

Control Reconfiguration of Dynamical Systems for Improved Performance via Reverse- and Forward-engineering

Han Shu, Xuan Zhang*, Na Li, and Antonis Papachristodoulou *Fellow, IEEE*

Abstract—This paper presents a control reconfiguration approach to improve the performance of two classes of dynamical systems. Motivated by recent research on re-engineering cyber-physical systems, we propose a three-step control retrofit procedure. First, we reverse-engineer a dynamical system to dig out an optimization problem it actually solves. Second, we forward-engineer the system by applying a corresponding faster algorithm to solve this optimization problem. Finally, by comparing the original and accelerated dynamics, we obtain the implementation of the redesigned part (the extra dynamics). As a result, the convergence rate/speed or transient behavior of the given system can be improved while the system control structure is maintained. Internet congestion control and distributed proportional-integral (PI) control, as applications in the two different classes of target systems, show the effectiveness of the proposed approach.

Index Terms—Control redesign, reverse-engineering, convex optimization, accelerated algorithm, dynamical systems.

I. INTRODUCTION

CYBER-physical systems (CPSs) integrate, coordinate and monitor the operations of both a physical process and the cyber world [1]. They are decisive in supporting fundamental infrastructures and smart applications including automotive systems, transportation systems, smart grids, etc. However, although those CPSs were advanced at the time when being constructed, they can be either economical inefficient or energy inefficient from today's viewpoint. For example, aging electricity distribution infrastructures are becoming less reliable and less efficient [2]. Also, there are societal and industrial needs for better control of CPSs. For example, the increasing penetration of renewable energy poses threats to the reliability of power grids: it is essential to design better control to quickly attenuate large fluctuations caused by these energy sources. Furthermore, autonomous vehicles and mobile robots, operated in an uncertain environment without complete information, require better control for more safety and reliability.

Usually, better control can be achieved from two perspectives. One way is rebuilding a new controller for the whole system. For example, [3] shows that by incorporating numerical modeling and simulation to design the controller for a photovoltaic system, system performance and robustness can be improved. The other way is to redesign the existing

controller by adding extra dynamics while maintaining the control structure of the existing controller, i.e., if the original controller is centralized/distributed, then the redesigned controller is also centralized/distributed. This is referred to as control reconfiguration and can be simply realized, for example, by using additional information produced by newly added sensors, without affecting the structure of the controlled system. For instance, [4] proposes a modification approach based on a penalty method for improving the performance and robustness to delays of Internet congestion control protocols, and [5] redesigns controllers by adding extra terms obtained based on continuous-time systems and Lyapunov functions to enhance stability and robustness. In general, there are tradeoffs between these two methods: rebuilding a new controller can achieve a better result by implementing state-of-the-art sensing, communication and computing technologies but with the expense of complexity and high investment, while redesigning the existing controller can be easier and more convenient though the effect may not be as good as rebuilding.

Instead of designing a new controller, this paper focuses on modifying the existing control from an optimization perspective to improve the performance of the whole system. Inspired by recent research on re-engineering typical CPSs [6]–[10], this work extends to study control of general CPSs based on a reverse- and forward-engineering framework, which serves as a tool to bridge the gap between engineering CPS applications and existing theoretical results on optimization. As will be shown later, the proposed framework shows great potential to handle large-scale system cases.

The idea of redesign using a reverse- and forward-engineering framework for optimality has been introduced over ten years ago [11]. From existing protocols designed based on engineering instincts, utility functions are implicitly determined and can be extracted via reverse-engineering. Other works consider various congestion control protocols as distributed algorithms for solving network utility maximization problems [6], [12], [13]. Based on the insights obtained from reverse-engineering, forward-engineering systematically improves the protocols [6], [11]. Recently, inspired by this idea, [7] connects automatic generation control (AGC) and economic dispatch by reverse-engineering AGC to improve power system economic efficiency, and [9], [14] develop a reverse- and forward-engineering framework to redesign control for improved efficiency, achieving optimal steady-state performance in network systems.

Nevertheless, little attention has been paid to improve system performance in terms of convergence rates/speeds and transient behaviors. This paper utilizes the reverse- and forward-engineering framework, together with several acceleration techniques, to systematically improve the performance

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N. Li is with the School of Engineering and Applied Sciences, Harvard University, Cambridge, MA, USA (email: nali@seas.harvard.edu).

A. Papachristodoulou is with the Department of Engineering Science, University of Oxford, Oxford, UK (email: antonis@eng.ox.ac.uk).

of two classes of dynamical systems in discrete time. These systems include but are not limited to existing protocols and controlled systems, e.g., Internet congestion control, distributed proportional-integral (PI) control.

The main contribution of this paper is twofold.

- A control reconfiguration approach is proposed to systematically improve the performance of two classes of general dynamical systems while maintaining the original control structure. This is realized by designing extra dynamics and adding it to the original control based on a reverse- and forward-engineering framework.
- Two standard acceleration methods are utilized in each class of dynamical systems for control reconfiguration and are theoretically proven to exhibit improved convergence rates.

The rest of this paper is organized as follows. Section II presents Class- \mathcal{O} ¹ and Class- \mathcal{S} ¹ as our target systems, together with two motivating examples. Section III presents the reconfiguration steps for systems in Class- \mathcal{O} and Class- \mathcal{S} and analyzes the convergence rates of the original and redesigned systems. Section IV provides simulation results of the motivating examples to illustrate the effectiveness of the reconfiguration framework. Section V concludes the paper.

Notations: Throughout this paper, we use upper case roman letters to denote matrices. Let $\text{diag}\{\star\}$ denote the diagonal matrix with corresponding entries \star on the main diagonal. Let $A \succeq 0$ ($A \succ 0$) denote that a square matrix is positive semi-definite (positive definite). For two symmetric matrices A_1 and A_2 , notation $A_1 \preceq A_2$ implies that $A_2 - A_1$ is positive semi-definite. Let $\langle \cdot, \cdot \rangle$ represent the Euclidean inner product, and let $\|\cdot\|$ denote the Euclidean norm for vectors and the spectrum norm for matrices. Denote by \mathbb{Z}^+ the set of positive integers, by \mathbb{R}^n the n -dimensional Euclidean space and by $A \in \mathbb{R}^{m \times n}$ an $m \times n$ real matrix. Let x^T and ∇f denote the transpose of x and the gradient of f (as a column vector). Let $\sigma_{\max}(B)/\sigma_{\min}(B)$ denote the maximum/minimum singular values of B . Let $\lambda(A)$ denote the eigenvalues of a square matrix A . Denote by \mathbf{H}_f the Hessian matrix of f with elements $\mathbf{H}_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

II. PRELIMINARIES

In this section, we introduce two special classes of dynamical systems as our focus, i.e., Class- \mathcal{O} and Class- \mathcal{S} , and present one typical example in each class as motivating applications. In particular, we show what kinds of motivations are required to determine whether or not a system belongs to these classes (when reverse-engineering works) [9], mainly for the discrete-time linear case.

