

# MOMENTS OF 2D PARABOLIC ANDERSON MODEL

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ABSTRACT. In this note, we use the Feynman-Kac formula to derive a moment representation for the 2D parabolic Anderson model in small time, which is related to the intersection local time of planar Brownian motions.

KEYWORDS: Feynman-Kac formula, renormalization, intersection local time.

## 1. INTRODUCTION

The aim of this note is to study the existence of moments of the solution to the parabolic Anderson model (PAM) in two spatial dimensions, formally given by

$$(1.1) \quad \partial_t u = \frac{1}{2} \Delta u + u \cdot \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

where  $\xi$  is the two dimensional spatial white noise, that is, a generalized Gaussian process with covariance  $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$ .

The equation is well-posed in dimension 1, but the product between  $u$  and  $\xi$  becomes ill-defined as soon as  $d \geq 2$ . For  $d = 2$ , the solution  $u$  is defined in [7, 8, 10] as the limit of a sequence of the regularized and renormalized equations. More precisely, fix a symmetric mollifier  $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^+$  with  $\rho(x) = \rho(-x)$  and  $\int \rho = 1$ . Let

$$\rho_\varepsilon(x) = \varepsilon^{-2} \rho(x/\varepsilon), \quad \xi_\varepsilon = \xi \star \rho_\varepsilon,$$

and consider the equation

$$(1.2) \quad \partial_t u_\varepsilon = \frac{1}{2} \Delta u_\varepsilon + (\xi_\varepsilon - C_\varepsilon) u_\varepsilon,$$

for some large constant  $C_\varepsilon$ . Then, for

$$(1.3) \quad C_\varepsilon = \frac{1}{\pi} \log \varepsilon^{-1},$$

the sequence of solutions  $\{u_\varepsilon\}$  converges in some weighted Hölder space in probability to a limit  $u$  that is independent of the mollification, see e.g. [10, Theorem 4.1], and we call this limit  $u$  the solution to 2D PAM. In  $d = 3$ , the mollifier  $\rho_\varepsilon(x) = \varepsilon^{-3} \rho(x/\varepsilon)$ , and the renormalization constant takes the form  $C_\varepsilon = c_1 \varepsilon^{-1} + c_2 \log \varepsilon^{-1} + O(1)$  [9].

So far, most of the results mentioned above focused on the existence of the solution and the convergence of the regularized PDE after renormalization. The statistical properties of  $u$  remains a challenge; see [1, 2, 4] for some relevant discussions. The goal of this note is to show that the  $n$ -th moment of the solution  $u$  to 2D PAM exists for small time, and we present a Feynman-Kac formula for  $\mathbb{E}[u^n]$ . The following is our main result.

**Theorem 1.1.** *There exists a universal constant  $\delta > 0$  such that for every  $n \in \mathbb{N}$ , the  $n$ -th moment of  $u$  exists for  $t \in (0, \frac{\delta}{n^2})$  with  $\mathbb{E}[u(t, x)^n]$  given by (1.10).*

**1.1. Heuristic argument.** We first give a heuristic derivation of  $\mathbb{E}[u(t, x)^n]$  by writing down a representation for  $\mathbb{E}[u_\varepsilon(t, x)^n]$  and passing to the limit formally.

Suppose  $u_\varepsilon(0, x) = u_0(x)$  for some continuous function  $u_0$  with  $\|u_0\|_\infty \leq 1$ , we write the solution to (1.2) by the Feynman-Kac formula

$$(1.4) \quad u_\varepsilon(t, x) = \mathbb{E}_{\mathbf{B}} \left[ u_0(x + B_t) \exp \left( \int_0^t \xi_\varepsilon(x + B_s) ds - C_\varepsilon t \right) \right].$$

Here,  $B = (B_t)_{t \geq 0}$  is a standard planar Brownian motion starting from the origin and independent of the white noise  $\xi$ , and  $C_\varepsilon$  is the constant defined in (1.3). We use  $\mathbb{E}_{\mathbf{B}}$  to denote the expectation with respect to  $B$ . We now proceed to calculating the  $n$ -th moment of  $u_\varepsilon(t, x)$ . First of all, the covariance function of  $\xi_\varepsilon$  satisfies

$$\mathbb{E}[\xi_\varepsilon(x)\xi_\varepsilon(y)] = R_\varepsilon(x - y) := \varepsilon^{-2}R\left(\frac{x - y}{\varepsilon}\right),$$

where  $R = \rho \star \rho$ , and  $\rho$  is the mollifier used to regularize the noise  $\xi$ . Next, one raises the expression (1.4) to the  $n$ -th power, and take a further expectation with respect to  $\xi_\varepsilon$ . Since  $B$  is independent of  $\xi_\varepsilon$ , one can interchange this expectation with the one with respect to the Brownian motions, and get

$$(1.5) \quad \mathbb{E}[u_\varepsilon(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k) \right].$$

