

Discovering T-dualities of little string theories

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ABSTRACT: We describe a general method for deducing T-dualities of little string theories, which are dualities between these theories that arise when they are compactified on circle. The method works for both untwisted and twisted circle compactifications of little string theories and is based on surface geometries associated to these circle compactifications. The surface geometries describe information about Calabi-Yau threefolds on which M-theory can be compactified to construct the corresponding circle compactified little string theories. Using this method, we deduce at least one T-dual, and in some cases multiple T-duals, for untwisted and twisted circle compactifications of most of the little string theories that can be described on their tensor branches in terms of a 6d supersymmetric gauge theory with a simple non-abelian gauge group, which are also known as rank-0 little string theories. This includes little string theories carrying $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (1, 0)$ supersymmetries. For many, but not all, circle compactifications of $\mathcal{N} = (1, 1)$ little string theories, we have T-dualities that realize Langlands dualities between affine Lie algebras. Along the way, we find another discrete theta angle distinct from 0 and π for an E-string node.

KEYWORDS: Field Theories in Higher Dimensions, M-Theory, String Duality, Supersymmetry and Duality

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1 Introduction

Little string theories (LSTs) are fascinating quantum-mechanical theories. They are expected to be UV complete, but are neither local quantum field theories (QFTs) nor fully-fledged string theories, sitting somewhere in between the two. Unlike string theories, gravity is not dynamical in these theories, and as a consequence they behave very similarly to local QFTs. The key property that differentiates LSTs from local QFTs is that LSTs admit T-dualities [1–3], which is the topic of this work.

All the known LSTs exist in six spacetime dimensions, and carry at least $\mathcal{N} = (1, 0)$ supersymmetry. Some very special LSTs carry $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (1, 1)$ supersymmetries. A uniform definition for all the known LSTs is provided as compactifications of F-theory on (a special class of) elliptically fibered Calabi-Yau threefolds. This includes both the standard F-theory compactifications [4], and also the recently explored *frozen* F-theory compactifications [5, 6]. These theories can also be classified using anomaly cancellation [7–9].

To study T-dualities¹ of such LSTs, we need to compactify the F-theory setups on circle, which are dual to M-theory compactified on the same elliptically fibered Calabi-Yau

¹See [4, 10–16] for recent works on little string T-dualities.

threefolds.² T-dualities thus express themselves as geometric isomorphisms between the corresponding threefolds.

In this paper, we study such geometric isomorphisms in terms of graphical structures that we call *surface geometries*, which capture the information associated to holomorphic curves and surfaces inside the threefold. Surface geometries have been immensely useful in the recent studies on 5d and 6d superconformal field theories (SCFTs) [17–45]. They were proposed as useful tools for these studies in [17], while the essential graphical calculus for these purposes was developed in [18–22, 25, 27, 28]. The present work is thus part of the vast progress [5, 17–90] made recently in understanding 5d and 6d supersymmetric UV complete theories — which so far has included SCFTs and KK theories; the present paper adds LSTs to this list.

The strategy for using surface geometries to determine T-dualities of an LST \mathfrak{T} can be summarized as follows:

1. Use the tensor branch description of \mathfrak{T} . This can be packaged in terms of a graph with nodes and edges. This graph is used in the following operations for studying T-duals of the untwisted compactification of \mathfrak{T} . For studying T-duals of a twisted compactification of \mathfrak{T} , one uses information regarding the twist to convert the graph for \mathfrak{T} into a new graph, which is then used in the following operations. Let us denote the circle compactified theory being studied as $\mathfrak{T}_{S^1}^\rho$, where ρ denotes the twist involved, which takes trivial value if the circle compactification being studied is untwisted.
2. To each node in the graph associated to $\mathfrak{T}_{S^1}^\rho$, we can associate a collection of Hirzebruch surfaces (i.e. compact Kahler surfaces with a choice of \mathbb{P}^1 fibration) glued to each other.
3. If two nodes are connected by an edge, we glue together the collections of Hirzebruch surfaces associated to the two nodes. By this point, we have described a surface geometry associated to $\mathfrak{T}_{S^1}^\rho$.
4. Some Hirzebruch surfaces contain multiple \mathbb{P}^1 fibers. Changing the choice of \mathbb{P}^1 fibers judiciously lets us recognize the resulting surface geometry as a surface geometry associated to $\widehat{\mathfrak{T}}_{S^1}^\rho$, i.e. a $\widehat{\rho}$ (which can take a trivial value for untwisted compactification) twisted circle compactification of an LST $\widehat{\mathfrak{T}}$. This means that we have a T-duality between LSTs \mathfrak{T} and $\widehat{\mathfrak{T}}$, where \mathfrak{T} is compactified with twist ρ and $\widehat{\mathfrak{T}}$ is compactified with twist $\widehat{\rho}$.

We illustrate this procedure by finding a T-dual for twisted and untwisted compactifications of many rank-0 LSTs describable on their tensor branches as 6d $\mathcal{N} = (1, 0)$ gauge theories with simple gauge algebras. More concretely, we study T-duals of the following systems:

- 6d theory with any simple gauge algebra \mathfrak{g} and a hyper in adjoint representation \mathbf{A} . This theory has $\mathcal{N} = (1, 1)$ supersymmetry. For $\mathfrak{g} = \mathfrak{su}(n)$, we can turn on a rational theta angle $\theta = 2\pi\frac{p}{q}$ [91], and for $\mathfrak{g} = \mathfrak{sp}(n)$, we can turn on a \mathbb{Z}_2 valued theta angle $\theta = 0, \pi$. We provide a T-dual for each untwisted compactification of these theories

²We also study twisted circle compactifications of LSTs, which involve turning on non-zero discrete holonomies along the compactification circle. In such cases, the M-theory threefold differs from the F-theory threefold.

for $\mathfrak{g} \neq \mathfrak{su}(n)$. On the other hand, for $\mathfrak{g} = \mathfrak{su}(n)$, we provide a T-dual for untwisted compactification only of $\theta = 0$ theory.³

For $\mathfrak{g} = \mathfrak{su}(n), \mathfrak{so}(2n), \mathfrak{e}_6$ we have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of such a Lie algebra \mathfrak{g} . For $\mathfrak{g} = \mathfrak{su}(n)$, we must have $\theta = 0, \pi$ for the twist to be allowed. We provide a T-dual for each such twisted compactification.

For $\mathfrak{g} = \mathfrak{so}(8)$, we have an additional possibility of twisting the compactification by the \mathbb{Z}_3 outer-automorphism of $\mathfrak{so}(8)$, and we provide a T-dual for this twisted compactification.

- 6d theory with $\mathfrak{su}(n \geq 3)$ gauge algebra, 2 hypers in antisymmetric irrep Λ^2 and 16 hypers in fundamental irrep F. We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(n)$. We provide a T-dual for each such compactification.
- 6d theory with $\mathfrak{su}(2)$ gauge algebra and 16 hypers in fundamental irrep F. We provide a T-dual for the untwisted compactification.
- 6d theory with $\mathfrak{su}(n \geq 3)$ gauge algebra, 1 hyper in antisymmetric irrep Λ^2 and 1 hyper in symmetric irrep S^2 . We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(n)$. We provide a T-dual only for the untwisted compactification.
- 6d theory with $\mathfrak{su}(6)$ gauge algebra, 1 hyper in three-index antisymmetric irrep Λ^3 and 18 hypers in fundamental irrep F. We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(6)$. We provide a T-dual for each such compactification.

Note that in the T-dual for the untwisted compactification, we find a hitherto unknown discrete theta angle for an E-string $\mathfrak{sp}(0)$ node, that is distinct from $\theta = 0, \pi$. We call this as a **new theta angle**. See equation (3.72). The distinction between the various theta angles is field-theoretically described in terms of different embeddings of the gauge symmetries into the full \mathfrak{e}_8 flavor symmetry of the E-string theory, leaving behind different residual flavor symmetries. In this case, while the $\theta = 0, \pi$ have residual flavor symmetries $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ respectively, the new theta angle has residual flavor symmetry $\mathfrak{su}(3)$. The residual flavor symmetries can be determined using the respective surface geometries by finding (-2) -curves inside the compact surfaces that have zero intersection with all the gluing curves, which means that they can be attached to \mathbb{P}^1 fibers of non-compact Hirzebruch surfaces describing (non-abelian) flavor symmetries [27, 28]. The intersection pattern of these (-2) -curves determines the intersection pattern of the non-compact Hirzebruch surfaces, thus determining the form of the flavor symmetry algebra.

- 6d theory with $\mathfrak{su}(6)$ gauge algebra, 1 half-hyper in three-index antisymmetric irrep Λ^3 , 1 hyper in antisymmetric irrep Λ^2 and 17 hypers in fundamental irrep F. We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(6)$. We provide a T-dual for each such compactification.

³We leave the discussion of T-dualities for non-zero theta angles to future works. It is an open question whether there exist surface geometries for these LSTs, or if some modifications are needed.

- 6d theory with $\mathfrak{so}(2n+1)$ gauge algebra for $3 \leq n \leq 6$, 2^{6-n} hypers in spinor irrep S and $2n-3$ hypers in vector irrep F . We provide a T-dual for the untwisted compactifications.
- 6d theory with $\mathfrak{so}(8)$ gauge algebra, 4 hypers in spinor irrep S , 4 hypers in cospinor irrep C and 4 hypers in vector irrep F . We have the possibility of twisting the compactification by either the \mathbb{Z}_2 outer-automorphism of $\mathfrak{so}(8)$ or by the \mathbb{Z}_3 outer-automorphism of $\mathfrak{so}(8)$. We provide a T-dual for each such compactification.
- 6d theory with $\mathfrak{so}(10)$ gauge algebra, 4 hypers in spinor irrep S and 6 hypers in vector irrep F . We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{so}(10)$. We provide a T-dual for each such compactification.
- 6d theory with $\mathfrak{so}(12)$ gauge algebra, 2 hypers in spinor irrep S and 8 hypers in vector irrep F . We provide a T-dual for the untwisted compactification.
- 6d theory with $\mathfrak{so}(12)$ gauge algebra, 1 hyper in spinor irrep S , 1 hyper in cospinor irrep C and 8 hypers in vector irrep F . We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{so}(12)$. We provide a T-dual for each such compactification.
- 6d theory with $\mathfrak{so}(14)$ gauge algebra, 1 hyper in spinor irrep S and 10 hypers in vector irrep F . We have the possibility of twisting the compactification by the \mathbb{Z}_2 outer-automorphism of $\mathfrak{so}(14)$. We provide a T-dual for each such compactification.
- 6d theory with $\mathfrak{sp}(n \geq 2)$ gauge algebra, 1 hyper in antisymmetric irrep Λ^2 and 16 hypers in fundamental irrep F . We provide a T-dual for the untwisted compactification.
- 6d theory with $\mathfrak{sp}(3)$ gauge algebra, 1 half-hyper in three-index antisymmetric irrep Λ^3 and 35 half-hypers in fundamental irrep F . We provide a T-dual for the untwisted compactification.
- 6d theory with \mathfrak{g}_2 gauge algebra and 10 hypers in 7-dimensional irrep F . We provide a T-dual for the untwisted compactification.

2 Surface geometries for circle compactifications of LSTs

In this paper, we will use the conventions and notations for surfaces that were introduced in [22].

2.1 Graphs associated to circle compactifications of LSTs

Consider, as in the introduction, a circle compactification $\mathfrak{T}_{S^1}^\rho$ of a LST \mathfrak{T} twisted by ρ . To such a circle compactification, one can associate a graph which essentially encodes the information regarding the twist and the low-energy theory arising on the tensor branch of the LST. The construction of the graph was explained in [22] for circle compactifications of 6d SCFTs, and carries over in a similar fashion to circle compactifications of LSTs.

