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DIRECT PRODUCTS OF FREE GROUPS IN $\mathrm{Aut}(F_N)$

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Abstract. We give a complete description of the embeddings of direct products of nonabelian free groups into $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$ when the number of direct factors is maximal. To achieve this, we prove that the image of each such embedding has a canonical fixed point of a particular type in the boundary of Outer space.

1 Introduction

Mapping class groups of surfaces of finite type and automorphism groups of free groups are central objects in geometric topology and group theory, and it is natural to study them in parallel. Nielsen-Thurston theory [Thu88] provides a potent geometric description of the individual elements of mapping class groups, and the train-track technology initiated by Bestvina and Handel [BH92] provides an equally potent description in the wilder setting of free group automorphisms. In pursuit of a more global understanding of these groups, one seeks insight from their actions on Teichmüller space and Outer space, as well as associated spaces such as the curve complex, in the case of mapping class groups, and free factor and splitting complexes in the case of $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$.

The need for geometric insights and invariants comes into particularly sharp focus when one is trying to elucidate the intricate subgroup structure of these groups, as we are in this article. One sees this clearly in the classification of abelian subgroups, which has many ramifications. From the Nielsen-Thurston theory, one knows that, up to finite-index, every abelian subgroup of the mapping class group is generated by combinations of Dehn twists in a collection of disjoint curves and, optionally, a pseudo-Anosov automorphism on each connected component of the complement of these curves [BLM83, Iva92]. This description provides a host of geometric invariants for studying the totality of abelian subgroups, starting with the stable laminations of the pseudo-Anosov pieces and compatibility conditions for the curve systems. With these in hand, one can organise the commensurability classes of maximal-rank abelian subgroups into a space that is closely related to the curve complex. This idea is central to Ivanov's proof [Iva97] of commensurator rigidity for mapping class groups: he proved that, with some low-genus exceptions, every isomorphism between finite-index subgroups of a mapping class group is the restriction of a conjugation in the ambient group. Although the situation in free groups is much more complicated, Feighn and Handel [FH11] succeeded in describing all abelian subgroups of $\mathrm{Out}(F_N)$, and this description was used by Farb and Handel [FH07] to establish commensurator rigidity for $\mathrm{Out}(F_N)$ in the case $N \geq 4$.

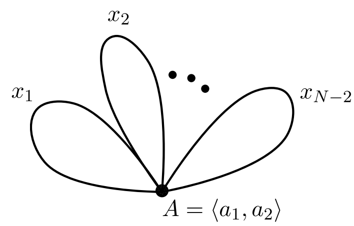

Figure 1: A collapsed rose with $N-2$ petals.

Ivanov's commensurator rigidity theorem was later extended by Bridson, Pettet and Souto [BPS11] to various subgroups of the mapping class group; they followed a similar template of proof but used *direct products of nonabelian free groups* in place of abelian subgroups, replacing the curve complex with a complex built from decompositions of the surface into subsurfaces of Euler characteristic -2 (cf. [BM19]). In the same spirit, by focussing on direct products of nonabelian free groups rather than abelian subgroups, Horbez and Wade [HW20] proved that $\mathrm{Out}(F_N)$ has commensurator rigidity for $N \geq 3$, as do many of its natural subgroups.

Our main purpose in this article is to give a complete classification of the maximal-rank direct products of free groups in $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$; we shall see that they are remarkably rigid. More generally, we will work with maximal commuting families of *full-sized* groups (groups that contain non-abelian free subgroups). In a companion to this paper [BW24], we shall use this classification, in harness with [BB23], to prove that $\mathrm{Aut}(F_N)$ and its Torelli subgroup are commensurator rigid if $N \geq 3$.

The most important step in our proof of the classification is a fixed-point theorem that we establish for the action of $\mathrm{Out}(F_N)$ on the space of free splittings of F_N (Theorem A). To motivate this theorem, we begin by describing an example of a subgroup of $\mathrm{Out}(F_N)$ that is a direct product of the maximal number of copies of F_2; Horbez and Wade [HW20] proved that this number is $2N-4$ (one less than the cohomological dimension).