Here,  $B^k, k = 1, \dots, n$  are independent Brownian motions, and  $\mathbb{E}_{\mathbf{B}}$  denotes the expectation with respect to these  $B^k$ 's. Also,  $I_n^\varepsilon(t)$  is given by

$$(1.6) \quad I_n^\varepsilon(t) = \sum_{k=1}^n \int_0^t \int_0^s R_\varepsilon(B_s^k - B_u^k) du ds + \sum_{1 \leq i < j \leq n} \int_0^t \int_0^s R_\varepsilon(B_s^i - B_u^j) ds du,$$

where  $R_\varepsilon(x) = \varepsilon^{-2}R(x/\varepsilon)$  converges to the Dirac function as  $\varepsilon \rightarrow 0$ . Note that we do not have the factor  $\frac{1}{2}$  in front of the first term since the integration is on the simplex rather than the square  $[0, t]^2$ . It is well known (see for example [3, Chapter 2]) that each term in the second term above (when  $i \neq j$ ) converges to the mutual intersection local time of Brownian motion, formally written as  $\int_{[0, t]^2} \delta(B_s^i - B_u^j) ds du$ . The first term above (when one has the same Brownian motion in the argument of  $R_\varepsilon$ ) unfortunately does not converge as  $\varepsilon \rightarrow 0$ , but it does when one subtracts its mean (see [12, 14, 15]). Thus, we define

$$(1.7) \quad \nu_\varepsilon(t) = \int_0^t \int_0^s \mathbb{E}_{\mathbf{B}}[R_\varepsilon(B_s - B_u)] du ds,$$

and for every  $t \geq 0$ , we have

$$(1.8) \quad I_n^\varepsilon(t) - n\nu_\varepsilon(t) \rightarrow \mathcal{X}_n(t)$$

in probability, where  $\mathcal{X}_n(t)$  is a linear combination of self- and mutual-intersection local times of planar Brownian motions, formally written as

$$(1.9) \quad \begin{aligned} \mathcal{X}_n(t) = & \sum_{k=1}^n \int_0^t \int_0^s \left( \delta(B_s^k - B_u^k) - \mathbb{E}_{\mathbf{B}}[\delta(B_s^k - B_u^k)] \right) duds \\ & + \sum_{1 \leq i < j \leq n} \int_0^t \int_0^s \delta(B_s^i - B_u^j) ds du. \end{aligned}$$

It is well known from [12] that  $\mathcal{X}_n(t)$  has exponential moments for small enough  $t$  (depending on  $n$ ). In order for the expression (1.5) to converge, one needs the divergent constant  $C_\varepsilon t$  coincides with  $\nu_\varepsilon(t)$ . A simple calculations shows that this is indeed the case up to an  $O(1)$  correction.

**Lemma 1.2.** *There exists constants  $\mu_1$  and  $\mu_2$  such that*

$$\nu_\varepsilon(t) - C_\varepsilon t \rightarrow t(\mu_1 + \mu_2 \log t)$$

as  $\varepsilon \rightarrow 0$ .

By (1.8) and Lemma 1.2, we have

$$\begin{aligned} I_n^\varepsilon(t) - nC_\varepsilon t &= I_n^\varepsilon(t) - n\nu_\varepsilon(t) + n(\nu_\varepsilon(t) - C_\varepsilon t) \\ &\rightarrow \mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t) \end{aligned}$$

in probability. If the families  $\{u_\varepsilon(t, x)^n\}$  and  $\{e^{I_n^\varepsilon(t) - nC_\varepsilon t}\}$  are both uniformly integrable, then we can pass both sides of (1.5) to the limit, and obtain

$$(1.10) \quad \mathbb{E}[u(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(\mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t)) \prod_{k=1}^n u_0(x + B_t^k) \right].$$

The rest of the note is to show the uniform integrability of  $\{u_\varepsilon(t, x)^n\}$  and  $\{e^{I_n^\varepsilon(t) - nC_\varepsilon t}\}$  for small time  $t$ , so (1.10) does hold.

