

# A NEW DEFINITION OF ROUGH PATHS ON MANIFOLDS

YOUNESS BOUTAIB AND TERRY LYONS

**ABSTRACT.** Smooth manifolds are not the suitable context for trying to generalize the concept of rough paths as quantitative estimates -which will be lost in this case- are key in this matter. Moreover, even with a definition of rough paths in smooth manifolds, rough differential equations can only be expected to be solved locally in such a case. In this paper, we first recall the foundations of the Lipschitz geometry, introduced in [3], along with the main findings that encompass the classical theory of rough paths in Banach spaces. Then we give what we believe to be a minimal framework for defining rough paths on a manifold that is both less rigid than the classical one and emphasized on the local behaviour of rough paths. We end by explaining how this same idea can be used to define any notion of coloured paths on a manifold.

## 1. REVIEW OF KEY ELEMENTS IN THE THEORY OF THE ROUGH PATHS

We start by defining rough paths and giving a reminder of all the notions that will be necessary to us in the rest of this work, one of which will be the extension of the notion of Lipschitz (Hölder) maps. Most of the results in this section are admitted without proofs as they are very basic and can be found, for example, in [2] and [9]. We merely give some examples and prove some of the statements to familiarize the reader with any possible new notions. In particular, we will attach a certain importance to associating a starting point to a geometric rough path.

### 1.1. Signatures of paths.

**1.1.1. The concept of the  $p$ -variation.** We introduce the notion of  $p$ -variation, crucial in both Stieltjes's and Young's integration theory and the rough paths theory. It is a way of measuring the amplitude of the oscillations of a path, independently of when said oscillations occur.

**Definition 1.1.** Let  $p \geq 1$ ,  $(E, \|\cdot\|)$  be a normed vector space and  $T > 0$ . Let  $x : [0, T] \rightarrow E$  be a continuous path. For a finite subdivision  $D = (t_i)_{0 \leq i \leq n}$  of  $[0, T]$ , we denote by  $\|x\|_{p,D}$  the quantity:

$$\|x\|_{p,D} := \left( \sum_{i=0}^{n-1} \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{\frac{1}{p}} = \left( \sum_D \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{\frac{1}{p}}$$

$x$  is said to have a finite  $p$ -variation over  $[0, T]$  if  $\{\|x\|_{p,D} \mid D \in \mathcal{D}_{[0,T]}\}$  has a finite supremum. In this case, this supremum is called the  $p$ -variation of  $x$  over  $[0, T]$  and is denoted by  $\|x\|_{p,[0,T]}$ . When  $p = 1$ , we say that the path  $x$  has bounded variation.

**Remark 1.2.** The previous definition can easily be adapted for a metric space  $(E, d)$  but we will not use this generalisation in the rest of these notes.

**Examples 1.3.** *Let  $T > 0$ :*

- (1) *Let  $E$  be a normed vector space. A  $\mathcal{C}^1$   $E$ -valued path  $x$  over  $[0, T]$  is of bounded variation over  $[0, T]$  and  $\|x\|_{1,[0,T]} \leq T\|x'\|_{\infty,[0,T]}$ .*
- (2) *Let  $E$  be a normed vector space and  $\alpha \leq 1$ . An  $\alpha$ -Hölder  $E$ -valued path over  $[0, T]$  has finite  $1/\alpha$ -variation over  $[0, T]$ .*
- (3) *Let  $\Omega$  be a probability space and  $B$  be a Brownian motion over  $[0, T]$ :*
  - *$B$  has finite  $p$ -variation over  $[0, T]$  almost surely, for any  $p > 2$ .*
  - *Seen as an  $L^2(\Omega)$ -valued path,  $B$  has finite 2-variation.*

**Proposition 1.4.** *Let  $p \geq 1$  and  $(E, \|\cdot\|)$  be a Banach space.*

- *The set  $\mathcal{V}^p([0, T], E)$  of all continuous paths from  $[0, T]$  to  $E$  that have a finite  $p$ -variation over  $[0, T]$  is a vector space when endowed with the natural operations of addition and multiplication by a scalar.*
- *The map:*

$$\begin{aligned} \|\cdot\|_{\mathcal{V}^p([0,T],E)} : \mathcal{V}^p([0,T],E) &\rightarrow \mathbb{R}^+ \\ x &\mapsto \|x\|_{p,[0,T]} + \sup_{t \in [0,T]} \|x_t\| \end{aligned}$$

*defines a norm on  $\mathcal{V}^p([0, T], E)$  called the  $p$ -variation norm.*

- *$(\mathcal{V}^p([0, T], E), \|\cdot\|_{\mathcal{V}^p([0,T],E)})$  is a Banach space.*
- *$\forall q \geq p \geq 1 : \mathcal{V}^p([0, T], E) \subseteq \mathcal{V}^q([0, T], E)$ .*

For a compact interval  $J$ , we define  $\Delta_J$  to be the simplex of all pairs  $(s, t) \in J^2$  such that  $s \leq t$ .

The manipulation of  $p$ -variations is often made easier by the introduction of the notion of controls.

**Definition 1.5.** *A function  $\omega : \Delta_{[0,T]} \rightarrow \mathbb{R}_+$  is said to be a control if it has the following properties:*

- *$\omega$  is continuous.*
- *$\omega$  is super-additive i.e.  $\omega(s, u) + \omega(u, t) \leq \omega(s, t)$  ,  $\forall 0 \leq s \leq u \leq t \leq T$ .*
- *$\omega(t, t) = 0$ ,  $\forall t \in [0, T]$ .*

Lemma 1.6 shows that to every path of finite  $p$ -variation, we can associate a “natural” control.

**Lemma 1.6.** *Let  $p \geq 1$ ,  $E$  be a normed vector space and  $T > 0$ . Let  $x : [0, T] \rightarrow E$  be a continuous path of finite  $p$ -variation over  $[0, T]$ . Then the function  $\omega$  defined on  $\Delta_{[0,T]}$  by  $\omega(s, t) = \|x\|_{p,[s,t]}^p$  is a control.*

Conversely, if we can find a control that upper-bound the  $p^{\text{th}}$  power of the increments of a continuous path then this path is necessarily of finite  $p$ -variation. This is what theorem 1.7 states. Consequently, it also gives an easy way to prove the finiteness of the  $p$ -variation of a path without having to go back to the definition.

**Theorem 1.7.** *Let  $p \geq 1$ ,  $E$  be a normed vector space and  $T > 0$ . Let  $x : [0, T] \rightarrow E$  be a continuous path. There exists a control  $\omega$  defined on  $\Delta_{[0,T]}$  such that for every  $(s, t) \in \Delta_{[0,T]}$  we have:  $\|x_t - x_s\|^p \leq \omega(s, t)$  if and only if  $x$  has a finite  $p$ -variation over  $[0, T]$  and is such that:*

$$\forall (s, t) \in \Delta_{[0,T]} : \|x\|_{p,[s,t]}^p \leq \omega(s, t)$$

We say in this case that the  $p$ -variation of  $x$  is controlled by  $\omega$ .

*Proof.* ( $\Rightarrow$ ) Let  $(s, t) \in \Delta_{[0, T]}$  and  $D \in \mathcal{D}_{[s, t]}$ . Using the super-additivity of the control, it is easy to show that:  $\|x\|_{p, D}^p \leq \omega(s, t)$ . By taking the supremum of the left-hand side term over all finite subdivisions of  $[s, t]$ ,  $x$  has then a finite  $p$ -variation over  $[0, T]$  controlled by  $\omega$ . ( $\Leftarrow$ ) The  $p$ -variation of  $x$  defines a natural control of its  $p$ -variation as stated in lemma 1.6.  $\square$

1.1.2. *The signature of a path.* We start by defining the tensor algebra of a vector space.

**Definition 1.8.** Let  $E$  be a vector space. For every  $n \in \mathbb{N}^*$ , let  $E^{\otimes n}$  be the space of homogeneous tensors of  $E$  of degree  $n$ . We use the convention:  $E^{\otimes 0} = \mathbb{R}$ . The set of formal series of tensors of  $E$ , denoted by  $T((E))$  is defined by the following:

$$T((E)) := \{\mathbf{a} = (a_n)_{n \in \mathbb{N}} \mid \forall n \in \mathbb{N} : a_n \in E^{\otimes n}\}$$

$T((E))$  has an algebra structure when endowed with the operations defined by the following: for  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  and  $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$  in  $T((E))$  and  $\lambda \in \mathbb{R}$ :

**(Addition):**  $\mathbf{a} + \mathbf{b} = (a_n + b_n)_{n \in \mathbb{N}}$ ,

**(Multiplication):**  $\mathbf{a} \otimes \mathbf{b} = \left( \sum_{k=0}^n a_k \otimes b_{n-k} \right)_{n \in \mathbb{N}}$ ,

**(Multiplication by a scalar):**  $\lambda \mathbf{a} = (\lambda a_n)_{n \in \mathbb{N}}$ .

**Proposition 1.9.** Let  $E$  be a vector space.  $(T((E)), +, \cdot, \otimes)$  is a non-commutative (assuming  $\dim(E) \geq 2$ ) unital algebra with unit  $\mathbf{1} = (1, 0, 0, \dots)$ . An element  $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$  in  $T((E))$  is invertible if and only if  $a_0 \neq 0$ . In this case, its inverse is given by the well-defined series:

$$\mathbf{a}^{-1} = \frac{1}{a_0} \sum_{n \geq 0} \left( \mathbf{1} - \frac{\mathbf{a}}{a_0} \right)^n$$

**Definition 1.10.** Let  $E$  be a Banach space and  $T > 0$ . Let  $x : [0, T] \rightarrow E$  be a path of bounded variation. For  $(s, t) \in \Delta_{[0, T]}$ , we define the following sequence by induction (well-defined by Stieltjes' integration theory):

$$\begin{cases} S^0(x)_{(s, t)} = 1 \\ S^n(x)_{(s, t)} = \int_{[s, t]} S^{n-1}(x)_{(s, u)} \otimes dx_u, \forall n \in \mathbb{N}^* \end{cases}$$

For every pair  $(s, t) \in \Delta_{[0, T]}$ , the sequence  $(S^n(x)_{(s, t)})_{n \in \mathbb{N}}$ , simply denoted  $S(x)$ , is called the signature of  $x$  over  $[s, t]$ . For  $N \in \mathbb{N}^*$ ,  $(S^n(x)_{(s, t)})_{n \leq N}$ , simply denoted  $S_N(x)$ , is called the truncated signature of  $x$  over  $[s, t]$  of degree  $N$ .

The label “signature”, implicitly implying the full characterization of a path, can be justified at many levels:

- Lyons and Hamblly in [5] show that the signature of a path of bounded variation fully and uniquely characterizes the path in question up to a tree-like equivalence.
- In the context of differential equations, the signature of the control signal is the only needed information to get the solution. This can be shown for example easily and explicitly in the case where the vector fields in the differential equation are linear and continuous (details in [2] for example).

**Lemma 1.11.** *Let  $E$  be a vector space. Let  $m \in \mathbb{N}$  be an integer and define*

$$B_m = \{\mathbf{a} = (a_n)_{n \in \mathbb{N}} \mid \forall i \in \{0, \dots, m\} \quad a_i = 0\}$$

*Then  $B_m$  is an ideal of  $T((E))$ .*

**Definition 1.12.** *Let  $E$  be a vector space. Let  $m \geq 0$  be an integer. The truncated tensor algebra of order  $m$  of  $E$ , denoted by  $T^{(m)}(E)$ , is the quotient algebra  $T((E))/B_m$ . We will denote the canonical homomorphism  $T((E)) \rightarrow T((E))/B_m$  by  $\pi_m$ . There is a natural identification between  $T^{(n)}(E)$  and  $\bigoplus_{0 \leq i \leq n} E^{\otimes i}$ .*

**Definition 1.13** (Action of the Symmetric Group on Tensors). *Let  $n \in \mathbb{N}^*$ ,  $\sigma \in \mathcal{S}_n$  and  $E$  be a vector space. We define the action of  $\sigma$  on the homogenous tensors of  $E$  of order  $n$  as a linear map by the following:*

$$\forall x_1, x_2, \dots, x_n \in E \quad \sigma(x_1 \otimes x_2 \otimes \dots \otimes x_n) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \dots \otimes x_{\sigma(n)}$$

**Definition 1.14.** *Let  $E$  be a vector space. A norm  $\|\cdot\|$  on  $T((E))$  is said to be admissible if the two following properties hold:*

- (1)  $\forall n \in \mathbb{N}^*, \forall \sigma \in \mathcal{S}_n, \forall x \in E^{\otimes n} \quad \|\sigma x\| = \|x\|.$
- (2)  $\forall n, m \in \mathbb{N}^*, \forall x \in E^{\otimes n}, \forall y \in E^{\otimes m} \quad \|x \otimes y\| \leq \|x\| \|y\|.$

We will assume in the rest of this paper that  $(E^{\otimes n})_{n \geq 1}$  are endowed with admissible norms. A more detailed discussion on certain basic properties of norms on tensor product spaces can be found in [1].

