, and thus of constant rank. It simply multiplies the dimension by a constant factor $\mathrm{rk} p / \max \mathrm{rk} = \mathrm{rk} p / \mathrm{dim} \mathrm{Gr}(s)_{*}E$. \square

This shows that the dimension is in fact a well-defined invariant of a family over a connected manifold.

Definition 2.3.2.19. We call a family $\mathcal{A} : U \rightarrow \mathbf{C}$ *étale*, *immersive* or *submersive*, if for all $x \in U$, there exist support lifts on some neighbourhood of x which are étale, immersive or submersive, respectively.

Lemma 2.3.2.20. *If \mathcal{M} is an effective orbifold, every submersive family is mobile.*

Proof. Let $\mathcal{S} : U \rightarrow \mathbf{C}$ be a submersive family. By assumption, it has a set of submersive local support lifts $\{\tilde{s}_i : U \rightarrow M\}$ around $x \in U$ to some chart $M \rightarrow \mathcal{M}$. By Corollary 2.3.2.16, \mathcal{S} locally splits as a direct sum corresponding to these lifts, and the idempotents picking out these summands can be chosen as locally constant projections in any trivialisations of the vector bundles involved. Let \mathcal{T} be the summand corresponding to a local support lift \tilde{s}_j . As it is submersive, the preimage of the generic region under \tilde{s}_j is nonempty in any neighbourhood of x . By Corollary 2.3.2.9, the projection in the generic region picks out a sub-vector bundle. As we can assume the projections are constant, this is true on an open neighbourhood of x in U . Hence, \mathcal{T} is the pushforward of this sub-vector bundle, and thus mobile. \square

We have now adequately prepared to prove the central result of this section.

Proposition 2.3.2.21 (Local Factorisation). *Let $\mathcal{A}, \mathcal{B} : U \rightarrow \mathbf{C}$ and $F \in \text{Hom}(\mathcal{A}, \mathcal{B})$, then $\forall x \in U$, there exists a neighbourhood V of x and an isomorphism of direct summands $\mathcal{A}' \subset \mathcal{A}|_V, \mathcal{B}' \subset \mathcal{B}|_V$, such that the restriction of the composite*

$$\mathcal{A}|_V \longrightarrow \mathcal{A}' \xrightarrow{\simeq} \mathcal{B}' \hookrightarrow \mathcal{B}|_V,$$

to the point x is equal to $F(x)$.

Proof. We pick an embedding of $\text{Hom}(\mathcal{A}, \mathcal{B})$ into smooth families of matrices as granted by Proposition 2.3.2.13. Define \bar{F} to be the constant extension of $F(x)$ under this embedding (which exists in an open neighbourhood of x). Now we use the image factorisation of the map $F(x)$: We pick a subspace $\mathcal{A}(x)' \subset \mathcal{A}(x)$ orthogonal to $\ker F(x)$. $F(x)$ is then an isomorphism between $\mathcal{A}'(x)$ and its image $\mathcal{B}(x)' := \text{im} F(x) \subset \mathcal{B}(x)$ under $F(x)$. Denote by p_A the projection to the summand $\mathcal{A}(x) \twoheadrightarrow \mathcal{A}'(x)$ and p_B the projection $\mathcal{B}(x) \twoheadrightarrow \mathcal{B}'(x)$. We may extend both these projections and their splittings locally. Denote by $\mathcal{A}', \mathcal{B}'$ the corresponding summands of $\mathcal{A}|_V$ and $\mathcal{B}|_V$. Now \bar{F} induces an isomorphism $\mathcal{A}' \rightarrow \mathcal{B}'$, because it does so pointwise. \square

Corollary 2.3.2.22. *If $F : \mathcal{A} \rightarrow \mathcal{B}$ is non-zero at $x \in U$, then \mathcal{A} and \mathcal{B} share a non-zero isomorphic summand in a neighbourhood of x .*

Lemma 2.3.2.23. *Given a family $\mathcal{S} : U \rightarrow \mathbf{C}$ with a single local support lift $\tilde{s} : U \rightarrow M$ to the standard chart $q : M \rightarrow [M/\Gamma]$ covering a quotient suborbifold $[M/\Gamma] \subset \mathcal{M}$, the germ of \tilde{s} is unique up to postcomposition with the Γ -action on M . Hence all mobile covers of \mathcal{S} are locally induced by the same map $s := q \circ \tilde{s} : U \rightarrow [M/\Gamma]$.*

Proof. Let $\mathcal{A} : U \rightarrow \mathbf{C}$ be a mobile cover of \mathcal{S} , obtained as the pushforward of a trivial vector bundle on U via s . Let $\mathcal{A}' : U \rightarrow \mathbf{C}$ be another mobile cover of \mathcal{S} , and $\tilde{s}' : U \rightarrow M$ an associated support lift. \mathcal{A} and \mathcal{A}' share a summand, so projection to the summand followed by inclusion gives a nowhere-zero map between them. Using the equivariantisation adjunction associated to the chart U ,

$$\text{Hom}(\mathcal{A}, \mathcal{A}') = \bigoplus_{g \in \Gamma} \text{Hom}(\text{Gr}(g \cdot \tilde{s}')^* \text{Gr}(\tilde{s})_* E, E'),$$

where E and E' are vector bundles on U . In order for this to have a section which is non-zero at x , there must be a summand, such that $g \cdot \tilde{s}' = \tilde{s}$ on an open neighbourhood of x . \square

Corollary 2.3.2.24. *Let $\mathcal{S} : U \rightarrow \mathbf{C}$ be a family with local support lifts $\{\tilde{s}_i : U \rightarrow M\}$ around $x \in U$ to some chart $M \rightarrow [M/\Gamma]$ covering a quotient suborbifold of \mathcal{M} , and let $\mathcal{T} \subset \mathcal{S}$ be a summand of \mathcal{S} , with a set of local support lifts $\{\tilde{t}_j : U \rightarrow M\}$ around x . Then each \tilde{t}_j is a Γ -translate of the germ of a lift \tilde{s}_i at x .*

Proof. \mathcal{T} decomposes into summands corresponding to the \tilde{t}_j on an open neighbourhood of x , so we can assume it has a single local support lift $\tilde{t} : U \rightarrow M$. Decomposing \mathcal{S} according to the support lifts \tilde{s}_i , the inclusion $\mathcal{T} \hookrightarrow \mathcal{S}$ must yield a nonzero map to at least one such summand. But this is nonzero at x , so by Corollary 2.3.2.22 they share a summand. This summand now has two local support lifts \tilde{t} and \tilde{s}_i . By Lemma 2.3.2.23, they are locally related by the Γ -action on M . \square

2.3.3 Grothendieck Group

For any orbisimple category $\underline{\mathbf{C}}$, the category of points $\mathbf{C} = \underline{\mathbf{C}}(*)$ is semisimple, so its *Grothendieck group* $\mathbf{K}(\mathbf{C})$ is free abelian with set of generators $\text{Irr } \mathbf{C}$. By Corollary 2.3.1.4, $\mathbf{K}(\mathbf{C})$ has a \mathbb{Z} -basis given by $\coprod(\text{Irr } \mathbf{Rep}^{\theta_x}(\Gamma_x))$.

Definition 2.3.3.1. Given an isomorphism class of points $[x] \in |\mathcal{M}|$, the *basic mobile at* $[x]$ is the isomorphism class

$$\mathbb{O}_x := [\text{Ind } x] = [x_*(\tau \cdot C^\infty)],$$

where x is some representative of $[x]$ and τ is any trivialisation of the gerbe $x^*\mathcal{G}$.¹² See Lemma 2.3.2.5 and Notation 2.3.2.6 for why the choice of trivialisation τ is irrelevant.

The *dimension* defined earlier is an isomorphism invariant, and is extended additively from simple objects, so it descends to the Grothendieck ring. By Lemma 2.3.1.3, $\mathbb{O}_x = [\mathbf{C}^{\theta_x}[\Gamma_x]]$, and so

$$\dim \mathbb{O}_x = \frac{\text{vdim } \mathbf{C}^{\theta_x}[\Gamma_x]}{|\Gamma_x|} = 1.$$

Basic mobile families 2.3.2.2 are pointwise of the form of Definition 2.3.3.1, and mobile families are direct sums thereof.

Corollary 2.3.3.2. *Mobile families have integer dimension.*

2.3.4 Lie Categories and Cauchy Completion

In this subsection, we show how the stack of twisted skyscraper sheaves $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ may be presented as the Cauchy completion of a Lie category. Lie categories are defined exactly as Lie groupoids, without the requirement of an inverse map for morphisms:

Definition 2.3.4.1. A *Lie category* is a category object internal to \mathbf{Man} whose source and target maps are submersions.

EXAMPLE 2.3.4.2. A smooth monoid R has an associated Lie category $[*/R]$ with object space $*$ and morphisms $\text{End}(\ast) = R$.

Smooth functors, smooth natural transformations and smooth equivalences are the internal versions (we spelled these out for Lie groupoids in Section 2.2.2). The 2-category of Lie groupoids embeds fully faithfully into the 2-category of Lie categories via the functor that forgets the inverse map.

A Lie category \mathbf{C}_\bullet defines a 2-presheaf

$$\text{Hom}(-, \mathbf{C}_\bullet) : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Cat}$$

sending a manifold M to the category of smooth functors $(M \rightrightarrows M) \rightarrow \mathbf{C}_\bullet$. The stack *presented by* \mathbf{C}_\bullet is the stackification of $\text{Hom}(-, \mathbf{C}_\bullet)$. When it is clear from context, we will denote the stack by \mathbf{C}_\bullet .

By [Rob12, Sec 8], the process of stackification is equivalent to inverting smooth equivalences, and the category of morphisms between two presented stacks is equivalent to the category of anafunctors between the presenting Lie categories.

¹²Of course, $C^\infty(\ast) = \mathbb{C}$ is just the canonical line bundle over the point.

EXAMPLE 2.3.4.3. Consider the stack $[* / R]$ from Example 2.3.4.2. An anafunctor $(M \rightrightarrows M) \rightarrow (R \rightrightarrows *)$ is a choice of cover $Y \rightarrow M$, and a transition function $Y^{[2]} \rightarrow R$ satisfying the 2-cocycle condition on $Y^{[3]}$. As every morphism in $Y^{[2]} \rightrightarrows Y$ is invertible, the map $Y^{[2]} \rightarrow R$ factors through $R^\times \rightarrow R$. Every anafunctor thus factors through the functor $[* / R^\times] \rightarrow [* / R]$, and the corresponding functor $[* / R^\times](M) \rightarrow [* / R](M)$ is faithful. However, it is not full in general: Recall that $[* / R^\times]$ is the stack of R^\times -bundles. Let $P, Q \in [* / R^\times](M)$ be two R^\times -bundles over M . Morphisms $P \rightarrow Q$ in $[* / R^\times](M)$ correspond to usual R^\times -bundle morphisms, while morphisms $P \rightarrow Q$ in $[* / R](M)$ are morphisms of the associated R -bundles $P \times_{R^\times} R \rightarrow Q \times_{R^\times} R$.

Let $\mathcal{G} \rightarrow \mathcal{M}$ be an orbifold equipped with a gerbe. We construct a Lie category whose associated stack recovers $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ once suitably completed under direct sums and direct summands. Pick a \mathbb{C}^\times -extension of Lie groupoids

$$\begin{array}{ccc} P & \longrightarrow & X_1 \\ \Downarrow & & \Downarrow \\ X_0 & \xrightarrow{\text{id}} & X_0, \end{array}$$

associated to a trivialisation of \mathcal{G} over $X_0 \rightarrow \mathcal{M}$ (see Section 2.2.4). The map of morphism spaces $P \rightarrow X_1$ is a \mathbb{C}^\times -bundle, compatible with the composition structure on both spaces. Denote the associated bundle $P_{\mathbb{C}} := P \times_{\mathbb{C}^\times} \mathbb{C} \rightarrow X_1$ where \mathbb{C}^\times acts on \mathbb{C} in the usual way. We build a Lie category $P_{\mathbb{C}} \rightrightarrows X_0$. Its source and target morphisms are the composites $P_{\mathbb{C}} \rightarrow X_1 \rightrightarrows X_0$, and the composition is induced by the composition on P and multiplication on \mathbb{C} :

$$P_{\mathbb{C}} \times_{X_0} P_{\mathbb{C}} \simeq (P \times_{X_0} P) \times_{(\mathbb{C}^\times \times \mathbb{C}^\times)} (\mathbb{C} \times \mathbb{C}) \rightarrow P \times_{\mathbb{C}^\times} \mathbb{C} = P_{\mathbb{C}}.$$

EXAMPLE 2.3.4.4. Consider the \mathbb{C}^\times -extension

$$\begin{array}{ccc} U \times \tilde{\Gamma} & \longrightarrow & U \times \Gamma \\ \Downarrow & & \Downarrow \\ U & \xrightarrow{\text{id}} & U, \end{array}$$

presenting a gerbe over the quotient orbifold $[U / \Gamma]$. Then the composition on $P_{\mathbb{C}} := U \times \tilde{\Gamma} \times_{\mathbb{C}^\times} \mathbb{C}$ is given by

$$\begin{aligned} P_{\mathbb{C}} \times_U P_{\mathbb{C}} &\rightarrow P_{\mathbb{C}} \\ ((u, \tilde{\gamma}, z), (u \cdot \gamma, \tilde{\gamma}', z')) &\mapsto (u \cdot \gamma \gamma', \tilde{\gamma} \tilde{\gamma}', z z'). \end{aligned}$$

Denote the stack over \mathcal{M} presented by the Lie groupoid $P_{\mathbb{C}} \rightrightarrows X_0$ by $\mathcal{G}_{\mathbb{C}} \rightarrow \mathcal{M}$. The map $\mathcal{G} \rightarrow \mathcal{G}_{\mathbb{C}}$ is the inclusion of the maximal groupoid-valued substack. It is represented by the map of Lie groupoids

$$\begin{array}{ccc} P & \longrightarrow & P_{\mathbb{C}} \\ \Downarrow & & \Downarrow \\ X & \xrightarrow{\text{id}_X} & X, \end{array}$$

where $P \rightarrow P_{\mathbb{C}}$ is the map from P to its associated bundle. An object in $\mathcal{G}_{\mathbb{C}}(U)$ is a map of stacks $U \rightarrow \mathcal{G}_{\mathbb{C}}$. As U is represented by a groupoid, this functor factors through $\mathcal{G} \hookrightarrow \mathcal{G}_{\mathbb{C}}$. The difference between \mathcal{G} and $\mathcal{G}_{\mathbb{C}}$ is seen on morphisms between objects (cf. Example 2.3.4.3). The data of an object $\bar{f} : U \rightarrow \mathcal{G}$ is equivalent to a pair $(f : U \rightarrow \mathcal{M}, \tau : \mathcal{I} \xrightarrow{\simeq} f^*\mathcal{G})$ of a map to \mathcal{M} and a trivialisaton of the pullback gerbe.¹³

Lemma 2.3.4.5. *Given a pair of objects $(f, \tau), (g, \kappa) \in \mathcal{G}_{\mathbb{C}}(U)$, the space of maps between them is empty if $f \not\cong g : U \rightarrow \mathcal{M}$. A choice of 2-cell $\alpha : f \xrightarrow{\simeq} g$ induces an isomorphism*

$$\mathrm{Hom}((f, \tau), (g, \kappa)) \simeq \Gamma(P_{\mathbb{C}}),$$

where $P_{\mathbb{C}}$ is the \mathbb{C} -bundle associated to the pair of trivialisations $(\alpha^*\tau, \kappa)$.

Proof. A morphism $(f, \tau) \rightarrow (g, \kappa)$ can only exist if there is a 2-cell $\alpha : f \xrightarrow{\simeq} g$ identifying the underlying maps to \mathcal{M} . A particular choice of such a 2-cell α induces an isomorphism

$$\alpha^* : f^*\mathcal{G} \xrightarrow{\simeq} g^*\mathcal{G}.$$

This way, we obtain a pair of trivialisations $(\alpha^*\tau, \kappa)$ of $g^*\mathcal{G}$. A homomorphism $\bar{f} \rightarrow \bar{g}$ in $\mathcal{G}(U)$ is the data of a section of the \mathbb{C}^{\times} -bundle P given by $(\alpha^*\tau, \kappa)$, while the space of homomorphisms in $\mathcal{G}_{\mathbb{C}}(U)$ is the space of sections of the associated \mathbb{C} -bundle $P_{\mathbb{C}}$. \square

NOTATION 2.3.4.6. We write $(\alpha, \sigma) : (f, \tau) \rightarrow (g, \kappa)$ to denote a representative $(\alpha : f \simeq g, \sigma \in \Gamma(P_{\mathbb{C}}))$, where α is used in the definition of $P_{\mathbb{C}}$.

Let $(f, \tau) \in \mathcal{G}_{\mathbb{C}}(U)$ and $\mathcal{F} \in \mathbf{Shv}_{\mathbb{C}^{\infty}}(U)$. Recall the map $\mathrm{Gr}(f) : U \rightarrow \mathcal{M} \times U$ is the graph of $f : U \rightarrow \mathcal{M}$. Denote by

$$(f, \tau) \cdot \mathcal{F} := \mathrm{Gr}(f)_*(\tau \cdot \mathcal{F}) \in \mathbf{Shv}^{p^*\mathcal{M}^{\mathcal{G}}}(\mathcal{M} \times U) = \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}(U)$$

the associated U -family of \mathcal{G} -twisted sheaves over \mathcal{M} . The assignment

$$((f, \tau), \mathcal{F}) \mapsto (f, \tau) \cdot \mathcal{F}$$

naturally extends to a map of stacks

$$\mathcal{G}_{\mathbb{C}} \times \mathbf{Shv}_{\mathbb{C}^{\infty}} \rightarrow \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}.$$

We describe the value of this map on a morphism

$$((\alpha, \sigma), \beta) : ((f, \tau), \mathcal{F}) \rightarrow ((g, \kappa), \mathcal{F}')$$

A 2-cell $\alpha : f \xrightarrow{\simeq} g$ induces a natural equivalence $\mathrm{Gr}(\alpha)_* : \mathrm{Gr}(f)_* \xrightarrow{\simeq} \mathrm{Gr}(g)_*$. A morphism of twisted sheaves $\alpha^*\tau \cdot \mathcal{F} \rightarrow \kappa \cdot \mathcal{F}'$ is given by a map of sheaves

$$\mathcal{F} \rightarrow (\alpha^*\tau)^{-1}\kappa \cdot \mathcal{F}' := P_{\mathbb{C}} \otimes_{\mathbb{C}^{\infty}} \mathcal{F}'.$$

A section $\sigma \in \Gamma(P_{\mathbb{C}})$ defines a morphism of twisted sheaves $\alpha^*\tau \cdot \mathcal{F} \rightarrow \kappa \cdot \mathcal{F}'$ with underlying map

$$\sigma \otimes \mathrm{id}_{\mathcal{F}} : \mathcal{F} \rightarrow P_{\mathbb{C}} \otimes_{\mathbb{C}^{\infty}} \mathcal{F}'.$$

¹³This follows from the universal property of the pullback $f^*\mathcal{G} = U \times_{\mathcal{M}} \mathcal{G}$.

The map $\mathcal{G}_{\mathbb{C}} \times \mathbf{Shv}_{\mathbb{C}^\infty} \rightarrow \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$ sends the morphism $((f, \tau), \beta)$ to the composite of natural transformations

$$\mathrm{Gr}(f)_*(\tau \cdot \mathcal{F}) \xrightarrow{\alpha} \mathrm{Gr}(g)_*(\tau \cdot \mathcal{F}) \xrightarrow{\sigma} \mathrm{Gr}(g)_*(\kappa \cdot \mathcal{F}) \xrightarrow{\beta} \mathrm{Gr}(g)_*(\kappa \cdot \mathcal{F}').$$

(We suppress decorations and simply denote the natural transformations by the datum that induces them.) By naturality, these maps of twisted sheaves are compatible with the pullback functors, which identifies this as a map of stacks. Applied to the distinguished object $C^\infty \in \mathbf{Shv}_{\mathbb{C}^\infty}(U)$, this yields a map

$$\begin{aligned} - \cdot C^\infty : \mathcal{G}_{\mathbb{C}} &\rightarrow \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}} \\ (f, \tau) &\mapsto (f, \tau) \cdot C^\infty. \end{aligned}$$

Lemma 2.3.4.7. *The map $- \cdot C^\infty$ lands in $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}} \subset \mathbf{Shv}_{\mathcal{M}}^{\mathcal{G}}$.*

Proof. We must show the condition in Definition 2.2.6.6 holds for $(f, \tau) \cdot C^\infty = \mathrm{Gr}(f)_*(\tau \cdot C^\infty)$. Let $Y \rightarrow \mathcal{M}$ be a cover equipped with a trivialisation κ of the gerbe. It pulls back to a cover

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & Y \\ p \downarrow & \lrcorner & \downarrow \pi \\ U & \xrightarrow{f} & \mathcal{M} \end{array}$$

of U , and the product cover $\pi \times p : Y \times \tilde{U} \rightarrow \mathcal{M} \times U$ is equipped with the trivialisation κ of the gerbe (pulled back from Y).

Base change (Lemma A.4.0.2) along

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\mathrm{Gr}(\tilde{f})} & Y \times \tilde{U} \\ p \downarrow & \lrcorner & \downarrow \pi \times p \\ U & \xrightarrow{\mathrm{Gr}(f)} & \mathcal{M} \times U \end{array}$$

gives an isomorphism

$$(\pi \times p)^* \mathrm{Gr}(f)_*(\tau \cdot C^\infty) \simeq \mathrm{Gr}(\tilde{f})_* p^*(\tau \cdot C^\infty) \simeq \mathrm{Gr}(\tilde{f})_*(p^* \tau \cdot p^* C^\infty).$$

The trivialisation κ of the gerbe over $Y \times \tilde{U}$ identifies the pushforward functor $\mathrm{Gr}(\tilde{f})_*$ with the usual pushforward of C^∞ -modules:

$$\mathrm{Gr}(\tilde{f})_* : \tau \cdot \mathcal{F} \mapsto \kappa \cdot \mathrm{Gr}(\tilde{f})_*(\mathcal{L} \otimes \mathcal{F}),$$

where \mathcal{L} denotes the \mathbb{C} -bundle associated to the pair of trivialisations $(\mathrm{Gr}(\tilde{f})^* \kappa, \tau)$ of the gerbe over \tilde{U} . This shows the pullback of $(f, \tau) \cdot C^\infty$ to $Y \times \tilde{U}$ is equivalent to

$$\mathrm{Gr}(\tilde{f})_*(p^* \tau \cdot p^* C^\infty) \simeq \kappa \cdot \mathrm{Gr}(\tilde{f})_*(\mathcal{L} \otimes p^* C^\infty),$$

the pushforward of the vector bundle $\mathcal{L} \otimes p^* C^\infty$ over \tilde{U} along the graph of the map $\tilde{f} : \tilde{U} \rightarrow Y$. This identifies $(f, \tau) \cdot C^\infty$ as a U -family of twisted skyscraper sheaves. \square

The map

$$- \cdot C^\infty : \mathcal{G}_{\mathbb{C}} \rightarrow \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$$

is not an equivalence: its component

$$\mathcal{G}_{\mathbb{C}}(U) \rightarrow \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$$

over $U \in \mathbf{Man}$ is faithful, but neither full nor essentially surjective.

The failure of the inclusion to be full may be blamed on the absence of a zero object in $\mathcal{G}_{\mathbb{C}}(U)$. The zero object in $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$ is given by the zero (twisted) sheaf. It plays the role of the empty direct sum, and the existence of zero morphisms makes $\mathrm{Hom}(X, Y)$ a vector space for all pairs of objects $X, Y \in \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$. (Cf. Lemma 2.3.4.5, which shows that Hom -spaces in $\mathcal{G}_{\mathbb{C}}(U)$ may be empty.) The zero object is preserved under all pullback functors, and so are zero morphisms between pairs of objects in $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$.

Denote by $\mathcal{G}_{\mathbb{C}}^+$ the minimal full substack of $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ which contains the image of $\mathcal{G}_{\mathbb{C}}$ as well as the zero sheaf. The stack $\mathcal{G}_{\mathbb{C}}^+$ is a full substack of a \mathbf{Vec}^∞ -enriched stack and thus \mathbf{Vec}^∞ -enriched itself. We now Cauchy complete this stack. Cauchy completion is the completion of a category under absolute colimits [Law73], ie. colimits which are preserved by any (\mathbf{Vec}^∞ -enriched) functor. As such, this completion may be performed pointwise: the pullback functors automatically extend to the Cauchy completions and preserve the added absolute colimits. For \mathbf{Vec}^∞ -enriched categories, Cauchy completion amounts to completing under direct sums and splitting idempotents. Denote by \mathbf{C}^\oplus the completion of a category \mathbf{C} under direct sums, and by $\mathrm{Kar}(\mathbf{C})$ its *Karoubi envelope*: the universal completion of \mathbf{C} where all idempotents split.

We denote by $\mathrm{Kar}(\mathcal{G}_{\mathbb{C}}^\oplus)$ the stack over \mathbf{Man} associated to the prestack

$$U \mapsto \mathrm{Kar} \left(\mathcal{G}_{\mathbb{C}}^+(U)^\oplus \right)$$

obtained from $\mathcal{G}_{\mathbb{C}}^+ \subset \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ by pointwise Cauchy completion.

Proposition 2.3.4.8. *The stack of skyscraper sheaves is the Cauchy completion*

$$\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}} \simeq \mathrm{Kar} \left(\mathcal{G}_{\mathbb{C}}^\oplus \right).$$

When $\mathcal{M} = M$ is a manifold, it suffices to complete under direct sums:

$$\mathbf{Sky}_M^{\mathcal{G}} \simeq \mathcal{G}_{\mathbb{C}}^\oplus.$$

Proof. First, note the stack $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ is indeed Cauchy complete: completeness under direct sum is obvious, and we proved Karoubi completeness in Corollary 2.3.2.17.

Now we show that $\mathcal{G}_{\mathbb{C}}^+$ indeed Cauchy completes to $\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$. This means every family $\mathcal{S} \in \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U)$ locally restricts to a section of $\mathrm{Kar}(\mathcal{G}_{\mathbb{C}}^\oplus)$: a summand of a sheaf

$$\bigoplus_{i=0}^r (f_i, \tau_i) \cdot C^\infty.$$

But this follows directly from the normal form for families of twisted skyscraper sheaves established in Corollary 2.3.2.16.

Now assume $\mathcal{M} = M$ is a manifold. Then M is an effective orbifold whose generic region (the region with trivial stabiliser) is all of M . We may now apply Corollary 2.3.2.9 to conclude that every family $\mathcal{A} \in \mathbf{Sky}_M^{\mathcal{G}}(U)$ is mobile, and hence a direct sum of objects in $\mathcal{G}_{\mathbb{C}}^+(U)$. \square

2.4 Orbifold Tensor Categories

Recall the notation introduced in Section 2.3: \mathbf{VCat} denotes the bicategory of \oplus -complete \mathbf{Vec}^∞ -enriched categories, and \mathbf{Vect} the \mathbf{VCat} -valued stack of vector bundles over the site \mathbf{Man} of smooth manifolds. We defined a *linear stack* to be a module over \mathbf{Vect} in the category of \mathbf{VCat} -valued stacks.

Definition 2.4.0.1. A *tensor stack* is a monoid object in the bicategory of linear stacks.

Recall also Definition 2.3.0.1: An orbisimple category is a linear stack $\underline{\mathbf{C}}$ which admits an equivalence $\underline{\mathbf{C}} \simeq \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ for some orbifold \mathcal{M} equipped with a gerbe \mathcal{G} . If \mathcal{M} can be chosen to be a manifold, such a stack is called a *manisimple category*.

Definition 2.4.0.2. An *orbifold/manifold tensor category* is a tensor stack whose underlying linear stack is an orbisimple/manisimple category.

We are primarily interested in the category $\mathbf{C} := \underline{\mathbf{C}}(*)$ assigned to $* \in \mathbf{Man}$. The remaining data keeps track of the smooth structure of \mathbf{C} . As $\mathbf{C} \cong \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(*)$, it is automatically semisimple abelian and Karoubi complete (Corollary 2.3.1.4).

Recall that a left dual for an object Y in a tensor category is a pair $(Y^\vee, \text{ev} : Y^\vee \otimes Y \rightarrow \mathbb{1})$ such that there exists a map $(\text{coev} : \mathbb{1} \rightarrow Y \otimes Y^\vee)$ satisfying the snake equations (cf. Definition 1.1.1.17). An object with both left and right duals is called *dualisable*, and a tensor category is *autonomous* if all of its objects are dualisable.

Definition 2.4.0.3. An orbifold/manifold tensor category $\underline{\mathbf{C}}$ is *orbifusion/manifusion* if the category $\mathbf{C} = \underline{\mathbf{C}}(*)$ assigned to the point is autonomous and the monoidal unit $\mathbb{1}$ is simple.

In Section 2.6, we will prove that this guarantees that the categories $\underline{\mathbf{C}}(U)$ an orbifusion category $\underline{\mathbf{C}}$ assigns to a manifold U are also autonomous – so long as the underlying orbifold \mathcal{M} may be chosen to be effective.

EXAMPLE 2.4.0.4. Using Pontryagin duality and Clifford Theory, one may show that the categories of finite-dimensional unitary representations of a virtually abelian discrete group naturally form an orbifusion category. Indeed, an extension of groups $\mathbb{Z}^n \rightarrow G \rightarrow H$ with H finite, yields

$$\mathbf{Rep}^u G = \mathbf{Rep}^u(\mathbb{Z}^n)^H \simeq \mathbf{Sky}(T^n)^H \simeq \mathbf{Sky}([T^n/H]),$$

where T^n denotes the n -torus. The monoidal structure is given by the usual tensor product of representations. We depicted the moduli space of simple objects for two such examples in Figure 2.1.

We spell out the components of a tensor stack explicitly. A tensor stack is a sextuple consisting of

- the underlying linear stack

$$\underline{\mathbf{C}} : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{VCat}$$

- two morphisms

$$\begin{aligned} \mathbb{1} &: \mathbf{Vect} \rightarrow \underline{\mathbf{C}} \\ \otimes &: \underline{\mathbf{C}} \boxtimes_{\mathbf{Vect}} \underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}} \end{aligned}$$

- and three natural isomorphisms

$$\begin{aligned} \alpha &: (- \otimes -) \otimes - \implies - \otimes (- \otimes -) \\ l &: (\mathbb{1} \otimes -) \implies \text{Id} \\ r &: (- \otimes \mathbb{1}) \implies \text{Id}, \end{aligned}$$

satisfying the triangle and pentagon equations familiar from the definition of a monoidal category (see also Section 1.1).

Evaluating the above data at an object $U \in \mathbf{Man}$ yields the data of a tensor category $(\mathbf{C}^U, \otimes^U, \mathbb{1}^U, \alpha^U, l^U, r^U)$ equipped with a $\mathbf{Vect}(U)$ -module structure (recall $\mathbf{C}^U = \underline{\mathbf{C}}(U)$). As \otimes is a natural transformation, it comes with naturality 2-isomorphisms for maps $f : V \rightarrow U$ in \mathbf{Man} :

$$\begin{array}{ccc} (\underline{\mathbf{C}} \boxtimes \underline{\mathbf{C}})(U) & \xrightarrow{\otimes^U} & \underline{\mathbf{C}}(U) \\ \downarrow f^* \boxtimes f^* & \nearrow \cong & \downarrow f^* \\ (\underline{\mathbf{C}} \boxtimes \underline{\mathbf{C}})(V) & \xrightarrow{\otimes^V} & \underline{\mathbf{C}}(V), \end{array}$$

which make the pullback functors monoidal. The \mathbf{Vect} -module structure further equips this with isomorphisms

$$b_U(X, P, Y) : (X \odot P) \otimes Y \xrightarrow{\cong} X \otimes (Y \odot P),$$

natural in $P \in \mathbf{Vect}(U)$ and $X, Y \in \mathbf{C}^U$. There are natural isomorphisms

$$f^*(P \odot X) \simeq f^*P \odot f^*X,$$

and the resulting square

$$\begin{array}{ccc} f^*((X \odot P) \otimes Y) & \xrightarrow{f^*b(X,P,Y)} & f^*(X \otimes (Y \odot P)) \\ \downarrow \cong & & \downarrow \cong \\ (f^*X \odot f^*P) \otimes f^*Y & \xrightarrow{b(f^*X, f^*P, f^*Y)} & f^*X \otimes (f^*Y \odot f^*P) \end{array}$$

commutes.

This data can be combined to yield a “global” tensor product, which we also denote by \otimes : let $U, V \in \mathbf{Man}$, then the global tensor product is the functor

$$\begin{aligned} \otimes &: \underline{\mathbf{C}}(U) \boxtimes \underline{\mathbf{C}}(V) \rightarrow \underline{\mathbf{C}}(U \times V) \\ (X, Y) &\mapsto p_1^*X \otimes^{U \times V} p_2^*Y. \end{aligned}$$

The associator and unitors are “globalised” in the same way. The pointwise tensor product \otimes^U of $\underline{\mathbf{C}}(U)$ can be recovered using the diagonal $\Delta_U : U \rightarrow U \times U$:

$$X \otimes^U Y = \Delta_U^* p_1^* X \otimes^U \Delta_U^* p_2^* Y = \Delta_U^* (p_1^* X \otimes^{U \times U} p_2^* Y) = \Delta_U^* (X \otimes Y).$$

2.4.1 Fusion Categories are Manifusion Categories

Let \mathbf{D} be a tensor category and denote by $\text{Irr } \mathbf{D}$ its set of isomorphism classes of simple objects. Recall the functor

$$- \boxtimes \mathbf{Vect} : \mathbf{VCat} \hookrightarrow \mathbf{LinSt},$$

which freely turns \mathbf{Vec} -modules into \mathbf{Vect} -modules (ie. linear stacks). This functor is monoidal, as shown in Lemma 2.2.7.8.

Corollary 2.4.1.1. *The functor $- \boxtimes \mathbf{Vect} : \mathbf{D} \mapsto \underline{\mathbf{D}}$ sends \mathbb{C} -linear \oplus -complete monoidal categories to tensor stacks.*

The value of the underlying linear stack at $U \in \mathbf{Man}$ is

$$\underline{\mathbf{D}} : U \mapsto \mathbf{D} \boxtimes \mathbf{Vect}(U).$$

Pick an equivalence of linear categories $\mathbf{Vec}^{\oplus \text{Irr } \mathbf{D}} \xrightarrow{\cong} \mathbf{D}$, corresponding to a choice of representatives for the set $\text{Irr } \mathbf{D}$. It induces an equivalence of linear stacks

$$\underline{\mathbf{D}} \xrightarrow{\cong} \mathbf{Vec}^{\oplus \text{Irr } \mathbf{D}} \boxtimes \mathbf{Vect} \simeq \mathbf{Vect}^{\oplus \text{Irr } \mathbf{D}}.$$

Let $M \in \mathbf{Man}$ be the discrete manifold $\text{Irr } \mathbf{D}$, a disjoint union of $|\text{Irr } \mathbf{D}|$ points. Then \mathbf{Sky}_M is isomorphic to $\mathbf{Vect}^{\oplus \text{Irr } \mathbf{D}}$ (see Example 2.2.6.7) as a linear stack. This identifies $\underline{\mathbf{D}}$ as a manisimple category (Definition 2.3.0.1). It also carries a monoidal structure induced by the monoidal structure of \mathbf{D} (see Corollary 2.4.1.1).

Corollary 2.4.1.2. *The assignment*

$$\mathbf{D} \mapsto \underline{\mathbf{D}}$$

fully faithfully embeds the bicategory of \mathbb{C} -linear \oplus -complete monoidal categories into that of manifold tensor categories.

By restriction, the same is true for fusion categories: they embed fully faithfully into manifusion categories. Hence, we don't distinguish between fusion categories and their associated manifusion categories.

The category of points of $\underline{\mathbf{D}}$ is

$$\underline{\mathbf{D}}(*) = \mathbf{D} \boxtimes \mathbf{Vect}(*) \simeq \mathbf{D}.$$

It thus follows that

- $\underline{\mathbf{D}}$ is a compact manifold tensor category iff $\text{Irr } \mathbf{D}$ is finite and
- $\underline{\mathbf{D}}$ is manifusion iff \mathbf{D} is fusion.

We describe the tensor product on $\underline{\mathbf{D}}$ explicitly. Pick a skeletal representative for \mathbf{D} , so that the set of simples $X \in \mathbf{D}$ is identified with the (discrete) manifold $M = \text{Irr } \mathbf{D}$. Let $X \in \mathbf{D}$ be simple and E a vector bundle over $U \in \mathbf{Man}$. Then we denote by (X, E) the U -family given by the sheaf in $\mathbf{Shv}_{\mathbb{C}^\infty}(M \times U) \cong \mathbf{Shv}_{\mathbb{C}^\infty}(\coprod_r U)$ given by the

pushforward of E onto the connected component corresponding to X . Every U -family of skyscraper sheaves can be globally decomposed into direct summands of this form. The tensor product \otimes^U is given by

$$(X, E) \otimes^U (Y, E') = (X \otimes Y, E \otimes E') = \bigoplus_{Z \in X \otimes Y} (Z, E \otimes E'),$$

where the sum is over simples Z in $X \otimes Y = \bigoplus Z$.

2.4.2 Categorical Group Rings

The motivating examples for the study of manifold categories are the categorified group rings $\mathbf{Vec}^\omega[G]$ built from a Lie group G and a cocycle ω representing a cohomology class $[\omega] \in H^3(\mathcal{B}G, \mathbb{C}^\times)$. We will build $\mathbf{Vec}^\omega[G]$ as a manifold tensor category by linearising the corresponding String 2-group G_ω .

We begin by recalling the construction of G_ω as a smooth 2-group after [SP11]. This is also spelt out in more detail in Section 3.3. The cocycle ω may be represented by a simplicial cover $Y_\bullet \rightarrow BG$ of the simplicial manifold BG , equipped with a triple

$$(\lambda, \mu, \omega) \in C^\infty(Y_1^{[3]}, \mathbb{C}^\times) \times C^\infty(Y_2^{[2]}, \mathbb{C}^\times) \times C^\infty(Y_3, \mathbb{C}^\times)$$

(we abuse notation and denote the third element of this triple by the same letter as the entire cocycle). This data may be used to build a smooth 2-group G_ω : The 2-cocycle $\lambda \in C^\infty(Y_1^{[3]}, \mathbb{C}^\times)$ encodes a \mathbb{C}^\times -gerbe over G with a trivialisation over Y_1 . As explained in Section 2.3.4, we may build a Lie category $Y_1^{[2]} \times \mathbb{C} \rightrightarrows Y_1$ from this data: the composition map is given by

$$\begin{aligned} Y_1^{[3]} \times \mathbb{C} \times \mathbb{C} &\rightarrow Y_1^{[2]} \times \mathbb{C} \\ (y, y', y'', z, z') &\mapsto (y, y'', z \cdot z' \cdot \lambda(y, y', y'')). \end{aligned}$$

Let \mathcal{G} be the gerbe over G presented by λ . We denote the associated stack over \mathbf{Man} by $\mathcal{G}_{\mathbb{C}}$. Recall from Section 2.3.4 that there is a natural map

$$- \cdot C^\infty : \mathcal{G}_{\mathbb{C}} \hookrightarrow \mathbf{Sky}_{\mathbb{C}}^{\mathcal{G}}.$$

The data μ and ω equip this stack with the structure of a monoid object in exact analogy with G_ω . We show that this structure extends along the inclusion above. Recall we denoted by $\mathcal{G}_{\mathbb{C}}^+ \subset \mathbf{Sky}_{\mathbb{C}}^{\mathcal{G}}$ the full substack on the essential image of the inclusion above and the zero object.

Lemma 2.4.2.1. *The monoid structure on $\mathcal{G}_{\mathbb{C}}$ extends to a monoid structure on $\mathcal{G}_{\mathbb{C}}^+$.*

Proof. We begin by showing the monoidal structure extends objectwise. Let $U \in \mathbf{Man}$ be a parameterising family. We extend the tensor product \otimes^U on $\mathcal{G}_{\mathbb{C}}(U)$ to a tensor product $\otimes^{U,+}$ on $\mathcal{G}_{\mathbb{C}}^+(U)$. On objects, this extension is straightforward: let $X, Y \in \mathcal{G}_{\mathbb{C}}^+(U)$ be a pair of objects over U . If X or Y are the zero object 0 , then $X \otimes^{U,+} Y = 0$. Otherwise, both X and Y are objects of $\mathcal{G}_{\mathbb{C}}(U)$, and the tensor product is given by $X \otimes^{U,+} Y = X \otimes^U Y$.

The value of $\otimes^{U,+}$ on morphisms is constructed pointwise: Consider a pair of morphisms $(\alpha : X \rightarrow X', \beta : Y \rightarrow Y')$. By restricting U enough, we may assume the objects involved are of the form

$$\begin{aligned} X &= (f, \tau) \cdot C^\infty & X' &= (f', \tau') \cdot C^\infty \\ Y &= (g, \kappa) \cdot C^\infty & Y' &= (g', \kappa') \cdot C^\infty \\ X \otimes Y &= (h, \lambda) \cdot C^\infty & X' \otimes Y' &= (h', \lambda') \cdot C^\infty. \end{aligned}$$

Note that for $u \in U$, the equalities $f(u) = f'(u)$ and $g(u) = g'(u)$ imply $h(u) = h'(u)$. Proposition 2.3.2.13 allows us to embed $\text{Hom}(X, X')$, $\text{Hom}(Y, Y')$ and $\text{Hom}(X \otimes Y, X' \otimes Y')$ into the space of smooth \mathbb{C} -valued functions. This (non-canonically) identifies α with an endomorphism $\bar{\alpha} \in \text{End}(X) \simeq C^\infty(U)$. Let $V_{f=f'}^\circ \subset U$ be the interior of the subset of U where $f = f'$. The endomorphism $\bar{\alpha}$ is 0 on the complement of $V_{f=f'}^\circ$. Similarly, β is identified with an endomorphism $\bar{\beta} \in \text{End}(Y)$ which is 0 on the complement of $V_{g=g'}^\circ$. Both $\bar{\alpha}$ and $\bar{\beta}$ are morphisms in $\mathcal{G}_\mathbb{C}(U)$, so the tensor product $\bar{\alpha} \otimes^U \bar{\beta} \in \text{End}(X \otimes Y)$ is well-defined. The resulting endomorphism is 0 outside the set $V_{f=f'}^\circ \cap V_{g=g'}^\circ$ where $f = f'$ and $g = g'$, and thus in particular 0 outside the set $V_{h=h'}^\circ$.

We may now use the technique above in reverse to identify the endomorphism $\bar{\alpha} \otimes^U \bar{\beta}$ with a morphism

$$\alpha \otimes^{U,+} \beta : X \otimes Y \rightarrow X' \otimes Y'.$$

The choices made above imply that this morphism is only canonical up to an invertible function $c \in C^\infty(U, \mathbb{C}^\times)$. As $\alpha \otimes^{U,+} \beta$ is 0 outside $W = V_{f=f'}^\circ \cap V_{g=g'}^\circ$, we need only fix the ambiguity there. But on W , both α and β restrict to morphisms in $\mathcal{G}_\mathbb{C}(W)$, and the tensor product is already defined. This completes the construction of $\otimes^{U,+}$.