## 1.2. Discussions.

*Remark 1.3.* The same argument leads to a similar result in  $d = 1$ , where we choose  $C_\varepsilon = 0$  and do not have the small time constraint. The renormalized self-intersection local time can be written as

$$\int_0^t \int_0^s (\delta(B_s - B_u) - \mathbb{E}_{\mathbf{B}}[\delta(B_s - B_u)]) duds = \frac{1}{2} \int_{\mathbb{R}} L_t(x)^2 dx - \frac{1}{2} \int_{\mathbb{R}} \mathbb{E}_{\mathbf{B}}[L_t(x)^2] dx,$$

with  $L_t(x)$  denoting the local time of 1D Brownian motion up to  $t$ .

*Remark 1.4.* For  $n = 1$ , the moment formula reads

$$\mathbb{E}[u(t, x)] = \mathbb{E}_{\mathbf{B}}[u_0(x + B_t) e^{\gamma([0, t]_{<}^2) + t(\mu_1 + \mu_2 \log t)}],$$

with  $\gamma([0, t]_{<}^2) = \int_0^t \int_0^s (\delta(B_s - B_u) - \mathbb{E}_{\mathbf{B}}[\delta(B_s - B_u)]) duds$  representing the self-intersection local time of  $B$ . It was proved in [13] that there exists  $t_0 > 0$  such that

$$\mathbb{E}_{\mathbf{B}}[e^{\gamma([0, t]_{<}^2)}] \begin{cases} < \infty & t < t_0, \\ = \infty & t > t_0. \end{cases}$$

Thus, it is natural to expect that the moments of  $u$  do not exist for large  $t$ , although we do not have a rigorous proof of it.

*Remark 1.5.* In [1], the authors defined the 2D Anderson Hamiltonian  $\mathcal{H} = -\Delta + \xi$  on the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  using para-controlled calculus. An interesting application is the exponential tail bounds for the ground state eigenvalue  $\Lambda_1$ . It was proved in [1, Proposition 5.4] that there exists  $C_1, C_2 > 0$  such that

$$e^{C_1 x} \leq \mathbb{P}[\Lambda_1 \leq x] \leq e^{C_2 x}$$

as  $x \rightarrow -\infty$ . Using the orthonormal eigenvectors of  $\mathcal{H}$ , denoted by  $\{e_n\}$ , we write the solution to PAM as

$$u(t, x) = \sum_{n=1}^{\infty} e^{-\Lambda_n t} \langle u_0, e_n \rangle e_n(x),$$

therefore,

$$\int_{\mathbb{T}^2} \mathbb{E}[|u(t, x)|^2] dx \leq \mathbb{E}[e^{-2\Lambda_1 t}] \int_{\mathbb{T}^2} |u_0(t, x)|^2 dx.$$

By the exponential tail bounds on  $\Lambda_1$ , it is clear the r.h.s. of the above display is only finite for small  $t$ , which is consistent with our result.

*Remark 1.6.* In the forthcoming article [5], the authors consider the 2D PAM with a small noise

$$(1.11) \quad \partial_t u = \Delta u + \beta u \cdot \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

They obtain an explicit chaos expansion of certain polymer measure associated with (1.11) for  $\beta \ll 1$ . In particular, this implies that the second moment of  $u$  exists for  $t \in [0, 1], x \in \mathbb{R}^2$  and  $\beta$  sufficiently small. The restriction of  $\beta \ll 1$  is equivalent with our small time restriction. Indeed, define

$$u_\beta(t, x) := u(t/\beta^2, x/\beta),$$

one sees that  $u_\beta$  satisfies (1.1), hence for  $u_\beta(t, x)$  to be square integrable, we need  $t/\beta^2 \leq 1$ , i.e.,  $t \leq \beta^2 \ll 1$ .

*Remark 1.7.* A simple calculation shows that the moments of the approximations to 3D PAM explode as  $\varepsilon \rightarrow 0$ , and indicates that the solution to 3D PAM may not have a moment. To see this, we consider the constant initial condition  $u_0 \equiv 1$ , so

$$\mathbb{E}[u_\varepsilon(t, x)] = e^{-C_\varepsilon t} \mathbb{E}_{\mathbf{B}} \left[ \exp \left( \int_0^t \int_0^s R_\varepsilon(B_s - B_u) du ds \right) \right],$$

where  $R_\varepsilon(x) = \varepsilon^{-3} R(x/\varepsilon)$ .