We will denote the nodes of the graph by i . Each node carries a non-negative integer k_i and a simple affine Lie algebra $\mathfrak{g}_i^{(q_i)}$ where \mathfrak{g}_i is a simple finite Lie algebra. The affine

algebra can be trivial, and there are two possible choices denoted by $\mathfrak{sp}(0)^{(1)}$ and $\mathfrak{su}(1)^{(1)}$. See [22] for more details.

An edge between two nodes can be directional or non-directional, and carries a positive integer label. Using the information about k_i and the edges, we can extract a matrix $[\Omega^{ij}]$ as described in [22]. For a circle compactification of a 6d SCFT $[\Omega^{ij}]$ is a positive-definite matrix, while for a circle compactification of an LST, $[\Omega^{ij}]$ is a positive semi-definite matrix with a 1-dimensional space of null vectors.

From the structure of the graph (including some additional decorations of nodes and edges), we can also extract a representation $R_{\mathfrak{g}}^\rho$ of a finite Lie algebra

$$\mathfrak{h} = \bigoplus_i \mathfrak{h}_i \tag{2.1}$$

where \mathfrak{h}_i is the subalgebra (which is tabulated in [22]) of \mathfrak{g}_i left invariant by the order q_i outer-automorphism used in the twist. The physical interpretation of $R_{\mathfrak{g}}^\rho$ is that the circle compactification $\mathfrak{T}_{S^1}^\rho$ of the 6d LST \mathfrak{T} admits a Coulomb branch phase in which at extremely low energies we find a 5d gauge theory with \mathfrak{h} gauge algebra and hypers transforming in representation $R_{\mathfrak{g}}^\rho$ of \mathfrak{h} .

2.2 Surface geometries from graphs

The information about the graph associated to $\mathfrak{T}_{S^1}^\rho$ can be converted into surface geometries describing various 5d $\mathcal{N} = 1$ supersymmetric Coulomb branch phases of $\mathfrak{T}_{S^1}^\rho$. Such a phase \mathcal{P} of $\mathfrak{T}_{S^1}^\rho$ can be constructed by compactifying M-theory on a smooth Calabi-Yau threefold $X_{\mathcal{P}}$. A 5d $\mathcal{N} = 1$ abelian gauge theory $\mathfrak{T}_{\mathcal{P}}$ with massive matter content describes the low-energy theory at a point in the interior of the phase \mathcal{P} . This gauge theory $\mathfrak{T}_{\mathcal{P}}$ can be identified as the low-energy theory obtained upon compactification of M-theory on $X_{\mathcal{P}}$.

Each $X_{\mathcal{P}}$ corresponds to a different surface geometry $\mathcal{G}_{\mathcal{P}}$. Surface geometries $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{G}_{\mathcal{P}'}$ for two different phases \mathcal{P} and \mathcal{P}' are related by a sequence of flop transitions. In this work, we will study phases for which the corresponding threefold $X_{\mathcal{P}}$ is a genus-one fibration. We will call such phases as *genus-one phases*. There always exists at least one genus-one phase for each $\mathfrak{T}_{S^1}^\rho$.

The surface geometry for any genus-one phase takes the following form. For each i , we have a collection of surfaces $S_{i,\alpha}$ where $0 \leq \alpha \leq r_i$ and r_i is the rank of \mathfrak{h}_i . Consider first the case when \mathfrak{h}_i is non-trivial, i.e. when $r_i > 0$. In this case, each $S_{i,\alpha}$ is a Hirzebruch surface with (typically non-generic) blowups. Let $f_{i,\alpha}$ be the \mathbb{P}^1 fiber of the Hirzebruch surface⁴ $S_{i,\alpha}$. These Hirzebruch surfaces must intersect such that we have

$$-f_{i,\alpha} \cdot S_{j,\beta} = \delta_{ij} C_{i,\alpha\beta} \tag{2.2}$$

where δ_{ij} is the Kronecker delta and $C_{i,\alpha\beta}$ is the Cartan matrix for $\mathfrak{g}_i^{(q_i)}$. Thus each $S_{i,\alpha}$ corresponds to a node in the Dynkin diagram of $\mathfrak{g}_i^{(q_i)}$, and our convention is that $S_{i,0}$ corresponds to the affine node, i.e. the nodes $S_{i,\alpha}$ for $\alpha \neq 0$ describe the Dynkin diagram of \mathfrak{h}_i .

⁴For us, a Hirzebruch surface is a surface with a chosen \mathbb{P}^1 fibration, which we denote by f . There might be other \mathbb{P}^1 fibers in the surface, which give different Hirzebruch surface descriptions for the same surface.

The curve

$$f_i := \sum_{\alpha} d_{i,\alpha} f_{i,\alpha} \quad (2.3)$$

where $d_{i,\alpha}$ is the row null vector of the Cartan matrix of $\mathfrak{g}_i^{(q_i)}$ with minimal positive integer values, is a degenerate genus-one fiber.

For trivial \mathfrak{h}_i , we have a single surface $S_{i,0}$ which arises only for the following cases:

- $\Omega^{ii} = 1$ without loop, in which case the surface $S_{i,0}$ is a Hirzebruch surface with 8 blowups, which admits an elliptic fibration. There are various ways of representing such a surface. The common representations used in this paper are as follows:
 - If we represent it as \mathbb{F}_0^8 , then the elliptic fiber is $f_i = 2e + 2f - \sum x_i$.
 - If we represent it as \mathbb{F}_1^8 , then the elliptic fiber is $f_i = 2h + f - \sum x_i$.
 - Finally, if we represent it as \mathbb{F}_2^8 , then the elliptic fiber is $f_i = 2h - \sum x_i$.

In these cases, we define a curve $E_{i,0} = f - x_1$ to be used later. The (trivial) algebra attached to such a node is $\mathfrak{g}_i^{(q_i)} = \mathfrak{sp}(0)^{(1)}$.

- $\Omega^{ii} = 2$ without loop, in which case the surface $S_{i,0}$ is a self-glued Hirzebruch surface which admits an elliptic fibration. In this paper, we typically use the following ways of representing this surface:
 - \mathbb{F}_0^2 with the two blowups x and y glued to each other. The elliptic fiber is $f_i = e + f - x - y$.
 - \mathbb{F}_0^2 with $e - x$ and $e - y$ glued to each other. The elliptic fiber is $f_i = f$.

In both of these cases, we define a curve $E_{i,0} = e$ to be used later. The (trivial) algebra attached to such a node is $\mathfrak{g}_i^{(q_i)} = \mathfrak{su}(1)^{(1)}$.

- $\Omega^{ii} = 2$ with a loop, in which case the surface $S_{i,0}$ is a self-glued Hirzebruch surface which in this paper, we represent this surface as \mathbb{F}_1^2 with the two blowups x and y glued to each other. It admits a special genus-one fiber $f_i = 2h + f - 2x - 2y$. We define a curve $E_{i,0} = e$ to be used later. The (trivial) algebra attached to such a node is $\mathfrak{g}_i^{(q_i)} = \mathfrak{su}(1)^{(1)}$.
- It is also possible to have $\Omega^{ii} = 1$ with a loop, which we do not discuss in this paper. The (trivial) algebra attached to such a node is $\mathfrak{g}_i^{(q_i)} = \mathfrak{sp}(0)^{(1)}$.

The curve f_i satisfies the property that

$$f_i \cdot S_{j,\beta} = 0 \quad (2.4)$$

for all i, j, β .

We have the property

$$f_{i,\alpha} \cdot S_j = 0 \quad (2.5)$$

for all i, α, j where for \mathfrak{h}_j non-trivial

$$S_j := \sum_{\beta} d_{j,\beta}^{\vee} S_{j,\beta} \tag{2.6}$$

and $d_{j,\beta}^{\vee}$ is a null column vector of the Cartan matrix of $\mathfrak{g}_j^{(q_j)}$ with minimal positive integer values, $S_i = S_{i,0}$ for trivial \mathfrak{h}_i , and $f_{i,0} := f_i$ (i.e. the elliptic fiber, not the \mathbb{P}^1 fiber of the Hirzebruch surface) for trivial \mathfrak{h}_i .

Additionally we have the property

$$-S_i \cdot E_{j,0} = \Omega^{ij} \tag{2.7}$$

where $E_{j,0} = e_{j,0}$ is the base curve of the Hirzebruch surface $S_{j,0}$ if \mathfrak{h}_j is non-trivial, and $E_{j,0}$ for \mathfrak{h}_j trivial was defined above.

The blowups living in the surfaces $S_{i,\alpha}$ for non-trivial \mathfrak{h}_i describe hypermultiplets charged under the gauge algebra $\mathfrak{h} = \oplus_i \mathfrak{h}_i$. The map between the blowups and representations of hypers is as follows. Pick a blowup x of a surface $S_{i,\alpha}$ such that \mathfrak{h}_i is non-trivial. It lives in a family \mathcal{F} of blowups, where some other blowup y of some other surface $S_{j,\beta}$ (such that \mathfrak{h}_j is non-trivial) is in the same family \mathcal{F} if and only if one of the following conditions hold:

- x is glued to y .
- x is glued to $f - y$.
- $f - x$ is glued to y .
- If $(i, \alpha) = (j, \beta)$, i.e. if x and y live in the same surface, and $f_{i,\alpha} - x - y$ is glued to $f_{k,\gamma}$ for some other surface $S_{k,\gamma}$.
- If $(i, \alpha) = (j, \beta)$, i.e. if x and y live in the same surface, and $x - y$ is glued to $f_{k,\gamma}$ for some other surface $S_{k,\gamma}$.

Such a family \mathcal{F} of blowups describes a hypermultiplet living in a representation $R_{\mathcal{F}}$ of \mathfrak{h} . The information about this representation can be described in terms of its weights which are obtained as follows. For every blowup $x \in \mathcal{F}$ compute its intersections

$$-x \cdot S_{k,\gamma} \tag{2.8}$$

with all surfaces $S_{k,\gamma}$ with $\gamma \neq 0$ for non-trivial \mathfrak{h}_k . These intersection numbers describe the Dynkin coefficients for a weight w_x of $R_{\mathcal{F}}$. Collect all weights w_x for all blowups $x \in \mathcal{F}$. Then $R_{\mathcal{F}}$ is recovered as the representation with minimal number of weights that contain the weights w_x for all $x \in \mathcal{F}$. The full hypermultiplet content carried by the 5d gauge theory associated to $\mathfrak{X}_{S^1}^{\rho}$ is recovered by combining the representations $R_{\mathcal{F}}$ for all families \mathcal{F} .

The intersection number

$$x \cdot S_{k,0} \tag{2.9}$$

for a blowup x living in surface $S_{i,\alpha}$ with non-trivial \mathfrak{h}_i , and for all k , is constrained by the property that

$$x \cdot S_k = 0 \tag{2.10}$$

Finally, the gluings of curves are constrained such that taking combinations of the gluing rules we find that a representative of a multiple of f_i is glued to a representative of a multiple of f_j if there is an edge between nodes i and j . More precisely,

- If there is an undirected edge carrying a label n between i and j , then the gluing rules imply the gluing

$$nf_i \sim nf_j \tag{2.11}$$

but not $mf_i \sim mf_j$ for any $m < n$.

- If there is a directed edge from i to j carrying label n , then the gluing rules imply the gluing

$$f_i \sim nf_j \tag{2.12}$$

2.3 T-dualities from local S-dualities

Consider a circle compactification $\mathfrak{T}_{1,S^1}^{\rho_1}$ of an LST \mathfrak{T}_1 and a circle compactification $\mathfrak{T}_{2,S^1}^{\rho_2}$ of an LST \mathfrak{T}_2 . We say that $\mathfrak{T}_{1,S^1}^{\rho_1}$ and $\mathfrak{T}_{2,S^1}^{\rho_2}$ are *T-dual* if we find a surface geometry \mathcal{G} that describes both $\mathfrak{T}_{1,S^1}^{\rho_1}$ and $\mathfrak{T}_{2,S^1}^{\rho_2}$.