**1.1.3. Algebraic and analytic properties of the signature.** We resume the notations of the previous subsection. Let  $S$  (respectively  $S_n$ , for  $n \in \mathbb{N}^*$ ) denote the map giving the signature (resp. the truncated signature of degree  $n$ ) over  $[0, T]$  of  $E$ -valued paths defined on  $[0, T]$  that have bounded variation. We give below three basic properties of signatures that will be key in defining rough paths in the next subsection.

For a vector space  $E$ , we identify linear forms on  $T((E))$  with the elements of the vector space  $T(E^*)$  (the tensor algebra of  $E^*$ ,  $E^*$  being the space of linear forms on  $E$ ). We start by what we will be calling the shuffle product property:

**Lemma 1.15.** *Let  $E$  be a Banach space. For every two linear forms  $e$  and  $f$  defined on  $T((E))$ , there exists a linear form denoted by  $e \sqcup f$  such that:*

$$\forall x \in \mathcal{V}^1([0, T], E) \quad e(S(x)) \cdot f(S(x)) = (e \sqcup f)(S(x))$$

*$e \sqcup f$  is called the shuffle product of  $e$  and  $f$ .*

**Definition 1.16.** *Let  $t \in [0, T]$  and  $x : [0, t] \rightarrow E$  and  $y : [t, T] \rightarrow E$  be any two paths. We call the concatenation of  $x$  and  $y$  the path denoted by  $x * y$  and defined on  $[0, T]$  by:*

$$\begin{cases} (x * y)_u = x_u & , \text{ if } u \in [0, t] \\ (x * y)_u = y_u - y_t + x_t & , \text{ if } u \in [t, T] \end{cases}$$

*We define in a similar way the concatenation of any two paths defined on two adjacent segments.*

**Remark 1.17.** *The concatenation operation is associative.*

The second property that is of interest to us is what we call the multiplicativity property. The following theorem is due to Chen ([4]). For a proof, see also [2]:

**Theorem 1.18.** *Let  $t \in [0, T]$ . Consider  $x \in \mathcal{V}^1([0, t], E)$  and  $y \in \mathcal{V}^1([t, T], E)$ . Then  $x * y \in \mathcal{V}^1([0, T], E)$  and:*

$$S(x * y)_{(0, T)} = S(x)_{(0, t)} \otimes S(y)_{(t, T)}$$

**Remark 1.19.** *Theorem 1.18 is usually used in the following way (which is equivalent to the previous formulation): for every path  $x \in \mathcal{V}^1([0, T], E)$  and  $s, u, t \in [0, T]$  such that  $s \leq u \leq t$ , we have:*

$$S(x)_{(s, t)} = S(x)_{(s, u)} \otimes S(x)_{(u, t)}$$

**Remark 1.20.** *For  $x \in \mathcal{V}^1([0, T], E)$  and for  $(s, t) \in \Delta_{[0, T]}$ ,  $S(x)_{(s, t)} \in \tilde{T}((E))$  and is therefore invertible. Consequently, using theorem 1.18:*

$$S(x)_{(s, t)} = (S(x)_{(0, s)})^{-1} \otimes S(x)_{(0, t)}.$$

*It is then equivalent to study the path  $u \mapsto S(x)_{(0, u)}$  on the interval  $[0, T]$  or  $(s, t) \mapsto S(x)_{(s, t)}$  on the simplex  $\Delta_{[0, T]}$ .*

Finally, the norms of successive terms in a signature of a path of bounded variation decay in a factorial way with respect to the 1-variation:

**Theorem 1.21.** *Let  $x$  be in  $\mathcal{V}^1([0, T], E)$ , then:*

$$\forall n \in \mathbb{N}^* \quad \forall (s, t) \in \Delta_{[0, T]} \quad \|S^n(x)_{(s, t)}\| \leq \frac{\|x\|_{1, [s, t]}^n}{n!}.$$

**1.2. Rough Paths.** The theory of rough paths generalizes the concept of signatures to more irregular paths and provides the tools to solving differential equations driven by these without having to build a whole new theory of integration for each one of them (as in Itô's calculus). The concept of rough paths finds its source in the signature and the main analytic and algebraic properties that it satisfies. The space of geometric rough paths is indeed simply defined as the completion of the set of signatures of paths with bounded variation under a suitably chosen metric similar to the  $p$ -variation metric for paths introduced in section 1.1.1. We introduce here the basic definitions and constructions.

### 1.2.1. Multiplicative functionals.

**Definition 1.22.** *Let  $E$  be a normed vector space and  $T > 0$ . Let  $X$  be a map on  $\Delta_{[0, T]}$  with values in  $T((E))$  (respectively in  $T^{(n)}(E)$ , with  $n \in \mathbb{N}^*$ ).  $X$  is said to be a multiplicative functional (resp. a multiplicative functional of degree  $n$ ) if the following holds:*

- (1)  $X$  is continuous.
- (2)  $\forall t \in [0, T] \quad X_{(t, t)} = \mathbf{1}$ .
- (3)  $X$  is multiplicative:  $\forall 0 \leq s \leq u \leq t \leq T \quad X_{(s, t)} = X_{(s, u)} \otimes X_{(u, t)}$ .

**Remark 1.23.** *When no confusion is possible, we may use the term multiplicative functional with no reference to its degree being finite or not.*

**Remark 1.24.** *We will use the notation  $X^i$  for the component of  $X$  of degree  $i$ .*

It is clear, by Chen's theorem 1.18, that, for every path  $x \in \mathcal{V}^1([0, T], E)$ ,  $S(x)$  is a multiplicative functional and that for every  $n \in \mathbb{N}^*$ ,  $S_n(X)$  is a multiplicative functional of degree  $n$ . Multiplicative functionals are a generalisation, then, of the Chen's multiplicativity property for signatures.

1.2.2.  $p$ -variation metric and controls.

**Definition 1.25.** Let  $E$  be a normed vector space and  $T > 0$ . Let  $p \geq 1$ . Let  $X$  be a map on  $\Delta_{[0,T]}$  with values in  $T((E))$  (respectively in  $T^{(n)}(E)$ , with  $n \in \mathbb{N}^*$ ) and  $\omega$  be a control over  $[0, T]$ .  $X$  is said to have a finite  $p$ -variation over  $[0, T]$  controlled by  $\omega$  if:

$$\forall i \in \mathbb{N}^* (\text{resp. } \forall i \in \llbracket 1, n \rrbracket) \quad \forall 0 \leq s \leq t \leq T \quad \|X_{(s,t)}^i\| \leq \frac{\omega(s, t)^{\frac{i}{p}}}{\beta_p \left(\frac{i}{p}\right)!}$$

where we write  $x!$  for  $\Gamma(x+1)$ , with  $\Gamma$  being the usual extension of the factorial function and:

$$\beta_p = p \left( 1 + \sum_{k=1}^{\infty} \left( \frac{2}{k} \right)^{\frac{[p]+1}{p}} \right)$$

If there exists a control such that the previous properties holds, we may say that  $X$  has a finite  $p$ -variation over  $[0, T]$  without mentionning the control. We denote by  $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$  the set of continuous paths defined from  $\Delta_{[0,T]}$  to  $T^{[p]}(E)$  that have finite  $p$ -variation.

We see, that for  $n = 1$ , the concept of  $p$ -variation in definition 1.25 is the same as the one introduced in subsection 1.1.1. By theorem 1.21, signatures have finite 1-variation. The  $p$ -variation control substitutes then the factorial decay property of signatures.

**Remark 1.26.** One can also easily notice that for  $1 \leq q \leq p$ , a multiplicative functional of finite  $q$ -variation is of finite  $p$ -variation.

The next result (the neo-classical inequality) will be of a technical use to us in a subsequent section and is crucial for the extension theorem for rough paths (see [6] for a complete proof or [9] for a loose version of the inequality).

**Lemma 1.27** (Neo-classical inequality).

$$\forall p \geq 1, \forall n \in \mathbb{N}, \forall a, b \in \mathbb{R}^+ \quad \frac{1}{p} \sum_{k=0}^n \frac{a^{\frac{k}{p}} b^{\frac{n-k}{p}}}{\left(\frac{k}{p}\right)! \left(\frac{n-k}{p}\right)!} \leq \frac{(a+b)^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}$$

The following lemma shows that a multiplicative functional of finite  $p$ -variation is determined by its terms of degree less than or equal to  $[p]$  (with  $p \in \mathbb{R}_+^*$ ). The extension theorem, which we state later, gives the reciprocal of this result: a multiplicative functional of degree  $[p]$  and of finite  $p$ -variation can be extended to a multiplicative functional (of an arbitrary degree) of finite  $p$ -variation.

**Lemma 1.28.** Let  $p \geq 1$  and  $n \in \mathbb{N}^*$  such that  $n \geq [p]$  and  $X$  and  $Y$  be two multiplicative functionals (resp. multiplicative functionals of degree  $n$ ) that have a finite  $p$ -variation over  $[0, T]$ . If  $\pi_{[p]}(X) = \pi_{[p]}(Y)$ , then  $X = Y$ .

**Theorem 1.29** (Extension theorem). Let  $p \geq 1$  and  $n \in \mathbb{N}^* \cup \{\infty\}$  such that  $n \geq [p]$ . Let  $E$  be a Banach space. Let  $X$  be a multiplicative functional of degree  $[p]$  in  $E$  that has a finite  $p$ -variation over  $[0, T]$  controlled by a control function  $\omega$ . There exists a unique multiplicative functional  $\tilde{X}$  of degree  $n$  that has a finite  $p$ -variation over  $[0, T]$  and such that  $\pi_{[p]}(X) = \pi_{[p]}(\tilde{X})$ . Furthermore, the  $p$ -variation of  $\tilde{X}$  is also controlled by  $\omega$ .

**Remark 1.30.** *Theorem 1.29 partially encompasses Young's integration theory (see [12]) in the sense that it allows the construction of the signature of a path of finite  $p$ -variation, when  $p < 2$ , using only the increments of the path.*

The signature of a path of bounded variation is, therefore, by theorem 1.29, the only multiplicative functional with finite 1-variation whose component of degree 1 corresponds to the increments of said path.

We are now ready to give a definition for rough paths:

**Definition 1.31.** *Let  $p \geq 1$  and  $E$  be a Banach space. A  $p$ -rough path is a multiplicative functional of degree  $[p]$  that has finite  $p$ -variation. The space of  $p$ -rough paths over  $[0, T]$  is denoted by  $\Omega_p^{[0,T]}(E)$ .*

**Definition 1.32.** *We define on  $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$  the  $p$ -variation metric, denoted by  $\tilde{d}_p$ , by the following:*

$$\tilde{d}_p(X, Y) = \max_{0 \leq i \leq [p]} \sup_{D \in \mathcal{D}_{[0,T]}} \left( \sum_D \|X_{(t_j, t_{j+1})}^i - Y_{(t_j, t_{j+1})}^i\|^{\frac{p}{i}} \right)^{\frac{1}{p}}$$

for all  $X, Y \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ . (For a subdivision  $D = (t_j)_{0 \leq j \leq n}$ :

$$\sum_D \|X_{(t_j, t_{j+1})}^i - Y_{(t_j, t_{j+1})}^i\|^{\frac{p}{i}} := \sum_{j=0}^{n-1} \|X_{(t_j, t_{j+1})}^i - Y_{(t_j, t_{j+1})}^i\|^{\frac{p}{i}})$$

**Definition 1.33.** *Let  $p \geq 1$  and  $E$  be a Banach space. Let  $X \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$  and  $(X(n))_{n \in \mathbb{N}}$  be a sequence of elements of  $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ . We say that  $(X(n))_{n \in \mathbb{N}}$  converges to  $X$  in the  $p$ -variation topology, if there exists a control function  $\omega$  and a sequence  $(a(n))_{n \in \mathbb{N}}$  converging to 0 such that:*

- (1)  $\omega$  controls the  $p$ -variation of  $X$  and of  $X(n)$  for each  $n \in \mathbb{N}$ .
- (2)  $\forall n \in \mathbb{N}, \forall (s, t) \in \Delta_{[0,T]}, \forall i \in \llbracket 1, [p] \rrbracket : \|X_{s,t}^i - X_{s,t}^i(n)\| \leq a(n)\omega(s, t)^{i/p}$ .