We fix a basis $\{a_1, a_2, x_1, \ldots, x_{N-2}\}$ of F_N and consider the direct product D of the $2N-4$ copies of F_2 in $\mathrm{Out}(F_N)$ obtained by multiplying the elements $x_1, \ldots, x_{N-2}$ on the left and right by elements of $\langle a_1, a_2 \rangle$. This group D fixes a graph-of-groups decomposition of F_N with a single vertex group given by $A = \langle a_1, a_2 \rangle$ and $N-2$ loops with trivial edge stabilizers (with $x_1, \ldots, x_{N-2}$ as the stable letters). The Bass–Serre tree associated to any such decomposition determines an open simplex in the boundary of Culler and Vogtmann's Outer space [CV86]; we call such a graph of groups a *collapsed rose with* $N-2$ *petals* (see Fig. 1).

Our first theorem shows that if a direct product of nonabelian free groups in $\mathrm{Out}(F_N)$ has the maximal number of factors, then it has a canonical fixed point of this type. Note that in the following theorem, and throughout, we do not assume that D is finitely generated.

Theorem A. *Let $N \geq 3$ and suppose $D < \mathrm{Out}(F_N)$ is generated by a commuting family of $2N-4$ full-sized subgroups. Then there is a unique collapsed rose with $N-2$ petals that is fixed by D.*

When a subgroup of $\mathrm{Out}(F_N)$ fixes a tree T, the preimage of this group in $\mathrm{Aut}(F_N)$ admits an action on T. With a small amount of extra work, the following result can be deduced from Theorem A.

Theorem B. *Let $N \geq 3$ and suppose $D < \mathrm{Aut}(F_N)$ is generated by a commuting family of $2N-3$ full-sized subgroups. Then, the image of D in $\mathrm{Out}(F_N)$ fixes a unique collapsed rose with $N-2$ petals, and D acts on the Bass–Serre tree of this collapsed rose with a unique global fixed point.*

In order to move from Theorems A and B to the precise algebraic description of the direct products of free groups that we seek, some more notation is required. We continue to work with a fixed basis $\{a_1, a_2, x_1, \ldots, x_{N-2}\}$ of F_N and let A be the free factor generated by a_1 and a_2. Let L_i be the free group of rank 2 in $\mathrm{Aut}(F_N)$ consisting of elements that send $x_i \mapsto ax_i$ for some $a \in A$ and fix all other basis elements. Similarly, we use R_i to denote the free group of right transvections of x_i by elements of A. Furthermore, we let $I(A)$ be the group of inner automorphisms generated by elements of A and let τ be the Nielsen automorphism mapping $a_1 \mapsto a_1a_2$ and fixing all other basis elements. We let L_i^τ, R_i^τ, and $I(A)^\tau$ be the respective subgroups of these groups that commute with τ (equivalently, the elements from A used in their associated transvections or inner automorphisms belong to $\mathrm{Fix}(\tau) \cap A = \langle a_1a_2a_1^{-1}, a_2\rangle$).

Theorem C. *Let $N \geq 3$ and suppose $D < \mathrm{Aut}(F_N)$ is a direct product of $2N-3$ nonabelian free groups. Then a conjugate of D is contained in one of the following groups.*

- $L_1 \times \cdots \times L_{N-2} \times R_1 \times \cdots \times R_{N-2} \times I(A)$
- $L_1^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times I(A)^\tau \times \langle \tau \rangle$
- $\langle L_1, \tau \rangle \times L_2^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times I(A)^\tau$
- $L_1^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times \langle I(A), \tau \rangle$

In the second case, if $D = D_1 \times \cdots \times D_{2N-3}$, then each summand D_i is contained in a unique subgroup of the form $\langle L_i^\tau, \tau\rangle$, $\langle R_i^\tau, \tau\rangle$, or $\langle I(A)^\tau, \tau\rangle$. In the remaining cases, the inclusion respects the visible direct product decompositions, up to permuting the summands.

