

Exponential Asymptotics in Unsteady and Three-Dimensional Flows



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To my family.

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Abstract

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The behaviour of free-surface gravity waves on small Froude number fluid flow past some obstacle cannot be determined using ordinary asymptotic power series methods, as the amplitude of the waves is exponentially small. An exponential asymptotic method is used by Chapman and Vanden-Broeck (2006) to consider the problem of two-dimensional, steady flow past a submerged obstacle in the small Froude number limit, finding that a steady downstream wavetrain is switched on rapidly across a curve known as a Stokes line. Here, equivalent wavetrains on three-dimensional and unsteady flow configurations are considered, and Stokes switching caused by the interaction between exponentially small free-surface components is shown to play an important role in both cases.

The behaviour of free-surface gravity waves is introduced by considering the problem of steady free-surface flow due to a line source. A steady wavetrain is shown to exist in the far field, and the behaviour of these waves is compared to existing numerical results. The problem of unsteady flow over a step is subsequently investigated, with the flow behaviour formulated in terms of Lagrangian coordinates so that the position of the free surface is fixed. Initially, the problem is linearized in the step-height, and the steady wavetrain is shown to spread downstream over time. The position of the wavefront is determined by considering the full Stokes structure present in the problem. The equivalent fully-nonlinear problem is then considered, with the position of the Stokes lines, and hence the wavefront, being determined numerically. Finally, linearized three-dimensional free-surface flow past an obstacle is considered in both the steady and unsteady case. The surface is shown to contain downstream longitudinal and transverse waves. These waves are shown to propagate downstream in the unsteady case, with the position of the wavefront again determined by considering the full Stokes structure of the problem.

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Chapter 1

Introduction

Mathematicians have long known that certain divergent series may, in the limit as some parameter becomes small or large, be truncated after a finite number of terms in order to provide useful approximations for the behaviour of some given expression. The modern understanding of these approximations originated with Henri Poincaré [92], who defined the asymptotic series representation of a function $f(x)$ as a series with coefficients a_r that satisfies

$$\lim_{x \rightarrow \infty} x^n \left\{ f(x) - \sum_{r=0}^n \frac{a_r}{x^r} \right\} \rightarrow 0, \quad (1.1)$$

for fixed, non-negative values of n .

This definition does not preclude the series from being divergent; however, judicious truncation of the asymptotic series at some finite value of n can produce very precise approximations of the exact function, if the truncation occurs earlier in the series than the point at which the addition of extra terms causes the error to increase. Indeed, a divergent asymptotic series truncated *optimally* will have a remainder that is exponentially small in the limit as $|x| \rightarrow \infty$ [9] (where this refers to a function that is exponentially decreasing in x , such as e^{-x} , for example).

The ability to obtain highly accurate approximations led to widespread acceptance of asymptotic methods in mathematics, and such techniques became particularly important in the study of physical phenomena. Unfortunately, as noted by Dingle [33], Poincaré's definition (1.1) contains a degree of non-uniqueness, as the addition of any exponentially small function of x to $f(x)$ will produce a different function that still satisfies (1.1). This limitation may seem inherent to the nature of asymptotic series approximations; however, there have been numerous studies, listed in [33], which have found that the appropriate exponentially small correction term in an asymptotic series obtained as the solution to some problem may be determined uniquely. Consequently, Poincaré's definition (1.1) is incomplete.

The problem of formally defining asymptotic series so as to allow for unique, exponentially small remainder terms is discussed at length by Dingle [33], and a more involved definition is

obtained which allows for an asymptotic series to be defined even down to the presence of these exponentially small terms. For applied mathematicians, however, a more pressing question is why such terms are necessary in the first place. After all, one of the most useful features of asymptotic series methods is that highly accurate approximations may be obtained by retaining only a small number of terms. To motivate the study of these exponentially small terms we turn to a physical problem that has interested mathematicians for over a century: the behaviour of water waves.

The theory of water waves may be traced back to Newton, who inferred by analogy that the frequency of waves on water of infinite depth was inversely proportional to the square root of the wavelength. For a more detailed review of this work, as well as numerous other early contributions to the theory of water waves by the likes of Lagrange, Laplace, Green and Airy, see [31]. Here, we restrict our attention to the problem of water waves caused by gravitational effects on free-surface flows. Typically, wave behaviour is represented in terms of a nondimensional quantity known as the Froude number, F , which represents the relative strength of inertial and gravitational forces in the flow configuration. In the case of flow past some submerged obstacle, the Froude number is given by

$$F = \frac{U}{\sqrt{gH}}, \quad (1.2)$$

where g is the acceleration due to gravity, and U and H are representative velocity and length scales respectively. We are particularly interested in the behaviour of free-surface gravity waves in the limit that the Froude number becomes small, or when gravitational forces dominate inertial forces.

Attempts to represent the flow behaviour using an asymptotic series in this limit lead to apparently paradoxical behaviour, in which free-surface gravity waves disappear in the low-Froude number limit. For example, Ogilvie [85] first noted that, in this limit for flow past a submerged obstruction or surface-piercing object, an asymptotic series in F failed to capture any wave behaviour present on the free surface. Peregrine [89] and Forbes [37] also found waveless free-surface behaviour while investigating a line source or sink in this limit.

However, numerical studies of free-surface gravity waves indicate that small Froude number flow is, in fact, not waveless. These studies include those performed by Vanden-Broeck and Tuck [117], and McCue and Forbes [76] on flow past a two-dimensional bow or stern in infinite and finite depth respectively; Vanden-Broeck *et al.* [116], and Mekias and Vanden-Broeck [78] on flow over a line source in infinite and finite depth respectively; Zhang and Zhu [122, 123] on flow past a submerged semiellipse; Dagan [32] on flow past a thin submerged obstacle; and Doctors and Dagan [34] for flow past a surface pressure distribution. In each of these investigations, the computed wave amplitude appears to be exponentially small in the limit that $F \rightarrow 0$.

This behaviour suggests that an asymptotic power series will be insufficient for capturing the behaviour of the free-surface waves. The wave behaviour is referred to as lying ‘beyond all orders’ of the asymptotic power series.

Any attempt to determine the behaviour of these waves will therefore rely on determining the behaviour of the exponentially small remainder term of the asymptotic power series in F . Fortunately, a range of techniques known as ‘exponential asymptotics’ or ‘hyperasymptotics’ were developed for precisely this purpose, such as in [9, 10, 22, 87]. These methods typically involve truncating the asymptotic series optimally, giving an exponentially small remainder term, and then rescaling the equations in order to investigate the behaviour of this term; these methods will be discussed more thoroughly in Section 1.1.

Using exponential asymptotic methods, Chapman and Vanden-Broeck [27] were able to obtain the behaviour of exponentially small free-surface waves for steady-state flow over a submerged two-dimensional obstacle in the small Froude number limit. The purpose of the current investigation is to extend these methods in order to determine the free-surface wave behaviour of unsteady (or time-varying) and three-dimensional flow problems.

1.1 Exponential asymptotics

We previously noted that free-surface gravity waves are known to be exponentially small in the small Froude number limit, and that this property renders them impossible to represent using ordinary asymptotic series methods. Indeed, using Poincaré’s definition of an asymptotic series (1.1), they cannot be uniquely determined at all. However, noting that exponentially small remainder terms *can* be uniquely determined as a function of the series terms themselves, and Dingle’s subsequent redefinition of asymptotic series, several studies developed methods that could be used to find these remainder terms in general.

Berry and Howls [10] described the optimal truncation of a divergent asymptotic series, producing a remainder which is exponentially small, as ‘superasymptotics’. They subsequently defined ‘hyperasymptotics’ (also known in the literature as ‘exponential asymptotics’ or ‘asymptotics beyond-all-orders’) as the systematic study of the exponentially small remainder itself. This process typically involves rescaling the system in question in such a fashion that the remainder may be expressed as a second asymptotic series (or even simply the first term in such a series), exponentially small compared even to the original, optimally-truncated series. A number of these techniques are discussed in the review by Boyd [14].

There are a number of reasons to be interested in the exponentially small terms of an asymptotic series. As we have previously noted in the context of free-surface gravity waves, exponentially

small terms may contain qualitative behaviour that is not found at any algebraic order in the small parameter. The review by Boyd [14] lists numerous problems with important features that are found only on an exponentially small scale, coming from areas such as fluid dynamics, quantum mechanics and condensed matter physics.

There are also other reasons to investigate these exponentially small remainder terms. In some cases, such as the Airy function [7] or the complementary error function [87], terms that are exponentially small in one part of the solution domain are dominant in other parts. Hence, to fully understand the global complex-plane behaviour of these functions, we must consider the effect of these contributions. Finally, and perhaps most strikingly, these exponentially small terms may influence the leading-order behaviour of the problem. This is typically seen in problems like the Saffman-Taylor fingering problems considered in [19, 21], which produce a family of solutions that satisfy the leading-order equations. In these problems, the correct leading-order behaviour may only be selected by considering the behaviour of the exponentially small terms.

While the free-surface fluid-flow problems presented in this thesis fall into the first of these categories, it is important to note that the effect of exponentially small contributions on a problem can be substantial, leading to the efforts that have been made in order to determine the behaviour of these contributions.

1.1.1 Stokes phenomenon

Using exponential asymptotic methods, Berry [7] was able to investigate the behaviour known as the Stokes phenomenon, which plays an important role in the current investigation. It was Stokes [96] who first observed that a function containing multiple exponentials in the complex plane can contain curves along which the behaviour of the subdominant exponential apparently changes discontinuously. These curves, known as Stokes lines, occur when one exponential term is maximally-subdominant compared to another.

In Figure 1.1, we illustrate the effect of Stokes lines on the asymptotic behaviour of the complementary error function as the argument becomes infinite. In this figure, the constant term changes from two to zero as the imaginary axis is crossed. However, the complementary error function, and indeed many functions which exhibit Stokes switching, is known to be entire. Therefore, it is impossible for there to be a discontinuous jump in the complex-plane behaviour of the function.

As the term switched on across a Stokes line is always exponentially subdominant to the leading asymptotic contribution, Stokes switching is impossible to investigate using ordinary asymptotic series methods. Hence, this apparent discontinuity remained unexplained until the advent of

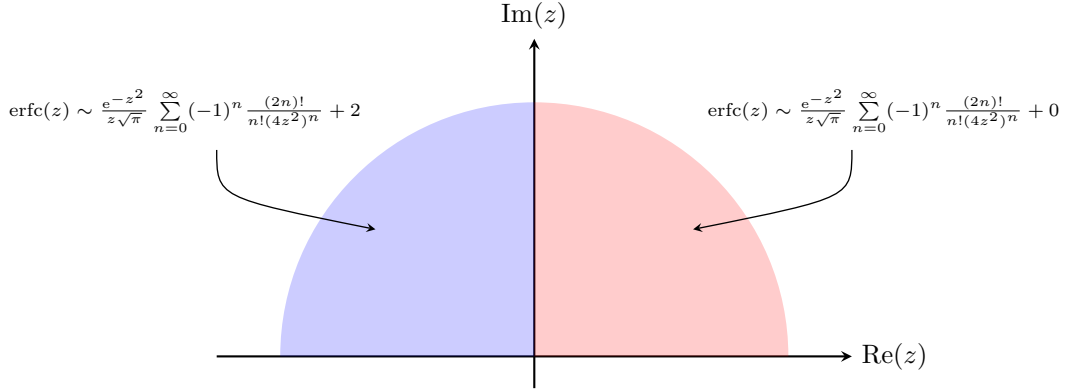


Figure 1.1: Asymptotic behaviour of $\text{erfc}(z)$ for $0 < \arg z < \pi$ as $|z| \rightarrow \infty$. In the blue region, the constant term is equal to two, while in the red region, the constant is equal to zero. There is a Stokes line following $\text{Im}(z) = 0$, across which the value of the constant changes.

exponential asymptotic methods. Using these ideas, Berry established that this apparent discontinuity in fact took the form of a smooth transition occurring in the neighbourhood of the Stokes line. In the examples investigated by Berry, this transition took the form of a rapidly-varying error function. This error-function smoothing is generic, but not completely universal [18].

Olde Daalhuis *et al.* [87] expanded on these ideas to develop a general method for determining Stokes switching behaviour in the solution to ordinary differential equations using optimal truncation and matched asymptotic expansions. This method requires an expression for the general behaviour of the asymptotic series terms for the solution, which restricts its application to linear ordinary differential equations for which this behaviour may be determined explicitly. However, the method was extended by Chapman *et al.* [22] so that it may be applied to nonlinear ordinary differential equations. These ideas are used extensively within the current investigation, and are described in detail in Section 1.3.

1.1.2 Second-generation Stokes phenomenon

The ideas of Olde Daalhuis *et al.* [87] and Chapman *et al.* [22] have been subsequently extended to investigate different classes of problem, such as difference-differential equations [66] and partial differential equations [12, 20]. Also in the latter category, Chapman and Mortimer [24], and Howls *et al.* [56] considered a partial differential equation governing time-varying dendritic crystal growth. The steady version of this problem gives an ordinary differential equation that demonstrates Stokes switching, which was examined using exponential asymptotic methods by Kruskal and Segur [68].

As noted, the unsteady crystal-growth problem takes the form of a partial differential equation

which contains behaviour more complicated than the Stokes switching addressed in [22, 87]. Specifically, the problem contained ‘new’ Stokes lines, described by Berk *et al.* [6], and ‘vanishing’ Stokes lines, described by Aoki *et al.* [4]. These features are explained in [24, 56], where they are described as ‘second-generation’ Stokes lines and ‘higher-order’ Stokes lines respectively.

The behaviour described as ‘second-generation Stokes switching’ occurs when the exponentially small remainder of an optimally-truncated asymptotic series *also* takes the form of a divergent asymptotic series, and hence contains further Stokes structure. This series may therefore be optimally-truncated and the remainder investigated using exponential asymptotic methods. In fact, it is entirely possible that a complicated hierarchy of Stokes lines may be present in the problem, such as those found in [12, 20].

It is important to note that, aside from the fact that the switching occurs within the remainder term, this behaviour is not distinct from the Stokes switching examined in [24, 56]. It is simply caused by the interaction between two subdominant contributions, rather than between the leading-order behaviour and a subdominant contribution. Consequently, it may be investigated by applying standard late-order term methods to the remainder term. It should also be noted that this behaviour may become the dominant contribution to the solution in some regions of the problem domain, and are hence not necessarily subdominant to the leading-order behaviour everywhere.

Higher-order Stokes lines may be found when multiple Stokes lines intersect at points known as ‘Stokes-crossing points’. These are Stokes lines across which we do not see the switching of an exponentially small contribution to the solution, but rather switching of the Stokes lines themselves. We illustrate this behaviour in Figure 1.2, taken from [24, 56]. This figure shows the Stokes structure of the time-varying crystal growth problem. The three Stokes lines intersect at a single point, and we see that two of the Stokes lines terminate at the intersection. The higher-order Stokes line is represented as a dashed line that also passes through the intersection point. It is this higher-order Stokes line that ‘switches off’ the Stokes lines, causing them to terminate at this point.

In Chapters 3–5, we will apply similar techniques to those of Chapman and Mortimer [24] to investigate partial differential equations arising from free-surface fluid flow problems. Each of these problems will have Stokes lines caused by the interaction between two contributions that are exponentially subdominant compared to the leading-order behaviour. This interaction will play an important role in determining how the surface waves propagate over time. Higher-order Stokes switching will only play a significant role in the unsteady three-dimensional problem considered in Chapter 5.

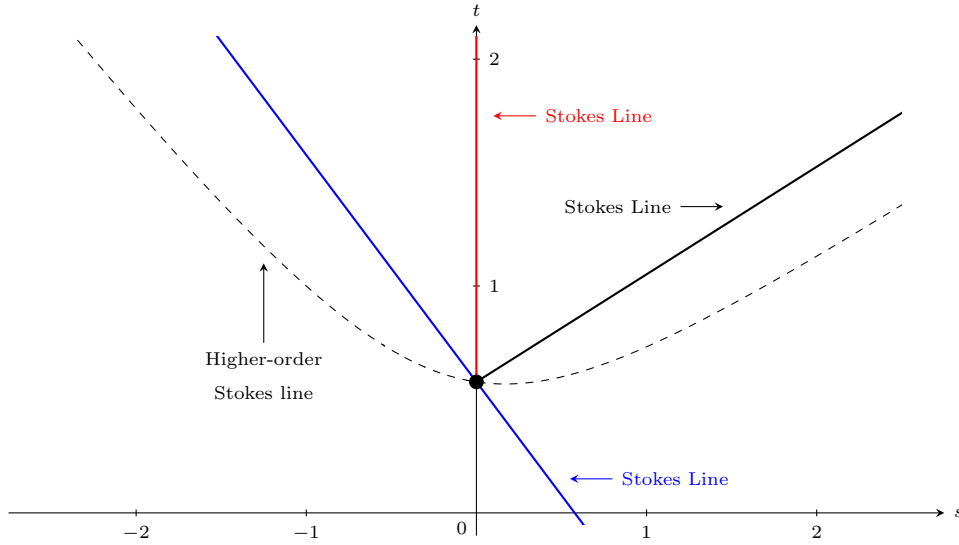


Figure 1.2: Stokes structure for the time-varying crystal growth problem investigated in [24, 56], which varies in s and t . The solid curves represent Stokes lines. The circle indicates the intersection of the Stokes lines, while the dashed curve represents a higher-order Stokes line that causes the red and black Stokes lines to terminate.

1.2 Free-surface gravity waves

Some of the earliest work on free-surface gravity waves was performed by Kelvin, who considered the downstream wave patterns produced by flow over a finite-depth submerged depression or elevation [62] as well as waves caused by flow past a ship represented as a point singularity in a moving frame [61]. The representation of obstacles in a potential flow path as singularities was used in a number of subsequent investigations, many of which involved linearizing the problem about the undisturbed solution, either through direct linearization of the boundary condition or by prescribing that the size of the disturbance be small compared to the depth of the obstacle.

A wide range of related problems have been addressed by linearizing the full flow system. For example, Gazdar determined flow behaviour past various submerged obstacles in a finite-depth stream [42] and under the surface of a flow with infinite depth [43], representing the obstructions using singularities. Lamb [70] investigated the behaviour of waves caused by flow past an infinite cylinder, as did Ursell [110]. Havelock formulated integral expressions for free-surface flow past a number of submerged obstacles, such as a spheroid [50] and a doublet [48], as well as an expression for flow past a pressure distribution [49]. King and Bloor found the behaviour of flow over a step [63] and submerged objects modelled as a source-sink pair [64]. Faltas *et al.* [35] investigated flow over a submerged trapezoid, and Abd-el-Malek and Masoud [1] considered flow over a ramp. King and Bloor [65], Boutros *et al.* [13], and Hanna [47] all examined free-surface behaviour for flow over an arbitrary base topography.

1.2.1 Unsteady flow

Each of the studies listed in Section 1.2 considers flow in a steady state, obtained either by background flow past a stationary obstacle or an obstacle moving at constant speed through the fluid. Unsteady models for free-surface waves have been studied at length, such as the Korteweg-de Vries (KdV) equation [67] for weakly nonlinear shallow water waves. Havelock [52] also directly considered the behaviour of surface waves caused by flow past a submerged cylinder starting from rest, formulating the wave resistance in terms of an integral equation. This investigation also noted that the formulation derived in [48] could be applied to unsteady flows with an obstacle represented as a combination of sources and sinks. John [58] used a Lagrangian formulation to find steady deep water waves; however, this method (later extended to more general flow configurations by Longuet-Higgins [73]) could also be applied to unsteady free-surface flows.

More recently, Cole [28] considered unsteady flow past a small bump, finding an integral expression for the behaviour of the surface for the linearized problem and solving the full nonlinear KdV equation numerically. These two regimes were then connected by deriving a quasi-linear theory. Abou-Dina [2] investigated unsteady gravity waves over a base topography caused by an initial prescribed free-surface position, using both the linearized theory and computing solutions to the full nonlinear formulation. Xue and Yue [120] computed the unsteady free-surface behaviour for flow due to an impulsively started point sink over a range of Froude numbers. Zhu and Zhang [124] also computed nonlinear flows over a submerged obstacle for a range of Froude numbers. Importantly, this investigation found that for subcritical flows ($F < 1$), a submerged obstacle moving with uniform velocity (equivalent to a stationary obstacle in a uniform flow field) produced a steady wavetrain that spreads over time. The behaviour of flow due to an impulsively-started submerged dipole moving with constant velocity may be seen in Figure 1.3 for $F = 0.5$.

Analytical studies of unsteady gravity waves have been performed in several asymptotic limits. Liu and Tao [71] formulated an integral expression for the far-field wave behaviour caused by an impulsively-started submerged body, while Lu [74] determined the far-field unsteady wave behaviour caused by an oscillating Stokeslet. Shen [95] found an asymptotic expression for unsteady long waves in a three-dimensional shallow channel with arbitrary base topography and small lateral velocity. Ockendon and Wilmott [84], and Wilmott [119] used matched asymptotic expansions to determine the behaviour of a free-surface flow past a small spheroid and arbitrary two-dimensional obstruction respectively, both in the case of arbitrary motion.

A number of asymptotic analyses have been performed in the small-time limit, such as the problem of flow due to an impulsively-started submerged cylinder [108, 109], a line source [106]

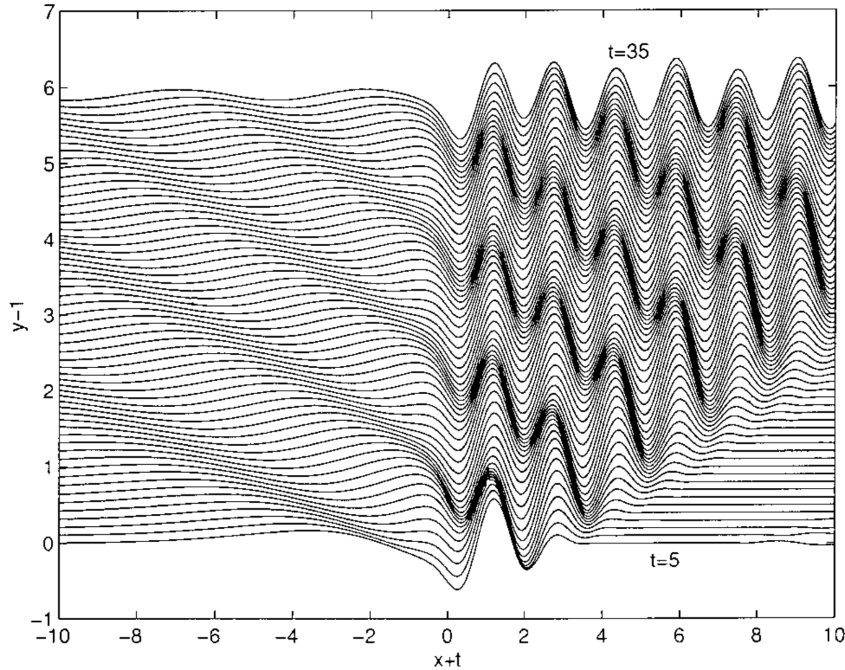


Figure 1.3: Numerically-calculated free surface for flow due to a submerged dipole with $F = 0.5$ at a depth of one-fifth of the undisturbed depth from the base, taken from [124]. The dipole is moving at a uniform velocity, having been impulsively started at $t = 0$. In this case, the undisturbed free-surface is positioned at $y = 1$. We see that a steady wavetrain is being generated in the wake of the obstacle. While there are other transient components to the flow present in this simulation, our analysis will concentrate on the propagation of steady surface waves in the wake of the submerged obstacle. Reprinted with permission from Zhang and Zhu [124]. Copyright 1997, American Institute of Physics.

or sink [97], and a point sink [107]. In addition to the small-time limit, Stokes, Hocking and Forbes [39] also considered the large-time asymptotic limit for an impulsively-started point sink submerged in a fluid. None of these investigations, however, determined the small Froude number wave behaviour of the flow configurations.

1.2.2 Nonlinear flow

The linearized unsteady problem considered in Chapter 3 contains two asymptotic limits: the limit of small Froude number, and of small obstacle size, denoted as δ . However, a formal power series in δ breaks down as $F \rightarrow 0$, because the first correction term in the small δ expansion overtakes the leading-order contribution in small F . Consequently, the two limits do not commute, and any results obtained by considering the linearized problem cannot simply be extended to the full nonlinear problem considered in Chapter 4.

This small Froude number non-uniformity is discussed by Salveson [94] and Dagan [32] for flow past a submerged obstacle, and Zhang and Zhu [122] for flow over submerged topography. To address this non-uniformity in their small Froude number analysis, Chapman and Vanden-

Broeck [27] determine the free-surface behaviour for both the linearized flow problem and the full nonlinear flow problem for steady flow over an obstruction. They then show that the two cases may be connected by a distinguished limit, when the small parameters in the problem balance appropriately.

The most substantial difficulty that arises in our analysis of the unsteady nonlinear free-surface behaviour is determining the position of the free boundary. In the linearized problem, the position of the surface is considered to follow a fixed, known curve; however, this is not possible when the submerged obstacle is not small. A common method of approaching this problem is to map the flow domain onto a fixed domain using conformal mapping techniques, such as in [40]. However, for reasons discussed in Section 3.2.2, this method is not possible in the unsteady case. Instead, we formulate the problem using Lagrangian coordinates, such as the formulations given in [16, 69, 91]. In our formulation, appropriate choice of the initial fluid particle-labelling allows both the free and fixed boundaries to be fixed, and the behaviour of the free surface to be determined.

1.2.3 Three-dimensional flow

One free-surface problem of particular importance to this investigation is that of flow past a submerged source in three dimensions. We specifically consider the case when the effect of the source is small, and therefore the problem may be linearized about the undisturbed solution. Much of the existing work on this topic involves formulating the surface behaviour in terms of a double integral using Green's functions. Havelock [51] formulated an integral expression by dividing the surface behaviour into separate near-field non-oscillatory and far-field oscillatory components. Alternative integral expressions of this form were found by Peters [90] and implicitly by Michell [79]. Noblesse [81, 82] demonstrated that these integral expressions, as well as several others, could be obtained directly from the original double integral methods using different transformations. Many of these results, as well as several alternative Green's function expressions, are given by Wehausen and Laitone [118].

While none of the listed investigations considered the low Froude number limit, there have been several studies into a related problem in this limit. Specifically, we refer to the problem of three-dimensional free-surface flow past a singularity located on the surface. This problem was first addressed by Kelvin [61], who determined the downstream wave behavior by linearizing the boundary condition. This investigation found that the downstream waves are restricted to a wedge with angle of approximately 19.5° from the direction of flow. The wedge contains two distinct classes of wave: longitudinal waves and transverse waves. Examples of these waves may be seen in Figure 1.4.

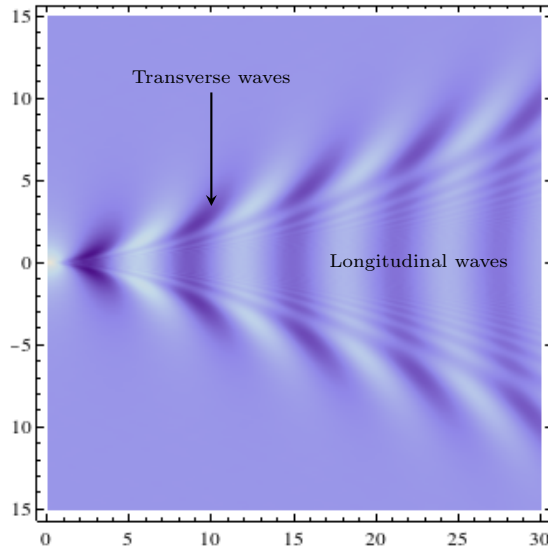


Figure 1.4: Computed wave patterns past a ship using the expression in [88]. Both the longitudinal and transverse wave patterns are visible in the downstream wake of the ship.

The problem of infinite-depth flow with a source on the free surface was considered in the small Froude number limit by studies such as that of Keller [59] and Brandsma [15]. These studies used geometric ray theory to determine the wave behaviour on the free surface, finding that the waves algebraically small in F were contained within a trailing downstream wedge. As in [61], the wake contained longitudinal and transverse waves. Geometric ray theory is not applicable when the source moves below the free surface; however, as the singularity responsible for the ray behaviour is no longer located on the surface and the waves become exponentially small in F .

1.2.4 Application of exponential asymptotics

Exponential asymptotic methods were first applied to the problem of gravity waves on free-surface flow over a submerged obstacle in the small Froude number limit by Chapman and Vanden-Broeck [27], who considered two-dimensional steady flow over a submerged obstacle. This obstacle led to the presence of a singularity in the flowpath. In this study, the authors used hodograph mapping methods, similar to that used in [40], to map first from the physical flowpath to a strip in the complex potential plane, and then to an upper-half mapped plane.

In the mapped system, the equations governing the system were formulated as a Cauchy integral equation in the body of the flow, and a differential equation governing the free-surface behaviour. The free surface was analytically-continued, so that exponential asymptotics could be used to find the Stokes lines and determine the quantity that is switched as they are crossed using the methodology given in [22, 87]. Using this method, an expression for the free-surface behaviour

was obtained, finding that the free-surface contains downstream train of exponentially small gravity waves switched on across a Stokes line.

In Chapter 2, we apply the same methodology to consider the closely-related problem of free-surface waves caused by steady two-dimensional flow due to a line source, which was omitted in [27]. Other free-surface gravity wave problems investigated using these ideas include flow past ships with a single corner [102] and with multiple corners [99], as well as flow over a submerged bump or trench with inclined edges [75]. In the latter of these studies, several configurations are produced with trapped-wave solutions, as the exponentially small gravity waves cancel in the far field.

A similar method was used by Chapman and Vanden-Broeck [26] to determine the behaviour of free-surface capillary waves in the small surface tension limit, which are also exponentially small. The two cases were combined by Trinh and Chapman [100, 101] for combined gravity-capillary waves in linearized and nonlinear flow configurations respectively.

Exponentially small gravity-capillary waves governed by the Korteweg-De Vries equation have been investigated in numerous studies, such as [17, 46, 45, 93], and more recently by Trinh [98], who used the exponential asymptotic methodology of [22, 87] to determine the exponentially small solitary wave behaviour on the surface. Van den Berg and King [112] also considered a two-parameter model water-wave equation, using exponential asymptotic methods to find an infinite number of solutions with homoclinic orbits, and hence solitary wave behaviour. These solutions are characterized by the fact that the solution does not contain exponentially small waves.

Finally, the use of exponential asymptotic methods has provided insight into a number of other problems from fluid dynamics. These include the selection of Saffman-Taylor finger width [19, 21] in the small-surface tension and kinetic undercooling parameter respectively, and the selection of admissible solutions for van der Waals-driven thin film rupture [25].

1.3 Methodology

This investigation will make extensive use of the exponential asymptotic technique developed by Olde Daalhuis *et al.* [87] and extended by Chapman *et al.* [22] for investigating the switching of exponentially small asymptotic contributions across Stokes lines. We recall that these terms are typically found in singular perturbation problems containing singularities, which produce divergent asymptotic series expansions.

The first step in this technique is to express the behaviour in question as an asymptotic power

series in the small variable, for example

$$f(x; \epsilon) \sim \sum_{r=0}^{\infty} \epsilon^r a_r(x) \quad \text{as } \epsilon \rightarrow 0.$$

Because the series is divergent, it may be truncated optimally in order to minimize the approximation error. To find the optimal truncation point, we follow the commonly-used strategy described by Boyd [14] which notes that for some given small parameter ϵ , the optimally-truncated series may usually be found by retaining up to the smallest term in the series expression, but no further. Boyd notes that this is not a universal law, and indeed that counter-examples may be constructed. However, it has been proven true for several classes of asymptotic series [30, 29, 86], and will prove sufficient for use in this investigation.

Unfortunately, determining the smallest term in the series, and consequently the optimal truncation point, is challenging. The method as described by Olde Daalhuis *et al.* [87] requires a general form for the asymptotic series coefficients $a_r(x)$. While such a form may be obtained for the solutions to some classes of problems, such as linear ordinary differential equations, in many other problems the series terms become algebraically intractable after even a small number of terms.

To overcome this restriction, Chapman *et al.* [22] observed that the optimal truncation point tends to become large in the asymptotic limit. Hence, knowledge of the behaviour of the *late-order terms* of the series (that is, the form of a_r in the limit that $r \rightarrow \infty$) is sufficient to truncate the asymptotic series optimally. Fortunately, the late-order terms of an asymptotic series often behave in predictable fashion, even if the general form of the series terms is too complicated to be found explicitly.

In singular perturbation problems, successive terms in the asymptotic series expansion are obtained by repeated differentiation of an earlier term in the series. Hence, the late-order terms of the resultant divergent asymptotic series diverge as the ratio between a factorial and the increasing power of a function χ , such as

$$a_r \sim \frac{A\Gamma(r + \gamma)}{\chi^{r+\gamma}} \quad \text{as } r \rightarrow \infty, \quad (1.3)$$

where Γ is the gamma function defined in [3], and A , γ and χ are functions that do not depend on r .

To determine the form of these late-order terms, we note Darboux's Theorem, given in Dingle [33] and outlined in Appendix A. This theorem implies that the form of the late-order terms of a series expansion at a given point of the complex plane is governed by the behaviour of the nearest singularity or singularities.

Chapman *et al.* [22] therefore propose that the form of the late-order terms may be determined by posing a factorial-over-power ansatz for the late-order terms, such as that given in (1.3), which is associated with a singularity in the complex plane. From Darboux's theorem, they conclude that the correct late-term behaviour may be represented as the sum of these ansatz expressions, each associated with a different singularity. The global behaviour of the functions A , γ and χ may be found by substituting this ansatz directly into the equations governing the asymptotic series terms, and then matching this behaviour to the local behaviour in the neighbourhood of the singularity under consideration. Consequently, the function χ will be equal to zero at the singularity in question.

Furthermore, they note that the late-order term behaviour given in (1.3) is related to applying a WKB ansatz of the form $Ae^{-\chi/\epsilon}$. Consequently, the behaviour of χ , or the 'singulant', will play an important role in understanding the Stokes line behaviour. Recalling that Stokes switching will occur when the exponential behaviour is maximally subdominant, we recognize that this will occur when the singulant is purely real and positive. This is therefore a convenient condition to determine the possible location of Stokes lines, noted by authors such as [33].

Once the form of the late-order terms is established, we may determine the optimal truncation point by finding the point at which terms in the series change from decreasing to increasing. To do so, we find the truncation value at which the ratio of consecutive terms is equal to one. Truncating the series at this point gives

$$f(x; \epsilon) = \sum_{r=0}^{N-1} \epsilon^r a_r(x) + R_N,$$

where N is the optimal truncation point, and R_N is the exponentially small remainder term.

Once the optimally-truncated remainder has been obtained, Olde Daalhuis *et al.* [87] substitute the truncated series expression back into the original problem, obtaining an equation for the exponentially small remainder term itself. The position of Stokes lines are found by determining the curves along which the remainder term varies rapidly, and the switching behaviour of this remainder is found by solving the remainder equation in the neighbourhood of Stokes lines.

In this thesis, we will investigate numerous problems which contain no singularities on the boundary itself; however, there *are* singularities present when the boundary is analytically-continued. We recall that the late-order terms in a divergent asymptotic series, and hence Stokes line behaviour, are governed by the behaviour of the nearest singularity in the complex plane. Consequently, analytic continuation is necessary for exponential asymptotic methods to be applied to problems restricted to a real-valued solution domain. In each of the free-surface fluid flow problems considered throughout this investigation, the singularities that govern the

Stokes line behaviour are found by analytically continuing the free surface into the complex plane.

The exponential asymptotic methodology of Olde Daalhuis *et al.* [87] and Chapman *et al.* [22] was developed for investigating ordinary differential equations. Chapman and Mortimer [24] extended these ideas so that they may be applied to partial differential equations. However, some extensions to the method were required in order to deal with the second-generation Stokes lines and higher-order Stokes lines described in Section 1.1.2.

Initially, the method is identical, with the Stokes switching being found by expanding the solution as an asymptotic power series, applying an ansatz to the late-order terms, optimally truncating the series, and then determining the behaviour of the exponentially small remainder R_N in the neighbourhood of the Stokes line. Further Stokes switching may be determined by finding an asymptotic expansion for the remainder, giving

$$R_N \sim e^{-\chi/\epsilon} \sum_{r=0}^{\infty} R_N^{(r)} \epsilon^r \quad \text{as } \epsilon \rightarrow 0,$$

where $R_N^{(0)}$ is the original remainder expression. By applying the method of [22, 87] to this asymptotic series, we can truncate optimally, giving

$$R_N = e^{-\chi/\epsilon} \sum_{r=0}^{M-1} \epsilon^r R_N^{(r)} + S_M,$$

where S_M is the new exponentially subdominant remainder term that may be investigated using standard exponential asymptotic techniques.

Obviously, it is possible to conceive of problems in which the remainder S_M may be expanded as another exponentially subdominant divergent asymptotic series, and so on.

While we will not directly calculate any higher-order Stokes switching in this study, it is worth noting that Chapman and Mortimer [24] determined that higher-order Stokes lines are found as a consequence of Stokes switching in the behaviour of the asymptotic series terms a_r themselves.

1.4 Thesis Structure

The body of this thesis is divided into four chapters, each of which addresses a different free-surface fluid flow configuration.

The first of these is Chapter 2, which is divided into two sections, each considering a preliminary problem relevant to the remainder of the thesis. The first of these problems involves solving Laplace's equation on a half-plane with a mixed boundary condition, with a particular emphasis on finding the solution behaviour on the boundary itself. The boundary condition is chosen such

that the problem is a singularly-perturbed system with a divergent asymptotic expansion. This problem serves as a demonstration of the exponential asymptotic technique that will be applied throughout this investigation.

Additionally, this problem contains a feature that will appear throughout the current investigation, in that the free-surface contains multiple possible late-order term contributions which are indistinguishable on the boundary of the problem. To determine which of these contributions is active on the problem boundary, we introduce a method that involves finding the complex rays of the singulant equation and matching the rays to the singularities in the analytically-continued leading-order expression.

The second problem in Chapter 2 involves determining the behaviour of exponentially small free-surface gravity waves due to a line source in the finite-depth flowpath in the small Froude number limit. This is a direct extension of the ideas contained in [27] to a case which was overlooked in that analysis. Using this analysis, we confirm the conjecture made by Mekias and Vanden-Broeck [78] that exponentially small gravity waves are present in the far field. We determine the amplitude of these waves, and compare this to numerical data obtained in [78], finding that the asymptotics accurately predict the amplitude of the waves in this problem.

In Chapter 3, we investigate unsteady linearized two-dimensional flow over a small step with a prescribed initial condition. We begin by developing a Lagrangian formulation for the flow problem, with the particle-labelling chosen such that both the free surface and the lower boundary follow known curves and hence fixing the domain of the problem. While this formulation is not strictly necessary for the problem considered in Chapter 3, it is important in the corresponding nonlinear problem.

We apply exponential asymptotic methods to the problem, and find that the free surface has two distinct exponentially small contributions in the small Froude number limit. The first of these is a downstream longitudinal wavetrain, identical to that found in the analysis of the steady system found in [27]. The second is an unsteady component that decays in time, caused by the initial disturbance to the flow system.

Recalling [24, 56], we determine that the system will also contain Stokes switching caused by the interaction between exponentially small terms. We therefore apply the methodology given in [24] to find this switching behaviour. Analysis of the switching terms shows that this behaviour switches off the downstream wavetrain in the far field, and that the switching point moves downstream over time. We therefore conclude that the unsteady flow problem contains stationary waves that propagate downstream in the small Froude number limit, and that they tend to the behaviour found in [27] for the steady case in the large-time limit.

In Chapter 4, we first consider the equivalent nonlinear unsteady problem for flow over a step of arbitrary height. As the problem is formulated in terms of appropriate Lagrangian coordinates, the boundaries are fixed and we can therefore apply exponential asymptotic techniques. To simplify the analysis, we prescribe the step height to be half the width of the channel, and obtain equations for the late-order terms. Due to the complexity of these equations, we cannot solve them analytically. Instead, we determine the position of the Stokes lines on the flow boundary numerically. From the position of these Stokes lines, we conclude that the surface contains exponentially small gravity waves which propagate over time in a similar manner to the linearized problem.

In Chapter 5, we determine the behaviour of free-surface gravity waves on infinite-depth steady and unsteady flow over a submerged source of arbitrary depth in the small Froude number limit. We consider only the linearized problem, which is a fixed-boundary problem in Eulerian coordinates. Using exponential asymptotic techniques, we find that the downstream wake consists of both longitudinal and transverse waves. The longitudinal waves are present in the entire downstream flow region; however, they are largest within a wedge that resembles the classic Kelvin wave pattern [61]. The transverse waves are switched on by the longitudinal wave contribution, and are restricted to two downstream wedge-like regions.

Furthermore, we show that as the depth of the source tends to zero, the downstream wave pattern tends to that obtained by Keller [59] in the small Froude number limit, with algebraically-small longitudinal and transverse waves within the downstream Kelvin wedge and exponentially small longitudinal waves outside this wedge.

In the time varying case, we find that the three exponentially small surface contributions are longitudinal waves, transverse waves, and an unsteady ripple that moves downstream with the flow and decays in time. By comparing the singularities related to all three contributions, we find each of the curves that satisfy the conditions required to be Stokes lines. Recalling that higher-order Stokes switching allows Stokes lines to terminate at intersection points, we are able to determine which Stokes lines are active, and hence the regions of the free surface containing longitudinal and transverse waves. These regions expand downstream at finite speed, tending to the steady case in the limit that $t \rightarrow \infty$.

Finally, in Chapter 6, we summarize our results and discuss future research directions that arise as a consequence of this investigation.

1.5 Statement of Originality

All work contained in this thesis is original, unless otherwise indicated in the text.

Chapter 2

Preliminary Work

2.1 Introduction

In this chapter, we will address two distinct problems which contain features of importance to subsequent chapters. In doing so we will demonstrate the exponential asymptotic techniques described in Chapter 1 that will be used throughout this investigation [22, 87] to determine exponentially small contributions within the solutions to free-surface flow problems. Additionally, we will solve a free-surface potential flow problem in the limit of small Froude number and show how these techniques may be applied to determine the behaviour of gravity waves on the free surface.

The first problem involves solving Laplace's equation on a half-plane in (x, y) with a mixed boundary condition on $y = 0$ that is singular at $x = \pm i$. The mixed boundary condition contains a small parameter multiplying the derivative term, and this system therefore constitutes a singular perturbation problem. As there are singularities present in the analytically-continued boundary behaviour, the solution to the system will contain Stokes lines originating at these singularities, across which exponentially small behaviour is switched. In order to treat this system in a similar fashion to the free-surface problems addressed later in this investigation, we will specifically obtain the behaviour of the solution along the boundary $y = 0$.

An important feature of this problem is that applying the exponential asymptotic techniques of [22, 87] directly to the differential equation leads to ambiguity when solving for the late-order terms. Specifically, we find four possible late-order contributions, two of which contribute to the late-order behaviour and two of which do not. These terms, however, may be grouped into pairs (one member of which contributes to the late-order behaviour and one which does not) that are indistinguishable on the boundary of the problem. Unfortunately, if the contributions are chosen incorrectly, the behaviour of the system on the boundary will be incorrect. In this

case, choosing the late-order contributions incorrectly leads to the presence of a singularity in the solution domain.

We require a method that permits us to determine which late-order terms are present in the problem, and which are not. We shall resolve this by introducing ideas from complex ray theory. By obtaining the ray structure of the singulants in the analytically-continued solution domain, we have sufficient information to compare the possible late-order contributions with the leading-order solution and determine which contributions will be present in the late-order terms. This method will be important in the analysis of Chapters 3–5, where similar ambiguity in the choice of late-order contributions is present.

The second problem involves determining the behaviour of free-surface gravity waves due to a submerged line source in the small Froude number limit. In Chapter 1, we discussed previous studies in which exponential asymptotic methods have been applied to determine the behaviour of exponentially small gravity waves on free-surface potential flows. Here, we demonstrate the methodology described by Chapman and Vanden-Broeck [27] for obtaining the behaviour of these gravity waves in a case overlooked by these previous investigations. Indeed, the authors in [27] remark that sources do not give rise to Stokes lines, a statement we aim to correct here.

The problem serves not only as an illustration of the manner in which the behaviour of free-surface gravity waves may be found using exponential asymptotic methods, but may also be thought of as preceding the work of Chapter 5. In Chapter 5, we consider the behaviour of free-surface flow over a point source in three dimensions, and hence it is of some importance to establish the correct behaviour for related systems in two dimensions.

2.2 Illustration of the methodology

To demonstrate the method of exponential asymptotics developed by Olde Daalhuis *et al.* [87] and extended by Chapman *et al.* [22], we will consider the partial differential equation described by

$$\nabla^2 \phi = 0, \quad y > 0 \quad (2.1)$$

$$\epsilon \phi_y + \phi = \frac{1}{1+x^2} \quad \text{on} \quad y = 0, \quad (2.2)$$

in the limit that $\epsilon \rightarrow 0$, where ϕ is bounded in the solution domain. The solution to this problem contains exponentially small waves, which we will investigate using exponential asymptotic methods. To fully characterize the waves in this problem, we will also need to impose a condition similar to the radiation condition, which will be discussed at the appropriate point in the subsequent analysis.

The boundary condition (2.2) was chosen since it contains two features which are present in the analysis of water waves that follows. Specifically, the problem is singularity perturbed, and the leading-order problem contains singularities on the boundary when it is analytically continued to $x \in \mathbb{C}$. These singularities, found at $x = \pm i$, play an important role in the analysis of the late-order terms of the solution and hence the exponentially small behaviour present in the problem.

We will first solve the problem by taking a Fourier transform of the system and evaluating the Fourier inversion integral using the method of steepest descents [5]. Subsequently, we will solve the problem using the late-order terms method developed in [22, 87] and described in Section 1.3 and show that the two solutions are equivalent. To accomplish this, we will introduce the idea of applying complex ray theory, described in [23], to determine which late-order contributions are present in the late-order analysis.

2.2.1 Fourier transform method

We define the Fourier transform and inverse Fourier transform as

$$\hat{\phi} = \int_{-\infty}^{\infty} \phi e^{ikx} dx, \quad \phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi} e^{-ikx} dk.$$

Taking the Fourier transform of the system given in (2.1)–(2.2) produces

$$\hat{\phi}_{yy} - k^2 \hat{\phi} = 0, \quad y > 0 \quad (2.3)$$

$$\epsilon \hat{\phi}_y + \hat{\phi} = \pi e^{-|k|} \quad \text{on} \quad y = 0. \quad (2.4)$$

Solving the differential equation in (2.3) using the boundary condition given in (2.4) and the requirement that $\hat{\phi}$ be bounded as $|y| \rightarrow \infty$ gives

$$\hat{\phi} = \frac{\pi e^{-|k|(1+y)}}{1 - \epsilon|k|}.$$

The inversion integral becomes

$$\phi = \frac{1}{2\pi} \int_0^{\infty} \frac{\pi e^{-k(1+y-ix)}}{1 - \epsilon k} dk + \frac{1}{2\pi} \int_{-\infty}^0 \frac{\pi e^{k(1+y+ix)}}{1 + \epsilon k} dk.$$

In order to apply the method of steepest descents, we set $\epsilon k = p$, resulting in

$$\phi = \frac{1}{2\epsilon} \int_0^{\infty} \frac{\pi e^{-p(1+y-ix)/\epsilon}}{1 - p} dp + \frac{1}{2\epsilon} \int_{-\infty}^0 \frac{\pi e^{p(1+y+ix)/\epsilon}}{1 + p} dp = I_1 + I_2, \quad (2.5)$$

where I_1 and I_2 refer to the individual integral expressions. The steepest descent path for I_1 will hence follow the curve satisfying $\arg(p) = -\arg(1 + iy - ix)$, while the second will follow the curve given by $\arg(p) = \arg(1 - iy + ix)$. These curves will be denoted as C_1 and C_2 respectively. Both I_1 and I_2 contain poles that contribute to the integral expression, at $p = 1$ and $p = -1$

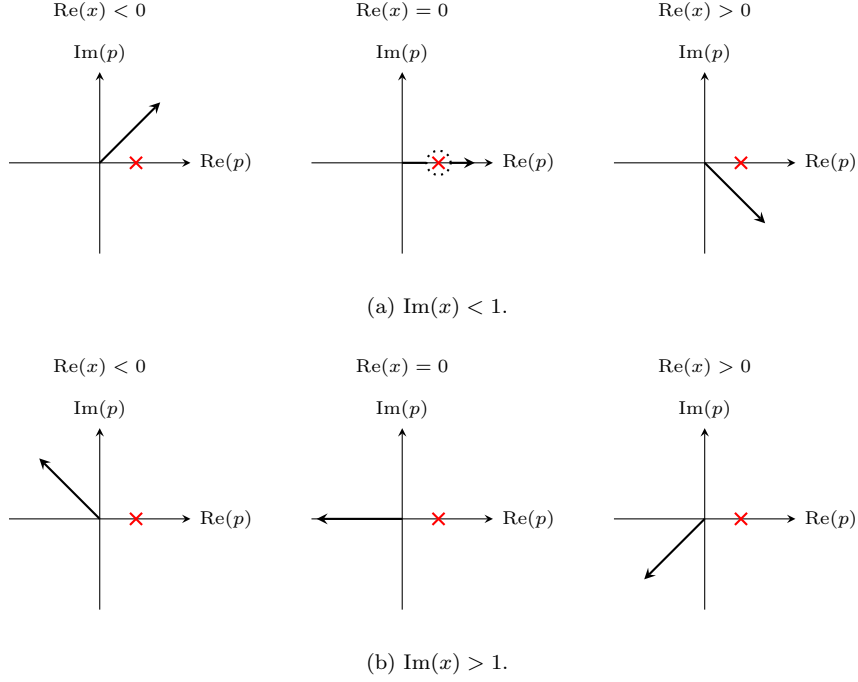


Figure 2.1: Typical steepest descent paths for I_1 in p -space for various complex values of x , given that $y \geq 0$, indicated in bold. If deforming the original path of integration along the real axis requires crossing the pole at $p = 1$, then this contribution will be picked up in the steepest descent analysis. When there are two possible integration paths, both are shown as dotted curves

respectively. It is possible to show that the contribution to the integral for I_2 are the complex conjugate of the contribution from I_1 . We may therefore restrict our attention to I_1 . Figure 2.1 (a) shows a typical example of C_1 . If x was to take complex values, we would find that this is only a typical contour for $\text{Im}(x) < 1$. For $\text{Im}(x) > 1$, no pole contribution would be present, as the integration contour takes the form shown in Figure 2.1 (b).

We restrict our analysis to $x \in \mathbb{R}$. Hence, the curve C_1 passes directly through the pole at $p = 1$ when $x = 0$. This is avoided by instead indenting the curve slightly above, or slightly below the pole. These two options are denoted in Figure 2.1 by dotted curves. If we choose to indent the curve above the pole, the contribution to the integral from the pole will be present as $x \rightarrow \infty$. Alternatively, if we choose to indent the curve below the pole, this contribution will instead be present as $x \rightarrow -\infty$. It is at this point in the analysis that we require a far-field condition to determine the inversion contour.

In typical fluid flow problems involving gravity waves, this far-field behaviour is given by the radiation condition [27] which states that there must be no upstream gravity waves in the solution to the problem. Conversely, in capillary wave problems, there must instead be no downstream capillary waves present in the solution [26]. In our present problem, we choose to specify that there is no pole contribution present as $x \rightarrow -\infty$, and therefore that C_1 is indented below the pole at $p = 1$.

Consequently, Figure 2.1 indicates that the pole will contribute to the integral expression for $\text{Re}(x) > 0$, given that $\text{Im}(x) < 1$. Therefore,

$$I_1 \sim I_{1,e} + \mathcal{S}I_{1,p},$$

where $I_{1,e}$ indicates the contribution from the end-point of the first integral, while $I_{1,p}$ denotes the contribution from the pole. The coefficient $\mathcal{S}$ is known as a Stokes switching parameter, and indicates whether the pole contribution is present in the solution. The Stokes parameter is often given by

$$\mathcal{S} = \begin{cases} 0 & \text{for } x < 0, \\ 1/2 & \text{for } x = 0, \\ 1 & \text{for } x > 0. \end{cases} \quad (2.6)$$

This definition leads to a discontinuity in the solution, which is impossible for an elliptic problem. In fact, we will find that the Stokes parameter instead switches rapidly in the neighbourhood of a curve in the solution domain, known as a *Stokes line* [7].

The end-point contribution $I_{1,e}$ may be evaluated in the usual way by expanding the integrand in (2.5) about $p = 0$ to give

$$\begin{aligned} I_{1,e} &\sim \frac{1}{2\epsilon} \int_0^\infty (1 + p + p^2 \dots) e^{-p(1+y+ix)/\epsilon} dp, \\ &\sim \frac{1}{2} \sum_{n=1}^\infty \frac{\Gamma(n+1)\epsilon^n}{[i(x-iy-i)]^n}. \end{aligned} \quad (2.7)$$

Similarly, we find that the contribution from $I_{2,e}$ is given by

$$I_{2,e} \sim \frac{1}{2} \sum_{n=0}^\infty \frac{\Gamma(n+1)\epsilon^n}{[-i(x+iy+i)]^n}. \quad (2.8)$$

To leading order, the combined contribution from the end-points (2.7)–(2.8) is therefore given by

$$I_e = I_{1,e} + I_{2,e} \sim \frac{1+y}{(1+y^2)+x^2}. \quad (2.9)$$

The pole contribution $I_{1,p}$ is simply found by applying the residue theorem to obtain

$$I_{1,p} = 2\pi i \text{Res} \left\{ \frac{1}{2\epsilon} \frac{e^{-p(1+y+ix)/\epsilon}}{1-p}, p=1 \right\} = \frac{\pi i}{\epsilon} e^{-(1+y+ix)/\epsilon}.$$

Knowing that $I_{2,p}$ is simply the complex conjugate of $I_{1,p}$, we find that

$$I_p = I_{1,p} + I_{2,p} \sim \frac{2\pi i}{\epsilon} e^{-(1+y)/\epsilon} \sin\left(\frac{x}{\epsilon}\right).$$

We see that the pole contribution is exponentially small in ϵ as $\epsilon \rightarrow 0$ for $y > -1$, while the end-point contributions (2.7)–(2.8) are algebraically small in this limit. Consequently, simply applying a naive power-series expansion to the original problem would fail to detect the pole

contribution in the solution domain ($y > 0$). Finally, the full asymptotic expression for the solution to (2.1)–(2.2) is given by

$$I \sim I_e + \mathcal{S}I_p \sim \frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\epsilon^n}{[\mathrm{i}(x - \mathrm{i}y - \mathrm{i})]^n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\epsilon^n}{[-\mathrm{i}(x + \mathrm{i}y + \mathrm{i})]^n} + \frac{2\pi\mathrm{i}\mathcal{S}}{\epsilon} e^{-(1+y)/\epsilon} \sin\left(\frac{x}{\epsilon}\right). \quad (2.10)$$

2.2.2 Late-order term analysis

2.2.2.1 Ansatz equations

While the method of steepest descents is able to be applied in order to determine exponentially small behaviour in many cases, it is only useful when the problem may be formulated as an integral, such as using transform methods or Green's functions. The method developed by Olde Daalhuis *et al.* [87] and extended by Chapman *et al.* [22], outlined in Section 1.3, avoids this requirement by directly analysing the late-order terms of the asymptotic series expansion, which may be obtained using standard power series methods.

We first express ϕ as a power series in ϵ , giving

$$\phi \sim \sum_{n=0}^{\infty} \phi^{(n)} \epsilon^n. \quad (2.11)$$

The resultant system of equations is given by

$$\nabla^2 \phi^{(n)} = 0, \quad y > 0 \quad (2.12)$$

$$\phi_y^{(n-1)} + \phi^{(n)} = 0 \quad \text{on} \quad y = 0, \quad (2.13)$$

$$\phi^{(0)} = \frac{1}{1+x^2} \quad \text{on} \quad y = 0. \quad (2.14)$$

During the steepest-descent analysis, we obtained an expression for the end-point contributions, given in (2.9), which may be identified as the first term in the series expression, giving

$$\phi^{(0)} = \frac{1+y}{(1+y)^2 + x^2}. \quad (2.15)$$

The domain of the solution is given by the upper-half x - y plane. Despite this restriction, we analytically continue the boundary of the problem such that $x \in \mathbb{C}$. The purpose of this continuation is to determine the influence of the *singularities* in the boundary data on the solution to the problem. Previous investigations on exponential asymptotics, such as [8, 11, 22, 68, 87], have made the important observation that exponentially small behaviour is determined by the behaviour of a system in the neighbourhood of singularities, as discussed in Chapter 1.

We recall that the reason for this observation is the important role played by the *late-order terms* of asymptotic series expansions in determining this exponentially small behaviour. By late-order terms, we mean the coefficients of $\phi^{(n)}$ in the limit that $n \rightarrow \infty$. Specifically, from [33], we know

that the form of the late-order terms may be obtained by a careful investigation of the problem in the neighbourhood of the contributing singularities. With the knowledge of the form of these late-order terms comes the possibility of *optimal truncation* of the asymptotic series, ensuring that the error term is exponentially small in ϵ [8, 87]. Subsequently investigating the remainder of this optimally-truncated series will then give the behaviour of the exponentially small terms that we seek to determine.

The boundary condition (2.2), as well as the behaviour of $\phi^{(0)}$ given in (2.14), shows that the solution contains singularities on the analytically-continued boundary of the problem, located at $x = \pm i$. It is these singularities that will determine the exponentially small asymptotic behaviour of ϕ . It is clear from the form of (2.13) that proceeding from $\phi^{(n-1)}$ to $\phi^{(n)}$ will require a differentiation step. Consequently, for each singularity contained in $\phi^{(n-1)}$, an equivalent singularity will be present at the same location in $\phi^{(n)}$, with the strength of the singularity increased by one. In fact, given that each term is obtained by successive differentiations, the strength of the singularity will increase at every order, causing the late-order terms to diverge as a factorial over a power. Consequently, we may define an WKB-type ansatz for the behaviour of the late-order terms, using the method described by Chapman *et al.* [22], giving

$$\phi^{(n)} \sim \frac{\Phi_1 \Gamma(n+1)}{\chi_1^{n+1}} + \frac{\Phi_2 \Gamma(n+1)}{\chi_2^{n+1}} \quad \text{as} \quad n \rightarrow \infty, \quad (2.16)$$

where terms of the form Φ_j and χ_j are functions of x and y , but not n . The terms χ_1 and χ_2 are referred to as singulants, and are defined such that they are equal to zero at $x = i$ and $x = -i$ respectively; consequently, the late-order terms (2.16) are singular at these points.

The behaviour of the late-order terms at any point in the domain of the problem will generally be dominated by one term from this ansatz, depending on the relative strength of the singularities. However, this is not necessarily true everywhere; in fact, we will find that both contributions have equal strength on the curve given by real x and $y = 0$.

In order to determine the form of the late-order terms, we will first consider the late-order contributions associated with the singularity at $x = i$, writing the ansatz locally as

$$\phi^{(n)} \sim \frac{\Phi \Gamma(n+\gamma)}{\chi^{n+\gamma}} \quad \text{as} \quad n \rightarrow \infty,$$

where $\chi = 0$ at $x = i$. Applying this ansatz expression to the series form of the governing equation (2.12) produces

$$\begin{aligned} & \frac{\Phi_{xx} \Gamma(n+\gamma)}{\chi^{n+\gamma}} - \frac{2\Phi_x \chi_x \Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}} - \frac{\Phi_{\chi xx} \Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}} + \frac{\Phi(\chi_x)^2 \Gamma(n+\gamma+2)}{\chi^{n+\gamma+2}} \\ & + \frac{\Phi_{yy} \Gamma(n+\gamma)}{\chi^{n+\gamma}} - \frac{2\Phi_y \chi_y \Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}} - \frac{\Phi_{\chi yy} \Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}} + \frac{\Phi(\chi_y)^2 \Gamma(n+\gamma+2)}{\chi^{n+\gamma+2}} + \dots = 0, \end{aligned} \quad (2.17)$$

where the omitted terms are subdominant as $n \rightarrow \infty$ in the neighbourhood of $x = i$. Applying the ansatz expression to the boundary condition for $n > 0$ given in (2.13) gives

$$\frac{\Phi\Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}} = -\frac{\Phi_y\Gamma(n+\gamma)}{\chi^{n+\gamma}} + \frac{\Phi\chi_y\Gamma(n+\gamma+1)}{\chi^{n+\gamma+1}}. \quad (2.18)$$

Equating the leading-order terms of (2.17) in the limit that $n \rightarrow \infty$ gives

$$\chi_x^2 + \chi_y^2 = 0,$$

giving the choice of two possible solutions

$$\chi_x = \pm i\chi_y. \quad (2.19)$$

Continuing to the next order in n gives

$$2\Phi_x\chi_x + 2\Phi_y\chi_y + \Phi\chi_{xx} + \Phi\chi_{yy} = 0. \quad (2.20)$$

Applying (2.19) to this expression gives

$$\pm i\Phi_x + \Phi_y = 0, \quad (2.21)$$

where the sign choice corresponds to the sign chosen in (2.19).

Equating the leading-order terms of the boundary equation (2.18) gives on $y = 0$

$$\chi_y = 1, \quad \Phi_y = 0.$$

From the relationships given in (2.19) and (2.21), we subsequently find that

$$\chi_x = \pm i, \quad \Phi_x = 0, \quad (2.22)$$

where again, the choice of sign corresponds to the sign chosen in (2.19). We will hence solve for χ using the method of characteristics, with boundary data given by (2.22) on $y = 0$. Choosing the case where $\chi_x = i\chi_y$ and $\chi = 0$ at $x = i$, applying the method of characteristics to solve the singular equation gives

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -i, \quad \frac{d\chi}{dt} = 0, \quad (2.23)$$

where t is a characteristic variable, with

$$x(0) = i, \quad y(0) = 0, \quad \chi(0) = 0.$$

Integrating directly and eliminating t gives

$$\chi = 0 \quad \text{on} \quad y = -i(x - i).$$

Hence

$$\begin{aligned}\chi &= f(y + ix), \\ \chi_y &= f'(y + ix).\end{aligned}$$

Applying the boundary condition $\chi_y = 1$ on $y = 0$ gives

$$\begin{aligned}f'(ix) &= 1, \\ f(ix) &= ix + C,\end{aligned}$$

which implies that

$$\chi = f(y + ix) = i(x - iy - i), \quad (2.24)$$

using the fact that $\chi = 0$ at $x = i$. Applying a similar analysis to the case when $\chi_x = -i\chi_y$ gives

$$\chi = f(y - ix) = -i(x + iy - i). \quad (2.25)$$

Hence, solving (2.20) using the condition that $\chi = 0$ at $x = i$, we find that there are two possible forms for the ansatz caused by the singularity at $x = i$, given by

$$\phi^{(n)} \sim \frac{\Phi \Gamma(n + \gamma)}{(\pm i(x - i) + y)^{n + \gamma}} \quad \text{as} \quad n \rightarrow \infty, \quad (2.26)$$

where Φ is constant on $y = 0$. To ensure these late-order terms are consistent with the leading-order singularity, we find $\gamma = 1$.

Consequently, there are two possible forms which these late-order terms can take. Performing a similar analysis instead considering the singularity at $x = -i$ gives the form of the ansatz caused by the corresponding singularity as

$$\phi^{(n)} \sim \frac{\Phi \Gamma(n + 1)}{(\pm i(x + i) + y)^{n + 1}} \quad \text{as} \quad n \rightarrow \infty,$$

Hence, there are four valid contributions to the late-order term expressions. From the solution obtained from the steepest-descent analysis, given in (2.10), we see that two of these contributions are present in the solution; the contribution associated with $\chi_x = i$ near $x = i$ and the contribution associated with $\chi_x = -i$ near $x = -i$. Without the application of integral solution methods, however, it seems impossible to completely determine which contributions are present in the late-order expression, and which are not.