Convergence in the  $p$ -variation topology and the  $p$ -variation metric are “almost” equivalent in the following way:

**Proposition 1.34.** *Let  $p \geq 1$  and  $E$  be a Banach space. Let  $X \in \mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$  and  $(X(n))_{n \in \mathbb{N}}$  be a sequence of elements of  $\mathcal{C}_{0,p}(\Delta_{[0,T]}, T^{[p]}(E))$ .*

- *if  $(X(n))_{n \in \mathbb{N}}$  converges to  $X$  in the  $p$ -variation metric, then the convergence also occurs in the  $p$ -variation topology.*
- *if  $(X(n))_{n \in \mathbb{N}}$  converges to  $X$  in the  $p$ -variation topology, then there exists a subsequence of  $(X(n))_{n \in \mathbb{N}}$  converging to  $X$  in the  $p$ -variation metric.*

The completeness of the space of rough paths is an important property. See [9] for a complete proof:

**Theorem 1.35.**  $(\Omega_p^{[0,T]}(E), \tilde{d}_p)$  is a complete metric space.

### 1.2.3. Geometric $p$ -rough paths.

**Definition 1.36.** *We define the set of geometric  $p$ -rough paths to be the closure of the set  $\{S_{[p]}(x) | x \in \mathcal{V}^1([0, T], E)\}$  in the  $p$ -variation metric. It is denoted by  $G\Omega_p^{[0,T]}(E)$ .*

Rough paths do not take into account the shuffle product property satisfied by signatures. We show that geometric rough paths do not have this “setback”:

**Proposition 1.37.** *Let  $E$  be a Banach space and  $T > 0$ . Let  $X$  be a geometric  $p$ -rough path over  $[0, T]$  in  $E$ . Then  $X$  is an element of  $\Omega_p^{[0, T]}(E)$  (in particular,  $X$  is multiplicative and has finite  $p$ -variation) and for every  $(s, t) \in \Delta_{[0, T]}$ ,  $X_{(s, t)}$  satisfy the shuffle product property.*

*Proof.* As the signatures of paths of bounded variation satisfy these algebraic and analytic properties, this proposition is a direct consequence of theorem 1.35 and the continuity of linear forms of  $T^{[p]}(E)$  in the  $p$ -variation metric.  $\square$

**Remark 1.38.** *It is interesting to see the analogy between the restriction of  $\tilde{d}_p$  on the space of geometric rough paths and the  $p$ -variation norm for paths taking their values in the Carnot group  $(G^{[p]}, \otimes)$  of the elements in the tensor algebra of degree at most  $[p]$  when endowed with the homogeneous norm:  $\|\cdot\| : (1, g_1, \dots, g_{[p]}) \mapsto \max_{1 \leq i \leq [p]} \|i!g_i\|^{1/i}$  (see [10]).*

Like for simple paths, we can define the concatenation of paths taking their values in the truncated tensor algebra:

**Definition 1.39.** *Let  $n \in \mathbb{N}^*$ . Let  $E$  be a vector space. Let  $s, u, t \in \mathbb{R}$  such that  $s \leq u \leq t$ . Let  $X$  (resp.  $Y$ ) be a functional defined on  $\Delta_{[s, u]}$  (resp. on  $\Delta_{[u, t]}$ ) with values in  $T^n(E)$ . We define the concatenation of  $X$  and  $Y$ , denoted  $X * Y$ , to be the functional over  $\Delta_{[s, t]}$  defined as follows: for  $(a, b) \in \Delta_{[s, t]}$*

$$(X * Y)_{(a, b)} = \begin{cases} X_{(a, b)} & , \quad \text{if } b \leq u \\ X_{(a, u)} \otimes Y_{(a, u)} & , \quad \text{if } a \leq u \leq b \\ Y_{(a, b)} & , \quad \text{if } u \leq a \end{cases}$$

The following theorem is straight-forward:

**Theorem 1.40.** *Let  $p \geq 1$ . Let  $E$  be a Banach space. Let  $s, u, t \in \mathbb{R}$  such that  $s \leq u \leq t$ . Let  $X$  (resp.  $Y$ ) be a functional defined on  $\Delta_{[s, u]}$  (resp. on  $\Delta_{[u, t]}$ ) with values in  $T^{[p]}(E)$ . Then:*

- *If  $X$  and  $Y$  are multiplicative functionals, then  $X * Y$  is a multiplicative functional;*
- *If  $X$  and  $Y$  have finite  $p$ -variation, then  $X * Y$  has finite  $p$ -variation;*
- *If  $X$  and  $Y$  are geometric  $p$ -rough paths, then  $X * Y$  is a geometric  $p$ -rough path.*

1.2.4. *The integral of Lipschitz one-forms along geometric  $p$ -rough paths.* When trying to make sense of integrals of one-forms along rough paths, it is very important to be able to control the smoothness (in terms of variation) of the image of the path under the one-form. It appears that Lipschitz maps, in the sense of Stein [11], are the appropriate type of maps to use in this frame:

**Definition 1.41.** *Let  $n \in \mathbb{N}$  and  $0 < \varepsilon \leq 1$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . We will use, when there is no ambiguity, the same notation  $\|\cdot\|$  to designate norms on  $E^{\otimes k}$ , for  $k \in \llbracket 1, n \rrbracket$ , and the norm on  $F$ . For every  $k \in \llbracket 0, n \rrbracket$ , let  $f^k : U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  be a map with values in the space of the symmetric  $k$ -linear mappings from  $E$  to  $F$ . The collection  $f = (f^0, f^1, \dots, f^n)$  is said to be Lipschitz of degree  $n + \varepsilon$  on  $U$  (or in short a  $\text{Lip} - (n + \varepsilon)$  map) if there exist a constant  $M$  and  $n + 1$  maps  $R_k : E \times E \rightarrow \mathcal{L}(E^{\otimes k}, F)$ ,  $k \in \llbracket 0, n \rrbracket$  (called the associated remainders) such that for all  $k \in \llbracket 0, n \rrbracket$ :*

$$\sup_{x \in U} \|f^k(x)\| \leq M$$



$$\forall x, y \in U, \forall v \in E^{\otimes k} : f^k(x)(v) = \sum_{j=k}^n f^j(y) \left( \frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!} \right) + R_k(x, y)(v),$$

$$\forall x, y \in U : \|R_k(x, y)\| \leq M \|x - y\|^{n+\varepsilon-k}.$$

The smallest constant  $M$  for which the properties above hold is called the Lipschitz- $(n + \varepsilon)$  norm of  $f$  and is denoted by  $\|f\|_{\text{Lip}-(n+\varepsilon)}$ .

**Remark 1.42.** On any open subset of  $U$  (and in particular on the interior of  $U$ ),  $f^1, \dots, f^n$  are the successive derivatives of  $f^0$ . However, these maps are not necessarily uniquely determined by  $f^0$  on an arbitrary subset of  $U$ .

If  $f^0 : U \rightarrow F$  is a map such that there exist  $f^1, \dots, f^n$  such that  $(f^0, f^1, \dots, f^n)$  is  $\text{Lip} - (n + \varepsilon)$ , we will often say that  $f^0$  is  $\text{Lip} - (n + \varepsilon)$  with no mention of  $f^1, \dots, f^n$ . We will call the functions  $R_0, R_1, \dots, R_n$  its associated remainders.

It will be useful to us, and contrary to custom in most studies of rough paths on Banach spaces, to attach a starting point to our geometric rough paths as we will be mostly dealing with integrals of rough paths and rough paths on manifolds; both of which require assigning a starting point to a rough path. In the next few sections, a geometric rough path will be a pair  $(x, X)$ , where  $x$  is called the starting point and  $X$  is a geometric rough path in the sense of definition 1.36 (a third and final convention on the definition will be introduced in the subsection 5.1). We will denote the “new space” of geometric rough paths with starting points in the same way as above, and when there is ambiguity, we will introduce an equivalence relation  $\sim$  on  $G\Omega_p^{[0,T]}(E)$  that makes two rough paths with the same increments equivalent, i.e.:

$$(x, X) \sim (y, Y) \Leftrightarrow X = Y$$

On  $G\Omega_p^{[0,T]}(E)$  we define a metric  $d_p$  as the product metric of  $\tilde{d}_p$  and the norm on  $E$ :

$$d_p((x, X), (y, Y)) = \max(\|x - y\|, \tilde{d}_p(X, Y))$$

**Theorem 1.43.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $E$  and  $F$  be two Banach spaces. Let  $\alpha : E \rightarrow \mathcal{L}_c(E, F)$  be a  $\text{Lip} - (\gamma - 1)$  one-form. There exists a unique continuous map:

$$I_\alpha : (G\Omega_p^{[0,T]}(E), d_p) \longrightarrow (G\Omega_p^{[0,T]}(F)/\sim, \tilde{d}_p)$$

such that, for all  $x \in \mathcal{V}^1([0, T], E)$ ,  $I_\alpha(x_0, S_{[p]}(x)) = S_{[p]}(\int \alpha(x) dx)$ . For a geometric  $p$ -rough path with a starting point  $(x_0, X)$ , we denote:

$$I_\alpha(x_0, X) = \int \alpha(x_0, X) dX$$

## 2. THE LIPSCHITZ GEOMETRY

This is a review of the most important findings of [3]:

**2.1. Lipschitz structures.** We first introduce the general definitions of Lipschitz manifolds and Lipschitz maps and one-forms on them.

**Definition 2.1.** Let  $\gamma \geq 1$ . Let  $n \in \mathbb{N}^*$  and let  $M$  be a  $n$ -topological manifold. Let  $I$  be a countable set and, for every  $i \in I$ ,  $U_i$  be an open subset of  $M$  and  $\phi_i : M \rightarrow \mathbb{R}^n$  be a map such that its restriction on  $U_i$  defines a homeomorphism. We say that  $((\phi_i, U_i))_{i \in I}$  is a Lipschitz- $\gamma$  atlas if the following properties are satisfied:

- $(U_i)_{i \in I}$  is a pre-compact locally finite cover of  $M$ ;
- There exists  $R > 0$  such that, for every  $i \in I$ :  $\phi_i(U_i) = B(0, 1)$  and  $\phi_i(M)$  is a compact subset contained in  $B(0, R)$ ;
- There exists  $\delta \in (0, 1)$ , such that  $(U_i^\delta)_{i \in I}$  covers  $M$ , where, for every  $i \in I$ :  $U_i^\delta = \phi_i^{-1}|_{U_i}(B(0, 1 - \delta))$ ;
- There exists  $L > 0$ , such that, for every  $i, j \in I$ ,  $\phi_j \circ (\phi_i|_{U_i})^{-1} : B(0, 1) \rightarrow \mathbb{R}^n$  is Lipschitz- $\gamma$  and  $\|\phi_j \circ (\phi_i|_{U_i})^{-1}\|_{Lip-\gamma} \leq L$ .

With the constants above, we will say that  $M$  is a Lipschitz- $\gamma$  manifold with constants  $(R, \delta, L)$ .

**Example 2.2.** As one would expect, finite-dimensional vector spaces can be endowed with a Lipschitz- $\gamma$  manifold structure of any degree  $\gamma \geq 1$ .