In this paper, we will only consider a subset of such T-dualities which can be described as follows. Let \mathcal{G}_1 be a surface geometry for a genus-one phase of $\mathfrak{T}_{1,S^1}^{\rho_1}$, which means that \mathcal{G}_1 comes with a preferred choice of \mathbb{P}^1 fibers for the Hirzebruch surfaces constituting \mathcal{G}_1 . Similarly, let \mathcal{G}_2 be a surface geometry for a genus-one phase of $\mathfrak{T}_{2,S^1}^{\rho_2}$, which means that \mathcal{G}_2 comes with a preferred choice of \mathbb{P}^1 fibers for the Hirzebruch surfaces constituting \mathcal{G}_2 . Now exchange e and f curves of a subset of degree-0 Hirzebruch surfaces in \mathcal{G}_1 , which can be thought of as performing *local S-dualities* at the location of these surfaces [20]. Assume \mathcal{G}_1 is converted to \mathcal{G}_2 after this exchange.⁵ Then, \mathcal{G}_1 and \mathcal{G}_2 are isomorphic as surface geometries, that is they describe the same surface geometry once we forget the special choice of \mathbb{P}^1 fibers, and we have found that $\mathfrak{T}_{1,S^1}^{\rho_1}$ is T-dual to $\mathfrak{T}_{2,S^1}^{\rho_2}$.

3 Examples of T-dualities

3.1 Gauge rank 1

At gauge rank 1 in 6d, we have three LSTs

- $\mathfrak{su}(2)$ with 16 hypers in fundamental. We represent this theory by the graph

$$\begin{array}{c} \mathfrak{su}(2) \\ 0 \end{array} \tag{3.1}$$

- $\mathfrak{su}(2)$ with a hyper in adjoint and discrete theta angle 0. We represent this theory by the graph

$$\begin{array}{c} \mathfrak{su}(2)_0 \\ 0_1 \end{array} \tag{3.2}$$

⁵As we will see in examples, one might also need to interpret some of the Hirzebruch surfaces as describing nodes with trivial \mathfrak{h}_i .

where the subscript 1 is placed to distinguish it from the graph associated to the previous LST, and indicates that the base curve used in F-theory construction of the LST has genus 1 (while the base curve for previous LST has genus 0). This theory actually has $\mathcal{N} = (1, 1)$ supersymmetry.

- $\mathfrak{su}(2)$ with a hyper in adjoint and discrete theta angle π . We represent this theory by the graph

$$\begin{array}{c} \mathfrak{su}(2)_\pi \\ 0_1 \end{array} \tag{3.3}$$

This theory also has $\mathcal{N} = (1, 1)$ supersymmetry.

Only untwisted circle compactification is possible for all three LSTs.

For the untwisted circle compactification of the first LST, a surface geometry is

$$\mathbf{0}_0^8 \xrightarrow{2e+2f-\sum x_i} \mathbf{1}_0^8 \tag{3.4}$$

Each surface can also be interpreted as a surface associated to a node with $\Omega = 1$ and trivial gauge algebra. Thus, we have the T-duality

$$\boxed{\begin{array}{ccc} \mathfrak{su}(2)^{(1)} & = & \mathfrak{sp}(0)^{(1)} \quad \mathfrak{sp}(0)^{(1)} \\ 0 & & 1 \text{ --- } 1 \end{array}} \tag{3.5}$$

For the untwisted circle compactification of the second LST, a surface geometry is

$$x \begin{array}{c} \mathbf{0}_0^{1+1} \\ \text{---} \\ y \end{array} \xrightarrow{2e+f-x-2y, f-x} 2 \xrightarrow{2e+f-x-2y, f-x} \mathbf{1}_0^{1+1} \begin{array}{c} x \\ \text{---} \\ y \end{array} \tag{3.6}$$

Each surface can also be interpreted as a surface associated to a node with $\Omega = 2$ and trivial gauge algebra. Thus, we have the T-duality

$$\boxed{\begin{array}{ccc} \mathfrak{su}(2)_0^{(1)} & = & \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\ 0_1 & & 2 \text{ --- } 2 \text{ --- } 2 \end{array}} \tag{3.7}$$

where we have a double edge because the gluing rules imply that $2e + 2f - 2x - 2y$ in S_0 is glued to $2e + 2f - 2x - 2y$ in S_1 , both of which are twice the elliptic fibers $e + f - x - y$. The right hand side denotes the untwisted circle compactification of the A_1 $\mathcal{N} = (2, 0)$ LST.

For the untwisted circle compactification of the third LST, a surface geometry is

$$x \begin{array}{c} \mathbf{0}_0^{1+1} \\ \text{---} \\ y \end{array} \xrightarrow{2e+f-x-2y, f-x} 2 \xrightarrow{2h-x-2y, f-x} \mathbf{1}_1^{1+1} \begin{array}{c} x \\ \text{---} \\ y \end{array} \tag{3.8}$$

The surface S_0 can also be interpreted as a surface associated to a node with $\Omega = 2$ and trivial gauge algebra, while the surface S_1 can also be interpreted as a surface associated to a node with value 2 (but $\Omega = 1$) and a loop along with trivial gauge algebra. Thus, we have the T-duality

$$\boxed{
 \begin{array}{c}
 \mathfrak{su}(2)_\pi^{(1)} \\
 0_1
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \leftarrow 2 \text{ --- } 2 \\
 \text{ } \quad \quad \quad \cup
 \end{array}
 }
 \tag{3.9}$$

where we have a directed edge with 2 in the middle because the gluing rules imply that twice the elliptic fiber $2e + 2f - 2x - 2y$ of S_0 is glued to the genus-one fiber $2h + f - 2x - 2y$ of S_1 . On the right hand side of the above equality, we have a twisted circle compactification of the $D_5 \mathcal{N} = (2, 0)$ LST. The twist is performed using a \mathbb{Z}_4 0-form symmetry of the $D_5 \mathcal{N} = (2, 0)$ LST whose action on the nodes of the affine $\mathfrak{so}(10)^{(1)}$ algebra is such that the affine node goes to spinor node, the spinor node goes to vector node, the vector node goes to cospinor node, the cospinor node goes to affine node, and the other two remaining (trivalent) nodes are exchanged with each other.

3.2 Gauge rank 2

At gauge rank 2 in 6d, we have the following LSTs and their possible circle compactifications

- $\mathfrak{su}(3)$ with 18 hypers in fundamental. We represent this theory by the graph

$$\begin{array}{c}
 \mathfrak{su}(3) \\
 0
 \end{array}
 \tag{3.10}$$

This theory can have twisted compactification by the \mathbb{Z}_2 outer-automorphism.

- $\mathfrak{su}(3)$ with a hyper in 2-index symmetric and a hyper in fundamental. We represent this theory by the graph

$$\begin{array}{c}
 \mathfrak{su}(\widehat{3}) \\
 0
 \end{array}
 \tag{3.11}$$

This theory can have twisted compactification by the \mathbb{Z}_2 outer-automorphism but we do not discuss its T-dual.

- $\mathfrak{su}(3)$ with a hyper in adjoint and rational theta angle $2\pi p/q$. We represent this theory by the graph

$$\begin{array}{c}
 \mathfrak{su}(3)_{2\pi p/q} \\
 0_1
 \end{array}
 \tag{3.12}$$

where the subscript 1 indicates that the base curve used in F-theory construction of the LST has genus 1. This theory actually has $\mathcal{N} = (1, 1)$ supersymmetry.

We only discuss T-duals for untwisted compactification of the $\theta = 0$ LST in the general rank subsection. A compactification twisted by \mathbb{Z}_2 outer-automorphism is possible only for $\theta = 0, \pi$, and we discuss both of these cases in this subsection.

- $\mathfrak{sp}(2)$ with a hyper in antisymmetric and 16 hypers in fundamental. We represent this theory by the graph

$$\begin{array}{c} \mathfrak{sp}(2) \\ 0 \end{array} \tag{3.13}$$

We will discuss two T-duals for the untwisted compactification of this LST. We discuss one in this subsection, and the other in the subsection on general gauge rank.

- \mathfrak{g}_2 with 10 hypers in irrep of dimension 7. We represent this theory by the graph

$$\begin{array}{c} \mathfrak{g}_2 \\ 0 \end{array} \tag{3.14}$$

First of all, consider the surface geometry

$$\begin{array}{c} \mathbf{1}_0^5 \\ \begin{array}{l} \nearrow^e \\ \searrow_e \end{array} \\ \begin{array}{l} h+2f-\sum x_i \\ \mathbf{2}_2^8 \\ e \end{array} \\ \begin{array}{l} \nearrow_e \\ \searrow_e \end{array} \\ \mathbf{0}_0^5 \end{array} \begin{array}{l} e+2f-\sum x_i \\ e+2f-\sum x_i \end{array} \tag{3.15}$$

which describes the untwisted compactification of the $\mathfrak{su}(3)$ theory with 18 fundamentals. Exchanging the \mathbb{P}^1 fibers e and f in the surfaces S_1 and S_0 , we can rewrite the surface geometry as

$$\begin{array}{c} \mathbf{1}_0^5 \\ \begin{array}{l} \nearrow^f \\ \searrow_f \end{array} \\ \begin{array}{l} h+2f-\sum x_i \\ \mathbf{2}_2^8 \\ e \end{array} \\ \begin{array}{l} \nearrow_e \\ \searrow_e \end{array} \\ \mathbf{0}_0^5 \end{array} \begin{array}{l} 2e+f-\sum x_i \\ 2e+f-\sum x_i \end{array} \tag{3.16}$$

The surfaces S_0 and S_1 now describe an $\mathfrak{sp}(1)^{(1)}$ and the surface S_2 can be recognized to be describing $\mathfrak{sp}(0)^{(1)}$, leading to the T-duality

$$\boxed{\begin{array}{ccc} \mathfrak{su}(3)^{(1)} & = & \mathfrak{sp}(0)^{(1)} \quad \mathfrak{sp}(1)^{(1)} \\ 0 & & 1 \text{ --- } 1 \end{array}} \tag{3.17}$$

Now consider another surface geometry for the untwisted compactification of the $\mathfrak{su}(3) + 18F$ theory

$$\begin{array}{c}
 \mathbf{1}_0^{4+1} \\
 \begin{array}{l}
 e+2f-\sum x_i \\
 h+2f-\sum x_i \\
 \mathbf{2}_2^8 \\
 e \\
 e+2f-\sum x_i
 \end{array}
 \begin{array}{l}
 / \\
 / \\
 \\
 \backslash \\
 \backslash
 \end{array}
 \begin{array}{l}
 | \\
 | \\
 | \\
 | \\
 |
 \end{array}
 \mathbf{0}_0^{4+1} \\
 \begin{array}{l}
 e-y \\
 \\
 e-y \\
 \\
 e-y
 \end{array}
 \end{array}
 \tag{3.18}$$

Now exchanging e and $f \mathbb{P}^1$ fibers in surfaces S_0 and S_1 leads to the surface geometry

$$\begin{array}{c}
 \mathbf{1}_0^{4+1} \\
 \begin{array}{l}
 2e+f-\sum x_i \\
 h+2f-\sum x_i \\
 \mathbf{2}_2^8 \\
 e \\
 2e+f-\sum x_i
 \end{array}
 \begin{array}{l}
 / \\
 / \\
 \\
 \backslash \\
 \backslash
 \end{array}
 \begin{array}{l}
 | \\
 | \\
 | \\
 | \\
 |
 \end{array}
 \mathbf{0}_0^{4+1} \\
 \begin{array}{l}
 f-y \\
 \\
 f-y \\
 \\
 f-y
 \end{array}
 \end{array}
 \tag{3.19}$$

The three surfaces now describe an $\mathfrak{sp}(2)^{(1)}$, leading to the T-duality

$$\boxed{
 \begin{array}{ccc}
 \mathfrak{su}(3)^{(1)} & = & \mathfrak{sp}(2)^{(1)} \\
 0 & & 0
 \end{array}
 }
 \tag{3.20}$$

where the right hand side is the untwisted compactification of the $\mathfrak{sp}(2) + \Lambda^2 + 16F$ LST.