In order to resolve this ambiguity, we first consider the nature of the late-order ansatz. Chapman *et al.* [22] note that the factorial over power form of the ansatz is related to application of WKB-type analysis to the late-order terms of the problem. Considering a simple late-order ansatz for a problem in one variable, such as

$$x_n \sim \frac{A \Gamma(n + \gamma)}{\chi^{n + \gamma}} \quad \text{as} \quad n \rightarrow \infty,$$

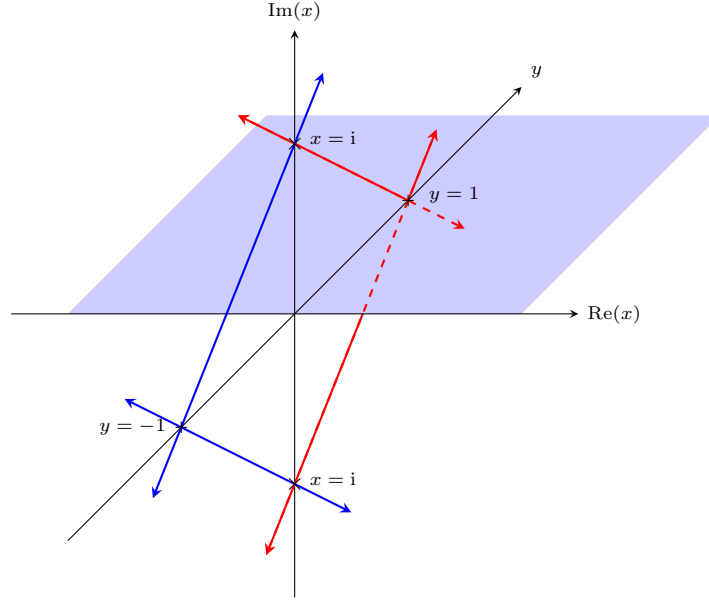


Figure 2.2: The path of the four possible rays that may contribute to the late-order terms, originating from $x = \pm i$. The ray paths passing through $y = -1$ are denoted by blue curves. These curves do not intersect the solution domain (given by $x, y \in \mathbb{R}$ and $y > 0$, and shaded in light blue). However, the ray paths passing through $y = 1$, represented by red curves do intersect the solution domain. As singularities cannot exist within this domain, these rays must not be active and hence do not contribute to the late-order terms.

we find that it would be associated with an exponential WKB ansatz with phase $-\chi/\epsilon$ and amplitude A , indicating that the late-order terms have an associated ray structure. As we are considering boundary data on an analytically continued free-surface, rather than simply restricted to the real domain, these rays will be complex [23].

2.2.2.2 Ray theory

Ideally, it should be possible to determine which late-order terms are present and which are not present directly from the late-term analysis, without resorting to integral transform methods. In order to achieve this, we must consider the ray behaviour of the late-order terms.

From the characteristic equations (2.23), we see that χ is constant along each ray. Hence, the late-order terms associated with each ray will be singular at all points along the characteristic that passes through the associated singularity. We may think of this behaviour as the singularity being ‘propagated’ along the rays originating at the singularity on the analytically-continued boundary of the problem. The path of the rays originating at $x = i$ may be seen in Figure 2.2.

By following the ray paths, shown in Figure 2.2, we see that one of the characteristics passes through the solution domain ($x, y \in \mathbb{R}$, $y \geq 0$), whereas the other does not. However, the solution to Laplace’s equation cannot contain singularities within the solution domain [83]. Consequently, the contributions associated with the first characteristic must not be present in

the late-order terms, given by (2.25). Hence, the only contribution to the late-order terms from the singularity at $x = i$ comes from terms with characteristics given by (2.24). Applying a similar analysis to the rays originating at $x = -i$ indicates that only late-order terms associated with the characteristics given by

$$\chi = f(y - ix) = y - ix + 1.$$

This choice of late-order term contributions is consistent with the result obtained using steepest-descent analysis, given in (2.10).

More generally, we compare the form of the resultant late-order terms with the leading-order solution (2.15). Each form of χ leads to the late-order terms being singular along a different ray. The leading-order expression is singular along the curves $x = \pm i(y + 1)$, which is also the expression for the blue ray paths in Figure 2.2. It was noted in [87] that the location of singularities remains constant throughout the late-order terms, and that the location of these singularities is given by the position of the singularities in the early orders. Importantly, the location of these singularities is given by the rays along which $\chi = 0$. Consequently, we may determine which expressions for χ are represented in the late-order terms of the problem by observing which of the possible forms of χ give rays that correspond to the singular curves in the leading-order behaviour.

Hence, we effectively compare the behaviour of the late-order terms as $n \rightarrow 0$ with the leading-order expression, selecting the late-order terms that give consistency between the expressions. In this problem, there were four possible singularity contributions; however, only the contributions that had singularities at $(x, y) = (0, -1)$ were consistent with the leading-order behaviour. This is similar to the method used in [22] to determine the strength of the singularities in the late-order terms, however we apply similar ideas to determine not only the strength of the singularities, but whether they contribute to the late-order terms at all.

2.2.2.3 Inner problem

At this stage, the late-order ansatz terms in (2.16) are almost completely determined. In order to determine the behaviour of Φ in (2.26), we consider the behaviour of the solution to the full problem given in (2.1)–(2.2) in the neighbourhood of the singularity at $x = i$. We note that we are interested in the behaviour of the solution only on $y = 0$, along which it is known that Φ is constant. The inner problem will be used to determine the value of this constant.

We define the inner variables near $x = i$ as

$$x = i + \epsilon \hat{x}, \quad y = \epsilon \hat{y}, \quad \phi = \hat{\phi}.$$

Applying this scaling to (2.1) gives

$$\hat{\phi}_{\hat{x}\hat{x}} + \hat{\phi}_{\hat{y}\hat{y}} = 0, \quad (2.27)$$

which the boundary condition gives (2.2) gives on $\hat{y} = 0$

$$\hat{\phi}_{\hat{y}} + \hat{\phi} = \frac{1}{2i\hat{x} + \epsilon\hat{x}^2} \sim \frac{1}{2i\hat{x}}, \quad (2.28)$$

as ϵ becomes small. Expressing the singulant obtained in (2.24) in these inner coordinates gives

$$\chi = \epsilon(\hat{x} - i\hat{y}).$$

Motivated by the form of the late-order ansatz (2.26), we therefore define a series expansion for $\hat{\phi}$ such that

$$\hat{\phi} \sim \sum_{n=0}^{\infty} \frac{\hat{\phi}^{(n)}}{(\hat{x} - i\hat{y})^{n+1}}, \quad (2.29)$$

where $\hat{\phi}^{(n)}$ is a constant to be determined. This series expression clearly satisfies the scaled governing equation (2.27) for any value chosen for the undetermined constants. Applying this series expression to the scaled boundary equation (2.28) gives the resultant expression on $\hat{y} = 0$

$$\sum_{n=0}^{\infty} \frac{\hat{\phi}^{(n)}}{(\hat{x} - i\hat{y})^{n+1}} = \frac{1}{2i\hat{x}} - i \sum_{n=0}^{\infty} \frac{(n+1)\hat{\phi}^{(n)}}{(\hat{x} - i\hat{y})^{n+2}}.$$

We may now simply set $\hat{y} = 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{\hat{\phi}^{(n)}}{\hat{x}^{n+1}} = \frac{1}{2i\hat{x}} - i \sum_{n=0}^{\infty} \frac{(n+1)\hat{\phi}^{(n)}}{\hat{x}^{n+2}}.$$

Matching terms in $\hat{x}$ as $\hat{x} \rightarrow \infty$ gives

$$\begin{aligned} \hat{\phi}^{(0)} &= \frac{1}{2i}, \\ \hat{\phi}^{(n)} &= -in\hat{\phi}^{(n-1)}. \end{aligned}$$

Solving this system gives

$$\hat{\phi}^{(n)} = \frac{\Gamma(n+1)}{2i^{n+1}}.$$

Converting back into the original expressions for ϕ and x , we find that the n th term of the inner-expansion, given by (2.29) takes the form

$$-\frac{\epsilon^n \Gamma(n+1)}{2i^{n+1}(x - iy - i)^{n+1}}. \quad (2.30)$$

Using (2.26), given the contributing singulant is given by (2.24), the n th term of the outer expansion as $x \rightarrow i$ takes the form

$$\frac{\epsilon^n \Phi \Gamma(n+1)}{i^{n+1}(x - iy - i)^{n+1}}. \quad (2.31)$$

Matching the outer limit of the inner solution, given in (2.30), with the inner limit of the outer solution, given in (2.31), gives $\Phi = -1/2$. Performing a similar analysis in neighbourhood of $x = -i$ instead gives $\Phi = 1/2$. Hence, we find that the two late-order contributions are given by

$$\phi^{(n)} \sim -\frac{\Gamma(n+1)}{2i^{n+1}(x - iy - i)^{n+1}}, \quad \phi^{(n)} \sim \frac{\Gamma(n+1)}{2(-i)^{n+1}(x + iy + i)^{n+1}}. \quad (2.32)$$

2.2.2.4 Stokes smoothing

Knowing that the form of the late-order terms of the series (2.11) on $y = 0$ take the form shown in (2.32), we may continue to follow the method described by Olde Daalhuis *et al.* [87] and outlined in Chapter 1. This method involves optimally-truncating the asymptotic series and obtaining an expression for the exponentially small remainder term. We will first consider the Stokes line originating at $x = i$, and then state the equivalent result for the Stokes line originating at $x = -i$, which may be obtained in a similar fashion.

Truncating the asymptotic series expansion for ϕ , given in (2.11), gives

$$\phi = \sum_{n=0}^{N-1} \phi^{(n)} \epsilon^n + R,$$

where R is the remainder, which will be exponentially small in ϵ when N is chosen such that the series is truncated optimally. Applying this truncated series to the governing equation given in (2.1) gives

$$\left(\sum_{n=0}^{N-1} \phi_{xx}^{(n)} \epsilon^n + R_{xx} \right) + \left(\sum_{n=0}^{N-1} \phi_{yy}^{(n)} \epsilon^n + R_{yy} \right) = 0,$$

giving

$$R_{xx} + R_{yy} = 0. \quad (2.33)$$

Applying the truncated series expression to the boundary expression obtained in (2.2) gives

$$\epsilon \left(\sum_{n=0}^{N-1} \phi_y^{(n)} \epsilon^n + R_y \right) + \left(\sum_{n=0}^{N-1} \phi^{(n)} \epsilon^n + R \right) = \frac{1}{1+x^2}, \quad \text{on } y = 0.$$

Knowing from (2.13) that $\phi^{(n)} = -\phi_y^{(n-1)}$, we may rewrite this as

$$\epsilon R_y + R = \epsilon^N \phi^{(N)}. \quad (2.34)$$

From Olde Daalhuis *et al.* [87] and Chapman *et al.* [22], we know that the quantity that undergoes switching when Stokes lines are crossed is a multiple of the exponentially small homogeneous solution to the remainder equations, in this case given by (2.33)–(2.34). Homogeneous solutions to these equations are given by

$$A_1 e^{-i(x-iy-i)/\epsilon}, \quad A_2 e^{i(x+iy+i)/\epsilon},$$

where A_1 and A_2 are arbitrary constants. Hence, the quantities switched as the Stokes lines are crossed will be a multiple of these two complex conjugate expressions, each of which is associated with one of the two late-order forms given in (2.32). We choose to consider the first of these contributions, noting that the second will produce the equivalent complex conjugate behaviour. We therefore allow the remainder expression to take the form

$$R = A(x, y) e^{-i(x-iy-i)/\epsilon}.$$

Applying this new expression to the governing remainder equation (2.33) gives

$$\frac{1}{\epsilon} [2iA_x + 2A_y - \epsilon(A_{xx} + A_{yy})] e^{-i(x-iy-i)} = 0,$$

which, to leading order as $\epsilon \rightarrow 0$ gives $A_y = -iA_x$. Consequently, (2.34) now becomes

$$\begin{aligned} -iA_x e^{-i(x-iy-i)/\epsilon} &= \epsilon^{N-1} \phi^{(N)}, \\ &\sim -\frac{\epsilon^{N-1} \Gamma(N+1)}{2i^{N+1} (x-iy-i)^{N+1}}, \end{aligned} \quad (2.35)$$

We now apply the convenient substitution $x-iy-i = re^{i\theta}$. In this coordinate system, N varies only in r , while the Stokes line follows a curve of constant θ , and hence the variation near the Stokes line is rapid as θ is varied. We therefore write

$$\frac{\partial}{\partial x} = -\frac{ie^{-i\theta}}{r} \frac{\partial}{\partial \theta}.$$

Hence, (2.35) becomes

$$-\frac{e^{-i\theta}}{r} A_\theta \sim -\frac{\epsilon^{N-1} \Gamma(N+1)}{2i^{N+1} (re^{i\theta})^{N+1}} \exp\left(\frac{ire^{i\theta}}{\epsilon}\right). \quad (2.36)$$

To proceed, we must determine the value of N which optimally truncates the asymptotic series expression, hence minimising the magnitude of R for a given ϵ . As discussed in Chapter 1, we may accomplish this by determining when ratio of the magnitude of successive terms in the asymptotic series is one, noting that the truncation point N is large. Hence, we determine the value of N such that

$$\left| \frac{\epsilon^{N+1} \phi^{(N+1)}}{\epsilon^N \phi^{(N)}} \right| \sim 1. \quad (2.37)$$

Applying the ansatz form for $\phi^{(n)}$ to this expression gives the optimal truncation point as $N \sim r/\epsilon$. An alternative approach to determine the optimal truncation point involves obtaining the asymptotic value of N that minimises the magnitude of the right-hand side of (2.36). We find this also gives $N \sim r/\epsilon$.

We therefore set $N = r/\epsilon + \alpha$, where $0 \leq \alpha < 1$, and α is chosen to ensure that N takes an integer value. Applying this result as well as Stirling's approximation for the gamma function [3] to (2.36) gives

$$A_\theta \sim \frac{\sqrt{2\pi r}}{2\epsilon^{3/2}} \exp\left(i\theta + \frac{r}{\epsilon} (ie^{i\theta} - 1) - \left(\frac{\pi i}{2} + i\theta\right) \left(\frac{r}{\epsilon} + \alpha + 1\right)\right). \quad (2.38)$$

The right-hand side is exponentially small in ϵ , except at $\theta = -\pi/2$, indicating that A varies rapidly across this curve. Hence, this must be the location of the Stokes line across which switching occurs. In order to determine the behaviour of the system in the neighbourhood of this Stokes line, we rescale (2.38) such that $\theta = \epsilon^{1/2} \hat{\theta}$. The resultant expression takes the form

$$\frac{\partial A}{\partial \hat{\theta}} \sim -\frac{i\sqrt{2\pi r}}{2\epsilon} e^{-r\hat{\theta}^2/2}.$$

Solving this by direct integration gives

$$A = -\frac{i\sqrt{2\pi}}{2\epsilon} \int_{-\infty}^{\hat{\theta}\sqrt{r}} e^{-t^2/2} dt, \quad (2.39)$$

which gives the jump as the Stokes line is crossed as

$$[R]_{-}^{+} = -\frac{\pi i}{\epsilon} e^{-i(x-iy-i)/\epsilon}. \quad (2.40)$$

Dingle [33] notes that the Stokes line will follow the curve along which the singulant is real and positive, which is equivalent to the statement that $\theta = -\pi/2$. It is easy to see that corresponds to the curve following $x = 0$ when $y > -1$ in the original coordinates. Performing a nearly identical analysis on the Stokes line originating at $x = -i$ shows that the corresponding Stokes line follows the same curve, and switches on the quantity

$$[R]_{-}^{+} = \frac{\pi i}{\epsilon} e^{i(x+iy+i)/\epsilon}. \quad (2.41)$$

We see that both exponentially small contributions are of the same magnitude in the limit that $\epsilon \rightarrow 0$ when x is real, and therefore neither dominates the other. Hence, at $x = 0$, the behaviour switched as the Stokes line is crossed is given by the sum of the two remainder terms found in (2.40)–(2.41), given by

$$[R]_{-}^{+} = \frac{2\pi i}{\epsilon} e^{-(1+y)/\epsilon} \sin\left(\frac{x}{\epsilon}\right),$$

which is identical to the exponentially small switching term given in (2.10). Applying a radiation condition in the same manner as Section 2.2.1 ensures that the exponentially small behaviour is present for $x > 0$ and absent for $x < 0$, hence producing the same exponentially small behaviour as was obtained using Fourier analysis.

Furthermore, we see from (2.39) that the variation as the Stokes line is crossed is smooth, varying rapidly in the neighbourhood of width $\mathcal{O}(\epsilon^{1/2})$ of the Stokes line itself, contrary to the commonly-used definition given in (2.6). This smooth switching behaviour is generic [7, 8], and was not found using the steepest-descent analysis. However, the error-function behaviour of the exponentially small switching in the neighbourhood of the Stokes line, while generic, is not universal [18].

Figure 2.3 (a) shows the exponentially small behaviour of the solution, illustrating the switching behaviour across the Stokes line following the curve $x = 0$. The Stokes line is shown to divide the solution region into two sections, with the exponentially small terms being present on the section corresponding to $\text{Re}(x) > 0$.

However, had we instead selected the late-order terms associated with the singulant term given in (2.25) and the corresponding complex conjugate term, the resultant Stokes structure would have been different to that obtained in the previous investigation. In this case, we see that the

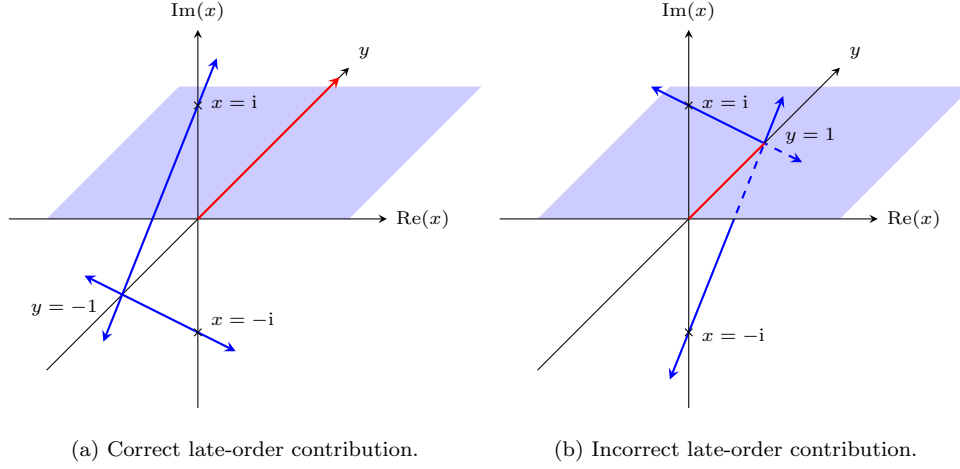


Figure 2.3: The Stokes line across which the exponentially small contributions are switched on in the solution to (2.1)–(2.2), when the late-order terms are chosen in a fashion that is (a) consistent with the leading-order behaviour, and (b) inconsistent with the leading-order behaviour. In each case, the Stokes line is shown in red, while the rays corresponding with the selected late-order terms are shown in blue. The singular behaviour originating at $x = \pm i$ is propagated along the rays, leading to a singularity in the solution domain at $(x, y) = (0, 1)$ for case (b). In case (a), however, the singularity instead propagates to the point $(x, y) = (0, -1)$, which is not part of the solution domain.

singulants are real and positive for $x = 0$ and $y < 1$. Consequently, the Stokes line would have followed the curve illustrated in Figure 2.3 (b), which follows the line in the solution domain beginning at $(x, y) = (0, 0)$ and terminating at the point $(x, y) = (0, 1)$, where the corresponding late-order terms are singular.

2.2.3 Discussion and conclusions

In this section, we constructed a simple differential equation which could be investigated using exponential asymptotic methods. The problem had a mixed boundary condition applied along the curve $y = 0$, which contained singularities at $x = \pm i$ when the boundary was analytically-continued into the complex plane. Specifically, we were interested in the behaviour of the solution on the boundary of the problem, which is analogous to determining the surface behaviour in a free-surface fluid flow system. The presence of singularities in the analytically-continued boundary condition ensured that the solution would exhibit Stokes switching, and therefore that exponentially small contributions to the solution behaviour would switch as Stokes lines in the solution were crossed.

To observe this Stokes switching, we first solved the system using Fourier transforms and approximating the resultant inversion integrals using the method of steepest descents. Using this method, we determined that exponentially small oscillations were switched on in the solution on the boundary as a Stokes line, intersecting the boundary at $x = 0$, was crossed. Applying the same method using late-order term analysis led to ambiguity, as there were four possible

late-order contributions, only two of which were active, and each of which was indistinguishable from another on the boundary itself. This issue did not arise in the integral formulation of the problem, as the two inactive contributions would have caused the Fourier integral to diverge, and therefore could not be present in the solution.

We resolved this ambiguity by examining the ray structure of the singulant, and hence determining the curves along which the late-order terms would be singular. This allowed us to determine which late-order contributions would cause the solution to display singular behaviour within the solution domain, and hence could not be active. More generally, we may use this ray structure to match the late-order contributions to equivalent contributions in the leading-order behaviour, allowing us to determine which late-order contributions are present or absent in a wide range of problems which contain this particular ambiguity. This method will be used in the free-surface fluid flow problems considered in Chapters 3–5.

Finally, we completed the exponential asymptotic analysis and determined the behaviour that was switched as the Stokes line was crossed. This behaviour was consistent with that previously obtained using Fourier transform methods, but was obtained without having to formulate the problem in terms of an integral expression.

2.3 Steady flow due to a source

The problem of steady free surface flow due to a line source has an interesting background. For the geometry of interest here, in which there is a stagnation point directly above the source, a number of authors apply formal series expansions in the limit that the Froude number vanishes (see [53, 78] for the finite-depth problem, [38, 41, 89, 116] for the corresponding infinite-depth problem). In each case the resulting asymptotic solutions are waveless, with the free surface displaying monotonic behaviour in the far field. Similar waveless solutions for small Froude numbers are computed numerically in [53] in the finite-depth case and by others in the infinite-depth case.

However, there is a great deal of numerical evidence that steady flows due to a line source *do* exhibit gravity waves on the free surface. In particular, the finite-depth problem is solved numerically in [54, 78, 113], with results showing a train of periodic waves appears on the free surface downstream from the source with amplitude increasing with Froude number. Mekias and Vanden-Broeck [78] explore this dependence by plotting numerical values of $\log(A)$ versus $-1/F^2$, where A is the (dimensionless) wave amplitude and the Froude number, also given in (1.2) is

$$F = \frac{U}{\sqrt{gH}}. \quad (2.42)$$

Here g is acceleration due to gravity, and U and H are some characteristic velocity and length scale respectively. They conjecture that

$$A \sim \text{const} \exp \left\{ -\frac{1}{F^2} \left(1 - \frac{y_s}{\pi} \right) \right\} \quad \text{as } F \rightarrow 0, \quad (2.43)$$

where y_s is the dimensionless height of the source above the horizontal bottom.

Chapman *et al.* [27] describe a framework for determining the form of exponentially small gravity waves for steady two-dimensional free surface flow problems in the limit that the Froude number F vanishes. The idea is to formulate the problem in terms of a formal power series in F^2 and note the series is divergent everywhere [116], with the divergence driven by singularities in the analytic continuation of the leading order term in the expansion. Anticipating factorial/power divergence, a WKB-type approach developed by [22] is applied to extract the form of the late-order terms. Following [33], the Stokes lines that emerge from the singularities are determined by equating the phases of each late order term with that of the subsequent term. Importantly, the periodic waves appear downstream from the point at which the Stokes lines intersect the free surface. The precise form of the exponentially small waves is determined via Stokes lines smoothing [8, 87].

A key result in [27] is that Stokes lines emerge from singularities that correspond to corners on solid boundaries of the flow field with in-fluid angles of greater than $2\pi/3$. The example of free surface flow over a step is treated in some detail, for which the Stokes lines emerge from the top corner (with in-fluid angle of $3\pi/2$) but not the stagnation point (with in-fluid angle of $\pi/2$). In other words, it is the top corner that gives rise to periodic waves on the free surface. The results complement the earlier study by [63], who apply numerical techniques to the fully nonlinear problem of free surface flow over a step and transform methods to the corresponding linear problem (which applies in the limit of vanishingly small step height).

By adapting the framework of Chapman and Vanden-Broeck [27] to hold for flows due to a line source, we apply techniques in exponential asymptotics to show that, for the finite-depth problem, there are periodic waves on the free surface for small values of the Froude number, and obtain an expression for these waves.

2.3.1 Governing equations

We consider steady irrotational flow of an inviscid, incompressible fluid caused by a submerged line source. Due to the symmetry in the system, this may be treated as a two-dimensional problem. In order to follow [27] as closely as possible, we set the channel to have finite-depth in which the mean velocity and channel height in the far field naturally approach the constant values U and H , respectively. It is assumed that there is a stagnation point on the free surface

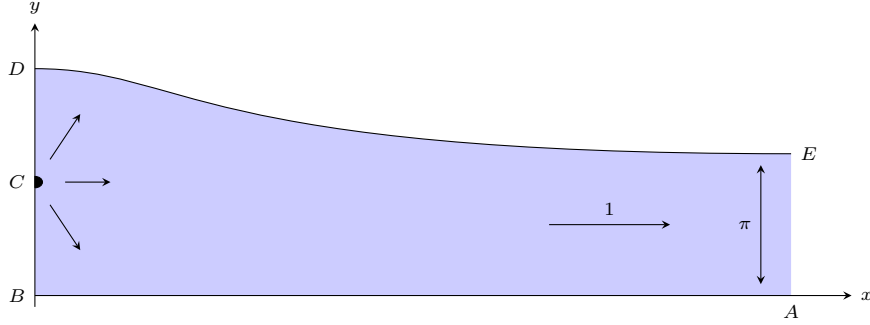


Figure 2.4: The dimensionless problem in the physical plane. The flow is symmetric about the y -axis, and so the left-hand side may be omitted. The shaded region denotes the flow region, with the solid circle representing the source.

directly above the source. In dimensional variables, the strength of the source is m , so that the flux of fluid in the far field is $m = UH$. Thus the Froude number in (2.42) is also defined by

$$F = \frac{m}{\sqrt{gH^3}}.$$

Motivated by the dimensionless variables used in [27], we immediately scale all lengths with respect to H/π and velocities with respect to U . Thus the problem depends on two dimensionless parameters, the Froude number F and the dimensionless height of the source above the bottom y_s . Again, in order to follow the framework in [27] as closely as possible, we introduce

$$\epsilon = \pi F^2, \quad (2.44)$$

which will act as our small parameter. Note that given the flow is symmetric about the y -axis, we treat only the fluid region for $x > 0$. A schematic of the resultant dimensionless system is shown in Figure 2.4.

As the flow is irrotational, the fluid velocity (u, v) is related to the velocity potential $\phi(x, y)$ by $(u, v) = \nabla\phi$. Enforcing incompressibility of the fluid thus requires that the potential satisfies Laplace's equation

$$\nabla^2\phi = 0$$

in the flow domain, except at the submerged source where it has the usual logarithmic singularity. On the free surface we have a dynamic boundary condition given by Bernoulli's equation

$$\frac{1}{2}\epsilon(|\nabla\phi|^2 - 1) + y = \pi. \quad (2.45)$$

On the boundaries of the flow region we prescribe a kinematic boundary condition such that

$$\frac{\partial\phi}{\partial n} = 0, \quad (2.46)$$

where n is the normal to the boundary.

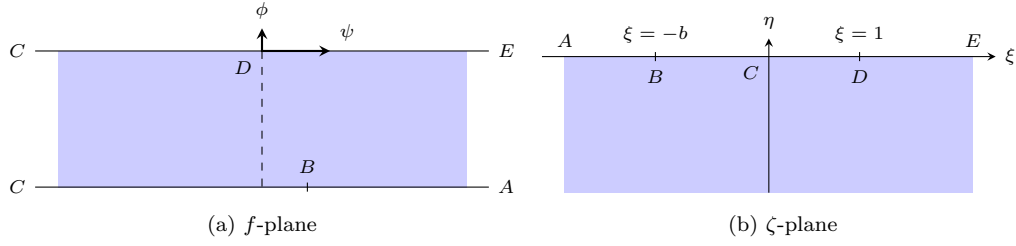


Figure 2.5: Flow region mapped into: (a) the complex potential plane; and (b) the ζ -plane.

The following reformulation of the problem follows [53] and [113]. We define $z = x + iy$ and the complex potential $f(z) = \phi + i\psi$, where ψ is the streamfunction. For definiteness we set $f = 0$ at the stagnation point D (see Figure 2.4), so that $\psi = 0$ on the free surface and $\psi = -\pi$ on the horizontal bottom. The complex velocity is defined as

$$\frac{df}{dz} = u - iv = q e^{-i\theta},$$

where q is the magnitude of the velocity vector, while θ is the angle a streamline makes with the horizontal; at this stage both q and θ are real. The flow region in the f -plane is given by an infinite strip in the region $-\pi \leq \psi \leq 0$. The second mapping $\zeta = \xi + i\eta = e^f$ transforms the flow region to the lower-half ζ -plane, with the source mapping to $\zeta = 0$, the bottom corner mapping to $\zeta = -b$ and the free surface located on $\eta = 0, \xi > 1$. These mappings are shown in Figure 2.5.

Since $q e^{-i\theta}$ is an analytic function of ζ , Cauchy's integral formula may be applied in the ζ -plane to give

$$\log q = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{\theta(s)}{s - \xi} ds,$$

where the dash through the integral sign indicates it is to be treated as a Cauchy Principal Value integral. For the problem shown in Figure 2.4, we have $\theta = 0$ for $\xi < -b$, $\theta = -\pi/2$ for $-b < \xi < 0$ and $\theta = \pi/2$ for $0 < \xi < 1$. Thus we have the integral equation

$$\log q = \frac{1}{2} \log \left(\frac{(\xi + b)(\xi - 1)}{\xi^2} \right) + \frac{1}{\pi} \int_1^{\infty} \frac{\theta(s)}{s - \xi} ds, \quad \xi > 1. \quad (2.47)$$

The kinematic boundary condition (2.46) is identically satisfied by setting $\psi = \text{constant}$ on the solid boundaries. To write Bernoulli's equation (2.45) in terms of our conformally mapped variable ζ , we apply the chain rule to give

$$\frac{dz}{d\zeta} = \frac{dz}{df} \frac{df}{d\zeta} = \frac{1}{\zeta q} e^{i\theta}.$$

Thus, on the free boundary $\eta = 0, \xi > 1$, we find

$$\frac{dx}{d\xi} = \frac{1}{\xi q} \cos \theta, \quad \frac{dy}{d\xi} = \frac{1}{\xi q} \sin \theta. \quad (2.48)$$

Taking the derivative of (2.45) with respect to ζ and applying (2.48) gives the dynamic condition

$$\epsilon \xi q^2 \frac{dq}{d\xi} = -\sin \theta, \quad \xi > 1. \quad (2.49)$$

To complete the formulation of the problem, we integrate (2.48) from the points B to C to give

$$y_s = - \int_{-b}^0 \frac{1}{\xi q} d\xi. \quad (2.50)$$

Thus we can now think of the two parameters in the problem as being the Froude number F and the conformally-mapped parameter $b \geq 0$; once the solution is determined, the height of the source above the bottom is then determined via (2.50). Note that $y_s \rightarrow 0$ as $b \rightarrow 0^+$; in this limit the point B merges with C , and some parts of the analysis need to be treated separately.

At this stage the governing equations (2.47) and (2.49) hold only on $\eta = 0$, $\xi > 1$; however, our analysis requires that we analytically continue (2.47) and (2.49) into the lower-half ξ -plane. Following [27], we shall use the notation $\zeta = \xi + i\eta$ to denote the complexified variable ξ , so that

$$\log q + i\theta = \frac{1}{2} \log \left(\frac{(\zeta + b)(\zeta - 1)}{\zeta^2} \right) + \frac{1}{\pi} \int_1^\infty \frac{\theta(s)}{s - \zeta} ds, \quad \eta \geq 0, \quad -\infty < \xi < \infty, \quad (2.51)$$

$$\epsilon \zeta q^2 \frac{dq}{d\zeta} = -\sin \theta, \quad \eta \geq 0, \quad -\infty < \xi < \infty. \quad (2.52)$$

Together, (2.51) and (2.52) provide governing equations for the analytically-continued free surface in the lower-half ζ -plane. We emphasise that we are now using ζ to be the independent variable for the complex functions $q(\zeta)$ and $\theta(\zeta)$, and not the conformally-mapped plane. Of course, (2.51) and (2.52) reduce to (2.47) and (2.49) for $\eta = 0$, $\xi > 1$, but otherwise, the solution of (2.51) and (2.52) in the ζ -plane does not provide the solutions for q and θ in the flow field.

2.3.2 General exponential asymptotics

We solve (2.51) and (2.52) in the limit $\epsilon \rightarrow 0$. This section illustrates the framework described in [27], while the subsequent sections contain the details for the specific problem of flow due to a line source.

2.3.2.1 Asymptotic power series

We write q and θ as the asymptotic expansions

$$q \sim \sum_{n=0}^{\infty} \epsilon^n q_n, \quad \theta \sim \sum_{n=0}^{\infty} \epsilon^n \theta_n, \quad \text{as } \epsilon \rightarrow 0. \quad (2.53)$$

The leading order solution for θ is $\theta_0 \equiv 0$, while the solution for q_0 depends on the geometry of the particular problem in question. The next few terms are calculated using

$$\theta_1 = -\zeta q_0^2 \frac{dq_0}{d\zeta}, \quad (2.54)$$

$$q_1 = i\zeta q_0^3 \frac{dq_0}{d\zeta} + \frac{q_0}{\pi} \int_1^\infty \frac{\theta_1(s)}{s-\zeta} ds, \quad (2.55)$$

$$\theta_2 = -\zeta q_0 \left(2q_1 \frac{dq_0}{d\zeta} + q_0 \frac{dq_1}{d\zeta} \right), \quad (2.56)$$

while subsequent terms in the series are found using the recurrence relationship

$$\frac{q_n}{q_0} - \frac{q_{n-1}q_1}{q_0^2} + \dots = -i\theta_n + \frac{1}{\pi} \int_1^\infty \frac{\theta_n(s)}{s-\zeta} ds, \quad (2.57)$$

$$\zeta q_0^2 \frac{dq_{n-1}}{d\zeta} + 2\zeta q_0 q_1 \frac{dq_{n-2}}{d\zeta} + 2\zeta q_0 q_{n-1} \frac{dq_0}{d\zeta} + \dots = -\theta_n + \dots \quad (n \geq 3), \quad (2.58)$$

where the ignored terms are subdominant in the limit $n \rightarrow \infty$ and will not be required in the subsequent analysis.

2.3.2.2 Late-order terms

We require the behaviour of q_n and θ_n in the limit $n \rightarrow \infty$. Following [27], we apply the ansatz

$$q_n \sim \frac{Q\Gamma(n+\gamma)}{\chi^{n+\gamma}}, \quad \theta_n \sim \frac{\Theta\Gamma(n+\gamma)}{\chi^{n+\gamma}} \quad (2.59)$$

to (2.57)–(2.58). Here, we note that the terms on the right-hand side of (2.57) will exponentially dominate the integral on the left-hand side. This point is noted in [27], with an argument made in [102]. We subsequently find as $n \rightarrow \infty$ that (2.58) may be written as

$$q_n \sim -iq_0\theta_n - iq_1\theta_{n-1} + \dots \quad (2.60)$$

Applying the late-order ansatz in (2.59) to (2.58) and (2.60) gives to leading order as $n \rightarrow \infty$

$$\frac{Q\Gamma(n+\gamma)}{\chi^{n+\gamma}} = -iq_0 \frac{\Theta\Gamma(n+\gamma)}{\chi^{n+\gamma}} + \dots, \quad \zeta q_0^2 \frac{d\chi}{d\zeta} \frac{Q\Gamma(n+\gamma)}{\chi^{n+\gamma}} = -\frac{\Theta\Gamma(n+\gamma)}{\chi^{n+\gamma}} + \dots,$$

where the omitted terms are small compared to those shown. Simplifying these equations gives

$$Q = -iq_0\Theta, \quad i\zeta q_0^3 \frac{d\chi}{dw} = -1. \quad (2.61)$$

Solving the second of these equations gives

$$\chi = i \int_{\zeta_0}^{\zeta} \frac{d\zeta}{\zeta q_0^3}, \quad (2.62)$$

where ζ_0 is the important singularity of q_0 (which turns out to be $\zeta_0 = 0$ for the problem here). At the subsequent order in n , we find that γ is constant, while matching at the next order in (2.58) gives

$$q_0^2 \frac{dQ}{d\zeta} - 2q_0 q_1 Q \frac{d\chi}{d\zeta} + 2q_0 q_1 Q = \frac{iq_1 Q}{q_0^2}.$$

This reduces to

$$\frac{1}{Q} \frac{dQ}{d\zeta} = \frac{2}{q_0} \frac{dq_0}{d\zeta} + \frac{3iq_1}{\zeta q_0^4},$$

which may be solved to obtain

$$\log Q = 2 \log q_0 + 3i \int \frac{q_1}{\zeta q_0^4} d\zeta + \text{constant}.$$

Rearranging this expression and using the relationship between Q and Θ given in (2.61) gives

$$\begin{aligned} Q &= \frac{\Lambda}{q_0^2} \exp \left(3i \int_{\zeta^*}^{\zeta} \frac{q_1 d\zeta}{\zeta q_0^4} \right) \\ &= \frac{\Lambda q_0^3(\zeta^*)}{q_0^5} \exp \left(\frac{3i}{\pi} \int_{\zeta^*}^{\zeta} \frac{1}{\zeta q_0^3} \int_1^{\infty} \frac{\theta_1(s) ds}{s - \zeta} d\zeta \right), \end{aligned} \quad (2.63)$$

$$\begin{aligned} \Theta &= \frac{i\Lambda}{q_0^3} \exp \left(3i \int_{\zeta^*}^{\zeta} \frac{q_1 d\zeta}{\zeta q_0^4} \right) \\ &= \frac{i\Lambda q_0^3(\zeta^*)}{q_0^6} \exp \left(\frac{3i}{\pi} \int_{\zeta^*}^{\zeta} \frac{1}{\zeta q_0^3} \int_1^{\infty} \frac{\theta_1(s) ds}{s - \zeta} d\zeta \right), \end{aligned} \quad (2.64)$$

where Λ is a constant that is determined by matching with the inner solution in the neighbourhood of the relevant singularity $\zeta = \zeta_0$, and the end-point ζ^* is arbitrary, provided that the resultant integrals converge (changing ζ^* changes the value of Λ).

2.3.2.3 Stokes line smoothing

Truncating (2.53) after N terms gives

$$\theta = \sum_{n=0}^{N-1} \epsilon^n q_n + R_N, \quad q = \sum_{n=0}^{N-1} \epsilon^n \theta_n + S_N.$$

Applying this truncated series to the (2.51) and (2.52) shows that the remainders satisfy

$$\frac{S_n}{q_0} - \frac{\epsilon q_1 S_N}{q_0^2} + \dots = -iR_N, \quad (2.65)$$

$$\epsilon \zeta q_0^2 \frac{dS_N}{d\zeta} + 2\epsilon \zeta q_0 \frac{dq_0}{d\zeta} S_N + 2\epsilon^2 \zeta q_0 q_1 \frac{dS_N}{d\zeta} + \epsilon^2 \zeta q_1 \frac{dq_0}{d\zeta} S_N - \epsilon^N \theta_N + \dots = -R_N. \quad (2.66)$$

The homogeneous versions of these equations have solutions $Qe^{-\chi/\epsilon}$ and $\Theta e^{-\chi/\epsilon}$ which motivates the ansatz

$$R_N = A Q e^{-\chi/\epsilon}, \quad S_N = A \Theta e^{-\chi/\epsilon},$$

where the unknown function $A(\zeta; \epsilon)$ is the Stokes multiplier. Applying these remainder forms, as well as the late-order terms ansatz given in (2.59), to (2.65)–(2.66) and simplifying gives

$$-iq_0^3 \zeta \frac{dA}{d\zeta} = \frac{\epsilon^{N-1} e^{\chi/\epsilon} \Gamma(N + \gamma)}{\chi^{N+\gamma}}.$$

Writing A as a function of χ gives

$$\frac{dA}{d\chi} = \frac{\epsilon^{N-1} e^{\chi/\epsilon} \Gamma(N + \gamma)}{\chi^{N+\gamma}}. \quad (2.67)$$

Minimising the right-hand side with respect to N gives the optimal truncation point as $N \sim |\chi|/\epsilon$ as $\epsilon \rightarrow 0$, and hence we set $N = |\chi|/\epsilon + \alpha$, where $0 \leq \alpha < 1$ and α is chosen such that N takes integer value. As N is a function of $|\chi|$, we set $\chi = \rho e^{i\vartheta}$ (where ϑ is not to be confused with the dependent variable θ). We therefore note that

$$\frac{d}{d\chi} = -\frac{ie^{-i\vartheta}}{\rho} \frac{d}{d\vartheta}.$$

Writing (2.67) in terms of these new variables and using Stirling's approximation eventually gives

$$\frac{dA}{d\vartheta} \sim \frac{\sqrt{2\pi}\rho^{1/2}}{\epsilon^{1/2+\gamma}} \exp\left(\frac{\rho}{\epsilon}(e^{i\vartheta} - 1) - i\vartheta\left(\frac{\rho}{\epsilon} - 1 + \rho + \gamma\right)\right).$$

The right-hand side of this expression is exponentially small except in the neighbourhood of $\vartheta = 0$. Hence, this curve describes the Stokes line, across which A varies rapidly. To examine the switching parameter in the neighbourhood of the Stokes line, we introduce local coordinates $\vartheta = \epsilon^{1/2}\bar{\vartheta}$, which gives

$$\frac{dA}{d\bar{\vartheta}} \sim \frac{i\sqrt{2\pi}\rho^{1/2}}{\epsilon^\gamma} e^{-\rho\bar{\vartheta}^2/2}.$$

Integrating this expression and returning to the outer coordinates, we find that

$$A = \frac{\sqrt{2\pi i}}{\epsilon^\gamma} \int_{-\infty}^{\sqrt{\rho}\vartheta/\epsilon} e^{-t^2/2} dt,$$

where the lower bound of the integral is chosen by matching the integral to the known upstream behaviour, where no waves are present. As the Stokes line is crossed, A varies smoothly and rapidly from 0 to $2\pi i/\epsilon^\gamma$, and therefore contributions are switched on of the form

$$\frac{2\pi i}{\epsilon^\gamma} Q e^{-\chi/\epsilon}, \quad \frac{2\pi i}{\epsilon^\gamma} \Theta e^{-\chi/\epsilon}, \quad (2.68)$$

for R_N and S_N , respectively, as $\epsilon \rightarrow 0$. As these contributions take the form of complex exponentials, they will cause wave behaviour on the free-surface.

In the above analysis, the governing equations (2.51)–(2.52) were derived by analytically continuing the free surface into the lower-half complex plane. By instead extending into the upper-half plane, we find a further contribution which is the complex conjugate of (2.68). Adding these together provides the leading-order behaviour of the exponentially small remainder on the free-surface, denoted θ_{exp} , given as

$$\theta_{\text{exp}} = \frac{4\pi|\Lambda|}{\epsilon^\gamma q_0^3} e^{-\text{Re}(\chi)/\epsilon} \cos\left(\arg \Theta - \frac{\text{Im}(\chi)}{\epsilon}\right). \quad (2.69)$$

2.3.3 Details for flow due to a line source

The framework summarised in the previous section was essentially taken directly from [27]. Here we consider the specific problem of flow due to a line source, which was not previously considered.

2.3.3.1 Early terms

Given that $\theta_0 \equiv 0$, substituting (2.53) into (2.51) reveals the important leading order solution

$$q_0 = \frac{(\zeta + b)^{1/2}(\zeta - 1)^{1/2}}{\zeta}, \quad (2.70)$$

while (2.54) provides the first correction term

$$\theta_1 = -\frac{((1-b)\zeta + 2b)(\zeta + b)^{1/2}(\zeta - 1)^{1/2}}{2\zeta^3}. \quad (2.71)$$

These terms are also calculated by [53]. By substituting the result

$$\begin{aligned} \int_1^\infty \frac{\theta_1(s)}{s - \zeta} ds = & \frac{1}{4}((1 - 6b + b^2)\zeta^2 + 8b\zeta(1 - b) + 8b^2) \frac{\arctan(b^{1/2})}{b^{1/2}\zeta^3} + \frac{3(1 - b)}{4\zeta} + \frac{b}{\zeta^2} \\ & - \frac{((1 - b)\zeta + 2b)(\zeta - 1)^{1/2}(\zeta + b)^{1/2}}{2\zeta^3} \log \left(\frac{1 - b - 2\zeta - 2(\zeta - 1)^{1/2}(\zeta + b)^{1/2}}{1 + b} \right) \end{aligned}$$

into (2.55)–(2.56) we can also determine the exact forms for the correction terms q_1 and θ_2 , but for brevity we do not include the full expressions.

Applying the expression for q_0 from (2.70) to (2.50) gives

$$y_s \sim \arctan \left(\frac{2b^{1/2}}{1 - b} \right) \quad \text{as } \epsilon \rightarrow 0, \quad (2.72)$$

which provides a leading order relationship between the parameter b and the height of the source above the bottom y_s .

2.3.3.2 Location of the Stokes lines

The leading order term (2.70) has a singularity at $\zeta = 0$ and, since the higher-order problems are linear, each of the q_n is also singular there. A Stokes line originates at $\zeta = 0$ and follows the path on which successive late-order terms θ_n and q_n have the same phase, which for our problem is when χ is real and positive (see [33]). Evaluating (2.62) with (2.70) gives the explicit expression

$$\chi = i \int_0^\zeta \frac{\zeta^2}{(\zeta + b)^{3/2}(\zeta - 1)^{3/2}} d\zeta \quad (2.73)$$

$$\begin{aligned} = & \frac{2i[(1 + b^2)\zeta + b(1 - b)]}{(1 + b)^2(\zeta + b)^{1/2}(\zeta - 1)^{1/2}} - \frac{2b^{1/2}(1 - b)}{(1 + b)^2} \\ & - i \log \left(\frac{1 - b - 2\zeta - 2(\zeta + b)^{1/2}(\zeta - 1)^{1/2}}{1 - b - 2ib^{1/2}} \right). \end{aligned} \quad (2.74)$$

The real part of χ is constant on the free surface, and takes the form

$$\text{Re}(\chi) = \pi - \arctan \left(\frac{2b^{1/2}}{1 - b} \right) + \frac{2b^{1/2}(1 - b)}{(1 + b)^2}, \quad \xi > 1. \quad (2.75)$$

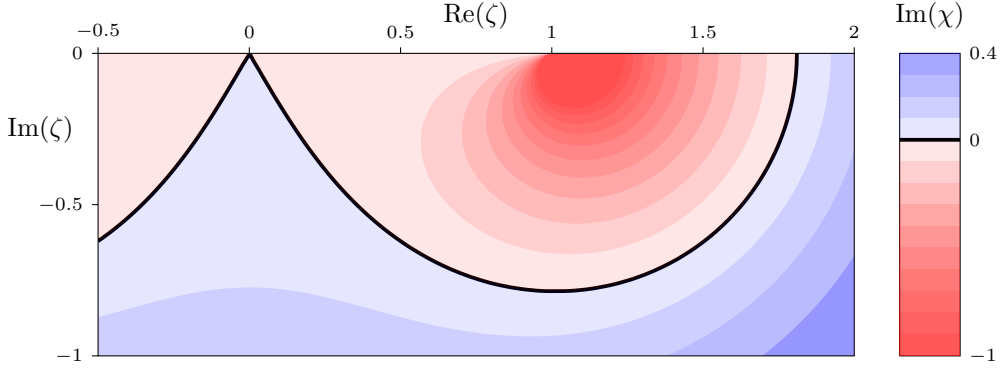


Figure 2.6: Contour plot for $\text{Im}(\chi)$ for $b = 1$ in the complex ζ -plane, with lighter shading indicating a more positive value. The Stokes line, given by $\text{Im}(\chi) = 0$, is shown in bold. The point at which it intersects the free surface (the real axis for $\xi > 1$) is the point at which an exponentially small contribution is switched on.

Significantly, $\text{Re}(\chi) > 0$ for $\xi > 1$, ensuring that Stokes switching may occur on the free surface (see [33]). It will be useful to combine (2.72) and (2.75) to give

$$\text{Re}(\chi) \sim \pi - y_s + \frac{1}{2} \sin 2y_s \quad \text{as } \epsilon \rightarrow 0, \quad \xi > 1.$$

A contour plot of the imaginary part of χ for $b = 1$ is provided in Figure 2.6 with the Stokes lines ($\text{Im}(\chi) = 0$) shown in bold. Since from (2.73) we have

$$\chi \sim -\frac{\zeta^3}{3b^{3/2}} \quad \text{as } \zeta \rightarrow 0, \quad (2.76)$$

the Stokes lines emerge from the singularity at ζ into the lower half-plane at angles $-\pi/3$ and $-2\pi/3$. The latter of these is cut off by the scale of the figure, and does not go on to intersect the free surface $\eta = 0$, $\xi > 1$. On the other hand, the former clearly does cross the real axis. For this particular value of b , the intersection point is $\xi_{SL} \approx 1.812$. More generally, the intersection point ξ_{SL} is a decreasing function of b , with $\xi_{SL} \rightarrow 3.2767$ as $b \rightarrow 0^+$, as shown in Figure 2.7(a).

By mapping this back onto the physical plane via (2.48) we can determine the point on the free surface that is crossed by the Stokes line, denoted by (x_{SL}, y_{SL}) . The dependence on x_{SL} on the height of the source above the bottom is provided in Figure 2.7(b), where we see that the waves switch on further downstream as y_s decreases.

2.3.3.3 Determining γ and Λ

To determine the value of γ in (2.69) we consider the behaviour of the early and late terms as $\zeta \rightarrow 0$ and ensure that they are consistent. We note that (2.70) gives

$$q_0 \sim \frac{ib^{1/2}}{\zeta} \quad \text{as } \zeta \rightarrow 0. \quad (2.77)$$

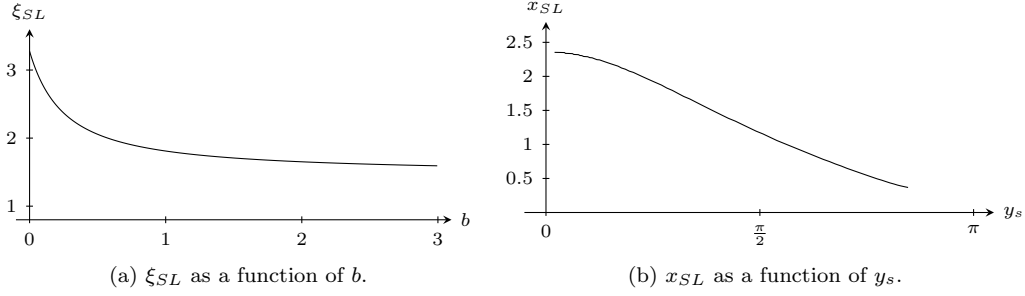


Figure 2.7: Values of (a) ξ_{SL} and (b) x_{SL} at which the Stokes line intersects the physical free surface as b , and consequently y_s , is varied.

By examining our governing equations we find that, as $\zeta \rightarrow 0$,

$$q_1 \sim -\frac{ib^2}{\zeta^4}, \quad q_2 \sim \frac{6ib^{7/2}}{\zeta^{7/2}}, \quad q_3 \sim -\frac{66ib^5}{\zeta^{10}},$$

and so on, so that, with the use of (2.76), we find

$$q_n \sim \frac{i(-1)^n \mu_n b^{(3n+1)/2}}{\zeta^{3n+1}} \quad \text{as } \zeta \rightarrow 0,$$

where μ_n are constants that depend on n . On the other hand, the inner limit of (2.59) is

$$q_n \sim -\frac{i\Lambda C(-3)^{n+\gamma} b^{(3n+3\gamma-5)/2} \Gamma(n+\gamma)}{\zeta^{3n+3\gamma-5}} \quad (2.78)$$

as $n \rightarrow \infty$ and $\zeta \rightarrow 0$, where

$$C = q_0^3(\zeta^*) \exp\left(\frac{3i}{\pi} \int_{\zeta^*}^0 \frac{1}{\zeta q_0^3} \int_1^\infty \frac{\theta_1(s) ds}{s-\zeta} d\zeta\right). \quad (2.79)$$

Thus matching the powers of ζ , it must be that $\gamma = 2$.

We can see from (2.78) that the series rearranges when $\zeta = \mathcal{O}(\epsilon^{1/3})$ corresponding to an inner region near the singularity. We determine Λ by matching (2.78) with this inner solution at leading order.

By observing the scalings in (2.76) and (2.77) we define the inner variables

$$q = \frac{ib^{1/2}}{\zeta} \bar{q}(\sigma), \quad \sigma = \frac{\zeta^3}{3\epsilon b^{3/2}}, \quad (2.80)$$

As argued in [23] and [102], (2.51) implies that $\log q + i\theta \sim \log q_0$ in the inner region, so that using the scalings (2.80), the second governing equation (2.52) becomes

$$-\frac{2p^2}{3\sigma} + p \frac{dp}{d\sigma} = p - 1, \quad (2.81)$$

where $p = \bar{q}^2$. The outer limit of (2.81) is obtained via a series expression for p in the limit that $|\sigma| \rightarrow \infty$:

$$p = \sum_{n=0}^{\infty} \frac{p_n}{(-\sigma)^n}, \quad p_0 = 1.$$

Applying this expression for p to (2.81) and matching terms as $|\sigma| \rightarrow \infty$ gives

$$p_{n+1} = \sum_{j=0}^n \left(j + \frac{2}{3} \right) p_j p_{n-j}. \quad (2.82)$$

With the values for p_n determined, we can write out the inner solution for q^2 in terms of the outer variables as

$$q^2 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} b (3b^{3/2})^n p_n \epsilon^n}{\zeta^{3n+2}}.$$

This outer limit of the inner solution is matched with inner limit (2.78) of the outer solution to give

$$\Lambda = -\frac{1}{18C} \lim_{n \rightarrow \infty} \frac{p_n}{\Gamma(n+2)} \approx -\frac{0.38}{18C}, \quad (2.83)$$

where the series in (2.82) was evaluated numerically to determine the behaviour of p_n in the large- n limit.

2.3.3.4 Computing the Wave Amplitude

From the preceding analysis, we have $\gamma = 2$, $|\Lambda|$ given by (2.83), and $q_0 \sim 1$, $\text{Im}(\chi) \sim -\log \xi$ as $\xi \rightarrow \infty$. Since, $\phi \sim \log \xi$ as $\xi \rightarrow \infty$ and $\phi \sim x$ as $x \rightarrow \infty$, we have in the far field

$$\theta_{\text{exp}} \sim 4\pi|\Lambda|\epsilon^{-2}e^{-\text{Re}(\chi)/\epsilon} \cos\left(\frac{x}{\epsilon} + \text{constant}\right) \quad \text{as } x \rightarrow \infty,$$

where $\text{Re}(\chi)$ is given in (2.75). Further, we can make the approximation $dy/dx \sim \theta$, so that written in terms of our Cartesian coordinates and the Froude number F , the wave train is

$$y_{\text{exp}} \sim 4|\Lambda|F^{-2}e^{-\text{Re}(\chi)/\pi F^2} \sin\left(\frac{x}{\pi F^2} + \text{constant}\right) \quad \text{as } x \rightarrow \infty. \quad (2.84)$$

It follows that the actual wave amplitude is of order $F^{-2}e^{-\beta/F^2}$, where the constant

$$\beta = 1 - \frac{y_s}{\pi} + \frac{1}{2\pi} \sin 2y_s.$$

2.3.3.5 The case $b = 0$

In the limit $b \rightarrow 0^+$, the height of the source $y_s \rightarrow 0$, and the corner point B merges with the source at C . As a result, the scalings change for $b = 0$.

The Stokes line calculations proceed as normal, with χ given by (2.74) with $b = 0$, except now we have

$$\chi \sim -\frac{2}{3}\zeta^{3/2} \quad \text{as } \zeta \rightarrow 0. \quad (2.85)$$

A contour plot of $\text{Im}(\chi)$ with $b = 0$ is shown in Figure 2.8, with the Stokes line $\text{Im}(\chi) = 0$ indicated by the bold curve. From (2.85) we see that one Stokes line emerges from $\zeta = 0$ into

the lower half-plane, at angle $-2\pi/3$ (rather than the two curves seen in Figure 2.6 for $b > 0$, emerging at angles $-\pi/3$ and $-2\pi/3$).

Some of the working from Section 2.3.3.3 changes. To determine γ we note that now $q_0 \sim i/\zeta^{1/2}$, $q_1 \sim -i/2\zeta^2$, $q_2 \sim 3i/2\zeta^{7/2}$, $\dots$, as $\zeta \rightarrow 0$, which leads to

$$q_n \sim \frac{i(-1)^n \mu_n}{\zeta^{(3n+1)/2}} \quad \text{as } \zeta \rightarrow 0,$$

where μ_n are constants that depend on n . The inner limit of (2.59) for $b = 0$ is

$$q_n \sim -\frac{i\Lambda C(-3/2)^{n+\gamma}\Gamma(n+\gamma)}{\zeta^{(3n+3\gamma-5)/2}}$$

as $n \rightarrow \infty$ and $\zeta \rightarrow 0$, where C is again given by (2.79). As a result, matching powers of ζ implies that once again $\gamma = 2$. To match with the inner region we note that the leading order behaviour of the n th term of q^2 is

$$2q_0 q_n \sim \frac{9(-1)^n \Lambda C(3/2)^n \Gamma(n+2)}{2\zeta^{3n/2+1}}$$

as $n \rightarrow \infty$ and $\zeta \rightarrow 0$.

The inner problem this time is for $\zeta = \mathcal{O}(\epsilon^{2/3})$. The relevant the inner variables are

$$q = \frac{i}{\zeta^{1/2}} \bar{q}(\sigma), \quad \sigma = \frac{2\zeta^{3/2}}{3\epsilon},$$

which again gives the differential equation (2.81) for $p = \bar{q}^2$. A near-identical analysis gives the inner solution for q^2 in terms of the outer variables as

$$q^2 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (3/2)^n p_n \epsilon^n}{\zeta^{3n/2+1}}.$$

Thus, matching gives

$$\Lambda = -\frac{2}{9C} \lim_{n \rightarrow \infty} \frac{p_n}{\Gamma(n+2)},$$

where the p_n and C are identical to those given in (2.83). Comparing this result to the equivalent expression for the general case, given in (2.83), we see that the amplitude of the downstream wavetrain is larger when $b = 0$ than would be expected from the general result, due to the fact that the source is against the boundary rather than located within the flow.

Finally, we note that the general case will transition smoothly to the $b = 0$ case in a region of width $b = \mathcal{O}(\epsilon^{2/3})$, at which point the singularity at $\zeta = -b$ enters the inner region of the problem. This transition region could be investigated in a similar manner to the distinguished limit investigation in [27]; however, this analysis is omitted here.

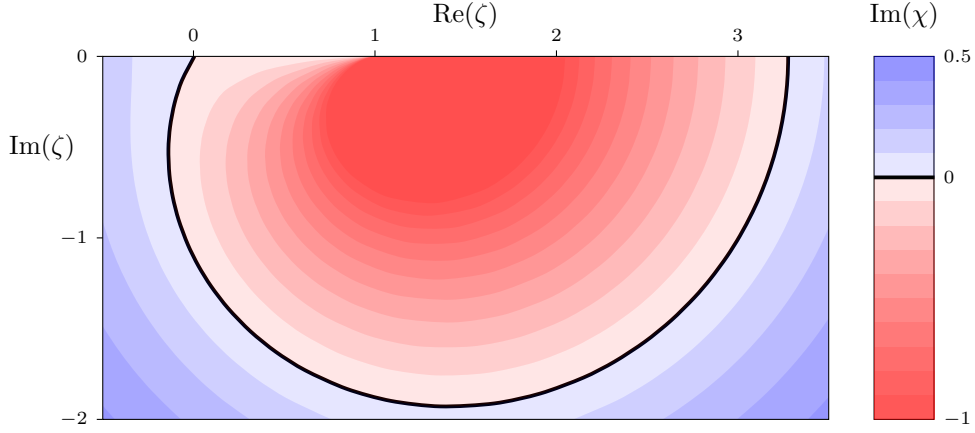


Figure 2.8: Contour plot for $\text{Im}(\chi)$ when $b = 0$ in the complex ζ -plane. The Stokes line, given by $\text{Im}(\chi) = 0$, is shown in bold. The point at which it intersects the free surface (the real axis for $\xi > 1$) is the point at which an exponentially small contribution is switched on.

2.3.4 Comparison of results

In Figure 2.9 we present free surface profiles computed by using our results for θ_1 , θ_2 , θ_{exp} , q_0 , q_1 and q_{exp} to integrate (2.48). The integral expression for C given in (2.79) was evaluated numerically by interpreting the expression as an integral over a box in the lower half-plane using MATLAB’s “quad” function, which applies an adaptive Simpson quadrature.

Note that due to symmetry we only show the right half of the flow field. In Figure 2.9(a), the profile is for $b = 3$ and $\epsilon = 0.1$, which from (2.72) gives the height of the source $y_s \approx 2\pi/3 \approx 2.09$. This example is representative of solutions for $b > 1$, which are for $y_s > \pi/2$ (that is, the source is located more than halfway up the channel). As explained in [53], the waveless part of the free surface profiles for $b > 1$ dip below the free stream level, reach a minimum value, and rise up before approaching the free stream level in the far field. On the other hand, the profile in Figure 2.9(b) is drawn for $b = 0.85$, which gives the height as $y_s \approx 1.49$. This profile is representative of solutions for $b < 1$ or $y_s < \pi/2$ (that, the case in which the source is located less than halfway up the channel). Here the waveless part of the profile remains above the free stream level with $\epsilon\theta_1 + \epsilon^2\theta_2 < 0$ for all $x > 0$.

In both profiles in Figure 2.9, the point at which the Stokes line intersects the free surface is marked with a circle. The downstream wavetrain is clearly visible to the right of this marker. In order to compare with the results in [78], the dependence of the amplitude A of this wavetrain on the Froude number is shown in Figure 2.10 for two different values of b (and hence y_s). These values of y_s correspond to those considered in [78], with the circles denoting the computed wave amplitudes from their study. The agreement between the asymptotic behaviour and the numerical solutions is reasonable for the values of F considered in this figure, becoming more accurate in the limit that $F \rightarrow 0$.

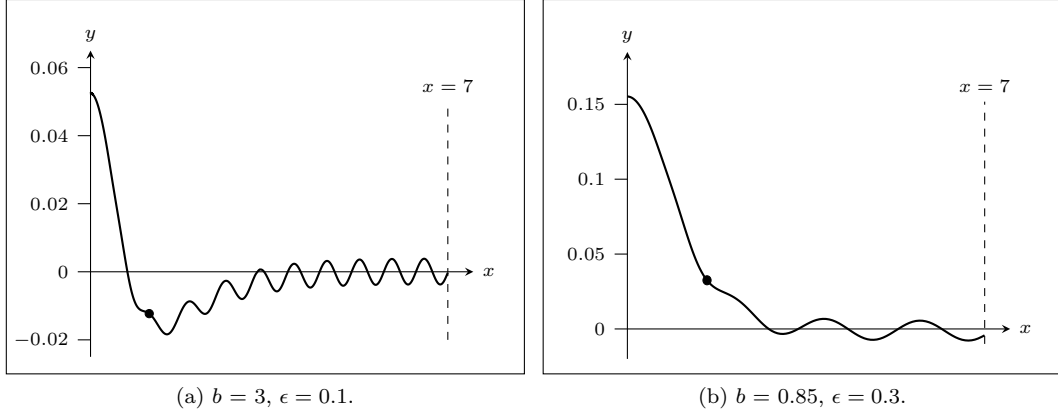


Figure 2.9: Free surface profile for a source with (a) $b = 3$ and $\epsilon = 0.1$ and (b) $b = 0.85$ and $\epsilon = 0.3$, computed using $\epsilon\theta_1 + \epsilon^2\theta_2 + \theta_{\text{exp}}$. The point at which the Stokes line intersects the free surface is indicated by a circle, with the wavetrain switched on to the right of this point.