Indeed, let  $\gamma \geq 1$ . Let  $V$  be a finite dimensional space and let  $(e_1, \dots, e_n)$  be a basis for  $V$ . Let  $\varphi$  be a Lip- $\gamma$  extension on  $V$  of  $Id_{B(0,1)}$  with support in  $B(0, 2)$ . For  $x \in V$ , let  $\varphi_x$  be the map  $(y \mapsto \varphi(y - x))$ . Then  $(\varphi_x, B(x, 1))_{x \in I}$  is a Lip- $\gamma$  atlas on  $V$ , where:

$$I = \left\{ \sum_{i=1}^n \frac{k_i}{2} e_i \mid k_1, \dots, k_n \in \mathbb{Z} \right\}$$

**Example 2.3.** For  $\gamma \geq 1$ , compact  $\mathcal{C}^{[\gamma]+1}$ -manifolds are Lip- $\gamma$  manifolds (see [3]).

**Definition 2.4.** Let  $\gamma_0, \gamma \in \mathbb{R}$  such that  $\gamma_0 \geq \gamma$ . Let  $M$  be a Lip- $\gamma_0$  manifold with an atlas  $\{(\phi_i, U_i), i \in I\}$  and  $E$  be a normed vector space. A map  $f : M \rightarrow E$  is said to be Lip- $\gamma$  if there exists a constant  $C$  such that, for every  $i \in I$ ,  $f \circ \phi_i|_{U_i}^{-1} : B(0, 1) \rightarrow E$  is Lip- $\gamma$  with a Lipschitz norm at most  $C$ . The smallest constant  $C$  for which this property holds is called the Lip- $\gamma$  norm of  $f$  and is denoted by  $\|f\|_{Lip-\gamma}$ .

**Definition 2.5.** Let  $\gamma_0, \gamma \in \mathbb{R}$  such that  $\gamma_0 \geq \gamma$  and  $d \in \mathbb{N}$ . Let  $M$  be a Lip- $\gamma_0$  manifold with an atlas  $\{(\phi_i, U_i), i \in I\}$  and  $E$  be a normed vector space. An  $E$ -valued one-form  $\alpha$  on  $M$  is said to be Lip- $\gamma$  if there exists a constant  $C$  such that, for every  $i \in I$ :

$$(\phi_i|_{U_i}^{-1})^* \alpha : B(0, 1) \rightarrow \mathcal{L}(\mathbb{R}^d, E)$$

is Lip- $\gamma$  with a Lipschitz norm at most  $C$ . The smallest constant  $C$  for which this property holds is called the Lip- $\gamma$  norm of  $\alpha$  and is denoted by  $\|\alpha\|_{Lip-\gamma}$ .

**2.2. Rough paths on a manifold.** In this subsection, we generalize the notion of rough paths to manifolds. As we don't have a natural notion of linearity and iterated integrals on a manifold, we have to consider a different approach than the one arising from the  $p$ -variational properties of paths and signatures. As we will hint to later, integrals of Lipschitz one-forms along rough paths do characterize the path; this is the first direction that we will take to define our rough paths. Additionally, in the absence of a natural translation, a rough path on a manifold comes attached with a starting point.

**Definition 2.6.** Let  $\gamma_0, p \in \mathbb{R}$  such that  $\gamma_0 > p \geq 1$  and  $T \geq 0$ . Let  $M$  be a Lip- $\gamma_0$  manifold and  $x \in M$ .  $X$  is a geometric  $p$ -rough path over  $[0, T]$  on  $M$  starting at  $x$ , if for every  $\gamma \in \mathbb{R}$  such that  $\gamma_0 \geq \gamma > p$  and every Banach space  $E$ , the following conditions are satisfied:

- (1)  $X$  maps Lip- $(\gamma - 1)$   $E$ -valued one-forms on  $M$  to  $E$ -valued geometric  $p$ -rough paths (in the classical sense).

- (2) For every Banach space  $F$  and every compactly supported Lip- $\gamma$  map  $\psi : M \rightarrow F$  and every  $E$ -valued Lip- $(\gamma - 1)$  one-form  $\alpha$  on  $F$  we have:

$$X(\psi^*\alpha) = \int \alpha(\psi(x), X(\psi_*))dX(\psi_*)$$

- (3) There exists a control  $\omega$  such that for every  $E$ -valued Lip- $(\gamma - 1)$  one-form  $\alpha$  on  $M$ ,  $X(\alpha)$  is controlled by  $\|\alpha\|_{Lip-(\gamma-1)}\omega$ , i.e.:

$$\forall (s, t) \in \Delta_{[0, T]}, \forall i \in \llbracket 1, [p] \rrbracket : \quad \|X(\alpha)_{(s, t)}^i\| \leq \frac{(\|\alpha\|_{Lip-(\gamma-1)}\omega(s, t))^{i/p}}{\beta_p(i/p)!}$$

Contrary to the classical case, in the context of manifolds, we do not need to make a difference between rough paths and geometric rough paths (as only the latter are defined). Consequently, we drop the word “geometric” when talking about geometric rough paths on manifolds. In the classical sense, geometric rough paths are determined by the values of the integral of compactly supported one-forms along them. In order to make the correspondance one-to-one between the concept of a classical geometric rough path on a finite-dimensional space and a rough path on the same space when endowed with its Lipschitz structure, we define the following equivalence relation:

**Definition 2.7.** Let  $\gamma_0, p \in \mathbb{R}$  such that  $\gamma_0 > p \geq 1$ . Let  $M$  be a Lip- $\gamma_0$  manifold. We say that two  $p$ -rough paths  $X$  and  $\tilde{X}$  on  $M$  are equivalent, and we write  $X \sim \tilde{X}$ , if they have the same starting point and if, for every  $\gamma \in \mathbb{R}$  such that  $\gamma_0 \geq \gamma > p$  and for every Banach space valued one-form  $\alpha$  on  $M$  that is compactly supported and Lip- $(\gamma - 1)$ , we have  $X(\alpha) = \tilde{X}(\alpha)$ .

We give now a hint on how to build a one-to-one correspondance between rough paths in the classical sense and in a Lip- $\gamma$  manifold, when said manifold is a finite-dimensional vector space  $V$ :

- Given a geometric  $p$ -rough path  $(x, X)$  on  $V$ , we define the rough path  $Z$  in the manifold to be the functional sending every Banach space-valued Lip- $(\gamma - 1)$  compactly supported one-form  $\alpha$  to the classical rough path:

$$Z(\alpha) = \int (x, X)d\alpha$$

- Conversely, given a rough path  $Z$  on  $V$  in the manifold sense, we are tempted to retrieve a rough path in the classical sense by integrating  $Z$  against  $dId_V$ . Since  $dId_V$  is not compactly supported and that  $Id_V$  is not Lip- $\gamma$  (which is key in the consistency condition), we can intuitively replace  $Id_V$  with a compactly supported Lip- $\gamma$  extension of its restriction on a set containing the “support” of  $Z$ . The notion of support is not yet defined at this point but we can still have a good guess at a set containing it by using the control of the  $p$ -variation of  $Z$ .

We can define the pushforward of rough paths by conveniently chosen Lipschitz maps:

**Lemma 2.8.** Let  $\gamma_0, p \in \mathbb{R}$  such that  $\gamma_0 > p \geq 1$ . Let  $M$  and  $N$  be Lip- $\gamma_0$  manifolds and  $f : M \rightarrow N$  be a Lip- $\gamma_0$  map such that there exists a constant  $C_f$  such that, for all  $\gamma \in (p, \gamma_0]$  and every Lip- $(\gamma - 1)$  Banach space valued one-form  $\alpha$  on  $M$ , we have:

$$\|f^*\alpha\|_{Lip-(\gamma-1)} \leq C_f \|\alpha\|_{Lip-(\gamma-1)}$$

Then  $f$  induces a pushforward  $f_*$  from  $p$ -rough paths on  $M$  to  $p$  rough-paths on  $N$  defined as follows: for every  $p$ -rough path  $X$  over  $[0, T]$  on  $M$  starting at  $x$ ,  $f_*X$  starts at  $f(x)$  and for every  $\text{Lip} - (\gamma - 1)$  Banach space valued one form  $\alpha$  on  $M$ , where  $\gamma \in (p, \gamma_0]$ ,  $f_*X(\alpha)$  is given by:

$$f_*X(\alpha) = X(f^*\alpha)$$

There exists a particular class of Lipschitz maps that induce pushforwards of rough paths<sup>1</sup>:

**Proposition 2.9.** *Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold and  $W$  be a Banach space. Let  $f : M \rightarrow W$  be a  $\text{Lip} - \gamma$  map and  $\alpha$  be a  $\text{Lip} - (\gamma - 1)$  Banach space valued one-form on  $W$ . Then  $f^*\alpha$  is  $\text{Lip} - (\gamma - 1)$  and there exists a constant  $C_\gamma$  depending only on  $\gamma$  such that:*

$$\|f^*\alpha\|_{\text{Lip}-(\gamma-1)} \leq C_\gamma \|\alpha\|_{\text{Lip}-(\gamma-1)} \|f\|_{\text{Lip}-\gamma} \max(\|f\|_{\text{Lip}-\gamma}^\gamma, 1)$$

Like in the classical case, we can define the concatenation of two rough paths:

**Definition 2.10.** *Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold. Let  $s \leq u \leq t$ . Let  $X$  (respectively  $Y$ ) be a  $p$ -rough path over  $[s, u]$  (resp.  $[u, t]$ ) on  $M$  with starting point  $x$  (resp.  $y$ ). We define the concatenation of  $X$  and  $Y$ , denoted by  $X * Y$ , to be the functional  $Z$  over  $[s, t]$  mapping every Banach space-valued  $\text{Lip} - (\gamma - 1)$  compactly supported one-form  $\alpha$  to the classical rough path  $X(\alpha) * Y(\alpha)$ .*

Unlike the classical case, the concatenation of two rough paths on a manifold is not necessarily a rough path. This is due to the fact that rough paths on a manifold come attached with a starting point and that we have no natural notion of translation. Therefore, for this concatenation to be a rough path, we have to make sure that the two rough paths in question have starting and “ending” points that agree in the following sense:

**Definition 2.11.** *Let  $\gamma, p, s, u, t \in \mathbb{R}$  such that  $\gamma > p \geq 1$  and  $s \leq u \leq t$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold. Let  $X$  (respectively  $Y$ ) be a  $p$ -rough path on  $M$  over  $[s, u]$  (resp.  $[u, t]$ ) with starting point  $x$  (resp.  $y$ ). We say that  $X$  has an end point consistent with the starting point  $y$  of  $Y$  if for every Banach valued compactly supported  $\text{Lip} - \gamma$  map  $f$  on  $M$ , we have:*

$$f(x) + X(f_*)_{s,u}^1 = f(y)$$

In this case, we can check the consistency condition for the concatenation of two rough paths and prove that it is also a rough path:

**Proposition 2.12.** *Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold. Let  $X$  (respectively  $Y$ ) be a  $p$ -rough path on  $M$  with starting point  $x$  (resp.  $y$ ). We assume that  $X$  has an end point consistent with the starting point of  $Y$ . Then  $X * Y$  is a  $p$ -rough path.*

The concatenation of rough paths is associative in the following case:

**Lemma 2.13.** *Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold. Let  $X, Y$  and  $Z$  be  $p$ -rough paths on  $M$  such that:*

- $X$  has an end point consistent with the starting point of  $Y$ ;
- $Y$  has an end point consistent with the starting point of  $Z$ .

Then:

- $X$  has an end point consistent with the starting point of  $Y * Z$ ;

---

<sup>1</sup>We correct the exponent in the inequality compared to the result that appeared in [3]

- $X * Y$  has an end point consistent with the starting point of  $Z$ ;
- $X * (Y * Z) = (X * Y) * Z$

In that case, we simply denote  $X * Y * Z := X * (Y * Z)$ .