Moving onto twisted compactifications of the $\mathfrak{su}(3) + 18F$ LST, we have the following surface geometry

$$\mathbf{0}_0^{1+1} \xrightarrow[e+f-x-y]{4e+4f-2\sum x_i} \mathbf{1}_0^8$$


$$\tag{3.21}$$

The surface S_0 can be recognized as a surface for $\mathfrak{su}(1)^{(1)}$ and the surface S_1 can be recognized as surface for $\mathfrak{sp}(0)^{(1)}$. From the gluing of the elliptic fibers, we read the precise edge between the two nodes, leading to the T-duality

$$\boxed{
 \begin{array}{ccc}
 \mathfrak{su}(3)^{(2)} & = & \mathfrak{su}(1)^{(1)} \quad \mathfrak{sp}(0)^{(1)} \\
 0 & & 2 \text{ --- } 2 \rightarrow 1
 \end{array}
 }
 \tag{3.22}$$

For the twisted compactification of the $\mathfrak{su}(3)_0 + \mathbf{A} \mathcal{N} = (1, 1)$ LST, we have the following surface geometry

$$\begin{array}{ccc}
 \mathbf{0}_0^{1+1} & \xrightarrow{e-y, f-x} & \mathbf{1}_0^{1+1} \\
 \text{circle} & \xrightarrow{2} & \text{circle} \\
 x & & x \quad y
 \end{array}
 \tag{3.23}$$

We can interpret both S_0 and S_1 as describing $\mathfrak{su}(2)^{(1)}$, leading to the T-duality

$$\boxed{
 \begin{array}{ccc}
 \mathfrak{su}(3)_0^{(2)} & = & \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 0_1 & & 2 \text{ --- } 4 \rightarrow 2
 \end{array}
 }
 \tag{3.24}$$

The right hand side is a circle compactification of $D_4 \mathcal{N} = (2, 0)$ LST twisted by the \mathbb{Z}_4 symmetry rotating the affine, spinor, cospinor and vector nodes of $\mathfrak{so}(8)^{(1)}$ Dynkin diagram into each other.

For the twisted compactification of the $\mathfrak{su}(3)_\pi + \mathbf{A} \mathcal{N} = (1, 1)$ LST, we have the following surface geometry

$$\begin{array}{ccc}
 \mathbf{0}_0^{1+1} & \xrightarrow{e-y, f-x} & \mathbf{1}_0^{1+1} \\
 \text{circle} & \xrightarrow{2} & \text{circle} \\
 x & & x \quad y
 \end{array}
 \tag{3.25}$$

In a similar way as above, we are led to the T-duality

$$\boxed{
 \begin{array}{ccc}
 \mathfrak{su}(3)_\pi^{(2)} & = & \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 0_1 & & 2 \text{ --- } 2 \rightarrow 2 \\
 & & \text{circle}
 \end{array}
 }
 \tag{3.26}$$

The right hand side is a circle compactification of $A_2 \mathcal{N} = (2, 0)$ LST twisted by the \mathbb{Z}_2 symmetry exchanging two adjacent nodes of the $\mathfrak{su}(3)^{(1)}$ Dynkin diagram.

Now, consider the following surface geometry

$$\begin{array}{ccc}
 & & \mathbf{1}_0^1 \\
 & \nearrow^{e-x, x} & \downarrow^{e+f, f-x} \\
 & & 2 \\
 \text{circle} & \xrightarrow{h+f-x-2y, f-x} & \text{circle} \\
 x & & x \quad y \\
 & \searrow_e & \downarrow^{e+f-x-2y-z, f-x} \\
 & & \mathbf{0}_0^{1+1+1} \\
 & & \text{circle} \\
 & & x \quad y
 \end{array}
 \tag{3.27}$$

describes the \mathbb{Z}_2 outer-automorphism twisted compactification of the $\mathfrak{su}(4)_0 + \mathbf{A} \mathcal{N} = (1, 1)$ LST. Each surface can be interpreted as describing $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of $A_3 \mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry exchanging the fundamental and anti-fundamental nodes, while leaving the other two nodes invariant of the Dynkin diagram of $\mathfrak{su}(4)^{(1)}$

$$\boxed{\begin{array}{c} \mathfrak{su}(4)_0^{(2)} \\ 0_1 \end{array} = \begin{array}{c} \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\ 2 \text{ --- } 2 \text{ --- } 2 \text{ --- } 2 \end{array}} \quad (3.34)$$

The various $\mathfrak{su}(1)^{(1)}$ on the right hand side form the Dynkin diagram of $\mathfrak{sp}(2)^{(1)}$. Note that $\mathfrak{su}(4)^{(2)}$ and $\mathfrak{sp}(2)^{(1)}$ are Langlands dual, i.e. their Dynkin diagrams are related by reversing the directions of arrows.

The surface geometry

$$\begin{array}{c} \mathbf{0}_0^{1+1} \xrightarrow{e-y, f-x} 2 \xrightarrow{2h-y-2x, f-y} \mathbf{1}_0^{1+1} \xrightarrow{2h-x-2y, f-x} 2 \xrightarrow{e-x, f-y} \mathbf{2}_0^{1+1} \\ x \text{ () } y \quad \quad \quad x \text{ () } y \quad \quad \quad x \text{ () } y \end{array} \quad (3.35)$$

describes the \mathbb{Z}_2 outer-automorphism twisted compactification of the $\mathfrak{su}(4)_\pi + \mathbf{A} \mathcal{N} = (1, 1)$ LST. Each surface can be interpreted as describing $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of $D_5 \mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry exchanging the affine and spinor nodes, the vector and cospinor nodes, and the two trivalent nodes of the Dynkin diagram of $\mathfrak{so}(10)^{(1)}$

$$\boxed{\begin{array}{c} \mathfrak{su}(4)_\pi^{(2)} \\ 0_1 \end{array} = \begin{array}{c} \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\ 2 \text{ --- } 2 \text{ --- } 2 \\ \text{ () } \end{array}} \quad (3.36)$$

The surface geometry

$$\begin{array}{c} \mathbf{0}_0^{4+2+2} \xrightarrow{2e+f-\sum x_i-\sum y_i} e \mathbf{1}_0^4 \xrightarrow{e+2f-\sum x_i} e \mathbf{2}_2^4 \xrightarrow{h-\sum x_i} 2e+f-\sum x_i \mathbf{3}_0^4 \\ f-y_1-y_2 \text{ () } y_i-z_i \text{ () } f \end{array} \quad (3.37)$$

describes untwisted circle compactification of the LST $\mathfrak{sp}(3) + \Lambda^2 + 16\mathbf{F}$. Exchanging e and f on surface S_1 leads to the following T-duality

$$\boxed{\begin{array}{c} \mathfrak{sp}(3)^{(1)} \\ 0 \end{array} = \begin{array}{c} \mathfrak{sp}(0)^{(1)} \quad \mathfrak{sp}(2)^{(1)} \\ 1 \text{ --- } 1 \end{array}} \quad (3.38)$$

The surface geometry

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & & f-x & \circlearrowleft & f-y \\
 \mathbf{0}_0^{1+5} & \xrightarrow{2e+f-x-\sum y_i} & \xrightarrow{e+2f-\sum x_i} & \mathbf{1}_0^4 & \xrightarrow{e} & \xrightarrow{e-x-y} & \mathbf{2}_0^{1+1} & \xrightarrow{e} & \xrightarrow{2e+2f-\sum x_i} & \mathbf{3}_0^8 \\
 & \underbrace{\hspace{10em}}_{x, f-x} & & & & & \underbrace{\hspace{10em}}_{x, y} & & & \\
 & & & & & & & & & 2
 \end{array} \\
 \end{array}
 \tag{3.39}$$

describes untwisted circle compactification of the LST $\mathfrak{sp}(3) + \frac{1}{2}\Lambda^3 + \frac{35}{2}F$. Exchanging e and f on surface S_1 leads to the following T-duality

$ \begin{array}{cccc} \mathfrak{sp}(\widehat{3})^{(1)} & = & \mathfrak{sp}(1)^{(1)} & \mathfrak{su}(1)^{(1)} & \mathfrak{sp}(0)^{(1)} \\ 0 & & 1 & \text{---} & 2 & \text{---} & 1 \end{array} $	(3.40)
---	--------

3.4 LSTs with \mathfrak{so} gauge algebra

3.4.1 Twisted compactifications

The surface geometry

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \mathbf{0}_{10} & \xrightarrow{e} & \xrightarrow{e+4f} & \mathbf{1}_0 & \xrightarrow{e} & \xrightarrow{3e+3f-2\sum x_i-\sum y_i} & \mathbf{2}_0^{4+4} \\
 & \underbrace{\hspace{10em}}_{f} & & & & & \underbrace{\hspace{10em}}_{x_i-y_i} \\
 & & & & & & & & & 4
 \end{array} \\
 \end{array}
 \tag{3.41}$$

describes compactification of LST $\mathfrak{so}(8) + 4F + 4S + 4C$ with a \mathbb{Z}_3 outer-automorphism twist. Exchanging e and f on S_1 leads to the T-duality

$ \begin{array}{cccc} \mathfrak{so}(8)^{(3)} & = & \mathfrak{su}(3)^{(2)} & \mathfrak{sp}(0)^{(1)} \\ 0 & & 3 & \text{---} & 3 & \text{---} & 1 \end{array} $	(3.42)
--	--------

where $\mathfrak{su}(3)^{(2)}$ is formed by the surfaces S_0 and S_1 .

The surface geometry

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \mathbf{0}_6 & \xrightarrow{e} & \xrightarrow{2e+4f-2\sum x_i} & \mathbf{1}_0^3 & \xrightarrow{e} & \xrightarrow{e+f-\sum x_i} & \mathbf{2}_2^{4+4} & \xrightarrow{2e+2f-\sum x_i-\sum y_i} & \xrightarrow{e+f-x-y} & \mathbf{3}_0^{1+1} \\
 & \underbrace{\hspace{10em}}_{f} & & & & & \underbrace{\hspace{10em}}_{x_i-y_i} & & & \underbrace{\hspace{10em}}_{\substack{x \\ y}} \\
 & & & & & & & & & 4
 \end{array} \\
 \end{array}
 \tag{3.43}$$

describes compactification of LST $\mathfrak{so}(8) + 4F + 4S + 4C$ with a \mathbb{Z}_2 outer-automorphism twist. Exchanging e and f on S_1 leads to the T-duality

$ \begin{array}{cccc} \mathfrak{so}(8)^{(2)} & = & \mathfrak{su}(1)^{(1)} & \mathfrak{sp}(0)^{(1)} & \mathfrak{su}(3)^{(2)} \\ 0 & & 2 & \text{---} & 1 & \text{---} & 2 \end{array} $	(3.44)
---	--------

where $\mathfrak{su}(3)^{(2)}$ is formed by the surfaces S_0 and S_1 .