Reverse-engineering seeks a proper optimization problem inherently from given dynamical equations. Different from the previous optimization-to-algorithm framework, it generates a reverse flow, i.e., algorithm-to-optimization.

Consider a linear time-invariant (LTI) system

$$x_{k+1} = Ax_k + Cw \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times p}$ and $w \in \mathbb{R}^p$ is the exogenous input, e.g., disturbances. Note that w can be constant or time-varying. In general, any given discrete-time LTI closed-loop system with either static feedback or dynamic feedback can be rearranged to fit (1).

¹Notation \mathcal{O} stands for *optimization algorithms* in contrast to \mathcal{S} for *saddle-point algorithms*.

A. Target Systems: Class- \mathcal{O}

Class- \mathcal{O} : System (1) belongs to Class- \mathcal{O} if there exists a convex function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$, such that the set $\{\nabla f(x) = \mathbf{0}\}$ is nonempty and system (1) is a gradient descent algorithm to solve an unconstrained convex optimization problem $\min_x f(x)$, i.e., $x_{k+1} = x_k - P\nabla f(x)|_{x=x_k}$.

For linear systems in Class- \mathcal{O} , the associated f must be a convex quadratic function, i.e., $f = \frac{1}{2}x^T Qx + x^T R(Cw) + s(w)$ for some matrices $Q \succeq 0$ and R , and $s(w)$ stands for terms only related to w . Note that f is regarded as a function of the variable x and w is treated as a parameter. Therefore, system (1) belongs to Class- \mathcal{O} if and only if there exist $P \succ 0$ and $Q \succeq 0$ such that $A = I_n - PQ$ and $R = -P^{-1}$ hold. This results in the following theorem [9].

Theorem 1. *Let w be constant in (1) and the set $\{(A - I_n)x + Cw = \mathbf{0}\}$ be nonempty. System (1) belongs to Class- \mathcal{O} if and only if system (1) is marginally or asymptotically stable², all the eigenvalues of $I_n - A$ are real numbers and $I_n - A$ is diagonalizable.*

B. Target Systems: Class- \mathcal{S}

Class- \mathcal{S} : System (1) belongs to Class- \mathcal{S} if there exists a function $f(x^{(1)}, x^{(2)}) : \mathbb{R}^n \rightarrow \mathbb{R}$ and positive definite matrices $P_{x^{(1)}}^3$, $P_{x^{(2)}}^3$ such that $\mathbf{H}_{f(x^{(1)})}^3 = \mathbf{0}$, $\mathbf{H}_{f(x^{(2)})} \succeq 0$, the saddle-point set $\{\nabla f(x) = \mathbf{0}\}$ is nonempty, and (1) is a primal-dual gradient algorithm to solve $\max_{x^{(1)}} \min_{x^{(2)}} f$, i.e., $x_{k+1} = x_k + \text{diag}\{P_{x^{(1)}}, -P_{x^{(2)}}\} \nabla f|_{x=x_k}$.

In the above definition, state x is partitioned into $x^{(1)}$ and $x^{(2)}$, and f is linear in $x^{(1)}$ (similar to that the Lagrangian function of a constrained optimization problem is linear in the dual variables). Accordingly, we rearrange system (1) as

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_A \begin{bmatrix} x_k^{(1)} \\ x_k^{(2)} \end{bmatrix} + Cw \quad (2)$$

where $x_k^{(1)} \in \mathbb{R}^{n_1}$, $x_k^{(2)} \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$. For linear systems in Class- \mathcal{S} , the associated function f must be a convex quadratic function, i.e.,

$$f = \frac{1}{2}x^T \underbrace{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}}_Q x + x^T R(Cw) + s(w) \quad (3)$$

where $Q_{11} = \mathbf{0} \in \mathbb{R}^{n_1 \times n_1}$, $Q_{22} \in \mathbb{R}^{n_2 \times n_2}$ is symmetric and positive semi-definite (f is linear in $x^{(1)}$ and convex in $x^{(2)}$), and $Q_{12} \in \mathbb{R}^{n_1 \times n_2}$. According to the definition of Class- \mathcal{S} , the following theorem is obtained.

Theorem 2. *Let w be constant in (2) and the set $\{(A - I_n)x + Cw = \mathbf{0}\}$ be nonempty. System (2) belongs to Class- \mathcal{S} if and only if the following conditions are satisfied: (i) system (2) is marginally or asymptotically stable; (ii) the eigenvalues of $A_{11} - I_{n_1}$ and $A_{22} - I_{n_2}$ are non-positive real; (iii) $A_{11} - I_{n_1}$*

²All the eigenvalues of A have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1 .

³Here $P_{x^{(1)}}/P_{x^{(2)}}$ is a positive definite matrix with dimension in accordance with $x^{(1)}/x^{(2)}$; $f(x^{(1)})$ means f is regarded as a function of the variable $x^{(1)}$ and $x^{(2)}$ is treated as a parameter.

and $A_{22} - I_{n_2}$ are diagonalizable with the diagonal canonical forms given by $A_{11} - I_{n_1} = J_1 \Lambda_1 J_1^{-1}$, $J_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_{22} - I_{n_2} = J_2 \Lambda_2 J_2^{-1}$, $J_2 \in \mathbb{R}^{n_2 \times n_2}$, and there exist V_1, V_2 that

$$\begin{aligned} (J_1^{-1})^T V_1 J_1^{-1} A_{12} + A_{21}^T (J_2^{-1})^T V_2 J_2^{-1} &= \mathbf{0} \\ V_1 \Lambda_1 &= \Lambda_1 V_1, \quad V_2 \Lambda_2 = \Lambda_2 V_2 \\ V_1 &\prec 0, \quad V_2 \prec 0. \end{aligned} \quad (4)$$

Note that there could be multiple optimization problems corresponding to the same dynamical system (1) in either Class- \mathcal{O} or Class- \mathcal{S} . The derivation procedure of those problems can be found in [9].

C. Motivating Applications

Many existing systems fall into the two classes. For example, consensus in multi-agent systems, primal Internet congestion control protocols and heating, ventilation and air-conditioning (HVAC) system control are in Class- \mathcal{O} (see [15] for a detailed description) while primal-dual Internet congestion control protocols, distributed PI control for single integrators and frequency control in power systems are in Class- \mathcal{S} . Here we present one typical example for each class.