Since our notion of rough paths does not attach, for the moment, an underlying path on the manifold to a rough path, we define a notion of support based on the images by rough paths of one-forms supported in all possible open sets:

**Definition 2.14.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold and  $U$  be an open subset of  $M$ . Let  $X$  be a  $p$ -rough path on  $M$ . We say that  $X$  misses  $U$  if for every Banach space-valued  $\text{Lip} - (\gamma - 1)$  compactly one-form  $\alpha$  on  $M$ , we have:

$$\text{supp}(\alpha) \subseteq U \Rightarrow X(\alpha) = 0$$

**Definition 2.15.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold and  $X$  be a  $p$ -rough path on  $M$  with starting point  $x$ . We call the support of  $X$  the closed subset of  $M$  defined by:

$$\text{supp}(X) = \{x\} \cup \left( M - \bigcup_{X \text{ misses } U} U \right)$$

Naturally, this notion of support is consistent with the definition of support for a classical rough path:

**Proposition 2.16.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$  and  $T > 0$ . Let  $V$  be a finite dimensional vector space endowed with its canonical structure of a  $\text{Lip} - \gamma$  manifold. Let  $(x, X)$  be a geometric  $p$ -rough path on  $V$  over  $[0, T]$  (in the classical sense), and let  $Z$  be the  $p$ -rough path associated to it in the manifold sense, i.e. for every Banach space-valued  $\text{Lip} - (\gamma - 1)$  compactly one-form  $\alpha$  on  $V$ :  $Z(\alpha) = \int \alpha(x, X) dX$ . Then:

$$\text{supp}(Z) = \{x + X_{0,T}^1 \mid t \in [0, T]\}$$

**Theorem 2.17.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold and  $X$  be a  $p$ -rough path on  $M$ . Then the support of  $X$  is compact.

A rough path on a manifold can be “localised” into rough paths that have their support contained in the domain of one chart at a time:

**Theorem 2.18.** Let  $\gamma, p \in \mathbb{R}$  such that  $\gamma > p \geq 1$  and  $T > 0$ . Let  $M$  be a  $\text{Lip} - \gamma$  manifold and  $X$  be a  $p$ -rough path over  $[0, T]$  on  $M$  with starting point  $x$ . Let  $\omega$  be a control of the  $p$ -variation of  $X$ . There exists  $N \in \mathbb{N}^*$  depending only on  $\omega(0, T)$  and a collection  $(x_i, X^i, s_i, (\phi_i, U_i))_{1 \leq i \leq N}$  such that:

- (1)  $0 \leq s_1 \leq \dots \leq s_N = T$ ;
- (2) For all  $i \in \llbracket 1, N \rrbracket$ ,  $(\phi_i, U_i)$  is a  $\text{Lip} - \gamma$  chart on  $M$ ;
- (3) For all  $i \in \llbracket 1, N \rrbracket$ ,  $X^i$  is a  $p$ -rough path over  $[s_{i-1}, s_i]$  on  $M$  (with  $s_0 = 0$ ) with starting point  $x_i$ ;
- (4) For all  $i \in \llbracket 1, N - 1 \rrbracket$ ,  $X^i$  has an end point consistent with the starting point of  $X^{i+1}$ ;
- (5) For all  $i \in \llbracket 1, N \rrbracket$ ,  $\text{supp}(X^i) \subseteq U_i$ ;
- (6)  $X = X_1 * \dots * X_N$ .

A sequence with such properties is called a localising sequence for  $X$ .

## 3. INTERVALS

We present here some basic topological properties of covers of intervals which will be of use when studying rough paths locally.

**Definition 3.1.** A collection of subsets  $(K_i)_{i \in I}$  of a topological space  $J$  is said to be *locally finite* if for each  $x \in J$  there exists a neighbourhood  $U_x$  of  $x$  such that at most finitely many  $K_i$ 's have non-empty intersection with  $U_x$ .

**Definition 3.2.** A collection  $(K_i)_{i \in I}$  of subsets of a topological space  $J$  is said to *cover*  $J$  (or is a *cover* for  $J$ ) if  $J = \cup_{i \in I} K_i$ .

**Definition 3.3.** A collection  $(K_i)_{i \in I}$  of compact subsets of a topological space  $J$  that is locally finite and covers  $J$  is called a *compact cover* for  $J$ .

The proof of the following lemma is straightforward:

**Lemma 3.4.** Each interval  $J$  of  $\mathbb{R}$  admits a compact cover.

**Lemma 3.5.** Let  $J$  be an interval and  $(K_i)_{i \in I}$  a compact cover for  $J$ . Then  $I$  is countable. Moreover, if  $J$  is compact, then  $I$  is finite.

*Proof.* • Let  $a, b \in \mathbb{R}$  such that  $a \leq b$  and let  $(K_i)_{i \in I}$  be a locally finite collection of compact intervals covering  $[a, b]$  and all intersecting  $[a, b]$  (note that this is a weaker assumption than  $(K_i)_{i \in I}$  being a compact cover for  $[a, b]$ ). Assume that  $I$  is infinite. Then, it is an easy exercise to construct two sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  such that:

$$\begin{cases} a_0 = a, b_0 = b \text{ and } \forall n \in \mathbb{N} : a_n \leq b_n; \\ (a_n)_{n \in \mathbb{N}} \text{ is non-decreasing and } (b_n)_{n \in \mathbb{N}} \text{ is non-increasing;} \\ \forall n \in \mathbb{N} : b_n - a_n = \frac{(b-a)}{2^n}; \\ \forall n \in \mathbb{N} : [a_n, b_n] \text{ intersects infinitely many } K_i\text{'s} \end{cases}$$

The two sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are adjacent, converge and have a common limit  $c \in [a, b]$ . Then any neighborhood of  $c$  intersects infinitely many  $K_i$ 's, which contradicts the fact that  $(K_i)_{i \in I}$  is locally finite. Therefore,  $I$  is finite.  $(K_i)_{i \in I}$  being a compact cover of  $[a, b]$  is then just a special case of the one discussed precendently.

• Let now  $J$  be an arbitrary interval and assume that  $(K_i)_{i \in I}$  is a compact cover for  $J$ . Then we can construct two sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  such that  $(a_n)_{n \in \mathbb{N}}$  is non-increasing and  $(b_n)_{n \in \mathbb{N}}$  is non-decreasing,  $a_n \leq b_n$  for all  $n \in \mathbb{N}$  and  $J = \cup_{n \in \mathbb{N}} [a_n, b_n]$ . For every  $n \in \mathbb{N}$ , define:

$$I_n = \{i \in I \mid K_i \cap [a_n, b_n] \neq \emptyset\}$$

By the discussion above,  $I_n$  is finite for every  $n \in \mathbb{N}$ . Moreover  $I = \cup_{n \in \mathbb{N}} I_n$ . Therefore  $I$  is countable. □

**Lemma 3.6.** If  $(K_i)_{i \in I}$  is a compact cover for an interval  $J$ , and if for each  $i \in I$ ,  $(K_{i_j})_{j \in I_i}$  is a compact cover for  $K_i$  then  $\{K_{i_j} \mid j \in I_i, i \in I\}$  is also a compact cover for  $J$ .

*Proof.* First note that:

$$\begin{aligned} \cup_{i \in I, j \in I_j} K_{i_j} &= \cup_{i \in I} (\cup_{j \in I_j} K_{i_j}) \\ &= \cup_{i \in I} K_i \\ &= J \end{aligned}$$

The  $K_{i_j}$ 's are all compact subsets of  $J$ .

Let  $x \in J$ . Let  $\mathcal{V}_{x,J}$  be a neighborhood of  $x$  in  $J$  that intersects finitely many  $K_i$ 's. As  $I_i$  is finite for every  $i \in I$  (lemma 3.5), then  $\mathcal{V}_{x,J}$  intersects finitely many  $K_{i_j}$ 's.  $\square$

**Definition 3.7.** Let  $(U_i)_{i \in I}$  and  $(V_j)_{j \in J}$  be two collections of sets. We say that  $(V_j)_{j \in J}$  is a refinement of  $(U_i)_{i \in I}$  if, for every  $j \in J$ , there exists  $i \in I$  such that  $V_j \subseteq U_i$ .

**Lemma 3.8.** Given any cover of an interval  $J$  by open sets  $(O_i)_{i \in I}$  there exists a compact cover  $\mathcal{K}$  for  $J$  that is a refinement of  $(O_i)_{i \in I}$ .

*Proof.* Let  $(K_h)_{h \in H}$  be a compact cover of  $J$ . Let  $h \in H$  and  $x \in K_h$ . As  $x \in J$ , then there exists  $i \in I$  and  $\alpha_x > 0$  such that  $(x - \alpha_x, x + \alpha_x) \subseteq O_i$ . Denote  $I_x = [x - \alpha_x/2, x + \alpha_x/2] \cap K_h$ . Then  $I_x$  is a compact subset of  $K_h$  that is contained in  $O_i$ . As  $K_h$  is compact and  $((x - \alpha_x/2, x + \alpha_x/2))_{x \in K_h}$  covers  $K_h$ , then there exists a finite subset  $P \subseteq K_h$  such that  $((x - \alpha_x/2, x + \alpha_x/2))_{x \in P}$  covers  $K_h$ . And therefore  $(I_x)_{x \in P}$  is a compact cover of  $K_h$ . We conclude using lemma 3.6.  $\square$

**Lemma 3.9.** Let  $(K_i)_{i \in I}$  be a compact cover for a compact interval  $[a, b]$ . Then there exists a (finite) subdivision  $(a_j)_{0 \leq j \leq n}$  of  $[a, b]$  such that:

$$\forall j \in \llbracket 0, n-1 \rrbracket, \exists i \in I \text{ such that: } [a_j, a_{j+1}] \subseteq K_i$$

*Proof.* It can be done by induction on the number of elements in  $I$  for example.  $\square$

#### 4. LOCALLY LIPSCHITZ MAPS

**Definition 4.1.** Let  $\gamma \in (0, +\infty[$ . Let  $E$  and  $F$  be two normed vector spaces,  $U$  be a subset of  $E$  and  $f : U \rightarrow F$  be a map. We say that  $f$  is locally Lipschitz- $\gamma$  if, for every  $x \in U$ , there exists a neighborhood  $\mathcal{V}_{x,U}$  of  $x$  in  $U$  such that  $f|_{\mathcal{V}_{x,U}}$  is Lipschitz- $\gamma$ .

We recall here two important results on Lipschitz maps. They can be found for example in [1] and [3]:

**Theorem 4.2.** Let  $E, F$  and  $G$  be three normed vector spaces. Let  $U$  be a subset of  $E$  and  $V$  be a subset of  $F$ . Let  $\gamma > 0$ . We assume that  $(E^{\otimes k})_{k \geq 1}$  and  $(F^{\otimes k})_{k \geq 1}$  are endowed with norms satisfying the projective property. Let  $f : U \rightarrow F$  and  $g : V \rightarrow G$  be two Lip- $\gamma$  maps such that  $f(U) \subseteq V$ . Then  $g \circ f$  is Lip- $\gamma$  and, if  $\gamma \geq 1$ , there exists a constant  $C_\gamma$  (depending only on  $\gamma$ ) such that:

$$\|g \circ f\|_{\text{Lip}-\gamma} \leq C_\gamma \|g\|_{\text{Lip}-\gamma} \max(\|f\|_{\text{Lip}-\gamma}^\gamma, 1)$$

**Lemma 4.3.** Let  $n \in \mathbb{N}$ ,  $0 < \varepsilon \leq 1$  and  $C \geq 0$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . Let  $f : U \rightarrow F$  be a map and for every  $k \in \llbracket 1, n \rrbracket$ , let  $f^k : U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  be a map with values in the space of the symmetric  $k$ -linear mappings from  $E$  to  $F$ . We consider the two following assertions:

(A1):  $(f, f^1, \dots, f^n)$  is Lip- $(n + \varepsilon)$  and  $\|f\|_{\text{Lip}-(n+\varepsilon)} \leq C$ .

(A2):  $f$  is  $n$  times differentiable, with  $f^1, \dots, f^n$  being its successive derivatives.  $\|f\|_\infty, \|f^1\|_\infty, \dots, \|f^n\|_\infty$  are upper-bounded by  $C$  and for all  $x, y \in U$ :  $\|f^n(x) - f^n(y)\| \leq C\|x - y\|^\varepsilon$ .

If  $U$  is open then (A1)  $\Rightarrow$  (A2). If, furthermore,  $U$  is convex then (A1)  $\Leftrightarrow$  (A2).

**Lemma 4.4.** Let  $\gamma \in (1, +\infty[$ . Let  $E$  and  $F$  be two normed vector spaces,  $U$  be an open subset of  $E$  and  $f : U \rightarrow F$  be a differentiable map. Then its derivative  $df$  is locally Lipschitz- $(\gamma - 1)$  if and only if  $f$  is locally Lipschitz- $\gamma$ .