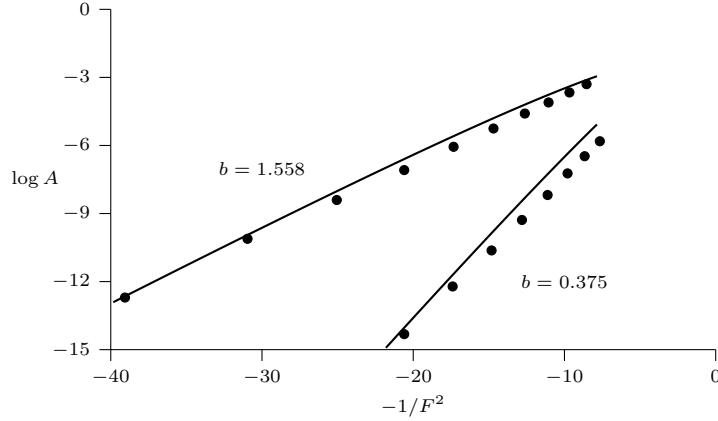


Figure 2.10: Plots of $\log A$ versus $-1/F^2$. From top to bottom, the curves are drawn for $b = 1.55$ ($y_s \approx 0.57\pi$) and 0.38 (0.35π). The circles represent data computed by [78].

2.3.5 Discussion and Conclusions

Using techniques in exponential asymptotics, we have provided a description of the problem of free surface flow due a line source in a fluid of finite depth in the limit of small Froude number. We found that the amplitude of the waves scales like $F^{-2}e^{-\beta/F^2}$ in the limit $F \rightarrow 0$, where β is a constant related to the height of the source above the bottom. The precise form of these waves is given in (2.84). We emphasise that it is not possible to extract this information solely from applying a formal power series, as the waves appear beyond all orders of an algebraic expansion in F^2 .

As mentioned in the Introduction, [78] conjecture that the wave amplitude in the far field is related to the Froude number via (2.43), which is incorrect for two reasons. The first is that it is out by a factor of F^2 . The simple explanation for this discrepancy is that Mekias and Vanden-Broeck plot numerical values of $\log A$ versus $-1/F^2$, noting the linear dependence in

the small Froude number limit. From (2.84) we have

$$\log A \sim -\frac{\beta}{F^2} - 2 \log F \quad \text{as } F \rightarrow 0$$

which, of course, agrees with the scalings observed by Mekias and Vanden-Broeck to leading order. The second reason is that Mekias and Vanden-Broeck conjecture that the constant β is given by $1 - y_s/\pi$, when our calculations suggest the correct value is $1 - y_s/\pi + \sin(2y_s)/2\pi$. These results agree for the special cases of $y_s = 0$ and $y_s = \pi/2$, corresponding to a source on the base of the channel and halfway up the channel respectively.

Our results suggest that for the numerical solutions presented in Hocking and Forbes [53], for example, the size of the Froude number is such that the downstream waves are too small to observe in practice. Finally, the analysis provides an explanation as to why solutions with waves do not arise from the application of a formal power series in F^2 : the exponentially small waves appear beyond all orders of the original series expansion and, as such, cannot be captured by terms that are algebraic in F^2 .

We emphasise that this work acts to fill in the gap left by [27], who overlooked the possibility of line sources giving rise to Stokes lines. The mistake arose because although $q_0 \sim \text{const } e^{-f}$ is not singular as a function of f , it is singular as a function of the mapped coordinate ζ .

As is well known, the governing equations for steady ideal fluid flow due to a line source are equivalent to those for flow due to a line sink (obtained by replacing ϕ with $-\phi$), as the velocity appears only as its square in Bernoulli's equation. However, the latter flow has the further restriction placed by the radiation condition, which does not allow gravity-driven water waves in the far field. As a consequence, since our solutions have waves on the free surface in the far field (although, admittedly the waves are exponentially small in the limit $F \rightarrow 0$), these are strictly applicable for source flow only.

Another way of interpreting our results is that if the flow in Figure 2.4 is right to left (that is, due to a line sink instead of line source), then the term (2.69) would appear to the left of the point at which the Stokes line meets the free surface. As $q_0 \rightarrow 0$ in the limit $x \rightarrow 0^+$ on the free surface, the magnitude of (2.69) would become large at the stagnation point (see [102] for a similar argument for flow past a semi-infinite stern). Thus our approach shows that there are no steady subcritical solutions for two-dimensional ideal fluid flow due to a line sink in a fluid of finite depth with a stagnation point on the free surface directly above the sink.

On the other hand, there is numerical evidence of steady solutions of a different type, with a cusp on the free surface instead of a stagnation point, such as [55, 77, 114, 115]. However, there is no zero-Froude-number solution with a cusp, and hence exponential asymptotic methods would not be of use in determining the behaviour of any free-surface wave behaviour.

Chapter 3

Linearized Two-Dimensional Unsteady Flow

3.1 Introduction and Background

Prior to the advent of computers, and the subsequent ubiquity of numerical algorithms for determining free-surface flow behaviour, the linearization of fluid flow problems in the limit as some submerged obstacle becomes small was a technique commonly applied in order to approximate flow behaviours. Studies considering free-surface flow around small obstacles include [36, 40, 63, 64, 65] for steady flow, and [28] for examples of unsteady flow. These studies were discussed in more detail in Chapter 1, and all share a common theme; in each case the resultant flow is obtained as a perturbation about the undisturbed solution.

The difficulties involved with finding the behaviour of gravity waves on free surface flows over an obstacle in the limit of small Froude number were discussed at length in Chapter 1. As noted in Section 2.3, Chapman and Vanden-Broeck considered the steady version of this problem and applied exponential asymptotic methods to determine the behaviour of gravity waves on the free surface of flow over an obstacle [24], both in the linearized and non-linear case. Furthermore, they determined that, in case of linearized flow over a small semi-infinite step, the results corresponded with those obtained by King and Bloor [63]. Unsurprisingly, the linearized problem exhibited much of the same Stokes switching behaviour as the fully nonlinear problem.

It is natural then, when considering the equivalent unsteady problem, to begin with the linearized case. Analysis of the linearized problem will allow us to determine what role the Stokes phenomenon has to play in the surface wave behaviour, and to understand how these waves may be calculated. The purpose of this chapter is therefore to understand the behaviour of free-surface waves that form on linearized two-dimensional potential flow over a small obstacle, with the nonlinear problem to follow in Chapter 4. For simplicity, we restrict our attention to

flow over a submerged semi-infinite step, although we shall subsequently make some comments regarding the equivalent analysis for more general topographies.

As we intend to eventually consider fully nonlinear free-surface flow over a step, we will formulate the problem in such a fashion that the boundaries are fixed. This will involve finding a Lagrangian formulation for the system, where the initial particle labelling is chosen such that the upper and lower boundaries of the problem both follow known, fixed curves. This particle labelling will be determined by solving an auxiliary steady fluid-flow problem, obtained by taking the initial flow configuration, setting the upper boundary to be rigid, and determining the steady flow behaviour in terms of the fluid potential using complex variable methods. Each point in the initial flow state from the unsteady problem corresponds to an equivalent point in the auxiliary problem with an associated complex potential. This complex potential value will be used as the particle label in the Lagrangian flow formulation, consequently setting the upper and lower boundaries of the flow to follow known, fixed curves as required.

Lagrangian methods have been applied to free-surface flow problems in previous investigations. John [58] found exact solutions to several problems by considering analytic functions of a general Lagrange labelling parameter. This analysis involved beginning with a prescribed analytic function and solving an inverse problem to determine the flow regime, although it was possible to prescribe some boundary conditions in some cases. Longuet-Higgins applied similar techniques to self-similar free-surface flows [72], and as the special case of a more general representation of free-surface flow in terms of a complex variable [73].

Pierson [91] and Buldakov *et al.* [16] both formulated two-dimensional free surface flows in terms of a Lagrangian variable, using the resultant expressions to obtain the behaviour of free-surface waves in the asymptotic limit that the surface variation is small. Both of these investigations considered the problem of waves travelling in deep water. Lagrangian formulations have also been applied to develop numerical methods for determining free-surface flow behaviour by following individual flow particles, such as in [103].

Upon linearizing the resultant system about the solution for an undisturbed channel, we will find the wave behaviour on the free surface using two separate methods. The first is to observe that, when expressed in Eulerian coordinates, the problem will now have a fixed upper boundary. Consequently, Fourier transforms will be used to solve the system. The second method is the late-order terms technique discussed in Chapter 1 and demonstrated in Chapter 2, found in [22, 87].

As this problem is a partial differential equation; however, we will find that Stokes lines caused by the interaction between subdominant free-surface contributions play an important role in determining the free-surface behaviour of the problem. Following [24], we will observe that

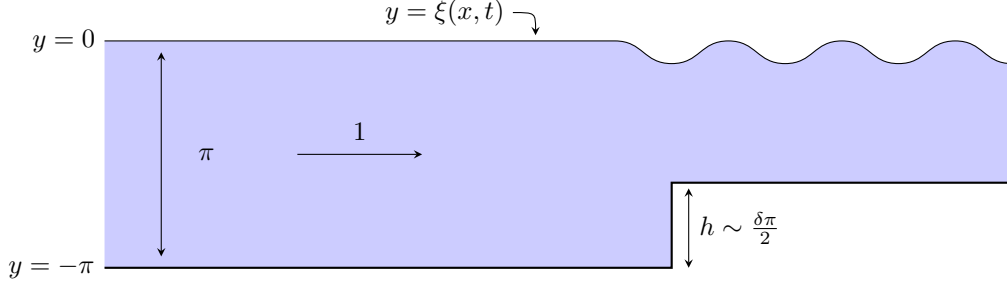


Figure 3.1: Prescribed fluid configuration for flow over a small step. The lower boundary is rigid, while the upper boundary is free, and governed by Bernoulli's equation.

these Stokes lines are caused by the Stokes switching of a second, subdominant asymptotic series expansion in the problem. We will determine the form of the late-order terms of this second series expansion, and hence obtain the free-surface wave behaviour caused by this Stokes switching. The combined effect of the Stokes switching associated with the leading-order expression and the Stokes switching between exponentially subdominant terms will form a complete expression for the behaviour of the free-surface waves on unsteady flow over a small, semi-infinite step.

3.2 Formulation

3.2.1 Formulation of the Eulerian problem

We consider a time-varying analogue of the specific case of irrotational, incompressible and inviscid two-dimensional free-surface flow over a step presented in [27]. The illustrated configuration has an upstream velocity given by U and a depth of L , with the upstream free-surface located at $y = 0$. A step of height h is located on the base of the flow at $x = 0$. The position of the free-surface is given by $\xi(x, t)$. To nondimensionalize the problem, we will scale velocities by U and lengths by L/π , with time and velocity potential scaled appropriately. The resultant dimensionless system is shown in Figure 3.1

We will be considering the small Froude number behaviour of the system, and hence define the small variable $\epsilon = \pi F^2$, where F is the Froude number as previously defined in (1.2).

In Section 2.3, we observed that the behaviour of two-dimensional steady flow could be described in terms of a fluid potential $\phi(x, y)$, provided that the flow was irrotational and incompressible. By allowing ϕ to vary in time, the velocity potential of a time-varying flow may similarly be defined, giving the governing expression for the flow as

$$\nabla^2 \phi = 0, \tag{3.1}$$

defined within the physical flow region. On the moving free surface, located at $y = \xi(x, t)$, the

kinematic boundary condition is given by

$$\phi_y = \phi_x \xi_x + \xi_t, \quad y = \xi(x, t). \quad (3.2)$$

On the fixed lower boundary, the flow passes over a submerged step, with an upstream depth of π and step height of h . Here, the kinematic boundary condition is given by

$$\frac{\partial \phi}{\partial \hat{n}} = 0,$$

where $\hat{n}$ is the unit vector normal to the base of the flow.

Applying the Bernoulli condition to obtain a dynamic boundary condition on the free surface gives

$$\epsilon[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) - \frac{1}{2}] + y = 0, \quad y = \xi(x, t), \quad (3.3)$$

where, as in Section 2.3, $\epsilon = \pi F^2$. The upstream flow conditions are given by

$$\lim_{x \rightarrow -\infty} \xi(x, t) = 0, \quad \lim_{x \rightarrow -\infty} \phi_x(x, y, t) = 0. \quad (3.4)$$

Finally, the initial conditions to the flow are defined to be

$$\xi(x, 0) = \xi_0(x), \quad \phi(x, y, 0) = \phi_0(x, y). \quad (3.5)$$

These initial conditions must be compatible with the upstream conditions in (3.4), and $\phi_0(x, y)$ must satisfy Laplace's equation. We will prescribe the flow at $t = 0$ to be given by the leading-order steady-state solution to the problem (that is, the solution for $\epsilon = 0$). Strictly speaking, we could define this by prescribing ξ and ξ_t at $t = 0$, and solving Laplace's equation would then give ϕ_0 . However, for simplicity, we instead specify the fluid potential, noting that it must satisfy Laplace's equation on the flow domain in order to be a valid initial condition.

The advantage to this choice of initial condition is that the leading-order behaviour as $\epsilon \rightarrow 0$ is steady, and may therefore easily be computed using complex variable methods. We therefore denote the fluid potential associated with the appropriate steady-state problem as $w_0 = \phi_0 + i\psi_0$. Following an identical analysis to that presented in [27, 63], we find that the leading-order behaviour in the limit as $\epsilon \rightarrow 0$ is obtained by solving

$$\frac{dw_0}{dz} = \left(\frac{e^{-w_0} + 1 + \delta}{e^{-w_0} + 1} \right)^{1/2}, \quad (3.6)$$

where δ is a parameter measuring the height of the step in the mapped plane, and is defined in this fashion in order to be consistent with [27]. Importantly, we note that this is distinct from the Dirac delta function, which will always be denoted as $\delta(x)$ in subsequent analysis to avoid confusion. The leading-order step height h is given by

$$h \sim \pi \left(1 - (1 + \delta)^{-1/2} \right), \quad \text{as } \epsilon \rightarrow 0. \quad (3.7)$$

Noting that δ is also small, this gives $h \sim \delta\pi/2$. Finally, we also note that the free surface is flat to leading order as $\epsilon \rightarrow 0$.

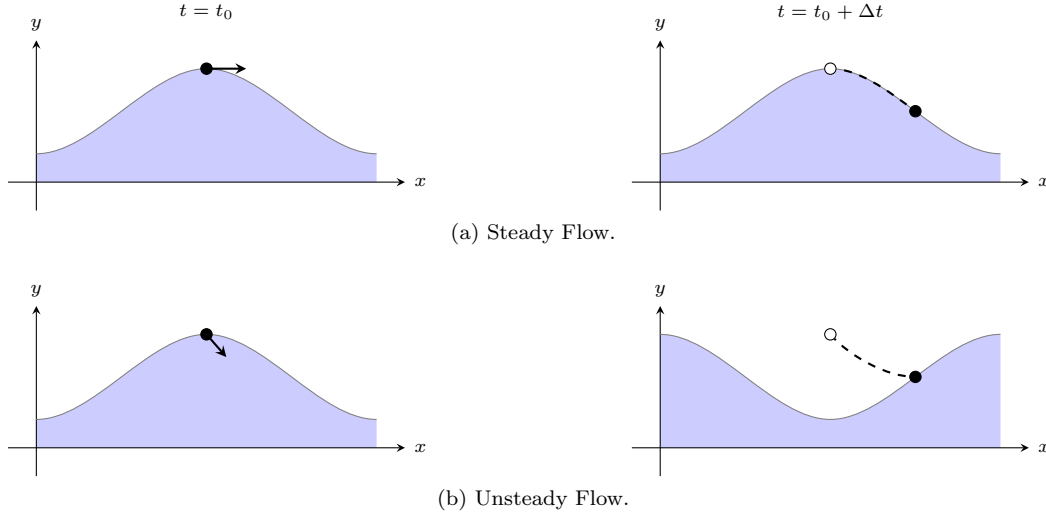


Figure 3.2: Position of an individual fluid particle on the boundary of (a) steady flow, and (b) unsteady flow at $t = t_0$ and $t = t_0 + \Delta t$. The fluid particle velocity at each point is denoted by an arrow, while the particle path in the intervening time segment is shown as a dashed curve. In the steady case, the fluid velocity component is parallel to the boundary, while in the unsteady case it is not.

3.2.2 Formulation of the Lagrangian problem

In Section 2.3, we defined the complex fluid potential $w = \phi + i\psi$ for steady flow due to a submerged source, and prescribed the boundary of the flow in terms of streamlines. This formulation fixed the position of the boundaries in complex w -space, and permitted complex variable methods to be applied to the problem. This is reasonable in the steady case, as the fluid particle paths follow streamlines, hence fixing the position of the free boundary. However, this property no longer holds in the unsteady case, in which the fluid particle paths are instead only instantaneously tangent to streamlines, as noted in [73] and more generally. Hence, the free surface is not a streamline. Figure 3.2 contains an illustration of the difference between the steady and unsteady cases.

Although we are unable to directly formulate the problem using the complex potential of the flow, we wish to retain some of the useful properties of such a representation, particularly the important property that both the rigid and free boundaries follow fixed curves. Observing that fluid particles on the boundaries of the flow will always stay on the boundaries, we instead consider a Lagrangian formulation for the flow. This formulation expresses each fluid particle in terms of a particle label and the amount of time that has elapsed, given by τ .

The utility of this formulation depends on the choice of particle label. We note that [16, 91] obtained equivalent fluid equations in Lagrangian form for a general particle labelling scheme; however, we will rederive these equations with a particular labelling in mind. A common choice for these coordinates is the initial position of the fluid particle, subsequently denoted here as

(a, b) . A schematic of this formulation is shown in Figure 3.3 (a). While this choice will fix the free boundary in the mapped plane, it is no longer possible to pose conditions on the rigid lower boundary without first knowing the velocity of the flow on the boundary itself. Given that the velocity of the flow is not known *a priori*, we find that the lower boundary of this problem is effectively not fixed at all, and we have instead simply moved the free boundary from the surface to the base of the flow. Clearly, an alternative particle labelling is desirable.

To determine appropriate particle labels, we instead pose an auxiliary problem. We consider the problem given by steady flow over a step with a rigid upper boundary, fixed at $y = 0$, in effect solving the steady equivalent of the initial condition with a rigid boundary in place of the free surface. As the problem is steady, a complex potential $w = \phi + i\psi$ may be defined on the flow region, with $\psi = 0$ defining the free surface, and $\psi = -\pi$ following the lower boundary of the problem.

Noting that each value of (a, b) in the initial state of the free-surface flow problem maps to a unique value of (ϕ, ψ) in this steady auxiliary problem, we label each point in the original problem in terms of the equivalent complex potential (with the particle-labelling subsequently denoted as (p, q) , to avoid confusion with standard complex potential representations for fluid flow). The free surface follows the curve $q = 0$, while the lower boundary also follows a fixed curve, given by $q = -\pi$. This formulation is illustrated in Figure 3.3 (b). Additionally, we denote $w = p + iq$ for convenience, but note that w is not to be confused with the complex potential of the leading-order solution, denoted by w_0 and seen in (3.6).

The equations for fluid flow in Lagrangian coordinates for potential flow are given by Lamb [69] as

$$\begin{aligned}\frac{\partial(x, y)}{\partial(a, b)} &= 1, \\ \frac{\partial(x_\tau, x)}{\partial(a, b)} + \frac{\partial(y_\tau, y)}{\partial(a, b)} &= \omega,\end{aligned}$$

where ω is the vorticity, and is hence equal to zero in this case. The first of these equations represents the incompressibility condition on the flow, while the second represents the irrotationality condition. Applying the mapping from (a, b) to (p, q) gives

$$\frac{\partial(x, y)}{\partial(p, q)} = \frac{\partial(a, b)}{\partial(p, q)}, \quad (3.8)$$

$$\frac{\partial(x_\tau, x)}{\partial(p, q)} + \frac{\partial(y_\tau, y)}{\partial(p, q)} = 0. \quad (3.9)$$

Lamb [69] also finds on the free surface that the dynamic condition takes the form

$$\frac{\partial P}{\partial a} + \frac{y_a}{\epsilon} = -x_{\tau\tau}x_a - y_{\tau\tau}y_a,$$

where P represents the pressure at a particular point on the surface, and again $\epsilon = \pi F^2$.

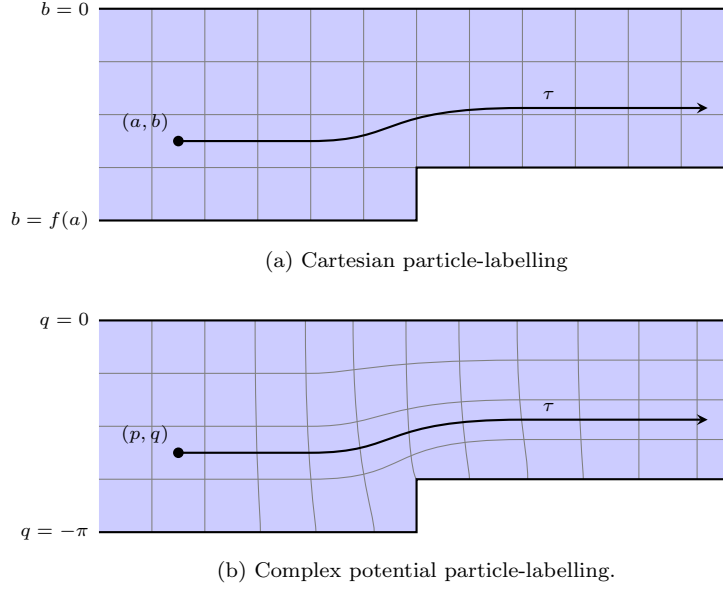


Figure 3.3: Schematic of Lagrangian coordinates using (a) initial position in Cartesian coordinates and (b) complex potential of initial position in the steady auxiliary problem as the particle labelling scheme. In the second scheme, the lower boundary occurs along constant q . Furthermore, analysis shows that the behaviour of the flow on the lower boundary in (a) depend upon how much time has elapsed, which is not the case for (b).

In the case of a free surface with P constant, this gives

$$\epsilon [x_{\tau\tau}x_a + y_{\tau\tau}y_a] + y_a = 0 \quad \text{on} \quad b = 0.$$

Applying the chain rule to obtain the boundary condition in terms of (p, q) gives

$$\epsilon [x_{\tau\tau}(x_p p_a + x_q q_a) + y_{\tau\tau}(y_p p_a + y_q q_a)] + (y_p p_a + y_q q_a) = 0, \quad (3.10)$$

which for the prescribed flat initial conditions becomes

$$\epsilon [x_{\tau\tau}x_p + y_{\tau\tau}y_p] + y_p = 0, \quad \text{on} \quad q = 0. \quad (3.11)$$

The kinematic condition is satisfied automatically on the free surface as a consequence of the mapping. On the lower boundary, the kinematic boundary condition requires that there is no flow through $q = -\pi$, and is hence given as

$$x_p y_\tau - x_\tau y_p = 0, \quad \text{on} \quad q = -\pi. \quad (3.12)$$

Information about the structure of the step appears in the problem as a consequence of the mapping between (a, b) and (p, q) . At this point, we note that equivalent equations to (3.8)–(3.9) and (3.11) may be found in [16, 69, 91] in terms of a general Lagrangian particle-labelling scheme, with the mapping between (a, b) and (p, q) left unspecified.

To evaluate the right-hand side of (3.8), we map from (a, b) to (p, q) using a Schwarz-Christoffel mapping. This mapping may be constructed by first mapping the physical flow-path seen in

Figure 3.3 (a) to a half-plane, and subsequently mapping this half-plane to a strip in the (p, q) plane, giving

$$a + ib = \int^{\exp(-w)} \left(\frac{s+1+\delta}{s+1} \right)^{1/2} ds.$$

The Jacobian of this transformation is given by

$$\frac{\partial(a, b)}{\partial(p, q)} = \left(\frac{e^{-2p} + 2(1+\delta)e^{-p} \cos q + (1+\delta)^2}{e^{-2p} + 2e^{-p} \cos q + 1} \right)^{1/2}.$$

Consequently, the system given in (3.8)–(3.9) becomes

$$x_p y_q - x_q y_p = \left(\frac{e^{-2p} + 2(1+\delta)e^{-p} \cos q + (1+\delta)^2}{e^{-2p} + 2e^{-p} \cos q + 1} \right)^{1/2}, \quad (3.13)$$

$$x_{\tau q} x_p - x_{\tau p} x_q + y_{\tau q} y_p - y_{\tau p} y_q = 0, \quad (3.14)$$

3.2.3 Linearizing the Lagrangian problem

The fully nonlinear problem defined by (3.11)–(3.14) with appropriate initial conditions will be considered in Chapter 4. However we first investigate the case where the step is small, even compared to the Froude number, F . Hence, we specify that $0 < \delta \ll \epsilon \ll 1$. Linearizing the problem in δ around the solution for unperturbed flow in a channel gives the appropriate coordinates as

$$x = p + \tau + \delta \bar{x}, \quad y = q + \delta \bar{y},$$

and retaining leading-order terms in δ gives the linearized problem

$$\bar{x}_p + \bar{y}_q = -\frac{1 + e^{-p} \cos(q)}{1 + 2e^{-p} \cos(q) + e^{-2p}}, \quad (3.15)$$

$$\bar{x}_{\tau q} - \bar{y}_{\tau p} = 0, \quad (3.16)$$

with the dynamic condition on the free boundary given as

$$\epsilon \bar{x}_{\tau\tau} + \bar{y}_p = 0 \quad \text{on } q = 0. \quad (3.17)$$

The kinematic boundary condition requires that

$$\bar{y}_\tau - \bar{y}_p = 0 \quad \text{on } q = -\pi. \quad (3.18)$$

It is important for subsequent analysis to have an expression for the leading-order behaviour of the flow. We define a series expression for $z = x + iy$ such that

$$z \sim z^{(0)} + \epsilon z^{(1)} + \dots \quad \text{as } \epsilon \rightarrow 0. \quad (3.19)$$

As the initial conditions were chosen to be the leading-order steady state, we know that the leading-order solution will simply be the initial condition itself, obtained in (3.6). Hence, this gives

$$\frac{dw_0}{dz^{(0)}} = \left(\frac{e^{-w_0} + 1 + \delta}{e^{-w_0} + 1} \right)^{1/2} = f(w_0). \quad (3.20)$$

Inverting this expression and linearizing for small δ gives

$$\frac{dz^{(0)}}{dw_0} \sim 1 - \frac{\delta}{2} \frac{1}{1 + e^{-w_0}} \quad \text{as } \delta \rightarrow 0,$$

which becomes upon integration

$$z^{(0)} \sim \phi_0 + i\psi_0 - \frac{\delta}{2} \log(1 + e^{\phi_0 + i\psi_0}). \quad (3.21)$$

Now, as the leading-order expression is steady, we know that particle paths will follow streamlines, and hence that $\psi_0 = q$. The chain rule therefore gives

$$\frac{d\phi_0}{d\tau} = \frac{dw_0}{dz^{(0)}} \frac{dz^{(0)}}{d\tau}.$$

The definition of the complex velocity is such that $dw_0/dz^{(0)} = u - iv$, where u and v are the components of the fluid velocity. Additionally, Lagrangian flow is defined in terms of the fluid velocity such that $dz^{(0)}/d\tau = u + iv$ [69]. Consequently, we find that

$$\frac{d\phi_0}{d\tau} = f(w_0)f^*(w_0), \quad (3.22)$$

where the star denotes complex conjugation. Linearizing the resultant expression in δ and integrating, noting that ϕ at $\tau = 0$ is given by p , gives

$$\tau \sim (\phi_0 - p) - \frac{\delta}{2} \log \left(\frac{1 + 2e^{\phi_0} \cos q + e^{2\phi_0}}{1 + 2e^p \cos q + e^{2p}} \right) \quad \text{as } \delta \rightarrow 0. \quad (3.23)$$

We see that, to leading order, $\phi_0 \sim p + \tau$. Applying this leading-order expression back into (3.23) gives

$$\phi_0 \sim p + \tau + \frac{\delta}{2} \log \left(\frac{1 + 2e^{p+\tau} \cos q + e^{2(p+\tau)}}{1 + 2e^p \cos q + e^{2p}} \right).$$

Using this expression in (3.21) gives

$$z^{(0)} \sim [p + \tau + iq] - \frac{\delta}{2} \log \left(\frac{1 + 2e^p \cos q + e^{2p}}{1 + e^{p+\tau-iq}} \right).$$

Defining $z = p + \tau + iq + \delta \bar{z}$, we find that the leading-order solution to the linearized problem is given by

$$\bar{z}^{(0)} = -\frac{1}{2} \log \left(\frac{1 + 2e^p \cos q + e^{2p}}{1 + e^{p+\tau-iq}} \right). \quad (3.24)$$

Importantly, we note that (3.24) is not only the leading-order behaviour, but also the initial condition for the flow. Hence, we may simply state that

$$\bar{z} = -\frac{1}{2} \log \left(\frac{1 + 2e^p \cos q + e^{2p}}{1 + e^{p+\tau-iq}} \right) \quad \text{at } \tau = 0. \quad (3.25)$$

We may now integrate the governing equation (3.16) with respect to τ and use this initial condition to evaluate the constant of integration. Hence this governing equation may be replaced by

$$\bar{x}_q - \bar{y}_p = \frac{e^{-p} \sin(q)}{1 + 2e^{-p} \cos(q) + e^{-2p}}. \quad (3.26)$$

Hence equations (3.15), (3.17)–(3.18), (3.25)–(3.26) fully describe the linearized problem in terms of Lagrangian coordinates, with appropriate initial and boundary conditions. Additionally, the leading-order behaviour in terms of ϵ is given by (3.24). In the steady problem [27], a radiation condition is imposed to determine which region of the free surface contains waves. In the unsteady problem, we instead state that the flow is waveless in the far field.

Late-order term analysis may now be applied to this system, in order to determine the behaviour of the free surface on the flow. Linearizing the problem allows for a second approach to be applied. The Eulerian form of the problem, given in (3.1)–(3.6), is made more difficult to solve by virtue of the presence of a free boundary. However, in the linearized case, this free boundary may be considered to lie along the line $y = 0$. Hence, the solution is defined in a fixed domain, and may be determined using Fourier analysis.

Consequently, we will solve the problem of linearized unsteady flow over a step in Eulerian coordinates using Fourier transforms, and subsequently in Lagrangian coordinates using late-order term methods. Comparing these results will show that the two methods produce equivalent results.

3.3 Method of steepest descents

3.3.1 Formulation of the integral expressions

We linearize the full Eulerian problem, given in (3.1)–(3.6), about the leading-order solution as $\delta \rightarrow 0$. This leading order solutions is simply undisturbed two-dimensional flow through a channel, and we hence define $\bar{\phi}(x, y, t)$ and $\bar{\xi}(x, t)$ such that

$$\phi = x + \delta \bar{\phi}, \quad \xi = \delta \bar{\xi}. \quad (3.27)$$

Linearizing (3.6) in this limit gives (after first determining the leading order expression in δ and applying it to the problem in an iterative fashion)

$$\frac{dw_0}{dz} \sim 1 + \frac{\delta}{2(1 + e^{-z})} + \mathcal{O}(\delta^2),$$

which indicates that

$$w_0 \sim z + \frac{\delta}{2} \log(1 + e^z) + \mathcal{O}(\delta^2),$$

and hence that the initial condition may be written as $w_0 = z + \delta \bar{w}_0$, where

$$\bar{w}_0 \sim \frac{1}{2} \log(1 + e^z) + \mathcal{O}(\delta). \quad (3.28)$$

The lower boundary may, when the problem is linearized by the step-height, be represented by the curve

$$y = -\pi + \frac{\delta\pi}{2} H(x),$$

where the expression for the step-height is given in (3.7), valid as $\epsilon \rightarrow 0$, and $H(x)$ represents the unit step function. This allows for a kinematic boundary condition to be explicitly represented along the curve $y = -\pi$ for the linearized problem. We note that this curve is a discontinuous distribution, and that the derivative will be a Dirac delta function, however this does not lead to any difficulties in the linearized problem.

Hence, we obtain the full linearized system

$$\bar{\phi}_{xx} + \bar{\phi}_{yy} = 0, \quad -\pi < y < 0, \quad (3.29)$$

$$\bar{\phi}_y = \frac{\pi}{2} \delta(x), \quad y = -\pi, \quad (3.30)$$

$$\bar{\phi}_y = \bar{\xi}_x + \bar{\xi}_t, \quad y = 0, \quad (3.31)$$

$$\epsilon(\bar{\phi}_x + \bar{\phi}_t) + \bar{\xi} = 0, \quad y = 0, \quad (3.32)$$

The upstream and initial conditions become

$$\lim_{x \rightarrow -\infty} \bar{\xi} = 0, \quad \lim_{x \rightarrow -\infty} \bar{\phi}_x = 0, \quad \bar{\phi}(x, y, 0) = \bar{\phi}_0(x, y), \quad \bar{\xi}(x, 0) = 0. \quad (3.33)$$

We define the Fourier transform such that

$$\hat{f}(k) = \int_{-\infty}^{\infty} \bar{f}(x) e^{-ikx} dx, \quad \bar{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk.$$

Taking the Fourier transform of the linearized equations (3.29)–(3.33) gives

$$\hat{\phi}_{yy} - k^2 \hat{\phi} = 0, \quad -\pi < y < 0, \quad (3.34)$$

$$\hat{\phi}_y = \frac{\pi}{2}, \quad y = -\pi, \quad (3.35)$$

$$\hat{\phi}_y = \hat{\xi}_t + ik\hat{\xi}, \quad y = 0, \quad (3.36)$$

$$\epsilon(\hat{\phi}_t + ik\hat{\phi}) + \hat{\xi} = 0, \quad y = 0. \quad (3.37)$$

The initial conditions become

$$\hat{\phi}(k, y, 0) = \hat{\phi}_0(k, y), \quad \hat{\xi}(k, 0) = 0, \quad (3.38)$$

where $\hat{\phi}_0(k, y)$ is given by the Fourier transform of the initial condition given in (3.28). The upstream conditions given in (3.33) will be applied during the Fourier inversion process, and will therefore remain unchanged.

Solving (3.34) in conjunction with the boundary condition (3.35) gives

$$\hat{\phi} = \frac{\pi}{2k} \sinh(k(y + \pi)) + B(k, t) \cosh(k(y + \pi)).$$

Applying this expression to (3.36)–(3.37) gives the system

$$\hat{\xi}_t = -ik\hat{\xi} + kB \sinh(\pi k) + \frac{\pi}{2} \cosh(\pi k), \quad (3.39)$$

$$B_t = -\frac{\hat{\xi}}{\epsilon \cosh(\pi k)} - ikB - \frac{i\pi}{2} \frac{\sinh(\pi k)}{\cosh(\pi k)}. \quad (3.40)$$

This expression may be treated as a system of ordinary differential equations, with initial conditions obtained from (3.38) given by

$$\hat{\xi}(k, 0) = 0, \quad B(k, 0) = \frac{\hat{\phi}_0(k, 0) - \pi \sinh(\pi k)/2k}{\cosh(\pi k)}. \quad (3.41)$$

Solving this system using variation of parameters gives $\hat{\xi}$ as

$$\hat{\xi} = \hat{\xi}_H + \hat{\xi}_{IC},$$

where

$$\begin{aligned} \hat{\xi}_H = \frac{i\epsilon\pi}{2} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} & \left[\frac{1}{2} \left(1 + \sqrt{\frac{\sinh(\pi k)}{\epsilon k \cosh(\pi k)}} \right) \exp \left(-ikt + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) \right. \\ & \left. + \frac{1}{2} \left(1 - \sqrt{\frac{\sinh(\pi k)}{\epsilon k \cosh(\pi k)}} \right) \exp \left(-ikt - it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) - 1 \right], \end{aligned}$$

and

$$\hat{\xi}_{IC} = -\frac{i\hat{\phi}_0(k, 0)}{2} \sqrt{\frac{\epsilon k \sinh \pi k}{\cosh \pi k}} \left[\exp \left(it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) - \exp \left(-it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) \right] e^{-ikt}. \quad (3.42)$$

Taking the inverse Fourier transform of $\hat{\xi}_H$ gives

$$\begin{aligned} \bar{\xi}_H = \frac{i\epsilon}{8} \int_{-\infty}^{\infty} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} & \left(1 + \sqrt{\frac{\sinh(\pi k)}{\epsilon k \cosh(\pi k)}} \right) \exp \left(ik(x - t) + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ & + \frac{i\epsilon}{8} \int_{-\infty}^{\infty} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} \left(1 - \sqrt{\frac{\sinh(\pi k)}{\epsilon k \cosh(\pi k)}} \right) \exp \left(ik(x - t) - it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ & - \frac{i\epsilon}{4} \int_{-\infty}^{\infty} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} e^{ikx} dk. \end{aligned} \quad (3.43)$$

To invert $\hat{\xi}_{IC}$, we require the Fourier transform of $\phi_0(k, 0)$. In Appendix B, we find that

$$\hat{\phi}_0(k, 0) = -\frac{\pi}{2k \sinh(\pi k)}.$$

Hence, applying taking the inverse Fourier transform of (3.42) gives

$$\bar{\xi}_{IC} = \frac{i}{8} \int_{-\infty}^{\infty} \sqrt{\frac{\epsilon}{k \cosh \pi k \sinh \pi k}} \left[\exp \left(it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) - \exp \left(-it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) \right] e^{-ikx} dk. \quad (3.44)$$

We now split the range of integration into two outer integrals and one inner integral. The range of the inner integral is given by $-\beta/\epsilon < k < \beta/\epsilon$, where $0 < \epsilon \ll \beta \ll 1$. The two outer integrals have ranges given by $-\infty < k < -\beta/\epsilon$ and $\beta/\epsilon < k < \infty$ respectively.

We now determine the behaviour of the outer integrals (3.43)–(3.44) using the method of steepest descents, setting $p = \epsilon k$. For $p > 0$ in the outer integral range, we find that

$$\frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} \sim \frac{2}{p-1} e^{-\pi p/\epsilon}, \quad \frac{\sinh(\pi k)}{\cosh(\pi k)} \sim 1,$$

whereas when $p < 0$,

$$\frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} \sim \frac{2}{p+1} e^{\pi p/\epsilon}, \quad \frac{\sinh(\pi k)}{\cosh(\pi k)} \sim -1.$$

Consequently, the Fourier inversion integral for ξ_H (3.43) over the outer integral range becomes

$$\bar{\xi}_H \sim I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where the outer integrals ($I_1 - I_6$) are given by

$$\begin{aligned} I_1 &= \frac{i}{4} \int_{\beta}^{\infty} \frac{1}{p-1} \left(1 + \frac{1}{\sqrt{p}}\right) \exp\left(\frac{1}{\epsilon} [ip\sigma_1 + it\sqrt{p}]\right) dp, \\ I_2 &= \frac{i}{4} \int_{\beta}^{\infty} \frac{1}{p-1} \left(1 - \frac{1}{\sqrt{p}}\right) \exp\left(\frac{1}{\epsilon} [ip\sigma_1 - it\sqrt{p}]\right) dp, \\ I_3 &= -\frac{i}{2} \int_{\beta}^{\infty} \frac{1}{p-1} \exp\left(\frac{1}{\epsilon} [ip(x + i\pi)]\right) dp, \\ I_4 &= \frac{i}{4} \int_{-\infty}^{-\beta} \frac{1}{p+1} \left(1 + \frac{i}{\sqrt{p}}\right) \exp\left(\frac{1}{\epsilon} [ip\sigma_2 - t\sqrt{p}]\right) dp, \\ I_5 &= \frac{i}{4} \int_{-\infty}^{-\beta} \frac{1}{p+1} \left(1 - \frac{i}{\sqrt{p}}\right) \exp\left(\frac{1}{\epsilon} [ip\sigma_2 + t\sqrt{p}]\right) dp, \\ I_6 &= -\frac{i}{2} \int_{-\infty}^{-\beta} \frac{1}{p+1} \exp\left(\frac{1}{\epsilon} [ip(x - i\pi)]\right) dp, \end{aligned}$$

with

$$\sigma_1 = x - t + i\pi, \quad \sigma_2 = x - t - i\pi.$$

The Fourier inversion integral for (3.44) over the outer integral range subsequently takes the form

$$\bar{\xi}_{IC} \sim I_7 + I_8 + I_9 + I_{10},$$

where

$$\begin{aligned} I_7 &= \frac{i}{4} \int_{\beta}^{\infty} \frac{1}{\sqrt{p}} \exp\left(\frac{1}{\epsilon} [ip\sigma_1 + it\sqrt{p}]\right) dp, \\ I_8 &= -\frac{i}{4} \int_{\beta}^{\infty} \frac{1}{\sqrt{p}} \exp\left(\frac{1}{\epsilon} [ip\sigma_1 - it\sqrt{p}]\right) dp, \\ I_9 &= -\frac{i}{4} \int_{-\infty}^{-\beta} \frac{i}{\sqrt{p}} \exp\left(\frac{1}{\epsilon} [ip\sigma_2 - t\sqrt{p}]\right) dp, \\ I_{10} &= \frac{i}{4} \int_{-\infty}^{-\beta} \frac{i}{\sqrt{p}} \exp\left(\frac{1}{\epsilon} [ip\sigma_2 + t\sqrt{p}]\right) dp, \end{aligned}$$

Noting that $0 < \epsilon \ll \beta \ll 1$, we may instead evaluate each of the integrals $I_1 - I_{10}$ over the ranges $0 < p < \infty$ and $-\infty < p < 0$ as appropriate.

We note, however, that determining the end-point contribution of these integrals near $p = 0$ requires considering the behaviour of the inner integral, and will therefore initially be obtained by considering the appropriate original integral expressions in (3.43) and (3.44).

3.3.2 Determining the integration contours

We first consider the integrals $I_1 - I_6$. In order to evaluate these integral, we must specify the location of the branch cuts for the square roots in I_1 , I_2 , I_4 and I_5 . Taking the branch cuts in I_1 and I_2 in opposite directions ensures that the sum $I_1 + I_2$ is purely real. Consequently, we choose the branch cuts to be the negative imaginary p -axis and the positive imaginary p -axis respectively. Similarly, we choose the negative imaginary p -axis to be the branch cut for I_4 and the positive imaginary p -axis to be the branch cut for I_5 , in order to ensure that $I_4 + I_5$ takes purely real values.

The integral contours may subsequently be deformed according to the method of steepest descents. Applying this method will produce contributions to the integrals from the end-points, the saddle points and the poles. The contributions from the end-points will always be present in the final integral expression.

The integrals I_1 and I_2 contain a saddle point located at p_s , given by

$$p_s = \frac{1}{4} \frac{t^2}{\sigma_1^2}. \quad (3.45)$$

The integrals I_4 and I_5 also contain a saddle point, located at

$$p_s = -\frac{1}{4} \frac{t^2}{\sigma_2^2}.$$

Finally, integrals I_3 and I_6 do not contain any saddle points.

Integrals $I_1 - I_3$ contain poles at $p = 1$, while integrals $I_4 - I_6$ contain poles at $p = -1$. Subsequent analysis will show that the integral contributions caused by the singularities in I_3 or I_6 will produce waves that propagate either forwards or backwards from the obstruction, depending on which way the contours are indented around the singularities when $x = 0$. At this point, we impose the upstream boundary conditions given in (3.33) and determine that the poles will produce no contribution as $x \rightarrow -\infty$. Consequently, we indent the contours I_3 and I_6 so that they pass below the pole when $x = 0$. This configuration will ensure that the waves produced by these poles propagate in the positive x -direction, and therefore ensure that the upstream behaviour is waveless, satisfying the far-field wave condition.

Examples of the resultant integration contours calculated for $t \geq 0$ may be seen in Figure 3.4 and Figure 3.5. Figure 3.4 shows the change in the contours as the point $x = 0$ is crossed. The only contribution change at this point comes in I_3 and I_6 , in which the poles do not contribute to the integral expression for $x < 0$, but do contribute for $x > 0$.

Figure 3.5 illustrates the change in the behaviour of the integral contours as the point $x = t$ is crossed. At this point, the saddle points in I_1 and I_4 change quadrant in complex p -space, and

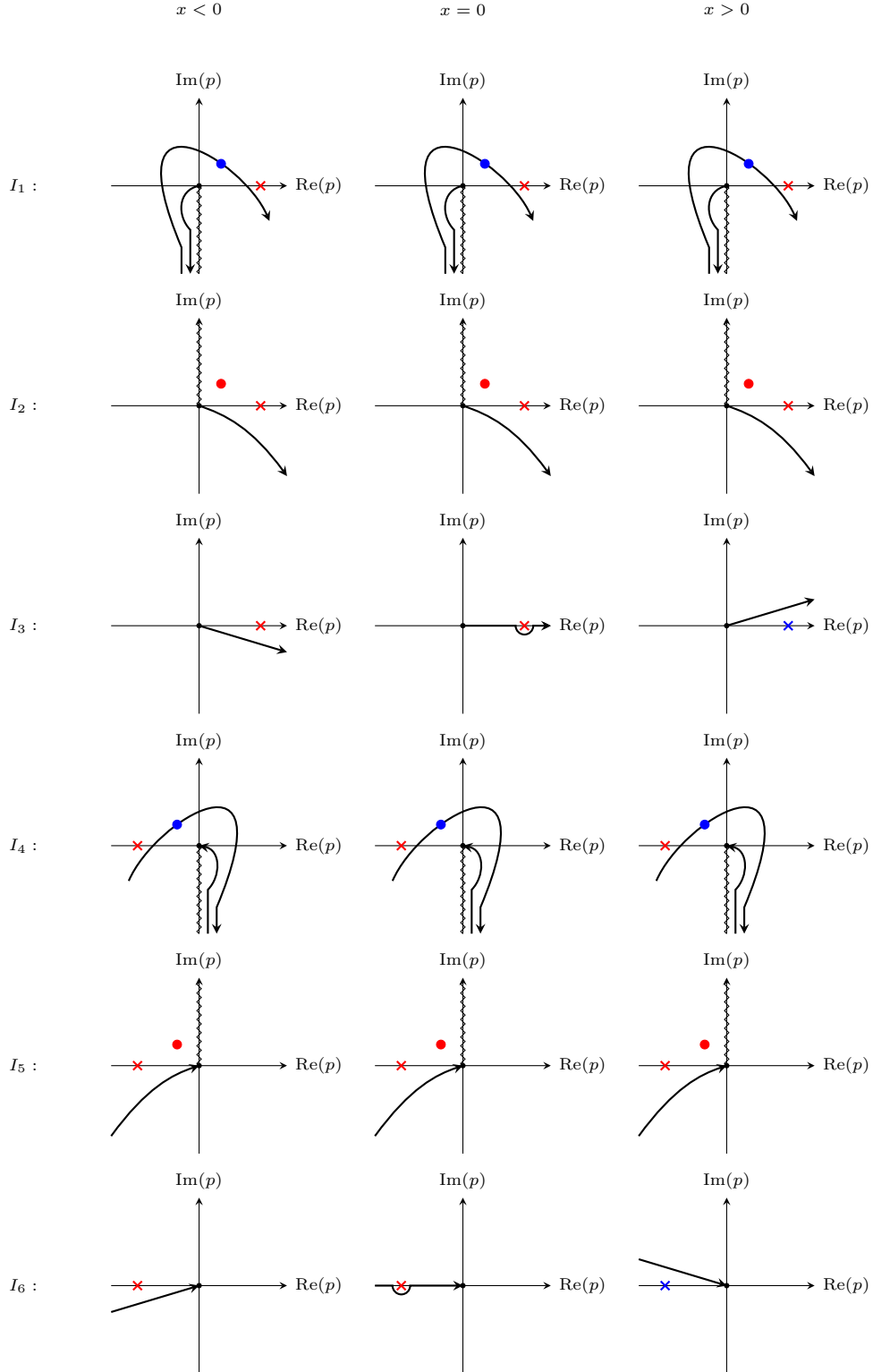


Figure 3.4: Steepest-descent contours for integrals $I_1 - I_6$ on either side of $x = 0$. Saddle points are indicated by blue circles if the contour passes through the point, and red circles the contour does not. Poles at $p = \pm 1$ are indicated by blue crosses if the pole contributes to the integral, and red crosses if the pole does not. Branch cuts are indicated by hatched lines. As $x = 0$ is crossed from negative to positive, the poles in I_3 and I_6 begin to contribute to the resultant integral expression.

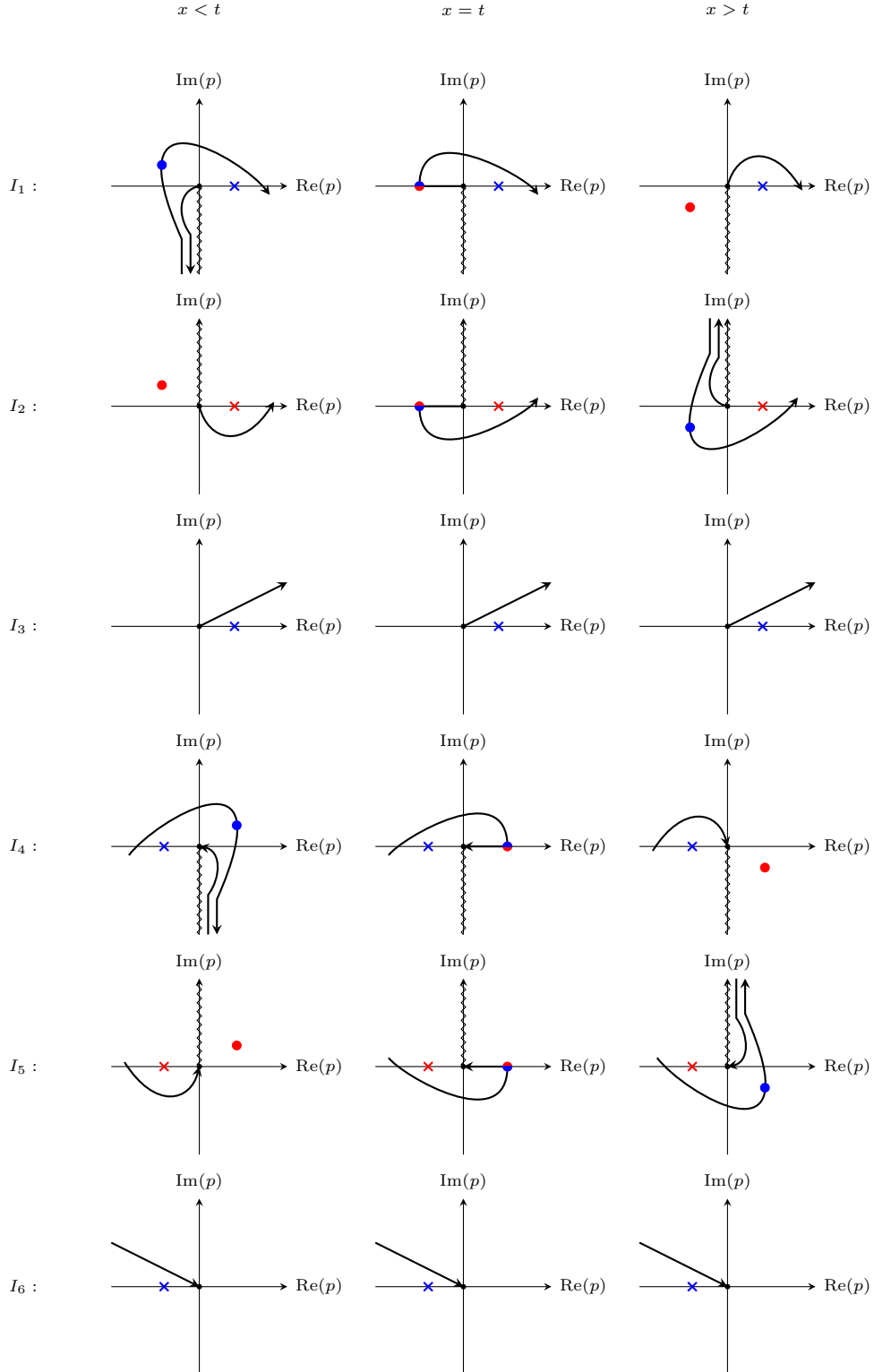


Figure 3.5: Steepest-descent contours for integrals I_1 - I_6 on either side of $x = t$. Saddle points are indicated by blue dots if the contour passes through the point, and red dots the contour does not. If the contour is switching between the two states, the saddle point is both red and blue. Poles at $p = \pm 1$ are indicated by blue crosses if the pole contributes to the integral, and red crosses if the pole does not. Branch cuts are indicated by hatched lines. As $x = 0$ is crossed from negative to positive, the poles in I_3 and I_6 begin to contribute to the solution.

no longer contribute to the integral expression. However, the saddle points in I_2 and I_5 also change quadrant so that they do contribute to the integral expression. Consequently, when $x = t$ is crossed, the saddle contributions from I_1 and I_4 are switched off, but the saddle contributions from I_2 and I_5 are switched on.

Importantly, we also see at some point within the range $0 < x < t$, the integration contours in I_1 and I_4 crossed the poles at $p = 1$ and $p = -1$ respectively. Hence there is some value of $x = x_s$ such that $0 < x_s < t$ where these poles switch from not contributing to the integral expression to producing a contribution. In order to determine the value of x_s for I_4 , we set

$$g(p) = ip\sigma_2 - t\sqrt{p}, \quad (3.46)$$

and note that the pole for this integral is located at $p = -1$. The pole is crossed when two conditions are satisfied; $\text{Im}(g(p_s)) = \text{Im}(g(p_p))$ and $\text{Re}(g(p_s)) < \text{Re}(g(p_p))$, where p_s and p_p are the locations of the saddle and pole respectively in complex p -space, and p_s is given in (3.45). Equation (3.46) gives

$$g(p_s) = \frac{i}{4} \frac{t^2}{\sigma_2}.$$

Hence, on the free surface (i.e. x and t real)

$$\text{Re}(g(p_s)) = -\frac{\pi t^2}{4[(x-t)^2 + \pi^2]}, \quad \text{Im}(g(p_s)) = \frac{t^2(x-t)}{4[(x-t)^2 + \pi^2]}.$$

Knowing that the pole is located at $p_p = -1$, (3.46) gives

$$\text{Re}(g(p_p)) = -\pi, \quad \text{Im}(g(p_p)) = -x.$$

The value of x at which the pole is crossed by the integral contour, denoted x_s , is therefore found by solving

$$\frac{t^2(x_s - t)}{4[(x_s - t)^2 + \pi^2]} = -x_s. \quad (3.47)$$

This equation has three solutions. The first is given by

$$x_1 = \frac{1}{6} \frac{t^2 - 12\pi^2 + (t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{2/3}}{(t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{1/3}} + \frac{2t}{3}.$$

The two remaining solutions are given by

$$\begin{aligned} x_{2,3} = & -\frac{1}{12} \frac{t^2 - 12\pi^2 + (t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{2/3}}{(t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{1/3}} \\ & \pm \frac{i\sqrt{3}}{12} \frac{t^2 - 12\pi^2 - (t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{2/3}}{(t^3 - 72t\pi^2 + 6\pi\sqrt{3(16\pi^4 + 44t^2\pi^2 - t^4)})^{1/3}} + \frac{2t}{3}. \end{aligned}$$

An analysis of these solutions shows that as $t \rightarrow \infty$,

$$\begin{aligned} x_1 & \sim t + \mathcal{O}(1/t), \\ x_2 & \sim t/2 + \pi + \mathcal{O}(1/t), \\ x_3 & \sim t/2 - \pi + \mathcal{O}(1/t). \end{aligned}$$

This behaviour is illustrated in Figure 3.6 (a). However, we also note that the contour from the saddle point will only cross the pole if the real part of $g(p_p)$ is less than the real part of $g(p_s)$. The curve along which these real parts are equal is given by

$$x = t \pm \frac{1}{2}\sqrt{t^2 - 4\pi^2},$$

and that Stokes switching may therefore only occur when

$$(x - t)^2 - \frac{1}{4}t^2 + \pi^2 > 0. \quad (3.48)$$

This curve is also illustrated in Figure 3.6 (a), with the region satisfying (3.48) shown as unshaded. From this figure, we see that switching will only occur on the smallest of the solutions at any given point in time. Consequently, region in which the pole contribution is active may be seen in Figure 3.6 (b). The region of the positive $x-t$ quadrant in which the pole contributions are active in I_1 , I_2 , I_4 and I_5 are unshaded. Subsequent analysis will show that these contributions precisely cancel the pole contributions from I_3 and I_6 , meaning that the unshaded region, denoted as R_1 , may be interpreted as the region in the positive $x-t$ quadrant in which waves are present on the free surface.

It is notable that these pole contributions are switched on by contours that pass through the saddle points, which are in turn switched on by the end-point contributions. This form of Stokes switching is described in [6] and discussed at length in [24, 56].

The integrals given in $I_7 - I_{10}$ follow identical contours to I_2 , I_3 , I_5 and I_6 respectively. Each of these integrals will contain an end-point contribution and a saddle-point contribution that has an identical Stokes structure to the corresponding previously-investigated integral. Importantly, these integrals do not have poles at $p = \pm 1$, and therefore will not generate any pole contributions in the final asymptotic expression of the solution.

3.3.3 Evaluating the integral contributions

3.3.3.1 End-point contributions

In order to determine the end-point contributions, we would have to scale the integration variable p by $1/\epsilon$, which is effectively undoing the transformation used to obtain $I_1 - I_{10}$. Hence, we instead return to the original integral expression given in (3.43). We begin by expanding the third integral in this expression as a series in ϵ and switching the order of summation and integration, giving

$$-\frac{i\epsilon}{4} \int_{-\infty}^{\infty} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} e^{ikx} dk \sim \frac{i}{4} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\epsilon^{n+1} k^n \cosh^n \pi k}{\sinh^{n+1} \pi k} e^{ikx} dk.$$

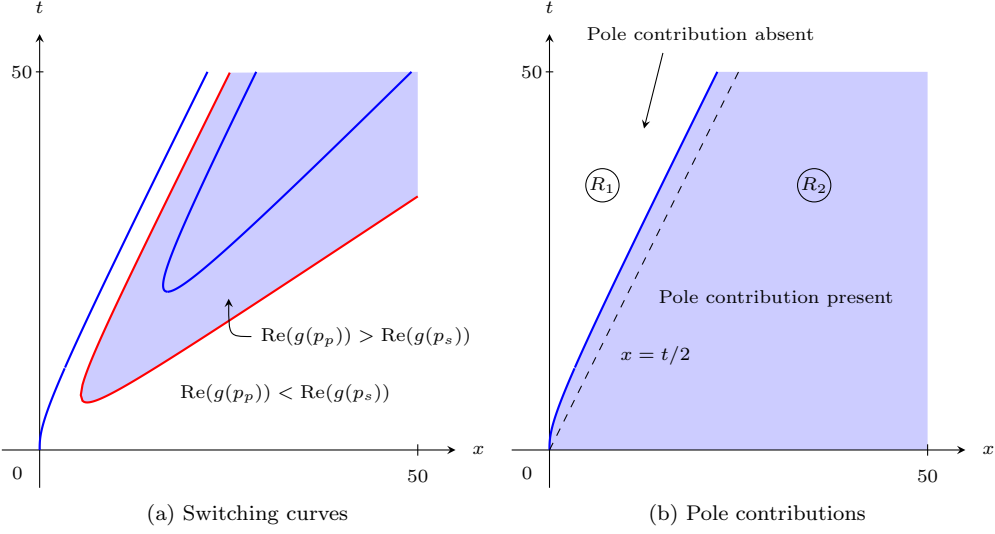


Figure 3.6: The curves in (a) illustrate the points at which the real and imaginary parts of $g(p_s)$ and $g(p_p)$ in I_1 , I_2 , I_5 and I_6 are equal. The blue curves indicate that the imaginary parts are equal, while the red curves indicate that the real parts are equal. The shaded region contains all points such that $\text{Re}(g(p_p)) > \text{Re}(g(p_s))$, hence indicating the region that the poles cannot be switched on by the saddle points, even when they have equal imaginary part. Pole contributions in these integrals are therefore switched on only as the far-left of the blue curves is crossed. The effect in the physical domain is illustrated in (b). Pole contributions are absent in the unshaded region R_1 , but switched on as the Stokes line, denoted by a blue curve, is crossed into the shaded region R_2 . Subsequent analysis will show that surface waves are present in the solution at points on the free surface that lie within R_1 .

This expression is an algebraic series in ϵ , which indicates that the exponentially small contributions come from elsewhere in the integral. The late-order terms in this series may be found by evaluating each integral in the series individually in the limit that $n \rightarrow \infty$. To perform this, we consider the $\mathcal{O}(\epsilon^{n+1})$ term, expanding it to give

$$\frac{i\epsilon^{n+1}}{4} \int_{-\infty}^{\infty} \frac{k^n \cosh^n \pi k}{\sinh^{n+1} \pi k} e^{ikx} dk = \frac{i\epsilon^{n+1}}{2} \int_{-\infty}^{\infty} \frac{k^n (e^{\pi k} + e^{-\pi k})^n}{(e^{\pi k} - e^{-\pi k})^{n+1}} e^{ikx} dk.$$

In the limit that $n \rightarrow \infty$, this may be divided into two integrals, giving

$$\frac{i\epsilon^{n+1}}{2} \int_{-\infty}^{\infty} \frac{k^n (e^{\pi k} + e^{-\pi k})^n}{(e^{\pi k} + e^{-\pi k})^{n+1}} e^{ikx} dk \sim -\frac{i\epsilon^{n+1}}{2} \int_{-\infty}^{-\eta} k^n e^{ik(x-i\pi)} dk + \frac{i\epsilon^{n+1}}{2} \int_{\eta}^{\infty} k^n e^{ik(x+i\pi)} dk, \quad (3.49)$$

where $\eta = \mathcal{O}(1/n)$. Considering only the latter of these integrals, we rewrite this as

$$\frac{i\epsilon^{n+1}}{2} \int_{\eta}^{\infty} k^n e^{ik(x+i\pi)} dk = \frac{i\epsilon^{n+1}}{2} \int_{\eta}^{\infty} e^{n \log k - ik(x+i\pi)} dk.$$

Defining $g(k) = n \log k - ik(x+i\pi)$ and solving $g'(k_s) = 0$ shows that this integral passes through a saddle-point at

$$k_s = \frac{in}{x + i\pi}. \quad (3.50)$$

This is a moveable-maximum problem, and the saddle-point contribution may be approximated by extending the range of integration and making the substitution

$$k = \frac{in}{x + i\pi} + n^{1/2} s,$$

giving the contribution as

$$\begin{aligned} \frac{i\epsilon^{n+1}}{2} \int_{\eta}^{\infty} e^{n \log k - ik(x+i\pi)} dk &\sim \frac{i\epsilon^{n+1}}{2} \left(\frac{n}{-i(x+i\pi)} \right)^n e^{-n} \int_{-\infty}^{\infty} n^{1/2} e^{s^2(x+i\pi)^2/2} ds, \\ &= \frac{i\epsilon^{n+1}}{2} \frac{\sqrt{2\pi n} n^n e^{-n}}{[-i(x+i\pi)]^{n+1}}, \\ &\sim \frac{i\epsilon^{n+1}}{2} \frac{\Gamma(n+1)}{[-i(x+i\pi)]^n}, \end{aligned}$$

where the final step identifies the expression as the first term in Stirling's formula for the gamma function. Similarly, considering the first integral in (3.49) gives a late-order contribution of

$$\frac{i\epsilon^{n+1}}{2} \int_{-\infty}^{-\eta} e^{n \log k - ik(x-i\pi)} dk \sim -\frac{i\epsilon^{n+1}}{2} \frac{\Gamma(n+1)}{[i(x-i\pi)]^{n+1}}.$$

These two expressions are complex conjugates, and therefore their sum will be purely real.