1) *Internet Congestion Control*: Internet congestion control regulates the data transfer and efficient bandwidth sharing between sources and links. Here we consider a standard primal congestion control algorithm [16], which belongs to Class- \mathcal{O} :

$$x_i(k+1) = x_i(k) + \varepsilon k_{x_i} (U'_i(x_i(k)) - q_i(k)) \quad (5a)$$

$$q_i(k) = \sum_{l=1}^M R_{li} p_l(k) \quad (5b)$$

$$p_l(k) = f_l(y_l(k)) \quad (5c)$$

$$y_l(k) = \sum_{i=1}^N R_{li} x_i(k) \quad (5d)$$

where ε is the step size, $U_i(x_i)$ is the utility function of user i , which is assumed to be a continuously differentiable, monotonically increasing, strictly concave function of the transmission rate x_i . R is a routing matrix describing the interconnection where $R_{li} = 1$ if user i uses link l ; otherwise $R_{li} = 0$. Also, $k_{x_i} > 0$ is the rate gain, $p_l > 0$ is the link price, q_i is the aggregate price along user i 's path, y_l is the total flow through link l , $f_l(y_l)$ is a barrier/penalty function satisfying $f_l(y_l), f'_l(y_l) > 0$ (usually, $f_l(y_l)$ forces the satisfaction of the inequality constraint $y_l \leq c_l$ where $c_l > 0$ is the link capacity of link l), N and M are the numbers of sources and links. The above dynamics can be rearranged in a vector form as

$$x(k+1) = x(k) + \varepsilon \text{diag}\{k_{x_i}\} (U'(x(k)) - R^T f(Rx(k))) \quad (6)$$

where $x(k), U'(x(k)) \in \mathbb{R}^N$, $f(Rx(k)) \in \mathbb{R}^M$ are the corresponding vector forms of $x_i(k), U'_i(x_i(k)), f_l(\sum_{i=1}^N R_{li} x_i(k))$.

Suppose $U_i(x_i)$ is quadratic and $f_l(y_l)$ is linear, i.e., $U(x) = -\frac{1}{2}Q \text{diag}\{x_i\}x + R_1 x + s_1$ and $f(Rx) = R_2 Rx + s_2$ where Q, R_1, R_2 are diagonal, positive definite constant matrices, and s_1, s_2 are constant vectors. Then (6) can be rearranged as

$$\begin{aligned} x(k+1) &= x(k) - \varepsilon \text{diag}\{k_{x_i}\} (Qx(k) + R^T R_2 Rx(k)) + \dots \\ &= (I_n - \varepsilon \text{diag}\{k_{x_i}\} (Q + R^T R_2 R)) x(k) + \dots \end{aligned} \quad (7)$$

where \dots denotes the remaining constant terms. Compared with (1), it is straightforward to notice that $I_n - A$ in (7) is $\varepsilon \text{diag}\{k_{x_i}\} (Q + R^T R_2 R)$. It satisfies the conditions listed in Theorem 1 and thus, the above dynamics (5) or (6) can be reverse-engineered as a gradient descent algorithm to solve

$$\min_{x_i \geq 0} - \sum_{i=1}^N U_i(x_i) + \sum_{l=1}^M \int_0^{\sum_{i=1}^N R_{li} x_i} f_l(\beta) d\beta. \quad (8)$$

In general, the reverse-engineering from (6) to the above optimization problem always works if $U_i(x_i)$ is concave. It is of interest to redesign the primal congestion control algorithm (6) for a faster convergence speed to improve data transfer.

2) *Distributed PI Control for Single Integrator Dynamics*: In the following, we propose distributed PI control for single integrators as a motivating example that belongs to Class- \mathcal{S} . Consider n agents with single integrator dynamics

$$\dot{y}_i = d_i + u_i$$

where d_i is a constant disturbance and u_i is the control input given by

$$\begin{aligned} u_i = - \sum_{j \in \mathcal{N}_i} \left(\rho_2 (y_i - y_j) + \rho_1 \int_0^t (y_i(\tau) - y_j(\tau)) d\tau \right) \\ - \delta (y_i - y_i(0)) \end{aligned} \quad (9)$$

where ρ_1, ρ_2, δ are positive constant parameters, $y_i(0)$ is the initial condition and \mathcal{N}_i is the set of neighbors of agent i . This controller drives agents to reach consensus under static disturbances and any initial condition according to Theorem 6 in [17]. Introduce the integral action $\dot{z}_i = y_i - y_n$ and rearrange the dynamics after discretizing in a vector form as

$$\underbrace{\begin{bmatrix} z(k+1) \\ y(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} I_{n-1} & \varepsilon_2 D \\ -\varepsilon_1 \rho_1 \tilde{L} & I_n - \varepsilon_1 \rho_2 L - \varepsilon_1 \delta I_n \end{bmatrix}}_A \underbrace{\begin{bmatrix} z(k) \\ y(k) \end{bmatrix}}_{x(k)} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \varepsilon_1 (d + \delta y(0)) \end{bmatrix}}_{Cw}$$

where $z(k) \in \mathbb{R}^{n-1}$, $y(k), d \in \mathbb{R}^n$, $x(k) \in \mathbb{R}^{2n-1}$, ε_1 and ε_2 are the step sizes and $D = [I_{n-1} \quad -\mathbf{1}]$ ($\mathbf{1}$ is a column vector of ones). $L \in \mathbb{R}^{n \times n}$ is the Laplacian of the connected agent network and $\tilde{L} \in \mathbb{R}^{n \times (n-1)}$ is obtained after removing the n th column of L . Compared with (2), it is straightforward to notice that $A_{11} - I_{n_1}$ is $\mathbf{0}$ and $A_{22} - I_{n_2}$ is $-\varepsilon_1 \rho_2 L - \varepsilon_1 \delta I_n$. They satisfy the conditions listed in Theorem 2 and therefore, the above dynamics can be reverse-engineered as a primal-dual gradient algorithm to solve

$$\begin{aligned} \max_{z \in \mathbb{R}^{n-1}} \min_{y \in \mathbb{R}^n} f \\ = \frac{\rho_2^2}{2} y^T L y + \rho_1 z^T \tilde{L}^T y + \frac{\delta}{2} y^T y - y^T d - y^T \delta y(0). \end{aligned} \quad (10)$$

It is of interest to redesign the distributed PI controller (9) to improve system performance.

D. Problem Setup

Motivated by the above examples, the desiderata is: for dynamical systems belonging to Class- \mathcal{O} and Class- \mathcal{S} , improve their performance (convergence rates/speeds and transient be-

haviors) through redesigning the existing protocols and controls while maintaining their original control structures, i.e., the amount and topology of input channels remain unchanged.