It remains to extend the associators and unitors. The associator

$$\alpha^{U,+}(X, Y, Z) : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

agrees with $\alpha^U(X, Y, Z)$ unless one of $X, Y, Z \in \mathcal{G}_\mathbb{C}^+$ is 0, in which case it is the zero morphism. The pentagon equation and naturality follow from the fact that any morphism commutes with a zero morphism. The unitors are dealt with in the same way.

The monoidal structure described above is manifestly compatible with the restriction functors, because the zero object and zero morphisms are preserved under restriction. This gives $\mathcal{G}_\mathbb{C}^+$ the structure of a monoid. \square

Proposition 2.4.2.2. *The monoid structure on $\mathcal{G}_\mathbb{C}$ induces on $\mathbf{Sky}_\mathbb{G}^\mathcal{G}$ the structure of a manifold category.*

Proof. The linear stack $\mathbf{Sky}_\mathbb{G}^\mathcal{G}$ is manifestly a manisimple category (Definition 2.3.0.1), so it suffices to equip it with the structure of a monoid in the bicategory of linear stacks.

Proposition 2.3.4.8 says that $\mathbf{Sky}_\mathbb{G}^\mathcal{G}$ is the direct sum completion of the stack $\mathcal{G}_\mathbb{C}^+ \subset \mathbf{Sky}_\mathbb{G}^\mathcal{G}$. Hence every family $\mathcal{A} : U \rightarrow \mathbf{Sky}_\mathbb{G}^\mathcal{G}$ locally decomposes as a direct sum of objects in $\mathcal{G}_\mathbb{C}^+$, and we may extend the monoid structure obtained from Lemma 2.4.2.1 linearly. This gives $\mathbf{Sky}_\mathbb{G}^\mathcal{G}$ the structure of a monoid in the bicategory of \mathbf{VCat} -valued stacks. To

upgrade this to a monoid structure in \mathbf{LinSt} , we must provide an invertible 2-cell s implementing compatibility between the monoid structure and the \mathbf{Vect} -module structure (see Section 2.2.7 for details). Its component at $U \in \mathbf{Man}$,

$$s_U(X \boxtimes Y, P) : (X \otimes Y) \odot P \xrightarrow{\simeq} X \otimes (Y \odot P)$$

must be natural in $P \in \mathbf{Vect}(U)$ and $X, Y \in \mathbf{Sky}_G^{\mathcal{G}}(U)$, and compatible with the restriction functors, associators, and unitors. The objects of $\mathbf{Sky}_G^{\mathcal{G}}(U)$ are locally generated under direct sum by objects in $\mathcal{G}_C^+(U)$, and the objects of $\mathbf{Vect}(U)$ by the object $C^\infty(U)$. Hence, it suffices to define the natural 2-cell on these objects. The morphism

$$s_U(X \boxtimes Y, C^\infty) : (X \otimes Y) \odot C^\infty(U) \xrightarrow{\simeq} X \otimes Y \xleftarrow{\simeq} X \otimes (Y \odot C^\infty(U)),$$

is the composite of unitors for the \mathbf{Vect} -action. It is thus automatically compatible with restriction functors. By naturality, it also commutes with the associators and unitors for the monoid structure. The other components of s are obtained by extending linearly, and hence share the above properties. This way, the monoid structure on $\mathbf{Sky}_G^{\mathcal{G}}$ is upgraded to the structure of a manifold tensor category.

The category of points is the category $\mathbf{Vec}^{\omega^\delta}[G^\delta]$ of G^δ -graded vector spaces, where G^δ denotes G with the discrete topology and ω^δ the pullback cocycle. Hence, duals exist pointwise (for a simple object \mathbb{C}_g , such a dual is provided by $\mathbb{C}_{g^{-1}}$), and the unit \mathbb{C}_e is a simple object. Hence this manifold tensor category is a manifold fusion category. \square

Definition 2.4.2.3. The *categorified group ring* $\mathbf{Vec}^\omega[G]$ associated to a Lie group G and a cocycle $\omega \in H^3(\mathcal{B}G, \mathbb{C}^\times)$ is the manifold fusion category obtained by linearising G_ω via the procedure outlined in Proposition 2.4.2.2.

In particular, $\mathbf{Vec}^\omega[G]$ is built such that the maximal sub-2-group recovers G_ω .

REMARK 2.4.2.4. Using the language of *multiplicative gerbes*¹⁴, we may describe the categorified group ring $\mathbf{Vec}^\omega[G]$ more directly. The group $H^3(\mathcal{B}G, \mathbb{C}^\times)$ is in bijection with equivalence classes of multiplicative gerbes over G [Wal10; SP11]. A multiplicative structure on a gerbe \mathcal{G} over G is a pair of an equivalence of gerbes

$$M : p_1^*\mathcal{G} \otimes p_2^*\mathcal{G} \rightarrow m^*\mathcal{G}$$

over $G \times G$, and an isomorphism α implementing associativity over $G \times G \times G$, satisfying a coherence condition over $G \times G \times G \times G$. Evaluated at $(g, h, k, l) \in G^{\times 4}$, this coherence condition is the pentagon equation. The equivalence M allows us to pass between $p_1^*\mathcal{G} \otimes p_2^*\mathcal{G}$ -twisted sheaves and $m^*\mathcal{G}$ -twisted sheaves (see Appendix A). We will denote its action on sheaves by a tensor product.

This structure allows us to equip $\mathbf{Sky}_G^{\mathcal{G}}$ (the underlying linear stack of $\mathbf{Vec}^\omega[G]$) with a monoid structure as follows: The convolution product of two objects $X, Y \in \mathbf{Vec}^\omega[G](*)$ is given by

$$X \otimes Y := m_*(M \otimes p_1^*X \otimes p_2^*Y),$$

¹⁴These multiplicative \mathbb{C}^\times -gerbes are usually called multiplicative *bundle* gerbes in the literature.

where $p_1, p_2, m : G \times G \rightarrow G$ are the obvious maps. The product is extended to U -families by multiplying all pullback and pushforward maps by id_U . This is manifestly compatible under pullback of twisted sheaves.

It remains to discuss the associator. We use the letter q to denote projection maps with domain $G^{\times 3}$. The following equalities follow from monoidality of pullback functors, smooth base change along

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m_{12} \times \text{id}} & G \times G \\ \downarrow q_{12} & & \downarrow p_1 \\ G \times G & \xrightarrow{m} & G \end{array}$$

and the projection formula $f_* \mathcal{F} \otimes \mathcal{G} = f_*(\mathcal{F} \otimes f^* \mathcal{G})$, which are proved in Appendix A:

$$\begin{aligned} (X \otimes Y) \otimes Z &= m_* (\mathbb{M} \otimes p_1^* m_* (\mathbb{M} \otimes p_1^* X \otimes p_2^* Y) \otimes p_2^* Z) \\ &= m_* (\mathbb{M} \otimes (m_{12} \times \text{id})_* (q_{12}^* \mathbb{M} \otimes q_1^* X \otimes q_2^* Y) \otimes p_2^* Z) \\ &= m_* (\mathbb{M} \otimes (m_{12} \times \text{id})_* (q_{12}^* \mathbb{M} \otimes q_1^* X \otimes q_2^* Y \otimes q_3^* Z)) \\ &= m_{123,*} ((m_{12} \times \text{id})^* \mathbb{M} \otimes q_{12}^* \mathbb{M} \otimes q_1^* X \otimes q_2^* Y \otimes q_3^* Z), \\ X \otimes (Y \otimes Z) &= m_{123,*} ((\text{id} \times m_{23})^* \mathbb{M} \otimes q_{23}^* \mathbb{M} \otimes q_1^* X \otimes q_2^* Y \otimes q_3^* Z). \end{aligned}$$

The isomorphism

$$\alpha : (m_{12} \times \text{id})^* \mathbb{M} \otimes q_{12}^* \mathbb{M} \xrightarrow{\cong} (\text{id} \times m_{23})^* \mathbb{M} \otimes q_{23}^* \mathbb{M}$$

now precisely encodes the associator.

2.4.3 Group Algebras in Tensor Stacks

An algebra object A in a tensor category \mathbf{C} induces a linear category of (right) modules over A , which we denote $\text{Mod}_A(\mathbf{C})$. In this section, we perform this construction in the world of tensor stacks. The central result is that when $\underline{\mathbf{C}}$ is an orbifold tensor category and A is a group algebra, then $\text{Mod}_A(\underline{\mathbf{C}})$ is naturally an orbisimple category.

Let A be an algebra object in $\mathbf{C} = \underline{\mathbf{C}}(*)$ (see Section 1.1.3). For each $U \in \mathbf{Man}$, this induces an algebra object A^U in \mathbf{C}^U , given by the constant family at A . In particular, this implies $f^* A^U \simeq A^V$ for all $f : V \rightarrow U$. We may form the 2-presheaf (valued in \mathbf{VCat}) of modules over A

$$\text{Mod}_A(\underline{\mathbf{C}}) : U \mapsto \text{Mod}_{A^U}(\mathbf{C}^U),$$

which objectwise assigns the category of right modules over the algebra.

Lemma 2.4.3.1. *The 2-presheaf $\text{Mod}_A(\underline{\mathbf{C}})$ is a stack.*

Proof. We prove that descent data of A -modules may be glued uniquely. Let $\pi : Y \rightarrow U$ be a cover. A descent datum for $\text{Mod}_A(\underline{\mathbf{C}})$ over Y is a triple

$$X \in \mathbf{C}^Y \quad \phi : p_1^* X \xrightarrow{\cong} p_2^* X \in \mathbf{C}^{Y^{[2]}} \quad m : X \otimes A^Y \rightarrow X,$$

where (X, ϕ) forms a descent datum for $\underline{\mathbf{C}}$, and m gives X the structure of an A_Y -module. The isomorphism ϕ must be compatible with m , in that

$$\begin{array}{ccc} p_1^*X \otimes A^{Y[2]} & \xrightarrow{p_1^*m} & p_1^*X \\ \phi \otimes \text{id}_A \downarrow & & \downarrow \phi \\ p_2^*X \otimes A^{Y[2]} & \xrightarrow{p_2^*m} & p_2^*X \end{array}$$

commutes. By naturality of the tensor product, the pair $(X \otimes A^Y, \phi \otimes \text{id}_A)$ forms a descent datum as well, and the commutative square above witnesses m as a morphism of descent data

$$m : (X \otimes A^Y, \phi \otimes \text{id}_A) \rightarrow (X, \phi).$$

As $\underline{\mathbf{C}}$ is a stack, there exists an object \tilde{X} , unique up to unique isomorphism, which glues the descent datum (X, ϕ) . The descent datum $(X \otimes A^Y, \phi \otimes \text{id}_A)$ is then glued by $\tilde{X} \otimes A^U$, and the morphism m lifts to a unique morphism

$$\tilde{m} : \tilde{X} \otimes A^U \rightarrow \tilde{X}.$$

As an action map, m is associative (and compatible with the unit, a datum which we suppress throughout). The condition of associativity involves a term $X \otimes A^Y \otimes A^Y$. This is part of a descent datum which glues to $\tilde{X} \otimes A^U \otimes A^U$. The functor sending objects of \mathbf{C}^U to their descent data over Y is an equivalence, so \tilde{m} satisfies the same equations as m . Hence, the pair (\tilde{X}, \tilde{m}) is a module over A^U in \mathbf{C}^U . It glues the descent datum uniquely up to unique isomorphism, which shows $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is a stack. \square

REMARK 2.4.3.2. The endomorphism

$$- \otimes A : \underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}},$$

is part of a monad [Bén67] on $\underline{\mathbf{C}}$ in \mathbf{LinSt} . Its unit and multiplication morphisms are induced by the inclusion of the unit $\mathbb{1} \hookrightarrow A$ and the monoid structure on $\underline{\mathbf{C}}$. Evaluated at $U \in \mathbf{Man}$, this monad is the usual monad associated to the algebra object $A^U \in \mathbf{C}^U$. The *Eilenberg-Moore object* of a monad in a bicategory [Str72; LS02] is the object of algebras over the monad. The stack $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is the Eilenberg-Moore object associated to $- \otimes A$.

The *free module map*

$$- \otimes A : \underline{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$$

sends an object $X \in \mathbf{C}^U$ to the object $X \otimes A \in \mathbf{Mod}_A(\underline{\mathbf{C}})^U = \mathbf{Mod}_{A^U}(\mathbf{C}^U)$, equipped with the free module structure induced by the algebra structure on A .

Lemma 2.4.3.3. *The stack $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is Karoubi-generated by the essential image of the free module map: The category $\mathbf{Mod}_{A^U}(\mathbf{C}^U)$ is the Karoubi completion of the essential image of*

$$- \otimes A^U : \mathbf{C}^U \rightarrow \mathbf{Mod}_{A^U}(\mathbf{C}^U).$$

Proof. First note that $\mathbf{Mod}_{A^U}(\mathbf{C}^U)$ is indeed Karoubi complete: Let $e \in \text{End}(X, m)$ be an idempotent of a module. Then $e : X \rightarrow X$ splits in \mathbf{C}^U by Corollary 2.3.2.17. The restriction of m to the summand picked out by X gives it the structure of a right A^U -module.

It remains to show that every module (Z, m) is a summand of a free module. The morphism

$$(Z \otimes A^U, m_{\text{free}}) \xrightarrow{m} (Z, m)$$

is split by the unit morphism (a datum we suppress throughout), and exhibits (Z, m) as such a summand. \square

Now, we specialise to group algebras. Let H be a finite group, and assume $\underline{\mathbf{C}}$ admits a linear equivalence $\underline{\mathbf{C}} \simeq \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$.

Lemma 2.4.3.4. *A map $\mathbf{Vec}[H] \rightarrow \underline{\mathbf{C}}$ induces an action of H on $\mathcal{G} \rightarrow \mathcal{M}$.*

Proof. Denote the image of $\mathbb{C}_h \subset \mathbb{C}[H] \in \mathbf{Vec}[H]$ by h . The object h is invertible, so $-\otimes h$ preserves the image of \mathcal{G} under the inclusion $\mathcal{G} \hookrightarrow \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ introduced in Section 2.3.4. Thus $-\otimes h$ descends to a functor on \mathcal{G} , and we obtain an action of H on \mathcal{G} . The action on \mathcal{G} descends to an action on \mathcal{M} : The map $\mathcal{G} \hookrightarrow \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ extends to a map $\mathcal{G} \times \mathcal{I} \hookrightarrow \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}} \boxtimes \mathbf{Vect}$, and the compatibility data between the H -action and the \mathbf{Vect} -module structure restrict to compatibility data between the H -action and the \mathcal{I} -action on \mathcal{G} . The orbifold \mathcal{M} is the quotient of \mathcal{G} by the \mathcal{I} -action, so the compatibility data is equivalent to descent data for the H -action. \square

We introduce notation for the H -action described in Lemma 2.4.3.4: The 1-cells of the H -action are given by diffeomorphisms

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{h_{\mathcal{G}}} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{h_{\mathcal{M}}} & \mathcal{M} \end{array}$$

for $h \in H$. The 2-cells

$$\begin{array}{ccc} & \mathcal{G} & \\ h \nearrow & \Downarrow \mu_{h,h'} & \searrow h' \\ \mathcal{G} & \xrightarrow{hh'} & \mathcal{G} \end{array}$$

implementing multiplicativity of the action are pulled back from the associator α on $\underline{\mathbf{C}}$ along the map $\mathcal{G} \rightarrow \underline{\mathbf{C}}$.

Associated to the H -action on $\mathcal{G} \rightarrow \mathcal{M}$ is the map of quotient stacks $[\mathcal{G}/H] \rightarrow [\mathcal{M}/H]$. This fibration of stacks defines a gerbe over $[\mathcal{M}/H]$ (see Definition 2.2.4.2). Indeed, the \mathcal{I} -action descends along $\mathcal{G} \rightarrow [\mathcal{G}/H]$ via the compatibility data between the H -action and the \mathcal{I} -action. The local triviality condition also holds: Let $Y \twoheadrightarrow \mathcal{M}$ be a cover of \mathcal{M} over which \mathcal{G} trivialises. Then $Y \twoheadrightarrow \mathcal{M} \twoheadrightarrow [\mathcal{M}/H]$ provides a cover over which $[\mathcal{G}/H]$ trivialises.

The category $\mathbf{Vec}[H]$ contains an algebra object $\mathbb{C}[H]$, and the map $\mathbf{Vec}[H] \rightarrow \underline{\mathbf{C}}$ induces the data of an algebra A in $\underline{\mathbf{C}}$, the image of $\mathbb{C}[H]$ under said map.

Lemma 2.4.3.5. *The map of stacks*

$$- \otimes A : \mathcal{G} \rightarrow \underline{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$$

factors through the quotient map

$$\mathcal{G} \rightarrow [\mathcal{G}/H].$$

Proof. We show this by equipping the map $- \otimes A : \mathcal{G} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$ with descent data for the cover $\mathcal{G} \rightarrow [\mathcal{G}/H]$. That is a family of 2-cells

$$\kappa_h : (- \otimes h) \otimes A \xrightarrow{\sim} - \otimes A$$

satisfying the cocycle condition

$$\begin{array}{ccc} ((- \otimes h) \otimes h') \otimes A & \xrightarrow{\kappa_{h'(- \otimes h)}} & (- \otimes h) \otimes A \\ \alpha_{-,h,h'} \otimes \text{id}_A \downarrow & & \downarrow \kappa_h(-) \\ (- \otimes (h \otimes h')) \otimes A & \xrightarrow{\kappa_{h \otimes h'}(-)} & - \otimes A \end{array}$$

Such a family of 2-cells is provided by

$$\kappa_h : (- \otimes h) \otimes A \xrightarrow{\alpha_{-,h,A}} - \otimes (h \otimes A) \xrightarrow{\text{id}_- \otimes \mu_h} - \otimes A,$$

where μ_h denotes the h -component of the multiplication on $A \simeq \bigoplus_{h \in H} h$. This family indeed satisfies the cocycle condition: for any $X \in \mathcal{G}(U)$, the diagram

$$\begin{array}{ccccc} ((X \otimes h) \otimes h') \otimes A & \xrightarrow{\alpha_{X \otimes h, h', A}} & (X \otimes h) \otimes (h' \otimes A) & \xrightarrow{(\text{id}_X \otimes \text{id}_h) \otimes \mu_{h'}} & (X \otimes h) \otimes A \\ \alpha_{X, h, h'} \otimes \text{id}_A \downarrow & & \downarrow \alpha_{X, h, h' \otimes A} & & \downarrow \alpha_{X, h, A} \\ & & X \otimes (h \otimes (h' \otimes A)) & \xrightarrow{\text{id}_X \otimes (\text{id}_h \otimes \mu_{h'})} & X \otimes (h \otimes A) \\ & & \uparrow \text{id}_X \otimes \alpha_{h, h', A} & & \downarrow \text{id}_X \otimes \mu_h \\ (X \otimes (h \otimes h')) \otimes A & \xrightarrow{\alpha_{X, h \otimes h', A}} & X \otimes ((h \otimes h') \otimes A) & \xrightarrow{\text{id}_X \otimes \mu_{h \otimes h'}} & X \otimes A \end{array}$$

commutes by the pentagon equation for α , the naturality of \otimes , and associativity of μ . \square

Lemma 2.4.3.5 constructs a map $[\mathcal{G}/H] \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$. As $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is \mathbf{VCat} -valued (and, in particular, has zero objects), we may extend this to a map $[\mathcal{G}/H]_{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$. Lemma 2.4.3.3 implies $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is further Karoubi complete, so we may extend further to a map of linear stacks $\mathbf{Sky}_{[\mathcal{M}/H]}^{[\mathcal{G}/H]} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$.

Proposition 2.4.3.6. *The map*

$$\mathbf{Sky}_{[\mathcal{M}/H]}^{[\mathcal{G}/H]} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$$

is an equivalence of linear stacks. Hence $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is orbisimple.

Proof. The essential image of $[\mathcal{G}/H] \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$ constructed in Lemma 2.4.3.5 contains the essential image of $-\otimes A : \mathcal{G} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$. Lemma 2.4.3.3 identifies $\mathbf{Mod}_A(\underline{\mathbf{C}})$ as the Cauchy completion of the essential image of the free module map $-\otimes A : \underline{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$. Meanwhile, Proposition 2.3.4.8 identifies $\mathbf{Sky}_{[\mathcal{M}/H]}^{[\mathcal{G}/H]}$ as the Cauchy completion of the essential image of $[\mathcal{G}/H]_{\mathbf{C}}$. This shows the map in the statement is essentially surjective.

Full faithfulness may be checked on the map $[\mathcal{G}/H]_{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$ – the property is preserved under Cauchy completion. Every object $X \in [\mathcal{G}/H]_{\mathbf{C}}(U)$ locally lifts to an object in $\mathcal{G}_{\mathbf{C}}(U)$. If \tilde{X} is such a lift, then so is $\tilde{X} \otimes h$. The map $[\mathcal{G}/H]_{\mathbf{C}} \rightarrow \mathbf{Mod}_A(\underline{\mathbf{C}})$ sends

$$X \mapsto \tilde{X} \otimes A.$$

We must prove that the map

$$\mathrm{Hom}_{[\mathcal{G}/H]_{\mathbf{C}}(U)}(X, Y) \rightarrow \mathrm{Hom}_{\mathbf{Mod}_A(\mathbf{C}^U)}(\tilde{X} \otimes A, \tilde{Y} \otimes A)$$

is an isomorphism for all $X, Y \in [\mathcal{G}/H]_{\mathbf{C}}$ with chosen lifts $\tilde{X}, \tilde{Y} \in \mathcal{G}_{\mathbf{C}}$. The right-hand side simplifies to

$$\mathrm{Hom}_{\mathbf{Mod}_A(\mathbf{C}^U)}(\tilde{X} \otimes A, \tilde{Y} \otimes A) \simeq \mathrm{Hom}_{\mathbf{C}^U}(\tilde{X}, \tilde{Y} \otimes A) \simeq \bigoplus_{h \in H} \mathrm{Hom}_{\mathbf{C}^U}(\tilde{X}, \tilde{Y} \otimes h),$$

exhibiting every morphism in $\mathbf{Mod}_A(\mathbf{C}^U)$ as a direct sum of morphisms of the form

$$\tilde{X} \otimes A \xrightarrow{\tilde{f} \otimes \mathrm{id}_A} (\tilde{Y} \otimes h) \otimes A \xrightarrow{\kappa_h(\tilde{Y})} \tilde{Y} \otimes A.$$

Here, $\tilde{f} : \tilde{X} \rightarrow \tilde{Y} \otimes h$ is a morphism in $\mathbf{C}(U)$, and $\kappa_h : (- \otimes h) \otimes A \rightarrow - \otimes A$ the obvious isomorphism (see also the proof of Lemma 2.4.3.5).

Applying the quotient functor $\mathcal{G} \rightarrow [\mathcal{G}/H]$ to \tilde{f} , we obtain a morphism $f : X \rightarrow Y$, whose image under $[\mathcal{G}/H] \rightarrow \mathbf{Mod}_A(\mathbf{C}^U)$ is $f \otimes A$. This shows fullness of the map.

To see the map is also faithful, we use the commutative square

$$\begin{array}{ccc} \mathcal{G}_{\mathbf{C}}(U) & \longrightarrow & \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}(U) \\ q \downarrow & & \downarrow (q \times \mathrm{id}_U)_* \\ [\mathcal{G}/H]_{\mathbf{C}}(U) & \longrightarrow & \mathbf{Sky}_{[\mathcal{M}/H]}^{[\mathcal{G}/H]}(U) \end{array}$$

to write

$$\begin{aligned} \mathrm{Hom}_{[\mathcal{G}/H]_{\mathbf{C}}(U)}(X, Y) &= \mathrm{Hom}_{[\mathcal{G}/H]_{\mathbf{C}}} \left((q \times \mathrm{id}_U)_* \tilde{X}, (q \times \mathrm{id}_U)_* \tilde{Y} \right) \\ &\simeq \mathrm{Hom}_{\mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}} \left((q \times \mathrm{id}_U)^* (q \times \mathrm{id}_U)_* \tilde{X}, \tilde{Y} \right) \\ &\simeq \bigoplus_{h \in H} \mathrm{Hom}_{\mathbf{C}^U}(\tilde{X} \otimes h, \tilde{Y}). \end{aligned}$$

This has the same rank (as a C^∞ -module) as $\mathrm{Hom}_{\mathbf{Mod}_A(\mathbf{C}^U)}(\tilde{X} \otimes A, \tilde{Y} \otimes A)$, so fullness implies faithfulness. \square

EXAMPLE 2.4.3.7. Let $\mathbf{Vec}^\omega[G]$ be a categorified group ring associated to a Lie group G and 3-cocycle $\omega \in H^3(\mathcal{B}G, \mathbb{C}^\times)$, as constructed in Section 2.4.2. Let H be a finite subgroup $i : H \hookrightarrow G$ and pick a trivialisation of the pullback 3-cocycle $i^*\omega$ on H . This data induces a map $\mathbf{Vec}[H] \rightarrow \mathbf{Vec}^\omega[G]$. Denote by $\theta \in H^2(G, \mathbb{C}^\times)$ the transgressed cocycle of ω , classifying the gerbe over G . The trivialisation of $i^*\omega$ induces descent data for the gerbe over G along the quotient map $G \twoheadrightarrow G/H$. This exhibits θ as the image of a class θ/H under the morphism $H^2(G/H, \mathbb{C}^\times) \rightarrow H^2(G, \mathbb{C}^\times)$.

The stack of modules $\mathbf{Mod}_{\mathbb{C}[H]}(\mathbf{Vec}^\omega[G])$ is a manisimple category, equivalent to the linear stack $\mathbf{Vec}^{\theta/H}[G/H]$ (we slightly abuse notation here: this stack does not have a monoidal product).

A particular example is provided by $SU(2)$ at level $k \in H^3(\mathcal{B}SU(2), \mathbb{C}^\times) \simeq \mathbb{Z}$, and the subgroup $i : \mathbb{Z}/2 \hookrightarrow SU(2)$. The relevant map on cohomology,

$$i^* : H^3(\mathcal{B}SU(2), \mathbb{C}^\times) \rightarrow H^3(\mathcal{B}\mathbb{Z}/2),$$

is the surjective map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$. Hence there is a map of tensor stacks $\mathbf{Vec}[\mathbb{Z}/2] \rightarrow \mathbf{Vec}^k[SU(2)]$ precisely when k is even.

Recall the underlying manifold of $SU(2)$ is S^3 . The quotient manifold $S^3/(\mathbb{Z}/2)$ is \mathbb{RP}^3 , the underlying manifold of $SO(3)$. Gerbes over \mathbb{RP}^3 are classified by the integers, and the pullback map

$$H^2(\mathbb{RP}^3, \mathbb{C}^\times) \rightarrow H^2(S^3, \mathbb{C}^\times)$$

is the map $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Now assume k is even, and pick a functor $\mathbf{Vec}[\mathbb{Z}/2] \rightarrow \mathbf{Vec}^k[SU(2)]$. By abuse of notation, we denote the manisimple category of $\mathbb{Z}/2$ -modules in $\mathbf{Vec}^k[SU(2)]$ by $\mathbf{Vec}^{k/2}[\mathbb{RP}^3]$ (Recall this is the notation we used for categorified group rings, but the manisimple category $\mathbf{Vec}^{k/2}[\mathbb{RP}^3]$ does not carry a monoidal structure.) In the next section, we show that when k is divisible by 4, the category $\mathbf{Vec}^{k/2}[\mathbb{RP}^3]$ naturally acquires the structure of a manifold category (and in fact turns into $\mathbf{Vec}^{k/4}[SO(3)]$).

2.4.4 Central Group Actions on Tensor Stacks

Recall that the Drinfel'd centre of a tensor category \mathbf{C} is the braided tensor category $\mathcal{Z}\mathbf{C}$ with

- objects: tuples $(X \in \mathbf{C}, \gamma : X \otimes - \rightarrow - \otimes X)$, s.t. γ satisfies the hexagon equation,
- morphisms $(X, \gamma) \rightarrow (X', \gamma')$: morphisms $f : X \rightarrow X'$ compatible with γ and γ' .

A natural isomorphism γ as above is called a *half-braiding* for X .

A *commutative algebra* in a tensor category \mathbf{C} is an algebra A in its Drinfel'd centre whose multiplication commutes with the braiding β in $\mathcal{Z}\mathbf{C}$:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\beta(A,A)} & A \otimes A \\ & \searrow m & \swarrow m \\ & & A \end{array}$$

In [Par95], it was shown that the category of modules over a commutative algebra A in \mathbf{C} is monoidal (subject to certain completeness conditions on \mathbf{C}). In this section, we extend this result to the context of orbifold tensor categories in the case of a finite group algebra.

We recall the necessary details from [Par95]. Let $(A, \gamma : A \otimes - \rightarrow - \otimes A)$ be a commutative algebra in \mathbf{C} . The half-braiding γ may be used to equip a right module M with a left module structure via

$$A \otimes M \xrightarrow{\gamma(M)} M \otimes A \rightarrow M.$$

The commutativity condition ensures that the left module structure is compatible with the right module structure, making M a bimodule. The tensor product of bimodules M, N is computed by the coequaliser (whose existence is assumed)

$$M \otimes_A N := \operatorname{colim}(M \otimes A \otimes N \rightrightarrows M \otimes N).$$

The tensor product of free right modules $M = X \otimes A, N = Y \otimes A$ is given by a free module

$$(X \otimes A) \otimes_A (Y \otimes A) \simeq (X \otimes Y) \otimes A.$$

Now we carry the above story over to our setup. As explained in [Str04b], the notion of Drinfel'd centre admits an internalisation to any braided monoidal bicategory. In the bicategory \mathbf{SmSt} of stacks over \mathbf{Man} , this is equivalent to the braided monoid

$$\mathcal{Z}\underline{\mathbf{C}} = \operatorname{End}(\underline{\mathbf{C}}\underline{\mathbf{C}})$$

of endomorphisms of $\underline{\mathbf{C}}$, considered as a $\underline{\mathbf{C}}\text{-}\underline{\mathbf{C}}$ -bimodule. It is equipped with a canonical monoidal morphism $\mathcal{Z}\underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}}$. The centre is computed by a limit. Hence the functor $\operatorname{ev}_U : \mathbf{SmSt} \rightarrow \mathbf{VCat}$ that evaluates a stack at $U \in \mathbf{Man}$ induces a monoidal comparison functor $(\mathcal{Z}\underline{\mathbf{C}})(U) \rightarrow \mathcal{Z}(\mathbf{C}^U)$. This functor should be thought of as including those half-braidings that are smooth.

Let $\mathbf{Vec}[H]$ denote the braided fusion category with simple objects H , equipped with the trivial associator and trivial braiding.

Definition 2.4.4.1. A central group action of a finite abelian group H on a tensor stack $\underline{\mathbf{C}}$ is a braided monoidal morphism of linear stacks $\mathbf{Vec}[H] \rightarrow \mathcal{Z}\underline{\mathbf{C}}$.

A central group action of H on $\underline{\mathbf{C}}$ induces a commutative algebra A in $\underline{\mathbf{C}}$, the image of the commutative algebra $\mathbf{C}[H]$ under the composite $\mathbf{Vec}[H] \rightarrow \mathcal{Z}\underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}}$. For each $U \in \mathbf{Man}$, we obtain a commutative algebra $A^U \in \mathbf{C}^U$. Forgetting the braiding, the algebra A is a group algebra in $\underline{\mathbf{C}}$. In Section 2.4.3, we constructed the stack of modules over a group algebra in an orbifold tensor category, and showed it is an orbisimple category (Proposition 2.4.3.6).

Theorem 2.4.4.2. Let A be the algebra induced by a central group action $\mathbf{Vec}[H] \rightarrow \mathcal{Z}\underline{\mathbf{C}}$ on an orbifold tensor category. The stack $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is naturally an orbifold tensor category.

Proof. Proposition 2.4.3.6 assures us that $\mathbf{Mod}_A(\underline{\mathbf{C}})$ is orbisimple. It remains to equip it with a monoidal structure.

We do this objectwise: the category

$$(\mathbf{Mod}_A(\underline{\mathbf{C}}))(U) = \mathbf{Mod}_{A^U}(\mathbf{C}^U)$$

is monoidal with tensor product \otimes_{A^U} as long as the necessary colimit in the diagram exists. The tensor product of free modules in $\mathbf{Mod}_{A^U}(\mathbf{C}^U)$ exists: it is given by

$$(X \otimes A^U) \otimes_{A^U} (Y \otimes A^U) \simeq (X \otimes Y) \otimes A^U.$$

By Lemma 2.4.3.3, $\mathbf{Mod}_{A^U}(\mathbf{C}^U)$ is Karoubi-generated by free modules. The monoidal structure on a category uniquely extends to its Karoubi completion: given a pair of objects $(X, e), (Y, f)$ picked out by idempotents $e : X \rightarrow X$ and $f : Y \rightarrow Y$, respectively, their tensor product is given by

$$(X, e) \otimes (Y, f) \simeq (X \otimes Y, e \otimes f).$$

This extends to a monoidal structure on the stack, because A is preserved under pullback along $f : V \rightarrow U$, and thus

$$f^*(- \otimes_{A^U} -) \simeq f^* - \otimes_{A^V} f^* - . \quad \square$$

Definition 2.4.4.3. We call $\mathbf{Mod}_A(\underline{\mathbf{C}})$, equipped with the structure of an orbifold tensor category constructed in Theorem 2.4.4.2 the *quotient of $\underline{\mathbf{C}}$ by H* , and denote it by $\underline{\mathbf{C}}^H$.

EXAMPLE 2.4.4.4. Let $\mathbf{Vec}^\omega[G]$ be a categorified group ring associated to a Lie group G and $\omega \in H^3(\mathcal{B}G, \mathbf{C}^\times)$. At certain ω , central subgroups $H \hookrightarrow Z(G) \rightarrow G$ admit a lift to a central group action $\mathbf{Vec}[H] \rightarrow \mathcal{L}\mathbf{Vec}^\omega[G]$. Such a functor induces the data of a 3-cocycle $\omega/H \in H^3(\mathcal{B}(G/H), \mathbf{C}^\times)$ which lifts ω against

$$H^3(\mathcal{B}(G/H), \mathbf{C}^\times) \rightarrow H^3(\mathcal{B}G, \mathbf{C}^\times).$$

The quotient of $\mathbf{Vec}^\omega[G]$ by H is given by the categorified group ring

$$(\mathbf{Vec}^\omega[G])^H \simeq \mathbf{Vec}^{\omega/H}[G/H].$$

As a particular example, the map $H^3(\mathcal{B}SO(3), \mathbf{C}^\times) \rightarrow H^3(\mathcal{B}SU(2), \mathbf{C}^\times)$ is the map $\mathbb{Z} \xrightarrow{4} \mathbb{Z}$. We show in Examples 2.4.5.5 and 3.6.1.4 that the inclusion of the centre $\mathbb{Z}/2 \hookrightarrow SU(2)$ admits a unique lift to a central group action $\mathbf{Vec}[\mathbb{Z}/2] \hookrightarrow \mathcal{L}\mathbf{Vec}^k[SU(2)]$ precisely when $k \in \mathbb{Z}$ is a multiple of 4. The associated quotient is the categorified group ring $\mathbf{Vec}^{k/4}[SO(3)]$. Its underlying manisimple category is $\mathbf{Vec}^{k/2}[\mathbb{RP}^3]$, see the end of Section 2.4.3.

2.4.5 Constructing Interpolated Fusion Categories

In this section, we explain how one may leverage the results of Section 2.4.4 to construct new families of manifold categories. We study the Drinfel'd centres of categorified group rings $\mathbf{Vec}^\omega[G]$ and fusion categories \mathbf{D} . The product $\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vect}} \mathbf{D}$ has centre

$$\mathcal{Z}(\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vect}} \mathbf{D}) \simeq \mathcal{Z}\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vect}} \mathcal{Z}\mathbf{D}.$$

As fusion categories embed into the bicategory of manifold categories, we have an equivalence

$$\mathcal{Z}(\mathbf{D}) \simeq \underline{\mathcal{Z}(\mathbf{D})},$$

so $\mathcal{Z}(\mathbf{D})$ may be computed treating \mathbf{D} as an ordinary fusion category. We find braided inclusions $\mathbf{Vec}^{\omega',\beta}[H] \rightarrow \mathbf{Vec}^\omega[G]$ (where H is a finite abelian group, and ω', β indicate the associator and braiding, respectively) first in $\mathcal{Z}(\mathbf{D})$, and then look for maps from the *oppositely* braided category $\mathbf{Vec}^{-\omega',-\beta}[H]$ to $\mathcal{Z}(\mathbf{Vec}^\omega[G])$. The diagonal embedding

$$\mathbf{Vec}[H] \hookrightarrow \mathbf{Vec}^{-\omega',-\beta}[H] \boxtimes \mathbf{Vec}^{\omega',\beta}[H] \rightarrow \mathcal{Z}(\mathbf{Vec}^\omega[G] \boxtimes \mathbf{D})$$

now yields a central H -action on $\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vect}} \mathbf{D}$. We denote the quotient by

$$\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vec}[H]} \mathbf{D} := (\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vect}} \mathbf{D})^H.$$

It is an orbifold tensor category whose underlying orbifold is given by $[G \times \mathrm{Irr}(\mathbf{D})/H]$. The *free algebra morphism*

$$- \otimes \mathbb{C}[H] : \mathbf{Vec}^\omega[G] \boxtimes \mathbf{D} \rightarrow \mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vec}[H]} \mathbf{D}$$

is given by tensoring with the group algebra of H , and equipping the result with the free algebra structure. It is monoidal, so the fusion rules and associator of its essential image are easy to determine from those on the domain. When the H -action is free, this functor is in fact essentially surjective, and the monoidal structure on the quotient is completely described by it.

When drawing pictures of the space of simple objects of interpolated fusion categories constructed as above, we mark the image of the objects of \mathbf{D} under the free algebra morphism. These simple objects then carry the same monoidal structure as the corresponding simples in \mathbf{D} . The remaining category “interpolates” between them. This is why we call the resulting categories interpolated fusion categories.

We now compute the Drinfel'd centres of categorified group rings. Let G be a compact connected Lie group. In Chapter 3, we compute the centre of its invertible part $G_\omega = (\mathbf{Vec}^\omega[G])^\times$ in the setting of smooth 2-groups. All of the material in this section is discussed in more detail there. To construct interpolated fusion categories, only the invertible part of the centre is of relevance to us.

Lemma 2.4.5.1.

$$(\mathcal{Z}\mathbf{Vec}^\omega[G])^\times = \mathcal{Z}(\mathbf{Vec}^\omega[G]^\times)$$

Proof. All simple objects of $\mathbf{Vec}^\omega[G]$ are invertible and vice versa. A half-braiding γ of $\mathbb{C}_g \in \mathbf{Vec}^\omega[G]$ determines a half-braiding for $\mathbb{C}_g \in \mathbf{Vec}^\omega[G]^\times$. By naturality, this data determines γ up to equivalence. \square

The invertible part of the Drinfel'd centre is a braided categorical group. We recall the characterisation of these by quadratic forms. Any braided categorical group is equivalent to $\mathbf{Vec}^\omega[H]^\times$ for some finite group H and cocycle $\omega \in H^3(\mathcal{B}H, \mathbb{C}^\times) \cong H^4(\mathcal{B}H, \mathbb{Z})$. If H is abelian, the tensor product is commutative at the level of isomorphism classes, so one may ask for braidings. A braiding β equips H with a quadratic form

$$\begin{aligned} q : H &\rightarrow \mathbb{C}^\times \\ h &\mapsto \beta_{\mathbb{C}_h, \mathbb{C}_h}, \end{aligned}$$

sending each group element to the self-braiding of the associated object, viewed as a morphism in $\text{End}(\mathbb{C}_{2h}) \cong \mathbb{C}$. A map $q : H \rightarrow \mathbb{C}^\times$ is called a *quadratic form* if $q(nx) = n^2q(x)$ and the associated polarisation form

$$\begin{aligned} \sigma : H \times H &\rightarrow \mathbb{C}^\times \\ (h, h') &\mapsto q(h + h') - q(h) - q(h') \end{aligned}$$

is bilinear. We denote the abelian group of quadratic forms by $\text{Quad}(H, \mathbb{C}^\times)$. By a classical Theorem of Eilenberg and MacLane [EM54], this assignment establishes an isomorphism between equivalence classes of braided categorical groups that categorify H and $\text{Quad}(H, \mathbb{C}^\times)$.

EXAMPLE 2.4.5.2. All quadratic forms on \mathbb{Z}/n are of the form

$$k \mapsto q^{k^2}$$

for some complex number q of unit length. In order for this to be a well defined map, q must satisfy $q^{n^2} = 1$. In order for it to be a quadratic form, it must satisfy $q^{2n} = 1$. This establishes a bijection between quadratic forms on \mathbb{Z}/n and n -th/ $2n$ -th roots of unity for odd/even n .

In the case $n = 2$, this means quadratic forms correspond to powers of $\exp(\pi/4) = \mathbf{i}$. These give the four braided monoidal structures on $\mathbb{Z}/2$: $\mathbb{Z}/2$ -graded vector spaces $\mathbf{Vec}_{\mathbb{Z}/2}$, the semion \mathbf{Semi} , super vector spaces \mathbf{sVec} and the anti-semion $\overline{\mathbf{Semi}}$.

The set of braidings on a particular fusion category $\mathbf{Vec}^\omega[H]$ is freely and transitively acted upon by the set of bicharacters $H \times H \rightarrow \mathbb{C}^\times$, denoted $\mathbf{Bilin}(H, \mathbb{C}^\times)$. The action is given by pointwise multiplication. The braidings form a (possibly empty) torsor for the group of bicharacters. In other words, there is an exact sequence beginning in

$$0 \rightarrow \mathbf{Bilin}(H, \mathbb{C}^\times) \rightarrow \mathbf{Quad}(H, \mathbb{C}^\times) \rightarrow H^3(\mathcal{B}H, \mathbb{C}^\times).$$

EXAMPLE 2.4.5.3. The bicharacters of \mathbb{Z}/n are given by

$$(i, j) \mapsto \tilde{q}^{ij}$$

for an n -th root of unity \tilde{q} . Combined with Example 2.4.5.2, this tells us that for odd n , $\mathbf{Vec}^\omega[\mathbb{Z}/n]$ only admits a braiding if ω is trivial, while for even n , there are n braidings for $\omega \in \{0, n/2\}$, where $n/2$ denotes the unique element of order 2 in $H^4(\mathcal{B}\mathbb{Z}/n, \mathbb{Z})$.

We finish this section by quickly recalling the centres computed in Chapter 3. Let G be a compact simply-connected Lie group and ω a cocycle in $H^3(\mathcal{B}G, \mathbb{C}^\times) = H^4(\mathcal{B}G, \mathbb{Z})$. Let $T_G \hookrightarrow G$ be a maximal torus of G , then ω defines a real-valued inner product $I_\omega : \mathfrak{t} \otimes \mathfrak{t} \rightarrow \mathbb{R}$ on the Lie algebra \mathfrak{t} of T_G . This identification is chosen such that the norm of every lift of the identity of G is an *even* integer. Pick a lift \bar{z} of each element $z \in Z(G)$ against the map $\mathfrak{t} \rightarrow T_G \rightarrow G$.