Since  $R(x)$  is continuous and  $R(0) > 0$ , without loss of generality we assume there exists  $\delta > 0$  such that  $R(x) > \delta > 0$  for  $|x| \leq 2$ . Thus, by considering the event that  $|B_s| < \varepsilon$  for all  $s \in [0, t]$ , we have

$$\mathbb{E}_{\mathbf{B}} \left[ \exp \left( \int_0^t \int_0^s R_\varepsilon(B_s - B_u) du ds \right) \right] \geq \exp \left( \frac{\delta t^2}{2\varepsilon^3} \right) \mathbb{P} \left[ \sup_{s \in [0, t]} |B_s| < \varepsilon \right].$$

The probability  $\mathbb{P}[\sup_{s \in [0, t]} |B_s| < \varepsilon]$  is bounded from below by  $e^{-c' t \varepsilon^{-2}}$  for some  $c' > 0$  depending on the dimension. When  $d = 3$ , the renormalization constant

$C_\varepsilon = c_1\varepsilon^{-1} + c_2|\log \varepsilon| + O(1)$ . It implies that for any  $t > 0, x \in \mathbb{R}^3$ , we have  $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[u_\varepsilon(t, x)] = \infty$ . The same discussion applies to  $d = 2$ , where

$$\mathbb{E}[u_\varepsilon(t, x)] \geq \exp\left(\frac{\delta t^2}{2\varepsilon^2} - \frac{c't}{\varepsilon^2} - C_\varepsilon t\right).$$

If  $t > 2c'/\delta$ , we also have  $\lim_{\varepsilon \rightarrow 0} \mathbb{E}[u_\varepsilon(t, x)] = \infty$ . Since we do not have a proof of  $\mathbb{E}[u(t, x)] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}[u_\varepsilon(t, x)]$  in  $d = 3$  or  $d = 2$  for large  $t$ , we only conjecture that  $\mathbb{E}[u(t, x)] = \infty$  in those cases.

*Remark 1.8.* When  $d = 2$ , the small time constraint for the existence of moments in our context also appears in [11, Theorem 4.1], where the usual product  $u \cdot \xi$  is replaced by the Wick product  $u \diamond \xi$ .

*Remark 1.9.* In [6], a similar result is derived for the random Schrödinger equation  $i\partial_t \phi + \frac{1}{2}\Delta \phi - \phi \cdot \xi = 0$ .

## 2. PROOF OF LEMMA 1.2 AND THEOREM 1.1

We denote  $[0, t]_{<}^n = \{0 \leq s_1 < \dots < s_n \leq t\}$ , and write  $a \lesssim b$  if  $a \leq Cb$  with some constant  $C$  independent of  $\varepsilon$ .

*Proof of Lemma 1.2.* By scaling property of Brownian motion, we have

$$R_\varepsilon(B_s - B_u) = \varepsilon^{-2} R\left(\frac{B_s - B_u}{\varepsilon}\right) \stackrel{\text{law}}{=} \varepsilon^{-2} R(B_{s/\varepsilon^2} - B_{u/\varepsilon^2}).$$

A change of variable  $(u/\varepsilon^2, s/\varepsilon^2) \mapsto (u, s)$  then yields

$$\nu_\varepsilon(t) = \varepsilon^2 \int_0^{t/\varepsilon^2} \int_0^s \mathbb{E}_{\mathbf{B}}[R(B_s - B_u)] du ds.$$

Now,  $B_s - B_u$  has the normal density  $x \mapsto (2\pi(s - u))^{-1} e^{-\frac{|x|^2}{2(s-u)}}$ . We then do another change of variable  $s - u \mapsto v$ , integrate  $s$  out, and rescale  $v \rightarrow v\varepsilon^2$ . This leads us to

$$\begin{aligned} \nu_\varepsilon(t) &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_0^t v^{-1} e^{-\frac{\varepsilon^2 |x|^2}{2v}} dv \right) dx - \frac{1}{2\pi} \int_0^t \left( \int_{\mathbb{R}^2} R(x) e^{-\frac{\varepsilon^2 |x|^2}{2v}} dx \right) dv \\ &:= \text{(i)} - \text{(ii)}. \end{aligned}$$