*Proof.* Notice that, on every ball  $B(x, \alpha) \subseteq U$  on which  $df$  is locally Lipschitz- $(\gamma - 1)$ , one can use the fundamental theorem of calculus to bound  $f$  and we deduce that the restriction of  $f$  on that set is Lipschitz- $\gamma$  using lemma 4.3. The converse is obvious.  $\square$

Local Lipschitzness is conserved under composition:

**Lemma 4.5.** *Let  $\gamma \in [1, +\infty[$ . Let  $E, F$  and  $G$  be normed vector spaces,  $U$  be a subset of  $E$  and  $V$  be a subset of  $F$ . Let  $f : U \rightarrow F$  and  $g : V \rightarrow G$  be two locally Lipschitz- $\gamma$  maps such that  $f(U) \subseteq V$ . Then  $g \circ f$  is locally Lipschitz- $\gamma$ .*

*Proof.* Direct consequence of theorem 4.2.  $\square$

Local Lipschitz paths conserve the smoothness, in the sense of variation, of paths:

**Theorem 4.6.** *Let  $p, \gamma \in [1, +\infty[$ . Let  $E$  and  $F$  be two normed vector spaces,  $U$  be a subset of  $E$  and  $J$  a compact interval. Let  $f : U \rightarrow F$  be a Lip- $\gamma$  map over  $U$  and  $x : J \rightarrow U$  a path with finite  $p$ -variation. Then  $f \circ x : J \rightarrow F$  is of finite  $p$ -variation.*

*Proof.* Let  $\omega$  be a control of the  $p$ -variation of  $x$  over  $J$ . Let  $t \in J$ . Let  $L_t$  be an open neighborhood of  $x_t$  in  $U$  such that  $f|_{L_t}$  is Lipschitz- $\gamma$ ; denote its Lip-1 norm by  $M_t$ . Let  $\mathcal{V}_{t,J}$  an open neighborhood of  $t$  in  $J$  such that  $x_u \in L_t$ , for all  $u \in \mathcal{V}_{t,J}$ . Let  $(K_j)_{j \in I}$  be a compact cover of  $J$  that is a refinement of  $(\mathcal{V}_{t,J})_{t \in J}$  (lemma 3.8). For every  $j \in I$ , let  $t_j \in J$  such that  $K_j \subseteq \mathcal{V}_{t_j,J}$ . Finally, let  $(a_i)_{0 \leq i \leq n}$  be a subdivision of  $J$  such that for all  $i \in \llbracket 0, n \rrbracket$ , there exists  $j_i \in I$  such that  $[a_i, a_{i+1}] \subseteq K_{j_i}$ . Denote  $M = \sup_{i \in \llbracket 0, n-1 \rrbracket} M_{t_{j_i}}$ . Let  $s, u \in J$ . Let  $q, r \in \llbracket 0, n \rrbracket$  such that  $a_q \leq s \leq \dots \leq u \leq a_r$ . Then we have, from the fact that  $f$  is Lip-1 on each of the  $L_{t_{j_i}}$  with a norm less than  $M_{t_{j_i}}$ , for  $i \in \llbracket 0, n-1 \rrbracket$ :

$$\begin{aligned} \|f(x_s) - f(x_u)\| &\leq \|f(x_s) - f(x_{a_{q+1}})\| + \sum_{k=q+1}^{r-2} \|f(x_{a_k}) - f(x_{a_{k+1}})\| + \\ &\quad \|f(x_{a_{r-1}}) - f(x_u)\| \\ &\leq M_{t_{j_q}} \|x_s - x_{a_{q+1}}\| + \sum_{k=q+1}^{r-2} M_{t_{j_k}} \|x_{a_k} - x_{a_{k+1}}\| + \\ &\quad M_{t_{j_{r-1}}} \|x_{a_{r-1}} - x_u\| \\ &\leq M(\omega(s, a_{q+1})^{1/p} + \sum_{k=q+1}^{r-2} \omega(a_k, a_{k+1})^{1/p} + \omega(a_{r-1}, u)^{1/p}) \end{aligned}$$

Which, using the super-additivity of  $\omega$ , gives the control:

$$\|f(x_s) - f(x_u)\|^p \leq M^p n^{p-1} \omega(s, t)$$

Therefore,  $f \circ x$  is of finite  $p$ -variation.  $\square$

## 5. ROUGH PATHS AND CATEGORIES

**5.1. A study of some properties rough paths.** For the sake of the developments we want to expose in the remaining of this paper, we take a final and a slightly different approach to rough paths that will highlight their local properties:

**Definition 5.1.** *Let  $E$  be a Banach space and  $p \geq 1$ . For an open subset  $U$  of  $E$ , a local geometric  $p$ -rough path in  $U$  is a triple  $(x, X, J)$  such that:*

- $J$  is an interval.
- $x$  is a  $U$ -valued path over  $J$ .



- $X$  is the limit in the  $p$ -variation metric of a sequence of truncated signatures of degree  $[p]$  of  $E$ -valued paths of bounded variation over  $J$  (i.e. a geometric  $p$ -rough path in the sense of definition 1.36:  $X \in G\Omega_p^J(E)/\sim$ ) which trace is  $x$ , i.e. for all  $(s, t) \in \Delta_J$ ,  $X_{s,t}^1 = x_t - x_s$ .

The set of local geometric  $p$ -rough paths in  $U$  will be denoted  $G\Omega_p(U)$ .

**Remark 5.2.** The trace of a geometric rough path is of finite  $p$ -variation.

**Lemma 5.3.** Let  $E$  be a normed space,  $p \in [1, +\infty[$  and  $J$  be an interval. Let  $X$  and  $Y$  be two multiplicative functionals on  $J$ . Let  $(K_i)_{i \in I}$  be a compact cover for  $J$  such that, for all  $i \in I$ ,  $X|_{K_i}$  and  $Y|_{K_i}$  are equal. Then  $X$  and  $Y$  are equal on  $J$ .

*Proof.* Let  $(a, b) \in \Delta_J$ . Then  $(K_i \cap [a, b])_{i \in S}$ , where  $S = \{i \in I \mid K_i \cap [a, b] \neq \emptyset\}$ , is a compact cover for  $[a, b]$ . Let  $(a_j)_{0 \leq j \leq n}$  be a subdivision of  $[a, b]$  such that for all  $j \in \llbracket 0, n-1 \rrbracket$ , there exists  $i \in S$  such that  $[a_j, a_{j+1}] \subseteq K_i$ . Then, by assumption:

$$\forall j \in \llbracket 0, n-1 \rrbracket : \quad X_{a_j, a_{j+1}} = Y_{a_j, a_{j+1}}$$

Therefore:

$$X_{a_0, a_1} \otimes \cdots \otimes X_{a_{n-1}, a_n} = Y_{a_0, a_1} \otimes \cdots \otimes Y_{a_{n-1}, a_n}$$

Which, by using the multiplicativity of  $X$  and  $Y$ , gives:  $X_{a,b} = Y_{a,b}$ . Since this holds for all  $(a, b) \in \Delta_J$ , then  $X = Y$ .  $\square$

Studying rough paths locally (on compact covers) is then enough to characterize the whole path. With this in mind, we can now see how to define the integral of a locally Lipschitz one-form  $\alpha$  along a geometric rough path  $(x, X)$ : we simply construct the integral locally on regions where  $\alpha$  is Lipschitz and ensure that the integral is the same on overlapping regions; which we can do easily by noting that this is indeed the fact for signatures of paths of bounded variation then taking the limit in the appropriate variation metric.

**Lemma 5.4.** Let  $E$  be a Banach space,  $p \in [1, +\infty[$  and  $J$  be a compact interval. Let  $X \in G\Omega_p^{[0,T]}(E)/\sim$  and let  $X(n)_{n \in \mathbb{N}}$  be a sequence of elements of  $G\Omega_p^{[0,T]}(E)/\sim$ . Suppose there exists a compact cover  $(K_i)_{i \in I}$  for  $J$  such that, for all  $i \in I$ ,  $(X(n)|_{K_i})_{n \in \mathbb{N}}$  converges to  $X|_{K_i}$  for  $\tilde{d}_p^{K_i}$ . Then  $(X(n))_{n \in \mathbb{N}}$  converges to  $X$  for  $\tilde{d}_p$ .

*Proof.* We start first with a basic inequality of how to control the distance between two geometric rough paths on a compact interval based on their distances on a compact cover of that interval. Let  $Y, Z \in G\Omega_p^{[0,T]}(E)/\sim$ . Let  $\omega$  be a control function that controls the  $p$ -variation of  $Y$  and  $Z$  and let  $a \in \mathbb{R}$  such that:

$$\forall (s, t) \in \Delta_{[0,T]}, \forall k \in \llbracket 0, [p] \rrbracket : \quad \|Y_{s,t}^k - Z_{s,t}^k\| \leq a \frac{\omega(s, t)^{k/p}}{\beta_p(k/p)!}$$

Let  $j \in \llbracket 1, [p] \rrbracket$ . Let  $s, t, u \in [0, T]$  such that  $s \leq t \leq u$ . Since  $Y$  and  $Z$  are multiplicative functionals, we have the following:

$$\begin{aligned} Y_{s,u}^j - Z_{s,u}^j &= \sum_{r=0}^j (Y_{s,t}^r \otimes Y_{t,u}^{j-r} - Z_{s,t}^r \otimes Z_{t,u}^{j-r}) \\ &= \sum_{r=0}^j ((Y_{s,t}^r - Z_{s,t}^r) \otimes Y_{t,u}^{j-r} + Z_{s,t}^r \otimes (Y_{t,u}^{j-r} - Z_{t,u}^{j-r})) \\ &= (Y_{s,t}^j - Z_{s,t}^j) + (Y_{t,u}^j - Z_{t,u}^j) + \sum_{r=1}^{j-1} ((Y_{s,t}^r - Z_{s,t}^r) \otimes Y_{t,u}^{j-r} + \\ &\quad Z_{s,t}^r \otimes (Y_{t,u}^{j-r} - Z_{t,u}^{j-r})) \end{aligned}$$

For  $(s, u) \in \Delta_{[0,T]}$ , define  $V_{s,u} = Y_{s,u}^j - Z_{s,u}^j$ . For a subdivision  $\mathcal{D} = (s_i)_{0 \leq i \leq r}$  of a compact sub-interval of  $[0, T]$ , define  $V_{s_0, s_r}^{\mathcal{D}} = \sum_{i=0}^{r-1} V_{s_i, s_{i+1}}$ .

Let  $(s, u) \in \Delta_{[0,T]}$  and let  $\mathcal{D} = (s_i)_{0 \leq i \leq r}$  be a subdivision of  $[s, u]$ . Define  $\mathcal{D}' = (t_0, t_2, \dots, t_q)$ . Then the previous identity along with the neo-classical inequality (lemma 1.27) shows that:

$$\begin{aligned} \|V_{s,u}^{\mathcal{D}} - V_{s,u}^{\mathcal{D}'}\| &\leq \frac{2a}{\beta_p(j/p)!} \omega(s_0, s_2)^{j/p} \\ &\leq \frac{2a}{\beta_p(j/p)!} \omega(s, u)^{j/p} \end{aligned}$$

By repeating the process finitely many times, we get the control:

$$\|V_{s,u}^{\mathcal{D}} - V_{s,u}\| \leq \frac{2a(r-1)}{\beta_p(j/p)!} \omega(s, u)^{j/p}$$