III. REDESIGN METHODOLOGY

In this section, we propose the redesign methodology for linear dynamical systems in Class- \mathcal{O} and Class- \mathcal{S} . In particular, we analyze the convergence rates of the original systems and the redesigned systems. The corresponding explicit forms of the extra dynamics in the redesigned systems are derived. For convenience, we only consider discrete-time cases here, and continuous-time cases will be discussed in a future paper.

Definition 1. Function class \mathcal{F}_L : $f \in \mathcal{F}_L$ means that the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and has L -Lipschitz continuous gradient ($L \geq 0$), i.e., $\forall x, y \in \mathbb{R}^n$, we have

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|. \quad (11)$$

Function class $\mathcal{S}_{\mu,L}$: $f \in \mathcal{S}_{\mu,L}$ means that f is μ -strongly convex ($\mu > 0$) and has L -Lipschitz continuous gradient ($L \geq 0$), i.e., $\forall x, y \in \mathbb{R}^n$, we have (11) and

$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\mu}{2} \|y - x\|^2. \quad (12)$$

A. Systems in Class- \mathcal{O}

1) *Convergence Rate of the Original Systems:* Any system (1) in Class- \mathcal{O} can be reverse-engineered as a gradient descent (GD) algorithm to solve an unconstrained convex optimization problem, so the convergence rate of system (1) in Class- \mathcal{O} follows that of a GD algorithm. Suppose the reverse-engineered optimization problem is

$$\min_{x \in \mathbb{R}^n} f(x) \quad (13)$$

where $x \in \mathbb{R}^n$ is the decision vector and f is a convex, scalar differentiable function of x . Assume that a global minimizer of problem (13) exists and is denoted by x^* .

The GD algorithm is the simplest algorithm to solve problem (13) in discrete-time [18], as shown in Algorithm 1, where $k = 0, 1, \dots, N$ is the time step. Theorems 3 and 4 summarize the convergence property of the GD algorithm/original system when the objective function $f \in \mathcal{F}_L$ or $f \in \mathcal{S}_{\mu,L}$.

Algorithm 1 Gradient descent algorithm

Setting: Choose appropriate positive step size ε , and $x_0 \in \mathbb{R}^n$.

$$x_{k+1} = x_k - \varepsilon \nabla f(x_k) \quad (14)$$

Theorem 3. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to \mathcal{F}_L and $0 < \varepsilon < 2/L$, then the convergence rate of this system is given by (See Theorem 2.1.14 in [18] for the proof)

$$f(x_k) - f(x^*) \leq \frac{2\|x_0 - x^*\|^2}{\varepsilon k} = O\left(\frac{1}{\varepsilon k}\right), \quad \forall k \in \mathbb{Z}^+.$$

Theorem 4. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to $\mathcal{S}_{\mu,L}$ and $0 < \varepsilon \leq \frac{2}{\mu+L}$, then the convergence rate of this system is given by [19]

$$\|x_k - x^*\| \leq (1 - \mu\varepsilon)^k \|x_0 - x^*\| = O(e^{-k\mu\varepsilon}), \quad \forall k \in \mathbb{Z}^+.$$

Corollary 1. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to $\mathcal{S}_{\mu,L}$ and the step size $\varepsilon = \frac{2}{\mu+L}$, then this system is with the optimal convergence rate given by [18]

$$\|x_k - x^*\| \leq \left(\frac{L - \mu}{L + \mu}\right)^k \|x_0 - x^*\| = O(e^{-k\frac{2\mu}{L+\mu}}), \quad \forall k \in \mathbb{Z}^+.$$

2) *Redesign via a Heavy Ball Method:* Once reverse-engineering a given system (1) in Class- \mathcal{O} as a GD algorithm to solve (13), it is natural to apply faster algorithms to solve this problem, which can result in a redesigned system formula with improved performance.

In Algorithm 1, the next point only depends on the current point like in a Markov chain. The heavy ball (HB) method, introduced by Polyak [20], utilizes a momentum term $x_k - x_{k-1}$ to incorporate the effect of second-order change, often leading to smoother trajectories and a faster convergence rate [20]. Algorithm 2 shows its iterations.

Algorithm 2 Heavy ball method

Setting: Choose appropriate positive step size ε , coefficient β and let $x_1 = x_0$ be an arbitrary initial condition.

$$x_{k+1} = x_k - \varepsilon \nabla f(x_k) + \beta(x_k - x_{k-1}) \quad (15)$$

For any system (1) in Class- \mathcal{O} , to improve system performance, apply the HB method to solve problem (13) to obtain

$$x_{k+1} = x_k - \varepsilon \nabla f(x_k) + \underbrace{\beta(x_k - x_{k-1})}_{\Delta u_k} \quad (16)$$

where $\Delta u_k = \beta(x_k - x_{k-1})$ is the explicit form of extra dynamics. It is clear that (16) is the sum of the GD formula (14) and extra dynamics Δu_k . Therefore, the redesigned system via the HB method is

$$x_{k+1} = Ax_k + Cw + \Delta u_k \quad (17)$$

where $\Delta u_k = \beta(x_k - x_{k-1})$. Theorem 5 summarizes the convergence property when applying the HB method to redesign.

Theorem 5. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to $\mathcal{S}_{\mu,L}$ and is twice-differentiable, the step size $\varepsilon = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}$

and $\beta = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}$, then the redesigned system via the HB method (17) is with the optimal convergence rate given by (See Section 3.2.1 in [20] for the proof): $\forall k \in \mathbb{Z}^+$,

$$\|x_k - x^*\| \leq \left(\frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}\right)^k \|x_0 - x^*\| = O(e^{-k\frac{2\sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}}).$$

According to Theorem 5, when implementing the HB method, constant coefficients can be adopted for simplification, i.e., $\varepsilon = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}$, $\beta = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}$.

3) *Redesign via an Accelerated Gradient Descent Method:* Similar to the HB method, an accelerated gradient descent (AGD) method also uses a momentum term. But it constructs an arbitrary auxiliary sequence y_k . AGD achieves the optimal convergence rate $O(1/\varepsilon k^2)$ when objectives belong to function class \mathcal{F}_L . Algorithm 3 shows its iterations.