Since  $R$  integrates to 1, it is clear that

$$\text{(ii)} \rightarrow \frac{t}{2\pi}$$

as  $\varepsilon \rightarrow 0$ . As for (i), a substitution of variable  $\frac{\varepsilon^2 |x|^2}{2v} \mapsto \lambda$  and then an integration by parts yields

$$\begin{aligned} \text{(i)} &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^\infty \lambda^{-1} e^{-\lambda} d\lambda \right) dx \\ &= \frac{t}{2\pi} \int_{\mathbb{R}^2} R(x) \left( \int_{\frac{\varepsilon^2 |x|^2}{2t}}^\infty e^{-\lambda} \log \lambda d\lambda - e^{-\frac{\varepsilon^2 |x|^2}{2t}} \log \left( \frac{\varepsilon^2 |x|^2}{2t} \right) \right) dx. \end{aligned}$$

It is clear from the above expression that as  $\varepsilon \rightarrow 0$ , the only divergent part of (i) is from the term  $\log(\varepsilon^2)$ , and a direct calculation shows

$$\nu_\varepsilon(t) - \frac{t}{\pi} \cdot |\log \varepsilon| \rightarrow \mu_1 t + \mu_2 t \log t$$

for some constant  $\mu_1, \mu_2$ .  $\square$

*Proof of Theorem 1.1.* Fix  $(t, x)$  and  $n$ , and recall that

$$(2.1) \quad \mathbb{E}[u_\varepsilon(t, x)^n] = \mathbb{E}_{\mathbf{B}} \left[ \exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k) \right],$$

where  $\mathbb{E}_{\mathbf{B}}$  is the expectation with respect to independent planar Brownian motions  $B^k$ 's, and  $I_n^\varepsilon$  is given by the expression (1.6). Note that  $u_\varepsilon(t, x)^n \rightarrow u(t, x)^n$  in probability, and that by (1.8) and Lemma 1.2, we have

$$I_n^\varepsilon(t) - nC_\varepsilon t \rightarrow \mathcal{X}_n(t) + nt(\mu_1 + \mu_2 \log t)$$

in probability. Thus, in view of (2.1), it suffices to show the uniform integrability of  $u_\varepsilon(t, x)^n$  and  $\exp(I_n^\varepsilon(t) - nC_\varepsilon t) \prod_{k=1}^n u_0(x + B_t^k)$ . This allows us to pass both sides of (2.1) to the limit and conclude Theorem 1.1.

To prove the uniform integrability, we bound the second moment of these two objects:

$$\mathbb{E}[|u_\varepsilon(t, x)|^{2n}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{I_{2n}^\varepsilon(t) - 2nC_\varepsilon t}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{I_{2n}^\varepsilon(t) - 2n\nu_\varepsilon(t)}],$$

and

$$\mathbb{E}_{\mathbf{B}} \left[ \left| e^{I_n^\varepsilon(t)} e^{-nC_\varepsilon t} \prod_{k=1}^n u_0(x + B_t^k) \right|^2 \right] \lesssim \mathbb{E}_{\mathbf{B}}[e^{2I_n^\varepsilon(t) - 2nC_\varepsilon t}] \lesssim \mathbb{E}_{\mathbf{B}}[e^{2(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}],$$

where we have used  $\|u_0\|_\infty \leq 1$ . Thus, it suffices to show that for every  $n$  and  $\theta$ , there exists  $t_0$  small enough such that  $\mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}]$  is uniformly bounded in  $\varepsilon$  for all  $t < t_0$ . To see this, using Hölder's inequality, we get

$$\mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}] \leq \prod_{k=1}^n \left[ \mathbb{E}_{\mathbf{B}} e^{\theta N[\beta_\varepsilon^k([0, t]_{<}^2) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon^k([0, t]_{<}^2)]} \right]^{\frac{1}{N}} \prod_{1 \leq i < j \leq n} \left( \mathbb{E}_{\mathbf{B}} e^{\theta N \alpha_\varepsilon^{i,j}([0, t]^2)} \right)^{\frac{1}{N}},$$

where  $N = \frac{n(n+1)}{2}$ , and we have used the notations

$$\beta_\varepsilon^k([0, t]_{<}^2) = \int_0^t \int_0^s R_\varepsilon(B_s^k - B_u^k) du ds, \quad \alpha_\varepsilon^{i,j}([0, t]^2) = \int_0^t \int_0^s R_\varepsilon(B_s^i - B_u^j) ds du.$$