Theorem 2.4.5.4 (Theorem 3.6.1.3). *The Drinfel'd centre of $\mathbf{Vec}^\omega[G]^\times$ is the braided categorical group with underlying group $Z(G)$ and quadratic form given by*

$$q : Z(G) \rightarrow \mathbb{C}^\times \\ z \mapsto \exp\left(\frac{1}{2}I_\omega(\bar{z}, \bar{z})\right),$$

where $\exp : \mathbb{R} \rightarrow \mathbb{C}^\times$ is shorthand for the exponential map $x \mapsto e^{i\pi x}$ with kernel $\mathbb{Z} \hookrightarrow \mathbb{R}$.

If G is also *simple*, then $H^4(\mathcal{B}G, \mathbb{Z}) = \mathbb{Z}$, with preferred generator the class inducing the smallest positive-definite inner product on its Lie algebra (and that of its maximal torus). For such groups, we use $k \in \mathbb{Z}$ to denote k times that preferred generator in its degree 4 cohomology.

EXAMPLE 2.4.5.5. As the level k varies, the Drinfel'd centre of $\mathbf{Vec}^k[\mathrm{SU}(n)]^\times$ displays all braided monoidal structures on $Z(\mathrm{SU}(n)) = \mathbb{Z}/n$. The braided categorical group corresponds to the quadratic form

$$q : \mathbb{Z}/n \rightarrow \mathbb{C}^\times \\ 1 \mapsto \exp\left(k \frac{n-1}{2n}\right).$$

For $\mathrm{SU}(2)$, the centre cycles through the 4 braided monoidal structures on $\mathbb{Z}/2$:

$$\mathcal{L}\mathbf{Vec}^k[\mathrm{SU}(2)]^\times = \begin{cases} \mathbf{Vec}[\mathbb{Z}/2]^\times & k \equiv 0 \pmod{4} \\ \mathbf{Semi}^\times & k \equiv 1 \pmod{4} \\ \mathbf{sVec}^\times & k \equiv 2 \pmod{4} \\ \overline{\mathbf{Semi}}^\times & k \equiv 3 \pmod{4}. \end{cases}$$

EXAMPLE 2.4.5.6. For odd Spin groups $\mathrm{Spin}(2n+1)$, the centre $\mathcal{L}\mathbf{Vec}^k[\mathrm{Spin}(2n+1)]^\times$ alternates between $\mathbf{Vec}[\mathbb{Z}/2]^\times$ (k even) and \mathbf{sVec}^\times (k odd).

Let T be a compact torus, and ω a cocycle in $H^4(\mathcal{B}T, \mathbb{Z})$. Denote by \mathfrak{t} its Lie algebra, and by $\tau = I_\omega(-, \cdot) : \mathfrak{t} \rightarrow \mathfrak{t}^*$ the homomorphism induced by the inner product corresponding to ω . Write $\Lambda = \mathrm{Hom}(T, U(1)) \subset \mathfrak{t}^*$ for the group of characters of T and $\Pi = \mathrm{Hom}(U(1), T) \subset \mathfrak{t}$ for its group of cocharacters.

Theorem 2.4.5.7 (Proposition 3.5.0.1). *The Drinfel'd centre of $\mathbf{Vec}^\omega[T]^\times$ is the braided categorical group with underlying group the quotient*

$$\frac{\mathfrak{t} \oplus \Lambda}{\Pi},$$

along the inclusion $\pi \mapsto (\pi, \tau(\pi))$, and quadratic form induced by

$$q : \mathfrak{t} \oplus \Lambda \rightarrow \mathbb{C}^\times \\ (x, \lambda) \mapsto \lambda(x) \exp\left(-\frac{1}{2}I_\omega(x, x)\right).$$

(See also [FHLT10] for a sketch proof of this.)

EXAMPLE 2.4.5.8. The Drinfel'd centre of $\mathbf{Vec}[S^1]^\times$ (with trivial associator) has underlying group

$$\pi_0 \mathcal{L}\mathbf{Vec}[S^1]^\times = \frac{\mathfrak{t}}{\Pi} \oplus \Lambda = T \oplus \Lambda = S^1 \oplus \mathbb{Z}$$

whose quadratic form simply pairs the character with the element of T : $q(x, \lambda) = \lambda(x)$.

The Drinfel'd centre of $\mathbf{Vec}^k[S^1]^\times$ for $k \neq 0$ splits as a direct sum $\mathbb{R} \oplus \Lambda/\Pi$. The second summand is a finite group isomorphic to $\mathbb{Z}/2k$. The braiding on it is encoded by the quadratic form

$$\begin{aligned} q : \mathbb{Z}/2k &\rightarrow \mathbb{C}^\times \\ 1 &\mapsto \exp\left(\frac{1}{4k}\right). \end{aligned}$$

Let G be any compact connected Lie group. It admits a cover $\pi : \tilde{G} \rightarrow G$, where $\tilde{G} = T \times G_i$ is a product of a torus T and simply-connected groups $\{G_i\}$. The cover can be chosen such that $Z := \ker(\tilde{G} \rightarrow G)$ is a finite subgroup of the centre of \tilde{G} . Let ω be a cocycle in $H^3(\mathcal{B}G, \mathbb{C}^\times)$. The cocycle pulled back to \tilde{G} , $\pi_*\omega$, decomposes as a product of a cocycle $\omega_T \in H^3(\mathcal{B}T, \mathbb{C}^\times)$ and cocycles $\omega_i \in H^3(\mathcal{B}G_i, \mathbb{C}^\times)$. The Drinfel'd centre of $\mathbf{Vec}^{\pi_*\omega}[\tilde{G}]^\times$ is the product

$$\mathcal{L}\mathbf{Vec}^{\pi_*\omega}[\tilde{G}]^\times = \left(\boxtimes_i \mathcal{L}\mathbf{Vec}^{\omega_i}[G_i]^\times\right) \boxtimes \mathcal{L}\mathbf{Vec}^{\omega_T}[T]^\times.$$

There is a braided inclusion $Z \hookrightarrow \mathcal{L}\mathbf{Vec}^{\pi_*\omega}[\tilde{G}]^\times$, where Z carries the trivial braiding. We denote by Z' the subcategory on objects that braid trivially with all of Z . This category has a braided Z -action.

Theorem 2.4.5.9 (Theorem 3.6.2.3). *The Drinfel'd centre of $\mathbf{Vec}^\omega[G]^\times$ is the quotient Z'/Z equipped with the quadratic form induced by that on Z' .*

EXAMPLE 2.4.5.10. For $\mathrm{SO}(4)$, we may pick the cover by $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ with kernel $\mathbb{Z}/2$. The degree 4 cohomology of $\mathrm{SO}(4)$ is generated by the Euler class χ and the first Pontryagin class p_1 , while $H^4(\mathcal{B}\mathrm{SU}(2), \mathbb{Z})$ is generated by the second Chern class c . The pullback map sends p_1 to the sum $2c_l + 2c_r$ of twice the generator in each factor, and χ to the difference $c_l - c_r$. For the cohomology class $\omega = ap_1 + b\chi$, one finds that the quadratic form on $Z(\mathrm{SU}(2) \times \mathrm{SU}(2))$ describing the braiding on $\mathcal{L}\mathbf{Vec}^\omega[\mathrm{SU}(2) \times \mathrm{SU}(2)]^\times$ is

$$\begin{aligned} q : \mathbb{Z}/2 \times \mathbb{Z}/2 &\rightarrow \mathbb{C}^\times \\ (1, 0) &\mapsto \mathbf{i}^{2a+b} \\ (0, 1) &\mapsto \mathbf{i}^{2a-b} \\ (1, 1) &\mapsto 1. \end{aligned}$$

The value on $(1, 1)$ is a mere reflection of the fact that this cohomology class on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ comes from a cohomology class on its quotient by the diagonal copy of $\mathbb{Z}/2$.

Whether the object corresponding to $(1, 0)$ braides trivially with $(1, 1)$ is equivalent to whether $q(1, 0) = q(0, 1)$. This happens if b is even, as then $2a + b \equiv 2a - b \pmod{4}$. As a result, the size of the centre varies with the choice of cohomology class:

$$\mathcal{L}\mathbf{Vec}^{a \cdot p_1 + b \cdot \chi}[\mathrm{SO}(4)]^\times = \begin{cases} \mathbf{Vec}[\mathbb{Z}/2]^\times & 2a + b \equiv 0 \pmod{4} \\ \mathbf{sVec}^\times & 2a + b \equiv 2 \pmod{4} \\ \mathbf{Vec}^\times & \text{else.} \end{cases}$$

Given the characterisation of braided 2-groups via quadratic forms, the construction of interpolated fusion categories reduces to computing and comparing quadratic forms on finite groups. If we find a braided 2-group $(H, q) \in \mathcal{L}\mathbf{D}$ in the centre of a fusion category \mathbf{D} , and the oppositely braided 2-group $(H, -q) \in \mathcal{L}\mathbf{Vec}^\omega[G]$ in the centre of a categorified group ring, we can form an interpolated fusion category $\mathbf{Vec}^\omega[G] \boxtimes_{\mathbf{Vec}[H]} \mathbf{D}$.

2.4.6 Interpolated Quantum Group Categories

Let G be a simple, simply-connected Lie group with Lie algebra \mathfrak{g} , denote by h^\vee its dual Coxeter number, and m the lacity. Choose a maximal torus $T \subset G$, with corresponding Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Denote by $\langle \cdot, \cdot \rangle$ the inner product on \mathfrak{t}^* , normalised such that short roots have squared length 2, and let $k \in \mathbb{Z}_{\geq 1}$. Set $\ell = m(h^\vee + k)$ and $q = \exp(1/(2\ell))$. Denote by $\mathbf{C}(\mathfrak{g}, k)$ the fusion category obtained as a semisimple quotient of the category of twisting modules for the small quantum group $u_q(\mathfrak{g})$. It is in fact a *braided* fusion category, and thus embeds into its Drinfel'd centre both via its braiding and its opposite braiding. Introductions to the construction and structure of these categories can be found in [Saw06; Sch18].

The simple objects of $\mathbf{C}(\mathfrak{g}, k)$ correspond to integral weights in the interior of the Weyl alcove, a certain convex subset of \mathfrak{t} depending on k . The centre $Z(G)$ injects into the group of isometries of the Weyl alcove. This map furnishes a bijection between $Z(G)$ and the set of highest weights sitting at those corners of the Weyl alcove that are isometric to the corner around the unit object 0 (with the exception of $\mathbf{C}(\mathfrak{e}_8, 2)$).

EXAMPLE 2.4.6.1. For $\mathrm{SU}(n)$, the Weyl alcove has precisely n corners isometric to the corner around 0. The corresponding highest weights are of the form $k\lambda_i$, where λ_i denotes the i -th fundamental weight. Let λ_1 denote a fundamental weight corresponding to either end of the Dynkin diagram of $\mathrm{SU}(n)$. Then $k\lambda_1 \otimes$ -generates a subcategory of $\mathbf{C}(\mathfrak{g}, k)$ which categorifies \mathbb{Z}/n .

The ribbon twist on the object $\lambda \in \mathbf{C}(\mathfrak{g}, k)$ is given by

$$\theta_\lambda = q^{\langle \lambda, \lambda + 2\rho \rangle},$$

where ρ denotes the sum of the fundamental weights. The categories $\mathbf{C}(\mathfrak{g}, k)$ are unitary modular tensor categories. As a result, the self-braiding of an invertible object is equal to its ribbon twist. Denote by I the basic inner product on \mathfrak{t} corresponding to the generator of $H^4(\mathcal{B}G, \mathbb{Z}) \cong \mathbb{Z}$, with normalisation such that short coroots have square norm 2. By [Hen17a, Prop 10], the twist on a highest weight $\lambda = k\omega$ can be calculated using

$$\langle k\lambda_i, k\lambda_i + 2\rho \rangle = \frac{k}{2} I(\lambda_i^\vee, \lambda_i^\vee).$$

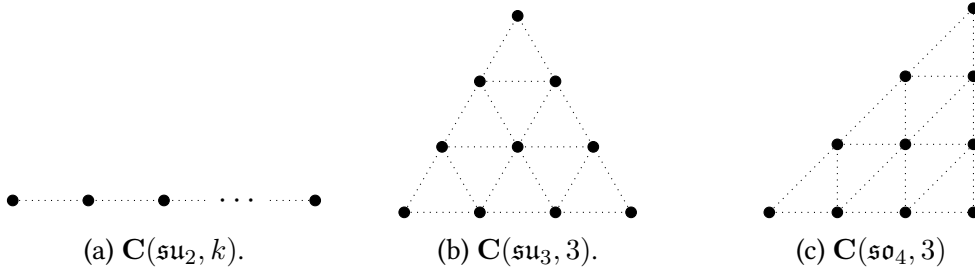


Figure 2.4: Simple objects of Quantum group categories. The dotted lines are visual aids.

The coweights $\lambda_i^\vee \in \mathfrak{t}$ are lifts of $z_i \in Z(G)$ against $\mathfrak{t} \rightarrow T \subset G$ (see Chapter 3 for more details). Under the bijection of $Z(G)$ with the highest weights λ_i , this induces the quadratic form

$$q : Z(G) \rightarrow U(1)$$

$$z_i \mapsto \exp\left(\frac{k}{2}I(\bar{z}_i, \bar{z}_i)\right).$$

EXAMPLE 2.4.6.2. For $SU(n)$, the self-twists are given by

$$\theta_{k\lambda_j} = q^{k^2\langle\lambda_j, \lambda_j\rangle + 2k\langle\lambda_j, \rho\rangle} = \exp\frac{k \cdot j \cdot (n-j)}{2n}.$$

Evaluating for our chosen generator λ_1 ,

$$\theta_{k\lambda_1} = \exp k \frac{n-1}{2n}.$$

The quadratic form on $Z(G)$ encoding the braiding on $\mathbf{C}(\mathfrak{g}, k)^\times$ is *equal* to the quadratic form that encodes $\mathcal{L}(\mathbf{Vec}^k[G])^\times$ (see Theorem 2.4.5.4), and hence the inverse of the quadratic form on $\mathcal{L}(\mathbf{Vec}^{-k}[G])^\times$. There is thus a braided embedding

$$\mathbf{Vec}[Z(G)] \hookrightarrow \mathcal{L}(\mathbf{Vec}^{-k}[G] \boxtimes_{\mathbf{Vect}} \mathbf{C}(\mathfrak{g}, k)),$$

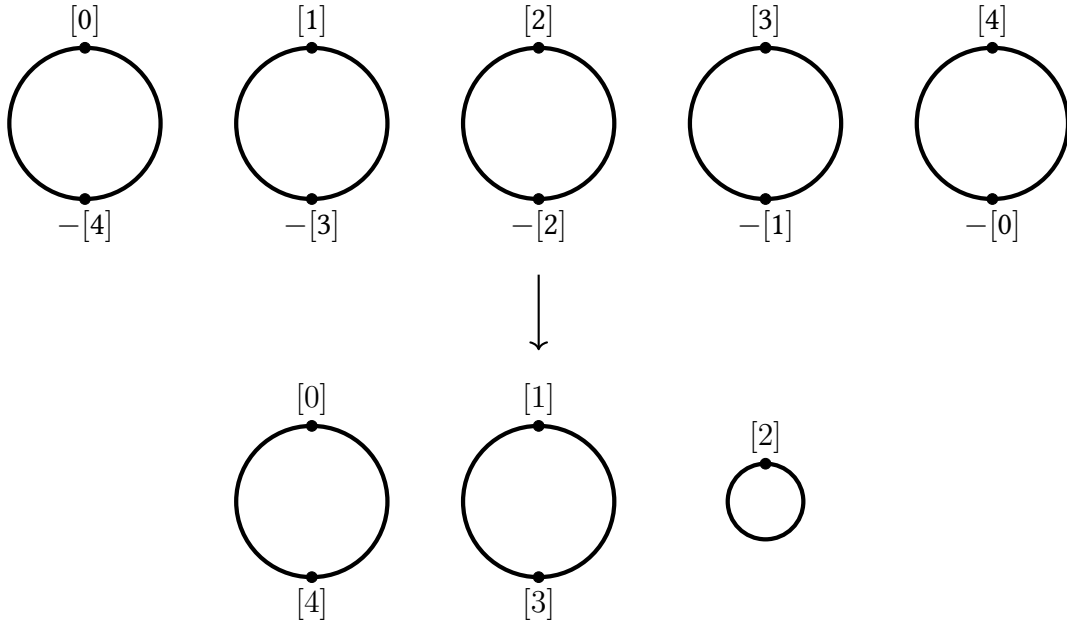
where $\mathbf{Vec}[Z(G)]$ carries the trivial braiding. The resulting interpolated fusion category

$$\mathbf{Vec}^{-k}[G] \boxtimes_{\mathbf{Vec}[Z(G)]} \mathbf{C}(\mathfrak{g}, k)$$

has set of simples a disjoint union of manifolds of the form G/H , where H is a finite central subgroup of G .

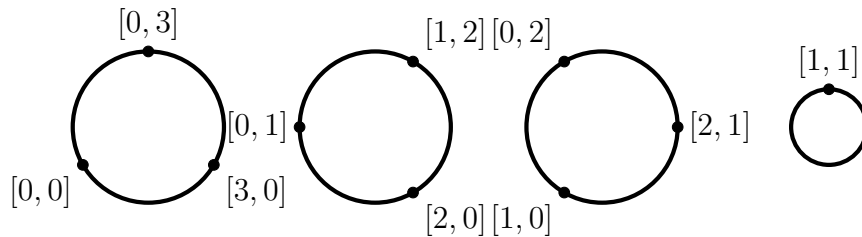
EXAMPLE 2.4.6.3. For $SU(2)$, the quantum group category at level k has $k + 1$ simple objects $[0] \dots [k]$ which we think of as arranged on a line (see Figure 2.4a). Tensoring with $Z(SU(2)) = \mathbb{Z}/2$ induces the flip action on this line. The simples of $\mathbf{Vec}^{-k}[SU(2)] \boxtimes \mathbf{C}(\mathfrak{su}_2, k)$ have underlying manifold the disjoint union of $k + 1$ copies of S^3 . When $k + 1$ is even, the diagonal action of $\mathbb{Z}/2$ is free on the set of connected components. The interpolated fusion category $\mathbf{Vec}^{-k}[SU(2)] \boxtimes_{\mathbf{Vec}[\mathbb{Z}/2]} \mathbf{C}(\mathfrak{su}_2, k)$ then has manifold of simples $\coprod_{k/2} S^3$. Antipodal points on each copy of S^3 correspond to pairs $[i]-[k-i]$. If $k + 1$ is odd, the object $[k/2] \in \mathbf{C}(\mathfrak{su}_2, k)$ is fixed under the $\mathbb{Z}/2$ -action. The set of simples of the interpolated fusion category then has one further component whose underlying

manifold is $S^3/(\mathbb{Z}/2) = \mathbb{R}P^3$. Any maximal torus $S^1 \hookrightarrow \mathrm{SU}(2)$ contains the centre $\mathbb{Z}/2$, and we can restrict along the embedding to obtain the manifold tensor subcategory $\mathrm{Vec}^k[S^1] \boxtimes_{\mathrm{Vec}[\mathbb{Z}/2]} \mathbf{C}(\mathfrak{su}_2, k)$. We indicate the construction as a quotient by depicting what happens to the manifold of simples for the case $k = 4$ below. At the top, we depict the manifold of simples of $\mathrm{Vec}^4[S^1] \boxtimes_{\mathrm{Vect}} \mathbf{C}(\mathfrak{su}_2, 4)$. We label the simples $[i]$ (short for $\mathbb{1} \boxtimes [i]$), as well as their image $-[i] := -1 \boxtimes [k - i]$ under the action of $-1 \in \mathbb{Z}/2$.



The same construction as in Example 2.4.6.3 can be repeated for $\mathbb{Z}(n) = Z(\mathrm{SU}(n)) \hookrightarrow \mathrm{SU}(n)$ for all n . It can also be restricted to any monoidal subcategory closed under the action of the centre.

EXAMPLE 2.4.6.4. The category $\mathbf{C}(\mathfrak{su}_3, k)$ has simples corresponding to points on a triangular lattice in a convex subset of \mathbb{R}^2 (see Figure 2.4b). We can interpolate using $\mathrm{SU}(3)$ at level $-k$ to obtain the manifold category $\mathrm{Vec}^{-k}[\mathrm{SU}(3)] \boxtimes_{\mathrm{Vec}[\mathbb{Z}/3]} \mathbf{C}(\mathfrak{su}_3, k)$. Below, we draw a subcategory obtained by the restricting along the embedding of a circle $S^1 \hookrightarrow \mathrm{SU}(3)$. The simple objects of $\mathbf{C}(\mathfrak{su}_3, k)$ are integral points in the weight lattice of \mathfrak{su}_3 . We label them by their coordinates.



2.4.7 Interpolated Tambara-Yamagami Categories

In [TY98], Tambara and Yamagami obtained a complete classification of fusion categories which contain only one non-invertible object. For convenience, we reproduce here the description of the Tambara-Yamagami categories given in Chapter 1, Example 1.1.1.26. The Tambara-Yamagami fusion category $\mathbf{TY}(A, \chi, \tau)$ associated to an abelian group A , a symmetric non-degenerate bicharacter $\chi : A \otimes A \rightarrow \mathbb{C}^\times$, and a square-root τ of $1/|A|$ has set of simple objects $A \coprod \{m\}$. In what follows, let $a, b, c \in A$. The fusion rules and non-trivial associators are:

$$\begin{aligned} a \otimes b &= a+b, & \alpha_{a,m,b} &= \chi(a,b) \text{id}_m : m \rightarrow m, \\ a \otimes m &= m \otimes a = m & \alpha_{m,a,m} &= \bigoplus_{b,c \in A} \chi(a,b) \delta_{b,c} \text{id}_b : \bigoplus_{b \in A} b \rightarrow \bigoplus_{c \in A} c \\ m \otimes m &= \bigoplus_{a \in A} a & \alpha_{m,m,m} &= \bigoplus_{b,c \in A} \tau \chi(b,c)^{-1} \text{id}_m : \bigoplus_{b \in A} m \rightarrow \bigoplus_{c \in A} m. \end{aligned}$$

The Drinfel'd centre of a Tambara-Yamagami category is computed in [Izu01; GNN09] – see also [Shi11] for a review. The centre contains a family of invertible objects $X_{a,\epsilon} = (a, \beta_{a,-})$ parameterised by an underlying object $a \in H$, and a square root ϵ of $\chi(a,a)$. The half-braiding has components

$$\begin{aligned} \beta_{a,b} &= \chi(a,b) \text{id}_{a+b} \\ \beta_{a,m} &= \epsilon \text{id}_m. \end{aligned}$$

The tensor product among them is given by $X_{a,\epsilon} \otimes X_{b,\eta} = X_{a+b, \epsilon\eta\chi(a,b)}$. The associator is inherited from \mathbf{TY} , and thus trivial. For a fixed $a \in H$, there are precisely two possible values of ϵ . Pick a set of numbers $\{\epsilon_a\}_{a \in H}$ squaring to $\epsilon_a^2 = \chi(a,a)$ such that $\epsilon_a \epsilon_b \chi(a,b) = \epsilon_{a+b}$. Then $\epsilon : H \rightarrow \mathbb{C}^\times$ is a *quadratic refinement* of χ : a quadratic form whose associated bilinear form is χ . The choice of such a quadratic refinement is unique up to multiplication by a homomorphism $s : H \rightarrow \mathbb{Z}/2$. A quadratic refinement of χ induces an inclusion of fusion categories

$$\begin{aligned} \mathbf{Vec}[H] &\hookrightarrow \mathcal{L}(\mathbf{TY}(H, \chi, \tau)) \\ a &\mapsto X_{a,\epsilon_a}. \end{aligned}$$

The induced braiding on $\mathbf{Vec}[H]$ is given by $\beta_{a,b} = \chi(a,b) \text{id}_{a+b}$, and so the quadratic form is the square of ϵ , $q(a) = \chi(a,a) = \epsilon_a^2$.

Now we mix and match quadratic forms to construct interpolated Tambara-Yamagami categories. Let $\mathbf{Vec}^{\omega',\beta}[H] \hookrightarrow \mathcal{L}(\mathbf{Vec}^\omega[G])$ be a braided inclusion. It can be used to interpolate a Tambara-Yamagami category with $\mathbf{TY}(H, \chi, \tau)$ if the quadratic form q induced by β on H is the square of a quadratic form ϵ whose associated bilinear form χ is non-degenerate.

EXAMPLE 2.4.7.1. Let $H = \mathbb{Z}/p$ (p an odd prime) be a finite subgroup of S^1 . It admits a monoidal map to the Drinfel'd centre of $\mathbf{Vec}^k[S^1]$ when k is a multiple of p . The induced quadratic form on \mathbb{Z}/p is given by $q : 1 \mapsto \exp(\frac{k/p}{p})$. This form is a square of another quadratic form iff k admits a square-root mod p . As p is odd, this is always the case. The bilinear form associated to the square-root is non-degenerate iff k is *not* a multiple of p^2 .

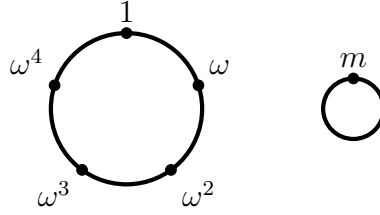
Hence, we get that the allowed levels $k \in H^4(\mathcal{B}S^1, \mathbb{Z}) = \mathbb{Z}$ for building the interpolated Tambara-Yamagami categories are

$$k \in p\mathbb{Z} \setminus p^2\mathbb{Z}.$$

Pick such a k , and pick a square root $\epsilon : \mathbb{Z}/p \rightarrow \mathbb{C}^\times$ of the induced quadratic form $q : n \mapsto \exp(n^2 \cdot k/p^2)$ on \mathbb{Z}/p . Denote by χ the bilinear form associated to ϵ . Lastly, pick $\tau \in \left\{ \frac{1}{\sqrt{p}}, -\frac{1}{\sqrt{p}} \right\}$. Then we may form an interpolated Tambara-Yamagami category

$$\mathbf{Vec}^{-k}[S^1] \boxtimes_{\mathbf{Vec}[\mathbb{Z}/p]} \mathbf{TY}(\mathbb{Z}/p, \chi, \tau).$$

Its manifold of simples is comprised of two circles, one interpolating the objects corresponding to \mathbb{Z}/p in $\mathbf{TY}(\mathbb{Z}/p, \chi, \tau)$, the other containing the image of the object $m \in \mathbf{TY}(\mathbb{Z}/p, \chi, \tau)$. The circles are to be thought of as having different sizes: tensoring with m induces the p -fold covering map $S^1 \rightarrow S^1$ from the unit circle to the circle containing m . An example for $p = 5$ is depicted below.



2.5 Structure of Orbifusion Categories

In this section we study the interplay of the smooth structure with the monoidal structure on an orbifusion category. We show that tensoring with objects induces a smooth action on the support of families of skyscraper sheaves. Further, we show in Corollary 2.5.1.5 that if the monoidal unit sits at a point with trivial stabiliser, every point on the underlying orbifold has trivial stabiliser. We also heavily constrain the structure in a neighbourhood of the monoidal unit: In Corollary 2.5.2.3, we show that the gerbe must admit a trivialisation over a neighbourhood of the support of the monoidal unit. Lemma 2.5.3.5 shows that the tensor product near the unit is encoded by a single smooth multiplication (up to isomorphism).

2.5.1 Tensor Products and Support Functions

Let $\underline{\mathbf{C}} \simeq \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ be an orbifold tensor category, and $\mathcal{S} : U \rightarrow \mathbf{C}$ a U -family. Recall Definition 2.3.2.10: a set of support lifts for \mathcal{S} with respect to an orbifold patch $q : M \rightarrow \mathcal{M}$ with a trivialisation τ of $q^*\mathcal{G}$ is a set of smooth maps $\{\tilde{s}_i : U \rightarrow M\}$, such that the induced mobile family

$$\bar{\mathcal{S}} := \mathrm{Ind}^{\tilde{s}_i^* \tau}(q \circ \tilde{s}_i) = \bigoplus_i \mathrm{Gr}(q \circ \tilde{s}_i)_*(\tilde{s}_i^* \tau \cdot C^\infty)$$

is a mobile cover for \mathcal{S} . This means it contains the family as a direct summand $\mathcal{S} \subset \bar{\mathcal{S}}$ and induces the same support function $\text{supp } \mathcal{S} = \text{supp } \bar{\mathcal{S}} : U \rightarrow |\mathcal{M}|$.

The Lemma below is a technical rephrasing of the following claim: The tensor product plays well with local support lifts, in that any support lifts of a tensor product $\mathcal{S} \otimes \mathcal{T}$ depend smoothly on any pair of support lifts for \mathcal{S} and \mathcal{T} .

Lemma 2.5.1.1. *Let $q : M \rightarrow \mathcal{M}$, $q' : M' \rightarrow \mathcal{M}$, $q'' : M'' \rightarrow \mathcal{M}$ be charts of \mathcal{M} that are disjoint unions of contractible spaces. Let $\mathcal{S} : U \rightarrow \mathbf{C}$, $\mathcal{T} : V \rightarrow \mathbf{C}$ be families of skyscraper sheaves, and fix a pair of points $x \in U$, $y \in V$. Assume \mathcal{S} and \mathcal{T} have single-valued local support lifts $\tilde{s} : U \rightarrow M$ and $\tilde{t} : U \rightarrow M'$ around x and y , respectively. Then any set of local support lifts $\{\tilde{r}_i : U \rightarrow M''\}$ of $\mathcal{S} \otimes \mathcal{T}$ around (x, y) , depends smoothly on the support lifts \tilde{s} and \tilde{t} : There are smooth functions $f_i : U \times V \rightarrow M''$ such that $\tilde{r}_i = f_i \circ (\tilde{s} \times \tilde{t})$.*

Proof. We can use the chart maps q, q' to induce mobile families

$$\mathcal{I} := (\text{Ind } q)^{\oplus n} : M \rightarrow \mathbf{C} \quad \mathcal{I}' := (\text{Ind } q')^{\oplus n'} : M' \rightarrow \mathbf{C},$$

where the integers n, n' are chosen large enough such that \mathcal{S} and \mathcal{T} have mobile covers given by

$$\mathcal{S} \subset \bar{\mathcal{S}} := \tilde{s}^* \mathcal{I} \quad \mathcal{T} \subset \bar{\mathcal{T}} := \tilde{t}^* \mathcal{I}',$$

and

$$\mathcal{S} \otimes \mathcal{T} \subset (\tilde{s} \times \tilde{t})^* (\mathcal{I} \otimes \mathcal{I}').$$

As we are working up to isomorphism, we need not pick explicit trivialisations of the pullback gerbes over M, M' and M'' (see Lemma 2.3.2.5 and Notation 2.3.2.6).

Given local support lifts $\{\tilde{f}_i : M \times M' \rightarrow M''\}$ of $\mathcal{I} \otimes \mathcal{I}'$ to some chart $q''' : M''' \rightarrow \mathcal{M}$, $(\tilde{s} \times \tilde{t})^* (\mathcal{I} \otimes \mathcal{I}')$ has local support lifts $\{\tilde{f}_i \circ (\tilde{s} \times \tilde{t}) : U \times V \rightarrow M''\}$. By Corollary 2.3.2.24, any local support lift of $\mathcal{S} \otimes \mathcal{T}$ to the chart M''' is a Γ -translate of one of the above. As their value agrees at x , they can also be lifted to the chart M'' . Both Γ -translation and the transition maps between charts are smooth, hence the composition of these maps with the maps \tilde{f}_i yield the desired smooth map $f_i : M \times M' \rightarrow M''$. \square

REMARK 2.5.1.2. Corollary 2.3.2.16 tells us that any family \mathcal{S} locally decomposes into summands with single-valued support, so the assumption in the Lemma above is not restrictive at all.

In particular, Lemma 2.5.1.1 says that the operations of left- or right-tensoring a family $\mathcal{S} : U \rightarrow \mathbf{C}$ with an object $X \in \mathbf{C}$ induce postcomposition with a smooth multivalued function (whose domain is U) on support lifts. Recall a family $\mathcal{S} : U \rightarrow \mathbf{C}$ was called étale-/submersive/immersive if it locally admits a set of support lifts which have the respective property (Definition 2.3.2.19).

Lemma 2.5.1.3. *Let $\mathcal{S} : U \rightarrow \mathbf{C}$ be an étale/submersive/immersive U -family, and $X \in \mathbf{C}$. Then $X \otimes \mathcal{S}$ is étale/submersive/immersive.*

Proof. It is enough to check this near a point $x \in U$. Corollary 2.3.2.16 tells us that $X \otimes \mathcal{S}$ locally decomposes into summands which admit a single local support lift. We will show that each of these is étale/submersive/immersive. Let $\mathcal{T} \subset X \otimes \mathcal{S}$ be a subfamily, with a single local support lift $s : U \rightarrow M$ to a chart $M \rightarrow \mathcal{M}$. Then the inclusion $\mathcal{T} \hookrightarrow X \otimes \mathcal{S}$ is a nowhere-zero morphism in

$$\mathrm{Hom}(X^\vee \otimes \mathcal{T}, \mathcal{S}) \simeq \mathrm{Hom}(\mathcal{T}, X \otimes \mathcal{S}),$$

so by Corollary 2.3.2.22, $X^\vee \otimes \mathcal{T}$ and \mathcal{S} share a summand on an open neighbourhood of x , which is étale/submersive/immersive by assumption. There must then exist a set of local support lifts $\{s'_j : U \rightarrow M'\}$ of $X^\vee \otimes \mathcal{T}$, which contains an étale/submersive/immersive map. By Lemma 2.5.1.1, this map is obtained by postcomposing the local support lift s with a smooth function $M \rightarrow M'$. As postcomposition by smooth maps cannot increase the rank, the map s must already be étale/submersive/immersive. This proves the assertion. \square

Lemma 2.5.1.4. *Let $\mathcal{S} : U \rightarrow \mathbf{C}$ be a mobile family in an effective orbifold tensor category and $X \in \mathbf{C}$, then both $X \otimes \mathcal{S}$ and $\mathcal{S} \otimes X$ are mobile.*

Proof. We check this is true at $x \in U$. Around x , \mathcal{S} decomposes as a direct sum of basic mobiles, so it suffices to show the statement if \mathcal{S} is a basic mobile. Without loss of generality, $\mathcal{S} \cong \mathrm{Ind} f$ for a smooth map $f : U \rightarrow \mathcal{M}$ with a local lift $\tilde{f} : U \rightarrow M$, to a chart $q : M \rightarrow \mathcal{M}$. The chart map q is étale, and thus the associated basic mobile $\mathrm{Ind} q$ is an étale family. We can compute the tensor product via a pullback:

$$X \otimes \mathcal{S} \cong X \otimes f^* \mathrm{Ind} q \cong f^*(X \otimes \mathrm{Ind} q).$$

Now we combine Lemma 2.5.1.3 with Lemma 2.3.2.20 (which is only valid in effective orbifold tensor categories) to conclude that $X \otimes \mathrm{Ind} q$ is étale, thus submersive, thus mobile. Being mobile is preserved under pullback, which proves $X \otimes \mathcal{S}$ is mobile. The proof for $\mathcal{S} \otimes X$ is identical. \square

We can now heavily constrain the structure of effective orbifold tensor categories.

Corollary 2.5.1.5. *If the unit $\mathbb{1}$ in an effective orbifold tensor category $\underline{\mathbf{C}}$ is supported at a point with trivial stabiliser $\Gamma = 1$, then $\underline{\mathbf{C}}$ is a manifold category.*

Proof. We prove this by contradiction. Assume $\mathbb{1}$ is supported at a point with trivial stabiliser, so $\mathbb{1}$ is mobile. Then Lemma 2.5.1.4 implies any object $X \in \mathbf{C}$ is mobile: A tensor product is mobile if either of the factors is, and $X \simeq \mathbb{1} \otimes X$.

Now assume there exists a point x of the underlying orbifold such that $\Gamma_x \neq 1$. Then there exists an object X with support x which is not mobile: The category $\mathbf{C}^{(x)}$ is equivalent to $\mathbf{Rep}^{\theta_x}(\Gamma_x)$ and the basic mobile at x corresponds (up to isomorphism) to the θ_x -twisted group ring (Lemma 2.3.1.3). As Γ_x is non-trivial, it always has non-trivial summands: This is a direct consequence of the decomposition given in Proposition 2.2.1.5. \square

2.5.2 Grothendieck Ring

The monoidal structure upgrades the Grothendieck group $K(\mathbf{C})$ to a ring. Its product takes the form

$$X_i \cdot X_j = \sum_{k \in I} n_{i,j}^k X_k,$$

where $n_{i,j}^k \in \mathbb{Z}_+$. The sum is evaluable because, for fixed (i, j) , only finitely many $n_{i,j}^k$ are nonzero. In the literature, this structure has been called a \mathbb{Z}_+ -based ring (see [EK95; Ost03a]). In our case, the so-called basis $\{X_i\}_{i \in I}$ is usually infinite. Duality data further equips the ring with an involution ${}^\vee : K(\mathbf{C}) \rightarrow K(\mathbf{C})$.

In an orbifusion category, the unit object $\mathbb{1}$ is required to be simple. We denote its support by $[0] := \text{supp}(\mathbb{1})$, and fix a representative $0 \in \mathcal{M}$ with stabiliser group $\Gamma := \Gamma_0$, and a 2-cocycle θ_0 representing the gerbe at the point 0.

Definition 2.5.2.1. The *mobile unit* \mathbb{O} is the basic mobile at $[0]$, (see Definition 2.3.3.1)

$$\mathbb{O} := \mathbb{O}_0.$$

In the following proofs, we perform calculations in the Grothendieck ring by picking lifts of its elements to the category. By abuse of notation, we will denote the lifts by the same symbols.

Lemma 2.5.2.2. *Let X be a simple mobile, and X^\vee its dual. $X^\vee \cdot X$ contains precisely one copy of \mathbb{O} and no other objects with support $[0]$:*

$$X^\vee \cdot X = \mathbb{O} + R,$$

for some $R \in K(\mathbf{C})$ and $[0] \notin \text{supp } R$.

Proof. By Lemma 2.5.1.4, $X^\vee \otimes X$ is mobile. Hence it splits as a direct sum $\mathbb{O}^{\oplus n} \oplus R$, where $n \geq 1$ and R has support disjoint from $[0]$. However, $\text{Hom}(\mathbb{O}, \mathbb{1}) \neq 0$, so \mathbb{O} must contain a summand isomorphic to $\mathbb{1}$. As X is simple, $\text{Hom}(X^\vee \otimes X, \mathbb{1}) \cong \text{Hom}(X, X)$ is one-dimensional. It follows that $n = 1$ and $\text{Hom}(\mathbb{O}, \mathbb{1})$ is one-dimensional. \square

By Lemma 2.3.1.3, $\mathbb{O} \cong \mathbf{Rep}^\theta(\Gamma) \cong \bigoplus_\chi \chi^{\oplus \text{vdim } \chi}$, so

$$\dim \text{Hom}(\mathbb{O}, \mathbb{1}) = \text{vdim } \mathbb{1} = |\Gamma| \dim \mathbb{1}.$$

Thus the above also proves the following:

$$\dim \mathbb{1} = \frac{1}{|\Gamma|}.$$

This means under an equivalence $\mathbf{C}^{(0)} \cong \mathbf{Rep}^\theta(\Gamma)$, $\mathbb{1}$ is always identified with a one-dimensional irreducible representation of Γ .

Corollary 2.5.2.3. *The gerbe \mathcal{G} is trivial at the point 0.*

Proof. Corollary 2.2.1.2 says that a 2-cocycle θ is trivial if and only if $\mathbf{Rep}^\theta(\Gamma)$ contains a 1-dimensional representation. \square

In other words, we may always take the representing cocycle θ_0 of \mathcal{G}_0 to be trivial (see the local classification of gerbes over orbifolds in Corollary 2.2.4.8). We will use this in Section 2.3.2 to treat the neighbourhood of $\mathbb{1}$ as being equipped with the trivial gerbe.

Lemma 2.5.2.4. *Let X be a simple object and $p \in \text{supp } X$, then*

$$\mathbb{O} \cdot X = |\Gamma| \cdot \dim X \cdot \mathbb{O}_p.$$

Proof. By Lemma 2.5.1.4, $\mathbb{O} \otimes X$ is mobile, so it can be expressed as a sum

$$\mathbb{O} \otimes X \cong \bigoplus_{[q] \in |\mathcal{M}|} \mathbb{O}_q^{\oplus n_q},$$

where we sum over a chosen set of pairwise distinct representatives of the equivalence classes of points in \mathcal{M} (such that p is one of them). We use dualisability to write

$$\text{Hom}(\mathbb{O} \otimes X, \mathbb{O}_q) \cong \text{Hom}(\mathbb{O}, \mathbb{O}_q \otimes X^\vee).$$

The object $\mathbb{O}_q \otimes X^\vee$ is again mobile, and thus of the form $\bigoplus_{[x] \in |\mathcal{M}|} \mathbb{O}_x^{\oplus m_x}$. The equality

$$\dim \text{Hom}(\mathbb{O}, \mathbb{O}_x) = |\Gamma| \cdot \dim \text{Hom}(\mathbb{1}, \mathbb{O}_x)$$

allows us to compute

$$\dim \text{Hom}(\mathbb{O}, \mathbb{O}_q \otimes X^\vee) = |\Gamma| \cdot \dim \text{Hom}(\mathbb{1}, \mathbb{O}_q \otimes X^\vee) = |\Gamma| \cdot \dim \text{Hom}(X, \mathbb{O}_q).$$

The latter is only non-zero when $q = p$, as otherwise \mathbb{O}_q cannot contain X . So $n_q = 0$ for $q \neq p$. The basic mobile \mathbb{O}_p contains X with multiplicity its vector space dimension, which is $\dim X \cdot |\Gamma_p|$, and hence

$$\dim \text{Hom}(\mathbb{O}, \mathbb{O}_p \otimes X^\vee) = |\Gamma| \cdot \dim X \cdot |\Gamma_p|.$$

Now n_p is given by

$$n_p = \dim \text{Hom}(\mathbb{O} \otimes X, \mathbb{O}_p) / \dim \text{Hom}(\mathbb{O}_p, \mathbb{O}_p) = |\Gamma| \cdot \dim X. \quad \square$$

EXAMPLE 2.5.2.5. For a basic mobile $X \in \mathbf{C}$, $X = \mathbb{O}_{\text{supp } X}$ in $\mathbf{K}(\mathbf{C})$, and $\dim X = 1$. We apply Lemma 2.5.2.4 to its simple summands to find

$$\mathbb{O} \cdot X = |\Gamma| \cdot X.$$

A particularly useful special case is

$$\mathbb{O} \cdot \mathbb{O} = |\Gamma| \cdot \mathbb{O}.$$

As the above plays well with sums, this is true for *all* mobile objects. In particular, the above is true for all objects $X \in \mathbf{C}$ with support in the generic region.