For any system (1) in Class- \mathcal{O} , we apply the AGD method to solve problem (13) obtained via reverse-engineering to

Algorithm 3 Accelerated gradient descent method

Setting: Choose appropriate positive step size ε , and let $x_1 = x_0$ be an arbitrary initial condition.

$$x_{k+1} = y_k - \varepsilon \nabla f(y_k) \quad (18a)$$

$$y_k = x_k + \beta_k (x_k - x_{k-1}) \quad (18b)$$

improve system performance. To show that the implementation is equivalent to introducing an extra part to the original GD dynamics, (18) is rearranged by interchanging the notations of x and y and combining the two equations:

$$x_{k+1} = x_k - \varepsilon \nabla f(x_k) + \underbrace{\beta_k (y_{k+1} - y_k)}_{\Delta u_k} \quad (19)$$

where $\Delta u_k = \beta_k (y_{k+1} - y_k) = \beta_k [x_k - \varepsilon \nabla f(x_k) - x_{k-1} + \varepsilon \nabla f(x_{k-1})]$. It is clear that (19) is the sum of the gradient descent formula (14) and extra dynamics Δu_k . Therefore, the redesigned system via the AGD method is

$$x_{k+1} = Ax_k + Cw + \Delta u_k \quad (20)$$

where $\Delta u_k = \beta_k (Ax_k + Cw - Ax_{k-1} - Cw')$ and w' is the value of w at time step $k-1$. Theorems 6 and 7 summarize the convergence property when applying the AGD method to redesign.

Theorem 6. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to \mathcal{F}_L , the step size $0 < \varepsilon \leq 1/L$ and $\beta_k = \frac{k-1}{k+2}$, then the convergence rate of the redesigned system via the AGD method (20) is (See Theorem 2.2.2 in [18] for the proof)

$$f(x_k) - f(x^*) \leq \frac{8 \|x_0 - x^*\|^2}{3\varepsilon(k+1)^2} = O\left(\frac{1}{\varepsilon k^2}\right), \quad \forall k \in \mathbb{Z}^+.$$

Theorem 7. For any system (1) in Class- \mathcal{O} , if the objective function f obtained via reverse-engineering belongs to $\mathcal{S}_{\mu,L}$, the step size $\varepsilon = \frac{1}{L}$ and $\beta_k = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$ (constant step scheme III in [18]), then the redesigned system via the AGD method (20) is with the optimal convergence rate [21]

$$\|x_k - x^*\| \leq \left(1 - \sqrt{\frac{\mu}{L}}\right)^k \|x_0 - x^*\| = O(e^{-k\frac{\sqrt{\mu}}{\sqrt{L}}}), \quad \forall k \in \mathbb{Z}^+.$$

According to Theorem 7, when implementing the AGD method, constant coefficients can be adopted for simplification, i.e., $\varepsilon = \frac{1}{L}$, $\beta_k = \frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}$ in Algorithm 3 to achieve the optimal convergence rate.

Note that although both Theorem 5 and Theorem 7 require $f \in \mathcal{S}_{\mu,L}$ and the constant coefficient β/β_k is related to μ , our HB/AGD-based redesign is not restricted to strongly convex functions (obtained via reverse-engineering) only. It is still effective for $f \in \mathcal{F}_L$, but the convergence rates are unclear when coefficients are constants [21]. In this case, we can adjust β/β_k manually. Note that increasing the value of β/β_k enlarges the influence of the momentum term $x_k - x_{k-1}$ in (16) and $y_{k+1} - y_k$ in (19).

TABLE I compares the convergence rates of the original and redesigned systems under HB and AGD methods. When the objective function f obtained via reverse-engineering is in \mathcal{F}_L , the redesigned system via the AGD method has a guaranteed

better convergence rate than the original system. When $f \in \mathcal{S}_{\mu,L}$, the optimal convergence rate of the redesigned system via the HB method is always better than that of the original system and the redesigned system via the AGD method.

TABLE I: Convergence rate comparison.

| f Classes | Original | Apply HB | Apply AGD |
|--|---|--|---|
| $f \in \mathcal{F}_L$ | $O\left(\frac{1}{\varepsilon k}\right)$ | \swarrow | $O\left(\frac{1}{\varepsilon k^2}\right)$ |
| $f \in \mathcal{S}_{\mu,L}$ ⁴ | $O(e^{-k\frac{2\mu}{L+\mu}})$ | $O(e^{-k\frac{2\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}})$ | $O(e^{-k\frac{\sqrt{\mu}}{\sqrt{L}}})$ |

B. Systems in Class- \mathcal{S}

1) *Convergence Rate of the Original Systems:* Any system (1) in Class- \mathcal{S} can be reverse-engineered as a primal-dual gradient (PDG) algorithm to solve a convex-concave saddle-point problem, so the convergence rate of system (1) or (2) follows that of a PDG algorithm. Specifically, we focus on $A_{11} = I_{n_1}$ in (2) so that the optimization problem obtained via reverse-engineering is

$$\max_{\lambda \in \mathbb{R}^m} \min_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda) = f(x) + \lambda^T (Bx - b) \quad (21)$$

where $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ is the Lagrangian multiplier vector (dual variable vector), $B \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Here we have rearranged (3) as above and replaced $x^{(1)}, x^{(2)}, n_1, n_2$ with λ, x, m, n . Notations f, n are abused in contrast to f, n in (3).

Assumption 1. Matrix B is full row rank.

The corresponding primal problem of (21) is an equality constrained convex optimization problem given by

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Bx = b. \quad (22)$$

A PDG algorithm is the simplest algorithm to solve problem (21), as shown in Algorithm 4.

Algorithm 4 First-order primal-dual gradient method

Setting: Choose appropriate positive step sizes $\varepsilon_1, \varepsilon_2$, and let x_0 and λ_0 be arbitrary initial conditions.

$$x_{k+1} = x_k - \varepsilon_1 (\nabla f(x_k) + B^T \lambda_k) \quad (23a)$$

$$\lambda_{k+1} = \lambda_k + \varepsilon_2 (Bx_k - b) \quad (23b)$$

Assume that a saddle point of \mathcal{L} exists and is denoted by (x^*, λ^*) , which satisfies the optimality conditions:

$$\nabla_x \mathcal{L} = \nabla f(x^*) + B^T \lambda^* = 0 \quad (24a)$$

$$\nabla_\lambda \mathcal{L} = Bx^* - b = 0. \quad (24b)$$

Furthermore, assume the objective function obtained via reverse-engineering $f \in \mathcal{S}_{\mu,L}$. Since the primal problem is strongly convex and the constraint is affine, strong duality holds. Therefore, x^* is unique and is an optimal solution of the primal problem (22). When Assumption 1 holds, λ^* is unique.

Introduce the conjugate function of $f(x)$: $f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}$ for all $y \in \mathbb{R}^n$, and (21) can be further

⁴For objectives in this class, we compare the optimal convergence rate for convenience.

expressed as $\min_{\lambda \in \mathbb{R}^m} f^*(-B^T \lambda) + \lambda^T b$. Theorem 8 summarizes the convergence rate of the PDG algorithm/original system.