By change of variables and the scaling property of the Brownian motion, we have

$$\beta_\varepsilon^k([0, t]_{<}^2) \stackrel{\text{law}}{=} t \beta_{\varepsilon/\sqrt{t}}^k([0, 1]_{<}^2), \quad \alpha_\varepsilon^{i,j}([0, t]^2) \stackrel{\text{law}}{=} t \alpha_{\varepsilon/\sqrt{t}}^{i,j}([0, 1]^2).$$

Then, Lemma A.1 implies that there exists  $\lambda, C > 0$  such that

$$t < \frac{\lambda}{\theta N} \Rightarrow \sup_{\varepsilon \in (0, 1)} \mathbb{E}_{\mathbf{B}}[e^{\theta(I_n^\varepsilon(t) - n\nu_\varepsilon(t))}] \leq C.$$

This completes the proof.  $\square$

# APPENDIX A. EXPONENTIAL MOMENTS OF INTERSECTION LOCAL TIME OF PLANAR BROWNIAN MOTIONS

Recall that  $R_\varepsilon(x) = \varepsilon^{-2}R(\frac{x}{\varepsilon})$ , we define

$$\alpha_\varepsilon(A) = \int_A R_\varepsilon(B_s^1 - B_u^2) ds du, \quad \beta_\varepsilon(A) = \int_A R_\varepsilon(B_s - B_u) ds du$$

for any set  $A \subset \mathbb{R}_+^2$ , and

$$X_\varepsilon = \beta_\varepsilon([0, 1]_\<^2) - \mathbb{E}_{\mathbf{B}}[\beta_\varepsilon([0, 1]_\<^2)], \quad Y_\varepsilon = \alpha_\varepsilon([0, 1]_\<^2).$$

**Lemma A.1.** *There exists universal constants  $\lambda, C > 0$  such that*

$$\sup_{\varepsilon \in (0,1)} (\mathbb{E}_{\mathbf{B}}[e^{\lambda X_\varepsilon}] + \mathbb{E}_{\mathbf{B}}[e^{\lambda Y_\varepsilon}]) \leq C.$$

The above result is standard. The case  $\varepsilon = 0$ , i.e., the exponential integrability of intersection local time, was addressed in the classical work [13]. We could not find a direct reference for  $\varepsilon > 0$ , though the proof follows essentially in the same line as the case of  $\varepsilon = 0$ . For the convenience of the reader, we present the details here.

*Proof.* We consider  $Y_\varepsilon$  first. Since  $R = \rho \star \rho$ , we can write

$$Y_\varepsilon = \int_{[0,1]^2} \int_{\mathbb{R}^2} \rho_\varepsilon(B_s^1 - x) \rho_\varepsilon(B_u^2 - x) dx ds du,$$

with  $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon)$ . For any  $n \in \mathbb{N}$ ,

$$\begin{aligned} \mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n] &= \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^{2n}} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^n \rho_\varepsilon(B_{s_k}^1 - x_k) \rho_\varepsilon(B_{u_k}^2 - x_k) \right] ds du \right) d\mathbf{x} \\ &= \int_{\mathbb{R}^{2n}} \left( \int_{[0,1]^n} \mathbb{E}_{\mathbf{B}} \left[ \prod_{k=1}^n \rho_\varepsilon(B_{s_k} - x_k) \right] ds \right)^2 d\mathbf{x}. \end{aligned}$$

By [3, (2.2.11)], we have

$$\mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n] = \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \rho_\varepsilon(z_k - x_k) \sum_{\sigma} \int_{[0,1]_\<^n} \prod_{k=1}^n p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) ds dz \right)^2 d\mathbf{x},$$

where  $p_t(x)$  is the density of  $N(0, t)$ ,  $[0, t]_\<^n = \{0 \leq s_1 < \dots < s_n \leq t\}$ , and  $\sum_{\sigma}$  denotes the summation over all permutations over  $\{1, \dots, n\}$ . If we denote

$$h(z_1, \dots, z_n) = \sum_{\sigma} \int_{[0,1]_\<^n} \prod_{k=1}^n p_{s_k - s_{k-1}}(z_{\sigma(k)} - z_{\sigma(k-1)}) ds, \quad Q_\varepsilon(z_1, \dots, z_n) = \prod_{k=1}^n \rho_\varepsilon(z_k),$$

