# A Matrix Formulation of Quantum Stochastic Calculus

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*Thesis submitted for the degree of Doctor of Philosophy at the University of Oxford*

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## Abstract

We develop the theory of chaos spaces and chaos matrices. A chaos space is a Hilbert space with a fixed, countably-infinite, direct-sum decomposition. A chaos matrix between two chaos spaces is a doubly-infinite matrix of bounded operators which respects this decomposition. We study operators represented by such matrices, particularly with respect to self-adjointness.

This theory is used to re-formulate the quantum stochastic calculus of Hudson and Parthasarathy. Integrals of chaos-matrix processes are defined using the Hitsuda-Skorokhod integral and Malliavin gradient, following Lindsay and Belavkin. A new way of defining adaptedness is developed and the consequent quantum product Itô formula is used to provide a genuine functional Itô formula for polynomials in a large class of unbounded processes, which include the Poisson process and Brownian motion.

A new type of adaptedness, known as $\Omega$-adaptedness, is defined. We show that quantum stochastic integrals of $\Omega$-adapted processes are well-behaved; for instance, bounded processes have bounded integrals. We solve the appropriate modification of the evolution equation of Hudson and Parthasarathy:

$$U(t) = I + \int_0^t E(s)\,\mathrm{d}\Lambda(s) + F(s)\,\mathrm{d}A(s) + G(s)U(s)\,\mathrm{d}A^\dagger(s) + H(s)U(s)\,\mathrm{d}s,$$

where the coefficients are time-dependent, bounded, $\Omega$-adapted processes acting on the whole Fock space. We show that the usual conditions on the coefficients, viz.

$$(E, F, G, H) = (W - I, L, -WL^*, iK + \tfrac{1}{2}LL^*)$$

where $W$ is unitary and $K$ self-adjoint, are necessary and sufficient conditions for the solution to be unitary. This is a very striking result when compared to the adapted case.

*To my parents*

# Contents

# Chapter 1

# Introduction

La dernière chose qu'on trouve en faisant un ouvrage,
est de savoir celle qu'il faut mettre la première.

Blaise Pascal, *Pensées*

The first achievement of this work is to provide the quantum stochastic calculus of Hudson and Parthasarathy with a functional Itô formula for processes of unbounded operators. To do so we find it useful to recast the theory in terms of infinite matrices. The construction of the matrix calculus gives us a tool to investigate non-adapted quantum stochastic differential equations. We obtain some significant results for existence and unitarity of these equations when the coefficients are $\Omega$-adapted, a modified form of adaptedness.

## 1.1 Introduction

Towards the end of the 1920s Norbert Wiener developed the mathematical foundations of the theory of Brownian motion, providing a rigorous definition of the measure which now bears his name (see [Wie]). A theory of integration against Brownian motion was developed and Kiyoshi Itô introduced in [Itô] the notion of stochastic differential equations (also known as SDE's) and a change of variable formula that is now known as the functional Itô formula for Brownian motion: given a function $f$ on the real line with continuous second derivative we have that

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\,\mathrm{d}W_s + \tfrac{1}{2}\int_0^t f''(W_s)\,\mathrm{d}s,$$

where $W$ is Brownian motion (the Wiener process). The second integral above is called the Itô correction term; its presence shows that integration with respect to

Brownian motion cannot be defined in a Lebesgue-Stieltjes framework. The body of theory that grew from these beginnings, which we call herein the classical theory of stochastic processes, has become an important part of twentieth-century mathematics. One can define integration against a large class of stochastic processes, called semimartingales, and the fundamental formula of the theory, the product Itô formula, takes the form

$$X_t Y_t = \int_0^t X_{s-}\, \mathrm{d}Y_s + \int_0^t Y_{s-}\, \mathrm{d}X_s + [X, Y]_t$$

where $[X, Y]$ is the bracket process of the semimartingales $X$ and $Y$. It is profitable to regard this as a generalisation of the formula for integration by parts, to which it reduces when $X$ and $Y$ have finite variation; it is the non-vanishing of the bracket that leads to the correction term in the functional Itô formula. For more details see [ReY] for Brownian motion and [Pro] for the general theory.

In their seminal paper [HuP] Hudson and Parthasarathy introduced an operator-theoretic generalisation of the classical theory, known as the quantum stochastic calculus. They construct an integration theory on Boson Fock space with four integrator processes (called creation, annihilation, preservation (or gauge) and time, and denoted $A^\dagger$, $A$, $\Lambda$ and $t$, respectively) which are related to well-known objects of quantum field theory. The inspiration for this comes, at least in part, from the Wiener-Itô isomorphism (also known as the Wiener-Itô-Segal isomorphism) between the $L^2$-space of Wiener measure and Boson Fock space: the Brownian motion process is represented by the sum of the creation and annihilation processes. Hudson and Parthasarathy's presentation shows much more than this; for instance the gauge process allows the realisation of the Poisson process in Fock space. A product Itô formula is shown to exist in this more general framework; in fact, the classical formula (at least for Brownian motion) can be seen to be a consequence of the chain rule and the physicists' notion of Wick ordering, which determine the bracket (or quantum Itô table) of the basic integrators (see [HuS]). From this starting point several generalisations suggest themselves.

The quantum product Itô formula of [HuP] avoids the need for multiplication of unbounded operators, or rather uses a weak multiplication defined in terms of the inner product. Whilst this is an elegant use of minimal machinery, it is too limiting for many situations. As shown by the proof of [RoW, Chapter IV, Theorem 32.8], the functional Itô formula for Brownian motion is determined by the product formula, induction to achieve a result for polynomials, and then polynomial approximation of $C^2$ functions. If we lack a method for composing operators we are unable to consider polynomials in them, so a functional Itô formula is unavailable to us, although sense can be given to functions and differentials of functions of self-adjoint operators, for example.

Attal and Meyer addressed the problem of composition in [AtM], using a classical interpretation of Fock space. That is, the quantum stochastic integrals are expressed in terms of integrals with respect to Brownian motion and time. This enabled Attal to give a polynomial Itô formula for regular quantum semimartingales, i.e., adapted processes of bounded operators with a well-behaved integral representation; see [Att]. This family was then provided with a functional Itô formula by Vincent-Smith in [Vin], which closely resembles the classical formula (in operator-theoretic dress). This is not sufficient to provide a proof of the polynomial Itô formula for Brownian motion or the Poisson process, both of which are represented by processes of unbounded operators. The following observation suggests a way forward, and is the starting point of this work.

The presentation of [HuP] regards Boson Fock space as the Hilbert space generated by a certain Gelfand pair: it is the closed linear span of a collection of linearly independent vectors, the exponential vectors, with respect to a particular inner-product. This is again a neat way of approaching the subject with minimal background requirements, but disregards the natural decomposition of Fock space into what are called chaos subspaces in the probabilists' language (or $n$-particle subspaces in the physicists'). That is, Boson Fock space has associated with it a positive operator with discrete spectrum consisting of the non-negative integers, called the number operator. The eigenspaces of this operator span the Fock space and with respect to this decomposition the basic processes of quantum stochastic calculus take a very simple form. When restricted to act between two chaos subspaces they become bounded operators; therefore, the operators themselves can be regarded as matrices of bounded operators with respect to this decomposition. Furthermore, these matrices are diagonal, so may be composed. Segal had observed the good behaviour of quantum field operators regarded as matrices on Boson Fock space in [Seg].

In the first part of this work we present the abstract theory of matrices of operators between spaces provided with chaos decompositions. We investigate certain subclasses of these chaos matrices, examine the operators that they induce, and obtain some useful results about boundedness and self-adjointness. For instance, diagonal matrices that are symmetric with respect to the natural matrix adjoint inherited from the Hilbert-space adjoint and that obey a certain growth condition are self-adjoint on their maximal domain (Theorem 2.2.8). The class of matrices that obey these restrictions include those corresponding to the Poisson process and Brownian motion. We believe this result to be new.

The quantum stochastic integrals of Hudson and Parthasarathy are defined using approximation by step processes, in the spirit of many approaches to Lebesgue (or Bochner-Lebesgue) integration theory. The estimates used to prove convergence of approximants are obtained by exploiting a notion of adaptedness. Lindsay and Belavkin discovered independently ([Lin], [Bel]) a way of defining the Hudson-Parthasarathy integrals without recourse to a limiting process, and furthermore this definition has no need of adaptedness assumptions. The technique involves the Hitsuda-Skorokhod integral and Malliavin gradient, both objects from classical stochastic calculus.

We define matrix versions of the gradient and the Hitsuda-Skorokhod integral, and use them to define quantum stochastic integrals of matrix processes. The theory fits in well with our matrix decomposition; integrals of uniformly diagonal processes remain diagonal, for example. The operators induced are shown to obey the inner-product identities that characterise the Hudson-Parthasarathy integrals.

It is natural to check that our new definitions agree with the formulation of Hudson and Parthasarathy. However, at this point we notice that the definition of adaptedness they give relies on the continuous tensor-product structure of Boson Fock space, to which the exponential-vector picture is very well suited: Boson Fock space is naturally isomorphic to the tensor product of a past and a future Fock space, and adapted processes are those that act trivially on the future, i.e., are ampliations by the identity of operators on the past space. It is not immediately clear what is the correct definition of adaptedness from the chaos-space perspective. If we compare matrix integrals of step processes defined *via* the Lindsay-Belavkin method to the matrix processes that result from the Hudson-Parthasarathy approach, we are led to a new definition.

Adaptedness is characterised in terms of commutation with the Malliavin gradient. This idea, which has been investigated independently by Attal and Lindsay [AtL], is a generalisation of the H-P definition that fits in well with our matrix formulation. We show the equivalence between the new definition and the old (Corollary 3.2.8 and Proposition 3.2.13). The link between the gradient and adaptedness has been observed in the classical context: compare [Mal, Lemma VII.2.1] with our Lemma 3.2.10.

Once we have decided on the correct notion of adaptedness the quantum product Itô formula is shown to exist at a local level (i.e., at the level of matrix components); it is a consequence of the inner-product identity obeyed by the Hitsuda-Skorokhod integral, and is a generalisation of the classical Skorokhod isometry. When the product of matrices exists in the appropriate topology this local formula gives rise to a true quantum product Itô formula for unbounded processes (in fact, our analysis allows us to obtain the formula in the bounded case as well). The polynomial Itô formula is then shown to exist for a large class of unbounded processes, including the Poisson process and Brownian motion; we believe this is the first explicit form of the quantum functional Itô formula for a general class of unbounded processes (cf. [Bel, Section 3]).

The one-dimensional theory of classical Brownian motion has a natural generalisation to $n$ dimensions, and so does quantum stochastic calculus; see [Eva], for example. The framework we have developed allows us to proceed from the one-dimensional case to the case of countable multiplicity with a minimum of effort; we sketch the details.

The latter part of this work concerns quantum stochastic differential equations. The paper [HuP] deals with dilations of quantum dynamical semigroups (i.e., semigroups of completely positive operators) *via* a quantum stochastic differential equation which can be regarded as a stochastic perturbation of the Schrödinger equation of quantum mechanics, the evolution equation:

$$U(t) = I + \int_0^t L_1(s)U(s)\,\mathrm{d}\Lambda(s) + L_2(s)U(s)\,\mathrm{d}A(s) + L_3(s)U(s)\,\mathrm{d}A^\dagger(s) + L_4(s)U(s)\,\mathrm{d}s.$$

Fock space is coupled with another Hilbert space, called the initial space (i.e., we form their tensor product) and the calculus on Fock space is used to provide 'noise' to influence the evolution of a system in the initial space. For the case considered by Hudson and Parthasarathy the driving coefficients $L_1, \ldots, L_4$ are constant and act trivially on the Fock space (and occur on the right, but this difference is not significant).

Although the physical motivations for this are clear, it is also the case that we may investigate quantum stochastic differential (or rather, integral) equations on the Fock space in their own right, as a generalisation of the classical theory of SDE's. Physically (or perhaps we should say heuristically) if the coefficients do not act trivially on Fock space, then the noise space is influenced by the initial space (the coupling is two-way) and we are considering propagators, a generalisation of evolutions produced by a (time-dependent) Hamiltonian. We are interested in conservative evolutions, i.e., unitary (or at least isometric) solutions of the evolution equation. Hudson and Parthasarathy give a necessary and sufficient condition for this when the equation involves bounded, constant coefficients acting only on the initial space. In our more general context Attal has provided an example to show that the question of unitarity of solutions is not a simple one to resolve.

We investigate this situation, using the Attal-Meyer extension of the quantum stochastic calculus to prove the existence of a solution to the evolution equation under mild integrability conditions on the time-dependent, adapted, bounded driving coefficients (Theorem 4.2.1). A necessary and sufficient condition for isometry is established; this is a straight-forward generalisation. We demonstrate two sufficient conditions for unitarity, neither of which is necessary. However, one of them is illuminating, not in the case when the QSDE has adapted coefficients, but in a related situation; that involving $\Omega$-adaptedness.

The term $\Omega$-adaptedness (pronounced 'vacuum-adaptedness') was coined by Lindsay (see [Lin, Example 4.3]). The interest in this notion is born out of a paper by Alicki and Fannes, [AlF], who were investigating the dilation of quantum dynamical semigroups using classical stochastic calculus. They examined the vector-valued equation

$$\phi_t = (1 - A_t V^*)\phi_0 + \int_0^t K\mathbb{E}_s(\phi_s)\,\mathrm{d}s + \int_0^t V\mathbb{E}_s(\phi_s)\,\mathrm{d}W_s$$

[AlF, Equation 12], where $K$ and $V$ are bounded operators acting on an initial space, $\mathbb{E}_s$ is conditional expectation at time $s$ with respect to the filtration generated by the Brownian motion $W$ and $A_t$ is the bounded operator with adjoint that acts on Wiener space as $f \mapsto \int_0^t \mathbb{E}_s(f)\,\mathrm{d}W_s$. They show that the condition

$$K + K^* = -V^*V$$

[AlF, Equation 3] is necessary and sufficient for unitarity of the associated semigroup. In [Vin2] Vincent-Smith noted that this idea could be generalised in a suitable quantum stochastic framework, and developed a non-adapted form of the calculus appropriate for this situation, predating the (more general) non-adapted calculus of Lindsay and Belavkin. He showed that the appropriate quantum stochastic differential equation is

$$U_t = I - \int_0^t V_s\mathbb{E}_s\,\mathrm{d}A_s + \int_0^t \mathrm{d}A_s^\dagger V_s\mathbb{E}_s U_s + \int_0^t K_s\mathbb{E}_s U_s\,\mathrm{d}s,$$

[Vin2, Equation 1.3], where $K$ and $V$ are now bounded, adapted processes with essentially-bounded norm, and if they satisfy the Alicki-Fannes unitarity condition pointwise almost everywhere then $U$ is unitary.

We abstract the idea of $\Omega$-adaptedness from the above and use our matrix formulation to examine quantum stochastic integrals of $\Omega$-adapted processes. The property of being $\Omega$-adapted is preserved by integration, and, strikingly, so is boundedness:

the quantum stochastic integral of a bounded, $\Omega$-adapted process is itself bounded (Theorem 3.3.5). We introduce a class of $\Omega$-adapted processes that parallels Attal's family of regular quantum semimartingales. We show that the evolution equation has a natural modification that corresponds to the Alicki-Fannes equation when the gauge integral is absent, and the A-F unitarity condition turns out to correspond to the Hudson-Parthasarathy condition for unitarity.

## 1.2 Notation

As a general rule, Roman indices (e.g., $i$, $j$, $k$) run over positive values, whereas Greek indices (e.g., $\lambda$, $\mu$, $\nu$) may also take the value zero. We shall use the *Evans delta* [Eva], denoted $\not\delta$, which is related to the Kronecker delta in the following manner:

$$\not\delta^{\lambda}_{\mu} := \delta^{\lambda}_{\mu}(1 - \delta^{\lambda}_{0}\delta^{0}_{\mu}) = \begin{cases} 1 & \lambda = \mu \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

If $X, Y$ are Banach spaces then we let $\mathcal{B}(X;Y)$ denote the Banach space of bounded (continuous) linear operators from $X$ to $Y$. A *linear operator* $(T, \mathcal{D}(T))$ *in* $X$ is a subspace $\mathcal{D}(T)$ of $X$ and a linear mapping $T : \mathcal{D}(T) \to Y$. We may sometimes write $(T, \mathcal{D}(T))$ as $T$ for brevity. If $(S, \mathcal{D}(S))$ and $(T, \mathcal{D}(T))$ are linear operators in $X$ we define their sum $(S+T, \mathcal{D}(S+T))$ and their product $(ST, \mathcal{D}(ST))$ by

$$\mathcal{D}(S+T) := \mathcal{D}(S) \cap \mathcal{D}(T) \ , \ (S+T)\xi := S\xi + T\xi$$

and

$$\mathcal{D}(ST) := \{\xi \in \mathcal{D}(T) \, : \, T\xi \in \mathcal{D}(S)\} \ , \ (ST)\xi := S(T\xi).$$

The class of continuous functions from a compact Hausdorff space $\Omega$ to a Banach space $X$ forms a Banach space and is denoted $C(\Omega; X)$. That is,

$$C(\Omega; X) := \{f : \Omega \to X \mid f \text{ continuous}\}$$

with norm

$$\|f\|_{C(\Omega;X)} := \sup \left\{ \|f(\omega)\|_X \, : \, \omega \in \Omega \right\}$$

and vector-space operations defined pointwise, is a Banach space (see [KaR, Example 1.7.2]).

If $X$ is a separable normed vector space and $p \in [1, \infty]$ then $L^p([0,1]; X)$ will denote the normed vector space of (equivalence classes of) strongly measurable functions on $[0,1]$ with $p$-integrable norm (and $L^p[0,1] := L^p([0,1]; \mathbb{C})$). We shall also use the Bochner-Lebesgue spaces $L^p([0,1];\mathcal{B}(X;Y)_s)$, where $X$ and $Y$ are separable Banach spaces and the '$s$' denotes strong, not uniform, measurability (see [HiP, p. 74]), i.e., the function $f : [0,1] \to \mathcal{B}(X;Y)$ is an element of (an equivalence class in) $L^p([0,1];\mathcal{B}(X;Y)_s)$ if and only if $t \mapsto f(t)\xi$ is strongly measurable for all $\xi \in X$ and

$$\|f\|_p := \left\{ \begin{array}{cc} \left(\int_0^1 \|f(t)\|^p_{\mathcal{B}(X;Y)}\,\mathrm{d}t\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \operatorname{ess\,sup}\left\{\|f(t)\|_{\mathcal{B}(X;Y)} : 0 \le t \le 1\right\} & p = \infty \end{array} \right\} < \infty;$$

measurability of $t \mapsto \|f(t)\|$ follows from Lemma A.1.2. These are Banach spaces for all $p \in [1,\infty]$: see Proposition A.2.1. The notation $\|\cdot\|_{p,t}$ is used as shorthand for $\|\cdot\chi_{[0,t]}\|_p$, the $L^p$-norm over $[0,t]$.

We follow the convention that inner products are anti-linear in their first argument and linear in their second. We denote the algebraic tensor product of two vector spaces $X$ and $Y$ by $X \underline{\otimes} Y$. If $X$ and $Y$ are Hilbert spaces then $X \otimes Y$ denotes their (Hilbert space) tensor product, and if $S$ and $T$ are bounded operators on $X$ and $Y$ respectively then $S \otimes T$ denotes the bounded operator on $X \otimes Y$ such that $S \otimes T(u \otimes v) = Su \otimes Tv$. The identity operator is denoted id, with a subscript to indicate the space on which it acts.

## 1.3 Acknowledgements

*γηράσκω δ' αἰεὶ πολλὰ διδασκόμενος.*

Solon

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# Chapter 2

# Operator Matrices

We introduce the notion of a chaos decomposition of a Hilbert space. Given two Hilbert spaces with chaos decompositions we introduce the class of operator matrices between them. This class is equipped with appropriate algebraic structure and interesting subclasses are defined. Operator matrices give rise to linear operators in a natural manner; relationships between the matrix adjoint and the Hilbert-space adjoint are explored. Those matrices giving rise to bounded operators are characterised and an equivalence between unitary matrices and unitary operators is obtained. Processes of operator matrices are defined and some terminology is established.

## 2.1 Chaos Decomposition

Let $(\Omega = C([0,1];\mathbb{R}), \mathcal{F}, \mathbb{P}^w)$ denote the probability space of Brownian motion over unit time and consider the associated real $L^2$-space. It is well-known that

$$L^2(\Omega, \mathcal{F}, \mathbb{P}^w; \mathbb{R}) = \mathbb{R}1 \oplus \bigoplus_{n \geq 1} I_n(L^2_{\text{sym}}([0,1]^n; \mathbb{R})),$$

where $1 : \omega \mapsto 1$ is the constant function and $I_n$ is the $n$-fold iterated Wiener integral

$$I_n(f) = \int_0^1 \cdots \int_0^1 f(t_1, \ldots, t_n) \, \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n}$$

($W = (W_t : t \in [0,1])$ is standard Brownian motion and the subscript 'sym' means that the function $f$ is invariant under permutations of its arguments). This is referred to as the *chaos decomposition* of Wiener space and the spaces in the direct sum as *chaos subspaces* (see [Mey, IV.2.1]). This example motivates our first definition.

**Definition 2.1.1** *Let* $\mathfrak{H}$ *and* $\mathfrak{h}_0, \mathfrak{h}_1, \ldots$ *be Hilbert spaces. We say that* $\mathfrak{H}$ *has* chaos decomposition $\{\mathfrak{h}_\mu : \mu \geq 0\}$ *if*

$$\mathfrak{H} = \bigoplus_{\mu \geq 0} \mathfrak{h}_\mu = \{\xi \in \prod_{\mu \geq 0} \mathfrak{h}_\mu : \|\xi\|^2 := \sum_{\mu \geq 0} \|\xi^\mu\|^2_{\mathfrak{h}_\mu} < \infty\}.$$

*The orthogonal projection onto the* $\mu$-th chaos subspace *is denoted* $E_\mu : \mathfrak{H} \to \mathfrak{h}_\mu : \xi \mapsto \xi^\mu$*. The term* chaos space *refers to* $\mathfrak{H}$ *with such a decomposition.*

*We may also think of the decomposition as an internal, rather than external, direct sum. That is, a chaos space is a Hilbert space* $\mathfrak{H}$ *and a family* $(E_\mu)_{\mu \geq 0} \subseteq \mathcal{B}(\mathfrak{H})$ *of mutually-orthogonal projections that span* $\mathfrak{H}$*, i.e.,* $E_\mu E_\nu = 0$ *if* $\mu \neq \nu$ *and* $\sum_{\mu \geq 0} E_\mu = \mathrm{id}_\mathfrak{H}$*, where convergence is in the strong operator topology. The maps* $E_\mu$ *are called* projection maps *and* $\mathfrak{h}_\mu := E_\mu \mathfrak{H}$ *is the* $\mu$*-th chaos space.*

There is a third way of viewing a chaos space. If $\mathfrak{H}$ is a Hilbert space equipped with a *number operator*, that is, a self-adjoint operator $N$ with spectrum $\sigma(N) = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$, then $\mathfrak{H}$ has a natural chaos-space structure. If $E$ is the resolution of the identity associated with $N$ then $(E_\mu := E(\{\mu\}))_{\mu \geq 0}$ is a family of projection maps and $\mathfrak{H} = \oplus_{\mu \geq 0} \mathfrak{h}_\mu$, where $\mathfrak{h}_\mu := E_\mu \mathfrak{H}$. We show below that every chaos space has such an operator.

The *finite particle subspace* of a chaos space $\mathfrak{H}$ is the algebraic sum of the constituent subspaces, i.e.,

$$\mathfrak{H}_{00} := \mathrm{lin} \cup_{\mu \geq 0} \mathfrak{h}_\mu = \{\xi \in \mathfrak{H} \, : \, |\xi| < \infty\},$$

where $|\cdot| : \mathfrak{H} \to \{0, 1, \ldots\} \cup \{\infty\}$ is a sub-additive function, the *rank* (or *degree*) of a vector, defined by

$$|\xi| := \inf\{\mu \geq 0 \, : \, E_\nu \xi = 0 \; \forall \, \nu \geq \mu\}$$

(we let $\inf \emptyset := \infty$). Note that $|0| = 0$ and $|\xi^\mu| = \mu + 1$ if $\xi^\mu \neq 0$. The elements of $\mathfrak{H}_{00}$ are called *finite particle vectors*. The *exponential subspace* of a chaos space $\mathfrak{H}$ is defined to be

$$\mathcal{E} := \{\xi \in \mathfrak{H} \,|\, \exists \, M \geq 0 : \|\xi^\mu\| \leq M^{\mu+1} (\mu!)^{-\frac{1}{2}} \; \forall \, \mu \geq 0\}.$$

It is easy to see that $\mathfrak{H}_{00}$ is a dense subspace of $\mathfrak{H}$, and it is readily verified that $\mathcal{E}$ is a subspace of $\mathfrak{H}$ that it contains $\mathfrak{H}_{00}$, so it is dense also.

**Examples**

1. We have already seen that Wiener space, $L^2(\Omega, \mathcal{F}, \mathbb{P}^w)$, has a natural chaos decomposition in terms of iterated integrals. More generally, a normal martingale $M$ (i.e., one such that $[M, M]_t = t$ for all $t$, where $[\cdot, \cdot]$ is the bracket process of classical stochastic calculus) has the *chaos representation property* if all the iterated integrals

$$J_n(f) := \int_0^1 \cdots \int_0^1 f(t_1, \ldots, t_n) \, \mathrm{d}M_{t_1} \cdots \mathrm{d}M_{t_n} \quad \forall \, f \in L^2_{\mathrm{sym}}([0, 1]^n)$$

for $n \geq 1$ and the constant functions are dense in the associated $L^2$-space. For example, the compensated Poisson process and the Azéma martingale are both normal martingales with this property. For more on this topic, see [Mey, IV.2.2] and references therein.

2. It is easy to see that any infinite-dimensional Hilbert space can be given a chaos decomposition, and so infinitely many (this is also true of finite-dimensional spaces if we allow trivial elements into the summand), but generally they will be of little interest; the decomposition should have some other motivation to be useful. Compare the fact that (up to isomorphism) there is one Hilbert space for each cardinality, but the problem at hand usually suggests an appropriate form in which to write the space.

3. The space $l^2(\{0, 1, 2, \ldots\})$ has a chaos decomposition in terms of the canonical basis $(\delta_\mu)_{\mu \geq 0}$. This is the simplest chaos decomposition and is useful for providing counter-examples. It is the (symmetric or full) Fock space over the one-dimensional Hilbert space $\mathbb{C}$; see the next example.

4. The three types of Fock space used in quantum theory all have natural chaos decompositions. Let $\mathcal{H}$ be a Hilbert space, let $\mathfrak{h}_0 = \mathbb{C}\Omega$, where $\Omega$ is a unit vector called the *vacuum vector*, and for $n \geq 1$ let $\mathfrak{h}_n = \mathcal{H}^{\otimes n}$, the $n$-fold tensor product of $\mathcal{H}$ with itself. The space $\mathfrak{H} = \bigoplus_{\mu \geq 0} \mathfrak{h}_\mu$ is called the *full Fock space* over $\mathcal{H}$, also known as the *Maxwell-Boltzmann* Fock space. If the tensor product is replaced by the symmetric tensor product we get the *Boson* (or *Bose-Einstein* or *symmetric*) *Fock space*, usually denoted $\mathfrak{F}_+(\mathcal{H})$:

$$\mathfrak{F}_+(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes_s n}.$$

Similarly, taking the antisymmetric tensor product yields the *Fermion* (or *Fermi-Dirac* or *antisymmetric*) *Fock space*

$$\mathfrak{F}_-(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes_a n}.$$

Note that as $\mathcal{H}^{\otimes_s n}$ lies naturally within $\mathcal{H}^{\otimes n}$, as does $\mathcal{H}^{\otimes_a n}$, we may regard $\mathfrak{F}_+(\mathcal{H})$ and $\mathfrak{F}_-(\mathcal{H})$ as subspaces of the full Fock space (cf. [BrR], [Mey], [Par]). The chaos decomposition of Wiener space gives a natural isomorphism between that space and Boson Fock space over $L^2[0, 1]$, the *Wiener-Itô isomorphism.*

4a. We may also obtain the Boson Fock space in the following manner. Let $\mathfrak{H}$ be the Hilbert space associated with the Gelfand pair $(\mathcal{H}, \exp)$, where $\mathcal{H}$ is some Hilbert space and exp is the kernel $\mathcal{H} \times \mathcal{H} \to \mathbb{C}; (u, v) \mapsto \exp\langle u, v\rangle$; i.e., $\mathfrak{H}$ is the Hilbert space with dense subspace $\mathcal{E}_0 := \operatorname{lin}\{\varepsilon(u) : u \in \mathcal{H}\}$ and inner product

$\langle \varepsilon(u), \varepsilon(v)\rangle_{\mathfrak{H}} = \exp\langle u, v\rangle$. (For the existence of such a space see [Par, Proposition 15.4].) Let $iN$ denote the generator of the strongly continuous, one-parameter unitary group $(U_t : t \in \mathbb{R})$, where $U_t$ is the unitary operator that acts on $\mathcal{E}_0$ as $\varepsilon(u) \mapsto \varepsilon(e^{it}u)$. The operator $N$ is self-adjoint (by Stone's theorem; see [Yos, Theorem 1, p. 345]) and the periodicity of the group means that the spectrum of $N$, $\sigma(N)$, is a subset of $\mathbb{Z}$: $\{1\} = \sigma(U_0) = \sigma(U_{2\pi}) \supseteq \sigma(e^{2\pi i N})$, by [Dav, Theorem 2.16], so if $\lambda \in \sigma(N)$ then $e^{2\pi i\lambda} = 1$. It is easy to see that the map $\mathbb{C} \ni z \mapsto \varepsilon(z\,u)$ is weakly holomorphic, so strongly holomorphic by Dunford's theorem ([HiP, Theorem 3.10.1(1)]): note that $z \mapsto \langle \xi, \varepsilon(z\,u)\rangle$ is holomorphic if $\xi \in \mathcal{E}_0$, so for general $\xi$, using the fact that locally uniform convergence preserves holomorphy and the continuity of $z \mapsto \|\varepsilon(z\,u)\|$. Thus $\mathcal{E}_0 \subseteq \mathcal{D}(N)$ and $\mathrm{d}^\nu\varepsilon(u) := \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\varepsilon(z\,u)\big|_{z=0}$ is an element of $\mathfrak{H}$ for all $\nu \geq 0$ (where the case $\nu = 0$ corresponds to $\varepsilon(0)$) with

$$\langle N\varepsilon(v),\, \mathrm{d}^\nu\varepsilon(u)\rangle = i\frac{\partial^{\nu+1}}{\partial t\,\partial z^\nu}\langle \varepsilon(e^{it}v), \varepsilon(z\,u)\rangle\bigg|_{t=z=0} = \nu\langle v, u\rangle^\nu = \nu\langle \varepsilon(v), d^\nu\varepsilon(u)\rangle.$$

This suggests that $d^\nu\varepsilon(u)$ is an eigenvector for $N$ with eigenvalue $\nu$, which may be verified directly. We claim that this is the entire spectrum of $N$, i.e, $N$ is positive. Note that if $\xi = \sum_{k=1}^n \lambda_k\varepsilon(u_k) \in \mathcal{E}_0$ then

$$\langle \xi, N\xi\rangle = \|\sum_{k=1}^n \lambda_k u_k \otimes \varepsilon(u_k)\|^2_{\mathcal{H}\otimes\mathfrak{H}} \geq 0,$$

so if we can show that $\mathcal{E}_0$ is a core for $N$ we are done. This follows from the invariance of the space $\mathcal{E}_0$ under $(U_t)$, by [Dav, Theorem 1.9], or it can be shown directly.

Note that $z \mapsto N\varepsilon(z\,u)$ is strongly holomorphic, by the same argument used above for $z \mapsto \varepsilon(z\,u)$, and $\mathrm{d}^\nu N\varepsilon(u) = \nu\,\mathrm{d}^\nu\varepsilon(u)$. If $\xi \in \mathcal{D}(N)$ is such that $\langle \xi, \varepsilon(u)\rangle + \langle N\xi, N\varepsilon(u)\rangle = 0$ for all $u \in \mathcal{H}$ then $z \mapsto \langle N\xi, N\varepsilon(z\,u)\rangle = -\langle \xi, \varepsilon(z\,u)\rangle$ is entire with derivatives

$$\frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\langle N\xi, N\varepsilon(z\,u)\rangle\bigg|_{z=0} = -\langle \xi,\, \mathrm{d}^\nu\varepsilon(u)\rangle.$$

Now $\mathrm{d}^\nu\varepsilon(u) \in \mathcal{D}(N^2)$ and

$$\langle \xi, N^2\,\mathrm{d}^\nu\varepsilon(u)\rangle = \langle N\xi, \nu\,\mathrm{d}^\nu\varepsilon(u)\rangle = \frac{\mathrm{d}^\nu}{\mathrm{d}z^\nu}\langle N\xi, N\varepsilon(z\,u)\rangle\bigg|_{z=0} = -\langle \xi,\, \mathrm{d}^\nu\varepsilon(u)\rangle.$$

Hence $(1+\nu^2)\langle \xi,\, \mathrm{d}^\nu\varepsilon(u)\rangle = 0$ for all $\nu \geq 0$, whence $\langle \xi, \varepsilon(u)\rangle = 0$ for all $u \in \mathcal{H}$, so $\xi = 0$ and $\mathcal{E}_0$ is a core for $N$, as claimed. Thus $N$ is a number operator, and it may be shown that it gives $\mathfrak{H}$ the chaos decomposition as in Example 4.

5. Given a Hilbert space $\mathcal{H}$ and a chaos space $\mathfrak{H}$ we can form their *product chaos space*; this is a chaos space $\tilde{\mathfrak{H}}$ isomorphic to their tensor product $\mathcal{H} \otimes \mathfrak{H}$:

$$\tilde{\mathfrak{H}} = \bigoplus_{\mu \geq 0} \tilde{\mathfrak{h}}_\mu, \quad \tilde{\mathfrak{h}}_\mu := \mathcal{H} \otimes \mathfrak{h}_\mu.$$

The space $\mathcal{H}$ is referred to as the *initial space*.

6. Given a chaos space $\mathfrak{H}$ there is a natural way to regard $L^2(M; \mathfrak{H})$ as a chaos space, where $(M, \mathcal{M}, \mu)$ is any ($\sigma$-finite) measure space;

$$L^2(M; \mathfrak{H}) \cong \bigoplus_{\mu \geq 0} L^2(M; \mathfrak{h}_\mu); \;\; f(t) = \sum_{\mu \geq 0} E_\mu f(t).$$

We use this decomposition without further comment.

□

As Boson Fock space is the chaos space we are most interested in, we prove here a proposition which will be useful later.

**Proposition 2.1.2** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{J}$ a closed subspace of $\mathcal{H}$. There exists a total family of linearly independent vectors called* exponential vectors*, indexed by $\mathcal{H}$, with linear span denoted*

$$\mathcal{E}_0 := \text{lin}\,\{\varepsilon(u) := \Omega + u + u^{\otimes 2}/\sqrt{2!} + \cdots = \sum_{\mu \geq 0} u^{\otimes \mu} (\mu!)^{-\frac{1}{2}} \,|\, u \in \mathcal{H}\} \subseteq \mathcal{E}(\mathfrak{F}_+(\mathcal{H})).$$

*There exists an isometric isomorphism $j : \mathfrak{F}_+(\mathcal{J}) \otimes \mathfrak{F}_+(\mathcal{J}^\perp) \to \mathfrak{F}_+(\mathcal{H})$ such that $j(\varepsilon(u) \otimes \varepsilon(v)) \mapsto \varepsilon(u+v)$ for all $u \in \mathcal{J}$, $v \in \mathcal{J}^\perp$. Furthermore, if $\phi \otimes \psi \in \mathfrak{F}_+(\mathcal{J}) \otimes \mathfrak{F}_+(\mathcal{J}^\perp)$ then*

$$\phi \otimes \psi \;\; = \;\; \sum_{\mu \geq 0} \sum_{\pi = 0}^{\mu} \phi^{\mu - \pi} \otimes \psi^\pi. \tag{2.1}$$

**Proof**

Everything but the last part is standard; see, e.g., [Par, Propositions 19.4 & 19.6]. For the final claim, note first that the continuity of the map $a \mapsto b \otimes a$ gives that

$$\phi \otimes \psi = \sum_{\mu \geq 0} \phi^\mu \otimes \sum_{\pi \geq 0} \psi^\pi = \sum_{\pi \geq 0} \sum_{\mu \geq 0} \phi^\mu \otimes \psi^\pi = \lim_{n \to \infty} \sum_{\pi = 0}^{n} \sum_{\mu \geq \pi} \phi^{\mu - \pi} \otimes \psi^\pi$$

and this equals

$$\lim_{n \to \infty} \sum_{\mu \geq 0} \sum_{\pi = 0}^{\mu \wedge n} \phi^{\mu - \pi} \otimes \psi^\pi = \sum_{\mu \geq 0} \sum_{\pi = 0}^{\mu} \phi^{\mu - \pi} \otimes \psi^\pi.$$

To see the last equality, note that

$$\sum_{\pi=0}^{\mu} \phi^{\mu-\pi} \otimes \psi^{\pi} \perp \sum_{\pi=0}^{\lambda} \phi^{\lambda-\pi} \otimes \psi^{\pi}$$

if $\lambda \neq \mu$ and $\phi^{\mu-\pi} \otimes \psi^{\pi} \perp \phi^{\mu-\lambda} \otimes \psi^{\lambda}$ if $\lambda \neq \pi$. Thus $\sum_{\mu \geq 0} \sum_{\pi=0}^{\mu} \phi^{\mu-\pi} \otimes \psi^{\pi}$ is an element of $\mathfrak{F}_+(\mathcal{J}) \otimes \mathfrak{F}_+(\mathcal{J}^{\perp})$, because

$$\sum_{\mu=0}^{n} \| \sum_{\pi=0}^{\mu} \phi^{\mu-\pi} \otimes \psi^{\pi} \|^2 = \sum_{\mu=0}^{n} \sum_{\pi=0}^{\mu} \|\phi^{\mu-\pi}\|^2 \|\psi^{\pi}\|^2 = \sum_{\pi=0}^{n} \sum_{\mu=\pi}^{n} \|\phi^{\mu-\pi}\|^2 \|\psi^{\pi}\|^2,$$

(reverse the order of summation) and this double series is bounded above by $\|\psi\|^2 \|\phi\|^2$ for all $n$. Furthermore

$$\| \sum_{\mu \geq 0} \sum_{\pi=0}^{\mu} \phi^{\mu-\pi} \otimes \psi^{\pi} - \sum_{\mu \geq 0} \sum_{\pi=0}^{\mu \wedge n} \phi^{\mu-\pi} \otimes \psi^{\pi} \|^2 = \| \sum_{\mu \geq n+1} \sum_{\pi=n+1}^{\mu} \phi^{\mu-\pi} \otimes \psi^{\pi} \|^2$$

and, using orthogonality and reversing the order of summation, we see that this equals $\sum_{\pi \geq n+1} \|\psi^{\pi}\|^2 \sum_{\mu \geq 0} \|\phi^{\mu}\|^2$, which converges to 0 as $n \to \infty$.

□

If we are dealing with the Boson Fock space over $L^2[0,1]$ the above gives the isomorphism $j_t : \mathfrak{H} \cong \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}$ for all $t \in [0,1]$, where $\mathfrak{H}_{t]} := \mathfrak{F}_+(L^2[0,t])$ and $\mathfrak{H}_{(t} := \mathfrak{F}_+(L^2(t,1])$. We let $u_{t]} := u\chi_{[0,t]} \in L^2[0,t]$ and $u_{(t} := u\chi_{(t,1]} \in L^2(t,1]$ for all $t \in [0,1]$, so that $j_t \varepsilon(u) = \varepsilon(u_{t]}) \otimes \varepsilon(u_{(t})$.

## 2.2 Operator Matrices

**Definition 2.2.1** *Let* $\mathfrak{H} = \bigoplus_{\mu \geq 0} \mathfrak{h}_{\mu}$ *and* $\mathfrak{J} = \bigoplus_{\mu \geq 0} \mathfrak{j}_{\mu}$ *be chaos spaces. An* operator matrix *(or* chaos matrix*) between* $\mathfrak{H}$ *and* $\mathfrak{J}$ *is a matrix* $A = (A^{\mu}_{\nu})_{\mu,\nu \geq 0}$ *such that* $A^{\mu}_{\nu} \in \mathcal{B}(\mathfrak{h}_{\nu}; \mathfrak{j}_{\mu})$ *for all* $\mu, \nu \geq 0$. *The collection of all such operator matrices is denoted* $\mathfrak{M}(\mathfrak{H}; \mathfrak{J})$.

If $\mathfrak{H} = \mathfrak{J}$ we will write $\mathfrak{M}(\mathfrak{H})$, and will sometimes omit explicit reference to the chaos spaces if this will not cause confusion. Another convention we adopt is to set $A^{\mu}_{\nu} := 0$ if $\mu < 0$ or $\nu < 0$.

We make $\mathfrak{M}(\mathfrak{H}; \mathfrak{J})$ a vector space in the obvious manner, i.e, by letting

$$(A + zB)^{\mu}_{\nu} := A^{\mu}_{\nu} + zB^{\mu}_{\nu} \quad \forall\, \mu, \nu \geq 0$$

for all $A, B \in \mathfrak{M}$ and $z \in \mathbb{C}$. There is a natural involution between $\mathfrak{M}(\mathfrak{H};\mathfrak{J})$ and $\mathfrak{M}(\mathfrak{J};\mathfrak{H})$ (the *matrix adjoint*) given by

$$(A^*)^\mu_\nu := (A^\nu_\mu)^* \quad \forall\, \mu, \nu \geq 0$$

for all $A$, where the $*$ on the right-hand side denotes the Hilbert-space adjoint.

Suppose $A \in \mathfrak{M}(\mathfrak{K};\mathfrak{J})$ and $B \in \mathfrak{M}(\mathfrak{H};\mathfrak{K})$ are such that $\sum_{\pi \geq 0} A^\mu_\pi B^\pi_\nu$ is strongly convergent in $\mathcal{B}(\mathfrak{h}_\nu; \mathfrak{j}_\mu)$ for all $\mu, \nu \geq 0$. We define the product $AB \in \mathfrak{M}(\mathfrak{H};\mathfrak{J})$ in the obvious manner, viz.

$$(AB)^\mu_\nu := \sum_{\pi \geq 0} A^\mu_\pi B^\pi_\nu \quad \forall\, \mu, \nu \geq 0.$$

If the convergence is not just strong, but holds in the uniform topology for all $\mu, \nu \geq 0$ then the product is compatible with the involution: $(AB)^* = B^* A^*$, in the sense that one exists if and only if the other does (and they are equal). Given $\lambda, \pi \geq 0$ we define a pointwise product by

$$(A_\lambda B^\pi)^\mu_\nu := \delta^\lambda_\pi A^\mu_\lambda B^\lambda_\nu \quad \forall\, \mu, \nu \geq 0,$$

so that $AB$ is the coordinate-wise strong limit of $\sum_{\pi \geq 0} A_\pi B^\pi$, if this limit exists.

