CRYSTALLINE COHOMOLOGY OF TOWERS OF CURVES

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Abstract. We investigate the geometry of *finite maps* and *correspondences* between curves, and construct canonical trace and pullback maps between Hyodo–Kato integral structures on de Rham cohomology of curves, which are functorial for finite morphisms of the generic fibres. This leads to a crystalline version of the étale cohomology of towers of modular curves considered by Hida and Ohta, whose ordinary part satisfies Λ-adic control and Eichler–Shimura theorems.

Contents

Introduction

We study finite maps between curves as geometric objects, and investigate their semi-stable models and functoriality of extra structures on various cohomology theories on the generic fibre.

A. Stable models of correspondences. The first part is preliminary, and mainly recalls language and techniques pertaining to semi-stable models of curves. As a by-product, we generalise and slightly strengthen work of Coleman [Col03] and Liu [Liu06] to *correspondences* between curves. More precisely, let $\mathbf{C} : Y_1 \leftarrow X \rightarrow Y_2$ be a correspondence between smooth, proper, geometrically connected curves X, Y_1 and Y_2 over a non-Archimedean field K, with finite maps.

Theorem A. *After passing to a finite separable extension L/K, we can find a correspondence*

$$\mathfrak{C} : \quad \mathcal{Y}_1 \xleftarrow{\ \pi_1\ } \mathcal{X} \xrightarrow{\ \pi_2\ } \mathcal{Y}_2$$

where $\mathcal{X}, \mathcal{Y}_1$, and $\mathcal{Y}_2$ are semi-stable models for $X_L, Y_{1,L}$, and $Y_{2,L}$ over the valuation ring of L, the morphisms π_1 and π_2 are finite, and $\mathfrak{C}$ restricts to $\mathbf{C}$ on the generic fibres of the curves.

If $\mathbf{C}$ is a hyperbolic correspondence, which may be achieved after adding a finite number of punctures, there is a canonical *stable* model which is minimal for the relation of domination. We prove that under certain conditions we may even find semi-stable models that are *skeletal*, see 2.4. The finiteness of the maps between the models is crucial, and is what gives us precise information about the spectral properties of various linearisations, such as for instance:

$$\mathbf{C}^*_{\text{ét}} : \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{1,\overline{K}}, \mathbf{Q}_l) \longrightarrow \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{2,\overline{K}}, \mathbf{Q}_l), \qquad \mathbf{C}^*_{\text{Nér}} : \Phi_1^{\text{Nér}} \longrightarrow \Phi_2^{\text{Nér}}.$$

As a toy example, we exhibit stable models at $\mathfrak{p}$ of various Hecke operators $T_{\mathfrak{l}}$ on quaternionic Shimura curves, both for $\mathfrak{l} \neq \mathfrak{p}$ and $\mathfrak{l} = \mathfrak{p}$. We also present a mild generalisation of the work on canonical subgroups of Goren–Kassaei [GK06] in Theorem 2.2.

B. Integral structures on de Rham cohomology. We then turn to the main result, which provides functorial integral structures on de Rham cohomology of smooth, proper curves X_K with semi-stable reduction over the integers R of a non-Archimidean local field K. These cohomology groups have a canonical filtration, as well as Frobenius and monodromy operators φ and N. A functorial R-structure of the Hodge filtration

$$0 \longrightarrow \mathrm{H}^0(X, \Omega^1_{X/K}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X/K) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow 0$$

was constructed by Cais [Cai09]. In this paper, we address functorial integral structures for the (φ, N)-module $\mathrm{H}^1_{\mathrm{dR}}(X)$ when X has semi-stable reduction over R. Hyodo–Kato cohomology provides us with a lattice over the Witt vectors W of the residue field, with operators φ and N, depending on a choice of uniformiser $\varpi \in R$. We show that these lattices, together with their concomitant operators φ and N, are in fact functorial for morphisms between generic fibres:

Theorem B. *Let $f : X \to Y$ be a finite map of smooth, proper, geometrically irreducible curves over K with semi-stable reduction, then there exist canonical trace and pullback maps of (φ, N)-modules*

$$\begin{array}{rcccc} f_* & : & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}^+_s/W) & \longrightarrow & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{Y}^+_s/W), \\ f^* & : & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{Y}^+_s/W) & \longrightarrow & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}^+_s/W), \end{array}$$

between cohomology of the special fibres of any semi-stable R-models $\mathcal{X}, \mathcal{Y}$, which recover the usual trace and pullback maps on de Rham cohomology of the generic fibres via the Hyodo–Kato isomorphism ρ_ϖ.

For instance, given a correspondence $\mathbf{C}$ as in the previous paragraph, it follows that the (φ, N)-modules over W provided by Hyodo–Kato cohomology are preserved by the de Rham linearisation

$$\mathbf{C}^*_{\mathrm{dR}} : \mathrm{H}^{\mathrm{i}}_{\mathrm{dR}}(Y_1/K) \longrightarrow \mathrm{H}^{\mathrm{i}}_{\mathrm{dR}}(Y_2/K).$$

In fact, the trace and pullback maps in Theorem B are already constructed on the level of de Rham–Witt complexes in the derived category. More precisely, we first establish a result that provides a canonical isomorphism between de Rham–Witt complexes of different semi-stable models. Then, for a certain choice of semi-stable models we obtain an extension $f : \mathcal{X} \to \mathcal{Y}$ to a *finite* map, and we construct pullback and trace maps on de Rham–Witt complexes $W\omega^\bullet$ and the extensions $W\widetilde{\omega}^\bullet$ used to define the monodromy operator, that make the following diagram commute:

$$\begin{array}{ccccccc} \mathrm{R}f_*W\omega^\bullet_{\mathcal{X}^+_s}[-1] & \longrightarrow & \mathrm{R}f_*W\widetilde{\omega}^\bullet_{\mathcal{X}^+_s} & \longrightarrow & \mathrm{R}f_*W\omega^\bullet_{\mathcal{X}^+_s} & \xrightarrow{+1} & \\ {\scriptstyle f^*[-1]}\downarrow\uparrow{\scriptstyle f_*[-1]} & & {\scriptstyle \widetilde{f}^*}\uparrow\downarrow{\scriptstyle \widetilde{f}_*} & & {\scriptstyle f^*}\downarrow\uparrow{\scriptstyle f_*} & & \\ W\omega^\bullet_{\mathcal{Y}^+_s}[-1] & \longrightarrow & W\widetilde{\omega}^\bullet_{\mathcal{Y}^+_s} & \longrightarrow & W\omega^\bullet_{\mathcal{Y}^+_s} & \xrightarrow{+1} & \end{array}$$

C. Log-crystalline cohomology of towers. The above construction of trace maps between Hyodo–Kato cohomology gives us a way to define crystalline analogues of completed cohomology and Eichler–Shimura cohomology. More precisely, consider a tower

$$\dots \xrightarrow{f_4} X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

of finite maps between smooth, proper, geometrically irreducible curves over $K/\mathbf{Q}_p$. Every curve X_r has a semi-stable model $\mathcal{X}_r$ over some finite K_r/K. By Theorem B, we may consider the limits

$$(1) \qquad \begin{cases} \widetilde{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty) & := & \varprojlim_n \varinjlim_{f_r^*} \mathrm{H}^1_{\mathrm{HK}}\left(\mathcal{X}_r^+ \otimes \overline{\mathbf{F}}_p, W/p^n W\right) \\ \widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty) & := & \varprojlim_{f_{r,*}} \mathrm{H}^1_{\mathrm{HK}}\left(\mathcal{X}_r^+ \otimes \overline{\mathbf{F}}_p, W\right) \end{cases}$$

Even though the field extensions K_r may quickly become highly ramified, the Hyodo–Kato cohomology is always defined over the ring of Witt vectors W of $\overline{\mathbf{F}}_p$. For towers of modular curves, these form crystalline analogues of completed cohomology, and Eichler–Shimura cohomology respectively.

To reassure us that these crystalline objects encode interesting arithmetic information, we investigate a special case. The U_p-ordinary part of the Eichler–Shimura cohomology for the $\Gamma_1(p^r)$ tower is a $\Lambda = \mathbf{Z}_p[\![1 + p\,\mathbf{Z}_p]\!]$-module. The following crystalline control and Eichler–Shimura theorems follow immediately from the work of Cais [Cai16a, Cai16b] who obtains crystalline objects via Dieudonné theory.

Theorem C. *The projection $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\mathrm{ord}}$ of crystalline Eichler–Shimura cohomology is a finite free Λ-module. There is a canonical decomposition of finite free Λ-modules*

$$\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\mathrm{ord}} \simeq \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}^\infty_\infty)'^{\mathrm{F-ord}} \oplus \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}^0_\infty)'^{\mathrm{V-ord}}.$$

Our approach provides a larger context for the ordinary cohomology studied by Cais, with a vast amount of non-ordinary spectral information encoded in it. Recent advances in the explicit determination of semi-stable models of towers of modular curves furthermore suggest a concrete geometric approach for studying this non-ordinary part, which will be addressed in future work.

1. Specialisation maps and formal fibres

We start by recalling the notion of semi-stable vertex sets of smooth quasi-projective curves. They provide us with a combinatorial tool to study semi-stable models and finite maps between them, via the specialisation map. The material in this section can be found in Baker–Payne–Rabinoff [BPR11] and Amini–Baker–Brugallé–Rabinoff [ABBR15] though we adopt the language of adic spaces.

1.1. **Notation.** Let K be an algebraically closed, complete, non-Archimedean field with topology induced by a non-trivial valuation $|\cdot|$ of rank 1. Let R be its valuation ring with maximal ideal $\mathfrak{m}$ and residue field k. For a curve $\mathcal{X}$ over R, we write $\mathcal{X}_s$ for its special fibre, and X for its generic fibre. We let $f : X \to Y$ be a finite morphism between smooth proper connected curves over K, and $X^{\mathrm{ad}}, Y^{\mathrm{ad}}$ the adic spaces associated to X, Y. We will abuse notation and write f for the induced map f^{ad} between these adic spaces. Given a point $x \in X^{\mathrm{ad}}$, we write $K(x)$ for the residue field of $\mathcal{O}_{X^{\mathrm{ad}},x}$ and $K(x)^+$ for the image of $\mathcal{O}^+_{X^{\mathrm{ad}},x}$, with residue field $k(x)$.

1.2. **Hyperbolic curves and stable reduction.** To include curves of small genus in our discussion, we will allow *punctures* $D_X \subset X(K)$ and $D_Y \subset Y(K)$, which are finite sets of Type-I points with $f^{-1}(D_Y) = D_X$. Recall that a punctured curve (X, D_X) is said to be *hyperbolic* if $\chi(X, D_X) < 0$, where $\chi(X, D_X) = 2 - 2g(X) - |D_X|$ is the Euler characteristic. A *semi-stable formal model* of (X, D_X) is an integral proper admissible formal R-scheme $\mathfrak{X}$ such that its adic generic fibre is isomorphic to X^{ad}, and moreover

- $\mathfrak{X}_s$ is a reduced connected curve over k with at most ordinary double points for singularities,
- all points in D_X reduce to distinct smooth points on $\mathfrak{X}_s$.

The category $\mathbf{Form}^{\mathrm{ss}}_X$ consists of semi-stable formal models of (X, D_X), together with an isomorphism between the adic generic fibre and X^{ad}. A morphism between two such models is a morphism of formal

R-schemes that induces the identity on X^{ad} via the chosen isomorphisms. When (X, D_X) is hyperbolic, there exists a terminal object, which we call the *stable formal model*.