2.5.3 Local Model

In this section, we study properties of the monoidal structure around the unit of an effective orbifusion category, using a (non-canonical) lift of the object $\mathbb{O} \in K(\mathbf{C})$ to the category \mathbf{C} . We fix an equivalence of linear stacks $\mathbf{C} \cong \mathbf{Sky}_{\mathcal{M}}^{\mathcal{G}}$ (where \mathcal{M} is effective), and pick a point $0 \in \mathcal{M}$ representing the support of $\mathbb{1} \in \mathbf{C}$. We call the stabiliser group of this point Γ .

We pick a chart $q : M \rightarrow [M/\Gamma] \hookrightarrow \mathcal{M}$ covering a quotient suborbifold around 0 , with a single preimage $x_0 := q^{-1}(0) \in M$. By Corollary 2.5.2.3, we may assume the restriction of the gerbe \mathcal{G} to $[M/\Gamma]$ is trivial. We pick once and for all a trivialisation τ of the pullback gerbe and use it to induce an étale family

$$\mathbb{A} := \text{Ind}^{\tau} q.$$

By abuse of notation, we denote its pullback over x_0 by

$$\mathbb{O} := x_0^* \mathbb{A} \in \mathbf{C}.$$

The object \mathbb{O} as defined above is a lift of the object we called $\mathbb{O} \in K(\mathbf{C})$ in Section 2.5.2 to \mathbf{C} . We pick an explicit inclusion

$$\mathbb{1} \hookrightarrow \mathbb{O}.$$

If Γ is trivial, this map is an isomorphism.

EXAMPLE 2.5.3.1. When \mathcal{M} is a manifold, \mathbb{A} is a U -family of dimension 1 whose support function $U \rightarrow \mathcal{M}$ is the embedding of a neighbourhood of $0 \in \mathcal{M}$.

EXAMPLE 2.5.3.2. Consider the orbisimple category $\mathbf{Sky}_{[\mathbb{R}/(\mathbb{Z}/2)]} = (\mathbf{Sky}_{\mathbb{R}})^{\mathbb{Z}/2}$, equipped with the monoidal structure coming from the Lie group structure on \mathbb{R} . The family \mathbb{A} may be represented by the \mathbb{R} -family of skyscraper sheaves whose restriction to $x \in \mathbb{R}$ is the skyscraper sheaf $\mathbb{C}_x \oplus \mathbb{C}_{-x}$, with $\mathbb{Z}/2$ -action given by swapping the two terms. The restriction of this representative along $0 : * \rightarrow [\mathbb{R}/(\mathbb{Z}/2)]$ yields the regular representation of $\mathbb{Z}/2$, a representative for \mathbb{O} .

The family \mathbb{A} and its tensor product with itself turn out to carry a lot of information. We briefly introduce some notation.

Definition 2.5.3.3. A family $\mathcal{S} : U \rightarrow \mathbf{C}$ is *simple at* $x \in U$ if there exists no open neighbourhood V of x , such that $\mathcal{S}|_V$ splits as a direct sum of two families. \mathcal{S} is simple if it is simple at all points $x \in U$.

Given Corollary 2.3.2.16, a family simple at x always admits a local support lift given by a single smooth function. In particular, this means the support at x is a single point.

Lemma 2.5.3.4. *Near $x \in U$, every family $\mathcal{S} : U \rightarrow \mathbf{C}$ simple at x is a direct summand of a pullback of $\mathbb{A} \otimes \mathcal{S}(x)$. I.e. there exists an open neighbourhood V of x and a smooth map $f : V \rightarrow M$, such that $\mathcal{S}|_V$ admits the structure of a summand of $f^* \mathbb{A} \otimes \mathcal{S}(x)$.*

Proof. $\mathbb{A} \otimes \mathcal{S}(x)$ is an étale M -family whose value at x is

$$\mathbb{A}(x) \otimes \mathcal{S}(x) \cong \mathbb{O} \otimes \mathcal{S}(x) \cong (\mathcal{S}(x))^{\oplus |\Gamma|}.$$

Thus local support lifts for $\mathbb{A} \otimes \mathcal{S}(x)$ and for \mathcal{S} can be chosen to land in the same chart $q_N : N \rightarrow \mathcal{M}$ around $\text{supp } \mathcal{S}(x)$. Let $\tilde{s} : U \rightarrow N$ be such a local support lift for \mathcal{S} , and $\tilde{t}_i : M \rightarrow N$ a local support lift for $\mathbb{A} \otimes \mathcal{S}(x)$. As \tilde{t}_i is étale, there exists a map $f : V \rightarrow M$ for some open neighbourhood V of $x \in U$, that factors \tilde{s} through \tilde{t}_i . Hence $f^* \mathbb{A} \otimes \mathcal{S}(x)$ locally contains $\text{Ind } q_N \circ \tilde{s}$ as a summand. By definition of a local support lift, $\text{Ind } q_N \circ \tilde{s}$ contains \mathcal{S} as a summand, which proves the assertion. \square

We will now study the tensor product of \mathbb{A} with itself. We do this by decomposing $\mathbb{A} \otimes \mathbb{A}$ into a direct sum of étale families indexed by Γ . Their local support lifts may be chosen to be translates of a multiplication on M . It will be convenient to pretend these lifts exist as maps $M \times M \rightarrow M$, when really they might only be defined on a smaller domain. We treat them as partial functions: their domain is denoted by $M \times M$, despite the functions only being defined on a subset $D \subset M \times M$. This domain will always be a neighbourhood of (x_0, x_0) .

Lemma 2.5.3.5. *There is a set of local support lifts of $\mathbb{A} \otimes \mathbb{A}$ given by partial maps*

$$\{\tilde{m}_g : M \times M \rightarrow M\}_{g \in \Gamma}$$

satisfying

- $\tilde{m}_g(x, y) = \tilde{m}_e(x, g \cdot y)$
- $\tilde{m}_e(x_0, x) = \tilde{m}_e(x, x_0) = x$.

Proof. First note that $\mathbb{A} \otimes \mathbb{A}$ must admit local support lifts with respect to the chart $M \rightarrow \mathcal{M}$. This is because $\mathbb{A}(x_0) \otimes \mathbb{A}(x_0) \cong \mathbb{O} \otimes \mathbb{O} \cong \mathbb{O}^{\oplus \Gamma}$ (Example 2.5.2.5), whose support is $[0]$, and thus covered by M . Pick such a set of local support lifts $\{\tilde{m}_i : M \times M \rightarrow M\}_{i \in I}$. By the equation above, these lifts satisfy $\tilde{m}_i(x_0, x_0) = x_0$. The lifts are further restricted by the condition

$$\mathbb{A}(x_0) \otimes \mathbb{A}(x) \cong \mathbb{A}(x) \otimes \mathbb{A}(x_0) \cong \mathbb{A}(x)^{\oplus \Gamma},$$

which yields pointwise equalities

$$q \circ \tilde{m}_i(x_0, x) = q \circ \tilde{m}_i(x, x_0) = q(x).$$

This forces the maps $\tilde{m}_i(x_0, -) : M \rightarrow M$ and $\tilde{m}_i(-, x_0) : M \rightarrow M$ to be given by translations

$$x \mapsto g \cdot x$$

for some $g \in \Gamma$. (This is true pointwise by the equalities above. The element g must be constant because Γ is finite.)

Lifts to M are only defined up to Γ -action, so we can modify all \tilde{m}_i such that

$$\tilde{m}_i(-, x_0) = \text{id}_M.$$

We denote by $g_i \in \Gamma$ the element such that $\tilde{m}_i(x_0, -) = g_i \cdot -$. The derivative of an \tilde{m}_i at (x_0, x_0) is the direct sum of the derivatives of id_M and $g_i \cdot -$. These are diffeomorphisms, hence the derivative is surjective, and all \tilde{m}_i are submersions in a neighbourhood

of (x_0, x_0) . Lemma 2.3.2.20 implies $\mathbb{A} \otimes \mathbb{A}$ is mobile in a neighbourhood of (x_0, x_0) , with each \tilde{m}_i contributing a mobile summand.

Now pick any single support lift \tilde{m}_i . We will use invariance of \mathbb{A} under the Γ -action to show that there is a set of support lifts $\{\tilde{m}_i(-, g \cdot -)\}_{g \in \Gamma}$. As $\mathbb{A} \otimes \mathbb{A}$ is mobile,

$$\mathbb{A} \otimes \mathbb{A} \cong \bigoplus_{i \in I} (\text{Ind}(q \circ \tilde{m}_i))^{\oplus n_i}$$

for some set of $n_i \in \mathbb{N}$. By counting dimensions at (x_0, x_0) , we know $\sum_i n_i = |\Gamma|$. The Γ -invariance $(g \cdot -)^* \mathbb{A} \cong \mathbb{A}$ implies

$$(\text{id}_M \otimes g \cdot -)^* \mathbb{A} \otimes \mathbb{A} \cong \mathbb{A} \otimes \mathbb{A},$$

and so

$$\bigoplus_i (\text{Ind}(q \circ \tilde{m}_i))^{\oplus n_i} \cong \bigoplus_i (\text{Ind}(q \circ \tilde{m}_i(-, g \cdot -)))^{\oplus n_i}$$

for all $g \in \Gamma$. In order for there to exist an isomorphism, the maps on the left-hand-side and on the right-hand-side must match up in an open neighbourhood of (x_0, x_0) by Lemma 2.3.2.21. This proves that all the n_i in the above expression are 1, the direct sum is indexed by Γ , and there is indeed a set of local support lifts $\{\tilde{m}_i(-, g \cdot -)\}_{g \in \Gamma}$. In particular, this includes a support lift \tilde{m}_e which satisfies

$$\tilde{m}_e(x, x_0) = \tilde{m}_e(x_0, x) = x. \quad \square$$

Define $m_g := q \circ \tilde{m}_g$, then over some neighbourhood of $(x_0, x_0) \in M \times M$,

$$\mathbb{A} \otimes \mathbb{A} \cong \bigoplus_{g \in \Gamma} \text{Ind}(m_g).$$

Lemma 2.5.3.6. *Locally around (x_0, x_0, x_0) ,*

$$\tilde{m}_g(\tilde{m}_h(-, -), -) = \tilde{m}_h(-, \tilde{m}_{h^{-1}g}(-, -)).$$

Proof. The associator $\alpha_{\mathbb{A}, \mathbb{A}, \mathbb{A}}$ is an isomorphism $(\mathbb{A} \otimes \mathbb{A}) \otimes \mathbb{A} \xrightarrow{\cong} \mathbb{A} \otimes (\mathbb{A} \otimes \mathbb{A})$. By Lemma 2.5.3.5, there are isomorphisms

$$\begin{aligned} (\mathbb{A} \otimes \mathbb{A}) \otimes \mathbb{A} &\cong \bigoplus_{h \in \Gamma} \text{Ind } m_h \otimes \mathbb{A} \cong \bigoplus_{h \in \Gamma} (\tilde{m}_h \times \text{id}_M)^* \mathbb{A} \otimes \mathbb{A} \\ &\cong \bigoplus_{g, h \in \Gamma} (\tilde{m}_h \times \text{id}_M)^* \text{Ind } m_g \cong \bigoplus_{g, h \in \Gamma} \text{Ind } m_g (\tilde{m}_h(-, -), -) \end{aligned}$$

and

$$\mathbb{A} \otimes (\mathbb{A} \otimes \mathbb{A}) \cong \bigoplus_{g, h \in \Gamma} \text{Ind } m_g (-, \tilde{m}_h(-, -)).$$

The two expressions witness that $(\mathbb{A} \otimes \mathbb{A}) \otimes \mathbb{A}$ has local support lifts $M \times M \times M \rightarrow M$ given by

$$\{\tilde{m}_{h,g} := \tilde{m}_g(\tilde{m}_h(-, -), -)\}_{g, h \in \Gamma} \quad \text{or} \quad \{\tilde{m}'_{h,g} := \tilde{m}_g(-, \tilde{m}_h(-, -))\}_{g, h \in \Gamma}.$$

We have fixed the ambiguity of the lifts by picking all of them such that

$$\tilde{m}_{h,g}(x, x_0, x_0) = \tilde{m}'_{h,g}(x, x_0, x_0) = x.$$

The values on $\{x_0\} \times M \times \{x_0\}$ and $\{x_0\} \times \{x_0\} \times M$ are

$$\begin{aligned} \tilde{m}_{h,g}(x_0, x, x_0) &= h \cdot x & \tilde{m}'_{h,g}(x_0, x, x_0) &= g \cdot x \\ \tilde{m}_{h,g}(x_0, x_0, x) &= g \cdot x & \tilde{m}'_{h,g}(x_0, x_0, x) &= gh \cdot x. \end{aligned}$$

$\{\tilde{m}_{h,g}\}$ and $\{\tilde{m}'_{h,g}\}$ are thus both sets of $\Gamma \times \Gamma$ distinct functions which don't equal each other on any open neighbourhood of (x_0, x_0, x_0) . Thus, they must match up in pairs under the isomorphism above. The value along the second and third factor of $M \times M \times M$ dictates

$$\tilde{m}_{h,g} = \tilde{m}'_{h^{-1}g,h}. \quad \square$$

NOTATION 2.5.3.7. We introduce more compact notation for the support lift labelled by the identity:

$$\mu := \tilde{m}_e.$$

We note an immediate Corollary of Lemma 2.5.3.6:

Corollary 2.5.3.8. *The partial function $\mu : M \times M \rightarrow M$ makes M into a partial smooth monoid with unit x_0 .*

This multiplication is locally group-like.

Lemma 2.5.3.9. *Any partial smooth monoid $(M, x_0 \in M, \mu : M \times M \rightarrow M)$ is group-like in a neighbourhood of its identity x_0 . Ie. there exists a smooth partial function $s : M \rightarrow M$ (which is defined on some neighbourhood of x_0), that acts as a left and right inverse.*

Proof. As μ restricts to the identity on $M \times \{x_0\}$ and $\{x_0\} \times M$, it is submersive at (x_0, x_0) . The Implicit Function Theorem then guarantees the existence of two (locally defined) maps $s, s' : M \rightarrow M$, such that $\mu(s(x), x) = \mu(x, s'(x)) = x_0$. The usual argument yields equality of left and right inverse:

$$s(x) = \mu(s(x), \mu(x, s'(x))) = \mu(\mu(s(x), x), s'(x)) = s'(x). \quad \square$$

2.6 Smooth Dualisability

In this section, we prove that pointwise dualisability in effective orbifusion categories automatically smoothly extends to families: All categories \mathbf{C}^U assigned to $U \in \mathbf{Man}$ by an effective orbifusion category $\underline{\mathbf{C}}$ are autonomous (Theorem 2.6.0.7). This is a categorification of the following Lemma from classical Lie Theory:

Lemma 2.6.0.1. *A manifold M equipped with a group structure whose multiplication map $m : M \times M \rightarrow M$ is smooth is a Lie group.*

Note that it is *not* necessary to require the inverse to be smooth.

Proof. The multiplication m is submersive everywhere, so $m^{-1}(e) \subset M \times M$ is a submanifold. The restriction of the projection map $M \times M \rightarrow M$ along $m^{-1}(e) \hookrightarrow M \times M$ is of constant rank: $p_1 : m^{-1}(e) \rightarrow M$ is equivariant with respect to the conjugation action of $x \in M$, and any two points in $m^{-1}(e)$ are related by the action of some $x \in M$. The rank at $(e, e) \in M \times M$ is full by Lemma 2.5.3.9, hence p_1 has full rank everywhere and is a diffeomorphism. The inverse map $p_2 \circ (p_1)^{-1} : M \rightarrow M$ gives M the structure of a Lie group. \square

We recall two lemmas from tensor category theory.

Lemma 2.6.0.2. *Duality data is unique up to unique isomorphism: Let X be an object in a tensor category and $(Y, \text{ev}), (Y', \text{ev}')$ two left duals for X , then there is a unique isomorphism $Y \rightarrow Y'$ compatible with the duality data.*

Proof. The proof is a categorification of the uniqueness of inverses in monoids. The isomorphism is furnished by the zig-zags $Y \rightarrow Y' \otimes X \otimes Y \rightarrow Y'$ and $Y' \rightarrow Y \otimes X \otimes Y' \rightarrow Y$. This morphism is manifestly compatible with the duality data. Uniqueness follows by composing any given isomorphism (compatible with the duality data) with the duality equations. They exhibit equality between the isomorphism and the zig-zag above. \square

Lemma 2.6.0.3. *Dualisability descends to direct summands: Let X be a left dualisable object in a Karoubi-complete tensor category, and $Y \subset X$ a direct summand. Then Y is dualisable.*

Proof. On the summand $Y \subset X$, the first duality equation for X restricts to

$$Y \rightarrow Y \otimes X^\vee \otimes Y \rightarrow Y = \text{id}_Y.$$

Using this, one shows that the composite

$$X^\vee \rightarrow X^\vee \otimes X \otimes X^\vee \rightarrow X^\vee \otimes Y \otimes X^\vee \rightarrow X^\vee \otimes X \otimes X^\vee \rightarrow X^\vee$$

is an idempotent. By Karoubi-completeness, X^\vee contains a summand Y^\vee , on which this idempotent is the identity. It is straightforward to check that the induced maps

$$\mathbb{1} \rightarrow X \otimes X^\vee \rightarrow Y \otimes Y^\vee \quad \text{and} \quad Y^\vee \otimes Y \rightarrow X^\vee \otimes X \rightarrow \mathbb{1}$$

identify Y^\vee as the left dual of Y . \square

Recall that we made a choice of chart $q : M \rightarrow \mathcal{X}$ around the support of the monoidal unit. Associated to these choices is a mobile family $\mathbb{A} = \text{Ind } q$, which we only fixed up to isomorphism so far. So as to be able to talk about maps into and out of \mathbb{A} , we simply pick a representative of the isomorphism class of \mathbb{A} . Its pullback over the point $x_0 = q^{-1}(\text{supp } \mathbb{1})$ is denoted $\mathbb{O} = \mathbb{A}(x_0)$, and equipped with the inclusion of a summand $\mathbb{1} \hookrightarrow \mathbb{O} \twoheadrightarrow \mathbb{1}$.

Recall that $\mathbb{A} \otimes \mathbb{A} \cong \bigoplus_{g \in \Gamma} \text{Ind}(m_{\tilde{g}})$ (Lemma 2.5.3.5). We pick an explicit isomorphism once and for all. Restricted to the origin (x_0, x_0) , this gives an explicit isomorphism $\mathbb{O} \otimes \mathbb{O} \xrightarrow{\cong} \mathbb{O}^{\oplus \Gamma}$. We denote the maps identifying the summand corresponding to $g \in \Gamma$ by

$$\mathbb{O} \xleftarrow{\iota_g} \mathbb{O} \otimes \mathbb{O} \xrightarrow{p_g} \mathbb{O}.$$

The following technical Lemma will be used in our proof of local dualisability.

Lemma 2.6.0.4. *The composites*

$$p_e \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \hookrightarrow \mathbb{O} \otimes \mathbb{O} \xrightarrow{p_e} \mathbb{O} \twoheadrightarrow \mathbb{1}$$

and

$$\iota_e \downarrow_{\mathbb{1}}^{\mathbb{1} \otimes \mathbb{1}} : \mathbb{1} \hookrightarrow \mathbb{O} \xrightarrow{\iota_e} \mathbb{O} \otimes \mathbb{O} \twoheadrightarrow \mathbb{1} \otimes \mathbb{1}$$

are nonzero.

Proof. Recall from Lemma 2.5.3.6 the two decomposition of $(\mathbb{A} \otimes \mathbb{A}) \otimes \mathbb{A}$, related by the associator $\alpha_{\mathbb{A}, \mathbb{A}, \mathbb{A}}$. This associator decomposes into components

$$\alpha_{\mathbb{A}, \mathbb{A}, \mathbb{A}}^{g, h} : \text{Ind } m_h(\tilde{m}_g(-, -), -) \xrightarrow{\cong} \text{Ind } m_g(-, \tilde{m}_{g^{-1}h}(-, -)).$$

We denote the restriction of $\alpha_{\mathbb{A}, \mathbb{A}, \mathbb{A}}^{g, h}$ to the origin (x_0, x_0, x_0) by

$$\alpha^{g, h} := \alpha_{\mathbb{A}, \mathbb{A}, \mathbb{A}}^{g, h}(x_0, x_0, x_0) : \mathbb{O} \xrightarrow{\cong} \mathbb{O}.$$

As indicated above, it is an isomorphism.

By definition, these components make the following diagram commute:

$$\begin{array}{ccccc} (\mathbb{O} \otimes \mathbb{O}) \otimes \mathbb{O} & \xrightarrow{p_g \otimes \text{id}} & \mathbb{O} \otimes \mathbb{O} & \xrightarrow{p_h} & \mathbb{O} \\ \downarrow \alpha_{\mathbb{O}, \mathbb{O}, \mathbb{O}} & & & & \downarrow \alpha^{g, h} \\ \mathbb{O} \otimes (\mathbb{O} \otimes \mathbb{O}) & \xrightarrow{\text{id} \otimes p_{g^{-1}h}} & \mathbb{O} \otimes \mathbb{O} & \xrightarrow{p_g} & \mathbb{O}. \end{array}$$

By restricting all maps appropriately, one obtains:

$$\begin{array}{ccccc} (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} & \xrightarrow{p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} \otimes \text{id}} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{p_h \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}} & \mathbb{1} \\ \downarrow \alpha_{\mathbb{1}, \mathbb{1}, \mathbb{1}} & & & & \downarrow \alpha^{g, h} \downarrow_{\mathbb{1}}^{\mathbb{1}} \\ \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{\text{id} \otimes p_{g^{-1}h} \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}} & \mathbb{1}. \end{array}$$

All objects in the diagram above are isomorphic to $\mathbb{1}$, which means all maps are multiples of the identity on $\mathbb{1}$ and both composition and tensoring are given by scalar multiplication. The components of the associator are isomorphisms, and thus non-zero, so the product $p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} \cdot p_h \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ coming from the top row is non-zero iff $p_{g^{-1}h} \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} \cdot p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ (coming from the bottom row) is non-zero. As $\sum \iota_g \circ p_g = \text{id}_{\mathbb{O} \otimes \mathbb{O}}$, at least one of the maps $p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ must be non-zero. But then $p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} \cdot p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ is also non-zero, which implies $p_e \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} \cdot p_g \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ is non-zero. We have now proved that $p_e \downarrow_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}}$ is not zero. Replacing the projections p_g above with their splittings ι_g , one proves $\iota_e \downarrow_{\mathbb{1}}^{\mathbb{1} \otimes \mathbb{1}}$ is also not zero. \square

Lemma 2.6.0.5. *The family $\mathbb{A} : M \rightarrow \mathbf{C}$ is locally dualisable at x_0 : There exists a neighbourhood V of x_0 , such that $\mathbb{A} \downarrow_V$ is dualisable as an object of \mathbf{C}^V .*

Proof. We will use the support lift $\mu = \tilde{m}_e$ introduced in Lemma 2.5.3.6. By Corollary 2.5.3.8, μ equips a neighborhood of x_0 with the structure of a smooth monoid. In Lemma 2.5.3.9 we showed that there exists a partial function $s : M \rightarrow M$ making (M, x_0, μ, s) into a smooth partial group. Below, we denote by c_{x_0} and c_0 the constant maps

$$\begin{aligned} c_{x_0} : M &\rightarrow M & c_0 : M &\rightarrow [M/\Gamma] \\ x &\mapsto x_0 & x &\mapsto 0, \end{aligned}$$

and write $\text{Gr}(s) = (s \times \text{id}_M) \circ \Delta_M$. By definition $c_0 = q \circ c_{x_0}$ and $c_{x_0} = \mu \circ \text{Gr}(s) = \mu \circ \text{swap} \circ \text{Gr}(s)$, where $\text{swap} : M \times M \rightarrow M \times M$ is the map swapping the two arguments. Pulling back via s yields a family

$$\mathbb{B} := s^* \mathbb{A} : M \rightarrow \mathbf{C}.$$

By Lemma 2.5.3.5, $\mathbb{A} \otimes \mathbb{A} \cong \bigoplus_g \text{Ind}(q \circ \mu \circ (\text{id} \times g \cdot -))$, and the M -family obtained as the tensor product

$$\mathbb{B} \otimes^M \mathbb{A} = \Delta_M^*(s^* \mathbb{A} \otimes \mathbb{A}) = \text{Gr}(s)^*(\mathbb{A} \otimes \mathbb{A})$$

contains a summand

$$\text{Gr}(s)^*(\mathbb{A} \otimes \mathbb{A}) \supset \text{Gr}(s)^* \text{Ind } q \circ \mu = \text{Ind } q \circ \mu \circ \text{Gr}(s) = \text{Ind } c_0.$$

$\text{Ind } c_0$ is isomorphic to the constant family at \mathbb{O} , which contains the constant family at $\mathbb{1}$ as a summand. This is the monoidal unit $\mathbb{1}^M$ of \mathbf{C}^M . In the same way, there is a summand

$$\mathbb{1}^M \subset (\text{swap} \circ \text{Gr}(s))^* \mathbb{A} \otimes \mathbb{A} \cong \mathbb{A} \otimes^M \mathbb{B}$$

Now \mathbb{B} is a candidate for the dual of \mathbb{A} , and the candidate duality maps are:

$$c : \mathbb{1}^M \hookrightarrow \mathbb{A} \otimes^M \mathbb{B} \qquad e : \mathbb{B} \otimes^M \mathbb{A} \twoheadrightarrow \mathbb{1}^M$$

given by inclusion of and projection to the summand. Denote the composites

$$\begin{aligned} \mathbb{A} &\xrightarrow{\cong} \mathbb{1}^M \otimes^M \mathbb{A} \xrightarrow{c \otimes^M \text{id}} (\mathbb{A} \otimes^M \mathbb{B}) \otimes^M \mathbb{A} \xrightarrow{\cong} \mathbb{A} \otimes^M (\mathbb{B} \otimes^M \mathbb{A}) \xrightarrow{\text{id} \otimes^M e} \mathbb{A} \otimes^M \mathbb{1}^M \xrightarrow{\cong} \mathbb{A} \\ \mathbb{B} &\xrightarrow{\cong} \mathbb{B} \otimes^M \mathbb{1}^M \xrightarrow{\text{id} \otimes^M c} \mathbb{B} \otimes^M (\mathbb{A} \otimes^M \mathbb{B}) \xrightarrow{\cong} (\mathbb{B} \otimes^M \mathbb{A}) \otimes^M \mathbb{B} \xrightarrow{e \otimes^M \text{id}} \mathbb{1}^M \otimes^M \mathbb{B} \xrightarrow{\cong} \mathbb{B} \end{aligned}$$

by $D_{\mathbb{A}} \in \text{End}(\mathbb{A})$ and $D_{\mathbb{B}} \in \text{End}(\mathbb{B})$. The duality equations for \mathbb{A} and \mathbb{B} can then be written as

$$D_{\mathbb{A}} = \text{id}_{\mathbb{A}} \qquad D_{\mathbb{B}} = \text{id}_{\mathbb{B}}.$$

Composition and tensor product are $C^\infty(M)$ -linear, thus multiplying $e \in \text{Hom}(\mathbb{B} \otimes^M \mathbb{A}, \mathbb{1}^M)$ by a function $f \in C^\infty(M)$ has the effect of multiplying both $D_{\mathbb{A}}$ and $D_{\mathbb{B}}$ by f . Thus \mathbb{A} and \mathbb{B} are duals if

$$f \cdot \begin{pmatrix} D_{\mathbb{A}} \\ D_{\mathbb{B}} \end{pmatrix} = \begin{pmatrix} \text{id}_{\mathbb{A}} \\ \text{id}_{\mathbb{B}} \end{pmatrix}$$

for some function $f \in C^\infty(M)$. At all $x \in M$ where $\mathbb{A}(x)$ is simple, $\text{Hom}(\mathbb{1}, \mathbb{A}(x) \otimes \mathbb{B}(x))$ and $\text{Hom}(\mathbb{B}(x) \otimes \mathbb{A}(x), \mathbb{1})$ are one-dimensional. $\mathbb{A}(x)$ and $\mathbb{B}(x)$ are in \mathbf{C} and thus dualisable, so there are evaluation and coevaluation maps in these Hom-spaces, and $c(x)$ and $e(x)$ are automatically non-zero multiples thereof. Hence, the equations $D_{\mathbb{A}}(x) = r(x) \cdot \text{id}_{\mathbb{A}}$, $D_{\mathbb{B}}(x) = r(x) \cdot \text{id}_{\mathbb{B}}$ hold at these points (for some $r(x) \in \mathbf{C}$). As \mathbb{A} is an étale family, $\mathbb{A}(x)$ is simple on a dense subset of M , so this property extends to all of M . There is then a function $r \in \text{Hom}(\mathbb{A}, \mathbb{A}) \cong C^\infty(M)$ satisfying

$$\begin{pmatrix} D_{\mathbb{A}} \\ D_{\mathbb{B}} \end{pmatrix} = r \cdot \begin{pmatrix} \text{id}_{\mathbb{A}} \\ \text{id}_{\mathbb{B}} \end{pmatrix}.$$

It remains to show that r is nowhere-zero in some neighbourhood of $x_0 \in M$. By smoothness of morphisms, this is equivalent to checking $D_{\mathbb{A}}(x_0) \neq 0$. The map $D_{\mathbb{A}}(x_0) \in \text{End}(\mathbb{O})$ decomposes into matrix components. We will check that the component

$$D_{\mathbb{A}}(x_0)|_{\mathbb{1}}^{\mathbb{1}} \in \text{End}(\mathbb{1})$$

is nonzero. It factors as a composition of maps in the one-dimensional vector space $\text{End}(\mathbb{1})$:

$$\mathbb{1} \xrightarrow{\cong} \mathbb{1} \otimes \mathbb{1} \xrightarrow{c(x_0)|_{\mathbb{1} \otimes \mathbb{1}} \otimes \text{id}} (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \xrightarrow{\cong} \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \xrightarrow{\text{id} \otimes e(x_0)|_{\mathbb{1} \otimes \mathbb{1}}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\cong} \mathbb{1}.$$

As the isomorphisms are clearly non-zero, the proof reduces to showing that

$$c(x_0)|_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1} \otimes \mathbb{1}} = \iota_e|_{\mathbb{1}}^{\mathbb{1} \otimes \mathbb{1}} : \mathbb{1} \rightarrow \mathbb{O} \otimes \mathbb{O} \rightarrow \mathbb{1} \otimes \mathbb{1}$$

and

$$e(x_0)|_{\mathbb{1} \otimes \mathbb{1}} = p_e|_{\mathbb{1} \otimes \mathbb{1}}^{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \hookrightarrow \mathbb{O} \otimes \mathbb{O} \rightarrow \mathbb{1}$$

are nonzero. This was done in Lemma 2.6.0.4. \square

Lemma 2.6.0.6. *Every family is locally dualisable. Given a U -family \mathcal{S} , for any point $x \in U$, there exists a neighbourhood V of x , such that $\mathcal{S}|_V$ is dualisable.*

Proof. Dualisability extends to direct sums, so we can assume \mathcal{S} is simple at x . Then by Lemma 2.5.3.4, \mathcal{S} is a summand of a pullback from $\mathbb{A} \otimes \mathcal{S}(x)$ on some neighbourhood V of x . $\mathbb{A} \otimes \mathcal{S}(x)$ has a dual $\mathcal{S}(x)^\vee \otimes \mathbb{A}^\vee$. They remain dual when pulled back via any smooth map. As \mathbf{C}^U is Karoubi-complete, Lemma 2.6.0.3 shows $\mathcal{S}|_V$ is dualisable. \square

The following is the central theorem of this section.

Theorem 2.6.0.7. *In an orbifusion category, \mathbf{C}^U is autonomous for all $U \in \text{Man}$.*

Proof. Let $\mathcal{S} : U \rightarrow \mathbf{C}$ be a U -family. By Lemma 2.6.0.6, there exists a cover $Y \twoheadrightarrow U$ such that \mathcal{S} is dualisable when pulled back to Y . Pick duality data $(\mathcal{T} : Y \rightarrow \mathbf{C}, \text{ev})$ for $\mathcal{S}|_Y$. Pulling back to the double- and triple- overlaps $Y^{[2]} := Y \times_U Y$ and $Y^{[3]} := Y \times_U Y \times_U Y$, we obtain duality data for $\mathcal{S}|_{Y^{[2]}}$ and $\mathcal{S}|_{Y^{[3]}}$. By Lemma 2.6.0.2, there exist unique isomorphisms between all of them. These yield canonical isomorphisms on $Y^{[2]}$ which commute on $Y^{[3]}$ by uniqueness. Thus local dualisability gives us a valid descent datum which can be glued to a global dual of \mathcal{S} on U . \square

Denote by $\underline{\mathbf{C}}^{\text{op}}$ the stack obtained from $\underline{\mathbf{C}}$ by pointwise replacing \mathbf{C}^U by $\mathbf{C}^{U,\text{op}}$.

Corollary 2.6.0.8. *An orbifusion category admits a smooth dualising functor*

$$(-)^\vee : \underline{\mathbf{C}} \rightarrow \underline{\mathbf{C}}^{\text{op}}.$$

Proof. By Theorem 2.6.0.7, there exist functors $(-)_U^\vee : \mathbf{C}^U \rightarrow \mathbf{C}^{U,\text{op}}$ that assign duals. Duals are unique up to unique isomorphism (Lemma 2.6.0.2), hence any two dualising functors are uniquely naturally isomorphic. There are thus unique natural isomorphisms

$$\xi_f : f^* \circ (-)_U^\vee \xrightarrow{\cong} (-)_V^\vee \circ f^*$$

for any $f : V \rightarrow U$ in \mathbf{Man} . By uniqueness, these natural isomorphisms satisfy the cocycle condition for pairs of composable morphisms $f \circ g : W \rightarrow V \rightarrow U$, so the data $\{(-)_U^\vee, \xi_f\}$ indeed assembles into a map of stacks. \square

Lemma 2.6.0.9. *All simple objects supported on the unit connected component of a manifold fusion category are invertible.*

Proof. Let $X \in \mathbf{C}$ be simple with support in the unit component of the underlying manifold M . Pick a path $p : [0, 1] \rightarrow M$ from $p(0) = \text{supp } \mathbb{1}$ to $p(1) = \text{supp } X$, and form the associated family $\text{Ind } p$. There are isomorphisms $\text{Ind } p(0) \cong \mathbb{1}$, $\text{Ind } p(1) \cong X$. Then, by Lemma 2.3.2.18,

$$\dim \mathbb{1} = \dim \mathbb{1} \otimes \mathbb{1} = \dim ((\text{Ind } p)^\vee \otimes \text{Ind } p) = \dim (X^\vee \otimes X).$$

As a result, $X^\vee \otimes X \cong \mathbb{1}$. \square

Corollary 2.6.0.10. *In a manifold fusion category, the isomorphism classes of objects supported on the unit component always form a Lie group.*

Proof. The Grothendieck group underlying the stack encodes a smooth manifold M , the tensor product an associative smooth multiplication map $m : M \times M \rightarrow M$, and pointwise invertibility implies there exists a smooth inverse $s : M \rightarrow M$ by Lemma 2.6.0.1. \square

Chapter 3

The Centre of the String 2-Group

3.1 Introduction

Let G be a compact connected Lie group and $k \in H^4(\mathcal{B}G, \mathbb{Z})$ a cohomology class. The String 2-group G_k is the central extension of G by the 2-group $[*/U(1)]$ classified by k . It has a close relationship to the level k extension of the loop group LG . In this chapter, we compute its Drinfel'd centre $\mathcal{Z}G_k$ in the context of smooth 2-groups. The result of this computation is interesting: we find that $\mathcal{Z}G_k$ recovers the invertible part of $\mathbf{Rep}^k LG$ when G is semisimple (as long as we exclude factors of E_8 at level 2).

Before stating the result in more detail, we introduce the model of the String 2-group and centre we use.

3.1.1 The String 2-Group

A 2-group [BL04] is a group object in the bicategory \mathbf{Cat} of categories. It is a monoidal groupoid $(\mathbf{C}, \mathbb{1}, \otimes, \omega)$ all of whose objects are invertible: for all $x \in \mathbf{C}$, there exists an object x^{-1} such that $x \otimes x^{-1} \cong x^{-1} \otimes x \cong \mathbb{1}$. The set of isomorphism classes of \mathbf{C} forms a group $(\pi_0 \mathbf{C}, \otimes)$, and the endomorphisms of $\mathbb{1}$ form an abelian group $(\pi_1 \mathbf{C}, \circ = \otimes)$.¹ Every object $x \in \pi_0 \mathbf{C}$ acts on $\pi_1 \mathbf{C}$ by conjugation: $f \mapsto \mathrm{id}_x \otimes f \otimes \mathrm{id}_{x^{-1}}$. This assembles into an action $\rho : \pi_0 \mathbf{C} \rightarrow \mathrm{Aut}(\pi_1 \mathbf{C})$. The associator of \mathbf{C} satisfies the pentagon equation

$$\rho(g)(\omega_{g',g'',g'''})\omega_{g,g'g'',g'''}\omega_{g',g'',g'''} = \omega_{gg',g'',g'''}\omega_{g,g',g''g'''}$$

for all $g, g', g'', g''' \in \pi_0 \mathbf{C}$. This is the equation of a group 3-cocycle in $Z^3(\mathcal{B}\pi_0 \mathbf{C}, \pi_1 \mathbf{C})$. Cohomologous 3-cocycles give rise to equivalent monoidal categories, and 2-groups are completely classified by the data $(\pi_0 \mathbf{C}, \pi_1 \mathbf{C}, \rho, [\omega] \in H^3(\mathcal{B}\pi_0 \mathbf{C}, \pi_1 \mathbf{C}))$ [Sin75; BL04]. The data of a (1-)group G induces a 2-group (also denoted G) with objects G , tensor product the multiplication on G , and only identity morphisms. An abelian group A also defines a 2-group $[*/A]$ with a single object $\mathbb{1}$, whose endomorphisms form the group $\pi_1 [*/A] = A$.²

¹There are two compatible group structures on $\pi_1 \mathbf{C}$ given by tensor product and composition. By the Eckmann-Hilton argument, these products agree and are commutative.

²The notation indicates that $[*/A]$ is the quotient stack associated to the trivial A -action on the point $*$.

Any 2-group \mathbf{C} is an extension of 2-groups of this type:

$$[*/\pi_1 \mathbf{C}] \rightarrow \mathbf{C} \rightarrow \pi_0 \mathbf{C}.$$

The data of this extension is encoded by the conjugation action ρ and the cohomology class $[\omega]$ of the associator. When ρ is trivial, one speaks of a *central extension*. Central extensions of G by $[* / A]$ are thus classified by $[\omega] \in H^3(\mathcal{B}G, A)$, where A carries the trivial G -module structure.

Smooth 2-groups are group objects in the bicategory of smooth groupoids (see Section 3.2). Just as for discrete 2-groups, there are smooth 2-groups G and $[* / A]$ associated to a Lie group G and an abelian Lie group A . Central extensions of G by $[* / A]$ are also classified by $H^3(\mathcal{B}G, A)$ [SP11], though one has to work with Segal-Mitchison cohomology [Seg70; Bry00] to make this precise. We recall Segal-Mitchison cohomology in Section 3.3. Let G be a compact connected Lie group. The short exact sequence of coefficients $\mathbb{Z} \rightarrow \mathbb{R} \xrightarrow{\exp} U(1)$ gives rise to an isomorphism $H^3(\mathcal{B}G, U(1)) \simeq H^4(\mathcal{B}G, \mathbb{Z})$. By abuse of notation, we write $k \in H^4(\mathcal{B}G, \mathbb{Z})$ to denote a $U(1)$ -valued 3-cocycle on G representing k . For simple G , $H^4(\mathcal{B}G, \mathbb{Z}) \cong \mathbb{Z}$, and k is often called a *level*.

The *String 2-group* G_k is the central extension of G by $[* / U(1)]$ classified by k . A large body of work has been devoted to constructing and understanding the String 2-groups, with particular interest in the case where G is simple simply-connected and k is a generator of $H^4(\mathcal{B}G, \mathbb{Z})$: The 2-group G_1 is a model for the universal 3-connected cover of the Lie group G . Such a 3-connected cover admits no incarnation as a finite-dimensional Lie group [SP11, Footnote 2], but it has been constructed variously as a topological group [Sto96; ST04], an infinite-dimensional Lie (2-)group [BL04; BSCS07; Hen08; NSW13], a diffeological 2-group [Wal12], and a smooth ∞ -group [FSS⁺12; Bun20]. We use the model of G_k as a *finite-dimensional* smooth 2-group given in [SP11].

Let $LG = C^\infty(S^1, G)$ denote the loop group of G . Transgression [BM94; Wal16] establishes a correspondence between String 2-groups $\{G_k\}$ and central extensions $\{LG_k\}$ of the loop group.³ The category $\mathbf{Rep}^k LG$ of *positive energy representations* of LG at level k is a linear braided monoidal category, defined when k satisfies a positivity condition. It is in fact a *modular tensor category*, and defines a 3-dimensional Topological Quantum Field Theory via the Reshetikhin-Turaev construction [RT91]. This is understood to be Chern-Simons theory with gauge group G at level k [Fre09], see [Hen17b] for an argument in the simply-connected case.

3.1.2 The Smooth Centre

The centre of a monoid M is the set of elements $z \in M$ such that $zm = mz$ for all $m \in M$. This concept admits a categorification to monoidal categories. The equality $zm = mz$ is replaced by the data of an isomorphism satisfying coherence conditions. The *Drinfel'd centre* $\mathcal{Z}\mathbf{C}$ of a monoidal category \mathbf{C} is the monoidal category whose objects are pairs (X, γ) , where $X \in \mathbf{C}$ and $\gamma : X \otimes - \rightarrow - \otimes X$ is a natural isomorphism satisfying the hexagon equation (recalled in Section 3.4). Such an isomorphism is called a

³The procedure requires a choice of connection on G_k . The correspondence between $\{G_k\}$ and $\{LG_k\}$ is bijective when G is simply-connected.

half-braiding for X . Just as the centre of a monoid is a commutative monoid, the Drinfel'd centre of \mathbf{C} is a braided monoidal category. The centre $\mathcal{Z}\mathbf{C}$ of a 2-group \mathbf{C} is again a 2-group (Lemma 3.4.1.4). The braiding β makes $\mathcal{Z}\mathbf{C}$ a *braided categorical group* [JS93]. The self-braidings $\beta_{x,x} \in \text{End}(x \otimes x) = \pi_1 \mathcal{Z}\mathbf{C}$ of objects $x \in \mathbf{C}$ assemble into a quadratic form [EGNO16, Ch 8.4]

$$\begin{aligned} q : \pi_0 \mathcal{Z}\mathbf{C} &\rightarrow \pi_1 \mathcal{Z}\mathbf{C} \\ x &\mapsto \beta_{x,x}. \end{aligned}$$

The map q encodes $\mathcal{Z}\mathbf{C}$ up to braided monoidal equivalence [EM54].

If \mathbf{C} is a monoidal category with a smooth structure, the Drinfel'd centre of the underlying monoidal category has a distinguished subcategory on objects with smooth half-braidings. In this chapter, we compute this smooth Drinfel'd centre for the String 2-groups G_k . The result is another instance of the close relationship between G_k and the associated central extension of the loop group, LG_k . We prove that for compact connected semisimple G and positive-definite $k \in H^4(\mathcal{B}G, \mathbb{Z})$, the Drinfel'd centre of G_k is

$$\mathcal{Z}G_k \simeq (\mathbf{Rep}^k LG)^\times,$$

the 2-group of invertible objects and invertible morphisms in $\mathbf{Rep}^k LG$, as long as we exclude factors of E_8 at level $k = 2$.⁴ The proof of the above statement is indirect: we compute the left hand side explicitly, and show the resulting braided categorical group agrees with that on the right hand side.