Importantly, we note from (3.50) that the position of the saddle satisfies $k_s = \mathcal{O}(n)$. Hence, we expect to find that for terms satisfying $n = \mathcal{O}(1/\epsilon)$ as $\epsilon \rightarrow 0$, the form of this contribution may be correctly obtained by considering the appropriate outer integral, I_3 , near $p = 0$, as the saddle point moves into the outer region.

We therefore consider I_3 near $p = 0$. Applying a Taylor expansion near $p = 0$, we find that

$$\bar{\xi}_{E,3} \sim \frac{i}{2} \int_0^{\infty} (1 + p + p^2 + \dots) \exp\left(\frac{1}{\epsilon} [ip(x+i\pi)]\right) dp,$$

where $\bar{\xi}_{E,j}$ is the end-point contribution from I_j . By swapping the order of the series and the integral and evaluating term-by-term, we find that

$$\bar{\xi}_{E,3} \sim \frac{i}{2} \sum_{n=1}^{\infty} \frac{\epsilon^n \Gamma(n)}{[-i(x+i\pi)]^n}. \quad (3.51)$$

Similarly, we find that

$$\bar{\xi}_{E,6} \sim -\frac{i}{2} \sum_{n=1}^{\infty} \frac{\epsilon^n \Gamma(n)}{[i(x-i\pi)]^n}. \quad (3.52)$$

As we expect, the resultant series has the correct form for the late-order terms.

To obtain the behaviour of the remaining end-point contributions, we return to (3.43) and consider the first integral. Expanding this as a series in ϵ gives

$$\begin{aligned} &\frac{i\epsilon}{8} \int_{-\infty}^{\infty} \frac{1}{k\epsilon \cosh \pi k - \sinh \pi k} \left(1 + \sqrt{\frac{\sinh(\pi k)}{\epsilon k \cosh(\pi k)}} \right) \exp\left(ik(x-t) + it\sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ &\sim \frac{i}{8} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\epsilon^{n+1} k^n \cosh^n \pi k}{\sinh^{n+1} \pi k} \exp\left(ik(x-t) + it\sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ &\quad + \frac{i}{8} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\epsilon^{n+1/2} k^{n-1/2} \cosh^{n-1/2} \pi k}{\sinh^{n+1/2} \pi k} \exp\left(ik(x-t) + it\sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk. \end{aligned}$$

We consider the $\mathcal{O}(\epsilon^{n+1})$ term, which takes the form

$$\frac{i\epsilon^{n+1}}{4} \int_{-\infty}^{\infty} \frac{k^n (e^{\pi k} + e^{-\pi k})^n}{(e^{\pi k} - e^{-\pi k})^{n+1}} \exp \left(ik(x-t) + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk.$$

When $k \gg 1/n$, this may be divided into two integrals, giving

$$\begin{aligned} & \frac{i\epsilon^{n+1}}{4} \int_{-\infty}^{\infty} \frac{k^n (e^{\pi k} + e^{-\pi k})^n}{(e^{\pi k} - e^{-\pi k})^{n+1}} \exp \left(ik(x-t) + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ & \sim -\frac{i}{4} \int_{-\infty}^{-\eta} \exp \left(n \log k + ik(x-t) + \pi k + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk \\ & \quad + \frac{i}{4} \int_{\eta}^{\infty} \exp \left(n \log k + ik(x-t) - \pi k + it \sqrt{\frac{k \sinh(\pi k)}{\epsilon \cosh(\pi k)}} \right) dk, \end{aligned}$$

where, again, $\eta = \mathcal{O}(1/n)$. As in the previous integral, we observe that the saddle point will satisfy $k = \mathcal{O}(n)$. Consequently, as $n \sim \mathcal{O}(1/\epsilon)$, these late-term contributions may be obtained by using an end-point analysis on the outer integral. This will be true of the remaining contributions in (3.43), as well as those of (3.44), and hence we may determine the late-term behaviour of these contributions simply by investigating end-points of the appropriate integral from $I_1 - I_{10}$.

To determine the late-order end-point contributions for the $\bar{\xi}_H$, we first consider I_1 and apply the transformation $\epsilon u = \sqrt{p}$. This gives

$$I_1 = \frac{i\epsilon}{2} \int_0^{\infty} \frac{\epsilon u}{\epsilon^2 u^2 - 1} \left(1 + \frac{1}{\epsilon u} \right) \exp(i\epsilon \sigma_1 u^2 + iut) du.$$

Expanding the leading part of the integrand as Taylor series about $u = 0$ gives

$$I_1 = -\frac{i\epsilon}{2} \int_0^{\infty} (1 + \epsilon u + \epsilon^2 u^2 + \epsilon^3 u^3 + \dots) \left(1 + i\epsilon \sigma_1 u^2 - \frac{1}{2} \epsilon^2 \sigma_1^2 u^4 + \dots \right) e^{iut} du. \quad (3.53)$$

Switching the order of the integral and the summation and integrating term-by-term (considering the dominant contribution to come near $u = 0$) produces

$$\bar{\xi}_{E,1} \sim -\frac{i}{2} \sum_{n=1}^{\infty} \left[\sum_{j=1}^n \frac{\Gamma(2n-j)}{\Gamma(n-j+1)} \left(-\frac{t}{\sigma_1} \right)^j \right] \left(-\frac{i\sigma_1 \epsilon}{t^2} \right)^n.$$

We see that this contribution is algebraic in ϵ .

Finally, we apply the transformation $\epsilon u = \sqrt{p}$ to the integral I_7 . This gives

$$I_7 = \frac{i\epsilon}{2} \int_0^{\infty} \exp(i\epsilon \sigma_1 u^2 + iut) du.$$

Again, expanding the leading part of the integrand near $u = 0$ gives

$$I_7 = \frac{i\epsilon}{2} \int_0^{\infty} \left(1 + i\epsilon \sigma_1 u^2 - \frac{1}{2} \epsilon^2 \sigma_1^2 u^4 + \dots \right) e^{iut} du.$$

Comparing this expression to (3.53), we see that it exactly cancels out several terms of that expansion, leaving

$$I_1 + I_7 = -\frac{i\epsilon}{2} \int_0^\infty (\epsilon u + \epsilon^2 u^2 + \epsilon^3 u^3 + \dots) \left(1 + i\epsilon\sigma_1 u^2 - \frac{1}{2}\epsilon^2\sigma_1^2 u^4 + \dots \right) e^{iut} du, \quad (3.54)$$

and hence that

$$\bar{\xi}_{E,1} + \bar{\xi}_{E,7} \sim -\frac{i}{2} \sum_{n=1}^\infty \left[\sum_{j=2}^n \frac{\Gamma(2n-j)}{\Gamma(n-j+1)} \left(-\frac{t}{\sigma_1} \right)^j \right] \left(-\frac{i\sigma_1\epsilon}{t^2} \right)^n. \quad (3.55)$$

Similar algebraic series expressions in ϵ may be found by examining $I_2 - I_6$ and $I_8 - I_{10}$ and combining the results appropriately.

3.3.3.2 Saddle point contributions

We now obtain the contribution to the integral expression for the free surface from the saddle points. In this analysis, we note the position of the square-root branch cuts, and take care to consider the appropriate branch in each case. Taking I_2 , we define

$$F_2(p) = \frac{i}{4} \frac{1}{p-1} \left(1 - \frac{1}{\sqrt{p}} \right),$$

$$g_2(p) = ip\sigma_1 - it\sqrt{p}.$$

The saddle point location may be found by solving $g_2'(p_s) = 0$, giving

$$p_s = \frac{1}{4} \frac{t^2}{\sigma_1^2}.$$

The leading term of the saddle contribution is given by

$$\bar{\xi}_{S,2} \sim \sqrt{-\frac{2\pi\epsilon}{g_1''(p_s)}} F_1(p_s) e^{g_1(p_s)/\epsilon} = -\frac{\sqrt{-i\pi\epsilon\sigma_1}}{2\sigma_1 + t} \exp\left(-\frac{it^2}{4\sigma_1\epsilon}\right), \quad (3.56)$$

where $\bar{\xi}_{S,j}$ denotes the saddle-point contribution from I_j . Using the same method, we find that $\bar{\xi}_{S,1} = -\bar{\xi}_{S,2}$, while the saddle points from I_4 and I_5 give a contribution

$$\bar{\xi}_{S,5} = -\bar{\xi}_{S,4} \sim -\frac{\sqrt{i\pi\epsilon\sigma_2}}{2\sigma_2 + t} \exp\left(\frac{it^2}{4\sigma_2\epsilon}\right). \quad (3.57)$$

We again note that the contributions from I_1 and I_2 are equal to the complex conjugate of the contribution from I_4 and I_5 , respectively. From Figure (3.5), we see that when the saddle contributions from I_1 and I_4 switch off at $x = t$, the saddle contributions from I_2 and I_5 switch on. Consequently, we find that the combined saddle contribution from the four integrals is equal to $\bar{\xi}_{S,1} + \bar{\xi}_{S,4}$ for $x < t$ and the negative of this expression for $x > t$.

Applying a similar analysis to the saddle points in $I_7 - I_{10}$ gives

$$\bar{\xi}_{S,8} = -\bar{\xi}_{S,7} \sim \frac{\sqrt{-i\pi\epsilon}}{2\sqrt{\sigma_1}} \exp\left(-\frac{it^2}{4\sigma_1\epsilon}\right), \quad (3.58)$$

and

$$\bar{\xi}_{S,10} = -\bar{\xi}_{S,9} \sim \frac{\sqrt{i\pi\epsilon}}{2\sqrt{\sigma_2}} \exp\left(\frac{it^2}{4\sigma_2\epsilon}\right). \quad (3.59)$$

Again, we note that the saddle contributions from I_7 and I_9 switch off at $x = t$, while the saddle contributions from I_8 and I_{10} switch on at this point. Therefore, the combined saddle contribution, equal to $\bar{\xi}_{S,7} + \bar{\xi}_{S,9}$, is present over the solution domain. Combining the saddle contributions from (3.56)–(3.59) gives the saddle-point contribution to the free surface, $\bar{\xi}_S$, as

$$\bar{\xi}_S \sim \frac{\sqrt{-i\pi\epsilon}}{2\sqrt{\sigma_1}} \left(\frac{t}{2\sigma_1+t}\right) \mathcal{S}_S \exp\left(-\frac{it^2}{4\sigma_1\epsilon}\right) + \frac{\sqrt{i\pi\epsilon}}{2\sqrt{\sigma_2}} \left(\frac{t}{2\sigma_2+t}\right) \mathcal{S}_S \exp\left(\frac{it^2}{4\sigma_2\epsilon}\right), \quad (3.60)$$

where

$$\mathcal{S}_S = \begin{cases} 1 & \text{if } x > t, \\ 0 & \text{if } x = t, \\ -1 & \text{if } x < t. \end{cases}$$

It is important to note that this is only the leading-order contribution from the saddle points. This fact will play an important role in the late-order terms analysis, as further Stokes switching will be found by considering the late-order terms of the series contribution that is equivalent to the saddle contributions.

3.3.3.3 Pole contributions

Finally, we determine the contribution from the poles in $I_1 - I_6$. Using Cauchy's integral formula (and noting that the integration contour is counter-clockwise with respect to the poles), we find that

$$\begin{aligned} \bar{\xi}_{P,1} &= -\pi e^{i(\sigma_1+t)/\epsilon}, & \bar{\xi}_{P,4} &= -\pi e^{-i(\sigma_2+t)/\epsilon} \\ \bar{\xi}_{P,2} &= 0, & \bar{\xi}_{P,5} &= 0 \\ \bar{\xi}_{P,3} &= \pi e^{i(\sigma_1+t)/\epsilon}, & \bar{\xi}_{P,6} &= \pi e^{-i(\sigma_2+t)/\epsilon}. \end{aligned}$$

where $\bar{\xi}_{P,j}$ refers to the pole contribution from I_j . Figure 3.4 shows that $\bar{\xi}_{P,3}$ and $\bar{\xi}_{P,6}$ contribute to the integral expression for the free surface when $x > 0$, giving

$$\bar{\xi}_{P,3} + \bar{\xi}_{P,6} = 2\pi e^{-\pi/\epsilon} \cos(x/\epsilon).$$

From the expression given in (3.47) and the Stokes behaviour shown in Figure 3.6 (b), we see that $\bar{\xi}_{P,1}$ and $\bar{\xi}_{P,4}$ contribute to the free surface expression when the x - t plane position is located within R_1 . The contribution from these poles is given by

$$\bar{\xi}_{P,1} + \bar{\xi}_{P,4} = -2\pi e^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right),$$

which exactly cancels the contribution given by $\bar{\xi}_{P,3}$ and $\bar{\xi}_{P,6}$. Consequently, the complete pole contribution to the free surface expression is given by

$$\bar{\xi}_P = 2\pi\mathcal{S}_Pe^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right), \quad (3.61)$$

where

$$\mathcal{S}_P = \begin{cases} 1 & \text{if } (x, t) \in R_1, \\ 0 & \text{otherwise,} \end{cases}$$

where the region R_1 is that shown in Figure 3.6. Consequently, exponentially small waves on the free surface generated by the pole contributions are found within the unshaded region of Figure 3.6 (b). This corresponds to the waves being switched on at $x = 0$, and switched whenever points satisfying the equation in (3.47) are crossed (that is, when they are switched across Stokes lines). The amplitude of these waves is consistent with the results obtained in [27, 63].

Combining the expression for the saddle point and pole contributions to the free surface, given in (3.60) and (3.61) respectively gives the exponentially small component of the free-surface behaviour as

$$\begin{aligned} \xi_{\text{exp}} \sim & \frac{\delta\sqrt{-i\pi\epsilon}}{2\sqrt{x-t+i\pi}} \left(\frac{t}{2(x+i\pi)-t} \right) \mathcal{S}_S \exp\left(-\frac{it^2}{4\epsilon(x-t+i\pi)}\right) \\ & + \frac{\delta\sqrt{i\pi\epsilon}}{2\sqrt{x-t-i\pi}} \left(\frac{t}{2(x-i\pi)-t} \right) \mathcal{S}_S \exp\left(\frac{it^2}{4\epsilon(x-t-i\pi)}\right) \\ & + 2\pi\delta\mathcal{S}_Pe^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right), \end{aligned} \quad (3.62)$$

where the free-surface position is expressed in terms of the original coordinates, prior to linearization. This expression is purely real, and contains the full algebraic series in ϵ , as well as the pole contributions and the leading-order contributions from the saddle points. The surface exhibits exponentially small gravity waves that propagate downstream from $x = 0$ with a wavefront positioned at $x \sim t/2 - \pi$ as $t \rightarrow \infty$.

3.4 Late-order term analysis

As noted previously, many problems, including the original nonlinear problem of time-varying flow over a step of arbitrary height, cannot be easily expressed in terms of an integral formulation. In these cases, the method of steepest descents cannot be used to determine the free-surface behaviour. The exponential asymptotic methods illustrated in Chapter 2, however, may still be applied in order to investigate the exponentially small waves on the free surface of the flow.

The linearized Eulerian problem given in (3.29)–(3.33) has fixed boundaries on $y = 0$ and $y = -\pi$. However, the objective of this analysis is to develop a method that may be applied to nonlinear flow problems, in addition to the linearized problem under consideration. To this

end, we consider the linearized version of the Lagrangian formulation given in (3.15)–(3.26), and establish that the results obtained by analysing the late-order terms of the asymptotic expansion for small Froude number are equivalent to those obtained in Section 3.3. This method will subsequently be applied to an example of non-linearized flow in Chapter 4.

3.4.1 Leading-order behaviour

The system of equations in (3.15), (3.17)–(3.18), (3.25)–(3.26) expresses the position of any fluid particle at Lagrangian time τ , denoted by (x, y) in terms of the time, as well as a coordinate pair indicating the position of the fluid particle at $\tau = 0$. This initial particle labelling, denoted as (p, q) is found by determining the velocity potential and streamfunction associated with the initial position in the auxiliary steady problem, as seen in Section 3.2. We begin by finding a power series expression for x and y in the small parameter ϵ , setting

$$x \sim \sum_{n=0}^{\infty} \epsilon^n x^{(n)}, \quad y \sim \sum_{n=0}^{\infty} \epsilon^n y^{(n)}. \quad (3.63)$$

Taking the real and imaginary parts of the leading-order behaviour given in (3.24) on the free surface $q = 0$ gives

$$x^{(0)} = \frac{1}{2} \log(1 + e^{p+\tau}) - \log(1 + e^p), \quad y^{(0)} = 0. \quad (3.64)$$

Applying the series expressions given in (3.63) to the governing equations (3.15, 3.26) gives

$$x_p^{(n)} + y_q^{(n)} = 0, \quad (3.65)$$

$$x_q^{(n)} - y_p^{(n)} = 0, \quad (3.66)$$

for $n > 0$. When $n = 0$, we find that the inhomogeneous right-hand side and the left-hand side containing the leading-order terms given in (3.64) are equal, and therefore the equation is satisfied. Applying the series expressions to the free-surface condition (3.17) gives

$$x_{\tau\tau}^{(n-1)} + y_p^{(n)} = 0, \quad n > 0. \quad (3.67)$$

The leading-order solution contains two singularities in the physical flow path. The first is found at $(p, q) = (-\tau, -\pi)$. This singular behaviour is caused by the presence of the step, and corresponds directly to the singularity present in the steady problem [27]. Additionally, there is singular behaviour present in $x^{(0)}$ at $(p, q) = (0, -\pi)$, which convects downstream with the flow and is a consequence of the sudden disturbance that occurs in the system at $\tau = 0$. This singularity is not present in the steady problem.

We now permit p to take complex values, with the intention of investigating the behaviour of the leading-order solution on the analytically-continued free surface, defined on the region $p \in \mathbb{C}$

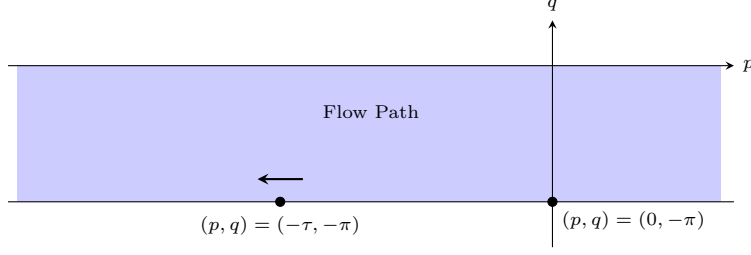


Figure 3.7: Singularities in the leading-order flow behaviour in the physical flow region, shaded in blue. Circles denote singularities contained within the leading-order flow behaviour.

and $q = 0$. Near the singularity at $p = -\tau + i\pi$, the leading order behaviour given in (3.24) takes the form

$$z^{(0)} \sim \frac{1}{2} \log(p + \tau - i\pi). \quad (3.68)$$

A similar expression may be found near the singularity at $p = -\tau - i\pi$, where

$$z^{(0)} \sim \frac{1}{2} \log(p + \tau + i\pi).$$

Near the remaining singularities at $p = \pm i\pi$, we find that

$$z^{(0)} \sim -\log(p - i\pi) \quad \text{as } p \rightarrow i\pi, \quad (3.69)$$

$$z^{(0)} \sim -\log(p + i\pi) \quad \text{as } p \rightarrow -i\pi. \quad (3.70)$$

While the local behaviour on the free surface for the two singularity pairs is of the same form, we see that the ray structure leading to this behaviour is different in each case. The expression for $z^{(0)}$ in (3.24) is singular along curves that satisfy

$$p + \tau - iq = \pm i\pi, \quad p + iq = \pm i\pi, \quad p - iq = \pm i\pi.$$

Figure 3.8 (a) illustrates the rays passing through the singularities at $p = -\tau \pm i\pi$. In the full leading-order flow behaviour (3.24), we see that these singularities are propagated along the rays following $p + \tau - iq = \pm i\pi$. Hence, there are two rays in the leading-order behaviour passing through these points along which the behaviour is singular. We therefore expect there to be two sets of associated late-order terms, each of which contains a singulant associated with one of the rays.

However, Figure 3.8 (b) shows the rays passing through the singularities at $p = \pm i\pi$. We see that the leading-order singularities following the rays $p + iq = \pm i\pi$ and $p - iq = \pm i\pi$ both pass through these two points on the free surface. Hence, there are four rays in total, two of which intersect the free surface at each of $p = \pm i\pi$. Hence, we expect four late-order terms associated with these singularities, rather than two.

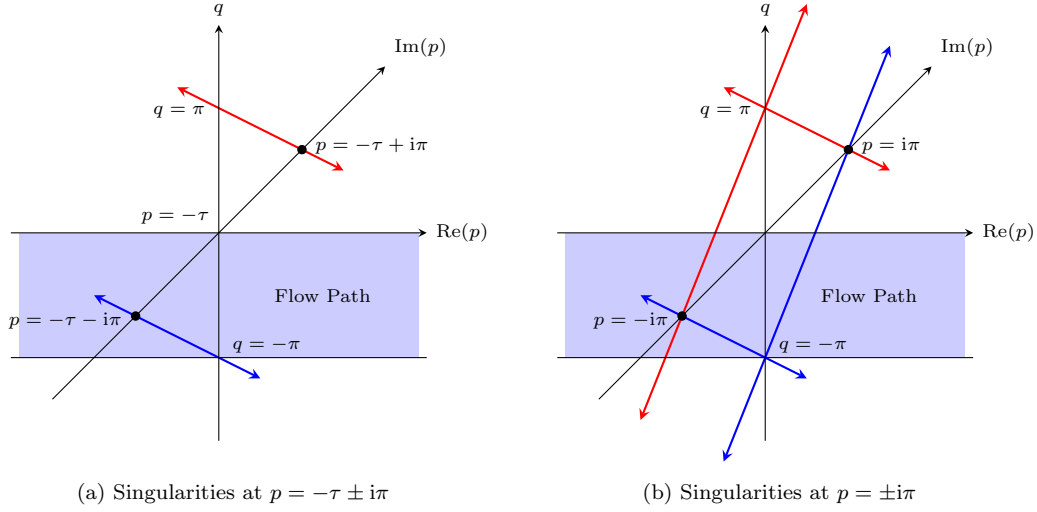


Figure 3.8: Curves (or rays) along which the leading order behaviour $z^{(0)}$ is singular, given that $p \in \mathbb{C}$. The curves in (a) are associated with the singularities at $p = -\tau \pm i\pi$, while those in (b) are associated with $p = \pm i\pi$. The rays originating at the physical singularity, located at $(p, q) = (0, -\tau - \pi)$ and $(0, -\pi)$ respectively, are represented by blue lines, while the rays originating at the reflected singularity, located at $(p, q) = (0, -\tau + \pi)$ and $(0, \pi)$ respectively, are shown as red lines. The points at which these lines intersect with the complex free surface ($q = 0$) are denoted by circles.

Noting the discussion on this topic in Chapter 2, we determine that this ray structure will be maintained in the late orders of the problem through the behaviour of the singulants. This is simple for the singularities on the complexified free surface found at $p = -\tau \pm i\pi$, as will find that of the four possible late-order term singulants, only two may contribute to the problem

Near the singularities at $p = i\pi$, however, we require the local late-order behaviour to be composed of two distinct contributions with equal magnitude. Consequently, the prefactors associated with each of these two late-order contributions must have equal magnitude near the appropriate singularity on the complex free surface. The same observation can be made regarding the late-order term behaviour near the singularity at $p = -i\pi$.

3.4.2 Formulation of ansatz equations

In order to determine the form of the late-order terms, we follow the method described in Chapter 1 and define a general ansatz for each of $x^{(n)}$ and $y^{(n)}$ as $n \rightarrow \infty$. Equation (3.67) indicates that $y^{(n)}$ is obtained by differentiating $x^{(n-1)}$ twice with respect to τ and integrating the resultant expression with respect to p , hence increasing the order of any singularities by one. The equations (3.65)–(3.66) imply that any singularities in $x^{(n)}$ and $y^{(n)}$ will be of the same order. Consequently, the order of the singularities increases by one in each subsequent term of the series. It is therefore reasonable to define late-order ansatz forms as

$$x^{(n)} \sim \frac{X\Gamma(n + \gamma)}{\chi^{n+\gamma}}, \quad y^{(n)} \sim \frac{Y\Gamma(n + \gamma)}{\chi^{n+\gamma}}, \quad (3.71)$$

where X , Y , γ and χ vary in p , q and τ , but not n . Each singularity on the analytically-continued free surface is investigated individually, with the resultant expressions added together, as described in [33]. Consequently, $\chi = 0$ at the position of the singularity in question. Applying these ansatz forms to the governing equations (3.65)–(3.66) and matching to leading order as $n \rightarrow \infty$ gives

$$X\chi_p + Y\chi_q = 0, \quad (3.72)$$

$$X\chi_q - Y\chi_p = 0. \quad (3.73)$$

Applying the ansatz forms from (3.71) to the free-surface condition (3.67) gives to leading order

$$X\chi_\tau^2 - Y\chi_p = 0, \quad \text{on } q = 0. \quad (3.74)$$

Matching (3.65)–(3.67) to the next order in n gives the result that γ is constant. Finally, continuing to match the governing equations (3.65)–(3.66) gives

$$X_p + Y_q = 0, \quad (3.75)$$

$$X_q - Y_p = 0, \quad (3.76)$$

while the next order of matching in (3.67) gives

$$-2X_\tau\chi_\tau - X\chi_{\tau\tau} - Y_p = 0 \quad \text{on } q = 0. \quad (3.77)$$

The system given in (3.72)–(3.73) only produces constant solutions for χ when $X = \pm iY$. Consequently, there are two possible late-order contributions from any singularity, subsequently denoted as χ_1 and χ_2 , and given by

$$X_1 = iY_1, \quad X_2 = -iY_2, \quad (3.78)$$

$$\chi_{1,p} = i\chi_{1,q}, \quad \chi_{2,p} = -i\chi_{2,q}, \quad (3.79)$$

$$\chi_{1,\tau}^2 = -i\chi_{1,p}, \quad \chi_{2,\tau}^2 = i\chi_{2,p}, \quad \text{on } q = 0, \quad (3.80)$$

with (3.75)–(3.77) also taking the appropriate form for each case.

Hence, (3.78)–(3.80) will be used to determine the form of the late-order terms of the power series, and hence the exponentially small contributions to the free-surface behaviour.

3.4.3 Analysis of the stationary singularities

We first consider the effect of the stationary singularities on the complexified free surface. These two singularities in question are located at $p = -\tau \pm i\pi$. We restrict the problem entirely to the free surface by setting the solution domain to be $q = 0$.

Using the condition that $\chi = 0$ at $p = -\tau \pm i\pi$, we apply Charpit's method to the equations for χ_1 and χ_2 given in (3.80). Letting $\chi_{1,p} = P_1$ and $\chi_{1,\tau} = T_1$, the resultant Charpit equations for χ_1 are given by

$$\frac{dp}{du} = i, \quad \frac{d\tau}{du} = 2T_1, \quad \frac{d\chi_1}{du} = T_1^2, \quad \frac{dP_1}{du} = 0, \quad \frac{dT_1}{du} = 0, \quad (3.81)$$

where u is the ray parameter. The initial conditions are given by

$$p(0) = -s - i\pi, \quad \tau(0) = s, \quad \chi(0) = 0, \quad T_1^2 + iP_1 = 0, \quad P_1 - T_1 = 0, \quad (3.82)$$

where the initial data is parametrized by s . The only nonzero solution to this system is given by

$$\chi_1 = -i(p + \tau - iq + i\pi), \quad (3.83)$$

where (3.79) was applied in order to determine how χ varies with q on the free surface. While it is strictly possible to set $q = 0$, the variation of this expression in q will be important in the context of specifying the ray structure of the late-order terms.

Applying a similar Charpit analysis to the second system in (3.78)–(3.80) gives

$$\chi_2 = i(p + \tau + iq + i\pi). \quad (3.84)$$

We see that if we set $q = 0$, $\chi_1 = -\chi_2$. Consequently, the two singularities provide contributions of equal magnitude at every point on the analytically-continued free surface. Hence, if we had not determined which singulant behaviour was consistent with the leading-order behaviour given in (3.24), as seen in Figure 3.8, it would be impossible to determine which form of the singulant is correct. However, in order to be consistent with the ray structure of the leading-order behaviour, we determine that the correct singulant form is (3.83).

Applying the singulant expressions given in (3.83)–(3.84) to (3.75)–(3.77) gives

$$X_1(p, q, \tau) = X_1 \left(p + \frac{1}{2}\tau - iq \right). \quad (3.85)$$

These equations imply that X_1 (and consequently Y_1) are constant along a set of characteristic curves. The Cauchy data for this problem will be specified by matching with the inner solution in the neighbourhood of $p + \tau = i\pi$. As each characteristic of the prefactors in (3.85) passes through this curve, we conclude that the values for X_1 and Y_1 are constants. To determine the value of these constants, we must obtain the behaviour of the late-order terms in the neighbourhood of the singularity.

To determine the value of γ , we ensure that the singularity strength in the late-order term behaviour is consistent with the known series terms. The dynamic boundary condition as given in (3.67) indicates that obtaining $y^{(1)}$ from $x^{(0)}$ involves differentiating the expression twice

with respect to τ and subsequently integrating with respect to p . The combined effect of this is to increase the order of the singularity, indicating that $y^{(1)}$ has a singularity of order one. To ensure that the ansatz (3.71) is consistent with this behaviour as $n \rightarrow 1$, given the form of χ_1 and χ_2 from (3.83)–(3.84), we find that $\gamma = 0$.

3.4.3.1 Inner problem

To determine the value of the ansatz prefactors, we define a system of inner coordinates near $p + \tau = i\pi$, given by

$$\epsilon\sigma_1 = -i(p + \tau - iq + i\pi), \quad \epsilon\sigma_2 = i(p + \tau + iq + i\pi), \quad \lambda = \tau,$$

where the free surface is found at $\sigma_1 = -\sigma_2$. We have retained the variation in q , as the equations involve derivatives in this variable. This gives

$$\frac{\partial}{\partial p} = \frac{i}{\epsilon} \frac{\partial}{\partial \sigma_1} - \frac{i}{\epsilon} \frac{\partial}{\partial \sigma_2}, \quad \frac{\partial}{\partial q} = -\frac{1}{\epsilon} \frac{\partial}{\partial \sigma_1} - \frac{1}{\epsilon} \frac{\partial}{\partial \sigma_2}, \quad \frac{\partial}{\partial \tau} = \frac{i}{\epsilon} \frac{\partial}{\partial \sigma_1} - \frac{i}{\epsilon} \frac{\partial}{\partial \sigma_2} + \frac{\partial}{\partial \lambda}.$$

The governing equations (3.15)–(3.26) become (keeping only the dominant terms as $\epsilon \rightarrow 0$)

$$-ix_{\sigma_1} + ix_{\sigma_2} - y_{\sigma_1} - y_{\sigma_2} = 0, \quad (3.86)$$

$$-x_{\sigma_1} - x_{\sigma_2} + iy_{\sigma_1} - iy_{\sigma_2} = 0. \quad (3.87)$$

The boundary condition (3.17) becomes

$$-x_{\sigma_1\sigma_1} - x_{\sigma_2\sigma_2} + 2x_{\sigma_1\sigma_2} - iy_{\sigma_1} + iy_{\sigma_2} = 0, \quad (3.88)$$

on $\sigma_1 = -\sigma_2$. The leading-order solutions (3.64) take the form

$$x^{(0)} \sim \frac{1}{2} \log(\sigma_1) + \frac{1}{2} \log(\epsilon),$$

$$y^{(0)} = 0.$$

Setting $x = \frac{1}{2} \log(\epsilon) + \hat{x}$ and $y = \hat{y}$ gives

$$\hat{x}^{(0)} \sim \frac{1}{2} \log(\sigma_1), \quad (3.89)$$

$$\hat{y}^{(0)} = 0. \quad (3.90)$$

Knowing the form of the late-order ansatz determined by the outer solutions, we prescribe the series expression

$$\hat{x} = \hat{x}^{(0)} + \sum_{n=1}^{\infty} \frac{A_n}{\sigma_1^n}, \quad \hat{y} = \hat{y}^{(0)} + \sum_{n=1}^{\infty} \frac{B_n}{\sigma_1^n}, \quad (3.91)$$

where A_n and B_n are constant. The first governing equation (3.86) becomes (noting that the leading order solutions $\hat{x}^{(0)}$ and $\hat{y}^{(0)}$ will satisfy the equation exactly, and may therefore be discarded)

$$i \sum_{n=1}^{\infty} \frac{nA_n}{\sigma_1^{n+1}} - \sum_{n=1}^{\infty} \frac{nB_n}{\sigma_1^{n+1}} = 0.$$

This expression may be matched in the limit that $\sigma_1 \rightarrow 0$ to give

$$iA_n + B_n = 0. \quad (3.92)$$

Applying the series expressions given in (3.91) to the second of the governing equations (3.87) gives the same relationship. The boundary condition (3.88) becomes

$$-\sum_{n=1}^{\infty} \frac{n(n+1)A_n}{\sigma_1^{n+2}} + i \sum_{n=1}^{\infty} \frac{nB_n}{\sigma_1^{n+1}} - \hat{x}_{\sigma_1 \sigma_1}^{(0)} = 0, \quad (3.93)$$

on $\sigma_1 + \sigma_2 = 0$. Again, we match this behaviour as $\sigma_1 \rightarrow 0$, giving

$$-(n-1)A_{n-1} + iB_n = 0, \quad n > 1,$$

From (3.92), this becomes

$$A_n = (n-1)A_{n-1}, \quad n > 1,$$

from which it is apparent that

$$A_n = -iB_n = \Gamma(n)A_1.$$

By similarly considering the leading-order behaviour of (3.93) as $|\sigma_1| \rightarrow \infty$, using (3.89), we find that $A_1 = -1/2$. Converting the resultant series expression to the original coordinate system and matching with the outer problem shows that $X_1 = -1/2$ and hence that $Y_1 = i/2$.

Performing a similar analysis near the singularity at $p = -\tau - i\pi$ and taking the sum of the late-order terms gives

$$x_S^{(n)} \sim -\frac{1}{2} \frac{\Gamma(n)}{[i(p + \tau - iq - i\pi)]^n} - \frac{1}{2} \frac{\Gamma(n)}{[-i(p + \tau - iq + i\pi)]^n}, \quad (3.94)$$

$$y_S^{(n)} \sim \frac{i}{2} \frac{\Gamma(n)}{[i(p + \tau - iq - i\pi)]^n} - \frac{i}{2} \frac{\Gamma(n)}{[-i(p + \tau - iq + i\pi)]^n}. \quad (3.95)$$

where $x_S^{(n)}$ and $y_S^{(n)}$ refer to the late-order contributions from the stationary singularities. On the free surface, given by $q = 0$, the contribution from the singularity at $p + \tau = i\pi$ is clearly equal to the conjugate of the contribution from the singularity at $p + \tau = -i\pi$, indicating that the sum of the terms will be real and positive on the free surface, as we would expect.

3.4.3.2 Stokes smoothing

Rather than consider both contributions in (3.94)–(3.95) separately, we will consider only one set of late-order term behaviours, noting that considering the other set produces the appropriate complex conjugate contribution. Specifically, we will consider the exponentially small behaviour switched on by the late-order terms caused by the singularity at $p = -\tau + i\pi$.

Hence we consider the only the contribution from

$$\epsilon^n x^{(n)} \sim -\frac{1}{2} \frac{\Gamma(n)}{[-i(p + \tau + i\pi)]^n}, \quad \epsilon^n y^{(n)} \sim -\frac{i}{2} \frac{\Gamma(n)}{[-i(p + \tau + i\pi)]^n},$$

and subsequently denote $\chi = -i(p + \tau + i\pi)$, $X = -1/2$ and $Y = -i/2$. Truncating the series expression given in (3.63) produces

$$x = \sum_{n=0}^{N-1} \epsilon^n x^{(n)} + R^{(N)}, \quad y = \sum_{n=0}^{N-1} \epsilon^n y^{(n)} + S^{(N)},$$

where $R^{(N)}$ and $S^{(N)}$ denote the truncation error. If the series are truncated optimally, these error terms will be exponentially small as $\epsilon \rightarrow 0$. Applying this truncated series expression to the governing equations (3.15), (3.26) gives

$$R_p^{(N)} + S_q^{(N)} = 0, \quad (3.96)$$

$$R_q^{(N)} - S_p^{(N)} = 0. \quad (3.97)$$

Applying the truncated series to the boundary equation (3.17) gives

$$\epsilon R_{\tau\tau}^{(N)} + S_p^{(N)} \sim \epsilon^N y_p^{(N)}. \quad (3.98)$$

The optimally-truncated switching behaviour will be exponentially small, and be a multiple of the homogeneous solution to (3.96)–(3.98). Applying a WKB ansatz to the homogeneous form of this system and comparing the result with the ansatz equations (3.78)–(3.80) shows that an appropriate form for the switching behaviour is given by

$$R^{(N)} \sim AXe^{-\chi/\epsilon}, \quad S^{(N)} \sim AYe^{-\chi/\epsilon}, \quad (3.99)$$

where A is the Stokes switching parameter that varies rapidly and smoothly in the neighbourhood of the Stokes line. The resultant expression for (3.98) (incorporating the ansatz expression for $y_p^{(n)}$ as $n \rightarrow \infty$ from (3.4.3.2) and noting that $X = iY$) takes the form

$$-iY(\epsilon A_{\tau\tau} - 2iA_\tau + iA_p)e^{-\chi/\epsilon} \sim -\epsilon^N \frac{\chi_p Y \Gamma(N+1)}{\chi^{N+1}}.$$

To leading order as $\epsilon \rightarrow 0$, this becomes

$$-2iA_\tau + iA_p \sim -\epsilon^N \frac{\Gamma(N+1)}{\chi^{N+1}} e^{\chi/\epsilon}.$$

We may set $u = \chi$. Additionally, we define v such that $2v_\tau - v_p = 0$. This gives the expression

$$A_u \sim e^{u/\epsilon} \epsilon^N \frac{\Gamma(N+1)}{u^{N+1}}. \quad (3.100)$$

Minimising this expression with respect to N in the limit that N is large gives the optimal truncation as

$$N = \frac{|\chi|}{\epsilon} + \alpha, \quad (3.101)$$

where $0 \leq \alpha < 1$, and α is chosen such that N is an integer. As N only depends on $|\chi|$, we again set $u = re^{i\theta}$, noting that $|\chi| = r$, giving

$$\frac{\partial}{\partial u} = -\frac{ie^{-i\theta}}{r} \frac{\partial}{\partial \theta}. \quad (3.102)$$

Upon applying this transformation, as well as applying the expression for the value of N given in (3.101), and Stirling's formula for the gamma function [3], the expression for A_u given in (3.100) becomes

$$A_\theta \sim i r e^{i\theta} \exp\left(\frac{r e^{i\theta}}{\epsilon}\right) \epsilon^{r/\epsilon + \alpha} \frac{\sqrt{2\pi} (r/\epsilon)^{r/\epsilon + \alpha + 1/2} e^{-r/\epsilon}}{(r e^{i\theta})^{r/\epsilon + \alpha + 1}}.$$

This equation simplifies to become

$$A_\theta \sim i \sqrt{\frac{2\pi r}{\epsilon}} \exp\left(\frac{r}{\epsilon} e^{i\theta} - \frac{r}{\epsilon} - i\theta \left(\frac{r}{\epsilon} + 2\alpha + 1\right) + i\theta\right).$$

The right-hand side of this expression is not exponentially small for $\theta = 0$, which corresponds to $p + \tau = 0$ on the free surface. Consequently, the Stokes line originating at $p = -\tau + i\pi$ passes through the physical free surface at $p = -\tau$. Setting $\theta = \epsilon^{1/2} \hat{\theta}$ gives

$$A_{\hat{\theta}} \sim i \sqrt{2\pi r} e^{-r \hat{\theta}^2/2},$$

which becomes upon integration

$$A \sim i \sqrt{2\pi} \int_{-\infty}^{\theta \sqrt{\tau/\epsilon}} e^{-t^2/2} dt + C,$$

where C is a constant to be determined using matched asymptotic expansions. Therefore, as the Stokes line is crossed, A varies rapidly (in a region of width $\mathcal{O}(\epsilon^{1/2})$) from 0 to 2π . From (3.99), we now see that

$$[S^{(N)}]_{-}^{+} = 2\pi i Y e^{-\chi/\epsilon}.$$

Noting that $Y = -i/2$, the quantity that is switched on as the Stokes line is crossed is given by

$$[S^{(N)}]_{-}^{+} = \pi \delta e^{-\pi/\epsilon} e^{i(p+\tau)/\epsilon}. \quad (3.103)$$

We recall that the late-order terms (3.94)–(3.95) also contain a contribution from the singularity at $p + \tau = -i\pi$. A similar analysis shows that the resultant expression for the switching behaviour is equal to the complex conjugate of the expression obtained in (3.103). Hence,

$$[S_{\text{TOT}}^{(N)}]_{-}^{+} = 2\pi e^{-\pi/\epsilon} \cos\left(\frac{p+\tau}{\epsilon}\right) \sim 2\pi \delta e^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right).$$

As there can be no upstream waves ($p + \tau < 0$), we therefore find that $S^{(N)}$ must be zero in this region, indicating that $C = 0$. Therefore, the form of the downstream waves ($p + \tau > 0$) is given by

$$S_{\text{TOT}}^{(N)} \sim 2\pi e^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right), \quad (3.104)$$

where $S_{\text{TOT}}^{(N)}$ switches on rapidly as the Stokes line at $p + \tau = 0$ is crossed from the upstream to the downstream region. This gives the amplitude of the downstream waves in the original coordinate system as

$$H \sim 2\pi \delta e^{-\pi/\epsilon}.$$

This amplitude is consistent with previous work that has been performed on the case of linearized steady-state flow over a step [27, 63], as well as the waves obtained by investigating the problem using the method of steepest descents (3.61). Hence, the contribution to the free surface behaviour from the singularities at $p = -\tau \pm i\pi$ is equivalent to the contribution to the steepest descent expression from the poles in I_3 and I_6 .

3.4.4 Analysis of the moving singularities

3.4.4.1 Late-order ansatz

We now consider the singularities in the complex free surface located at $p = \pm\pi i$. Again, we will apply Charpit's method to the equations for χ_1 and χ_2 given in (3.80). However, if the Charpit system is formulated in the same manner as (3.81)–(3.82), we find that the derivative terms, P_1 and T_1 , are zero. This indicates that the Cauchy data is prescribed along a characteristic curve, and hence the information never reaches any other part of the solution. Similar behaviour was exhibited by the partial differential equation investigated within [24].

In a similar manner to this previous work, we prescribe a slightly different condition. Instead of parametrizing the initial data as a curve in s , we instead prescribe that each characteristic should pass through the point at which the initial disturbance occurs. In this case, the initial disturbance to the complex free surface occurs at $p = \pm i\pi$ and $\tau = 0$. The variable T_1 is instead unspecified, hence producing a system parametrized in terms of the ray variable u as well as T_1 , given by

$$\frac{dp}{du} = i, \quad \frac{d\tau}{du} = 2T_1, \quad \frac{d\chi_1}{du} = T_1^2, \quad \frac{dP_1}{du} = 0,$$

where u is the ray parameter. The initial conditions are now given by

$$p(0) = -i\pi, \quad \tau(0) = 0, \quad \chi(0) = 0, \quad T_1^2 + iP_1 = 0.$$

Solving this system of differential equations (and noting (3.79) in order to determine the variation in q) gives

$$\chi_1 = \frac{i}{4} \frac{\tau^2}{p - iq + i\pi}. \quad (3.105)$$

The corresponding Charpit system for χ_2 gives

$$\chi_2 = -\frac{i}{4} \frac{\tau^2}{p + iq + i\pi}. \quad (3.106)$$

For simplicity of notation, we shall define $p - iq + i\pi = \lambda_1$ and $p + iq + i\pi = \lambda_2$. Applying these singular expressions to (3.75)–(3.77) gives

$$X_1 = \frac{G_1(\tau/\lambda_1)}{\sqrt{\lambda_1}}, \quad X_2 = \frac{G_2(\tau/\lambda_2)}{\sqrt{\lambda_2}},$$

where G_1 and G_2 are arbitrary functions of a single complex variable. From (3.78), this gives

$$Y_1 = -\frac{iG_1(\tau/\lambda_1)}{\sqrt{\lambda_1}}, \quad Y_2 = \frac{iG_2(\tau/\lambda_2)}{\sqrt{\lambda_2}},$$

Assuming that $G_j(\nu) \sim g_j \nu^{k_j}$, where g_j is some constant, as $\lambda_j \rightarrow 0$ for $j = 1, 2$, the late-order terms take the general form as $\lambda_j \rightarrow 0$ given by

$$\begin{aligned} x_1^{(n)} &\sim g_1 \left(\frac{\tau}{\lambda_1} \right)^{k_1} \frac{\Gamma(n + \gamma_1)}{\lambda_1^{1/2}} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{n+\gamma_1}, \\ x_2^{(n)} &\sim g_2 \left(\frac{\tau}{\lambda_2} \right)^{k_2} \frac{\Gamma(n + \gamma_2)}{\lambda_2^{1/2}} \left(\frac{4i\lambda_2}{\tau^2} \right)^{n+\gamma_2}. \end{aligned}$$

The contribution to the late-order terms from the singularity at $p = -i\pi$ is therefore given by $x_1^{(n)} + x_2^{(n)}$. In order to determine the value of γ , we consider the behaviour near the singularity (as $\sigma_j \rightarrow 0$) and ensure that it is consistent when $n \rightarrow 0$ with the leading-order expression. As noted in the analysis of the stationary singularities, the behaviour of $x^{(0)}$ near $p = i\pi$ contains only logarithmic singularities at $p = \pm i\pi$, and no singularities at $\tau = 0$. At the next order, $x^{(1)}$ contains poles of order one at $p = \pm i\pi$, and so on. To ensure that the ansatz behaviour as $n \rightarrow 0$ near $p = -i\pi$ is consistent with the early-order terms in the series, the powers of τ and λ must be zero at $n = 0$. This gives

$$k_1 = k_2 = -1, \quad \gamma_1 = \gamma_2 = -1/2.$$

In order to simplify subsequent analysis, we define $\bar{G}_j = (\tau/\lambda_j)G_j$, giving the ansatz expressions as

$$x_1^{(n)} \sim \frac{\bar{G}_1 \Gamma(n - 1/2)}{(i\tau^2/4\lambda_1)^n}, \quad x_2^{(n)} \sim \frac{\bar{G}_2 \Gamma(n - 1/2)}{(-i\tau^2/4\lambda_2)^n}. \quad (3.107)$$

The form of $\bar{G}_j$ may be obtained by matching the ansatz form to the behaviour of the system near $\tau = 0$, $p = -i\pi$.

3.4.4.2 Inner problem

In order to consider the problem in the neighbourhood of the singularity at $p = i\pi$, we define the inner coordinates

$$\tau = \epsilon\alpha, \quad -4i\lambda_1 = \epsilon\sigma_1, \quad 4i\lambda_2 = \epsilon\sigma_2,$$

with the free surface defined on $\sigma_1 = -\sigma_2$. It follows that

$$\bar{G}_1 = \bar{G}_1 \left(\frac{\alpha}{\sigma_1} \right), \quad \bar{G}_2 = \bar{G}_2 \left(\frac{\alpha}{\sigma_2} \right), \quad \chi_1 = \frac{\epsilon\alpha^2}{\sigma_1}, \quad \chi_2 = \frac{\epsilon\alpha^2}{\sigma_2}, \quad (3.108)$$

and

$$\frac{\partial}{\partial p} = -\frac{4i}{\epsilon} \frac{\partial}{\partial \sigma_1} + \frac{4i}{\epsilon} \frac{\partial}{\partial \sigma_2}, \quad \frac{\partial}{\partial q} = -\frac{4}{\epsilon} \frac{\partial}{\partial \sigma_1} - \frac{4}{\epsilon} \frac{\partial}{\partial \sigma_2}, \quad \frac{\partial}{\partial \tau} = \frac{1}{\epsilon} \frac{\partial}{\partial \alpha}.$$

The governing equations (3.15,3.26) now take the form

$$\frac{4i}{\epsilon}x_{\sigma_1} - \frac{4i}{\epsilon}x_{\sigma_2} - \frac{4}{\epsilon}y_{\sigma_1} - \frac{4}{\epsilon}y_{\sigma_2} = \frac{1}{2} \left[\frac{1}{1 - \exp(-i\epsilon\sigma_1/4)} + \frac{1}{1 - \exp(i\epsilon\sigma_2/4)} \right], \quad (3.109)$$

$$-\frac{4}{\epsilon}x_{\sigma_1} - \frac{4}{\epsilon}x_{\sigma_2} - \frac{4i}{\epsilon}y_{\sigma_1} + \frac{4i}{\epsilon}y_{\sigma_2} = \frac{i}{2} \left[\frac{1}{1 - \exp(-i\epsilon\sigma_1/4)} - \frac{1}{1 - \exp(i\epsilon\sigma_2/4)} \right]. \quad (3.110)$$

The free-surface condition (3.17) becomes

$$\frac{1}{\epsilon}x_{\alpha\alpha} + \frac{4i}{\epsilon}y_{\sigma_1} - \frac{4i}{\epsilon}y_{\sigma_2} = 0 \quad \text{on} \quad \sigma_1 + \sigma_2 = 0. \quad (3.111)$$

The leading-order behaviour on the free surface given in (3.64) becomes

$$\begin{aligned} x^{(0)} &= -\log \left(1 - \exp \left(\frac{i\epsilon\sigma_1}{4} \right) \right) + \frac{1}{2} \log \left(1 - \exp \left(\frac{i\epsilon\sigma_1}{4} + \epsilon\alpha \right) \right), \\ &= -\log \left(1 - \exp \left(-\frac{i\epsilon\sigma_2}{4} \right) \right) + \frac{1}{2} \log \left(1 - \exp \left(-\frac{i\epsilon\sigma_2}{4} + \epsilon\alpha \right) \right), \end{aligned} \quad (3.112)$$

$$y^{(0)} = 0, \quad (3.113)$$

where we recall that $\sigma_1 = -\sigma_2$ on the free surface. As $\epsilon \rightarrow 0$, (3.109)–(3.110) become

$$2ix_{\sigma_1} - 2ix_{\sigma_2} - 2y_{\sigma_1} - 2y_{\sigma_2} \sim \frac{i}{\sigma_1} - \frac{i}{\sigma_2}, \quad (3.114)$$

$$-2x_{\sigma_1} - 2x_{\sigma_2} - 2iy_{\sigma_1} + 2iy_{\sigma_2} \sim -\frac{1}{\sigma_1} - \frac{1}{\sigma_2}, \quad (3.115)$$

while the boundary equation (3.111) becomes

$$x_{\alpha\alpha} + 4iy_{\sigma_1} - 4iy_{\sigma_2} = 0 \quad \text{on} \quad \sigma_1 + \sigma_2 = 0. \quad (3.116)$$

The leading-order behaviour (3.112)–(3.113) near the singularity is therefore

$$\begin{aligned} x^{(0)} &\sim -\log \left(-\frac{i\epsilon\sigma_1}{4} \right) + \frac{1}{2} \log \left(-\frac{i\epsilon\sigma_1}{4} - \alpha\epsilon \right) = -\log \left(\frac{i\epsilon\sigma_2}{4} \right) + \frac{1}{2} \log \left(\frac{i\epsilon\sigma_2}{4} - \alpha\epsilon \right), \\ y^{(0)} &= 0. \end{aligned}$$

Setting $x = \frac{1}{2} \log \epsilon + \hat{x}$, $y = \hat{y}$ gives

$$\hat{x}^{(0)} \sim -\log \left(-\frac{i\sigma_1}{4} \right) + \frac{1}{2} \log \left(-\frac{i\sigma_1}{4} - \alpha \right) = -\log \left(\frac{i\sigma_2}{4} \right) + \frac{1}{2} \log \left(\frac{i\sigma_2}{4} - \alpha \right), \quad (3.117)$$

$$\hat{y}^{(0)} = 0, \quad (3.118)$$

while (3.114)–(3.116) remain unchanged. Knowing the inner limit of the late-order terms ansatz given in (3.108), we set

$$\hat{x} \sim \hat{x}^{(0)} + \sum_{n=1}^{\infty} \frac{A_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{B_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n}, \quad \hat{y} \sim \hat{y}^{(0)} + \sum_{n=1}^{\infty} \frac{C_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{D_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n}.$$

In the equivalent expression for the stationary singularity, given in (3.91), there was only one late-order contribution which was dominant in the neighbourhood of the singularity, whereas

here, there are two terms in the series for $\hat{x}$ and $\hat{y}$. This is due to the ray behaviour seen in Figure 3.8, in which we see that there are two singular rays which intersect at each of $p = \pm i\pi$.

Applying this series expression to the governing fluid equations (3.117)–(3.118) gives

$$\sum_{n=1}^{\infty} \left\{ -4i \left[-\frac{\alpha}{\sigma_1^2} \frac{A'_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{nA_n(\alpha/\sigma_1)}{\sigma_1(\alpha^2/\sigma_1)^n} \right] + 4i \left[-\frac{\alpha}{\sigma_2^2} \frac{B'_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n} + \frac{nB_n(\alpha/\sigma_2)}{\sigma_2(\alpha^2/\sigma_2)^n} \right] \right. \\ \left. - \left[-\frac{\alpha}{\sigma_1^2} \frac{C'_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{nC_n(\alpha/\sigma_1)}{\sigma_1(\alpha^2/\sigma_1)^n} \right] - \left[-\frac{\alpha}{\sigma_2^2} \frac{D'_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n} + \frac{nD_n(\alpha/\sigma_2)}{\sigma_2(\alpha^2/\sigma_2)^n} \right] \right\} = 0.$$

Matching terms as α/σ_1 and α/σ_2 respectively become small gives (for $n \geq 1$)

$$0 = -i \left[\frac{\alpha}{\sigma_1} A'_n \left(\frac{\alpha}{\sigma_1} \right) - nA_n \left(\frac{\alpha}{\sigma_1} \right) \right] - \left[\frac{\alpha}{\sigma_1} C'_n \left(\frac{\alpha}{\sigma_1} \right) - nC_n \left(\frac{\alpha}{\sigma_1} \right) \right], \quad (3.119)$$

$$0 = i \left[\frac{\alpha}{\sigma_2} B'_n \left(\frac{\alpha}{\sigma_2} \right) - nB_n \left(\frac{\alpha}{\sigma_2} \right) \right] - \left[\frac{\alpha}{\sigma_2} D'_n \left(\frac{\alpha}{\sigma_2} \right) - nD_n \left(\frac{\alpha}{\sigma_2} \right) \right]. \quad (3.120)$$

Applying the series expansion to the boundary equation (3.116) gives for $n \geq 1$

$$0 = \sum_{n=1}^{\infty} \left\{ \left[\frac{A''_n(\alpha/\sigma_1)}{\sigma_1^2(\alpha^2/\sigma_1)^n} - \frac{4nA'_n(\alpha/\sigma_1)}{\alpha\sigma_1(\alpha^2/\sigma_1)^n} + \frac{2n(2n+1)A_n(\alpha/\sigma_1)}{\alpha^2(\alpha^2/\sigma_1)^n} \right] \right. \\ + \left[\frac{B''_n(\alpha/\sigma_2)}{\sigma_2^2(\alpha^2/\sigma_2)^n} - \frac{4nB'_n(\alpha/\sigma_2)}{\alpha\sigma_2(\alpha^2/\sigma_2)^n} + \frac{2n(2n+1)B_n(\alpha/\sigma_2)}{\alpha^2(\alpha^2/\sigma_2)^n} \right] \\ \left. + 4i \left[-\frac{\alpha}{\sigma_1^2} \frac{C'_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{nC_n(\alpha/\sigma_1)}{\sigma_1(\alpha^2/\sigma_1)^n} \right] - 4i \left[-\frac{\alpha}{\sigma_2^2} \frac{D'_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n} + \frac{nD_n(\alpha/\sigma_2)}{\sigma_2(\alpha^2/\sigma_2)^n} \right] \right\}. \quad (3.121)$$

It is impossible to distinguish between α/σ_1 and α/σ_2 on the free surface, as $\sigma_1 = -\sigma_2$, and the magnitude of both will become small at the same rate. Hence, in order to obtain two separate equations from (3.121), we cannot simply match as the two variables tend become large independently. Instead, we recall the reasoning in Section 3.4.1, which claimed that both contributions must have the same magnitude at all orders of n in order to be consistent with the leading-order behaviour of the problem.

This condition can only be satisfied if (3.121) is split into

$$0 = \sum_{n=1}^{\infty} \left\{ \left[\frac{A''_n(\alpha/\sigma_1)}{\sigma_1^2(\alpha^2/\sigma_1)^n} - \frac{4nA'_n(\alpha/\sigma_1)}{\alpha\sigma_1(\alpha^2/\sigma_1)^n} + \frac{2n(2n+1)A_n(\alpha/\sigma_1)}{\alpha^2(\alpha^2/\sigma_1)^n} \right] \right. \\ \left. + 4i \left[-\frac{\alpha}{\sigma_1^2} \frac{C'_n(\alpha/\sigma_1)}{(\alpha^2/\sigma_1)^n} + \frac{nC_n(\alpha/\sigma_1)}{\sigma_1(\alpha^2/\sigma_1)^n} \right] \right\}, \\ 0 = \sum_{n=1}^{\infty} \left\{ \left[\frac{B''_n(\alpha/\sigma_2)}{\sigma_2^2(\alpha^2/\sigma_2)^n} - \frac{4nB'_n(\alpha/\sigma_2)}{\alpha\sigma_2(\alpha^2/\sigma_2)^n} + \frac{2n(2n+1)B_n(\alpha/\sigma_2)}{\alpha^2(\alpha^2/\sigma_2)^n} \right] \right. \\ \left. - 4i \left[-\frac{\alpha}{\sigma_2^2} \frac{D'_n(\alpha/\sigma_2)}{(\alpha^2/\sigma_2)^n} + \frac{nD_n(\alpha/\sigma_2)}{\sigma_2(\alpha^2/\sigma_2)^n} \right] \right\},$$

which ensures that $|A_n| = |B_n|$ and $|C_n| = |D_n|$ (and hence the weighting of the respective ray contributions) is equal for all orders of n , and therefore consistent with the complex ray structure

of the problem, and hence the leading-order behaviour $z^{(0)}$. Matching these expressions as $|\alpha/\sigma_1|$ and $|\alpha/\sigma_2|$ become small gives

$$\left[\frac{\alpha^2}{\sigma_1^2} A''_{n-1} \left(\frac{\alpha}{\sigma_1} \right) - 4(n-1) \frac{\alpha}{\sigma_1} A'_{n-1} \left(\frac{\alpha}{\sigma_1} \right) + (2n-2)(2n-1) A_{n-1} \left(\frac{\alpha}{\sigma_1} \right) \right] - 4i \left[\frac{\alpha}{\sigma_1} C'_n \left(\frac{\alpha}{\sigma_1} \right) - n C_n \left(\frac{\alpha}{\sigma_1} \right) \right] = 0, \quad (3.122)$$

$$\left[\frac{\alpha^2}{\sigma_2^2} B''_{n-1} \left(\frac{\alpha}{\sigma_2} \right) - 4(n-1) \frac{\alpha}{\sigma_2} B'_{n-1} \left(\frac{\alpha}{\sigma_2} \right) + (2n-2)(2n-1) B_{n-1} \left(\frac{\alpha}{\sigma_2} \right) \right] + 4i \left[\frac{\alpha}{\sigma_2} D'_n \left(\frac{\alpha}{\sigma_2} \right) - n D_n \left(\frac{\alpha}{\sigma_2} \right) \right] = 0. \quad (3.123)$$

We will first consider (3.120) and (3.123), setting $\beta = \alpha/\sigma_1$. This gives the equations the series coefficients as

$$\begin{aligned} 0 &= [\beta A'_n(\beta) - n A_n(\beta)] - i [\beta C'_n(\beta) - n C_n(\beta)], \\ 0 &= [\beta^2 A''_{n-1}(\beta) - 4(n-1) \beta A'_{n-1}(\beta) + (2n-2)(2n-1) A_{n-1}(\beta)] + 4i [\beta C'_n(\beta) - n C_n(\beta)]. \end{aligned}$$

Solving this system gives

$$\begin{aligned} A_n &= -\frac{\beta^n}{4} \int \left\{ \beta^{-n+1} A''_{n-1}(\beta) - 4(n-1) \beta^{-n} A'_{n-1}(\beta) \right. \\ &\quad \left. + (2n-2)(2n-1) \beta^{-n-1} A_{n-1}(\beta) \right\} d\beta + K_A^{(n)} \beta^n, \end{aligned} \quad (3.124)$$

where $K_A^{(n)}$ is a constant determined by the initial conditions. Typically, this involves determining the small-time behaviour of the system, and matching the expression at each order of n . In the interests of expediency, however, we omit this step and instead simply determine the appropriate value of $K_A^{(n)}$ by comparing the expression with the corresponding contribution from the steepest descent analysis in Section 3.3.

In this case, the equivalent expression in the steepest descents analysis is given by $\bar{\xi}_{E,1} + \bar{\xi}_{E,7}$, seen in (3.55). Specifically, this gives the late-order terms of the free-surface position, or $y^{(n)}$. By comparing this expression with C_n (and hence A_n) at any order of n , we see that specifying $K_A^{(n)}$ is equivalent to specifying the coefficient of the series term associated with $j = n$ at each order of ϵ . Consequently, we find that the values for $K_A^{(n)}$ which give the correct late-order term behaviour are given by

$$\begin{aligned} K_A^{(1)} &= 0, \\ K_A^{(n)} &= \frac{i}{2} \frac{\Gamma(n)}{(-4)^n}, \quad n > 1. \end{aligned}$$

In practice, if we had not performed the steepest-descents analysis previously, we would instead require an expression for the small-time behaviour of the system near the singularity in order to obtain these constants.

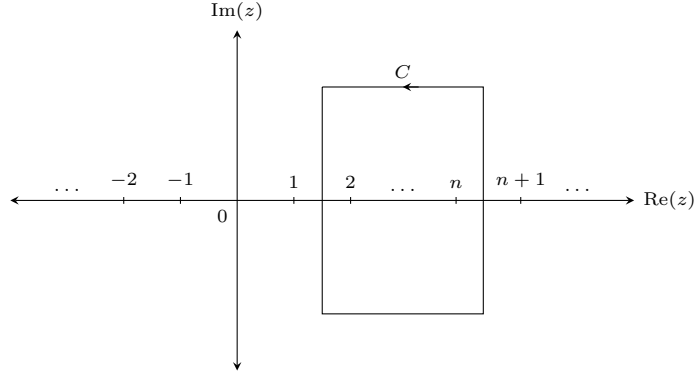


Figure 3.9: Integral contour for (3.126) in the complex- z plane.

Consequently, the iterative integral expression (3.124) gives

$$A_n = -\frac{1}{2} \sum_{j=2}^n \frac{(-i)^j \Gamma(2n-j)}{4^{n-j} \Gamma(n-j+1)} \beta^j. \quad (3.125)$$

While this expression is sufficient to match with the inner limit of the outer solution given in (3.107), it would be convenient to have a simple closed-form expression for B_n as $n \rightarrow \infty$. In order to obtain such an expression, we follow [24] and express the summation in (3.125) as a closed line integral in the complex plane, given by

$$A_n = -\frac{1}{2} \frac{\Gamma(n)}{4^n} \frac{1}{2\pi i} \oint_C \frac{\Gamma(2n-z)}{\Gamma(n)\Gamma(n-z+1)} (4i\beta)^z \frac{\pi}{\sin \pi z} dz, \quad (3.126)$$

where the contour C is any closed curve that contains all integers z such that $2 \leq z \leq n$. Consequently, we define C such that it has one end-point fixed at $z = 3/2$, and the other fixed at $z = n + 1/2$, as seen in Figure 3.9. It may be shown that the integrand decays rapidly away from these end-points. Hence, the asymptotic expression for the integral contains contributions from the two end-points, as well as any saddle points relevant to the problem.