Finally, we note that for curves there is no essential difference between working with semi-stable formal models, or algebraic semi-stable models, as follows from the following well-known result.

Lemma 1.1. *Let X/K be a proper smooth connected curve. Then completion along the special fibre defines an equivalence between semi-stable R-models of X and semi-stable formal R-models of X.*

1.3. **Wide open disks and annuli.** We follow Coleman in defining a wide open disk to be the complement of the set $|t| = 1$ in $\mathrm{Spa}(K\langle t\rangle, R\langle t\rangle)$. A wide open annulus is the complement in a wide open disk of the set $|t| \leq p^{-w}$ for some $w \in \mathbf{R}_{>0} \cup \{\infty\}$ which we call the *width* of the annulus. We note that a wide open disk possesses exactly one Type-V point which is not the specialisation of any Type-II point. This Type-V point is called the *apex point* of the open disk. Similarly, wide open annuli have exactly 2 such points, which we also call apex points.

1.4. **The specialisation map.** Let $\mathfrak{X}$ be an admissible formal R-scheme with generic fibre X^{ad}. The specialisation map is a canonical morphism of locally ringed topological spaces

$$\mathrm{sp}_{\mathfrak{X}} : \left(X^{\mathrm{ad}}, \mathcal{O}^{+}_{X^{\mathrm{ad}}}\right) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), \tag{2}$$

whose fibres are called *formal fibres*. When $\mathfrak{X}$ is semi-stable, it is possible to determine the nature of the formal fibres by combining the work of Bosch-Lütkebohmert [BL85, Propositions 2.2 and 2.3] and Berkovich [Ber90, Proposition 2.4.4]. We obtain the following theorem, see also [BPR11, Theorem 4.6].

Theorem 1.2 (Bosch–Lütkebohmert, Berkovich)**.** *Let ξ be a point of $\mathfrak{X}_s$. Then*

- *ξ is a generic point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ consists of a single Type-II point of X^{ad},*
- *ξ is a smooth closed point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ is a wide open disk,*
- *ξ is an ordinary double point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ is a wide open annulus.*

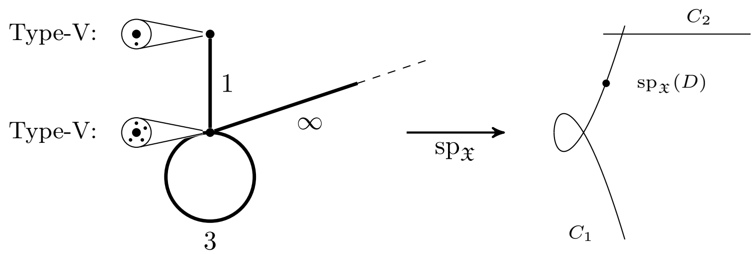
1.5. **Semi-stable vertex sets and skeleta.** A *semi-stable vertex set* of a smooth, proper, punctured curve (X, D_X) is a finite set V of Type-II points of X^{ad} such that

- the space $X^{\mathrm{ad}}\backslash V$ is a disjoint union of wide open disks and finitely many wide open annuli,
- the points in D_X belong to distinct wide open disks in $X^{\mathrm{ad}}\backslash V$.

For a Type-II point x in a semi-stable vertex set V, call its *valency* the number of apex points of wide open annuli in $X^{\mathrm{ad}}\backslash V$ in the topological closure of x. The category $\mathbf{Vert}^{\mathrm{ss}}_X$ consists of semi-stable vertex sets of (X, D_X), where morphisms are given by inclusion. Theorem 1.2 allows us to attach to a semi-stable formal model $\mathfrak{X}$ the finite set $V_{\mathfrak{X}} := \{\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)\}_{\xi}$, where ξ ranges over the generic points of the irreducible components of $\mathfrak{X}_s$. It follows from Theorem 1.2 that $V_{\mathfrak{X}}$ is a semi-stable vertex set for (X, D_X), and we obtain a functor from $\mathbf{Form}^{\mathrm{ss}}_X$ to $\mathbf{Vert}^{\mathrm{ss}}_X$. This is an anti-equivalence, as proved in [BPR11, Theorem 4.11].

Theorem 1.3. *The functor $\mathbf{Form}^{\mathrm{ss}}_X \to \mathbf{Vert}^{\mathrm{ss}}_X \ : \ \mathfrak{X} \mapsto V_{\mathfrak{X}}$ induces an anti-equivalence of categories.*

To a semi-stable vertex set V for (X, D_X), we associate its *skeleton* Σ_V, which is the set of points of X^{ad} that are not contained in a wide open disk which is disjoint from $V \cup D_X$. As an example, consider the projective closure of $y^2 = x^3 + x^2 + p^3$ as a formal scheme over $\mathcal{O}_{\mathbf{C}_p}$, and let $\mathfrak{X}$ be the blow-up at a smooth point of the special fibre. Set X to be the generic fibre of $\mathfrak{X}$, and $D_X = \{(0, 1, 0)\}$ the point at infinity. Then $\mathfrak{X}$ is a semi-stable model of (X, D_X) whose special fibre consists of a nodal curve C_1 and a projective line C_2, crossing transversally. The skeleton can be visualised as



The Type-II points in the corresponding semi-stable vertex set $V_{\mathfrak{X}}$ have valencies 1 and 4, and are both of genus 0. The widths of the wide open annuli in $X^{\mathrm{ad}}\backslash V_{\mathfrak{X}}$ are $1, 3$ and ∞.

The following theorem is proved in [ABBR15, Theorem 5.13].

Theorem 1.4. *Let $\mathfrak{X}, \mathfrak{Y}$ be semi-stable formal models of (X, D_X) and (Y, D_Y), then f extends to a morphism $\mathfrak{X} \to \mathfrak{Y}$ if and only if $f^{-1}(V_{\mathfrak{Y}}) \subseteq V_{\mathfrak{X}}$. This extension is finite if and only if $f^{-1}(V_{\mathfrak{Y}}) = V_{\mathfrak{X}}$.*

We will be interested in base fields K_0 that are not necessarily algebraically closed or complete. Let K_0 with a non-trivial non-Archimedean valuation of rank 1 and valuation ring R_0. Let $K = \widehat{\overline{K}_0}$ be the completion of an algebraic closure of K_0, which is itself algebraically closed, and let R be the valuation ring of K. The following lemma is proved in [ABBR15, Lemma 5.5].

Lemma 1.5. *Let $f : X \to Y$ be a finite morphism between smooth, proper, geometrically connected curves over K_0, with semi-stable R_0-models $\mathcal{X}$ and $\mathcal{Y}$ respectively. Suppose that f_K extends to a finite morphism $\mathcal{X}_R \to \mathcal{Y}_R$, then f extends uniquely to a finite morphism $\mathcal{X} \to \mathcal{Y}$ defined over R_0.*

2. Stable models of correspondences and their skeleta

As a toy application of these analytic techniques, we prove an analogue for correspondences of the stable reduction theorem of Deligne–Mumford [DM69]. This generalises the results for finite maps proved by Coleman [Col03] and Liu [Liu06]. We also prove a stronger *skeletal* version under some additional hypotheses, and discuss examples coming from Hecke operators on Shimura curves.

2.1. **Definitions.** Let K be a field equipped with a non-trivial non-Archimedean valuation of rank 1, whose valuation ring R has maximal ideal $\mathfrak{m}$. A *punctured correspondence* is a diagram

(3)
$$\mathbf{C} : \quad (Y_1, D_1) \xleftarrow{\ \pi_1\ } (X, D_X) \xrightarrow{\ \pi_2\ } (Y_2, D_2)$$

where X, Y_1, Y_2 are smooth, proper, geometrically connected K-curves, π_1, π_2 are finite K-morphisms; and D_X, D_1, D_2 are finite sets of K-points with $\pi_1^{-1}(D_1) = D_X = \pi_2^{-1}(D_2)$. A punctured correspondence $\mathbf{C}$ is said to be *hyperbolic* if its objects are hyperbolic punctured curves in the sense of 1.2.

A *semi-stable R-model* of $\mathbf{C}$ is a diagram $\mathfrak{C} : \mathcal{Y}_1 \leftarrow \mathcal{X} \to \mathcal{Y}_2$ with finite morphisms, together with isomorphisms $\mathcal{X}_K \simeq X$ as well as $\mathcal{Y}_{1,K} \simeq Y_1$ and $\mathcal{Y}_{2,K} \simeq Y_2$, so that their formal completions along the special fibre are semi-stable formal R-models for $(X, D_X), (Y_1, D_1)$ and (Y_2, D_2) in the sense of 1.2 and $\mathfrak{C}$ restricts to $\mathbf{C}$ via the given isomorphisms. A morphism $\mathfrak{C} \longrightarrow \mathfrak{C}'$ is a commutative diagram

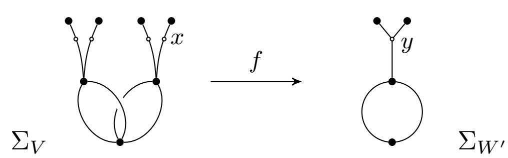
(4)
$$\begin{array}{ccccccc} & & \mathcal{X} & \dashrightarrow & \mathcal{X}' & & \\ & \swarrow & & \searrow \ \swarrow & & \searrow & \\ \mathcal{Y}_1 & \dashrightarrow & \mathcal{Y}_1' & & \mathcal{Y}_2 & \dashrightarrow & \mathcal{Y}_2' \end{array}$$

where the dashed arrows are morphisms of semi-stable models as in 1.2. We say that $\mathfrak{C} \longrightarrow \mathfrak{C}'$ is dominant if the dashed arrows in the above diagram are dominant. A model is called *stable* if it is minimal with respect to the relation of domination. Clearly, the stable model of a hyperbolic correspondence is unique up to isomorphism if it exists.

2.2. **Stable models of Galois morphisms.** Before coming to a proof of potentially stable reduction of correspondences, we prove a lemma about Galois maps $f : X \longrightarrow Y$ to which the general case will be reduced. The weaker statement, not insisting that the extension of f should be finite, was proved by Liu–Lorenzini [LL99, Proposition 4.4] when R is a DVR. A proof for $K = \mathbf{C}_p$ is given in Coleman [Col03, Section 3]. We note also that this result is false without the assumption that f is Galois.

Lemma 2.1. *Let $f : (X, D_X) \longrightarrow (Y, D_Y)$ be a finite Galois morphism of smooth, proper, geometrically connected, hyperbolic punctured curves over K. Assume (X, D_X) has stable model $\mathcal{X}$ over R. Then after a finite separable base change, there exists a unique semi-stable model $\mathcal{Y}$ of (Y, D_Y) such that f extends to a finite $\mathcal{X} \longrightarrow \mathcal{Y}$.*

Proof. Extend scalars to $\overline{K}^{\wedge}$, and let $V \subset X^{\mathrm{ad}}$ be the stable vertex set of (X, D_X) and set $W = f(V)$. By [ABBR15, Theorem 5.25], the set W contains the stable vertex set V of (Y, D_Y). There is a minimal semi-stable vertex set $W' \subset Y^{\mathrm{ad}}$ for (Y, D_Y) containing W, see [ABBR15, Lemma 3.15]. Pick any element $y \in W' \backslash W$, and any element $x \in f^{-1}(y)$. By construction, y must lie on a path between two vertices in W, and is therefore contained in $f(\Sigma_V)$. Since f is Galois, the automorphism group of X^{ad} over Y^{ad} acts transitively on the fibres, so that we must have $x \in \Sigma_V$. Since $y \in W' \backslash W$, we know that y must be of valency at least 3 in $\Sigma_{W'}$, whereas x must be of valency 2 in Σ_V.