Question. *Can all of $\mathbf{Rep}^k LG$ be recovered as a generalised centre of the corresponding String 2-group G_k ?*

3.1.3 Statement of Results

Let G be a compact simple simply-connected Lie group, and \mathfrak{g} its complexified Lie algebra. There is a unique smallest positive-definite form $I : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ which is Ad_G -invariant and satisfies $I(X, X) \in 2\mathbb{Z}$ for all coroots of \mathfrak{g} (a coroot is an element $X \in \mathfrak{g}$ satisfying $e^{2\pi i X} = 1$).⁵ We use $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ to denote the map $w \mapsto e^{2\pi i w}$. Its restriction $\exp : \mathbb{R} \rightarrow U(1)$ has kernel $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Theorem. *The Drinfel'd centre of the String 2-group G_k is the braided categorical group with $\pi_0 \mathcal{Z}G_k = Z(G)$, $\pi_1 \mathcal{Z}G_k = U(1)$, and braided monoidal structure encoded by the quadratic form*

$$\begin{aligned} q : Z(G) &\rightarrow U(1) \\ z &\mapsto \exp \frac{k}{2} I(\bar{z}, \bar{z}). \end{aligned}$$

Here, $\bar{z} \in \mathfrak{g}$ denotes an arbitrary lift of $z \in Z(G)$ to the Lie algebra \mathfrak{g} of G .

⁴ $\mathbf{Rep}^{k=2} LE_8$ contains a non-trivial invertible object, while $\mathcal{Z}E_{8,2}$ does not. The invertibility of the non-identity object in $\mathbf{Rep}^{k=2} LE_8$ may be viewed as an accident of low level. See for example the computation in [Fuc91].

⁵For $SU(n)$, I is the trace of the product of matrices: $I(X, Y) = \text{tr } XY$.

The inner product $\frac{1}{2}I(\bar{z}, \bar{z})$ is a real number. Its values on different lifts of $z \in Z(G)$ differ by integers. By composing with \exp , we get a well-defined map $Z(G) \rightarrow U(1)$.

After a choice of maximal torus of G , every element of the centre lifts to an element of the coweight lattice (see Section 3.6). Their norms under I can be found e.g. in [Bou94]. We list the results for Lie groups with non-trivial centre in Table 3.1. The centreless Lie groups E_8, F_4, G_2 all have trivial Drinfel'd centre.

Type	G_k	$Z(G)$	$q(z) = \exp\left(\frac{k}{2}I(\bar{z}, \bar{z})\right)$
A_{n-1}	$SU(n)_k$	$\mathbb{Z}/n\langle\omega_1\rangle$	$q(\omega_1) = \exp\left(\frac{k \cdot (n-1)}{2n}\right)$
B_n	$\text{Spin}(2n+1)_k$	$\mathbb{Z}/2\langle\omega_1\rangle$	$q(\omega_1) = \exp\left(\frac{k}{2}\right) = (-1)^k$
C_n	$\text{Sp}(n)_k$	$\mathbb{Z}/2\langle\omega_n\rangle$	$q(\omega_n) = \exp\left(\frac{k \cdot n}{2}\right) = (-1)^{k \cdot n}$
D_{2n+1}	$\text{Spin}(4n+2)_k$	$\mathbb{Z}/4\langle\omega_{2n+1}\rangle$	$q(\omega_{2n+1}) = \exp\left(\frac{k \cdot (2n+1)}{8}\right)$
$D_{2n \geq 4}$	$\text{Spin}(4n)_k$	$\mathbb{Z}/2\langle\omega_{2n-1}, \omega_{2n}\rangle$	$q(\omega_{2n-1}) = q(\omega_{2n}) = \exp\left(\frac{k \cdot n}{4}\right) = \mathbf{i}^{k \cdot n}$ $q(\omega_{2n-1} + \omega_{2n}) = \exp\left(\frac{k}{2}\right) = (-1)^k$
E_6	$E_{6,k}$	$\mathbb{Z}/3\langle\omega_1\rangle$	$q(\omega_1) = \exp\left(\frac{2k}{3}\right)$
E_7	$E_{7,k}$	$\mathbb{Z}/2\langle\omega_7\rangle$	$q(\omega_1) = \exp\left(\frac{k}{4}\right) = \mathbf{i}^k$

Table 3.1: The Drinfel'd centre of G_k (displayed for groups with nontrivial centre). We denote by ω_i the i th coweight. The conventions for the numbering are taken from [Bou94].

A crucial step in our proof of the above theorem is the calculation of the centre of String 2-groups for $G = T$ a torus. They are also called *categorical tori* [Gan18]. Let $\mathfrak{t} = \text{Lie}(T)$ be the Lie algebra of a torus T , $\Lambda = \text{Hom}(T, U(1)) \subset \mathfrak{t}^*$ its character lattice and $\Pi = \text{Hom}(U(1), T) \subset \mathfrak{t}$ its cocharacter lattice. The group $H^4(\mathcal{B}T, \mathbb{Z})$ is the group of symmetric bilinear forms $\langle \cdot, \cdot \rangle : \mathfrak{t} \otimes \mathfrak{t} \rightarrow \mathbb{R}$, such that $\langle \pi, \pi \rangle \in 2\mathbb{Z}$ for all $\pi \in \Pi$. To describe the String 2-group associated to such a bilinear form, we pick a (not-necessarily-symmetric) bilinear form $J : \mathfrak{t} \otimes \mathfrak{t} \rightarrow \mathbb{R}$ which restricts to an integral form on $\Pi \otimes \Pi$ and satisfies $J(x, y) + J(y, x) = -\langle x, y \rangle$ for all $x, y \in \mathfrak{t}$.

The computation of the smooth Drinfel'd centre of the associated categorical torus T_J was sketched in [FHLT10]. We provide a proof of the following statement in Section 3.5.

Proposition. *The Drinfel'd centre of T_J has Lie group of objects*

$$\pi_0 \mathcal{L}T_J = (\Lambda \oplus \mathfrak{t})/\Pi,$$

where the inclusion $\Pi \hookrightarrow \Lambda \oplus \mathfrak{t}$ is $\pi \mapsto (-J(\cdot, \pi) - J(\pi, \cdot), \pi)$. Further, $\pi_1 \mathcal{L}T_J = U(1)$, and the braided monoidal structure is encoded by the quadratic form

$$q : (\Lambda \oplus \mathfrak{t})/\Pi \rightarrow U(1)$$

$$[\lambda, x] \mapsto \lambda(x) \exp(J(x, x)).$$

Every compact connected Lie group G fits into a short exact sequence $Z \hookrightarrow \tilde{G} \twoheadrightarrow G$. Here $\tilde{G} = T \times \prod_i G_i$ is a product of simple simply-connected groups G_i with a torus T , and $Z \subset Z(\tilde{G})$ is a finite subgroup of its centre [MT91, Cor V.5.31]. A central extension of G by $[*/U(1)]$ pulls back to a central extension of \tilde{G} by $[*/U(1)]$, giving the String

2-group \tilde{G}_k . It is classified by the pullback degree 4 cohomology class in $H^4(\mathcal{B}\tilde{G}, \mathbb{Z}) \simeq H^4(\mathcal{B}T, \mathbb{Z}) \times \prod_i H^4(\mathcal{B}G_i, \mathbb{Z})$ and we write $k = (J, \{k_i\})$.

Theorem. *The group $Z = \ker(\tilde{G} \rightarrow G)$ admits a unique lift to $\mathcal{Z}\tilde{G}_k$, and the quadratic form q is trivial on Z . The Drinfel'd centre of G_k is the subquotient*

$$\pi_0 \mathcal{Z}G_k = Z^\perp / Z,$$

where Z^\perp denotes the subgroup of $\pi_0 \mathcal{Z}\tilde{G}_k$ on elements satisfying $q(x+z) = q(x)$ for all $z \in Z$. The quadratic form on $\pi_0 \mathcal{Z}\tilde{G}_k$ descends to a quadratic form on $\pi_0 \mathcal{Z}G_k$, and this quadratic form describes the braided monoidal structure of $\mathcal{Z}G_k$.

The Drinfel'd centre of the covering String 2-group $\tilde{G}_k = \times_i G_{i,k_i} \times T_J$ is given by

$$\pi_0 \mathcal{Z}\tilde{G}_{(J, \{k_i\})} = (\Lambda \oplus \mathfrak{t}) / \Pi \times \prod_i G_{k_i},$$

$\pi_1 \mathcal{Z}\tilde{G}_k = U(1)$ and quadratic form the product of the quadratic forms q_{k_i} and q_J computed previously.

In the simply-connected case, every element of the centre admits a half-braiding, but in general the subset of the centre admitting a half-braiding varies with k . We present the result for the case $G = \mathrm{SO}(4)$ here (the computation is sketched in more detail in Example 3.6.2.5). The relevant cohomology group $H^4(\mathcal{B}\mathrm{SO}(4), \mathbb{Z})$ has two generators: the first Pontryagin class p_1 and the Euler class χ . For the cohomology class $k = a \cdot p_1 + b \cdot \chi$, the Drinfel'd centre is given by

$$\mathcal{Z}\mathrm{SO}(4)_k = \begin{cases} \mathbf{Vec}_{\mathbb{Z}/2}^\times & 2a + b \equiv 0 \pmod{4} \\ \mathbf{sVec}^\times & 2a + b \equiv 2 \pmod{4} \\ \mathbf{Vec}^\times & \text{else.} \end{cases}$$

Here, $\mathbf{Vec}_{\mathbb{Z}/2}^\times$ denotes the trivially braided 2-group $[*/U(1)] \times \mathbb{Z}/2$, \mathbf{sVec}^\times is *super*- $\mathbb{Z}/2$ with non-trivial self-braiding of $-1 \in \mathbb{Z}/2$, and $\mathbf{Vec}^\times = [*/U(1)]$ is the trivial Drinfel'd centre whose only object is the monoidal unit. As the notation indicates, they are the maximal braided sub-2-groups of the braided fusion categories $\mathbf{Vec}[\mathbb{Z}/2]$ and \mathbf{sVec} (see Example 1.1.2.5).

3.1.4 Structure

Section 3.2 is a quick introduction to groupoids with smooth structure. We review smooth 2-groups in Section 3.3, recalling in particular the model for the String 2-groups given in [SP11]. In Section 3.4, we discuss centres of smooth 2-groups as a special case of the notion of centre defined in [Str04b]. We compute this centre for categorical tori in Section 3.5 and for the String 2-groups in Section 3.6. We end by showing that $\mathcal{Z}G_k = (\mathbf{Rep}^k LG)^\times$ for G compact connected semisimple.

3.2 Smooth Groupoids

We use \mathbf{Man} to denote the category of paracompact smooth manifolds and smooth maps. We equip it with the structure of a site by declaring covers to be surjective submersions $f : Y \twoheadrightarrow M$, and denote the n -fold fibre product of a cover along itself by $Y^{[n]} := Y \times_M Y \times_M \cdots Y$.

3.2.1 Lie Groupoids

A *Lie groupoid* G_\bullet is a groupoid object in \mathbf{Man} . Its data are two manifolds G_1, G_0 with a pair of surjective submersions $s, t : G_1 \rightrightarrows G_0$, and smooth maps

$$\circ : G_1 \times_{G_0} G_1 \rightarrow G_1 \quad \text{id}_- : G_0 \rightarrow G_1 \quad (-)^{-1} : G_1 \rightarrow G_1,$$

implementing composition, identities and inverses, respectively. A *smooth functor* $G_\bullet \rightarrow H_\bullet$ is a pair of smooth maps $G_0 \rightarrow H_0, G_1 \rightarrow H_1$ that preserve identities and composition. A *smooth natural transformation* between two smooth functors is a smooth map $G_0 \rightarrow H_1$ that makes the usual naturality diagrams commute – see [MP97] for an introduction.

EXAMPLE 3.2.1.1. Every manifold M defines a Lie groupoid $M \rightrightarrows M$, with only identity morphisms.

EXAMPLE 3.2.1.2. A Lie group G has an associated Lie groupoid $[*/G] := G \rightrightarrows *$.

A smooth functor $F_\bullet : G_\bullet \rightarrow H_\bullet$ is *fully faithful* if

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ \downarrow (s,t) & & \downarrow (s,t) \\ G_0 \times G_0 & \xrightarrow{F_0 \times F_0} & H_0 \times H_0 \end{array}$$

is a pullback square, and it is *essentially surjective* if $s \circ p_2 : G_0 \times_{H_0} H_1 \rightarrow H_0$ is a surjective submersion (the pullback $G_0 \times_{H_0} H_1$ is formed along the maps $F_0 : G_0 \rightarrow H_0$ and $t : H_1 \rightarrow H_0$). A fully faithfully essentially surjective functor F_\bullet is a *smooth equivalence* [MP97]. Given a submersion $f : Y \rightarrow G_0$ from a manifold to the space of objects of a Lie groupoid G_\bullet , the *pullback groupoid* f^*G_\bullet is the groupoid $f^*G_1 \rightrightarrows Y$, with manifold of morphisms given by the pullback

$$\begin{array}{ccc} f^*G_1 & \longrightarrow & G_1 \\ \downarrow & \lrcorner & \downarrow (s,t) \\ Y \times Y & \xrightarrow{f \times f} & G_0 \times G_0. \end{array}$$

It is constructed such that the natural functor $f^*G_\bullet \rightarrow G_\bullet$ is fully faithful. When f is also surjective (and hence a cover), the functor $f^*G_\bullet \rightarrow G_\bullet$ is a smooth equivalence.

EXAMPLE 3.2.1.3. The *Čech groupoid* f^*M associated to a cover $f : Y \twoheadrightarrow M$ is $Y^{[2]} \rightrightarrows Y$ with composition given by the map $Y^{[2]} \times_Y Y^{[2]} = Y^{[3]} \rightarrow Y^{[2]}$ projecting out the middle factor. This Lie groupoid is equivalent to $M \rightrightarrows M$.

A crucial example for our computation is the Lie groupoid built from a smooth Čech 2-cocycle (Example 3.2.1.4). We briefly recall how to compute smooth Čech cohomology $\check{H}^*(M, A)$ of a manifold M with coefficients in an abelian Lie group A . The cochain complex computing Čech cohomology with respect to a cover $Y \rightrightarrows M$ is given by

$$C_Y^p(M, A) := C^\infty(Y^{[p+1]}, A)$$

with differential d_{Cech} the alternating sum over the pullbacks along the maps $\delta_i : Y^{[p]} \rightarrow Y^{[p-1]}$ that project out the i -th factor:

$$\delta_i^* : C^\infty(Y^{[p-1]}, A) \rightarrow C^\infty(Y^{[p]}, A).$$

Any two covers $Y, Z \rightrightarrows M$ admit a common refinement $Y \times_M Z \rightrightarrows M$. The assignment of cohomology groups assembles into a (contravariant) functor from the category of covers of M into the category of graded abelian groups. Čech cohomology of M is the colimit over this diagram. A cover is *good* if $Y^{[p]}$ is homotopy equivalent to a discrete space for all p , ie. if each fibre product is a disjoint union of open balls. The collection of good covers is a cofinal subset of the poset of covers of M . It is well-known that Čech cohomology with respect to any good cover computes sheaf cohomology for a paracompact manifold. Hence, the colimit we used to define Čech cohomology recovers sheaf cohomology, and can be computed using *any* good cover.

EXAMPLE 3.2.1.4. A cocycle representing $[\lambda] \in \check{H}^2(M, A)$ is a map $\lambda : Y^{[3]} \rightarrow A$ for some cover $Y \rightrightarrows M$. The *principal $[*/A]$ -bundle classified by λ* is the Lie groupoid $E^\lambda : Y^{[2]} \times A \rightrightarrows Y$, with source and target given by the two projections $Y^{[2]} \rightrightarrows Y$, and composition given by

$$\begin{aligned} (Y^{[2]} \times A) \times_Y (Y^{[2]} \times A) &\rightarrow (Y^{[2]} \times A) \\ ((y_0, y_1, a), (y_1, y_2, b)) &\mapsto (y_0, y_2, a + b + \lambda(y_0, y_1, y_2)). \end{aligned}$$

The cocycle condition ensures that this is associative (a condition checked on $Y^{[4]}$).

Lie groupoids, smooth functors and smooth natural transformations form a 2-category which we denote \mathbf{LieGpd} . This 2-category of groupoids internal to \mathbf{Man} is not a good 2-category of smooth groupoids, because smooth functors are too strict to implement the principle of equivalence: A smooth equivalence $f^*G_\bullet \rightarrow G_\bullet$ usually does not admit an inverse. Any sufficiently generic cover $f : \coprod_i U_i \rightrightarrows M$ provides a counterexample.

The standard way to proceed is to invert these morphisms by force. We follow [Pro96] in writing $\mathbf{LieGpd}[\mathbf{W}^{-1}]$ to denote the localisation of \mathbf{LieGpd} at smooth equivalences. Every 1-morphism in $\mathbf{LieGpd}[\mathbf{W}^{-1}]$ can be represented by an *anafunctor* [Rob12]: a span

$$G_\bullet \xleftarrow{\simeq} f^*G_\bullet \rightarrow H_\bullet,$$

whose left leg is a smooth equivalence. Morphisms between anafunctors are defined as natural transformations on a common refinement of the involved covers [Rob12]. In particular, if two anafunctors have the same left leg $G_\bullet \xleftarrow{\simeq} f^*G_\bullet$, then all morphisms between them are represented by natural transformations between their right legs $f^*G_\bullet \rightarrow H_\bullet$. The introduction of anafunctors manifestly inverts smooth equivalences. They can be viewed as smooth functors defined on some cover of the source.

EXAMPLE 3.2.1.5. The category of anafunctors from M to $[*/G]$ is equivalent to the category of G -principal bundles over M . The subcategory of smooth functors $M \rightarrow [*/G]$ is that of topologically trivial bundles.

3.2.2 Differentiable Stacks

Every Lie groupoid G_\bullet defines a functor of bicategories $\text{Hom}(-, G_\bullet) : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Gpd}$, sending a manifold M to the groupoid of smooth functors $(M \rightrightarrows M) \rightarrow G_\bullet$. Given any functor $\mathcal{F} : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Gpd}$, there is a natural way to evaluate it on the Čech groupoid (Example 3.2.1.3) associated to a cover $f : Y \rightarrow M$ (we suppress natural 2-cells in the diagram):

$$\mathcal{F}(f^*M) := \lim (\mathcal{F}(Y) \rightrightarrows \mathcal{F}(Y^{[2]}) \rightrightarrows \mathcal{F}(Y^{[3]})) .$$

The limit on the right hand side is the groupoid of descent data for \mathcal{F} with respect to Y (see Section 1.2.2). For $\mathcal{F} = \text{Hom}(-, G_\bullet)$, this is the groupoid of smooth functors $\mathcal{F}(f^*M) = \text{Hom}(f^*M, G_\bullet)$. The fact that smooth functors are too strict is reflected in the fact that the restriction functor $f^* := \text{Hom}(f, G_\bullet) : \text{Hom}(M, G_\bullet) \rightarrow \text{Hom}(f^*M, G_\bullet)$ is generally not an equivalence of categories.

We restate the definition of a stack.

Definition 3.2.2.1. A *stack* (of groupoids over the site of manifolds) is a functor of bicategories $\mathcal{F} : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Gpd}$ satisfying *descent*: It sends coproducts to products $\mathcal{F}(\coprod_i M_i) \simeq \prod_i \mathcal{F}(M_i)$ and for every cover $f : Y \rightarrow M$, the restriction functor $\mathcal{F}(f) : \mathcal{F}(M) \rightarrow \mathcal{F}(f^*M)$ is an equivalence of groupoids.

For the remainder of this chapter, we use the word *stack* to mean ‘stack of groupoids over the site of manifolds’. We denote the bicategory of stacks by

$$\mathbf{SmSt} \subset [\mathbf{Man}^{\text{op}}, \mathbf{Gpd}] .$$

Recall that there is a stackification functor $\#$ which turns 2-presheaves into stacks (see Section 1.2).

Definition 3.2.2.2. The *stack presented by G_\bullet* is the stackification of the associated 2-presheaf $\text{Hom}(-, G_\bullet)$.

The assignment $G_\bullet \mapsto \text{Hom}(-, G_\bullet)^\#$ defines a functor $\mathbf{LieGpd} \rightarrow \mathbf{SmSt}$.

Definition 3.2.2.3. The category \mathbf{DiffSt} of *differentiable stacks* is the full subcategory of \mathbf{SmSt} on the essential image of $\mathbf{LieGpd} \rightarrow \mathbf{SmSt}$.

Theorem 3.2.2.4. [Pro96] *The functor $\mathbf{LieGpd} \rightarrow \mathbf{DiffSt}$, sending a Lie groupoid G_\bullet to the stackification of $\text{Hom}(-, G_\bullet)$ induces an equivalence*

$$\mathbf{LieGpd}[\mathbf{W}^{-1}] \simeq \mathbf{DiffSt} .$$

This tells us that inverting weak equivalences has the same effect as stackification. It allows us to use both the explicit description of $\mathbf{LieGpd}[\mathbf{W}^{-1}]$ and the nice formal properties of $\mathbf{DiffSt} \subset \mathbf{SmSt}$. We denote the stack presented by a Lie groupoid G_\bullet by

the same symbol, $G_\bullet : \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Gpd}$. Its value $G_\bullet(M)$ on a manifold M is the groupoid of morphisms $M \rightarrow G_\bullet$ in $\mathbf{LieGpd}[\mathbf{W}^{-1}]$, which can be computed as the groupoid ofanafunctors $M \rightarrow G_\bullet$.

Every manifold M defines a functor $\text{ev}_M : \mathbf{DiffSt} \rightarrow \mathbf{Gpd}$, given by evaluation on M . In particular, ev_* sends each stack \mathcal{X} to its *groupoid of points* $\mathcal{X}(*)$. We view the remaining data of a differentiable stack as equipping $\mathcal{X}(*)$ with a smooth structure.

3.3 Smooth 2-Groups

3.3.1 From Discrete to Smooth 2-Groups

A (*discrete*) 2-group is a monoidal category $(\mathbf{C}, \otimes, \mathbb{1})$ with the property that every morphism in \mathbf{C} has an inverse, and every object $x \in \mathbf{C}$ has a *weak inverse*: an object $x^{-1} \in \mathbf{C}$ satisfying $x^{-1} \otimes x \cong \mathbb{1} \cong x \otimes x^{-1}$. Up to equivalence, a 2-group \mathbf{C} is characterised by 4 invariants (see e.g. [BL04]):

1. $\pi_0 \mathbf{C}$, the group of isomorphism classes of objects
2. $\pi_1 \mathbf{C}$, the abelian group of endomorphisms of $\mathbb{1}$
3. $\rho : \pi_0 \mathbf{C} \rightarrow \text{Aut}(\pi_1 \mathbf{C})$, the conjugation action $x \mapsto \text{id}_x \otimes - \otimes \text{id}_{x^{-1}}$
4. $[\omega] \in H^3(\mathcal{B}\pi_0 \mathbf{C}, \pi_1 \mathbf{C})$, the cohomology class of the associator.

We can build the 2-group associated to this data explicitly: The underlying groupoid is

$$\pi_0 \mathbf{C} \times_{\rho} \pi_1 \mathbf{C} \rightrightarrows \pi_0 \mathbf{C},$$

with both source and target morphisms given by projection to $\pi_0 \mathbf{C}$.

The zero section $\pi_0 \mathbf{C} \rightarrow \pi_0 \mathbf{C} \times_{\rho} \pi_1 \mathbf{C}$ serves as the assignment of the identity morphism to each object. This canonically identifies the endomorphism group of each object $g \in \pi_0 \mathbf{C}$ as $\text{End}(g) = \pi_1 \mathbf{C}$. There are no morphisms between objects corresponding to different elements $g \neq g' \in \pi_0 \mathbf{C}$. (Note that in the model of *smooth* 2-groups we use, it will not be possible to realise the String 2-groups in such a way.) The composition of endomorphisms is group multiplication in $\pi_1 \mathbf{C}$. The tensor product on objects and morphisms is given by the group multiplication in $\pi_0 \mathbf{C}$ and $\pi_0 \mathbf{C} \times_{\rho} \pi_1 \mathbf{C}$, respectively. The unitor isomorphisms are trivial. An associator for the tensor structure must pick out an endomorphism $\omega_{g,g',g''} \in \text{End}(gg'g'') = \pi_1 \mathbf{C}$ for each triple $g, g', g'' \in \pi_0 \mathbf{C}$. Any cocycle $\omega : \pi_0 \mathbf{C}^{\times 3} \rightarrow \pi_1 \mathbf{C}$ representing the cohomology class $[\omega]$ will do.

The notion of a *smooth* 2-group is the categorification of that of a Lie group (a smooth monoid with a smooth map sending each element to its inverse). One may avoid writing down conditions on the map $g \mapsto g^{-1}$ by encoding the existence of the inverse indirectly:

Definition 3.3.1.1. A *Lie group* is a unital monoid (G, μ) in the 1-category of smooth manifolds, such that the map $(p_1, \mu) : G \times G \rightarrow G \times G$ is a diffeomorphism.

The smooth inverse appearing in the traditional definition of Lie groups can be recovered as

$$G \xrightarrow{i_1} G \times G \xrightarrow{(p_1, \mu)^{-1}} G \times G \xrightarrow{p_2} G.$$

This definition now categorifies verbatim.

Definition 3.3.1.2. A *group object* in a bicategory with finite products is a unital monoid (G_\bullet, \otimes) such that

$$(p_1, \otimes) : G_\bullet \times G_\bullet \rightarrow G_\bullet \times G_\bullet$$

is an equivalence.⁶

EXAMPLE 3.3.1.3. A *discrete 2-group* is a group object in the bicategory of groupoids. That is a monoidal category in which each object and morphism is invertible.

Definition 3.3.1.4. [SP11] A *smooth 2-group* is a group object in \mathbf{DiffSt} .⁷

EXAMPLE 3.3.1.5. Every Lie group G defines a smooth 2-group. It has underlying Lie groupoid $G \rightrightarrows G$, with tensor product given by multiplication.

EXAMPLE 3.3.1.6. Let A be a Lie group. We denote by $[*/A]$ the Lie groupoid $A \rightrightarrows *$. The multiplication on A equips it with a smooth 2-group structure iff A is abelian.

EXAMPLE 3.3.1.7. The 2-category of discrete 2-groups is the full subcategory of the 2-category of smooth 2-groups on those whose underlying Lie groupoid is discrete.

EXAMPLE 3.3.1.8. A *strict smooth 2-group* is a category object in the category of Lie groups. Up to equivalence, one may always bring the underlying Lie groupoid into the form $G \rtimes_\rho H \rightrightarrows G$. Here, G, H are Lie groups equipped with a G -action $\rho : G \rightarrow \text{Aut}(H)$, and a target homomorphism $t_0 : H \rightarrow G$. (This data (G, H, t, ρ) satisfies further conditions and is known as a *crossed module*.) The source map can be chosen to be projection to G , and the target map is given by $t : (g, h) \mapsto t_0(h)g$. Composition is multiplication in H (i.e. $(t_0(h)g, h') \circ (g, h) = (t_0(h'h)g, h'h)$), tensor product is multiplication in the object and morphism groups, and all the other data is trivial. This example is spelt out in great detail in [Por08].

3.3.2 String 2-Groups and Segal-Mitchison Cohomology

String 2-groups are smooth 2-groups associated to a Lie group G and a cohomology class. The underlying discrete 2-group of a String 2-group is a 2-group with $\pi_1 = U(1)$ and trivial action $\rho : \pi_0 \rightarrow \text{Aut}(\pi_1)$. These discrete String 2-groups are classified by their underlying “object group” $\pi_0 = G^\delta$ and the class of the associator $k \in H^3(\mathcal{B}G^\delta, U(1))$ – the latter is a class in discrete group cohomology. To understand String 2-groups as *smooth 2-groups*, we must talk about Lie group cohomology.

Globally smooth group cohomology [Sta78; Bla85] $H_{\text{sm}}^\bullet(\mathcal{B}G, A)$ of a Lie group G with coefficients in a smooth G -module A is computed just as in the discrete case, except cocycles $G^{\times n} \rightarrow A$ are required to be smooth maps. This cohomology theory is *not* very well-behaved. A short exact sequence of coefficients does not give rise to a long exact sequence in globally smooth group cohomology. Further, the group $H_{\text{sm}}^2(\mathcal{B}G, A)$, which ordinarily classifies extensions of G by A , only detects those extensions $A \rightarrow E \rightarrow G$ where E is topologically a trivial A -bundle over G . One may fix this problem by only

⁶By a monoid object, we mean the maximally weak notion, often called a pseudomonoid.

⁷In [SP11], smooth 2-groups are defined internal to a bicategory denoted \mathbf{Bibun} . \mathbf{LieGpd} includes fully faithfully into \mathbf{Bibun} , and this induces an equivalence $\mathbf{DiffSt} \simeq \mathbf{LieGpd}[\mathbf{W}^{-1}] \simeq \mathbf{Bibun}$ [Pro96].

requiring smoothness in a neighbourhood of the identity of G . The resulting cohomology theory is called *locally smooth cohomology*. More geometric is the cohomology theory introduced by Segal in [Seg70] and recalled below. It is equivalent to locally smooth cohomology by [WW15].

Lie group cohomology of G is the cohomology of its *classifying space* $\mathcal{B}G$. We use the model of $\mathcal{B}G$ as a simplicial manifold: the manifold of q -simplices is $\mathcal{B}G_q = G^q$ and the face maps $d_i : G^q \rightarrow G^{q-1}$ are given by

$$d_i : (g_1, \dots, g_q) \mapsto \begin{cases} (g_2, \dots, g_q) & i = 0 \\ (\dots, g_i g_{i+1}, \dots) & 0 < i < q \\ (g_1, \dots, g_{q-1}) & i = q. \end{cases}$$

This simplicial manifold contains more homotopical information than the topological space $|\mathcal{B}G|$ obtained from $\mathcal{B}G$ by geometric realisation. The difference is detected by Segal-Mitchison cohomology unless the coefficient group is discrete (see Example 3.3.2.2).

A *simplicial cover* $Y_\bullet \rightarrow \mathcal{B}G$ is a collection of covers $Y_q \rightarrow \mathcal{B}G_q$, together with simplicial maps (which we will also denote by d_i) such that the corresponding squares in the diagram below commute.

$$\begin{array}{ccccccc} Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & Y_3 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ * & \longleftarrow & G & \longleftarrow & G \times G & \longleftarrow & G \times G \times G & \longleftarrow & \dots \end{array}$$

Associated to a simplicial cover is a Čech-simplicial double complex [Bry00] (recall that $Y^{[p]}$ denotes the p -fold fibre product of Y over M):

$$C^{p,q}(\mathcal{B}G, A) = C^\infty(Y_q^{[p+1]}, A) \quad d = d_{\check{C}ech} + d_{\text{simp}},$$

where $d_{\check{C}ech}$ denotes the Čech differential and d_{simp} is the alternating sum of the pullback maps

$$d_i^* : C^\infty(Y_q^{[p]}, A) \rightarrow C^\infty(Y_{q+1}^{[p]}, A).$$

A simplicial cover is *good* if $Y_i \rightarrow \mathcal{B}G_i$ is good for all i . For good simplicial covers of $\mathcal{B}G$, we may always choose Y_0 to be a point.

Definition 3.3.2.1. *Segal-Mitchison cohomology* of G with coefficients in an abelian Lie group A is the cohomology of the Čech-simplicial complex associated to a good simplicial cover $Y_\bullet \rightarrow \mathcal{B}G$.

This is independent of the good cover chosen [Bry00]. Segal-Mitchison cohomology can be defined more generally with coefficients in any smooth G -module, but we will only need the case with trivial action.

EXAMPLE 3.3.2.2. For certain coefficients A , Segal-Mitchison cohomology reduces to other cohomology theories [Seg70; Bry00]:

- If A is a vector space, it agrees with globally smooth group cohomology.

- If A is discrete, it is the (singular) cohomology of the topological space $|\mathcal{B}G|$.

Segal-Mitchison cohomology associates to any short exact sequence of coefficients a long exact sequence of cohomology groups. The exponential exact sequence $\mathbb{Z} \hookrightarrow \mathbb{R} \rightarrow U(1)$ gives rise to a long exact sequence

$$\cdots \rightarrow H^p(\mathcal{B}G, \mathbb{R}) \rightarrow H^p(\mathcal{B}G, U(1)) \xrightarrow{\delta} H^{p+1}(\mathcal{B}G, \mathbb{Z}) \rightarrow H^{p+1}(\mathcal{B}G, \mathbb{R}) \rightarrow \cdots .$$

Note that $H^\bullet(\mathcal{B}G, \mathbb{R})$ is cohomology with respect to the sheaf of smooth \mathbb{R} -valued functions, not the sheaf of locally constant \mathbb{R} -valued functions. In particular, it is *not* the same as singular cohomology of $|\mathcal{B}G|$ with coefficients in discrete \mathbb{R} and thus does not see the usual real characteristic classes. When G is compact, $H^\bullet(\mathcal{B}G, \mathbb{R})$ vanishes in positive degrees [Bla85] (see also [SP11, Cor 97]), so the map δ is an isomorphism for $p > 0$. In particular, $H^3(\mathcal{B}G, U(1)) \simeq H^4(\mathcal{B}G, \mathbb{Z})$. If G is also simply-connected and simple, then G is in fact 2-connected and $\pi_3 G = \mathbb{Z}$ [MT91, Thm VI.4.17][Noo09]. The group $H^4(\mathcal{B}G, \mathbb{Z})$ can be computed as the simplicial cohomology of the geometric realisation $|\mathcal{B}G|$ (see Example 3.3.2.2). This topological space has homotopy groups $\pi_{i < 4} |\mathcal{B}G| = 0$, $\pi_4 |\mathcal{B}G| = \mathbb{Z}$. The Hurewicz isomorphism and Universal Coefficient Theorem imply

$$H^3(\mathcal{B}G, U(1)) \simeq H^4(\mathcal{B}G, \mathbb{Z}) \cong \mathbb{Z}.$$

From a Segal-Mitchison 3-cocycle, one may build a smooth 2-group [SP11]. We recall this construction now. Let G be a Lie group, A an abelian Lie group, and $k \in H^3(\mathcal{B}G, A)$ a class in Segal-Mitchison cohomology. A Segal-Mitchison cocycle representing a cohomology class in $H^3(\mathcal{B}G, A)$ is a simplicial cover $Y_\bullet \rightarrow \mathcal{B}G$ and a triple⁸

$$(\lambda, \mu, \omega) \in C^\infty(Y_1^{[3]}, A) \times C^\infty(Y_2^{[2]}, A) \times C^\infty(Y_3, A).$$

Recall the Lie groupoid $E^\lambda : Y_1^{[2]} \times A \rightrightarrows Y_1$ associated to the Čech 2-cocycle λ from Example 3.2.1.4. Using the map $(d_0, d_2) : Y_2 \rightarrow Y_1 \times Y_1$, we can form the pullback groupoid $F^\lambda = (d_0, d_2)^*(E^\lambda \times E^\lambda) = Y_2^{[2]} \times A \times A \rightrightarrows Y_2$. The tensor structure on E^λ is given by the anafunctor

$$E^\lambda \times E^\lambda \xleftarrow{\simeq} F^\lambda \rightarrow E^\lambda,$$

where the map $F^\lambda \rightarrow E^\lambda$ is given by $d_1 : Y_2 \rightarrow Y_1$ on objects, and by

$$\begin{aligned} Y_2^{[2]} \times A \times A &\rightarrow Y_1^{[2]} \times A \\ (v_0, v_1, a, b) &\mapsto (d_1(v_0), d_1(v_1), a + b + \mu(v_0, v_1)), \end{aligned}$$

on morphisms. One may further pull back along $(d_0 d_0, d_2 d_0, d_1 d_0) : Y_3 \rightarrow Y_1^{\times 3}$ to obtain another Lie groupoid H^λ . Then $(- \otimes -) \otimes -$ and $- \otimes (- \otimes -)$ are represented by spans $(E^\lambda)^{\times 3} \xleftarrow{\simeq} H^\lambda \rightarrow E^\lambda$, and $\omega \in C^\infty(Y_3, A)$ defines a smooth natural transformation between them. The cocycle condition ensures that the above indeed defines a smooth 2-group.

⁸The Čech-simplicial complex has four groups in cohomological degree 3, but the cocycle condition implies that the component in $C^\infty(Y_0^{[4]}, A)$ is trivial.

There is a notion of *central extension* of smooth 2-groups [SP11, Defn 83] paralleling the definition for groups. Equivalence classes of central extensions of G by $[*/A]$ are in bijective correspondence with elements of $H^3(\mathcal{B}G, A)$, and a representative of this equivalence class may be built from a cocycle as above. The smooth String 2-group G_k is the central extension of G by $[*/U(1)]$ corresponding to $k \in H^3(\mathcal{B}G, U(1))$ [SP11, Thm 100].

3.4 The Smooth Centre

The Drinfel'd centre of a monoidal category $(\mathbf{C}, \otimes, \mathbf{1}, \omega)$ is the category whose objects are pairs (X, γ) , where $X \in \mathbf{C}$ and $\gamma : X \otimes - \xrightarrow{\cong} - \otimes X$ is a *half-braiding*: a natural isomorphism satisfying the *hexagon equation*

$$\omega_{Y,Z,X} \circ \gamma_{Y \otimes Z} \circ \omega_{X,Y,Z} = (\gamma_Y \otimes \text{id}_Z) \circ \omega_{Y,X,Z} \circ (\text{id}_Y \otimes \gamma_Z).$$

It is the analogue of the centre of a monoid in the world of monoidal categories. We study the corresponding notion in the context of smooth 2-groups.

3.4.1 Drinfel'd Centres of Smooth 2-Groups

In [Str04b], the notion of the Drinfel'd centre was internalised to any braided monoidal bicategory (and thus in particular to the symmetric monoidal bicategory of stacks). Let \mathcal{B} be a symmetric monoidal bicategory with product \boxtimes and braiding b , and (\mathbf{C}, \otimes) a monoid object in \mathcal{B} .

Definition 3.4.1.1. Let $U \in \mathcal{B}$. A U -family of *centre pieces* for (\mathbf{C}, \otimes) is a pair (u, γ) where $u : U \rightarrow \mathbf{C}$ is a morphism and

$$\gamma : \otimes \circ (u \boxtimes \text{id}_{\mathbf{C}}) \xrightarrow{\cong} \otimes \circ (\text{id}_{\mathbf{C}} \boxtimes u) \circ b_{U, \mathbf{C}}$$

an invertible 2-cell satisfying the hexagon equation (phrased internal to \mathcal{B}). We think of u as a U -point of \mathbf{C} and call γ a *half-braiding* for u . A *morphism of centre pieces* $(u, \gamma) \rightarrow (u', \gamma')$ is a morphism $u \rightarrow u'$ which commutes with the half-braidings. We denote by $\mathcal{CP}(U, \mathbf{C})$ the category of U -families of centre pieces for (\mathbf{C}, \otimes) .

Half-braidings may be pulled back along maps $U' \rightarrow U$, and $\mathcal{CP}(-, \mathbf{C})$ admits the structure of a 2-presheaf over \mathbf{Man} .

Definition 3.4.1.2. The centre $\mathcal{Z}\mathbf{C}$ of a monoid \mathbf{C} in a braided monoidal bicategory \mathcal{B} is the representing object for the 2-presheaf $\mathcal{CP}(-, \mathbf{C}) : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$.

In [Pir21], this centre was computed explicitly in the bicategory of crossed modules (see Example 3.3.1.8). We now specialise to $\mathcal{B} = \mathbf{SmSt}$, the bicategory of stacks. The 2-Yoneda Lemma (Lemma 1.2.2.7) says that for any stack \mathcal{F} and manifold M , the category $\mathcal{F}(M)$ is naturally equivalent to the category of 1-morphisms $M \rightarrow \mathcal{F}$. Thus, a morphism $u : M \rightarrow A$ is equivalently an object of the category $A(M)$, which makes Definition 3.4.1.1 a parameterised version of the ordinary Drinfel'd centre in this case.

Theorem 3.4.1.3. [Str04b] *The centre $\mathcal{Z}\mathbf{C}$ of a monoid $\mathbf{C} \in \mathcal{B}$ exists if \mathcal{B} is finitely complete and closed. Denote the internal Hom of \mathcal{B} by $[\cdot, \cdot]$, then the centre is the limit*

$$\mathcal{Z}\mathbf{C} = \lim (\mathbf{C} \rightrightarrows [\mathbf{C}, \mathbf{C}] \rightleftarrows [\mathbf{C} \boxtimes \mathbf{C}, \mathbf{C}]).^9$$

The centre is equipped with a monoidal product \otimes , a monoidal morphism $\mathcal{Z}\mathbf{C} \rightarrow \mathbf{C}$, and a braiding $\beta : \otimes \xrightarrow{\simeq} \otimes \circ b_{\mathcal{Z}\mathbf{C}, \mathcal{Z}\mathbf{C}}$.

The bicategory \mathbf{SmSt} is finitely complete [Str82], so in particular fulfills the assumptions of the above theorem. Any smooth 2-group $G \in \mathbf{DiffSt} \subset \mathbf{SmSt}$ thus has a centre $\mathcal{Z}G \in \mathbf{SmSt}$. The functor $\text{ev}_U : \mathbf{SmSt} \rightarrow \mathbf{Gpd}$ which evaluates a stack on a manifold U induces a braided monoidal comparison functor $(\mathcal{Z}G)(U) \rightarrow \mathcal{Z}(G(U))$. This functor is the inclusion of the smooth half-braidings and smooth maps of the associated centre pieces.

The monoidal groupoid $G(U)$ is a 2-group, and hence every centre piece (X, γ) has an inverse $(iX, i\gamma_i^{-1})$ [EGNO16, Ch 7.13], where $i : G(U) \rightarrow G(U)$ is a functor assigning inverses.

We would like to show that this inverse exists as a smooth half-braiding in $(\mathcal{Z}G)(U)$. One can pick a global inverse functor $i_{\text{glob}} : G \rightarrow G$ and directly show this, but the diagrams involved are unwieldy due to the presence of coherences. By uniqueness of inverses (up to isomorphism), it suffices to show the existence of inverses in $(\mathcal{Z}G)(U)$.