Similarly, the surface geometry

$$\begin{array}{ccccc}
 \mathbf{1}_2 & \xrightarrow{e} & \mathbf{2}_0^4 & \xrightarrow{e} & \mathbf{0}_2 \\
 & \searrow f & & \nearrow f & \\
 & & \mathbf{3}_2^{2+2+2+2} & & \\
 & & \downarrow e & & \\
 & & \mathbf{4}_0^{1+1} & & \\
 & & \uparrow e+f-x-y & & \\
 & & \downarrow 2h-\sum x_i-\sum y_i-\sum z_i-\sum w_i & & \\
 & & \downarrow y & & \\
 & & \mathbf{4}_0^{1+1} & & \\
 & & \uparrow x & &
 \end{array}
 \tag{3.53}$$

describes untwisted compactification of LST $\mathfrak{so}(9) + 5F + 4S$. Exchanging e and f on S_2 leads to the T-duality

$$\boxed{
 \begin{array}{cccc}
 \mathfrak{so}(9)^{(1)} & = & \mathfrak{su}(1)^{(1)} & \mathfrak{sp}(0)^{(1)} & \mathfrak{su}(4)^{(2)} \\
 0 & & 2 & \text{---} & 1 & \text{---} & 2
 \end{array}
 }
 \tag{3.54}$$

where $\mathfrak{su}(4)^{(2)}$ is formed by the surfaces S_0, S_1 and S_2 .

In a similar fashion, we can deduce the following T-dualities

$$\boxed{
 \begin{array}{cccc}
 \mathfrak{so}(11)^{(1)} & = & \mathfrak{su}(1)^{(1)} & \mathfrak{sp}(0)^{(1)} & \mathfrak{su}(6)^{(2)} \\
 0 & & 2 & \text{---} & 1 & \text{---} & 2
 \end{array}
 }
 \tag{3.55}$$

$$\boxed{
 \begin{array}{cccc}
 \mathfrak{so}(13)^{(1)} & = & \mathfrak{su}(1)^{(1)} & \mathfrak{sp}(0)^{(1)} & \mathfrak{su}(8)^{(2)} \\
 0 & & 2 & \text{---} & 1 & \text{---} & 2
 \end{array}
 }
 \tag{3.56}$$

The surface geometry

$$\begin{array}{ccccc}
 \mathbf{1}_0^2 & \xrightarrow{e} & \mathbf{2}_0^{2+2+2+2} & \xrightarrow{e} & \mathbf{4}_0^2 \\
 f-x_i \downarrow & & & & \downarrow f-x_i \\
 2 & & & & 2 \\
 f-x_i \downarrow & & & & \downarrow f-x_i \\
 \mathbf{0}_0^2 & \xrightarrow{e} & \mathbf{3}_0^2 & \xrightarrow{e} & \mathbf{3}_0^2 \\
 & \nearrow e+f-\sum y_i-\sum w_i & & \searrow e+f-\sum z_i-\sum w_i & \\
 & & & &
 \end{array}
 \tag{3.57}$$

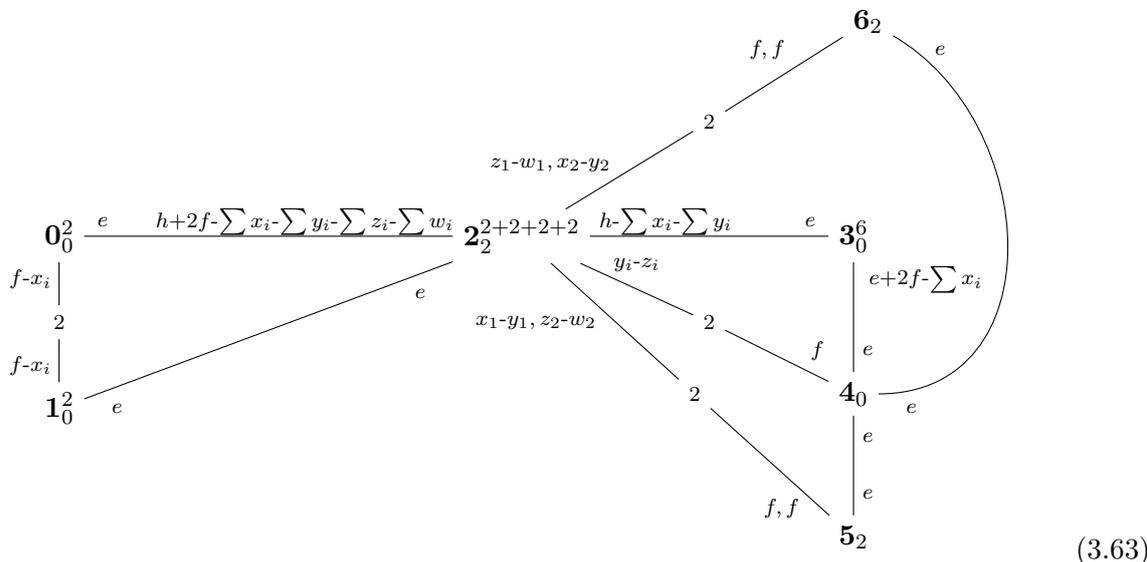
describes untwisted compactification of LST $\mathfrak{so}(8) + 4F + 4S + 4C$. Exchanging e and f on S_0, S_1, S_3 and S_4 leads to the T-duality

$$\boxed{
 \begin{array}{cccc}
 \mathfrak{so}(8)^{(1)} & = & \mathfrak{su}(2)^{(1)} & \mathfrak{sp}(0)^{(1)} & \mathfrak{su}(2)^{(1)} \\
 0 & & 2 & \text{---} & 1 & \text{---} & 2
 \end{array}
 }
 \tag{3.58}$$

where one $\mathfrak{su}(2)^{(1)}$ is formed by the surfaces S_0 and S_1 , and the other $\mathfrak{su}(2)^{(1)}$ is formed by the surfaces S_3 and S_4 .

where $\mathfrak{su}(6)^{(2)}$ is formed by the surfaces S_3, S_4, S_5 and S_6 , and $\mathfrak{su}(2)^{(1)}$ is formed by the surfaces S_0 and S_1 . The subscript on $\mathfrak{sp}(0)$ denotes that theta angle is 0, which means that the flavor symmetry attached to the node is $\mathfrak{u}(2)$.

The surface geometry

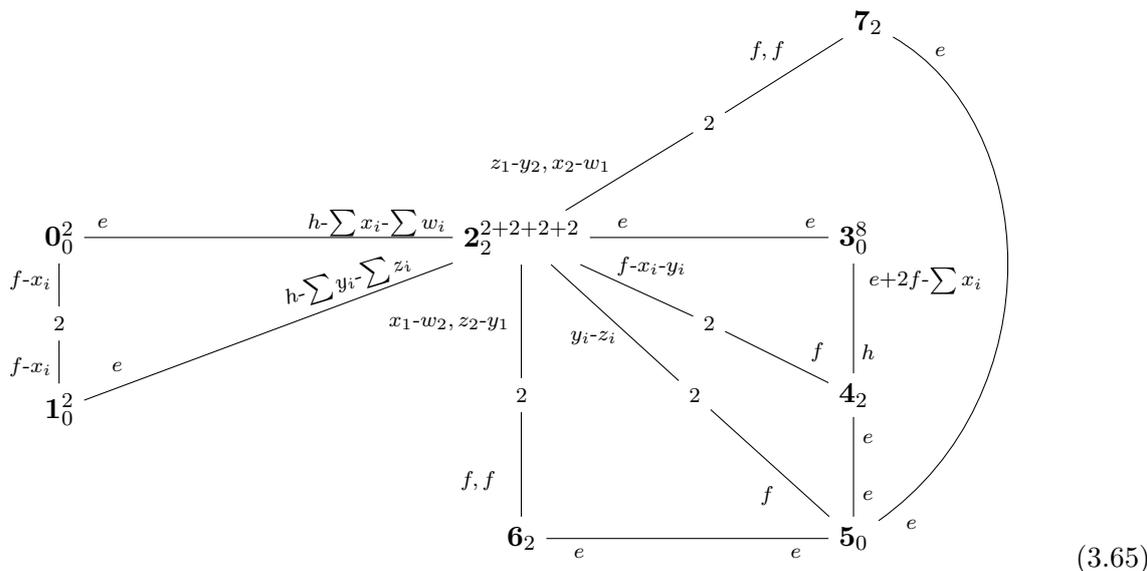


describes untwisted compactification of LST $\mathfrak{so}(12) + 8F + S + C$. Exchanging e and f on S_0, S_1 and S_3 leads to the T-duality

$$\boxed{\begin{array}{c} \mathfrak{so}(\widehat{12})^{(1)} \\ 0 \end{array}} = \begin{array}{c} \mathfrak{su}(2)^{(1)} \\ 2 \end{array} \text{---} \begin{array}{c} \mathfrak{sp}(0)_\pi^{(1)} \\ 1 \end{array} \text{---} \begin{array}{c} \mathfrak{su}(6)^{(2)} \\ 2 \end{array} \quad (3.64)$$

where $\mathfrak{su}(6)^{(2)}$ is formed by the surfaces S_3, S_4, S_5 and S_6 , and $\mathfrak{su}(2)^{(1)}$ is formed by the surfaces S_0 and S_1 . The subscript on $\mathfrak{sp}(0)$ denotes that theta angle is π , which means that the flavor symmetry attached to the node is $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$.

The surface geometry



The surface geometry

$$\begin{array}{c}
 \mathbf{3}_0^{8+1+1} \xrightarrow{2e+f-\sum x_i-y-z} \mathbf{2}_0 \xrightarrow{e+2f} \mathbf{1}_2^{1+1} \xrightarrow{e} \mathbf{0}_2^{1+1} \\
 \begin{array}{c}
 \text{---} \xrightarrow{e} \text{---} \xrightarrow{e} \text{---} \\
 \text{---} \xrightarrow{f-x, f-y} \text{---} \xrightarrow{f-x, f-y} \text{---} \\
 \text{---} \xrightarrow{f-y, y} \text{---} \xrightarrow{f-y, y} \text{---} \\
 \text{---} \xrightarrow{f-z, z} \text{---} \xrightarrow{f-z, z} \text{---}
 \end{array}
 \end{array}
 \quad (3.77)$$

describes twisted compactification of LST $\mathfrak{su}(6) + 18F + \Lambda^3$. Exchanging e and f on S_2 leads to the T-duality

$$\boxed{
 \begin{array}{c}
 \mathfrak{su}(\tilde{6})^{(2)} \\
 0
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{sp}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \text{ --- } 1 \text{ --- } 2
 \end{array}
 }
 \quad (3.78)$$

where $\mathfrak{sp}(1)^{(1)}$ is formed by the surfaces S_2 and S_3 .