**Definition 2.2.2** *For $\pi \geq 0$ the class of* $(2\pi + 1)$-diagonal matrices *is denoted $\mathfrak{D}_\pi$, and defined by*

$$\mathfrak{D}_\pi := \{A \in \mathfrak{M} \mid A^\mu_\nu = 0 \ \forall\, \mu, \nu : |\mu - \nu| > \pi\};$$

*the collection of all diagonal matrices is $\mathfrak{D} := \cup_{\pi \geq 0} \mathfrak{D}_\pi$.*

*The class of* sparse matrices *is denoted $\mathfrak{F}$ and equals $\cup_{\pi \geq 0} \mathfrak{F}_\pi$, where*

$$\mathfrak{F}_\pi := \{A \in \mathfrak{M} \mid A^\mu_\nu = 0 \ \forall\, \mu, \nu : \mu > \pi \text{ or } \nu > \pi\} \subset \mathfrak{D}_\pi.$$

It is easily seen that $\mathfrak{D}$ is a subspace of $\mathfrak{M}$, invariant under $*$ (i.e., if $A \in \mathfrak{D}(\mathfrak{H};\mathfrak{J})$ then $A^* \in \mathfrak{D}(\mathfrak{J};\mathfrak{H})$) and furthermore

$$\mathfrak{D}_\mu \mathfrak{D}_\nu := \{AB \,:\, A \in \mathfrak{D}_\mu(\mathfrak{J};\mathfrak{K}), B \in \mathfrak{D}_\nu(\mathfrak{H};\mathfrak{J})\} \subseteq \mathfrak{D}_{\mu+\nu}(\mathfrak{H};\mathfrak{K}).$$

In particular $\mathfrak{D}(\mathfrak{H})$ is a unital $*$-algebra over $\mathbb{C}$, with unit $I := (\delta^\mu_\nu \mathrm{id}_{\mathfrak{h}_\mu}) \in \mathfrak{D}_0$.

The diagonal matrices $\mathfrak{D}(\mathfrak{H};\mathfrak{J})$ have a left action on $\mathfrak{M}(\mathfrak{K};\mathfrak{H})$ and right action on $\mathfrak{M}(\mathfrak{J};\mathfrak{K})$ given by multiplication. In particular $\mathfrak{M}(\mathfrak{H})$ is a $\mathfrak{D}(\mathfrak{H})$-bimodule, even a $*$-bimodule: $(AB)^* = B^* A^*$ and $(BA)^* = A^* B^*$ for all $A \in \mathfrak{D}(\mathfrak{H})$ and $B \in \mathfrak{M}(\mathfrak{H})$.

If we observe the inclusion

$$\mathfrak{F}_\mu(\mathfrak{J};\mathfrak{K}) \mathfrak{F}_\nu(\mathfrak{H};\mathfrak{J}) \subseteq \mathfrak{F}_{\mu \vee \nu}(\mathfrak{H};\mathfrak{K})$$

it is apparent that $\mathfrak{F}(\mathfrak{H})$ is a (non-unital) $*$-algebra. In fact,

$$\mathfrak{F}_\mu \mathfrak{D}_\nu \subseteq \mathfrak{F}_{\mu+\nu} \ , \ \mathfrak{D}_\nu \mathfrak{F}_\mu \subseteq \mathfrak{F}_{\mu+\nu},$$

where $\mathfrak{F}$ and $\mathfrak{D}$ are between the appropriate spaces, so we have that $\mathfrak{F}(\mathfrak{H})$ is a 2-sided $*$-ideal of $\mathfrak{D}(\mathfrak{H})$: compare this to the class of bounded operators with finite rank over some Hilbert space.

**Definition 2.2.3** *Let $A \in \mathfrak{M}(\mathfrak{H};\mathfrak{J})$. The linear operator $\hat{A}$ defined by*

$$\begin{aligned} \mathcal{D}(\hat{A}) &:= \{\xi \in \mathfrak{H} : \sum_{\mu \geq 0} \| \sum_{\nu \geq 0} A^\mu_\nu \xi^\nu \|^2 < \infty\}, \\ \hat{A}\xi &:= \sum_{\mu \geq 0} \sum_{\nu \geq 0} A^\mu_\nu \xi^\nu \end{aligned}$$

*is called the* operator induced by $A$.

It is implicit in the definition that $\sum_{\nu \geq 0} A^\mu_\nu \xi^\nu$ must converge for all $\mu \geq 0$ if $\xi \in \mathcal{D}(\hat{A})$. (If $A$ is diagonal, for instance, or $\xi$ is a finite particle vector, this is automatically the case.) Our definition is in line with the usual convention for matrices acting on (column) vectors: we have

$$\begin{pmatrix} A^0_0 & A^0_1 & \cdots \\ A^1_0 & A^1_1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi^0 \\ \xi^1 \\ \vdots \end{pmatrix} = \begin{pmatrix} A^0_0\xi^0 + A^0_1\xi^1 + \cdots \\ A^1_0\xi^0 + A^1_1\xi^1 + \cdots \\ \vdots \end{pmatrix};$$

the upper (or *row*) index corresponds to the label of the "output space", the lower (or *column*) index to that of the "input space".

Matrix addition and operator addition are compatible, in the sense that $\hat{A} + \hat{B} \subseteq \widehat{A+B}$, because

$$\| \sum_{\nu=0}^{n} A^\mu_\nu \xi^\nu + B^\mu_\nu \xi^\nu \|^2 \leq 2 \| \sum_{\nu=0}^{n} A^\mu_\nu \xi^\nu \|^2 + 2 \| \sum_{\nu=0}^{n} B^\mu_\nu \xi^\nu \|^2$$

for all $n$. Furthermore the left action of $\mathfrak{D}$ is compatible with operator multiplication, i.e., $\hat{A}\hat{B} \subseteq \widehat{AB}$ for all $A \in \mathfrak{D}$ and $B \in \mathfrak{M}$. To see this, suppose that $\xi \in \mathcal{D}(\hat{A}\hat{B})$ and $A \in \mathfrak{D}_\kappa$;

$$\sum_{\nu \geq 0} A^\mu_\nu (\hat{B}\xi)^\nu = \sum_{\nu=0}^{\kappa+\mu} \sum_{\pi \geq 0} A^\mu_\nu B^\nu_\pi \xi^\pi = \sum_{\pi \geq 0} \sum_{\nu=0}^{\kappa+\mu} A^\mu_\nu B^\nu_\pi \xi^\pi = \sum_{\pi \geq 0} (AB)^\mu_\pi \xi^\pi$$

and

$$\sum_{\mu \geq 0} \| \sum_{\nu \geq 0} (AB)^\mu_\nu \xi^\nu \|^2 = \sum_{\mu \geq 0} \| \sum_{\nu \geq 0} A^\mu_\nu (\hat{B}\xi)^\nu \|^2 < \infty,$$

so $\xi \in \mathcal{D}(\widehat{AB})$. This working shows also that $\hat{A}\hat{B} = \widehat{AB}$ if $\mathcal{D}(\widehat{AB}) \subseteq \mathcal{D}(\hat{A}\hat{B})$.

**Definition 2.2.4** *Let* $A \in \mathfrak{M}$*;* $A$ *is* row-square-summable *if*

$$\sum_{\mu \geq 0} \|A^\mu_\nu\|^2 < \infty$$

*for all* $\nu \geq 0$.

Any diagonal matrix is row-square-summable, and finite linear combinations of row-square-summable matrices retain this property. If $A$ is row-square-summable then $\mathcal{D}(\hat{A}) \supseteq \mathfrak{H}_{00}$, as

$$\sum_{\mu \geq 0} \| \sum_{\nu=0}^{|\xi|} A^\mu_\nu \xi^\nu \|^2 \leq \sum_{\mu \geq 0} \sum_{\nu=0}^{|\xi|} \|A^\mu_\nu\|^2 \|\xi\|^2.$$

For such matrices we define the linear operator $\check{A}$ by

$$\mathcal{D}(\check{A}) := \mathfrak{H}_{00} \ , \ \ \check{A}\xi = \hat{A}\xi.$$

The following theorem demonstrates the connexion between the matrix adjoint and Hilbert-space adjoint. Its origins lie in [Sto, pp. 90–91].

**Theorem 2.2.5** *Let* $A \in \mathfrak{M}(\mathfrak{H}; \mathfrak{J})$ *be row-square-summable. We have that*

$$\check{A}^* = \widehat{A^*} \ \text{and} \ \hat{A}^* \subseteq \widehat{A^*}.$$

*In particular* $\widehat{A^*}$ *is closed. Furthermore* $\check{A}$ *is symmetric if and only* $A$ *is symmetric, i.e.,* $A = A^*$*, and* $\check{A}$ *is essentially self-adjoint if and only if* $\hat{A}$ *is self-adjoint.*

**Proof**

Let $\xi \in \mathcal{D}(\widehat{A^*})$ and $\eta \in \mathcal{D}(\check{A})$;

$$\langle \widehat{A^*}\xi, \eta \rangle = \sum_{\mu=0}^{|\eta|} \sum_{\nu \geq 0} \langle (A^*)^\mu_\nu \xi^\nu, \eta^\mu \rangle = \sum_{\nu \geq 0} \sum_{\mu=0}^{|\eta|} \langle \xi^\nu, A^\nu_\mu \eta^\mu \rangle = \langle \xi, \check{A}\eta \rangle,$$

so $\xi \in \mathcal{D}(\check{A}^*)$ and $\check{A}^*\xi = \widehat{A^*}\xi$, i.e., $\widehat{A^*} \subseteq \check{A}^*$.

Now suppose $\xi \in \mathcal{D}(\check{A}^*)$; there exists $\theta \in \mathfrak{H}$ such that $\langle \xi, \check{A}\eta \rangle = \langle \theta, \eta \rangle$ for all $\eta \in \mathfrak{H}_{00}$. Taking $\eta = \eta^\mu \in \mathfrak{h}_\mu$ we see that

$$\langle \theta^\mu, \eta^\mu \rangle = \sum_{\nu \geq 0} \langle \xi^\nu, A^\nu_\mu \eta^\mu \rangle,$$

and so conclude that $\theta^\mu = \sum_{\nu \geq 0} (A^*)^\mu_\nu \xi^\nu$, as this series is convergent:

$$\| \sum_{\nu=\lambda}^{\pi} (A^*)^\mu_\nu \xi^\nu \| \leq (\sum_{\nu=\lambda}^{\pi} \|A^\nu_\mu\|^2)^{\frac{1}{2}} \|\xi\| \to 0$$

as $\lambda, \pi \to \infty$. We know that $\theta \in \mathfrak{H}$, so

$$\sum_{\mu \geq 0} \| \sum_{\nu \geq 0} (A^*)^\mu_\nu \xi^\nu \|^2 = \sum_{\mu \geq 0} \|\theta^\mu\|^2 < \infty.$$

Thus $\xi \in \mathcal{D}(\widehat{A^*})$ and $\widehat{A^*}\xi = \check{A}^*\xi$, i.e., $\check{A}^* \subseteq \widehat{A^*}$, and so we have equality. The fact that $\widehat{A^*}$ is closed follows immediately, as we have exhibited it as the adjoint of the operator $\check{A}$. To see that $\hat{A}^* \subseteq \widehat{A^*}$, note that $\check{A} \subseteq \hat{A}$ implies that $\hat{A}^* \subseteq \check{A}^* = \widehat{A^*}$.

To see the equivalence between symmetric matrices and operators, observe that

$$\begin{aligned} \check{A} \text{ is symmetric } &\Leftrightarrow \langle \check{A}\xi, \eta \rangle = \langle \xi, \check{A}\eta \rangle \ \forall \xi, \eta \in \mathfrak{H}_{00} \\ &\Leftrightarrow \langle A^\mu_\nu \xi^\nu, \eta^\mu \rangle = \langle \xi^\nu, A^\nu_\mu \eta^\mu \rangle \ \forall \mu, \nu \geq 0, \eta^\mu \in \mathfrak{h}_\mu, \xi^\nu \in \mathfrak{h}_\nu \\ &\Leftrightarrow A^\mu_\nu = (A^\nu_\mu)^* \ \forall \mu, \nu \geq 0, \end{aligned}$$

i.e., $A = A^*$.

Finally, if $\hat{A}$ is self-adjoint then $\check{A}$ is symmetric ($\check{A} \subseteq \hat{A} = \hat{A}^* \subseteq \check{A}^*$), whence $A = A^*$ and so $\check{A}$ is essentially self-adjoint:

$$\check{A}^* = \widehat{A^*} = \hat{A} = \hat{A}^* = \widehat{A^*}^* = \check{A}^{**}.$$

Conversely, if $\check{A}$ is essentially self-adjoint then $\check{A}$ is symmetric (if $\eta \in \mathcal{D}(\check{A})$ then $\langle \check{A}^*\xi, \eta \rangle = \langle \xi, \check{A}\eta \rangle$ for all $\xi \in \mathcal{D}(\check{A}^*)$, i.e., $\check{A} \subseteq \check{A}^{**} = \check{A}^*$), so $A = A^*$ and

$$\hat{A} = \widehat{A^*} = \check{A}^* = \check{A}^{**} = \widehat{A^*}^* = \hat{A}^*,$$

i.e., $\hat{A}$ is self-adjoint.

□

**Example**

Let $\mathfrak{H} = l^2(\{0, 1, 2, \ldots\})$ and define $A \in \mathfrak{M}(\mathfrak{H})$ by

$$A^\mu_\nu := \begin{cases} \nu & \mu = 0 \\ 0 & \mu \neq 0 \end{cases}.$$

(We may write elements of $\mathfrak{M}(\mathfrak{H})$ as matrices of scalars as the chaos spaces are one-dimensional.) A is row-square-summable, but $A^*$ is not. $\check{A}$ is not closable: to see this note that $\delta_{\mu+1}/(\mu+1) \to 0$ as $\mu \to \infty$, but $\check{A}\delta_{\mu+1}/(\mu+1) = \delta_0$ for all $\mu \geq 0$. This example can be found in [KaR, Example 2.7.3, p. 156].

□

Matrices with entries lying on one diagonal are particularly well-behaved with respect to the adjoint.

**Corollary 2.2.6** *If $A \in \mathfrak{D}$ lies on a diagonal, i.e., there exists $l \in \mathbb{Z}$ such that $A^\mu_\nu = \delta^{\mu+l}_\nu A^\mu_\nu$ for all $\mu, \nu \geq 0$ then $\hat{A}^* = \widehat{A^*}$. In particular, $\hat{A}$ is self-adjoint if and only if $A$ is symmetric, i.e., $A = A^*$ (in which case $l = 0$, i.e., $A \in \mathfrak{D}_0$).*

**Proof**

Let $\xi \in \mathcal{D}(\hat{A})$, $\eta \in \mathcal{D}(\widehat{A^*})$. Then

$$\langle \hat{A}\xi, \eta\rangle = \sum_{\mu \geq 0} \langle A^\mu_{\mu+l} \xi^{\mu+l}, \eta^\mu \rangle = \sum_{\mu \geq 0} \langle \xi^{\mu+l}, (A^*)^{\mu+l}_\mu \eta^\mu\rangle = \langle \xi, \widehat{A^*}\eta\rangle$$

which shows that $\widehat{A^*} \subseteq \hat{A}^*$. Theorem 2.2.5 gives the reverse inclusion, so $\hat{A}^* = \widehat{A^*}$. Thus if $A = A^*$ then $\hat{A}^* = \widehat{A^*} = \hat{A}$. Conversely, suppose $\hat{A} = \hat{A}^*$. Then $\check{A}$ is essentially self-adjoint, so symmetric, so $A = A^*$ by the above.

□

In [Seg, pp. 109–110], Segal introduces a $*$-algebra of operators over Fock space, the elements of which are called *semi-graded*. These have matrix representations which are intermediate between diagonal and row-square-summable. He deduces properties of such operators which are contained in our Theorem 2.2.5.

**Definition 2.2.7** *A matrix $A$ has* growth of order $a$ *(where $a$ is a positive real number) if there exists a constant $C > 0$ such that*

$$\|A^\mu_\nu\| \leq C((\mu \vee \nu) + 1)^a \quad \forall\, \mu, \nu \geq 0.$$

*A matrix with growth of order $a$ for some $a$ is said to have* polynomial growth *or to be* polynomially bounded*. The collection of all diagonal matrices with polynomial growth is denoted $\mathfrak{D}_{\mathrm{pb}}$:*

$$\mathfrak{D}_{\mathrm{pb}} := \{A \in \mathfrak{D} \,|\, \exists\, C, a > 0 : \|A^\mu_\nu\| \leq C((\mu \vee \nu) + 1)^a \;\forall\, \mu, \nu \geq 0\}.$$

If $A$ has growth of order $a$ then it has growth of all higher orders. It is easy to verify that $\mathfrak{D}_{\mathrm{pb}}(\mathfrak{H})$ is a $*$-algebra.

Diagonal matrices with growth of order 1 are self-adjoint when they 'look' self-adjoint; in more correct language we have the following theorem.

**Theorem 2.2.8** *Let $A$ be a symmetric, diagonal matrix with growth of order 1. The elements of $\mathfrak{H}_{00}$ are analytic vectors for the operator $\check{A}$, which is essentially self-adjoint.*

**Proof**

Note first that $\check{A}$ is symmetric (by Theorem 2.2.5) and $\mathfrak{H}_{00}$ is stable under the action of this operator. We wish to show that the finite particle vectors are analytic for $\check{A}$, i.e., for all $\xi \in \mathfrak{H}_{00}$ the series

$$\sum_{n \geq 1} \|\check{A}^n \xi\| z^n / n!$$

has non-zero radius of convergence. Nelson's analytic vector theorem (see [ReS, Theorem X.39, pp. 201–202]) gives that $\check{A}$ is essentially self-adjoint.

Let $\xi \in \mathfrak{H}_{00}$ be fixed from now on, and suppose that $A$ is $(2\kappa+1)$-diagonal and $C > 0$ is such that $\|A^\mu_\nu\| \leq C((\mu \vee \nu) + 1)$ for all $\mu, \nu \geq 0$. It is easy to see that $\check{A}^n \xi = \widehat{A^n} \xi$, and $|\check{A}\eta| \leq |\eta| + \kappa$ for any $\eta \in \mathfrak{H}_{00}$, so $|\check{A}^n \xi| \leq |\xi| + n\kappa$. Thus

$$\begin{aligned}
\|\check{A}^n \xi\|^2 &= \sum_{\mu=0}^{n\kappa+|\xi|} \| \sum_{\nu=0}^{|\xi|} (A^n)^\mu_\nu \xi^\nu \|^2 \\
&\leq (n\kappa + |\xi| + 1)(2\kappa+1)^{2n-2} C^{2n} \prod_{l=1}^{n} (l\kappa + |\xi| + 1)^2 (|\xi| + 1) \|\xi\|^2;
\end{aligned}$$

this follows from the Cauchy-Schwarz-Buniakowski inequality and the following estimate:

$$\max_{\substack{\mu=0,\ldots,n\kappa+\sigma; \\ \nu=0,\ldots,\sigma}} \{\|(A^n)^\mu_\nu\|\} \leq (2\kappa+1)^{n-1} C^n \prod_{l=1}^{n} (l\kappa + \sigma + 1)$$

for all $\sigma \geq 0$. The case $n = 1$ is immediate, so suppose the inequality holds for $n = p \geq 1$. If $\mu \in \{0, \ldots, (p+1)\kappa + \sigma\}$ and $\nu \in \{0, \ldots, \sigma\}$ we have that

$$\begin{aligned}
\|(A^{p+1})^\mu_\nu\| &\leq \sum_{\pi=0}^{\sigma+\kappa} \|(A^p)^\mu_\pi\| \, \|A^\pi_\nu\| \\
&\leq (2\kappa+1)^{p-1} C^p \prod_{l=1}^{p} (l\kappa + \sigma + \kappa + 1) \, (2\kappa+1) C (\sigma + \kappa + 1) \\
&= (2\kappa+1)^p C^{p+1} \prod_{l=1}^{p+1} (l\kappa + \sigma + 1),
\end{aligned}$$

so the estimate holds, by induction. The limit ratio test completes the proof, as

$$((n+1)\kappa + |\xi| + 1)^{\frac{1}{2}} (2\kappa+1) C ((n+1)\kappa + |\xi| + 1) / (n\kappa + |\xi| + 1)^{\frac{1}{2}} (n+1) \to C(2\kappa+1)\kappa$$

as $n \to \infty$.

□

**Note**

The above proof can be modified to show that any symmetric, $(2\kappa+1)$-diagonal matrix $A$ that satisfies the growth condition

$$\lim_{n\to\infty} \frac{1}{n+1} \max\{\|A_\nu^\mu\| : \mu \geq 0; \nu = 0, \ldots, n\kappa + \sigma\} < \infty \quad \forall \sigma \geq 0$$

gives rise to a self-adjoint operator $\hat{A}$ with $\mathfrak{H}_{00}$ as analytic vectors.

**Example**

Let $\mathfrak{H} = l^2(\{0,1,2,\ldots\})$ and define $A = A^* \in \mathfrak{D}_1$ by setting $A_{\mu+1}^\mu = A_\mu^{\mu+1} = (\mu+1)^a$ for all $\mu \geq 0$ and $A_\nu^\mu = 0$ elsewhere: $a$ is a fixed real number. That is,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 2^a & 0 & 0 & \cdots \\ 0 & 2^a & 0 & 3^a & 0 & \cdots \\ 0 & 0 & 3^a & 0 & 4^a & \ddots \\ 0 & 0 & 0 & 4^a & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is not difficult to see that $\|(\check{A})^n\delta_0\| \geq (n!)^a$, and more generally $\|(\check{A})^n\delta_\mu\| \geq \left(\frac{(n+\mu)!}{\mu!}\right)^a$ for any $\mu \geq 0$. In particular, if $a > 1$ then

$$\|(\check{A})^n\delta_\mu\||z|^n/n! \geq \left(\frac{(n+\mu)!}{\mu!}\right)^a \frac{|z|^n}{n!} \to \infty$$

as $n \to \infty$, for $z \neq 0$, so $\delta_\mu$ is not an analytic vector for any $\mu \geq 0$. Since $A$ has growth of at least $a$ we see that the growth condition of Theorem 2.2.8 cannot be weakened.

□

**Definition 2.2.9** *A matrix $A$ is said to have* rapid decay *if there exist positive constants $C$, $a$ and $b$ such that*

$$\|A_\nu^\mu\| \leq C((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} \quad \forall \mu, \nu \geq 0.$$

*The collection of all such matrices is denoted $\mathfrak{R}$.*

This definition is motivated by thinking of the diagonal matrices as those matrices having 'instant decay' at some fixed distance from the main diagonal. The properties of matrices with rapid decay are summed up in the next proposition.

**Proposition 2.2.10** *The class of matrices with rapid decay, $\mathfrak{R}$, is a vector subspace of $\mathfrak{M}$, and the involution is a bijection between $\mathfrak{R}(\mathfrak{H};\mathfrak{J})$ and $\mathfrak{R}(\mathfrak{J};\mathfrak{H})$. Multiplication of matrices with rapid decay is well-defined, associative and compatible with the involution. In particular $\mathfrak{R}(\mathfrak{H})$ generates a $*$-algebra which contains $\mathfrak{D}_{\mathrm{pb}}(\mathfrak{H})$ as a $*$-subalgebra (in fact, diagonal matrices with polynomial growth are matrices with rapid decay). Elements of $\mathfrak{R}$ are row-square-summable and induce operators with domains that include the exponential subspace.*

**Proof**

Verification that $\mathfrak{R}$ is a vector space and that it behaves as claimed with respect to the involution is straightforward, so we omit it. Let $n \geq 2$ and suppose $A_1 \in \mathfrak{R}(\mathfrak{H}_2;\mathfrak{H}_1), \ldots, A_n \in \mathfrak{R}(\mathfrak{H}_{n+1};\mathfrak{H}_n)$; we may find positive constants $C$, $a$ and $b$ such that

$$\|(A_i)^\mu_\nu\| \leq C((\mu \vee \nu)+1)^a |\mu-\nu|!^{-b} \quad \forall\, \mu,\nu \geq 0$$

for $i = 1,\ldots,n$. Since

$$\begin{aligned}
&\sum_{\pi_1 \geq 0} \cdots \sum_{\pi_{n-1}\geq 0} \|(A_1)^{\pi_0}_{\pi_1}\| \, \|(A_2)^{\pi_1}_{\pi_2}\| \cdots \|(A_n)^{\pi_{n-1}}_{\pi_n}\| \\
&\quad \leq \; C^n \sum_{\pi_1 \geq 0} \cdots \sum_{\pi_{n-1}\geq 0} ((\pi_0 \vee \pi_1)+1)^a \cdots ((\pi_{n-1}\vee \pi_n)+1)^a \\
&\qquad\qquad \times |\pi_0-\pi_1|!^{-b} \cdots |\pi_{n-1}-\pi_n|!^{-b}
\end{aligned}$$

is convergent for all $\pi_0, \pi_n \geq 0$, by Lemma A.3.1, the multiple series

$$(A_1 \cdots A_n)^{\pi_0}_{\pi_n} = \sum_{\pi_1 \geq 0} \cdots \sum_{\pi_{n-1}\geq 0} (A^1)^{\pi_0}_{\pi_1} \cdots (A_n)^{\pi_{n-1}}_{\pi_n}$$

converges absolutely. Thus multiplication of matrices with rapid decay is well-defined, and independent of the order in which the summation is carried out, i.e., associative. That multiplication is compatible with the involution follows from the fact that the series converges in the uniform topology. Thus

$$\mathrm{lin}\{A_1 \cdots A_n \,:\, n \geq 1; A_i \in \mathfrak{R}(\mathfrak{H}) \ (i=1,\ldots,n)\}$$

is an associative $*$-algebra. To see that $\mathfrak{R}$ contains $\mathfrak{D}_{\mathrm{pb}}$, suppose $A$ is $(2\kappa+1)$-diagonal; there exist positive constants $C$ and $a$ such that

$$\|A^\mu_\nu\| \leq \begin{cases} C((\mu\vee\nu)+1)^a & |\mu-\nu| \leq \kappa \\ 0 & |\mu-\nu| > \kappa \end{cases} .$$

Hence for any $b > 0$ we have that

$$\|A^\mu_\nu\| \leq C(\kappa!)^b((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} \quad \forall \mu, \nu \geq 0$$

and so $A$ has rapid decay, as claimed.

To see row-square-summability, note that

$$\begin{aligned} \sum_{\mu \geq 0} \|(A_1)^\mu_\nu\|^2 &\leq \sum_{\mu \geq 0} C^2((\mu \vee \nu) + 1)^{2a} |\mu - \nu|!^{-2b} \\ &\leq K + C^2 \sum_{\mu \geq \nu} (\mu + 1)^{2a} (\mu - \nu)!^{-2b}, \end{aligned}$$

where $K$ is finite and the final series is convergent by the limit ratio test.

Finally, if $\xi \in \mathcal{E}$ then

$$\sum_{\nu \geq 0} \|(A_1)^\mu_\nu \xi^\nu\| \leq C \sum_{\nu \geq 0} ((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} M^{\nu+1} (\nu!)^{-\frac{1}{2}}$$

(where $M > 0$ is a constant) and this is convergent for all $\mu \geq 0$, by the limit ratio test. Furthermore

$$\begin{aligned} &\sum_{\mu \geq 0} \| \sum_{\nu \geq 0} (A_1)^\mu_\nu \xi^\nu \|^2 \\ &\leq C^2 \sum_{\mu \geq 0} \sum_{\nu \geq 0} ((\mu \vee \nu) + 1)^{2a} |\mu - \nu|!^{-2b} (\nu!)^{-\frac{1}{2}} \sum_{\nu \geq 0} M^{2\nu+2} (\nu!)^{-\frac{1}{2}} \end{aligned}$$

using the Cauchy-Schwarz-Buniakowski inequality. The latter series is convergent, as is the double series, which is equal to

$$\begin{aligned} &\sum_{\nu \geq 0} \sum_{\mu = 0}^{\nu} (\nu + 1)^{2a} (\nu - \mu)!^{-2b} (\nu!)^{-\frac{1}{2}} + \sum_{\nu \geq 0} \sum_{\mu > \nu} (\mu + 1)^{2a} (\mu - \nu)!^{-2b} (\nu!)^{-\frac{1}{2}} \\ &\leq \sum_{\nu \geq 0} (\nu + 1)^{2a} (\nu!)^{-\frac{1}{2}} \sum_{\mu \geq 0} (\mu!)^{-2b} + \sum_{\nu \geq 0} (\nu + 1)^{2a} (\nu!)^{-\frac{1}{2}} \sum_{\mu \geq 1} (\mu + 1)^{2a} (\mu!)^{-2b} \end{aligned}$$

using the inequality $(x + y + 1)^{2a} \leq (x + 1)^{2a} (y + 1)^{2a}$ for $x, y \geq 0$. Every series in sight is convergent, by the limit ratio test, so the proof is complete. □

**Definition 2.2.11** *A matrix $A \in \mathfrak{M}(\mathfrak{H}; \mathfrak{J})$ is* bounded *if $\hat{A}$, the operator it induces, is an element of $\mathcal{B}(\mathfrak{H}; \mathfrak{J})$. The class of bounded matrices is denoted $\mathfrak{B}(\mathfrak{H}; \mathfrak{J})$.*

Note that the sparse matrices are bounded: if $A \in \mathfrak{F}_\kappa$ then

$$\|\hat{A}\theta\|^2 = \sum_{\mu=0}^{\kappa} \| \sum_{\nu=0}^{\kappa} A^\mu_\nu \xi^\nu \|^2 \leq (\sum_{\mu=0}^{\kappa} \sum_{\nu=0}^{\kappa} \|A^\mu_\nu\|^2) \|\xi\|^2$$

for all $\theta \in \mathfrak{H}$, so $\|\hat{A}\| \leq (\sum_{\mu,\nu=0}^{\kappa} \|A_\nu^\mu\|^2)^{\frac{1}{2}}$. In fact, every bounded operator has a representation as an operator matrix, and the correspondence is structure-preserving: in particular there is a $*$-algebra isomorphism between $\mathcal{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{H})$.

**Theorem 2.2.12** *The map $m = m_{\mathfrak{H};\mathfrak{J}} : \mathcal{B}(\mathfrak{H};\mathfrak{J}) \to \mathfrak{B}(\mathfrak{H};\mathfrak{J})$ given by*

$$m(S)_\nu^\mu = F_\mu S E_\nu^* \quad \forall\, \mu, \nu \geq 0$$

*(where $E_\mu$ and $F_\mu$ are the projection maps on $\mathfrak{H}$ and $\mathfrak{J}$, respectively) is a vector-space isomorphism and such that*

$$m_{\mathfrak{J};\mathfrak{K}}(T) m_{\mathfrak{H};\mathfrak{J}}(S) = m_{\mathfrak{H};\mathfrak{K}}(TS) \quad \forall\, S \in \mathcal{B}(\mathfrak{H};\mathfrak{J}), T \in \mathcal{B}(\mathfrak{J};\mathfrak{K})$$

*and*

$$m_{\mathfrak{H};\mathfrak{J}}(S)^* = m_{\mathfrak{J};\mathfrak{H}}(S^*) \quad \forall\, S \in \mathcal{B}(\mathfrak{H};\mathfrak{J})$$

*In particular $m_{\mathfrak{H};\mathfrak{H}}$ is an algebra $*$-isomorphism. Furthermore $\widehat{m(S)} = S$ for all $S \in \mathcal{B}(\mathfrak{H};\mathfrak{J})$ and $m(\hat{A}) = A$ for all $A \in \mathfrak{B}(\mathfrak{H};\mathfrak{J})$, i.e., $m$ and $\hat{\cdot}|_{\mathfrak{B}}$ are mutually inverse.*

**Proof**

Linearity, multiplicativity and commutation with the adjoint are immediate if we note that $\sum_{\pi \geq 0} F_\pi^* F_\pi = \mathrm{id}_{\mathfrak{J}}$ in the strong sense. If $\xi \in \mathfrak{H}$ then

$$\sum_{\mu \geq 0} \sum_{\nu \geq 0} m(S)_\nu^\mu \xi^\nu = \sum_{\mu \geq 0} F_\mu S \sum_{\nu \geq 0} E_\nu^* \xi^\nu = \sum_{\mu \geq 0} (S\xi)^\mu = S\xi,$$

so $\widehat{m(S)} = S$. In particular $S = 0$ if $m(S) = 0$, so $m$ is injective. Finally, if $A \in \mathfrak{B}$ then

$$\langle \xi, \widehat{m(\hat{A})} \eta \rangle = \langle \xi, \hat{A} \eta \rangle;$$

taking suitable $\xi$ and $\eta$ shows that $F_\mu \hat{A} E_\nu^* = A_\nu^\mu$, so $m(\hat{A}) = A$ and $m$ is surjective. □

A consequence of the above is that isometric operators give rise to isometric matrices, i.e., those matrices $V$ such that $V^*V = I$. Similar statements hold for co-isometric and unitary operators. It is useful that the converse also holds.

**Proposition 2.2.13** *A matrix $V \in \mathfrak{M}$ is* isometric, *i.e., $V^*V = I$, if and only if the operator $\hat{V}$ is isometric. The statement remains true if the word 'isometric' is replaced by* co-isometric, *i.e., $VV^* = I$, or* unitary, *$V^*V = VV^* = I$.*

**Proof**

We prove the unitary case; the others are then clear. Suppose $V \in \mathfrak{B}$ gives rise to a unitary operator. Then

$$V^*V = m(\hat{V})^* m(\hat{V}) = m(\hat{V}^*) m(\hat{V}) = m(\hat{V}^*\hat{V}) = m(\mathrm{id}) = I$$

and similarly $VV^* = I$. For the converse, suppose $V \in \mathfrak{M}$ is unitary, and let $\xi \in \mathfrak{H}_{00}$:

$$\|\xi\|^2 = \langle \xi, \widehat{V^*V}\xi \rangle = \sum_{\mu=0}^{|\xi|} \sum_{\nu=0}^{|\xi|} \sum_{\pi \geq 0} \langle \xi^\mu, (V_\mu^\pi)^* V_\nu^\pi \xi^\nu \rangle = \sum_{\pi \geq 0} \| \sum_{\mu=0}^{|\xi|} V_\mu^\pi \xi^\mu \|^2,$$

so $\mathcal{D}(\hat{V})$ contains $\mathfrak{H}_{00}$, on which it is isometric. Extend $\hat{V}|_{\mathfrak{H}_{00}}$ to an isometry $W \in \mathcal{B}(\mathfrak{H}; \mathfrak{J})$. If $\xi \in \mathfrak{j}_\mu, \eta \in \mathfrak{h}_\nu$ then

$$\langle F_\mu^* \xi, W E_\nu^* \eta \rangle = \langle F_\mu^* \xi, \hat{V} E_\nu^* \eta \rangle = \langle \xi, V_\nu^\mu \eta \rangle$$

so $V = m(W)$. Hence $\hat{V} = W$ and $\hat{V}$ is isometric. The same argument shows that $\widehat{V^*}$ is isometric, but

$$\widehat{V^*} = \widehat{m(\hat{V})^*} = \widehat{m(\hat{V}^*)} = \hat{V}^*,$$

so $\hat{V}$ is unitary.

□

It is possible to characterise bounded matrices in terms of their truncations to sparse matrices. This is the content of the next proposition, which is a simplification of [KaR, Proposition 2.6.13].

**Proposition 2.2.14** *Let $A \in \mathfrak{M}$ and for all $\pi \geq 0$ let $A^{(\pi)}$ be the bounded linear operator that acts on $\oplus_{\mu=0}^{\pi} \mathfrak{h}_\mu$ in the obvious manner, i.e.,*

$$A^{(\pi)}\xi = \sum_{\mu=0}^{\pi} \sum_{\nu=0}^{\pi} A_\nu^\mu \xi^\nu.$$

*The norms of these operators are non-decreasing, i.e., $\|A^{(\pi)}\| \leq \|A^{(\kappa)}\|$ if $\pi \leq \kappa$ and $A \in \mathfrak{B}$ if and only if $\sup_{\pi \geq 0} \|A^{(\pi)}\| < \infty$, in which case*

$$\|\hat{A}\| = \sup_{\pi \geq 0} \|A^{(\pi)}\| = \lim_{\pi \geq 0} \|A^{(\pi)}\|.$$

**Proof**

For the first claim, let $\xi \in \oplus_{\mu=0}^{\pi} \mathfrak{h}_\mu$ and define $\eta \in \oplus_{\mu=0}^{\kappa} \mathfrak{h}_\mu$ by

$$\eta^\mu = \begin{cases} \xi^\mu & (\mu \leq \pi) \\ 0 & (\mu > \pi) \end{cases}.$$

Then

$$\|A^{(\pi)}\xi\|^2 = \sum_{\mu=0}^{\pi} \|\sum_{\nu=0}^{\pi} A^\mu_\nu \xi^\nu\|^2 \leq \sum_{\mu=0}^{\kappa} \|\sum_{\nu=0}^{\kappa} A^\mu_\nu \eta^\mu\|^2 = \|A^{(\kappa)}\eta\|^2 \leq \|A^{(\kappa)}\|^2\|\eta\|^2,$$

but $\|\eta\| = \|\xi\|$, so $\|A^{(\pi)}\| \leq \|A^{(\kappa)}\|$, as claimed.

Now suppose that $M := \sup_{\pi\geq 0} \|A^{(\pi)}\|$ is finite. If $\mu \geq 0$, $\pi \geq \lambda \geq 0$ and $\xi \in \mathfrak{H}$ then

$$\|\sum_{\nu=\lambda}^{\pi} A^\mu_\nu \xi^\nu\|^2 \leq \sum_{\mu=0}^{\pi} \|\sum_{\nu=\lambda}^{\pi} A^\mu_\nu \xi^\nu\|^2 \leq \|A^{(\pi)}\|^2 \|P_{[\lambda,\pi]}\xi\|^2 \leq M^2 \sum_{\nu=\lambda}^{\pi} \|\xi^\nu\|^2,$$

which converges to 0 as $\lambda, \pi \to \infty$. (Here $P_{[\lambda,\pi]}$ denotes the orthogonal projection from $\mathfrak{H}$ onto $\oplus_{\mu=\lambda}^{\pi} \mathfrak{h}_\mu$.) Hence $\sum_{\nu\geq 0} A^\mu_\nu \xi^\nu$ is Cauchy, and so convergent, for all $\mu \geq 0$. Furthermore, this working shows that

$$\sum_{\mu=0}^{\pi} \|\sum_{\nu\geq 0} A^\mu_\nu \xi^\nu\|^2 \leq M^2\|\xi\|^2 \quad \forall\, \pi \geq 0$$

so $\mathcal{D}(\hat{A}) = \mathfrak{H}$ and $\|\hat{A}\xi\| \leq M\|\xi\|$, i.e., $A \in \mathfrak{B}$ and $\|\hat{A}\| \leq M$.

Finally, suppose $A \in \mathfrak{B}$ and let $\pi \geq 0$; if $\xi \in \mathfrak{H}$ then

$$\|A^{(\pi)}\xi\|^2 = \sum_{\mu=0}^{\pi} \|\sum_{\nu=0}^{\pi} A^\mu_\nu \xi^\nu\|^2 \leq \sum_{\mu\geq 0} \|\sum_{\nu=0}^{\pi} A^\mu_\nu \xi^\nu\|^2 = \|\hat{A}P_{[0,\pi]}\xi\|^2 \leq \|\hat{A}\|^2\|\xi\|^2,$$

so $\|A^{(\pi)}\| \leq \|\hat{A}\|$ for all $\pi$. Thus $\sup_{\pi\geq 0} \|A^{(\pi)}\| \leq \|\hat{A}\|$, but then since this supremum is finite the working above shows the opposite inequality. Hence $\|\hat{A}\| = \sup_{\pi\geq 0} \|A^{(\pi)}\|$ for all $A \in \mathfrak{B}$.

□

The simplest special case of the above involves matrices that have entries only on the main diagonal.

**Corollary 2.2.15** *If $A \in \mathfrak{D}_0$ then $A \in \mathfrak{B}$ if and only if $\sup_{\mu=0}^{\infty} \|A^\mu_\mu\| < \infty$, in which case $\|A\| = \sup_{\mu=0}^{\infty} \|A^\mu_\mu\|$.*

### Proof

This is a consequence of the theorem and the fact that $\|A^{(\pi)}\| = \max_{\mu=0}^{\pi} \|A^\mu_\mu\|$. To see the latter, note that if $\xi \in \oplus_{\mu=0}^{\pi} \mathfrak{h}_\mu$ then

$$\|A^{(\pi)}\xi\|^2 = \sum_{\mu=0}^{\pi} \|A^\mu_\mu \xi^\mu\|^2 \leq \max_{\mu=0}^{\pi} \|A^\mu_\mu\|^2\|\xi\|^2,$$

so $\|A^{(\pi)}\| \leq \max_{\mu=0}^{\pi} \|A^\mu_\mu\|$. For the reverse inequality, let $\nu$ be such that $\|A^\nu_\nu\| = \max_{\mu=0}^{\pi} \|A^\mu_\mu\|$. Let $\epsilon > 0$ and choose a unit vector $\xi^\nu \in \mathfrak{h}_\nu$ such that $\|A^\nu_\nu \xi^\nu\| > \|A^\nu_\nu\| - \epsilon$. Define $\xi \in \oplus_{\mu=0}^{\pi} \mathfrak{h}_\mu$ by setting $\xi^\mu = 0$ if $\mu \neq \nu$: we see that

$$\|A^{(\pi)}\xi\| = \|A^\nu_\nu \xi^\nu\| > \|A^\nu_\nu\| - \epsilon.$$

Since $\epsilon$ is arbitrary it follows that $\|A^{(\pi)}\| \geq \|A^\nu_\nu\| = \max_{\mu=0}^{\pi} \|A^\mu_\mu\|$.

□

The following proposition allows us to form 'product matrices' which are compatible with the tensor product of operators.

**Proposition 2.2.16** *Let $\mathcal{H}$ be a Hilbert space, $S \in \mathcal{B}(\mathcal{H})$ and $A \in \mathfrak{M}(\mathfrak{H})$. Define $B \in \mathfrak{M}(\tilde{\mathfrak{H}})$, where $\tilde{\mathfrak{H}}$ is the product chaos space, by*

$$B^\mu_\nu = S \otimes A^\mu_\nu \qquad \forall\, \mu, \nu \geq 0.$$

*The operator $\hat{B}$ is such that $\mathcal{D}(\hat{B}) \supseteq \mathcal{H} \underline{\otimes} \mathcal{D}(\hat{A})$ and $\hat{B}(f \otimes \xi) = Sf \otimes \hat{A}\xi$ for all $f \in \mathcal{H}$ and $\xi \in \mathcal{D}(\hat{A})$. Furthermore, if $A \in \mathfrak{B}(\mathfrak{H})$ then $B \in \mathfrak{B}(\tilde{\mathfrak{H}})$ and $\hat{B} = S \otimes \hat{A}$.*

**Proof**

If $0 \leq \lambda \leq \pi$ and $B$ is defined as above then

$$\sum_{\nu=\lambda}^{\pi} B^\mu_\nu (f \otimes \xi)^\nu = Sf \otimes \sum_{\lambda=\nu}^{\pi} A^\mu_\nu \xi^\nu \to 0$$

as $\lambda, \pi \to \infty$, and

$$\sum_{\mu=0}^{\pi} \|\sum_{\nu \geq 0} B^\mu_\nu (f \otimes \xi)^\nu\|^2 = \|Sf\|^2 \sum_{\mu=0}^{\pi} \|\sum_{\nu \geq 0} A^\mu_\nu \xi^\nu\|^2 \leq \|Sf\|^2 \|\hat{A}\xi\|^2,$$

so $f \otimes \xi \in \mathcal{D}(\hat{B})$ for any $f \in \mathcal{H}$, $\xi \in \mathcal{D}(\hat{A})$. Thus $B$ is well-defined on $\mathcal{H} \underline{\otimes} \mathcal{D}(\hat{A})$, and acts as claimed. If $A \in \mathfrak{B}(\mathfrak{H})$ then $S \otimes \hat{A}$ has matrix representation $C \in \mathfrak{B}(\tilde{\mathfrak{H}})$, and

$$\langle f \otimes \xi, \hat{C}(g \otimes \eta)\rangle = \langle f, Sg\rangle \langle \xi, \hat{A}\eta\rangle = \langle f \otimes \xi, \hat{B}(g \otimes \eta)\rangle$$

for all $f, g \in \mathcal{H}$ and $\xi, \eta \in \mathfrak{H}$. Taking suitable $\xi, \eta \in \mathfrak{H}_{00}$ gives that $B = C$, as required.

□

In particular, if $S \in \mathcal{B}(\mathcal{H})$ then $S \otimes \mathrm{id}_{\mathfrak{H}}$ has a matrix representation in $\mathfrak{B}(\tilde{\mathfrak{H}}) \cap \mathfrak{D}_0(\tilde{\mathfrak{H}})$.