Lemma 3.4.1.4. *The Drinfel'd centre of a group object in \mathbf{SmSt} is a group object in \mathbf{SmSt} .*

Proof. This is completely formal. The inclusion of group objects in \mathbf{SmSt} into monoid objects in \mathbf{SmSt} is a reflective localisation of $(2, 1)$ -categories [SP11, Cor 59-Thm 61], and thus of $(\infty, 1)$ -categories. Reflective localisations of $(\infty, 1)$ -categories reflect limits [Wil13] so the limit exists in the bicategory of 2-groups, and agrees with the limit calculated in the bicategory of monoid objects. \square

Corollary 3.4.1.5. *The centre $\mathcal{Z}G$ of any 2-group in \mathbf{DiffSt} is a 2-group object in \mathbf{SmSt} . Given a global inverse $i : G \rightarrow G$, the inverse of a centre piece $(X, \gamma) \in (\mathcal{Z}G)(U)$ is given by $(i_U X, i_U \gamma_i^{-1})$.*

Proof. Lemma 3.4.1.4 guarantees the existence of an inverse to each centre piece (X, γ) . Under the functor $\text{ev}_U : \mathbf{SmSt} \rightarrow \mathbf{Gpd}$, this inverse must be isomorphic to $(iX, i\gamma_i^{-1})$ for any choice of inverse map i . \square

Lemma 3.4.1.6. *Let \mathbf{C} be a 2-group. Then $\pi_1 \mathcal{Z}\mathbf{C} = (\pi_1 \mathbf{C})^{\pi_0 \mathbf{C}}$, the invariants under the conjugation action ρ .*

Proof. The trivial centre piece (the monoidal unit of $\mathcal{Z}\mathbf{C}$) is the unit morphism $\mathbb{1} : * \rightarrow \mathbf{C}$, with half-braiding $\gamma_{\mathbb{1}}$ built as a composite of unitor morphisms $\mathbb{1} \otimes - \rightarrow - \rightarrow - \otimes \mathbb{1}$. The condition for a morphism $f \in \text{End}(\mathbb{1}) = \pi_1 \mathbf{C}$ to be an endomorphism of $(\mathbb{1}, \gamma_{\mathbb{1}})$ is

$$\gamma_{\mathbb{1}}(x) \circ (f \otimes \text{id}_x) = (\text{id}_x \otimes f) \circ \gamma_{\mathbb{1}}(x)$$

for all $x \in \mathbf{C}$. Tensoring with $\text{id}_{x^{-1}}$ shows that this is equivalent to f being conjugation-invariant. \square

⁹The morphisms in the diagram are induced by the monoid structure of \mathbf{C} . We suppress the 2-cells.

The central extensions we are interested in have trivial conjugation action, and so $\pi_1 \mathcal{L}\mathbf{C} = \pi_1 \mathbf{C}$ in this case.

3.4.2 Braided 2-Groups and Quadratic Forms

Recall that a braiding β on a monoid $(\mathbf{B}, \otimes, \dots)$ is a 2-morphism $\beta : - \otimes - \rightarrow - \otimes^{\text{op}} -$ which is compatible with units and satisfies the hexagon equation in both variables (see Section 1.1.5).

Definition 3.4.2.1. A *braided smooth 2-group* is a braided monoid $(\mathbf{B}, \otimes, \omega, l, r, \beta) \in \text{DiffSt}$ whose underlying monoid $(\mathbf{B}, \otimes, \omega, l, r)$ is a smooth 2-group.

A *discrete* braided 2-group is a braided monoidal groupoid whose underlying monoidal groupoid is a 2-group. They are also known as *braided categorical groups* [JS93]. The existence of a braiding on \mathbf{B} forces $\pi_0 \mathbf{B}$ to be abelian and the action $\rho : \pi_0 \mathbf{B} \rightarrow \text{Aut}(\pi_1 \mathbf{B})$ to be trivial.

Theorem 3.4.2.2. [EM54] *Equivalence classes of discrete braided categorical groups \mathbf{B} with $\pi_0 \mathbf{B} = A, \pi_1 \mathbf{B} = B$ are in one-to-one correspondence with quadratic forms $q : A \rightarrow B$. Under this correspondence, $q(a)$ is the self-braiding $\beta_{a,a} \in \text{End}(a) \cong B$ of $a \in A$.*

A quadratic form is a map $q : A \rightarrow B$ such that $q(n \cdot a) = n^2 \cdot q(a)$ for all $n \in \mathbb{N}, a \in A$, and the associated form

$$\begin{aligned} \sigma_q : A \times A &\rightarrow B \\ (a, b) &\mapsto q(a + b) / (q(a)q(b)) \end{aligned}$$

is bilinear. We conjecture that Theorem 3.4.2.2 also holds true in the smooth case. We will only show one half of this statement, namely that braided 2-groups are captured by their quadratic form. Denote the bicategory of smooth braided 2-groups with $\pi_0 = A, \pi_1 = B$ for two abelian Lie groups A, B by $\mathbf{B2G}(A, B)$. It is easy to check that the quadratic form associated to any such braided 2-group is smooth, and that the Baer sum of braided 2-groups corresponds to pointwise product of the associated quadratic forms.

Lemma 3.4.2.3. *The homomorphism*

$$\mathbf{B2G}(A, B) \rightarrow \text{Quad}^{\text{sm}}(A, B)$$

that sends a braided 2-group to its quadratic form is injective. We assume the group of connected components of A is finitely generated.

Proof. We need to show that there is only one (braided) equivalence class of braided 2-groups that give the *trivial* quadratic form $q : A \rightarrow B$. The associator is detected by degree three Segal-Mitchison cohomology $H^3(\mathcal{B}A, B)$ [SP11, Thm 99]. This cohomology group *injects* into the corresponding cohomology group of discrete groups [WW15]. By Theorem 3.4.2.2, the associator of the underlying discrete 2-group is trivial up to equivalence. As a result, we can assume that the associator of the smooth 2-group is trivial, so we are considering braidings on $A \times [* / B]$. The hexagon equations demand that the braiding

be a bilinear map $\beta : A \otimes A \rightarrow B$. As $q(a) = \beta(a, a)$ is trivial, β is alternating. A braiding on $A \times [*/B]$ is equivalent to the trivial braiding if there is a monoidal functor whose underlying functor is the identity and whose monoidal structure 2-cell $\eta : A \times A \rightarrow B$ makes the square

$$\begin{array}{ccc} a \otimes a' & \xrightarrow{\eta(a, a')} & a \otimes a' \\ \beta(a, a') \downarrow & & \downarrow \beta^{\text{triv}}(a, a')=1 \\ a' \otimes a & \xrightarrow{\eta(a', a)} & a' \otimes a \end{array}$$

commute for all $a, a' \in A$. Monoidality of the functor demands that η is a bilinear map. Hence, β is equivalent to the trivial braiding if there is a bilinear map $\eta : A \times A \rightarrow B$ such that $\beta(a, a') = \eta(a, a') - \eta(a', a)$.

It remains to show that any alternating bilinear form β admits such a trivialisation η . The group A splits as a product of a discrete group K and an abelian Lie group H . Let \mathfrak{h} be the Lie algebra of H , and $x, x' \in \mathfrak{h}$, $k, k' \in K$. The value of a bilinear form on $(k \cdot \exp(\alpha x), k' \cdot \exp(\alpha' x')) \in A^{\times 2} = (K \times H)^{\times 2}$ is completely determined by its values on pairs of elements of k, k', x, x' (where evaluating on a Lie algebra element takes the derivative). Pick a set of generators I_K for K and $I_{\mathfrak{h}}$ of the Lie algebra \mathfrak{h} of H , as well as an ordering on $I = I_K \cup I_{\mathfrak{h}}$. This allows encoding bilinear forms completely in an $I \times I$ matrix whose entries are given by evaluating on the generators corresponding to the row and column. As β is alternating, it is represented by an antisymmetric matrix. We may simply pick η to be its upper triangular half. \square

Let $\text{Quad}^{\text{sm}}(A, B) \rightarrow \text{H}^3(\mathcal{B}A, B)$ be the homomorphism extracting the monoidal structure of a braided categorical group from the quadratic form. There exist explicit formulae that recover an associator from the data of a quadratic form, see [Qui98; Bra20]. Taking monoidal equivalence classes gives the above map. Its kernel is the group of braidings for the 2-group with trivial associator. When the associator is trivial, the hexagon equations reduce to character equations of the map $\beta : A \times A \rightarrow B$ in each variable. Thus braidings are exactly bilinear maps $A \otimes A \rightarrow B$. The image of the map $\text{Quad}^{\text{sm}}(A, B) \rightarrow \text{H}^3(\mathcal{B}A, B)$ is called *soft cohomology* $\text{H}_{\text{soft}}^3(\mathcal{B}A, B) \subset \text{H}^3(\mathcal{B}A, B)$ in [DS18]. Soft cohomology is the group of (equivalence classes of) associators that can be part of a *braided* monoidal structure. In summary, we get a sequence of groups, exact at $\text{Quad}^{\text{sm}}(A, B)$:

$$\text{Bilin}(A, B) \hookrightarrow \text{Quad}^{\text{sm}}(A, B) \twoheadrightarrow \text{H}_{\text{soft}}^3(\mathcal{B}A, B) \hookrightarrow \text{H}^3(\mathcal{B}A, B).$$

EXAMPLE 3.4.2.4. We work out the case $A = \mathbb{Z}/n$, $B = U(1)$. The relevant cohomology group is $\text{H}^3(\mathcal{B}\mathbb{Z}/n, U(1)) \simeq \text{H}^4(\mathcal{B}\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$. The group of bilinear forms $\mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n \rightarrow U(1)$ is $\text{Bilin}(\mathbb{Z}/n, U(1)) = \mathbb{Z}/n$, generated by a primitive n -th root of unity. Quadratic forms $q : \mathbb{Z}/n \rightarrow U(1)$ are determined by the value on a generator: $q(k) = q(1)^{k^2}$. This defines a quadratic form iff $q(1) \in \mathbb{Z}/(2n, n^2)$, so

$$\text{Quad}(\mathbb{Z}/n, U(1)) = \begin{cases} \mathbb{Z}/n & n \text{ odd} \\ \mathbb{Z}/2n & n \text{ even.} \end{cases}$$

The exact sequence of groups introduced above implies

$$H_{\text{soft}}^3(\mathcal{B}\mathbb{Z}/n, U(1)) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}/2 & n \text{ even.} \end{cases}$$

For odd n , only the 2-group with trivial associator admits a braiding, while for even n , the 2-group corresponding to $[n/2] \in H^3(\mathcal{B}\mathbb{Z}/n, U(1)) = \mathbb{Z}/n$ also does.

NOTATION 3.4.2.5. A quadratic form $\mathbb{Z}/2 \rightarrow U(1)$ must send the generator of $\mathbb{Z}/2$ to a fourth root of unity, $q(1) = \mathbf{i}^k$. We borrow notation from the world of tensor categories to denote the corresponding braided 2-group $(\mathbb{Z}/2, q)$ as below (cf. Example 1.1.2.5).

$$\begin{array}{c|cccc} q(1) & 1 & \mathbf{i} & -1 & -\mathbf{i} \\ \hline (\mathbb{Z}/2, q) & \mathbf{Vec}_{\mathbb{Z}/2}^\times & \mathbf{Semi}^\times & \mathbf{sVec}^\times & \mathbf{Semi}^\times \end{array}$$

3.5 The Centres of Categorical Tori

A *categorical torus* \mathcal{T} [Gan18] is a central extension

$$[* / U(1)] \rightarrow \mathcal{T} \rightarrow T$$

of a (compact) torus T by $[* / U(1)]$. They are classified by $H^3(\mathcal{B}T, U(1)) \simeq H^4(\mathcal{B}T, \mathbb{Z})$. Let T be a torus, $\Lambda = \text{Hom}(T, U(1))$ its group of characters, $\Pi = \Lambda^\vee = \pi_1 T$ its group of cocharacters and $\mathfrak{t} = \text{Lie}(T)$ its Lie algebra, identified as the universal cover of T via $\Pi \hookrightarrow \mathfrak{t} \xrightarrow{\text{exp}} T$. The topological space $|\mathcal{B}T|$ is homotopy equivalent to an r -fold product of $\mathbb{C}\mathbb{P}^\infty$'s, where r is the rank of the torus T . The cohomology ring of $\mathcal{B}T$ is naturally identified with $H^*(\mathcal{B}T, \mathbb{Z}) = H^*(|\mathcal{B}T|, \mathbb{Z}) = \text{Sym}^*(\Lambda)$, where Λ is placed in degree 2. We identify the group $H^4(\mathcal{B}T, \mathbb{Z}) = \text{Sym}^2(\Lambda)$ with the group of symmetric bilinear forms $I : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ such that for all $\pi \in \Pi$, $I(\pi, \pi) \in 2\mathbb{Z}$. An element $\lambda \in \Lambda = \text{Hom}(T, U(1))$ induces a map of Lie algebras $D_e \lambda : \mathfrak{t} \rightarrow \mathbb{R}$. Then we send $\lambda_1 \otimes \lambda_2 \in \Lambda^{\otimes 2}$ to the symmetric bilinear form $I : (x, y) \mapsto D_e \lambda_1(x) \cdot D_e \lambda_2(y) + D_e \lambda_2(x) \cdot D_e \lambda_1(y)$.

Given a class $I \in H^4(\mathcal{B}T, \mathbb{Z})$, we will use τ to denote the induced map $\Pi \rightarrow \Lambda = \Pi^\vee$, given by $\tau(\pi) = I(\pi, -)$. We further pick a (not necessarily symmetric) bilinear form J on \mathfrak{t} such that J restricts to a \mathbb{Z} -valued form on Π , and $I = -(J + J^t)$, where J^t denotes the transpose of J . We recall a construction of the categorical torus T_J given in [Gan18]. It is a strict smooth 2-group (Example 3.3.1.8). The underlying Lie groupoid is

$$\mathfrak{t} \ltimes (\Pi \times U(1)) \rightrightarrows \mathfrak{t},$$

with arrows $(x, \pi, w) = \left(x \xrightarrow{w} x + \pi \right)$. Composition and tensor product are given by

$$\begin{aligned} \left(x + \pi \xrightarrow{w'} x + \pi + \pi' \right) \circ \left(x \xrightarrow{w} x + \pi \right) &= \left(x \xrightarrow{ww'} x + \pi + \pi' \right) \\ \left(x \xrightarrow{w} x + \pi \right) \otimes \left(x' \xrightarrow{w'} x + \pi' \right) &= \left(x + x' \xrightarrow{ww' \exp(J(\pi, x'))} x + x' + \pi + \pi' \right). \end{aligned}$$

The associator and unitor cells are trivial. This categorical torus is classified up to equivalence by the symmetric bilinear form $I = -(J + J^t) \in H^4(\mathcal{B}G, \mathbb{Z})$ [Gan18, Thm 4.1]. The paper [FHLT10] contains a sketch of calculation of the smooth Drinfel'd centre of T_J . We complement this with a proof based on the construction above.

Proposition 3.5.0.1. *The Drinfel'd centre of T_J has underlying Lie groupoid*

$$\mathcal{Z}T_J = ((\mathfrak{t} \oplus \Lambda) \ltimes (\Pi \times U(1))) \rightrightarrows \mathfrak{t} \oplus \Lambda,$$

with arrows $(x, \lambda, \pi, w) : (x, \lambda) \rightarrow (x + \pi, \lambda + \tau(\pi))$. The braiding is given by

$$\beta_{[x,\lambda],[x',\lambda']} = \lambda(x') \exp(J(x', x)).$$

Proof. A half-braiding on an object $x \in \mathfrak{t}$ is a 2-cell $\gamma : x \otimes - \rightarrow - \otimes x$ (subject to the hexagon equation). Tensoring with x on either side is a smooth functor $T_J \rightarrow T_J$. Hence each such 2-cell γ is represented by a smooth natural transformation: a smooth map $\gamma : \mathfrak{t} \rightarrow \mathfrak{t} \ltimes (\Pi \times U(1))$, which sends $y \in \mathfrak{t}$ to an endomorphism of $x + y = y + x$. In the absence of associators, the hexagon equation simplifies to the character equation $\gamma_{y+z} = \gamma_z \gamma_y$. The condition that γ be natural gives it the form

$$\gamma_y = \lambda(y) \cdot \exp(J(y, x)),$$

where $\lambda \in \Lambda$ is a smooth character of T . This allows a parameterisation of objects of $\mathcal{Z}T_J(*)$ by pairs $(x, \lambda) \in \mathfrak{t} \oplus \Lambda$.

Morphisms $(x, \lambda) \rightarrow (x', \lambda')$ are morphisms $x \rightarrow x'$ which are compatible with the braidings. The morphism $(x, \pi, w) \in \mathfrak{t} \ltimes (\Pi \times U(1))$ is a morphism of centre pieces $(x, \lambda) \rightarrow (x + \pi, \lambda')$ precisely when

$$\lambda'(y) = \lambda(y) \cdot \exp(-J(y, \pi) - J(\pi, y)) = (\lambda + \tau(\pi))(y).$$

for all $y \in \mathfrak{t}$.

So far, we have calculated $\mathcal{Z}T_J(*)$. To deduce the smooth structure, note that any object $\tilde{x} \in T_J(V)$ can be represented by a smooth map $\tilde{x} : V \rightarrow \mathfrak{t}$, and the corresponding half-braidings are represented by smooth maps $\tilde{\gamma} : V \times \mathfrak{t} \rightarrow \mathfrak{t} \ltimes (\Pi \times U(1))$. Naturality and the hexagon equation can now be checked pointwise.

The braiding on the centre can be computed in $\mathcal{Z}(T_J(*))$, where it takes the usual form $\beta_{(x,\gamma),(x',\gamma')} = \gamma_{x'}$ [EGNO16, Ch 8.5]. This completes the proof. \square

As mandated by Lemma 3.4.1.6, $\pi_1 \mathcal{Z}T_J = U(1)$. The object group of the centre is

$$\pi_0 \mathcal{Z}T_J = \frac{\mathfrak{t} \oplus \Lambda}{\Pi},$$

with $\Pi \ni \pi \mapsto (\pi, \tau(\pi)) \in \mathfrak{t} \oplus \Lambda$. We end this section by computing the *maximal compact subgroup* of $\pi_0 \mathcal{Z}T_J$. The cohomology class corresponding to the symmetric bilinear form $I = -(J + J^t)$ induces the maps $\tau : \Pi \rightarrow \Lambda$ and $\tau_{\mathbb{R}} := \tau \otimes \mathbb{R} : \mathfrak{t} \rightarrow \mathfrak{t}^*$. The case where $\tau_{\mathbb{R}}$ is an isomorphism (such levels are referred to as *non-degenerate*) was already analysed in [FHLT10]. The map $(\mathfrak{t} \oplus \Lambda)/\Pi \rightarrow \Lambda/\Pi$ is split by $s : \lambda \mapsto (\tau_{\mathbb{R}}^{-1} \lambda, \lambda)$. This furnishes an

isomorphism $\pi_0 \mathcal{L}T_J \xrightarrow{\cong} \mathfrak{t} \oplus (\Lambda/\Pi)$. Note that Λ/Π is finite, so $\pi_0 \mathcal{L}T_J$ is a direct sum of $\mathbb{R}^{\text{rk } \tau_{\mathbb{R}}}$ with a finite abelian group.

For general level, we pick a splitting $\Pi \cong \Pi_{\ker} \oplus \Pi_{\text{coim}}$, where $\Pi_{\ker} := \ker \tau$. We tensor with \mathbb{R} to obtain $\mathfrak{t} \cong \mathfrak{t}_{\ker} \oplus \mathfrak{t}_{\text{coim}}$. Now $\tau_{\mathbb{R}}$ restricts to an isomorphism $\tau_{\text{im}} : \mathfrak{t}_{\text{coim}} \xrightarrow{\cong} \text{im } \tau_{\mathbb{R}} \subset \mathfrak{t}^*$. We denote the intersection $(\text{im } \tau_{\mathbb{R}}) \cap \Lambda$ by $\Lambda_{\text{im}_{\mathbb{R}}}$. Lastly, we pick a decomposition $\Lambda \cong \Lambda_{\text{im}_{\mathbb{R}}} \oplus \Lambda_{\text{coker}_{\mathbb{R}}}$. These isomorphisms assemble into

$$\pi_0 \mathcal{L}T_J \xrightarrow{\cong} \Lambda_{\text{coker}_{\mathbb{R}}} \oplus \frac{\mathfrak{t}_{\ker}}{\Pi_{\ker}} \oplus \frac{\mathfrak{t}_{\text{coim}} \oplus \Lambda_{\text{im}_{\mathbb{R}}}}{\Pi_{\text{coim}}} \cong \Lambda_{\text{coker}_{\mathbb{R}}} \oplus T_{\ker} \oplus \mathfrak{t}_{\text{coim}} \oplus \Lambda_{\text{im}_{\mathbb{R}}}/\Pi_{\text{coim}},$$

where the second isomorphism uses that τ is non-degenerate when viewed as a map $\Pi_{\text{coim}} \rightarrow \Lambda_{\text{im}_{\mathbb{R}}}$. As before, $\Lambda_{\text{im}_{\mathbb{R}}}/\Pi_{\text{coim}}$ is a finite group. Both $\Lambda_{\text{coker}_{\mathbb{R}}}$ and $\mathfrak{t}_{\text{coim}}$ are free, and $T_{\ker} := \mathfrak{t}_{\ker}/\Pi_{\ker}$ is a compact torus. One may now read off the maximal compact subgroup as $T_{\ker} \oplus \Lambda_{\text{im}_{\mathbb{R}}}/\Pi_{\text{coim}}$. We are justified in calling this *the* maximal compact subgroup: any element of the free subgroup $\Lambda_{\text{coker}_{\mathbb{R}}} \oplus \mathfrak{t}_{\text{coim}}$ generates a non-compact group, so all compact subgroups of $\pi_0 \mathcal{L}T_J$ must intersect trivially with it – the maximal compact subgroup we computed above is the orthogonal complement of $\Lambda_{\text{coker}_{\mathbb{R}}} \oplus \mathfrak{t}_{\text{coim}}$. Using the explicit maps above, it is straightforward to compute the induced braiding on it. We call the resulting braided categorical group the maximal compact sub-2-group.

Proposition 3.5.0.2. *The maximal compact sub-2-group of $\mathcal{L}T_J$ is the braided 2-group with*

$$\pi_0 \text{Cpct } \mathcal{L}T_J = T_{\ker} \oplus \Lambda_{\text{im}}/\Pi_{\text{coim}},$$

$\pi_1 \mathcal{L}T_J = U(1)$, and braiding encoded by the quadratic form

$$\bar{q}([t_{\ker}, \lambda]) = \lambda(t_{\ker} + \tau_{\text{im}}^{-1} \lambda) \exp(J(\tau_{\text{im}}^{-1} \lambda, \tau_{\text{im}}^{-1} \lambda)).$$

Proof. This is a straightforward computation. The component T_{\ker} does not contribute in the exponential because

$$J(t_{\ker}, t_{\ker}) = \frac{1}{2} \tau(t_{\ker})(t_{\ker}) = 0 \quad \text{and} \quad J(t_{\ker}, \tau_{\text{im}}^{-1} \lambda) + J(\tau_{\text{im}}^{-1} \lambda, t_{\ker}) = 0. \quad \square$$

We record one important feature of this subgroup.

Lemma 3.5.0.3. *The map*

$$u : \pi_0 \text{Cpct } \mathcal{L}T_J \rightarrow T$$

that forgets the half-braiding is injective.

Proof. The decomposition of \mathfrak{t} descends to a decomposition of the torus T . Then u is the direct sum of injective maps

$$\pi_0 \text{Cpct } \mathcal{L}T_J = T_{\ker} \oplus \Lambda_{\text{im}}/\Pi_{\text{coim}} \xrightarrow{\text{id} \oplus \tau_{\text{im}}^{-1}} T_{\ker} \oplus \mathfrak{t}_{\text{coim}}/\Pi_{\text{coim}} = T_{\ker} \oplus T_{\text{coim}}. \quad \square$$

3.6 The Centres of String 2-Groups

In this chapter, we compute the centre \mathcal{Z} for the String groups G_k . We do this using obstruction theory. In particular, we establish an exact sequence of groups in Proposition 3.6.1.1, which relates the group $\pi_0 \mathcal{Z} G_k$ to the ordinary centre $Z(G)$ and the group cohomology of G . When G is simply-connected, this sequence allow us to deduce that every element of the ordinary centre $z \in Z(G)$ admits a unique lift to the Drinfel'd centre $\mathcal{Z} G_k$. This lift restricts to an element in $\mathcal{Z} T_k$, where T_k is a maximal 2-torus for G_k . The computations of Section 3.5 allow us to deduce the resulting braided structure. We treat the case of non-simply-connected G in Section 3.6.2. We pick a simply-connected covering group $\pi : \tilde{G} \rightarrow G$ and check which centre pieces in $\mathcal{Z} \tilde{G}_{\pi^* k}$ descend to centre pieces of $\mathcal{Z} G_k$.

In Section 3.6.3, we show the Drinfel'd centre $\mathcal{Z} G_k$ agrees with the invertible part of $\text{Rep}^k LG$ (if G is semisimple and there is no factor of E_8 at level 2).

3.6.1 Simply-connected Lie Groups

Let G be a compact simple simply-connected Lie group, pick a maximal torus $T \hookrightarrow G$, and let $\mathfrak{t}, \mathfrak{t}^*, \Lambda, \Pi$ be as in Section 3.5. The Lie algebra $\mathfrak{t} \subset \mathfrak{g} = \text{Lie}(G)$ is also called a *Cartan algebra* for G . The *Weyl group* $W = N(T)/T$, where $N(T)$ denotes the normaliser of T in G , acts on T by conjugation.

Recall from Section 3.5 that $H^4(\mathcal{B}T, \mathbb{Z})$ is identified with the group of \mathbb{R} -valued symmetric bilinear forms I on \mathfrak{t} satisfying $I(\pi, \pi) \in 2\mathbb{Z}$ for all $\pi \in \Pi \subset \mathfrak{t}$. Borel [Bor53; Bor54] identified $H^4(\mathcal{B}G, \mathbb{Z})$ as the Weyl-invariant part $H^4(\mathcal{B}G, \mathbb{Z}) = H^4(\mathcal{B}T, \mathbb{Z})^W$, see [Tod87] for a review. We identify $H^4(\mathcal{B}T, \mathbb{Z})^W$ with the group of those inner products I as above that are invariant under the W -action on \mathfrak{t} . This identification holds true for general compact connected Lie groups [Hen17a, Thm 6].

For G compact simple and simply-connected, $H^4(\mathcal{B}G, \mathbb{Z}) = H^4(\mathcal{B}T, \mathbb{Z})^W \cong \mathbb{Z}$ has a distinguished generator: the basic positive-definite Weyl-invariant inner product $I : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$, normalised such that short coroots have norm squared 2. We denote by $k \in \mathbb{Z}$ the cohomology class corresponding to $k \cdot I \in H^4(\mathcal{B}G, \mathbb{Z})$. As before, we denote the map induced by $k \cdot I$ on Π by $\tau : \Pi \rightarrow \Lambda$ – recall it sends $\pi \mapsto k \cdot I(\pi, -)$. Every non-zero cohomology class k induces an isomorphism $\tau_{\mathbb{R}} = \tau \otimes \mathbb{R} : \mathfrak{t} \rightarrow \mathfrak{t}^*$.

One may pull back the extension of G by $[*/U(1)]$ along the inclusion of a maximal torus $T \hookrightarrow G$ to obtain a maximal 2-torus. In [Gan18], Ganter shows that the maximal 2-torus of G_k is the categorical torus T_{J_k} associated to a (non-symmetric) bilinear form $J_k : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$ such that $J_k + J_k^t = -k \cdot I$.

Proposition 3.6.1.1. *Let H_ω denote the central extension of H by $[*/A]$ (A an abelian Lie group) corresponding to $\omega \in H^3(\mathcal{B}H, A)$. Then there is an exact sequence*

$$0 \rightarrow H^1(\mathcal{B}H, A) \rightarrow \pi_0 \mathcal{Z} H_\omega \rightarrow Z(H) \rightarrow H^2(\mathcal{B}H, A).$$

Proof. Each centre piece $(g, \gamma) \in \mathcal{Z} H_\omega(*)$ must satisfy $g \in Z(H)$, otherwise $gx \not\cong xg$ for some $x \in H$. In the discrete case, one can work with a skeletal representative of H_ω .

The hexagon equation evaluated at $(x, y) \in G \times G$ is then

$$(d\gamma)(x, y) := \frac{\gamma_{xy}}{\gamma_x \gamma_y} = \frac{\omega_{x,g,y}}{\omega_{g,x,y} \omega_{x,y,g}} =: \omega(x, y|g),$$

so γ is a 1-cochain whose coboundary is $\omega(-, -|g)$. It is straightforward to check that $\omega(-, -|g)$ is indeed a 2-cocycle. The map $Z(H) \rightarrow H^2(\mathcal{B}H, A)$ assigns to each element the equivalence class of the corresponding cocycle: $g \mapsto [\omega(-, -|g)]$. The element g admits a half-braiding precisely if $\omega(-, -|g)$ is a coboundary, which proves exactness at $Z(H)$.

The kernel of the map $\pi_0 \mathcal{L}H_\omega \rightarrow Z(H)$ is the group of half-braidings for the identity element of H . The associator ω can be chosen to be trivial whenever at least one of the entries is the identity, and the hexagon equation becomes the equation of an A -valued character on H . These are precisely the elements of $H^1(\mathcal{B}H, A)$.

We now port the above proof to the smooth case. Recall that the Lie groupoid modelling H_ω is of the form $L \rightrightarrows Y$, where $Y \rightarrow H$ is a surjective submersion and $L \rightarrow Y^{[2]}$ is an A -bundle. We pick a cover $\kappa : V \rightarrow Y$ such that the line bundle L trivialises over $V^{[2]}$, and replace H_ω by the equivalent groupoid $\kappa^* H_\omega = V^{[2]} \times A \rightrightarrows V$. Then we pick a cover $\pi : W \rightarrow V \times V \rightarrow Y \times Y$ such that all six functors/vertices in the hexagon equation for γ (ie. $(g \otimes -) \otimes -, g \otimes (- \otimes -), (- \otimes -) \otimes g \dots$) are representable by smooth functors $\pi^*(H_\omega \times H_\omega) \rightarrow \kappa^* H_\omega$. Each 2-morphism/edge in the hexagon equation $(\omega_{z,-,-}, \gamma_{-\otimes-}, \dots)$ is then represented by a smooth natural transformation. Each pair of functors F_i, F_j gives a map $f_{ij} : W \rightarrow V^{[2]}$, and a smooth natural transformation $F_i \Rightarrow F_j$ is a section of the pullback bundle $f_{ij}^*(V^{[2]} \times A)$. Under the choices we made, these bundles are all trivial. The hexagon axiom reduces to the same equation as above, $d\gamma = \omega(-, -|g)$, except it is now an equation in Segal-Mitchison cohomology. The pair $(V \rightarrow H, W \rightarrow H \times H)$ forms the first two steps of a simplicial cover of $\mathcal{B}H$. The maps $\gamma : V \rightarrow A, \omega(-, -|g) : W \rightarrow A$ represent Segal-Mitchison cochains. \square

Corollary 3.6.1.2. *Each element $z \in Z(G)$ admits a unique half-braiding over G_k :*

$$\pi_0 \mathcal{L}G_k = Z(G).$$

Proof. Compactness of G implies $H^*(\mathcal{B}G, U(1)) \simeq H^{*+1}(\mathcal{B}G, \mathbb{Z})$. The connectivity assumptions further imply $H^2(\mathcal{B}G, \mathbb{Z}) = H^3(\mathcal{B}G, \mathbb{Z}) = 0$ (see Section 3.3). The exact sequence of Proposition 3.6.1.1 shortens to an isomorphism. \square

Every half-braiding for $z \in Z(G)$ over G_k restricts to a half-braiding of z over T_{J_k} . We thus get a restriction functor $r : \mathcal{L}G_k \rightarrow \mathcal{L}T_{J_k}$. To describe this restriction functor explicitly, we recall how the centre $Z(G)$ of a Lie group lifts to \mathfrak{t} – see Chapter 13 of [Hal15] for proofs of the following facts. The centre $Z(G)$ includes into any maximal torus T of G . It lifts to \mathfrak{t} as the dual Φ^\vee of the *root lattice* $\Phi \subset \mathfrak{t}^*$ of G . For simply-connected G , the lattice Φ^\vee agrees with the *coweight lattice* (see e.g. [KKJ05]), and thus elements of $Z(G)$ lift to coweights in $\Phi^\vee \subset \mathfrak{t}$. The centre of a simply-connected compact Lie group may be computed from the coweight and cocharacter lattice as $Z(G) = \Phi^\vee / \Pi$.

Theorem 3.6.1.3. *The Drinfel'd centre of G_k is the braided categorical group specified by $\pi_0 \mathcal{L}G_k = Z(G)$, $\pi_1 \mathcal{L}G_k = U(1)$ and the quadratic form*

$$q : Z(G) \rightarrow U(1)$$

$$z \mapsto \exp\left(\frac{k}{2}I(\bar{z}, \bar{z})\right),$$

where \bar{z} denotes any lift of $z \in Z(G)$ to \mathfrak{t} .

Proof. As $\pi_0 \mathcal{L}G_k = Z(G)$ is finite, the map $r : \mathcal{L}G_k \rightarrow \mathcal{L}T_{J_k}$ must land in the maximal compact subgroup $\text{Cpct } \mathcal{L}T_{J_k}$, computed in Proposition 3.5.0.2. It fits into the commutative diagram

$$\begin{array}{ccc} \pi_0 \mathcal{L}G_k & \xrightarrow{r} & \pi_0 \text{Cpct } \mathcal{L}T_{J_k} \\ \downarrow \simeq & \nearrow r_0 & \downarrow u \\ Z(G) & \xrightarrow{\quad} & T, \end{array}$$

where the left hand map is an isomorphism by Corollary 3.6.1.2, and the right hand map u is injective by Lemma 3.5.0.3. The map $r_0 : Z(G) \rightarrow \pi_0 \text{Cpct } \mathcal{L}T_{J_k}$ is uniquely fixed by the requirement that the bottom right triangle commute.

For $k = 0$, the map u is the identity on T . The braiding is trivial on all of T , and restricts to the trivial braiding on $Z(G)$. For $k \neq 0$, the maximal compact subgroup is Λ/Π , u is equal to $\tau_{\mathbb{R}}^{-1} : \Lambda/\Pi \rightarrow \mathfrak{t}/\Pi = T$, and r_0 is the section

$$Z(G) = \Phi^\vee/\Pi \xrightarrow{\tau/\Pi} \Lambda/\Pi.$$

Denote a lift of $z \in Z(G)$ to Φ^\vee by \bar{z} . The quadratic form \bar{q} computed in Proposition 3.5.0.2 pulls back to

$$q(z) = \bar{q}(\tau\bar{z}) = \tau(\bar{z})(\bar{z}) \exp J_k(\bar{z}, \bar{z}) = \exp(k \cdot I(\bar{z}, \bar{z}) + J_k(\bar{z}, \bar{z})) = \exp\left(\frac{k}{2}I(\bar{z}, \bar{z})\right). \quad \square$$

Each lift \bar{z} in this formula is a coweight. The norm of a coweight may be computed as the norm of the corresponding weight of the *dual root datum*. In the realm of compact simple simply-connected Lie groups, dualising root data simply exchanges the odd-dimensional Spin groups $B_n = \text{Spin}(2n+1)$ and the symplectic Lie groups $C_n = \text{Sp}(2n)$. All other groups are fixed by this operation. The norm is computed using the inner product on the dual of the Cartan of the dual root datum. It is normalised such that short roots have length squared 2. The values of the length squared of weights under this product can be read off from the explicit expansion for weights in terms of roots given in [Bou94]. In Table 3.1, we list the results of this computation.

EXAMPLE 3.6.1.4. The Drinfel'd centre of $\text{SU}(2)_k$ is given by (see Notation 3.4.2.5)

$$\mathcal{L}\text{SU}(2)_k = \begin{cases} \mathbf{Vec}_{\mathbb{Z}/2}^\times & k \equiv 0 \pmod{4} \\ \mathbf{Semi}^\times & k \equiv 1 \pmod{4} \\ \mathbf{sVec}^\times & k \equiv 2 \pmod{4} \\ \overline{\mathbf{Semi}}^\times & k \equiv 3 \pmod{4}. \end{cases}$$

3.6.2 Non-simply-connected Lie Groups

Any compact connected Lie group G fits into a short exact sequence

$$Z \hookrightarrow \tilde{G} \xrightarrow{\pi} G,$$

where the middle term is a product $\tilde{G} = T \times \prod_i G_i$ of a torus T with simply-connected simple Lie groups G_i , and $Z \hookrightarrow Z(\tilde{G})$ is a finite central subgroup [MT91, Cor V.5.31]. The degree 4 cohomology of \tilde{G} decomposes as

$$H^4(\mathcal{B}\tilde{G}, \mathbb{Z}) = H^4(\mathcal{B}T, \mathbb{Z}) \oplus \prod_i H^4(\mathcal{B}G_i, \mathbb{Z}).$$

Hence any degree 4 cohomology class of \tilde{G} can be represented by a cocycle which is a product of cocycles pulled back from the individual factors.

Lemma 3.6.2.1. *Let H, H' be a pair of Lie groups and ω, ω' cocycles representing associators. Denote by $\bar{\omega} := p_H^* \omega + p_{H'}^* \omega'$ the product cocycle on $H \times H'$. The centre $\mathcal{Z}(H \times H')_{\bar{\omega}}$ is the braided abelian group $(\pi_0 \mathcal{Z} H_\omega \times \pi_0 \mathcal{Z} H'_{\omega'}, \bar{q})$, with quadratic form*

$$\bar{q} : \pi_0 \mathcal{Z} H_\omega \times \pi_0 \mathcal{Z} H'_{\omega'} \rightarrow U(1) = \pi_1 \mathcal{Z}(H \times H')_{\bar{\omega}}$$

given by the pointwise product of the quadratic forms on $\mathcal{Z} H_\omega$ and $\mathcal{Z} H'_{\omega'}$.

Proof. We prove this in the discrete setup. The argument is carried over to the smooth setting exactly as in the proof of Proposition 3.6.1.1. A centre piece for $(H \times H')_{\bar{\omega}}$ is a tuple of elements (h, h') , equipped with a half-braiding $\gamma : H \times H' \rightarrow U(1)$ satisfying the hexagon equation

$$d\gamma = \bar{\omega}(-, -|(h, h')).$$

We will now show that every such half-braiding is a product of half-braidings in H_ω and $H'_{\omega'}$ and vice versa. The cocycle $\bar{\omega}$ splits as a product of ω and ω' , and thus so does $\bar{\omega}(-, -|(h, h'))$. The hexagon equation implies that the half-braiding splits as a product

$$\gamma_{(x, x')} = \gamma_{(x, e)} \gamma_{(e, x')} =: \gamma_x \gamma'_{x'}.$$

The hexagon equation for γ is now equivalent to the hexagon equations $d\gamma = \omega(-, -|h)$ and $d\gamma' = \omega'(-, -|h')$. Hence $\pi_0 \mathcal{Z}(H \times H')_{\bar{\omega}} = \pi_0 \mathcal{Z} H_\omega \times \pi_0 \mathcal{Z} H'_{\omega'}$ is indeed the product. The quadratic form is given by the pointwise product because the half-braidings are. \square

The central extension G_k of G corresponding to $k \in H^4(\mathcal{B}G, \mathbb{Z})$ pulls back to an extension $\tilde{G}_{\pi^* k}$, where $\pi^* k = (J, \{k_i\}) \in H^4(\mathcal{B}T, \mathbb{Z}) \oplus \prod_i H^4(\mathcal{B}G_i, \mathbb{Z})$. Its Drinfel'd centre $\mathcal{Z}\tilde{G}_{\pi^* k}$ is the braided 2-group with

$$\pi_0 \mathcal{Z}\tilde{G}_{(J, \{k_i\})} = \mathcal{Z}(T \times \prod_i G_{i, k_i}) = (\Lambda_T \oplus \mathfrak{t}_T) / \Pi_T \times \prod_i Z(G_{i, k_i}),$$

$\pi_1 \mathcal{Z}\tilde{G}_{(J, \{k_i\})} = U(1)$, and quadratic form given by the pointwise product $q = q_J \times \prod_i q_{k_i}$. The finite central subgroup $Z = \ker \pi$ lifts uniquely to $\pi_0 \mathcal{Z}\tilde{G}$: it must land in the maximal

compact subgroup, which injects into \tilde{G} as $\pi_0 \text{Cpct } \mathcal{L}T_J \times \Pi_i Z(G_i) \hookrightarrow T \times \Pi_i Z(G_i)$ by Lemma 3.5.0.3.

Choose a maximal torus $\tilde{T} \hookrightarrow \tilde{G}$, then $\tilde{T}/Z \hookrightarrow \tilde{G}/Z = G$ is a maximal torus for G . The Lie algebras of these two tori may both be identified with their common universal cover, which we denote $\tilde{\mathfrak{t}}$. We write $\tilde{\Pi}$ for the fundamental group of \tilde{T} , embedded as a lattice in $\tilde{\mathfrak{t}}$. The fundamental group $\tilde{\Pi}_Z$ of \tilde{T}/Z is an extension

$$\tilde{\Pi} \hookrightarrow \tilde{\Pi}_Z \twoheadrightarrow Z.$$

Considered as a lattice in $\tilde{\mathfrak{t}}$, it is the preimage of $Z \subset \tilde{T}$ in $\tilde{\mathfrak{t}}$:

$$\begin{array}{ccc} \tilde{\Pi}_Z & \hookrightarrow & \tilde{\mathfrak{t}} \\ \downarrow & \lrcorner & \downarrow \\ Z & \hookrightarrow & \tilde{T}. \end{array}$$

The Weyl groups of \tilde{G} and G under these choices of maximal tori are canonically isomorphic and the Weyl-actions on $\tilde{\mathfrak{t}}$ agree.

Recall that we identified $H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ with the group of symmetric bilinear forms I on $\tilde{\mathfrak{t}}$ which are Weyl-invariant and satisfy $I(\tilde{\pi}, \tilde{\pi}) \in 2\mathbb{Z}$ for all $\tilde{\pi} \in \tilde{\Pi}$. Under this identification with bilinear forms, the map $\pi^* : H^4(\mathcal{B}G, \mathbb{Z}) \rightarrow H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ restricts the bilinear form along $\tilde{\pi}$ (see also [Hen17a]). The quotient map π induces the identity on the Lie algebra $\tilde{\mathfrak{t}}$, and preserves the Weyl-action. Hence, the image of the map $\pi^* : H^4(\mathcal{B}G, \mathbb{Z}) \rightarrow H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ consists precisely of those symmetric bilinear forms on $\tilde{\mathfrak{t}}$ that satisfy the even integrality condition not only for $\tilde{\Pi}$, but for $\tilde{\Pi}_Z$.