Theorem 8. *For any system (1) in Class-S, if the objective function f in (21) obtained via reverse-engineering belongs to $\mathcal{S}_{\mu,L}$, Assumption 1 holds and step sizes $\varepsilon_1, \varepsilon_2$ satisfy $0 < \varepsilon_1 \leq \frac{2}{L+\mu}$, $0 < \varepsilon_2 \leq \frac{2}{\sigma_{\min}^2(B)/L + \sigma_{\max}^2(B)/\mu}$ and $c < 1$, then this system converges to the unique saddle point (x^*, λ^*) exponentially. Let $a_k = \|x_k - \nabla f^*(-B^T \lambda_k)\|$, $b_k = \|\lambda_k - \lambda^*\|$ and define a potential function $V_k = \gamma a_k + b_k$, then for some constants c, γ that depend on $\varepsilon_1, \varepsilon_2$, we have*

$$V_{k+1} \leq cV_k, \quad \forall k \in \mathbb{Z}^+. \quad (25)$$

where $\gamma > 0$ and $c = \max\{c_1, c_2\}$ with $c_1 = 1 - \frac{\mu\varepsilon_1}{\varepsilon_2\sigma_{\max}^2(B) + \frac{\varepsilon_2\sigma_{\max}^2(B)}{\gamma}}$ and $c_2 = 1 - \frac{\varepsilon_2\sigma_{\min}^2(B)}{L} + \frac{\varepsilon_2\gamma\sigma_{\max}^2(B)}{\mu^2}$.

Proof. See the Appendix in [22]. \square

Note that the potential function V_k decreases at a geometric rate and the error of $\|x_k - x^*\|$ and $\|\lambda_k - \lambda^*\|$ are bounded by V_k : $\|\lambda_k - \lambda^*\| \leq V_k$ and $\|x_k - x^*\| \leq \|x_k - \nabla f^*(-B^T \lambda_k)\| + \|\nabla f^*(-B^T \lambda_k) - x^*\| \leq a_k + \frac{\sigma_{\max}(B)}{\mu} b_k \leq \max\left\{\frac{1}{\gamma}, \frac{\sigma_{\max}(B)}{\mu}\right\} V_k$. Therefore, as V_k approaches zero, (x_k, λ_k) approaches the saddle point (x^*, λ^*) .

Corollary 2. *When the step sizes $\varepsilon_1 = \frac{2}{L+\mu}$, $\varepsilon_2 = \left(\frac{\sigma_{\max}^2(B)}{\mu} + \frac{\sigma_{\max}(B)}{\gamma} + \frac{\sigma_{\min}^2(B)}{L} - \frac{\gamma\sigma_{\max}^2(B)}{\mu^2}\right)^{-1} \frac{2\mu}{L+\mu}$, the optimal convergence rate is attained for the original system. Also, suppose $\gamma = \frac{\mu^2\sigma_{\min}^2(B)}{2L\sigma_{\max}^2(B)}$ and we have $c \leq 1 - \frac{1}{\kappa^3(4\tau^2+2\tau+1)}$, where $\kappa = \frac{L}{\mu}$ is the condition number and $\tau = \frac{\sigma_{\max}^2(B)}{\sigma_{\min}^2(B)}$.*

Proof. See the proof in [22], Corollaries 2&3. \square

Corollary 2 shows that the optimal convergence rate of the original system is related to the condition number κ and τ . A large κ implies a slow convergence rate since the iterations oscillate back and forth [23]. Thus, we utilize Augmented Lagrangian (AL) and hat-x methods to change the convergence rate by changing the value of κ of the objective. Applying those methods to solve problem (21) obtained via reverse-engineering results in a redesigned system formula with improved performance.

2) Redesign via an Augmented Lagrangian Method: Adding the square of the equality constraints as penalty terms can change the condition number of the original problem while the optimal solution stays unchanged. This method is known as the AL method. The corresponding AL function for problem (22) is

$$\mathcal{L}_a = f(x) + \lambda^T(Bx - b) + \frac{\alpha}{2} \|Bx - b\|^2 \quad (26)$$

where $\alpha > 0$ is a scalar. Equation (26) is the Lagrangian for

$$\min_{x \in \mathbb{R}^n} f(x) + \frac{\alpha}{2} \|Bx - b\|^2 \quad \text{s.t. } Bx = b \quad (27)$$

which has the same minimum and optimal solution as the original problem (22) [24].

For any system (1) in Class-S, applying the AL method to solve the problem obtained via reverse-engineering leads to a redesigned system with improved performance:

$$x_{k+1} = x_k - \varepsilon_1 (\nabla f(x_k) + B^T \lambda_k) - \underbrace{\varepsilon_1 \alpha B^T (Bx_k - b)}_{\Delta u_k}$$

where $\Delta u_k = -\varepsilon_1 \alpha B^T (Bx_k - b)$. Note that the extra part is added to primal variables only.

Let $g(x) = f(x) + \frac{\alpha}{2} \|Bx - b\|^2$ and we compare the condition numbers of f and g . The Hessian matrix of $g(x)$ is $\mathbf{H}_g = \mathbf{H}_f + \alpha B^T B$ and $\lambda_{\min}(\mathbf{H}_g) I_n \preceq \mathbf{H}_g \preceq \lambda_{\max}(\mathbf{H}_g) I_n$. The condition number of $g(x)$, denoted by κ_g , is $\frac{\lambda_{\max}(\mathbf{H}_g)}{\lambda_{\min}(\mathbf{H}_g)}$. Denote by κ_0 the condition number of f and $\kappa_0 = L/\mu$. For a specific problem, we are able to obtain the numerical form of matrix \mathbf{H}_g . In this case, we can calculate κ_g and κ_0 precisely. If $\kappa_g < \kappa_0$, applying the AL method is able to improve the convergence rate; otherwise, α should be set to zero to avoid the influence of penalty terms. Note that matrix B has a significant influence on the effectiveness of this method. The matrix product $B^T B$ will not change the topological structure of the original system as the redesigned system can be rearranged as $x_{k+1} = x_k - \varepsilon_1 \nabla f(x_k) - \varepsilon_1 B^T (\lambda_k + \alpha(Bx_k - b))$ where $Bx_k - b$ can be obtained from the original update of λ .

Another benefit of applying the AL method is the convexification when the objective f obtained via reverse-engineering is not strongly convex in \mathbb{R}^n .