In Sect. 7 we shall prove that every direct product of $2N-4$ nonabelian free groups in $\mathrm{Out}(F_N)$ is contained in the image of one of the subgroups listed in Theorem C. Furthermore, the enveloping groups are canonical in the following sense:

Theorem D. *Let $N \geq 3$ and suppose $D < \mathrm{Aut}(F_N)$ is generated by a set of $2N-3$ pairwise-commuting, full-sized subgroups. Then D is maximal with respect to inclusion among such subgroups of $\mathrm{Aut}(F_N)$ if and only if D is conjugate to one of the groups in the first, third, or fourth bullet points in Theorem C.*

Theorem C places each summand $D_i < D$ inside a group of the form F_2 or $F_2 \times \mathbb{Z}$ or $M_\tau = F_2 \rtimes_\tau \mathbb{Z}$. It is obvious that every free subgroup of F_2 or $F_2 \times \mathbb{Z}$ is contained in a maximal free subgroup. In Sect. 8, we shall prove the less obvious fact that this is also the case in M_τ.

Corollary E. *Let $N \geq 3$ and let $\mathcal{D}$ be the family of subgroups of either* $\mathrm{Aut}(F_N)$ *or* $\mathrm{Out}(F_N)$ *that are direct products of $2N-3$ or $2N-4$ nonabelian free groups, respectively. Then every $D \in \mathcal{D}$ is contained in a maximal element (with respect to inclusion).*

A further consequence of Theorem C is that the centralizer of the direct product D is cyclic, and when it is non-trivial it is generated by a Nielsen automorphism. This yields the following rigidity result, which plays a crucial role in [BW24].

Theorem F. *Let Γ be a finite-index subgroup of* $\mathrm{Aut}(F_N)$ *with $N \geq 3$ and let $f\colon \Gamma \to$* $\mathrm{Aut}(F_N)$ *be an injective homomorphism. Every power of a Nielsen automorphism is mapped to a power of a Nielsen automorphism under f.*

1.1 Techniques and proofs. In the remainder of this introduction, we shall describe the structure of this paper and sketch some of the main ideas that go into the proofs of the main results. Throughout, we assume $N \geq 3$. For ease of exposition, we restrict to the case where D is a direct product of free groups.

Our first goal is to establish the existence of the fixed point described in Theorem A. Let D be a direct product of $2N-4$ nonabelian free groups in $\mathrm{Out}(F_N)$ and let $\widetilde{D}$ be its preimage in $\mathrm{Aut}(F_N)$. Our starting point is Theorem 6.1 from [HW20], where actions of $\mathrm{Out}(F_N)$ on relative free factor complexes were used to show that D fixes a one-edge nonseparating free splitting of F_N. Lemma 2.7 tells us that a simplicial tree T in the boundary of Outer space will be fixed by D if and only if the F_N-action on T extends to an action of $\widetilde{D}$ on T (where F_N is identified with the group of inner automorphisms in $\widetilde{D}$). We apply this to the one-edge splitting fixed by D, blowing-up the action of $\widetilde{D}$ on the Bass–Serre tree to obtain the action on a collapsed rose with $N-2$ petals that we seek; this blow-up, which is described in Sect. 6, is constructed using a *graph of actions* in the sense of Levitt [Lev94]. A key point is to argue that the stabilizer in $\widetilde{D}$ of a vertex $v \in T$ acts on a collapsed rose with $N-3$ petals, and that the adjacent edge stabilizers $\widetilde{D}_e$ for the one-edge splitting are elliptic in this new action; this last property implies the edges of the old tree can be glued onto the new tree in a coherent fashion.

The uniqueness of the fixed point described in Theorem A is tackled separately. By work of Guirardel and Horbez [GH21, Sect. 6], if there were two collapsed roses with $N-2$ petals fixed by D then they would belong to the same deformation space, and a folding path between these two collapsed roses would have to be fixed by D. However, D is too large to fix any graph of groups decomposition of F_N with more than one vertex group, which means the folding path must be trivial. Details are given in Sect. 5.