It is shown in the proof of [ABBR15, Theorem 5.25] that this can not occur, and hence the set $W = f(V)$ is semi-stable. Since f is Galois, we necessarily have $f^{-1}(W) = V$.

Theorem 1.4 then assures the existence of a finite morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of semi-stable formal models over the ring of integral elements in $\overline{K}^{\wedge}$, where $\mathfrak{X}$ is in fact stable. We obtain algebraic models $\mathcal{X}$ and $\mathcal{Y}$ by Lemma 1.1, and both $\mathcal{X}$ and $\mathcal{Y}$ descend to the integral closure of R in some finite separable extension of K, see [ABBR15, Lemma 5.4]. Finally, the morphism f descends to a finite morphism over the same field by Lemma 1.5. □

2.3. **Potentially stable reduction for correspondences.** We now show that the theorem of Deligne–Mumford [DM69, Corollary 2.7] on potentially semi-stable reduction of smooth proper curves can be bootstrapped to an analogous statement for hyperbolic correspondences.

Theorem A. *Let $\mathbf{C}$ be a hyperbolic punctured correspondence over K. There is a finite separable extension of K over which $\mathbf{C}$ has a stable R-model.*

Proof. Change scalars to $\overline{K}^{\wedge}$. Consider the Galois closure $g : (\widetilde{X}, \widetilde{D}) \to (X, D_X)$ of both π_1 and π_2, where we set $\widetilde{D} = g^{-1}(D_X)$. We see that $(\widetilde{X}, \widetilde{D})$ is hyperbolic, and as such there is a unique stable vertex set $\widetilde{V} \subset \widetilde{X}^{\mathrm{ad}}$. Set $V := g(\widetilde{V})$ and $W_i = \pi_i(V)$, which are semi-stable vertex sets by Lemma 2.1. The uniqueness of $\widetilde{V}$ implies that it is preserved by the action of Galois. This action is transitive on fibres, so that we must have $V = \pi_i^{-1}(W_i)$. It follows that π_i extends to a finite morphism between the corresponding formal models. Using Lemma 1.1, this yields three semi-stable curves over the ring of integral elements in $\overline{K}^{\wedge}$, which descend to a finite separable extension of K by [ABBR15, Lemma 5.4]. We then get a semi-stable model $\mathfrak{C}$ from Lemma 1.5.

To construct a minimal such model, consider the stable vertex set $S \subset X^{\mathrm{ad}}$ of (X, D_X) and the smallest semi-stable vertex sets $T_i \supseteq \pi_i(S)$. Now iterate the following steps: Enlarge S to contain $\pi_i^{-1}(T_i)$, and then enlarge the sets T_i to contain $\pi_i(S)$. This procedure terminates, as all newly introduced Type-II points are necessarily contained in the finite semi-stable vertex sets corresponding to the semi-stable model $\mathfrak{C}$ constructed above via the Galois closure. This yields a semi-stable model $\mathfrak{C}'$ for $\mathbf{C}$ which is minimal with respect to the relation of domination. □

We note that in general, the stable model of $\mathbf{C}$ does **not** consist of the stable models of its objects. As can be seen below for the case of Hecke operators at p, typically some extra components appear. These could be considered as manifestations of the internal geometry of a correspondence $\mathbf{C}$.

2.4. **Skeleta of correspondences.** Given a finite morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of semi-stable curves, it is not true that $f^{-1}(\Sigma_{\mathfrak{Y}}) = \Sigma_{\mathfrak{X}}$, see [ABBR15, Remark 5.23]. When this holds, we say that f is *skeletal*. A semi-stable model $\mathfrak{C}$ of a punctured correspondence $\mathbf{C}$ is *skeletal* if π_1 and π_2 are both skeletal. A skeletal model of $\mathbf{C}$ effectively decomposes it as a sum of correspondences between open disks and open annuli, and may therefore be considered a non-Archimedean triangulation of $\mathbf{C}$ that is particularly convenient for analysing its spectral properties. Not every semi-stable model $\mathfrak{C}$ of $\mathbf{C}$ is skeletal, and the question arises whether it is always possible to modify $\mathfrak{C}$ to make it skeletal.

Let V, W_1, W_2 be the semi-stable vertex sets for the objects of the stable model $\mathfrak{C}$, so that in particular $\pi_1^{-1}(W_1) = \pi_2^{-1}(W_2) = V$. We can now attempt to adjust $\mathfrak{C}$ by doing the following:

(1) As is shown in [ABBR15, Remark 4.19], $\pi_i(\Sigma_V)$ is the union of Σ_{W_i} and a finite set of edges. These edges are disjoint, by the proof of [ABBR15, Proposition 5.25]. Enlarge the sets W_i to contain the endpoints of these edges, then enlarge the set V to contain $\pi_i^{-1}(W_i)$.
(2) Enlarge the set W_1 to contain $\pi_1(V)$, then
(3) enlarge the set V to contain $\pi_1^{-1}(W_1)$, then
(4) enlarge the set W_2 to contain $\pi_2(V)$, then
(5) enlarge the set V to contain $\pi_2^{-1}(W_2)$, then go back to step (2).

The question now becomes whether this procedure always terminates. The answer is *no*, as the example of the Hecke operator $T_{\mathfrak{p}}$ in section 2.6 shows. However, the answer is *yes* if we assume that either

(i) π_1, π_2 are Galois, or
(ii) either π_1 or π_2 is the identity.

First, if either π_1 or π_2 is the identity, there is no iteration and the procedure ends after the first step. Assume now that π_1, π_2 are Galois. Call Σ the stable skeleton of X, then any vertex of V introduced in step (1) must lie exactly in the middle of some edge in Σ, since it subdivides that edge into two edges which are Galois conjugate, and therefore have the same widths. After step (1), the skeleta of all three curves remain constant, as Σ is preserved by both Galois groups G_1 and G_2. Any subsequently introduced vertex then lies exactly in the middle of an edge of Σ, which is furthermore a $G_1 \times G_2$-translate of an edge that was previously subdivided in half. This shows not only that the procedure terminates, but that in fact the minimal width that occurs remains constant after step (1). This allows for some control over the ramification of the field extension we need to make $\mathfrak{C}$ skeletal.

Theorem A'. *Let $\mathbf{C}$ be a hyperbolic punctured correspondence over K, and assume that either (i) both morphisms are Galois, or (ii) one of the morphisms is the identity. Given semi-stable models $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ for the objects of $\mathbf{C}$, there is a finite separable extension of K over which $\mathbf{C}$ has a unique minimal **skeletal** semi-stable model whose objects dominate $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ pairwise.*

Skeletal semi-stable models $f : \mathcal{X} \longrightarrow \mathcal{Y}$ are especially convenient, since they decompose the map into a finite collection of maps between components in the special fibres, and a finite collection of finite maps of annuli. Étale locally around a singularity of $\mathcal{X}_s$, a skeletal morphism f is of the form

$$R[x,y]/(xy - \varpi^{ne}) \longrightarrow R[x,y]/(xy - \varpi^{n}) \ : \ (x,y) \longmapsto (x^e, y^e) \tag{5}$$

for some $n, e \geq 1$, see for instance Mochizuki [Moc95, §3.9]. The ramification index e is constant along both components of $\mathcal{X}_s$ passing through the singularity, as it is equal to the quotient of the widths of the annuli corresponding to the singularities, see [ABBR15, Theorem 4.23].

2.5. **Hecke operators away from $\mathfrak{p}$.** As a toy example, we now consider stable models of certain Hecke operators. Let F be a totally real number field, and B/F a quaternion algebra which is split at exactly one infinite prime. We fix a prime $\mathfrak{p}$ above p at which B is split, and a sufficiently small level structure away from $\mathfrak{p}$. We denote X^B for the corresponding Shimura curve, and $X_0^B(\mathfrak{p})$ for the Shimura curve with additional Iwahori level structure at $\mathfrak{p}$.

Let $\mathfrak{q}$ be coprime to both $\mathfrak{p}$ and the implicit tame level. By the work of Deligne–Rapoport, Carayol, and Buzzard [DR73, Car86, Buz97] we obtain a skeletal semi-stable model $\mathfrak{T}_\mathfrak{q}$ over $\mathcal{O}_{F,\mathfrak{p}}^{\mathrm{sh}}$ for the Hecke operators $T_\mathfrak{q}$ on $X_0^B(\mathfrak{p})$, with skeleton as depicted in Figure 1.

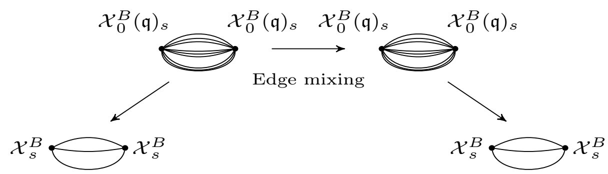


FIGURE 1. The stable skeleton of $\mathfrak{T}_\mathfrak{q}$

This gives geometric description of a special case of the Jacquet–Langlands correspondence. We have the weight-monodromy filtration

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(\widetilde{\mathcal{X}}_{\overline{s}}, \mathbf{Q}_l) \longrightarrow \mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_l) \longrightarrow \mathrm{H}^1(\Gamma, \mathbf{Q}_l)(-1) \longrightarrow 0, \tag{6}$$

where Γ is the dual graph of the special fibre of $\mathcal{X}_0^B(\mathfrak{p})$, $\widetilde{\mathcal{X}}_{\overline{s}}$ denotes the normalisation of its geometric special fibre, and $l \neq p$ is a prime. The space of p-new forms is identified as a Hecke module with the top graded piece, and is therefore isomorphic to the set of supersingular points. The action of $T_\mathfrak{q}$ is therefore described by the edge mixing, which one may compute as in Mestre–Oesterlé [Mes86] and Dembélé–Voight [DV13]. Via the monodromy pairing, we likewise recover that $T_\mathfrak{q}$ acts on the Néron component group as multiplication by $\mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{q}) + 1$.

2.6. **Hecke operators at $\mathfrak{p}$.** The morphisms $\pi_1, \pi_2 : \mathcal{X}_0^B(\mathfrak{p}) \longrightarrow \mathcal{X}^B$ defining $T_\mathfrak{p}$ on X^B are finite flat, and hence define a semi-stable model for $T_\mathfrak{p}$. If we attempt to make this model skeletal using the procedure outlined in 2.4, we keep introducing components which accumulate at the ordinary components. Figure 2 depicts the first few iterations of this process.

Assume henceforth that $B = M_2(\mathbf{Q})$ and consider U_p on $X_0(Np^2)$ where $(N, p) = 1$. To increase the number of cases in which U_p is hyperbolic, we puncture the modular curves at the cusps. From the work of Edixhoven [Edi90], we easily recover the stable skeleton, which is pictured in Figure 3. The four outer components are quotients of Igusa curves, whereas Z is a hyperelliptic curve. The degrees of the induced finite maps of annuli are indicated whenever they are different from 1.