Now, take  $\Delta = (t_i)_{0 \leq i \leq q}$  and  $\mathcal{D} = (s_i)_{0 \leq i \leq r}$  to be two subdivisions of  $[0, T]$ . Let  $i \in \llbracket 0, q-1 \rrbracket$ . Define the subdivision  $\mathcal{D} \cap [t_i, t_{i+1}]$  of  $[t_i, t_{i+1}]$  to be  $(m_l) := (t_i, s_{\tilde{r}}, \dots, s_{\tilde{r}+n}, t_{i+1})$ , where  $\tilde{r}$  and  $n$  are such that  $s_{\tilde{r}-1} < t_i \leq s_{\tilde{r}}$  and  $s_{\tilde{r}+n} \leq t_{i+1} < s_{\tilde{r}+n+1}$ . Then we have:

$$\begin{aligned} \|V_{t_i, t_{i+1}}\|^{p/j} &\leq \left( \|V_{t_i, t_{i+1}}^{\mathcal{D} \cap [t_i, t_{i+1}]} - V_{t_i, t_{i+1}}\| + \|V_{t_i, t_{i+1}}^{\mathcal{D} \cap [t_i, t_{i+1}]} \| \right)^{p/j} \\ &\leq \left( \|V_{t_i, t_{i+1}}^{\mathcal{D} \cap [t_i, t_{i+1}]} - V_{t_i, t_{i+1}}\| + \sum_{\mathcal{D} \cap [t_i, t_{i+1}]} \|Y_{m_l, m_{l+1}}^j - Z_{m_l, m_{l+1}}^j\| \right)^{p/j} \\ &\leq (r+1)^{p/j-1} \left( \|V_{t_i, t_{i+1}}^{\mathcal{D} \cap [t_i, t_{i+1}]} - V_{t_i, t_{i+1}}\|^{p/j} + \right. \\ &\quad \left. \sum_{\mathcal{D} \cap [t_i, t_{i+1}]} \|Y_{m_l, m_{l+1}}^j - Z_{m_l, m_{l+1}}^j\|^{p/j} \right) \\ &\leq (r+1)^{p/j-1} \left( \left( \frac{2a(r-1)}{\beta_p(j/p)!} \right)^{p/j} \omega(t_i, t_{i+1}) + \right. \\ &\quad \left. \sum_{\mathcal{D} \cap [t_i, t_{i+1}]} \|Y_{m_l, m_{l+1}}^j - Z_{m_l, m_{l+1}}^j\|^{p/j} \right) \end{aligned}$$

Finally:

$$\sum_{\Delta} \|V_{t_i, t_{i+1}}\|^{p/j} \leq (r+1)^{p/j-1} \left( \left( \frac{2a(r-1)}{\beta_p(j/p)!} \right)^{p/j} \omega(0, T) + \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i, s_{i+1}]}(Y, Z)^p \right)$$

and therefore

$$\tilde{d}_p(Y, Z) \leq \max_{1 \leq j \leq [p]} (r+1)^{p/j-1} \left( \left( \frac{2a(r-1)}{\beta_p(j/p)!} \right)^{p/j} \omega(0, T) + \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i, s_{i+1}]}(Y, Z)^p \right)^{1/p}$$

Let  $\mathcal{D} = (s_i)_{0 \leq i \leq r}$  be a subdivision of  $[0, T]$  such that for each  $i \in \llbracket 0, r-1 \rrbracket$ , there exists  $j \in I$  such that  $[s_i, s_{i+1}] \subseteq K_j$ . Let  $\omega$  be a control function over  $[0, T]$  and  $(a_n)_{n \in \mathbb{N}}$  a sequence of non-negative numbers converging to 0 such that:

- $X$  and  $X(n)$ , for every  $n \in \mathbb{N}$ , are controlled in  $p$ -variation by  $\omega$ ;
- $\forall n \in \mathbb{N}, \forall (s, t) \in \Delta_{[0, T]}, \forall i \in \llbracket 1, [p] \rrbracket : \quad \|X(n)_{s, t}^i - X_{s, t}^i\| \leq a(n) \frac{\omega(s, t)^{i/p}}{\beta_p(i/p)!}$

Note that such a control and sequence exist by proposition 1.34 and the fact that the convergences occur on finitely many intervals. Then

$$\begin{aligned} \tilde{d}_p(X, X(n)) &\leq \max_{1 \leq j \leq [p]} (r+1)^{p/j-1} \cdot \\ &\quad \left( \sum_{i=0}^{r-1} \tilde{d}_p^{[s_i, s_{i+1}]}(X, X(n))^p + \left( \frac{2a(n)(r-1)}{\beta_p(j/p)!} \right)^{p/j} \omega(0, T) \right)^{1/p} \end{aligned}$$

Which trivially converges to 0.  $\square$

We can now show that the integral of a locally Lipschitz one-form along geometric rough paths is, as expected, continuous when varying the path.

**Theorem 5.5.** *Let  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $E$  and  $F$  be two Banach spaces and  $U$  be an open subset of  $E$ . Let  $\alpha : U \rightarrow \mathcal{L}(E, F)$  be a locally Lip- $(\gamma - 1)$  one-form. Then, the map:*

$$\begin{aligned} I_\alpha : (G\Omega_p^{[0, T]}(U), d_p) &\longrightarrow (G\Omega_p^{[0, T]}(F)/\sim, \tilde{d}_p) \\ (x, X) &\mapsto \int \alpha(x(0), X) dX \end{aligned}$$

is continuous.

*Proof.* Let  $(x, X) \in G\Omega_p^{[0, T]}(U)$  and let  $(x(n), X(n))_{n \in \mathbb{N}}$  be a sequence of elements of  $G\Omega_p^{[0, T]}(U)$  converging to  $(x, X)$  in  $d_p$ .

Let  $t \in [0, T]$ . Let  $r_t > 0$  such that  $B(x_t, r_t) \subseteq U$  and  $\alpha$  is Lip- $(\gamma - 1)$  on  $B(x_t, r_t)$ . Let  $\eta_t > 0$  such that, for all  $s \in [0, T]$ , if  $|s - t| < \eta_t$  then  $x_s \in B(x_t, r_t/3)$ . Finally, denote by  $O_t$  the open interval  $(t - \eta_t, t + \eta_t)$ . Then  $(O_t)_{t \in [0, T]}$  defines a subdivision  $\sigma = (s_i)_{0 \leq i \leq n}$  on  $[0, T]$  such that, for all  $i \in \llbracket 0, n-1 \rrbracket$ , there exists  $t_i \in [0, T]$  such that  $[s_i, s_{i+1}] \subseteq O_{t_i}$  (combination of the lemmas 3.8 and 3.9).

Let now  $i \in \llbracket 0, n-1 \rrbracket$  and  $\varepsilon > 0$ . The map:

$$\begin{aligned} I_\alpha^i : (G\Omega_p^{[s_i, s_{i+1}]}(B(x_{t_i}, r_{t_i})), d_p^i) &\longrightarrow (G\Omega_p^{[s_i, s_{i+1}]}(F)/\sim, \tilde{d}_p^i) \\ (y, Y) &\mapsto \int \alpha(y(s_i), Y) dY \end{aligned}$$

is continuous. Let  $\gamma_i > 0$  such that, for  $(y, Y) \in G\Omega_p^{[s_i, s_{i+1}]}(B(x_{t_i}, r_{t_i}))$ :

$$\|y(s_i) - x(s_i)\| \vee d_p^i(Y, X_{[s_i, s_{i+1}]}) \leq \gamma_i \Rightarrow \tilde{d}_p^i(I_\alpha^i(y(s_i), Y), I_\alpha^i(x(s_i), X_{[s_i, s_{i+1}]})) \leq \varepsilon$$

Let  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ :

$$n \geq N \Rightarrow \|x(n)(0) - x(0)\| \vee d_p^i(X(n), X) \leq \gamma_i \wedge (r_{t_i}/3)$$

Let  $n \in \mathbb{N}$  such that  $n \geq N$ , then we have:

$$\begin{aligned} \|x(n)(s_i) - x(s_i)\| &\leq \|x(n)(0) - x(0)\| + \|X(n)_{0, s_i}^1 - X_{0, s_i}^1\| \\ &\leq \|x(n)(0) - x(0)\| + d_p(X(n), X) \\ &\leq \gamma_i \end{aligned}$$

Furthermore, for all  $u \in [s_i, s_{i+1}]$ :

$$\begin{aligned} \|x(n)(u) - x(t_i)\| &\leq \|x(n)(0) - x(0)\| + \|X(n)_{0,u}^1 - X_{0,u}^1\| + \|x(u) - x(t_i)\| \\ &< \|x(n)(0) - x(0)\| + d_p(X(n), X) + r_{t_i}/3 \\ &< r_{t_i} \end{aligned}$$

Meaning, in particular, that  $x(n)([s_i, s_{i+1}]) \subseteq B(x_{t_i}, r_{t_i})$ . Therefore:

$$\tilde{d}_p^i \left( \int \alpha(x(n)(s_i), X(n)) dX(n), \int \alpha(x(s_i), X) dX \right) \leq \varepsilon$$

Which implies that:

$$\int \alpha(x(n)(s_i), X(n)) dX(n)_{|[s_i, s_{i+1}]} \xrightarrow[n \rightarrow \infty]{\tilde{d}_p^i} \int \alpha(x(s_i), X) dX_{|[s_i, s_{i+1}]}$$

We conclude using lemma 5.4. □

**Lemma 5.6.** *Let  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $E$  and  $F$  be two Banach spaces and  $U$  be an open subset of  $E$ . and  $J$  an interval. Let  $f : U \rightarrow F$  be a locally Lip- $\gamma$  map over  $U$  and  $x : J \rightarrow U$  a path with bounded variation. Then:*

$$(f(x), \int df(x, S_{[p]}(x)) dS_{[p]}(x), J) = (f(x), S_{[p]}(f(x)), J)$$

*Proof.* First notice that  $S_{[p]}(f(x))$  is well-defined since  $f(x)$  has bounded variation by theorem 4.6. Let  $s, t \in J$  such that  $s \leq t$ :

- (1) As  $df$  is continuous and  $x$  is of bounded variation, then  $\int df(x, S_{[p]}(x)) dS_{[p]}(x)$  has finite 1-variation and is equal to the Stieltjes integral  $\int df(x) dx$ . Therefore:

$$\left( \int df(x, S_{[p]}(x)) dS_{[p]}(x) \right)_{u,v}^1 = f(x_v) - f(x_u)$$

for all  $(u, v) \in \Delta_{[s,t]}$ .

- (2)  $S_{[p]}(f(x))$  has finite 1-variation and  $S_{[p]}(f(x))_{u,v}^1 = f(x_v) - f(x_u)$  for all  $(u, v) \in \Delta_{[s,t]}$ .

Two multiplicative functionals that have finite 1-variation and which terms of the 1<sup>st</sup> degree agree are equal by lemma 1.28. Therefore, the sought identity stands. □

**5.2. A functorial rule.** To highlight the minimalist framework on which we can define the notion of rough paths, we will be using the language of categories. Let  $E$  be a Banach space and  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $\mathcal{C}$  be a category whose objects are open subsets of  $E$ . The arrows of  $\mathcal{C}$  between two objects  $U$  and  $V$  are locally Lipschitz- $\gamma$  maps between  $U$  and  $V$ , and if  $U = V$ , the identity map (which is also locally Lipschitz- $\gamma$ ) is also an arrow; the set of such arrows will be denoted  $\text{hom}_{\mathcal{C}}(U, V)$ .

For  $U$  and  $V$  objects of  $\mathcal{C}$  and  $f \in \text{hom}_{\mathcal{C}}(U, V)$ , we denote by  $f_*$  the following map:

$$\begin{aligned} f_* : G\Omega_p(U) &\longrightarrow G\Omega_p(V) \\ (x, X, J) &\longmapsto (f(x), \int df(x, X) dX, J) \end{aligned}$$

**Theorem 5.7.** *The rule that assigns to every object  $U$  in  $\mathcal{C}$  the object  $G\Omega_p(U)$  and to every morphism  $f \in \text{hom}_{\mathcal{C}}(U, V)$ , where  $U$  and  $V$  are objects in  $\mathcal{C}$ , the map  $f_*$ , is functorial.*

*Proof.* For every open subsets  $U, V$  and  $W$  of  $E$  that are objects in  $\mathcal{C}$ , we need to prove the following:

- (1)  $(Id_U)_* = Id_{G\Omega_p(U)}$ ;

$$(2) \quad \forall f \in \text{hom}_{\mathcal{C}}(U, V), \forall g \in \text{hom}_{\mathcal{C}}(V, W) : (g \circ f)_* = g_* \circ f_*$$

Let  $U, V$  and  $W$  be open subsets of  $E$  that are objects in  $\mathcal{C}$ . Let  $f \in \text{hom}_{\mathcal{C}}(U, V)$  and  $g \in \text{hom}_{\mathcal{C}}(V, W)$ . Let  $x$  be a  $U$ -valued path over  $J$  with bounded variation. By lemma 5.6

$$\begin{aligned} f_*(x, S_{[p]}(x), J) &= (f(x), \int df(x, S_{[p]}(x)) dS_{[p]}(x), J) \\ &= (f(x), S_{[p]}(f(x)), J) \end{aligned}$$

as  $f(x)$  has bounded variation (lemma 4.6), we similarly have:

$$g_*(f(x), S_{[p]}(f(x)), J) = (g \circ f(x), S_{[p]}(g \circ f(x)), J)$$

and as  $g \circ f$  is locally Lip- $\gamma$ :

$$(g \circ f(x), S_{[p]}(g \circ f(x)), J) = (g \circ f(x), \int d(g \circ f)(x, S_{[p]}(x)) dS_{[p]}(x), J)$$

Therefore

$$((g \circ f)_*)|_{G\Omega_1(U)} = (g_* \circ f_*)|_{G\Omega_1(U)}$$

As both  $(g \circ f)_*$  and  $g_* \circ f_*$  are continuous in the  $p$ -variation metric (theorem 5.5) and as  $G\Omega_1(U)$  is dense in  $G\Omega_p(U)$  for this metric then we deduce that  $(g \circ f)_* = g_* \circ f_*$ . The first assertion regarding the identity map can be proved using a similar argument.  $\square$

**5.3. A definition of rough paths on manifolds.** Based on our findings so far, we are now able to give a minimal approach for defining rough paths on a manifold.