then  $\mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n]$  equals to

(A.1)

$$\begin{aligned} \int_{\mathbb{R}^{2n}} |Q_\varepsilon \star h(x_1, \dots, x_n)|^2 d\mathbf{x} &\leq \left( \int_{\mathbb{R}^{2n}} Q_\varepsilon(x_1, \dots, x_n) d\mathbf{x} \right)^2 \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x} \\ &= \int_{\mathbb{R}^{2n}} |h(x_1, \dots, x_n)|^2 d\mathbf{x} = \mathbb{E}_{\mathbf{B}}[\alpha([0, 1]_\<^2)^n], \end{aligned}$$

where  $\alpha([0, 1]^2)$  is the mutual-intersection local time formally written as

$$\alpha([0, 1]^2) = \int_0^1 \int_0^1 \delta(B_s^1 - B_u^2) ds du,$$

and we used the Le Gall's moment formula in the second line of (A.1). By [13], we have

$$\mathbb{E}_{\mathbf{B}}[\exp(\mu\alpha([0, 1]^2))] < C$$

for some  $\mu > 0$ , hence we only need to choose  $\lambda = \mu$  to get

$$\mathbb{E}_{\mathbf{B}}[e^{\lambda Y_\varepsilon}] = \sum_{n=0}^{\infty} \frac{\lambda^n \mathbb{E}_{\mathbf{B}}[Y_\varepsilon^n]}{n!} \leq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \mathbb{E}_{\mathbf{B}}[|\alpha([0, 1]^2)|^n] = \mathbb{E}_{\mathbf{B}}[e^{\mu\alpha([0, 1]^2)}] < \infty.$$

Next, we consider  $X_\varepsilon$ . We define the triangle approximation of  $\{(u, s) : 0 \leq u < s \leq 1\}$ :

$$A_l^k = \left[ \frac{2l}{2^{k+1}}, \frac{2l+1}{2^{k+1}} \right) \times \left[ \frac{2l+1}{2^{k+1}}, \frac{2l+2}{2^{k+1}} \right), \quad l = 0, 1, \dots, 2^{k-1}, k = 0, 1, \dots$$

We will use the following three properties:

- (i) Fix any  $k$ ,  $\{\beta_\varepsilon(A_l^k)\}_{l=0, \dots, 2^{k-1}}$  are i.i.d. random variables.
- (ii)  $\beta_\varepsilon(A_l^k) \stackrel{\text{law}}{=} 2^{-(k+1)} \beta_{\varepsilon 2^{(k+1)/2}}([0, 1] \times [1, 2]) \stackrel{\text{law}}{=} 2^{-(k+1)} \alpha_{\varepsilon 2^{(k+1)/2}}([0, 1]^2)$
- (iii)  $\sup_{\varepsilon > 0} \mathbb{E}_{\mathbf{B}}[e^{\lambda \alpha_\varepsilon([0, 1]^2)}] \leq C$  for some  $\lambda, C > 0$ .

By (iii) and a Taylor expansion, there exists  $C > 0$  such that for sufficiently small  $\lambda$

$$(A.2) \quad \sup_{\varepsilon > 0} \mathbb{E}_{\mathbf{B}}[e^{\lambda(\alpha_\varepsilon([0, 1]^2) - \mathbb{E}_{\mathbf{B}}[\alpha_\varepsilon([0, 1]^2)])}] \leq e^{C\lambda^2}.$$

We fix the constants  $\lambda, C$  from now on, and write

$$X_\varepsilon = \sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}}[\beta_\varepsilon(A_l^k)]).$$

Fix  $a \in (0, 1)$  and define a sequence of constants

$$b_1 = 2\lambda, \quad b_N = 2\lambda \prod_{j=2}^N (1 - 2^{-a(j-1)}), \quad N = 2, 3, \dots,$$



we have

$$\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \\
& \leq \left( \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \right)^{1-2^{-a(N-1)}} \\
& \quad \times \left( \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N \sum_{l=0}^{2^N-1} (\beta_\varepsilon(A_l^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^N)) \right] \right)^{2^{-a(N-1)}} \\
& \leq \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \\
& \quad \times \left( \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N (\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_0^N)) \right] \right)^{2^{N-a(N-1)}}.
\end{aligned}$$