3.6 General gauge rank

3.6.1 $\mathcal{N} = (1, 0)$ examples

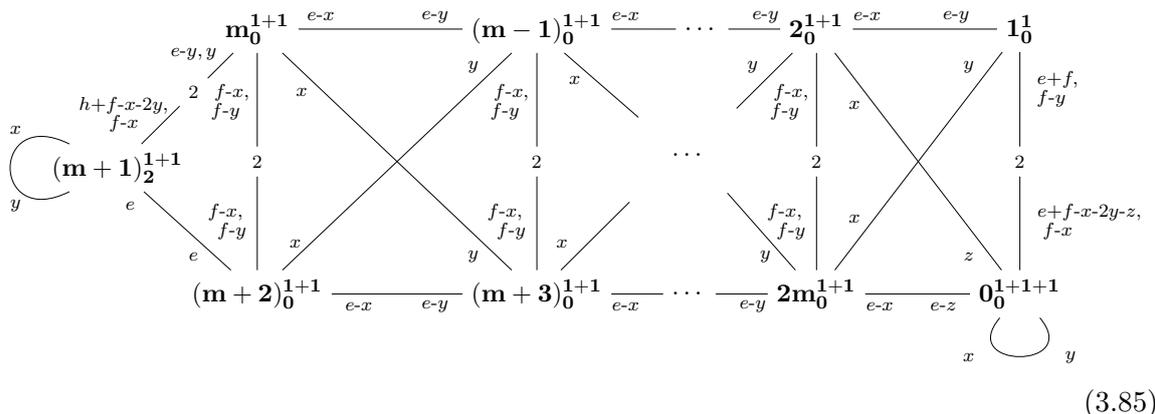
The surface geometry

$$\begin{array}{c}
 (\mathbf{m} + 1)_1^8 \\
 \left| \begin{array}{c} 2h+f-\sum x_i \\ e+f-x-y \end{array} \right. \\
 \mathbf{m}_0^{1+1} \xrightarrow{e-y, f-x} \mathbf{2} \xrightarrow{e-x, f-y} (\mathbf{m} - 1)_0^{1+1} \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} \mathbf{2}_0^{1+1} \xrightarrow{e-y, f-x} \mathbf{2} \xrightarrow{e-x, f-y} \mathbf{1}_0^{1+1} \\
 \begin{array}{c}
 \text{---} \xrightarrow{e-y, f-x} \text{---} \xrightarrow{e-y, f-x} \text{---} \\
 \text{---} \xrightarrow{e-x, f-y} \text{---} \xrightarrow{e-x, f-y} \text{---} \\
 \text{---} \xrightarrow{e-y, f-x} \text{---} \xrightarrow{e-y, f-x} \text{---} \\
 \text{---} \xrightarrow{e-x, f-y} \text{---} \xrightarrow{e-x, f-y} \text{---}
 \end{array}
 \end{array}
 \quad (3.79)$$

can be interpreted as describing untwisted compactification of $\mathfrak{sp}(m + 1) + \Lambda^2 + 16F$ LST. Equivalently, the surfaces S_0 and S_{m+1} can be interpreted as describing $\mathfrak{sp}(0)^{(1)}$ and the other surfaces describing $\mathfrak{su}(1)^{(1)}$, leading to the T-duality

$$\boxed{
 \begin{array}{c}
 \mathfrak{sp}(m + 1)^{(1)} \\
 0
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{sp}(0)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \dots \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{sp}(0)^{(1)} \\
 1 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 1 \\
 \underbrace{\hspace{10em}}_{m \geq 1}
 \end{array}
 }
 \quad (3.80)$$

The surface geometry



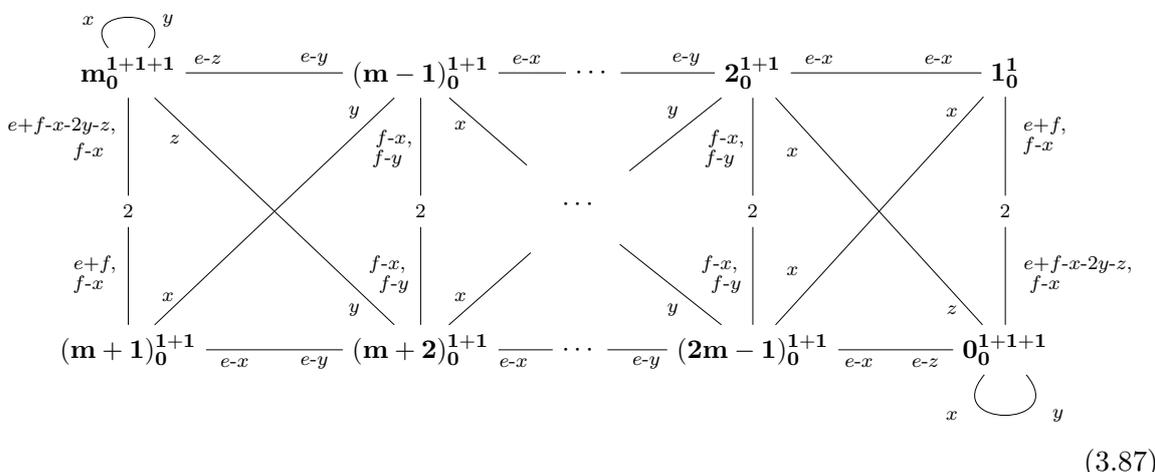
(3.85)

describes untwisted compactification of LST $\mathfrak{su}(2m+1) + \mathbb{S}^2 + \Lambda^2$. Exchanging e and f on all surfaces except S_{m+1} leads to the T-duality

$$\boxed{\begin{array}{c} \mathfrak{su}(\widehat{2m+1})^{(1)} \\ 0 \end{array} = \begin{array}{c} \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(2)^{(1)} \quad \mathfrak{su}(2)^{(1)} \quad \mathfrak{su}(2)^{(1)} \\ 2 \leftarrow 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 2 \\ \underbrace{\hspace{10em}}_{m-1 \geq 1} \end{array} \quad \text{loop}} \quad (3.86)$$

where various $\mathfrak{su}(2)^{(1)}$ without a loop are formed by pairs of surfaces (S_i, S_{2m+2-i}) for $2 \leq i \leq m$, and (S_0, S_1) forms $\mathfrak{su}(2)^{(1)}$ with loop.

The surface geometry



(3.87)

describes untwisted compactification of LST $\mathfrak{su}(2m) + \mathbb{S}^2 + \Lambda^2$. Exchanging e and f on all surfaces leads to the T-duality

$$\boxed{\begin{array}{c} \mathfrak{su}(\widehat{2m})^{(1)} \\ 0 \end{array} = \begin{array}{c} \mathfrak{su}(2)^{(1)} \quad \mathfrak{su}(2)^{(1)} \quad \mathfrak{su}(2)^{(1)} \quad \mathfrak{su}(2)^{(1)} \\ \text{loop} \text{ --- } 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 2 \text{ --- } \text{loop} \\ \underbrace{\hspace{10em}}_{m-2 \geq 0} \end{array} \quad (3.88)$$

where various $\mathfrak{su}(2)^{(1)}$ without a loop are formed by pairs of surfaces (S_i, S_{2m+1-i}) for $2 \leq i \leq m-1$, and (S_0, S_1) and (S_m, S_{m+1}) form $\mathfrak{su}(2)^{(1)}$ with loop.

The surface geometry

$$(3.89)$$

describes untwisted compactification of LST $\mathfrak{su}(2m+2) + 16F + 2\Lambda^2$. Exchanging e and f on all surfaces except S_0 and S_{m+1} leads to the T-duality

$$\mathfrak{su}(2m+2)^{(1)}_0 = \mathfrak{sp}(0)^{(1)}_1 \text{ --- } \mathfrak{su}(2)^{(1)}_2 \text{ --- } \dots \text{ --- } \mathfrak{su}(2)^{(1)}_2 \text{ --- } \mathfrak{sp}(0)^{(1)}_1$$

$\underbrace{\hspace{10em}}_{m \geq 1}$

$$(3.90)$$

where various $\mathfrak{su}(2)^{(1)}$ are formed by pairs of surfaces (S_i, S_{2m+2-i}) for $1 \leq i \leq m$.

The surface geometry

$$(3.91)$$

describes untwisted compactification of LST $\mathfrak{su}(2m+1) + 16F + 2\Lambda^2$. Exchanging e and f on all surfaces except S_{m+1} leads to the T-duality

$$\mathfrak{su}(2m+1)^{(1)}_0 = \mathfrak{sp}(0)^{(1)}_1 \text{ --- } \mathfrak{su}(2)^{(1)}_2 \text{ --- } \dots \text{ --- } \mathfrak{su}(2)^{(1)}_2 \text{ --- } \mathfrak{sp}(1)^{(1)}_1$$

$\underbrace{\hspace{10em}}_{m-1 \geq 1}$

$$(3.92)$$

where various $\mathfrak{su}(2)^{(1)}$ are formed by pairs of surfaces (S_i, S_{2m+2-i}) for $2 \leq i \leq m$, and $\mathfrak{sp}(1)^{(1)}$ is formed by S_0 and S_1 .

The surface geometry

$$\begin{array}{ccccccc}
 \mathbf{m}_0^{4+1} & \xrightarrow{e+2f-\sum x_i-y} & \xrightarrow{e-y} & (\mathbf{m}-1)_0^{1+1} & \xrightarrow{e-x} & \dots & \xrightarrow{e-y} & \mathbf{2}_0^{1+1} & \xrightarrow{e-x} & \xrightarrow{e+2f-\sum x_i-y} & \mathbf{1}_0^{4+1} \\
 \begin{array}{c} e-y \\ | \\ y \\ | \\ e-y \end{array} & \begin{array}{c} \diagdown \\ y \\ \diagup \\ y \end{array} & \begin{array}{c} \diagup \\ f-x, \\ f-y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ x \\ \diagup \\ x \end{array} & \dots & \begin{array}{c} \diagup \\ y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ f-x, \\ f-y \\ \diagup \\ y \end{array} & \begin{array}{c} \diagup \\ x \\ \diagdown \\ x \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ y \\ \diagup \\ y \end{array} & \begin{array}{c} e-y \\ | \\ e-y \end{array} \\
 (\mathbf{m}+1)_0^{4+1} & \xrightarrow{e+2f-\sum x_i-y} & \xrightarrow{e-y} & (\mathbf{m}+2)_0^{1+1} & \xrightarrow{e-x} & \dots & \xrightarrow{e-y} & (\mathbf{2m}-1)_0^{1+1} & \xrightarrow{e-x} & \xrightarrow{e+2f-\sum x_i-y} & \mathbf{0}_0^{4+1}
 \end{array}
 \tag{3.93}$$

describes untwisted compactification of LST $\mathfrak{su}(2m) + 16F + 2\Lambda^2$. Exchanging e and f on S_0 and S_1 leads to a description of the untwisted compactification of LST $\mathfrak{sp}(2m-1) + 16F + \Lambda^2$. If we also exchange e and f on S_m and S_{m+1} , then we are lead to a surface geometry in which surfaces S_1 to S_m describe an $\mathfrak{sp}(m-1)^{(1)}$, and surfaces S_{m+1} to S_{2m-1} and S_0 describe an $\mathfrak{sp}(m-1)^{(1)}$. In total, we have the following T-dualities

$ \begin{array}{ccccc} \mathfrak{su}(2m)^{(1)} & = & \mathfrak{sp}(2m-1)^{(1)} & = & \mathfrak{sp}(m-1)^{(1)} & \mathfrak{sp}(m-1)^{(1)} \\ 0 & & 0 & & 1 & \text{-----} & 1 \end{array} $	(3.94)
---	--------

Similarly, the surface geometry

$$\begin{array}{ccccccc}
 \mathbf{m}_0^{1+1+1+2} & \xrightarrow{e+2f-x-y-z-\sum w_i} & \xrightarrow{e-y} & (\mathbf{m}-1)_0^{1+1} & \xrightarrow{e-x} & \dots & \xrightarrow{e-y} & \mathbf{2}_0^{1+1} & \xrightarrow{e-x} & \xrightarrow{e+2f-y-\sum x_i} & \mathbf{1}_0^{4+1} \\
 \begin{array}{c} e-y-z \\ | \\ z-x, \\ y \end{array} & \begin{array}{c} \diagdown \\ x \\ \diagup \\ x \end{array} & \begin{array}{c} \diagup \\ y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ f-x, \\ f-y \\ \diagup \\ x \end{array} & \dots & \begin{array}{c} \diagup \\ y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ f-x, \\ f-y \\ \diagup \\ y \end{array} & \begin{array}{c} \diagup \\ x \\ \diagdown \\ x \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ y \\ \diagup \\ y \end{array} & \begin{array}{c} e-y \\ | \\ e-y \end{array} \\
 (\mathbf{m}+1)_0^6 & \xrightarrow{e+2f-\sum x_i} & \xrightarrow{e} & (\mathbf{m}+2)_0^1 & \xrightarrow{e-x} & \dots & \xrightarrow{e-y} & (\mathbf{m}+3)_0^{1+1} & \xrightarrow{e-x} & \dots & \xrightarrow{e-y} & \mathbf{2m}_0^{1+1} & \xrightarrow{e-x} & \xrightarrow{e+2f-y-\sum x_i} & \mathbf{0}_0^{4+1} \\
 \begin{array}{c} e \\ | \\ f, \\ f-x \end{array} & \begin{array}{c} \diagdown \\ x \\ \diagup \\ x \end{array} & \begin{array}{c} \diagup \\ f-x, \\ f-y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ f-x, \\ f-y \\ \diagup \\ y \end{array} & \dots & \begin{array}{c} \diagup \\ y \\ \diagdown \\ y \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ f-x, \\ f-y \\ \diagup \\ y \end{array} & \begin{array}{c} \diagup \\ x \\ \diagdown \\ x \end{array} & \begin{array}{c} | \\ 2 \\ | \end{array} & \begin{array}{c} \diagdown \\ y \\ \diagup \\ y \end{array} & \begin{array}{c} e-y \\ | \\ e-y \end{array}
 \end{array}
 \tag{3.95}$$

describes untwisted compactification of LST $\mathfrak{su}(2m+1) + 16F + 2\Lambda^2$. Exchanging e and f on S_0 and S_1 leads to a description of the untwisted compactification of LST $\mathfrak{sp}(2m) + 16F + \Lambda^2$. If we also exchange e and f on S_m and S_{m+1} , then we are lead to a surface geometry in which surfaces S_1 to S_m describe an $\mathfrak{sp}(m-1)^{(1)}$, and surfaces S_{m+1} to S_{2m} and S_0 describe an $\mathfrak{sp}(m)^{(1)}$. In total, we have the following T-dualities.