Theorem 9. *Assume that \mathbf{H}_f is positive definite on the nullspace of $B^T B$, i.e., $y^T \mathbf{H}_f y > 0$ for all $y \neq 0$ with $y^T B^T B y = 0$. Then there exists a scalar $\bar{\alpha}$ such that for $\alpha > \bar{\alpha}$, we have (See Theorem 4.2 in [25] for the proof)*

$$\mathbf{H}_f + \alpha B^T B \succ 0.$$

3) Redesign via a Hat-x method: Another way is to introduce a free variable $\hat{x} \in \mathbb{R}^n$ to prevent a dramatic change of x . This leads to

$$\min_{x, \hat{x} \in \mathbb{R}^n} f(x) + \frac{\alpha}{2} \|x - \hat{x}\|^2 \quad \text{s.t. } Bx = b \quad (28)$$

where α is a scalar. This problem is equivalent to

$$\min_{z \in \mathbb{R}^{2n}} h(z) \quad \text{s.t. } \bar{B}z = b$$

where $z = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$, $h(z) = f(x) + \frac{\alpha}{2} z^T \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} z$, $\bar{B} = \begin{bmatrix} B & 0 \end{bmatrix} \in \mathbb{R}^{m \times 2n}$. Suppose Assumption 1 holds, then \bar{B} is also full row rank and the maximal and minimal singular values of \bar{B} are the same as those of B . The Lagrangian is

$$\mathcal{L}_h = f(x) + \lambda^T(Bx - b) + \frac{\alpha}{2} \|x - \hat{x}\|^2.$$

Applying an optimality condition, we have

$$\nabla_x \mathcal{L}_h = \nabla f(x^*) + B^T \lambda^* + \alpha(x^* - \hat{x}^*) = 0 \quad (29a)$$

$$\nabla_{\hat{x}} \mathcal{L}_h = \alpha(\hat{x}^* - x^*) = 0 \quad (29b)$$

$$\nabla_{\lambda} \mathcal{L}_h = Bx^* - b = 0. \quad (29c)$$

Then $\hat{x}^* = x^*$. Therefore, x^*, λ^* in the optimal solution $(x^*, \hat{x}^*, \lambda^*)$ of (28) are the same as that of (22).

For any system (1) in Class-S, apply the hat-x method to solve the optimization problem obtained via reverse-engineering to improve system performance. The iterations of the primal variables are

$$x_{k+1} = x_k - \varepsilon_1 (\nabla f(x_k) + A^T \lambda_k) - \underbrace{\varepsilon_1 \alpha (x_k - \hat{x}_k)}_{\Delta u_k}$$

$$\hat{x}_{k+1} = \hat{x}_k + \varepsilon_1 \alpha (x_k - \hat{x}_k)$$

where $\Delta u_k = -\varepsilon_1 \alpha (x_k - \hat{x}_k)$. Note that the extra part is added to the primal variables only.

Denote by κ_h and \mathbf{H}_h the condition number and the Hessian matrix of $h(z)$. Then $\kappa_h = \frac{\lambda_{\max}(\mathbf{H}_h)}{\lambda_{\min}(\mathbf{H}_h)}$. Next, we compare κ_0 and κ_h . To do this, we first obtain the range of κ_h , and then compare the lower and upper bounds of κ_h with κ_0 .

Theorem 10. Assume the objective obtained via reverse-engineering in (22) $f \in \mathcal{S}_{\mu,L}$, then $\kappa_h \leq \kappa_h \leq \bar{\kappa}_h$, where $\underline{\kappa}_h = \frac{2\alpha + \mu + \sqrt{\mu^2 + 4\alpha^2}}{2\alpha + L - \sqrt{L^2 + 4\alpha^2}}$ and $\bar{\kappa}_h = \frac{2\alpha + L + \sqrt{L^2 + 4\alpha^2}}{2\alpha + \mu - \sqrt{\mu^2 + 4\alpha^2}}$.

Proof. See the proof in [22], Proposition 2. \square

Now we can compare the range of κ_h with κ_0 . Let $M = \kappa_h - \kappa_0$, $w = \frac{\mu}{L}$ and $v = \frac{\alpha}{L}$ with $0 < w \leq 1$ and $v > 0$, then $M = \frac{2v + w + \sqrt{w^2 + 4v^2}}{2v + 1 - \sqrt{1 + 4v^2}} - \frac{1}{w}$. When w tends to 0, $M < 0$; otherwise, $M > 0$. On the other hand, $\bar{\kappa}_h - \kappa_0 = \frac{1}{4\alpha\mu} [(2\alpha + \sqrt{L^2 + 4\alpha^2})(2\alpha + \mu + \sqrt{\mu^2 + 4\alpha^2}) + L(\mu + \sqrt{\mu^2 + 4\alpha^2} - 2\alpha)]$. Since $\mu + \sqrt{\mu^2 + 4\alpha^2} > 2\alpha$, $\bar{\kappa}_h > \kappa_0$. To conclude, κ_h can be smaller than κ_0 , depending on specific problems. Moreover, the hat-x method increases the dimension of the original system and works like a low-pass filter. This method could slow down system dynamics and smooth the trajectories, as demonstrated in Section IV-B.

C. Reconfiguration Steps

To summarize, the redesign approach is consist of the following three steps:

- 1) **Reverse-engineering:** For a given system (1), apply Theorem 1 or 2 to reverse-engineer it as a GD algorithm (14) or a PDG algorithm (23) for solving an unconstrained convex optimization problem (13) or a saddle-point problem (21), i.e., system dynamics (1) can be rewritten as the form (14) or (23).
- 2) **Acceleration:** Apply an HB/AGD method or an AL/hat-x method to solve the optimization problem obtained in Step 1), which results in a redesigned system.
- 3) **Implementation:** Rearrange the redesigned system and compare it with the original system (14)/(23) or (1) to obtain the implementation of the extra dynamics.

This approach can be implemented by any dynamical system belonging to Class- \mathcal{O} or Class- \mathcal{S} to systematically improve its performance. With the extra dynamics added to the original controllers, the convergence rates/speeds and transient behaviors will be improved while the original control structures remain. Also, this approach is able to handle network systems as shown in Section IV, whose scale can be large.

IV. EXAMPLES REVISITED

A. Internet Congestion Control

Following the redesign procedure, applying Algorithm 2 and Algorithm 3 to solve problem (8), we obtain the same form of redesigned dynamics but with different $\Delta u(k)$ given by

$$x(k+1) = x(k) + \varepsilon \text{diag}\{k_{x_i}\} (U'(x(k)) - R^T f(Rx(k))) + \Delta u(k).$$

For HB-based redesign, $\Delta u(k) = \beta(x(k) - x(k-1))$; while for AGD-based redesign, $\Delta u(k) = \beta_k \left\{ x(k) - x(k-1) + \varepsilon \text{diag}\{k_{x_i}\} [U'(x(k)) - q(k) - U'(x(k-1)) + q(k-1)] \right\}$, where q is the corresponding vector form of q_i and $q = R^T f(Rx)$.