**Examples**

1. Every chaos space has the *number operator* $\hat{N}$; define $N \in \mathfrak{M}$ by $N^\mu_\nu = \mu\delta^\mu_\nu \mathrm{id}_{\mathfrak{h}_\mu}$. The domain of this operator

$$\mathcal{D}(\hat{N}) = \{\xi \in \mathfrak{H} : \sum_{\mu \geq 1} \mu^2 \|\xi^\mu\|^2 < \infty\}$$

is frequently the domain of other operators of interest. The number operator has spectrum $\sigma(\hat{N}) = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}$. To see this, note that $\ker(\mu \,\mathrm{id}_{\mathfrak{H}} - \hat{N}) = \mathfrak{h}_\mu$ for $\mu \geq 0$, and if $z \in \mathbb{C} \setminus (\{0\} \cup \mathbb{N})$ then $R_z(N) := (\,(z-\mu)^{-1}\delta^\mu_\nu \mathrm{id}_{\mathfrak{h}_\mu})_{\mu,\nu \geq 0}$ is an element of $\mathfrak{B}$, since $\sup_{\mu \geq 0} |z-\mu|^{-1} < \infty$. (For reasons of typography and aesthetics we write $\widehat{R_z(N)}$ as $\hat{R}_z(N)$, and take similar liberties when it suits us.) As $\hat{R}_z(N)(z\,\mathrm{id}_{\mathfrak{H}} - \hat{N}) \subset (z\,\mathrm{id}_{\mathfrak{H}} - \hat{N})\hat{R}_z(N) = \mathrm{id}_{\mathfrak{H}}$, we see that $z \notin \sigma(\hat{N})$, as claimed.

For the rest of the examples we consider the chaos space $\mathfrak{H}$ to be $\mathfrak{F}_+(\mathcal{H})$, Boson Fock space over the Hilbert space $\mathcal{H}$.

2. If $T \in \mathcal{B}(\mathcal{H})$ then let $s_n(T) \in \mathcal{B}(\mathcal{H}^{\otimes n})$ be the operator

$$T \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \mathrm{id}_{\mathcal{H}} \otimes T \otimes \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} + \cdots + \mathrm{id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}} \otimes T.$$

It is easy to see that $s_n(T)(\mathcal{H}^{\otimes_s n}) \subseteq \mathcal{H}^{\otimes_s n}$, and we denote the restriction of $s_n(T)$ to $\mathcal{H}^{\otimes_s n}$ by the same symbol. The matrix defined by

$$\mathrm{d}\Gamma(T)^\mu_\nu := \not\delta^\mu_\nu s_\mu(T) = \begin{cases} s_\mu(T) & \mu = \nu \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

is called the *differential second quantisation* of $T$ (see [BrR, p. 8]). When $T = \mathrm{id}_{\mathcal{H}}$ we have the (matrix representing the) number operator. Note that if $T$ is self-adjoint then $\mathrm{d}\Gamma(T)$ is symmetric, as $s_n(T)^* = s_n(T^*)$, and the induced operator is self-adjoint, by Corollary 2.2.6. Furthermore, if $\xi \in \mathcal{H}$ is a unit vector then it is easy to see that

$$\|s_n(T)\xi^{\otimes n}\|^2 = n\|T\xi\|^2 + (n^2 - n)|\langle \xi, T\xi \rangle|^2 \geq n^2 |\langle \xi, T\xi \rangle|^2$$

by the Cauchy-Schwarz-Buniakowski inequality. Hence if $T$ is self-adjoint $\|s_n(T)\| = n\|T\|$, so we see that $\mathcal{D}(\widehat{\mathrm{d}\Gamma}(T)) \supseteq \mathcal{D}(\hat{N})$. For general $T$ we have that $\sqrt{n}\,\|T\| \leq \|s_n(T)\| \leq n\|T\|$ and the lower bound may be obtained. (Consider, e.g., $\mathfrak{H} = \mathfrak{F}_+(\mathbb{C}^2)$ and $s_2(T)$, where $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$).

3. Let $T \in \mathcal{B}(\mathcal{H})$ and define

$$\Gamma(T)^\mu_\nu := \delta^\mu_\nu T^{\otimes \nu} = \begin{cases} T^{\otimes \nu} & \mu = \nu \\ 0 & \text{otherwise} \end{cases},$$

where $T^{\otimes 0} := \mathrm{id}_{\mathfrak{h}_0}$. It is clear that $\Gamma(T)$ is a (strictly) diagonal matrix, called the *second quantisation* of $T$, and

$$\Gamma(T)\Gamma(S) = \Gamma(TS)\ ,\ \Gamma(T)^* = \Gamma(T^*) \quad \forall\, S, T \in \mathcal{B}(\mathcal{H}).$$

The relationship with the previous example is as follows. If $T$ is any (not necessarily bounded) self-adjoint operator in $\mathcal{H}$ then $(U_t := \exp(itT))_{t\in\mathbb{R}}$ is a strongly-continuous, one-parameter group of unitary operators. The second quantisation of this group also has this form, and so has a self-adjoint generator (by Stone's theorem). If the operator $T$ is bounded then this generator coincides with the differential second quantisation defined previously.

From now on, all examples are on Boson Fock space over $L^2[0,1]$; $\mathfrak{H} = \mathfrak{F}_+(L^2[0,1])$.

4. For $t \in [0,1]$ let $m_{\chi_{[0,t]}} \in \mathcal{B}(\mathcal{H})$ be the operator that acts on $\mathcal{H} = L^2[0,1]$ as multiplication by the indicator function of the interval $[0,t]$; $\hat{\Gamma}(m_{\chi_{[0,t]}})$ is the operator of conditional expectation 'up to time $t$', and we write $\Gamma(m_{\chi_{[0,t]}}) = \mathbb{E}_{t]}$ and $\Gamma(m_{\chi_{[0,t]}})^\mu_\mu = \mathbb{E}^\mu_{t]}$. It is not difficult to show that

$$(\mathbb{E}^i_{t]} f)(t_1, \ldots, t_i) \;=\; \prod_{l=1}^{i} \chi_{[0,t]}(t_l) f(t_1, \ldots, t_i). \tag{2.2}$$

$\hat{\mathbb{E}}_{t]}$ is the orthogonal projection onto $\mathfrak{H}_{t]} = \mathfrak{F}_+(L^2[0,t])$, considered as a subspace of $\mathfrak{H}$, and similarly $\mathbb{E}^i_{t]}$ is the orthogonal projection onto $L^2[0,t]^{\otimes_s i} \subseteq \mathfrak{h}_i$. It is clear that this example generalises to any space $\mathcal{H}$ equipped with an increasing family of subspaces $(\mathcal{H}_t)_{0\le t\le 1}$, and that the interval $[0,1]$ can be replaced by any totally ordered (or even partially ordered) set.

5. Define $\nabla \in \mathfrak{D}_1(\mathfrak{H}; L^2([0,1];\mathfrak{H}))$ by setting $\nabla^\mu_\nu = 0$ if $\nu \neq \mu+1$ and

$$(\nabla^\mu_{\mu+1} f)(t)(t_1, \ldots, t_\mu) := \sqrt{\mu+1}\, f(t_1, \ldots, t_\mu, t).$$

The inner-product identity

$$\langle \nabla^\mu_{\mu+1} f, \nabla^\mu_{\mu+1} g\rangle = (\mu+1)\langle f, g\rangle \tag{2.3}$$

gives us that $\|\nabla^\mu_{\mu+1}\| = \sqrt{\mu+1}$, and in particular

$$\mathcal{D}(\hat{\nabla}) = \{\xi \in \mathfrak{H} : \sum_{\mu\ge 0} (\mu+1)\|\xi^{\mu+1}\|^2 < \infty\} = \mathcal{D}(\hat{N}^{\frac{1}{2}}).$$

We define $\mathcal{S} \in \mathfrak{D}_1(L^2([0,1];\mathfrak{H});\mathfrak{H})$ by setting $\mathcal{S}^\mu_\nu = 0$ if $\nu + 1 \neq \mu$ and

$$(\mathcal{S}^{\mu+1}_\mu f)(t_1,\ldots,t_{\mu+1}) := \frac{1}{\sqrt{\mu+1}} \sum_{k=1}^{\mu+1} f(t_k)(t_1,\ldots,\widehat{t_k},\ldots,t_{\mu+1}),$$

where the $\widehat{\ }$ denotes omission. It is easy to verify that $\mathcal{S} = \nabla^*$, so by Corollary 2.2.6 the operators $\hat{\nabla}$ and $\hat{\mathcal{S}}$ are mutually adjoint. The operator $\hat{\nabla}$ is called the *Malliavin gradient*, and $\hat{\mathcal{S}}$ is the *Hitsuda-Skorokhod integral* (see [NuZ], [Sko]). After some work it can be shown that

$$\langle \mathcal{S}^{\mu+1}_\mu f, \mathcal{S}^{\mu+1}_\mu g \rangle = \langle f, g \rangle + \int_0^1 \int_0^1 \langle \nabla^{\mu-1}_\mu [f(t)](s), \nabla^{\mu-1}_\mu [g(s)](t) \rangle \, \mathrm{d}s \, \mathrm{d}t. \tag{2.4}$$

(We give a demonstration of this in the proof of Proposition 3.1.2.) The identity (2.3) gives that $\mathcal{S}\nabla = N$, and $\|\mathcal{S}^{\mu+1}_\mu f\|^2 \leq \|(\nabla^\mu_{\mu+1})^*\|^2 \|f\|^2 = (\mu+1)\|f\|^2$, so $\mathcal{D}(\hat{N}^{\frac{1}{2}}) \subseteq \mathcal{D}(\hat{\mathcal{S}})$. Hence if $\xi \in \mathcal{D}(\hat{N})$ then $\hat{\nabla}\xi \in \mathcal{D}(\hat{N}^{\frac{1}{2}}) \subseteq \mathcal{D}(\hat{\mathcal{S}})$ and so $\hat{\mathcal{S}}\hat{\nabla} = \hat{N}$ (a result of Gaveau and Trauber [GaT]). The identity (2.4) is the source of Itô's formula; it is the matrix version of what Lindsay calls the *quantum Skorokhod isometry* (see [Lin, Theorem 2.2, p. 72], [NuZ, Proposition 3.1] and [Sko, Equation 14]).

6. The creation and annihilation operators of quantum field theory (see [BrR], [ReS, Section X.7]) can be defined in the following manner. Given $f \in L^2[0,1]$ and $g \in \mathfrak{h}_\mu$ let $f \otimes g \in L^2([0,1];\mathfrak{h}_\mu)$ be defined by

$$(f \otimes g)(t)(t_1,\ldots,t_\mu) := f(t) g(t_1,\ldots,t_\mu).$$

The linear mapping $g \mapsto f \otimes g$ is continuous for fixed $f$ and $\|f \otimes g\| = \|f\| \, \|g\|$. We define $A^\dagger(f) \in \mathfrak{D}_1(\mathfrak{H})$ by

$$A^\dagger(f)^\mu_\nu := \begin{cases} g \mapsto \mathcal{S}^{\nu+1}_\nu (f \otimes g) & \mu = \nu + 1 \\ 0 & \text{otherwise} \end{cases}.$$

We know that $\|\mathcal{S}^{\nu+1}_\nu (f \otimes g)\| \leq \sqrt{\mu+1}\, \|f\| \, \|g\|$, so $A^\dagger(f)$ has growth of order $\frac{1}{2}$. Taking $g = f^{\otimes \nu} : (t_1,\ldots,t_\nu) \mapsto f(t_1) \cdots f(t_\nu)$ and using the inner-product identity (2.4) we see that

$$\|S^{\nu+1}_\nu (f \otimes g)\|^2 = \|f\|^{2\nu+2} + \nu \int_{[0,1]^{\nu+1}} |f(t_1)|^2 \cdots |f(t_{\nu+1})|^2 \, \mathrm{d}\mathbf{t} = (\nu+1)\|f\|^{2\nu+2},$$

so $\|A^\dagger(f)^{\nu+1}_\nu\| = \sqrt{\nu+1}\|f\|$. The operator $\hat{A}^\dagger(f)$ is called the *creation operator of strength $f$*.

We define $A(f) := A^\dagger(f)^*$; the operator $\hat{A}(f)$ is the *annihilation operator of strength $f$*. (If $\xi \in \mathfrak{h}_\mu$ then $\hat{A}^\dagger(f)\xi \in \mathfrak{h}_{\mu+1}$ and $\hat{A}(f)\xi \in \mathfrak{h}_{\mu-1}$ (if $\mu \geq 1$).) The operators $\hat{A}(f)$ and $\hat{A}^\dagger(f)$ are mutually adjoint, by Corollary 2.2.6. Furthermore, if $\Phi(f) := A(f) + A^\dagger(f)$ then $\Phi(f)$ has growth of order $\frac{1}{2}$ and is symmetric, so $\hat{\Phi}(f)$ is self-adjoint. It can be shown that

$$\hat{\Phi}(f)\Big|_{\mathcal{E}_0} = \tilde{m}_{\int_0^1 \Re f(s)\,\mathrm{d}W(s)} + \hat{\Gamma}(i)\tilde{m}_{\int_0^1 \Im f(s)\,\mathrm{d}W(s)}\hat{\Gamma}(-i),$$

where $\tilde{m}_f$ is the Fock space version of the operator in Wiener space that acts on the exponential vectors as multiplication by $f$ (see [Mey, pp. 71–73]) and $\hat{\Gamma}(i)$ is the *Fourier-Wiener transform* (see [Hid, pp. 182–184]).

The *gauge operator of strength H*, denoted $\hat{\Lambda}(H)$, is the differential second quantisation of $H \in \mathcal{B}(L^2[0,1])$: $\Lambda(H) := \mathrm{d}\Gamma(H)$. We know that $\Lambda(H)$ has growth 1, and as $\Pi(f) := A(f) + \Lambda(m_f) + A^\dagger(f)$ is a symmetric matrix (where $f \in L^\infty([0,1];\mathbb{R})$ and $m_f$ is the operator of multiplication by $f$) we see that $\hat{\Pi}(f)$ is self-adjoint. This operator is unitarily equivalent to multiplication by $\int_0^1 f(s)\,\mathrm{d}P(s)$, where $P$ is the compensated Poisson process, in the appropriate $L^2$ space; see [HuP, Section 6]. □

## 2.3 Matrix Processes

Throughout this section we assume that any chaos space $\mathfrak{H}$ is separable (equivalently, that $\mathfrak{h}_\mu$ is separable for all $\mu \geq 0$).

**Definition 2.3.1** *A* matrix process *is a mapping $F : [0,1] \to \mathfrak{M}$ such that $t \mapsto F(t)^\mu_\nu \xi$ is strongly measurable for all $\xi \in \mathfrak{h}_\nu$ and all $\mu, \nu \geq 0$.*

Addition and scalar multiplication of matrix processes are defined pointwise, and the resultant vector space is denoted $\mathfrak{L}$. The measurability conditions and separability imply that $t \mapsto \|F(t)^\mu_\nu\|$ is measurable for all $\mu, \nu \geq 0$ (see Lemma A.1.2), and so for $p \in [1, \infty]$ we define

$$\mathfrak{L}^p := \{F \in \mathfrak{L} \,:\, t \mapsto \|F(t)^\mu_\nu\| \in L^p[0,1] \;\forall\, \mu, \nu \geq 0\}.$$

These are vector spaces (inheriting this property from $L^p[0,1]$) and we have the inclusions $\mathfrak{L}^{\text{step}} \subseteq \mathfrak{L}^p \subseteq \mathfrak{L}^q \subseteq \mathfrak{L}$ if $p \geq q$, by Hölder's inequality, where

$$\mathfrak{L}^{\text{step}} := \{F \in \mathfrak{L} \,:\, t \mapsto F(t)^\mu_\nu \text{ is a step function } \forall\, \mu, \nu \geq 0\}.$$

(A function $f$ on $[0,1]$ is a *step function* if there exists a partition $\tau = \{0 = t_0 < t_1 < \cdots < t_{n+1} = 1\}$ such that $f(t) = f(t_\mu)$ for all $t \in [t_\mu, t_{\mu+1})$.)

Note that if $F \in \mathfrak{L}$ is *weakly continuous*, i.e., $t \mapsto F(t)^\mu_\nu$ is weakly continuous for all $\mu, \nu \geq 0$, then $F \in \mathfrak{L}^\infty$, by the uniform boundedness theorem.

The involution between $\mathfrak{M}(\mathfrak{H};\mathfrak{J})$ and $\mathfrak{M}(\mathfrak{J};\mathfrak{H})$ induces an involution $*$ on $\mathfrak{L}$ defined pointwise:

$$* : \mathfrak{L}(\mathfrak{H};\mathfrak{J}) \to \mathfrak{L}(\mathfrak{J};\mathfrak{H}); F^*(t) = (F(t))^*.$$

Measurability of $F^*$ follows from separability and Pettis' measurability theorem ([Pet, Theorem 1.1], [Yos, pp. 131–132]).

If $F \in \mathfrak{L}(\mathfrak{J};\mathfrak{K})$ and $G \in \mathfrak{L}(\mathfrak{H};\mathfrak{J})$ are such that $F(t)G(t)$ exists for all $t \in [0,1]$ then the *product process* $FG \in \mathfrak{L}(\mathfrak{H};\mathfrak{K})$ is equal to the pointwise product: $(FG)(t) = F(t)G(t)$. Measurability follows from Lemma A.1.3.

**Definition 2.3.2** *A process F determines an* induced process $\hat{F}$, *where* $\hat{F}(t) := \widehat{F(t)}$. *The* domain *of the induced process, denoted* $\mathcal{D}(\hat{F})$, *is defined by*

$$\mathcal{D}(\hat{F}) := \bigcap_{0 \leq t \leq 1} \mathcal{D}(\hat{F}(t)).$$

*A process is said to be an* HP-process *if the induced process admits the exponential subspace, i.e.,* $\mathcal{D}(\hat{F}) \supseteq \mathcal{E}$.

If $\xi \in \mathcal{D}(\hat{F})$ then $t \mapsto \hat{F}(t)\xi$ is strongly measurable; if $\eta \in \mathfrak{J}$ then

$$t \mapsto \langle \eta, \hat{F}(t)\xi \rangle = \sum_{\mu \geq 0} \sum_{\nu \geq 0} \langle \eta^\mu, F(t)^\mu_\nu \xi^\nu \rangle$$

is the limit of measurable functions, so $t \mapsto \hat{F}(t)\xi$ is weakly, hence strongly, measurable, by Pettis' theorem.

Now we introduce some topology.

**Definition 2.3.3** *Let* $\mathcal{L}$ *be the space obtained from* $\mathfrak{L}$ *by identifying processes that are equal almost everywhere, i.e.,* $\mathcal{L} = \mathfrak{L}/\mathfrak{N}$, *where* $\mathfrak{N} := \{F \in \mathfrak{L} : F(t) = 0 \text{ a.e. } t \in [0,1]\}$ *is the collection of* null processes. *Similarly we define* $\mathcal{L}^p = \mathfrak{L}^p/\mathfrak{N}$ *and we introduce semi-norms* $\| \cdot {}^\mu_\nu \|_p$ *on* $\mathcal{L}^p$ *by setting*

$$\|[F]^\mu_\nu\|_p := \begin{cases} \left(\int_0^1 \|F(t)^\mu_\nu\|^p \, \mathrm{d}t\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \text{ess sup}\Big\{ \|F(t)^\mu_\nu\| : 0 \leq t \leq 1 \Big\} & p = \infty \end{cases}$$

*for all* $\mu, \nu \geq 0$. *The components* $[F]^\mu_\nu$ *are elements of the Bochner-Lebesgue space* $L^p([0,1]; \mathcal{B}(\mathfrak{h}_\nu; \mathfrak{j}_\mu)_s)$ *and so the locally convex topology defined by these seminorms* *makes* $\mathcal{L}^p$ *a Fréchet space, i.e., complete and with a translation-invariant metric, e.g.,*

$$d([F],[G]) = \max\left\{ \frac{\| [F-G]^\mu_\nu \|_p}{2^{\mu+\nu}(1+\| [F-G]^\mu_\nu \|_p)} \ : \ \mu,\nu \geq 0 \right\}.$$

*If* $F, G \in \mathfrak{L}$ *are such that* $F \in [G]$ *then* $F$ *is said to be a* version *of* $G$*.*

The involution is continuous for this topology, and is in fact a linear homeomorphism between $\mathcal{L}^p(\mathfrak{H};\mathfrak{J})$ and $\mathcal{L}^p(\mathfrak{J};\mathfrak{H})$, because $\| [F^*]^\mu_\nu \|_p = \| [F]^\nu_\mu \|_p$.

We may regard $\mathcal{L}$ (and $\mathcal{L}^p$ for any $p \in [0,1]$) as an $L^\infty[0,1]$-bimodule, by setting

$$(fF)(t)^\mu_\nu := f(t)F(t)^\mu_\nu =: (Ff)(t)^\mu_\nu \quad \forall\, t \in [0,1].$$

That is, we embed $L^\infty[0,1]$ into $\mathcal{L}^\infty$ in the obvious manner. More generally, we have the following.

**Definition 2.3.4** *The vector space of* uniformly $(2\pi+1)$-diagonal processes *is*

$$\mathfrak{L}_{\mathfrak{D}_\pi} := \{F \in \mathfrak{L} \ : \ F(t) \in \mathfrak{D}_\pi \ \forall\, t \in [0,1]\},$$

*and the union of all these is the algebra of* uniformly diagonal processes*, denoted* $\mathfrak{L}_{\mathfrak{D}} := \cup_{\pi \geq 0} \mathfrak{L}_{\mathfrak{D}_\pi}$*. This contains the algebra of* uniformly diagonal, polynomially bounded processes

$$\begin{aligned} \mathfrak{L}_{\mathfrak{D}_{\rm pb}} \ := \ & \{F \in \mathfrak{L}_{\mathfrak{D}} \,|\, \exists\, C : [0,1] \to [0,\infty), a > 0 \ : \\ & \qquad \| F(t)^\mu_\nu \| \leq C(t)((\mu \vee \nu)+1)^a \ \forall\, t \in [0,1]\}. \end{aligned}$$

*If the function* $C$ *in this definition is* $p$*-integrable (where* $p \in [1,\infty]$*) then we say that* $F \in \mathfrak{L}^p_{\mathfrak{D}_{\rm pb}}$*, which is a subspace of* $\mathfrak{L}^p_{\mathfrak{D}} := \mathfrak{L}_{\mathfrak{D}} \cap \mathfrak{L}^p$*.*

We have that $\mathfrak{L}^p_{\mathfrak{D}_\kappa} \mathfrak{L}^q_{\mathfrak{D}_\pi} \subseteq \mathfrak{L}^r_{\mathfrak{D}_{\kappa+\pi}}$ for $p$, $q$ and $r$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, by Hölder's inequality. Furthermore, this multiplication is continuous. More formally, if $\mathcal{L}_{\mathfrak{D}_\pi} := \mathfrak{L}_{\mathfrak{D}_\pi}/\mathfrak{N}$ and $\mathcal{L}^p_{\mathfrak{D}_\pi} := \mathfrak{L}^p_{\mathfrak{D}_\pi}/\mathfrak{N}$ (where here $\mathfrak{N} := \{F \in \mathfrak{L}_{\mathfrak{D}_\pi} \ : \ F(t) = 0 \ a.e.\ t \in [0,1]\}$) then we have the following proposition.

**Proposition 2.3.5** *Let* $p, q, r \in [1,\infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*, and suppose that* $\pi \geq 0$*. The left and right actions of* $\mathcal{L}^p_{\mathfrak{D}_\pi}$ *on* $\mathcal{L}^q$ *are continuous; the maps*

$$\lambda_{[F]} : \mathcal{L}^q \to \mathcal{L}^r; [G] \mapsto [FG]$$

*and*

$$\rho_{[F]} : \mathcal{L}^q \to \mathcal{L}^r; [G] \mapsto [GF]$$

*are continuous for all* $[F] \in \mathcal{L}^p_{\mathfrak{D}_\pi}$*.*

**Proof**

Note that

$$\|[FG]^\mu_\nu\|_r \leq \sum_{\kappa=0}^{\mu+\pi} \|[F]^\mu_\kappa\|_p [G]^\kappa_\nu\|_q < \infty,$$

by Hölder's inequality, so the map $\lambda_{[F]}$ is well-defined. Replacing $[F]$ by $[F_n - F]$, where $([F_n])_{n\geq 1} \subseteq \mathcal{L}^p_{\mathfrak{D}_\pi}$ is a sequence converging to $[F]$ in $\mathcal{L}^p$ gives continuity. To see that the right action $\rho_{[F]}$ is well-defined and continuous, we may use a similar argument, exchanging the rôles of $F$ and $G$, or alternatively the fact that the involution is continuous and the identity $\rho_{[F]}(\cdot) = \lambda_{[F^*]}(\cdot)^*$. □

We introduce processes of matrices with rapid decay; they are processes with values in $\mathfrak{R}$, such that the growth and decay constants $a$ and $b$ are independent of $t$.

**Definition 2.3.6** *A process $F$ is an element of $\mathfrak{L}_\mathfrak{R}$, the class of* processes of matrices with rapid decay*, if there exist positive constants $a$ and $b$ and a function $C : [0,1] \to [0,\infty)$ such that*

$$\|F(t)^\mu_\nu\| \leq C(t)((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} \quad \forall\, t \in [0,1].$$

*If the function $C$ is $p$-integrable for some $p \in [1,\infty]$ then $F \in \mathfrak{L}^p_\mathfrak{R}$.*

The properties of these classes are summarised in the next proposition.

**Proposition 2.3.7** *The space $\mathfrak{L}^p_\mathfrak{R}$ is a subspace of $\mathfrak{L}^p$ that contains $\mathfrak{L}^p_{\mathfrak{D}_{\mathrm{pb}}}$, and the involution is a bijection between $\mathfrak{L}_{\mathfrak{R}(\mathfrak{H};\mathfrak{J})}$ and $\mathfrak{L}_{\mathfrak{R}(\mathfrak{J};\mathfrak{H})}$. Multiplication of any finite number of elements of $\mathfrak{L}_\mathfrak{R}$ is well-defined, associative and compatible with the involution. If $(F_i)_{i=1}^n \subseteq \mathfrak{L}_\mathfrak{R}$ is a family of processes such that $F_i \in \mathfrak{L}^{p_i}_\mathfrak{R}$ for all $i$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p}$ for some $p \in [1,\infty]$ then $F_1 \cdots F_n \in \mathfrak{L}^p$ and the product converges absolutely, i.e.,*

$$\sum_{\pi_1 \geq 0} \cdots \sum_{\pi_{n-1}\geq 0} \|[F_1]^\mu_{\pi_1}\|_{p_1} \cdots \|[F_n]^{\pi_{n-1}}_\nu\|_{p_n} < \infty$$

*for all $\mu, \nu \geq 0$. If $F \in \mathfrak{L}_\mathfrak{R}$ the domain of the induced process contains the exponential subspace $\mathcal{E}$, i.e., $F$ is an HP-process. If $F \in \mathfrak{L}_\mathfrak{R}$ is strongly continuous, i.e., $t \mapsto F(t)^\mu_\nu$ is strongly continuous for all $\mu, \nu \geq 0$, then $t \mapsto \hat{F}(t)\xi$ is continuous for all $\xi \in \mathcal{E}$.*

**Proof**

Everything except the last statement of this proposition follows from the proof of Proposition 2.2.10, Hölder's inequality and the fact that if $F \in \mathfrak{L}^p_\mathfrak{R}$ then there exist positive constants $C$, $a$ and $b$ such that

$$\|[F]^\mu_\nu\|_p \leq C((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} \;\; \forall\, \mu, \nu \geq 0.$$

For the proof of the last statement, let $\xi \in \mathcal{E}$; there exists $M > 0$ such that $\|\xi^\nu\| \leq M^{\nu+1}(\nu!)^{-\frac{1}{2}}$ for all $\nu \geq 0$. Let $a, b > 0$ and $C : [0,1] \to [0,\infty)$ be such that $\|C\|_\infty < \infty$ and

$$\|F(t)^\mu_\nu\| \leq C(t)((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} \quad \forall\, \mu, \nu \geq 0$$

for all $t \in [0,1]$. The strong continuity means that we may replace '$C(t)$' by '$\|C\|_\infty$' on the right-hand side of the above inequality, so

$$\sum_{\nu \geq 0} \|F(\cdot)^\mu_\nu \xi^\nu\|_{C([0,1];\mathfrak{h}_\mu)} \leq \|C\|_\infty \sum_{\nu \geq 0} ((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} M^{\nu+1} (\nu!)^{-\frac{1}{2}} < \infty.$$

Hence $t \mapsto f^\mu(t) := \sum_{\nu \geq 0} F(t)^\mu_\nu \xi$ is in $C([0,1];\mathfrak{h}_\mu)$, and furthermore

$$\sum_{\mu \geq 0} \|f^\mu\|_{C([0,1];\mathfrak{H})} \leq \|C\|_\infty \sum_{\mu \geq 0} \sum_{\nu \geq 0} ((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b} M^{\nu+1} (\nu!)^{-\frac{1}{2}},$$

and this double series is seen to be convergent, working as in the proof of Proposition 2.2.10. Hence $t \mapsto \sum_{\mu \geq 0} f^\mu(t) = \hat{F}(t)\xi$ is continuous, as claimed.

□

In the same way that we defined $\mathfrak{L}_\mathfrak{D}$ we introduce the algebra of bounded processes.

**Definition 2.3.8** *The algebra of* bounded processes*, denoted* $\mathfrak{L}_\mathfrak{B}$*, is defined by*

$$\mathfrak{L}_\mathfrak{B} := \{F \in \mathfrak{L} \,:\, F(t) \in \mathfrak{B} \;\forall\, t \in [0,1]\},$$

*and for* $p \in [1,\infty]$ *we define* $\mathfrak{L}^p_\mathfrak{B}$ *and* $\mathcal{L}^p_\mathfrak{B}$ *in the following manner:*

$$\mathfrak{L}^p_\mathfrak{B} := \{F \in \mathfrak{L}_\mathfrak{B} \,:\, t \mapsto \|\hat{F}(t)\| \in L^p[0,1]\}, \;\; \mathcal{L}^p_\mathfrak{B} := \mathfrak{L}^p_\mathfrak{B}/\mathfrak{N},$$

*where* $\mathfrak{N} := \{F \in \mathfrak{L}_\mathfrak{B} \,:\, F(t) = 0 \text{ a.e } t \in [0,1]\}$.

Measurability of the map $t \mapsto \|\hat{F}(t)\|$ follows from the measurability of $t \mapsto \|\hat{F}(t)\xi\|$ for all $\xi \in \mathfrak{H}$ and Lemma A.1.2. We have the inclusion $\mathfrak{L}^p_\mathfrak{B} \subseteq \mathfrak{L}^p$, as for all $\mu, \nu \geq 0$

$$\|F(t)^\mu_\nu\| \;=\; \|F_\mu \hat{F}(t) E^*_\nu\| \leq \|\hat{F}(t)\|. \tag{2.5}$$

It is easy to see that $\mathcal{L}^p_\mathfrak{B}$ is isomorphic to $L^p([0,1];\mathcal{B}(\mathfrak{H};\mathfrak{J})_s)$, and that the topology it inherits from this isomorphism is stronger than that it inherits from $\mathcal{L}^p$, by the inequality (2.5).

**Note**

All of the above is valid if the interval $[0,1]$ is replaced by any $\sigma$-finite measure space $(M,\mathcal{M},m)$; the appropriate seminorms become $\|\cdot{}^{\mu}_{\nu}\chi_{M_n}(\cdot)\|_p$ (for $\mu,\nu \geq 0$ and $n \geq 1$), where the sets $M_n$ have finite measure and exhaust $M$, i.e., $M = \cup_{n\geq 1} M_n$.

**Examples**

1. The *conditional expectation process* $\mathbb{E} \in \mathfrak{L}^{\infty}_{\mathfrak{B}} \cap \mathfrak{L}^{\infty}_{\mathfrak{D}_0}$ is defined by $\mathbb{E}(t) := \mathbb{E}_{t]}$; we prefer the latter notation. Since

$$\|\mathbb{E}^i_{t]} f - \mathbb{E}^i_{s]} f\|^2 = \int_{s\wedge t}^{s\vee t} \cdots \int_{s\wedge t}^{s\vee t} |f(t_1,\ldots,t_i)|^2 \, \mathrm{d}\mathbf{t},$$

we see that $t \mapsto \mathbb{E}(t)^{\mu}_{\nu} f$ is continuous, so strongly measurable.

2. We define the four *integrator processes* of quantum stochastic calculus, $A$, $A^{\dagger}$, $\Lambda$ and $T$, in the following manner. Let

$$A(t) := A(\chi_{[0,t]}), \quad A^{\dagger}(t) := A^{\dagger}(\chi_{[0,t]}), \quad \Lambda(t) := \mathrm{d}\Gamma(m_{\chi_{[0,t]}})$$

(where $m_{\chi_{[0,t]}}$ is the multiplication operator $f \mapsto \chi_{[0,t]} f$), and

$$T(t) := (t\delta^{\mu}_{\nu} \mathrm{id}_{\mathfrak{h}_{\mu}})_{\mu,\nu \geq 0}.$$

Since $t \mapsto X^{\mu}_{\nu}(t)\xi$ is easily seen to be weakly continuous for all $X \in \{A, A^{\dagger}, \Lambda, T\}$ we see that the measurability condition is fulfilled, while the norms above show that $A$, $A^{\dagger}$, $\Lambda$ and $T$ are elements of $\mathfrak{L}^{\infty}_{\mathfrak{D}_{\mathrm{pb}}}$.

3. One achievement of the Hudson-Parthasarathy calculus is the unification of the Boson and Fermion fields ([HuP2]). This is obtained by using the *parity process* (also known as the *continuous Jordan-Wigner transformation* or *reflection process*).

Define the *parity process* $P \in \mathfrak{L}^{\infty}_{\mathfrak{B}} \cap \mathfrak{L}^{\infty}_{\mathfrak{D}_0}$ by $P(t) := \Gamma((-1)^{\chi_{[0,t]}})$, the second quantisation of the self-adjoint operator $f \mapsto -\chi_{[0,t]} f + \chi_{(t,1]} f$. Explicitly $P^0_0 \equiv \mathrm{id}_{\mathbb{C}}$ and for $i \geq 1$,

$$(P(t)^i_i f)(t_1,\ldots,t_i) := \prod_{j=1}^{i} (-1)^{\chi_{[0,t]}(t_j)} f(t_1,\ldots,t_i).$$

Since $(-1)^{\chi_{[0,s]}}(-1)^{\chi_{[0,t]}} = (-1)^{\chi_{(s\wedge t, s\vee t]}}$ we see that $P(s)P(t) = P(t)P(s)$ for all $s,t \in [0,1]$ and $P^2 = I$, so $P$ is unitary. We also note that

$$P(t)^{\mu}_{\mu} \varepsilon(u)^{\mu} = \varepsilon(-\chi_{[0,t]} u + \chi_{(t,1]} u)^{\mu}$$

from which it is simple to verify measurability.

□

# Chapter 3

# Quantum Stochastic Calculus

From now on we work on $\mathfrak{H} = \mathfrak{F}_+(L^2[0,1])$, the Boson Fock space over $L^2[0,1]$. Thus $\mathfrak{L} = \mathfrak{L}(\mathfrak{H};\mathfrak{H})$, etc.

## 3.1 Quantum Stochastic Integrals

We define matrix versions of the four integrals of quantum stochastic calculus (which we will also refer to as QS integrals, for brevity) using the Malliavin gradient; this technique was first introduced by Lindsay [Lin] and Belavkin [Bel]. We deduce basic properties of the integrals, including an integration by parts formula which is a precursor to the local form of the quantum Itô formula. The operator processes induced by these integrals are shown to coincide with the usual definitions of QS integrals ([HuP], [AtM]) when the integrands are processes of matrices with rapid decay or bounded processes. We prove a form of the dominated convergence theorem for QS integrals and demonstrate their independence: this latter result is crucial for the study of quantum stochastic differential equations.

Let $\alpha, \beta \in \{0,1\}$ and define $p = p(\alpha,\beta) := 2(2-\alpha-\beta)^{-1} \in \{1,2,\infty\}$. Given $\mu \geq 0$ and $\xi \in \mathfrak{h}_\mu$ let

$$\nabla_t^\alpha \xi := \begin{cases} (\nabla_\mu^{\mu-1}\xi)(t) & \alpha = 1 \\ \xi & \alpha = 0 \end{cases}$$

for all $t \in [0,1]$. The four quantum stochastic integrals are defined in the next proposition.

**Proposition 3.1.1** *Let $\alpha, \beta \in \{0,1\}$. For all $F \in \mathfrak{L}^p$ there exists a unique process $I_\beta^\alpha(F) \in \mathfrak{L}^\infty$ such that*

$$\langle \xi, I_\beta^\alpha(F)(t)_\nu^\mu \eta \rangle = \int_0^t \langle \nabla_s^\beta \xi, F(s)_{\nu-\alpha}^{\mu-\beta} \nabla_s^\alpha \eta \rangle \,\mathrm{d}s \qquad (3.1)$$

*for all $t \in [0,1]$, $\mu, \nu \geq 0$ and $\xi \in \mathfrak{h}_\mu, \eta \in \mathfrak{h}_\nu$. The mapping $I^\alpha_\beta : \mathfrak{L}^p \to \mathfrak{L}^\infty$ is linear, satisfies the identity $I^\alpha_\beta(F)^* = I^\beta_\alpha(F^*)$ and is such that $I^\alpha_\beta(F) = 0$ if $F(t) = 0$ for almost all $t \in [0,1]$. Furthermore*

$$\|I^\alpha_\beta(F)(t)^\mu_\nu\|_\infty \;\leq\; \mu^{\frac{\beta}{2}}\nu^{\frac{\alpha}{2}}\|[F]^{\mu-\beta}_{\nu-\alpha}\|_p \tag{3.2}$$

*for all $t \in [0,1]$, so $I^\alpha_\beta : \mathcal{L}^p \to \mathcal{L}^\infty$ is continuous.*

**Proof**

Clearly, if such a process exists it is unique. For existence note that

$$\left|\int_0^t \langle \nabla^\beta_s \xi, F(s)^{\mu-\beta}_{\nu-\alpha}\nabla^\alpha_s \eta\rangle \, \mathrm{d}s\right| \leq \|[F]^{\mu-\beta}_{\nu-\alpha}\|_p \mu^{\frac{\beta}{2}}\nu^{\frac{\alpha}{2}}\|\xi\|\,\|\eta\|,$$

so we define $I^\alpha_\beta(F)(t)^\mu_\nu$ to be the element of $\mathcal{B}(\mathfrak{h}_\nu;\mathfrak{h}_\mu)$ corresponding to the sesquilinear form on the right-hand side of (3.1). The continuity of the map $t \mapsto \langle \xi, I^\alpha_\beta(F)(t)^\mu_\nu \eta\rangle$ and Pettis' theorem [Pet] gives the required strong measurability and the bound above shows that $I^\alpha_\beta(F) \in \mathfrak{L}^\infty$. The remainder of the proposition is easily verified.

□

For suitable integrands (i.e., HP-processes that are *adapted*: see Section 3.2) our matrix integrals correspond to the quantum stochastic integrals defined by Hudson and Parthasarathy ([HuP]):

$$\Big(I^0_0(\cdot), I^1_0(\cdot), I^0_1(\cdot), I^1_1(\cdot)\Big) \sim \Big(\int_0 \cdot \,\mathrm{d}t, \int_0 \cdot \,\mathrm{d}A, \int_0 \cdot \,\mathrm{d}A^\dagger, \int_0 \cdot \,\mathrm{d}\Lambda\Big).$$

We use the present notation to emphasise the fact that our integrals are defined without adaptedness restrictions and without the use of a limiting procedure. Our choice of indices agrees with Evans [Eva] (but unfortunately, not with Holevo [Hol2]); recall the tendency of lower indices to become raised when crossing an inner product.

If we wish to see the matrices representing $I^\alpha_\beta(F)$ explicitly, it follows from the inner-product identity (3.1) that

$$I^\alpha_\beta(F)(t)^\mu_\nu \;=\; \xi \mapsto \begin{cases} \int_0^t F(s)^\mu_\nu \xi \, \mathrm{d}s & (\alpha = \beta = 0) \\ \int_0^t F(s)^\mu_{\nu-1}(\nabla^{\nu-1}_\nu \xi)(s)\, \mathrm{d}s & (\alpha = 1, \beta = 0) \\ \mathcal{S}^\mu_{\mu-1}\Big((\chi_{[0,t]}F)(\cdot)^{\mu-1}_\nu \xi\Big) & (\alpha = 0, \beta = 1) \\ \mathcal{S}^\mu_{\mu-1}\Big((\chi_{[0,t]}F)(\cdot)^{\mu-1}_{\nu-1}(\nabla^{\nu-1}_\nu \xi)(\cdot)\Big) & (\alpha = \beta = 1) \end{cases} \tag{3.3}$$

for all $\mu, \nu \geq 0$, $t \in [0,1]$ and $\xi \in \mathfrak{h}_\mu$. From this, or the fact that

$$\begin{aligned}\langle \xi, [I_\beta^\alpha(F)(t) - I_\beta^\alpha(F)(s)]_\nu^\mu \eta\rangle &= \int_{s\wedge t}^{s\vee t} \langle \nabla_s^\beta \xi, F(s)_{\nu-\alpha}^{\mu-\beta} \nabla_s^\alpha \eta\rangle \, \mathrm{d}s \\ &= \langle \xi, I_\beta^\alpha(F\chi_{(s\wedge t, s\vee t]})(r)_\nu^\mu \eta\rangle \qquad (3.4)\end{aligned}$$

for any $r \geq s\vee t$, we see that the QS integrals behave in the usual manner for 'integrals up to $t$'. We could use this observation to define $I_\beta^\alpha(F)(A) := I_\beta^\alpha(F\chi_A)(1)$ for any measurable set $A \subseteq [0,1]$, but choose not to pursue this generalisation.

The following inner-product identities (or, if one prefers, integration-by-parts formulae) are the source of all our results: they contain the quantum (and so classical) Itô formulae.