$ \begin{array}{ccccc} \mathfrak{su}(2m+1)^{(1)} & = & \mathfrak{sp}(2m)^{(1)} & = & \mathfrak{sp}(m)^{(1)} & \mathfrak{sp}(m-1)^{(1)} \\ 0 & & 1 & & 1 & \text{-----} & 1 \end{array} $	(3.96)
---	--------

3.6.2 $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ LSTs

For untwisted compactification of $\mathfrak{su}(m)$ $\mathcal{N} = (1, 1)$ LST with $\theta = 0$, we have the surface geometry

$$\begin{array}{c}
 \begin{array}{c} x \quad y \\ \frown \quad \smile \\ \mathbf{m}_0^{1+1} \end{array} \\
 \begin{array}{c} e-y, f-x \\ | \\ 2 \end{array} \quad \begin{array}{c} e-x, f-y \\ | \\ 2 \end{array} \\
 \begin{array}{c} e-x, f-y \\ | \\ \mathbf{(m-1)}_0^{1+1} \end{array} \quad \begin{array}{c} e-y, f-x \quad \dots \quad e-x, f-y \\ | \quad \quad \quad | \\ 2 \quad \quad \quad 2 \end{array} \quad \begin{array}{c} e-y, f-x \\ | \\ \mathbf{1}_0^{1+1} \end{array} \quad \begin{array}{c} e-x, f-y \\ | \\ 2 \end{array} \quad \begin{array}{c} e-y, f-x \\ | \\ \mathbf{0}_0^{1+1} \end{array} \\
 \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array} \quad \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array} \quad \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array}
 \end{array}
 \tag{3.97}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with untwisted compactification of A_{m-1} $\mathcal{N} = (2, 0)$ LST

$$\boxed{
 \begin{array}{c}
 \mathfrak{su}(m)_0^{(1)} \\
 0_1 = \begin{array}{c} \overbrace{\mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)}}^{m-1} \\ 2 \quad \dots \quad 2 \\ \diagdown \quad \diagup \\ 2 \\ \mathfrak{su}(1)^{(1)} \end{array}
 \end{array}
 }
 \tag{3.98}$$

For \mathbb{Z}_2 outer-automorphism twisted compactification of $\mathfrak{su}(2m)$ $\mathcal{N} = (1, 1)$ LST with $\theta = 0$, we have the surface geometry

$$\begin{array}{c}
 \begin{array}{c} \mathbf{0}_0^{1+1} \quad x \quad y \\ \frown \quad \smile \end{array} \\
 \begin{array}{c} e-y, f-x \\ | \\ 2 \end{array} \\
 \begin{array}{c} e-x, f-y \\ | \\ \mathbf{1}_0^{1+1} \end{array} \quad \begin{array}{c} e-y, f-x \quad \dots \quad e-x, f-y \\ | \quad \quad \quad | \\ 2 \quad \quad \quad 2 \end{array} \quad \begin{array}{c} e-y, f-x \\ | \\ \mathbf{2}_0^{1+1} \end{array} \quad \begin{array}{c} e-x, f-y \\ | \\ 2 \end{array} \quad \begin{array}{c} e-y, f-x \\ | \\ \mathbf{(m-1)}_0^{1+1} \end{array} \\
 \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array} \quad \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array} \quad \begin{array}{c} x \quad y \\ \frown \quad \smile \end{array}
 \end{array}
 \tag{3.99}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of D_{m+1} $\mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry exchanging the

spinor and cospinor nodes

$$\begin{array}{c}
 \mathfrak{su}(1)^{(1)} \\
 \downarrow 2 \\
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \dots \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 \leftarrow 2 \quad \leftarrow 2 \quad \dots \quad \leftarrow 2 \quad \leftarrow 2 \rightarrow 2 \\
 \underbrace{\hspace{10em}}_{m-2 \geq 1}
 \end{array}
 \quad = \quad
 \begin{array}{c}
 \mathfrak{su}(2m)_0^{(2)} \\
 0_1
 \end{array}
 \quad (3.100)$$

where the various $\mathfrak{su}(1)^{(1)}$ on the right hand side form the Dynkin diagram of $\mathfrak{so}(2m+1)^{(1)}$. Note that $\mathfrak{su}(2m)^{(2)}$ and $\mathfrak{so}(2m+1)^{(1)}$ are Langlands dual, i.e. their Dynkin diagrams are related by reversing the directions of arrows.

For \mathbb{Z}_2 outer-automorphism twisted compactification of $\mathfrak{su}(2m)$ $\mathcal{N} = (1, 1)$ LST with $\theta = \pi$, we have the surface geometry

$$\begin{array}{c}
 \mathbf{0}_0^{1+1} \begin{array}{c} \curvearrowright^x \\ \curvearrowleft^y \end{array} \\
 \begin{array}{c} e-y, f-x \\ \downarrow 2 \\ e-x, f-y \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \curvearrowright^x \\ \curvearrowleft^y \end{array} \xrightarrow{e-y, f-x} 2 \xrightarrow{e-x, f-y} \mathbf{2}_0^{1+1} \begin{array}{c} \curvearrowright^x \\ \curvearrowleft^y \end{array} \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} (\mathbf{m}-1)_0^{1+1} \begin{array}{c} \curvearrowright^x \\ \curvearrowleft^y \end{array} \\
 \begin{array}{c} \downarrow 2h-x-2y, f-x \\ \downarrow 2 \\ e-y, f-x \end{array} \\
 \mathbf{m}_1^{1+1} \begin{array}{c} \curvearrowright^x \\ \curvearrowleft^y \end{array}
 \end{array}
 \quad (3.101)$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of D_{2m+1} $\mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry exchanging the spinor node with affine node and cospinor node with vector node (the action on other nodes is fixed by these requirements)

$$\begin{array}{c}
 \mathfrak{su}(1)^{(1)} \\
 \downarrow 2 \\
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \dots \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 \leftarrow 2 \quad \leftarrow 2 \quad \dots \quad \leftarrow 2 \quad \leftarrow 2 \rightarrow \curvearrowright \\
 \underbrace{\hspace{10em}}_{m-2 \geq 1}
 \end{array}
 \quad = \quad
 \begin{array}{c}
 \mathfrak{su}(2m)_\pi^{(2)} \\
 0_1
 \end{array}
 \quad (3.102)$$

For \mathbb{Z}_2 outer-automorphism twisted compactification of $\mathfrak{su}(2m+1)$ $\mathcal{N} = (1, 1)$ LST with

$\theta = 0$, we have the surface geometry

$$\begin{array}{ccc}
 \mathbf{0}_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array} & & \begin{array}{c} x \\ \text{---} \\ y \end{array} \mathbf{m}_0^{1+1} \\
 \begin{array}{c} e-x, f-y \\ | \\ 2 \\ | \\ 2e+f-2x-y, f-y \end{array} & & \begin{array}{c} | \\ 2e+f-x-2y, f-x \\ | \\ 2 \\ | \\ e-y, f-x \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array} & \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} & (\mathbf{m}-1)_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array}
 \end{array} \tag{3.103}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of D_{2m+2} $\mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_4 symmetry that sends affine node to spinor node, spinor node to vector node, vector node to cospinor node, and cospinor node to vector node

$$\mathfrak{su}(2m+1)_0^{(2)} = \mathfrak{su}(1)^{(1)} \text{---} \mathfrak{su}(1)^{(1)} \text{---} \underbrace{\mathfrak{su}(1)^{(1)} \text{---} \dots \text{---} \mathfrak{su}(1)^{(1)}}_{m-1 \geq 1} \text{---} \mathfrak{su}(1)^{(1)}$$

(3.104)

For \mathbb{Z}_2 outer-automorphism twisted compactification of $\mathfrak{su}(2m+1)$ $\mathcal{N} = (1, 1)$ LST with $\theta = \pi$, we have the surface geometry

$$\begin{array}{ccc}
 \mathbf{0}_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array} & & \begin{array}{c} x \\ \text{---} \\ y \end{array} \mathbf{m}_1^{1+1} \\
 \begin{array}{c} e-x, f-y \\ | \\ 2 \\ | \\ 2e+f-2x-y, f-y \end{array} & & \begin{array}{c} | \\ 2h-x-2y, f-x \\ | \\ 2 \\ | \\ e-y, f-x \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array} & \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} & (\mathbf{m}-1)_0^{1+1} \begin{array}{c} \text{---} x \\ \text{---} y \end{array}
 \end{array} \tag{3.105}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of A_{2m} $\mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry that leaves affine node invariant but acts on other nodes as \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(2m+1)$

$$\mathfrak{su}(2m+1)_\pi^{(2)} = \mathfrak{su}(1)^{(1)} \text{---} \mathfrak{su}(1)^{(1)} \text{---} \underbrace{\mathfrak{su}(1)^{(1)} \text{---} \dots \text{---} \mathfrak{su}(1)^{(1)}}_{m-1 \geq 1} \text{---} \mathfrak{su}(1)^{(1)}$$

(3.106)

For untwisted compactification of $\mathfrak{so}(2m+1) \mathcal{N} = (1, 1)$ LST, we have the surface geometry

$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{0}_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array} \\
 \begin{array}{c} e-y, f-x \\ | \\ 2 \end{array} \\
 \begin{array}{c} e-x, f-y \\ | \\ 2 \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array} \xrightarrow{e-y, f-x} 2 \xrightarrow{e-x, f-y} \mathbf{2}_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array} \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} (\mathbf{m}-\mathbf{1})_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array}
 \end{array}
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 \mathbf{m}_0^{1+1} \begin{array}{c} \curvearrowleft \\ x \\ y \end{array} \\
 \begin{array}{c} e-y, f-x \\ | \\ 2 \end{array} \\
 \begin{array}{c} 2e+f-x-2y, f-x \\ | \\ 2 \end{array} \\
 (\mathbf{m}-\mathbf{1})_0^{1+1} \begin{array}{c} \curvearrowleft \\ x \\ y \end{array}
 \end{array}
 \end{array}
 \tag{3.107}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of $D_{2m} \mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry exchanging the spinor node with affine node and cospinor node with vector node

$$\begin{array}{c}
 \mathfrak{so}(2m+1)^{(1)} \\
 \mathbf{0}_1 = \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \\
 \begin{array}{c} 2 \\ | \end{array} \\
 \mathfrak{su}(1)^{(1)} \begin{array}{c} \text{---} \\ 2 \end{array} \mathfrak{su}(1)^{(1)} \begin{array}{c} \text{---} \\ 2 \end{array} \dots \begin{array}{c} \text{---} \\ 2 \end{array} \mathfrak{su}(1)^{(1)} \begin{array}{c} \text{---} \\ 2 \end{array} \mathfrak{su}(1)^{(1)} \\
 \underbrace{\hspace{10em}}_{m-2 \geq 1}
 \end{array}
 \end{array}
 \tag{3.108}$$

The various $\mathfrak{su}(1)^{(1)}$ on the right hand side form the Dynkin diagram of $\mathfrak{su}(2m)^{(2)}$. Note that $\mathfrak{so}(2m+1)^{(1)}$ and $\mathfrak{su}(2m)^{(2)}$ are Langlands dual, i.e. their Dynkin diagrams are related by reversing the directions of arrows.