This contribution from the end-point at $z = 3/2$ may be written as

$$I_{E,1} \sim -\frac{1}{2} \frac{\Gamma(n)}{4^n} \frac{1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \frac{\Gamma(2n-z)}{\Gamma(n)\Gamma(n-z+1)} (4i\beta)^z \frac{\pi}{\sin \pi z} dz,$$

where $I_{E,j}$ refers to the end-point contribution near $z = 3/2$ for $j = 1$ and $z = n + 1/2$ for $j = 2$. The integrand of each gamma function is always large and positive in the region of interest. Using Stirling's formula to approximate the gamma functions for large n therefore gives

$$I_{E,1} \sim -\frac{1}{2} \frac{\Gamma(n)}{4^n} \frac{1}{2\pi i} \int_{3/2-\infty i}^{3/2+\infty i} \frac{2^{-1-z} 4^n}{\sqrt{n\pi}} (4i\beta)^z \frac{\pi}{\sin \pi z} dz.$$

By closing the integrand either to the right or the left and evaluating this as a summation of

residues, we find

$$\begin{aligned} I_{E,1} &\sim -\frac{\Gamma(n-1/2)}{4\sqrt{\pi}} \sum_{j=1}^{\infty} (-2i\beta)^j - \frac{i\beta\Gamma(n-1/2)}{2\sqrt{\pi}}, \\ &= \frac{i\beta\Gamma(n-1/2)}{2\sqrt{\pi}(1+2i\beta)} - \frac{i\beta\Gamma(n-1/2)}{2\sqrt{\pi}}, \end{aligned} \quad (3.127)$$

where the sum was taken of all residues such that $z > 0$, and the value of the residue at $z = 1$ subtracted from the result. The contribution was expressed in this form so that the Stokes switching could be calculated for each of the two terms individually.

Setting $z = u + n$, the contribution from the end-point at $z = n + 1/2$ may be written as

$$I_{E,1} \sim -\frac{1}{2} \frac{\Gamma(n)}{4^n} \frac{1}{2\pi i} \int_{1/2-\infty i}^{1/2+\infty i} \frac{\Gamma(n-u)}{\Gamma(n)\Gamma(1-u)} (4i\beta)^{u+n} \frac{\pi(-1)^n}{\sin \pi u} du.$$

We may expand $\Gamma(n-u)$ and $\Gamma(n)$ using Stirling's formula, but not $\Gamma(1-u)$, as the integrand is not necessarily large. This gives

$$I_{E,1} \sim -\frac{1}{2} \frac{\Gamma(n)}{4^n} \frac{1}{2\pi i} (-4i\beta)^u \int_{1/2-\infty i}^{1/2+\infty i} \frac{1}{\Gamma(1-u)} \left(\frac{4i\beta}{n}\right)^n \frac{\pi}{\sin \pi u} du.$$

Closing the contour to the right gives a pole contribution of zero, as the poles usually found at $u \in \mathbb{Z}$ are removed by the zeros present in $1/\Gamma(1-u)$ for $u \in \mathbb{Z}^+$. The same is true when the integral is closed to the left, as the contribution from the poles found within $0 \leq z < n$ in (3.126) precisely cancels the contribution from the poles found in the region $z < 0$. Consequently, there is no contribution from the second end-point in the final integral expression at leading order for large n .

Finally, setting $z = nv$ and using a similar method to that found in [24], we see that the integral (3.126) may be written as

$$\int_C F(v; \beta, n) e^{nf(v; \beta, n)} dv,$$

where

$$f(v; \beta, n) = \begin{cases} (2-v)\log(2-v) - (1-v)\log(1-v) + v\log(4i\beta) + i\pi v & \text{Im}(v) > 0, \\ (2-v)\log(2-v) - (1-v)\log(1-v) + v\log(4i\beta) - i\pi v & \text{Im}(v) < 0. \end{cases}$$

This transformation sets the end-points of the integral at $v = 0$ and $v = 1$. Solving $f'(v_s) = 0$ shows that there is a saddle point present in the integral, located at

$$v_s = \frac{2 + 4i\beta}{1 + 4i\beta}.$$

Comparing the imaginary part of $f(v_s)$ with the imaginary parts of $f(0)$ and $f(1)$ shows that the steepest descent path does not pass through the saddle point for any $\tau > 0$. Consequently, the expression given in (3.127) is the asymptotic behaviour of B_n as $n \rightarrow \infty$. Converting this into the outer coordinates of the problem gives

$$A_n \sim \frac{1}{4} \frac{\Gamma(n-1/2)}{\sqrt{\pi}} \frac{\tau}{\tau + 2\lambda_1} - \frac{\tau}{8\lambda_1\sqrt{\pi}}. \quad (3.128)$$

Matching this to the form of the late-order terms in the outer solution given in (3.108), we find that

$$\overline{G}_1 = \frac{1}{4\sqrt{\pi}} \frac{\tau}{\tau + 2\lambda_1} - \frac{\tau}{8\lambda_1\sqrt{\pi}}. \quad (3.129)$$

A similar analysis performed on (3.119) and (3.122) gives

$$\overline{G}_2 = \frac{1}{4\sqrt{\pi}} \frac{\tau}{\tau + 2\lambda_2} - \frac{\tau}{8\lambda_2\sqrt{\pi}}.$$

Hence, the ansatz expressions in (3.107) for the late order series terms have been completely determined, and Stokes smoothing may be applied in order to determine the behaviour that is switched as the Stokes lines are crossed.

Combining the resultant ansatz expressions gives late-order terms that may be expressed as

$$\epsilon^n x_1^{(n)} + \epsilon^n x_2^{(n)} \sim \frac{\overline{G}_1 \Gamma(n - 1/2)}{(it^2/4\lambda_1)^n} + \frac{\overline{G}_2 \Gamma(n - 1/2)}{(-it^2/4\lambda_2)^n}. \quad (3.130)$$

We note, however, that on the free surface ($q = 0$), we have $\overline{G}_1 = \overline{G}_2$. Hence, we simplify this to give

$$\epsilon^{2n} x_1^{(n)} + \epsilon^{2n} x_2^{(n)} \sim \frac{2\overline{G}_1 \Gamma(2n - 1/2)}{(it^2/4\lambda_1)^{2n}}. \quad (3.131)$$

Similarly, we find that

$$\epsilon^{2n+1} y_1^{(n)} + \epsilon^{2n+1} y_2^{(n)} \sim \frac{2i\overline{G}_1 \Gamma(2n + 1/2)}{(it^2/4\lambda_1)^{2n+1}}. \quad (3.132)$$

We also note that the late-order terms in the neighbourhood of the singularity at $p = i\pi$ will produce a late-order contribution that is the complex conjugate of the late-order terms given in (3.131)–(3.132), and hence that the combined expression is purely real.

3.4.4.3 Stokes smoothing

We will consider the Stokes contribution switched on by terms given in (3.131)–(3.132). Additionally, there will be a contribution switched on by corresponding terms associated with the singularity at $p = i\pi$ on the complex free surface, providing a further contribution which is equal to the complex conjugate of the contribution caused by the singularity at $p = -i\pi$. Hence the overall contribution will be equal to twice the real part of the contribution under consideration here.

Furthermore, we note that the expressions given for $\overline{G}_1$ found in (3.129) contain two distinct terms. The Stokes switching behaviour associated with these two terms will be considered individually, with the resultant expressions subsequently being combined in order to obtain a single expression for the exponentially small behaviour caused by the singularities at $p = -i\pi$.

The late-order terms for the second contribution from the singularity at $p = -i\pi$ were shown to take the form

$$\epsilon^{2n} x^{(n)} \sim \epsilon^{2n} x_A^{(n)} + \epsilon^{2n} x_B^{(n)}, \quad \epsilon^{2n+1} y^{(n)} \sim \epsilon^{2n+1} y_A^{(n)} + \epsilon^{2n+1} y_B^{(n)},$$

where

$$x_A^{(n)} \sim \frac{1}{2} \frac{\Gamma(2n-1/2)}{\sqrt{\pi}} \frac{\tau}{\tau+2\lambda_1} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{2n}, \quad x_B^{(n)} \sim -\frac{1}{4} \frac{\Gamma(2n-1/2)}{\sqrt{\pi}} \frac{\tau}{\lambda_1} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{2n}, \quad (3.133)$$

$$y_A^{(n)} \sim \frac{i}{2} \frac{\Gamma(2n+1/2)}{\sqrt{\pi}} \frac{\tau}{\tau+2\lambda_1} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{2n+1}, \quad y_B^{(n)} \sim -\frac{i}{4} \frac{\Gamma(2n+1/2)}{\sqrt{\pi}} \frac{\tau}{\lambda_1} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{2n+1}. \quad (3.134)$$

To obtain the optimally-truncated series, we set

$$x_A \sim \sum_{n=0}^{N-1} \epsilon^{2n} x_A^{(n)} + R_A^{(N)}, \quad y_A \sim \sum_{n=0}^{N-1} \epsilon^{2n+1} y_A^{(n)} + S_A^{(N)}.$$

Applying this series expression to the governing equations (3.15), (3.26) gives $R^{(N)} = -iS^{(N)}$.

The equation on the free surface (3.17) becomes

$$\epsilon \sum_{n=0}^{N-1} \epsilon^{2n} x_{A,\tau\tau}^{(n)} + \epsilon R_{A,\tau\tau}^{(N)} + \sum_{n=0}^{N-1} \epsilon^{2n+1} y_{A,\lambda_1}^{(n)} + S_{A,\lambda_1}^{(N)} = 0,$$

where the derivative is expressed in terms of λ_1 in order to simplify subsequent analysis. Eliminating terms gives

$$\epsilon R_{A,\tau\tau}^{(N)} + S_{A,\lambda_1}^{(N)} \sim \epsilon^{2N} y_{A,\lambda_1}^{(N)}. \quad (3.135)$$

Using similar reasoning to the analysis of the stationary singularity contributions, and recalling that $X_1 = iY_1$, we set the remainders to be of the form

$$R_A^{(N)} = iA(\lambda_2, \tau) \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right), \quad S_A^{(N)} = A(\lambda_1, \tau) \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right),$$

giving the expression

$$-\frac{i}{2\lambda_1} (2i\tau A_\tau + iA + 2i\lambda_1 A_{\lambda_1} - 2\lambda_1 \epsilon A_{\tau\tau}) e^{-\chi/\epsilon} \sim \epsilon^{2N} y_{\lambda_1}^{(N)}. \quad (3.136)$$

Solving the homogeneous version of (3.136) gives

$$A = \frac{1}{\sqrt{\lambda_1}} B\left(\frac{\tau}{\lambda_1}\right).$$

Noting this, we set

$$A = \frac{1}{\sqrt{\lambda_1}} B(u, v), \quad (3.137)$$

where

$$u = \frac{i\tau^2}{4\lambda_1}, \quad v = \frac{\tau}{\lambda_1}. \quad (3.138)$$

Using the asymptotic expression for $y_A^{(N)}$ given in (3.134), the equation becomes

$$\frac{i}{4} \frac{\tau^2}{\lambda_1^{5/2}} B_u e^{-u/\epsilon} \sim \epsilon^{2N} \left[\frac{i}{2} \frac{\Gamma(2N+1/2)}{\sqrt{\pi}} \frac{\tau}{2\lambda_1 + \tau} \left(-\frac{4i\lambda_1}{\tau^2} \right)^{2N+1} \right]_{\lambda_1}.$$

Changing coordinates and rearranging gives

$$B_u \sim \epsilon^{2N} g(v) \frac{\Gamma(2N+1/2)}{u^{2N+1/2} \sqrt{\pi}} e^{u/\epsilon}. \quad (3.139)$$

where

$$g(v) = \frac{i\sqrt{-i}}{2+v}. \quad (3.140)$$

Minimizing the absolute value of the right-hand side of (3.139) with respect to N in the limit that $N \rightarrow \infty$ gives $N \sim |\chi|/2\epsilon$. Setting $u = re^{i\theta}$ and expressing the optimal truncation as $N = r/\epsilon + \alpha$, where $0 \leq \alpha < 1$ to ensure that N is an integer, gives

$$B_\theta \sim ire^{i\theta} \epsilon^{r/\epsilon+2\alpha} g(v) \frac{\Gamma(r/\epsilon+2\alpha+1/2)}{\sqrt{\pi}(re^{i\theta})^{r/\epsilon+2\alpha+1/2}} \exp\left(\frac{re^{i\theta}}{\epsilon}\right),$$

This reduces to

$$B_\theta \sim ig(v) \sqrt{2r} \exp \left[i\theta - \frac{r}{\epsilon} - i\theta \left(\frac{r}{\epsilon} + 2\alpha - \frac{1}{2} \right) + \frac{re^{i\theta}}{\epsilon} \right].$$

This variation is exponentially small, except where $\theta = 0$, meaning that $\theta = 0$ corresponds to Stokes lines in the problem. Furthermore, we see that $\theta = 0$ corresponds to $\tau = 0$ as well as $p = 0$, indicating that the Stokes lines are given by these curves. We set $\theta = \epsilon^{1/2} \bar{\theta}$, and obtain

$$B_{\bar{\theta}} \sim ig(v) \sqrt{2r\epsilon} \exp \left(-\frac{r^2}{2} \bar{\theta}^2 \right). \quad (3.141)$$

Integrating this gives

$$B \sim ig(v) \sqrt{2r\epsilon} \left(\int_{-\infty}^{\bar{\theta}} e^{-r\bar{\theta}^2/2} d\bar{\theta} + C \right), \quad (3.142)$$

where C is a constant typically determined by matching. In this case, however, we note that if this contribution is active at $\tau = 0$, it must be equal to zero at this point, as it is not present in the initial condition. However, as previously noted, $\tau = 0$ corresponds to $\bar{\theta} = 0$, and therefore C is defined such that

$$B \sim ig(v) \sqrt{2r\epsilon} \int_0^{\bar{\theta}} e^{-r\bar{\theta}^2/2} d\bar{\theta}.$$

Therefore, when the Stokes line is crossed, noting (3.140), we find that B switches such that

$$[B]_{-}^{+} \sim 2ig(v) \sqrt{\pi\epsilon} = -\frac{2\sqrt{-i\pi\epsilon}}{2+v} = -\frac{2\lambda_1 \sqrt{-i\pi\epsilon}}{2\lambda_1 + \tau}.$$

Recalling (3.137), we therefore find that

$$[A]_{-}^{+} \sim -\frac{2\sqrt{-i\pi\epsilon\lambda_1}}{2\lambda_1 + \tau}.$$

Consequently, the quantity switched as the Stokes line is crossed is given by

$$[S_A^{(N)}]_{-}^{+} = -\frac{2\sqrt{-i\pi\epsilon\lambda_1}}{2\lambda_1 + \tau} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right).$$

As in the previous contribution, the combined late-order term behaviour is equal to twice times the real part of this expression (as there is a second contribution caused by the singularity at $p = i\pi$ producing complex conjugate behaviour), or

$$[S_{A,\text{TOT}}^{(N)}]_{-}^{+} \sim -\frac{2\sqrt{-i\pi\epsilon\lambda_1}}{2\lambda_1 + \tau} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) - \frac{2\sqrt{i\pi\epsilon\lambda_1^*}}{2\lambda_1^* + \tau} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right),$$

where λ_1^* is the complex conjugate of λ_1 . Knowing that this must equal zero at the origin (where $\tau = 0$), we find that the contribution is given by

$$S_{A,\text{TOT}}^{(N)} \sim \begin{cases} -\frac{\sqrt{-i\pi\epsilon\lambda_1}}{2\lambda_1 + \tau} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) - \frac{\sqrt{i\pi\epsilon\lambda_1^*}}{2\lambda_1^* + \tau} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right), & \theta > 0, \\ \frac{\sqrt{-i\pi\epsilon\lambda_1}}{2\lambda_1 + \tau} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) + \frac{\sqrt{i\pi\epsilon\lambda_1^*}}{2\lambda_1^* + \tau} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right), & \theta < 0, \end{cases} \quad (3.143)$$

where switching occurs smoothly and rapidly in the neighbourhood of $\theta = 0$. From the definition of θ , we see that this corresponds to the curves $\tau = 0$ and $p = 0$. This agrees precisely with the saddle-point contributions found in (3.58)–(3.59), which was determined for the case $t > 0$, or $\tau > 0$ in Lagrangian coordinates.

Performing a similar analysis the late terms defined by $x_B^{(n)}$ and $y_B^{(n)}$ gives

$$[S_{B,\text{TOT}}^{(N)}]_{-}^{+} \sim \frac{2\sqrt{-i\pi\epsilon}}{2\sqrt{\lambda_1}} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) + \frac{2\sqrt{i\pi\epsilon}}{2\sqrt{\lambda_1^*}} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right).$$

Again, observing that the value of $S_{B,\text{TOT}}^{(N)}$ must be zero on $\tau = 0$, we find that

$$S_{B,\text{TOT}}^{(N)} \sim \begin{cases} \frac{\sqrt{-i\pi\epsilon}}{2\sqrt{\lambda_1}} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) + \frac{\sqrt{i\pi\epsilon}}{2\sqrt{\lambda_1^*}} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right), & \theta > 0, \\ -\frac{\sqrt{-i\pi\epsilon}}{2\sqrt{\lambda_1}} \exp\left(-\frac{i\tau^2}{4\lambda_1\epsilon}\right) - \frac{\sqrt{i\pi\epsilon}}{2\sqrt{\lambda_1^*}} \exp\left(\frac{i\tau^2}{4\lambda_1^*\epsilon}\right), & \theta < 0. \end{cases} \quad (3.144)$$

On the free surface, $\lambda_1 = p + i\pi$ and $\lambda_1^* = p - i\pi$. Hence, the sum of these two expressions, denoted $S_{\text{TOT}}^{(N)}$ is given by

$$S_{\text{TOT}}^{(N)} \sim \frac{\sqrt{-i\pi\epsilon}}{2\sqrt{p+i\pi}} \left(\frac{\tau}{2(p+i\pi)+\tau} \right) \exp\left(-\frac{i\tau^2}{4(p+i\pi)\epsilon}\right) + \frac{\sqrt{i\pi\epsilon}}{2\sqrt{p-i\pi}} \left(\frac{\tau}{2(p-i\pi)+\tau} \right) \exp\left(\frac{i\tau^2}{4(p-i\pi)\epsilon}\right), \quad (3.145)$$

for $\theta > 0$, and

$$S_{\text{TOT}}^{(N)} \sim -\frac{\sqrt{-i\pi\epsilon}}{2\sqrt{p+i\pi}} \left(\frac{\tau}{2(p+i\pi)+\tau} \right) \exp\left(-\frac{i\tau^2}{4(p+i\pi)\epsilon}\right) - \frac{\sqrt{i\pi\epsilon}}{2\sqrt{p-i\pi}} \left(\frac{\tau}{2(p-i\pi)+\tau} \right) \exp\left(\frac{i\tau^2}{4(p-i\pi)\epsilon}\right), \quad (3.146)$$

for $\theta < 0$. Given that $\theta = \arg(\chi)$, we see that when $\tau > 0$, $\theta > 0$ for $p > 0$ and $\theta < 0$ for $p < 0$. As $x \sim p + \tau$ in the linearized problem, this therefore gives the same result as the equivalent terms in the steepest descents analysis, found in (3.60).

3.4.5 Complete Stokes structure

The prefactor for late-order terms related to the moving singularity, given in (3.129) was found by taking the limit of a series expression (3.125) as $n \rightarrow \infty$, giving (3.128). However, this limit was found by performing an asymptotic analysis on the integral (3.126). Consequently, we find that the limiting expression given in (3.128), and hence the prefactor of the late-order terms, is actually the first term of a second divergent asymptotic expansion. This second asymptotic expansion will also display Stokes switching, as described in Chapter 1.

In a similar fashion to the analysis of exponentially subdominant Stokes switching found in [24], we will determine the behaviour of the exponentially subdominant terms in this second expansion by optimally-truncating this second asymptotic expansion, and determining the behaviour in the neighbourhood of the Stokes lines. We define the second asymptotic series, with (3.143) giving the form of the leading-order term in this series, given on $q = 0$ by

$$S_{A,TOT}^{(N)} \sim \sum_{n=0}^{\infty} A^{(n)}(\sigma_2, \tau) \epsilon^{n+1/2} \exp\left(\frac{i\tau^2}{4\epsilon\sigma_2}\right), \quad (3.147)$$

where $\sigma_2 = p - i\pi$ (correspondingly, $\sigma_1 = p + i\pi$), and

$$A^{(0)}(\sigma_2, \tau) = -\frac{\sqrt{i\pi\sigma_2}}{2\sigma_2 + \tau}. \quad (3.148)$$

We do not include the behaviour found in (3.144), however a separate analysis of this contribution shows that it does not generate a series expression, and hence does not produce late-order terms. For ease of notation, $S_{A,TOT}^{(N)}$ will subsequently be denoted $S^{(N)}$. Applying this expression to the homogeneous form of (3.135), which is valid for the combined sum of the two contributions on the free surface, gives

$$-i\epsilon S_{\tau\tau}^{(N)} + S_{\sigma_2}^{(N)} = 0, \quad (3.149)$$

giving

$$-iA_{\tau\tau}^{(n-1)} + \frac{\tau}{\sigma_2} A_{\tau}^{(n)} + \frac{1}{2\sigma_2} A^{(n)} + A_{\sigma_2}^{(n)} = 0. \quad (3.150)$$

Using scaling arguments, the solution may be expressed in the form

$$A^{(n)} = \frac{1}{\sigma_2^{n+1/2}} f_n\left(\frac{\tau}{\sigma_2}\right).$$

The equation (3.150) then reduces to (setting $\eta = \tau/\sigma_1$)

$$nf_n(\eta) + if_{n-1}''(\eta) = 0, \quad (3.151)$$

while the expression for $A^{(0)}$ given in (3.148) in this coordinate system becomes

$$f_0(\eta) = -\frac{\sqrt{i\pi}}{\eta+2}. \quad (3.152)$$

Following [24], we are interested only in the late-order expression for f_n . Hence, we prescribe a late-order ansatz as $n \rightarrow \infty$ of the form

$$f_n \sim \frac{F(\eta)\Gamma(n+\gamma)}{V(\eta)^{n+\gamma}}.$$

Equation (3.151) now gives (to leading order as $n \rightarrow \infty$)

$$\frac{nF\Gamma(n+\gamma)}{V^{n+\gamma}} + \frac{iV_\eta^2 F\Gamma(n+\gamma+1)}{V^{n+\gamma+1}} = 0,$$

which simplifies in the limit that $n \rightarrow \infty$ to give

$$V + iV_\eta^2 = 0,$$

where $V(-2) = 0$ in order for the solution to be consistent with (3.152). Solving this equation for $V(\eta)$ gives

$$V(\eta) = \frac{i}{4}(\eta+2)^2,$$

which gives

$$V(\eta) = \frac{1}{\sigma_2} \left[\frac{i\tau^2}{4\sigma_2} + i(\tau + \sigma_2) \right] = \frac{1}{\sigma_2} \left[i(p + \tau - i\pi) - \left(-\frac{i\tau^2}{4(p - i\pi)} \right) \right].$$

This is clearly the difference between the singulants of the two late-order terms generated by the stationary and moving singularities respectively, seen by comparing this expression to (3.94) and (3.131) respectively. Taking (3.151) to the next order as $n \rightarrow \infty$ now gives

$$\gamma F + 2iF_\eta V_\eta + iV_{\eta\eta} F = 0,$$

which gives

$$G = \Lambda(\eta+2)^{\gamma-1/2}.$$

This implies that

$$f_n \sim \frac{\Lambda(\eta+2)^{\gamma-1/2}\Gamma(n+\gamma)}{[\frac{i}{4}(\eta+2)^2]^{n+\gamma}},$$

and matching this expression with f_0 as $n \rightarrow 0$ in the neighbourhood of $\eta = -2$ gives $\gamma = 1/2$ and $\Lambda = -i/2$. Hence,

$$A^{(n)} = -\frac{i\Gamma(n+1/2)}{2 \left[\frac{i\sigma_2}{4} \left(\frac{\tau}{\sigma_2} + 2 \right)^2 \right]^{n+1/2}}.$$

Truncating the series for $S^{(N)}$ given in (3.147) produces

$$S^{(N)} = \sum_{n=0}^{N-1} A^{(n)}(\sigma_2, \tau) \epsilon^{n+1/2} \exp\left(\frac{i\tau^2}{4\sigma_2}\right) + Q,$$

which gives, when used in (3.149),

$$-i\epsilon Q_{\tau\tau} + Q_{\sigma_2} = \sum_{n=0}^{N-1} \epsilon^{n+1/2} \left[i\epsilon A_{\tau\tau}^{(n)} - \frac{\tau}{\sigma_2} A_{\tau}^{(n)} - \frac{1}{2\sigma_2} A^{(n)} - A_{\sigma_2}^{(n)} \right] \exp\left(\frac{i\tau^2}{4\epsilon\sigma_2}\right). \quad (3.153)$$

Noting (3.150), this becomes

$$\begin{aligned} -i\epsilon Q_{\tau\tau} + Q_{\sigma_2} &= \epsilon^{N+1/2} [iA_{\tau\tau}^{(N-1)}] \exp\left(\frac{i\tau^2}{4\epsilon\sigma_2}\right) \\ &+ \epsilon^{N-1/2} \left[iA_{\tau\tau}^{(N-2)} - \frac{\tau}{\sigma_2} A_{\tau}^{(N-1)} - \frac{1}{2\sigma_2} A^{(N-1)} - A_{\sigma_2}^{(N-1)} \right] \exp\left(\frac{i\tau^2}{4\epsilon\sigma_2}\right). \end{aligned} \quad (3.154)$$

In order to detect the Stokes switching behaviour, we set

$$Q = C(\sigma_2, \tau) e^{-i(\sigma_2 + \tau)/\epsilon}.$$

The homogeneous system $-i\epsilon Q_{\tau\tau} + Q_{\sigma_2} = 0$ is solved when $C = C(2\sigma_2 + \tau)$. We therefore set

$$C = C \left[\frac{i\sigma_2}{4} \left(\frac{\tau}{\sigma_2} + 2 \right)^2, 2\sigma_2 + \tau \right] = C(u, v).$$

Equation (3.154) now becomes

$$\begin{aligned} \frac{4iu^2}{v^2} C_u &\sim \left(\epsilon^{N+1/2} \left[\frac{i u_{\tau}^2 \Lambda \Gamma(N+3/2)}{u^{N+3/2}} + \dots \right] + \epsilon^{N-1/2} \left[\frac{i u_{\tau}^2 \Lambda \Gamma(N+1/2)}{u^{N+1/2}} \right. \right. \\ &\left. \left. + \left(\frac{\tau}{\sigma_2} \right) \frac{u_{\tau} \Lambda \Gamma(N+1/2)}{u^{N+1/2}} + \frac{u_{\sigma_2} \Lambda \Gamma(N+1/2)}{u^{N+1/2}} + \dots \right] \right) \exp\left(\frac{i\tau^2}{4\epsilon\sigma_2} + \frac{i(\sigma_2 + \tau)}{\epsilon}\right). \end{aligned}$$

This reduces to

$$\begin{aligned} \frac{4iu^2}{v^2} C_u &\sim \epsilon^{N-1/2} \frac{\Lambda \Gamma(N+1/2)}{u^{N+1/2}} \left[\frac{i u_{\tau}^2 \epsilon N}{u} + i u_{\tau}^2 + \frac{\tau}{\sigma_2} u_{\tau} + u_{\sigma_1} \right] e^{u/\epsilon}, \\ C_u &\sim \epsilon^{N-1/2} \frac{\Lambda \Gamma(N+1/2)}{u^{N+1/2}} \left[\frac{\epsilon N}{u} \right] e^{u/\epsilon}. \end{aligned}$$

It is simple to show that optimal truncation occurs when $N \sim |u|/\epsilon$. Setting the usual transformation $u = r e^{i\theta}$ and truncating at $N = r/\epsilon + \alpha$ gives

$$\begin{aligned} C_{\theta} &\sim i r e^{i\theta} \epsilon^{r/\epsilon + \alpha - 1/2} \frac{\Lambda \Gamma(r/\epsilon + \alpha + 1/2)}{(r e^{i\theta})^{r/\epsilon + \alpha + 3/2}} (r) \exp\left(\frac{r e^{i\theta}}{\epsilon}\right), \\ &\sim i r^2 e^{i\theta} \epsilon^{r/\epsilon + \alpha - 1/2} \exp\left(\frac{r e^{i\theta}}{\epsilon}\right) \frac{\Lambda \sqrt{2\pi} (r/\epsilon)^{r/\epsilon + \alpha} e^{-r/\epsilon}}{(r e^{i\theta})^{r/\epsilon + \alpha + 3/2}}, \\ &= i \sqrt{\frac{2\pi r}{\epsilon}} \Lambda \exp\left(i\theta - i\theta \left(\frac{r}{\epsilon} + \alpha + \frac{3}{2}\right) + \frac{r}{\epsilon} \left(e^{i\theta} - \frac{r}{\epsilon}\right)\right). \end{aligned}$$

Setting $\theta = \epsilon^{1/2} \bar{\theta}$ and noting that $\Lambda = -i/2$ gives

$$C_{\bar{\theta}} \sim \sqrt{2\pi r e}^{-r \bar{\theta}^2}.$$

Integrating this expression shows that the variation as the Stokes line is crossed is given by

$[C]_{-}^{\pm} = 2\pi$, giving

$$[Q]_{-}^{\pm} \sim \pi e^{-i(\sigma_2 + \tau)/\epsilon}$$

Recalling again that the complex conjugate expression is also switched as this Stokes line is crossed, the complete switching behaviour is given by

$$[Q_{\text{TOT}}]_{-}^{+} \sim 2\pi e^{-\pi/\epsilon} \cos\left(\frac{p+\tau}{\epsilon}\right) \sim 2\pi e^{-\pi/\epsilon} \cos\left(\frac{x}{\epsilon}\right).$$

We observe that this will switch the quantity that was switched on by the stationary Stokes lines, given in (3.104). By comparing the behaviour of the singulants from the stationary and moving singularities, we see that switching occurs along curves that satisfy

$$\operatorname{Re}\left[-\frac{i\tau^2}{4\sigma_2}\right] < \operatorname{Re}[i(\sigma_2 + \tau)], \quad \operatorname{Im}\left[-\frac{i\tau^2}{4\sigma_2}\right] = \operatorname{Im}[i(\sigma_2 + \tau)].$$

The latter of these conditions is equivalent to (3.47) when expressed in terms of Eulerian coordinates, and hence that the waves present on the free surface of the flow are restricted to the region denoted R_1 in Figure 3.6 (b). Furthermore, we move from $\theta > 0$ to $\theta < 0$ as the relevant Stokes line is crossed, which means that this Stokes contribution precisely cancels the downstream stationary wavetrain.

Hence, the combined exponentially small behaviour on the free surface may be written as

$$\begin{aligned} \xi_{\text{exp}} \sim & \frac{\delta\sqrt{-i\pi\epsilon}}{2\sqrt{p+i\pi}} \left(\frac{\tau}{2(p+i\pi)+\tau} \right) \mathcal{S}_S \exp\left(-\frac{i\tau^2}{4\epsilon(p+i\pi)}\right) \\ & + \frac{\delta\sqrt{i\pi\epsilon}}{2\sqrt{p-i\pi}} \left(\frac{\tau}{2(p-i\pi)+\tau} \right) \mathcal{S}_S \exp\left(\frac{i\tau^2}{4\epsilon(p-i\pi)}\right) \\ & + 2\pi\delta\mathcal{S}_P e^{-\pi/\epsilon} \cos\left(\frac{p+\tau}{\epsilon}\right), \end{aligned}$$

where $\mathcal{S}_S$ and $\mathcal{S}_P$ are defined as in Section 3.3, with the added condition that instead of switching instantaneously, the switching is smooth and restricted to the neighbourhood of the Stokes lines in a region of width $\mathcal{O}(\epsilon^{1/2})$. Noting that $x \sim p + \tau$ in the linearized problem, we see that this gives the same exponentially small behaviour as that obtained using steepest descents, given in (3.62).

3.5 Discussion and Conclusions

In this chapter, we have calculated the behaviour of waves on the free surface of flow over a small step. This was performed by linearizing the problem in the step height, and then considering the problem using two methods. The first of these methods was to apply Fourier transforms to the Eulerian system, made possible by the fact that linearization causes the free boundary to be fixed at $y = 0$. The second was to determine the late-order terms of the series solution to the Lagrangian system, and apply optimal truncation to look directly at the terms that were switched on across Stokes lines in the problem.

Both methods showed that free-surface waves are switched on across an ordinary Stokes line at $x = 0$ on the physical free surface (corresponding to $p = -\tau$ in the Lagrangian coordinates). These waves are subsequently switched downstream along Stokes lines that follow curves given by the smallest real solution to (3.47). This results in a compact region of the flow containing gravity waves between $x = 0$ and $x \sim t/2$ for large t . The resultant surface waves are consistent with those obtained by Chapman & Vanden-Broeck [27] in the steady case, with the time-varying behaviour tending to the steady behaviour as $t \rightarrow \infty$.

This problem allows for a number of generalisations. Care must be taken, however, as this often has the effect of changing the mapping between the physical coordinates and the complex potential in the auxiliary problem. The right-hand side of the first governing equation, (3.15) will therefore take a different form, although depending on the choice of initial condition, this may not be difficult to determine.

We may consider a more general small submerged obstacle without substantially changing the governing equations. For example, in the case of an inclined step, we find that

$$\frac{dw_0}{dz} = \left(\frac{e^{-w_0} + 1 + \delta}{e^{-w_0} + 1} \right)^{\theta_I/2\pi},$$

where θ_I is the inclination of the slope. The leading-order free-surface behaviour in Lagrangian coordinates becomes

$$x^{(0)} = \frac{\theta_I}{2\pi} \log(1 + e^{p+\tau}) - \frac{\theta_I}{\pi} \log(1 + e^p), \quad y^{(0)} = 0.$$

Investigating the late-order terms of this series, we find that the waves on the free surface have amplitude

$$H \sim 4\theta_I \delta e^{-\pi/\epsilon}.$$

In this problem, the expression for the singulants do not change, so the region of the free surface containing waves is the same as the example studied in this investigation. We may also consider combinations of inclined steps and bumps, in a similar fashion to [75], and it would be of some interest to determine the presence of trapped wave solutions due to the interaction between waves from the contributing singularities. More general base topographies could be considered, such as smoothly varying boundaries; however, care must be taken to ensure that these topographies still permit the problem to be linearized about the undisturbed channel solution.

Another natural extension is that of more general initial conditions. In this case, the dynamic boundary condition given in (3.10) also changes, as many of the terms present in (3.10) will not necessarily equal zero, as they do in the previous analysis. Nonetheless, if we consider only problems where the leading-order solution is given by the steady-state solution, which is

required in order for the linearization to be valid, we obtain the linearized condition given in (3.17). Consequently, a full late-order term analysis could be applied to the resultant system.

However, the analysis performed in Section 3.3 indicates that the behaviour of the surface waves themselves is unaffected by the initial condition. This analysis is valid for any initial condition that may be written in the linearized form given in (3.27). The initial condition has no effect on the pole contributions, which are responsible for the behaviour of the waves on the free surface. From this, we conclude that the surface wave behaviour is generic for linearized problems.

Finally, in Section 3.2.2, the fully nonlinear problem of flow over a step was formulated as a fixed-boundary problem in Lagrangian coordinates. The application of late-order term analysis to this nonlinear problem is considered in Chapter 4, as well as the limitations of the method in this case.

Chapter 4

Nonlinear Two-Dimensional Unsteady Flow

4.1 Introduction and Background

In Chapter 3, we considered the problem of determining the behaviour of unsteady surface waves on free-surface flow over a submerged step. We specified that the non-dimensionalized step-height was small compared to the Froude number, and therefore that the problem could be linearized in this small height. In this chapter, we will consider flow over a submerged step when the step-height may no longer be considered small, and therefore the problem cannot be linearized in this fashion.

In Chapter 1, we described a number of numerical studies into fully-nonlinear free-surface flows over an obstacle which considered the small Froude number behaviour. In a number of these studies [32, 94, 122] it was noted that small-disturbance size and small Froude number limits do not commute, which is to be expected given that the former is a regular perturbation to the undisturbed problem, while the latter is a singular perturbation. This is known as the small Froude number nonuniformity. As a consequence of this nonuniformity, we cannot simply extend the results obtained in Chapter 3 to the equivalent full nonlinear problem.

Indeed, in their study on steady gravity waves, Chapman and Vanden-Broeck [27] compared the results of the exponential asymptotic analysis for the full nonlinear problem to those of King and Bloor [63] for linearized flow over a step, finding that the results were not consistent. To completely describe the behaviour of steady gravity waves flowing over a step, they therefore solved the linearized problem separately, and then found the distinguished limit that connects the two cases smoothly when the small parameters balance appropriately (in this case, $\delta = \mathcal{O}(F^2)$, where δ is defined as in Chapter 3). While we will not be considering the distinguished limit problem here, this difference does emphasize the importance of considering both the linearized

problem and the fully nonlinear problem.

Each of the previous studies into exponentially small gravity waves on free-surface flows using exponential asymptotic methods [27, 75, 102] applied conformal mapping techniques in order to map the flow region to a half-plane, with the position of the free-surface fixed. This allowed the subsequent exponential asymptotic analysis to be performed on a specified domain.

In Section 3.2.2, we discussed the reasons that conformal mapping techniques cannot be directly applied to unsteady problems, due to the difference between pathlines and streamlines in unsteady flow. To address this, we developed a Lagrangian formulation of flow over a step which fixed the position of both the free-surface and the rigid base in the mapped plane. This is not strictly necessary for the linearized problem, as the boundary of the physical flow region is fixed anyway, but will be of use in the nonlinear problem.

In this chapter, we express the problem of nonlinear flow over a step of arbitrary height using the Lagrangian coordinate system developed in Section 3.2.2. We expand the flow behaviour as a power series in $\epsilon = \pi F^2$, and find an equation for the leading-order behaviour which is solved explicitly for one particular value of the step-height parameter. We determine the behaviour of the relevant singularities in the problem, one set of which is associated with the top of the step while the other is associated with the initial disturbance. This allows us to define an ansatz for the late-order terms in the asymptotic series associated with each of these singularities, and hence determine the behaviour of the singulants in this problem.

We solve the singulant equation numerically to determine the position of Stokes lines on the free surface, and use this information to determine the region of the free surface on which exponentially small gravity waves are present. Finally, we optimally truncate the asymptotic series and determine the surface behaviour that is switched on as the Stokes line is crossed.

4.2 Formulation

4.2.1 Governing equations

The governing equations for fully nonlinear flow over a step of arbitrary height were given in Chapter 3; however, we will restate them here for convenience. We consider potential flow over a step of arbitrary height, with distances normalized by the upstream depth L/π , and velocities by the upstream velocity U . The system is illustrated in Figure 4.1.

The steady leading-order behaviour was obtained in an identical fashion to the completely steady problem considered in [27], and a similar fashion to that used in Chapter 2 in the analysis of flow due to a source. This involved determining the steady leading-order behaviour in terms of the complex potential w , and subsequently mapping the problem into the upper-half ζ plane,

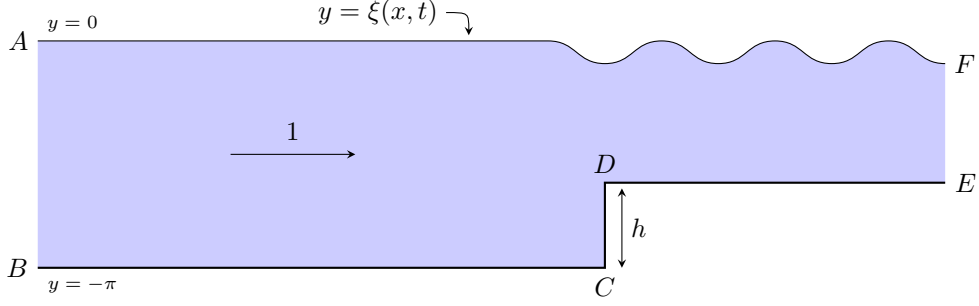


Figure 4.1: Prescribed fluid configuration for flow over a step. The lower boundary is rigid, while the upper boundary is free, and governed by Bernoulli's equation. The nondimensionalized step height may no longer be considered a small parameter in the problem.

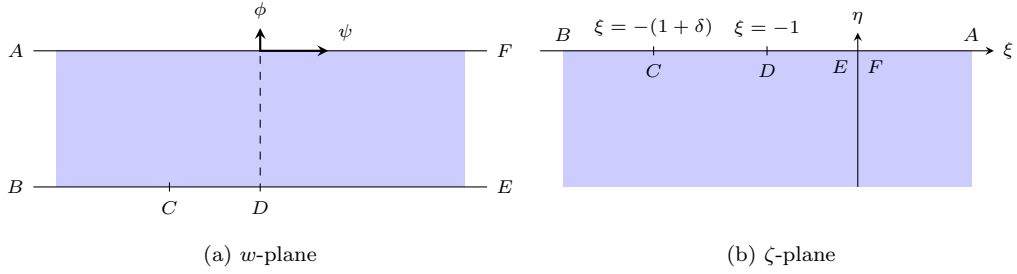


Figure 4.2: Flow region mapped into: (a) the complex potential plane; and (b) the ζ -plane.

using the mapping $\zeta = e^{-w}$. In this auxiliary mapped plane, the position of the base of the step is found at $\zeta = -1$, while the top of the step is located at $\zeta = -(1 + \delta)$, where $\delta > 0$. By mapping these points back into the complex potential plane, we see that the top of the step is therefore located at $w = -\log(-(1 + \delta))$. This is introduced into the problem explicitly by the initial condition (3.6). These mappings are illustrated in Figure 4.2.

In the current analysis of time-varying problems, we again note that we are not treating the full time-varying problem, but rather a steady problem associated with the initial flow configuration, allowing us to obtain a useful particle-labelling scheme.

For the linearized problem, δ , and hence the height of the step, was subsequently taken to be a small parameter. Here, we no longer consider the the step height to be small. Consequently, we define the position of the top of the step simply as $w = -\log(-b)$, where $b > 1$.

We define the small parameter ϵ in the same fashion as in previous chapters, so that $\epsilon = \pi F^2$. The governing equations (3.13)–(3.14) become

$$x_p y_q - x_q y_p = \left[\frac{(b + e^{p+iq})(b + e^{p-iq})}{(1 + e^{p+iq})(1 + e^{p-iq})} \right]^{1/2}, \quad (4.1)$$

$$x_{\tau q} x_p - x_{\tau p} x_q + y_{\tau q} y_p - y_{\tau p} y_q = 0. \quad (4.2)$$

We find that the Bernoulli condition on the free boundary, given in (3.11) becomes

$$\epsilon(x_{\tau\tau}x_p + y_{\tau\tau}y_p) + y_p = 0 \quad \text{on } q = 0, \quad (4.3)$$

and the appropriate kinematic boundary condition on the base, given by (3.12), becomes

$$x_py_\tau - x_\tau y_p = 0, \quad \text{on } q = -\pi. \quad (4.4)$$

The upstream flow is uniform, giving

$$\lim_{p \rightarrow -\infty} x = p + \tau, \quad \lim_{p \rightarrow -\infty} y = q.$$

The flow state is initially given by the leading-order steady-state solution to the problem in the limit that $\epsilon \rightarrow 0$. This is represented as

$$x(p, q, 0) = x_0(p, q), \quad y(p, q, 0) = y_0(p, q). \quad (4.5)$$

where x_0 and y_0 describe the appropriate leading-order steady state behaviour in the limit that $\epsilon \rightarrow 0$. In the case of flow over a step, we will find that this has a flat upper boundary, giving $y_0(p, 0) = 0$. This leading-order expression may be obtained in terms of the complex potential of the steady leading-order behaviour, denoted as $w_0 = \phi_0 + i\psi_0$, by solving

$$\frac{dw_0}{dz} = \left(\frac{e^{-w_0} + b}{e^{-w_0} + 1} \right)^{1/2}, \quad (4.6)$$

which was given in (3.20). In the previous chapter we specified that $0 < \delta \ll \epsilon \ll 1$, and hence equations (4.1)–(4.6) were linearized in small δ prior to the late-order terms analysis. Here, we instead allow b to take arbitrary value, with ϵ being the only small parameter present in the problem. Finally, we impose that the flow must be flat in the far field, with no waves present.

4.2.2 Series expansion

We expand the dependent variables as asymptotic power series in ϵ , giving

$$x \sim \sum_{n=0}^{\infty} \epsilon^n x^{(n)}, \quad y \sim \sum_{n=0}^{\infty} \epsilon^n y^{(n)} \quad \text{as } \epsilon \rightarrow 0, \quad (4.7)$$

where we note that $z = x + iy$, and hence define $z^{(n)}$ as $x^{(n)} + iy^{(n)}$. Applying these series expressions to the governing equations (4.1)–(4.2) and matching orders of ϵ in the asymptotic limit gives a system of recurrence relations, taking the general form

$$\sum_{j=0}^n \left[x_p^{(n)} y_q^{(n-j)} - x_q^{(n)} y_p^{(n-j)} \right] = 0, \quad (4.8)$$

$$\sum_{j=0}^n \left[x_{p\tau}^{(n)} x_q^{(n-j)} - x_{q\tau}^{(n)} x_p^{(n-j)} + y_{p\tau}^{(n)} y_q^{(n-j)} - y_{q\tau}^{(n)} y_p^{(n-j)} \right] = 0, \quad (4.9)$$

for $n > 0$. The boundary conditions in (4.3)–(4.4) become

$$\sum_{j=0}^n \left[x_{\tau\tau}^{(n-1)} x_p^{(n-j-1)} - y_p^{(n-1)} y_{\tau\tau}^{(n-j-1)} \right] + y_p^{(n)} = 0, \quad q = 0, \quad (4.10)$$

$$\sum_{j=0}^n \left[x_p^{(n)} y_{\tau}^{(n-j)} - x_{\tau}^{(n)} y_p^{(n-j)} \right] = 0, \quad q = -\pi, \quad (4.11)$$

for $n \geq 0$. Finally, the inhomogeneous leading-order behaviour satisfies the equations

$$\begin{aligned} x_p^{(0)} y_q^{(0)} - x_q^{(0)} y_p^{(0)} &= \left[\frac{(b + e^{p+iq})(b + e^{p-iq})}{(1 + e^{p+iq})(1 + e^{p-iq})} \right]^{1/2}, \\ x_{p\tau}^{(0)} x_q^{(0)} - x_{q\tau}^{(0)} x_p^{(0)} + y_{p\tau}^{(0)} y_q^{(0)} - y_{q\tau}^{(0)} y_p^{(0)} &= 0. \end{aligned}$$

We recall that the initial condition is chosen to be the leading-order steady-state behaviour, and consequently that the leading-order expression is steady. Hence, we may obtain $x^{(0)}$ and $y^{(0)}$ using complex variable methods. Setting $z^{(0)} = x^{(0)} + iy^{(0)}$, we rewrite (4.6) in terms of this leading-order behaviour and rearrange the equation to give

$$\frac{dz^{(0)}}{dw_0} = \left(\frac{e^{-w_0} + 1}{e^{-w_0} + b} \right)^{1/2}.$$

Solving this equation exactly gives

$$z^{(0)} = w_0 - \log \Delta_1 + b^{-1/2} \log \Delta_2 + C_1, \quad (4.12)$$

where

$$\begin{aligned} \Delta_1 &= (1 + b) e^{w_0} + 1 + e^{w_0} (1 + e^{-w_0})^{1/2} (b + e^{-w_0})^{1/2}, \\ \Delta_2 &= 2 e^{w_0} + (1 + b) + 2 e^{w_0} b^{1/2} (1 + e^{-w_0})^{1/2} (b + e^{-w_0})^{1/2}, \\ C_1 &= 2 \log 2 - b^{-1/2} \log (1 + b + 2b^{1/2}), \end{aligned}$$

where C_1 is chosen in order to ensure that the upstream boundary condition is satisfied, giving uniform flow in the upstream far field. This gives $z^{(0)}$ in terms of the steady state velocity potential and streamfunction. It remains to express these in terms of the Lagrangian variables.

Since the leading-order behaviour is steady, fluid particles will always remain on the same streamline, giving $\psi_0 = q$. In order to determine an expression for ϕ_0 , we apply the equation given in (3.22), which gives

$$\frac{dw_0}{d\tau} = \left(\frac{b + \exp^{-(\phi_0+iq)}}{1 + \exp^{-(\phi_0+iq)}} \right)^{1/2} \left(\frac{b + \exp^{-(\phi_0-iq)}}{1 + \exp^{-(\phi_0-iq)}} \right)^{1/2}.$$

This is difficult to solve over the entire flow region, but on the free surface following the curve $q = 0$, it reduces to

$$\frac{d\phi_0}{d\tau} = \left(\frac{b + \exp^{-\phi_0}}{1 + \exp^{-\phi_0}} \right), \quad \phi|_{\tau=0} = p,$$

Inverting this equation produces

$$\frac{d\tau}{d\phi_0} = \left(\frac{1 + \exp^{-\phi_0}}{b + \exp^{-\phi_0}} \right), \quad \tau|_{\phi=p} = 0.$$

Solving this differential equation gives

$$\tau = \frac{\phi_0 - p}{b} + \frac{b-1}{b} \log \left(\frac{b + e^{-p}}{b + e^{-\phi_0}} \right).$$

Rearranging this gives

$$\phi_0 = b\tau + p - (b-1) \log(\kappa), \quad (4.13)$$

where κ is a nonzero solution to

$$1 + be^p(1 - \kappa) - \kappa^b e^{-b\tau} = 0. \quad (4.14)$$

Unfortunately, we cannot solve this polynomial in general, although exact solutions may be found for certain values of b . For example, setting $b = 2$ allows (4.13) to be evaluated, giving

$$\phi_0 = -\log \left(-1 + \sqrt{1 + 2e^{-2\tau-p} + e^{-2\tau-2p}} \right). \quad (4.15)$$

Exact solutions for ϕ_0 may also be found for a range of other values for b , and are included in Appendix C. For simplicity, we will consider only the case $b = 2$, corresponding to a step of leading-order height $\pi/2$, or half the undisturbed flow depth.

From (4.12), we find

$$x^{(0)} = \phi_0 - \frac{1}{2} \left(\log \Delta_1 + \log \Delta_1^* + 2^{-1/2} \log \Delta_2 + 2^{-1/2} \log \Delta_2^* \right) + C_1, \quad (4.16)$$

$$y^{(0)} = q + \frac{i}{2} \left(\log \Delta_1 - \log \Delta_1^* + 2^{-1/2} \log \Delta_2 - 2^{-1/2} \log \Delta_2^* \right), \quad (4.17)$$

where

$$\begin{aligned} \Delta_1^* &= 3e^{\phi_0-iq} + 1 + e^{\phi_0-iq}(1 + e^{-(\phi_0-iq)})^{1/2}(2 + e^{-(\phi_0-iq)})^{1/2}, \\ \Delta_2^* &= 2e^{\phi_0-iq} + 3 + 2e^{\phi_0-iq}2^{1/2}(1 + e^{-(\phi_0-iq)})^{1/2}(2 + e^{-(\phi_0-iq)})^{1/2}, \end{aligned}$$

where ϕ_0 is given in (4.15). Finally, we note that $\Delta_j = \Delta_j^*$ on the free surface ($q = 0$) for $j = 1, 2$. Hence, $y^{(0)} = 0$ for $q = 0$, and the free surface is flat to leading order, as we would expect.

4.2.3 Singularity locations

We recall that the divergence of the asymptotic series in ϵ is caused by singularities on the analytically-continued free surface. These singularities are present even in early terms of the series. We therefore allow the variable p to take complex values, as in Chapter 3.

The leading-order behaviour on the free-surface, obtained by taking $q = 0$ in (4.16)–(4.17), contains singularities at $p = p_s$, where

$$p_s = -\log(2) \pm i\pi. \quad (4.18)$$

In the neighbourhood of this singularity, we find that

$$x^{(0)} \sim -\frac{1}{\sqrt{2}} \log(p - p_s), \quad (4.19)$$

$$\phi_0 \sim -\log(p - p_s). \quad (4.20)$$

We also note that the expression for ϕ_0 given in (4.15) contains a square-root term, and hence a branch point singularity where the radicand is equal to zero, and hence the surface behaviour is non-analytic. Hence, we determine that there will be singularities in the late-order behaviour at

$$p_s(\tau) = -2\tau + \log(-1 \pm \sqrt{1 - \exp 2\tau}), \quad (4.21)$$

where $\text{Re}(p_s(\tau)) = -\tau$. In (4.21), choosing the positive sign indicates the singularity above the real axis, while choosing the negative sign gives the position of the singularity below. Choosing the positive sign, we find that

$$x^{(0)} \sim X_0 - (2\sqrt{2} - 3) \left(\frac{2(1 - e^{2\tau} - \sqrt{1 - e^{2\tau}})}{(\sqrt{1 - e^{2\tau}} - 1)^2} \right)^{1/2} (p - p_s)^{1/2}, \quad (4.22)$$

$$\phi_0 \sim \Phi_0 + \left(\frac{2(1 - e^{2\tau} - \sqrt{1 - e^{2\tau}})}{(\sqrt{1 - e^{2\tau}} - 1)^2} \right)^{1/2} (p - p_s)^{1/2}, \quad (4.23)$$

where X_0 and Φ_0 are constants independent of p and τ , and choosing the negative sign would give equivalent complex conjugate values. We see that the first singularity pair, given in (4.18), is constant in τ . This indicates that it is always associated with the same fluid particle, and therefore convected with the flow. The second singularity pair, given in (4.21), has a real part given by $\text{Re}(p_s) = -\tau$. We therefore expect this singularity to behave in a similar fashion to the stationary singularity seen in the linearized problem, remaining above the step. The behaviour of the four singularities is shown in Figure 4.3.

Using the method of Olde Daalhuis *et al.* [87], knowledge of the leading-order behaviour of the problem and the behaviour of the solution in the neighbourhood of the singularities is sufficient to determine the exponentially small behaviour in question.

4.3 Late-order term analysis

The next stage in the problem is to use the system of recurrence relations given in (4.8)–(4.10) to determine the form of the late-order terms of the series given in (4.7). To accomplish this, we

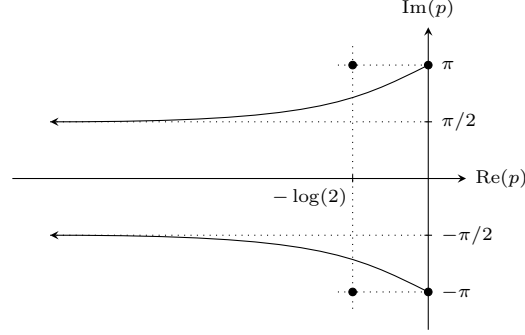


Figure 4.3: Position of early-order singularities on the analytically-continued free surface. The positions of the singularities at $\tau = 0$ are shown as circles. The first singularity pair is stationary, while the second moves as τ is varied, following the path indicated by the arrows.

again follow Chapman *et al.* [22] and define an ansatz for the late-order terms, taking the form

$$x^{(n)} \sim \frac{X\Gamma(n+\gamma)}{\chi^{n+\gamma}}, \quad y^{(n)} \sim \frac{Y\Gamma(n+\gamma)}{\chi^{n+\gamma}} \quad \text{as } n \rightarrow \infty. \quad (4.24)$$

Applying this expression to the recurrence relationships given in (4.8)–(4.9) and taking the leading-order behaviour in the limit that $n \rightarrow \infty$ gives

$$-X\chi_p y_q^{(0)} - Y\chi_q x_p^{(0)} + X\chi_q y_p^{(0)} + Y\chi_p x_q^{(0)} = 0, \quad (4.25)$$

$$-X\chi_\tau \chi_p x_q^{(0)} + X\chi_\tau \chi_q x_p^{(0)} - Y\chi_\tau \chi_p y_q^{(0)} + Y\chi_\tau \chi_q y_p^{(0)} = 0. \quad (4.26)$$

This system only has nonzero solutions when

$$X = \pm iY, \quad (4.27)$$

$$\chi_q y_p^{(0)} - \chi_p y_q^{(0)} = \pm i(\chi_p x_q^{(0)} - \chi_q x_p^{(0)}), \quad (4.28)$$

where the upper and lower sign choices correspond, which shall function as the convention for the remainder of this chapter unless otherwise noted. The Bernoulli condition as expressed in (4.10) gives

$$X\chi_\tau^2 x_p^{(0)} - Y\chi_\tau^2 y_p^{(0)} - Y\chi_p = 0 \quad \text{on } q = 0.$$

Noting that $y_p^{(0)}$ is zero on the free boundary, this reduces to

$$-\chi_p \pm i x_p^{(0)} \chi_\tau^2 = 0. \quad (4.29)$$

Continuing to the next order in n as $n \rightarrow \infty$, (4.10) gives on $q = 0$

$$2x_p^{(0)} \chi_\tau X_\tau \pm iX_p + [x_p^{(0)} \chi_{\tau\tau} + x_{\tau\tau}^{(0)} \chi_p + \chi_\tau^2 (x_p^{(1)} \pm i y_p^{(1)})]X = 0. \quad (4.30)$$

The characteristics of this equation are given by

$$\frac{d\tau}{du} = 2\chi_\tau x_p^{(0)}, \quad \frac{dp}{du} = \pm i, \quad \frac{dX}{du} = x_p^{(0)} \chi_{\tau\tau} + x_{\tau\tau}^{(0)} \chi_p + \chi_\tau^2 (x_p^{(1)} \pm i y_p^{(1)}),$$

where u is the characteristic parameter. Cauchy data will be obtained by matching the solution in the inner limit of the singularities in the analytically-continued free surface. The position of these singularities are given for the stationary and moving singularities in (4.21) and (4.18) respectively. The paths of the prefactor characteristics are therefore given by

$$\frac{d\tau}{dp} = \mp 2ix_p^{(0)}\chi_\tau.$$

4.3.1 Local singulant behaviour

4.3.1.1 Stationary singularity

We may determine the behaviour of the singulant terms in the neighbourhood of the relevant singularities. We first consider the stationary singularity, the location of which is given in (4.21). The local behaviour for $x^{(0)}$ and ϕ was given in (4.22)–(4.23), and will be denoted here as

$$x^{(0)} \sim X_0 + X_1(\tau)(p - p_s(\tau))^{1/2}, \quad (4.31)$$

$$\phi_0 \sim \Phi_0 + \Phi_1(\tau)(p - p_s(\tau))^{1/2}, \quad (4.32)$$

where the full form of the prefactor terms X_1 and Φ_1 are those given in (4.22)–(4.23). We set the form of χ as

$$\chi \sim A(\tau)(p - p_s(\tau))^\alpha. \quad (4.33)$$

Applying these expressions to the boundary condition on $q = 0$ given in (4.29) produces

$$\alpha A(\tau)(p - p_s(\tau))^{\alpha-1} \pm \frac{i}{2} X_1(\tau)(p - p_s(\tau))^{-1/2} [-\alpha p'_s(\tau) A(\tau)(p - p_s(\tau))^{\alpha-1}]^2 + \dots = 0,$$

where the omitted terms are small as $p \rightarrow p_s(\tau)$. These terms only balance when $\alpha = 3/2$, which gives

$$A(\tau) = \pm \frac{4i}{3X_1(\tau)(p'_s(\tau))^2}.$$

Therefore, on the free surface ($q = 0$) in the neighbourhood of the singularity at $p = p_s(\tau)$, the singulant behaviour is given by

$$\chi \sim \pm \frac{4i(p - p_s(\tau))^{3/2}}{3X_1(\tau)(p'_s(\tau))^2}. \quad (4.34)$$

4.3.1.2 Moving singularity

We now determine the behaviour of the singulant terms in the neighbourhood of the moving singularities, located at $p_s = -\log(2) \pm i\pi$. In this case, the local behaviour of $x^{(0)}$ and ϕ is given in (4.19) and (4.20) respectively. In this case, however, setting the local behaviour of χ to take the form prescribed in (4.33) will not produce a nonzero solution. Instead, we set

$$\chi \sim B(\tau) [\log(p - p_s)]^\beta.$$

Applying this local behaviour to the boundary condition on $q = 0$ given in (4.29) gives

$$-\frac{B(\tau)\beta [\log(p - p_s)]^{\beta-1}}{(p - p_s)} \pm \frac{i\sqrt{2}(B'(\tau))^2 [\log(p - p_s)]^{2\beta}}{2(p - p_s)} + \dots = 0,$$

where the omitted terms are small as $p \rightarrow p_s(\tau)$. Balancing terms in this limit gives $\beta = -1$, and an expression for $B(\tau)$ given as

$$B(\tau) \pm \frac{i\sqrt{2}}{2}(B'(\tau))^2 = 0.$$

In the full Charpit analysis, we will set $\chi = 0$ at $p = p_s$ and $\tau = 0$. The latter condition may be used to uniquely solve this differential equation, giving

$$B(\tau) = \pm \frac{i\sqrt{2}\tau^2}{4}.$$

Therefore, on the free surface ($q = 0$) in the neighbourhood of the singularity at $p = p_s$, the singulant behaviour is given by

$$\chi \sim \pm \frac{i\sqrt{2}\tau^2}{4\log(p - p_s)}. \quad (4.35)$$

4.4 Finding the Stokes lines

From the behaviour of the singulant, we may determine potential locations for Stokes lines on the analytically-continued free surface. Specifically, from [33], we know that Stokes lines may be present when

$$\text{Re}(\chi) > 0, \quad \text{Im}(\chi) = 0.$$

However, due to the possible existence of higher-order Stokes switching [22, 56], these conditions are not sufficient for a Stokes line to exist. Furthermore, we know that additional Stokes lines may exist when

$$\text{Im}(\chi_1) - \text{Im}(\chi_2) = 0,$$

where χ_1 and χ_2 are the singulants related to late-order terms from two distinct singularity contributions. When switching of this nature occurs, the behaviour associated with the singulant with greater real part on the Stokes line is the term that switches across the Stokes line (that is, the more subdominant of the two late-order behaviours).

4.4.1 Charpit analysis

Unfortunately, the singulant equation given in (4.29) is not simple to solve. We therefore turn to a numerical approach. The singulant equation will be formulated in terms of ray equations with appropriate boundary conditions using Charpit's method. The Charpit equations will then be integrated numerically in order to determine the behaviour of each singulant on the free surface.

Setting $\chi_p = P$ and $\chi_\tau = T$, expressing (4.29) in Charpit form gives

$$\begin{aligned}\frac{dp}{du} &= -1, & \frac{d\tau}{du} &= \pm 2ix_p^{(0)}T, & \frac{d\chi}{du} &= \pm ix_p^{(0)}T^2, \\ \frac{dT}{du} &= \mp ix_{p\tau}^{(0)}T^2, & \frac{dP}{du} &= \mp ix_{pp}^{(0)}T^2,\end{aligned}\tag{4.36}$$

where u is the Charpit parameter. Since the last equation decouples, only the first four of these equations will be required in order to determine the position of the Stokes lines.