**Proposition 3.1.2** *Let $\alpha, \beta, \gamma, \delta \in \{0,1\}$, $p = 2(2-\alpha-\beta)^{-1}, q = 2(2-\gamma-\delta)^{-1} \in \{1, 2, \infty\}$ and $F \in \mathfrak{L}^p, G \in \mathfrak{L}^q$. For all $\mu, \nu, \pi \geq 0$ we have that*

$$\begin{aligned}\langle I_\beta^\alpha(F)(t)_\nu^\mu \xi, I_\delta^\gamma(G)(t)_\pi^\mu \eta\rangle &= \int_0^t \int_0^t \langle \nabla_s^\delta F(r)_{\nu-\alpha}^{\mu-\beta} \nabla_r^\alpha \xi, \nabla_r^\beta G(s)_{\pi-\gamma}^{\mu-\delta} \nabla_s^\gamma \eta\rangle \, \mathrm{d}r \, \mathrm{d}s \\ &\quad + \delta_\delta^\beta \int_0^t \langle F(s)_{\nu-\alpha}^{\mu-1} \nabla_s^\alpha \xi, G(s)_{\pi-\gamma}^{\mu-1} \nabla_s^\gamma \eta\rangle \, \mathrm{d}s \qquad (3.5)\end{aligned}$$

*for all $t \in [0,1]$ and $\xi \in \mathfrak{h}_\nu, \eta \in \mathfrak{h}_\pi$. In particular $t \mapsto I_\beta^\alpha(F)(t)_\nu^\mu \xi$ is continuous for all $\xi \in \mathfrak{h}_\nu$.*

**Proof**

The only non-trivial case of this proposition is when $\beta = \delta = 1$. We assume this, and further take $t = 1$ (replacing $F$ by $F\chi_{[0,t]}$ and similarly with $G$, if necessary). We have that

$$\begin{aligned}\langle I_1^\alpha(F)(1)_\nu^\mu \xi, I_1^\gamma(G)(1)_\pi^\mu \eta\rangle &= \int_0^1 \langle \nabla_s I_1^\alpha(F)(1)_\nu^\mu \xi, G(s)_{\pi-\gamma}^{\mu-1} \nabla_s^\gamma \eta\rangle \, \mathrm{d}s \\ &= \langle I_1^\alpha(F)(1)_\nu^\mu \xi, \mathcal{S}_{\mu-1}^\mu [G(\cdot)_{\pi-\gamma}^{\mu-1} \nabla^\gamma \eta(\cdot)]\rangle \\ &= \int_0^1 \langle F(s)_{\nu-\alpha}^{\mu-1} \nabla_s^\alpha \xi, \nabla_s \mathcal{S}_{\mu-1}^\mu [G(\cdot)_{\pi-\gamma}^{\mu-1} \nabla^\gamma \eta(\cdot)]\rangle \, \mathrm{d}s \\ &= \langle \mathcal{S}_{\mu-1}^\mu [F(\cdot)_{\nu-\alpha}^{\mu-1} \nabla^\alpha \xi(\cdot)], \mathcal{S}_{\mu-1}^\mu [G(\cdot)_{\pi-\gamma}^{\mu-1} \nabla^\gamma \eta(\cdot)]\rangle\end{aligned}$$

The conclusion follows directly from the identity (2.4), which we prove now. Let $f, g \in \mathfrak{h}_\mu$;

$$\begin{aligned}&\langle \mathcal{S}_\mu^{\mu+1} f, \mathcal{S}_\mu^{\mu+1} g\rangle \\ &= \frac{1}{\mu+1} \int_{[0,1]^{\mu+1}} \sum_{k,l=1}^{\mu+1} \overline{f}(t_k)(t_1, \ldots, \widehat{t_k}, \ldots, t_{\mu+1}) g(t_l)(t_1, \ldots, \widehat{t_l}, \ldots, t_{\mu+1}) \, \mathrm{d}\mathbf{t}\end{aligned}$$

$$= \int_{[0,1]^{\mu+1}} \overline{f}(t_1)(t_2,\ldots,t_{\mu+1}) g(t_1)(t_2,\ldots,t_{\mu+1})\, \mathrm{d}\mathbf{t}$$

$$+ \mu \int_{[0,1]^{\mu+1}} \overline{f}(t_1)(t_2,\ldots,t_{\mu+1}) g(t_{\mu+1})(t_1,\ldots,t_\mu)\, \mathrm{d}\mathbf{t}$$

$$= \int_0^1 \langle f(s), g(s)\rangle \,\mathrm{d}s + \int_0^1 \int_0^1 \langle \nabla_\mu^{\mu-1}[f(r)](s), \nabla_\mu^{\mu-1}[g(s)](r)\rangle \,\mathrm{d}r\,\mathrm{d}s.$$

□

It is easy to see that QS integrals of uniformly diagonal processes retain this property, as if $F \in \mathfrak{L}^p_{\mathfrak{D}_\pi}$ then, by (3.2),

$$\|I^\alpha_\beta(F)(t)^\mu_\nu\| \leq \mu^{\frac{\beta}{2}} \nu^{\frac{\alpha}{2}} \|[F]^{\mu-\beta}_{\nu-\alpha}\|_p = 0 \text{ if } |\mu - \nu| > \pi + 1.$$

It is also clear that if $F$ has growth of order $a$ then $I^\alpha_\beta(F)$ has growth of order $a + \frac{\alpha}{2} + \frac{\beta}{2}$, from the same estimate. This behaviour generalises to the case of quantum stochastic integrals of processes of matrices with bounded decay.

**Proposition 3.1.3** *Let $\alpha, \beta \in \{0,1\}$, $p = 2(2-\alpha-\beta)^{-1}$ and suppose $F \in \mathfrak{L}^p_{\mathfrak{R}}$. The QS integral $I^\alpha_\beta(F) \in \mathfrak{L}^\infty_{\mathfrak{R}}$, so $\mathcal{D}(\hat{I}^\alpha_\beta(F)) \supseteq \mathcal{E}_0$ and $t \mapsto \hat{I}^\alpha_\beta(F)(t)\varepsilon(u)$ is continuous for all $u \in L^2[0,1]$. Furthermore*

$$\langle \varepsilon(u), \hat{I}^\alpha_\beta(F)(t)\varepsilon(v)\rangle = \int_0^t \langle \varepsilon(u), \hat{F}(s)\varepsilon(v)\rangle \overline{u}(s)^\beta v(s)^\alpha \,\mathrm{d}s \tag{3.6}$$

*for all $u, v \in L^2[0,1]$ and $t \in [0,1]$.*

**Proof**

The fact that $I^\alpha_\beta(F) \in \mathfrak{L}^\infty_{\mathfrak{R}}$ follows from the estimate (3.2) and the inequality

$$|\mu - \beta - \nu + \alpha|!^{-b} \leq ((\mu \vee \nu) + 1)^b |\mu - \nu|!^{-b}$$

which holds for all $\mu, \nu \geq 0$, $\alpha, \beta \in \{0,1\}$ and $b > 0$. If $\alpha = \beta$ or $\mu = \nu$ this is clear; otherwise

$$|\mu - \beta - \nu + \alpha|!^{-b} = (|\mu - \nu| \pm 1)!^{-b} \leq |\mu - \nu|!^{-b}$$

if the positive case is taken, and for the negative case

$$(|\mu - \nu| - 1)!^{-b} = |\mu - \nu|!^{-b} |\mu - \nu|^b \leq |\mu - \nu|!^{-b} ((\mu \vee \nu) + 1)^b.$$

The claims about the domain of the induced process and continuity follow from Proposition 2.3.7. For the final part, note that

$$\begin{aligned}
&\sum_{\mu\geq 0}\sum_{\nu\geq 0}\int_0^t |\langle \varepsilon(u)^\mu, F(s)^\mu_\nu \varepsilon(v)^\nu\rangle|\,|u(s)|^\beta |v(s)|^\alpha \,\mathrm{d}s \\
&\leq \sum_{\mu\geq 0}\sum_{\nu\geq 0} \|[F]^\mu_\nu\|_p \|u\|^\beta \|v\|^\alpha \|u\|^\mu \|v\|^\nu (\mu!\nu!)^{-\frac{1}{2}} \\
&\leq \|C\|_p \|u\|^\beta \|v\|^\alpha \sum_{\mu\geq 0}\sum_{\nu\geq 0} ((\mu\vee\nu)+1)^a |\mu-\nu|!^{-b} \|u\|^\mu \|v\|^\nu (\mu!\nu!)^{-\frac{1}{2}} \\
&< \infty
\end{aligned}$$

because the double sum is equal to

$$\begin{aligned}
&\sum_{\mu\geq 0}\sum_{\nu=0}^{\mu} (\mu+1)^a (\mu-\nu)!^{-b} \|u\|^\mu \|v\|^\nu (\mu!\nu!)^{-\frac{1}{2}} \\
&\qquad + \sum_{\mu\geq 0}\sum_{\nu>\mu} (\nu+1)^a (\nu-\mu)!^{-b} \|u\|^\mu \|v\|^\nu (\mu!\nu!)^{-\frac{1}{2}} \\
&\leq \sum_{\mu\geq 0} (\mu+1)^a \|u\|^\mu (\mu!)^{-\frac{1}{2}} (1\vee\|v\|)^\mu \sum_{\pi\geq 0} (\pi!)^{-b} \\
&\qquad + \sum_{\mu\geq 0} \|u\|^\mu \|v\|^\mu (\mu!)^{-\frac{1}{2}} \sum_{k\geq 1} (k+\mu+1)^a (k!)^{-b} \|v\|^k (k+\mu)!^{-\frac{1}{2}} \\
&\leq \sum_{\mu\geq 0} (\mu+1)^a (\|u\|(1\vee\|v\|))^\mu (\mu!)^{-\frac{1}{2}} \sum_{\pi\geq 0} (\pi!)^{-b} \\
&\qquad + \sum_{\mu\geq 0} (\|u\|\,\|v\|)^\mu (\mu!)^{-\frac{1}{2}} (\mu+1)^a \sum_{k\geq 1} (k+1)^a (k!)^{-b} \|v\|^k
\end{aligned}$$

(since $(x+y+1)^a \leq (x+1)^a (y+1)^a$ for $x, y \geq 0$) and all these series converge, by the limit ratio test. Hence we may exchange the order of integration and summation in the expression for $\langle \varepsilon(u), \hat{I}^\alpha_\beta(F)(t)\varepsilon(v)\rangle$ to get the equality (3.6). □

Our next two propositions concern QS integrals of bounded processes. On suitable domains the integrals have the form suggested by that of their components. Note that if $p \geq 2$ we may regard $F \in \mathfrak{L}^p_{\mathfrak{B}}$ as an element of $\mathfrak{B}(\mathfrak{H}; L^2([0,1];\mathfrak{H}))$ by setting

$$F(\cdot)^\mu_\nu \xi : t \mapsto (F^\mu_\nu \xi)(t) := F(t)^\mu_\nu \xi \in L^2([0,1];\mathfrak{h}_\mu),$$

so $\hat{F}\xi = t \mapsto \hat{F}(t)\xi$. Recall that $\hat{\nabla}$ is a linear map from $\mathfrak{H}$ to $L^2([0,1];\mathfrak{H})$ with domain $\mathcal{D}(\hat{N}^{\frac{1}{2}})$. If $F \in \mathfrak{L}^\infty_{\mathfrak{B}}$ we let $\hat{F}\hat{\nabla}$ denote the linear operator from $\mathcal{D}(\hat{N}^{\frac{1}{2}})$ to $L^2([0,1];\mathfrak{H})$ defined by their pointwise product, i.e., $\hat{F}\hat{\nabla}\xi(t) := \hat{F}(t)\hat{\nabla}_t\xi$, where $\hat{\nabla}_t\xi := (\hat{\nabla}\xi)(t)$. The product is measurable by Lemma A.1.3, and it is simple to verify that

$$\|\hat{F}\hat{\nabla}\xi\| = (\int_0^1 \|\hat{F}(t)\hat{\nabla}_t\xi\|^2 \,\mathrm{d}t)^{\frac{1}{2}} \leq \|\hat{F}\|_\infty \|\hat{N}^{\frac{1}{2}}\xi\|.$$

**Proposition 3.1.4** *Let $F \in \mathfrak{L}^p_{\mathfrak{B}}$, where $p = 2(2-\alpha-\beta)^{-1}$ and $\alpha, \beta \in \{0,1\}$. The QS integral $\hat{I}^\alpha_\beta(F)(1)$ has the form*

$$\xi \mapsto \begin{cases} \int_0^1 \hat{F}(t)\xi \, \mathrm{d}t & \forall \xi \in \mathfrak{H} \quad (\alpha=\beta=0) \\ \int_0^1 \hat{F}(t)\hat{\nabla}_t\xi \, \mathrm{d}s & \forall \xi \in \mathcal{D}(\hat{N}^{\frac{1}{2}}) \quad (\alpha=1,\beta=0) \\ \hat{\mathcal{S}}(\hat{F}\xi) & \forall \xi \in \mathcal{D}(\hat{\mathcal{S}}(\hat{F})) \quad (\alpha=0,\beta=1) \\ \hat{\mathcal{S}}(\hat{F}\hat{\nabla}\xi) & \forall \xi \in \mathcal{D}(\hat{\mathcal{S}}(\hat{F}\hat{\nabla})) \quad (\alpha=\beta=1) \end{cases}.$$

**Proof**

Let $\xi \in \mathfrak{H}$; the map $t \mapsto \hat{F}(t)\xi$ is strongly measurable and $\int_0^1 \|\hat{F}(t)\xi\| \, \mathrm{d}t \le \|\hat{F}\|_1\|\xi\|$, so $\xi \mapsto \int_0^1 \hat{F}(t)\xi \, \mathrm{d}t$ is a bounded operator. Now

$$\int_0^1 \hat{F}(t)\xi \, \mathrm{d}t = \sum_{\mu\ge 0} E_\mu \int_0^1 \hat{F}(t)\xi \, \mathrm{d}t = \sum_{\mu\ge 0} \int_0^1 \sum_{\nu \ge 0} F(t)^\mu_\nu \xi^\nu \, \mathrm{d}t$$

and $\xi \mapsto \int_0^1 E_\mu \hat{F}(t)\xi \, \mathrm{d}t$ is continuous, so

$$\sum_{\nu\ge 0} \int_0^1 E_\mu \hat{F}(t) E^*_\nu \xi^\nu \, \mathrm{d}t = \int_0^1 \sum_{\nu\ge 0} F(t)^\mu_\nu \xi^\nu \, \mathrm{d}t,$$

whence

$$\int_0^1 \hat{F}(t)\xi \, \mathrm{d}t = \sum_{\mu\ge 0}\sum_{\nu\ge 0} \int_0^1 F(t)^\mu_\nu \xi^\nu \, \mathrm{d}t = \hat{I}^0_0(F)(1)\xi.$$

Now suppose $\xi \in \mathcal{D}(\hat{N}^{\frac{1}{2}})$; the discussion before the proposition gives that $t \mapsto \hat{F}(t)\hat{\nabla}_t\xi$ is strongly measurable, and the estimate $\int_0^1 \|\hat{F}(t)\hat{\nabla}_t\xi\| \, \mathrm{d}t \le \|\hat{F}\|_2 \|\hat{N}^{\frac{1}{2}}\xi\|$ shows that it is an element of $L^1([0,1];\mathfrak{H})$. Thus $\int_0^1 \hat{F}(t)\hat{\nabla}_t\xi \, \mathrm{d}t$ exists and, because the series $\sum_{\nu\ge 0} F(\cdot)^\mu_{\nu-1}(\nabla^{\nu-1}_\nu \xi)(\cdot)$ is Cauchy, so convergent, in $L^1([0,1];\mathfrak{h}_\mu)$, we see that

$$\begin{aligned} \sum_{\mu\ge 0} \int_0^1 \sum_{\nu\ge 0} F(t)^\mu_{\nu-1}(\nabla^{\nu-1}_\nu \xi^\nu)(t) \, \mathrm{d}t &= \sum_{\mu\ge 0}\sum_{\nu\ge 0} \int_0^1 F(t)^\mu_{\nu-1}(\nabla^{\nu-1}_\nu \xi^\nu)(t) \, \mathrm{d}t \\ &= \sum_{\mu\ge 0}\sum_{\nu\ge 0} \hat{I}^0_1(F)(1)\xi. \end{aligned}$$

Since $\hat{F} \in \mathcal{B}(\mathfrak{H}; L^2([0,1];\mathfrak{H}))$ is continuous, the sum $\sum_{\nu\ge 0} F(\cdot)^\mu_\nu \xi^\nu$ is convergent in $L^2([0,1];\mathfrak{h}_\mu)$, to $E_\mu \hat{F}\xi$. Thus $\mathcal{S}^{\mu+1}_\mu(\sum_{\nu\ge 0} F(\cdot)^\mu_\nu \xi^\nu) = \sum_{\nu\ge 0} \mathcal{S}^{\mu+1}_\mu(F(\cdot)^\mu_\nu \xi^\nu)$ for all $\mu \ge 0$. In particular, if $\hat{F}\xi \in \mathcal{D}(\hat{\mathcal{S}})$ then

$$\hat{\mathcal{S}}(\hat{F}\xi) = \sum_{\mu\ge 0} \mathcal{S}^{\mu+1}_\mu(\sum_{\nu\ge 0} F(\cdot)^\mu_\nu \xi^\nu) = \sum_{\mu\ge 0}\sum_{\nu\ge 0} \mathcal{S}^{\mu+1}_\mu(F(\cdot)^\mu_\nu \xi^\nu) = \hat{I}^0_1(F)(1)\xi.$$

Finally, if $\xi \in \mathcal{D}(\hat{F}\hat{\nabla})$ then

$$\|\sum_{\nu=\lambda}^{\pi} F(\cdot)^{\mu}_{\nu-1}(\nabla^{\nu-1}_{\nu}\xi^{\nu})(\cdot)\|_{L^2([0,1];\mathfrak{h}_\mu)} \leq \|\hat{F}\|_\infty (\sum_{\nu=\lambda}^{\pi} \nu\|\xi^\nu\|^2)^{\frac{1}{2}} \to 0$$

as $\lambda, \pi \to \infty$, so if $\hat{F}\hat{\nabla}\xi \in \mathcal{D}(\hat{\mathcal{S}})$ then

$$\hat{\mathcal{S}}(\hat{F}\hat{\nabla}\xi) = \sum_{\mu\geq 0}\sum_{\nu\geq 0} \mathcal{S}^{\mu+1}_{\mu}\Big(F(\cdot)^{\mu}_{\nu-1}(\nabla^{\nu-1}_{\nu}\xi^{\nu})(\cdot)\Big) = \hat{I}^1_1(F)(1)\xi.$$

□

In the Hudson-Parthasarathy calculus the domains of quantum stochastic integrals automatically include the exponential vectors. As we see above, this is not the case for our non-adapted calculus; as Lindsay points out ([Lin, p. 66]) we exchange the assumption of adaptedness for restrictions on the 'growth' of processes. If our processes are suitably well-behaved, we get the usual inner-product identity of quantum stochastic calculus, as Proposition 3.1.3 and the next proposition demonstrate. We introduce the notation $\sum_{(\alpha,\beta)}$ as an abbreviated form of the double sum $\sum_{\alpha=0}^{1}\sum_{\beta=0}^{1}$.

**Proposition 3.1.5** *Let $F^{\alpha}_{\beta} \in \mathfrak{L}^p_{\mathfrak{B}}$ for $\alpha, \beta \in \{0,1\}$, where $p = 2(2-\alpha-\beta)^{-1}$, and suppose*

$$M := \sum_{(\alpha,\beta)} I^{\alpha}_{\beta}(F^{\beta}_{\alpha}) \in \mathfrak{L}^\infty$$

*is such that $\mathcal{D}(\hat{M}) \supseteq \mathcal{E}_0$. Then for all $u, v \in L^2[0,1]$ and $t \in [0,1]$ we have that*

$$\langle\varepsilon(u), \hat{M}(t)\varepsilon(v)\rangle = \sum_{(\alpha,\beta)} \int_0^t \langle\varepsilon(u), \hat{F}^{\beta}_{\alpha}(s)\varepsilon(v)\rangle \overline{u}(s)^{\beta} v(s)^{\alpha}\,\mathrm{d}s.$$

**Proof**

Note that

$$\begin{aligned}
&\sum_{(\alpha,\beta)} \int_0^t \sum_{\mu\geq 0}\sum_{\nu\geq 0} |\langle\varepsilon(u)^{\mu-\beta}, F^{\beta}_{\alpha}(s)^{\mu-\beta}_{\nu-a}\varepsilon(v)^{\nu-a}\rangle\overline{u}(s)^{\beta}v(s)^{\alpha}|\,\mathrm{d}s \\
&\leq \sum_{(\alpha,\beta)} \int_0^t \|\hat{F}^{\beta}_{\alpha}(s)\|\,|u(s)|^{\beta}|v(s)|^{\alpha}\,\mathrm{d}s \sum_{\mu\geq 0}\|u\|^{\mu}(\mu!)^{-\frac{1}{2}} \sum_{\nu\geq 0}\|v\|^{\nu}(\nu!)^{-\frac{1}{2}} < \infty
\end{aligned}$$

as

$$\int_0^t \|\hat{F}^{\beta}_{\alpha}(s)\|\,|u(s)|^{\beta}|v(s)|^{\alpha}\,\mathrm{d}s \leq \|\hat{F}^{\beta}_{\alpha}\|_p\|u\|^{\beta}\|v\|^{\alpha}.$$

Hence, by Tonelli's theorem,

$$\begin{aligned}
&\langle \varepsilon(u), \hat{M}(t)\varepsilon(v)\rangle \\
&= \sum_{\mu\geq 0}\sum_{\nu\geq 0}\sum_{(\alpha,\beta)} \langle \varepsilon(u)^{\mu}, I^{\alpha}_{\beta}(F^{\beta}_{\alpha})(t)^{\mu}_{\nu}\varepsilon(v)^{\nu}\rangle \\
&= \sum_{\mu\geq 0}\sum_{\nu\geq 0}\sum_{(\alpha,\beta)} \int_0^t \langle \varepsilon(u)^{\mu-\beta}, F^{\beta}_{\alpha}(s)^{\mu-\beta}_{\nu-\alpha}\varepsilon(v)^{\nu-\alpha}\rangle \overline{u}(s)^{\beta} v(s)^{\alpha}\,\mathrm{d}s \\
&= \sum_{(\alpha,\beta)} \int_0^t \sum_{\mu\geq 0}\sum_{\nu\geq 0} \langle \varepsilon(u)^{\mu}, F^{\beta}_{\alpha}(s)^{\mu}_{\nu}\varepsilon(v)^{\nu}\rangle \overline{u}(s)^{\beta} v(s)^{\alpha}\,\mathrm{d}s \\
&= \sum_{(\alpha,\beta)} \int_0^t \langle \varepsilon(u), \hat{F}^{\beta}_{\alpha}(s)\varepsilon(v)\rangle \overline{u}(s)^{\beta} v(s)^{\alpha}\,\mathrm{d}s
\end{aligned}$$

as claimed.

□

The following is the matrix-quantum-stochastic analogue of Lebesgue's dominated convergence theorem.

**Theorem 3.1.6 (QS Integral DCT)** *Let $\alpha, \beta \in \{0,1\}$ and suppose $F \in \mathfrak{L}^p$ and $(F_n)_{n\geq 1} \subseteq \mathfrak{L}^p$, where $p = 2(2-\alpha-\beta)^{-1}$, are such that*

*(i) $F_n(t)^{\mu}_{\nu} \to F(t)^{\mu}_{\nu}$ in the weak operator topology a.e. $\forall\, \mu, \nu \geq 0$*
*and (ii) $t \mapsto \sup_{n\geq 1} \|F_n(t)^{\mu}_{\nu}\| \in L^p[0,1]$ $\forall\, \mu, \nu \geq 0$.*

*Then $I^{\alpha}_{\beta}(F_n)^{\mu}_{\nu}(t)$ converges to $I^{\alpha}_{\beta}(F)^{\mu}_{\nu}(t)$ in the weak operator topology for all $\mu, \nu \geq 0$ and all $t \in [0,1]$.*

**Proof**

Let $\mu, \nu \geq 0$, $t \in [0,1]$ and $\xi \in \mathfrak{h}_{\mu}$, $\eta \in \mathfrak{h}_{\nu}$; we know that

$$|\langle \xi, I^{\alpha}_{\beta}(F_n - F)(t)^{\mu}_{\nu}\eta\rangle| \leq \int_0^t |\langle \nabla^{\beta}_s \xi, (F_n - F)(s)^{\mu-\beta}_{\nu-\alpha}\nabla^{a}_s \eta\rangle|\,\mathrm{d}s$$

and, by $(i)$, the integrand converges to zero almost everywhere as $n \to \infty$. Now

$$|\langle \nabla^{\beta}_s \xi, (F_n - F)(s)^{\mu-\beta}_{\nu-\alpha}\nabla^{\alpha}_s \eta\rangle| \leq (\sup_{n\geq 1} \|F_n(s)^{\mu-\beta}_{\nu-\alpha}\| + \|F(s)^{\mu-\beta}_{\nu-\alpha}\|)\, \|\nabla^{\beta}_s \xi\|\, \|\nabla^{\alpha}_s \eta\|$$

and the right-hand side of this inequality is an element of $L^1[0,1]$ when regarded as a function of $s$. To see this, note that if $\alpha + \beta = 2$ say,

$$\begin{aligned}
&\int_0^1 (\sup_{n\geq 1} \|F_n(s)^{\mu-1}_{\nu-1}\| + \|F(s)^{\mu-1}_{\nu-1}\|)\, \|\nabla_s \xi\|\, \|\nabla_s \eta\|\,\mathrm{d}s \\
&\leq \Big(\Big\| \sup_n \|F_n(\cdot)^{\mu-1}_{\nu-1}\| \Big\|_{\infty} + \Big\| \|F(\cdot)^{\mu-1}_{\nu-1}\| \Big\|_{\infty}\Big)\Big(\int_0^1 \|\nabla_s \xi\|^2\,\mathrm{d}s \int_0^1 \|\nabla_s \eta\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \\
&< \infty
\end{aligned}$$

by $(ii)$; the other cases are similar, using the Cauchy-Schwarz-Buniakowski inequality as appropriate. Lebesgue's dominated convergence theorem shows that $I^\alpha_\beta(F_n)(t)^\mu_\nu$ tends to $I^\alpha_\beta(F)(t)^\mu_\nu$ in the weak operator topology for all $t \in [0,1]$, as claimed.

$\square$

We are indebted to [Lin2] for the idea behind the proof of the following proposition, which shows the 'independence' of the QS integrals; if their sum is zero then the integrands are zero, up to equivalence of null processes.

**Proposition 3.1.7** *For all $\alpha, \beta \in \{0,1\}$ let $F^\alpha_\beta \in \mathfrak{L}^p$, where $p = 2(2-\alpha-\beta)^{-1}$. The sum*

$$\sum_{(\alpha,\beta)} I^\alpha_\beta(F^\beta_\alpha) = 0$$

*if and only if $[F^\beta_\alpha] = 0$ for all $\alpha, \beta \in \{0,1\}$.*

**Proof**

Let $L^{\mathrm{step}}([0,1]; \mathbb{Q} + i\mathbb{Q})$ denote the $(\mathbb{Q} + i\mathbb{Q})$-linear span of the set of indicator functions of closed subintervals of $[0,1]$ with rational endpoints, i.e.,

$$\{\sum_{j=1}^n r_j \chi_{[p_j,q_j]} \,:\, n \in \mathbb{N}; r_j \in \mathbb{Q} + i\mathbb{Q}; p_0 < q_0 < \cdots < p_n < q_n \in \mathbb{Q} \cap [0,1]\}.$$

It is easy to see that

$$L(t) := \{f \in L^{\mathrm{step}}([0,1]; \mathbb{Q} + i\mathbb{Q}) \,:\, f = 0 \text{ on a neighbourhood of } t\}$$

is dense in $L^2[0,1]$ for all $t \in [0,1]$, as is

$$L(t)' := \{f \in L^{\mathrm{step}}([0,1]; \mathbb{Q} + i\mathbb{Q}) \,:\, f \neq 0 \text{ on a neighbourhood of } t\},$$

and so

$$C^n_t := \{f^{\otimes n} \,:\, f \in L(t)\} \quad \text{and} \quad D^n_t := \{f^{\otimes n} \,:\, f \in L(t)'\}$$

are total in $\mathfrak{h}_n$. If $\sum_{(\alpha,\beta)} I^\alpha_\beta(F^\beta_\alpha) = 0$ then for all $\mu, \nu \geq 0$, $t \in [0,1]$, $\xi \in \mathfrak{h}_\mu$ and $\eta \in \mathfrak{h}_\nu$ we see that

$$0 = \langle \xi, \sum_{(\alpha,\beta)} I^\alpha_\beta(F^\beta_\alpha)(t)^\mu_\nu \eta \rangle = \int_0^t \sum_{(\alpha,\beta)} \langle \nabla^\beta_s \xi, F^\beta_\alpha(s)^{\mu-\beta}_{\nu-\alpha} \nabla^\alpha_s \eta \rangle \,\mathrm{d}s$$

and so

$$S(\xi,\eta) := \sum_{(\alpha,\beta)} \langle \nabla^\beta_s \xi, F^\beta_\alpha(s)^{\mu-\beta}_{\nu-\alpha} \nabla^\alpha_s \eta \rangle = 0$$

for almost all $s \in [0,1]$. If we let $\xi \in C_s^\mu$, $\eta \in C_s^\nu$ then

$$0 = S(\xi, \eta) = \langle \xi, F_0^0(s)_\nu^\mu \eta \rangle$$

whence $F_0^0(s)_\nu^\mu = 0$ by the totality of $C_s^\mu$ and $C_s^\nu$. Next, taking $\xi \in C_s^\mu$ and $\eta \in D_s^\nu$ we see that

$$0 = S(\xi, \eta) = \langle \xi, F_1^0(s)_{\nu-1}^\mu \nabla_s \eta \rangle,$$

so $F_1^0(s)_{\nu-1}^\mu \psi = 0$ for all $\psi \in D_s^{\nu-1}$. As this set is total, $F_1^0(s)_{\nu-1}^\mu = 0$. Similarly we show that $F_0^1(s)_\nu^{\mu-1} = 0$, so

$$0 = S(\xi, \eta) = \langle \nabla_s \xi, F_1^1(s)_{\nu-1}^{\mu-1} \nabla_s \eta \rangle$$

and $F_1^1(s)_{\nu-1}^{\mu-1} = 0$.

The converse is trivial.

□

**Examples**

1. The processes $\Lambda$, $A$, $A^\dagger$ and $T$ are equal to the quantum stochastic integrals $I_1^1(I)$, $I_0^1(I)$, $I_1^0(I)$ and $I_0^0(I)$, respectively, as may be easily verified. For instance,

$$\langle \xi, I_1^0(I)(t)_\nu^\mu \eta \rangle = \int_0^t \langle \nabla_s \xi, \eta \rangle \, \mathrm{d}s = \langle \nabla_\mu^{\mu-1} \xi, \chi_{[0,t]} \otimes \eta \rangle = \langle \xi, \mathcal{S}_\nu^{\nu+1}(\chi_{[0,t]} \otimes \eta) \rangle,$$

so $I_1^0(I) = \mathcal{S}(\chi_{[0,t]} \otimes \cdot) = A^\dagger(\chi_{[0,t]})$, as claimed.

2. The inner-product identity (3.5) allows us to calculate the commutator of $A$ and $A^\dagger$ (recall that the *commutator* of two elements $x$ and $y$ of a ring is $[x, y] := xy - yx$);

$$\begin{aligned}
\langle \xi, \Big( I_0^1(I)(s) I_1^0(I)(t) \Big)_\nu^\mu \eta \rangle &= \langle I_1^0(I)(s)_\nu^{\mu+1} \xi, I_1^0(I)(t)_\nu^{\nu+1} \eta \rangle \\
&= \int_0^t \int_0^s \langle \nabla_p \xi, \nabla_r \eta \rangle \, \mathrm{d}r \, \mathrm{d}p + \int_0^{s \wedge t} \langle \xi, \eta \rangle \, \mathrm{d}p \\
&= \langle I_0^1(I)(t)_\mu^{\mu-1} \xi, I_0^1(I)(s)_\nu^{\nu-1} \eta \rangle + \langle \xi, T(s \wedge t)_\nu^\mu \eta \rangle
\end{aligned}$$

so $[A(s), A^\dagger(t)] = T(s \wedge t)$ for all $s, t \in [0,1]$. This shows that $A$ and $A^\dagger$ satisfy one of the canonical commutation relations (see [BrR]). The others ($[A(s), A(t)] = [A^\dagger(s), A^\dagger(t)] = 0$) may be deduced immediately from Corollary 3.4.5.

3. The identity (3.5) also yields the following:

$$\begin{aligned}
\langle \xi, [I_1^1(I) I_1^1(I)](t)_\nu^\mu \eta \rangle &= \langle I_1^1(I)(t)_\mu^\mu \xi, I_1^1(I)(t)_\nu^\nu \eta \rangle \\
&= \int_0^t \int_0^t \langle \nabla_s \nabla_r \xi, \nabla_r \nabla_s \eta \rangle \, \mathrm{d}r \, \mathrm{d}s + \int_0^t \langle \nabla_s \xi, \nabla_s \eta \rangle \, \mathrm{d}s \\
&= 2 \int_0^t \langle \nabla_s \xi, I_1^1(I)(s)_{\nu-1}^{\mu-1} \nabla_s \eta \rangle \, \mathrm{d}s + \langle \xi, I_1^1(I)(t)_\nu^\mu \eta \rangle
\end{aligned}$$

(it is easy to verify that $\nabla_r \nabla_s = \nabla_s \nabla_r$ in the appropriate sense) which implies that $I_1^1(I)^2 = I_1^1(2I_1^1(I) + I)$, or, using a suggestive differential notation we may write this as (cf. Poisson process)

$$\mathrm{d}(\Lambda^2) = (2\Lambda + I)\,\mathrm{d}\Lambda.$$

4. Let $B = I_0^1(P)$, where $P$ is the parity process. The estimates

$$\|B(t)^\mu_\nu\| = \|I_0^1(P)^\mu_\nu\| \leq \begin{cases} 0 & \mu \neq \nu - 1 \\ \nu^{\frac{1}{2}} t^{\frac{1}{2}} & \mu = \nu - 1 \end{cases}$$

show that $B(t)$ is super-diagonal for all $t$. In particular it is row-square-summable, so $\mathcal{D}(\hat{B}) \supseteq \mathfrak{H}_{00}$. We have also that $B^*(t)$ is sub-diagonal for all $t$ and $\widehat{B^*} = \hat{B}^*$ by Corollary 2.2.6, so $\mathcal{D}(\hat{B}^*) \supseteq \mathfrak{H}_{00}$. We shall see later that $B$ and $B^*$ give rise to bounded processes that satisfy the canonical anticommutation relations (see [BrR]). □

## 3.2 Adaptedness

We introduce the idea of adaptedness. Our approach is motivated by examining integrals of step processes against the four integrator processes. The conditions on the integrand for it to give a 'good' integral lead to a characterisation of adaptedness in terms of commutativity with the gradient, and this generalises the notion of adaptedness used by Hudson and Parthasarathy; we show the relationship between the H-P form of adaptedness and our version. We define the notion of martingale and show that three of the QS integrals of adapted processes are martingales.

### ∇-Adaptedness

Let us first introduce some notation. If $0 \leq s \leq t \leq 1$ then $\int_s^t \nabla_p \,\mathrm{d}p \in \mathfrak{D}_1$ is the super-diagonal matrix with entries given by

$$\Big(\int_s^t \nabla_p \,\mathrm{d}p\Big)^{\mu-1}_\mu : \xi \mapsto \int_s^t (\nabla^{\mu-1}_\mu \xi)(p)\,\mathrm{d}p.$$

The adjoint matrix is sub-diagonal, with entries

$$\Big(\Big(\int_s^t \nabla_p \,\mathrm{d}p\Big)^*\Big)^{\mu+1}_\mu : \xi \mapsto \mathcal{S}^{\mu+1}_\mu(\chi_{(s,t]} \otimes \xi).$$

This representation shows immediately that $\|(\int_s^t \nabla_p \,\mathrm{d}p)^{\mu-1}_\mu\| \leq (t-s)^{\frac{1}{2}}\sqrt{\mu}$. We seek to avoid a surfeit of indices by omitting them when the context makes them superfluous, e.g., if $\xi \in \mathfrak{h}_\mu$ then $\int_s^t \nabla_p \xi \, \mathrm{d}p = (\int_s^t \nabla_p \, \mathrm{d}p)_\mu^{\mu-1} \xi$. It is easy to verify that if $u \in L^2[0,1]$ then

$$(\int_s^t \nabla_p \, \mathrm{d}p) \varepsilon(u)^{\mu+1} = \int_s^t u(p) \, \mathrm{d}p \, \varepsilon(u)^\mu$$

and

$$(\int_s^t \nabla_p \, \mathrm{d}p)^* \varepsilon(u)^\mu = D_x \varepsilon(u + x\chi_{(s,t]})^{\mu+1} := \frac{\mathrm{d}}{\mathrm{d}x} \varepsilon(u + x\chi_{(s,t]})^{\mu+1} \Big|_{x=0}$$

for all $\mu \geq 0$.

**Definition 3.2.1** *A process $F \in \mathfrak{L}$ is* $\nabla$-adapted *(pronounced "gradient-adapted") or* adapted to $\nabla$, *if for all $s, t \in [0,1]$ such that $s \leq t$,*

$$\int_s^t \nabla_p \, \mathrm{d}p \, F(s) = F(s) \int_s^t \nabla_p \, \mathrm{d}p.$$

*Equivalently, for all $\mu, \nu \geq 0$, $s \in [0,1]$ and $\xi \in \mathfrak{h}_\nu$,*

$$\nabla_t F(s)_\nu^{\mu+1} \xi = F(s)_{\nu-1}^\mu \nabla_t \xi \quad (a.e.\ t \geq s).$$

*We say that $F$ is* adapted *if both $F$ and $F^*$ are $\nabla$-adapted.*

Note that if $F$ is $\nabla$-adapted then $F(r)$ commutes with $\int_s^t \nabla_p \, \mathrm{d}p$ for all $r \leq s$, as may be seen by writing $\int_s^t \nabla_p \, \mathrm{d}p$ as $\int_r^t \nabla_p \, \mathrm{d}p - \int_r^s \nabla_p \, \mathrm{d}p$.

It is immediate that $t \mapsto zI$ is adapted for any $z \in \mathbb{C}$. Addition, scalar multiplication and multiplication of processes preserve $\nabla$-adaptedness, but the involution does not; an example of a $\nabla$-adapted process with non-$\nabla$-adapted adjoint is given below.

**Proposition 3.2.2** *Let $F, G \in \mathfrak{L}$ be $\nabla$-adapted and let $z \in \mathbb{C}$. Then $F + zG$ is $\nabla$-adapted, as is $FG$ if the product exists. If $F \in \mathfrak{L}^p$ for $p = 2(2-\alpha-\beta)^{-1}$ (where $\alpha, \beta \in \{0,1\}$) then $I_\beta^\alpha(F)$ is $\nabla$-adapted.*

**Proof**

The fact that addition and scalar multiplication preserve $\nabla$-adaptedness is immediate. For the product, note that, as multiplication of bounded operators is separately continuous in the strong operator topology,

$$\begin{aligned}
\int_s^t \nabla_p \, \mathrm{d}p \sum_{\pi \geq 0} F(s)_\pi^{\mu+1} G(s)_\nu^\pi &= \sum_{\pi \geq 0} \int_s^t \nabla_p \, \mathrm{d}p \, F(s)_\pi^{\mu+1} G(s)_\nu^\pi \\
&= \sum_{\pi \geq 1} F(s)_{\pi-1}^\mu \int_s^t \nabla_p \, \mathrm{d}p \, G(s)_\nu^\pi \\
&= \sum_{\pi \geq 0} F(s)_\pi^\mu G(s)_{\nu-1}^\pi \int_s^t \nabla_p \, \mathrm{d}p
\end{aligned}$$

for all $\mu, \nu \geq 0$ and $t \geq s \in [0,1]$.

For preservation under integration, note that if $u, v \in L^2[0,1]$ then

$$\begin{aligned}
&\langle \varepsilon(u)^\mu, I^\alpha_\beta(F)(s)^\mu_{\nu-1} \int_s^t \nabla_p \, \mathrm{d}p \, \varepsilon(v)^\nu \rangle \\
= \;& \int_s^t v(p) \, \mathrm{d}p \int_0^s \langle \varepsilon(u)^{\mu-\beta}, F(r)^{\mu-\beta}_{\nu-1-\alpha} \varepsilon(v)^{\nu-1-\alpha} \rangle \overline{u}(r)^\beta v(r)^\alpha \, \mathrm{d}r \\
= \;& \int_0^s \langle \varepsilon(u)^{\mu-\beta}, F(r)^{\mu-\beta}_{\nu-1-\alpha} \int_s^t \nabla_p \, \mathrm{d}p \, \varepsilon(v)^{\nu-\alpha} \rangle \overline{u}(r)^\beta v(r)^\alpha \, \mathrm{d}r \\
= \;& \int_0^s \langle \varepsilon(u)^{\mu-\beta}, \int_s^t \nabla_p \, \mathrm{d}p \, F(r)^{\mu+1-\beta}_{\nu-\alpha} \varepsilon(v)^{\nu-\alpha} \rangle \overline{u}(r)^\beta v(r)^\alpha \, \mathrm{d}r \\
= \;& \int_0^s D_x \langle \varepsilon(u + x\chi_{(s,t]})^{\mu+1-\beta}, F(r)^{\mu+1-\beta}_{\nu-\alpha} \varepsilon(v)^{\nu-\alpha} \rangle \overline{u}(r)^\beta v(r)^\alpha \, \mathrm{d}r \\
= \;& D_x \langle \varepsilon(u + x\chi_{(s,t]})^{\mu+1}, I^\alpha_\beta(F)(s)^{\mu+1}_\nu \varepsilon(u)^\nu \rangle \\
= \;& \langle (\int_s^t \nabla_p \, \mathrm{d}p)^* \varepsilon(u)^\mu, I^\alpha_\beta(F)(s)^{\mu+1}_\nu \varepsilon(v)^\nu \rangle
\end{aligned}$$

and so $\nabla$-adaptedness holds.

□

We say that an element $[F] \in \mathcal{L}$ is $\nabla$-*adapted* if there exists a version of $F$ that is $\nabla$-adapted. If there is also some $G \in [F]$ such that $G^*$ is $\nabla$-adapted, then $[F]$ is *adapted*. That is, $[F]$ is adapted if both $[F]$ and $[F^*]$ are $\nabla$-adapted. Since $I^\alpha_\beta(F) = I^\alpha_\beta(G)$ if $[F] = [G]$, if $[F]$ is ($\nabla$-)adapted then $I^\alpha_\beta(F)$ is ($\nabla$-)adapted. The following proposition deals with the ($\nabla$-)adaptedness of weakly-continuous processes that have ($\nabla$-)adapted versions.

**Proposition 3.2.3** *Let $F \in \mathfrak{L}$ be a weakly continuous process, i.e., $t \mapsto F(t)^\mu_\nu$ is weakly continuous for all $\mu, \nu \geq 0$. If $[F]$ is $\nabla$-adapted then $F$ is $\nabla$-adapted. If $[F]$ is adapted then $F$ is adapted.*

**Proof**

Let $G$ be a $\nabla$-adapted version of $F$. Let $s \in [0,1)$, $t \in (s,1]$ and choose $(s_n)_{n\geq 1} \subseteq [0,t)$ such that $s_n \to s$ as $n \to \infty$ and $F(s_n) = G(s_n)$ for all $n \geq 1$. We have that

$$F(s_n) \int_{s_n}^t \nabla_p \, \mathrm{d}p = G(s_n) \int_{s_n}^t \nabla_p \, \mathrm{d}p = \int_{s_n}^t \nabla_p \, \mathrm{d}p \, G(s_n) = \int_{s_n}^t \nabla_p \, \mathrm{d}p \, F(s_n)$$

for all $n \geq 1$, using the $\nabla$-adaptedness of $G$. The pointwise product of a uniformly continuous function and a weakly continuous function is weakly continuous (which may be seen from the uniform boundedness theorem and the estimate

$$|\langle x, [f(t)g(t) - f(s)g(s)]y \rangle| \leq |\langle f(t)^* x, [g(t) - g(s)]y \rangle| + \|x\| \, \|f(t) - f(s)\| \, \|g(s)y\|$$

with suitable $x$, $y$, $f$ and $g$), so we may take limits in the above and conclude that $F$ is $\nabla$-adapted.

If $F$ has an adapted version, then $F$ is $\nabla$-adapted, by the first part. Since $F$ is weakly continuous, $F^*$ is weakly continuous, and has a $\nabla$-adapted version, so $F^*$ is $\nabla$-adapted, again by the first part. Thus $F$ is adapted, as claimed.

□

**Proposition 3.2.4** *Let $([F_n])_{n\geq 1} \subseteq \mathcal{L}^p$ be convergent to $[F] \in \mathcal{L}^p$, where $p \in [1,\infty]$. If $[F_n]$ is $\nabla$-adapted for all $n \geq 1$ then so is $[F]$.*

**Proof**

For any $\mu, \nu \geq 0$ we have that $[F_n]_\nu^{\mu+1} \to [F]_\nu^{\mu+1}$ in $L^p([0,1]; \mathcal{B}(\mathfrak{h}_\nu; \mathfrak{h}_{\mu+1})_s)$ and $[F_n]_{\nu-1}^\mu \to [F]_{\nu-1}^\mu$ in a similar fashion, so (passing to a subsequence if necessary) we may assume that, for all $\mu, \nu \geq 0$, $F_n(s)_\nu^{\mu+1} \to F(s)_\nu^{\mu+1}$ and $F_n(s)_{\nu-1}^\mu \to F(s)_{\nu-1}^\mu$ for almost all $s \in [0,1]$. Since multiplication of bounded operators is continuous, we see that

$$\begin{aligned} F(s)_{\nu-1}^\mu \int_s^t \nabla_p \,\mathrm{d}p &= \lim_{n\to\infty} F_n(s)_{\nu-1}^\mu \int_s^t \nabla_p \,\mathrm{d}p \\ &= \lim_{n\to\infty} \int_s^t \nabla_p \,\mathrm{d}p \, F_n(s)_\nu^{\mu+1} \\ &= \int_s^t \nabla_p \,\mathrm{d}p \, F(s)_\nu^{\mu+1} \end{aligned}$$

for all $t \geq s$ and for almost all $s$. Setting $F(s) = 0$ on the exceptional set gives the required $\nabla$-adapted version of $F$.