**Definition 5.8.** Let  $\gamma \geq 1$ . Let  $M$  be a topological manifold. We say that  $M$  is a locally Lip- $\gamma$   $n$ -manifold if it has an atlas such that any two charts in that atlas  $(U, \phi)$  and  $(V, \psi)$  such that  $U \cap V \neq \emptyset$ , the map  $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \mathbb{R}^n$  is locally Lip- $\gamma$ .

**Examples 5.9.** • A Lip- $\gamma$  manifold is a locally Lip- $\gamma$  manifold.

- Any topological space homeomorphic to an open subset of the Euclidean space is a locally Lip- $\gamma$  manifold. In particular, finite-dimensional vector spaces are locally Lip- $\gamma$  manifolds.

**Definition 5.10.** Let  $n \in \mathbb{N}^*$  and  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $M$  be a locally Lip- $\gamma$   $n$ -manifold. A local  $p$ -rough path on  $M$  over an interval  $J$  is a collection  $(x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$  of  $p$ -rough paths on  $\mathbb{R}^n$  satisfying the following conditions:

- $(J_i)_{i \in I}$  is a compact cover for  $J$ ;
- For every  $i \in I$ ,  $(\phi_i, U_i)$  is a locally Lip- $\gamma$  chart on  $M$ .
- $\forall i \in I : (x_i, X_i, J_i) \in G\Omega_p(\phi_i(U_i))$ ;
- **(Consistency condition)** If  $i, k \in I$  such that  $J_i \cap J_k \neq \emptyset$ , then we have:

$$(\phi_k \circ \phi_i^{-1})_*(x_i, X_i, J_i \cap J_k) = (x_k, X_k, J_i \cap J_k)$$

We identify similar local rough paths in the following way:

**Definition 5.11.** Let  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $M$  be a locally Lip- $\gamma$  manifold and  $J$  an interval. Two local  $p$ -rough paths on  $M$  over  $J$   $A = (x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$  and  $B = (x_i, X_i, J_i, (\phi_i, U_i))_{i \in J}$  are said to be equivalent if  $A \cup B$  is also a local  $p$ -rough path.

This, of course, defines an equivalence relation. We will be henceforth only considering the equivalence classes associated to this relation.

**Theorem 5.12.** *Let  $M$  be a Lipschitz- $\gamma$  manifold and  $J$  a compact interval. There is a one-to-one mapping between rough paths on  $M$  and equivalence classes of local rough paths on  $M$ .*

*Proof.* • A rough path on a manifold has a localising sequence by theorem 2.18. The pushforwards of the elements of this localising sequence under the associated chart maps define a local rough path: every rough path in this sequence has an end point consistent with the starting point of the following rough path, which is equivalent in our case to the consistency condition in definition 5.10, where the intersections of successive intervals are reduced to single points.

- Conversely, the compact cover for a compact interval  $J$  defines a subdivision  $(a_i)_{1 \leq i \leq n}$ . We can construct then a representative of a given equivalence class of local rough paths that is of the form  $(x_i, X_i, [a_i, a_{i+1}], (\phi_i, U_i))_{1 \leq i \leq n-1}$ . For each  $i \in \llbracket 1, n-1 \rrbracket$ , define  $Z_i = (\phi_i^{-1})_*(x_i, X_i)$ . Now let  $Z = Z_1 * \dots * Z_n$ . Then  $Z$  is a rough path on  $M$ .  $\square$

**Definition 5.13.** *Let  $p, \gamma \in [1, +\infty[$  such that  $\gamma > p$ . Let  $M$  be a locally Lip- $\gamma$  manifold. A local rough path  $(x_i, X_i, J_i, (\phi_i, U_i))_{i \in I}$  on an interval  $J$  is said to be a rough path extension for the path  $x : J \rightarrow M$  if the following holds:*

- $\forall i \in I : x(J_i) \subseteq U_i$ ;
- $\forall i \in I : x_i = \phi_i \circ x|_{J_i}$ .

*If such a rough path exists, we say then that  $x$  admits a rough path extension.*

In the same way as in definition 5.11, we will consider that two rough path extensions of a given path to be the same if their unions is also a rough path extension of that path.

## 6. COLOURED PATHS ON MANIFOLDS

The procedure detailed in the previous sections is more general and can be extended to define any notion of “colouring” already existing on the Euclidean space to a manifold, assuming that we can find a suitable functorial rule. Indeed, the roughness of a path can be seen as a colour: an extra bit of information that cannot necessarily be learned by looking at the base path only.

Let  $n \in \mathbb{N}^*$ . Let  $\mathcal{C}$  be a category whose objects are the open subsets of  $\mathbb{R}^n$  and for which inclusion maps are arrows. We can define a notion of “coloured” charts and atlas over any  $n$ -topological manifold by assuming that the transition maps are arrows in the category  $\mathcal{C}$ . We will call such a manifold a “coloured manifold”.

**Definition 6.1.** *Let  $M$  be a topological manifold. We say that  $M$  is a coloured manifold if it has an atlas such that for any two charts in that atlas  $(U, \phi)$  and  $(V, \psi)$  such that  $U \cap V \neq \emptyset$ , we have  $\psi \circ \phi^{-1} \in \text{hom}_{\mathcal{C}}(\phi(U \cap V), \psi(U \cap V))$ .*

Suppose now that we have a notion of “coloured paths” on  $\mathbb{R}^n$  that have base paths underlying them which we will be calling traces (rough paths are an example). Denote by  $T(U)$  the sets of coloured paths whose traces lie in an open subset  $U$  of  $\mathbb{R}^n$ . For each arrow  $f$  between the objects  $U$  and  $V$  of  $\mathcal{C}$ , assume there exists a map  $f_*$  between  $T(U)$  and  $T(V)$  such that the trace of the image of each coloured path on  $U$  is the image by  $f$  of its trace. Such a map will be called a “colouring map”. We can now define coloured paths on a

coloured manifold in the same way as in definition 5.10 and the existence of coloured path extensions for manifold-based paths as in definition 5.13.

**Definition 6.2.** Let  $n \in \mathbb{N}^*$ . Let  $M$  be a coloured  $n$ -manifold. A coloured path on  $M$  over an interval  $J$  is a collection  $(X_i, J_i, (\phi_i, U_i))_{i \in I}$  of coloured paths on  $\mathbb{R}^n$  satisfying the following conditions:

- $(J_i)_{i \in I}$  is a compact cover for  $J$ ;
- For every  $i \in I$ ,  $(\phi_i, U_i)$  is in the coloured atlas of  $M$ .
- $\forall i \in I$  :  $X_i \in T(\phi_i(U_i))$  and  $X_i$  is defined over  $J_i$ ;
- **(Consistency condition)** If  $i, k \in I$  such that  $J_i \cap J_k \neq \emptyset$ , then we have:

$$(\phi_k \circ \phi_i^{-1})_*(X_i|_{J_i \cap J_k}) = X_k|_{J_i \cap J_k}$$

**Definition 6.3.** Let  $n \in \mathbb{N}^*$ . Let  $M$  be a coloured  $n$ -manifold. A coloured path  $(X_i, J_i, (\phi_i, U_i))_{i \in I}$  on an interval  $J$  is said to be a coloured path extension for the path  $x : J \rightarrow M$  if the following holds:

- $\forall i \in I$  :  $x(J_i) \subseteq U_i$ ;
- $\forall i \in I$  :  $\text{trace}(X_i) = \phi_i \circ x|_{J_i}$ .

If such a coloured path exists, we say then that  $x$  admits a coloured path extension.

But for these definitions to make sense on their own and also be consistent with the definitions of coloured paths on the Euclidean space (seen now as a coloured manifold), the rule that assigns  $T(U)$  to  $U$  and  $f_*$  to  $f$ , for  $U$  object in  $\mathcal{C}$  and  $f$  arrow in  $\mathcal{C}$ , must be functorial.

**Example 6.4.** In the study of rough paths made in the previous sections, rough paths can be seen as coloured paths, locally Lipschitz maps as arrows and locally Lipschitz manifolds as coloured manifolds. The maps that assign to each rough path its image by a locally Lipschitz map are colouring maps.

**Example 6.5.** On a more basic and trivial level, continuous paths can also be seen as coloured paths. Let  $M$  be a topological manifold. If we take continuous maps as arrows, then a continuous atlas will be a covering of  $M$  with charts such that the transition maps are continuous. As it happens, a topological manifold comes naturally with a continuous atlas! The colouring map associated to an arrow  $f$  is a map that assigns to every continuous path  $x$  the path  $f(x)$ . Then our new definition of a continuous map on  $M$  can be identified with the classical one which relies only on the topology on  $M$ . Conversely, every continuous path on  $M$  in the classical sense can be seen as the concatenation of pushforwards of continuous paths on the Euclidean space.

**Example 6.6.** Building on the previous example, we now aim to give an analogous definition of smooth maps on a manifold. Classically defined, smooth atlases and smooth manifolds correspond exactly to coloured atlases and coloured manifolds, when the arrows are taken to be smooth maps. The associated functorial rule is the same as earlier by replacing continuity with smoothness. Finally, one can see the definitions of smooth maps in the classical sense and when using coloured paths are equivalent.

## REFERENCES

- [1] Boutaib, Y. (2015) *On Lipschitz maps and their flows*. arXiv:1510.07614.

- [2] Caruana, M., Lyons, T.J. and Lévy T. (2007) *Differential Equations Driven by Rough Paths*. Volume 1908 of Lecture Notes in Mathematics. Springer, Berlin.
- [3] Cass, T., Litterer, C and Lyons, T.J. (2012) Rough paths on manifolds. *Interdiscip. Math. Sci.* 12, 33–88.
- [4] Chen, K. (1958) Integration of paths - a faithful representation of paths by non-commutative formal power series. *Trans. Am. Math. Soc.* 89, 395–407.
- [5] Hambly, B. and Lyons, T.J. (2010) Uniqueness for the signature of a path of bounded variation and the reduced path group. *Ann. of Math.* 171, no. 1, 109–167.
- [6] Hara, K. and Hino, M. (2010) Fractional order Taylor’s series and the neo-classical inequality. *Bull. Lond. Math. Soc.* 42, no. 3, 467–477.
- [7] Lee, J.M. (2002) *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics, Springer.
- [8] Lyons, T.J. (1998) Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* 14, no. 2, 215–310.
- [9] Lyons, T.J. and Qian, Z. (2002) *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford Science Publications, Oxford.
- [10] Lyons, T.J and Nicolas Victoir, N. (2007) An extension theorem to rough paths. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24, no. 5, 835–847.
- [11] Stein, E.M. (1970) *Singular integrals and differentiability properties of functions*. Princeton University Press.
- [12] Young, L.C. (1936) An inequality of Hölder type connected with Stieltjes integration *Acta Math.* 67, 251–282.

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD.

*E-mail address:* `boutaib@maths.ox.ac.uk`

OXFORD-MAN INSTITUTE OF QUANTITATIVE FINANCE. UNIVERSITY OF OXFORD.

*E-mail address:* `terry.lyons@oxford-man.ox.ac.uk`