4.4.1.1 Stationary singularity

We first consider the singulant caused by the stationary singularity, the location of which is given in (4.21). In this case, the initial conditions are given by

$$\begin{aligned}p(0) &= p_s(s), & \tau(0) &= s, & \chi(0) &= 0, \\ -P(0) \pm ix_{0,p}^{(0)}T^2(0) &= 0, & p'_s(s)P(0) + T(0) &= 0,\end{aligned}\tag{4.37}$$

where s parametrises the Cauchy data, $x_0^{(0)}$ denotes to the value of $x^{(0)}$ when $p = p(0)$ and $\tau = \tau(0)$, and p_s is defined in (4.21). Solving the final two equations gives

$$T(0) = \mp \frac{i}{x_{0,p}^{(0)}p'_s(s)}.\tag{4.38}$$

In conjunction with the system of ordinary differential equations given in (4.36), this system fully defines the behaviour of the singulant associated with the stationary singularity. Potential Stokes lines may now be found by searching the solution for points on which

$$\text{Re}(\chi) > 0, \quad \text{Im}(p) = \text{Im}(\tau) = \text{Im}(\chi) = 0.\tag{4.39}$$

4.4.1.2 Moving singularity

We now consider the Charpit formulation for the singulant caused by the moving singularity, the location of which is given in (4.18). As in the linearized case, we again find that the standard Charpit formulation will lead to boundary data being prescribed along a characteristic curve, and hence that the system is not uniquely determined. Specifically, we find that $T = P = 0$, and therefore that boundary information never propagates away from this curve. Instead, we again prescribe the rays to pass through the singularity at the point of the initial disturbance, given by $p = p_s$ and $\tau = 0$. This eliminates the reliance on s , but instead allows $T(0)$ to be specified arbitrarily. Hence, this gives the initial conditions as

$$p(0) = -\log(2) \pm i\pi, \quad \tau(0) = 0, \quad \chi(0) = 0, \quad ix_{0,p}^{(0)}T^2(0) = 0,\tag{4.40}$$

where $T(0)$ remains unspecified. For ease of notation, we shall subsequently refer to this parameter using the subscript T_0 .

Hence, rather than being parametrized in terms of s and t , as in the case of the stationary singularity, we find that this system is parametrized in terms of T_0 and t . Again, in conjunction with the system of ordinary differential equations given in (4.36), this system may be used to determine the behaviour of the singulant associated with the moving singularity in the flow. Potential Stokes lines satisfy the same conditions as in the previous case, given in (4.39).

4.4.1.3 Complete Stokes structure

Finding the curve across which Stokes switching between the two exponentially subdominant contributions requires the evaluation of the both singulant associated with the stationary singularity, using (4.36)–(4.38), and the singulant associated with the moving singularity, using (4.36) and (4.40). This may be thought of as one system parametrized in terms of s , T_0 , t_1 and t_2 , where the subscript indicates whether the parameter is associated with the first or the second Charpit system.

The Stokes line may be found by searching the solutions to both systems for points which satisfy

$$\begin{aligned} \operatorname{Re}(p_1) &= \operatorname{Re}(p_2), & \operatorname{Re}(\tau_1) &= \operatorname{Re}(\tau_2), \\ \operatorname{Im}(p_1) &= \operatorname{Im}(p_2) = 0, & \operatorname{Im}(\tau_1) &= \operatorname{Im}(\tau_2) = 0, & \operatorname{Im}(\chi_1) &= \operatorname{Im}(\chi_2). \end{aligned}$$

If the dominant contribution (that is, the contribution for which χ has smaller real part) is active at the location of a curve satisfying these conditions, the curve will be a Stokes line, and Stokes switching will occur in the subdominant contribution.

4.4.2 Numerical Investigation

4.4.2.1 First-generation Stokes lines

In order to determine the points where the Stokes lines intersect the free surface, we first develop a method to solve the ordinary differential equation systems given in (4.36) with the boundary data specified in (4.37) and (4.40). The first system describes the behaviour associated with the stationary singularity, and the second describes the behaviour associated with the moving singularity. These systems will subsequently be denoted as system one and two respectively.

There are four unknown parameters in these systems. In system one, the four parameters are the real and imaginary parts of s and u respectively. In system two, the four parameters are instead the real and imaginary parts of u and T_0 . In both cases, we may eliminate one of these parameters by integrating the first equation in (4.36), giving

$$p = -u + p(0).$$

This allows for u to be expressed in terms of p and $p(0)$ (where $p(0)$ is a function of s in system one, and a constant in system two). Knowing that p must be real, we are able to instead parametrize system one and two using the real-valued quantity p in place of the real and imaginary parts of u , reducing the number of parameters in the problem by one. To avoid confusion with the path of the integral contour the complex p -domain, we will refer to this quantity as p_{end} . We shall subsequently refer to the real and imaginary parts of s and T_0 as $s = s_R + is_I$ and $T_0 = T_{0R} + iT_{0I}$ respectively.

Solving these systems involves integrating the ray equations (4.36) along a curve connecting the boundary data (prescribed at the location of the singularity on the complexified free surface, or p_s) to the point p_{end} on the physical free surface. However, noting that the ray equations contain $x_p^{(0)}$ and $x_{p\tau}^{(0)}$, we see that these terms will become singular at p_s . To avoid this difficulty, rather than integrating along a curve beginning at p_s , we instead integrate from a point in the neighbourhood of the singularity, denoted p_{start} . The appropriate boundary data is obtained using local values of $x^{(0)}$ and χ , given in (4.22) and (4.34) for the stationary singularity, and (4.19) and (4.35) for the moving singularity.

We now numerically integrate the ray equations along a curve in the complex p -domain connecting p_{start} to p_{end} . For simplicity, we integrate along a line parallel to the imaginary p -axis until the real axis is reached, and then along the real axis to the point $p = p_{\text{end}}$. Unfortunately, as the partial derivatives of $x^{(0)}$ contain numerous branch cuts within the parameter space, we must take care to check that the correct branch is chosen when this occurs.

Using this algorithm for solving the ray equations numerically given a set of parameters, we search for points in the solutions to systems one and two that satisfy the conditions given in (4.39) using parameter sweeps. In system one, we select values for the s_R and s_I and solve the system of differential equations for a range of values for p until a point is found satisfying $\text{Im}(\chi) = 0$ and $\text{Re}(\chi) > 0$, subsequently denoted p_{Stokes} (it is certainly possible that multiple points satisfying this condition may exist, or none at all; however, this does not cause difficulties in the current investigation, if care is taken). This process is repeated over a prescribed range of the complex s -plane, so that each point is associated with a single value for p_{Stokes} indicating the position of the Stokes surface associated with $s = s_R + is_I$.

We proceed by determining the value of τ for each of the resultant points in the s -space. Hence, each value for s is associated with a value of p_{Stokes} and τ on which $\text{Im}(\chi) = 0$. In many cases τ will be complex-valued, and hence the points on the Stokes surface corresponding to these values of s do not intersect the physical free surface. However, some values of s produce values of τ that satisfy $\text{Im}(\tau) = 0$. Consequently, these values of s are associated with points on the physical free-surface, with corresponding real values for p_{Stokes} and τ . In Figure 4.4 (a), we show

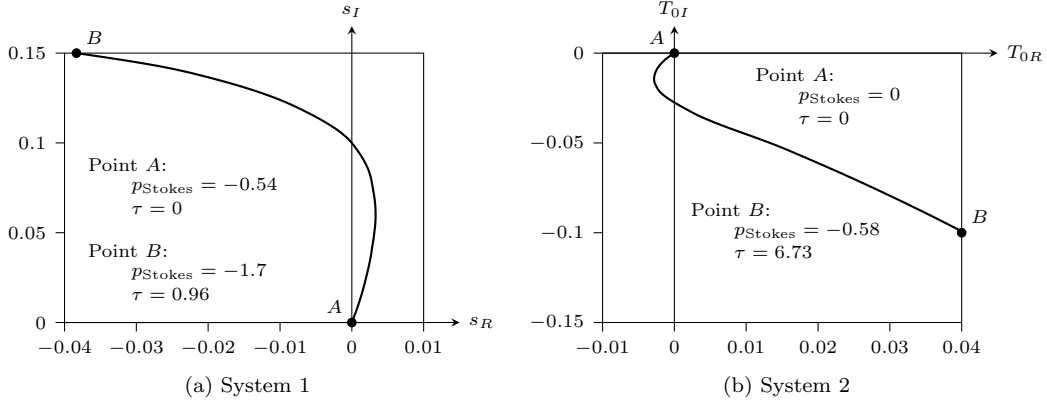


Figure 4.4: Curves in (a) complex s -space and (b) complex T -space along which the Stokes surface intersects the physical free surface in system 1 and system 2 respectively. Each point on the curve has an associated value for p_{Stokes} and τ indicating the point on the free-surface where this intersection occurs. These values are displayed for selected points.

the path in a portion of the complex s -plane along which $\text{Im}(\tau) = 0$ is satisfied. Once this path is known, the point at which the Stokes surface intersects the physical free surface may easily be obtained by finding the values associated with p_{Stokes} and τ along that curve.

A similar technique is used to find Stokes lines within system two, with the problem being parametrized in terms of T_{0R} and T_{0I} rather than s_R and s_I . In Figure 4.4 (b), we illustrate the curve in T_0 -space along which $\text{Im}(\tau) = 0$, and hence correspond to a curve along which the Stokes surface intersects the free surface.

With the position of the Stokes lines known in p and s , it is a simple matter to obtain the value for $\text{Re}(\tau)$ along this curve. Hence, we are able to describe the position of the Stokes line in p - τ space, shown in Figure 4.5 (a). Using (4.16), we find the position of the potential Stokes line in x - t space, shown in Figure 4.5 (b). We see that the Stokes line associated with the stationary singularity intersects the free surface approximately above the step, while the intersection point caused by the Stokes line associated with the moving singularity propagates downstream. This is similar to the equivalent Stokes structure in the linear problem given in Chapter 3. By comparing the linear and nonlinear results, we infer that the potential Stokes lines in the nonlinear problem are active for $\tau > 0$, and that the Stokes line associated with the stationary singularity generates free-surface gravity waves that propagate downstream from the step, while the Stokes line associated with the moving singularity produces an unsteady free-surface contribution that is active everywhere and decays with time.

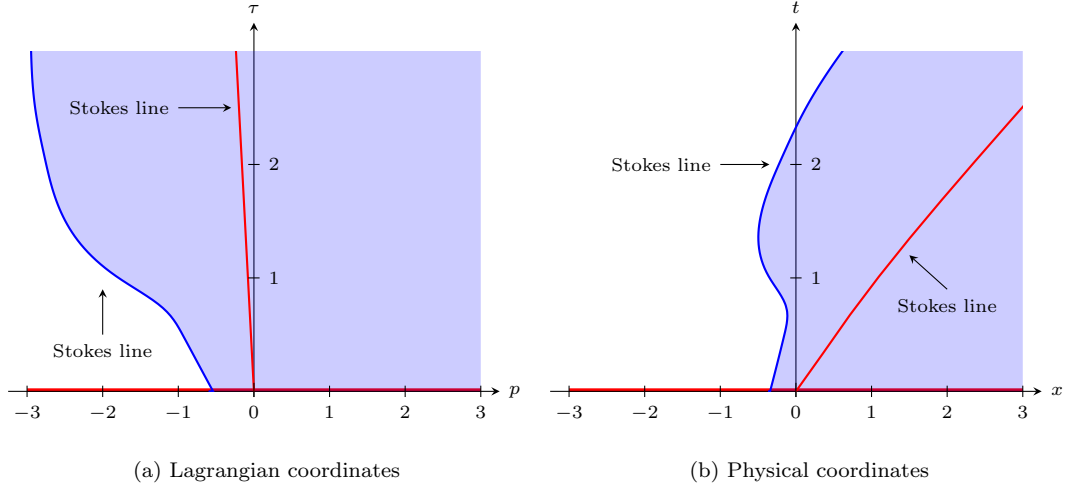


Figure 4.5: First-generation Stokes lines in the (a) Lagrangian and (b) physical coordinate systems. The blue curves are the Stokes lines associated with the stationary singularity, while the red curves are the Stokes lines associated with the propagating singularity. Comparing this figure with the Stokes structure obtained by investigating the linearized problem, we infer that gravity waves are switched on as the blue curve is crossed from left to right. In the absence of any interaction between the exponentially subdominant contributions, this would imply that surface waves are present in the shaded region of the figures.

4.4.2.2 Complete Stokes structure

The position of the Stokes line associated with the interaction between the exponentially small stationary and moving free-surface contributions may be found by integrating the ray equations as previously described. However, we now solve both system one and system two simultaneously, finding solutions for a prescribed p that satisfy

$$\operatorname{Re}(\tau_1) = \operatorname{Re}(\tau_2) = 0, \quad \operatorname{Im}(\tau_1) = \operatorname{Im}(\tau_2) = 0, \quad \operatorname{Im}(\chi_1) = \operatorname{Im}(\chi_2).$$

Fortunately, we are able to obtain one point satisfying these conditions by noting when the two Stokes lines cross in Figure 4.5. We then perform a parameter sweep over a range of values for s_R , S_I , T_{0R} and T_{0I} for specified values of p , searching for solutions that satisfy all of these conditions.

The resultant position of the Stokes line may be seen in p - τ space in Figure 4.6 (a), and in x - t space in Figure 4.6 (b). Knowing that Stokes lines cause the subdominant contribution to switch, we conclude that the waves that are switched on across the first Stokes line will be switched off again as this Stokes line is crossed. As this Stokes line moves in time, this will take the form of a propagating wavefront, similar to that obtained in Chapter 3 for the linearized problem. Therefore, we find that the free-surface gravity waves will be restricted to the shaded regions in Figure 4.6.

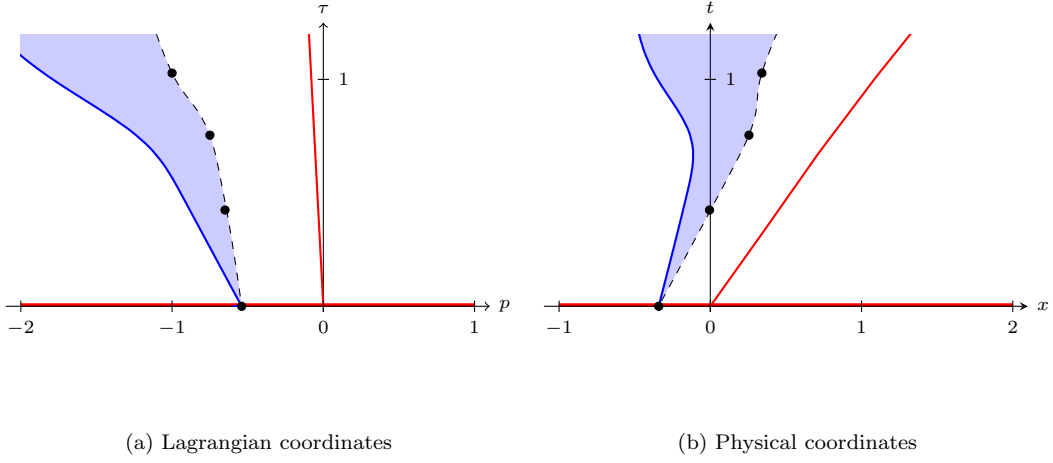


Figure 4.6: Stokes structure for the nonlinear problem of free-surface flow over a step in (a) Lagrangian and (b) physical coordinates. The Stokes lines caused by the leading-order behaviour are as described in Figure 4.6, while the dashed curve represents the Stokes lines caused by interaction between the two exponentially subdominant contributions. Noting that these Stokes lines will switch off the subdominant of the two free-surface contributions, we determine that the exponentially small gravity waves are switched on as the blue Stokes line is crossed, and are switched off again across the dashed Stokes line. This leads to the propagation of the steady free-surface waves in time. The shaded region indicates the region in which waves are present in the solution.

4.5 Optimal truncation and Stokes smoothing

Typically, we would now solve the prefactor equation given in (4.30) using the local behaviour in the neighbourhood of the singularities as boundary data for the characteristic equations. We omit this step, as any solution to the prefactor equation would require knowledge of $x^{(1)}$ and $y^{(1)}$, which would be extremely difficult to obtain analytically.

However, even without an analytical expression for the prefactor or even the singulant, we may still optimally truncate the asymptotic power series prescribed in (4.7) and apply Stokes smoothing in the vicinity of Stokes lines to determine the form of the resultant free-surface behaviour in terms of Φ and χ .

Truncating the series expression in (4.7) gives

$$x = \sum_{n=0}^{N-1} \epsilon^n x^{(n)} + R^{(N)}, \quad y = \sum_{n=0}^{N-1} \epsilon^n y^{(n)} + S^{(N)},$$

where $R^{(N)}$ and $S^{(N)}$ represent the remainder term of the series after truncation. Applying this series expression to the governing equations (4.1)–(4.2) gives

$$y_q^{(0)} R_p^{(N)} + x_p^{(0)} S_q^{(N)} - y_p^{(0)} R_q^{(N)} - x_q^{(0)} S_p^{(N)} + \dots = 0, \quad (4.41)$$

$$\begin{aligned} x_{\tau q}^{(0)} R_p^{(N)} + x_p^{(0)} R_{\tau q}^{(N)} - x_{\tau p}^{(0)} R_q^{(N)} - x_q^{(0)} R_{\tau p}^{(N)} \\ + y_{\tau q}^{(0)} S_p^{(N)} + y_p^{(0)} S_{\tau q}^{(N)} - y_{\tau p}^{(0)} S_q^{(N)} - y_q^{(0)} S_{\tau p}^{(N)} + \dots = 0, \end{aligned} \quad (4.42)$$

where the omitted terms are small as $\epsilon \rightarrow 0$. Finally, the kinematic boundary condition (4.4) becomes

$$x_p^{(0)} S_\tau^{(N)} + y_\tau^{(0)} R^{(p)} - x_\tau^{(0)} S_p^{(N)} - y_p^{(0)} R_\tau^{(N)} + \dots = 0, \quad (4.43)$$

on $q = 0$, where the omitted terms are again small as $\epsilon \rightarrow 0$. Applying this series expression to the free-surface equation (4.3) and applying the recurrence relation (4.10) to eliminate terms leaves

$$\begin{aligned} \sum_{n=0}^{N-1} \epsilon^{n+1} \left(x_p^{(n)} R_{\tau\tau}^{(N)} + x_{\tau\tau}^{(n)} R_p^{(N)} + S_{\tau\tau}^{(N)} y_p^{(n)} + S_p^{(N)} y_{\tau\tau}^{(n)} \right) + \epsilon \left(R_p^{(N)} R_{\tau\tau}^{(N)} + S_p^{(N)} S_{\tau\tau}^{(N)} \right) \\ + \epsilon^N \left(x_{\tau\tau}^{(N-1)} x_p^{(0)} + x_{\tau\tau}^{(0)} x_p^{(N-1)} + y_{\tau\tau}^{(N-1)} y_p^{(0)} + y_{\tau\tau}^{(0)} y_p^{(N-1)} \right) + y_p^{(0)} + S_p^{(N)} = 0. \end{aligned}$$

We simplify this expression further by retaining only the largest contributions in the limit that $N \rightarrow \infty$, using (4.24), as well as the largest terms containing exponentially small remainders. This gives

$$\epsilon \left[x_p^{(0)} R_{\tau\tau}^{(N)} + x_{\tau\tau}^{(0)} R_p^{(N)} \right] + \epsilon^2 \left[x_p^{(1)} R_{\tau\tau}^{(N)} + y_p^{(1)} S_{\tau\tau}^{(N)} \right] + S_p^{(N)} \sim -\epsilon^N x_p^{(0)} x_{\tau\tau}^{(N-1)}, \quad (4.44)$$

on $q = 0$. Furthermore, in the limit that $n \rightarrow \infty$, keeping only the dominant terms in the recurrence relation (4.10) implies that

$$x_p^{(0)} x_{\tau\tau}^{(n-1)} \sim -y_p^{(n)} \quad \text{as } n \rightarrow \infty,$$

and hence that (4.44) may be rewritten as

$$\epsilon \left[x_p^{(0)} R_{\tau\tau}^{(N)} + x_{\tau\tau}^{(0)} R_p^{(N)} \right] + \epsilon^2 \left[x_p^{(1)} R_{\tau\tau}^{(N)} + y_p^{(1)} S_{\tau\tau}^{(N)} \right] + S_p^{(N)} \sim \epsilon^N y_p^{(N)}, \quad (4.45)$$

on $q = 0$.

By considering the homogeneous version the full system given in (4.41)–(4.44) in the limit that $\epsilon \rightarrow 0$, and applying an exponential WKB ansatz in a similar fashion to the equivalent step in Chapter 3, we find that the exponentially small remainder term may be written as

$$R^{(N)} \sim AX e^{-\chi/\epsilon}, \quad S^{(N)} \sim AY e^{-\chi/\epsilon}, \quad (4.46)$$

where, as usual, A is the Stokes switching parameter that varies rapidly in the neighbourhood of Stokes lines on the complexified free surface. Applying these expressions for the remainder terms to (4.45), as well as the late-order ansatz given in (4.24), we find after some cancellation using (4.29) and (4.30) that

$$A_p Y - 2x_p^{(0)} A_\tau X \chi_\tau \sim -\epsilon^N e^{\chi/\epsilon} \frac{Y \chi_p \Gamma(N+1+\gamma)}{\chi^{N+1+\gamma}}.$$

Recalling that $X = \pm iY$, we divide throughout by Y to obtain the convenient form

$$A_p \mp 2ix_p^{(0)} \chi_\tau A_\tau \sim -\epsilon^N e^{\chi/\epsilon} \frac{\chi_p \Gamma(N+1+\gamma)}{\chi^{N+1+\gamma}}, \quad (4.47)$$

Using the method of characteristics, we find that the solution varies along rays satisfying

$$\frac{d\tau}{dp} = \mp 2ix_p^{(0)}\chi_\tau,$$

We define a new coordinate system (u, v) such that

$$u = \chi, \quad v_p \mp 2ix_p^{(0)}\chi_\tau v_\tau = 0.$$

In these coordinates, (4.47) becomes

$$A_u\chi_p - 2iA_u\chi_\tau^2 \sim -\epsilon^N e^{\chi/\epsilon} \chi_p \frac{\Gamma(N+1+\gamma)}{u^{N+1+\gamma}}.$$

Using the eikonal equation (4.29), this becomes

$$A_u \sim \epsilon^N e^{u/\epsilon} \frac{\Gamma(N+1+\gamma)}{u^{N+1+\gamma}}.$$

By minimising the right-hand side of this expression with respect to N in the limit that $\epsilon \rightarrow 0$, we find that the optimal truncation point is given by $N \sim |\chi|/\epsilon$. Setting $u = re^{i\theta}$, $N = r/\epsilon + \alpha$ with $0 \leq \alpha < 1$, and applying Stirling's formula gives

$$A_\theta \sim \frac{i}{\epsilon^\gamma} \sqrt{\frac{2\pi r}{\epsilon}} \exp \left[\frac{r}{\epsilon} (e^{i\theta} - 1) - i\theta \left(\frac{r}{\epsilon} + \alpha + 1 + \gamma \right) + i\theta \right].$$

The right-hand side is exponentially small except along the Stokes lines, which follow $\theta = 0$. Setting $\theta = \epsilon^{1/2}\hat{\theta}$, we obtain

$$A_{\hat{\theta}} \sim \frac{i\sqrt{2\pi r} \exp(-r\hat{\theta}^2/2)}{\epsilon^\gamma}.$$

Hence we find that

$$A = A_0 \frac{i\sqrt{2\pi}}{\epsilon^\gamma} \int_{-\infty}^{\hat{\theta}\sqrt{r}} e^{-t^2/2} dt,$$

and consequently that the quantity switched on as the Stokes line is crossed is given by

$$[A]_-^+ = \frac{2\pi i}{\epsilon^\gamma}.$$

Using (4.46), we therefore find that the contribution to the free-surface behaviour obtained as the Stokes line is crossed is given by

$$[S_{\text{TOT}}]_-^+ = \frac{2\pi i Y}{\epsilon^\gamma} e^{-\chi/\epsilon} + \text{c.c.}, \quad (4.48)$$

where we have included the complex conjugate contribution, obtaining purely real behaviour on the free-surface of the flow.

4.6 Discussion and conclusions

In this section, we attempted to extend the results obtained in Chapter 3 for free-surface flow linearized about small step height to the full nonlinear problem for arbitrary step height. We formulated the system in terms of Lagrangian coordinates chosen so that the free boundary followed $q = 0$ and the fixed boundary followed $q = -\pi$ in the mapped plane. The problem could therefore be treated as a fixed boundary problem in this mapped plane.

Unfortunately, this formulation produced a system which could not be explicitly solved for the leading-order behaviour except for a set of isolated values of the step-height. In order to proceed with the analysis, we set the step-height parameter to $b = 2$. Examining the leading-order behaviour showed that there were two singularity pairs on the complexified free surface, given in (4.18) and (4.21). The first of these was associated with the singularity in the flowpath at the top of the step, while the second propagated with the flow and was caused by the initial disturbance at $t = 0$.

We subsequently followed [22] and prescribed an ansatz for the late-order terms of the asymptotic series governing the flow behaviour. Equations were obtained for the singulant and prefactor in this ansatz. Due to the cumbersome nature of the expressions for the leading-order behaviour these could not be solved analytically, although we were able to obtain the local behaviour of the singulants in the neighbourhood of the singularity. Instead, we determined the position of the Stokes lines on the free surface by solving the singulant equation numerically with appropriate boundary data and searching for points on the curve satisfying $\text{Im}(\chi) = 0$ and $\text{Re}(\chi) > 0$.

In Chapter 3, we found that the travelling wavefront for the linearized case was caused by Stokes switching across which the unsteady free-surface contribution switched off the free-surface gravity waves. We therefore used similar numerical methods to obtain a numerical approximation for the equivalent Stokes line in the nonlinear case. This enabled us to obtain the full Stokes structure on the free surface, presented in Figure 4.6. Specifically, we were able to show the expanding wave region on the surface.

It is possible that further numerical work could be performed such as finding the behaviour of χ , and therefore the phase of the surface waves, over the region in which these waves are present. However, this is not a simple task, as the ray equations given in (4.36) contain numerous branch cuts and hence care must be taken to ensure that the integration path remains on the correct sheet. The numerical results obtained in this chapter were produced by performing parameter sweeps; however, the development of more sophisticated numerical methods for tracking Stokes lines given the appropriate ray equations would be of some interest.

Finally, we optimally-truncated the asymptotic series for the flow behaviour and found an expression for the free-surface behaviour switched across Stokes lines in terms of the singulant and prefactor.

Any extensions to this work would be limited by the difficulty of finding useful expressions for the leading-order behaviour of the system. We did note in Appendix C several values of b for which closed-form leading-order behaviour can be obtained. Additionally, it is possible to obtain leading-order behaviour for some cases of a downwards step in the flowpath. It is likely that closed-form leading order solutions may be obtained in some cases for other geometries, such as flow over an inclined step (as in [75]) or possibly even past a surface-piercing object (as in [102]). Nonetheless, it seems that the Lagrangian formulation developed in Section 3.2.2 of limited utility in the investigation of free-surface flows due to the complicated nature of the resultant algebraic expressions.

Chapter 5

Three-Dimensional Flow

5.1 Introduction

In this chapter, we consider the problem of linearized three-dimensional flow past a singularity in the case of small Froude number, considering both steady and unsteady cases. We will apply exponential asymptotic analysis in order to describe the gravity waves present on the free surface when the Froude number is small. Linearizing the problem allows us to fix the position of the free surface, and we may therefore simply consider the problem in Eulerian coordinates.

Linearized three-dimensional flow problems have been studied in a number of previous investigations, many of which are described in Chapter 1. Some of the earliest work was performed by Kelvin [61], who first showed using the method of stationary phase that a moving pressure source the surface of the flow produced downstream gravity waves within a wedge with angle of approximately 19.3° from the horizontal. This wedge contained two distinct classes of wave behaviour. The first was a wave field that propagated backwards along the direction of the motion of the source, subsequently described as longitudinal waves. The second was a wave field that propagated approximately perpendicularly to the direction of motion, known as transverse waves. The two classes of waves are illustrated in Figure 5.1. Keller [59] showed that this was true in the small F limit using geometric ray theory, which was possible due to the fact that the surface waves are not exponentially small for a point disturbance on the surface of the flow.

The problem of linearized flow past a point obstacle submerged underneath the surface has also been investigated at length. These investigations typically involve formulating the solution as an integral using Green's functions with a linearized free-surface condition. The most well-known of these is the integral formulation for steady flow past a submerged point source obtained by Havelock [51] and also given in Wehausen and Laitone [118]. Alternative integral expressions were obtained by Peters [90] and Michell [79]. These studies, as well as others, are discussed in Chapter 1. The numerical evaluation of the resultant integral expressions forms the basis

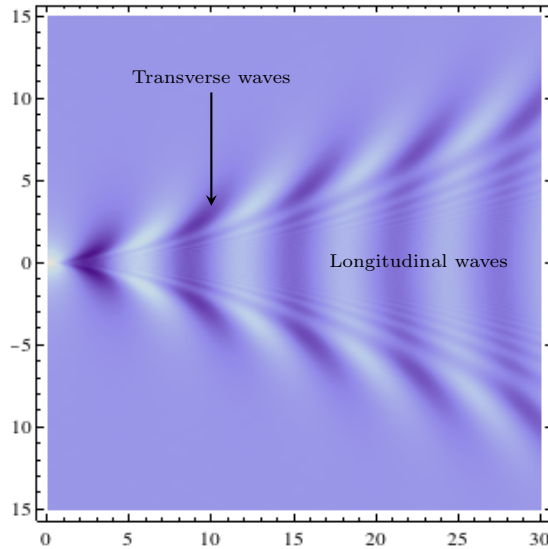


Figure 5.1: Computed wave patterns past a ship using the expression given in [88]. Both the longitudinal and transverse wave patterns are visible in the downstream wake of the ship. This figure was also shown in Chapter 1; however, we reproduce it here for convenience.

of many algorithms for determining the behaviour of free-surface flows in three dimensions, for example in [80, 105]. In Sections 5.2.6 and 5.3.5, we shall formulate and evaluate similar integral expressions for the free-surface behaviour to compare with the results of our asymptotic analysis.

In this investigation, we are interested in determining the small Froude number behaviour of the free surface. As in the analysis of the two-dimensional problem [27], the gravity waves will be exponentially small in the Froude number. We therefore determine the behaviour of these waves by applying the exponential asymptotic techniques of [87] and [22] directly to the governing equations of the system.

The chapter is divided into two sections. The first section is concerned with steady flow over a submerged point source, linearized by the strength of the source. Using exponential asymptotic methods, we will show that the exponentially small longitudinal waves are switched on by the leading order behaviour, and are present in the entire downstream region of the flow. We will subsequently derive an expression for the behaviour of these waves, and compare it to a numerical solution of the linearized system. Furthermore, we will show that the longitudinal wave contribution switches on exponentially subdominant transverse waves, which are restricted to two wedge-like regions in the downstream region. Finally, we will show that these results are consistent with those of Keller [59] as the source approaches the surface of the flow.

The second section involves considering an identical flow geometry, but in the unsteady case. Here, we prescribe an initial waveless behaviour on the flow, and allow it to evolve over $t > 0$. As in Chapter 3, we will find that there are unsteady late-order contributions caused by the

disturbance at $t = 0$. By examining the Stokes structure of the resultant system, we will show that both the longitudinal and transverse waves are switched by this unsteady contribution, and spread downstream as time increases, and tend to the steady-state behaviour as $t \rightarrow \infty$.

5.2 Steady Flow

5.2.1 Formulation

We consider the steady-state problem of mean flow past a submerged point source in three dimensions. In order that the problem may be linearized about the undisturbed solution, we consider the strength of the source to be small. We note that adjusting the problem to consider flow over some other variety of singularity simply requires adjusting the prescribed behaviour of the flow near the singularity. Hence, the analysis performed in this section is not substantially altered by changing the behaviour of the singularity, such as considering flow over a submerged doublet.

5.2.1.1 Full problem

We consider a mean three-dimensional potential free-surface flow of infinite depth with a submerged point source at depth h and upstream flow velocity U . The Froude number is given as

$$F = \frac{U}{\sqrt{gL}},$$

where g is gravity and L is an appropriate length scale, as defined in Chapter 1 in (1.2). We normalize the velocities in the problem by the upstream flow velocity. Usually we would normalize the lengths in the problem by h , but here we will instead use an unspecified length scale, so that the behaviour of the surface as h is varied becomes more easily apparent. Additionally, using this scaling, the problem becomes equivalent in the limit that $h \rightarrow 0$ to that posed in previous work by Keller [59]. We may therefore compare results from this previous study with those found in the current investigation.

As in the two-dimensional case, the flow potential $\phi(x, y, z)$ is governed by Laplace's equation with dynamic and kinematic boundary conditions on the free-surface. We set the Froude number to be small, defining $\epsilon = F^2$ and requiring $0 < \epsilon \ll 1$. The source is chosen to have strength δ in the scaled system, where $0 < \delta \ll \epsilon$. This allows the problem to be linearized in δ , and formulated as a fixed-boundary problem in Eulerian coordinates that may be investigated using a power series approach in small ϵ . The position of the free surface is given by $z = \xi(x, y)$. This configuration may be seen in Figure 5.2.

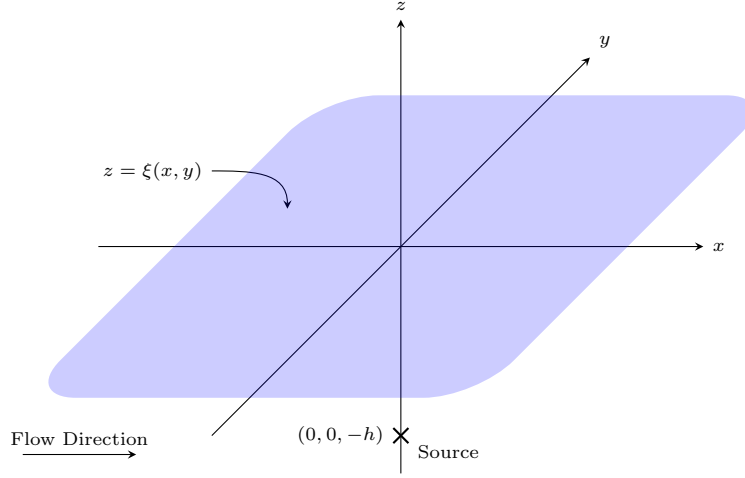


Figure 5.2: Prescribed fluid configuration for three-dimensional flow over a source. The shaded region represents the position of the free surface, and the cross represents the position of the source. The flow region lies below the free surface, and the mean flow is moving from left to right, with flow velocity U in the unscaled problem.

The governing equation is given by

$$\nabla^2 \phi = 0, \quad -\infty < z < \xi(x, y). \quad (5.1)$$

On $z = \xi(x, y)$, the kinematic and dynamic boundary conditions are, respectively,

$$\xi_x \phi_x + \xi_y \phi_y = \phi_z, \quad z = \xi(x, y), \quad (5.2)$$

$$\frac{1}{2} \epsilon (|\nabla \phi|^2 - 1) + \xi = 0, \quad z = \xi(x, y). \quad (5.3)$$

The far-field conditions are

$$\phi_x \rightarrow 1 \quad \text{as} \quad x \rightarrow -\infty, \quad (5.4)$$

$$\phi_x \rightarrow 1 \quad \text{as} \quad y \rightarrow \pm\infty, \quad (5.5)$$

$$\phi_x \rightarrow 1 \quad \text{as} \quad z \rightarrow -\infty. \quad (5.6)$$

and the potential becomes singular near $(x, y, z) = (0, 0, -h)$ as

$$\phi \sim \frac{\delta}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}}. \quad (5.7)$$

The nature of the submerged obstacle is represented in the problem by this potential condition. Hence, changing the form of the singularity (to a doublet, for example) causes this local behaviour to change. In Section 5.2.7, we will demonstrate this by instead considering flow over a submerged doublet, rather than a source.

Finally, we require a radiation condition, which ensures that the gravity waves may only be present downstream from the singularity.

We previously noted that as $h \rightarrow 0$ and the singularity moves onto the surface of the flow, this problem tends to the ship wave problem investigated by Keller [59] using geometric ray theory. Consequently, the problem of a submerged source or obstacle may be thought of as an extension to this work which incorporates the theory of complex rays [23]. In this case, the singularity is now located on the complexified free surface, and it is this singular behaviour that generates the downstream surface waves.

5.2.1.2 Linearization

Linearizing about the undisturbed solution in terms of δ involves setting

$$\phi = x + \delta\bar{\phi}, \quad \xi = \delta\bar{\xi}.$$

At leading order in δ , the governing equation (5.1) simply becomes

$$\nabla^2 \bar{\phi} = 0, \quad -\infty < z < 0. \quad (5.8)$$

The free-surface boundary conditions now apply on $z = 0$, with the linearized versions of (5.2) and (5.3) respectively taking the form

$$\bar{\phi}_z - \bar{\xi}_x = 0, \quad z = 0, \quad (5.9)$$

$$\epsilon \bar{\phi}_x + \bar{\xi} = 0, \quad z = 0. \quad (5.10)$$

The far-field conditions (5.4)–(5.6) imply that $\bar{\phi} \rightarrow 0$ as $x \rightarrow -\infty$, $y \rightarrow \pm\infty$ and $z \rightarrow -\infty$. Finally, the singularity condition given in (5.7) becomes

$$\bar{\phi} \sim \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} \quad \text{as } (x, y, z) \rightarrow (0, 0, -h). \quad (5.11)$$

The system described in (5.8)–(5.11) completely specifies the linearized version of the three-dimensional problem shown in Figure (5.2), with the addition of a radiation condition, specifying that gravity waves may only propagate downstream. The bars in these equations will subsequently be omitted, in the understanding that all subsequent analysis on this problem is to be performed on the linearized system, rather than the full nonlinear problem.

Finally, we analytically continue the free surface such that $x, y \in \mathbb{C}$, with the free surface still defined as the surface satisfying $z = 0$. This does not change the form of equations (5.8)–(5.11), but it does mean that the two-dimensional physical free surface is a subset of a four-dimensional complexified free surface.

5.2.1.3 Series expression

In order for exponential asymptotic techniques to be applied, the fluid potential and free-surface position must be expressed in terms of a power series in ϵ , given by

$$\bar{\phi} \sim \sum_{n=0}^{\infty} \epsilon^n \phi^{(n)}, \quad \bar{\xi} \sim \sum_{n=0}^{\infty} \epsilon^n \xi^{(n)}. \quad (5.12)$$

Applying this expression to (5.8) gives

$$\nabla^2 \phi^{(n)} = 0, \quad -\infty < z < 0. \quad (5.13)$$

The boundary conditions on the free-surface become

$$\phi_z^{(n)} - \xi_x^{(n)} = 0, \quad z = 0, \quad (5.14)$$

$$\phi_x^{(n-1)} + \xi^{(n)} = 0 \quad z = 0, . \quad (5.15)$$

The far-field behaviour tends to zero at all orders of n , and the singularity condition (5.11) becomes

$$\phi^{(0)} \sim \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} \quad \text{as } (x, y, z) \rightarrow (0, 0, -h). \quad (5.16)$$

The leading-order solution to (5.13)–(5.16) may be found using the method of images [83], and is obtained by taking the sum of the fundamental solution for a source located at $(0, 0, -h)$ and the fundamental solution for a source reflected across the free surface and located at $(0, 0, h)$. This gives

$$\phi^{(0)} = \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \frac{1}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}}, \quad \xi^{(0)} = 0. \quad (5.17)$$

Using (5.15), we therefore find that

$$\xi^{(1)} = \frac{x}{2\pi(x^2 + y^2 + h^2)^{3/2}}. \quad (5.18)$$

Through repeated iterations of (5.14)–(5.15), we find that the position of singularities in subsequent terms of the series (5.12) remains constant, while the power of the singularity increases at each order, as we expect for such a problem.

5.2.2 Late-Order Ansatz

In order to optimally truncate the asymptotic series prescribed in (5.12), we must determine the form of the late-order terms in the same fashion as previous chapters. As usual, we choose an appropriate ansatz for the late-order terms of factorial-over-power form [22], giving

$$\phi^{(n)} \sim \frac{\Phi(x, y, z)\Gamma(n + \gamma)}{\chi(x, y, z)^{n+\gamma}}, \quad \xi^{(n)} \sim \frac{\Xi(x, y)\Gamma(n + \gamma)}{\chi(x, y, 0)^{n+\gamma}}. \quad (5.19)$$

We note that γ could potentially also be a function of x, y and z ; however, this will not be the case in this problem. Additionally, in order that the singularities are consistent with the leading-order solution (5.17), we note that the singulant, χ , satisfies

$$\chi = 0 \quad \text{on} \quad x^2 + y^2 + (z \pm h)^2 = 0, \quad (5.20)$$

where the sign chosen depends upon the singularity under consideration. Recalling that x and y are permitted to take complex values, we see that this defines a surface on which $\chi = 0$. Irrespective of which singularity is under consideration, this surface intersects the complexified free surface at points satisfying $x^2 + y^2 + h^2 = 0$.

It is important to note that the expression for $\xi^{(n)}$ is restricted to $z = 0$, as it describes the free-surface position. This does not pose a problem for the subsequent analysis, but does ensure that care must be taken at each stage to determine whether we are considering the full flow region, or just the free surface.

5.2.2.1 Calculating the singulant

Applying the ansatz expressions in (5.19) to the governing equation (5.13) and taking the first two orders as $n \rightarrow \infty$ gives

$$\chi_x^2 + \chi_y^2 + \chi_z^2 = 0, \quad (5.21)$$

$$2\Phi_x \chi_x + 2\Phi_y \chi_y + 2\Phi_z \chi_z = -(\chi_{xx} + \chi_{yy} + \chi_{zz}), \quad (5.22)$$

while the boundary conditions on $z = 0$ become to leading order

$$-\chi_x \Phi + \Xi = 0, \quad (5.23)$$

$$-\chi_z \Phi + \chi_x \Xi = 0. \quad (5.24)$$

The system in (5.23)-(5.24) has nonzero solutions when

$$\chi_x^2 = \chi_z.$$

Combining this result with (5.21) gives an eikonal equation for χ on the free surface,

$$\chi_x^4 + \chi_x^2 + \chi_y^2 = 0. \quad (5.25)$$

We recall that the leading-order behaviour (5.17) contains singularities at $x^2 + y^2 + (z \pm h)^2 = 0$, and that these singularities must be present in the late-order terms of the series. This means that the singulant is equal to zero on the complexified free surface at all points satisfying the condition that $x^2 + y^2 + h^2 = 0$. Using this boundary condition, (5.25) may be solved using Charpit's method. By setting $h = 0$, we obtain the formulation given in [59] for the application

of geometric ray theory to surface waves generated by a thin ship, given there in terms of the wave vector $\mathbf{k} = \nabla s$, where s is the phase function of the surface waves and is equivalent to $i\chi$ in our notation. The behaviour of the phase function found in [59] will be compared to the results of the current analysis for χ in the limit that $h \rightarrow 0$ in Section 5.2.5.

Solving (5.25) using Charpit's method requires setting $\chi_x = P$ and $\chi_y = Q$. In this case, we define u to be the characteristic variable, and parametrize the initial data using s . The system to be solved is subsequently given by

$$\frac{dx}{du} = 4P^3 + 2P, \quad \frac{dy}{du} = 2Q, \quad \frac{d\chi}{du} = 2P^4, \quad (5.26)$$

where P and Q satisfy

$$P^4 + P^2 + Q^2 = 0, \quad P + \frac{dy_0}{ds}Q = 0, \quad (5.27)$$

and the initial condition specified in (5.20) is parametrized as

$$x_0 = s, \quad y_0 = \pm i\sqrt{s^2 + h^2}, \quad \chi_0 = 0. \quad (5.28)$$

We note that this is only one of many possible parameterizations, and indeed it may seem more natural to use

$$x_0 = ih \sin(s), \quad y_0 = ih \cos(s), \quad \chi_0 = 0. \quad (5.29)$$

However, the conditions in (5.28) ensure that the analysis required to obtain the solution is restricted to elementary algebraic manipulations. Were the conditions given in (5.29) used, the same results could be obtained; however, the intervening analysis would be more complicated due to the presence of trigonometric functions.

With some algebraic manipulation, we find that the possible ray contributions are given by

$$\chi = \frac{2h^4 u}{s^4}, \quad (5.30)$$

where u is one of

$$u = \pm \frac{h^3(s - x)}{s(2h^2 + s^2)} \quad (5.31)$$

and s is any one of the four solutions to

$$(x^2 + y^2)s^4 + 4xh^2s^3 + (h^2x^2 + 4h^2y^2 + 4h^4)s^2 + 4h^4xs + (4y^2h^4 + 4h^6) = 0. \quad (5.32)$$

This produces eight possible expressions for the singulant (the four solutions of (5.32) and the choice of sign in (5.30)), corresponding to eight possible sets of late-order behaviour in the problem. Contour plots illustrating the behaviour of these singulant terms are presented in Figures 5.3-5.4 for $h = 1$. We have labelled the singulants according to their behaviour on the physical free surface, where χ_{Lj} for $j = 1, 2, 3, 4$ are associated with longitudinal waves on the

free surface (seen in Figure 5.5), while χ_{Tj} for $j = 1, 2, 3, 4$ are associated with transverse waves (seen in Figure 5.9).

Each of these expressions satisfies $\text{Im}(\chi) = 0$ along the curve $x = 0$; however, only the behaviour illustrated in Figure 5.3 satisfies $\text{Re}(\chi) > 0$ along this curve. Given that both of these conditions must be satisfied along a first-generation Stokes line [33], we see that Stokes switching is only possible in these four cases.

The initial condition (5.17) contains two singularities, each of which we expect will correspond with one of the eight possible singulant expressions. Importantly, we know that any free-surface behaviour switched on across Stokes lines must be exponentially small in the entire region of the free surface which it is active. This is only possible if $\text{Re}(\chi) > 0$ for all of $x > 0$. This condition is satisfied by the singulants illustrated in Figure 5.3, but not those shown in Figure 5.4.

Knowing that any contribution switched across the Stokes lines must only be present in the downstream region due to the radiation condition, we therefore determine that the free-surface contains an exponentially small contribution associated with late-order terms having singulant behaviour illustrated in Figure 5.3 that is switched on as the curve $x = 0$ is crossed from the negative to the positive side. We also note that while the remaining contributions are not switched on by the leading-order behaviour, it is still possible that they may produce contributions to the free-surface behaviour due to the interaction between exponentially small contributions. This possibility will be considered in Section 5.2.4.

Additionally, while the values of χ in Figures 5.3-5.4 were calculated for $h = 1$, the conclusions drawn regarding ray contributions are valid for all h . This is a consequence of the observation that the depth of the source is the only length parameter in the problem, and may therefore be scaled freely to take arbitrary value.

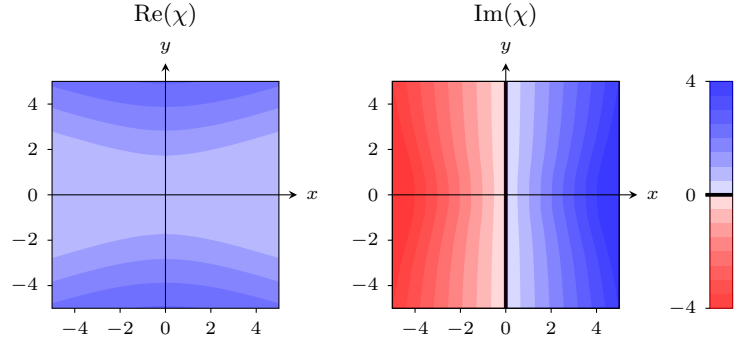
5.2.2.2 Finding the prefactor

In order to obtain a complete expression for the steady downstream longitudinal free-surface waves, we require an expression for the prefactors, Φ and Ξ . To find the prefactor equation, we consider the next order in (5.14)–(5.15) as $n \rightarrow \infty$. From (5.23), we obtain

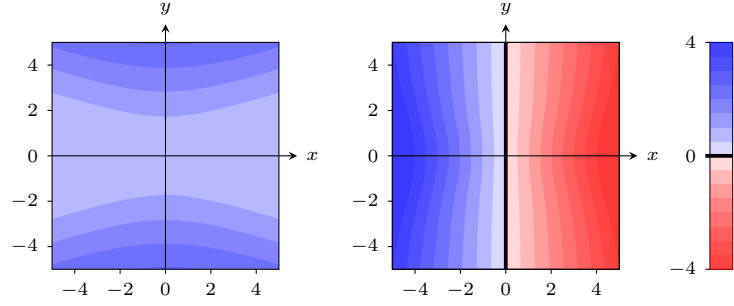
$$\Xi = \chi_x \Phi. \quad (5.33)$$

Care must be taken at this stage of the analysis. In the prescribed ansatz for the late order terms, (5.19), the prefactors are actually leading-order terms in a power series as $n \rightarrow \infty$, or

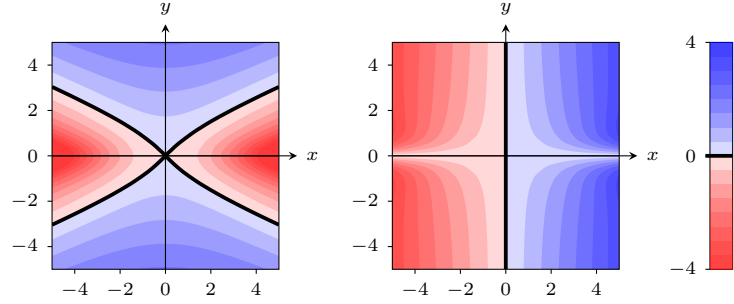
$$\Phi = \Phi_0 + \frac{1}{n}\Phi_1 + \dots, \quad \Xi = \Xi_0 + \frac{1}{n}\Xi_1 + \dots \quad (5.34)$$



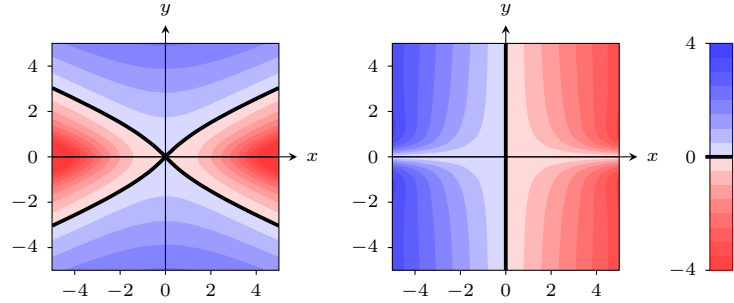
(a) Contour plots for χ_{L1}



(b) Contour plots for χ_{L2}

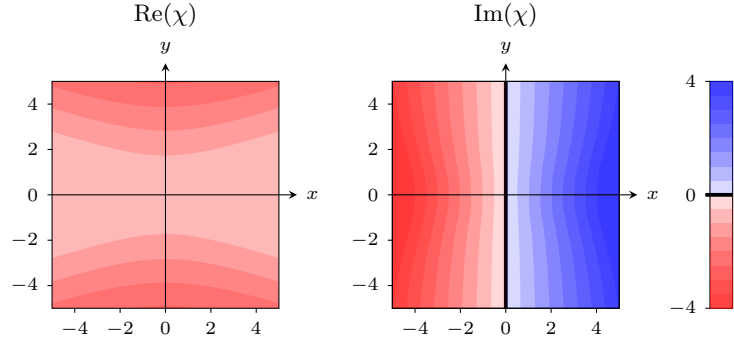


(c) Contour plots for χ_{T3}

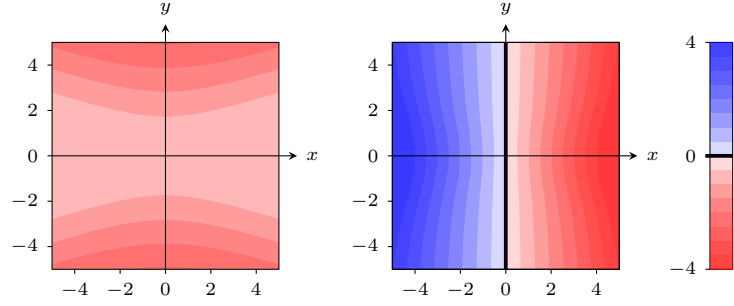


(d) Contour plots for χ_{T4}

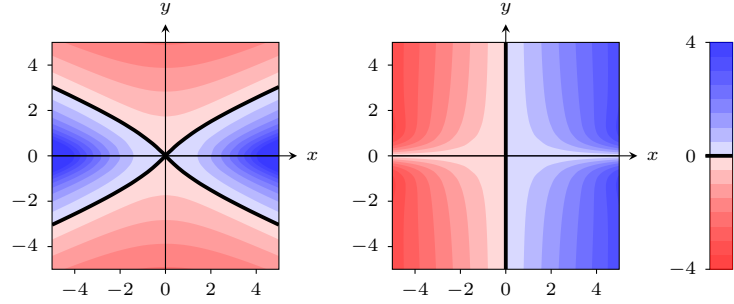
Figure 5.3: Singulants obtained by solving (5.25) with boundary data (5.28). This figure shows the four singulants that satisfy both $\text{Im}(\chi) = 0$ and $\text{Re}(\chi) > 0$ along the curve $x = 0$. Hence, the associated late-order contributions associated with these singulants may lead to Stokes switching behaviour on the free surface.



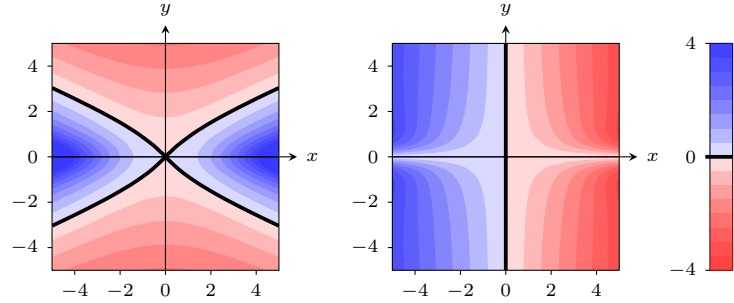
(a) Contour plots for χ_{L3}



(b) Contour plots for χ_{L4}



(c) Contour plots for χ_{T1}



(d) Contour plots for χ_{T2}

Figure 5.4: Singulants obtained by solving (5.25) with boundary data (5.28). This figure shows the four singulants that satisfy $\text{Im}(\chi) = 0$ on $x = 0$ but have $\text{Re}(\chi) < 0$ along this line. Hence there can be no Stokes switching caused by the interaction between the leading-order contribution and the contributions generated by the associated late-order terms.

This was not important in the leading-order equations given in (5.21)–(5.24), and hence was not considered in the associated analysis. However, these terms do become relevant when considering the next order in (5.14)–(5.15) as $n \rightarrow \infty$. Applying the late-order ansatz to (5.21)–(5.24) and taking the next order as $n \rightarrow \infty$ now gives

$$\begin{aligned} -\chi_x \Phi_1 + \Phi_{0,x} + \Xi_1 &= 0, \\ -\chi_z \Phi_1 + \Phi_{0,z} + \chi_x \Xi_1 - \Xi_{0,x} &= 0. \end{aligned}$$

This system only has nontrivial solutions when

$$\chi_x \Phi_{0,x} - \Phi_{0,z} + \Xi_{0,x} = 0.$$

Having found the solvability condition we now omit the subscripts and denote $\Xi_0 = \Xi$ and $\Phi_0 = \Phi$, as we are only interested in finding the leading-order behaviour of the prefactor. This gives

$$\chi_x \Phi_x = \Phi_z - \Xi_x. \quad (5.35)$$

Combining (5.35), as well as (5.33), with (5.22) gives

$$(4\chi_x^3 + 2\chi_x)\Phi_x + 2\chi_y\Phi_y = -(\chi_{xx} + \chi_{yy} + 6\chi_x^2\chi_{xx})\Phi. \quad (5.36)$$

This equation may be solved using the method of characteristics, giving the ray equations (with characteristic variable u) as

$$\frac{dx}{du} = 4\chi_x^3 + 2\chi_x, \quad \frac{dy}{du} = 2\chi_y, \quad \frac{d\Phi}{du} = -(\chi_{xx} + \chi_{yy} + 6\chi_x^2\chi_{xx})\Phi. \quad (5.37)$$

The first two equations govern the ray paths, and are identical to those produced by applying Charpit's method to the singulant equation, given in (5.26). This allows (5.37) to be rewritten in terms of the Charpit variables as

$$\frac{dx}{du} = 4P^3 + 2P, \quad \frac{dy}{du} = 2Q, \quad \frac{d\Phi}{du} = -(P_s s_x + Q_s s_y + 6P^2 P_s s_x)\Phi, \quad (5.38)$$

noting that P and Q are independent of t . From the Charpit analysis performed in the previous section, we know that

$$P = \pm \frac{h}{s}, \quad Q = \pm \frac{ih\sqrt{s^2 + h^2}}{s^2}, \quad (5.39)$$

where the combination of signs depends on the chosen value of y_0 and the chosen root of (5.27). Integrating the ray equations in (5.37) gives

$$x = (4P^3 + 2P)u + s, \quad y = 2Qu \pm i\sqrt{s^2 + h^2}. \quad (5.40)$$

At this stage, we choose the positive sign, as it is this choice that leads to the expressions for χ that govern the real free-surface waves, represented in Figure 5.3. Taking the partial derivative

of these equations with respect to x produces a system of equations for s_x and u_x , which may be solved to give

$$s_x = \frac{s^4(h^2 + s^2)}{h^2(6hus^2 + 4h^3u + s^4)}, \quad (5.41)$$

Similarly, taking the partial derivative of the equations given in (5.40) with respect to y and solving the resultant system gives

$$s_y = -\frac{is^3(2h^2 + s^2)\sqrt{h^2 + s^2}}{h^2(6hus^2 + 4h^3u + s^4)}, \quad (5.42)$$

Applying the expressions given in (5.39), (5.41) and (5.42) to the third equation in (5.38) gives a differential equation in u that may be solved to give

$$\Phi(s, u) = \Phi(s, 0) \left(\frac{s^4}{6hus^2 + 4h^3u + s^4} \right)^{1/2}. \quad (5.43)$$

Selecting the corresponding expressions for s and u in terms of x and y from (5.31) and (5.32) gives the solution in terms of the physical coordinates x and y . To find an expression for $\Phi(s, 0)$, the behaviour of the system in the neighbourhood of $u = 0$ must be computed.

5.2.2.3 Inner problem

To solve the inner problem, we first consider the behaviour of the singulant caused by the singularity at $x^2 + y^2 + (z + h)^2 = 0$ near the singularity itself. Setting

$$\chi \sim f(x, y, z)(x^2 + y^2 + (z + h)^2)^\alpha \quad \text{as} \quad x^2 + y^2 + (z + h)^2 \rightarrow 0,$$

and equating dominant terms in (5.25), we find that $\alpha = 1$ and

$$f(x, y, z) = \pm \frac{h}{2x^2}. \quad (5.44)$$

Similarly, considering the behaviour of the singulant caused by the singularity at $x^2 + y^2 + (z - h)^2 = 0$ (denoted χ_{L2}) near the singularity also gives $\alpha = 1$ and

$$f(x, y, z) = \pm \frac{h}{2x^2}. \quad (5.45)$$

In the prefactor equation (5.43), we see that the unknown coefficient is a function of s . From (5.40), it follows that $s \sim x$ near the singularity at $t = 0$. Hence, we define a system of inner coordinates given by

$$\epsilon\sigma_1 = \frac{h}{2x^2} (x^2 + y^2 + (z - h)^2), \quad \epsilon\sigma_2 = \frac{h}{2x^2} (x^2 + y^2 + (z + h)^2), \quad \lambda = x. \quad (5.46)$$

The partial derivatives subsequently become

$$\begin{aligned} \frac{\partial}{\partial x} &= \left[\frac{hx}{\lambda^2\epsilon} - 2\lambda\sigma_1 \right] \frac{\partial}{\partial\sigma_1} + \left[\frac{hx}{\lambda^2\epsilon} - 2\lambda\sigma_2 \right] \frac{\partial}{\partial\sigma_2} + \frac{\partial}{\partial\lambda}, \\ \frac{\partial}{\partial y} &= \left[\frac{hy}{\lambda^2\epsilon} \right] \frac{\partial}{\partial\sigma_1} + \left[\frac{hy}{\lambda^2\epsilon} \right] \frac{\partial}{\partial\sigma_2}, \\ \frac{\partial}{\partial z} &= \left[\frac{h(z+h)}{\lambda^2\epsilon} \right] \frac{\partial}{\partial\sigma_1} + \left[\frac{h(z-h)}{\lambda^2\epsilon} \right] \frac{\partial}{\partial\sigma_2}. \end{aligned}$$

To leading order, the linearized governing equation (5.8) becomes (omitting the bars)

$$\sigma_1 \phi_{\sigma_1 \sigma_1} - \sigma_2 \phi_{\sigma_2 \sigma_2} + \lambda \phi_{\lambda \sigma_2} - \lambda \phi_{\lambda \sigma_1} = 0,$$

where terms containing derivatives with respect to both σ_1 and σ_2 were disregarded due to the form of the inner expansion, (5.50). Furthermore, we find that

$$z \sim \frac{\epsilon(\sigma_1 - \sigma_2)}{4h},$$

and hence disregarding terms that are small compared to the leading-order behaviour leaves

$$2\lambda^2 \sigma_1 \phi_{\sigma_1 \sigma_1} + 2\lambda^2 \sigma_2 \phi_{\sigma_2 \sigma_2} - h \phi_{\lambda \sigma_2} + h \phi_{\lambda \sigma_1} = 0.$$

Similarly, the boundary conditions (5.9)-(5.10) become

$$\frac{h}{\lambda} \phi_{\sigma_1} - \frac{h}{\lambda} \phi_{\sigma_2} - \xi_{\sigma_1} - \xi_{\sigma_2} = 0 \quad \text{on} \quad \sigma_1 = -\sigma_2, \quad (5.47)$$

$$\frac{h}{\lambda} \phi_{\sigma_1} + \frac{h}{\lambda} \phi_{\sigma_2} + \xi = 0 \quad \text{on} \quad \sigma_1 = -\sigma_2. \quad (5.48)$$

Finally, by expressing the leading-order behaviour (5.17) in terms of the local variables, we find that

$$\phi^{(0)} \sim \frac{h^{1/2}}{4\sqrt{2}\pi\epsilon^{1/2}\lambda^{1/2}} \frac{1}{\sigma_1^{1/2}} + \frac{h^{1/2}}{4\sqrt{2}\pi\epsilon^{1/2}\lambda^{1/2}} \frac{1}{\sigma_2^{1/2}}, \quad \xi^{(0)} = 0. \quad (5.49)$$

We now define the series expansion near the singularity on the complexified free surface as

$$\phi \sim \sum_{n=0}^{\infty} \left[\frac{a_n(\lambda)\Gamma(n+1/2)}{\sigma_1^{n+1/2}} + \frac{b_n(\lambda)\Gamma(n+1/2)}{\sigma_2^{n+1/2}} \right], \quad \xi \sim \sum_{n=0}^{\infty} \left[\frac{2c_n(\lambda)\Gamma(n+1/2)}{\sigma_1^{n+1/2}} \right], \quad (5.50)$$

where the latter expression is only valid on the free-surface itself, where $\sigma_1 = \sigma_2$. The factor of two is included for subsequent algebraic convenience, and has no effect on the solution to the problem as c_n is still unknown at this stage. From (5.49), we have

$$a_0(\lambda) = \frac{h^{1/2}}{4\sqrt{2}\pi\lambda^{1/2}}, \quad b_0(\lambda) = \frac{h^{1/2}}{4\sqrt{2}\pi\lambda^{1/2}}, \quad c_0 = 0. \quad (5.51)$$

We are interested in the behaviour of the terms on the complexified free surface in the neighbourhood of the singularity at $x^2 + y^2 + h^2 = 0$, or $\sigma_1 = \sigma_2 = 0$. Consequently, we apply the series expression to (5.47), giving on $\sigma_1 = \sigma_2$

$$\begin{aligned} -\frac{h}{\lambda} \sum_{n=0}^{\infty} \frac{(n+1/2)a_n\Gamma(n+3/2)}{\sigma_1^{n+3/2}} + \frac{h}{\lambda} \sum_{n=0}^{\infty} \frac{(n+1/2)b_n\Gamma(n+3/2)}{\sigma_1^{n+3/2}} \\ - \frac{h}{\lambda} \sum_{n=0}^{\infty} \frac{2(n+1/2)c_n\Gamma(n+3/2)}{\sigma_1^{n+3/2}} = 0. \end{aligned}$$

Matching as $\sigma_1 \rightarrow 0$ gives

$$\frac{h}{\lambda}(a_n - b_n) + 2c_n = 0, \quad n \geq 0. \quad (5.52)$$

Applying the series expansion to (5.48) gives on the free surface

$$-\frac{h}{\lambda} \sum_{n=0}^{\infty} \frac{(n+1/2)a_n \Gamma(n+3/2)}{\sigma_1^{n+3/2}} - \frac{h}{\lambda} \sum_{n=0}^{\infty} \frac{(n+1/2)b_n \Gamma(n+3/2)}{\sigma_1^{n+3/2}} + \sum_{n=0}^{\infty} \frac{2c_n \Gamma(n+1/2)}{\sigma_1^{n+1/2}} = 0,$$

Matching terms similarly gives

$$-\frac{h}{\lambda}(a_n + b_n) + 2c_{n+1} = 0, \quad n \geq 0. \quad (5.53)$$

We are primarily interested in the behaviour on the complexified free-surface; however, restricting the domain in this fashion means that it is impossible to distinguish between the contributions from the series in σ_1 and the series in σ_2 . Again, we notice that the two contributions have equal magnitude in (5.49), and hence using the same arguments as in previous chapters, we specify that $|a_n| = |b_n|$ in order to maintain consistency with the lower-order terms in the asymptotic series. This may only be accomplished if we divide the two equations given in (5.52)–(5.53) into four equations such that

$$\begin{aligned} \frac{ha_n}{\lambda} - c_n &= 0, & -\frac{h(n+1/2)a_n}{\lambda} + c_{n+1} &= 0, \\ -\frac{hb_n}{\lambda} - c_n &= 0, & -\frac{h(n+1/2)b_n}{\lambda} + c_{n+1} &= 0. \end{aligned}$$

We will consider only the equations involving a_n and c_n , simply noting that this system implies that $b_n = (-1)^n a_n$. Eliminating c_n from this system gives

$$a_{n+1} = (n+1/2)a_n = \frac{a_0 \Gamma(n+1/2)}{\Gamma(1/2)}.$$

Hence, using the expression for a_0 given in (5.51), we may match the local series expression given in (5.50) with the prefactor given in (5.43). Noting that λ is the local expression for s in the outer solution, we find that

$$\Phi(s, 0) = \frac{h^{1/2}}{4\pi s^{1/2} \sqrt{2\pi}}. \quad (5.54)$$

Hence, we are able to completely describe the late-order behaviour of terms in (5.12).