□

Since the involution is continuous on $\mathcal{L}^p$, if we strengthen the $\nabla$-adapted hypothesis in the statement of the theorem above and require adaptedness, then we may strengthen the conclusion in the same manner. In particular, this remark allows us to conclude that the collection of $p$-integrable, adapted processes form a closed subspace of $\mathcal{L}^p$, for all $p \in [1,\infty]$. We find below an explicit form for the orthogonal projection (conditional expectation) onto these subspaces.

**Step Integrals**

To motivate the definition of $\nabla$-adaptedness, we consider step integrals with respect to the processes $A$, $A^\dagger$, $T$ and $\Lambda$. Given $F \in \mathfrak{L}^{\mathrm{step}}$ we define $\int_0^1 F \,\mathrm{d}A$ by

$$(\int_0^1 F \,\mathrm{d}A)_\nu^\mu := \sum_{\pi=0}^{n-1} F(t_\pi)_{\nu-1}^\mu A(t_\pi, t_{\pi+1}]_\nu^{\nu-1}$$

where $F$ is assumed to be *locally subordinate* to the partition $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$, i.e., $t \mapsto F(t)^{\mu+\alpha}_{\nu+\beta}$ is subordinate to this partition for all $\alpha, \beta \in \{0, \pm 1\}$, and $A(t_\pi, t_{\pi+1}] := A(\chi_{(t_\pi, t_{\pi+1}]})$. For $\xi \in \mathfrak{h}_\mu$ and $\eta \in \mathfrak{h}_\nu$,

$$\begin{aligned}
\langle \xi, (\int_0^1 F \,\mathrm{d}A)^\mu_\nu \eta\rangle &= \sum_{\pi=0}^{n-1} \langle \xi, F(t_\pi)^\mu_{\nu-1} A(t_\pi, t_{\pi+1}]^{\nu-1}_\nu \eta\rangle \\
&= \sum_{\pi=0}^{n-1} \langle \mathcal{S}^\nu_{\nu-1}(\chi_{(t_\pi, t_{\pi+1}]} \otimes F^*(t_\pi)^{\nu-1}_\mu \xi), \eta\rangle \\
&= \sum_{\pi=0}^{n-1} \int_{t_\pi}^{t_{\pi+1}} \langle \xi, F(t_\pi)^\mu_{\nu-1} \nabla_s \eta\rangle \,\mathrm{d}s \\
&= \langle \xi, I^1_0(F)(1)^\mu_\nu \eta\rangle
\end{aligned}$$

so $\int_0^1 F \,\mathrm{d}A = I^1_0(F)(1)$. Taking adjoints we see that $I^0_1(F)(1) = \int_0^1 \mathrm{d}A^\dagger F$ for any $F \in \mathfrak{L}^{\mathrm{step}}$, where

$$(\int_0^1 \mathrm{d}A^\dagger F)^\mu_\nu := \sum_{\pi=0}^{n-1} A^\dagger(t_\pi, t_{\pi+1}]^\mu_{\mu-1} F(t_\pi)^{\mu-1}_\nu.$$

In the same manner it may be shown that $I^0_0(F)(1) = \int_0^1 F \,\mathrm{d}T = \int_0^1 \mathrm{d}T\, F$, where

$$(\int_0^1 F \,\mathrm{d}T)^\mu_\nu := \sum_{\pi=0}^{n-1} F(t_\pi)^\mu_\nu (T(t_{\pi+1})^\nu_\nu - T(t_\pi)^\nu_\nu)$$

and

$$(\int_0^1 \mathrm{d}T\, F)^\mu_\nu := \sum_{\pi=0}^{n-1} (T(t_{\pi+1})^\mu_\mu - T(t_\pi)^\mu_\mu) F(t_\pi)^\mu_\nu.$$

In order to discuss integrals with respect to the remaining process, and to look at integrals where the integrators occur on the 'wrong side', we need $\nabla$-adaptedness.

**Proposition 3.2.5** *Let $F \in \mathfrak{L}^{\mathrm{step}}$ be $\nabla$-adapted. Then*

$$\int_0^1 F \,\mathrm{d}A = \int_0^1 \mathrm{d}A\, F \text{ and } \int_0^1 \mathrm{d}A^\dagger F = \int_0^1 F \,\mathrm{d}A^\dagger,$$

*where $(\int_0^1 \mathrm{d}A\, F)^\mu_\nu = \sum_\pi A(t_\pi, t_{\pi+1}]^\mu_{\mu+1} F(t_\pi)^{\mu+1}_\nu$, etc. Furthermore*

$$\int_0^1 \mathrm{d}\Lambda\, F = I^1_1(F)(1) = \int_0^1 F \,\mathrm{d}\Lambda,$$

*where*

$$(\int_0^1 F \,\mathrm{d}\Lambda)^\mu_\nu := \sum_{\pi=0}^{n-1} F(t_\pi)^\mu_\nu \Lambda(t_\pi, t_{\pi+1}]^\nu_\nu$$

*and*

$$(\int_0^1 \mathrm{d}\Lambda\, F)^\mu_\nu := \sum_{\pi=0}^{n-1} \Lambda(t_\pi, t_{\pi+1}]^\mu_\mu F(t_\pi)^\mu_\nu.$$

**Proof**

In both statements the second equality follows from the first by taking adjoints. Now, assuming $F$ is locally subordinate to the partition $\{0 = t_0 < \cdots < t_n = 1\}$, if $\xi \in \mathfrak{h}_\mu$, $\eta \in \mathfrak{h}_\nu$ and $F \in \mathfrak{L}^{\text{step}}$ is $\nabla$-adapted, then

$$\begin{aligned}
\langle \xi, (\int_0^1 F\,\mathrm{d}A)^\mu_\nu \eta\rangle &= \sum_{\pi=0}^{n-1} \int_{t_\pi}^{t_{\pi+1}} \langle \xi, F(t_\pi)^\mu_{\nu-1} \nabla_s \eta\rangle\,\mathrm{d}s \\
&= \sum_{\pi=0}^{n-1} \int_{t_\pi}^{t_{\pi+1}} \langle \xi, \nabla_s F(t_\pi)^{\mu+1}_\nu \eta\rangle\,\mathrm{d}s \\
&= \sum_{\pi=0}^{n-1} \langle \xi, A(t_\pi, t_{\pi+1}]^\mu_{\mu+1} F(t_\pi)^{\mu+1}_\nu \eta\rangle \\
&= \langle \xi, (\int_0^1 \mathrm{d}A\, F)^\mu_\nu \eta\rangle.
\end{aligned}$$

For $\Lambda$, we have that

$$\begin{aligned}
\langle \xi, I^1_1(F)(1)^\mu_\nu \eta\rangle &= \sum_{\pi=0}^{n-1} \int_{t_\pi}^{t_{\pi+1}} \langle \nabla_s \xi, F(t_\pi)^{\mu-1}_{\nu-1} \nabla_s \eta\rangle\,\mathrm{d}s \\
&= \sum_{\pi=0}^{n-1} \int_{t_\pi}^{t_{\pi+1}} \langle \nabla_s \xi, \nabla_s F(t_\pi)^\mu_\nu \eta\rangle\,\mathrm{d}s \\
&= \sum_{\pi=0}^{n-1} \int_{[0,1]^\mu} \sum_{k=1}^{\mu} \chi_{(t_\pi, t_{\pi+1}]}(x_k) \overline{\xi}(\mathbf{x}) (F(t_\pi)^\mu_\nu \eta)(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= \sum_{\pi=0}^{n-1} \langle \xi, s_\mu(m_{\chi_{(t_\pi, t_{\pi+1}]}}) F(t_\pi)^\mu_\nu \eta\rangle
\end{aligned}$$

which shows that $\int_0^1 \mathrm{d}\Lambda\, F = I^1_1(F)(1)$, as claimed.

□

**HP-Adaptedness**

Recall [HuP, Section 3] that Hudson and Parthasarathy deal with processes of operators in $\mathfrak{H}$ that admit the exponential vectors as a common domain. They say that such a process $F'$ is adapted if, for all $t \in [0, 1]$, $F'(t)$ is the ampliation by the identity of a linear operator defined on the exponential vectors in $\mathfrak{H}_{t]} = \mathfrak{F}_+(L^2[0,t])$. That is, for all $u \in L^2[0,1]$

$$\begin{cases} F'(t)\varepsilon(u_{t]}) \in \mathfrak{H}_{t]} \\ F'(t)\varepsilon(u) \text{ ``=''} \ [F'(t)\varepsilon(u_{t]})] \otimes \varepsilon(u_{(t}) \end{cases},$$

where we use the isomorphism $j_t : \mathfrak{H} = \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}$ (see [HuP, Definition 3.1, p. 305]). With this in mind, we make the following definition.

**Definition 3.2.6** *An HP-process $F \in \mathfrak{L}$ is* adapted in the sense of Hudson and Parthasarathy*, or* HP-adapted*, if, for all $u, v \in L^2[0,1]$ and all $t \in [0,1]$*

$$\langle \varepsilon(u), \hat{F}(t)\varepsilon(v)\rangle \quad = \quad \langle \varepsilon(u_{t]}), \hat{F}(t)\varepsilon(v_{t]})\rangle \langle \varepsilon(u_{(t}), \varepsilon(v_{(t})\rangle. \tag{3.7}$$

This identity allows us to investigate the behaviour of $F$ at the coordinate level.

**Lemma 3.2.7** *Let $F \in \mathfrak{L}$ give rise to an HP-process that is HP-adapted. Then*

$$\langle \varepsilon(u)^\mu, F(t)^\mu_\nu \varepsilon(v)^\nu \rangle = \sum_{\pi=0}^{\mu\wedge\nu} \langle \varepsilon(u_{t]})^{\mu-\pi}, F(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\rangle \langle \varepsilon(u_{(t})^\pi, \varepsilon(v_{(t})^\pi\rangle \tag{3.8}$$

*for all $\mu, \nu \geq 0$, $u, v \in L^2[0,1]$ and $t \in [0,1]$.*

**Proof**

Note that for all $z, w \in \mathbb{C}$,

$$P(z,w) := \langle \varepsilon(\overline{z}\, u), \hat{F}(t)\varepsilon(w\, v)\rangle = \sum_{\lambda\geq 0}\sum_{\pi \geq 0} z^\lambda w^\pi \langle \varepsilon(u)^\lambda, F(t)^\lambda_\pi \varepsilon(v)^\pi\rangle$$

is convergent, so we may differentiate to find that

$$(\nu!)^{-1}\partial_w^\nu (\mu!)^{-1}\partial_z^\mu P(0,0) = \langle \varepsilon(u)^\mu, F(t)^\mu_\nu \varepsilon(v)^\nu\rangle$$

(where $\partial_z^m = \frac{\partial^m}{\partial z^m}$ and $\partial_z^0 = 1$, etc.). However, as $\hat{F}$ is HP-adapted, by (3.7)

$$\begin{aligned} P(z,w) &= \langle \varepsilon(\overline{z}\, u_{t]}), \hat{F}(t)\varepsilon(w\, v_{t]})\rangle \langle \varepsilon(\overline{z}\, u_{(t}), \varepsilon(w\, v_{(t})\rangle \\ &= \sum_{\lambda\geq 0}\sum_{\pi\geq 0} z^\lambda w^\pi \langle \varepsilon(u_{t]})^\lambda, F(t)^\lambda_\pi \varepsilon(v_{t]})^\pi\rangle \sum_{\rho\geq 0}(zw)^\rho \langle \varepsilon(u_{(t})^\rho, \varepsilon(v_{(t})^\rho\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} &\langle \varepsilon(u)^\mu, F(t)^\mu_\nu \varepsilon(v)^\nu\rangle \\ &= \ (\nu!)^{-1}\partial_w^\nu \sum_{\kappa=0}^{\mu}\sum_{\pi\geq 0} w^\pi \langle \varepsilon(u_{t]})^{\mu-\kappa}, F(t)^{\mu-\kappa}_\pi \varepsilon(v_{t]})^\pi\rangle w^\kappa \langle \varepsilon(u_{(t})^\kappa, \varepsilon(v_{(t})^\kappa\rangle\big|_{w=0} \end{aligned}$$

by Leibniz' rule, and this expression is equal to

$$\sum_{\kappa=0}^{\mu\wedge\nu}\langle\varepsilon(u_{t]})^{\mu-\kappa}, F(t)_{\nu-\kappa}^{\mu-\kappa}\varepsilon(v_{t]})^{\nu-\kappa}\rangle\langle\varepsilon(u_{(t})^{\kappa}, \varepsilon(v_{(t})^{\kappa}\rangle,$$

as claimed.

□

Taking $F \equiv I$ in Lemma 3.2.7 gives the formula

$$\langle\varepsilon(u)^{\mu}, \varepsilon(v)^{\mu}\rangle = \sum_{\pi=0}^{\mu}\langle\varepsilon(u)^{\mu-\pi}, \varepsilon(v)^{\mu-\pi}\rangle\,\langle\varepsilon(u)^{\pi}, \varepsilon(v)^{\pi}\rangle \quad \forall\, u, v \in L^2[0,1], \mu \geq 0$$

which is simple to verify by hand. A more important consequence of the above is the following.

**Corollary 3.2.8** *If $F \in \mathfrak{L}$ is an HP-adapted, HP-process then $F$ is adapted.*

**Proof**

If $u, v \in L^2[0,1]$, $\mu, \nu \geq 0$ and $t \in [0,1]$ then we know that

$$\langle\varepsilon(u)^{\mu}, \int_s^t \nabla_p \,\mathrm{d}p\, F(s)_{\nu}^{\mu+1}\varepsilon(v)^{\nu}\rangle = D_x\langle\varepsilon(u + x\chi_{(s,t]})^{\mu+1}, F(s)_{\nu}^{\mu+1}\varepsilon(v)^{\nu}\rangle$$

and by the identity (3.8), this is equal to

$$\begin{aligned} & D_x \sum_{\pi=0}^{(\mu+1)\wedge\nu}\langle\varepsilon(u_{s]})^{\mu+1-\pi}, F(s)_{\nu-\pi}^{\mu+1-\pi}\varepsilon(v_{s]})^{\nu-\pi}\rangle\langle\varepsilon(u_{(s} + x\chi_{(s,t]})^{\pi}, \varepsilon(v_{(s})^{\pi}\rangle \\ = & \int_s^t v(p)\,\mathrm{d}p \sum_{\pi=0}^{\mu\wedge(\nu-1)}\langle\varepsilon(u_{s]})^{\mu-\pi}, F(s)_{\nu-1-\pi}^{\mu-\pi}\varepsilon(v_{s]})^{\nu-1-\pi}\rangle\langle\varepsilon(u_{(s})^{\pi}, \varepsilon(v_{(s})^{\pi}\rangle \\ = & \int_s^t v(p)\,\mathrm{d}p\,\langle\varepsilon(u)^{\mu}, F(s)_{\nu-1}^{\mu}\varepsilon(v)^{\nu}\rangle \\ = & \langle\varepsilon(u)^{\mu}, F(s)_{\nu-1}^{\mu}\int_s^t \nabla_p\,\mathrm{d}p\,\varepsilon(v)^{\nu}\rangle \end{aligned}$$

and so $F$ is $\nabla$-adapted. The same manner of working gives the $\nabla$-adaptedness of $F^*$.

□

Thus HP-adaptedness implies adaptedness. To prove the reverse implication (and so their equivalence for HP-processes) we examine the projection map onto the subspace of adapted processes.

**The Adapted Projection**

For $\mu \geq 0$, $\pi = 0, \ldots, \mu$ and $t \in [0,1]$ let us define $P^{0,0} = \mathrm{id}_{\mathbb{C}}$ and if $\mu > 0$

$$\begin{aligned} P_t^{\mu,\pi} \quad &: \quad \mathfrak{h}_\mu \to (\mathfrak{h}_{t]})_{\mu-\pi} \otimes (\mathfrak{h}_{(t})_\pi \subseteq \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}; \\ &\quad \theta \mapsto \binom{\mu}{\pi}^{\frac{1}{2}} \underbrace{\mathbb{E}_{t]} \otimes \cdots \otimes \mathbb{E}_{t]}}_{(\mu-\pi)} \otimes \underbrace{\mathbb{E}_{(t} \otimes \cdots \otimes \mathbb{E}_{(t}}_{(\pi)} \iota(\theta), \end{aligned}$$

where $\iota$ is the natural embedding from $\mathfrak{h}_\mu$ into $\mathfrak{h}^{\otimes \mu}$ and $\mathbb{E}_{(t} = I - \mathbb{E}_{t]}$ is the orthogonal projection onto $L^2(t,1] \subseteq L^2[0,1]$. The constant is chosen with the following identity in mind:

$$P_t^{\mu,\pi} \varepsilon(u)^\mu = \varepsilon(u_{t]})^{\mu-\pi} \otimes \varepsilon(u_{(t})^\pi.$$

It is clear that $P_t^{\mu,\pi} \in \mathcal{B}(\mathfrak{h}_\mu; \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t})$ with norm at most $\binom{\mu}{\pi}^{\frac{1}{2}}$ and easy to see that $\mathrm{Im}\, P_t^{\mu,\pi} \perp \mathrm{Im}\, P_t^{\mu,\lambda}$ if $\lambda \neq \pi$. In particular,

$$\begin{aligned} \langle \sum_{\pi=0}^{\mu} P_t^{\mu,\pi} \varepsilon(u)^\mu, \sum_{\lambda=0}^{\mu} P_t^{\mu,\lambda} \varepsilon(v)^\mu \rangle &= \sum_{\pi=0}^{\mu} \langle P_t^{\mu,\pi} \varepsilon(u)^\mu, P_t^{\mu,\pi} \varepsilon(v)^\mu \rangle \\ &= \sum_{\pi=0}^{\mu} \langle \varepsilon(u_{t]})^{\mu-\pi}, \varepsilon(v_{t]})^{\mu-\pi} \rangle \langle \varepsilon(u_{(t})^\pi, \varepsilon(v_{(t})^\pi \rangle \\ &= \langle \varepsilon(u)^\mu, \varepsilon(v)^\mu \rangle, \end{aligned}$$

so $\sum_{\pi=0}^{\mu} P_t^{\mu,\pi}$ is an isometry. Furthermore if $j_t : \mathfrak{H} \to \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}$ is the isometric isomorphism that acts on exponential vectors as $\varepsilon(u) \mapsto \varepsilon(u_{t]}) \otimes \varepsilon(u_{(t})$ (see the remark after Proposition 2.1.2) then

$$\begin{aligned} j_t(\varepsilon(u)^\mu) &= \frac{1}{\mu!} \frac{\mathrm{d}^\mu}{\mathrm{d}z^\mu} j_t(\varepsilon(z\,u)) \Big|_{z=0} \\ &= \frac{1}{\mu!} \frac{\mathrm{d}^\mu}{\mathrm{d}z^\mu} \sum_{\lambda \geq 0} z^\mu \varepsilon(u_{t]})^\lambda \otimes \sum_{\nu \geq 0} z^\nu \varepsilon(u_{(t})^\nu \Big|_{z=0} \\ &= \sum_{\pi=0}^{\mu} \varepsilon(u_{t]})^{\mu-\pi} \otimes \varepsilon(u_{(t})^\pi, \end{aligned}$$

by Leibniz' rule. Hence $\sum_{\pi=0}^{\mu} P_t^{\mu,\pi} = j_t|_{\mathfrak{h}_\mu}$ and $j_t = \sum_{\mu \geq 0} \sum_{\pi=0}^{\mu} P_t^{\mu,\pi} E_\mu$, where the series is strongly convergent.

**Definition 3.2.9** *For $F \in \mathfrak{L}$ the* adapted projection *of $F$ is denoted $\breve{F}$ and defined by*

$$\breve{F}(t)_\nu^\mu = \sum_{\pi=0}^{\mu \wedge \nu} (P_t^{\mu,\pi})^* (F(t)_{\nu-\pi}^{\mu-\pi} \otimes \mathrm{id}_{\mathfrak{h}_\pi}) P_t^{\nu,\pi}.$$

*The mapping $F \mapsto \breve{F}$ is also referred to as the adapted projection.*

We see that

$$\begin{aligned}
|\langle \phi^\mu, \breve{F}(t)^\mu_\nu \psi^\nu\rangle| &= |\sum_{\pi=0}^{\mu\wedge\nu} \langle P_t^{\mu,\pi}\phi^\mu, (F(t)^{\mu-\pi}_{\nu-\pi} \otimes \mathrm{id}_{\mathfrak{h}_\pi}) P_t^{\nu,\pi}\psi^\nu\rangle| \\
&\le (\sum_{\pi=0}^{\mu} \|P_t^{\mu,\pi}\phi^\mu\|^2)^{\frac{1}{2}} (\sum_{\pi=0}^{\nu} \|F(t)^{\mu-\pi}_{\nu-\pi}\|^2 \|P_t^{\nu,\pi}\psi^\nu\|^2)^{\frac{1}{2}} \\
&\le \|\phi^\mu\| \max\{\|F(t)^{\mu-\pi}_{\nu-\pi}\| \,:\, \pi = 0,\ldots,\mu\wedge\nu\} \|\psi^\nu\|
\end{aligned}$$

by the Cauchy-Schwarz-Buniakowski inequality and orthogonality. Thus $\breve{F}(t)^\mu_\nu$ is a bounded linear operator with norm

$$\|\breve{F}(t)^\mu_\nu\| \le \max_{\pi=0,\ldots,\mu\wedge\nu} \{\|F(t)^{\mu-\pi}_{\nu-\pi}\|\}.$$

Measurability is clear from the equality (cf. (3.8))

$$\langle \varepsilon(u)^\mu, \breve{F}(t)^\mu_\nu \varepsilon(v)^\nu\rangle = \sum_{\pi=0}^{\mu\wedge\nu} \langle \varepsilon(u_{t]})^{\mu-\pi}, F(t)^{\mu-\pi}_{\nu-\pi} \varepsilon(v_{t]})^{\nu-\pi}\rangle \langle \varepsilon(u_{(t})^\pi, \varepsilon(v_{(t})^\pi\rangle \tag{3.9}$$

and the measurability of $\mathbb{E}$ and $F$, by Lemma A.1.3, so $\breve{F} \in \mathfrak{L}$. This equality also shows that $[\breve{F}] = [\breve{G}]$ if $[F] = [G]$, so $\breve{}$ is a mapping from $\mathcal{L}$ to itself. Furthermore, the above shows that if $F \in \mathfrak{L}^p$ then so is $\breve{F}$. In fact, suppose $F \in \mathfrak{L}^p_{\mathfrak{R}}$, i.e., there exists a non-negative, $p$-integrable function $C$ and $a, b > 0$ such that, for all $\mu, \nu \ge 0$

$$\|F(t)^\mu_\nu\| \le C(t)((\mu \vee \nu) + 1)^a |\mu - \nu|!^{-b}$$

for all $t \in [0,1]$. Then

$$\begin{aligned}
\|\breve{F}(t)^\mu_\nu\| &\le \max_{\pi=0,\ldots,\mu\wedge\nu} \{\|F(t)^{\mu-\pi}_{\nu-\pi}\|\} \\
&\le \max_{\pi=0,\ldots,\mu\wedge\nu} \{C(t)((\mu-\pi)\vee(\mu-\pi)+1)^a |\mu-\pi-\nu+\pi|!^{-b}\} \\
&= C(t) \max_{\pi=0,\ldots,\mu\wedge\nu} \{((\mu\vee\nu)-\pi+1)^a\} |\mu-\nu|!^{-b} \\
&= C(t)((\mu\vee\nu)+1)^a |\mu-\nu|!^{-b},
\end{aligned}$$

that is, $\breve{F} \in \mathfrak{L}^p_{\mathfrak{R}}$. To see that the mapping $F \mapsto \breve{F}$ is a projection, note that (3.9) implies that

$$\langle \varepsilon(u_{t]})^\mu, \breve{F}(t)^\mu_\nu \varepsilon(v_{t]})^\nu\rangle = \langle \varepsilon(u_{t]})^\mu, F(t)^\mu_\nu \varepsilon(v_{t]})^\nu\rangle,$$

so $(\breve{F})^{\breve{}} = \breve{F}$, and also we have that $(\breve{F})^* = (F^*)^{\breve{}}$, as

$$(\breve{F}(t)^\mu_\nu)^* = \sum_{\pi=0}^{\mu\wedge\nu} (P_t^{\nu,\pi})^* [(F(t)^{\mu-\pi}_{\nu-\pi})^* \otimes \mathrm{id}_{\mathfrak{h}_\pi}] P_t^{\mu,\pi} = (F^*)^{\breve{}}(t)^\nu_\mu.$$

Most importantly, $\breve{F}$ is adapted; this follows from the identity (3.9) and the proof of Corollary 3.2.8.

There is a link between the orthogonal projection $\mathbb{E}_{s]}$ and the operators $\int_s^t \nabla_p \, \mathrm{d}p$ $(t \geq s)$, as the following lemma makes clear. This link is crucial to our proof of the equivalence between our form of adaptedness and the definition of Hudson and Parthasarathy. For an occurrence of this lemma in the classical stochastic calculus see [Mal, Lemma VII.2.1].

**Lemma 3.2.10** *If $\mu \geq 0$, $\xi \in \mathfrak{h}_\mu$ and $s \in [0,1]$ then $(\mathbb{E}^\mu_{s]})^\perp \xi = 0$ if and only if $\int_s^t \nabla_p \, \mathrm{d}p \, \xi = 0$ for all $t \in (s,1]$.*

**Proof**

We wish to show that

$$\ker (\mathbb{E}^\mu_{s]})^\perp = \bigcap_{s<t\leq 1} \ker \int_s^t \nabla_p \, \mathrm{d}p.$$

The proposition is trivial if $\mu = 0$, so suppose $\mu = i \geq 1$. Note first that

$$\xi \in \ker (\mathbb{E}^i_{s]})^\perp \Leftrightarrow (I - \mathbb{E}^i_{s]})\xi = 0 \Leftrightarrow \xi = \mathbb{E}^i_{s]}\xi \Leftrightarrow \xi \in \operatorname{Im} \mathbb{E}^i_{s]},$$

and $\operatorname{lin} \{u^{\otimes i} : u \in L^2[0,1]\}$ is dense in $\mathfrak{h}_i$, so $\operatorname{lin} \{u^{\otimes i}_{s]} : u \in \mathfrak{h}_1\} = \mathbb{E}^i_{s]} \operatorname{lin} \{u^{\otimes i}\}$ is dense in $\operatorname{Im} \mathbb{E}^i_{s]}$, being the image of a dense set under a continuous surjection. It is easy to see that if $\xi \in \operatorname{lin} \{u^{\otimes i}_{s]}\}$ then $\int_s^t \nabla_p \, \mathrm{d}p \, \xi = 0$ for $t > s$, so

$$\ker (\mathbb{E}^i_{s]})^\perp \subseteq \bigcap_{s<t\leq 1} \ker \int_s^t \nabla_p \, \mathrm{d}p.$$

Let $\xi$ be in $\ker \int_s^t \nabla_p \, \mathrm{d}p$ for all $t > s$, and suppose $\xi$ is in $(\ker(\mathbb{E}^i_{s]})^\perp)^\perp = (\operatorname{Im} \mathbb{E}^i_{s]})^\perp$, so that $\langle \xi, \mathbb{E}^i_{s]} \psi \rangle = 0$ for all $\psi \in \mathfrak{h}_i$. If we can prove that $\xi = 0$, i.e, $\|\xi\| = 0$, we are done. We may write

$$\begin{aligned}
\|\xi\|^2 &= \int_{[0,1]^i} |\xi(t_1, \ldots, t_i)|^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_i \\
&= \int_{[0,s]^i} |\xi(\mathbf{t})|^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_i \\
&\quad + \sum_{k=1}^i \binom{i}{k} \int_0^s \cdots \int_0^s \int_s^1 \cdots \int_s^1 |\xi(\mathbf{t})|^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_{i-k} \, \mathrm{d}t_{i-k+1} \, \mathrm{d}t_i,
\end{aligned}$$

and the first integral equals $\|\mathbb{E}^i_{s]}\xi\|^2 = \langle \xi, \mathbb{E}^i_{s]}\xi \rangle$, so it vanishes. For almost any $(i-1)$-tuple $(t_1, \ldots, t_{i-1}) \in [0,1]^{i-1}$ we have that

$$0 = (\int_s^t \nabla_p \, \mathrm{d}p \, \xi)(t_1, \ldots, t_{i-1}) = \sqrt{i} \int_s^t \xi(t_1, \ldots, t_{i-1}, p) \, \mathrm{d}p$$

for all $t \in (s,1]$, so $\int_A \xi(t_1,\ldots,t_{i-1},p)\,\mathrm{d}p = 0$ for all measurable $A \subseteq (s,1]$. Thus $\xi(t_1,\ldots,t_{i-1},p) = 0$ for almost all $p \in (s,1]$, and so $\int_s^1 |\xi(t_1,\ldots,t_{i-1},p)|^2\,\mathrm{d}p = 0$. Hence $\|\xi\|^2 = 0$, which is what we wished to show.

□

**Proposition 3.2.11** *If $F \in \mathfrak{L}$ then $\breve{F}\mathbb{E} = \mathbb{E}F\mathbb{E} = \mathbb{E}\breve{F}$, i.e.,*

$$\breve{F}(t)^\mu_\nu \mathbb{E}^\nu_{t]} = \mathbb{E}^\mu_{t]} F(t)^\mu_\nu \mathbb{E}^\nu_{t]} = \mathbb{E}^\mu_{t]} \breve{F}(t)^\mu_\nu$$

*for all $\mu,\nu \geq 0$ and all $t \in [0,1]$. If $F$ is $\nabla$-adapted then $\mathbb{E}F\mathbb{E} = F\mathbb{E}$ and so $\breve{F}\mathbb{E} = F\mathbb{E}$. If $F$ is adapted, then $\breve{F}\mathbb{E} = F\mathbb{E} = \mathbb{E}F = \mathbb{E}\breve{F}$.*

**Proof**

The first statement follows from (3.9), which implies that

$$\langle \varepsilon(u)^\mu, \breve{F}(t)^\mu_\nu \varepsilon(v_{t]})^\nu \rangle = \langle \varepsilon(u_{t]})^\mu, F(t)^\mu_\nu \varepsilon(v_{t]})^\nu \rangle = \langle \varepsilon(u_{t]})^\mu, \breve{F}(t)^\mu_\nu \varepsilon(v)^\nu \rangle.$$

If $F$ is $\nabla$-adapted, let $s \leq t$;

$$\int_s^t \nabla_p \,\mathrm{d}p\, F(s)^\mu_\nu \varepsilon(u_{s]})^\nu = F(s)^{\mu-1}_{\nu-1} \int_s^t \nabla_p \,\mathrm{d}p\, \varepsilon(u_{s]})^\nu = 0,$$

so by Lemma 3.2.10 $(\mathbb{E}^\mu_{s]})^\perp F(s)^\mu_\nu \mathbb{E}^\nu_{s]} = 0$, i.e., $\mathbb{E}^\mu_{s]} F(s)^\mu_\nu \mathbb{E}^j_{s]} = F(s)^\mu_\nu \mathbb{E}^\nu_{s]}$. If $F^*$ is also $\nabla$-adapted then, because $\mathbb{E} \in \mathfrak{L}_{\mathfrak{D}}$,

$$F\mathbb{E} = (\mathbb{E}F\mathbb{E})^{**} = (\mathbb{E}F^*\mathbb{E})^* = (F^*\mathbb{E})^* = \mathbb{E}F.$$

□

**Corollary 3.2.12** *If $F \in \mathfrak{L}$ is such that $F$ is adapted then $F = \breve{F}$.*

**Proof**

We wish to show that $F(t)^\mu_\nu = \breve{F}(t)^\mu_\nu$ for all $\mu,\nu \geq 0$ and $t \in [0,1]$: we proceed by induction on $\pi = \max\{\mu,\nu\}$. For $\pi = 0$ the proposition follows immediately from the definition of $\breve{F}$.

From the above we know that $F = (\mathbb{E} + \mathbb{E}^\perp)F = \mathbb{E}\breve{F} + \mathbb{E}^\perp F$, so it suffices to show that $(\mathbb{E}^\mu_{t]})^\perp F(t)^\mu_\nu = (\mathbb{E}^\mu_{t]})^\perp \breve{F}(t)^\mu_\nu$ for all $t \in [0,1]$. Now if $\alpha,\beta \in \{0,1\}$ and $s \leq t$ then

$$\int_s^t \nabla_p \,\mathrm{d}p\, F(s)^{\mu+\alpha}_{\nu+\beta} = F(s)^{\mu+\alpha-1}_{\nu+\beta-1} \int_s^t \nabla_p \,\mathrm{d}p = \breve{F}(s)^{\mu+\alpha-1}_{\nu+\beta-1} \int_s^t \nabla_p \,\mathrm{d}p = \int_s^t \nabla_p \,\mathrm{d}p\, \breve{F}(s)^{\mu+\alpha}_{\nu+\beta}$$

and Lemma 3.2.10 gives the result.

□

We have amassed sufficient information to prove the equivalence of adaptedness and HP-adaptedness for HP-processes.

**Proposition 3.2.13** *If $F \in \mathfrak{L}$ is an HP-process then $\breve{F}$ is an HP-process, and*

$$\langle \varepsilon(u), \widehat{\breve{F}}(t)\varepsilon(v)\rangle = \langle \varepsilon(u_{t]}), \widehat{\breve{F}}(t)\varepsilon(v_{t]})\rangle\langle \varepsilon(u_{(t}), \varepsilon(v_{(t})\rangle$$

*for all $t \in [0,1]$, i.e., $\breve{F}$ is HP-adapted.*

**Proof**

We know that $\mathbb{E}F\mathbb{E} = \breve{F}\mathbb{E}$, by Proposition 3.2.11, so

$$\sum_{\nu\geq 0} \breve{F}(t)^{\mu}_{\nu}\varepsilon(v_{t]})^{\nu} = \mathbb{E}^{\mu}_{t]}\sum_{\nu\geq 0} F(t)^{\mu}_{\nu}\varepsilon(v_{t]})^{\nu}$$

is convergent for all $\mu \geq 0$ and

$$\sum_{\mu\geq 0}\|\sum_{\nu\geq 0}\breve{F}(t)^{\mu}_{\nu}\varepsilon(u_{t]})^{\nu}\|^2 \leq \sum_{\mu\geq 0}\|\sum_{\nu\geq 0}F(t)^{\mu}_{\nu}\varepsilon(u_{t]})^{\nu}\|^2 < \infty,$$

so $\varepsilon(v_{t]}) \in \mathcal{D}(\widehat{\breve{F}}(t))$ for all $u \in \mathfrak{h}$.

The inner-product identity (3.9) and the fact that $\mathbb{E}\breve{F}\mathbb{E} = \mathbb{E}F\mathbb{E}$ yields the following:

$$\begin{aligned}
\langle \varepsilon(u)^{\mu}, \breve{F}(t)^{\mu}_{\nu}\varepsilon(v)^{\nu}\rangle &= \sum_{\pi=0}^{\mu\wedge\nu}\langle \varepsilon(u_{t]})^{\mu-\pi}\otimes\varepsilon(u_{(t})^{\pi}, \left(\breve{F}(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\right)\otimes\varepsilon(v_{(t})^{\pi}\rangle \\
&= \langle j_t(\varepsilon(u)^{\mu}), \sum_{\pi=0}^{\mu\wedge\nu}\left(\breve{F}(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\right)\otimes\varepsilon(v_{(t})^{\pi}\rangle,
\end{aligned}$$

by orthogonality, so

$$\breve{F}(t)^{\mu}_{\nu}\varepsilon(v)^{\nu} = j_t^{-1}(\sum_{\pi=0}^{\mu\wedge\nu}\left(\breve{F}(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\right)\otimes\varepsilon(v_{(t})^{\pi}).$$

However, by Proposition 2.1.2

$$\begin{aligned}
\widehat{\breve{F}}(t)\varepsilon(v_{t]})\otimes\varepsilon(v_{(t}) &= \sum_{\mu\geq 0}\sum_{\pi=0}^{\mu}(\widehat{\breve{F}}(t)\varepsilon(v_{t]}))^{\mu-\pi}\otimes\varepsilon(v_{(t})^{\pi} \\
&= \sum_{\mu\geq 0}\sum_{\pi=0}^{\mu}\sum_{\nu\geq\pi}\left(\breve{F}(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\right)\otimes\varepsilon(v_{(t})^{\pi} \\
&= \sum_{\mu\geq 0}\sum_{\nu\geq 0}\sum_{\pi=0}^{\mu\wedge\nu}\left(\breve{F}(t)^{\mu-\pi}_{\nu-\pi}\varepsilon(v_{t]})^{\nu-\pi}\right)\otimes\varepsilon(v_{(t})^{\pi}
\end{aligned}$$

so $j_t^{-1}(\widehat{\breve{F}}(t)\varepsilon(v_{t]}) \otimes \varepsilon(v_{(t})) = \widehat{\breve{F}}(t)\varepsilon(v)$, whence $\mathcal{E}_0 \subseteq \mathcal{D}(\widehat{\breve{F}})$ and

$$\langle \varepsilon(u), \widehat{\breve{F}}(t)\varepsilon(v)\rangle = \langle j_t(\varepsilon(u)), \widehat{\breve{F}}(t)\varepsilon(v_{t]}) \otimes \varepsilon(v_{(t})\rangle = \langle \varepsilon(u_{t]}), \widehat{\breve{F}}(t)\varepsilon(v_{t]})\rangle \langle \varepsilon(u_{(t}), \varepsilon(v_{(t})\rangle,$$

as claimed.

□

### Martingales

Three of the integrator processes, creation, annihilation and preservation (i.e., $A^\dagger$, $A$ and $\Lambda$) are often referred to as operator martingales. This term may be defined as follows.

**Definition 3.2.14** *The process $F \in \mathfrak{L}$ is a* martingale *if it is adapted and such that*

$$\mathbb{E}_{s]}F(t)\mathbb{E}_{s]} = F(s)\mathbb{E}_{s]} \tag{3.10}$$

*for all $s, t \in [0,1]$ such that $s \leq t$. Equivalently*

$$\langle \varepsilon(u_{s]})^\mu, F(t)^\mu_\nu \varepsilon(v_{s]})^\nu\rangle = \langle \varepsilon(u)^\mu, F(s)^\mu_\nu \varepsilon(v_{s]})^\nu\rangle$$

*if $s \leq t$, for all $u, v \in L^2[0,1]$ and $\mu, \nu \geq 0$.*

**Proposition 3.2.15** *Let $\alpha, \beta \in \{0,1\}$, $p = 2(2-\alpha-\beta)^{-1}$ and suppose $F \in \mathfrak{L}^p$ is adapted. Then $I^\alpha_\beta(F)$ is a martingale if $\alpha + \beta > 0$.*

**Proof**

The adaptedness of $I^\alpha_\beta(F)$ follows from Proposition 3.2.2. To prove the martingale property (3.10) holds, let $\mu, \nu \geq 0$, let $s, t \in [0,1]$ be such that $s \leq t$ and let $u, v \in L^2[0,1]$. We have that

$$\begin{aligned}\langle \varepsilon(u_{s]})^\mu, I^\alpha_\beta(F)(t)^\mu_\nu \varepsilon(v_{s]})^\nu\rangle &= \int_0^s \langle \varepsilon(u_{s]})^{\mu-\beta}, F(r)^{\mu-\beta}_{\nu-\alpha}\varepsilon(v_{s]})^{\nu-\alpha}\rangle \overline{u}^\beta(r) v^\alpha(r)\,\mathrm{d}r \\ &= \langle \varepsilon(u_{s]})^\mu, I^\alpha_\beta(F)(s)^\mu_\nu \varepsilon(v_{s]})^\nu\rangle,\end{aligned}$$

and $\mathbb{E}_{s]}I^\alpha_\beta(F)(s)\mathbb{E}_{s]} = I^\alpha_\beta(F)(s)\mathbb{E}_{s]}$, by Proposition 3.2.11.

□

Given $D \in \mathfrak{M}$ we define the *martingale associated with $D$* to be $\breve{\mathbb{E}}D$, the adapted projection of the process $t \mapsto \mathbb{E}_{t]}D$. Martingales that are also bounded processes have well-behaved norm, as shown by the following proposition.

**Proposition 3.2.16** *Let $M \in \mathfrak{L}_{\mathfrak{B}}$ be a martingale. Then $M \in \mathfrak{L}^\infty_{\mathfrak{B}}$ and $t \mapsto \|\hat{M}(t)\|$ is non-decreasing.*

**Proof**

Since $M(t) \in \mathfrak{B}$ for all $t \in [0,1]$, we have an HP-process, so we may write $\hat{M}(t) = j_t^{-1}(M'(t) \otimes \mathrm{id}_{\mathfrak{H}_{(t}})j_t$, where $M'(t) \in \mathcal{B}(\mathfrak{H}_{t]})$. Thus

$$\|\hat{M}(t)\| = \|M'(t)\| = \sup\{\|M'(t)\hat{\mathbb{E}}_{t]}\theta\| \,:\, \theta \in \mathfrak{H}, \|\theta\| = 1\},$$

and if $s \le t$, the martingale property (3.10) gives that

$$\|M'(s)\hat{\mathbb{E}}_{s]}\theta\|^2 = \|\hat{M}(s)\hat{\mathbb{E}}_{s]}\theta\|^2 = \|\hat{\mathbb{E}}_{s]}\hat{M}(t)\hat{\mathbb{E}}_{s]}\theta\|^2 \le \|\hat{M}(t)\|^2\|\theta\|^2,$$

as required.

□

**Examples**

1. The processes $\Lambda$, $A$, $A^\dagger$ and $T$ defined above all give rise to HP-adapted, HP-processes. This follows from the fact that they may be written as quantum stochastic integrals of the identity, the fact that the identity is adapted and an element of $\mathfrak{L}^\infty_{\mathfrak{R}}$, and Propositions 3.1.3 & 3.2.2.

2. The parity process $P$ is adapted, as it is symmetric and

$$\begin{aligned} \int_s^t \nabla_p \,\mathrm{d}p\, P(s)^{\mu+1}_{\mu+1}\varepsilon(u)^{\mu+1} &= \int_s^t u(p)\,\mathrm{d}p\,\varepsilon(-\chi_{[0,s]}u + \chi_{(s,1]}u)^\mu \\ &= P(s)^\mu_\mu \int_s^t \nabla_p \,\mathrm{d}p\,\varepsilon(u)^{\mu+1} \end{aligned}$$

if $s \le t \in [0,1]$, and furthermore

$$\begin{aligned} \int_0^s \nabla_p \,\mathrm{d}p\, P(t)^{\mu+1}_{\mu+1}\varepsilon(u)^{\mu+1} &= -\int_0^s u(p)\,\mathrm{d}p\,\varepsilon(-\chi_{[0,t]}u + \chi_{(t,1]}u)^\mu \\ &= -P(t)^\mu_\mu \int_0^s \nabla_p \,\mathrm{d}p\,\varepsilon(u)^{\mu+1}, \end{aligned}$$

i.e., $P(t)$ anticommutes with $\int_0^s \nabla_p \,\mathrm{d}p$ if $s \le t$.