Section 4 contains an analysis of the stabilizers in $\mathrm{Aut}(F_N)$ of collapsed roses and similar graphs. This analysis plays a significant role in the proof of Theorem A, and it renders the deduction of Theorem B straightforward, as we shall see at the end of Sect. 6. The analysis of stabilizers of collapsed roses also provides a crucial bridge from Theorem A to Theorem C. In particular, with Theorem A in hand, Proposition 4.1 essentially reduces Theorem C to an analysis of the ways in which a direct product of $k+1$ nonabelian free groups can embed in

$$M_k(F_2) := F_2^k \rtimes \mathrm{Aut}(F_2),$$

where the action of $\mathrm{Aut}(F_2)$ in this semidirect product is diagonal. These embeddings are described in Theorem 3.8; the required algebra is surprisingly delicate. Given that our main results involve free groups of higher rank, it seems incongruous that special features of $\mathrm{Aut}(F_2)$ should play a crucial role at this stage of the proof, but nevertheless this is the case. A key fact that makes many arguments in Sect. 3 work is that powers of Nielsen transformations are the only automorphisms of F_2 that have nonabelian fixed subgroups [CT96]. This special property lies behind the appearance of the Nielsen transformation τ in Theorem C.

2 Product rank, splittings, and automorphic lifts

2.1 Direct products of free groups and commuting families of full-sized groups.

A group G has *product rank* $\mathrm{rk}_F(G) = k$ if k is the largest integer such that G contains a direct product of k nonabelian free groups (possibly $\mathrm{rk}_F(G) = \infty$). To understand how product rank behaves with respect to homomorphisms between groups, we make use of the following standard lemma.

Lemma 2.1. *Let K be a normal subgroup of a direct product of nonabelian free groups $G_1 \times G_2 \times \cdots \times G_k$. Suppose that $\mathrm{rk}_F(K) = l$. Then, after reordering the factors, K is a normal subgroup of $G_1 \times G_2 \times \cdots \times G_l \times 1 \times \cdots \times 1$.*

Proof. Let $g = (g_1, g_2, \dots, g_k) \in K$ and without loss of generality assume $g_1 \neq 1$. Let $h \in G_1$ be an element that does not commute with g_1. Conjugation by $(h, 1, 1, \dots, 1)$ shows that $(hg_1h^{-1}, g_2, g_3, \dots, g_k) \in K$, so $(hg_1h^{-1}g_1^{-1}, 1, 1, \dots, 1) \in K$. Hence if K has a nontrivial projection to a factor then it intersects that factor in an infinite normal (hence nonabelian) subgroup. As $\mathrm{rk}_F(K) = l$, it intersects exactly l factors. $\square$

The following easy consequence of Lemma 2.1 is a variation on [HW20, Lemma 6.3].

Lemma 2.2. *If H is a finite-index subgroup of G then $\mathrm{rk}_F(H) = \mathrm{rk}_F(G)$. If*

$$1 \to N \to G \to Q \to 1$$

is an exact sequence of groups then $\mathrm{rk}_F(G) \leq \mathrm{rk}_F(N) + \mathrm{rk}_F(Q)$.

Recall that a group is *full-sized* if it contains a nonabelian free subgroup.

PROPOSITION 2.3. *Let G be a group generated by a set $G_1, \dots, G_k$ of commuting full-sized subgroups. Then $\mathrm{rk}_F(G) \geq k$. Furthermore, suppose K is a normal subgroup of G with $\mathrm{rk}_F(K) = l < k$. Then, after permuting the G_i, the groups $G_{l+1}/K, \dots, G_k/K$ form a commuting family of full-sized subgroups in G/K.*

Proof. As each G_i is full-sized, each G_i contains a group $D_i \cong F_2$. Let

$$\phi\colon D_1 \times D_2 \times \cdots \times D_k \to G$$

be the homomorphism induced by mapping each D_i into G. We claim that ϕ is injective. Indeed, if $K = \ker\phi$ is nontrivial then there exists i such that K has a nontrivial projection to D_i, and as in the proof of Lemma 2.1, the intersection $K \cap D_i$ is nontrivial. However, each D_i injects into G under ϕ, which is a contradiction. Hence ϕ is injective and $\mathrm{rk}_F(G) \geq k$. Now suppose that K is a normal subgroup of G with $\mathrm{rk}_F(K) \leq l$. Let $D = \langle D_1, \dots, D_k \rangle$, which is isomorphic to $D_1 \times D_2 \times \cdots \times D_k$ by the work above. Let $K' = K \cap D$. Then by Lemma 2.2, after reordering the G_i the group K' is a normal subgroup of $D_1 \times D_2 \times \cdots \times D_l \times 1 \times \cdots \times 1$, and D_i embeds in G_i/K for $i = l+1, \dots, k$. Hence these quotient groups are also full-sized. □