5.2.3 Stokes Smoothing

Upon obtaining the form of the late-order terms, given by (5.19), the asymptotic series given in (5.12) may be truncated to give

$$\phi = \sum_{n=0}^{N-1} \epsilon^n \phi^{(n)} + R^{(N)}, \quad \xi = \sum_{n=0}^{N-1} \epsilon^n \xi^{(n)} + S^{(N)},$$

where N will be chosen in order to minimize the remainder of the asymptotic approximations, given by $R^{(N)}$ and $S^{(N)}$. Applying this series expression to (5.13) gives

$$\nabla^2 R^{(N)} = 0, \quad (5.55)$$

while the dynamic boundary conditions (5.9)–(5.10) become

$$R_z^{(N)} - S_x^{(N)} = 0, \quad (5.56)$$

$$\epsilon R_x^{(N)} + S^{(N)} = \epsilon^N \xi^{(N)}, \quad (5.57)$$

having made use of the relationship in (5.15) and the fact that $\xi^{(0)} = 0$. The homogeneous form of (5.55)–(5.57) is satisfied as $\epsilon \rightarrow 0$ by

$$R^{(N)} \sim \Phi e^{-\chi_{L1}/\epsilon}, \quad S^{(N)} \sim \Xi e^{-\chi_{L1}/\epsilon},$$

obtained by applying an ansatz to the homogeneous system (5.55)–(5.57). The choice of χ_{L1} simply ensures we are obtaining the correct late-order term behaviour. For the remainder of this section, we shall omit the subscript and simply refer to this singulant as χ . We therefore set the remainder terms for the inhomogeneous problem to take the form

$$R^{(N)} = A\Phi e^{-\chi/\epsilon}, \quad S^{(N)} = A\Xi e^{-\chi/\epsilon}, \quad (5.58)$$

where A is the usual Stokes switching parameter. Applying these expressions to (5.57) while noting (5.33) gives

$$\epsilon[A_x\Phi + A\Phi_x]e^{-\chi/\epsilon} = \epsilon^N \xi^{(N)}.$$

Applying the late-order ansatz gives

$$\epsilon[A_x\Phi + A\Phi_x]e^{-\chi/\epsilon} \sim \epsilon^N \frac{\Xi\Gamma(N+1/2)}{\chi^{N+1/2}}.$$

Motivated by the homogeneous solution, we set $u = \chi$, $v = y$ and apply (5.33) to obtain

$$A_u\chi_x\Phi + A\Phi_u\chi_x = \epsilon^{N-1}e^{u/\epsilon} \frac{\chi_x\Phi\Gamma(N+1/2)}{u^{N+1/2}},$$

which becomes

$$A_u + \frac{A\Phi_u}{\Phi} = \epsilon^{N-1}e^{u/\epsilon} \frac{\Gamma(N+1/2)}{u^{N+1/2}}.$$

Taking the derivative of the right-hand side of this expression and equating to zero as in the previous chapters, we find that the optimal truncation point is given by $N \sim |\chi|/\epsilon$ in the limit that $\epsilon \rightarrow 0$. As usual, we change coordinates such that $u = re^{i\theta}$. The effect of this transformation on the partial derivative was given in (2.2.2.4). Consequently, we set $N = r/\epsilon + \alpha$ so that it takes integer value. Using Stirling's formula on the resultant expression gives

$$A_\theta + \frac{A\Phi_\theta}{\Phi} \sim \frac{i\sqrt{2\pi r}}{\epsilon} \exp\left(\frac{r}{\epsilon}(e^{i\theta} - 1) - i\theta\left(\frac{r}{\epsilon} + \alpha - \frac{1}{2}\right)\right).$$

This variation is exponentially small, except in the neighbourhood of the Stokes line, given by $\theta = 0$. To investigate the behaviour in the vicinity of the Stokes line itself, we set $\theta = \epsilon^{1/2}\hat{\theta}$.

Knowing that the prefactor Φ does not vary rapidly in the neighbourhood of the Stokes line, and hence that $\Phi_{\hat{\theta}}$ is small compared to $A_{\hat{\theta}}$, we obtain

$$A_{\hat{\theta}} \sim i \sqrt{\frac{2\pi r}{\epsilon}} e^{-r\hat{\theta}^2/2}.$$

Upon integration, this becomes

$$A \sim i \sqrt{\frac{2\pi}{\epsilon}} \int_{-\infty}^{\theta\sqrt{r/\epsilon}} e^{-t^2/2} dt + C,$$

where C is the constant to be determined using matched asymptotic expansions. As the Stokes line is crossed, we therefore find that the value of A varies rapidly from 0 to $2\pi i/\epsilon^{1/2}$. Knowing that there are no waves present in the upstream far field, we determine that the constant is zero, and hence waves are present only downstream. From (5.58), we therefore find that the variation in the free surface is given by

$$[S^{(N)}]_{-}^{+} = \frac{2\pi i \Xi}{\epsilon^{1/2}} e^{-\chi/\epsilon}. \quad (5.59)$$

As in previous chapters, we observe that there are two active contributions to the late-order terms, and that performing a similar analysis on the contribution associated with χ_{L2} gives late-order terms taking the complex conjugate value of those obtained in (5.59). Hence, the full late-order expression is given by

$$[S_{\text{TOT}}^{(N)}]_{-}^{+} = -\frac{4\pi}{\epsilon^{1/2}} \left[\text{Re}(\Xi) \sin\left(\frac{\text{Im}(\chi)}{\epsilon}\right) + \text{Im}(\Xi) \cos\left(\frac{\text{Im}(\chi)}{\epsilon}\right) \right] e^{-\text{Re}(\chi)/\epsilon}. \quad (5.60)$$

where a bar indicates complex conjugation. As there are no waves present upstream, we find that gravity waves of the form given in (5.60) are present on the free-surface of the flow. Figure 5.5 illustrates the behaviour of these waves for $\epsilon = 0.15$, $h = 1$. Along the curve $y = 0$, we find that the far-field behaviour is given by

$$\xi_{\text{exp}} \sim \left(\frac{h}{8\pi\epsilon x}\right)^{1/2} e^{-h/\epsilon} \cos\left(\frac{x}{\epsilon}\right). \quad (5.61)$$

We recall from Figure 5.3 that the Stokes line intersects the free surface along the curve $x = 0$. Despite this, we see that the waves appear to be restricted to a wedge behind the source with an angle to the horizontal of approximately the Kelvin angle. We find that the apparently-flat regions of the surface actually contain exponentially small wave behaviour which switches when the Stokes line is crossed; however, their comparative size renders them unable to be seen.

Many previous investigations, such as that of Kelvin [61], and more recently in the small F limit by Keller [59], have found that when a source is present on the surface of the flow, the resultant Kelvin waves will be restricted to a wedge with angles of 19.3° to the horizontal. Keller demonstrated this result using geometric ray theory by showing that the solution only contained real rays within this wedge. In Section 5.2.5, we will demonstrate that, as in the case for flow

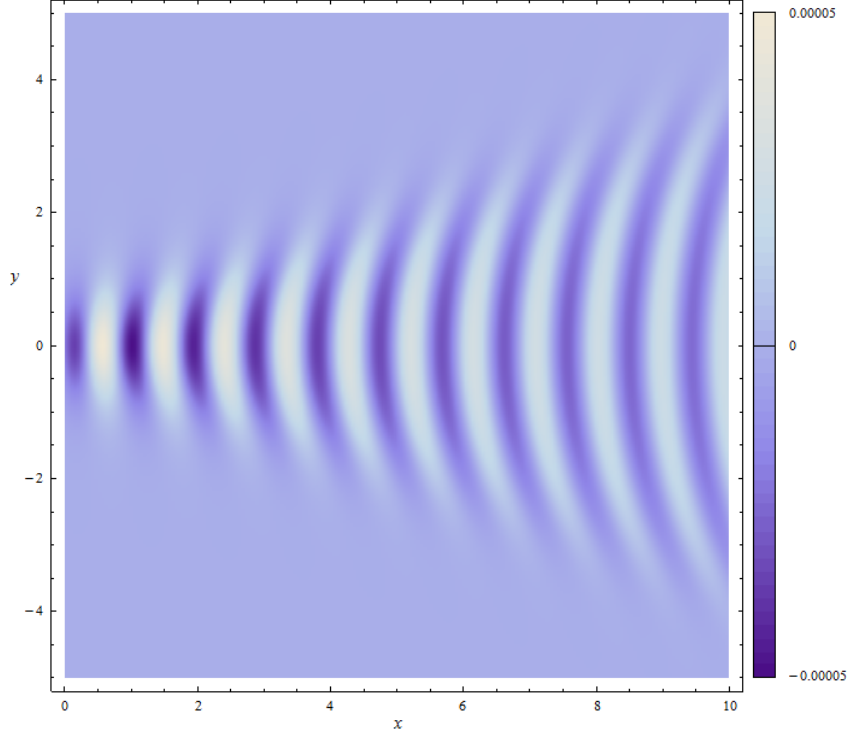


Figure 5.5: Exponentially small contribution to the free-surface behaviour with $\epsilon = 0.15$ and $h = 1$, given in (5.60). The waves switches on rapidly across the curve $x = 0$ and generates waves on the positive side of the Stokes line. These waves appear to be restricted to a wedge-shaped region of the domain, but are in fact present for all $x > 0$. In the regions of the surface that appear flat, the waves are simply exponentially small compared to the visible waves within the wedge.

over a submerged source, there exist exponentially subdominant waves outside of this wedge region when $x > 0$ caused by complex rays [23] that were not considered in the geometric ray analysis.

5.2.4 Complete Stokes structure

5.2.4.1 Stokes Structure

In Figure 5.3, four possible contributions were shown. We saw that χ_{L1} and χ_{L2} were switched on by the leading-order behaviour, and produce a downstream train of longitudinal gravity waves. Here, we will show that this contribution generates further Stokes lines, across which contributions from χ_{T1} and χ_{T2} will be switched on. These contributions will produce transverse waves on the free-surface.

Stokes switching will occur on curves satisfying $\text{Im}(\chi_{L1}) = \text{Im}(\chi_{T1})$ and $\text{Re}(\chi_{L1}) < \text{Re}(\chi_{T1})$. Along these curves, the same conditions will also be satisfied by χ_{L2} and χ_{T2} , which are the respective complex conjugate expressions.

To fully illustrate the Stokes behaviour in this problem, we examine the behaviour of the Stokes

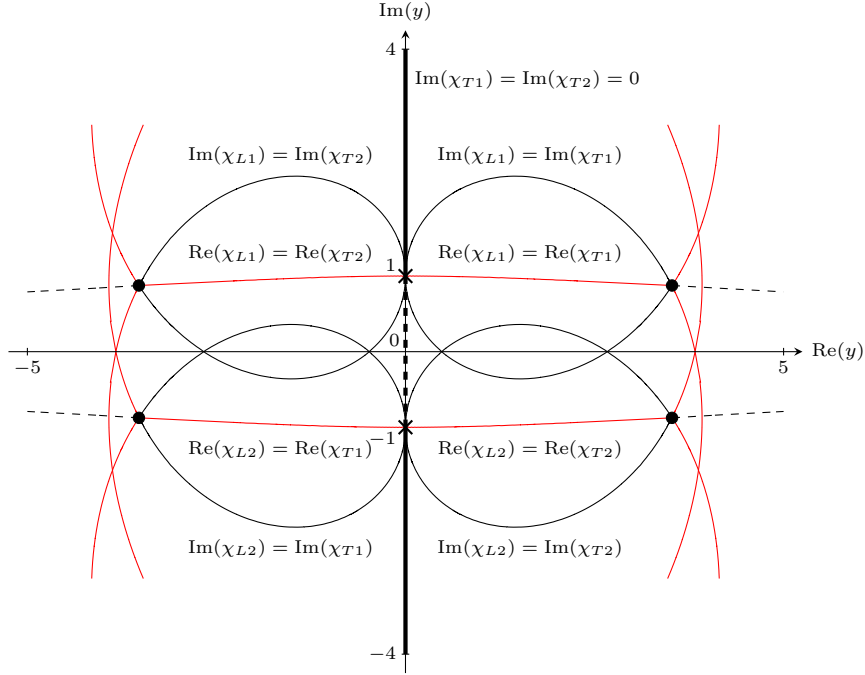


Figure 5.6: Stokes structure for flow over a source of depth $h = 1$, where $x = 10$ and $y \in \mathbb{C}$. Turning points and singularities are represented by circles and crosses respectively. Active Stokes lines are shown as solid black curves, while inactive Stokes lines are shown as dashed black curves. Only anti-Stokes lines that are relevant to the Stokes structure of the problem are illustrated, represented as red curves. Stokes and anti-Stokes lines that follow the real and imaginary axis are shown in bold for clarity.

lines for $y \in \mathbb{C}$ and fixed x . In Figure 5.6, we see that pattern of Stokes lines and anti-Stokes lines present in the analytically-continued solution to the problem at $x = 10$. The solution contains four turning points, denoted as circles, and two singularities, denoted as crosses. From each turning point emerges three possible Stokes lines, shown in black, and three anti-Stokes lines, shown in red. Two of the Stokes lines emerging from each turning point end at a singularity, and the remaining Stokes line continues towards $\text{Re}(y) \rightarrow \pm\infty$ without crossing the real axis. We will subsequently show these Stokes lines to be inactive, and they hence are illustrated in Figure 5.6 as dashed black curves to emphasize this distinction.

Finally, there is a Stokes line following $\text{Re}(y) = 0$, across which χ_{T1} and χ_{T2} are switched by the leading-order behaviour, although we shall subsequently show in Figure 5.7 that they are not active for $-1 < \text{Im}(y) < 1$.

In Figure 5.7, we consider one turning point in isolation, noting which of the interacting contributions are dominant or subdominant at each point. In the neighbourhood of the caustic, the interaction of the singular terms has the Stokes structure of an Airy function, with three anti-Stokes lines and three Stokes lines (one of which is inactive) intersecting. There must also be a branch cut across which the labels of χ_{L1} and χ_{T1} switch, which we position so that it does not intersect the physical free surface.

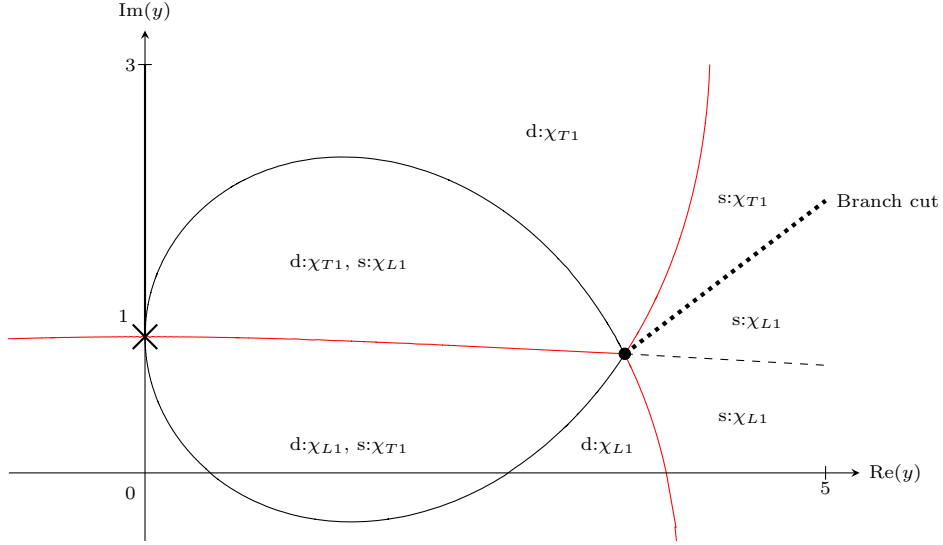


Figure 5.7: Stokes switching in the first quadrant about a caustic. Active Stokes lines are shown as solid black curves, while the inactive Stokes line is shown as a dashed curve. There is a Stokes line present on the imaginary axis, shown as a thick black curve. The red lines indicate anti-Stokes lines, and the dotted line represents a branch cut. In the neighbourhood of the caustic, the singulant behaves as an Airy function. In each region, the active singulant contributions are indicated, and allocated the letter 's' or 'd' to denote either subdominance or dominance with respect to the other contribution. The real axis represents the free surface, and we see that the contribution associated with χ_{L1} is present on the entire free surface, while that associated with χ_{T1} is restricted to a finite region.

Between the two active Stokes lines, both χ_{L1} and χ_{T1} contribute to the free-surface behaviour. Below this region, only the contribution associated with χ_{L1} is present. Above this region, the contribution associated with χ_{T1} is present, although it is switched off in the second quadrant by the Stokes line following $\text{Im}(y) = 0$. Hence the free-surface contribution associated with χ_{L1} is present on the entire free surface, while the contribution associated with χ_{T1} is restricted to a finite region. Similar behaviour is exhibited by interacting singulant contributions in each quadrant, as seen in Figure 5.6, meaning that all four contributions are active between the two active Stokes lines, but only exponentially small contributions associated with χ_{L1} and χ_{L2} are present over the full downstream free surface for $x > 0$.

Figure 5.8 illustrates the location of these Stokes lines for a range of values of h . These contributions will be restricted to two separate regions on either side of $y = 0$. As h increases the waves switch on further downstream, and as h tends to zero, the downstream wave region tends to a wedge. For each of these values for h , $\text{Re}(\chi_{T1})$ and $\text{Re}(\chi_{T2})$ are both positive over the downstream wave region in which they contribute to the free-surface behaviour. Consequently, the transverse waves on the free-surface are always exponentially small in this region. In the limit that $h \rightarrow 0$ this is no longer true, as $\text{Re}(\chi_{T1}) = \text{Re}(\chi_{T2}) = 0$ inside the downstream wave region; this case will be considered in more detail in Section 5.2.5.

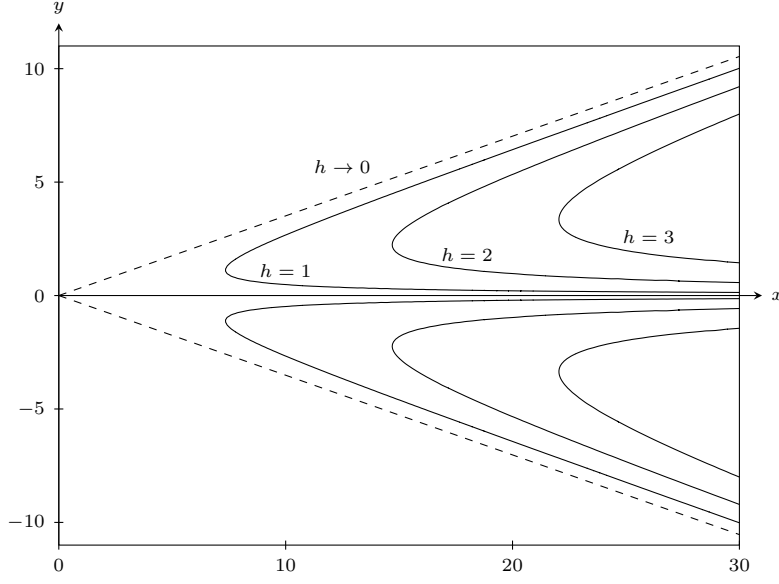


Figure 5.8: Curves along which $\text{Im}(\chi_{L1}) = \text{Im}(\chi_{T1})$ and $\text{Re}(\chi_{L1}) < \text{Re}(\chi_{T1})$ for a range of values of h . These curves are Stokes lines, and exponentially small transverse wave behaviour is switched on as they are crossed from the upstream to the downstream region. The transverse waves are exponentially subdominant to the longitudinal waves over the entire region in which they are active. As h tends to zero, the region in which transverse waves are present tends to a wedge, as predicted by Keller [59]. As h increases, the waves are switched on further downstream.

5.2.4.2 Surface Behaviour

In Figure 5.9 we show the phase of free-surface waves (given by $\text{Im}(\chi_{T1,2})$) that switch across the Stokes lines illustrated in Figure 5.8 for $h = 1$. It is apparent here that these are transverse waves, moving outwards from the x -axis.

Finally, in Figure 5.10, we illustrate the behaviour of the free surface for $h = 1$ by taking $\xi \approx \epsilon \xi^{(1)} + \epsilon^2 \xi^{(2)} + \xi_{\text{exp}}$, where the final term represents the exponentially small contribution to the free surface behaviour given in (5.60). We have not included the transverse waves in this figure as they are exponentially small even compared to the longitudinal waves, and they would not be visible on this scale.

5.2.5 Varying the source depth

By comparing the behaviour of χ as h is varied, we may draw conclusions about the behaviour of the longitudinal wavetrain as the depth of the source is varied. We therefore take cross-sections of $\chi_{L1,2}$ along the line $x = 3$ for various values of h , as illustrated in Figure 5.11. From this figure, we see that the $\text{Re}(\chi_{L1,2})$ decreases as h is decreased. Consequently, the amplitude of the longitudinal waves on the free surface decreases as the depth of the source increases.

The effect of the source depth on the location of the Stokes lines across which the transverse waves are switched was exhibited in Figure 5.8. Here, we noted that as the depth increases, the

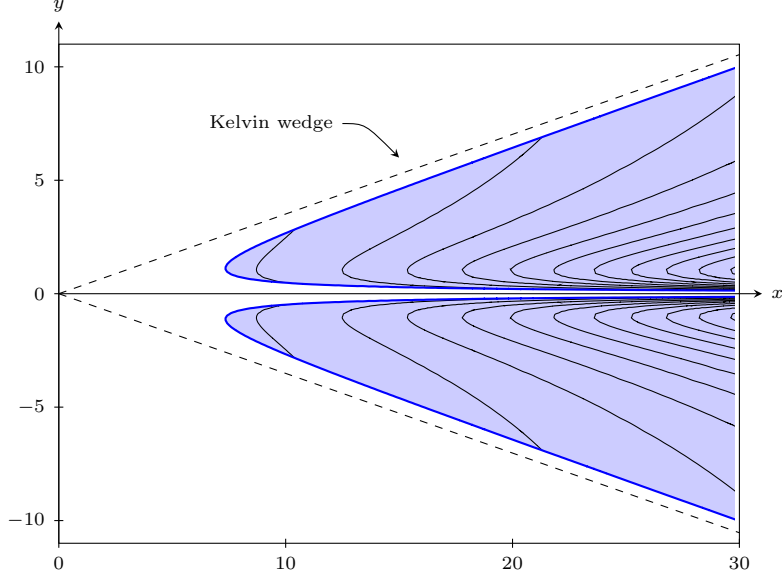


Figure 5.9: Equal-phase lines for the free-surface behaviour associated with χ_{T1} and χ_{T2} for $h = 1$, represented as black curves. This behaviour takes the form of transverse waves which are subdominant to the longitudinal waves illustrated in Figure 5.5, and is switched on across Stokes lines, shown as blue curves. The contribution is active on the shaded region of the free surface. For reference, the Kelvin wedge is illustrated as a dashed curve.

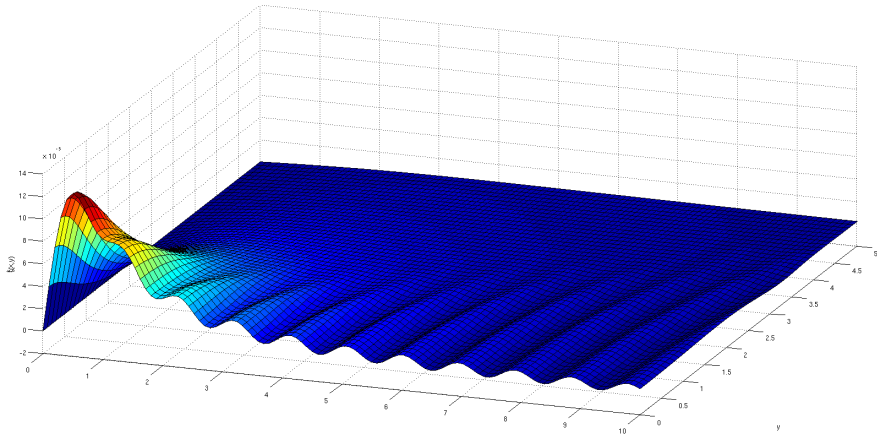


Figure 5.10: Illustration of the free-surface behaviour for flow past a point source with $h = 1$ and $\epsilon = 0.15$ with the leading-order behaviour removed, where the resultant surface is approximated as $\xi \approx \epsilon \xi^{(1)} + \epsilon^2 \xi^{(2)} + \xi_{\text{exp}}$. The steady downstream wavetrain is clearly visible in this flow regime.

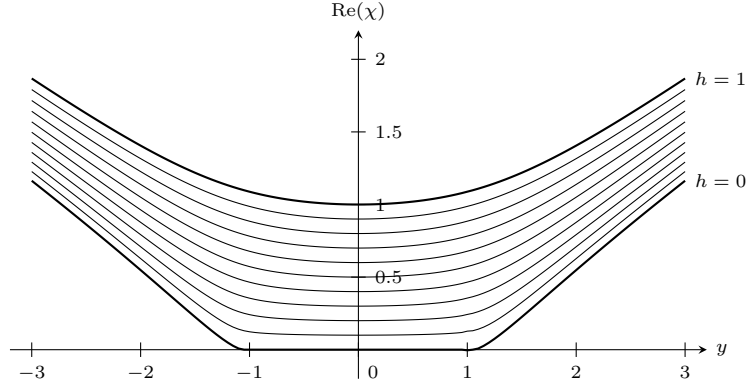


Figure 5.11: Cross-section of $\text{Re}(\chi_{1,2})$ along the line $x = 10$ for $h = 0, 0.1, \dots, 1$. As the depth of the source decreases, the size of the singulant also decreases, and hence the waves become larger in amplitude. At $h = 0$, the singulant may be divided into a purely real part in a finite region about $y = 0$, and complex parts that lie outside this region. This is consistent with previous work on Kelvin waves [59, 61].

transverse waves are switched on further downstream, and that this switching region tends to a wedge in the limit that $h \rightarrow 0$. The effect of varying the source depth on the amplitude of the free-surface transverse waves is dependent on the specified location; however, it is important to note that $\text{Re}(\chi_{T1,2})$ is greater than $\text{Re}(\chi_{L1,2})$ in regions where transverse waves are present on the free surface, with the exception of the zero-depth case.

5.2.5.1 Zero source depth

The problem of ship waves caused by a source on the surface of the flow has been posed in previous investigations, such as [59, 61], in which it was found that the surface waves will be algebraic in a wedge defined by $|y/x| < \arctan(2^{3/2})$ and $x > 0$. Hence, this implies that the singulant will be purely imaginary in this region. Solving (5.26) in conjunction with the first condition given in (5.27) and the boundary conditions

$$x_0 = 0, \quad y_0 = 0, \quad \chi_0 = 0,$$

we find that for $h = 0$, the singulant is given by

$$\chi = \frac{\alpha^3 x}{2\alpha^2 + 1},$$

where α is a solution to

$$4y^2\alpha^4 + (x^2 + 4y^2)\alpha^2 + (x^2 + y^2) = 0.$$

We find that α has purely imaginary solutions for $|y/x| < \arctan(2^{3/2})$, and therefore that χ is also purely imaginary in this region. Noting that the corresponding free-surface contributions are only switched on in the downstream wave region, this is therefore consistent with the ship-wave behaviour obtained in previous investigations of flow past a source positioned on the surface. In

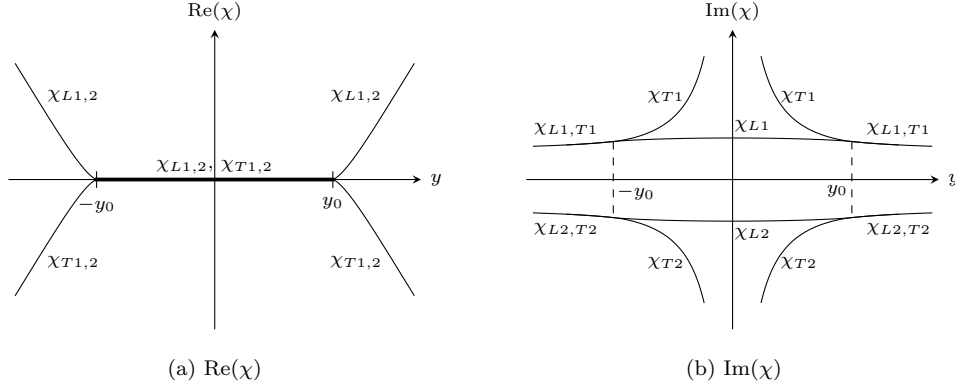


Figure 5.12: Cross-section of (a) $\text{Re}(\chi)$ and (b) $\text{Im}(\chi)$ for each singulant expression with $h = 0$ and $x > 0$. The value y_0 is given by $y_0 = x \arctan(2^{3/2})$. Each singulant is purely imaginary in the range $-y_0 < y < y_0$, meaning that the surface behaviour is not exponentially small. Outside of this wedge, we see that any free-surface contribution due to $\chi_{L1,2}$ would be exponentially small, while any contribution due to $\chi_{T1,2}$ would be exponentially large.

Figure 5.13, we illustrate the behaviour of each of the four singulant contributions at $h = 0$, for a fixed value of $x > 0$. As noted, all four contributions are purely imaginary within the range $|y/x| < \arctan(2^{3/2})$, which implies that any resultant waves present on the free surface within this wedge are of algebraic order.

Figure 5.13 illustrates the full Stokes structure of the problem for fixed $x > 0$, with y permitted to take complex values. In comparing this structure to that illustrated in Figure 5.7, we see that the turning points have coalesced on the physical free-surface. In order for this behaviour to be consistent with the $h > 0$ case, we determine that the contribution associated with $\chi_{T1,2}$ is present between the two turning points, and hence is restricted to the wedge on the free surface defined by $|y/x| < \arctan(2^{3/2})$. This is consistent with the limiting behaviour of the Stokes lines demonstrated in Figure 5.8.

The contribution associated with $\chi_{L1,2}$, however, is present for all $x > 0$, switching in the same manner as the corresponding contributions for $h > 0$. The resultant waves are of algebraic order within the region defined by $|y/x| < \arctan(2^{3/2})$, but exponentially small outside of this wedge.

These results are consistent with those of Ursell [111], who determined that near the curve $y = 0$, the linearized problem would display infinitely decreasing wavelength. This is confirmed in Figure 5.12, where $\text{Im}(\chi) \rightarrow \infty$ along $y = 0$. Furthermore, Ursell showed that the wave behaviour near the critical angle took the form of Airy function. This may be seen in Figure 5.13, showing that the turning points intersect the free-surface on the critical angle when $h = 0$.

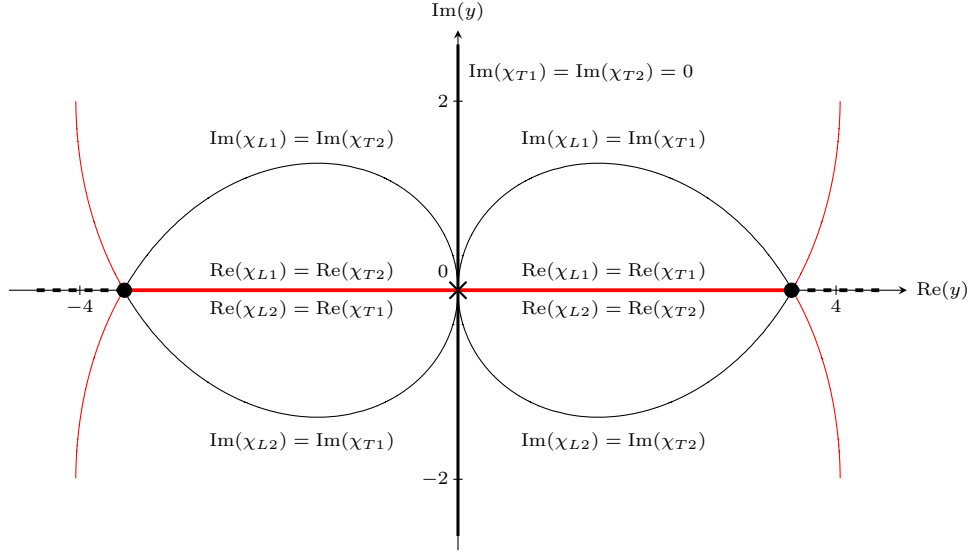


Figure 5.13: Stokes structure for flow over a source with $h = 0$, for $x > 0$ and $y \in \mathbb{C}$. Turning points and singularities are represented by circles and crosses respectively. Active Stokes lines are shown as solid black curves, while inactive Stokes lines are shown as dashed black curves. Only anti-Stokes lines that are relevant to the Stokes structure of the problem are illustrated, represented as red curves.

5.2.6 Numerical calculations

5.2.6.1 Formulating an integral expression

In order to verify the asymptotic results obtained in Section 5.2.3, we will compare them to numerical solutions to the system described in (5.8)–(5.11). These solutions will be obtained by computing the solution to a double integral formulation for the free-surface behaviour. We will formulate the integral expression here, but we note that many similar integral expressions have been found in previous work, such as [81]. We set

$$\phi = \frac{1}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}} + \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \hat{\phi}.$$

The resultant governing equation is given by

$$\nabla^2 \hat{\phi} = 0, \tag{5.62}$$

with boundary conditions given by

$$\epsilon \hat{\phi}_x + \xi = -\epsilon \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right]_x, \quad z = 0, \tag{5.63}$$

$$\hat{\phi}_z - \xi_x = 0, \quad z = 0. \tag{5.64}$$

Finally, the far-field conditions determine that $\hat{\phi} \rightarrow 0$ far from the source. Taking the Fourier transform of (5.62) with respect to x and y (with the resultant expression given in terms of the Fourier variables k and l respectively) gives

$$\bar{\bar{\phi}}_{zz} - \rho^2 \bar{\bar{\phi}} = 0,$$

where $\rho = \sqrt{k^2 + l^2}$. Noting that the solution must decay in the limit that $z \rightarrow -\infty$, this differential equation may be solved to give

$$\bar{\bar{\phi}} = A(k, l)e^{\rho z}. \quad (5.65)$$

Taking the Fourier transform of the boundary conditions (5.63)-(5.64) gives

$$\begin{aligned} \epsilon i k \bar{\bar{\phi}} + \bar{\bar{\xi}} &= -i k \epsilon \mathcal{F}_{xy} \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right], & z = 0, \\ \bar{\bar{\phi}}_z - i k \bar{\bar{\xi}} &= 0, & z = 0. \end{aligned}$$

Using the expression given in (5.65), this becomes

$$\epsilon i k A + \bar{\bar{\xi}} = -i k \epsilon \mathcal{F}_{xy} \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right], \quad (5.66)$$

$$\rho A - i k \bar{\bar{\xi}} = 0. \quad (5.67)$$

Calculating the Fourier transform of the right-hand side of the first equation gives

$$\mathcal{F}_{xy} \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} e^{-ikx - ily} dx dy.$$

Representing (x, y) and (k, l) in terms of polar coordinates (r, θ) and (ρ, ϕ) respectively gives

$$\begin{aligned} \mathcal{F}_{xy} \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right] &= \int_0^{\infty} \int_0^{2\pi} \frac{r}{2\pi(r^2 + h^2)^{1/2}} e^{-ir\rho \cos(\theta - \phi)} d\theta dr, \\ &= \int_0^{\infty} \frac{r J_0(\rho r)}{(r^2 + h^2)^{1/2}} dr, \\ &= \frac{e^{-\rho h}}{\rho}, \end{aligned}$$

where the final line was obtained using the appropriate integral identity from [44].

Solving (5.66) - (5.67) for the free-surface behaviour therefore gives

$$\bar{\bar{\xi}} = \frac{i k \epsilon e^{-\rho h}}{\rho - \epsilon k^2}.$$

Taking the inverse Fourier transform, we find that

$$\xi = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i k \epsilon e^{-\rho h}}{\rho - \epsilon k^2} e^{ikx + ily} dk dl. \quad (5.68)$$

We note that this integral is singular along the curve $\rho = \epsilon k^2$. In order to evaluate this integral numerically, we would prefer this curve to follow a straight line. Hence, we first transform into polar coordinates as before, giving

$$\xi = \frac{1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} \frac{i \rho \cos \phi \epsilon e^{-\rho h}}{1 - \epsilon \rho \cos^2 \phi} e^{ir\rho \cos(\theta - \phi)} d\phi d\rho. \quad (5.69)$$

Setting $\rho = u / \cos^2 \phi$, we finally obtain

$$\xi = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \frac{i u \epsilon}{(1 - \epsilon u) \cos^3 \phi} \exp \left(\frac{i u r \cos(\theta - \phi) - u h}{\cos^2 \phi} \right) du d\phi, \quad (5.70)$$

where we see that the integrand is singular along $u = 1/\epsilon$ and hence that the inner integral may be treated as a Cauchy principal value integral about this point, which we have noted explicitly. Additionally, we note that there are removable singularities along $\phi = \pi/2$ and $\phi = 3\pi/2$, which do not cause any numerical difficulties.

This integral is substantially more complicated than the forms given in (5.68)–(5.69). However, the fact that the integrand is singular along a straight line ensures that it is more simple to solve using standard numerical integration in two dimensions.

5.2.6.2 Comparison of results

In order to evaluate (5.70) numerically for given values of h and ϵ , we truncate the integral domain in the u -direction and split it into two regions, given by

$$\begin{aligned} D1 &= \{(u, \phi) : u \in (0, 1/\epsilon - \Delta), \phi \in (0, 2\pi)\}, \\ D2 &= \{(u, \phi) : u \in (1/\epsilon + \Delta, u_\infty), \phi \in (0, 2\pi)\}, \end{aligned}$$

where Δ is a small constant used to approximate the Cauchy principal value integral, and u_∞ is the truncation point. Typical values used in this investigation are $\Delta = 1 \times 10^{-5}$ and $u_\infty = 50$. On each domain, we evaluate the two-dimensional integral numerically using the inbuilt MATLAB integrator, which uses two-dimensional quadrature techniques.

Figure 5.14 compares the far-field amplitude predicted by the asymptotic analysis along the line $y = 0$ with numerical results. Specifically, we compare the amplitude of the expression in (5.61), denoted by H , with the numerically-calculated solution for a range of values of ϵ . It is clear that the asymptotic results provide an accurate approximation for the exact wave behaviour.

5.2.7 Changing the singularity

Extending the results that were obtained for flow past a point source in order to consider flow past similar obstacles simply a different analysis of the inner problem, considered in (5.2.2.3). The late-order ansatz given in (5.19) is therefore largely unchanged, with the singulant taking the same form, and the prefactor form (5.43) differing only in terms of the function obtained from the local analysis, given in (5.54), and the value of γ . Consequently, the late-order term behaviour will always take the form

$$[S_{\text{TOT}}^{(N)}]_{-}^{+} = \Phi(s, 0) \frac{2\pi i \chi_x}{\epsilon^\gamma} \left(\frac{s^4}{6hus^2 + 4h^3u + s^4} \right)^{1/2} e^{-\chi/\epsilon} + \text{c.c.}, \quad (5.71)$$

where expressions for s , u and χ were given in Section 5.2.2.1, and again, χ denotes χ_{L1} . This behaviour always switches across a Stokes line that follows the curve $x = 0$, producing

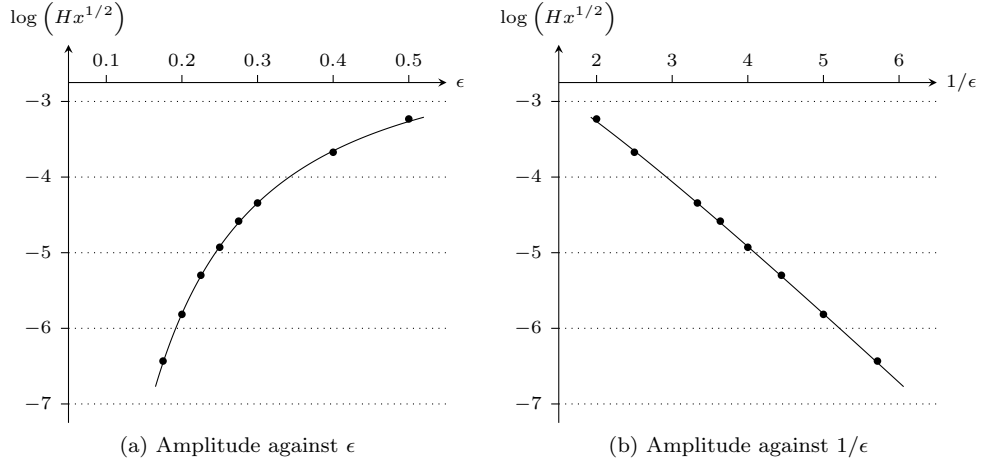


Figure 5.14: Comparison of asymptotic and numerical values for the far-field gravity wave amplitude along $y = 0$, scaled by the strength of the source, for varying values of (a) ϵ and (b) $1/\epsilon$. The source is fixed at depth $h = 1$. The solid line represents the asymptotic value for the amplitude, given in (5.61), multiplied by $x^{1/2}$ so that it takes a constant value in the far field. The slope of the line in (b) is equal to $-h$, which may be seen by noting the value of the singulant in the far field. The circles represent the equivalent value obtained from numerical solutions to the exact problem. The asymptotic results provide an accurate approximation to the exact solution.

downstream waves, and the Stokes switching proceeds in the same fashion as found in Section 5.2.4.

For example, to consider flow past a dipole, we replace the singularity condition given in (5.11). In cylindrical coordinates, the dipole condition is given by

$$\bar{\phi} \rightarrow \frac{\cos \theta}{4\pi(x^2 + y^2 + (z + h)^2)} \quad \text{as } (r, z) \rightarrow (0, -h).$$

Writing this in Cartesian coordinates, we obtain

$$\bar{\phi} \rightarrow \frac{y}{4\pi\sqrt{x^2 + y^2}} \frac{1}{x^2 + y^2 + (z + h)^2} \quad \text{as } (x, y, z) \rightarrow (0, 0, -h).$$

We find that the leading-order solution, equivalent to that given in (5.17), instead becomes

$$\phi^{(0)} = \frac{y}{4\pi\sqrt{x^2 + y^2}} \left[\frac{1}{x^2 + y^2 + (z + h)^2} - \frac{1}{x^2 + y^2 + (z - h)^2} \right].$$

Matching the order of the singularity in the late-order ansatz to this leading-order expression gives $\gamma = 1$. As noted, the form of the singulant and the global contribution to the prefactor are both identical to the problem of flow over a source.

To solve the inner problem, we express $\phi^{(0)}$ in terms of the local coordinates given in (5.46). In the neighbourhood of the singularity, this gives

$$\phi^{(0)} \sim \sqrt{\frac{h^2 + \lambda^2}{h^2}} \frac{h}{8\pi\epsilon\lambda^2\sigma_1} - \sqrt{\frac{h^2 + \lambda^2}{h^2}} \frac{h}{8\pi\epsilon\lambda^2\sigma_2}. \quad (5.72)$$

We now define series of a similar form to that given in (5.50), adjusting the power of the singularities to be consistent with (5.72). A similar series analysis to that performed in Section 5.2.2.3 eventually gives

$$a_{n+1} = a_0 \Gamma(n+1),$$

where

$$a_0(\lambda) = \sqrt{\frac{h^2 + \lambda^2}{h^2}} \frac{h}{8\pi\lambda^2},$$

and consequently implies that

$$\Phi(s, 0) = \sqrt{\frac{h^2 + s^2}{h^2}} \frac{h}{8\pi s^2}. \quad (5.73)$$

Applying these results to the general expression given in (5.71) gives the quantity that switches across the Stokes line as

$$[S_{\text{TOT}}^{(N)}]_{-}^{+} = \frac{i\chi_{L1,x}}{4\epsilon s^2} \left(\frac{(h^2 + s^2)s^4}{6hus^2 + 4h^3u + s^4} \right)^{1/2} e^{-\chi_{L1}/\epsilon} + \text{c.c.}.$$

The location of the Stokes lines are not affected by the inner problem, and hence the Stokes lines follow the curve $x = 0$, with the waves being switched on such that they are present downstream from this line.

Furthermore, as we have effectively calculated the Green's function for a point singularity, we could easily extend this analysis to determine the free-surface behaviour for flow over any distribution of singularities.

5.3 Unsteady Flow

In this section we will again consider linearized three-dimensional flow over a submerged source; however, we will instead consider the unsteady problem rather than the steady-state problem considered in Section 5.2. We will consider an identical flow configuration to that shown in Figure 5.2, with the state of the flow prescribed at $t = 0$. As in Chapter 3, we will set this initial condition to be the leading-order steady state of the flow, which was given in (5.17).

We will determine the behaviour of the singulants associated with the late-order terms caused by the initial flow disturbance. These terms are equivalent to the behaviour found in Section 3.4.4 in the analysis of the two-dimensional problem. These singulants will be compared with those associated with both the longitudinal and transverse waves to determine the full Stokes structure present on the free surface. The location of these Stokes lines will be used to determine the regions of the free surface that contain these waves as time is varied.

5.3.1 Formulation

5.3.1.1 Governing Equations

In the unsteady case, the governing equation of the flow does not change, and hence is given by

$$\nabla^2 \phi = 0, \quad -\infty < z < \xi(x, y). \quad (5.74)$$

The kinematic and dynamic boundary conditions are now respectively

$$\xi_x \phi_x + \xi_y \phi_y + \xi_t = \phi_z, \quad z = \xi(x, y), \quad (5.75)$$

$$\frac{1}{2} \epsilon (\phi_t + |\nabla \phi|^2 - 1) + \xi = 0, \quad z = \xi(x, y). \quad (5.76)$$

The far-field conditions are

$$\phi_x \rightarrow 1 \quad \text{as} \quad x \rightarrow -\infty, \quad (5.77)$$

$$\phi_x \rightarrow 1 \quad \text{as} \quad y \rightarrow \pm\infty, \quad (5.78)$$

$$\phi_x \rightarrow 1 \quad \text{as} \quad z \rightarrow -\infty. \quad (5.79)$$

The source is located at $(x, y, z) = (0, 0, -h)$ at all points in time, and therefore the potential becomes singular near this point for all t , giving

$$\phi \rightarrow \frac{\delta}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} \quad \text{as} \quad (x, y, z, t) \rightarrow (0, 0, -h, t). \quad (5.80)$$

We denote the initial flow conditions as $\phi = \phi_0$ and $\xi = \xi_0$. For simplicity, we set the initial condition to be the leading-order solution to the steady-state problem, given in (5.17). We therefore have

$$\phi_0 = \frac{\delta}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \frac{\delta}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}}, \quad \xi_0 = 0. \quad (5.81)$$

As the problem is unsteady, instead of a radiation condition, we again impose that the flow is flat in the far field.

5.3.1.2 Linearization

Linearising the problem about the undisturbed solution for small δ again involves setting

$$\phi = x + \delta \bar{\phi}, \quad \xi = \delta \bar{\xi}.$$

The governing equation (5.74) again becomes

$$\nabla^2 \bar{\phi} = 0, \quad -\infty < z < 0. \quad (5.82)$$

As in the steady analysis, the free-surface boundary conditions now apply on $z = 0$, with the linearized versions of (5.75) and (5.76) respectively taking the form

$$\bar{\phi}_z - (\bar{\xi}_x + \bar{\xi}_t) = 0, \quad z = 0, \quad (5.83)$$

$$\epsilon(\bar{\phi}_x + \bar{\phi}_t) + \bar{\xi} = 0, \quad z = 0. \quad (5.84)$$

The far-field conditions (5.77)–(5.79) again become $\bar{\phi} \rightarrow 0$ as $x \rightarrow -\infty$, $y \rightarrow \pm\infty$ and $z \rightarrow -\infty$. The singularity condition (5.80) gives for all t

$$\bar{\phi} \rightarrow \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} \quad \text{as } (x, y, z, t) \rightarrow (0, 0, -h, t). \quad (5.85)$$

The initial conditions given in (5.81) now take the form

$$\bar{\phi}_0 = \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \frac{1}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}}, \quad \bar{\xi}_0 = 0. \quad (5.86)$$

5.3.1.3 Series Expression

Again, the fluid potential and free-surface position are expressed in terms of a power series in ϵ , given by

$$\bar{\phi} \sim \sum_{n=0}^{\infty} \epsilon^n \phi^{(n)}, \quad \bar{\xi} \sim \sum_{n=0}^{\infty} \epsilon^n \xi^{(n)}. \quad (5.87)$$

Applying this expression to (5.82) gives an identical governing equation to the steady case,

$$\nabla^2 \phi^{(n)} = 0, \quad -\infty < z < 0, \quad n \geq 0. \quad (5.88)$$

The boundary conditions on the free surface (5.83) - (5.84) become

$$\phi_z^{(n)} - (\xi_x^{(n)} + \xi_t^{(n)}) = 0, \quad z = 0, \quad n \geq 0 \quad (5.89)$$

$$(\phi_x^{(n-1)} + \phi_t^{(n-1)}) + \xi^{(n)} = 0, \quad z = 0, \quad n > 0, \quad (5.90)$$

where the latter condition is replaced by $\xi^{(0)} = 0$ on the boundary for the $n = 0$ case. The far-field conditions are enforced at all orders of n , and the singularity condition (5.85) for all t is given by

$$\phi^{(0)} \rightarrow \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} \quad \text{as } (x, y, z, t) \rightarrow (0, 0, -h, t).$$

As this initial condition (5.86) is simply the leading-order steady state solution, the leading-order behaviour remains steady. Therefore, the leading-order behaviour is simply given by

$$\phi^{(0)} = \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \frac{1}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}}, \quad \xi^{(0)} = 0.$$

5.3.1.4 Unsteady Singularity

In order for the system to exhibit unsteady behaviour, we note that there must be another singularity that is not immediately apparent in the leading-order behaviour. In Chapter 3, we found that the unsteady exponentially small behaviour on the free surface was generated by a singularity that was located at the disturbance in the flowpath at $t = 0$ and subsequently propagated with the flow. In order to demonstrate that the same behaviour may be found in the three-dimensional problem, we will identify the lowest-order asymptotic contribution in which this singularity may be found.

To find this contribution, we consider the homogeneous problem without enforcing the source condition. This allows us to use standard Fourier methods, and to isolate the effect of the initial disturbance on the flow. To emphasize this distinction, we will formulate this problem in terms of $\hat{\phi}$ and $\hat{\xi}$, where these represent the flow behaviour caused by the initial disturbance only, neglecting the continuing effect of the source.

The governing equation remains

$$\nabla^2 \hat{\phi} = 0, \quad -\infty < z < 0. \quad (5.91)$$

The boundary conditions become

$$\hat{\phi}_z - (\hat{\xi}_x + \hat{\xi}_t) = 0, \quad z = 0, \quad (5.92)$$

$$\epsilon(\hat{\phi}_x + \hat{\phi}_t) + \hat{\xi} = 0, \quad z = 0, \quad (5.93)$$

and $\hat{\phi}$ decays in the far field of the domain. The initial conditions on the free surface are given on $z = 0$ by

$$\phi_0 = \frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}}, \quad \xi_0 = 0 \quad (5.94)$$

Taking the Fourier transform in x and y , where k and l are the corresponding Fourier variables, of the governing equation (5.91)

$$\bar{\bar{\phi}}_z - \rho^2 \bar{\bar{\phi}} = 0,$$

where $\rho = \sqrt{k^2 + l^2}$ and two bars denotes taking the double Fourier transform. Noting that $\bar{\bar{\phi}}$ decays as $z \rightarrow -\infty$, we obtain

$$\bar{\bar{\phi}} = A(k, l, t)e^{\rho z}. \quad (5.95)$$

Taking the Fourier transform of the boundary conditions (5.92)–(5.93) and applying (5.95) to the resultant expression gives

$$\epsilon i k A + \epsilon A_t + \bar{\bar{\xi}} = 0, \quad z = 0, \quad (5.96)$$

$$\rho A - i k \bar{\bar{\xi}} - \bar{\bar{\xi}}_t = 0, \quad z = 0. \quad (5.97)$$

Finally, taking the Fourier transform of the initial condition on the free surface (5.94), using the appropriate integral identity from [44], gives

$$\bar{\bar{\phi}}_0 = A(k, l, 0) = -\frac{e^{-\rho h}}{\rho}, \quad \bar{\bar{\xi}}_0 = 0$$

Using these initial conditions on A and $\bar{\bar{\xi}}$, we solve (5.96)–(5.97) to obtain

$$\bar{\bar{\xi}} = \frac{i\epsilon^{1/2}}{4\rho^{1/2}} \left[e^{it\sqrt{\rho/\epsilon}} - e^{-it\sqrt{\rho/\epsilon}} \right] e^{-\rho h - ikt}.$$

Taking the inverse Fourier transform, we find

$$\hat{\xi} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{i\epsilon^{1/2}}{4\rho^{1/2}} \left[e^{it\sqrt{\rho/\epsilon}} - e^{-it\sqrt{\rho/\epsilon}} \right] e^{-\rho h - ikt} e^{ikx + il y} dk dl.$$

Setting (x, y) and (k, l) in terms of polar coordinates (r, θ) and (ρ, ϕ) respectively gives

$$\hat{\xi} = \int_0^{\infty} \int_0^{2\pi} \frac{i\sqrt{\epsilon\rho}}{16\pi^2} \left[e^{it\sqrt{\rho/\epsilon}} - e^{-it\sqrt{\rho/\epsilon}} \right] e^{-\rho h - it\rho \cos \phi} e^{i\rho r \cos(\theta - \phi)} d\phi d\rho.$$

Evaluating the inner integral gives

$$\hat{\xi} = \int_0^{\infty} \frac{i\sqrt{\epsilon\rho}}{8\pi} \left[e^{it\sqrt{\rho/\epsilon}} - e^{-it\sqrt{\rho/\epsilon}} \right] e^{-\rho h} J_0(\rho(r - t)) d\rho,$$

where J_0 is the Bessel function of the first kind. Setting $\rho = \epsilon u^2$, we obtain the form

$$\hat{\xi} = \int_0^{\infty} \frac{i\epsilon^2 u^2}{4\pi} e^{-\epsilon u^2 h} J_0(\epsilon u^2(r - t)) e^{iut} du - \int_0^{\infty} \frac{i\epsilon^2 u^2}{4\pi} e^{-\epsilon u^2 h} J_0(\epsilon u^2(r - t)) e^{-iut} du.$$

Making the transformation $\rho = \epsilon u^2$ and applying Laplace's method to find the contribution to $\hat{\xi}$ from the end-point of these integrals at $u = 0$, denoted by $\hat{\xi}_E$, gives

$$\hat{\xi}_E \sim -\frac{\epsilon^2}{\pi t^2}, \quad (5.98)$$

which is clearly singular at $t = 0$. Consequently, the free-surface behaviour contains a singularity at $t = 0$ which only appears in terms that are $\mathcal{O}(\epsilon^2)$. This is the singularity that generates unsteady variation on the free surface. We emphasize that this is not a full representation of the asymptotic contributions at this order, but rather a single contribution that contains the singular behaviour which produces unsteady exponentially small behaviour on the free surface.

5.3.2 Late-Order Ansatz

Again, we define an ansatz for terms in the series (5.87) as $n \rightarrow \infty$

$$\phi^{(n)} \sim \frac{\Phi(x, y, z, t) \Gamma(n + \gamma)}{\chi(x, y, z, t)^{n + \gamma}}, \quad \xi^{(n)} \sim \frac{\Xi(x, y, t) \Gamma(n + \gamma)}{\chi(x, y, 0, t)^{n + \gamma}}, \quad (5.99)$$

which is identical to (5.99), aside from the possibility that the coefficients may vary in t . Again, it is simple to show that γ is constant. Additionally, we again note that $\chi = 0$ at the location of the singularity under consideration on the analytically-continued free surface.

Applying the ansatz expressions in (5.99) to the governing equation (5.88) and taking the first two orders as $n \rightarrow \infty$ gives

$$\chi_x^2 + \chi_y^2 + \chi_z^2 = 0, \quad (5.100)$$

$$2\Phi_x \chi_x + 2\Phi_y \chi_y + 2\Phi_z \chi_z = -(\chi_{xx} + \chi_{yy} + \chi_{zz}), \quad (5.101)$$

while the boundary conditions (5.89)–(5.90) become at leading order

$$\begin{aligned} -(\chi_x \Phi + \chi_t \Phi) + \Xi &= 0, & z = 0, \\ -\chi_z \Phi + (\chi_x \Xi + \chi_t \Xi) &= 0, & z = 0. \end{aligned}$$

This system of equations has nonzero solutions when

$$(\chi_x + \chi_t)^2 = \chi_z. \quad (5.102)$$

Combining this result with (5.100) gives the eikonal equation, valid on the free surface, as

$$(\chi_x + \chi_t)^4 + \chi_x^2 + \chi_y^2 = 0. \quad (5.103)$$

In order to find a useful form for the prefactor equation, we again consider the full prefactor series in n as $n \rightarrow \infty$ and take the boundary conditions to the next order in n . The resultant system only has nontrivial solutions when

$$\Phi_z = (\chi_x + \chi_t)(\Phi_x + \Phi_t) + (\Xi_x + \Xi_t).$$

Applying this expression, as well as (5.102), to the prefactor equation (5.101) allows for the resultant expression to be rearranged into

$$\begin{aligned} (2\chi_x + 4(\chi_x + \chi_t)^3) \Phi_x + 2\chi_y \Phi_y + 4(\chi_x + \chi_t)^3 \Phi_t = \\ -(\chi_{xx} + \chi_{yy} + 6(\chi_x + \chi_t)^2(\chi_{xx} + 2\chi_{xt} + \chi_{tt})) \Phi, \end{aligned} \quad (5.104)$$

which is valid on the free surface of the flow. As in the steady problem, this equation has the same ray structure as the singulant equation (5.103). We will not solve the prefactor equation for this problem, but some discussion of the method that would be used to do so may be found in Appendix D.

5.3.3 Solving the Singulant Equation

5.3.3.1 Steady Singularity

Solving (5.103) using Charpit's method requires setting $\chi_x = P$, $\chi_y = Q$ and $\chi_t = R$. In this case, we define u to be the characteristic variable, and parametrize the initial data $(x_0, y_0, \chi_0$ and $t_0)$ in terms of two variables, r and s . The Charpit system consequently takes the form

$$\frac{dx}{du} = 4(P + R)^3 + 2P, \quad \frac{dy}{du} = 2Q, \quad \frac{dt}{du} = 4(P + R)^3, \quad \frac{d\chi}{du} = -2P^2 - 2Q^2, \quad (5.105)$$

where P and Q satisfy

$$(P + R)^4 + P^2 + Q^2 = 0, \quad (5.106)$$

$$\frac{\partial x_0}{\partial s} P + \frac{\partial y_0}{\partial s} Q + \frac{\partial t_0}{\partial s} R = 0, \quad (5.107)$$

$$\frac{\partial x_0}{\partial r} P + \frac{\partial y_0}{\partial r} Q + \frac{\partial t_0}{\partial r} R = 0. \quad (5.108)$$

Initially, we consider the contribution by the singularity located at the source. This gives

$$x_0 = s, \quad y_0 = \pm i(s^2 + h^2)^{1/2}, \quad t_0 = r, \quad \chi_0 = 0. \quad (5.109)$$

Hence (5.107)-(5.108) become

$$P \pm \frac{isQ}{(s^2 + h^2)^{1/2}} = 0, \quad R = 0. \quad (5.110)$$

Comparing (5.105)–(5.110) to the equivalent analysis for the steady problem, given in (5.26)–(5.28), we see that the current Charpit analysis will produce an identical result to that obtained in the analysis of the steady problem. Noting that χ is therefore steady, we find that the prefactor equation (5.104) reduces to that obtained in the steady case, shown in (5.36). The inner analysis proceeds in an identical fashion, and we therefore find that the exponentially small contribution to the unsteady free-surface behaviour switched on as a consequence of the source is identical to that found in the steady problem, described in Section 5.2.