3. For a $\nabla$-adapted, HP process that is not HP-adapted, let $H \in \mathfrak{L}_{\mathfrak{D}_1}$ be defined by

$$H(t)^\mu_\nu := \begin{cases} H(t)^\mu_{\mu+1} & \nu = \mu + 1 \\ 0 & \text{otherwise} \end{cases},$$

where $H(t)^\mu_{\mu+1}$ is acts as

$$(H(t)^\mu_{\mu+1}f)(t_1, \ldots, t_\mu) := \sqrt{\mu+1}\int_t^1 f(t_1, \ldots, t_{\mu+1})\,\mathrm{d}t_{\mu+1};$$

by the Cauchy-Schwarz-Buniakowski inequality $\|H(t)^{\mu}_{\mu+1}\| \leq (1-t)^{\frac{1}{2}}\sqrt{\mu+1}$. It is easy to see that $H \in \mathfrak{L}_{\mathfrak{D}_{\mathrm{pb}}}$, so $H$ is an HP-process. Since

$$\hat{H}(t)\varepsilon(u) = \sum_{\mu\geq 0} H(t)^{\mu}_{\mu+1}\varepsilon(u)^{\mu+1} = \sum_{\mu\geq 0}\int_t^1 u(s)\,\mathrm{d}s\,\varepsilon(u)^{\mu} = \int_t^1 u(s)\,\mathrm{d}s\,\varepsilon(u),$$

we see that $\hat{H}(t)\varepsilon(u_{t]}) = 0$, so $H$ is not HP-adapted, but

$$\begin{aligned}
\int_s^t \nabla_p\,\mathrm{d}p\,H(s)^{\mu}_{\mu+1}\varepsilon(u)^{\mu+1} &= \int_s^1 u(r)\,\mathrm{d}r\int_s^t \nabla_p\,\mathrm{d}p\,\varepsilon(u)^{\mu} \\
&= \int_s^1 u(r)\,\mathrm{d}r\int_s^t u(p)\,\mathrm{d}p\,\varepsilon(u)^{\mu-1} \\
&= \int_s^t u(p)\,\mathrm{d}p\,H(s)^{\mu-1}_{\mu}\varepsilon(u)^{\mu} \\
&= H(s)^{\mu-1}_{\mu}\int_s^t \nabla_p\,\mathrm{d}p\,\varepsilon(u)^{\mu+1}
\end{aligned}$$

so $H$ is $\nabla$-adapted.

□

## 3.3 Ω-Adapted Processes

In this section we study the class 'Ω-adapted' processes, a term coined by Lindsay ([Lin, p. 76]). This notion was introduced into quantum stochastic calculus by Vincent-Smith ([Vin2]), who generalised a method of Alicki and Fannes for the dilation of quantum dynamical semigroups using classical Brownian motion ([AlF]). We show that Ω-adaptedness is preserved under algebraic operations and limiting procedures, and that QS integrals of Ω-adapted, bounded processes are themselves bounded. We use this property to define a $*$-algebra of processes similar to the regular quantum semimartingales of Attal ([AtM], [Att]).

**Definition 3.3.1** *A process $F \in \mathfrak{L}$ is* Ω-adapted*, pronounced "vacuum-adapted", if $F = \mathbb{E}F\mathbb{E}$, i.e.,*

$$F(t)^{\mu}_{\nu} = \mathbb{E}^{\mu}_{t]}F(t)^{\mu}_{\nu}\mathbb{E}^{\nu}_{t]} \quad \forall\, t \in [0,1]$$

*for all $\mu, \nu \geq 0$. Equivalently $F\mathbb{E} = F = \mathbb{E}F$.*

Suppose $F$ gives rise to an HP-process. If $F$ is Ω-adapted

$$\langle\varepsilon(u), \hat{F}(t)\varepsilon(v)\rangle = \langle\varepsilon(u_{t]}), \hat{F}(t)\varepsilon(v_{t]})\rangle,$$

i.e., $\hat{\mathbb{E}}_{t]}\hat{F}(t)\hat{\mathbb{E}}_{t]} = \hat{F}(t)$ on $\mathcal{E}_0$. This is unlike an HP-adapted process, where

$$\hat{F}(t)\varepsilon(u) \text{ ``='' } \hat{F}(t)\varepsilon(u_{t]}) \otimes \varepsilon(u_{(t});$$

if $F$ is $\Omega$-adapted

$$\hat{F}(t)\varepsilon(u) \text{ ``='' } \hat{F}(t)\varepsilon(u_{t]}) \otimes \varepsilon(0),$$

that is, $\hat{F}$ maps the future to the vacuum (hence '$\Omega$'). This property can be used to define $\Omega$-adaptedness in the Hudson-Parthasarathy framework, as the exponential vectors are mapped into themselves under the action of $\hat{\mathbb{E}}_{t]}$.

**Proposition 3.3.2** *The collection of $\Omega$-adapted processes is a $*$-vector space. Furthermore $\Omega$-adaptedness is preserved by multiplication and integration, i.e., if $F, G \in \mathfrak{L}$ are $\Omega$-adapted processes such that the product $FG$ exists then $FG$ is $\Omega$-adapted, and if $F \in \mathfrak{L}^p$ (where $p = 2(2-\alpha-\beta)^{-1}$ for some $\alpha, \beta \in \{0,1\}$) then $I^\alpha_\beta(F)$ is $\Omega$-adapted.*

**Proof**

That $\Omega$-adaptedness is preserved by addition and scalar multiplication is clear, and if $F$ is $\Omega$-adapted then

$$F^* = (\mathbb{E}F\mathbb{E})^* = \mathbb{E}^*F^*\mathbb{E}^* = \mathbb{E}F^*\mathbb{E}$$

as $\mathbb{E} \in \mathfrak{L}_{\mathfrak{D}}$, so we have a $*$-vector space as claimed. Now suppose that $G$ is also $\Omega$-adapted and $FG$ exists; if $\xi \in \mathfrak{h}_\nu$ then

$$\begin{aligned}
(FG)(t)^\mu_\nu\xi &= \sum_{\pi\geq 0} F(t)^\mu_\pi G(t)^\pi_\nu \xi \\
&= \sum_{\pi\geq 0} \mathbb{E}^\mu_{t]} F(t)^\mu_\pi (\mathbb{E}^\pi_{t]})^2 G(t)^\pi_\nu \mathbb{E}^\nu_{t]}\xi \\
&= \mathbb{E}^\mu_{t]} \sum_{\pi\geq 0} F(t)^\mu_\pi G(t)^\pi_\nu \mathbb{E}^\nu_{t]}\xi \\
&= \mathbb{E}^\mu_{t]}(FG)(t)^\mu_\nu \mathbb{E}^\nu_{t]}\xi
\end{aligned}$$

so $FG = \mathbb{E}FG\mathbb{E}$. Finally, if $F \in \mathfrak{L}^p$ is $\Omega$-adapted then

$$\begin{aligned}
\langle\varepsilon(u)^\mu, I^\alpha_\beta(F)(t)^\mu_\nu\varepsilon(v)^\nu\rangle &= \int_0^t \langle\varepsilon(u)^{\mu-\beta}, F(s)^{\mu-\beta}_{\nu-\alpha}\varepsilon(v)^{\nu-\alpha}\rangle \overline{u}^\beta(s)v^\alpha(s)\,\mathrm{d}s \\
&= \int_0^t \langle\varepsilon(u_{s]})^{\mu-\beta}, F(s)^{\mu-\beta}_{\nu-\alpha}\varepsilon(v_{s]})^{\nu-\alpha}\rangle \overline{u}^\beta(s)v^\alpha(s)\,\mathrm{d}s
\end{aligned}$$

and replacing '$\varepsilon(u)$' by '$\varepsilon(u_{t]})$' at the start (and similarly for $\varepsilon(v)$') we get the same expression. Thus $I^\alpha_\beta(F) = \mathbb{E}I^\alpha_\beta(F)\mathbb{E}$, as required.

□

Given any adapted process $F$, we may obtain an $\Omega$-adapted process by applying $\mathbb{E}$; $\mathbb{E}F = F\mathbb{E}$ is $\Omega$-adapted by Proposition 3.2.11.

The next proposition shows that $\Omega$-adaptedness is preserved by the pointwise weak convergence of processes. An immediate consequence of this is that if $([F_n])_{n\geq 1} \subseteq \mathcal{L}^p$ is convergent, to $[F]$, say, and each $[F_n]$ has a $\Omega$-adapted version then $[F]$ has a $\Omega$-adapted version.

**Proposition 3.3.3** *Let* $(F_n)_{n\geq 1} \subseteq \mathfrak{L}$ *be a sequence of* $\Omega$*-adapted processes and suppose there exists* $F \in \mathfrak{L}$ *such that, for all* $\mu, \nu \geq 0$ *and* $u, v \in L^2[0,1]$,

$$\langle \varepsilon(u)^\mu, F_n(t)^\mu_\nu \varepsilon(v)^\nu \rangle \to \langle \varepsilon(u)^\mu, F(t)^\mu_\nu \varepsilon(v)^\nu \rangle$$

*as* $n \to \infty$ *for all* $t \in [0,1]$. *Then* $F$ *is* $\Omega$*-adapted.*

**Proof**

This is immediate from the fact that

$$\langle \varepsilon(u)^\mu, F_n(t)^\mu_\nu \varepsilon(v)^\nu \rangle = \langle \varepsilon(u_{t]})^\mu, F_n(t)^\mu_\nu \varepsilon(v_{t]})^\nu \rangle.$$

□

Our interest in $\Omega$-adapted processes is explained by the following lemma; a consequence is the boundedness of the QS integral of a $\Omega$-adapted, bounded process. (Compare the *adapted gradient* of Attal and Lindsay; see [AtL] and [Att2, Theorem I.3, p. 9]). Using the notation developed before Proposition 3.1.4, the lemma states that $\hat{\mathbb{E}}\hat{\nabla}$ extends to a bounded operator, such that $\|\hat{\mathbb{E}}\hat{\nabla}\xi\|^2_{L^2([0,1];\mathfrak{H})} = \|\xi\|^2 - \|\xi^0\|^2$ for all $\xi \in \mathfrak{H}$.

**Lemma 3.3.4** *Let* $j \geq 1$ *and* $f \in \mathfrak{h}_j$; *for all* $t \in [0,1]$

$$\|\mathbb{E}^j_{t]} f\|^2 = \int_0^t \|\mathbb{E}^{j-1}_{s]} \nabla_s f\|^2 \,\mathrm{d}s.$$

*Thus*

$$\|\hat{\mathbb{E}}_{t]}\xi\|^2 = \|\xi^0\|^2 + \int_0^t \|\hat{\mathbb{E}}_{s]}\hat{\nabla}_s \xi\|^2 \,\mathrm{d}s := \|\xi^0\|^2 + \int_0^t \sum_{j\geq 1} \|\mathbb{E}^{j-1}_{s]} \nabla_s \xi^j\|^2 \,\mathrm{d}s$$

*for any* $\xi \in \mathfrak{H}$ *and* $t \in [0,1]$.

**Proof**

We see that

$$\begin{aligned}\int_0^t \|\mathbb{E}_{s]}^{j-1}\nabla_s f\|^2 \,\mathrm{d}s &= \int_0^t \int_{[0,1]^{j-1}} j|\prod_{l=1}^{j-1}\chi_{[0,t_j]}(t_l) f(t_1,\ldots,t_j)|^2 \,\mathrm{d}t_1\cdots \mathrm{d}t_j \\ &= j\int_0^t\int_0^{t_j}\cdots\int_0^{t_j}|f(t_1,\ldots,t_j)|^2\,\mathrm{d}t_1\cdots\mathrm{d}t_j \\ &= \sum_{l=1}^{j}\int_{A_l}|f(t_1,\ldots,t_j)|^2\,\mathrm{d}\mathbf{t},\end{aligned}$$

where

$$A_l := \{(t_1,\ldots,t_j)\in[0,t]^j \,:\, t_k\le t_l\ (k=1,\ldots,j)\}.$$

Note that $\cup_{l=1}^j A_l = [0,t]^j$ and $A_l\cap A_m$ has measure zero if $l\neq m$, so

$$\begin{aligned}\int_0^t \|\mathbb{E}_{s]}^{j-1}\nabla_s f\|^2\,\mathrm{d}s &= \int_{[0,t]^j}|f(t_1,\ldots,t_j)|^2\,\mathrm{d}\mathbf{t} \\ &= \int_{[0,1]^j}|\prod_{l=1}^{j}\chi_{[0,t]}(t_l)f(t_1,\ldots,t_j)|^2\,\mathrm{d}\mathbf{t} \\ &= \|\mathbb{E}_{t]}^j f\|^2,\end{aligned}$$

as claimed. The last part is immediate from the Lebesgue series theorem (Beppo Levi's theorem).

□

Taking $\xi=\varepsilon(u)$ in the above, we see that

$$\|\varepsilon(u_{t]})\|^2 = 1+\int_0^t |u(s)|^2\|\varepsilon(u_{s]})\|^2\,\mathrm{d}s : \tag{3.11}$$

this formula is familiar from the theory of Wiener space. If $u\in L^2[0,1]$ and $W$ is standard Brownian motion then $\mathfrak{z}(u) := \exp(\int_0^\cdot u(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^\cdot u(s)^2\,\mathrm{d}s)$ is the *Brownian exponential*, which satisfies the stochastic differential equation

$$\mathfrak{z}(u)(t) = 1+\int_0^t u(s)\mathfrak{z}(u)(s)\,\mathrm{d}W_s,$$

(see [McK, p.33]) so

$$\mathbb{E}[|\mathfrak{z}(u)(t)|^2] = 1+\int_0^t |u(s)|^2\mathbb{E}[|\mathfrak{z}(u)(s)|^2]\,\mathrm{d}s,$$

by the classical Itô product formula for Brownian motion. Since $\mathbb{E}_{t]}\mathfrak{z}(u)(1)=\mathfrak{z}(u)(t)$ and $\varepsilon(u)\leftrightarrow\mathfrak{z}(u)(1)$ *via* the Wiener-Itô isomorphism, this is the same expression as (3.11).

**Theorem 3.3.5** *Let $\alpha, \beta \in \{0, 1\}$ and suppose $F \in \mathfrak{L}^p_{\mathfrak{B}}$ is $\Omega$-adapted (where $p = 2(2 - \alpha - \beta)^{-1}$). Then $I^\alpha_\beta(F) \in \mathfrak{L}^\infty_{\mathfrak{B}}$ and*

$$\|\hat{I}^\alpha_\beta(F)(t) - \hat{I}^\alpha_\beta(F)(s)\| \leq \|\hat{F}\chi_{(s,t]}\|_p$$

*for all $s, t \in [0, 1]$ such that $s \leq t$.*

**Proof**

Let $\lambda, \rho \geq 0$, $t \in [0, 1]$ and $\xi \in \mathfrak{H}$. From (3.5) and the fact that $\nabla_s \mathbb{E}_{r]} f = 0$ for $s \geq r$, we see that

$$\begin{aligned}
&\sum_{\mu=0}^{\lambda} \| \sum_{\nu=0}^{\rho} I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu \|^2 \\
&= \sum_{\mu=0}^{\lambda} \sum_{\nu,\pi=0}^{\rho} \langle I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu, I^\alpha_\beta(F)(t)^\mu_\pi \xi^\pi \rangle \\
&= \sum_{\mu=0}^{\lambda} \sum_{\nu,\pi=0}^{\rho} \int_0^t \int_0^t \langle \nabla^\beta_s F(r)^{\mu-\beta}_{\nu-\alpha} \nabla^\alpha_r \xi^\nu, \nabla^\beta_r F(s)^{\mu-\beta}_{\pi-\alpha} \nabla^\alpha_s \xi^\pi \rangle \,\mathrm{d}r \,\mathrm{d}s \\
&\qquad\qquad + \not\delta^\beta_\beta \int_0^t \langle F(s)^{\mu-1}_{\nu-\alpha} \nabla^\alpha_s \xi^\nu, F(s)^{\mu-1}_{\pi-\alpha} \nabla^\alpha_s \xi^\pi \rangle \,\mathrm{d}s \\
&= \delta^0_\beta \| \int_0^t \sum_{\mu=0}^{\lambda} \sum_{\nu=0}^{\rho} F(s)^\mu_{\nu-\alpha} \nabla^\alpha_s \xi^\nu \,\mathrm{d}s \|^2 + \delta^1_\beta \int_0^t \| \sum_{\mu=0}^{\lambda} \sum_{\nu=0}^{\rho} F(s)^{\mu-1}_{\nu-\alpha} \nabla^\alpha_s \xi^\nu \|^2 \,\mathrm{d}s \\
&\leq \delta^0_\beta \| \int_0^t \hat{F}(s) \sum_{\nu=0}^{\rho} E^*_{\nu-\alpha} (\mathbb{E}^{\nu-1}_{s]})^\alpha \nabla^\alpha_s \xi^\nu \,\mathrm{d}s \|^2 \\
&\qquad\qquad + \delta^1_\beta \int_0^t \| \hat{F}(s) \sum_{\nu=0}^{\rho} E^*_{\nu-\alpha} (\mathbb{E}^{\nu-1}_{s]})^\alpha \nabla^\alpha_s \xi^\nu \|^2 \,\mathrm{d}s.
\end{aligned}$$

The last inequality is obtained from the identities $F = F\mathbb{E}^\alpha$ and $F^\mu_\nu = E_\mu \hat{F} E^*_\nu$. Examining each case separately, using the Cauchy-Schwarz-Buniakowski inequality and Lemma 3.3.4, we see that

$$\sum_{\mu=0}^{\lambda} \| \sum_{\nu=0}^{\rho} I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu \|^2 \leq \|\hat{F}\chi_{[0,t]}\|^2_p \sum_{\nu=0}^{\rho} \|\xi^\nu\|^2.$$

In particular,

$$\| \sum_{\nu=\lambda}^{\rho} I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu \|^2 \leq \|\hat{F}\|^2_p \sum_{\nu=\lambda}^{\rho} \|\xi^\nu\|^2 \to 0$$

as $\lambda, \rho \to \infty$, so $\sum_{\nu \geq 0} I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu \in \mathfrak{h}_\mu$ for all $\mu \geq 0$ and

$$\sum_{\mu=0}^{\infty} \| \sum_{\nu \geq 0} I^\alpha_\beta(F)(t)^\mu_\nu \xi^\nu \|^2 \leq \|\hat{F}\chi_{[0,t]}\|^2_p \|\xi\|^2,$$

so $\mathcal{D}(\hat{I}^\alpha_\beta(F)(t)) = \mathfrak{H}$ and $\|\hat{I}^\alpha_\beta(F)(t)\| \leq \|\hat{F}\chi_{[0,t]}\|_p$. Replacing $F$ by $F\chi_{(s,t]}$ gives the inequality in the statement of the proposition, by (3.4).

□

**Definition 3.3.6** *A* $\Omega$-adapted bounded quantum semimartingale *(henceforth* $\Omega$-semimartingale*, for the sake of brevity) is a process* $M$ *of the form*

$$M = zI + I^1_1(E) + I^1_0(F) + I^0_1(G) + I^0_0(H)$$

*where the integrands are* $\Omega$*-adapted, bounded processes satisfying the appropriate integrability conditions, i.e.,*

$$(E, F, G, H) \in \mathfrak{L}^\infty_{\mathfrak{B}} \times \mathfrak{L}^2_{\mathfrak{B}} \times \mathfrak{L}^2_{\mathfrak{B}} \times \mathfrak{L}^1_{\mathfrak{B}},$$

*and* $z \in \mathbb{C}$ *(the* initial value*). We may denote such a process using the differential notation*

$$M(0) = zI, \ \ \mathrm{d}M = E\,\mathrm{d}\Lambda + F\,\mathrm{d}A + G\,\mathrm{d}A^\dagger + H\,\mathrm{d}t.$$

Note that

$$\|\hat{M}(t) - \hat{M}(s)\| \leq \|\hat{E}\chi_{(s,t]}\|_\infty + \|\hat{F}\chi_{(s,t]}\|_2 + \|\hat{G}\chi_{(s,t]}\|_2 + \|\hat{H}\chi_{(s,t]}\|_1$$

for all $s, t \in [0,1]$ such that $s \leq t$, by Theorem 3.3.5, so $t \mapsto \|\hat{M}(t)\|$ is essentially bounded. It is easy to see that the class of $\Omega$-semimartingales is a $*$-vector space; we shall see later (Corollary 3.4.9) that it is in fact a $*$-algebra.

## 3.4 Quantum Itô Formulae

The following is the 'local' version of the quantum Itô product formula of Hudson and Parthasarathy ([HuP, Theorem 4.4]). It is a consequence of the integration-by-parts formula (3.5) and $\nabla$-adaptedness.

**Proposition 3.4.1** *Let* $\alpha, \beta, \gamma, \delta \in \{0,1\}$ *and let* $F \in \mathfrak{L}^p$*,* $G \in \mathfrak{L}^q$ *(where* $p = 2(2-\alpha-\beta)^{-1}$ *and* $q = 2(2-\gamma-\delta)^{-1}$*) be* $\nabla$*-adapted. Then*

$$\begin{aligned}&\langle I^\alpha_\beta(F)(t)^\mu_\nu \xi, I^\gamma_\delta(G)(t)^\mu_\pi \eta\rangle \\ &= \ \langle \xi, \Big( I^\beta_\alpha(F^*_{\mu-\beta} I^\gamma_\delta(G)^{\mu-\beta}) + I^\gamma_\delta(I^\beta_\alpha(F^*)_{\mu-\delta} G^{\mu-\delta}) + \delta^\beta_\delta I^\gamma_\alpha(F^*_{\mu-\beta} G^{\mu-\delta})\Big)(t)^\nu_\pi \eta\rangle\end{aligned}$$

*for all* $\mu, \nu, \pi \geq 0$*,* $\xi \in \mathfrak{h}_\nu$*,* $\eta \in \mathfrak{h}_\pi$ *and* $t \in [0,1]$*.*

**Proof**

Firstly, note that

$$\|[F^*_{\mu-\beta} I^\gamma_\delta(G)^{\mu-\beta}]^{\nu-\alpha}_{\pi-\beta}\|_p \le (\mu-\beta)^{\frac{\delta}{2}}(\pi-\beta)^{\frac{\gamma}{2}}\|[F]^{\mu-\beta}_{\nu-\alpha}\|_p\|[G]^{\mu-\beta-\delta}_{\pi-\beta-\gamma}\|_q,$$

$$\|[I^\beta_\alpha(F^*)_{\mu-\delta}G^{\mu-\delta}]^{\nu-\delta}_{\pi-\gamma}\|_q \le (\nu-\delta)^{\frac{\alpha}{2}}(\mu-\delta)^{\frac{\beta}{2}}\|[F]^{\mu-\delta-\beta}_{\nu-\delta-\alpha}\|_p\|[G]^{\mu-\delta}_{\pi-\gamma}\|_q,$$

and if $\beta=\delta=1$, $r^{-1} := 2^{-1}(2-\gamma-\alpha) = p^{-1}+q^{-1}$ then

$$\|[F^*_{\mu-\beta}G^{\mu-\delta}]^{\nu-\alpha}_{\pi-\gamma}\|_r \le \|[F]^{\mu-\beta}_{\nu-\alpha}\|_p\|[G]^{\mu-\delta}_{\pi-\gamma}\|_q,$$

by Hölder's inequality, so everything above is well-defined. The result follows from the identity (3.5):

$$\begin{aligned}
&\langle I^\alpha_\beta(F)(t)^\mu_\nu\xi, I^\gamma_\delta(G)(t)^\mu_\pi\eta\rangle \\
&= \int_0^t\int_0^t \langle \nabla^\delta_s F(r)^{\mu-\beta}_{\nu-\alpha}\nabla^\alpha_r\xi, \nabla^\beta_r G(s)^{\mu-\delta}_{\pi-\gamma}\nabla^\gamma_s\eta\rangle\,\mathrm{d}r\,\mathrm{d}s \\
&\qquad\qquad + \not{\delta}^\beta_\delta\int_0^t \langle F(s)^{\mu-\beta}_{\nu-\alpha}\nabla^\alpha_s\xi, G(s)^{\mu-\delta}_{\pi-\gamma}\nabla^\gamma_s\eta\rangle\,\mathrm{d}s \\
&= \int_0^t\int_0^r \langle \nabla^\delta_s F(r)^{\mu-\beta}_{\nu-\alpha}\nabla^\alpha_r\xi, G(s)^{\mu-\delta-\beta}_{\pi-\gamma-\beta}\nabla^\beta_r\nabla^\gamma_s\eta\rangle\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^t\int_0^s \langle F(r)^{\mu-\beta-\delta}_{\nu-\alpha-\delta}\nabla^\delta_s\nabla^\alpha_r\xi, \nabla^\beta_r G(s)^{\mu-\delta}_{\pi-\gamma}\nabla^\gamma_s\eta\rangle\,\mathrm{d}r\,\mathrm{d}s \\
&\qquad\qquad + \not{\delta}^\beta_\delta\int_0^t \langle \nabla^\alpha_s\xi, F^*(s)^{\nu-\alpha}_{\mu-\beta}G(s)^{\mu-\delta}_{\pi-\gamma}\nabla^\gamma_s\eta\rangle\,\mathrm{d}s \\
&= \int_0^t \langle F(r)^{\mu-\beta}_{\nu-\alpha}\nabla^\alpha_r\xi, I^\gamma_\delta(G)(r)^{\mu-\beta}_{\pi-\beta}\nabla^\beta_r\eta\rangle\,\mathrm{d}r \\
&\quad + \int_0^t \langle I^\alpha_\beta(F)(s)^{\mu-\delta}_{\nu-\delta}\nabla^\delta_s\xi, G(s)^{\mu-\delta}_{\pi-\gamma}\nabla^\gamma_s\eta\rangle\,\mathrm{d}s + \not{\delta}^\beta_\delta\langle\xi, I^\gamma_\alpha(F^*_{\mu-\beta}G^{\mu-\delta})(t)^\nu_\pi\eta\rangle \\
&= \langle\xi, \Big(I^\beta_\alpha(F^*_{\mu-\beta}I^\gamma_\delta(G)^{\mu-\beta}) + I^\gamma_\delta(I^\beta_\alpha(F^*)_{\mu-\delta}G^{\mu-\delta}) + \not{\delta}^\beta_\delta I^\gamma_\alpha(F^*_{\mu-\beta}G^{\mu-\delta})\Big)(t)^\nu_\pi\eta\rangle.
\end{aligned}$$

We used in the third equality above the fact that $\nabla_r\nabla_s\eta = \nabla_s\nabla_r\eta$ for almost all $r,s\in[0,1]$ (and similarly for $\xi$) which may be readily verified, as we remarked in Example 3, Section 3.1.

□

We say that $(F,G)\in\mathfrak{L}\times\mathfrak{L}$ is an *adapted pair* if $F^*$ and $G$ are $\nabla$-adapted. For example, $(F,G)$ is an adapted pair if both $F$ and $G$ are adapted, and $F$ is adapted if and only if $(F,F)$ is an adapted pair. Using this expression we may rewrite the proposition above in the following manner.

**Corollary 3.4.2** *Let* $\alpha, \beta, \gamma, \delta \in \{0, 1\}$*, and let* $(F, G) \in \mathfrak{L}^p \times \mathfrak{L}^q$ *(where* $p = 2(2 - \alpha - \beta)^{-1}$ *and* $q = 2(2 - \gamma - \delta)^{-1}$*) be an adapted pair. The equality*

$$I^\alpha_\beta(F)_\pi I^\gamma_\delta(G)^\pi = I^\alpha_\beta(F_{\pi-\alpha} I^\gamma_\delta(G)^{\pi-\alpha}) + I^\gamma_\delta(I^\alpha_\beta(F)_{\pi-\delta} G^{\pi-\delta}) + \not{\delta}^\alpha_\delta I^\gamma_\beta(F_{\pi-\alpha} G^{\pi-\delta}) \quad (3.12)$$

*holds for all* $\pi \geq 0$.

Before we prove the quantum Itô product formula for matrix processes, we introduce the following terminology (or rather, clarify the meaning of the following phrase). Given $p \in [1, \infty]$ and processes $F$ and $G$, we say that $[FG]$ *converges in* $\mathcal{L}^p$ if $\sum_{\pi \geq 0} [F_\pi G^\pi]$ has a limit in $\mathcal{L}^p$. Note that if this is the case then $F$ and $G$ have versions such that their pointwise product exists.

**Theorem 3.4.3 (QIF)** *Let* $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ *and let* $(F, G) \in \mathfrak{L}^p \times \mathfrak{L}^q$ *(where* $p = 2(2 - \alpha - \beta)^{-1}$ *and* $q = 2(2 - \gamma - \delta)^{-1}$*) be an adapted pair. Then*

$$I^\alpha_\beta(F) I^\gamma_\delta(G) = I^\alpha_\beta(F I^\gamma_\delta(G)) + I^\gamma_\delta(I^\alpha_\beta(F) G) + \not{\delta}^\alpha_\delta I^\gamma_\beta(FG)$$

*whenever the integrands on the right-hand side are convergent in the appropriate* $\mathcal{L}$*-space, i.e.,* $[F I^\gamma_\delta(G)]$ *converges in* $\mathcal{L}^p$*,* $[I^\alpha_\beta(F) G]$ *converges in* $\mathcal{L}^q$ *and (if* $\alpha = \delta = 1$*)* $[FG]$ *converges in* $\mathcal{L}^r$*, where* $r := 2(2 - \gamma - \beta)^{-1}$.

**Proof**

From (3.12) we see that the product $(I^\alpha_\beta(F) I^\gamma_\delta(G))^\mu_\nu$ exists and is equal to the appropriate expression, because

$$\begin{aligned}
&\sum_{\pi \geq 0} I^\alpha_\beta(F)^\mu_\pi I^\gamma_\delta(G)^\pi_\nu \\
&= \lim_{n \to \infty} \sum_{\pi=0}^{n} I^\alpha_\beta(F_{\pi-\alpha} I^\gamma_\delta(G)^{\pi-\alpha})^\mu_\nu + I^\gamma_\delta(I^\alpha_\beta(F)_{\pi-\delta} G^{\pi-\delta})^\mu_\nu + \not{\delta}^\alpha_\delta I^\gamma_\beta(F_{\pi-\alpha} G^{\pi-\delta})^\mu_\nu \\
&= \lim_{n \to \infty} I^\alpha_\beta(\sum_{\pi=0}^{n-\alpha} F_\pi I^\gamma_\delta(G)^\pi)^\mu_\nu + I^\gamma_\delta(\sum_{\pi=0}^{n-\delta} I^\alpha_\beta(F)_\pi G^\pi)^\mu_\nu + \not{\delta}^\alpha_\delta I^\gamma_\beta(\sum_{\pi=0}^{n-\alpha} F_\pi G^\pi)^\mu_\nu \\
&= I^\alpha_\beta(F I^\gamma_\delta(G))^\mu_\nu + I^\gamma_\delta(I^\alpha_\beta(F) G)^\mu_\nu + \not{\delta}^\alpha_\delta I^\gamma_\beta(FG)^\mu_\nu,
\end{aligned}$$

by the continuity of $I^\alpha_\beta$, $I^\gamma_\delta$ and $I^\gamma_\beta$.

□

**Corollary 3.4.4** *Let* $\alpha, \beta, \gamma, \delta \in \{0, 1\}$*,* $p = 2(2 - \alpha - \beta)^{-1}$ *and* $q = 2(2 - \gamma - \delta)^{-1}$*. If* $(F, G) \in \mathfrak{L}^p_{\mathfrak{R}} \times \mathfrak{L}^q_{\mathfrak{R}}$ *is an adapted pair then*

$$I^\alpha_\beta(F) I^\gamma_\delta(G) = I^\alpha_\beta(F I^\gamma_\delta(G)) + I^\gamma_\delta(I^\alpha_\beta(F) G) + \not{\delta}^\alpha_\delta I^\gamma_\beta(FG).$$

**Proof**

As $I_\delta^\gamma(G) \in \mathfrak{L}_{\mathfrak{R}}^\infty$ if $G \in \mathfrak{L}_{\mathfrak{R}}^q$, by Proposition 3.1.3, we may conclude from Proposition 2.3.7 that $[FI_\delta^\gamma(G)]$, $[I_\beta^\alpha(F)G]$ and $\not\int_\delta^\alpha[FG]$ converge in the appropriate $\mathcal{L}$-space, so the product Itô formula holds by the above.

□

**Corollary 3.4.5** *Let $\alpha, \beta, \gamma, \delta \in \{0,1\}$ and let $p = 2(2-\alpha-\beta)^{-1}$ and $q = 2(2-\gamma-\delta)^{-1}$. If $(F,G) \in \mathfrak{L}^p \times \mathfrak{L}^q$ is an adapted pair and such that*

$$\sum_{\pi \geq 0} \|[F]_\pi^\mu\|_p \, \pi^{\frac{\delta}{2}} \|[G]_{\nu-\gamma}^{\pi-\delta}\|_q < \infty, \quad \sum_{\pi \geq 0} \|[F]_{\pi-\alpha}^{\mu-\beta}\|_p \, \pi^{\frac{\alpha}{2}} \|[G]_\nu^\pi\|_q < \infty$$

*and, if $\alpha = \delta = 1$,*

$$\sum_{\pi \geq 0} \|[F]_\pi^\mu\|_p \|[G]_\nu^\pi\|_q < \infty$$

*for all $\mu, \nu \geq 0$ then*

$$I_\beta^\alpha(F) I_\delta^\gamma(G) = I_\beta^\alpha(F I_\delta^\gamma(G)) + I_\delta^\gamma(I_\beta^\alpha(F)G) + \not\int_\delta^\alpha I_\beta^\gamma(FG).$$

**Proof**

We have that

$$\sum_{\pi \geq 0} \|[F_\pi I_\delta^\gamma(G)^\pi]_\nu^\mu\|_p \leq \sum_{\pi \geq 0} \|[F]_\pi^\mu\|_p \|[I_\delta^\gamma(G)]_\nu^\pi\|_\infty \leq \sum_{\pi \geq 0} \|[F]_\pi^\mu\|_p \, \pi^{\frac{\delta}{2}} \nu^{\frac{\gamma}{2}} \|[G]_{\nu-\gamma}^{\pi-\delta}\|_q$$

and similarly for the other terms.

□

Note that the conditions for this corollary hold if either $F$ or $G$ are uniformly diagonal, so for any polynomials in the processes $\Lambda$, $A$, $A^\dagger$ and $T$. More generally we have the following (cf. [Vin, Lemma 4.1]), the quantum polynomial Itô formula for processes of matrices with rapid decay.

**Theorem 3.4.6** *Let $(E,F,G,H) \in \mathfrak{L}_{\mathfrak{R}}^\infty \times \mathfrak{L}_{\mathfrak{R}}^2 \times \mathfrak{L}_{\mathfrak{R}}^2 \times \mathfrak{L}_{\mathfrak{R}}^1$ be a quadruple of adapted processes and let*

$$M := I_1^1(E) + I_0^1(F) + I_1^0(G) + I_0^0(H).$$

*For $n \geq 1$ there exist adapted processes $E_n$, $F_n$, $G_n$ and $H_n$ such that*

$$M^n = I_1^1(E_n) + I_0^1(F_n) + I_1^0(G_n) + I_0^0(H_n),$$

*and these processes have the following form:*

$$\begin{aligned} E_n &= (M+E)^n - M^n, \\ F_n &= \sum_{\alpha+\beta=n-1} M^\alpha F (M+E)^\beta, \\ G_n &= \sum_{\alpha+\beta=n-1} (M+E)^\alpha G M^\beta, \\ H_n &= \sum_{\alpha+\beta=n-1} M^\alpha H M^\beta \ + \sum_{\alpha+\beta+\gamma=n-2} M^\alpha F (M+E)^\beta G M^\gamma, \end{aligned}$$

*where the summation is taken over non-negative integers.*

**Proof**

Note that $M \in \mathfrak{L}^\infty_{\mathfrak{R}}$, by Proposition 3.1.3, so the processes $E_n$, $F_n$, $G_n$ and $H_n$ are well-defined for all $n$ and satisfy the appropriate integrability conditions, by Proposition 2.3.7. Adaptedness follows from Propositions 2.3.7 and 3.2.2.

The statement of the proposition holds for $n = 1$, so assume it is true for $n = k$. The conditions of Corollary 3.4.5 are satisfied if we take $F = X_k$ and $G = Y$ (where $X, Y \in \{E, F, G, H\}$); this follows from Lemma A.3.1. After some algebraic manipulation we see that

$$\begin{aligned} M^k M &= I^1_1(E_k M + M^k E + E_k E) + I^1_0(F_k M + M^k F + F_k E) \\ &\quad + I^0_1(G_k M + M^k G + E_k G) + I^0_0(H_k M + M^k H + F_k G) \end{aligned}$$

and each of the integrands simplifies to the required expression.

□

**Examples**

In the following we use the differential notation

$$\mathrm{d}M = E\,\mathrm{d}\Lambda + F\,\mathrm{d}A + G\,\mathrm{d}A^\dagger + H\,\mathrm{d}t$$

to mean that $M(t) = M(0) + I^1_1(E)(t) + I^1_0(F)(t) + I^0_1(G)(t) + I^0_0(H)(t)$ for all $t \in [0,1]$, and also linearity to make sense of expressions such as $N\,\mathrm{d}M$:

$$N\,\mathrm{d}M := NE\,\mathrm{d}\Lambda + NF\,\mathrm{d}A + NG\,\mathrm{d}A^\dagger + NH\,\mathrm{d}t.$$

1. Let $E \equiv H \equiv 0$, $F \equiv G \equiv I$; for any polynomial $p \in \mathbb{C}[z]$

$$\mathrm{d}(p(M)) = p'(M)\,\mathrm{d}M + \tfrac{1}{2}p''(M)\,\mathrm{d}t,$$

the well-known Itô formula for Brownian motion.

2. Let $E \equiv F \equiv G \equiv H \equiv I$; for any polynomial $p \in \mathbb{C}[z]$

$$\mathrm{d}(p(M)) = (p(M+I) - p(M))\,\mathrm{d}M,$$

which is the Itô formula for the standard Poisson process of intensity 1.

3. The local Itô formula allows us to prove that $B = I_0^1(P)$ and $B^* = I_1^0(P)$ are bounded processes. We see from (3.12) that

$$\begin{aligned}
\|I_0^1(P)(t)_{\mu+1}^{\mu}\xi^{\mu+1}\|^2 &= \langle \xi^{\mu+1}, \Big(I_0^1(I_1^0(P)_\mu P^\mu) + I_1^0(P_\mu I_0^1(P)^\mu)\Big)(t)_{\mu+1}^{\mu+1}\xi^{\mu+1}\rangle \\
&= 2\Re \int_0^t \langle \xi^{\mu+1}, I_1^0(P)(s)_\mu^{\mu+1} P(s)_\mu^\mu \nabla_s \xi^{\mu+1}\rangle \,\mathrm{d}s \\
&= 2\Re \int_0^t \int_0^s \langle \nabla_r \xi^{\mu+1}, P(r)_\mu^\mu P(s)_\mu^\mu \nabla_s \xi^{\mu+1}\rangle \,\mathrm{d}r\,\mathrm{d}s \\
&= \int_0^t \int_0^t \langle P(s)_\mu^\mu \nabla_r \xi^{\mu+1}, P(r)_\mu^\mu \nabla_s \xi^{\mu+1}\rangle \,\mathrm{d}r\,\mathrm{d}s,
\end{aligned}$$

where we have used the facts that $P^* = P$ and $P(r)P(s) = P(s)P(r)$. Since $P^2 = I$ we have also that

$$\begin{aligned}
\|I_1^0(P)_\mu^{\mu+1}\xi^\mu\|^2 &= \langle \xi^\mu, \Big(I_1^0(I_0^1(P)_\mu P^\mu) + I_0^1(P_\mu I_1^0(P)^\mu) + I_0^0(P_\mu P^\mu)\Big)(t)_\mu^\mu \xi^\mu\rangle \\
&= 2\Re \int_0^t \langle \nabla_s \xi^\mu, I_0^1(P)(s)_\mu^{\mu-1} P(s)_\mu^\mu \xi^\mu\rangle \,\mathrm{d}s + \int_0^t \langle \xi^\mu, \xi^\mu\rangle \,\mathrm{d}s \\
&= 2\Re \int_0^t \int_0^s \langle P(r)_{\mu-1}^{\mu-1} \nabla_s \xi^\mu, \nabla_r P(s)_\mu^\mu \xi^\mu\rangle \,\mathrm{d}r\,\mathrm{d}s + t\|\xi^\mu\|^2 \\
&= -\|I_0^1(P)(t)_\mu^{\mu-1}\xi^\mu\|^2 + t\|\xi^\mu\|^2,
\end{aligned}$$

where the last equality holds as $P(s)$ anticommutes with $\nabla_r$ if $r \leq s$. Hence

$$\sum_{\mu \geq 0} \|\sum_{\nu \geq 0} B(t)_\nu^\mu \xi^\nu\|^2 = \sum_{\mu \geq 0} \|I_0^1(P)(t)_{\mu+1}^\mu \xi^{\mu+1}\|^2 \leq t \sum_{\mu \geq 0} \|\xi^\mu\|^2,$$

so $\xi \in \mathcal{D}(\hat{B})$ for any $\xi \in \mathfrak{H}$ and $\|\hat{B}(t)\xi\| \leq \sqrt{t}\,\|\xi\|$, so $B(t) \in \mathfrak{B}$ for all $t \in [0,1]$. As $\mathfrak{B}$ is a $*$-algebra the same statement holds for $B^*$. This working (or rather, a slight variation) shows that

$$B(s)B^*(t) + B^*(t)B(s) = T(s \wedge t) \quad \forall\, s,t \in [0,1]$$

i.e., that $B$ and $B^*$ obey (one of the) canonical anticommutation relations (see [BrR]).

□

Our methods allow us to reproduce the quantum Itô product formula for bounded processes (see [HuP, Theorem 4.5], [AtM, Théorème 4], [Att, Theorem 3]).

**Proposition 3.4.7** *For $\alpha, \beta \in \{0,1\}$ let $F^\alpha_\beta, G^\alpha_\beta \in \mathfrak{L}^p_{\mathfrak{B}}$, where $p := 2(2-\alpha-\beta)^{-1}$, be adapted processes and let*

$$M := \sum_{(\alpha,\beta)} I^\alpha_\beta(F^\beta_\alpha) \text{ and } N := \sum_{(\alpha,\beta)} I^\alpha_\beta(G^\beta_\alpha).$$

*If $M, N \in \mathfrak{L}^\infty_{\mathfrak{B}}$ then*

$$MN = \sum_{(\alpha,\beta)} I^\alpha_\beta(MG^\beta_\alpha) + I^\alpha_\beta(F^\beta_\alpha N) + I^\alpha_\beta(F^\beta_1 G^1_\alpha).$$

**Proof**

From the definitions of $M$ and $N$, the identity (3.12) yields the equality

$$\begin{aligned}&\sum_{\pi=0}^{n} M^\mu_\pi N^\pi_\nu \\ &= \sum_{(\alpha,\beta)} I^\alpha_\beta(\sum_{\pi=0}^{n-\beta} M_\pi (G^\beta_\alpha)^\pi)^\mu_\nu + I^\alpha_\beta(\sum_{\pi=0}^{n-\alpha}(F^\beta_\alpha)_\pi N^\pi)^\mu_\nu + I^\alpha_\beta(\sum_{\pi=0}^{n-1}(F^\beta_1)_\pi (G^1_\alpha)^\pi)^\mu_\nu.\end{aligned}$$

The left-hand side converges strongly to $(MN)^\mu_\nu$ almost everywhere. Now for all $t \in [0,1]$

$$\begin{aligned}\|\sum_{\pi=0}^{n-\beta} M(t)^{\mu-\beta}_\pi G^\beta_\alpha(t)^\pi_{\nu-a}\| &= \|\sum_{\pi=0}^{n-\beta} E_{\mu-\beta}\hat{M}(t)E^*_\pi E_\pi \hat{G}^\beta_\alpha(t) E^*_{\nu-\alpha}\| \\ &\leq \|\hat{M}(t)\|\,\|\hat{G}^\beta_\alpha(t)\|,\end{aligned}$$

and $\sum_{\pi=0}^{n-\beta} M(t)^{\mu-\beta}_\pi G^\beta_\alpha(t)^\pi_{\nu-\alpha}$ converges strongly to $(MG)(t)^{\mu-\beta}_{\nu-\alpha}$. Hence, by Proposition 3.1.6,

$$\sum_{(\alpha,\beta)} I^\alpha_\beta(\sum_{\pi=0}^{n-\beta} M_\pi (G^\beta_\alpha)^\pi) \xrightarrow{w.o.} \sum_{(\alpha,\beta)} I^\alpha_\beta(MG^\beta_\alpha)$$

as $n \to \infty$. The two other terms may be handled similarly.