For untwisted compactification of $\mathfrak{sp}(m) \mathcal{N} = (1, 1)$ LST with $\theta = 0$, we have the surface geometry

$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{0}_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array} \\
 \begin{array}{c} 2e+f-2x-y, f-y \\ | \\ 2 \end{array} \\
 \begin{array}{c} e-x, f-y \\ | \\ 2 \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array} \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} (\mathbf{m}-\mathbf{1})_0^{1+1} \begin{array}{c} \curvearrowright \\ x \\ y \end{array}
 \end{array}
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 \mathbf{m}_0^{1+1} \begin{array}{c} \curvearrowleft \\ x \\ y \end{array} \\
 \begin{array}{c} 2e+f-x-2y, f-x \\ | \\ 2 \end{array} \\
 \begin{array}{c} e-y, f-x \\ | \\ 2 \end{array} \\
 (\mathbf{m}-\mathbf{1})_0^{1+1} \begin{array}{c} \curvearrowleft \\ x \\ y \end{array}
 \end{array}
 \end{array}
 \tag{3.109}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of $D_{m+2} \mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry that sends affine

node to vector node and spinor node to cospinor node

$$\boxed{
 \begin{array}{c}
 \mathfrak{sp}(m)_0^{(1)} \\
 0_1
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \leftarrow 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 2 \rightarrow 2 \\
 \underbrace{\hspace{10em}}_{m-1 \geq 1}
 \end{array}
 }
 \tag{3.110}$$

The various $\mathfrak{su}(1)^{(1)}$ on the right hand side form the Dynkin diagram of $\mathfrak{so}(2m+2)^{(2)}$. Note that $\mathfrak{sp}(m)^{(1)}$ and $\mathfrak{so}(2m+2)^{(2)}$ are Langlands dual, i.e. their Dynkin diagrams are related by reversing the directions of arrows.

For untwisted compactification of $\mathfrak{sp}(m)$ $\mathcal{N} = (1, 1)$ LST with $\theta = \pi$, we have the surface geometry

$$\begin{array}{ccc}
 \begin{array}{c}
 \mathbf{0}_0^{1+1} \text{ --- } \begin{array}{c} x \\ \circlearrowleft \\ y \end{array} \\
 \left. \begin{array}{c} 2e+f-2x-y, f-y \\ 2 \end{array} \right| \\
 e-x, f-y \text{ --- } \mathbf{1}_0^{1+1} \text{ --- } \frac{e-y, f-x}{\dots} \dots \frac{e-x, f-y}{\dots} \text{ --- } (\mathbf{m}-1)_0^{1+1} \\
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array}
 \end{array}
 &
 &
 \begin{array}{c}
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array} \text{ --- } \mathbf{m}_1^{1+1} \\
 \left. \begin{array}{c} 2h-x-2y, f-x \\ 2 \end{array} \right| \\
 e-y, f-x \text{ --- } (\mathbf{m}-1)_0^{1+1} \\
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array}
 \end{array}
 \end{array}
 \tag{3.111}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of D_{2m+3} $\mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_4 symmetry that sends affine node to spinor node, spinor node to vector node, vector node to cospinor node, and cospinor node to vector node

$$\boxed{
 \begin{array}{c}
 \mathfrak{sp}(m)_\pi^{(1)} \\
 0_1
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \leftarrow 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } \begin{array}{c} \circlearrowleft \\ 2 \end{array} \\
 \underbrace{\hspace{10em}}_{m-1 \geq 1}
 \end{array}
 }
 \tag{3.112}$$

For untwisted compactification of $\mathfrak{so}(2m)$ $\mathcal{N} = (1, 1)$ LST, we have the surface geometry

$$\begin{array}{ccc}
 \begin{array}{c}
 \mathbf{0}_0^{1+1} \text{ --- } \begin{array}{c} x \\ \circlearrowleft \\ y \end{array} \\
 \left. \begin{array}{c} e-y, f-x \\ 2 \end{array} \right| \\
 e-x, f-y \text{ --- } \mathbf{2}_0^{1+1} \text{ --- } \frac{e-y, f-x}{\dots} \dots \frac{e-x, f-y}{\dots} \text{ --- } (\mathbf{m}-2)_0^{1+1} \text{ --- } \frac{e-y, f-x}{2} \text{ --- } \frac{e-x, f-y}{2} \text{ --- } (\mathbf{m}-1)_0^{1+1} \\
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array}
 \end{array}
 &
 &
 \begin{array}{c}
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array} \text{ --- } \mathbf{m}_0^{1+1} \\
 \left. \begin{array}{c} e-x, f-y \\ 2 \end{array} \right| \\
 e-y, f-x \text{ --- } (\mathbf{m}-2)_0^{1+1} \text{ --- } \frac{e-y, f-x}{2} \text{ --- } \frac{e-x, f-y}{2} \text{ --- } (\mathbf{m}-1)_0^{1+1} \\
 \begin{array}{c} x \\ \circlearrowleft \\ y \end{array}
 \end{array}
 \end{array}
 \tag{3.113}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with untwisted compactification of $D_m \mathcal{N} = (2, 0)$ LST

$$\begin{array}{c}
 \mathfrak{so}(2m)^{(1)} \\
 0_1
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \quad 2 \\
 | \quad | \\
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \dots \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 2 \\
 \underbrace{\hspace{10em}}_{m-3 \geq 1}
 \end{array}
 \tag{3.114}$$

For \mathbb{Z}_2 outer-automorphism twisted compactification of $\mathfrak{so}(2m+2) \mathcal{N} = (1, 1)$ LST, we have the surface geometry

$$\begin{array}{ccc}
 \mathbf{0}_0^{1+1} \begin{array}{c} \textcircled{x} \\ y \end{array} & & \begin{array}{c} x \\ \textcircled{y} \end{array} \mathbf{m}_0^{1+1} \\
 \begin{array}{c} e-x, f-y \\ | \\ 2 \\ | \\ 2e+f-2x-y, f-y \end{array} & & \begin{array}{c} | \\ e-y, f-x \\ | \\ 2 \\ | \\ 2e+f-x-2y, f-x \end{array} \\
 \mathbf{1}_0^{1+1} \begin{array}{c} \textcircled{x} \\ y \end{array} & \xrightarrow{e-y, f-x} \dots \xrightarrow{e-x, f-y} & (\mathbf{m}-\mathbf{1})_0^{1+1} \begin{array}{c} \textcircled{x} \\ y \end{array}
 \end{array}
 \tag{3.115}$$

in which each surface can be interpreted as describing an $\mathfrak{su}(1)^{(1)}$, which leads to a T-duality with compactification of $A_{2m-1} \mathcal{N} = (2, 0)$ LST twisted by \mathbb{Z}_2 symmetry that leaves affine node invariant but acts on other nodes as \mathbb{Z}_2 outer-automorphism of $\mathfrak{su}(2m)$

$$\begin{array}{c}
 \mathfrak{so}(2m+2)^{(2)} \\
 0_1
 \end{array}
 =
 \begin{array}{c}
 \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \quad \mathfrak{su}(1)^{(1)} \\
 2 \text{ --- } 2 \text{ --- } 2 \text{ --- } \dots \text{ --- } 2 \text{ --- } 2 \\
 \underbrace{\hspace{10em}}_{m-1 \geq 2}
 \end{array}
 \tag{3.116}$$

The various $\mathfrak{su}(1)^{(1)}$ on the right hand side form the Dynkin diagram of $\mathfrak{sp}(m)^{(1)}$. Note that $\mathfrak{so}(2m+2)^{(2)}$ and $\mathfrak{sp}(m)^{(1)}$ are Langlands dual, i.e. their Dynkin diagrams are related by reversing the directions of arrows.

3.7 Other T-dualities between $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$

In a similar way, the reader can derive remaining T-dualities of compactifications of $\mathcal{N} = (1, 1)$ LSTs, for which we quote the results below:

- The untwisted compactification of $\mathfrak{e}_n \mathcal{N} = (1, 1)$ LST is T-dual to untwisted compactification of $E_n \mathcal{N} = (2, 0)$ LST, for each $6 \leq n \leq 8$.

- The \mathbb{Z}_2 outer-automorphism twisted compactification of \mathfrak{e}_6 $\mathcal{N} = (1, 1)$ LST is T-dual to compactification of E_6 $\mathcal{N} = (2, 0)$ LST twisted by a \mathbb{Z}_2 symmetry exchanging two of the three single valent nodes. The $\mathfrak{su}(1)^{(1)}$ involved in the compactification of the $\mathcal{N} = (2, 0)$ theory arrange themselves in the form of Dynkin diagram of $\mathfrak{f}_4^{(1)}$ which is Langlands dual to the affine algebra $\mathfrak{e}_6^{(2)}$ related to the compactification of the T-dual $\mathcal{N} = (1, 1)$ LST.
- The untwisted compactification of \mathfrak{f}_4 $\mathcal{N} = (1, 1)$ LST is T-dual to compactification of E_7 $\mathcal{N} = (2, 0)$ LST twisted by a \mathbb{Z}_2 symmetry exchanging two of the three single valent nodes. The $\mathfrak{su}(1)^{(1)}$ involved in the compactification of the $\mathcal{N} = (2, 0)$ theory arrange themselves in the form of Dynkin diagram of $\mathfrak{e}_6^{(2)}$ which is Langlands dual to the affine algebra $\mathfrak{f}_4^{(1)}$ related to the compactification of the T-dual $\mathcal{N} = (1, 1)$ LST.
- The untwisted compactification of \mathfrak{g}_2 $\mathcal{N} = (1, 1)$ LST is T-dual to compactification of E_6 $\mathcal{N} = (2, 0)$ LST twisted by a \mathbb{Z}_3 symmetry rotating the three single valent nodes. The $\mathfrak{su}(1)^{(1)}$ involved in the compactification of the $\mathcal{N} = (2, 0)$ theory arrange themselves in the form of Dynkin diagram of $\mathfrak{so}(8)^{(3)}$ which is Langlands dual to the affine algebra $\mathfrak{g}_2^{(1)}$ related to the compactification of the T-dual $\mathcal{N} = (1, 1)$ LST.
- The \mathbb{Z}_3 outer-automorphism twisted compactification of $\mathfrak{so}(8)$ $\mathcal{N} = (1, 1)$ LST is T-dual to compactification of D_4 $\mathcal{N} = (2, 0)$ LST twisted by a \mathbb{Z}_3 symmetry rotating three out of four single valent nodes. The $\mathfrak{su}(1)^{(1)}$ involved in the compactification of the $\mathcal{N} = (2, 0)$ theory arrange themselves in the form of Dynkin diagram of $\mathfrak{g}_2^{(1)}$ which is Langlands dual to the affine algebra $\mathfrak{so}(8)^{(3)}$ related to the compactification of the T-dual $\mathcal{N} = (1, 1)$ LST.

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