5.3.3.2 Unsteady Singularity

In Section 5.3.1.4, we determined that the solution contains asymptotic contributions with a singularity at $t = 0$, such as the contribution shown in (5.98). In a similar fashion to [24], as well as the analysis for the two-dimensional unsteady problem in Chapter 3, we impose that the rays all pass through the point of the flow disturbance, $x^2 + y^2 + h^2 = 0$, at $t = 0$. Hence, the boundary conditions for the Charpit system described in (5.105)–(5.108) become

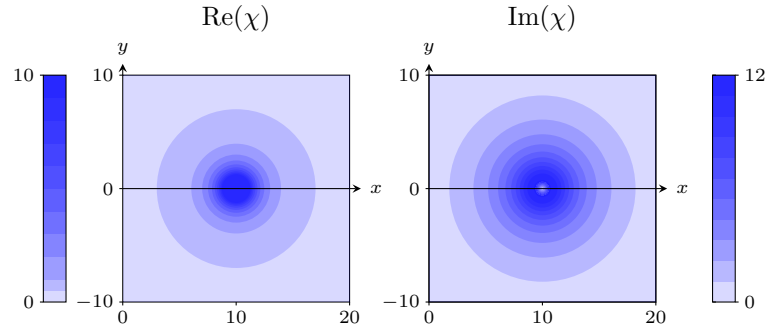
$$x_0 = s, \quad y_0 = \pm i(s^2 + h^2)^{1/2}, \quad t_0 = 0, \quad \chi_0 = 0. \quad (5.111)$$

In this case, we may simply omit (5.108) from the system, instead allowing R to be arbitrary. Solving this system gives

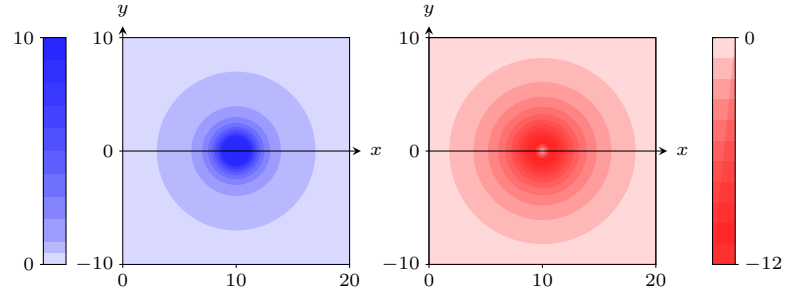
$$\chi_{U1} = -\frac{t^2 \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}}}{4h \left(iy + \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}} \right)}, \quad \chi_{U2} = -\frac{t^2 \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}}}{4h \left(-iy + \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}} \right)}. \quad (5.112)$$

$$\chi_{U3} = \frac{t^2 \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}}}{4h \left(iy + \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}} \right)}, \quad \chi_{U4} = \frac{t^2 \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}}}{4h \left(-iy + \sqrt{h^2 - \frac{h^2(x-t)^2}{(x-t)^2 + y^2}} \right)}. \quad (5.113)$$

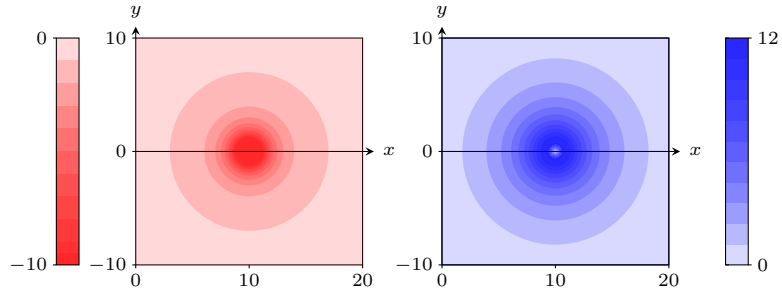
The real and imaginary parts of each of these singulants are shown in Figure 5.15.



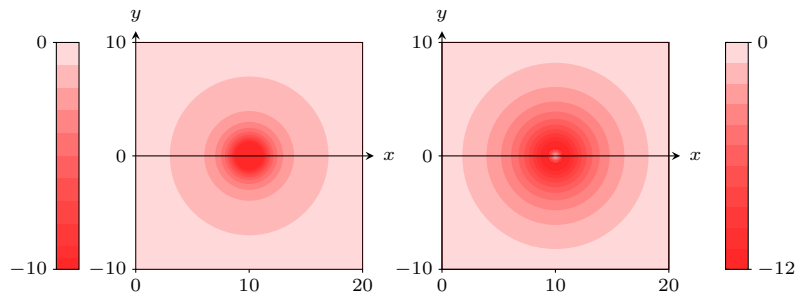
(a) Contour plots for χ_{U1}



(b) Contour plots for χ_{U2}



(c) Contour plots for χ_{U3}



(d) Contour plots for χ_{U4}

Figure 5.15: Contour plots of the real and imaginary parts of χ_{U1} to χ_{U4} for $h = 1$ and $t = 10$. Each of the four singulant expressions satisfies $\text{Im}(\chi) = 0$ only at $x = t$. Only the singulants shown in (a) and (b) have $\text{Re}(\chi) > 0$ on the free surface, and hence lead to exponentially small free-surface contributions. These contributions are therefore always present, while the free-surface contributions associated with (c) and (d) are not present.

In Figure 5.15, we see that each of these singulants satisfies $\text{Im}(\chi) = 0$ at the point $x = t$ only. Hence they produce free-surface behaviour that is either active everywhere or inactive everywhere on the free surface. We see that only χ_{U1} and χ_{U2} satisfy $\text{Re}(\chi) > 0$, and hence produce exponentially small free-surface behaviour. We therefore conclude that late-order terms associated with χ_{U1} and χ_{U2} are present in the problem, while late-order terms associated with χ_{U3} and χ_{U4} are not.

Finally, we note that there are four more solutions to the system (5.105)–(5.108) with boundary data (5.111). We neglect these solutions as they cannot possibly lead to an exponentially small contribution to the free-surface behaviour, and hence cannot be active on the free surface.

5.3.4 Complete Stokes structure

5.3.4.1 Longitudinal Waves

We first consider the interaction between the unsteady free-surface contribution described in Section 5.3.3.2. In Figure 5.16, we show the curves along which χ_{L1} and χ_{U1} have equal real and imaginary parts, giving the location of anti-Stokes lines and Stokes lines respectively. From this figure, we see that there are two potential Stokes lines on the free surface, separated by an anti-Stokes line. On the left-hand side of the anti-Stokes line, $\text{Re}(\chi_{L1}) > \text{Re}(\chi_{U1})$, and therefore longitudinal waves are switched across the Stokes line in this region. However, on the right-hand side of the anti-Stokes line, $\text{Re}(\chi_{U1}) > \text{Re}(\chi_{L1})$. Consequently, the unsteady behaviour does not switch the longitudinal waves as the Stokes line in this region of the free surface is crossed.

We therefore conclude that longitudinal waves are switched on across the Stokes line along $x = 0$ as in Section 5.2, and switched off as the Stokes line under investigation here is crossed. Figure 5.17 illustrates the wave region for a range of values of t with $h = 1$. We see that the wavefront tends to $x \rightarrow \infty$ as $t \rightarrow \infty$. In fact, by comparing the behaviour of χ_{L1} and χ_{U1} as $x \rightarrow \infty$, it may be seen that the position of the wavefront along the curve $y = 0$ (denoted x_f) is given by

$$x_f \sim \frac{t}{2} \quad \text{as} \quad t \rightarrow \infty.$$

We note that this wave propagation behaviour is similar to the that exhibited by the two-dimensional unsteady problem described in Section 3.4.4.

5.3.4.2 Transverse Waves

Determining the unsteady behaviour of the free-surface transverse waves is more complicated, as they may interact with both the unsteady free-surface behaviour and the steady longitudinal free-surface waves. We expect that as $t \rightarrow \infty$, the behaviour of the transverse waves will tend to that described in Section 5.2.4.

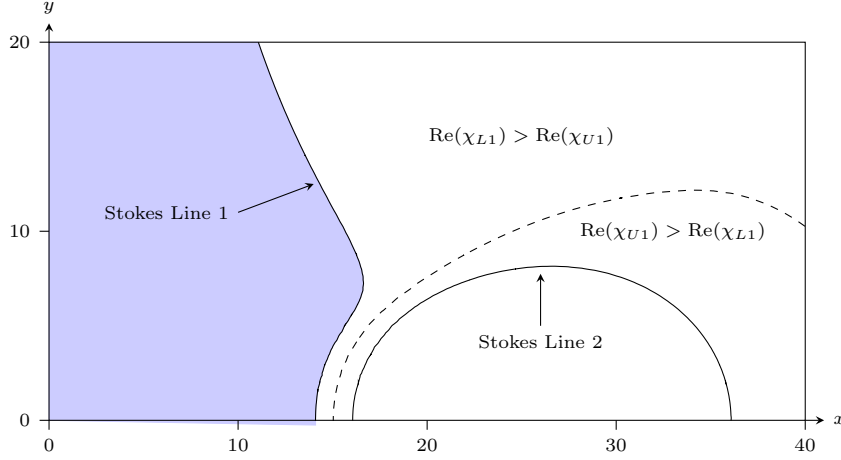


Figure 5.16: This figure illustrates Stokes and anti-Stokes lines caused by interactions between the late-order terms corresponding to χ_{L1} and χ_{U1} for $t = 30$ and $h = 1$. The solid lines indicate Stokes lines, while the dashed line indicates an anti-Stokes line. The unsteady free-surface contribution associated with χ_{U1} is always present. In Section 5.2, it was shown that steady longitudinal waves associated with χ_{L1} are switched across as Stokes line following $x = 0$. Here, we find that this behaviour is switched off by the unsteady contribution as Stokes line 1 is crossed. Stokes line 2 must be inactive, as the longitudinal contribution would dominate the unsteady contribution if it were present in this region. Hence, the longitudinal waves are present only in the shaded region of the free surface. An identical figure would be obtained by considering interactions between χ_{L2} and χ_{U2}

In Figure 5.18, we illustrate all of the potential Stokes lines involving the transverse wave contribution for $t = 35$ and $h = 1$, as well as relevant anti-Stokes lines. We recall that the unsteady contribution is present everywhere on the free surface, while the longitudinal wave contribution is present only to the left of the Stokes line caused by the interaction between the two. The Stokes line across which transverse waves are switched by longitudinal waves is identical to that obtained in the steady case found in Section 5.2.4. There are two possible Stokes lines across which transverse waves may be switched by the unsteady behaviour. However, as in the longitudinal case, these two possible Stokes lines are separated by an anti-Stokes line. By comparing the real parts of the singulants in each of these regions, we see that the unsteady behaviour may only switch the transverse wave behaviour on the left-hand side of the anti-Stokes line.

In Figure 5.21, there are two points at which the Stokes lines all intersect. In order for the transverse waves to be switched in such a fashion that they occupy a finite region of the free surface and tend to the steady solution as $t \rightarrow \infty$, some of the Stokes lines must switch off at these intersection points, or Stokes crossing points, due to higher-order Stokes switching [24, 56]. Hence, the region of the free-surface containing transverse waves must be either that shown in Figure 5.19 (a) or (b). To determine which case is correct, we set $\text{Re}(y) = 3$ and examine the Stokes behaviour when y is permitted to take complex values, shown in Figure 5.20. We see that if the second case is true, the Stokes line in Figure 5.21 on the right-hand side of the anti-Stokes line must be active, which is not possible. Hence, transverse waves are restricted to the region

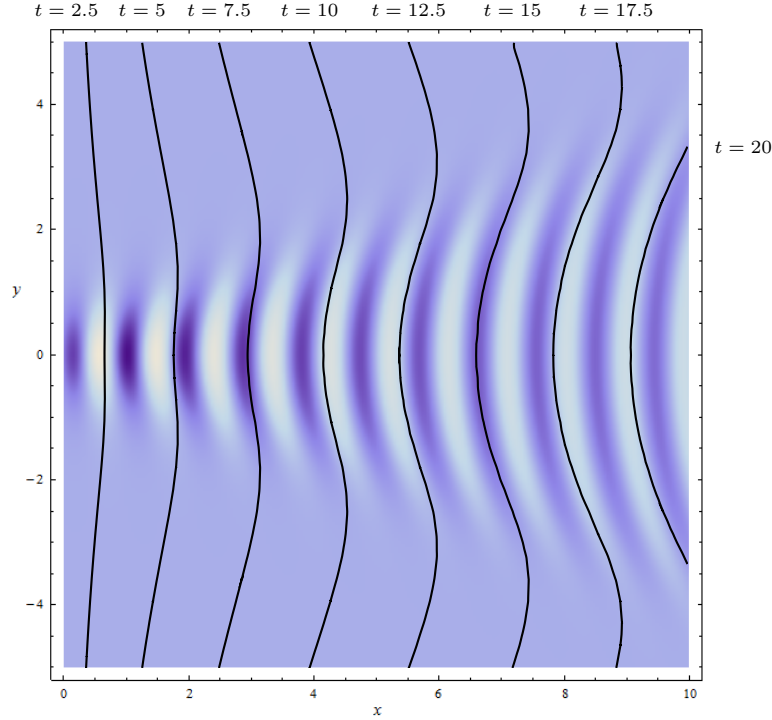


Figure 5.17: Position of the Stokes line associated with the longitudinal wavefront as t is varied, shown as bold curves. The steady free-surface longitudinal waves from Figure 5.5 are shown; however, only those to the left of the wavefront at a given time are present on the surface in the unsteady case. We also note that the waves are switched on across the curve $x = 0$ for all time.

illustrated in Figure 5.19 (a).

Finally, in Figure 5.21 we illustrate the behaviour of this region as t increases. The transverse wave region clearly expands over time, and does tend to the steady state behaviour obtained in Section 5.2.4.

5.3.5 Numerical comparison

5.3.5.1 Integral Formulation

In order to verify the unsteady free-surface behaviour, we compare the position of the longitudinal wavefront to a numerical solution for the free-surface behaviour. We obtain this free-surface behaviour by formulating the solution to (5.74)–(5.81) in terms of a double integral which may be evaluated using standard numerical methods. Again, while we will formulate the double integral expression here, we note that similar integral expressions have been obtained in previous investigations, such as [52].

As in the steady case, we set

$$\phi = \frac{1}{4\pi(x^2 + y^2 + (z - h)^2)^{1/2}} + \frac{1}{4\pi(x^2 + y^2 + (z + h)^2)^{1/2}} + \hat{\phi}.$$

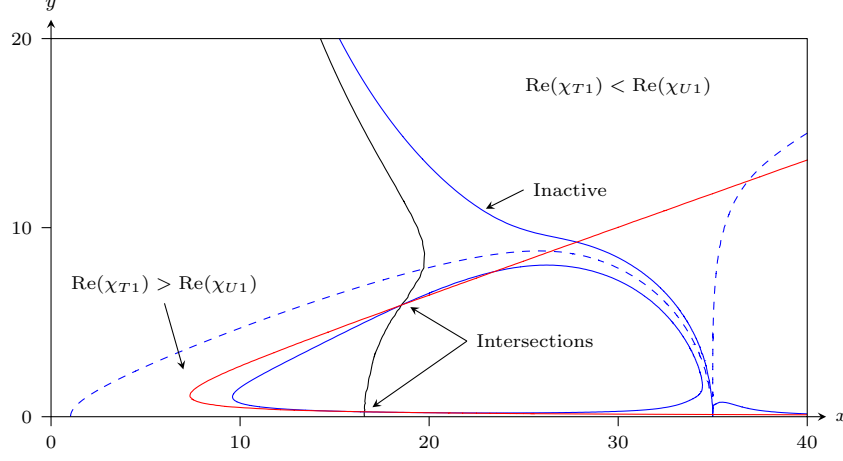


Figure 5.18: Possible Stokes lines on the free-surface for unsteady flow over a source of depth $h = 1$ at $t = 35$. The black curve corresponds to $\text{Im}(\chi_{L1}) = \text{Im}(\chi_{U1})$. The red curve corresponds to $\text{Im}(\chi_{L1}) = \text{Im}(\chi_{T1})$. The solid blue curve corresponds to $\text{Im}(\chi_{T1}) = \text{Im}(\chi_{U1})$. The dashed blue curve corresponds to the anti-Stokes line given by $\text{Re}(\chi_{T1}) = \text{Re}(\chi_{U1})$. The steady free-surface transverse-wave contribution associated with χ_{T1} can only be switched on by the unsteady free-surface contribution associated with χ_{U1} when $\text{Re}(\chi_{T1}) > \text{Re}(\chi_{U1})$, and hence the labelled Stokes line cannot be active.

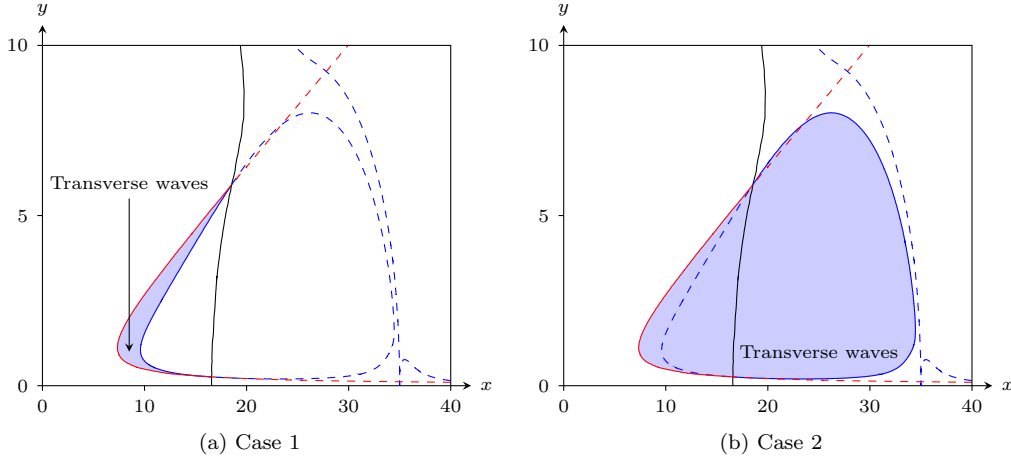


Figure 5.19: The region of the free surface in Figure 5.18 containing transverse waves must be either (a) Case 1 or (b) Case 2. These are the only possibilities that restrict the transverse waves to a finite region of the free surface and tend to the steady behaviour obtained in Section 5.2.4 as $t \rightarrow \infty$. In each case, active Stokes lines are represented as solid curves, while relevant inactive Stokes lines are shown as dashed curves.

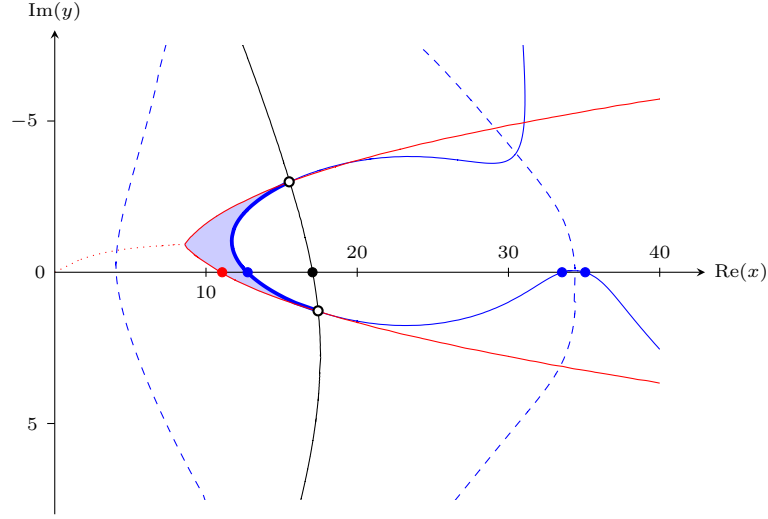


Figure 5.20: Stokes structure on the complexified free surface associated with Figure 5.18, along the curve $y = 3$, with Stokes lines that are known to be inactive not shown. White circles represent the Stokes crossing points at which Stokes lines are switched off, noting that there are other intersection points present where this does not occur. Blue, red and black circles represent points at which the blue, red and black Stokes lines intersect the physical free surface. The two blue Stokes lines which appear unconnected in Figure 5.18 are in fact shown to be part of a single Stokes surface. Hence, knowing that the Stokes line cannot be active at points when it crosses an anti-Stokes line, we conclude that the bold region of this Stokes line is active, and hence that the transverse wave contribution to the free-surface behaviour is active in the shaded region. This corresponds to Case 1 in Figure 5.19. Finally, the red Stokes lines intersect at a caustic. The inactive Stokes line originating from this caustic is illustrated here, and shown as a dotted curve.

The resultant governing equation is given by

$$\nabla^2 \hat{\phi} = 0, \quad (5.114)$$

with boundary conditions given by

$$\epsilon(\hat{\phi}_x + \hat{\phi}_t) + \xi = -\epsilon \left[\frac{1}{2\pi(x^2 + y^2 + h^2)^{1/2}} \right]_x, \quad z = 0, \quad (5.115)$$

$$\hat{\phi}_z - (\xi_x + \xi_t) = 0, \quad z = 0. \quad (5.116)$$

Finally, the far-field conditions determine that $\hat{\phi} \rightarrow 0$ far from the source, and the initial condition is simply

$$\hat{\phi} = 0, \quad \xi = 0, \quad \text{at} \quad t = 0. \quad (5.117)$$

Taking the Fourier transform of (5.114) with respect to x and y (with the resultant expression given in terms of the Fourier variables k and l respectively) gives

$$\bar{\bar{\phi}}_{zz} - \rho^2 \bar{\bar{\phi}} = 0,$$

where $\rho = \sqrt{k^2 + l^2}$. Noting that the solution must decay in the limit that $z \rightarrow -\infty$, this differential equation may be solved to give

$$\bar{\bar{\phi}} = A(k, l, t) e^{\rho z}. \quad (5.118)$$

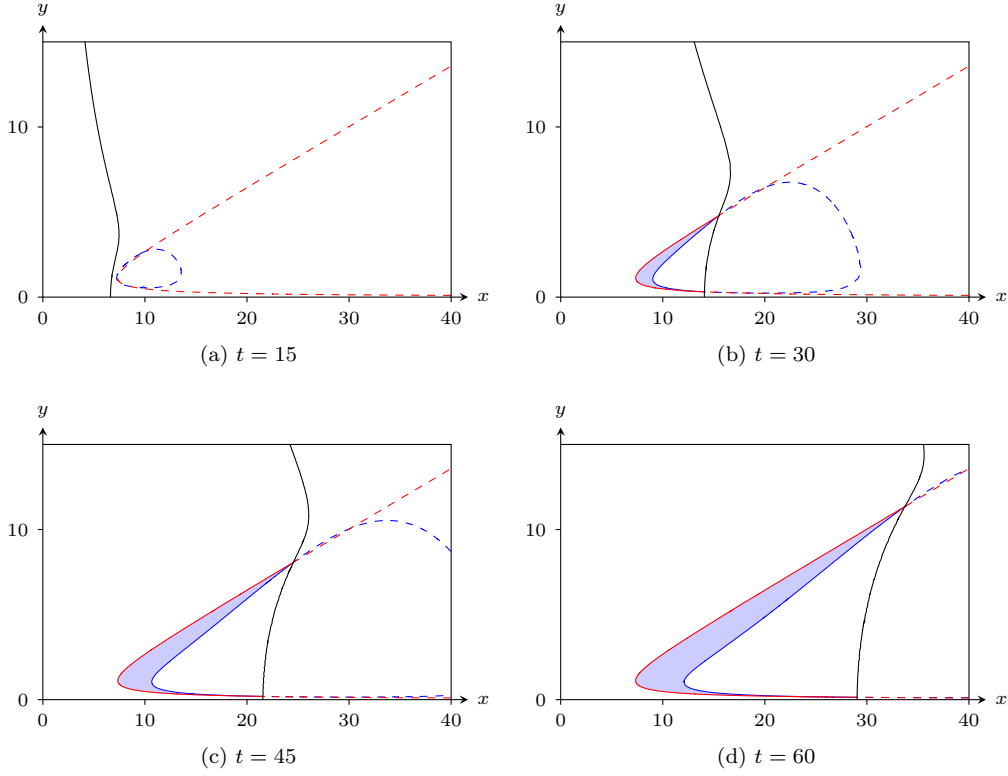


Figure 5.21: The region of the free-surface containing an exponentially small transverse wave contribution as t is varied. Transverse waves are present in the shaded regions only.

Taking the Fourier transform of the boundary conditions (5.115)–(5.116) and using the expression in (5.118) gives

$$\epsilon i k A + \epsilon A_t + \bar{\bar{\xi}} = -\frac{i k \epsilon e^{-\rho h}}{\rho}, \quad (5.119)$$

$$\rho A - i k \bar{\bar{\xi}} - \bar{\bar{\xi}}_t = 0, \quad (5.120)$$

which is identical to (5.66)–(5.67) except for the inclusion of time-varying terms. From the initial conditions in (5.117), we see that

$$A = 0, \quad \bar{\bar{\xi}} = 0, \quad \text{at} \quad t = 0,$$

and may therefore solve (5.119)–(5.120) to obtain

$$\bar{\bar{\xi}} = \frac{i k \epsilon e^{-\rho h}}{2 \rho (\rho - \epsilon k^2)} \left\{ \left[(\rho + k \sqrt{\epsilon \rho}) e^{i t \sqrt{p/\epsilon}} + (\rho - k \sqrt{\epsilon \rho}) e^{-i t \sqrt{p/\epsilon}} \right] e^{-i k t} - 2 \rho \right\}.$$

We formulate the inverse Fourier transform integral, again representing (x, y) and (k, l) in terms of polar coordinates (r, θ) and (ρ, ϕ) , giving

$$\begin{aligned} \bar{\bar{\xi}} = \frac{1}{4 \pi^2} \int_0^{2 \pi} \int_0^\infty \frac{i \epsilon \rho \cos(\phi) e^{-\rho h}}{2 (1 - \epsilon \rho \cos^2(\phi))} \left\{ \left[(1 + \cos(\phi) \sqrt{\epsilon \rho}) e^{i t \sqrt{p/\epsilon}} \right. \right. \\ \left. \left. + (1 - \cos(\phi) \sqrt{\epsilon \rho}) e^{-i t \sqrt{p/\epsilon}} \right] e^{-i t \rho \cos(\phi)} - 2 \rho \right\} e^{i \rho r \cos(\theta - \phi)} d \rho d \phi. \end{aligned} \quad (5.121)$$

This integral is singular along the curve $1 - \epsilon \rho \cos^2(\phi)$. To simplify the numerical evaluation of this double integral, we again set the singular curve to follow a straight line. In this case, we apply the transformation $u = \rho \cos^2(\phi)$, giving

$$\begin{aligned} \bar{\bar{\xi}} = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\infty \frac{i\epsilon u e^{-uh/\cos^2(\phi)}}{\cos^3(\phi)(1-\epsilon u)} & \left\{ \left[\left(1 + \cos(\phi) \sqrt{\frac{\epsilon u}{\cos^2(\phi)}} \right) e^{it\sqrt{u/\epsilon \cos^2(\phi)}} \right. \right. \\ & \left. \left. + \left(1 - \cos(\phi) \sqrt{\frac{\epsilon u}{\cos^2(\phi)}} \right) e^{-it\sqrt{u/\epsilon \cos^2(\phi)}} \right] e^{-itu/\cos(\phi)} - 2 \right\} e^{iur \cos(\theta-\phi)/\cos^2(\phi)} du d\phi. \end{aligned} \quad (5.122)$$

As before, this integral is substantially more complicated than the form given in (5.121). Again, the fact that the integrand is singular along a straight line allows the double integral to be solved using standard numerical integration in two dimensions.

5.3.5.2 Comparison of results

In order to evaluate (5.122) numerically for given values of h and ϵ , we again truncate the integral domain in the u -direction and split it into two regions, given by

$$\begin{aligned} D1 &= \{(u, \phi) : u \in (0, 1/\epsilon - \Delta), \phi \in (0, 2\pi)\}, \\ D2 &= \{(u, \phi) : u \in (1/\epsilon + \Delta, u_\infty), \phi \in (0, 2\pi)\}, \end{aligned}$$

where Δ is a small constant used to approximate the Cauchy principal value integral, and u_∞ is the truncation point. Typical values used in this investigation are $\Delta = 1 \times 10^{-5}$ and $u_\infty = 500$. On each domain, we evaluate the two-dimensional integral numerically using the inbuilt MATLAB integrator, which uses two-dimensional quadrature techniques. A number of sample surface profiles along $y = 0$ may be seen in Figure 5.23.

In these profiles, we have labelled the unsteady contribution, taking the form of an expanding ripple. The propagating longitudinal wavetrain is apparent, and we note that the transverse waves are not present along the curve $y = 0$. In Figure 5.23, we compare the position of the wavetrain obtained numerically with the behaviour predicted by the asymptotic solution. In Section 5.3.4.1, we predicted that the longitudinal waves would switch off across a Stokes line that intersects the free surface, in an inner region with width of $\mathcal{O}(\epsilon^{1/2})$. The point at which this Stokes line intersects the free-surface is indicated in Figure 5.23 by a dashed line. We see that there is a strong agreement between the asymptotic prediction for the position of the propagating wavefront and that obtained in the numerical solution.

Finally, in Figure 5.24, we illustrate the numerically-obtained free-surface behaviour for flow over a source with depth $h = 1$ and $\epsilon = 0.15$, at $t = 10$. Both the propagating ripple caused by the initial disturbance and the longitudinal waves are visible in this figure. From Figure 5.21

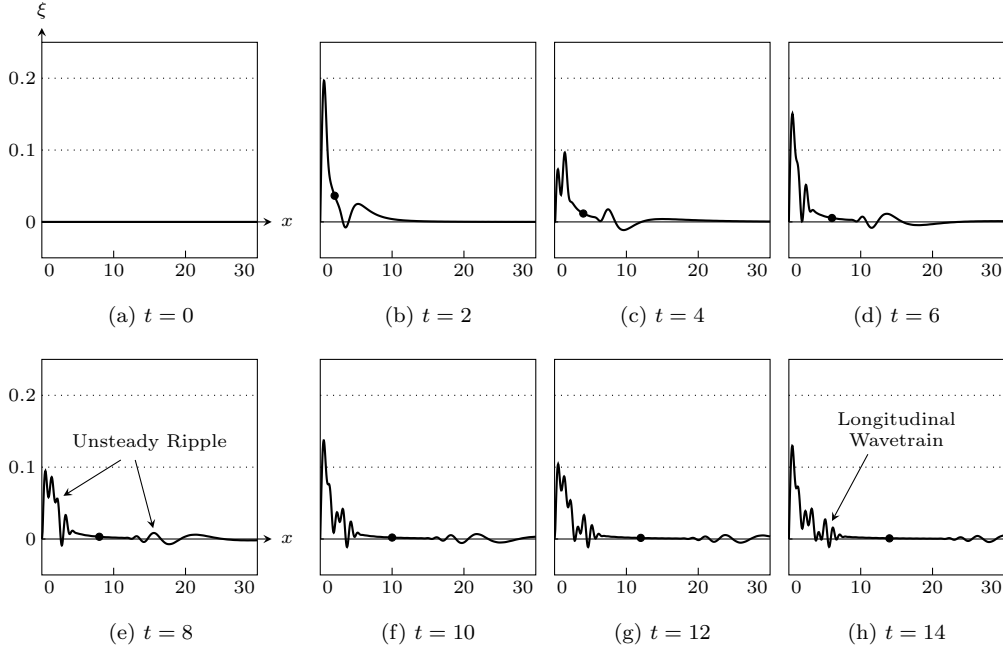


Figure 5.22: Numerically-calculated profiles of the free surface associated with flow over a source of depth $h = 1$ with $\epsilon = 0.15$ along $y = 0$ as t is varied, with the leading-order behaviour removed. The unsteady contribution, labelled in (e), takes the form of an expanding ripple which both moves downstream along the line $y = 0$ and propagates outwards with the flow velocity. This behaviour has the effect of the upstream front of the ripple remaining at $x = 0$ and the downstream front being located about $x = 2t$. The center of the expanding ripple (where $\text{Im}(\chi_U) = 0$) is denoted by a black circle in each figure. The longitudinal wave contribution is apparent by $t = 10$, and is indicated in (h).

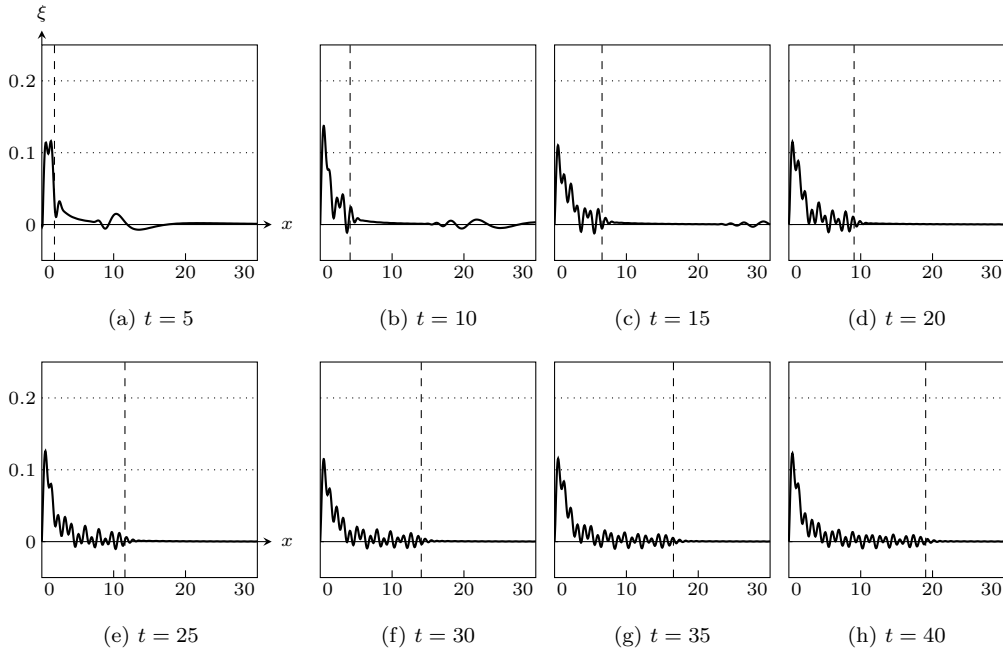


Figure 5.23: Comparison between the numerically-obtained free-surface for flow over a source of depth $h = 1$ with $\epsilon = 0.15$ (with the leading-order behaviour removed) and the position of the wavefront predicted using exponential asymptotic analysis. The dashed line indicates the point at which the Stokes line intersects the free-surface along $y = 0$. The asymptotic analysis predicts that the longitudinal waves are switched off smoothly in a region of width $\mathcal{O}(\epsilon^{1/2})$ as this point is crossed. The computed wavefront agrees strongly with the asymptotic results.

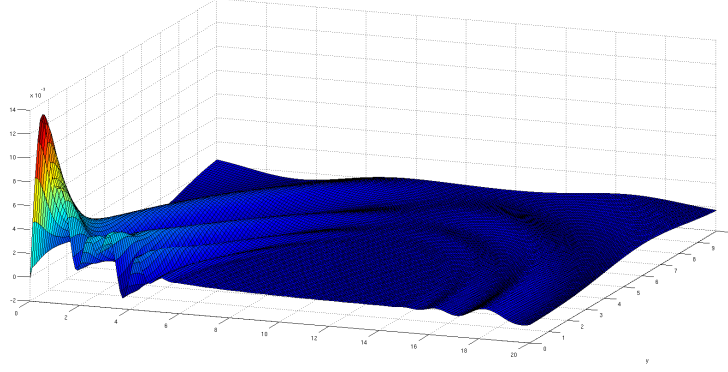


Figure 5.24: Numerically-calculated free surface for flow over a source of depth $h = 1$ with $\epsilon = 0.15$ at $t = 10$ in three dimensions, with the leading-order behaviour removed.

(a), we see that transverse waves are not yet present on the free surface at $t = 10$; however, they are too small to be visible on this scale even when sufficient time has elapsed for these waves to be present on the surface.

5.4 Discussion and conclusions

In this chapter, we calculated the behaviour of waves on the free surface of flow over a point source in three dimensions. We considered the source to be weak, and therefore linearized the problem about the undisturbed solution. We first considered the steady problem, in which the flow behaviour was assumed not to vary in time, and subsequently the unsteady problem, in which the flow was started at $t = 0$ from a prescribed initial condition. As in the analysis in Chapters 3 and 4, we set this initial condition to be the leading-order steady-state behaviour in order to simplify the subsequent analysis. We subsequently applied the exponential asymptotic techniques of [22, 24, 87] in order to determine the behaviour of the resultant free-surface waves. Again, we found that Stokes switching caused by the interaction between exponentially subdominant free-surface contributions plays an important role in the behaviour of the flow, both in the steady and unsteady case.

In the analysis of the steady problem, we found that longitudinal waves were switched on across the curve $x = 0$, and were present in the entire right-half plane. From a late-order terms analysis, we found the form of these waves in general, and obtained a simple expression for the wave amplitude in the far-field along $y = 0$. The behaviour of these waves may be seen in Figure 5.5. While the longitudinal wave contribution is present in the entire right-half plane, the amplitude of the waves decays away from the x -axis so that the visible waves appear to be restricted to a wedge.

The far-field amplitude of these longitudinal waves was compared to numerical solutions to the linearized equations in Figure 5.14. The numerical results were obtained by formulating the solution to the linearized system as an integral equation using methods similar to those given in [51, 81] and evaluating the integral numerically. The comparison showed good agreement between the asymptotic and numerical results.

By comparing the singulants associated with the transverse wave and longitudinal wave contributions, we found that transverse waves were switched on across Stokes lines containing approximately wedge-shaped regions on either side of the curve $y = 0$, as seen in Figure 5.8. While we did not compute the prefactor associated with this behaviour, knowledge of the singulant is sufficient to determine the phase of the waves produced by this contribution. Hence, we have determined that, in the steady-state problem, the leading-order behaviour switches on the downstream longitudinal waves, and these waves switch on a subdominant downstream transverse wavetrain.

We then showed that, in the limit $h \rightarrow 0$, this analysis agrees with the small Froude number behaviour found by Keller [59] when the singularity is located on the free-surface. In this case, the longitudinal waves are only algebraically-small (having $\text{Re}(\chi) = 0$) within a downstream wedge bounded by the curves $y = \pm x \arctan(2^{3/2})$, and exponentially small outside of this wedge. Furthermore, the wedge also contains algebraically-small transverse wave behaviour which is not present outside of the wedge. This is caused by the existence of Airy-type behaviour along the edge of the wedge region, which was noted by Ursell [111]. Finally, we observed that the nature of the point singularity could easily be changed, with the only impact on the late-order terms coming about due to the local matching of the prefactor near the singularity. This was demonstrated when the behaviour of free-surface longitudinal waves over a doublet was obtained.

The unsteady problem was then formulated, and it was observed that allowing the problem to vary in time produced another set of singularities in the problem, found at $t = 0$. As in [24] and the analysis of Chapter 3, we determine that this singularity is associated with the initial disturbance to the flow. Therefore, the rays of the singulant must all pass through the singularity (located at $x^2 + y^2 + (z \pm h)^2 = 0$) at $t = 0$. This produced a third set of exponentially small wave behaviour present everywhere on the free surface, denoted as the unsteady free-surface contribution. By comparing the singulants of the unsteady, longitudinal and transverse wave behaviours, we were able to determine the region surface on which the longitudinal and transverse waves were present at any point in time.

Finally, we compared the position of the longitudinal wavefront along $y = 0$ predicted by the asymptotics with numerical solutions to the unsteady flow problem. The numerical solutions

were again obtained by formulating an integral expression for the surface behaviour and computing the solution to the integrals. As in the steady case, the asymptotic and numerical results showed good agreement.

It would be interesting to consider other linearized geometries, such as flow past a finite line source, or a narrow submerged obstruction. It would be expected that flow past a finite line source would tend to the results found in Chapter 3 as the length of the source became large. As long as leading-order solutions could be obtained, this analysis should not require substantial changes to the methods described in this chapter.

In Chapter 4, we considered the nonlinear equivalent to the two-dimensional unsteady linearized problem examined in Chapter 3. Similarly, the most natural extension to the work of this chapter is to consider the behaviour of nonlinear three-dimensional flows. In addressing this problem, however, we again find that the boundary is not fixed in Eulerian coordinates. In Chapter 4, we circumvented this difficulty by formulating the problem in terms of Lagrangian coordinates, where the particle labels were given in terms of the solution to an auxiliary steady problem in terms of the complex potential.

Unfortunately, while Lagrangian formulations for three-dimensional flow problems are well-known [69, 121] and used frequently in numerical methods that involve the tracking of fluid particles [103], determining an appropriate particle-labelling scheme is substantially more difficult in three dimensions where complex variable methods may no longer be applied. One possible solution to this difficulty is to consider the use of Clebsch potentials, such as in [57]. However, Clebsch potential formulations are degenerate in the zero vorticity case, which is precisely the case which we hope to consider. Other complex potential-type formulations for three dimensional potential flow exist [60], and it is possible that these may be applicable to the problem under investigation.

Chapter 6

Conclusions

In this study, we applied exponential asymptotic methods in order to extend the results of Chapman and Vanden-Broeck [27] for gravity waves on steady free-surface flow in two dimensions to more general flow configurations. We considered linearized and fully nonlinear unsteady two-dimensional flows, as well as steady and unsteady three-dimensional flows. In each of these cases, we described the downstream wavetrain caused by flow over some submerged obstacle in the small Froude number limit. As the waves were exponentially small, we applied the exponential asymptotic technique developed by Olde Daalhuis *et al.* [87] and extended by Chapman *et al.* [22]. Furthermore, in each case, we found that Stokes switching caused by the interaction between exponentially subdominant free-surface contributions played an important role in describing the free-surface behaviour, which we obtained using the technique developed by Chapman and Mortimer [24]. In unsteady problems, these Stokes lines determine the position of the propagating wavefront, while in three-dimensional problems, they determine the location of the various classes of waves present on the surface.

6.1 Review

In Chapter 2, we considered two problems of relevance to the remainder of the thesis. The first of these problems was solving a differential equation constructed to share several features with the free-surface flow problems considered in subsequent chapters. The system satisfied Laplace's equation in the upper half-plane, with a mixed boundary condition imposed along the curve $y = 0$. The boundary condition was chosen to contain a singular perturbation, as well as to contain singularities in the analytic continuation of the boundary, located at $x = \pm i$. We chose specifically to determine the behaviour of the solution on the boundary itself, which is equivalent to finding the free-surface behaviour in the fluid flow problems addressed later in this investigation.

As the system produced a singular perturbation, the resultant asymptotic series expansion was divergent, containing Stokes lines between the two singularities on analytically-continued boundary. This problem served as an illustration of the exponential asymptotic methodology developed by Olde Daalhuis *et al.* [87] and extended by Chapman *et al.* [22], which were used to determine the behaviour that was switched as the Stokes line was crossed. In this problem, as in the the fluid flow problems studied throughout this investigation, we found that there was ambiguity in the form of the late-order terms. In this case, there were four possible late-order term contributions, two of which were present in the problem. However, on the boundary itself, these active contributions were indistinguishable from the inactive contributions.

To resolve this ambiguity, we solved the complex ray equations for the singular behaviour. Consequently, we noted that two of the late-order contributions led to singularities being present in the solution domain, which is impossible for an elliptic problem. More generally, we observed that in situations such as this, we may resolve the ambiguity by comparing the form of the late-order singulants (and hence, the late-order terms themselves) with the leading-order behaviour and ensuring the two are consistent. This is a similar process to determining the power of the late-order singularity in [22], except we also determine which singularities are present in the first place.

The second problem in Chapter 2 was chosen to illustrate the behaviour of exponentially small gravity waves on free-surface flows with small Froude number. Specifically, we consider the problem of flow due to a line source, overlooked in the general analysis of [27]. In this problem, as in [27], we applied exponential asymptotic late-order term methods to determine that exponentially small gravity waves are switched on in the far field as Stokes lines are crossed. We determined the form of these waves, and compared the wave amplitude to numerical results from [78], finding good agreement between the numerical and asymptotic results that improved as the Froude number became small.

In Chapter 3, we began our investigation into unsteady free-surface fluid flow over a submerged step. However, the general method for determining free-surface gravity waves given in [27] relies upon complex variable techniques that cannot be applied to the unsteady problem. Instead, we fixed the boundaries by formulating the problem in terms of Lagrangian coordinates, as in [16, 69]. Using Cartesian particle-labeling would have fixed the free-surface, but this would no longer be true of the flow base. Instead, we considered the initial flow configuration and solved it as if it were a steady, fixed-boundary problem, obtaining the fluid potential and streamfunction. We then used these quantities to uniquely label each fluid particle at $t = 0$, producing a Lagrangian system in which the boundaries were fixed. This was not strictly necessary for the linearized

problem considered in the remainder of Chapter 3, however it became important for the full nonlinear version of the problem addressed in Chapter 4.

We subsequently used exponential asymptotic methods to determine the late-order contributions to the asymptotic series behaviour, finding contributions associated with the singularity at the top of the step, and the initial disturbance to the flow at $t = 0$. The singularities associated with the step generated Stokes lines which intersected the free-surface above the step itself, while the singularities associated with the initial disturbance produced Stokes lines that propagated downstream with the flow. Both of these sets of late-order terms led to exponentially small waves on the free surface of the flow, which exhibited switching behaviour as the Stokes lines were crossed. The first gave the downstream wavetrain observed in the steady case in [27], while the second produced transient behaviour which was present on the entirety of the free surface, and decayed in time.

Following [24], we determined the Stokes switching caused by the interaction between the two different contributions. This analysis produced Stokes lines which intersected the free-surface at $x \sim t/2$ as $t \rightarrow \infty$, across which the downstream wavetrain was switched off. We therefore found that these waves do not appear instantaneously, but rather propagate downstream from the singularity, tending to the steady-state behaviour as $t \rightarrow \infty$. These results were compared with the asymptotic solution obtained by solving the linearized equations using Fourier transforms, and using the method of steepest descents to evaluate the Fourier inversion integral.

In Chapter 4, we considered fully nonlinear unsteady flow over a submerged step, using the Lagrangian formulation developed in Chapter 3. We found that the system could not be explicitly solved for the leading-order behaviour except for particularly convenient values for the step-height parameter b . To proceed with the analysis, we set $b = 2$, and obtained the resultant leading-order behaviour. We then followed [27], posing an ansatz for the late-order terms, and obtained equations governing the singulant and prefactor in this ansatz. While we were not able to solve these equations analytically, we used numerical methods to determine the position of the Stokes lines on the free-surface. Having obtained the Stokes structure, we were able to determine the region of the free-surface containing exponentially small gravity waves in this problem. Finally, we determined the optimal truncation point and performed Stokes smoothing in order to determine the behaviour of the free-surface waves, with the result expressed in terms of the singulant and prefactor.

Chapter 5 was divided into two sections. The first section considered steady free-surface flow over a submerged point source. As we were considering only the linearized case for a small flow disturbance, we formulated the problem in terms of Eulerian coordinates, giving a fixed-boundary problem. We found that the asymptotic series behaviour switched on downstream

longitudinal waves across a Stokes line following $x = 0$, which were present in the entire downstream region. These waves were largest within a wedge-shaped region that originated above the singularity. Furthermore, we found that the longitudinal waves switched on a further transverse wave contribution across Stokes lines, which was exponentially subdominant to both the original asymptotic series behaviour and the longitudinal waves themselves. We then determined an expression for the prefactor associated with the longitudinal waves, and used Stokes switching to determine the downstream wave behaviour. We subsequently determined the far-field wave amplitude along $y = 0$, and compared this with the equivalent numerical free-surface behaviour, obtaining good agreement between the two.

We compared these results in the limit that the source approaches the surface to those obtained by Keller [59], who considered the problem of low Froude number flow past a source positioned on the surface itself. Keller used geometric ray theory to show that the free-surface contained algebraically-small longitudinal and transverse gravity waves within the Kelvin wedge behind the source. In our analysis, we found that as the source reaches the free-surface, both classes of waves *do* become algebraically small within the Kelvin wedge, and that exponentially small longitudinal waves are present within the downstream region outside of the wedge. Furthermore, we noted that this analysis may easily be adjusted to consider different singularities simply by changing the inner problem, and illustrated how this could be performed for flow past a doublet. We also observed that, as we are effectively finding the Green's function for flow past a singularity, the results could easily be extended to be applied to any distribution of point singularities.

The second problem considered in Chapter 5 was that of unsteady linearized flow past a submerged source. As in the two-dimensional case considered in Chapter 3, this introduced a new exponentially small free-surface contribution associated with the initial disturbance to the system at $t = 0$. By comparing the singulants associated with the unsteady, longitudinal and transverse wave contributions, we determined the flow region in which each of the wave contributions could be found at a given time. We first considered only the unsteady and longitudinal wave interactions, finding that the longitudinal wavefront position was given by $x \sim t/2$ as $t \rightarrow \infty$, tending to the steady-state behaviour found earlier in the chapter. Subsequently, we illustrated the full Stokes structure of the problem, and, noting that higher-order Stokes switching can only occur at Stokes line intersection points, determined the free-surface region that contained transverse waves at any given time. This region also tended to the appropriate transverse wave region obtained from the steady-state problem.

Finally, we compared the position of the longitudinal wavefront along $y = 0$ predicted by the asymptotic analysis with a numerical simulation of the free-surface behaviour for a range of

values of t . This comparison found good agreement between the analytical and numerical results.

6.2 Further work

The analysis performed in this investigation can be extended in a number of ways, many of which involve determining the behaviour of flow over geometries not considered in this analysis.

In Chapter 2, we considered free-surface waves due to flow over a line source in finite depth. The two most natural ways to extend this problem are flow over a line source in fluid of infinite depth, and flow over a point source in fluid of either infinite or finite depth. In the first case, we could omit the second conformal mapping, and perform the exponential asymptotic analysis directly on the complex potential representation of the flow system, as the domain in this representation is an entire half-plane. While the latter two cases are three-dimensional problems, they may be represented in two-dimensional axisymmetric form, and could therefore be investigated using similar techniques to those used here.

The investigation of unsteady flow problems performed in Chapters 3 and 4 could be extended to consider a range of different flow geometries. We discussed some extensions of the linearized results in Chapter 3, such as flow over an small, inclined step. We could also consider singularities of a different nature, such as a line source or a smoothed step, or bases that generate multiple singularities. Indeed, the base could be prescribed to have any number of submerged small steps and dips, or even small, smooth variations, without invalidating the linearization process. It would be interesting to consider the interaction between the unsteady free-surface contributions associated with multiple singularities, such as the trapped waves found in [75].

In the unsteady, nonlinear flow problem considered in Chapter 4, it would be of interest to determine the first-order correction terms to the flow, and subsequently the prefactor behaviour. This would result in the downstream wavetrain behaviour being completely determined, using the form of the Stokes-switching behaviour obtained in (4.48).

It would be of great interest if this analysis could be extended to a range of geometries, however this possibility is limited by the necessity of determining the leading-order behaviour in the Lagrangian formulation, which may not necessarily be possible. Indeed, even the leading-order behaviour for flow over a step can only be found for particularly convenient values of b . Furthermore, as the formulation of the Lagrangian governing equations is dependent on the flow geometry, changing the geometry would require re-evaluation of the governing equations themselves. Nonetheless, this does not completely preclude the possibility that similar methods may be applied to other geometries; at the very least, it would be interesting to consider flow over a downwards step, or a step with a value of b given in Appendix C.

The analysis of Chapter 5 could also be extended to a number of previously unconsidered infinite-depth geometries, such as flow over a finite line source. Furthermore, we could represent multiple obstacles as a distribution of singularities, and hence determine the manner in which their wave patterns interfere. Furthermore, we could suggest arrangements that minimize wave resistance, such as the ‘Weinblum’ catamaran or other multihull configurations discussed in [105, 104]. We could also obtain equivalent free-surface behaviour for finite-depth flow configurations using similar techniques.

We previously discussed the possibility of considering the equivalent fully nonlinear three-dimensional steady and unsteady flow problem; however, this is more complicated than the equivalent two-dimensional extension. Equivalent equations for the Lagrangian formulation of three-dimensional flow are well-known [69, 121], and given an appropriate particle-labeling, it is likely that the asymptotic analysis may be performed in a similar fashion to that given in Chapter 4. However, in the two-dimensional analysis of the nonlinear problem, the particle-labeling was found using complex variable techniques which are not valid in three dimensions. Hence, we require a particle-labeling scheme that fixes both the free-surface and the position of the obstacle.

One possible solution is to consider a particle-labeling scheme that utilizes Clebsch potentials [57]. However, Clebsch potentials become degenerate in the zero vorticity case, which means that this application would not be trivial. Other complex potential-type formulations for three-dimensional potential flow exist [60], and it is possible that these may allow the fluid particles to be labeled in an appropriate fashion.

Finally, we note that each of the flow problems considered within this investigation neglected surface tension. It would be interesting to consider equivalent results for capillary waves, such as [26], or even combining gravity and surface tension effects to consider gravity-capillary waves, as in [100, 101]. Each of the listed studies considers only the steady, two-dimensional case. However, it is likely that similar ideas to those used in this study could be applied to consider more general flow configurations.

While we have determined free-surface gravity wave behaviour in a range of flow configurations, we emphasize that many open problems and unsolved configurations remain. However, the techniques developed by this investigation for unsteady and three-dimensional flows, as well as the important observation that the interaction between exponentially small free-surface contributions produces Stokes lines that play an important role in each of these problems, should be of use in any further investigation into low Froude number free-surface flows.

Appendix A

Darboux's Theorem

Here we summarize the explanation of Darboux's Theorem given by Dingle [33]. This result is important for determining the behaviour of late-order terms in an asymptotic series, as it states that the behaviour of late-order series terms at any point in the complex plane is governed by the nearest singularities in the complex plane.

We are given some function $\phi(f)$ which converges within a circle in the complex plane centered on zero, and may therefore be expanded in this circle as a Taylor-Maclaurin series given by

$$\phi(f) = \sum_{r=0}^{\infty} a_r f^r. \quad (\text{A.1})$$

Let f_i represent a singular point lying on or outside the circle of convergence. This singularity may be a pole or branch point, but not an essential singularity. Near the singularity, we obtain

$$\phi(f) \sim \phi_i(f)(f_i - f)^{-p_i}, \quad (\text{A.2})$$

where p_i is the order of the singularity, and the coefficient $\phi_i(f)$ may be expanded as a Taylor series about f_i , giving

$$\phi_i(f) = \phi_i(f_i) - (f_i - f)\phi'_i(f_i) + \frac{(f_i - f)^2}{2}\phi''_i(f_i) - \dots$$

Hence, if we expand (A.2) to determine the coefficient of f^r and compare this with the series (A.1), we find that the contribution to a_r from the singularity at $f = f_i$ is given by

$$\frac{\Gamma(r + p_i)}{\Gamma(r + 1)\Gamma(p_i)f_i^{r+p_i}} \left\{ \phi_i - \frac{p_i - 1}{r + p_i - 1} f_i \phi'_i + \dots \right\},$$

for late coefficients such that $r \gg p_i$. The behaviour of a_r in the limit that $r \rightarrow \infty$ will contain contributions from every singularity in the complex plane; however, the factor f_i^r in the denominator ensures that the dominant contribution will come from those singularities for which $|f_i|$ is smallest, or those lying on the circle of convergence.

Consequently, to determine the late-order terms of an asymptotic series at any point, we need only determine the contribution from the nearest singularities in the complex plane.

Appendix B

Finding the Fourier Transform of the Initial Condition

Taking the Fourier transform of the initial condition given in (3.29) on $y = 0$, we obtain

$$\hat{\phi}_0(k, 0) = \int_{-\infty}^{\infty} \frac{1}{2} \log(1 + e^x) e^{ikx} dx. \quad (\text{B.1})$$

In order for the integral in this expression to converge in the upper half-plane, we require $0 < \text{Im}(k) < 1$. The integrand has poles at $x = \pm(2n + 1)\pi i$ for integer values of n . Due to the logarithm term, the integrand also has branch cuts originating from each of these poles, which shall be defined to follow the paths in Figure B.1, although the result of the integral will be independent of the orientation of the cuts. To evaluate the integral, we deform the integral contour into the upper half-plane, as shown by the contour C_1 in Figure B.1. This deformation is valid for $\text{Re}(k) > 0$. For $\text{Re}(k) < 0$, the contour would instead be deformed into the lower half-plane, shown by the contour C_2 in Figure B.1. Evaluating the integral along this contour will give an identical result.

We see that contributions to the integral will arise due to the integration along the branch cuts. Along the branch originating at $x = -(2n + 1)\pi i$, we consider the sum of the integral on either side of the cut. Using $(2n + 1)\pi i^+$ to indicate a point just above the cut, and $(2n + 1)\pi i^-$ to indicate a point just below it, we see that this contribution is given by

$$\begin{aligned} \int_{\infty + (2n+1)\pi i^+}^{(2n+1)\pi i^+} \left[\frac{1}{2} \log(|1 + e^x|) - \frac{\pi i}{2} \right] e^{ikx} dx \\ + \int_{(2n+1)\pi i^-}^{\infty + (2n+1)\pi i^-} \left[\frac{1}{2} \log(|1 + e^x|) + \frac{\pi i}{2} \right] e^{ikx} dx, \end{aligned}$$

which simply becomes

$$\int_{\infty + (2n+1)\pi i}^{(2n+1)\pi i} -\pi i e^{ikx} dx = \frac{\pi e^{(2n+1)k\pi}}{k}.$$

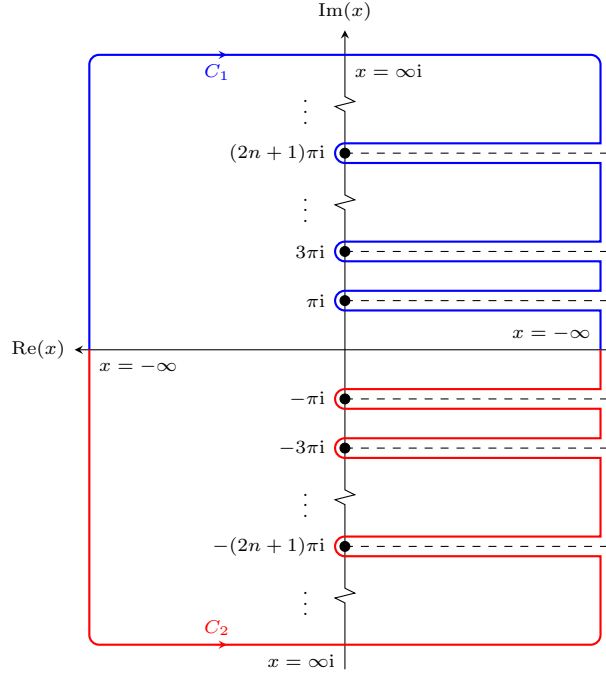


Figure B.1: Possible contour deformations for (B.1). The blue contour in the upper-half plane C_1 is the appropriate choice for $\text{Re}(k) > 0$, while the red contour C_2 in the lower-half plane is the appropriate choice for $\text{Re}(k) < 0$. Branch cuts are indicated by dashed lines and singularities are indicated by filled circles.

The sum of the branch-cut contributions is therefore given by

$$\sum_{n=0}^{\infty} \frac{\pi e^{(2n+1)k\pi}}{k} = -\frac{\pi}{2k \sinh(\pi k)}.$$

Consequently, we find that (B.1) evaluates to become

$$\hat{\phi}_0(k, 0) = -\frac{\pi}{2k \sinh(\pi k)}.$$

Appendix C

Exact solutions for the leading-order flow potential

Here, we provide exact solutions to (4.14) for a range of discrete values of b . Exact solutions may, in general, be found for any value of b that causes the polynomial in (4.14) to have integer order less than or equal to four. We consider only values of b that are associated with upward steps in the flowpath (that is, $b > 1$); however, exact solutions also exist for many values of b that lead to downward steps ($0 < b < 1$). This list is not exhaustive; an exact solution for α was calculated for $b = 4/3$; however, it was too cumbersome to be represented here. It is possible that there may exist other values of b such that exact solutions to (4.14) may be obtained.

Exact solutions for κ when $b = 3/2, 2, 3$ and 4 are shown in Table C.1. The values given in the table may be used to obtain the steady-state fluid potential for each case using (4.13).

Table C.1: Values of α for different values of b

b	κ	Parameters
4	$-\frac{1}{2}\beta^{1/2} + \frac{1}{2}\sqrt{\frac{8e^{p+4\tau}}{\beta^{1/2}} - \beta}$	$\beta = \frac{\alpha}{3 \times 2^{1/3}} - \frac{4 \times 2^{1/3} e^{4\tau} (1 + 4e^p)}{\alpha}$ $\alpha = \left(432 e^{2p+8\tau} + \sqrt{186624 e^{4p+16\tau} + 6912 e^{12\tau} (1 + 4e^p)^3} \right)^{1/3}$
3	$-\frac{3 \times 2^{1/3} e^{p+3\tau}}{\alpha} + \frac{\alpha}{3 \times 2^{1/3}}$	$\alpha = \left(27 e^{3\tau} + 81 e^{p+3\tau} + \sqrt{2916 e^{3p+9\tau} + (27 e^{3\tau} + 81 e^{p+3\tau})^2} \right)^{1/3}$
2	$-e^{p+2\tau} + e^{2\tau} \sqrt{e^{2p} + e^{-2\tau}} + 2e^{-2\tau+p}$	
3/2	$\frac{1}{2\alpha}(\alpha^2 + e^{2p} - \alpha e^p) e^{3\tau/2}$	$\alpha = 4 e^{-3\tau} + 6 e^{p-3\tau} - e^{3p} + \frac{2 e^{3\tau/2} \sqrt{4 e^{-3\tau} + 12 e^{-3\tau+p} - 2 e^{3p} + 9 e^{-3\tau+2p} - 3 e^{4p}}}{}$

Appendix D

Finding the prefactor for unsteady three-dimensional flow

For unsteady, three-dimensional flow, the equation for the prefactor was given in (5.104). As in the steady case, we obtain a form using the method of characteristics. Defining u as the characteristic variable, the ray equation formulation of this problem becomes

$$\frac{dx}{du} = 4(\chi_x + \chi_t)^3 + 2\chi_x, \quad \frac{dy}{du} = 2\chi_y, \quad \frac{dt}{du} = 4(\chi_x + \chi_t)^3, \quad (\text{D.1})$$

$$\frac{d\Phi}{du} = -(\chi_{xx} + \chi_{yy} + 6(\chi_x + \chi_t)^2(\chi_{xx} + 2\chi_{xt} + \chi_{tt}))\Phi, \quad (\text{D.2})$$

with boundary conditions are parametrized by s and given by

$$x_0 = s, \quad y_0 = \pm i(s^2 + h^2)^{1/2}, \quad t_0 = 0, \quad \Phi_0 = \Phi_0(s),$$

where $\Phi_0(s)$ is obtained by considering the behaviour in the neighbourhood of the singularity. Again, we will choose the positive sign for t_0 , and note that the opposite choice would give the complex conjugate contribution. We notice that the equations governing the ray behaviour are identical to those of the singulant equation (5.105), hence giving

$$\frac{dx}{du} = 4(P + R)^3 + 2P, \quad \frac{dy}{du} = 2Q, \quad \frac{dt}{du} = 4(P + R)^3, \quad (\text{D.3})$$

$$\frac{d\Phi}{du} = -(P_x + Q_y + 6(P + R)^2(P_x + 2R_x + R_t))\Phi. \quad (\text{D.4})$$

We may rewrite (D.4) in terms of independent variables as

$$\frac{d\Phi}{du} = -(P_s s_x + P_R R_x + Q_s s_y + Q_R R_y + 6(P + R)^2(P_s s_x + P_R R_x + 2R_x + R_t))\Phi, \quad (\text{D.5})$$

where the form of P and Q are obtained as a product of the analysis in Section 5.2.2.1. To determine s_x and R_x , we integrate (D.3) to obtain

$$\begin{aligned} x &= (4(P(R, s) + R)^3 + 2P(R, s))u + s, \\ y &= 2Q(R, s)u + i(s^2 + h^2)^{1/2}, \\ t &= 4(P(R, s) + R)^3u. \end{aligned}$$

Taking the partial derivative of each equation with respect to x gives a system of equations taking the form

$$\begin{aligned} 1 &= (1 + 2uP_s(1 + 6(P + R)^2))s_x + (2P_R + 12(P + R)^2(P_R + 1))uR_x + (4(P + R)^3 + 2P)u_x, \\ 0 &= \left(2Q_s u + \frac{is}{(s^2 + h^2)^{1/2}}\right)s_x + 2Q_R uR_x + 2Q u_x, \\ 0 &= 3uP_s s_x + 3(P_R + 1)uR_x + (P + R)u_x, \end{aligned}$$

which may be solved to find s_x , R_x and u_x in terms of R , s and u . Similarly, taking the partial derivatives with respect to y and t give similar systems of equations that may be solved to obtain s_y , R_y and u_y , and s_t , R_t and u_t respectively. Consequently, (D.5) may be written in the form

$$\frac{d\Phi}{du} = g(s, R, u)\Phi,$$

which may in principle be solved using direct integration along each ray, with the arbitrary constant being related to the inner solution in the neighbourhood of the singularity. In practice, however, this integral is too complicated to be evaluated analytically. Hence, we leave the prefactor in integral form, giving

$$\Phi(s, R, u) = \Phi_0(s, R, 0) \exp \left(\int_0^u g(s, R, u') du' \right),$$

where the form of s , R and u were computed when solving the singulant equation in Section 5.3.3.2, and $\Phi_0(s, R, 0)$ is found by matching this outer solution with the inner expansion near the singularity.

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