□

Note that if $F^\alpha_\beta$ and $M$ are as in the proposition above, to show that $M \in \mathfrak{L}^\infty_{\mathfrak{B}}$ it suffices to demonstrate that $(I^1_1(F^1_1) + I^1_0(F^0_1) + I^0_1(F^1_0))(t) \in \mathfrak{B}$ for all $t \in [0,1]$, by Propositions 3.1.4, 3.2.15 and 3.2.16.

### $\Omega$-Adapted Processes

We turn now to integrals of $\Omega$-adapted processes, which have a quite different Itô formula to that obeyed by integrals of processes that are adapted.

**Proposition 3.4.8** *Let $\alpha, \beta, \gamma, \delta \in \{0,1\}$ and let $F \in \mathfrak{L}^p$, $G \in \mathfrak{L}^q$ (where $p = 2(2-\alpha-\beta)^{-1}$ and $q = 2(2-\gamma-\delta)^{-1}$) be $\Omega$-adapted. The identity*

$$I^\alpha_\beta(F)_\pi I^\gamma_\delta(G)^\pi = \delta^\alpha_0 I^0_\beta(F_\pi I^\gamma_\delta(G)^\pi) + \delta^0_\delta I^\gamma_0(I^\alpha_\beta(F)_\pi G^\pi) + \hat{\delta}^\alpha_\delta I^\gamma_\beta(F_{\pi-1}G^{\pi-1})$$

*holds for all $\pi \geq 0$. Furthermore, if $F \in \mathfrak{L}^p_{\mathfrak{B}}$ and $G \in \mathfrak{L}^q_{\mathfrak{B}}$ then*

$$I^\alpha_\beta(F) I^\gamma_\delta(G) = \delta^\alpha_0 I^0_\beta(F I^\gamma_\delta(G)) + \delta^0_\delta I^\gamma_0(I^\alpha_\beta(F) G) + \hat{\delta}^\alpha_\delta I^\gamma_\beta(FG). \tag{3.13}$$

**Proof**

Recall that

$$(\mathbb{E}^i_{t]}f)(t_1,\ldots,t_i) = \prod_{l=1}^{i} \chi_{[0,t]}(t_l) f(t_1,\ldots,t_i),$$

so that $\nabla_s \mathbb{E}_{t]} f = 0$ if $s > t$. From the identity (3.5) we see that

$$\begin{aligned}
&\langle \xi, I^\alpha_\beta(F)(t)^\mu_\pi I^\gamma_\delta(G)(t)^\pi_\nu \eta\rangle \\
&= \langle I^\beta_\alpha(F^*)(t)^\pi_\mu \xi, I^\gamma_\delta(G)(t)^\pi_\nu \eta\rangle \\
&= \int_0^t \int_0^s \langle \nabla^\delta_s F^*(r)^{\pi-\alpha}_{\mu-\beta} \nabla^\beta_r \xi, \nabla^\alpha_r G(s)^{\pi-\delta}_{\nu-\gamma} \nabla^\gamma_s \eta\rangle \, \mathrm{d}r \, \mathrm{d}s \\
&\qquad + \int_0^t \int_0^r \langle \nabla^\delta_s F^*(r)^{\pi-\alpha}_{\mu-\beta} \nabla^\beta_r \xi, \nabla^\alpha_r G(s)^{\pi-\delta}_{\nu-\gamma} \nabla^\gamma_s \eta\rangle \, \mathrm{d}s \, \mathrm{d}r \\
&\qquad\qquad + \hat{\delta}^\alpha_\delta \int_0^t \langle F^*(s)^{\pi-\alpha}_{\mu-\beta} \nabla^\beta_s \xi, G(s)^{\pi-\delta}_{\nu-\gamma} \nabla^\gamma_s \eta\rangle \, \mathrm{d}s \\
&= \delta^0_\delta \int_0^t \langle \xi, I^\alpha_\beta(F)(s)^\mu_\pi G(s)^\pi_{\nu-\gamma} \nabla^\gamma_s \eta\rangle \, \mathrm{d}s \\
&\qquad + \delta^\alpha_0 \int_0^t \langle F^*(r)^\pi_{\mu-\beta} \nabla^\beta_r \xi, I^\gamma_\delta(G)(r)^\pi_\nu \eta\rangle \, \mathrm{d}r \\
&\qquad\qquad + \hat{\delta}^\alpha_\delta \int_0^t \langle \nabla^\beta_s \xi, F(s)^{\mu-\beta}_{\pi-1} G(s)^{\pi-1}_{\nu-\gamma} \nabla^\gamma_s \eta\rangle \, \mathrm{d}s \\
&= \delta^0_\delta \langle \xi, I^\gamma_0(I^\alpha_\beta(F)_\pi G^\pi)(t)^\mu_\nu \eta\rangle + \delta^\alpha_0 \langle \xi, I^0_\beta(F_\pi I^\gamma_\delta(G)^\pi)(t)^\mu_\nu \eta\rangle \\
&\qquad\qquad + \hat{\delta}^\alpha_\delta \langle \xi, I^\gamma_\beta(F_{\pi-1}G^{\pi-1})(t)^\mu_\nu \eta\rangle
\end{aligned}$$

which gives us the first result. The statement about bounded processes is immediate, all the terms are bounded.

□

We may use this proposition to provide the product Itô formula for $\Omega$-semimartingales.

**Corollary 3.4.9** *Let $M$ and $M'$ be $\Omega$-semimartingales, with initial values $z$ and $z'$, respectively, and integral representations*

$$\mathrm{d}M = E\,\mathrm{d}\Lambda + F\,\mathrm{d}A + G\,\mathrm{d}A^\dagger + H\,\mathrm{d}t$$

*and*

$$\mathrm{d}M' = E'\,\mathrm{d}\Lambda + F'\,\mathrm{d}A + G'\,\mathrm{d}A^\dagger + H'\,\mathrm{d}t.$$

*The process $MM'$ is a $\Omega$-semimartingale, with initial value $zz'$ and integral representation*

$$\begin{aligned} \mathrm{d}(MM') &= (zE' + z'E + EE')\,\mathrm{d}\Lambda + (MF' + z'F + FE')\,\mathrm{d}A \\ &\quad + (zG' + GM' + EG')\,\mathrm{d}A^\dagger + (MH' + HM' + FG')\,\mathrm{d}t. \end{aligned} \tag{3.14}$$

**Proof**

The fact that $MM'(0) = zz'I$ is trivial, and the integrands in the supposed integral representation of $MM'$ are easily seen to be well-defined and to satisfy the appropriate integrability conditions. The fact that they are $\Omega$-adapted follows from observing that $MF' = (M - zI)F' + zF'$, etc. Finally, if we let

$$(E, F, G, H) = (F_1^1, F_1^0, F_0^1, F_0^0) \text{ and } (E', F', G', H') = (G_1^1, G_1^0, G_0^1, G_0^0),$$

Proposition 3.4.8 yields the equality

$$\begin{aligned} &(M - zI)(M' - z'I) \\ &= \sum_{(\alpha,\beta)} I_\beta^\alpha(F_\alpha^\beta) \sum_{(\gamma,\delta)} I_\delta^\gamma(G_\gamma^\delta) \\ &= \sum_{(\alpha,\beta)} \sum_{(\gamma,\delta)} \delta_0^\alpha I_\beta^0(F_0^\beta I_\delta^\gamma(G_\gamma^\delta)) + \delta_\delta^0 I_0^\gamma(I_\beta^\alpha(F_\alpha^\beta) G_\gamma^0) + \not{\delta}_\delta^\alpha I_\beta^\gamma(F_1^\beta G_\gamma^1) \\ &= \sum_{\beta=0}^1 I_\beta^0(F_0^\beta(M' - z'I)) + \sum_{\alpha=0}^1 I_0^\alpha((M - zI)G_\alpha^0) + \sum_{(\alpha,\beta)} I_\beta^\alpha(F_1^\beta G_\alpha^1) \\ &= -z'(M - zI) + z' \sum_{\beta=0}^1 I_\beta^1(F_1^\beta) - z(M' - z'I) + z \sum_\alpha I_1^\alpha(G_\alpha^1) \\ &\qquad + \sum_{\beta=0}^1 I_\beta^0(F_0^\beta M') + \sum_{\alpha=0}^1 I_0^\alpha(MG_\alpha^0) + \sum_{(\alpha,\beta)} I_\beta^\alpha(F_1^\beta G_\alpha^1). \end{aligned}$$

Translating this into the notation used in the statement of the proposition gives

$$\begin{aligned} MM' - zz'I &= I_1^1(z'E + zE' + EE') + I_0^1(z'F + MF' + FE') \\ &\quad + I_1^0(zG' + GM' + EG') + I_0^0(HM' + MH' + FG'), \end{aligned}$$

as claimed.

□

This proposition shows that the class of $\Omega$-semimartingales forms a $*$-algebra when equipped with the obvious algebraic structure. We have the following polynomial Itô formula; cf. [Vin, Lemma 4.1] and Theorem 3.4.6 above.

**Proposition 3.4.10** *Let $M$ be a $\Omega$-semimartingale, with integral representation*

$$\mathrm{d}M = E\,\mathrm{d}\Lambda + F\,\mathrm{d}A + G\,\mathrm{d}A^\dagger + H\,\mathrm{d}t$$

*and initial value $z$. For $n \geq 1$ we have that*

$$M^n = z^n I + I_1^1(E_n) + I_0^1(F_n) + I_1^0(G_n) + I_0^0(H_n), \tag{3.15}$$

*where $E_n$, $F_n$, $G_n$ and $H_n$ are bounded, $\Omega$-adapted processes given by*

$$E_n = (zI + E)^n - z^n I, \tag{3.16}$$

$$F_n = \sum_{\alpha+\beta=n-1} M^\alpha F (zI + E)^\beta, \tag{3.17}$$

$$G_n = \sum_{\alpha+\beta=n-1} (zI + E)^\alpha G M^\beta, \tag{3.18}$$

$$H_n = \sum_{\alpha+\beta=n-1} M^\alpha H M^\beta + \sum_{\alpha+\beta+\gamma=n-2} M^\alpha F (zI + E)^\beta G M^\gamma, \tag{3.19}$$

*the summation being taken over non-negative integers.*

**Proof**

The proposition is seen to be true for the case $n = 1$. We proceed by induction; suppose $M^k$ has the form above. Then $M^k$ is a $\Omega$-semimartingale, as is $M$, whence so is their product $M^{k+1} = M^k M$, and

$$\begin{aligned} \mathrm{d}(M^{k+1}) &= (zE_k + z^k E + E_k E)\,\mathrm{d}\Lambda + (zF_k + M^k F + F_k E)\,\mathrm{d}A^\dagger \\ &\quad + (z^k G + G_k M + E_k G)\,\mathrm{d}A + (H_k M + M^k H + F_k G)\,\mathrm{d}t \end{aligned}$$

by the product formula (3.13). Simplifying gives the identities (3.16)-(3.19) above; for example

$$\begin{aligned}
&M^kH + H_kM + F_kG \\
&= M^kH + \sum_{\alpha+\beta=k-1} M^\alpha H M^{\beta+1} + \sum_{\alpha+\beta+\gamma=k-2} M^\alpha F(zI+E)^\beta G M^{\gamma+1} \\
&\qquad + \sum_{\alpha+\beta=k-1} M^\alpha F(zI+E)^\beta G \\
&= \sum_{\alpha+\beta=k} M^\alpha H M^\beta + \sum_{\alpha+\beta+\gamma=k-1} M^\alpha F(zI+E)^\beta G M^\gamma
\end{aligned}$$

as claimed.

□

## 3.5 Multi-Dimensional Calculus

We show that the theory above generalises in a natural manner to the case of countable multiplicity: the base space $L^2[0,1]$ is replaced by $L^2([0,1];\mathfrak{k})$, where $\mathfrak{k}$ is some separable Hilbert space. The formulae developed above have the same appearance in this situation and their proofs require little, if any, modification.

### The Gradient

Let $\mathfrak{k}$ denote a separable Hilbert space, and let $(e_i)_{i=1}^N$ be a maximal orthonormal sequence in $\mathfrak{k}$ (Hilbert basis), where $N \in \mathbb{N} \cup \{\infty\}$. By convention $\{1,\dots,N\}$ has its usual meaning if $N$ is finite, and denotes $\mathbb{N}$ if $N$ is countably infinite. A similar meaning is given to expressions like $k = 1,\dots,N$. We consider the Fock space $\mathfrak{H} = \mathfrak{F}_+(L^2([0,1];\mathfrak{k}))$; the case $N = 1$ corresponds to the situation discussed above.

**Proposition 3.5.1** *Let $n \geq 1$ and let $f \in \mathfrak{k}$. The map*

$$\nabla_0(f)_n^{n-1} : \phi^{\otimes n} \mapsto \sqrt{n}\langle f, \phi\rangle_{\mathfrak{k}} \phi^{\otimes n-1} \tag{3.20}$$

*extends to a bounded linear operator $\nabla(f)_n^{n-1}$ from $\mathfrak{h}_n$ to $L^2([0,1];\mathfrak{h}_{n-1})$, such that $\|\nabla(f)\| = \sqrt{n}\,\|f\|$. In particular, $\nabla(f) \in \mathfrak{D}_1(\mathfrak{H}; L^2([0,1];\mathfrak{H}))$ has growth of order $\frac{1}{2}$, where $\nabla(f)_\nu^\mu := 0$ if $\mu \neq \nu - 1$.*

**Proof**

To prove this we split $\nabla(f)$ into three maps,

$$\mathfrak{h}_n \xrightarrow{T_1} \mathfrak{h} \otimes \mathfrak{h}_{n-1} \xrightarrow{T_2} L^2([0,1]) \otimes \mathfrak{h}_{n-1} \xrightarrow{T_3} L^2([0,1];\mathfrak{h}_{n-1}),$$

where $\mathfrak{h} := \mathfrak{h}_1 = L^2([0,1];\mathfrak{k})$. Consider first the map $T_1$, whose action on the dense subspace $\mathrm{lin}\{\phi^{\otimes n} : \phi \in \mathfrak{h}\}$ is given by

$$T_1 : \phi^{\otimes n} \mapsto \phi \otimes \phi^{\otimes n-1} \in \mathfrak{h} \otimes \mathfrak{h}_{n-1}$$

and linearity. It is easy to verify that $T_1$ is well-defined, and since

$$\langle T_1 \phi^{\otimes n}, T_1 \psi^{\otimes n} \rangle = \langle \phi \otimes \phi^{\otimes n-1}, \psi \otimes \psi^{\otimes n-1} \rangle = \langle \phi, \psi \rangle^n = \langle \phi^{\otimes n}, \psi^{\otimes n} \rangle,$$

we see that $T_1$ is an isometry, and so extends to an isometry from $\mathfrak{h}_n$ to $\mathfrak{h} \otimes \mathfrak{h}_{n-1}$. Now let $T_2' : \mathfrak{h} \to L^2([0,1]); \phi \mapsto \langle f, \phi \rangle$. As

$$\|T_2' \phi\|^2 = \int_0^1 |\langle f, \phi(s) \rangle|^2 \,\mathrm{d}s \le \|f\|^2 \int_0^1 \|\phi(s)\|^2 \,\mathrm{d}s = \|f\|^2 \|\phi\|^2$$

we see that $T_2'$ has norm at most $\|f\|$; taking $\phi \equiv f$ shows that this upper bound is attained. We define $T_2 : \mathfrak{h} \otimes \mathfrak{h}_{n-1} \to L^2([0,1]) \otimes \mathfrak{h}_{n-1}$ by setting $T_2 = \sqrt{n} T_2' \otimes \mathrm{id}_{\mathfrak{h}_{n-1}}$. The map $T_3$ acts on $\mathrm{lin}\{f \otimes \xi : f \in L^2(T), \xi \in \mathfrak{h}_{n-1}\}$ by

$$(T_3 f \otimes \xi)(s) = f(s) \xi$$

and linearity, and as

$$\langle T_3 f \otimes \xi, T_3 g \otimes \zeta \rangle = \int_T \overline{f}(s) g(s) \,\mathrm{d}s \langle \xi, \zeta \rangle = \langle f \otimes \xi, g \otimes \zeta \rangle,$$

we see that $T_3$ is an isometry (in fact, it is the natural Hilbert-space isomorphism $L^2[0,1] \otimes \mathfrak{h}_{n-1} \cong L^2([0,1]; \mathfrak{h}_{n-1})$).

Since $\nabla_0(f)_n^{n-1} = T_3 T_2 T_1$ on $\mathrm{lin}\{\phi^n\}$, we let $\nabla(f)_n^{n-1} = T_3 T_2 T_1$ and note that $\|\nabla(f)\| = \sqrt{n}\,\|f\|$, as claimed.

□

We denote the adjoint of the gradient by $\overset{*}{\nabla}(f)$. The action of $\overset{*}{\nabla}(f)_\mu^{\mu+1}$ on the set $\{u(\cdot)\phi^{\otimes \mu} : u \in L^2[0,1], \phi \in \mathfrak{h}\}$ is given by

$$\overset{*}{\nabla}(f)_\mu^{\mu+1} u(\cdot) \phi^{\otimes \mu} = \frac{1}{\sqrt{\mu+1}} \frac{\mathrm{d}}{\mathrm{d}x} (\phi + x u(\cdot) f)^{\otimes \mu+1} \bigg|_{x=0}, \tag{3.21}$$

and this defines $\nabla(f)_\mu^{\mu+1}$ everywhere, as this set is total in $\mathfrak{h}_\mu$. (As we remarked in the proof above, $L^2[0,1] \otimes \mathfrak{h}_\lambda \cong L^2([0,1]; \mathfrak{h}_\lambda)$ *via* the identification $(u \otimes \xi)(t) := u(t)\xi$, and $\mathcal{H}_0 \underline{\otimes} \mathcal{K}_0$ is dense in $\mathcal{H} \otimes \mathcal{K}$ if $\mathcal{H}_0$ and $\mathcal{K}_0$ are total subsets of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively.)

The $N$-dimensional analogue of the quantum Skorokhod isometry (2.4) is the following.

**Proposition 3.5.2** *If $\mu \geq 0$, $f, g \in \mathfrak{k}$ and $\Phi, \Psi \in L^2([0,1]; \mathfrak{h}_\mu)$ then*

$$\begin{aligned} \langle \overset{*}{\nabla} (f)_\mu^{\mu+1} \Phi, \overset{*}{\nabla} (g)_\mu^{\mu+1} \Psi \rangle &= \langle f, g \rangle \langle \Phi, \Psi \rangle \\ &\quad + \int_0^1 \int_0^1 \langle \nabla(g)_\mu^{\mu-1}[\Phi(r)](s), \nabla(f)_\mu^{\mu-1}[\Psi(s)](r) \rangle \,\mathrm{d}r\,\mathrm{d}s. \end{aligned} \tag{3.22}$$

**Proof**

Without loss of generality we may assume that $\Phi = u(\cdot)\phi^{\otimes\mu}$ and $\Psi = v(\cdot)\psi^{\otimes\mu}$, where $u, v \in L^2[0,1]$ and $\phi, \psi \in L^2([0,1]; \mathfrak{k})$. Using (3.21) we see that the left-hand side of (3.22) equals

$$\begin{aligned} (\mu+1)^{-1} \frac{\partial^2}{\partial x\, \partial y} \langle \phi + xu(\cdot)f, \psi + yv(\cdot)g \rangle^{\mu+1} \Big|_{x=y=0} \\ &= \frac{\partial}{\partial x} \langle \phi + xu(\cdot)f, v(\cdot)g \rangle \langle \phi + xu(\cdot)f, \psi \rangle^\mu \Big|_{x=0} \\ &= \langle u(\cdot)f, v(\cdot)g \rangle \langle \phi, \psi \rangle^\mu + \mu \langle \phi, v(\cdot)g \rangle \langle u(\cdot)f, \psi \rangle \langle \phi, \psi \rangle^{\mu-1}, \end{aligned}$$

and this is equal to the right-hand side of (3.22). □

**QS Integrals**

To make our notation similar to that used earlier, let $\nabla^a := \nabla(e_a)$ and $\nabla_t^a \xi := (\nabla^a \xi)(t)$ for $a = 1, \ldots, N$. We define $\alpha'$ by

$$\alpha' := \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha \geq 1 \end{cases}$$

for $\alpha \in \{0, \ldots, N\}$. The following proposition defines the $(N+1)^2$ QS integrals in the same manner as Proposition 3.1.1 did for the one-dimensional case.

**Proposition 3.5.3** *Let $\alpha, \beta \in \{0, \ldots, N\}$. For all $F \in \mathfrak{L}^p$ there exists a unique process $I_\beta^\alpha(F) \in \mathfrak{L}^\infty$ such that*

$$\langle \xi, I_\beta^\alpha(F)(t)_\nu^\mu \eta \rangle = \int_0^t \langle \nabla_s^\beta \xi, F(s)_{\nu-\alpha'}^{\mu-\beta'} \nabla_s^\alpha \eta \rangle \,\mathrm{d}s \tag{3.23}$$

*for all $t \in [0,1]$, $\mu, \nu \geq 0$ and $\xi \in \mathfrak{h}_\mu, \eta \in \mathfrak{h}_\nu$. The mapping $I_\beta^\alpha : \mathfrak{L}^p \to \mathfrak{L}^\infty$ is linear, satisfies the identity $I_\beta^\alpha(F)^* = I_\alpha^\beta(F^*)$ and is such that $I_\beta^\alpha(F) = 0$ if $F(t) = 0$ for almost all $t \in [0,1]$. Furthermore*

$$\| I_\beta^\alpha(F)(t)_\nu^\mu \|_\infty \leq \mu^{\frac{\beta'}{2}} \nu^{\frac{\alpha'}{2}} \| [F]_{\nu-\alpha'}^{\mu-\beta'} \|_p \tag{3.24}$$

*for all $t \in [0,1]$, so $I_\beta^\alpha : \mathcal{L}^p \to \mathcal{L}^\infty$ is continuous.*

**Adaptedness**

Let $a \in \{1, \ldots, N\}$ and for $0 \leq s \leq t \leq 1$ let $\int_s^t \nabla_p^a \, \mathrm{d}p \in \mathfrak{D}_1$ be the super-diagonal matrix with entries given by

$$(\int_s^t \nabla_p^a \, \mathrm{d}p)_\mu^{\mu-1} : \xi \mapsto \int_s^t (\nabla(e_a)_\mu^{\mu-1}\xi)(p) \, \mathrm{d}p.$$

We see from this that $\|(\int_s^t \nabla_p \, \mathrm{d}p)_\mu^{\mu-1}\| \leq (t-s)^{\frac{1}{2}}\sqrt{\mu}$.

**Definition 3.5.4** *The process $F \in \mathfrak{L}$ is $\nabla$-adapted if, for all $a \in \{1, \ldots, N\}$ and $s \in [0,1]$*

$$\int_s^t \nabla_p^a \, \mathrm{d}p \, F(s) = F(s) \int_s^t \nabla_p^a \, \mathrm{d}p \quad \forall\, t \in [s,1].$$

*Equivalently, for all $a \in \{1, \ldots, N\}$, $\mu, \nu \geq 0$, $s \in [0,1]$ and $f \in \mathfrak{h}_\nu$*

$$\nabla_t^a F(s)_\nu^{\mu+1} f = F(s)_{\nu-1}^\mu \nabla_t^a f \quad (a.e. \ t \geq s).$$

This definition is the natural generalisation of the one-dimensional case, and is exactly what is required for the proof of the local Itô formula.

**Proposition 3.5.5** *Let $\alpha, \beta, \gamma, \delta \in \{0, \ldots, N\}$ and let $F \in \mathfrak{L}^p$, $G \in \mathfrak{L}^q$ (where $p = 2(2-\alpha'-\beta')^{-1}$ and $q = 2(2-\gamma'-\delta')^{-1}$) be $\nabla$-adapted. Then*

$$\begin{aligned} &\langle I_\beta^\alpha(F)(t)_\nu^\mu \xi, I_\delta^\gamma(G)(t)_\pi^\mu \eta \rangle \\ &\quad = \langle \xi, \Big( I_\alpha^\beta(F^*_{\mu-\beta'} I_\delta^\gamma(G)^{\mu-\beta'}) + I_\delta^\gamma(I_\alpha^\beta(F^*)_{\mu-\delta'} G^{\mu-\delta'}) + \hat{\delta}_\delta^\beta I_\alpha^\gamma(F^*_{\mu-\beta'} G^{\mu-\delta'}) \Big)(t)_\pi^\nu \eta \rangle \end{aligned}$$

*for all $\mu, \nu, \pi \geq 0$, $\xi \in \mathfrak{h}_\nu$, $\eta \in \mathfrak{h}_\pi$ and $t \in [0,1]$.*

**Proof**

This follows the one-dimensional proof, which relies on the integration-by-parts formula (3.5). To prove the $N$-dimensional version of this, we note that the only difficulty occurs when $\beta' = \delta' = 1$, and then we use the generalisation of the quantum Skorokhod isometry, (3.22).

□

# Chapter 4

# Quantum Stochastic Differential Equations

We consider solutions of the quantum stochastic differential equation (henceforth QSDE)

$$U(0) = \mathrm{id}_{\tilde{\mathfrak{H}}},\ \mathrm{d}U(s) = U(s)E(s)\,\mathrm{d}\Lambda + U(s)F(s)\,\mathrm{d}A(s) + U(s)G(s)\,\mathrm{d}A^\dagger(s) + U(s)H(s)\,\mathrm{d}s$$

where the coefficients $E$, $F$, $G$ and $H$ are time-dependent, bounded, adapted processes acting on the whole Fock space. We provide sufficient conditions for a solution to exist in terms of integrability requirements for the coefficients. The solution is shown to be isometric if certain conditions are satisfied, which are a straightforward generalisation of the conditions of [HuP, Section 7]. We recall Attal's example of an isometric but non-unitary process with coefficients that satisfy the necessary conditions for unitarity, showing that these conditions are not sufficient. We note two sufficient conditions for unitarity, neither of which is necessary.

We investigate a modified version of the evolution equation, where the coefficients are bounded, $\Omega$-adapted processes. We demonstrate the existence of a unique solution. Conditions that are necessary for the solution to be unitary are found and we show that, in contrast to the adapted case, these conditions are also sufficient for the unitarity of the solution.

## 4.1 Preliminaries

Throughout this section we deal with $\tilde{\mathfrak{H}}$, Boson Fock space equipped with the initial space $\mathcal{H}$. We introduce enough of the standard theory of quantum stochastic calculus for our needs; more details can be found in [HuP], [AtM], [Mey] and [Par].

We let $\mathcal{E}_{\mathrm{lb}}$ denote the linear span of the set of exponential vectors corresponding to elements of $L^\infty[0,1]$, i.e., $\mathcal{E}_{\mathrm{lb}} := \operatorname{lin}\{\varepsilon(u) : u \in L^\infty[0,1]\}$, and let $\tilde{\mathcal{E}}_{\mathrm{lb}}$ denote the algebraic tensor product of the initial space with $\mathcal{E}_{\mathrm{lb}}$; this space is dense in $\tilde{\mathfrak{H}}$. A *process* $F = (F(t) : t \in [0,1])$ is a family of operators with common domain including $\tilde{\mathcal{E}}_{\mathrm{lb}}$ such that $t \mapsto F(t) f \otimes \varepsilon(u)$ is strongly measurable for all $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$. A process is *adapted* if it satisfies the equality

$$\langle f \otimes \varepsilon(u), F(t) g \otimes \varepsilon(v) \rangle \;=\; \langle f \otimes \varepsilon(u_{t]}), F(t) g \otimes \varepsilon(v_{t]}) \rangle \langle \varepsilon(u_{(t}), \varepsilon(v_{(t}) \rangle \qquad (4.1)$$

for all $f \otimes \varepsilon(u), g \otimes \varepsilon(v) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$ and all $t \in [0,1]$ (cf. Section 3.2). If $E$, $F$, $G$ and $H$ are adapted processes that satisfy

$$\int_0^1 |u(s)|^2 \|E(s) f \otimes \varepsilon(u)\|^2 + |u(s)|\, \|F(s) f \otimes \varepsilon(u)\| + \|G(s) f \otimes \varepsilon(u)\|^2 + \|H(s) f \otimes \varepsilon(u)\| \,\mathrm{d}s \;<\; \infty \qquad (4.2)$$

for all $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$ then the quantum stochastic integrals

$$M(t) := \int_0^t E(s)\,\mathrm{d}\Lambda(s) + F(s)\,\mathrm{d}A(s) + G(s)\,\mathrm{d}A^\dagger(s) + H(s)\,\mathrm{d}s$$

are well-defined as the unique adapted process with domain $\tilde{\mathcal{E}}_{\mathrm{lb}}$ such that

$$\langle f \otimes \varepsilon(u), M(t) g \otimes \varepsilon(v) \rangle = \int_0^t \langle f \otimes \varepsilon(u), (\overline{u} v E + v F + \overline{u} G + H)(s) g \otimes \varepsilon(v) \rangle \,\mathrm{d}s.$$

The following is a slight extension of a special case of Hudson and Parthasarathy's quantum Itô formula [HuP, Theorem 4.4], which is sufficient for our needs. For $p \in [1,\infty]$ we say that a process $X$ is *p-integrable* if $t \mapsto \|X(t) f \otimes \varepsilon(u)\|$ is an element of $L^p[0,1]$ for all $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$.

**Lemma 4.1.1** *Let $E$, $F$ and $G$ be square-integrable, adapted processes and let $H$ be an integrable, adapted process. The process $M$ given by*

$$M(t) = \int_0^t E(s)\,\mathrm{d}\Lambda(s) + F(s)\,\mathrm{d}A(s) + G(s)\,\mathrm{d}A^\dagger(s) + H(s)\,\mathrm{d}s$$

*is well-defined and such that, for all $t \in [0,1]$ and $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$,*

$$\|M(t) f \otimes \varepsilon(u)\|^2 \;=\; 2\Re \int_0^t \langle M(s) f \otimes \varepsilon(u), (|u|^2 E + u F + \overline{u} G + H)(s) f \otimes \varepsilon(u) \rangle \,\mathrm{d}s + \int_0^t \|(uE + G)(s) f \otimes \varepsilon(u)\|^2 \,\mathrm{d}s. \qquad (4.3)$$

**Proof**

The conditions on the integrands are sufficient to ensure they satisfy (4.2) for all $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$, so $M$ is well-defined. If $H \equiv 0$ the norm equality is immediate from [HuP, Theorem 4.4]. Otherwise, let $N = \int E \,\mathrm{d}\Lambda + F \,\mathrm{d}A + G \,\mathrm{d}A^\dagger$:

$$\begin{aligned} \|M(t)f \otimes \varepsilon(u)\|^2 &= \|N(t)f \otimes \varepsilon(u)\|^2 + \|\int_0^t H(s)f \otimes \varepsilon(u)\,\mathrm{d}s\|^2 \\ &\quad + 2\Re \int_0^t \langle N(t)f \otimes \varepsilon(u), H(s)f \otimes \varepsilon(u)\rangle \,\mathrm{d}s. \end{aligned}$$

If we rewrite the right-hand side of this equality to give us what we want, we have a remainder term

$$\begin{aligned} &2\Re \int_0^t \langle (M(t) - M(s))f \otimes \varepsilon(u), H(s)f \otimes \varepsilon(u)\rangle \,\mathrm{d}s \\ &\quad - 2\Re \int_0^t \int_0^s \langle H(r)f \otimes \varepsilon(u), (|u|^2 E + uF + \overline{u}G + H)(s)f \otimes \varepsilon(u)\rangle \,\mathrm{d}r \,\mathrm{d}s, \end{aligned}$$

and the first term in this expression is equal to

$$2\Re \int_0^t \int_s^t \langle (|u|^2 E + uF + \overline{u}G + H)(r)f \otimes \varepsilon(u), H(s)f \otimes \varepsilon(u)\rangle \,\mathrm{d}r \,\mathrm{d}s$$

using the adaptedness of $M$ and $H$ (and the techniques in Appendix A.4). This shows that the remainder term is zero and so (4.3) holds.

□

It is a consequence of the quantum Itô formula ([HuP, Corollary 4.2], [Att, Lemma 14]) that $t \mapsto M(t)f \otimes \varepsilon(u)$ is continuous for all $f \otimes \varepsilon(u) \in \tilde{\mathcal{E}}_{\mathrm{lb}}$.

## 4.2 The Evolution Equation

In [HuP, Section 7] Hudson and Parthasarathy solve the *evolution equation*

$$U(t) = \mathrm{id}_{\tilde{\mathfrak{H}}} + \int_0^t U(s)(L_1 \,\mathrm{d}\Lambda(s) + L_2 \,\mathrm{d}A(s) + L_3 \,\mathrm{d}A^\dagger(s) + L_4 \,\mathrm{d}s),$$

where the *driving coefficients* $L_1, \ldots, L_4$ are (ampliations of) bounded operators on the initial space $\mathcal{H}$. They provide necessary and sufficient conditions on these coefficients for the solution to be unitary: we say that the coefficients satisfy the *usual conditions* (which are necessary and sufficient for unitarity in this case) if

$$(L_1, L_2, L_3, L_4) = (w - \mathrm{id}_{\mathcal{H}}, l, -wl^*, ik - \tfrac{1}{2}ll^*)$$

where $w$, $l$ and $k$ are bounded operators on $\mathcal{H}$, with $w$ unitary and $k$ is self-adjoint. This problem may be extended in two ways.

One approach is to consider unbounded operators on $\mathcal{H}$; this has been dealt with in depth by many authors (see, e.g., [App], [Eva], [Fag], [MoS], [Vin3]) and will not concern us here. The other alternative is to let the driving coefficients act, not just on $\mathcal{H}$, but on the whole space $\tilde{\mathfrak{H}}$, and to consider time-dependent instead of constant coefficients, i.e., processes. The evolution equation becomes

$$\begin{aligned} U(t) &= \mathrm{id}_{\tilde{\mathfrak{H}}} + \int_0^t E(s)U(s)\,\mathrm{d}\Lambda(s) + F(s)U(s)\,\mathrm{d}A(s) \\ &\qquad + G(s)U(s)\,\mathrm{d}A^\dagger(s) + H(s)U(s)\,\mathrm{d}s \end{aligned} \tag{4.4}$$

(we have the driving coefficients on the left for technical reasons, e.g., domain considerations). This problem has received little attention in the literature (but see [Hol2]), however the following proposition is probably well-known; we learnt of it from [Att3] (the proof is our own, but follows a standard pattern of reasoning).

**Theorem 4.2.1** *If $E, G \in L^2([0,1]; \mathcal{B}(\tilde{\mathfrak{H}})_s)$ and $F, H \in L^1([0,1]; \mathcal{B}(\tilde{\mathfrak{H}})_s)$ are adapted processes then the evolution equation (4.4) admits a unique solution on the domain $\tilde{\mathcal{E}}_{\mathrm{lb}}$. If the driving coefficients satisfy the* usual conditions

$$(E, F, G, H) = (W - \mathrm{id}_{\tilde{\mathfrak{H}}}, L, -WL^*, iK - \tfrac{1}{2}LL^*) \tag{4.5}$$

*where $W(t)$ is unitary and $K(t)$ is self-adjoint for almost all $t$ (which implies that $E \in L^\infty([0,1]; \mathcal{B}(\tilde{\mathfrak{H}})_s)$ and $F \in L^2([0,1]; \mathcal{B}(\tilde{\mathfrak{H}})_s)$) then the solution is isometric.*

**Proof**

Let $X_0 \equiv \mathrm{id}_{\tilde{\mathfrak{H}}}$ and for $\nu \geq 0$ define $X_{\nu+1}$ to be the process

$$X_{\nu+1}(t) = \int_0^t E(s)X_\nu(s)\,\mathrm{d}\Lambda(s) + F(s)X_\nu(s)\,\mathrm{d}A(s) + G(s)X_\nu(s)\,\mathrm{d}A^\dagger(s) + H(s)X_\nu(s)\,\mathrm{d}s.$$

The conditions on the driving coefficients $E$, $F$, $G$ and $H$, together with the strong continuity of $X_\nu$ on $\tilde{\mathcal{E}}_{\mathrm{lb}}$, ensure this definition is a good one. From (a slight extension of) [Att, Lemma 14] we have the estimate

$$\begin{aligned} \|X_{\nu+1}(t)f \otimes \varepsilon(u)\| &\leq C \int_0^t (\|E(s)\|^2 + \|G(s)\|^2)\|X_\nu(s)f \otimes \varepsilon(u)\|^2\,\mathrm{d}s \\ &\qquad + C\Big(\int_0^t (\|F(s)\| + \|H(s)\|)\|X_\nu(s)f \otimes \varepsilon(u)\|\,\mathrm{d}s\Big)^2 \end{aligned}$$

where $C = (\|u\|_\infty^2 \vee 1)\exp(5\|u\|_\infty^2)$, so setting

$$\alpha(s) = \|E(s)\|^2 + \|G(s)\|^2 + 2(\|F(s)\| + \|H(s)\|)\int_0^s \|F(r)\| + \|H(r)\|\,\mathrm{d}r$$

we have that

$$\begin{aligned}
\|X_{\nu+1}(\cdot)f\otimes\varepsilon(u)\|^2_{C([0,t];\tilde{\mathfrak{H}})} &\leq C\int_0^t \alpha(s)\,\|X_\nu(\cdot)f\otimes\varepsilon(u)\|^2_{C([0,s];\tilde{\mathfrak{H}})}\,\mathrm{d}s \qquad (4.6)\\
&\leq C^\nu\Big(\int_0^t \alpha(s)\,\mathrm{d}s\Big)^\nu/\nu!\\
&\leq C^\nu(\|E\|_2^2 + \|F\|_1^2 + \|G\|_2^2 + \|H\|_1^2)^\nu/\nu!.
\end{aligned}$$

Thus $U = \sum_{\nu\geq 0} X_\nu$ is a strongly continuous process on $\tilde{\mathcal{E}}_{\mathrm{lb}}$, and it is easily verified that it satisfies (4.4). Uniqueness follows by taking the difference $U - V$ of two processes that obey the evolution equation and using the inequality (4.6), with '$X_{\nu+1}$' and '$X_\nu$' replaced by '$U - V$'. The proof of isometry follows from Lemma 4.1.1. Since the process $U$ is strongly continuous and the usual conditions imply that $F \in L^2([0,1];\mathcal{B}(\mathfrak{H})_s)$, Lemma 4.1.1 gives (after some algebra) that $\|U(t)f\otimes\varepsilon(u)\|^2$ is equal to

$$\|f\otimes\varepsilon(u)\|^2 + \int_0^t \langle U(s)f\otimes\varepsilon(u), (|u|^2C_1 + uC_2 + \overline{u}C_3 + C_4)(s)U(s)f\otimes\varepsilon(u)\rangle\,\mathrm{d}s.$$

Here

$$\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\end{pmatrix} = \begin{pmatrix} E + E^* + E^*E\\ F + G^* + G^*E\\ G + F^* + E^*G\\ H + H^* + G^*G\end{pmatrix},$$

and it is easy to see that the usual conditions cause the $C_i$ to vanish. In fact we can see from this that $W$ being isometric (together with $K$ self-adjoint) is a necessary and sufficient condition for the solution to be isometric.

□

It is a direct consequence of the quantum Itô formula for bounded processes that the conditions (4.5) are necessary for unitarity. They are not sufficient, however, as the following example demonstrates.

**Example**

For all $t \in [0,1]$ let $K(t) := \tilde{m}_{\operatorname{sgn} W(t)}$ be the Fock space operator corresponding to the bounded operator that acts on Wiener space as multiplication by $\operatorname{sgn} W(t)$, the sign of Brownian motion $W$ at time $t$. It is immediate that $K(t)^2 = \mathrm{id}_{\mathfrak{H}}$ and

$K(t) = K(t)^*$, so $K(t)$ is unitary, and measurability of the process $K$ follows from the joint measurability of $[0,1]\times\Omega \ni (t,\omega) \mapsto W(t,\omega)$. By Theorem 4.2.1 there exists an isometric process $T$ that satisfies the QSDE

$$T(t) = \mathrm{id}_{\mathfrak{H}} + \int_0^t (K(s) - \mathrm{id}_{\mathfrak{H}})T(s)\,\mathrm{d}\Lambda(s).$$

It may be shown that $T(1)$ corresponds to the martingale-preserving endomorphism of Wiener space

$$W \mapsto \widetilde{W};\ \widetilde{W}(t) := \int_0^t \operatorname{sgn} W(s)\,\mathrm{d}W(s)$$

(see [Att2, II.2]) called the *Lévy transform*, which is non-invertible (by Tanaka's formula and consideration of the filtration generated by $\widetilde{W}$: see [ReY, Corollary VI.2.2]). Thus $T(1)$, and furthermore $T(t)$ for all $t \neq 0$, is not unitary. This example is due to [Att2, II.2.4, pp. 56–57].

□

We see from this that a naïve generalisation of the conditions of [HuP] is not possible, and that the problem of unitarity is a ticklish one. We present two sufficient conditions, neither of which are necessary, as is easily verified.

Given an isometric solution $U$, unitarity of $U$ is equivalent to the regular quantum semimartingale $Z = UU^* - \mathrm{id}_{\tilde{\mathfrak{H}}}$ being identically zero. Using the quantum Itô formula we see that $Z$ has the representation

$$\begin{aligned} Z(0) &= 0, \\ \mathrm{d}Z &= (WZW^* - Z)\,\mathrm{d}\Lambda + (LZW^* - ZLW^*)\,\mathrm{d}A \\ &\quad + (WZL^* - WL^*Z)\,\mathrm{d}A^\dagger + (i[K,Z] - \tfrac{1}{2}\{LL^*, Z\} + LZL^*)\,\mathrm{d}t, \end{aligned} \tag{4.7}$$

where $[x,y] := xy - yx$ is the commutator of $x$ and $y$ and $\{x,y\} := xy + yx$ the anticommutator.

**Definition 4.2.2** *A process of bounded operators $F$ is* obvious *if there exists a partition $\tau = \{0 = t_0 < \cdots < t_n = 1\}$ such that*

$$F(t) \in j_{t_{k-1}}^{-1}(\mathcal{B}(\tilde{\mathfrak{H}}_{t_{k-1}]}) \otimes \mathrm{id}_{\mathfrak{H}_{(t_{k-1}}})j_{t_{k-1}} \qquad \forall\, t \in [t_{k-1}, t_k),\ k = 1, \ldots, n \tag{4.8}$$

*where, for $s \in [0,1]$,*

$$j_s^{-1}(\mathcal{B}(\tilde{\mathfrak{H}}_{s]}) \otimes \mathrm{id}_{\mathfrak{H}_{(s}})j_s := \{j_s^{-1}(A \otimes \mathrm{id}_{\mathfrak{H}_{(s}})j_s \,:\, A \in \mathcal{B}(\tilde{\mathfrak{H}}_{s]})\}.$$

It is clear that being obvious implies adaptedness. Conversely, if $F$ is a bounded, adapted process and $G$ is defined by

$$G(t) = \begin{cases} F(0) & t \in [0, \epsilon) \\ F(t-\epsilon) & t \in [\epsilon, 1] \end{cases},$$

where $\epsilon > 0$ is fixed, then $G$ is obvious. This extreme form of previsibility (cf. the classical theory of stochastic processes) is sufficient to guarantee unitarity.