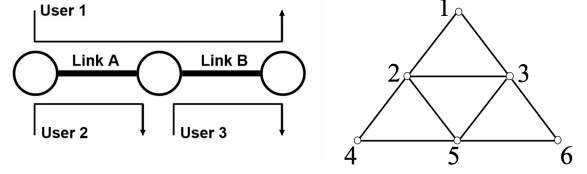


Fig. 1: Left: A 2-link network shared by 3 users. Right: A regular network of 6 agents.

Consider a 2-link network as shown in Fig. 1. Choose utility functions as $U_i(x_i) = \log x_i$, $k_{x_i} = 0.1$, $\beta = 0.54$ and $\beta_k = 0.6$. Choose the penalty function as in [12]: $f_l(y_l) = \frac{(y_l - c_l + \sigma)^+}{\sigma^2}$, where σ is a small positive number and $(y_l - c_l + \sigma)^+ = \max\{y_l - c_l + \sigma, 0\}$. Initially, the capacities of links A and B are 2, 4 respectively and they change to 3, 1 after some time to simulate a sudden change of the link capacity. As shown in Fig. 2, the redesigned dynamics via HB and AGD methods perform better than the original dynamics.

B. Distributed PI Control for Single Integrator Dynamics

Problem (10) obtained via reverse-engineering is equal to

$$\begin{aligned} \min_{y \in \mathbb{R}^n} f &= \frac{\rho_2}{2} y^T L y + \frac{\delta}{2} y^T y - y^T d - \delta y^T y(0) \\ \text{s.t. } \rho_1 \tilde{L}^T y &= \mathbf{0}. \end{aligned}$$

Since $\mathbf{H}_f = \rho_2 L + \delta I_n \succ 0$, $f \in \mathcal{S}_{L,u}$. Following the redesign procedure, applying the Augmented Lagrangian and hat-x methods, we obtain the same form of redesigned dynamics but with different $\Delta u(k)$ given by

$$\begin{aligned} y(k+1) &= y(k) - \varepsilon_1 (\rho_2 L y(k) + \delta y(k) - d - \delta y(0) \\ &\quad + \rho_1 \tilde{L} z(k)) + \Delta u(k) \\ z(k+1) &= z(k) + \varepsilon_2 D y(k). \end{aligned}$$

For the AL-based redesign, $\Delta u(k) = -\varepsilon_1 \alpha \tilde{L}^T y(k)$; while for the hat-x-based redesign, $\Delta u(k) = -\varepsilon_1 \alpha (y(k) - \hat{y}(k))$, where $\hat{y}(k+1) = \hat{y}(k) + \varepsilon_1 \alpha (y(k) - \hat{y}(k))$.

Consider a network of 6 agents with the topology shown in Fig. 1. In addition, communication delay is considered in this network since this could happen in reality. Let the constant disturbance $d = [0, 2, 0, 0, 0, 0]^T$, the initial condition $x(0) = [5, -6, 8, 2, -4, 0]^T$, the integral gain $\rho_1 = 10$, the static gain $\rho_2 = 0.5$, $\delta = 1$. As shown in Fig. 3, redesigned dynamics via the AL and hat-x methods perform better than the original dynamics, while the redesigned dynamics via hat-x is the most smooth.

V. CONCLUSION AND FUTURE WORK

This paper has proposed a control reconfiguration approach to improve the performance of two classes of dynamical systems. This approach is to firstly reverse-engineer a given dynamical system as a gradient descent or a primal-dual gradient algorithm to solve a convex optimization problem. Then, by utilizing several acceleration techniques, the extra control term is obtained and added to the original control structure. Under this retrofit procedure, system performance is improved while the control structure remains, as demonstrated by both theoretical results and practical applications.

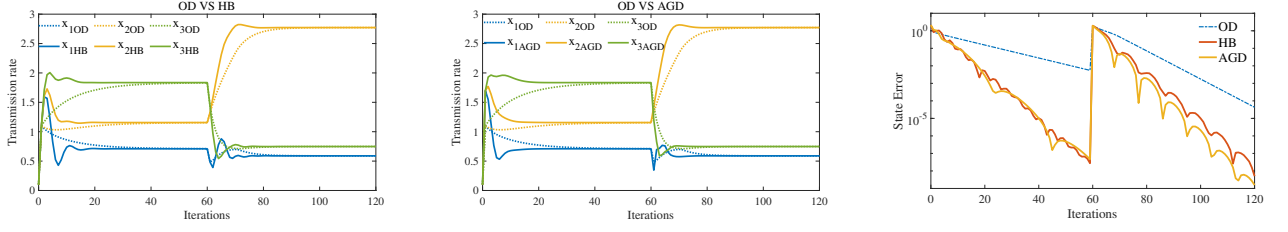


Fig. 2: Internet congestion control. Left & Middle: Simulation results (“OD” stands for original dynamics; “HB” and “AGD” stand for redesigned dynamics via HB and AGD). Right: State error measured by $\|x - x^*\|_2$.

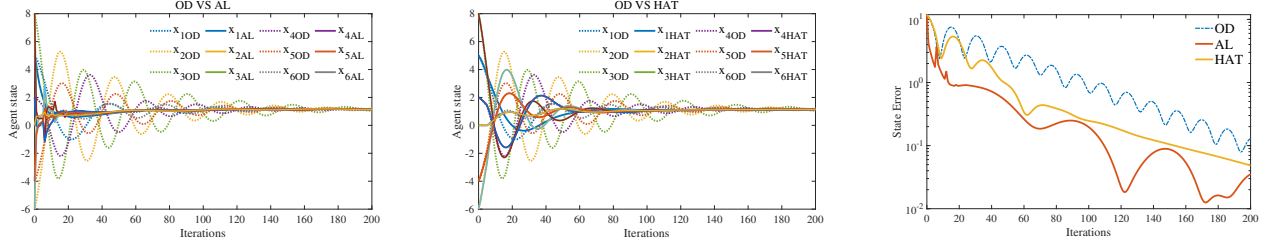


Fig. 3: Distributed PI control. Left & Middle: Simulation results (“OD” stands for original dynamics; “AL” and “HAT” stand for redesigned dynamics via AL and hat-x). Right: State error measured by $\|y - y^*\|_2$.

In the future, we will investigate the implementation of the redesign in a continuous-time setting as well as the framework for nonlinear systems. Also, the case when $\mathbf{H}_{f(x^{(1)})} \preceq 0$ in the definition of Class- \mathcal{S} will be studied. Last but not least, we will consider redesigning to improve other properties of systems, for example, robustness to delays.

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