We see that U_p acts on the edges as Frobenius, so that in particular we obtain from the weight-monodromy filtration the well-known fact that

$$U_p^2 = p^{k-2} \quad \text{on} \quad \mathrm{H}^1_{\text{ét}}\left(X_0(p),\ \mathrm{Sym}^{k-2}\,\mathbf{Q}_l\right)^{p\text{-new}} \tag{7}$$

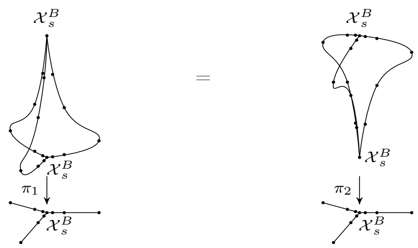


FIGURE 2. The stable model $\mathfrak{T}_{\mathfrak{p}}$

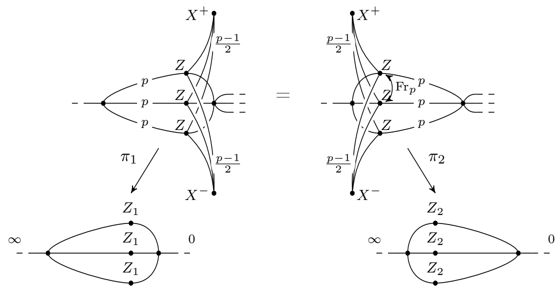


FIGURE 3. The stable skeleton of $\mathfrak{U}_p$

The stable model of a hyperbolic correspondence typically does *not* consist of the stable models of its components, as the above examples show. In general, additional (but canonical) components appear whose significance is not always clear. In the case of the stable model $\mathfrak{U}_p$ above, the additional components Z_1 and Z_2 have a clear moduli interpretation. An elliptic curve over $\mathbf{C}_p$ is said to be *too supersingular* if it does not have a canonical subgroup in the sense of Katz–Lubin. It is *nearly too supersingular* if it is p-isogenous to such a curve. It is clear that $\mathrm{sp}^{-1}(Z_1^{\mathrm{sm}})$ and $\mathrm{sp}^{-1}(Z_2^{\mathrm{sm}})$ classify those elliptic curves over $\mathbf{C}_p$ with Hasse invariants $p/(p+1)$ and $1/(p+1)$ respectively, which therefore classify too supersingular, resp. nearly too supersingular, elliptic curves with $\Gamma_0(N)$-structure.

2.7. **Canonical subgroups for curves.** In the theory of p-adic modular forms, it is important to identify the maximal section of forgetful maps between certain Shimura curves. The underlying geometric mechanism is most easily phrased in our language of skeletal stable models, leading to the following generalisation of Goren–Kassaei [GK06, Theorem 3.9].

Theorem 2.2. *Assume K is complete with respect to its valuation, and let*

$$\mathfrak{g} : \mathcal{X} \longrightarrow \mathcal{Y}$$

be a skeletal finite morphism of semi-stable curves over R, with generic fibre $g^{\mathrm{ad}} : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$. Assume that $\mathfrak{g}$ is an isomorphism on some component Z of $\mathcal{X}_s$. Then g^{ad} is an isomorphism on $\mathrm{sp}_{\mathcal{X}}^{-1}(Z)$.

Proof. By Goren–Kassaei [GK06, Proposition 3.1] the section $\mathfrak{g}(Z) \to Z$ gives rise to a section

$$s : \mathrm{sp}_{\mathcal{Y}}^{-1}(\mathfrak{g}(Z)^{\mathrm{sm}}) \longrightarrow X^{\mathrm{ad}}$$

as R is complete and thus Henselian. By Coleman–Gouvêa–Jochnowitz [CGJ95, Lemma 6], we may extend s to a section $s^\dagger : U \to X^{\mathrm{ad}}$, for $U \subset Y^{\mathrm{ad}}$ some open subspace strictly containing the domain of s. Since $\mathfrak{g}$ is skeletal, the induced map $g^{\mathrm{ad}} : X^{\mathrm{ad}} \backslash V_{\mathcal{X}} \longrightarrow Y^{\mathrm{ad}} \backslash V_{\mathcal{Y}}$ decomposes as a collection of finite maps of annuli and finite maps of disks. The degree on the collection of wide open annuli $\mathrm{sp}_{\mathcal{X}}^{-1}(Z^{\mathrm{sing}})$ is equal to the ramification index of $\mathfrak{g}$ at the corresponding singular point. Therefore the annuli are mapped isomorphically onto their image by g^{ad}. The inverse of g^{ad} on these annuli agrees with $s^\dagger$ on the intersection with the open set U, and hence it glues with $s^\dagger$ to produce a section of g^{ad} on the image of $\mathrm{sp}_{\mathcal{X}}^{-1}(Z)$ under g^{ad}. $\square$

We obtain a section of $\mathfrak{g}$ onto the union of $\mathrm{sp}_{\mathcal{X}}^{-1}(Z)$ over all components Z on which $\mathfrak{g}$ induces an isomorphism. The restriction of this section onto any connected component A is *maximal* in the sense that it does not extend to a section onto any connected open properly containing A. Indeed, any such open must contain an element in the semi-stable vertex set of $\mathcal{X}$ corresponding to a component of $\mathcal{X}_s$ on which $\mathfrak{g}$ is not an isomorphism, which means that the corresponding map on residue fields cannot be an isomorphism. Theorem 2.2 is hence a version of the result of Goren–Kassaei, where all the assumptions on the shapes of the dual graphs made in *loc. cit.* may be dropped.

3. De Rham–Witt complexes

In what follows, we come to the main subject of this paper, and construct trace maps for crystalline lattices in de Rham cohomology of curves. Our main tool will be the de Rham–Witt complexes on the special fibres of semi-stable models, and this section recalls some results in the literature which we will need in the sequel. It contains no new results.

Henceforth, K will denote a finite extension of $\mathbf{Q}_p$, with ring of integers R and residue field k.

3.1. **Logarithmic schemes.** We will use logarithmic structures in the étale topology. Suppose X is a scheme, and $\mathcal{M}_X$ a sheaf of commutative monoids on $X_{\text{ét}}$ defining a logarithmic structure with exponential map $\alpha : \mathcal{M} \longrightarrow \mathcal{O}_X$. We write X^+ for the logarithmic scheme defined by this data, and conversely, if we are given a logarithmic scheme X^+ we write X for its underlying scheme, and $\mathcal{M}_X$ for its sheaf of monoids. The sheaf of groups associated to $\mathcal{M}_X$ is denoted by $\mathcal{M}_X^{\text{gp}}$.

We choose a uniformiser ϖ of R. The fs logarithmic structure on Spec R with pre-logarithmic chart

$$(\mathbf{N} \longrightarrow R : 1 \longmapsto \varpi), \tag{8}$$

defines a logarithmic scheme which we refer to as R^+. Let $W = W(k)$ be the ring of Witt vectors of the residue field k of R, K_0 its field of fractions, and σ the lift to W of the Frobenius map $x \mapsto x^p$ on k. Set $W_n = W/p^nW$. We denote k^+ for the logarithmic scheme Spec k with pre-logarithmic chart

$$(\mathbf{N} \longrightarrow k : 1 \longmapsto 0).$$

If X is a k-scheme and $n \geq 1$, then we denote $W_n(X)$ for the scheme over W_n with the same underlying topological space as X, and with structure sheaf $W_n\,\mathcal{O}_X$ of p-typical Witt vectors over $\mathcal{O}_X$ of length n. For a section $x \in \mathcal{O}_X$ we denote $[x]$ for its Teichmüller lift $(x, 0, \ldots, 0) \in W_n\mathcal{O}_X$, and $[x]_i = V^i[x]$ for the vector $(0, \ldots, x, \ldots, 0)$ with x as its i-th entry.

3.2. **De Rham–Witt complexes.** As we are interested in comparisons with de Rham cohomology, it will be most convenient to introduce crystalline cohomology using the de Rham–Witt complex, as opposed to working directly with the log-crystalline topos. The main reference here is [HK94].

Assume now that X^+ is a proper, logarithmically smooth scheme over k^+ of Cartier type (see [HK94, §2.12]), then Hyodo and Kato define for any $n \geq 1$ a complex of sheaves of $W_n\mathcal{O}_X$-modules $W_n\omega^\bullet_{X^+}$. We briefly recall the construction, and some basic properties. Denote W_n^+ for the scheme Spec W_n with its canonical lift logarithmic structure, which has a chart given by

$$\left(\mathbf{N} \oplus \mathrm{Ker}[W_n(k)^\times \longrightarrow k^\times] \to W_n \;:\; (a, w) \longmapsto (0, w)\right), \tag{9}$$

and let

$$u^{\text{cris}}_{X^+/W_n^+} : \left(X^+/W_n^+\right)^\sim_{\text{cris}} \longrightarrow (X)^\sim_{\text{ét}} \tag{10}$$

be the natural morphism from the log-crystalline topos to the étale topos of X, see [Kat89, Section 5]. Then define for any $q \geq 0$ the de Rham–Witt sheaf

$$W_n\omega^q_{X^+} := \mathrm{R}^q u^{\text{cris}}_{X^+/W_n^+,*}\mathcal{O},$$

which is made into a complex $W_n\omega^\bullet_{X^+}$ via a certain Bockstein homomorphism d constructed on the right hand side. It has the structure of a differential graded algebra, endowed with operators

$$\begin{cases} \pi : & W_{n+1}\omega^\bullet_{X^+} \longrightarrow W_n\omega^\bullet_{X^+} \\ F : & W_{n+1}\omega^\bullet_{X^+} \longrightarrow W_n\omega^\bullet_{X^+} \\ V : & W_n\omega^\bullet_{X^+} \longrightarrow W_{n+1}\omega^\bullet_{X^+}, \end{cases}$$

satisfying the relations $FV = VF = p$ and $FdV = d$. The map π is called the *restriction map*. It is surjective and commutes with F and V. The absolute Frobenius map on X^+ induces a morphism of differential graded algebras $\varphi : W_n\omega^\bullet_{X^+} \to W_n\omega^\bullet_{X^+}$, and we have $\varphi = p^iF$ on $W_n\omega^i_{X^+}$. Finally, there is a canonical, functorial, inverse system of isomorphisms

$$\mathrm{R}u^{\mathrm{cris}}_{X^+/W_n^+,*}\mathcal{O} \xrightarrow{\sim} W_n\omega^\bullet_{X^+}. \tag{11}$$

On cohomology, one constructs through this isomorphism the *higher Cartier operators*

$$C^{-n} \ : \ W_n\omega^q_{X^+} \xrightarrow{\sim} \mathcal{H}^q\left(W_n\omega^\bullet_{X^+}\right). \tag{12}$$

Multiplication by p on de Rham–Witt complexes induces an exact triangle which is frequently useful in dévissage arguments. More precisely, it is shown in Hyodo–Kato [HK94, Corollary 4.5] that there is an exact triangle

$$W_{n-1}\omega^\bullet_{X^+} \xrightarrow{\cdot p} W_n\omega^\bullet_{X^+} \longrightarrow W_1\omega^\bullet_{X^+} \xrightarrow{+1}. \tag{13}$$

Apart from the above structures, which all have counterparts in the non-logarithmic setting, we get an additional feature in the form of a canonical morphism

$$\mathrm{dlog} : \mathcal{M}_X \longrightarrow W_n\omega^1_{X^+} \qquad \text{such that} \qquad d[\alpha(m)] = [\alpha(m)]\ \mathrm{dlog}\ m.$$

3.3. **Hyodo–Kato cohomology.** The Hyodo–Kato cohomology of X^+ is the hypercohomology

$$\mathrm{H}^i_{\mathrm{HK}}(X^+/W) = \mathrm{H}^i(X, W\omega^\bullet_{X^+}), \qquad \text{where} \qquad W\omega^\bullet_{X^+} = \varprojlim_n W_n\omega^\bullet_{X^+},$$

which is a finitely generated W-module. The absolute Frobenius map induces a σ-linear endomorphism

$$\varphi : \mathrm{H}^i_{\mathrm{HK}}(X^+/W) \longrightarrow \mathrm{H}^i_{\mathrm{HK}}(X^+/W)$$

The Hyodo–Kato cohomology groups come equipped with a monodromy operator N, as follows. For any $n \geq 1$, there exists a canonical extension

$$0 \longrightarrow W_n\omega^\bullet_{X^+}[-1] \longrightarrow W_n\widetilde{\omega}^\bullet_{X^+} \longrightarrow W_n\omega^\bullet_{X^+} \longrightarrow 0, \tag{14}$$

where $W_n\widetilde{\omega}^\bullet_{X^+}$ is also a $W_n\,\mathcal{O}_X$-complex, which comes equipped with operators F, V and π with the same properties as before, and the extension is compatible with these operators. One defines

$$N : \mathrm{H}^i_{\mathrm{HK}}(X^+/W) \longrightarrow \mathrm{H}^i_{\mathrm{HK}}(X^+/W)$$

to be the connecting homomorphism on cohomology obtained from taking the inverse limit over n of the extension (14). The operators φ and N satisfy the relation $p\varphi N = N\varphi$.