Now consider the unique lift of Z to $\pi_0 \mathcal{L}\tilde{G}_{\pi^*k}$. The quadratic form on $\pi_0 \mathcal{L}\tilde{G}_{\pi^*k}$ restricts to

$$\begin{aligned} q : Z &\rightarrow U(1) \\ z &\mapsto \exp \frac{1}{2} I(\tilde{z}, \tilde{z}). \end{aligned}$$

It vanishes on $z \in Z$ if and only if $I(\tilde{z}, \tilde{z}) \in 2\mathbb{Z}$ for all lifts $\tilde{z} \in \tilde{\mathfrak{t}}$. As discussed above, the preimage of Z under π is precisely $\tilde{\Pi}_Z$, so this is equivalent to the integrality condition for $I \in H^4(\mathcal{B}G, \mathbb{Z})$ and we arrive at the following:

Lemma 3.6.2.2. *Let $Z \hookrightarrow \tilde{G} \twoheadrightarrow G$ be as above. Then $\tilde{k} \in H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ is in the image of the map $H^4(\mathcal{B}G, \mathbb{Z}) \hookrightarrow H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ if and only if the quadratic form on $\pi_0 \mathcal{L}\tilde{G}_{\tilde{k}}$ vanishes when restricted along the unique lift $Z \hookrightarrow \pi_0 \mathcal{L}\tilde{G}_{\tilde{k}}$.*

A description of the maps $H^4(\mathcal{B}G, \mathbb{Z}) \hookrightarrow H^4(\mathcal{B}\tilde{G}, \mathbb{Z})$ for quotients $G = \tilde{G}/Z$ of simple simply-connected Lie groups may be found in [GW09, Table 1].

Consider the closed subgroup

$$Z^\perp := \left\{ x \in \pi_0 \mathcal{L}\tilde{G} \mid q(x+z) = q(x), \forall z \in Z \right\}$$

of objects in $\mathcal{L}\tilde{G}$ that braid trivially with every element of Z . By abuse of notation, we also denote by this the braided smooth sub-2-group of $\mathcal{L}\tilde{G}$ obtained by pulling back along

the inclusion $Z^\perp \hookrightarrow \pi_0 \mathcal{L}\tilde{G}$. By Lemma 3.6.2.2, q vanishes on Z , so in particular $Z \subset Z^\perp$. The quotient Z^\perp/Z inherits a smooth structure, because Z^\perp is an embedded Lie subgroup of $\pi_0 \mathcal{L}\tilde{G}$, and Z is a normal subgroup of Z^\perp .

Theorem 3.6.2.3. *The Drinfel'd centre of G_k is the smooth braided categorical group*

$$\mathcal{L}G_k = (Z^\perp/Z, q|_{Z^\perp}).$$

Proof. Let $(a, \gamma) \in \mathcal{L}G_k$ be a centre piece for G_k . Pick a lift $\bar{a} \in Z(\tilde{G})$ of $a \in Z(G)$ against $\pi : \tilde{G} \rightarrow G$. Pulling back the half-braiding, we obtain a new centre piece $(\bar{a}, \pi^*\gamma) \in \mathcal{L}\tilde{G}_{\pi^*k}$. Indeed, the hexagon equation $d\gamma = \omega(-, -|a)$ is preserved under pullback. The lifts of a form a Z -torsor, and the induced quadratic form q on the group of lifts is invariant under the Z -action: $q(\bar{a} + z) = \pi^*\gamma_{\bar{a}+z} = \gamma_a = q(\bar{a})$ for all $z \in Z$. This construction extends to a smooth functor $\mathcal{L}G_k \rightarrow Z^\perp/Z$, which preserves the quadratic form. We now build the inverse functor.

The pullback cocycle $\pi^*\omega$ representing the associator on \tilde{G}_{π^*k} can be chosen to be equivariant under translation by Z in all variables. The hexagon equation implies that for any centre piece $(\bar{a}, \gamma) \in \mathcal{L}\tilde{G}_{\pi^*k}$, $z \in Z$, and $\bar{x}, \bar{x}' \in \tilde{G}$,

$$\gamma_{\bar{x}+z}/\gamma_{\bar{x}} = \gamma_{\bar{x}'+z}/\gamma_{\bar{x}'}$$

The projection $\pi : \tilde{G} \rightarrow G$ defines a simplicial cover $\pi_\bullet : \mathcal{B}\tilde{G}_\bullet \rightarrow \mathcal{B}G_\bullet$, and γ is a Segal-Mitchison cochain for G defined with respect to this cover. One may check that γ satisfies the cocycle condition precisely if it is Z -equivariant: $\gamma_{\bar{x}+z} = \gamma_{\bar{x}}$ for all $\bar{x} \in \tilde{G}, z \in Z$. By the above equation, it is in fact enough to check it for a single $\bar{x} \in \tilde{G}$. The Z -equivariance of $\pi^*\omega$ implies that any Z -translate $\bar{a} + z$ of \bar{a} admits γ as a half-braiding. The quadratic form q sends $(\bar{a}, \gamma) \mapsto \gamma_{\bar{a}}$ and $(\bar{a} + z, \gamma) \mapsto \gamma_{\bar{a}+z}$. But $(\bar{a}, \gamma) \in Z^\perp$ implies $\gamma_{\bar{a}+z} = \gamma_{\bar{a}}$. Hence, the functor $Z^\perp \rightarrow \mathcal{L}G_k$ descends to a smooth functor $Z^\perp/Z \rightarrow \mathcal{L}G_k$, which is manifestly inverse to the construction in the first half of the proof. \square

REMARK 3.6.2.4. The above calculation admits a more abstract description in the setting of additive monoidal categories. Denote by \mathbf{C}^\oplus the direct-sum completion of a (smooth) monoidal category \mathbf{C} . Then G_k^\oplus is a monoidal module category over $\tilde{G}_{\pi^*k}^\oplus$, obtained by taking modules over the algebra corresponding to $Z \subset \tilde{G}$. The category of modules over an algebra receives a monoidal structure precisely when the algebra is commutative, which happens here if and only if the quadratic form vanishes on Z . The Drinfel'd centre of the category \mathbf{C}_A of modules over a commutative algebra $A \in \mathcal{L}\mathbf{C}$ was computed in [Sch01] (under completeness conditions which are satisfied here): it is the category $\mathcal{L}\mathbf{C}_A = \text{loc}_A \mathcal{L}\mathbf{C}$ of *local* A -modules (Definition 1.1.3.4) in $\mathcal{L}\mathbf{C}$. In the case at hand, $\mathcal{L}G_k^\oplus = \text{loc}_Z \mathcal{L}\tilde{G}_{\pi^*k}^\oplus = (Z^\perp/Z, q|_{Z^\perp})^\oplus$. The statement about 2-groups can be recovered by restricting to simple objects.

EXAMPLE 3.6.2.5. Let $G = \text{SO}(4)$. It is the quotient $\text{Spin}(4)/Z$, where $Z = \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 = Z(\text{Spin}(4))$ is the diagonal copy of $\mathbb{Z}/2$ in the centre (under the decomposition $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$). We use Theorem 3.6.2.3 to compute the Drinfel'd centre of G_k , for $k \in H^4(\mathcal{B}\text{SO}(4), \mathbb{Z}) \subset H^4(\mathcal{B}\text{Spin}(4), \mathbb{Z})$. The isomorphism $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$ allows us to identify the relevant cohomology group as freely generated by the

second Chern class in each factor: $H^4(\mathcal{B}\text{Spin}(4), \mathbb{Z}) = H^4(\mathcal{B}\text{SU}(2) \times \mathcal{B}\text{SU}(2), \mathbb{Z}) = \mathbb{Z}\langle c_l, c_r \rangle$. We write the cohomology class of the associator in this basis as $\underline{k} = k_l \cdot c_l + k_r \cdot c_r$. By Theorem 3.6.1.3, the Drinfel'd centre of $\text{Spin}(4)$ at level \underline{k} is given by

$$\mathcal{Z}\text{Spin}(4)_{\underline{k}} = \left(\begin{array}{l} \mathbb{Z}/2 \times \mathbb{Z}/2, q : \\ \quad (-1, 1) \mapsto \exp \frac{k_l}{4} \\ \quad (1, -1) \mapsto \exp \frac{k_r}{4} \\ \quad (-1, -1) \mapsto \exp \frac{k_l + k_r}{4} \end{array} \right).$$

The quadratic form q is trivial on Z iff $k_l + k_r \equiv 0 \pmod{4}$. This equation cuts out the subspace of cohomology classes in the image of $H^4(\mathcal{B}\text{SO}(4), \mathbb{Z}) \hookrightarrow H^4(\mathcal{B}\text{Spin}(4), \mathbb{Z})$. The Drinfel'd centre of G_{k_l, k_r} is generated by $[(-1, 1)] = [(1, -1)] \in Z(\text{Spin}(4))/Z$ if $(-1, 1) \in Z^\perp$, otherwise it is trivial. This is equivalent to the condition that k_l be even. In conclusion (see Notation 3.4.2.5),

$$\mathcal{Z}\text{SO}(4)_{k_l, k_r \in 4\mathbb{Z} - k_l} = \begin{cases} \mathbf{Vec}_{\mathbb{Z}/2}^\times & k_l \equiv 0 \pmod{4} \\ \mathbf{sVec}^\times & k_l \equiv 2 \pmod{4} \\ \mathbf{Vec}^\times & \text{else.} \end{cases}$$

The computations in [ČV98] relate the classes c_l and c_r to the more familiar first Pontryagin class p_1 and Euler class χ , which generate $H^4(\mathcal{B}\text{SO}(4), \mathbb{Z})$. This allows the rephrasing of the above result that we gave in Section 3.1.3

3.6.3 Comparison to Loop Group Representations

We end this chapter by comparing $\mathcal{Z}G_k$ (as computed by Theorems 3.6.1.3 and 3.6.2.3) to $(\mathbf{Rep}^k LG)^\times$, the maximal sub-2-group of the category of positive energy representations of the loop group.

Let G be a semisimple compact connected Lie group, ie. a Lie group of the form $G = (\prod_i G_i)/Z$, where all G_i are compact simple simply-connected Lie groups and Z is a finite central subgroup of the product. Let $k \in H^4(\mathcal{B}G, \mathbb{Z})$ be a cohomology class corresponding to a positive-definite bilinear form I (under the identification made in Section 3.6). Below, we show that the braided 2-groups computed in the above subsections are equal to $(\mathbf{Rep}^k LG)^\times$, as long as we explicitly exclude any factors of E_8 at level $k = 2$.

We use the model of $\mathbf{Rep}^k LG$ as representations of a unitary vertex operator algebra: $\mathbf{Rep}V_{G,k}$ – see [Hen17a, Eq (2)] for a definition of $V_{G,k}$ and its modules and [Hen17b, Sect 3] for a discussion of alternative models of $\mathbf{Rep}^k LG$ and their relation. More than just tensor categories, these categories are *unitary modular tensor categories* [Gui19]. When we write $(\mathbf{Rep}^k LG)^\times$, we mean the maximal *unitary* sub-2-group: we only retain \otimes -invertible objects and unitary morphisms. This way, we obtain a 2-group with

$$\pi_1(\mathbf{Rep}^k LG)^\times = U(1)$$

(rather than \mathbb{C}^\times). By Lemma 3.4.1.6, the fundamental group $\pi_1 \mathcal{Z}G_k = U(1)$ agrees with that of $(\mathbf{Rep}^k LG)^\times$. It remains to compare π_0 and the induced quadratic form. We will need the following result.

Lemma 3.6.3.1. *Let X be an invertible object in a unitary modular tensor category. Its self-braiding $\beta_{X,X}$ (considered as a complex number via the canonical isomorphism $\text{End}(X \otimes X) = \mathbb{C}$ sending $\text{id}_{X \otimes X} \mapsto 1$) is equal to its ribbon twist θ_X .*

Proof. This is best understood using string diagrams. The values of $\beta_{X,X}$ and θ_X are captured by the following equations (all strands are coloured with the object X).

$$\begin{array}{c} \text{Diagram 1: Crossing of two strands} \\ \text{Diagram 2: Ribbon twist of a strand} \end{array} = \beta_{X,X} \cdot \text{Vertical Line} \quad \text{and} \quad \begin{array}{c} \text{Diagram 3: Crossing of two strands} \\ \text{Diagram 4: Ribbon twist of a strand} \end{array} = \theta_X \cdot \text{Vertical Line}$$

The left hand sides of these equations become equal upon taking the trace – they both form the figure eight. The traces of the right hand sides must thus also be equal:

$$\beta_{X,X} \cdot \text{Tr}(\text{id}_{X \otimes X}) = \theta_X \cdot \text{Tr}(\text{id}_X).$$

In a unitary tensor category, $\text{Tr}(\text{id}_Y) = 1$ if Y is invertible. Invertibility of X and $X \otimes X$ now implies $\beta_{X,X} = \theta_X$. \square

Armed with this Lemma, we proceed as before: we first compute the simply-connected case, and then deal with quotients.

Theorem 3.6.3.2. *For a compact simply-connected Lie group G and positive-definite level $k \in H^4(\mathcal{B}G, \mathbb{Z})$, there is a braided equivalence*

$$\mathcal{L}G_k \simeq (\mathbf{Rep}V_{G,k})^\times,$$

unless G contains a factor of E_8 to which k restricts as level 2.

Proof. The group of invertible objects of $\mathbf{Rep}V_{G,k}$ was identified as

$$\pi_0(\mathbf{Rep}V_{G,k})^\times = Z(G)$$

in [Li01, Prop 2.20] (see also [Hen17a, Prop 7]). This only fails when G contains a factor of E_8 and k restricts to level 2 on that factor. The category $\mathbf{Rep}V_{E_8,2}$ contains an invertible object of order 2, despite $Z(E_8)$ being trivial. Excluding this case, $(\mathbf{Rep}V_{G,k})^\times$ is tensor equivalent to the underlying 2-group of $\mathcal{L}G_k$, and it remains to compare the self-braidings.

By Lemma 3.6.3.1, the self-braiding on an invertible object is equal to its ribbon twist. The values of these are well known: On an invertible object corresponding to $z \in Z(G)$,

$$\beta_{z,z} = \theta_z = \exp\left(\frac{1}{2}I(\bar{z}, \bar{z})\right),$$

where \bar{z} again denotes a lift of z to a Cartan of G (see [Hen17a, Prop 10]). This formula is the same as that computed for simply-connected groups in Theorem 3.6.1.3 and Lemma 3.6.2.1: the quadratic form on $\pi_0(\mathbf{Rep}^k LG)^\times$ agrees with that on $\pi_0 \mathcal{L}G_k$. \square

To show the above result also holds for non-simply-connected semisimple compact Lie groups G , we use the relation between $(\mathbf{Rep}V_{G,k})^\times$ and $(\mathbf{Rep}V_{\tilde{G},\tilde{k}})^\times$, where \tilde{G} denotes the universal cover of G and \tilde{k} the pullback cohomology class. Recall that G is of the form \tilde{G}/Z where $Z \hookrightarrow \tilde{G}$ is a finite central subgroup.

Theorem 3.6.3.3. *Let $G = \tilde{G}/Z$ be a semisimple compact connected Lie group and $k \in H^4(\mathcal{B}G, \mathbb{Z})$ positive-definite (such that \tilde{G} contains no factor of E_8 to which k pulls back as level 2). Then there is a braided equivalence*

$$\mathcal{L}G_k = (\mathbf{Rep}V_{G,k})^\times.$$

Proof. By [HKL15, Thm 3.4] and [CKM17], there is a braided equivalence

$$\mathbf{Rep}V_{G,k} = \mathrm{loc}_Z \mathbf{Rep}V_{\tilde{G},\tilde{k}},$$

where the right-hand side is the category of local Z -modules in $\mathbf{Rep}V_{\tilde{G},\tilde{k}}$. (We recalled this notion in Definition 1.1.3.4.) The maximal sub-2-group of the right-hand side may be computed by first restricting to the additive subcategory on invertible objects, taking local modules there, and then taking the maximal sub-2-group. (See also Remark 3.6.2.4 for a computation of $\mathcal{L}G_k$ using this method.) The additive subcategory on invertibles is a pointed braided fusion category with simple objects $Z(\tilde{G})$ and braiding described by a quadratic form q (which agrees with that on $\mathcal{L}\tilde{G}_{\tilde{k}}$ by Theorem 3.6.3.2). The category of local Z -modules is described in [DS18, Sect 3.1]: the local modules form a pointed braided fusion category with group of simple objects Z^\perp/Z (where Z^\perp is defined as before Theorem 3.6.2.3). The quadratic form on Z^\perp/Z is given by the restriction of q to Z^\perp . Thus the maximal sub-2-group is

$$(\mathrm{loc}_Z \mathbf{Rep}V_{\tilde{G},\tilde{k}})^\times = (Z^\perp/Z, q|_{Z^\perp}).$$

This agrees with the result of Theorem 3.6.2.3, completing the comparison. \square

Appendix A

Twisted Sheaf Theory for Étale Stacks

A smooth map of manifolds $f : M \rightarrow N$ induces an adjunction of C^∞ -modules

$$f^* : \mathbf{Shv}_{C^\infty}(N) \rightleftarrows \mathbf{Shv}_{C^\infty}(M) : f_*$$

We study the analogous situation, generalised in two directions: We work over *étale stacks* instead of just manifolds, and we work with twisted sheaves rather than ordinary sheaves of C^∞ -modules. We generalise the functors f_* and f^* to this setup, and show that they are again adjoint functors (Proposition A.4.0.4).

Étale stacks and their categories of (non-twisted) sheaves were previously studied in [Car10; Car11; Car13]. Treatments of twisted sheaves over manifolds and schemes may be found in [C00] and [Lie07; Lie08], but we could not find a proof of the pullback-pushforward adjunction in these references.

The authors of [BSS07] introduce a derived pullback-pushforward adjunction of sheaves over a gerbe. However, their notion of sheaves over a gerbe differs from that of sheaves twisted by that gerbe.

NOTATION A.0.0.1. We denote by \mathbf{Vec}^∞ the category of vector spaces over \mathbb{C} , and by $\mathbf{Vec} \subset \mathbf{Vec}^\infty$ its subcategory of finite-dimensional vector spaces. We write \mathbf{VCat} for the 2-category of \mathbf{Vec}^∞ -enriched categories that are complete under finite direct sums. We use the word sheaf to refer to a \mathbf{Vec}^∞ -valued sheaf. The sheaf C^∞ is the sheaf of smooth \mathbb{C} -valued functions over the site \mathbf{Man} . We freely use the term *line bundle* to refer to a sheaf of C^∞ -modules which is locally equivalent to C^∞ . In this chapter, we use the word stack to refer to a stack over the site \mathbf{Man} of smooth manifolds, equipped with the surjective submersion topology. We will be concerned with both stacks valued in groupoids and stacks valued in \mathbf{VCat} . Given a map $Y \twoheadrightarrow X$, we will denote the iterated fibre products $Y \times_X Y \cdots Y$ by $Y^{[n]}$, and the simplicial projection maps by $p_{ij\dots k} : Y^{[m]} \rightarrow Y^{[n]}$.

A.1 The Étale Site of an Étale Stack

Definition A.1.0.1. A map of stacks $\mathcal{Y} \rightarrow \mathcal{X}$ is *representable* if for any map $U \rightarrow \mathcal{X}$ whose codomain is a manifold, the pullback $U \times_{\mathcal{X}} \mathcal{Y}$ is representable by a manifold. We call $\mathcal{Y} \rightarrow \mathcal{X}$ *representably étale/submersive/surjective* if it is representable and, for any

map $U \rightarrow \mathcal{X}$ with domain a manifold, the pullback map $U \times_{\mathcal{X}} \mathcal{Y} \rightarrow U$ is étale/submersive/surjective as a map of manifolds.¹

The pasting law for pullback squares implies that these properties are preserved under pullback along any map of stacks $\mathcal{Z} \rightarrow \mathcal{X}$. Using further that the property of being étale/submersive/surjective is preserved under composition, it is easy to see that the set of representably étale/submersive/surjective maps are closed under composition.

Definition A.1.0.2. An *atlas* for a stack \mathcal{X} is a representably surjective submersion $Y \rightarrow \mathcal{X}$ whose codomain Y is a manifold. If the map is further representably étale, we call it an *étale atlas*.

Definition A.1.0.3. A *differentiable stack* is a stack which admits an atlas. A differentiable stack is an *étale stack* if the atlas may be chosen to be étale.

NOTATION A.1.0.4. We denote the bicategory of differentiable stacks by \mathbf{DiffSt} and that of étale stacks by \mathbf{EtSt} .

EXAMPLE A.1.0.5. Every manifold M is an étale stack via the identity atlas $M \rightarrow M$.

EXAMPLE A.1.0.6. Let X be a manifold with an action by a Lie group G . The quotient stack $[X/G]$ admits an atlas $X \rightarrow [X/G]$, and is thus differentiable. The map $X \rightarrow [X/G]$ is an étale atlas iff G is discrete. However, $[X/G]$ may be an étale stack even if G is nondiscrete: let $X = G$ with the usual right action, then the quotient stack is $[G/G] \simeq *$.

A differentiable stack \mathcal{X} has an associated bicategory $\mathbf{DiffSt}/\mathcal{X}$, the overcategory of \mathcal{X} . Its objects are maps

$$\mathcal{Y} \xrightarrow{y} \mathcal{X},$$

its 1-morphisms $y \rightarrow y'$ are pairs (f, ϕ) as below

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}' \\ & \searrow y & \swarrow y' \\ & & \mathcal{X} \end{array} \quad \begin{array}{c} \phi \\ \longleftarrow \\ \longleftarrow \end{array}$$

and its 2-morphisms $(f, \phi) \rightarrow (f', \phi')$ are 2-cells

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}' \\ & \searrow \alpha \downarrow & \swarrow f' \\ & & \mathcal{Y}' \end{array}$$

such that $\phi = \phi' \circ \alpha$.

We now build the étale overcategory of an étale stack \mathcal{M} in two steps. First, we restrict to the full subcategory of $\mathbf{DiffSt}/\mathcal{M}$ on objects $y : \mathcal{Y} \rightarrow \mathcal{M}$ where y is representably étale. Then we form the homotopy 1-category, modding out by 2-cells. We spell out the result in the following definition.

¹We use the word *étale* to mean a local diffeomorphism.

Definition A.1.0.7. The *étale overcategory* $\widetilde{\text{Et}}(\mathcal{M})$ of an étale stack \mathcal{M} has

- objects: representably étale maps $y : \mathcal{X} \rightarrow \mathcal{M}$
- morphisms ($y \rightarrow y'$): pairs $(f, \phi) : y \rightarrow y'$ as above, modulo the equivalence relation generated by 2-cells $-(f, \phi) \sim (f', \phi')$ if there exists a 2-cell $\alpha : f \rightarrow f'$ such that

$$\begin{array}{ccc}
 \mathcal{Y} & \xrightarrow{f} & \mathcal{Y} \\
 \swarrow y & \xleftarrow{\phi} & \searrow y' \\
 & \mathcal{X} &
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{Y} & \xrightarrow{f} & \mathcal{Y} \\
 \swarrow y & \xleftarrow{f'} & \searrow y' \\
 & \mathcal{X} &
 \end{array}
 \begin{array}{c}
 \downarrow \alpha \\
 \downarrow \alpha \\
 \downarrow \alpha
 \end{array}$$

We equip the étale overcategory with the structure of a site by declaring covers of $y : \mathcal{Y} \rightarrow \mathcal{M}$ to be collections $\{(f_i, \phi_i) : y_i \rightarrow y\}_{i \in I}$ such that the map $\coprod f_i$ is a representably surjective submersion.

The property of being representably étale may be checked on an étale atlas:

Lemma A.1.0.8. *A map $f : \mathcal{X} \rightarrow \mathcal{Y}$ of étale stacks is representably étale if there exists an étale atlas $Y \rightarrow \mathcal{Y}$ such that $Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow Y$ is an étale map of manifolds.*

Proof. Let $g : U \rightarrow \mathcal{Y}$ be a map of stacks with codomain a manifold. We must show that the map $g^*f : U \times_{\mathcal{Y}} \mathcal{X} \rightarrow U$ is étale. Consider the cube of pullback squares

$$\begin{array}{ccccc}
 & & U \times_{\mathcal{Y}} X & \longrightarrow & X \\
 & \swarrow & \downarrow & & \downarrow \\
 U \times_{\mathcal{Y}} Y & \longleftarrow & Y & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 & \swarrow & U \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
 & \swarrow & \downarrow & & \downarrow \\
 U & \longleftarrow & \mathcal{Y} & \longrightarrow & \mathcal{Y}
 \end{array}$$

The unlabeled maps in the diagram below

$$\begin{array}{ccc}
 U \times_{\mathcal{Y}} X & \longrightarrow & U \times_{\mathcal{Y}} Y \\
 \downarrow & & \downarrow \\
 U \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{g^*f} & U
 \end{array}$$

are étale because they are pullbacks of representably étale maps. The 2-out-of-3 property for étale maps of manifolds implies that g^*f is étale, which concludes the proof. \square

Lemma A.1.0.9. *Let*

$$\begin{array}{ccc}
 \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}' \\
 \swarrow y & \xleftarrow{\phi} & \searrow y' \\
 & \mathcal{M} &
 \end{array}$$

be a morphism in $\widetilde{\text{Et}}(\mathcal{M})$. Then f is a representably étale map of étale stacks.

Proof. An étale atlas $M \twoheadrightarrow \mathcal{M}$ pulls back to a representably étale map $M \times_{\mathcal{M}} \mathcal{Y} \rightarrow \mathcal{Y}$. As $y : \mathcal{Y} \rightarrow \mathcal{M}$ is representable, $M \times_{\mathcal{M}} \mathcal{Y}$ is a manifold and thus forms an étale atlas for \mathcal{Y} . Hence \mathcal{Y} is an étale stack, and the same is true for \mathcal{Y}' . Pulling the diagram in the statement back along $\pi^* : M \twoheadrightarrow \mathcal{M}$, we obtain

$$\begin{array}{ccc} M \times_{\mathcal{M}} \mathcal{Y} & \xrightarrow{\pi^* f} & M \times_{\mathcal{M}} \mathcal{Y}' \\ & \searrow \pi^* y & \swarrow \pi^* y' \\ & M & \end{array}$$

Both $\pi^* y$ and $\pi^* y'$ are étale by assumption, hence so is $\pi^* f$. But $\pi^* f$ is the pullback of f along the étale atlas map $M \times_{\mathcal{M}} \mathcal{Y}' \rightarrow \mathcal{Y}'$, so we may use Lemma A.1.0.8 to finish the proof. \square

REMARK A.1.0.10. Let $(x : \mathcal{X} \rightarrow \mathcal{M}) \in \widetilde{\text{Et}}(\mathcal{M})$, and consider the overcategory $\widetilde{\text{Et}}(\mathcal{M})/x$. Lemma A.1.0.9 implies that forgetting the map to \mathcal{M} is a functor

$$\widetilde{\text{Et}}(\mathcal{M})/x \rightarrow \widetilde{\text{Et}}(\mathcal{X}).$$

Definition A.1.0.11. The *étale site* of an étale stack \mathcal{M} is the subsite $\text{Et}(\mathcal{M}) \subset \widetilde{\text{Et}}(\mathcal{M})$ on those objects $x : X \rightarrow \mathcal{M}$ where X is a manifold.

Note that there are no 2-cells between two maps $f, f' : X \rightarrow Y'$, so every 1-morphism in $\text{Et}(\mathcal{M})$ admits a unique representative (f, ϕ) , and there is a forgetful functor $\text{Et}(\mathcal{M}) \rightarrow \mathbf{Man}$. The notion of covers on $\text{Et}(\mathcal{M})$ agrees with the one pulled back from \mathbf{Man} via this functor: A collection $\{(f_i, \phi_i) : x_i \rightarrow x\}_{i \in I}$ is a cover in $\text{Et}(\mathcal{M})$ iff the map $\coprod f_i$ is a surjective submersion. This is equivalent to asking $\{f_i\}_{i \in I}$ to be a cover in \mathbf{Man} . By Lemma A.1.0.9, all the maps f_i are automatically étale, so this reduces to the condition that the union be surjective.

Definition A.1.0.12. Let $Y \twoheadrightarrow \mathcal{M}$ be an étale atlas. We denote the *site associated to the atlas Y* by $\widetilde{\text{Et}}(Y/\mathcal{M})$. Its objects are étale maps $(\tilde{x} : \mathcal{X} \rightarrow Y)$ with source an étale stack \mathcal{X} , and morphisms $\tilde{x} \rightarrow \tilde{x}'$ are morphisms of the composites

$$x : \mathcal{X} \xrightarrow{\tilde{x}} Y \twoheadrightarrow \mathcal{M}$$

in $\widetilde{\text{Et}}(\mathcal{M})$. We denote by $\text{Et}(Y/\mathcal{M})$ the full subsite on objects $(\tilde{x} : X \rightarrow Y) \in \widetilde{\text{Et}}(Y/\mathcal{M})$ where the source X is a manifold.

These sites $\text{Et}(Y/\mathcal{M})$ are built so as to make the functors $\text{Et}(Y/\mathcal{M}) \rightarrow \text{Et}(\mathcal{M})$ and $\widetilde{\text{Et}}(Y/\mathcal{M}) \rightarrow \widetilde{\text{Et}}(\mathcal{M})$ fully faithful. We have introduced 4 sites associated to an étale stack, and now show that for the purposes of studying sheaves and stacks over \mathcal{M} , we can freely move between these sites.

Lemma A.1.0.13. *The functors in the diagram below*

$$\begin{array}{ccc} & \widetilde{\text{Et}}(Y/\mathcal{M}) & \\ \swarrow & \nearrow & \searrow \\ \text{Et}(Y/\mathcal{M}) & & \widetilde{\text{Et}}(\mathcal{M}) \\ \searrow & \nearrow & \\ & \text{Et}(\mathcal{M}) & \end{array}$$

are inclusions of dense subsites.

Proof. As all the functors are fully faithful inclusions of subsites, it suffices to prove $\text{Et}(Y/\mathcal{M})$ is dense in $\widetilde{\text{Et}}(\mathcal{M})$. We need to check that every object $(x : \mathcal{X} \rightarrow \mathcal{M}) \in \widetilde{\text{Et}}(\mathcal{M})$ admits a cover by objects in $\text{Et}(Y/\mathcal{M})$. Form the pullback of $Y \rightarrow \mathcal{M}$ with x . Then the map $Y \times_{\mathcal{M}} \mathcal{X} \rightarrow \mathcal{X}$ is an étale atlas for \mathcal{X} . Indeed, the map is representably étale because it is pulled back from the representably étale map $Y \rightarrow \mathcal{M}$, and the fibre product is a manifold because x is representably étale. In particular, $Y \times_{\mathcal{M}} \mathcal{X} \rightarrow Y$ is an object in $\text{Et}(Y/\mathcal{M})$, and together with the 2-cell ϕ

$$\begin{array}{ccc} Y \times_{\mathcal{M}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & \searrow \phi & \downarrow x \\ Y & \longrightarrow & \mathcal{M} \end{array}$$

provided by the universal property of the pullback, it covers $x : \mathcal{X} \rightarrow \mathcal{M}$ in $\widetilde{\text{Et}}(\mathcal{M})$. \square

By the Comparison Lemma (Lemma 1.2.1.3), this means that sheaf and stack categories over all these sites are canonically equivalent. We henceforth refer to sheaves/stacks over any of these sites as sheaves/stacks over \mathcal{M} .

We equip \mathcal{M} with the (\mathbf{Vec}^∞ -valued) structure sheaf C^∞ of smooth \mathbb{C} -valued functions. This sheaf is pulled back from \mathbf{Man} via the forgetful functor $\text{Et}(\mathcal{M}) \rightarrow \mathbf{Man}$.

NOTATION A.1.0.14. We write $\mathbf{Shv}(\mathcal{M})$ to denote the category of \mathbf{Vec}^∞ -valued sheaves over \mathcal{M} and $\mathbf{Shv}_{C^\infty}(\mathcal{M})$ to denote its category of C^∞ -modules.

In what follows, we show that $\widetilde{\text{Et}}(\mathcal{M})$ admits certain limits. To clarify the following section, we explicitly describe the category of maps $\mathcal{W} \rightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ from a stack \mathcal{W} into a fibre product of stacks: An object of $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(\mathcal{W})$ is the datum of three maps $f_{\mathcal{X}}, f_{\mathcal{Y}}, f_{\mathcal{Z}}$ and two 2-cells $\phi_{\mathcal{X}}, \phi_{\mathcal{Y}}$ as below.

$$\begin{array}{cccc} \mathcal{W} & \mathcal{W} & \mathcal{W} & \mathcal{W} \longrightarrow \mathcal{Y} \\ \downarrow f_{\mathcal{X}} & \downarrow f_{\mathcal{Y}} & \downarrow f_{\mathcal{Z}} & \downarrow \phi_{\mathcal{X}} \searrow \phi_{\mathcal{Y}} \downarrow \\ \mathcal{X} & \mathcal{Y} & \mathcal{Z} & \mathcal{X} \longrightarrow \mathcal{Z} \end{array}$$

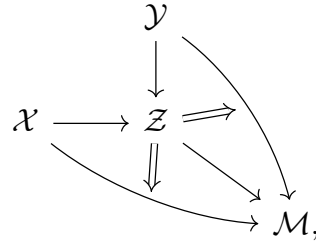
A morphism between two objects is a triple of 2-cells $\alpha_{\mathcal{X}/\mathcal{Y}/\mathcal{Z}} : f_{\mathcal{X}/\mathcal{Y}/\mathcal{Z}} \rightarrow f'_{\mathcal{X}/\mathcal{Y}/\mathcal{Z}}$ such that both

$$f_{\mathcal{X}} \left(\begin{array}{ccc} \mathcal{W} & & \\ \downarrow \alpha_{\mathcal{X}} & \searrow f'_{\mathcal{Z}} & \\ \mathcal{X} & \xrightarrow{\phi'_{\mathcal{X}}} & \mathcal{Z} \end{array} \right) = \begin{array}{ccc} \mathcal{W} & \xrightarrow{f'_{\mathcal{Z}}} & \mathcal{Y} \\ \downarrow f_{\mathcal{X}} & \searrow \alpha_{\mathcal{Z}} \nearrow & \downarrow \\ \mathcal{X} & \xrightarrow{\phi_{\mathcal{X}}} & \mathcal{Z} \end{array}$$

and the analogous equation involving $\alpha_{\mathcal{Y}}, \phi_{\mathcal{Y}}$ and $\phi'_{\mathcal{Y}}$ hold. The canonical morphisms from $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ to \mathcal{X}, \mathcal{Y} and \mathcal{Z} are the obvious forgetful maps.

REMARK A.1.0.15. One may also describe the category $\text{Hom}(\mathcal{W}, \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ as consisting of pairs $\mathcal{W} \rightarrow \mathcal{X}, \mathcal{W} \rightarrow \mathcal{Y}$, equipped with a single 2-cell relating the two resulting composites $\mathcal{W} \rightarrow \mathcal{Z}$. There is a map from the stack as we described it above to this one, given by forgetting the data $\mathcal{W} \rightarrow \mathcal{Z}$ and composing the 2-cells $\phi_{\mathcal{X}}$ and $\phi_{\mathcal{Y}}$ to obtain a single 2-cell. It is straightforward to check that this map is an equivalence.

Let $\mathcal{X} \rightarrow \mathcal{M}$ and $\mathcal{Y} \rightarrow \mathcal{M}$ be objects in $\widetilde{\text{Et}}(\mathcal{M})$. Then $\mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \rightarrow \mathcal{M}$ is again an object of $\widetilde{\text{Et}}(\mathcal{M})$, because the property of being representably étale is preserved under pullback and closed under composition. Similarly, given a pair of maps with common target in $\widetilde{\text{Et}}(\mathcal{M})$



the pullback $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{M}$ is an object in $\widetilde{\text{Et}}(\mathcal{M})$. The map $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{X}$ is representably étale because it is pulled back from $\mathcal{Y} \rightarrow \mathcal{Z}$, which is representably étale by Lemma A.1.0.9. The map to \mathcal{M} is (equivalent to the map) obtained from it by composing with the representably étale map $\mathcal{X} \rightarrow \mathcal{M}$.

Lemma A.1.0.16. *The site $\widetilde{\text{Et}}(\mathcal{M})$ has binary products and pullbacks, computed by*

$$\begin{array}{ccc} \mathcal{X} & & \mathcal{Y} \\ \downarrow & \times & \downarrow \\ \mathcal{M} & & \mathcal{M} \end{array} \simeq \begin{array}{c} \mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

and

$$\text{lim} \begin{array}{ccc} & \mathcal{Y} & \\ & \downarrow g & \\ \mathcal{X} & \xrightarrow{f} \mathcal{Z} & \rightrightarrows \beta \\ & \downarrow \alpha & \\ & \mathcal{M} & \end{array} \simeq \begin{array}{c} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

respectively.

Proof. We begin by proving the statement for binary products: A morphism $(\mathcal{W} \rightarrow \mathcal{M}) \rightarrow (\mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \rightarrow \mathcal{M})$ is a pair of a morphism $\mathcal{W} \rightarrow \mathcal{X} \times_{\mathcal{M}} \mathcal{Y}$ and a 2-cell ϕ . Each such morphism is canonically isomorphic (via ϕ itself) to one where the composite $\mathcal{W} \rightarrow \mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \rightarrow \mathcal{M}$ is equal to $\mathcal{W} \rightarrow \mathcal{M}$ and the 2-cell ϕ is the trivial 2-cell. The remaining data is precisely a pair of maps $\mathcal{W} \rightarrow \mathcal{X}, \mathcal{W} \rightarrow \mathcal{Y}$, and a pair of 2-cells between the composites $\mathcal{W} \rightarrow \mathcal{M}$. This identifies $(\mathcal{X} \times_{\mathcal{M}} \mathcal{Y} \rightarrow \mathcal{M})$ as the product.

A morphism $(\mathcal{W} \rightarrow \mathcal{M}) \rightarrow (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{M})$ is equivalent to the data of three maps and three 2-cells as below.

$$\begin{array}{ccc} \mathcal{W} & \mathcal{W} & \mathcal{W} \\ \downarrow & \downarrow & \downarrow \\ \mathcal{X} & \mathcal{Y} & \mathcal{Z} \end{array} \quad \begin{array}{ccc} \mathcal{W} & \xrightarrow{\phi_{\mathcal{Y}}} \mathcal{Y} & \\ \downarrow & \swarrow \phi_{\mathcal{X}} & \downarrow \\ \mathcal{X} & \xrightarrow{\phi_{\mathcal{X}}} \mathcal{Z} & \end{array} \quad \begin{array}{ccc} \mathcal{W} & \xrightarrow{\kappa} \mathcal{Z} & \\ \downarrow & \swarrow \downarrow & \downarrow \\ & \mathcal{M} & \end{array}$$

The universal property of the pullback says that a map into the pullback of $(f, \alpha) : (\mathcal{X} \rightarrow \mathcal{Z})/\mathcal{M}$ with $(g, \beta) : (\mathcal{Y} \rightarrow \mathcal{Z})/\mathcal{M}$ is the data of two maps and two 2-cells

$$\begin{array}{ccc} \mathcal{W} & & \mathcal{W} \\ \downarrow & & \downarrow \\ \mathcal{X} & & \mathcal{Y} \end{array} \quad \begin{array}{ccc} \mathcal{W} & \xrightarrow{\lambda_y} & \mathcal{Y} \\ \downarrow & \swarrow \lambda_x & \downarrow \\ \mathcal{X} & \xrightarrow{\lambda_x} & \mathcal{M} \end{array}$$

such that

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\phi_x} & \mathcal{X} & \xrightarrow{\alpha} & \mathcal{Z} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{M} & & \end{array} \quad \simeq \quad \begin{array}{ccc} \mathcal{W} & \xrightarrow{\phi_y} & \mathcal{Y} & \xrightarrow{\beta} & \mathcal{Z} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{M} & & \end{array}$$

Given the data of a map into $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{M})$ as above, one may obtain such data by forgetting the map $\mathcal{W} \rightarrow \mathcal{Z}$ and setting

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\lambda_y} & \mathcal{Y} \\ \downarrow & \swarrow & \downarrow \\ \mathcal{X} & \xrightarrow{\lambda_x} & \mathcal{M} \end{array} \quad := \quad \begin{array}{ccccc} \mathcal{W} & \xrightarrow{\phi_y} & \mathcal{Y} & & \\ \downarrow & \swarrow & \downarrow & \xrightarrow{\beta} & \\ & & \mathcal{Z} & \xrightarrow{\beta} & \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathcal{X} & \xrightarrow{\phi_x} & \mathcal{M} & & \end{array}$$

It is straightforward to check that the 2-cells satisfy the required condition, and that this assignment is an isomorphism. This identifies $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{M})$ as the pullback. \square

A.2 The Untwisted Pullback-Pushforward Adjunction

Let $f : \mathcal{N} \rightarrow \mathcal{M}$ be a map of étale stacks. It defines a functor

$$\begin{aligned} f^{-1} : \widetilde{\text{Et}}(\mathcal{M}) &\rightarrow \widetilde{\text{Et}}(\mathcal{N}) \\ (\mathcal{X} \rightarrow \mathcal{M}) &\mapsto (\mathcal{X} \times_{\mathcal{M}} \mathcal{N} \rightarrow \mathcal{N}). \end{aligned}$$

Lemma A.2.0.1. *The functor f^{-1} preserves covers, pullbacks and binary products.*

Proof. The functor f^{-1} amounts to computing a pullback in **DiffSt**. Covers are collections of jointly surjective representable submersions, a condition which is preserved under pullback.

By Lemma A.1.0.16, both pullbacks and binary products in $\widetilde{\text{Et}}(\mathcal{M})$ may be computed as pullbacks in **DiffSt**. Limits commute with limits, in particular, pullback squares pull back to pullback squares along f . Hence f^{-1} preserves pullbacks and binary products. \square

Let $y : \mathcal{Y} \rightarrow \mathcal{N}$ be an object in $\widetilde{\text{Et}}(\mathcal{M})$. The arrow category $y \downarrow f^{-1}$ is the category of pairs $(x : \mathcal{X} \rightarrow \mathcal{M}, (g, \phi) : y \rightarrow f^{-1}(x))$. It may equivalently be thought of as the category of equivalence classes of diagrams

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{g} & \mathcal{X} \\ y \downarrow & \swarrow \phi & \downarrow x \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M}, \end{array}$$

where x and f are fixed and the equivalence relation is generated by the obvious action of 2-cells

$$\mathcal{Y} \begin{array}{c} \xrightarrow{g'} \\ \Downarrow \gamma \\ \xrightarrow{g} \end{array} \mathcal{X}.$$

Lemma A.2.0.2. *The category $y \downarrow f^{-1}$ is cofiltered, ie. every finite diagram in $y \downarrow f^{-1}$ admits a cone.*

Proof. As any finite diagram may be built out of finite products and equalisers, it is enough to show that these admit cones. Hence we need only check cones over the empty diagram, binary products, and equalisers.

The category $\widetilde{\text{Et}}(\mathcal{M})$ has both binary products and pullbacks by Lemma A.1.0.16. Lemma A.2.0.1 says that f^{-1} preserves these. Since equalisers may be built out of binary products and pullbacks, the same is true for equalisers.