Since  $\beta_\varepsilon(A_0^N) \stackrel{\text{law}}{=} 2^{-(N+1)} \alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2)$ , we have

$$\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N (\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_0^N)) \right] \\
& = \mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N 2^{-(N+1)} (\alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2) - \mathbb{E}_{\mathbf{B}} \alpha_{\varepsilon 2^{(N+1)/2}}([0, 1]^2)) \right].
\end{aligned}$$

Using the fact that  $2^{a(N-1)} b_N 2^{-(N+1)} < \lambda$  and (A.2), we derive for all  $\varepsilon > 0$  that

$$\mathbb{E}_{\mathbf{B}} \exp \left[ 2^{a(N-1)} b_N (\beta_\varepsilon(A_0^N) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_0^N)) \right] \leq e^{C b_N^2 2^{-2N+2a(N-1)}},$$

so there exists  $C' > 0$  such that

$$\begin{aligned}
& \mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \\
& \leq \mathbb{E}_{\mathbf{B}} \exp \left[ b_{N-1} \sum_{k=0}^{N-1} \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] e^{C' 2^{(a-1)N}}.
\end{aligned}$$

Iterating the above inequality, we get

$$\mathbb{E}_{\mathbf{B}} \exp \left[ b_N \sum_{k=0}^N \sum_{l=0}^{2^k-1} (\beta_\varepsilon(A_l^k) - \mathbb{E}_{\mathbf{B}} \beta_\varepsilon(A_l^k)) \right] \leq \exp(C'(1 - 2^{a-1})^{-1})$$

Since  $b_N \rightarrow b_\infty$  for some  $b_\infty > 0$ , we have

$$\mathbb{E}_{\mathbf{B}}[\exp(b_\infty X_\varepsilon)] \leq \exp(C'(1 - 2^{a-1})^{-1}),$$

which completes the proof.  $\square$

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## REFERENCES

- [1] R. ALLEZ AND K. CHOUK, *The continuous Anderson hamiltonian in dimension two*, arXiv preprint arXiv:1511.02718, (2015).
- [2] G. CANNIZZARO AND K. CHOUK, *Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential*, arXiv preprint arXiv:1501.04751, (2015).
- [3] X. CHEN, *Random walk intersections: Large deviations and related topics*, no. 157, American Mathematical Soc., 2010.
- [4] K. CHOUK, J. GAIRING, AND N. PERKOWSKI, *An invariance principle for the two-dimensional parabolic Anderson model with small potential*, arXiv preprint arXiv:1609.02471, (2016).
- [5] D. ERHARD AND N. ZYGOURAS, *private communication*.
- [6] Y. GU, T. KOMOROWSKI, AND L. RYZHIK, *The Schrödinger equation with spatial white noise: the average wave function*, arXiv preprint arXiv:1706.01351, (2017).
- [7] M. GUBINELLI, P. IMKELLER, AND N. PERKOWSKI, *Paracontrolled distributions and singular PDEs*, in Forum of Mathematics, Pi, vol. 3, Cambridge Univ Press, 2015, p. e6.
- [8] M. HAIRER, *A theory of regularity structures*, Inventiones mathematicae, 198 (2014), pp. 269–504.
- [9] M. HAIRER AND C. LABBÉ, *Multiplicative stochastic heat equations on the whole space*, arXiv preprint arXiv:1504.07162, (2015).
- [10] —, *A simple construction of the continuum parabolic Anderson model on  $\mathbf{R}^2$* , Electronic Communications in Probability, 20 (2015).
- [11] Y. HU, *Chaos expansion of heat equations with white noise potentials*, Potential Analysis, 16 (2002), pp. 45–66.
- [12] J.-F. LE GALL, *Some properties of planar Brownian motion*, in Ecole d’Eté de Probabilités de Saint-Flour XX-1990, Springer, 1992, pp. 111–229.
- [13] —, *Exponential moments for the renormalized self-intersection local time of planar Brownian motion*, in Séminaire de Probabilités XXVIII, Springer, 1994, pp. 172–180.
- [14] S. VARADHAN, *Appendix to euclidean quantum field theory by k. symanzik*, Local Quantum Theory. Academic Press, Reading, MA, 1 (1969).
- [15] M. YOR, *Precisions sur l’existence et la continuité des temps locaux d’intersection du mouvement Brownien dans  $\mathbf{R}^2$* , in Séminaire de Probabilités XX 1984/85, Springer, 1986, pp. 532–542.

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