We have the following explicit presentation, proved by Nakkajima [Nak05, Theorem 11.1]:

$$\begin{aligned} W_n\omega^1_{X^+} &\simeq \left(W_n\mathcal{O}_X \oplus (W_n\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}_X^{\mathrm{gp}}/(\mathbf{N}\oplus k^\times)^{\mathrm{gp}})\right)/\sim, \\ W_n\widetilde{\omega}^1_{X^+} &\simeq \left(W_n\mathcal{O}_X \oplus (W_n\mathcal{O}_X \otimes_{\mathbf{Z}} \mathcal{M}_X^{\mathrm{gp}}/k^\times)^{\mathrm{gp}}\right)/\sim, \end{aligned} \tag{15}$$

where the equivalence relation $\sim$ is generated by the relations

$$p^i([\alpha(m)]_i, 0) \sim (0, [\alpha(m)]_i \otimes m), \qquad m \in \mathcal{M}_X^{\mathrm{gp}},\ 0 \leq i \leq n-1.$$

The presentation for $W_n\omega^1_{X^+}$ already appears in [HK94, Proposition 4.6]. Though we only need the result for $q = 1$, we note that the statement in *loc. cit.* contains a mistake in the presentation of the de Rham–Witt sheaves $W_n\omega^q_{X^+}$ when $q \geq 2$, as was noticed, and corrected, by Nakkajima. We refer the interested reader to Nakkajima [Nak05] for a very thorough treatment of de Rham–Witt theory, containing a plethora of useful facts about de Rham–Witt complexes.

Finally, we note that the de Rham–Witt complexes are contravariantly functorial in the sense that if $f : X^+ \to Y^+$ is a k^+-morphism, there is an adjunction map $\mathcal{O}_{Y^+/W_n} \to \mathrm{R}f_*\mathcal{O}_{X^+/W_n}$ on the log-crystalline topoi of X^+ and Y^+, giving us via the isomorphism (11) a natural pullback map

$$f^* : W\omega^\bullet_{Y^+} \longrightarrow \mathrm{R}f_*W\omega^\bullet_{X^+}$$

which for curves is given via the obvious map on the presentation 15, see [Nak05, Lemma 9.1].

3.4. **Relation with de Rham cohomology.** We now discuss the relation between the de Rham complex of a semi-stable model of a curve, and the de Rham–Witt complex of its special fibre. Let R be the ring of integral elements in a finite extension K of $\mathbf{Q}_p$, of ramification degree e, and let R^+ be the logarithmic scheme defined by (8). The following theorem was proved by Berthelot–Ogus [BO83] in the case of good reduction, and in general by Hyodo–Kato [HK94, Theorem 5.1].

Theorem 3.1 (Berthelot–Ogus, Hyodo–Kato)**.** *Let $\mathcal{X}^+ \to R^+$ be proper fs logarithmically smooth such that $\mathcal{X}_s^+ \to k^+$ is of Cartier type. Then there are canonical isomorphisms*

$$\begin{array}{llcll} \rho_\varpi^K: & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W) \otimes_W K & \simeq & \mathrm{H}^1_{\mathrm{dR}}(X/K) & \\ \rho_\varpi^R: & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W) \otimes_W R & \simeq & \mathrm{H}^1_{\mathrm{dR}}(\mathcal{X}^+/R^+) & (\textit{if } \ e \leq p-1). \end{array}$$

More precisely, Hyodo–Kato show the existence of an isomorphism

$$\rho_\varpi \ : \ \mathbf{Q} \otimes \left\{ R_n \otimes^L_{W_n} W_n\omega^\bullet_{\mathcal{X}_s^+} \right\}_n \xrightarrow{\sim} \mathbf{Q} \otimes \left\{ \Omega^\bullet_{\mathcal{X}_n^+/R_n^+} \right\}_n$$

of projective systems in $D((\mathcal{X}_s^+)_{\text{ét}})$, where $\mathbf{Q} \otimes -$ denotes the image in the isogeny category.

Although we will not need it, let us say a few words about this isomorphism. When $\mathcal{X}$ is a relative curve, we may find a global deformation $\mathcal{Z}^+$ which is log smooth over Spf $W[\![t]\!]^+$ such that the local structure at a singular point of $\mathcal{X}_s$ is given by $W[\![x,y,t]\!]/(xy - t^n)$. We obtain a Cartesian diagram

$$\begin{array}{ccc} \mathcal{X}^+ & \longrightarrow & \mathcal{Z}^+ \\ \downarrow & & \downarrow \\ \mathrm{Spf}\ R^+ & \xrightarrow[t \mapsto \varpi]{} & \mathrm{Spf}\ W[\![t]\!]^+ \end{array}$$

and a Frobenius map $t \mapsto t^p$. Let R_n^{PD} be the PD-envelope of R_n in $W[t]$, then the crystalline complex of $\mathcal{X}_s^+/R_n^{\mathrm{PD}}$, which is closely related to the de Rham complex of $\mathcal{X}^+$, arises as the fibre of an F-crystal on $W[t]$ at $t = \varpi$. Hyodo–Kato prove a rigidity theorem that shows that this crystal is constant over $W\langle t \rangle$. After successive applications of Frobenius, which is an isogeny, we can shrink the disk and get our "de Rham" fibre sufficiently close to $t = 0$ to obtain the isomorphism from the rigidity theorem. The need to conjugate by powers of Frobenius is the underlying reason that in general, the comparison isomorphism in Theorem 3.1 requires us to invert p.

4. Crystalline lattices in de Rham cohomology

Let $f : X \longrightarrow Y$ be a finite map between proper, smooth, geometrically connected curves over a finite extension K of $\mathbf{Q}_p$. If X and Y have semi-stable reduction, Theorem 3.1 endows the de Rham cohomology groups $\mathrm{H}^1_{\mathrm{dR}}(X/K)$ and $\mathrm{H}^1_{\mathrm{dR}}(Y/K)$ with W-lattices with operators φ and N, depending on the choice of a uniformiser ϖ of the valuation ring $R \subset K$. These W-lattices are provided by the Hyodo–Kato, or log-crystalline, cohomology groups

$$\mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W) \quad \text{and} \quad \mathrm{H}^1_{\mathrm{HK}}(\mathcal{Y}_s^+/W) \qquad \text{with } (\varphi, N)\text{-structure}$$

of the special fibres of the minimal regular semi-stable models $\mathcal{X}$ and $\mathcal{Y}$ over R.

There exist canonical trace and pullback maps on de Rham cohomology

$$f_* : \mathrm{H}^1_{\mathrm{dR}}(X/K) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(Y/K) \qquad \text{and} \qquad f^* : \mathrm{H}^1_{\mathrm{dR}}(Y/K) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(X/K), \tag{16}$$

In this section, we show that the W-lattices provided by Hyodo–Kato cohomology are preserved by f_* and f^*. We do this on the level of the derived category, by constructing canonical trace and pullback maps between between the de Rham–Witt complexes, for any choices of semi-stable models $\mathcal{X}$ and $\mathcal{Y}$. We show that the induced maps on cohomology preserve the (ϕ, N)-structure, and that they recover the maps f_* and f^* on de Rham cohomology through the Hyodo–Kato isomorphism.

4.1. **Logarithmic structures on semi-stable curves.** In order to prove our main theorem, we will make use of the existence of semi-stable models of finite maps. In the case of bad reduction, this essentially always forces the semi-stable models of the curves to be non-regular. We start by defining the logarithmic structures we wish to consider on such curves.

Assume $\mathcal{X}$ is a (not necessarily regular) semi-stable model for X over R. There is a natural *divisorial logarithmic structure* on $\mathcal{X}$, defined by $\mathcal{M}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \cap j_*\mathcal{O}_X^\times$, where $j : X \hookrightarrow \mathcal{X}$ is the inclusion of the generic fibre. We may find étale local charts on $\mathcal{X}$ at a singular point of the special fibre given by

$$\begin{array}{rcl} \frac{1}{n}\Delta(\mathbf{N}) + \mathbf{N}^2 & \longrightarrow & R[x,y]/(xy-\varpi^n) \\ \frac{1}{n}\Delta(c) + (a,b) & \longmapsto & x^a y^b \varpi^c, \end{array} \tag{17}$$

where $\Delta : \mathbf{N} \to \mathbf{N}^2$ is the diagonal embedding and the sum $\frac{1}{n}\Delta(\mathbf{N}) + \mathbf{N}^2$ is taken inside $\mathbf{Q}^2$. It is shown in [AI12, Lemma 3.1] that the induced structure morphism $\mathcal{X}^+ \to R^+$ is logarithmically smooth. We record some more properties in the following lemma, most importantly that for this logarithmic structure, the curve is of Cartier type [Kat89, Definition 4.8].

Lemma 4.1. *The logarithmic scheme $\mathcal{X}^+$ is fine and saturated, and $\mathcal{X}^+ \to \operatorname{Spec} R^+$ is logarithmically smooth and integral with special fibre $\mathcal{X}_s^+ \to \operatorname{Spec} k^+$ of Cartier type.*

Proof. It is clear that $\mathcal{X}^+$ is fine and saturated. The logarithmic structure is trivial away from the singularities of the special fibre, and étale locally around such a singularity, we have a chart

$$\frac{1}{n}\Delta \ : \ \mathbf{N} \ \longrightarrow \ \frac{1}{n}\Delta(\mathbf{N}) + \mathbf{N}^2$$

for the structural morphism. The induced map on groups has trivial kernel, with cokernel isomorphic to $\mathbf{Z}$. As both are torsion-free, f is logarithmically smooth. It follows from [Kat89, Corollary 4.4] that f is integral. Let $\mathrm{Fr}_k : \operatorname{Spec} k^+ \to \operatorname{Spec} k^+$ be the absolute Frobenius on the logarithmic point, which on logarithmic structures $\mathrm{Fr}_k^{-1}\mathbf{N} \simeq \mathbf{N} \to \mathbf{N}$ is just multiplication by p. The relative Frobenius map $\mathcal{X}_s^+ \to \mathcal{X}_s^{(p)+}$ is therefore given on charts by the multiplication by p map

$$\frac{1}{np}\Delta + \mathbf{N}^2 \ \xrightarrow{\cdot p} \ \frac{1}{n}\Delta + \mathbf{N}^2 .$$

This map on monoids is the restriction to the first quadrant of the induced map on groups. Relative Frobenius is therefore an exact morphism, and hence the special fibre is of Cartier type. □

The divisorial logarithmic is functorial, in the following sense. Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite map between semi-stable R-models for smooth proper geometrically connected curves X, Y over K. If we endow these models with the logarithmic structure of the previous subsection, the map f is a morphism of logarithmic schemes via the natural map

$$f^{-1}\mathcal{M}_{\mathcal{Y}} = f^{-1}\left(\mathcal{O}_{\mathcal{Y}} \cap j_*\mathcal{O}_Y^\times\right) \ \longrightarrow \ \mathcal{O}_{\mathcal{X}} \cap j_*\mathcal{O}_X^\times.$$

The induced map $f_s : \mathcal{X}_s^+ \to \mathcal{Y}_s^+$ is a finite morphism of fs smooth logarithmic schemes over k^+.