**Proposition 4.2.3** *If the driving coefficients satisfy the usual conditions and are obvious then the solution to the evolution equation is unitary.*

**Proof**

We assume that $W$, $K$ and $L$ are obvious with respect to the same partition, $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$. We fix $u, v \in L^\infty[0,1]$ and define $\theta : \mathcal{H}^2 \times [0,1] \to \mathbb{C}$ by

$$\begin{aligned} \theta(f,g)(t) &:= \langle f \otimes \varepsilon(u), Z(t) g \otimes \varepsilon(v) \rangle, \\ &= \langle U^*(t) f \otimes \varepsilon(u), U^*(t) g \otimes \varepsilon(v) \rangle - \langle f \otimes \varepsilon(u), g \otimes \varepsilon(v) \rangle \end{aligned}$$

where $Z$ is the regular quantum semimartingale $UU^* - \mathrm{id}_{\tilde{\mathfrak{H}}}$. If $t \in [0, s_1)$ then we may write $K(t) = K_0(t) \otimes \mathrm{id}_{\mathfrak{H}}$ for some $K_0(t) \in \mathcal{B}(\mathcal{H})$, and similarly for $L$ and $W$. By (4.7) we see that

$$\begin{aligned} \theta(f,g)(t) = \int_0^t \Big( &(\theta(W_0^*(s)f, W_0^*(s)g)(s) - \theta(f,g)(s))\overline{u}(s)v(s) \\ &+ (\theta(L_0^*(s)f, W_0^*(s)g)(s) - \theta(f, L_0(s)W_0^*(s)g)(s))v(s) \\ &+ (\theta(W_0^*(s)f, L_0^*(s)g)(s) - \theta(L_0(s)W_0^*(s)f, g)(s))\overline{u}(s) \\ &- \theta(iK_0(s)f, g)(s) - \theta(f, iK_0(s)g)(s) + \theta(L_0^*(s)f, L_0^*(s)g))(s) \\ &- \theta(\tfrac{1}{2}L_0(s)L_0^*(s)f, g)(s) - \theta(f, \tfrac{1}{2}L_0(s)L_0^*(s)g)(s) \Big)\, \mathrm{d}s \end{aligned}$$

so

$$|\theta(f,g)(t)| \le (\|u\|_\infty \vee 1)(\|v\|_\infty \vee 1) \int_0^t \sum_{(X,Y)} |\theta(X_0(s)f, Y_0(s)g)(s)|\, \mathrm{d}s \tag{4.9}$$

where the sum is taken over all pairs in

$$\{(\mathrm{id}_{\tilde{\mathfrak{H}}}, \mathrm{id}_{\tilde{\mathfrak{H}}}), (W^*, W^*), (L^*, W^*), (W^*, L^*), (\mathrm{id}_{\tilde{\mathfrak{H}}}, LW^*), (LW^*, \mathrm{id}_{\tilde{\mathfrak{H}}}), (iK, \mathrm{id}_{\tilde{\mathfrak{H}}}), (\mathrm{id}_{\tilde{\mathfrak{H}}}, iK), (\tfrac{1}{2}LL^*, \mathrm{id}_{\tilde{\mathfrak{H}}}), (\mathrm{id}_{\tilde{\mathfrak{H}}}, \tfrac{1}{2}LL^*), (L^*, L^*)\}.$$

Iterating this we see that

$$
\begin{aligned}
|\theta(f,g)(t)| &\leq C^n \int_{\Delta_n(t)} \sum_{(\mathbf{X},\mathbf{Y})} |\theta(X_0^n(t_n)\cdots X_0^1(t_1)f, Y_0^n(t_n)\cdots Y_0^1(t_1)g)(t_n)|\,\mathrm{d}\mathbf{t} \\
&\leq 2C^n \int_{\Delta_n(t)} \sum_{(\mathbf{X},\mathbf{Y})} \prod_{j=1}^{n} \|X_0^j(t_j)\|\,\|Y_0^j(t_j)\|\,\mathrm{d}\mathbf{t}\,\|f\otimes\varepsilon(u)\|\,\|g\otimes\varepsilon(v)\| \\
&\leq \frac{2C^n}{n!}\Big(\int_0^t \sum_{(X,Y)} \|X_0(s)\|\,\|Y_0(s)\|\,\mathrm{d}s\Big)^n \|f\otimes\varepsilon(u)\|\,\|g\otimes\varepsilon(v)\|
\end{aligned}
$$

where $C = (\|u\|_\infty \vee 1)(\|v\|_\infty \vee 1)$, $\Delta_n(t)$ is the region $\{(t_1,\ldots,t_n) : 0 < t_n < \cdots < t_1 < t\}$ and the sum $\sum_{(\mathbf{X},\mathbf{Y})}$ is the $n$-fold sum $\sum_{(X^1,Y^1)}\cdots\sum_{(X^n,Y^n)}$. Letting $n\to\infty$ we see that $\theta(f,g)\equiv 0$ on $[0,s_1)$, and so $U$ is co-isometric there. Since $t\mapsto Z(t)u\otimes\varepsilon(f)$ is a continuous map, $U$ is co-isometric on $[0,s_1]$.

To complete the proof we now regard $\tilde{\mathfrak{H}}_{s_k]}$ as the initial space. That is, we assume that $Z\equiv 0$ on $[0,s_k]$ and consider the bounded, adapted process on $\tilde{\mathfrak{H}}_{s_k]}\otimes\mathfrak{H}_{1-s_k]}$ defined by

$$
Z_k(s) = \tilde{\Phi}(s_k)Z(s+s_k)\tilde{\Phi}(s_k)^*
$$

for $s\in[0,s_{k+1}-s_k)$ (and, say, zero elsewhere), where $\tilde{\Phi}(t)$ is the isometric isomorphism between $\tilde{\mathfrak{H}}$ and $\tilde{\mathfrak{H}}_{t]}\otimes\mathfrak{H}_{1-t]}$ that acts on $\tilde{\mathcal{E}} = \mathcal{H}\,\underline{\otimes}\,\mathcal{E}_0$ as

$$
f\otimes\varepsilon(u)\mapsto\Big(f\otimes\varepsilon(u_{t]})\Big)\otimes\varepsilon(u(\cdot+t))
$$

(see Appendix A.4). Note that if $s\in[0,s_{k+1}-s_k)$

$$
\begin{aligned}
Z_k(s) &= \tilde{\Phi}(s_k)Z(s_k)\tilde{\Phi}(s_k)^* \\
&\quad + \tilde{\Phi}(s_k)\int_{s_k}^{s+s_k} E(Z)(r)\,\mathrm{d}\Lambda(r) + F(Z)(r)\,\mathrm{d}A(r) \\
&\qquad + G(Z)(r)\,\mathrm{d}A^\dagger(r) + H(Z)(r)\,\mathrm{d}r\,\tilde{\Phi}(s_k)^*
\end{aligned}
$$

where $E(Z) = WZW^* - Z$, etc. Lemma A.4.1 gives that

$$
Z_k(s) = \int_0^s \tilde{\Phi}(s_k)E(Z)(r+s_k)\tilde{\Phi}(s_k)^*\,\mathrm{d}\Lambda(r) + \cdots + \tilde{\Phi}(s_k)H(Z)(r+s_k)\tilde{\Phi}(s_k)^*\,\mathrm{d}r.
$$

Writing $W(r+s_k) = W_k(r)\otimes\mathrm{id}_{\mathfrak{H}_{(s_k}}$ we have

$$
\tilde{\Phi}(s_k)E(Z)(r+s_k)\tilde{\Phi}(s_k)^* = W_k'(r)Z_k(r)(W_k')^*(r) - Z_k(r) =: E_k(Z_k)(r),
$$

where $W_k'(r) = W_k(r)\otimes\mathrm{id}_{\mathfrak{H}_{1-s_k]}}$, and similarly for the other integrands. Thus

$$
\begin{aligned}
Z_k(s) &= \int_0^s E_k(Z_k)(r)\,\mathrm{d}\Lambda(r) + F_k(Z_k)(r)\,\mathrm{d}A(r) \\
&\qquad + G_k(Z_k)(r)\,\mathrm{d}A^\dagger(r) + H_k(Z_k)(r)\,\mathrm{d}r
\end{aligned}
$$

for $s \in [0, s_{k+1} - s_k)$, so same argument as above shows that $Z^{(k)} \equiv 0$ on $[0, s_{k+1} - s_k]$. Hence $Z \equiv 0$ on $[s_k, s_{k+1}]$ and so everywhere, whence $U$ is a unitary process, as claimed.

□

The following proposition gives unitarity of an isometric solution if the map $t \mapsto U(t)$ is (norm) continuous. This seems to be a very strong requirement for the adapted case. For Ω-adapted QSDE's, however, the subject of our next discussion, this idea does allows us to establish unitarity.

**Proposition 4.2.4** *Let $\phi : [0,1] \to \mathcal{B}(\mathcal{H})$ be a isometry-valued function, where $\mathcal{H}$ is a Hilbert space. If $\phi(0)$ is unitary and there exists a partition of $[0,1]$, $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$, such that*

$$\|\phi(s) - \phi(t_{k-1})\| < 1 \quad \forall\, s \in (t_{k-1}, t_k],\ k = 1, \ldots, n$$

*then $\phi(t)$ is unitary for all $t \geq 0$.*

**Proof**

Suppose $\phi(t)$ is unitary for all $t \in [0, t_k]$. If $s \in (t_k, t_{k+1}]$ then

$$\|1 - \phi(t_k)^* \phi(s)\| = \|\phi(t_k)^*(\phi(t_k) - \phi(s))\| \leq \|\phi(t_k) - \phi(s)\| < 1,$$

so $\phi(t_k)^*\phi(s)$ is invertible, by the basic criterion for invertibility, and so $\phi(s)$ is invertible, as the units form a group. Since $\phi(0)$ is unitary by assumption, we are done.

□

## 4.3 Ω-Adapted QSDE's

Now we study the *modified evolution equation*

$$U = U_0 + \sum_{\beta=0}^{1} I^0_\beta(L^\beta_0 U) + \sum_{\beta=0}^{1} I^1_\beta(L^\beta_1) = U_0 + \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha U^{1-\alpha}), \tag{4.10}$$

where the processes $L^\alpha_\beta \in \mathfrak{L}^p_{\mathfrak{B}}$ are Ω-adapted and $U_0 \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is such that $U_0 - I$ is Ω-adapted. In the notation used for the first part of this section, this equation is written as

$$U(t) = U_0(t) + \int_0^t L^1_1(s)\,\mathrm{d}\Lambda(s) + L^0_1(s)\,\mathrm{d}A(s) + L^1_0(s)U(s)\,\mathrm{d}A^\dagger(s) + L^0_0(s)U(s)\,\mathrm{d}s.$$

We show that a solution always exists and if the coefficients have the usual form and the initial condition $U_0 = I$ then the solution is unitary.

**Theorem 4.3.1** *For each* $(\alpha, \beta) \in \{0,1\}^2$ *let* $L^\alpha_\beta \in \mathfrak{L}^p_{\mathfrak{B}}$ *be an* $\Omega$*-adapted process (where* $p = 2(2-\alpha-\beta)^{-1}$*) and let* $U_0 \in \mathfrak{L}^\infty_{\mathfrak{B}}$ *be such that* $U_0 - I$ *is* $\Omega$*-adapted. There exists a unique process* $U \in \mathfrak{L}^\infty_{\mathfrak{B}}$ *such that* $U - I$ *is* $\Omega$*-adapted and*

$$U = U_0 + \sum_{(\alpha,\beta)} I^\alpha_\beta (L^\beta_\alpha U^{1-\alpha}).$$

**Proof**

Let

$$U^{(0)} := U_0 + \sum_{\beta=0}^{1} I^1_\beta(L^\beta_1), \quad U^{(n)} := U^{(0)} + \sum_{\beta=0}^{1} I^0_\beta(L^\beta_0 U^{(n-1)}) \quad (n \geq 1).$$

Note that $U^{(0)} \in \mathfrak{L}^\infty_{\mathfrak{B}}$ and if $U^{(k)} \in \mathfrak{L}^\infty_{\mathfrak{B}}$ then

$$U^{(k+1)} = U^{(0)} + \sum_{\beta=0}^{1} I^0_\beta(L^\beta_0 U^{(k)}) \in \mathfrak{L}^\infty_{\mathfrak{B}},$$

so $U^{(\nu)}$ is well-defined for all $\nu \geq 0$. Furthermore $U^{(0)} - I$ is $\Omega$-adapted, and if $U^{(k)} - I$ is $\Omega$-adapted then

$$U^{(k+1)} - I = U^{(0)} - I + \sum_{\beta} I^0_\beta(L^\beta_0(U^{(k)} - I) + L^\beta_0)$$

is $\Omega$-adapted. Hence $U^{(\nu)} - I$ is $\Omega$-adapted for all $\nu \geq 0$. Let

$$\Delta U^{(0)} := U^{(0)}, \quad \Delta U^{(n)} := U^{(n)} - U^{(n-1)} = \sum_{\beta=0}^{1} I^0_\beta(L^\beta_0 \Delta U^{(n-1)}) \quad (n \geq 1).$$

It is clear that $\Delta U^{(\nu)} \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is $\Omega$-adapted for $\nu \geq 0$, and so by Theorem 3.3.5

$$\begin{aligned}
\|\widehat{\Delta U}^{(n)}\|^2_{\infty,t} &\leq 2\|\hat{I}(L^0_0 \Delta U^{(n-1)})\|^2_{\infty,t} + 2\|\hat{I}(L^1_0 \Delta U^{(n-1)})\|^2_{\infty,t} \\
&\leq 2\int_0^1 \alpha(s) \|\widehat{\Delta U}^{(n-1)}\|^2_{\infty,s}\, \mathrm{d}s \\
&\leq \frac{2^n}{n!}(\int_0^1 \alpha(s)\, \mathrm{d}s)^n \|\hat{U}^{(0)}\|^2_\infty
\end{aligned}$$

for all $n \geq 1$, where

$$\alpha(s) := \|\hat{L}^1_0(s)\|^2 + 2\|\hat{L}^0_0(s)\| \int_0^s \|\hat{L}^0_0(r)\|\, \mathrm{d}r.$$

Hence

$$\sum_{\nu \geq 0} \|\widehat{\Delta U}^{(\nu)}\|_\infty \leq \|\hat{U}^{(0)}\|_\infty \sum_{\nu\geq 0} 2^{\frac{\nu}{2}}(\|\hat{L}^0_1\|^2_2 + \|L^0_0\|^2_1)^{\frac{\nu}{2}}/\sqrt{\nu!} < \infty,$$

so $[U^{(n)}]$ converges in $\mathcal{L}^\infty_{\mathfrak{B}}$ to some $[U']$, and $[U'-I]$ has an $\Omega$-adapted version, by the comment after Proposition 3.3.3. Thus if $U := U_0 + \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha(U')^{1-\alpha})$ then $U-I$ is $\Omega$-adapted, and since

$$\|\hat{I}^0_\beta(L^\beta_0(U'-U^{(n)}))\|_\infty \leq \|[\hat{L}^\beta_0]\|_p \|[\hat{U}'-\hat{U}^{(n)}]\|_\infty \to 0$$

as $n\to\infty$, we see that $U \in [U']$ satisfies the integral equation

$$U = U_0 + I^1_1(L^1_1) + I^1_0(L^0_1) + I^0_1(L^1_0 U) + I^0_0(L^0_0 U).$$

For uniqueness, suppose $V \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is such that $V-I$ is $\Omega$-adapted and satisfies

$$V = U_0 + \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha V^{1-\alpha})$$

Let $Z := U - V = \sum_{\beta=0}^1 I^0_\beta(L^\beta_0 Z)$; $Z$ is $\Omega$-adapted, so

$$\|\hat{Z}\|^2_{\infty,t} \leq 2\int_0^t \alpha(s)\|\hat{Z}\|^2_{\infty,s}\,\mathrm{d}s \leq \frac{2^n}{n!}\Big(\int_0^t \alpha(s)\,\mathrm{d}s\Big)^n \|\hat{Z}\|^2_{\infty,t},$$

so $U \equiv V$ as claimed.

□

When $U_0 \equiv I$ the usual conditions are necessary and sufficient for the unitarity of the solution.

**Theorem 4.3.2** *For each $(\alpha,\beta)\in\{0,1\}^2$ let $L^\alpha_\beta \in \mathfrak{L}^p_{\mathfrak{B}}$ be an $\Omega$-adapted process. If $U \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is the unique process such that $U-I$ is $\Omega$-adapted and*

$$U = I + \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha U^{1-\alpha})$$

*then $U$ is unitary if and only if*

$$(L^1_1, L^0_1, L^1_0, L^0_0) = (W-I, L, -WL^*, iK - \tfrac{1}{2}LL^*),$$

*where $L\in\mathfrak{L}^2_{\mathfrak{B}}$, $K\in\mathfrak{L}^1_{\mathfrak{B}}$ is almost-everywhere self-adjoint and $W\in\mathfrak{L}^\infty_{\mathfrak{B}}$ is almost-everywhere unitary.*

**Proof**

Since $U$ is a $\Omega$-semimartingale we have that

$$U^*U - I = \sum_{(\alpha,\beta)} I^\alpha_\beta((U^*)^{1-\beta}(L^\beta_\alpha + (L^\alpha_\beta)^* + (L^1_\beta)^* L^1_\alpha)U^{1-\alpha}) \qquad (4.11)$$

and

$$UU^* - I \;=\; \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha(UU^*)^{1-\alpha} + (UU^*)^{1-\beta}(L^\alpha_\beta)^* + L^\beta_1(L^\alpha_1)^*), \qquad (4.12)$$

by (3.4.9). Thus if $U$ is unitary then, by (3.1.7), the integrands in (4.11) and (4.12) must be zero almost everywhere. After a little algebra we see that $L^\alpha_\beta$ must have the usual form, i.e.,

$$(L^1_1, L^0_1, L^1_0, L^0_0) = (W - I, L, -WL^*, iK - \tfrac{1}{2}LL^*),$$

where $L \in \mathfrak{L}^2_{\mathfrak{B}}$, $K \in \mathfrak{L}^1_{\mathfrak{B}}$ is self-adjoint (a.e.) and $W \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is unitary (a.e.).

For the converse, note first that (4.11) gives that $U$ is isometric, and if $Z = UU^* - I$ then

$$Z = I^1_0(-ZLW^*) + I^0_1(-WL^*Z) + I^0_0(i[K, Z] - \tfrac{1}{2}\{LL^*, Z\})$$

(where $[x, y] := xy - yx$ and $\{x, y\} := xy + yx$). Since $Z$ has this representation and is $\Omega$-adapted, $t \mapsto \hat{Z}(t)$ is norm continuous, and so uniformly norm-continuous on $[0, 1]$. Hence there exists a partition $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ such that

$$\|\hat{Z}(s_{k-1}) - \hat{Z}(t)\| < 1 \;\; \forall\, t \in (s_{k-1}, s_k],\; (k = 1, \ldots, n).$$

Suppose that $Z(s_{k-1}) = 0$; then $\|\hat{Z}(t)\| < 1$ for $t \in (s_{k-1}, s_k]$ so $\hat{Z}(t) + I$ is invertible for $t$ in this interval. We know that $U$ is isometric, so $U$ is unitary on $(s_{k-1}, s_k]$:

$$\hat{U}^*\hat{U} = I \Rightarrow \hat{U}^*\hat{U}\hat{U}^* = \hat{U}^* \Rightarrow \hat{U}^* = \hat{U}^*(\hat{U}\hat{U}^*)^{-1} \Rightarrow \hat{U}\hat{U}^* = \hat{U}\hat{U}^*(\hat{U}\hat{U}^*)^{-1} = I.$$

Thus $Z \equiv 0$ on $(s_{k-1}, s_k]$. Since $Z(0) = 0$ we see that $Z \equiv 0$, i.e., $U$ is unitary, on $[0, 1]$.

□

In the above, the requirement that $K$ must be $\Omega$-adapted is not a great restriction; take any self-adjoint, bounded process $K' \in \mathfrak{L}^1_{\mathfrak{B}}$ and let $K = \mathbb{E}K'\mathbb{E}$. The fact that $W$ must be unitary and $W - I$ must be $\Omega$-adapted is a constraint, but there are still plenty of such processes. Indeed $W \equiv I$ is one, and then Theorem 4.3.2 provides plenty more.

### Motivation

Suppose $U \in \mathfrak{L}^\infty_{\mathfrak{B}}$ is such that $U - I$ is $\Omega$-adapted and satisfies the evolution equation, i.e.,

$$U = I + \sum_{(\alpha,\beta)} I^\alpha_\beta(L^\beta_\alpha U),$$

where the driving coefficients $L^\alpha_\beta \in \mathfrak{L}^p_{\mathfrak{B}}$ are $\Omega$-adapted. If $U$ is a unitary solution then we find that these coefficients satisfy the following equations:

$$\begin{aligned}
0 &= L^0_0 + (L^0_0)^* + (L^0_1)(L^0_1)^*; \\
0 &= L^0_1 + (L^1_0)^* U^* + (L^1_0)^* L^1_1; \\
0 &= L^1_1 U + U^*(L^1_1)^* + L^1_1 U U^*(L^1_1)^*; \\
0 &= L^1_1 U + U^*(L^1_1)^* + U^*(L^1_1)^* L^1_1 U.
\end{aligned}$$

From these we deduce that

$$(L^1_1, L^0_1, L^1_0, L^0_0) = ((W - I)U^*, L, -WU^*L^*, iK - \tfrac{1}{2}LL^*),$$

where $K$ is a self-adjoint process and $W$ a unitary one, and if we write $M = LU$ we see that the evolution equation takes the form

$$U = I + \sum_{(\alpha,\beta)} I^\alpha_\beta (M^\beta_\alpha U^{1-\alpha})$$

where

$$(M^1_1, M^0_1, M^1_0, M^0_0) = (W - I, M, -WM^*, iK - \tfrac{1}{2}MM^*).$$

This is our justification for studying the modified evolution equation (4.10).

# Chapter 5

# References

*Felix qui potuit rerum cognoscere causas.*

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# Appendix A

# Technical Results

## A.1 Measure Theory

**Proposition A.1.1** *Let $X$ and $Y$ be Banach spaces with $X$ separable. There exists a sequence of unit vectors $(x_n)_{n\geq 1} \subseteq X$ such that $\|\Phi\| = \sup\{\|\Phi x_n\| : n \geq 1\}$ for all $\Phi \in \mathcal{B}(X;Y)$.*

**Proof**

It is not difficult to see that $\{x \in X : \|x\| = 1\}$ is separable; one way is to note that $X \setminus \{0\}$ is separable and $x \mapsto x/\|x\|$ is a continuous surjection. Let $(x_n)_{n\geq 1}$ be a dense subset of $\{x \in X : \|x\| = 1\}$. Let $\Phi \in \mathcal{B}(X;Y)$, let $\epsilon > 0$ and choose $x \in X$ such that $\|x\| = 1$ and $\|\Phi x\| > \|\Phi\| - \epsilon/2$. There exists $n \geq 1$ such that $\|x - x_n\| < \epsilon/2\|\Phi\|$, so

$$\|\Phi\| < \|\Phi x\| + \epsilon/2 < \|\Phi(x - x_n)\| + \|\Phi x_n\| + \epsilon/2 < \|\Phi x_n\| + \epsilon,$$

i.e., $\|\Phi x_n\| > \|\Phi\| - \epsilon$. Since this holds for all $\epsilon$ we see that

$$\|\Phi\| = \sup\{\|\Phi x_n\| : n \geq 1\}$$

as claimed.

□

**Lemma A.1.2 ([Din, p. 102, Proposition 17])** *Let $X$ and $Y$ be Banach spaces, $X$ separable, and suppose $\Phi : [0,1] \to \mathcal{B}(X;Y)$ is such that $t \mapsto \Phi(t)x$ is strongly measurable for all $x \in X$. Then $t \mapsto \|\Phi(t)\|$ is measurable.*

**Proof**

Let $(x_n)_{n\geq 1}$ be as in Proposition A.1.1 and note that

$$\|\Phi(t)\| = \lim_{n\to\infty} \max_{k=1}^{n} \|\Phi(t)x_k\|.$$

Since $t \mapsto \Phi(t)x_k$ is strongly measurable for all $k$, $t \mapsto \|\Phi(t)x_k\|$ is measurable. Thus $\|\Phi(\cdot)\|$ is the pointwise limit of measurable functions, hence measurable.

□

**Lemma A.1.3 (Cf. [Vin, Lemma 3.3], [Din, p. 102, Proposition 16])** *Let $X$ and $Y$ be separable Banach spaces, let $\phi : [0,1] \to X$ be strongly measurable and let $\Phi : [0,1] \to \mathcal{B}(X;Y)$ be such that $t \mapsto \Phi(t)x$ is strongly measurable for all $x \in X$. Then $\Phi\phi : [0,1] \to Y; t \mapsto \Phi(t)\phi(t)$ is strongly measurable.*

**Proof**

Since $\phi$ is strongly measurable, there exist step functions $(\phi_n)$ such that $\phi_n(t) \to \phi(t)$ almost everywhere. The map

$$t \mapsto \Phi(t)\phi_n(t) = \sum_k \Phi(t)\phi_n(t_k)\chi_{[t_k,t_{k+1})}(t),$$

where $\phi_n$ is supposed to be subordinate to the partition $\{t_k\}$, is strongly measurable. By Pettis' measurability theorem [Pet], $t \mapsto \xi(\Phi(t)\phi_n(t))$ is measurable for all $\xi \in Y^*$, so

$$t \mapsto \xi(\Phi(t)\phi(t)) \overset{a.e.}{=} \lim_{n\to\infty} \xi(\Phi(t)\phi_n(t))$$

is measurable. A further application of Pettis' theorem gives us that $\Phi\phi$ is strongly measurable, as claimed.

□

## A.2 Bochner-Lebesgue Spaces

**Proposition A.2.1** *Let $X$ and $Y$ be separable Banach spaces. The Bochner-Lebesgue space $L^p([0,1];\mathcal{B}(X;Y)_s)$ is complete.*

**Proof**

It is easily verified that $L^p([0,1];\mathcal{B}(X;Y)_s)$ is a normed vector space, where we identify functions that are equal almost everywhere. Suppose $(f_n)_{n\geq 1}$ is a Cauchy sequence; passing to a subsequence $(f_{n_k})_{k\geq 1}$ we may assume that $\|f_{n_{k+1}} - f_{n_k}\|_p < 2^{-k}$. For $m \geq 1$ let

$$g_m(t) := \|f_{n_1}(t)\| + \sum_{k=1}^{m} \|f_{n_{k+1}}(t) - f_{n_k}(t)\|$$

and define

$$g(t) := \|f_{n_1}(t)\| + \sum_{k=1}^{\infty} \|f_{n_{k+1}}(t) - f_{n_k}(t)\| \in [0,\infty].$$

Then $t \mapsto g(t) = \sup_{m\geq 1} g_m(t)$ is measurable, as $g_m$ is by Lemma A.1.2. Since

$$\|g\|_p \leq \|f_{n_1}\|_p + \sum_{k\geq 1} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + 1,$$

we see that $g \in L^p[0,1]$. Thus $g(t)$ is finite for almost all $t \in [0,1]$, so

$$\lim_{m\to\infty} f_{n_{m+1}} = f_{n_1} + \lim_{m\to\infty} \sum_{k=1}^{m} f_{n_{k+1}} - f_{n_k}$$

exists almost everywhere; let $f(t)$ equal this limit when it exists, and set $f(t)$ to be zero otherwise. As $t \mapsto f_{n_{m+1}}(t)\xi$ is strongly measurable for all $\xi \in X$ and any $m$, $f$ has this property as well. In particular, $t \mapsto \|f(t)\|$ is measurable and

$$\|f\|_p \leq \|f_{n_1}\|_p + \sum_{k\geq 1} \|f_{n_{k+1}} - f_{n_k}\|_p < \infty,$$

so $f \in L^p([0,1]; \mathcal{B}(X;Y)_s)$. Finally, note that

$$\|f - f_{n_m}\|_p = \|\sum_{k\geq m} f_{n_{k+1}} - f_{n_k}\|_p \leq \sum_{k\geq m} 2^{-k},$$

so $f_{n_m} \to f$ as $m \to \infty$, as required.

□

## A.3 A Multiple Series

**Lemma A.3.1** *Let $n \geq 1$ and suppose $\pi_0, \pi_{n+1} \in \{0,1,2,\ldots\}$ and $a, b > 0$. The series*

$$\begin{aligned} S(z_1,\ldots,z_n) \;&:=\; \sum_{\pi_1\geq 0}\cdots\sum_{\pi_n\geq 0} ((\pi_0 \vee \pi_1)+1)^a \cdots ((\pi_n \vee \pi_{n+1})+1)^a \\ &\qquad\qquad \times |\pi_0 - \pi_1|!^{-b} \cdots |\pi_n - \pi_{n+1}|!^{-b} z_1^{\pi_1} \cdots z_n^{\pi_n} \end{aligned}$$

*is absolutely convergent for any $z_1,\ldots,z_n \in \mathbb{C}$.*

**Proof**

Let $z = \max\{1, |z_1|, \ldots, |z_n|\}$ and note that

$$\begin{aligned} S(|z_1|,\ldots,|z_n|) \;\leq\; &\sum_{\pi_1\geq 0}\cdots\sum_{\pi_n\geq 0} (|\pi_0 - \pi_1|+1)^a (\pi_0+1)^a \cdots (|\pi_n - \pi_{n+1}|+1)^a \\ &\qquad \times (\pi_n+1)^\alpha |\pi_0 - \pi_1|!^{-b} \cdots |\pi_n - \pi_{n+1}|!^{-b} z^{k_1+\cdots+k_n}, \end{aligned}$$

using that fact that $x \vee y = ((x-y) \vee 0) + y \leq |x-y| + y$ and $(x+y+1) \leq (x+1)(y+1)$ for any $x, y \geq 0$. Writing $\pi_1 + 1 = \pi_1 - \pi_0 + \pi_0 + 1$, and more generally

$$\pi_i + 1 = \pi_i - \pi_{i-1} + \pi_{i-1} - \cdots - \pi_0 + \pi_0 + 1$$

for $i \geq 2$, yields the inequality

$$(\pi_0 + 1) \cdots (\pi_n + 1) \leq (\pi_0 + 1)^{n+1} (|\pi_1 - \pi_0| + 1)^n \cdots (|\pi_n - \pi_{n-1}| + 1),$$

so that

$$\begin{aligned} S(|z_1|, \ldots, |z_n|) \quad \leq \quad & \sum_{\pi_1 \geq 0} \cdots \sum_{\pi_n \geq 0} \frac{(|\pi_0 - \pi_1| + 1)^{a(n+1)}}{|\pi_0 - \pi_1|!^b} \cdots \frac{(|\pi_n - \pi_{n+1}| + 1)^a}{|\pi_n - \pi_{n+1}|!^b} \\ & \times z^{\pi_1 + \cdots + \pi_n} (\pi_0 + 1)^{a(n+1)}. \end{aligned}$$

If we let

$$\lambda_0 := \pi_0, \;\; \lambda_1 := \pi_1 - \pi_0, \;\; \ldots, \;\; \lambda_{n+1} := \pi_{n+1} - \pi_n$$

(which gives a bijection from $\mathbb{Z}^{n+1}$ to itself) then

$$\pi_1 + \cdots + \pi_n = (\lambda_0 + \lambda_1) + \cdots + (\lambda_0 + \lambda_1 + \cdots \lambda_n) \leq (n+1)|\lambda_0| + n|\lambda_1| + \cdots + |\lambda_n|,$$

so

$$S(|z_1|, \ldots, |z_n|) \leq M_1 \sum_{\lambda_1 \in \mathbb{Z}} \cdots \sum_{\lambda_{n+1} \in \mathbb{Z}} \frac{(|\lambda_1| + 1)^{a(n+1)} z^{n|\lambda_1|}}{|\lambda_1|!^b} \cdots \frac{(|\lambda_{n+1}| + 1)^a}{|\lambda_{n+1}|!^b},$$

where $M_1 := z^{\pi_0(n+1)} (\pi_0 + 1)^{a(n+1)}$. Hence

$$S(|z_1|, \ldots, |z_n|) \leq M_2 \prod_{\mu=0}^{n} \sum_{\nu \geq 0} \frac{(\nu + 1)^{a(\mu+1)} z^{\mu\nu}}{(\nu!)^b},$$

where $M_2 := 2^{n+1} z^{(n+1)\pi_0} (\pi_0 + 1)^{a(n+1)}$, and this is finite by the limit ratio test:

$$\left( \frac{\nu + 2}{\nu + 1} \right)^{a(\mu+1)} z^{\mu} (\nu + 1)^{-b} \to 0$$

as $\nu \to \infty$, for all $\mu \geq 0$.

□

## A.4 Time Shifts and QS Integrals

For $t \in [0,1]$ the map $\tilde{\Phi}(t) : \tilde{\mathfrak{H}} \to \tilde{\mathfrak{H}}_{t]} \otimes \mathfrak{H}_{1-t]}$ which acts as

$$f \otimes \varepsilon(u) \mapsto \Big( f \otimes \varepsilon(u_{t]}) \Big) \otimes \varepsilon(u(\cdot + t))$$

is an isometric isomorphism, as can be seen by considering the map $\tilde{\jmath}_t : \tilde{\mathfrak{H}} \to \tilde{\mathfrak{H}}_{t]} \otimes \mathfrak{H}_{(t}$ which acts as

$$u \otimes \varepsilon(u) \mapsto \Big( f \otimes \varepsilon(u_{t]}) \Big) \otimes \varepsilon(u_{(t})$$

and the second quantisation of the isomorphism between $L^2(t,1]$ and $L^2[0,1-t]$ given by $u \mapsto u(\cdot + t)$.

**Lemma A.4.1** *Let $t \in [0,1]$.*

*(i) If $F$ is an adapted process in $\tilde{\mathfrak{H}}$ then $s \mapsto \tilde{\Phi}(t)F(s+t)\tilde{\Phi}(t)^*$ is an adapted process in $\tilde{\mathfrak{H}}_{t]} \otimes \mathfrak{H}_{1-t]}$.*

*(ii) Let $q, r \in [0,1]$ be such that $q \geq r$. If $F$ is an adapted process such that $\int_{r+t}^{q+t} F(s)\, \mathrm{d}\Xi(s)$ exists, where $\Xi(s) \in \{\Lambda(s), A(s), A^\dagger(s), s\}$, then*

$$\tilde{\Phi}(t) \int_{r+t}^{q+t} F(s)\, \mathrm{d}\Xi(s) \tilde{\Phi}(t)^* = \int_r^q \tilde{\Phi}(t) F(s+t) \tilde{\Phi}(t)^*\, \mathrm{d}\Xi(s)$$

*where the second integral is in $\tilde{\mathfrak{H}}_{t]} \otimes \mathfrak{H}_{1-t]}$, that is, $\tilde{\mathfrak{H}}_{t]}$ is regarded as the initial space.*

**Proof**

It is easily verified that $s \mapsto \tilde{\Phi}(t)F(s+t)\tilde{\Phi}(t)^*$ is adapted as a process in $\tilde{\mathfrak{H}}_{t]} \otimes \mathfrak{H}_{1-t]}$. For *(ii)*, note that

$$\begin{aligned}
&\langle f \otimes \varepsilon(u), \int_{r+t}^{q+t} F(s)\, \mathrm{d}\Xi(s) g \otimes \varepsilon(v) \rangle \\
&= \int_{r+t}^{q+t} \langle f \otimes \varepsilon(u), F(s) g \otimes \varepsilon(v) \rangle \overline{u}(s)^\beta v(s)^\alpha\, \mathrm{d}s \\
&= \int_r^q \langle f \otimes \varepsilon(u), F(s+t) g \otimes \varepsilon(v) \rangle \overline{u}(s+t)^\beta v(s+t)^\alpha\, \mathrm{d}s \\
&= \int_r^q \langle \tilde{\Phi}(t)^* f \otimes \varepsilon(u_{t]}) \otimes \varepsilon(u(\cdot+t)), F(s+t)\tilde{\Phi}(t)^* g \otimes \varepsilon(v_{t]}) \otimes \varepsilon(v(\cdot+t)) \rangle \\
&\qquad\qquad \times \overline{u}(s+t)^\beta v(s+t)^\alpha\, \mathrm{d}s \\
&= \int_r^q \langle f \otimes \varepsilon(u_{t]}) \otimes \varepsilon(u(\cdot+t)), \tilde{\Phi}(t) F(s+t)\tilde{\Phi}(t)^* g \otimes \varepsilon(v_{t]}) \otimes \varepsilon(v(\cdot+t)) \rangle \\
&\qquad\qquad \times \overline{u}(s+t)^\beta v(s+t)^\alpha\, \mathrm{d}s \\
&= \langle \tilde{\Phi}(t) f \otimes \varepsilon(u), \int_r^q \tilde{\Phi}(t) F(s+t) \tilde{\Phi}(t)^*\, \mathrm{d}\Xi(s) \tilde{\Phi}(t) g \otimes \varepsilon(v) \rangle
\end{aligned}$$

if $\alpha, \beta \in \{0,1\}$ are chosen appropriately, so

$$\int_{r+t}^{q+t} F(s)\,\mathrm{d}\Xi(s) = \tilde{\Phi}(t)^* \int_r^q \tilde{\Phi}(t) F(s+t) \tilde{\Phi}(t)^*\,\mathrm{d}\Xi(s) \tilde{\Phi}(t)$$

which is equivalent to the above.

□

*Iamque opus exegi, quod nec Iovis ira, nec ignis,*
*Nec porterit ferrum, nec edax abolere vetustas.*

Ovid, *Metamorphoses*

# Appendix B

# Addenda

## B.1 Row-Square-Summability

We know that a row-square-summable matrix gives rise to an operator with domain containing the finite particle vectors. The following example demonstrates that row-square-summability is far from being a necessary condition for this to occur.

### Example

For each $\pi \geq 0$ let $\mathfrak{h}_\pi$ be an infinite-dimensional Hilbert space containing the orthonormal set $\{e_0^\pi, e_1^\pi, \ldots\}$ and let $\mathfrak{H} = \bigoplus_{\pi \geq 0} \mathfrak{h}_\pi$. Define $A \in \mathfrak{M}$ by $A_\nu^\mu = |e_\mu^\mu\rangle\langle e_\mu^\nu|$, where we use the Dirac "bra-ket" notation:

$$|e_\mu^\mu\rangle\langle e_\mu^\nu| : \mathfrak{h}_\nu \to \mathfrak{h}_\mu;\ \xi^\nu \mapsto \langle e_\mu^\nu, \xi^\nu\rangle e_\mu^\mu.$$

It is easy to see that $\|A_\nu^\mu\| = 1$ for all $\mu, \nu \geq 0$, so $A$ is not row-square-summable, but $\mathcal{D}(\hat{A}) \supseteq \mathfrak{H}_{00}$:

$$\sum_{\mu \geq 0} A_\nu^\mu \xi^\nu = \sum_{\mu \geq 0} \langle e_\mu^\nu, \xi^\nu\rangle e_\mu^\mu,$$

and this is convergent, since $\sum_{\mu \geq 0} |\langle e_\mu^\nu, \xi^\nu\rangle|^2 \leq \|\xi^\nu\|^2$, so

$$\|\sum_{\mu=m}^{n} A_\nu^\mu \xi^\nu\|^2 = \sum_{\mu=m}^{n} |\langle e_\mu^\nu, \xi^\nu\rangle|^2 \to 0$$

as $m, n \to \infty$.

□

The following remark demonstrates more of the behaviour of row-square-summable matrices. It may be used to show that the number operator on any chaos space is positive; cf. Example 2.1.4a.

Recall that a linear operator $T$ is *closed* if its graph, $\mathcal{G}(\hat{T}) := \{(\xi, \hat{T}\xi) : \xi \in \mathcal{D}(\hat{T})\}$, is a closed subspace of $\mathfrak{H} \oplus \mathfrak{H}$. A *core* for a closed operator $\hat{T}$ is a subspace $\mathcal{D}_0 \subseteq \mathcal{D}(\hat{T})$ such that the graph of $\hat{T}$ restricted to $\mathcal{D}_0$ is dense in $\mathcal{G}(\hat{T})$.

**Proposition B.1.1** *Let $A \in \mathfrak{M}$ be row-square-summable. If $\hat{A}$ is closed then $\mathfrak{H}_{00}$ is a core for $\hat{A}$ if and only if $\hat{A}^* = \widehat{A^*}$. In particular, if $A$ is symmetric then $\hat{A}$ is self-adjoint if and only if $\mathfrak{H}_{00}$ is a core for $\hat{A}$.*

**Proof**

Suppose that $\xi \in \mathcal{D}(\hat{A})$ is such that $(\xi, \hat{A}\xi)$ is orthogonal to the graph of $\check{A}$ (i.e., $\hat{A}$ restricted to $\mathfrak{H}_{00}$), so that

$$\langle \xi^\mu, \eta^\mu \rangle + \sum_{\pi \geq 0} \langle (\hat{A}\xi)^\pi, A^\pi_\mu \eta^\mu \rangle = \langle \xi^\mu, \eta^\mu \rangle + \langle \hat{A}\xi, \hat{A}\eta^\mu \rangle = 0 \quad \forall\, \eta^\mu \in \mathfrak{h}_\mu, \mu \geq 0.$$

As

$$\sum_{\pi \geq 0} \|(A^*)^\mu_\pi (\hat{A}\xi)^\pi\| \leq (\sum_{\pi \geq 0} \|A^\pi_\mu\|^2)^{\frac{1}{2}} \|\hat{A}\xi\| < \infty$$

we see that, for all $\mu \geq 0$,

$$\xi^\mu + \sum_{\pi \geq 0} (A^*)^\mu_\pi (\hat{A}\xi)^\pi = 0,$$

and so $\hat{A}\xi \in \mathcal{D}(\widehat{A^*})$, because $\xi \in \mathfrak{H}$. Hence, using the fact that $\hat{A}^* = \widehat{A^*}$,

$$0 = \langle \xi, \xi \rangle + \langle \widehat{A^*}\hat{A}\xi, \xi \rangle = \|\xi\|^2 + \|\hat{A}\xi\|^2,$$

and so $\xi = 0$, as required.

For the converse, suppose that $\mathfrak{H}_{00}$ is a core for $\hat{A}$. We know that $\widehat{A^*} = \check{A}^* \supseteq \hat{A}^*$, by Theorem 2.2.5, and we wish to demonstrate the reverse inclusion. Let $\xi \in \mathcal{D}(\check{A}^*)$; there exists $\psi \in \mathfrak{H}$ such that

$$\langle \xi, \check{A}\eta \rangle = \langle \psi, \eta \rangle \quad \forall\, \eta \in \mathfrak{H}_{00}.$$

Let $\theta \in \mathcal{D}(\hat{A})$; as $\mathfrak{H}_{00}$ is a core for $\hat{A}$ we may find $(\eta_n)_{n \geq 1} \subseteq \mathfrak{H}_{00}$ such that $\eta_n \to \theta$ and $\hat{A}\eta_n \to \hat{A}\theta$ as $n \to \infty$, so

$$\langle \xi, \hat{A}\theta \rangle = \lim_{n \to \infty} \langle \xi, \hat{A}\eta_n \rangle = \lim_{n \to \infty} \langle \psi, \eta_n \rangle = \langle \psi, \theta \rangle.$$

Thus $\xi \in \mathcal{D}(\hat{A}^*)$, with $\hat{A}^*\xi = \psi$, and we have the desired inclusion.

□

## B.2 References

Professor R. L. Hudson has kindly drawn our attention to the following:

**Ω-Adaptedness**

R. L. Hudson & P. Krée, Quantum stochastic calculus for Hilbert Schmidt processes, in K. D. Elworthy & J.-C. Zambrini (ed.), *Stochastic analysis, path integration and dynamics (Warwick 1987)*, Pitman Research Notes in Mathematics Series **200**, pp. 83–93, Longman Scientific & Technical (1989).

**Obvious Processes**

R. L. Hudson & K. R. Parthasarathy, Generalised Weyl operators, in A. Truman & D. Williams (ed.), *Stochastic Analysis and Applications (Proceedings, Swansea 1983)*, Lecture Notes in Mathematics **1095**, pp. 45–50, Springer-Verlag (1984).