Giving a cone over the empty diagram amounts to showing $y \downarrow f^{-1}$ is non-empty. The category always contains the object

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f \circ y} & \mathcal{M} \\ y \downarrow & \swarrow \text{id} & \downarrow \text{id} \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M}. \end{array}$$

A cone over a pair of objects is given by their binary product as illustrated below

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \times_{\mathcal{M}} \mathcal{X}' & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{X} & \longrightarrow & \mathcal{X}' \\ & \downarrow y & & \downarrow & \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M} & & \end{array}$$

Similarly, the cone over a parallel pair of morphisms is provided by their equaliser. \square

A classical result of topos theory (see [Joh02, Rem C.2.3.7] or [Shu12, Sec 4]) says that a functor f^{-1} which satisfies the conditions proved in Lemma A.2.0.1 and Lemma A.2.0.2 induces a pair of adjoint functors

$$f_0^* : \mathbf{Shv}(\mathcal{M}) \rightleftarrows \mathbf{Shv}(\mathcal{N}) : f_{*,0}$$

between the categories of \mathbf{Vec}^∞ -valued sheaves.² The pushforward functor $f_{*,0}$ precomposes $\mathcal{F} \in \mathbf{Shv}(\mathcal{N})$ with f^{-1}

$$f_{*,0}\mathcal{F} : (\mathcal{X} \rightarrow \mathcal{M}) \mapsto \mathcal{F}(\mathcal{X} \times_{\mathcal{M}} \mathcal{N})$$

and the pullback f_0^* takes $\mathcal{F}' \in \mathbf{Shv}(\mathcal{M})$ to the sheafification of the presheaf

$$\text{pre}f_0^*\mathcal{F}' : (y : \mathcal{Y} \rightarrow \mathcal{N}) \mapsto \text{colim}_{x \in (y \downarrow f^{-1})^{\text{op}}} \mathcal{F}'(x).$$

By Lemma A.2.0.2, the colimit in the above expression is filtered and thus commutes with finite limits. Sheafification similarly commutes with finite limits [Joh02, Lem A.4.4.6]. As a result, f_0^* not only preserves colimits (as a left adjoint), it also preserves finite limits.

Recall that the category $\mathbf{Shv}(\mathcal{M})$ is equipped with a ring object $C_{\mathcal{M}}^\infty$, which sends an object $(x : \mathcal{X} \rightarrow \mathcal{M}) \in \widetilde{\text{Et}}(\mathcal{M})$ to the ring of smooth \mathbb{C} -valued functions on \mathcal{X} . There is a natural map $f_\# : C_{\mathcal{M}}^\infty \rightarrow f_{*,0}C_{\mathcal{N}}^\infty$ with components

$$f_\#(\mathcal{X} \rightarrow \mathcal{M}) : C^\infty(\mathcal{X}) \rightarrow C^\infty(\mathcal{X} \times_{\mathcal{M}} \mathcal{N})$$

given by restriction of functions along the map $\mathcal{X} \times_{\mathcal{M}} \mathcal{N} \rightarrow \mathcal{X}$.

The data $(f_0^* \dashv f_{*,0}, f_\#) : (\mathbf{Shv}(\mathcal{N}), C_{\mathcal{N}}^\infty) \rightarrow (\mathbf{Shv}(\mathcal{M}), C_{\mathcal{M}}^\infty)$ is known as a *morphism of ringed topoi*. It is a standard result of topos theory [SP, Chapter 03A4]) that a morphism of ringed topoi gives rise to an adjunction of module categories, so we obtain the following.

Proposition A.2.0.3. *A map $f : \mathcal{N} \rightarrow \mathcal{M}$ of étale stacks gives rise to an adjoint pair of functors*

$$f^* : \mathbf{Shv}_{C^\infty}(\mathcal{M}) \rightleftarrows \mathbf{Shv}_{C^\infty}(\mathcal{N}) : f_*.$$

The left adjoint is equipped with a canonical monoidal structure

$$f^*(\mathcal{F} \otimes_{C_{\mathcal{M}}^\infty} \mathcal{F}') \simeq f^*(\mathcal{F}) \otimes_{C_{\mathcal{N}}^\infty} f^*(\mathcal{F}').$$

These functors are defined as follows. Using $f_\#$, the image of a $C_{\mathcal{N}}^\infty$ -module \mathcal{F} under $f_{*,0}$ naturally becomes a $C_{\mathcal{M}}^\infty$ -module. The resulting functor $f_* : \mathbf{Shv}_{C^\infty}(\mathcal{N}) \rightarrow \mathbf{Shv}_{C^\infty}(\mathcal{M})$ has a left adjoint

$$\begin{aligned} f^* : \mathbf{Shv}_{C^\infty}(\mathcal{M}) &\rightarrow \mathbf{Shv}_{C^\infty}(\mathcal{N}) \\ \mathcal{F} &\mapsto C_{\mathcal{N}}^\infty \otimes_{f_0^* C_{\mathcal{M}}^\infty} f_0^* \mathcal{F}. \end{aligned}$$

The tensor product in the expression above involves the morphism $f^\# : f_0^* C_{\mathcal{M}}^\infty \rightarrow C_{\mathcal{N}}^\infty$, which corresponds to $f_\#$ via the adjunction $(f_0^* \dashv f_{*,0})$.

For a map between manifolds $f : M \rightarrow N$ (viewed as a map of étale stacks), the adjunction $(f^* \dashv f_*)$ reduces to the usual adjunction of C^∞ -module categories over manifolds.

²We add the subscript 0 to distinguish these functors from the analogous functors induced on the categories of C^∞ -modules introduced below.

Lemma A.2.0.4. *Consider a diagram*

$$\begin{array}{ccc} & \mathcal{N} & \\ f \nearrow & \Downarrow \gamma & \searrow g \\ \mathcal{M} & \xrightarrow{h} & \mathcal{O} \end{array}$$

of étale stacks. There are natural equivalences

$$g_* \circ f_* \simeq h_* \qquad f^* \circ g^* \simeq h^*.$$

Proof. As adjoints compose, it is enough to show this for the pushforwards. The pushforwards send a sheaf to its postcomposition with $f^{-1}/g^{-1}/h^{-1}$, so it suffices to provide a natural transformation

$$\eta_\gamma : g^{-1} \circ f^{-1} \xrightarrow{\simeq} h^{-1}.$$

Its component $\eta_\gamma(\mathcal{X} \rightarrow \mathcal{M})$ is given by the natural map

$$\mathcal{M} \times_{\mathcal{N}} \mathcal{N} \times_{\mathcal{O}} \mathcal{X} \xrightarrow{\simeq} \mathcal{M} \times_{\mathcal{O}} \mathcal{X}.$$

The 2-cell γ may be used to build the relevant 2-cells between the composites to \mathcal{M} . This makes the above map a morphism in $\widetilde{\text{Et}}(\mathcal{M})$. \square

A.3 The Stack of Twisted Sheaves

Lemma A.3.0.1. *The functor*

$$\begin{aligned} \mathbf{Shv}_{\mathbf{C}^\infty} &: \mathbf{Man}^{\text{op}} \rightarrow \mathbf{VCat} \\ U &\mapsto \mathbf{Shv}_{\mathbf{C}^\infty}(U) \\ (f : U \rightarrow V) &\mapsto (f^* : \mathbf{Shv}_{\mathbf{C}^\infty}(V) \rightarrow \mathbf{Shv}_{\mathbf{C}^\infty}(U)) \end{aligned}$$

is a stack (valued in linear, \oplus -complete categories).

Proof. We check the gluing condition. The functor $\mathbf{Shv}_{\mathbf{C}^\infty}$ clearly sends coproducts to products, so it is enough to check gluing for a surjective submersion $Y \twoheadrightarrow U$. We exhibit an inverse to the functor sending a sheaf to its corresponding descent datum over Y . Given a descent datum $(\mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y), \phi : p_1^* \mathcal{F} \rightarrow p_2^* \mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y^{[2]}))$, we construct a sheaf $\bar{\mathcal{F}} \in \mathbf{Shv}_{\mathbf{C}^\infty}(U)$. Consider the full subcategory $\mathbf{Op}(Y/U) \rightarrow \mathbf{Op}(U)$ on open sets $V \subset U$ equipped with a factorisation

$$V \xrightarrow{v} Y \rightarrow U.$$

It is a dense subsite of $\mathbf{Op}(U)$. The sheaf $\bar{\mathcal{F}}$ assigns to $V \xrightarrow{v} Y$ the global sections of the pullback sheaf $v^* \mathcal{F}$ over V . Given an inclusion $\iota : W \hookrightarrow V$, and two chosen lifts $w : W \rightarrow Y, v : V \rightarrow Y$, there is an induced map $(w, v \circ \iota) : W \rightarrow Y^{[2]}$. The restriction map $\iota^* : \bar{\mathcal{F}}(v) \rightarrow \bar{\mathcal{F}}(w)$ is given by $(w, v \circ \iota)^* \phi$. Functoriality of this assignment is equivalent to the cocycle condition satisfied by ϕ . \square

Now let \mathcal{M} be an étale stack. Recall there is a forgetful functor $\text{Et}(\mathcal{M}) \rightarrow \mathbf{Man}$. We can restrict $\mathbf{Shv}_{\mathbb{C}^\infty}$ along this functor to obtain the stack $\mathbf{Shv}_{\mathbb{C}^\infty}$ over $\text{Et}(\mathcal{M})$. This stack extends to $\widetilde{\text{Et}}(\mathcal{M})$ via the dense inclusion $\text{Et}(\mathcal{M}) \rightarrow \widetilde{\text{Et}}(\mathcal{M})$.

REMARK A.3.0.2. We may describe the resulting stack over $\widetilde{\text{Et}}(\mathcal{M})$ explicitly. For each isomorphism class of morphisms $[(f, \phi)]$, pick a representative (f, ϕ) and send it to the pullback functor f^* . Let $(f, \phi), (g, \kappa)$ be composable morphisms as above, and denote the chosen representative of the class $[(f, \phi) \circ (g, \kappa)]$ by (h, λ) . Then there exists a unique 2-cell $\gamma : f \circ g \rightarrow h$ that identifies $(f, \phi) \circ (g, \kappa) \xrightarrow{\cong} (h, \lambda)$. This 2-cell induces a natural transformation of the associated pullback functors. That natural transformation is the value of the compositor $\eta_{(f, \phi), (g, \kappa)}$ which is part of the data of $\mathbf{Shv}_{\mathbb{C}^\infty}$.

Let \mathcal{X} be a differentiable stack and consider the bicategory $\mathbf{DiffSt}/\mathcal{X}$. Denote $\mathcal{I}_{\mathcal{X}} := \mathcal{X} \times [*/\mathbb{C}^\times]$. The category of maps $(\mathcal{W} \rightarrow \mathcal{X}) \rightarrow (\mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{X})$ in $\mathbf{DiffSt}/\mathcal{X}$ is the category of \mathbb{C}^\times -bundles over \mathcal{W} . The tensor product of \mathbb{C}^\times -bundles gives $\mathcal{I}_{\mathcal{X}}$ the structure of a group object in \mathbf{DiffSt} . The following definition is a special case of [SP11, Def 70]:

Definition A.3.0.3. A \mathbb{C}^\times -gerbe over \mathcal{X} is a stack $\mathcal{G} \in \mathbf{DiffSt}/\mathcal{X}$, equipped with a right action of $\mathcal{I}_{\mathcal{X}}$, locally equivalent to $\mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{X}$: There exists a representable submersion $f : \mathcal{Y} \rightarrow \mathcal{X}$ such that the pullback $\mathcal{G} \times_{\mathcal{X}} \mathcal{Y} \in \mathbf{DiffSt}/\mathcal{Y}$ is $[*/\mathbb{C}^\times]$ -equivariantly equivalent to $\mathcal{I}_{\mathcal{Y}}$ with the standard $\mathcal{I}_{\mathcal{Y}}$ -action.

From now on, we refer to \mathbb{C}^\times -gerbes simply as gerbes and suppress the subscript from the trivial gerbe \mathcal{I} when the base is clear. We say a gerbe \mathcal{G} is *trivial* if it admits an equivalence $\mathcal{I} \xrightarrow{\cong} \mathcal{G}$.

Let $\mathcal{G} \rightarrow \mathcal{M}$ be a gerbe over an étale stack. The gerbe \mathcal{G} defines a groupoid-valued stack over the étale site of \mathcal{M} . It assigns to $x : \mathcal{X} \rightarrow \mathcal{M}$ the category of trivialisations of $x^*\mathcal{G} = \mathcal{G} \times_{\mathcal{M}} \mathcal{X}$:

$$\begin{aligned} \mathcal{G} : \widetilde{\text{Et}}(\mathcal{M})^{\text{op}} &\rightarrow \mathbf{Gpd} \\ (x : \mathcal{X} \rightarrow \mathcal{M}) &\mapsto \text{Hom}(x^*\mathcal{I}, x^*\mathcal{G}). \end{aligned}$$

For the trivial gerbe \mathcal{I} , this stack is the stack of \mathbb{C}^\times -bundles, which is equivalent to the stack of complex line bundles (and invertible maps between them). The sheaf of sections of a complex line bundle is a C^∞ -module, so we obtain a map of stacks (valued in categories) $\mathcal{I} \rightarrow \mathbf{Shv}_{\mathbb{C}^\infty}$. The stack \mathcal{I} acts on $\mathbf{Shv}_{\mathbb{C}^\infty}$ by composing with the tensor product of C^∞ -modules:

$$\mathcal{I} \times \mathbf{Shv}_{\mathbb{C}^\infty} \rightarrow \mathbf{Shv}_{\mathbb{C}^\infty} \times \mathbf{Shv}_{\mathbb{C}^\infty} \rightarrow \mathbf{Shv}_{\mathbb{C}^\infty}.$$

The 2-cells exhibiting this as an action are induced by the associator 2-cells for the tensor product on $\mathbf{Shv}_{\mathbb{C}^\infty}$.

Every trivialisation $\tau \in \text{Hom}(\mathcal{I}, \mathcal{G})$ over \mathcal{M} has an inverse τ^{-1} . Given another trivialisation κ of \mathcal{G} , the composite $\tau^{-1}\kappa$ is an endomorphism of \mathcal{I} , and thus equivalent to the data of a line bundle $\mathcal{L}_{\tau, \kappa}$ over \mathcal{M} .

NOTATION A.3.0.4. We write $\tau^{-1}\kappa \cdot \mathcal{F}$ to denote the action of the line bundle $\mathcal{L}_{\tau, \kappa}$ on a C^∞ -module sheaf $\mathcal{F} \in \mathbf{Shv}_{\mathbb{C}^\infty}(\mathcal{M})$.

Let \mathcal{G} be a trivial gerbe over \mathcal{M} . We define the category $\mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$: Its objects are pairs $(\tau : \mathcal{I} \rightarrow \mathcal{G}, \mathcal{F} \in \mathbf{Shv}_{C^\infty}(\mathcal{M}))$, which we denote by $\tau \cdot \mathcal{F}$. The Hom-sets are given by

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathcal{G}}(\mathcal{M})}(\tau \cdot \mathcal{F}, \tau' \cdot \mathcal{F}') = \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(\mathcal{M})}(\mathcal{F}, \tau^{-1}\tau' \cdot \mathcal{F}').$$

A pair of morphisms

$$\tau \cdot \mathcal{F} \xrightarrow{\phi} \tau' \cdot \mathcal{F}' \xrightarrow{\varphi} \tau'' \cdot \mathcal{F}''$$

is given by morphisms of sheaves $\bar{\phi} : \mathcal{F} \rightarrow \tau^{-1}\tau' \cdot \mathcal{F}'$ and $\bar{\varphi} : \mathcal{F}' \rightarrow \tau'^{-1}\tau'' \cdot \mathcal{F}''$. Their composite $\varphi \circ \phi : \tau \cdot \mathcal{F} \rightarrow \tau'' \cdot \mathcal{F}''$ is given by the morphism of sheaves

$$\mathcal{F} \xrightarrow{\bar{\phi}} \tau^{-1}\tau' \cdot \mathcal{F}' \xrightarrow{\bar{\varphi}} \tau^{-1}\tau' \cdot \tau'^{-1}\tau'' \cdot \mathcal{F}'' \xrightarrow{\cong} \tau^{-1}\tau'' \cdot \mathcal{F}''.$$

Denote by $\lambda : \mathcal{G}' \xrightarrow{\cong} \mathcal{G}$ a morphism of gerbes (necessarily an isomorphism). It induces an equivalence

$$\begin{aligned} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) &\xrightarrow{\cong} \mathbf{Shv}^{\mathcal{G}'}(\mathcal{M}) \\ \tau \cdot \mathcal{F} &\mapsto \lambda\tau \cdot \mathcal{F} \end{aligned}$$

with inverse given by λ^{-1} . In particular, a trivialisaton τ of \mathcal{G} induces an equivalence

$$\mathbf{Shv}^{\mathcal{I}}(\mathcal{M}) \xrightarrow{\cong} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}).$$

Lemma A.3.0.5. *Denote by τ_0 the canonical trivialisaton of \mathcal{I} . The functor*

$$\begin{aligned} \tau_0 \cdot - : \mathbf{Shv}_{C^\infty}(\mathcal{M}) &\rightarrow \mathbf{Shv}^{\mathcal{I}}(\mathcal{M}) \\ \mathcal{F} &\mapsto \tau_0 \cdot \mathcal{F} \end{aligned}$$

is an equivalence.

Proof. Full faithfulness uses the natural equivalence $\tau_0^{-1}\tau_0 \simeq C^\infty$:

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathcal{G}}(\mathcal{M})}(\tau_0 \cdot \mathcal{F}, \tau_0 \cdot \mathcal{F}') := \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(\mathcal{M})}(\mathcal{F}, \tau_0^{-1}\tau_0 \cdot \mathcal{F}') \simeq \mathrm{Hom}_{\mathbf{Shv}_{C^\infty}(\mathcal{M})}(\mathcal{F}, \mathcal{F}').$$

The functor is also essentially surjective: the object $\kappa \cdot \mathcal{F} \in \mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ is equivalent to $\tau_0 \cdot (\tau_0^{-1}\kappa \cdot \mathcal{F})$ via the map of sheaves

$$\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F} \xrightarrow{\cong} \kappa^{-1}\tau_0\tau_0^{-1}\kappa \cdot \mathcal{F}. \quad \square$$

We can compose the functor $\tau_0 \cdot -$ with the equivalence $\mathbf{Shv}^{\mathcal{I}}(\mathcal{M}) \xrightarrow{\cong} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ induced by a trivialisaton τ to obtain an equivalence

$$\begin{aligned} \tau \cdot - : \mathbf{Shv}_{C^\infty}(\mathcal{M}) &\xrightarrow{\cong} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \\ \mathcal{F} &\mapsto \tau \cdot \mathcal{F}. \end{aligned}$$

Let $f : \mathcal{N} \rightarrow \mathcal{M}$ be a map of étale stacks. A trivialisaton τ of $\mathcal{G} \rightarrow \mathcal{M}$ pulls back to a trivialisaton $f^*\tau$ of $f^*\mathcal{G} \rightarrow \mathcal{N}$. We define a pullback functor (keeping the same symbol as for C^∞ -modules)

$$\begin{aligned} f^* : \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) &\rightarrow \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \\ \kappa \cdot \mathcal{F} &\mapsto f^*\kappa \cdot f^*\mathcal{F}. \end{aligned}$$

We denote this pullback functor by the same functor as for sheaves of C^∞ -modules. For any choice of trivialisation τ , the square

$$\begin{array}{ccc} \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) & \xrightarrow{f^*} & \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \\ \tau \cdot - \uparrow & & \uparrow f^* \tau \cdot - \\ \mathbf{Shv}_{C^\infty}(\mathcal{M}) & \xrightarrow{f^*} & \mathbf{Shv}_{C^\infty}(\mathcal{N}) \end{array}$$

commutes. Now Lemma A.2.0.4 implies that a 2-cell $\gamma : g \circ f \xrightarrow{\cong} h$ induces a natural equivalence

$$f^* \circ g^* \simeq h^*.$$

Let $\pi : M \rightarrow \mathcal{M}$ be an atlas such that the pullback gerbe $\pi^*\mathcal{G}$ is trivial. We define a \mathbf{VCat} -valued 2-presheaf

$$\begin{aligned} \mathbf{Shv}^{\mathcal{G}} : \mathrm{Et}(M/\mathcal{M})^{\mathrm{op}} &\rightarrow \mathbf{VCat} \\ (x : X \rightarrow M) &\mapsto \mathbf{Shv}^{x^*\mathcal{G}}(X) \\ ((f, \phi) : x \rightarrow y) &\mapsto (f^* : \mathbf{Shv}^{y^*\mathcal{G}}(Y) \rightarrow \mathbf{Shv}^{x^*\mathcal{G}}(X)). \end{aligned}$$

REMARK A.3.0.6. We choose $\mathrm{Et}(M/\mathcal{M})$ because it is a dense subsite of $\widetilde{\mathrm{Et}}(\mathcal{M})$ where every object admits a trivialisation of the pullback gerbe. One could instead define this presheaf directly on the (dense) subsite of objects over which the pullback gerbe admits a trivialisation.

Lemma A.3.0.7. *The functor $\mathbf{Shv}^{\mathcal{G}}$ is a stack over $\mathrm{Et}(M/\mathcal{M})$.*

Proof. Let $\tilde{x} : X \rightarrow M$ be an object of $\mathrm{Et}(M/\mathcal{M})$, and $\tilde{y} : Y \rightarrow M$ an object covering \tilde{x} via a morphism $(f, \phi) : \tilde{y} \rightarrow \tilde{x}$. Note in particular, that this identifies $f : Y \rightarrow X$ as an atlas of X . A descent datum for $\mathbf{Shv}^{\mathcal{G}}$ over the cover $(f, \phi) : \tilde{y} \rightarrow \tilde{x}$ is given by a pair $(\kappa \cdot \mathcal{F} \in \mathbf{Shv}^{\tilde{y}^*\pi^*\mathcal{G}}(Y), \varphi : p_1^*(\kappa \cdot \mathcal{F}) \xrightarrow{\cong} p_2^*(\kappa \cdot \mathcal{F}))$, where $p_1, p_2 : Y^{[2]} \rightarrow Y$ are the two projections. The isomorphism φ satisfies a cocycle condition over $Y^{[3]}$.

Without loss of generality, we may fix the trivialisation used in the descent datum to be the pullback of a chosen trivialisation τ of \mathcal{G} over X : $\kappa = f^*\tau$. The isomorphism φ then corresponds to an isomorphism of sheaves $p_1^*\mathcal{F} \xrightarrow{\cong} (p_1^*f^*\tau^{-1}p_2^*f^*\tau) \cdot p_2^*\mathcal{F} \simeq p_2^*\mathcal{F}$.³

We now describe a twisted sheaf $\tilde{x}^*\tau \cdot \bar{\mathcal{F}} \in \mathbf{Shv}^{\tilde{x}^*\pi^*\mathcal{G}}(X)$ which is obtained by gluing this descent datum. This is analogous to the construction in the proof of Lemma A.3.0.1. We view $\bar{\mathcal{F}} \in \mathbf{Shv}_{C^\infty}(X)$ as a sheaf over the site $\mathrm{Et}(Y/X)$ associated to the atlas $f : Y \rightarrow X$. The sheaf $\bar{\mathcal{F}}$ assigns to an object

$$(u : U \rightarrow Y) \mapsto \mathcal{F}(u) = u^*\mathcal{F}(U),$$

the global sections of the pullback sheaf $u^*\mathcal{F}$ over U . A morphism $v \rightarrow u$ in $\mathrm{Et}(Y/X)$ is a map $g : V \rightarrow U$ such that the composites

$$f \circ v = f \circ u \circ g : V \rightarrow X$$

³The last equivalence uses that τ^{-1} and τ are pulled back along the same map, and thus cancel.

are equal. As a result, the two maps $u \circ g, v : V \rightarrow Y$ assemble into a map $(u \circ g, v) : V \rightarrow Y^{[2]}$. We pull back φ along this map to obtain an isomorphism of sheaves

$$g^*u^*\mathcal{F} \simeq (u \circ g)^*\mathcal{F} \xrightarrow{\sim} v^*\mathcal{F}$$

over V . The sheaf $\bar{\mathcal{F}}$ assigns to the morphism g the map

$$u^*\mathcal{F}(U) = \mathcal{F}(u : U \rightarrow Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(u \circ g : V \rightarrow Y) = (g^*u^*\mathcal{F})(V) \xrightarrow{\sim} v^*\mathcal{F}(V).$$

The fact that this assignment is functorial is equivalent to the cocycle condition satisfied by φ over $Y^{[3]}$.

We have constructed a functor

$$\mathbf{Desc}(Y/X) \rightarrow \mathbf{Shv}^{\mathcal{G}}(X),$$

where $\mathbf{Desc}(Y/X)$ denotes the category of descent data of $\mathbf{Shv}^{\mathcal{G}}$ for the cover $(f, \phi) : \tilde{y} \rightarrow \tilde{x}$ – recall these were objects $\tilde{y} : Y \rightarrow M$ and $\tilde{x} : X \rightarrow M$. It is straightforward to check the composite $\mathbf{Shv}^{\mathcal{G}}(X) \rightarrow \mathbf{Desc}(Y/X) \rightarrow \mathbf{Shv}^{\mathcal{G}}(X)$ is equivalent to the identity functor. The composite $\mathbf{Desc}(Y/X) \rightarrow \mathbf{Shv}^{\mathcal{G}} \rightarrow \mathbf{Desc}(Y/X)$ is given by

$$(\mathcal{F}, \varphi) \mapsto \tau \cdot \bar{\mathcal{F}} \mapsto (f^*\bar{\mathcal{F}}, \bar{\varphi} = \text{id}_{p^*\bar{\mathcal{F}}}),$$

where $\bar{\mathcal{F}}$ is the sheaf constructed above, and $p : Y^{[2]} = Y \times_X Y \rightarrow X$ is the canonical projection. The sheaf $f^*\bar{\mathcal{F}}$ sends

$$(u : U \rightarrow Y) \mapsto \bar{\mathcal{F}}(f \circ u : U \rightarrow Y \rightarrow X) = \mathcal{F}(u' : U \rightarrow Y)$$

where $u' : U \rightarrow Y$ is a choice of factorisation of $f \circ u$. The induced map $(u, u') : U \rightarrow Y^{[2]}$ may be used to pull back φ and obtain an isomorphism

$$u^*\mathcal{F} \xrightarrow{\sim} u'^*\mathcal{F}$$

These isomorphisms assemble into an equivalence of descent data $(\mathcal{F}, \varphi) \rightarrow (f^*\bar{\mathcal{F}}, \text{id})$ with components

$$\mathcal{F}(u : U \rightarrow Y) = u^*\mathcal{F}(U) \xrightarrow{\sim} u'^*\mathcal{F}(U) = \mathcal{F}(u' : U \rightarrow Y) = f^*\bar{\mathcal{F}}(u : U \rightarrow Y).$$

The fact that this defines a morphism of descent data is again a consequence of the cocycle condition satisfied by φ . \square

Definition A.3.0.8. Let $\mathcal{G} \rightarrow \mathcal{M}$ be a gerbe over an étale stack. We call $\mathbf{Shv}^{\mathcal{G}}$ the *stack of \mathcal{G} -twisted sheaves over \mathcal{M}* . A *\mathcal{G} -twisted sheaf over \mathcal{M}* is a global section $\mathcal{F} \in \mathbf{Shv}^{\mathcal{G}}(\text{id}_{\mathcal{M}})$.

REMARK A.3.0.9. The stack $\mathbf{Shv}^{\mathcal{G}}$ over \mathcal{M} is equipped with a map

$$\begin{aligned} \mathcal{G} \times \mathbf{Shv}_{\mathbf{C}^\times} &\rightarrow \mathbf{Shv}^{\mathcal{G}} \\ (\tau, \mathcal{F}) &\mapsto \tau \cdot \mathcal{F} \end{aligned}$$

and natural equivalences between the two composites

$$\mathcal{G} \times \mathcal{I} \times \mathbf{Shv}_{\mathbf{C}^\times} \rightrightarrows \mathcal{G} \times \mathbf{Shv}_{\mathbf{C}^\times} \rightarrow \mathbf{Shv}^{\mathcal{G}},$$

whose components are isomorphisms $(\tau \cdot \mathcal{L}) \cdot \mathcal{F} = \tau \cdot (\mathcal{L} \cdot \mathcal{F})$. These equivalences satisfy a cocycle condition which exhibits $\mathbf{Shv}^{\mathcal{G}}$ as the colimit

$$\mathcal{G} \times_{\mathcal{I}} \mathbf{Shv}_{\mathbf{C}^\times} = \text{colim } \mathcal{G} \times \mathcal{I} \times \mathcal{I} \times \mathbf{Shv}_{\mathbf{C}^\times} \rightrightarrows \mathcal{G} \times \mathcal{I} \times \mathbf{Shv}_{\mathbf{C}^\times} \rightrightarrows \mathcal{G} \times \mathbf{Shv}_{\mathbf{C}^\times}.$$

This is akin to the construction of a line bundle $P \times_{\mathbf{C}^\times} \mathbf{C}$ associated to a \mathbf{C}^\times -bundle P .

A.4 The Twisted Pullback-Pushforward Adjunction

Let $\mathcal{G} \rightarrow \mathcal{M}$ be a trivial gerbe over an étale stack and $f : \mathcal{N} \rightarrow \mathcal{M}$ a map of étale stacks. Pick a trivialisaton τ of \mathcal{G} . We define a functor

$$\begin{aligned} f_*^\tau : \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) &\rightarrow \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \\ \kappa \cdot \mathcal{F} &\mapsto \tau \cdot f_*((f^*\tau^{-1})\kappa \cdot \mathcal{F}). \end{aligned}$$

Lemma A.4.0.1. *The functor f_*^τ is right adjoint to f^* .*

Proof. The equivalences $\tau \cdot - : \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{M}) \rightarrow \mathbf{Shv}^{\mathcal{G}}(\mathcal{M})$ and $f^*\tau \cdot - : \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{N}) \rightarrow \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N})$ identify the functors f^* and f_*^τ with the pullback and pushforward functors of Proposition A.2.0.3. \square

The adjoint to a functor is unique up to natural isomorphism, so we drop the superscript τ from f_*^τ . Below, we transport this adjunction to the case of non-trivial gerbes. If the gerbe \mathcal{G} is non-trivial, there is no \mathcal{G} -twisted sheaf over \mathcal{M} of the form $\tau \cdot \mathcal{F}$. We can instead describe them by descent data. A twisted sheaf is then given by the data of a cover $\pi : Y \twoheadrightarrow \mathcal{M}$ with a trivialisaton τ of $\pi^*\mathcal{G}$, a sheaf \mathcal{F} over Y , and an isomorphism of sheaves $\varphi : p_1^*\mathcal{F} \rightarrow p_1^*\tau^{-1}p_2^*\tau \cdot p_2^*\mathcal{F}$ over $Y^{[2]}$, such that

$$\begin{array}{ccc} p_1^*\mathcal{F} & \xrightarrow{p_{13}^*\varphi} & p_1^*\tau^{-1}p_3^*\tau \cdot p_3^*\mathcal{F} \\ p_{12}^*\varphi \downarrow & & \wr \downarrow \\ p_1^*\tau^{-1}p_2^*\tau \cdot p_2^*\mathcal{F} & \xrightarrow{p_{1^*\tau^{-1}p_2^*\tau \cdot p_{23}^*\varphi}} & p_1^*\tau^{-1}p_2^*\tau \cdot p_2^*\tau^{-1}p_3^*\tau \cdot p_3^*\mathcal{F} \end{array}$$

commutes.

By Lemma A.3.0.7, $\mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \simeq \mathbf{Desc}(Y/\mathcal{M})$ where $\pi : Y \twoheadrightarrow \mathcal{M}$ is any cover such that $\pi^*\mathcal{G}$ is trivial. This cover pulls back to a cover $f^*Y = Y \times_{\mathcal{M}} \mathcal{N} \twoheadrightarrow \mathcal{N}$ and trivialisations τ of $\pi^*\mathcal{G}$ over Y pull back to trivialisations $f^*\tau$ of $f^*\pi^*\mathcal{G}$ over f^*Y .⁴ Pullbacks commute, so there are equivalences $(f^*Y)^{[n]} \simeq f^*(Y^{[n]})$, and $f : \mathcal{N} \rightarrow \mathcal{M}$ extends to a map of simplicial diagrams

$$\begin{array}{ccccccc} f^*Y^{[3]} & \rightrightarrows & f^*Y^{[2]} & \rightrightarrows & f^*Y & \longrightarrow & \mathcal{N} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y^{[3]} & \rightrightarrows & Y^{[2]} & \rightrightarrows & Y & \longrightarrow & \mathcal{M}. \end{array}$$

In other words, the map f commutes with the simplicial maps $p_{ij\dots k} : Y^{[m]} \rightarrow Y^{[n]}$, and the squares formed by corresponding maps in the two rows are pullback squares.

We establish the adjunction

$$\mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \rightleftarrows \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N})$$

⁴Note that, even when $Y \twoheadrightarrow \mathcal{M}$ is an atlas, this is not necessarily true for $f^*Y \twoheadrightarrow \mathcal{N}$: the pullback f^*Y is not representable by a manifold unless $f : \mathcal{M} \rightarrow \mathcal{N}$ is a representable map.

through a pair of functors

$$f^* : \mathbf{Desc}(Y/\mathcal{M}) \rightleftarrows \mathbf{Desc}(f^*Y/\mathcal{N}) : f_*$$

Denote by \mathcal{L} the line bundle corresponding to $p_2^*\tau^{-1}p_1^*\tau$ over $Y^{[2]}$. It is equipped with a natural equivalence $p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \simeq p_{13}^*\mathcal{L}$ over $Y^{[3]}$. A descent datum in $\mathbf{Desc}(Y/\mathcal{M})$ is uniquely equivalent to a descent datum of the form

$$(\mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y), \varphi : p_1^*\mathcal{F} \xrightarrow{\sim} \mathcal{L} \cdot p_2^*\mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(Y^{[2]})),$$

and similarly for $\mathbf{Desc}(f^*Y/\mathcal{N})$.

We define the pullback functor by

$$\begin{aligned} f^* : \mathbf{Desc}(Y/\mathcal{M}) &\rightarrow \mathbf{Desc}(f^*Y/\mathcal{N}) \\ (\mathcal{F}, \varphi) &\mapsto (f^*\mathcal{F}, \tilde{f}^*\varphi), \end{aligned}$$

where $\tilde{f}^*\varphi$ is the composite

$$p_1^*f^*\mathcal{F} \simeq f^*p_1^*\mathcal{F} \xrightarrow{f^*\varphi} f^*(\mathcal{L} \cdot p_2^*\mathcal{F}) \xrightarrow{\sim} f^*\mathcal{L} \cdot f^*p_2^*\mathcal{F} \simeq f^*\mathcal{L} \cdot p_2^*f^*\mathcal{F}.$$

(The first and last equivalences come from the fact that p_i commutes with f , the third equivalence is given by the monoidal structure of f^* .) Up to composition by natural equivalences, the cocycle condition for $\tilde{f}^*\varphi$ is the image of the cocycle condition for φ under the functor $f^* : \mathbf{Shv}_{\mathbf{C}^\infty}(Y^{[3]}) \rightarrow \mathbf{Shv}_{\mathbf{C}^\infty}(f^*Y^{[3]})$, and thus $(f^*\mathcal{F}, \tilde{f}^*\varphi)$ is indeed a valid descent datum.

Lemma A.4.0.2 (Base change). *Let*

$$\begin{array}{ccc} f^*\mathcal{Y} & \xrightarrow{p} & \mathcal{X} \\ f \downarrow & \lrcorner & \downarrow f \\ \mathcal{Y} & \xrightarrow{p} & \mathcal{Z} \end{array}$$

be a pullback square of étale stacks where p is representably étale. Then the natural transformation

$$p^*f_* \rightarrow p^*f_*p_*p^* \simeq p^*p_*f_*p^* \rightarrow f_*p^*$$

is an equivalence of functors $\mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{X}) \rightarrow \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{Y})$.

Proof. The component of this natural transformation on a sheaf $\mathcal{F} \in \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{X})$ is given by a morphism $p^*f_*\mathcal{F} \rightarrow f_*p^*\mathcal{F}$ in $\mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{Y})$. It is enough to exhibit this as an isomorphism on an étale atlas of \mathcal{Y} .

Pick an étale atlas $Z \rightarrow \mathcal{Z}$ and an étale atlas $X \rightarrow \mathcal{X}$ fine enough such that the map $f : \mathcal{X} \rightarrow \mathcal{Z}$ is represented by a map $\bar{f} : X \rightarrow Z$. This yields atlases $Y := p^*Z \rightarrow \mathcal{Y}$ and $f^*Y = f^*p^*Z \rightarrow f^*\mathcal{Y}$. The resulting pullback square of manifolds

$$\begin{array}{ccc} f^*Y & \xrightarrow{\bar{p}} & X \\ \bar{f} \downarrow & \lrcorner & \downarrow \bar{f} \\ Y & \xrightarrow{\bar{p}} & Z \end{array}$$

maps to the pull back square in the statement via representably étale maps. It also satisfies the property that \bar{p} is étale, and thus the classical base change theorem implies

$$\bar{p}^* \bar{f}_* \rightarrow \bar{f}_* \bar{p}^*$$

is a natural equivalence in $\mathbf{Shv}_{C^\infty}(Y)$. But this is exactly the image of the natural transformation

$$p^* f_* \rightarrow f_* p^*$$

under the conservative functor $\mathbf{Shv}_{C^\infty}(\mathcal{Y}) \rightarrow \mathbf{Shv}_{C^\infty}(Y)$. \square

Lemma A.4.0.3 (Projection Formula). *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of étale stacks, $\mathcal{F} \in \mathbf{Shv}_{C^\infty}(\mathcal{X})$ a sheaf of C^∞ -modules, $\mathcal{L} \in \mathbf{Shv}_{C^\infty}(\mathcal{Y})$ a line bundle. Then the natural map*

$$\mathcal{L} \otimes f_* \mathcal{F} \rightarrow f_*(f^* \mathcal{L} \otimes \mathcal{F})$$

is an isomorphism. This isomorphism is compatible with the monoidal structure: given another line bundle $\mathcal{L}' \in \mathbf{Shv}_{C^\infty}(\mathcal{Y})$,

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{L}' \otimes f_* \mathcal{F} & \longrightarrow & \mathcal{L} \otimes f_*(f^* \mathcal{L}' \otimes \mathcal{F}) \\ \downarrow & & \downarrow \\ f_*(f^*(\mathcal{L} \otimes \mathcal{L}') \otimes \mathcal{F}) & \simeq & f_*(f^* \mathcal{L} \otimes f^* \mathcal{L}' \otimes \mathcal{F}) \end{array}$$

commutes.

Proof. The first part of the proof proceeds analogously to the proof of Lemma A.4.0.2: pick an atlas $Y \twoheadrightarrow \mathcal{Y}$. Then the fact that the natural map is an isomorphism follows from the classical projection formula for sheaves over manifolds.

Consider the following tiling of the diagram in the statement:

$$\begin{array}{ccccc} \mathcal{L} \otimes \mathcal{L}' \otimes f_* \mathcal{F} & \longrightarrow & \mathcal{L} \otimes f_* f^* \mathcal{L}' \otimes f_* \mathcal{F} & \longrightarrow & \mathcal{L} \otimes f_*(f^* \mathcal{L}' \otimes \mathcal{F}) \\ \downarrow & & \downarrow & \text{naturality of } \otimes & \downarrow \\ & & f_* f^* \mathcal{L} \otimes f_* f^* \mathcal{L}' \otimes f_* \mathcal{F} & \rightarrow & f_* f^* \mathcal{L} \otimes f_*(f^* \mathcal{L}' \otimes \mathcal{F}) \\ & & \downarrow & & \downarrow \\ f_* f^*(\mathcal{L} \otimes \mathcal{L}') \otimes f_* \mathcal{F} & \cong & f_*(f^* \mathcal{L} \otimes f^* \mathcal{L}') \otimes f_* \mathcal{F} & \xrightarrow{\text{lax monoidality of } f_*} & f_*(f^* \mathcal{L} \otimes f^* \mathcal{L}' \otimes \mathcal{F}) \\ \downarrow & \text{naturality of lax monoidal structure} & & & \downarrow \\ f_*(f^*(\mathcal{L} \otimes \mathcal{L}') \otimes \mathcal{F}) & \xrightarrow{\cong} & & & f_*(f^* \mathcal{L} \otimes f^* \mathcal{L}' \otimes \mathcal{F}). \end{array}$$

The labelled squares commute for the reason indicated, while the empty square commutes because the lax monoidal structure of f_* is induced by the monoidal structure of f^* . \square

The pushforward functor of twisted sheaves is given by

$$\begin{aligned} f_* : \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) &\rightarrow \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \\ (\mathcal{F}, \varphi) &\mapsto (f_*\mathcal{F}, \tilde{f}_*\varphi), \end{aligned}$$

where $\tilde{f}_*\varphi$ is the composite

$$p_1^*f_*\mathcal{F} \simeq f_*p_1^*\mathcal{F} \xrightarrow{f_*\varphi} f_*(f^*\mathcal{L} \otimes p_2^*\mathcal{F}) \xrightarrow{\simeq} \mathcal{L} \otimes f_*p_2^*\mathcal{F} \simeq \mathcal{L} \otimes p_2^*f_*\mathcal{F},$$

making use of base change (Lemma A.4.0.2) and the projection formula (Lemma A.4.0.3). The cocycle condition for $\tilde{f}_*\varphi$ follows from the cocycle condition for φ and the compatibility of the projection formula with the tensor product (Lemma A.4.0.3).

Proposition A.4.0.4. *The functors defined above are adjoint*

$$f^* : \mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \rightleftarrows \mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) : f_*.$$

Proof. The functors f^*, f_* we defined between the categories $\mathbf{Shv}^{\mathcal{G}}(\mathcal{M}) \simeq \mathbf{Desc}(Y/\mathcal{M})$ and $\mathbf{Shv}^{f^*\mathcal{G}}(\mathcal{N}) \simeq \mathbf{Desc}(f^*Y/\mathcal{N})$ make the diagram

$$\begin{array}{ccc} \mathbf{Desc}(Y/\mathcal{M}) & \begin{array}{c} \xleftarrow{f_*} \\ \xrightarrow{f^*} \end{array} & \mathbf{Desc}(f^*Y/\mathcal{N}) \\ \downarrow & & \downarrow \\ \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{M}) & \begin{array}{c} \xleftarrow{f_*} \\ \xrightarrow{f^*} \end{array} & \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{N}) \end{array}$$

commute, where the vertical functors discard the isomorphism φ of a descent datum (\mathcal{F}, φ) . These functors are faithful: the map

$$\mathrm{Hom}_{\mathbf{Desc}}((\mathcal{F}, \varphi), (\mathcal{F}', \varphi')) \hookrightarrow \mathrm{Hom}_{\mathbf{Shv}_{\mathbf{C}^\infty}}(\mathcal{F}, \mathcal{F}')$$

includes those maps of sheaves $\mathcal{F} \rightarrow \mathcal{F}'$ that make the diagram

$$\begin{array}{ccc} p_1^*\mathcal{F} & \longrightarrow & p_1^*\mathcal{F}' \\ \downarrow \varphi & & \downarrow \varphi' \\ p_2^*\mathcal{F} & \longrightarrow & p_2^*\mathcal{F}' \end{array}$$

commute. This condition is manifestly preserved by the functors

$$f^* : \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{M}) \rightleftarrows \mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{N}) : f_*,$$

hence the natural isomorphism

$$\mathrm{Hom}_{\mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{M})}(f^*\mathcal{F}, \mathcal{F}') \simeq \mathrm{Hom}_{\mathbf{Shv}_{\mathbf{C}^\infty}(\mathcal{N})}(\mathcal{F}, f_*\mathcal{F}')$$

downstairs restricts to a natural isomorphism upstairs, proving the statement. \square

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