4.2. **Adjusting semi-stable models.** We will frequently need to pass between different semi-stable models, and we now relate their de Rham–Witt complexes. The crucial property that gives us the flexibility we need is the fact that semi-stable models have *rational* singularities, and we deduce several properties of de Rham–Witt complexes by reducing via a dévissage argument to the following:

Lemma 4.2. *Let $\pi : \mathcal{Z} \to \mathcal{X}$ be a birational map between semi-stable models of X. Then the adjunction map and natural pullback map yield (quasi-)isomorphisms*

$$\begin{array}{rcl} \mathrm{R}\pi_*\pi^!\Omega^1_{\mathcal{X}^+/R^+} & \xrightarrow{\sim} & \Omega^1_{\mathcal{X}^+/R^+} \\ \Omega^\bullet_{\mathcal{X}^+/R^+} & \xrightarrow{\sim} & \mathrm{R}\pi_*\Omega^\bullet_{\mathcal{Z}^+/R^+}. \end{array}$$

Proof. First, it is a classical result that the sheaves of logarithmic differentials $\Omega^1_{\mathcal{X}^+/R^+}$ and $\Omega^1_{\mathcal{Z}^+/R^+}$ are the relative dualising sheaves of $\mathcal{X}$ and $\mathcal{Z}$ over $\mathrm{Spec}(R)$. For a general statement, see for instance Tsuji [Tsu99b, Theorem 2.21]. Both statements then follow from the theory of rational singularities, see Cais [Cai09, Proposition 4.6]. □

Any two semi-stable models may be related by a chain of blow-ups and blow-downs, and hence it suffices to investigate how these two operations affect de Rham–Witt complexes. More precisely, we need to investigate maps which are one of the two following types:

- **Type A:** The blow-up of the singularity $xy = \varpi^n$ at (x, y, ϖ) for $n \geq 2$,
- **Type B:** The blow-up of a smooth point in the special fibre.

In both cases π is a logarithmic blow-up along a coherent ideal. The work of Vidal [Vid04, Théorème 2.4.3.1] proves for more general logarithmic blow-ups that the Hyodo–Kato lattice in de Rham cohomology is unaffected by this. We give a proof that uses the specifics of the case at hand.

Theorem 4.3. *Let $\pi : \mathcal{Z}_s^+ \longrightarrow \mathcal{X}_s^+$ be a blow-up of type A or B as above. Then the induced maps*

$$\begin{array}{rccc} \psi_n : & W_n\omega^\bullet_{\mathcal{X}_s^+} & \longrightarrow & \mathrm{R}\pi_* W_n\omega^\bullet_{\mathcal{Z}_s^+} \\ \widetilde{\psi}_n : & W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+} & \longrightarrow & \mathrm{R}\pi_* W_n\widetilde{\omega}^\bullet_{\mathcal{Z}_s^+} \end{array} \tag{18}$$

are W_n-linear isomorphism in $D^+(\mathcal{X}_{s,\text{ét}})$ for every n.

Proof. From (13) we get a commutative diagram of exact triangles

$$\begin{array}{ccccccc} W_{n-1}\omega^\bullet_{\mathcal{X}_s^+} & \xrightarrow{\cdot p} & W_n\omega^\bullet_{\mathcal{X}_s^+} & \longrightarrow & W_1\omega^\bullet_{\mathcal{X}_s^+} & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \psi_{n-1}} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_1} & & \\ \mathrm{R}\pi_* W_{n-1}\omega^\bullet_{\mathcal{Z}_s^+} & \xrightarrow{\cdot p} & \mathrm{R}\pi_* W_n\omega^\bullet_{\mathcal{Z}_s^+} & \longrightarrow & \mathrm{R}\pi_* W_1\omega^\bullet_{\mathcal{Z}_s^+} & \xrightarrow{+1} & \end{array}$$

from which we see that by a dévissage argument we are reduced to proving ψ_1 is an isomorphism. When $n = 1$ the de Rham–Witt complex is the logarithmic de Rham complex of the special fibre, so that ψ_1 is the reduction of the natural map

$$\Omega^\bullet_{\mathcal{X}^+/R^+} \longrightarrow \mathrm{R}\pi_*\Omega^\bullet_{\mathcal{Z}^+/R^+}. \tag{19}$$

It follows from Lemma 4.2 that ψ_1, and hence by induction ψ_n for all $n \geq 1$, is an isomorphism.

From the explicit presentation (15) and [Nak05, Theorem 11.1], we see that the canonical pullback maps give rise to a commutative diagram of exact triangles

$$\begin{array}{ccccccc} W_n\omega^\bullet_{\mathcal{X}_s^+} & \longrightarrow & W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+} & \longrightarrow & W_n\omega^\bullet_{\mathcal{X}_s^+} & \xrightarrow{+1} & \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \widetilde{\psi}_n} & & \downarrow{\scriptstyle \psi_n} & & \\ \mathrm{R}\pi_* W_n\omega^\bullet_{\mathcal{Z}_s^+} & \longrightarrow & \mathrm{R}\pi_* W_n\widetilde{\omega}^\bullet_{\mathcal{Z}_s^+} & \longrightarrow & \mathrm{R}\pi_* W_n\omega^\bullet_{\mathcal{Z}_s^+} & \xrightarrow{+1} & \end{array}$$

Since ψ_n is an isomorphism, it follows that $\widetilde{\psi}_n$ is an isomorphism. □

4.3. **Poincaré duality.** We now discuss a duality result for logarithmic de Rham–Witt complexes. A general theory in the smooth case was initiated by Ekedahl [Eke85]. An extension to *regular* semistable families was proved by Hyodo [Hyo91], from which we may deduce the following result.

Theorem 4.4 (Ekedahl, Hyodo)**.** *Let $\mathcal{X}_s^+ \longrightarrow k^+$ be as above, and denote the structural morphism of $W_n(\mathcal{X}_s)$ by $a_n : W_n(\mathcal{X}_s) \longrightarrow \mathrm{Spec}\ W_n$. Then there is a canonical isomorphism*

$$W_n\omega^1_{\mathcal{X}_s^+} \quad \simeq \quad a_n^! W_n[-1]. \tag{20}$$

Furthermore, we have canonical isomorphisms

$$\begin{array}{rcl} W_n\omega^\bullet_{\mathcal{X}_s^+} & \xrightarrow{\sim} & \mathrm{R}\mathcal{H}om(W_n\omega^\bullet_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-1]), \\ W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+} & \xrightarrow{\sim} & \mathrm{R}\mathcal{H}om(W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-2]). \end{array} \tag{21}$$

induced by a commutative diagram of perfect pairings:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W\omega^0_{\mathcal{X}_s^+} & \longrightarrow & W\widetilde{\omega}^1_{\mathcal{X}_s^+} & \longrightarrow & W\omega^1_{\mathcal{X}_s^+} & \longrightarrow & 0 \\ & & \times & & \times & & \times & & \\ 0 & \longleftarrow & W\omega^1_{\mathcal{X}_s^+} & \longleftarrow & W\widetilde{\omega}^1_{\mathcal{X}_s^+} & \longleftarrow & W\omega^0_{\mathcal{X}_s^+} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & W\omega^1_{\mathcal{X}_s^+} & = & W\omega^1_{\mathcal{X}_s^+} & = & W\omega^1_{\mathcal{X}_s^+} & & \end{array} \tag{22}$$

Proof. Let $\pi : \mathcal{Z}_s^+ \longrightarrow \mathcal{X}_s^+$ be the special fibre of the minimal desingularisation $\mathcal{Z}^+$ of $\mathcal{X}^+$, endowed with the logarithmic structure induced by its special fibre. Since $\mathcal{Z}$ is regular semi-stable, the results follow for $\mathcal{Z}_s^+$ by [Hyo91, Theorem 3.1/3.2]. We now deduce them for $\mathcal{X}_s^+$.

Using the higher Cartier morphisms (12) and the isomorphism (18) we get a composite map

$$\begin{array}{ccc} W_n\omega^1_{\mathcal{X}_s^+} & \overset{\upsilon_n}{\dashrightarrow} & \mathrm{R}\pi_* W_n\omega^1_{\mathcal{Z}_s^+} \\ \downarrow {\scriptstyle C^{-n}} & & \uparrow {\scriptstyle \pi_*(C^n)} \\ \mathcal{H}^1\left(W_n\omega^\bullet_{\mathcal{X}_s^+}\right) & & \pi_*\mathcal{H}^1(W_n\omega^\bullet_{\mathcal{Z}_s^+}) \\ & \searrow^{\mathcal{H}^1(\psi_n)} \quad \nearrow^{\mathrm{can}} & \\ & \mathrm{R}^1\pi_* W_n\omega^\bullet_{\mathcal{Z}_s^+} & \end{array}$$

where the map $\pi_*(C^n)$ is obtained from the higher Cartier isomorphism C^n via the natural map $\pi_* W_n\omega^1_{\mathcal{Z}_s^+} \longrightarrow \mathrm{R}\pi_* W_n\omega^1_{\mathcal{Z}_s^+}$. It was shown in [Nak05, Lemma 9.1] that the map υ_n is given by the obvious map on the presentation (15). We may now define a morphism by composition

$$\begin{array}{rcrcl} \Upsilon_n & : & W_n\omega^1_{\mathcal{X}_s^+} & \xrightarrow{\upsilon_n} & \mathrm{R}\pi_* W_n\omega^1_{\mathcal{Z}_s^+} \\ & & & \xrightarrow{\sim} & \mathrm{R}\pi_*(a_n \circ \pi)^! W_n[-1] = \mathrm{R}\pi_*\pi^!(a_n^! W_n[-1]) \\ & & & \xrightarrow{\mathrm{Tr}} & a_n^! W_n[-1] \end{array}$$

Since the Cartier morphisms are a functorial inverse system of isomorphisms [Nak05, Theorem 7.19], we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} W_{n-1}\omega^1_{\mathcal{X}_s^+} & \xrightarrow{\cdot p} & W_n\omega^1_{\mathcal{X}_s^+} & \longrightarrow & W_1\omega^1_{\mathcal{X}_s^+} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Upsilon_{n-1}} & & \downarrow{\scriptstyle \Upsilon_n} & & \downarrow{\scriptstyle \Upsilon_1} & & \\ a_{n-1}^! W_{n-1}[-1] & \xrightarrow{\cdot p} & a_n^! W_n[-1] & \longrightarrow & a_1^! W_1[-1] & \xrightarrow{+1} & \end{array}$$

Let us first investigate the map Υ_n when $n = 1$. In this case, the de Rham–Witt sheaf is just the sheaf of logarithmic differentials $\Omega^1_{\mathcal{X}_s^+/k^+}$, and it suffices to show that the two morphisms

$$\upsilon_1 : \Omega^1_{\mathcal{X}_s^+/k^+} \longrightarrow \mathrm{R}\pi_*\Omega^1_{\mathcal{Z}_s^+/k^+}, \qquad \mathrm{Tr} : \mathrm{R}\pi_*\pi^!\Omega^1_{\mathcal{X}_s^+/k^+} \longrightarrow \Omega^1_{\mathcal{X}_s^+/k^+}$$

are isomorphisms. The latter immediately follows from Lemma 4.2. To see that υ_1 is an isomorphism, we note that $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Z}_s} = 0$, so that the Grothendieck spectral sequence

$$\mathrm{R}^p\pi_*\mathcal{H}^q(\Omega^\bullet_{\mathcal{Z}_s^+/k^+}) \quad \Rightarrow \quad \mathrm{R}^{p+q}\pi_*\Omega^\bullet_{\mathcal{Z}_s^+/k^+}$$

degenerates at E_2 and the map 'can' in the definition of υ_1 is an isomorphism. Étale locally, the morphism π has a chart $g : P \to Q$ such that g^{gp} is bijective, and such that the induced map

$$\mathcal{Z}_s \longrightarrow \mathcal{X}_s \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$$

is an isomorphism, so that π is log-étale. It follows from [Kat89, Proposition 3.12] that

$$\Omega^1_{\mathcal{Z}_s^+/k^+} = \pi^*\Omega^1_{\mathcal{X}_s^+/k^+}.$$

Therefore, by the projection formula, we have

$$\pi_*\Omega^1_{\mathcal{Z}_s^+/k^+} \longrightarrow \mathrm{R}\pi_*\Omega^1_{\mathcal{Z}_s^+/k^+} \simeq (\pi_*\,\mathcal{O}_{\mathcal{Z}_s} \longrightarrow \mathrm{R}\pi_*\,\mathcal{O}_{\mathcal{Z}_s}) \otimes \Omega^1_{\mathcal{X}_s^+/k^+}$$

The natural map on the right hand side is an isomorphism, since $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{Z}_s} = 0$. We conclude that Υ_1, and hence Υ_n for all $n \geq 1$ by dévissage, is an isomorphism.

For the duality statement, we deduce formally that if $q = 0, 1$ we have

$$\begin{aligned} \mathrm{R}\mathcal{H}om(W_n\omega^{1-q}_{\mathcal{X}_s^+}, a_n^! W_n) &\simeq \mathrm{R}\mathcal{H}om(\mathrm{R}\pi_* W_n\omega^{1-q}_{\mathcal{Z}_s^+}, a_n^! W_n) \\ &\simeq \mathrm{R}\pi_*\mathrm{R}\mathcal{H}om(W_n\omega^{1-q}_{\mathcal{Z}_s^+}, (a_n \circ \pi)^! W_n) \\ &\simeq \mathrm{R}\pi_* W_n\omega^q_{\mathcal{Z}_s^+} \simeq W_n\omega^q_{\mathcal{X}_s^+} \end{aligned}$$

and similarly for $W_n\widetilde{\omega}^q_{\mathcal{X}_s^+}$, so that we obtain the desired diagram of perfect pairings (22) from the corresponding statement for $\mathcal{Z}_s^+$ via (18). This induces maps

$$\begin{aligned} W_n\omega^\bullet_{\mathcal{X}_s^+} &\longrightarrow \mathrm{R}\mathcal{H}om(W_n\omega^\bullet_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-1]), \\ W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+} &\longrightarrow \mathrm{R}\mathcal{H}om(W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-2]). \end{aligned}$$

which must both be isomorphisms. Indeed, take for instance the first map, which fits in a commutative diagram of exact triangles

$$\begin{array}{ccc} W_n\omega^1_{\mathcal{X}_s^+}[-1] & \longrightarrow & \mathrm{R}\mathcal{H}om(W_n\omega^0_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-1]) \\ \downarrow & & \downarrow \\ W_n\omega^\bullet_{\mathcal{X}_s^+} & \longrightarrow & \mathrm{R}\mathcal{H}om(W_n\omega^\bullet_{\mathcal{X}_s^+}, W_n\omega^1_{\mathcal{X}_s^+}[-1]) \\ \downarrow & & \downarrow \\ W_n\omega^0_{\mathcal{X}_s^+} & \longrightarrow & \mathrm{R}\mathcal{H}om(W_n\omega^1_{\mathcal{X}_s^+}[-1], W_n\omega^1_{\mathcal{X}_s^+}[-1]) \\ \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} \end{array}$$

Since the top and bottom maps are isomorphisms, we conclude the same for the middle map. Similarly for $W_n\widetilde{\omega}^\bullet_{\mathcal{X}_s^+}$, whence we establish (21). □

4.4. **Construction of trace maps.** A trace map f_* on logarithmic de Rham–Witt complexes may now be constructed in the usual fashion, see Gros [Gro85, Section II.1] for the smooth case. We start by noting that Ekedahl duality (20) implies that $W\omega^1_{\mathcal{X}_s^+} \simeq W(f)^! W\omega^1_{\mathcal{Y}_s^+}$, so that

$$\begin{aligned} \gamma_1 : \mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+} &\simeq \mathrm{R}f_*\mathrm{R}\mathcal{H}om(W\omega^\bullet_{\mathcal{X}_s^+}, W\omega^1_{\mathcal{X}_s^+}[-1]) \\ &\simeq \mathrm{R}f_*\mathrm{R}\mathcal{H}om(W\omega^\bullet_{\mathcal{X}_s^+}, W(f)^! W\omega^1_{\mathcal{Y}_s^+}[-1]), \end{aligned}$$

where the first isomorphism follows by auto-duality (21). The adjunction maps

$$\mathrm{Tr}_n : \mathrm{R}W_n(f)_* W_n(f)^! \longrightarrow 1$$

on $W_n(\mathcal{Y}_s)$ commute with restriction in n. We obtain by composition a map

$$\begin{aligned} \gamma_2 : \mathrm{R}f_*\mathrm{R}\mathcal{H}om(W\omega^\bullet_{\mathcal{X}_s^+}, W(f)^! W\omega^1_{\mathcal{Y}_s^+}[-1]) &\longrightarrow \mathrm{R}\mathcal{H}om(\mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+}, \mathrm{R}f_* W(f)^! W\omega^1_{\mathcal{Y}_s^+}[-1]) \\ &\longrightarrow \mathrm{R}\mathcal{H}om(\mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+}, W\omega^1_{\mathcal{Y}_s^+}[-1]). \end{aligned}$$

The natural pullback morphism $f^* : W\omega^\bullet_{\mathcal{Y}_s^+} \to \mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+}$ induces a map

$$\gamma_3 : \mathrm{R}\mathcal{H}om(\mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+}, W\omega^1_{\mathcal{Y}_s^+}[-1]) \longrightarrow \mathrm{R}\mathcal{H}om(W\omega^\bullet_{\mathcal{Y}_s^+}, W\omega^1_{\mathcal{Y}_s^+}[-1]) \simeq W\omega^\bullet_{\mathcal{Y}_s^+},$$

where the last isomorphism is the auto-duality (21). Putting everything together, we define

$$f_* := \gamma_3 \circ \gamma_2 \circ \gamma_1 : \mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+} \longrightarrow W\omega^\bullet_{\mathcal{Y}_s^+}. \tag{23}$$

4.5. **Functoriality of integral structures on de Rham cohomology.** We now compile the results established in the previous sections to prove the main result of this paper, providing trace and pullback maps between the de Rham–Witt complexes of semi-stable models $\mathcal{X}$ and $\mathcal{Y}$.

Theorem B. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite map of semi-stable models of smooth, proper, geometrically irreducible curves over K. There exist canonical pullback and trace maps (f^*, f_*) and $(\widetilde{f}^*, \widetilde{f}_*)$ that make the following diagram of exact triangles commute:*

$$\begin{array}{ccccccc}
\mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+}[-1] & \longrightarrow & \mathrm{R}f_* W\widetilde{\omega}^\bullet_{\mathcal{X}_s^+} & \longrightarrow & \mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+} & \xrightarrow{+1} & \\
{\scriptstyle f^*[-1]}\Big\downarrow\Big\uparrow{\scriptstyle f_*[-1]} & & {\scriptstyle \widetilde{f}^*}\Big\downarrow\Big\uparrow{\scriptstyle \widetilde{f}_*} & & {\scriptstyle f^*}\Big\downarrow\Big\uparrow{\scriptstyle f_*} & & \\
W\omega^\bullet_{\mathcal{Y}_s^+}[-1] & \longrightarrow & W\widetilde{\omega}^\bullet_{\mathcal{Y}_s^+} & \longrightarrow & W\omega^\bullet_{\mathcal{Y}_s^+} & \xrightarrow{+1} &
\end{array} \tag{24}$$

As a consequence, we obtain that when $f : X \to Y$ is a finite map of smooth, proper, geometrically irreducible curves over K with semi-stable reduction, then for any semi-stable models $\mathcal{X}, \mathcal{Y}$ there exist canonical pullback and trace maps of (φ, N)-modules

$$\begin{array}{rlll}
f^* : & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{Y}_s^+/W) & \longrightarrow & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W), \\
f_* : & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W) & \longrightarrow & \mathrm{H}^1_{\mathrm{HK}}(\mathcal{Y}_s^+/W).
\end{array}$$

For any uniformiser $\varpi \in R$, they induce the usual trace and pullback maps on de Rham cohomology of the generic fibres via the Hyodo–Kato isomorphism ρ^K_ϖ of Theorem 3.1.

Proof. From the presentation (15) and [Nak05, Theorem 11.1], we see that the pullback maps

$$\begin{array}{rlll}
f^* & : W\omega^\bullet_{\mathcal{Y}_s^+} & \longrightarrow & \mathrm{R}f_* W\omega^\bullet_{\mathcal{X}_s^+} \\
f^* & : W\widetilde{\omega}^\bullet_{\mathcal{Y}_s^+} & \longrightarrow & \mathrm{R}f_* W\widetilde{\omega}^\bullet_{\mathcal{X}_s^+}
\end{array}$$

make (24) commute. In the construction of the trace map f_* in 4.4, we may replace $W\omega^\bullet$ with $W\widetilde{\omega}^\bullet$, using (21). This yields a trace map $\widetilde{f}_*$, which by (22) makes (24) commute.

By Theorem 4.3, this induces trace and pullback maps between the Hyodo–Kato cohomology groups of *any* semi-stable models $\mathcal{X}, \mathcal{Y}$. The trace map f_* on de Rham–Witt complexes commutes with F and V, so that the resulting map on cohomology commutes with the Frobenius operator φ. By the commutativity of (24), it also commutes with the monodromy operator N. Finally[1], since we constructed f_* as the Poincaré dual of f^* it follows from the compatibility results of Tsuji [Tsu99a, §4.4] that f_* recovers the usual trace map on de Rham cohomology. □

4.6. **Remark.** Let X be a smooth, proper, geometrically irreducible curve over K with semi-stable reduction. Choose a uniformiser $\varpi \in R$, then we have two canonical lattices

$$\mathrm{H}^1_{\mathrm{HK}}(\mathcal{X}_s^+/W) \subseteq_\varpi \mathrm{H}^1_{\mathrm{dR}}(X/K), \qquad \mathrm{H}^1_{\mathrm{dR}}(\mathcal{X}/R) \subseteq \mathrm{H}^1_{\mathrm{dR}}(X/K). \tag{25}$$

The W-lattice of Hyodo–Kato cohomology is independent of the choice of semi-stable model $\mathcal{X}$, and is functorial in finite morphisms in X. The same is true for the R-lattice of de Rham cohomology of $\mathcal{X}$. The question arises how the two integral structures are related. It follows from Theorem 3.1 that

[1]We are very grateful to the anonymous referee for making this observation.

whenever the ramification index e of R is less than p, both lattices coincide after tensoring with R. In highly ramified situations, it is unclear how the two relate. This comparison becomes especially mysterious when one considers the cohomology of towers of curves, and we are forced to abandon any geometric relation to de Rham cohomology completely.

5. Crystalline cohomology of towers of curves

We now define the log-crystalline cohomology of towers of curves, and make some brief comments about it. We do not explicate the link with the de Rham cohomology of towers considered by Cais [Cai16a]. Therefore, the difficulty of having to choose appropriate sequences of uniformisers and Hyodo–Kato isomorphisms, and dealing with growing ramification of the base ring, is avoided by working purely with the collection of special fibres, endowed with their divisorial logarithmic structures.

By abandoning the effort to compare this new crystalline cohomology group of a tower to its de Rham cohomology, which is known to contain a great deal of arithmetic information by the work of Cais [Cai16a], we introduce (a priori) some doubts about having defined a meaningful object. In an attempt to alleviate such doubts, we show that the U_p-ordinary part of the crystalline cohomology of the tower $\{X_1(Np^r)\}_r$ is related to the work of Cais [Cai16a, Cai16b]. His arguments then show crystalline analogues of the Λ-adic control theorem, and the Λ-adic Eichler–Shimura decomposition.

5.1. **Towers of curves.** A *tower* of curves is a chain

$$\dots \xrightarrow{f_4} X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \tag{26}$$

of finite maps between smooth, proper, geometrically irreducible curves over a finite extension of $\mathbf{Q}_p$. For any such tower of curves, we may apply Theorem B to find trace and pullback maps between Hyodo–Kato cohomology groups attached to semi-stable models over some finite field extension. As a consequence of Theorem 4.3, we see that we may define these trace maps for any choice of semi-stable models over some finite field extension. Taking limits, we obtain two cohomology groups

$$\begin{aligned} \widetilde{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty) &:= \varprojlim_n \varinjlim_{f_r^*} \mathrm{H}^1_{\mathrm{HK}}\left(\mathcal{X}_r^+ \otimes \overline{\mathbf{F}}_p, W_n\right) \\ \widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty) &:= \varprojlim_{f_{r,*}} \mathrm{H}^1_{\mathrm{HK}}\left(\mathcal{X}_r^+ \otimes \overline{\mathbf{F}}_p, W\right) \end{aligned} \tag{27}$$

Here, $W = W(\overline{\mathbf{F}}_p)$ is the ring of integers in the completion of the maximal unramified extension of $\mathbf{Q}_p$. The former will be called the log-crystalline "completed" cohomology of our tower, whereas the latter will be called the log-crystalline (or simply crystalline) "Eichler–Shimura" cohomology of our tower.

5.2. **Crystalline completed cohomology for Shimura curves.** Let F be a totally real number field, and B_F a quaternion algebra split at exactly one infinite place and some finite place $\mathfrak{p}$. We let X_r be the Shimura curve for B with sufficiently small level structure away from $\mathfrak{p}$, as well as full $\mathfrak{p}^r$-level at $\mathfrak{p}$. The forgetful maps $f_r : X_r \to X_{r-1}$ form a tower, giving rise to the crystalline "*completed cohomology*" group $\widetilde{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)$. The full level structure at $\mathfrak{p}$ induces an action of $\mathrm{GL}_2(F_\mathfrak{p})$ on $\widetilde{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)[1/p]$. Extracting non-trivial information about this representation would require a precise description of semi-stable models of the X_r, and we will not attempt to analyse it here. See 5.5 for some further comments on this.

5.3. **Crystalline Eichler–Shimura cohomology for modular curves.** Consider the modular curves $X_1(p^r)$, with an implicit fixed tame level Γ that is small enough, as well as $\Gamma_1(p^r)$-structure at p. These curves are defined over $\mathbf{Q}_p$, and classify pairs (E, ψ_r) where E is a (generalised) elliptic curve over a $\mathbf{Q}_p$-scheme with Γ-level structure, and $\psi_r : \mu_{p^r} \hookrightarrow E[p^r]$ is a point of order p^r. We have a tower

$$\dots \longrightarrow X_1(p^r) \xrightarrow{\pi_2} \dots \longrightarrow X_1(p^2) \xrightarrow{\pi_2} X_1(p), \qquad \pi_2(E, \psi_r) = (E/\psi_r(\mu_p), \psi_r/\psi_r(\mu_p)) \tag{28}$$

We work with quotient degeneracy maps, rather than forgetful degeneracy maps, to make the trace map Hecke equivariant. We will consider the crystalline "*Eichler–Shimura*" cohomology

$$\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty) = \varprojlim_{\pi_{2,*}} \mathrm{H}^1_{\mathrm{HK}}\left(\mathcal{X}_r^+ \otimes \overline{\mathbf{F}}_p, W\right). \tag{29}$$

This group should be thought of as a crystalline analogue of the cohomologies of towers in the étale and de Rham settings considered by Ohta [Oht00] and Cais [Cai16a] respectively. It follows from Theorem B that crystalline Eichler–Shimura cohomology is a $W[\![\mathbf{Z}_p^\times]\!]$-module via the diamond operators on the generic fibres, and is equipped with an action of the completed Hecke algebra $\mathfrak{H} = \varprojlim_{\pi_2} \mathfrak{H}(X_r)$. It has an action of the Galois group $\Gamma = \mathrm{Gal}(K_\infty / \mathbf{Q}_p)$, where K_∞ is the compositum of all the finite extensions $K_r / \mathbf{Q}_p$ where the X_r obtain semi-stable reduction. Note that this is strictly larger than the cyclotomic p-extension of $\mathbf{Q}_p$, and hence Γ has $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \simeq \mathbf{Z}_p$ as a proper quotient. For precise information on the fields K_r, see Krir [Kri96].

5.4. **Ordinary Eichler–Shimura cohomology for modular curves.** It is not a priori clear that the crystalline Eichler–Shimura cohomology $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)$ is an interesting invariant to attach to a tower of curves. As a sanity check, we investigate the U_p-ordinary part for the $\Gamma_1(p^r)$-tower of modular curves, and prove analogues of the main results in Hida theory. The ordinary setting is very well-studied, so we frequently omit details which are readily found in [MW86, Cai16a, Cai16b].

Let $e^{\mathrm{ord}} = \lim U_p^{n!} \in \mathfrak{H}$ be the ordinary projector in the completed Hecke algebra $\mathfrak{H}$ generated by T_p, U_l, and the diamond operators $\langle d \rangle$. The diamond operators make $\mathfrak{H}$ into a module over $\mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!]$. For any module M over $\mathfrak{H}$, we write M^{ord} for $e^{\mathrm{ord}}M$, and M' for the projection onto the part where $\mu_{p-1} \subset \mathbf{Z}_p^\times$ acts non-trivially. Note that $M'^{\,\mathrm{ord}}$ is naturally a module over $\Lambda = \mathbf{Z}_p[\![1 + p\,\mathbf{Z}_p]\!]$.

The key observation, which is due to Mazur–Wiles [MW86], is that the natural map

$$\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\,\mathrm{ord}} \longrightarrow \varprojlim_{\pi_{2,*}} \left(\mathrm{H}^1_{\mathrm{cris}}\left(\mathrm{Ig}_r^\infty \otimes \overline{\mathbf{F}}_p, W\right)'^{\,\mathrm{ord}} \oplus \mathrm{H}^1_{\mathrm{cris}}\left(\mathrm{Ig}_r^0 \otimes \overline{\mathbf{F}}_p, W\right)'^{\,\mathrm{ord}}\right) \tag{30}$$

is an isomorphism, where Ig_r^∞ and Ig_r^0 are the unique distinct irreducible components of the stable model of X_r intersecting the cuspidal divisors ∞ and 0. As the Igusa curves Ig_r are defined over $\mathbf{F}_p$, we may naturally consider $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\,\mathrm{ord}}$ as a module over Λ. Let us define the following Λ-modules:

$$\begin{cases} \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_\infty^\infty)'^{\,\mathrm{F\text{-}ord}} & := \varprojlim_{\pi_{2,*}} \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_r^\infty, \mathbf{Z}_p)'^{\,\mathrm{F\text{-}ord}}, \\ \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_\infty^0)'^{\,\mathrm{V\text{-}ord}} & := \varprojlim_{\pi_{2,*}} \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_r^0, \mathbf{Z}_p)'^{\,\mathrm{V\text{-}ord}}. \end{cases} \tag{31}$$

where the superscripts denote the parts which are ordinary for the Frobenius operator F and Verschiebung operator V respectively. Using the above observation of Mazur–Wiles, the following crystalline analogue of Hida theory for $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\mathrm{ord}}$ follows immediately from Cais [Cai16a, Cai16b].

Theorem C. *The projection $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\mathrm{ord}}$ of crystalline Eichler–Shimura cohomology is a finite free Λ-module. There is a canonical decomposition of finite free Λ-modules*

$$\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)'^{\mathrm{ord}} \simeq \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_\infty^\infty)'^{\mathrm{F-ord}} \oplus \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_\infty^0)'^{\mathrm{V-ord}}.$$

Proof. By the isomorphism (30), we reduce to an analysis of the U_p-ordinary part of the crystalline cohomology of two Igusa towers. The correspondence U_p decomposes as:

$$U_p : \quad \mathrm{Ig}_r^\infty \xleftarrow{\ \mathrm{Id}\ } \mathrm{Ig}_r^\infty \xrightarrow{\ \mathrm{Frob}_p\ } \mathrm{Ig}_r^\infty \quad + \quad \mathrm{Ig}_r^0 \xleftarrow{\ \mathrm{Frob}_p\ } \mathrm{Ig}_r^0 \xrightarrow{\ \mathrm{Id}\ } \mathrm{Ig}_r^0$$

These two correspondences are interchanged under the Atkin–Lehner involution, see Cais [Cai16a, Propositions B.9 and B.25]. This allows us to identify the U_p-ordinary part of $\mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}_r^\infty)'$ with the

Frobenius-ordinary part, and the U_p-ordinary part of $\mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}^0_r)'$ with the Verschiebung-ordinary part. The Λ-freeness of the Frobenius- and Veschiebung-ordinary parts

$$\mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}^{\infty}_{\infty})'^{\,\mathrm{F\text{-}ord}}, \qquad \text{and} \qquad \mathrm{H}^1_{\mathrm{cris}}(\mathrm{Ig}^{\infty}_{0})'^{\,\mathrm{V\text{-}ord}}$$

was proved in [Cai16b, Theorems 1.2.1 and 1.2.3], so the result follows from (30). □

5.5. **The non-ordinary part of Eichler–Shimura cohomology.** In the discussion above, we applied the ordinary projector, after which most information encoded in the tower of semi-stable models is lost. In addition, this did not give us an object for which a clear need existed, as Cais [Cai16b] has already constructed a crystalline analogue of Eichler–Shimura cohomology in the ordinary case, using Dieudonné theory. Clearly, the main interest of $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)$ lies in its non-ordinary part.

One is particularly led to wonder what can be said about the *Lubin–Tate tower*, which is the supersingular part of the tower. The crystalline cohomology of this part can be understood via a detailed description of the geometry of the Lubin–Tate tower, which was recently achieved by Weinstein [Wei16], resulting in a complete determination of the supersingular components in the special fibres of the stable models of the modular curves in the tower, as well as their incidence structure. This suggests a concrete approach for the study of the non-ordinary part of $\widehat{\mathrm{H}}^1_{\mathrm{HK}}(X_\infty)$, which we will pursue in the future.

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