2.2 The product rank of $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$. Fixing a basis $\mathcal{B} = \{a_1, a_2, x_1, \dots, x_{N-2}\}$ for F_N, one obtains a direct product of $2N-4$ free groups of rank 2 in $\mathrm{Aut}(F_N)$ as follows. For $i = 1, \dots, N-2$ let L_i be the subgroup consisting of automorphisms of the form $[x_i \mapsto wx_i,\ x_j \mapsto x_j\,(j \neq i)]$, where w is a word in the free group on $\{a_1, a_2\}$, and let R_i be the subgroup consisting of automorphisms of the form $[x_i \mapsto x_i w,\ x_j \mapsto x_j\,(j \neq i)]$. Each L_i and R_i is a free group of rank 2, and these subgroups generate a direct product $D_{\mathcal{B}} = L_1 \times R_1 \times \cdots \times L_{N-2} \times R_{N-2} < \mathrm{Aut}(F_N)$. As $D_{\mathcal{B}}$ contains no inner automorphisms, it injects into $\mathrm{Out}(F_N)$. Theorem 6.1 of [HW20] shows that $\mathrm{Out}(F_N)$ does not contain a direct product of $2N-3$ nonabelian free groups if $N > 2$, thus

$$\mathrm{rk}_F(\mathrm{Out}(F_N)) = 2N - 4$$

for $N \geq 3$. (Note that the virtual cohomological dimension of $\mathrm{Out}(F_N)$, which gives an upper bound on product rank, is $2N-3$.)

The conjugations of F_N by a_1 and a_2 generate a further free subgroup $I(A) < \mathrm{Aut}(F_N)$ that commutes with $D_{\mathcal{B}}$. As $I(A) \cap D_{\mathcal{B}}$ is trivial, we get

$$\mathrm{rk}_F(\mathrm{Aut}(F_N)) \geq 2N - 3,$$

and by Lemma 2.2 we must have equality when $N \geq 3$. Since $\mathrm{Aut}(F_2)$ does not contain a direct product of two nonabelian free groups [Gor04] (see also Corollary 3.4 (4) below), we have equality in the case $N = 2$ as well.

We summarize this discussion for later use:

Proposition 2.4 ([HW20], Theorem 6.1). *For every $N \geq 2$ we have*

$$\mathrm{rk}_F(\mathrm{Aut}(F_N)) = 2N - 3.$$

For every $N \geq 3$ we have

$$\mathrm{rk}_F(\mathrm{Out}(F_N)) = 2N - 4.$$

Since $\mathrm{Out}(F_2) \cong \mathrm{GL}(2, \mathbb{Z})$ is virtually free, $\mathrm{rk}_F(\mathrm{Out}(F_2)) = 1$.

2.3 Splittings and their stabilizers. A *splitting* of a group G is a minimal, simplicial left action on a tree. (The terminology comes from the fact that the quotient graph of groups splits G in terms of amalgamated free products and HNN extensions [Ser80].) The splitting is said to be *free* if all edge stabilizers are trivial. Two splittings T and T' are deemed *equivalent* if there is a G-equivariant simplicial isomorphism from T to T'. The trees that we consider are not allowed to have vertices of valence two. (The quotient graph of groups may still have vertices v of valence two, in which case the vertex group G_v will be nontrivial.) We say that T' is a *collapse* of T if the action of G on T' is obtained by equivariantly collapsing a forest in T. Going in the opposite direction, we say that T is a *refinement* of T' if T' is a collapse of T. Two splittings are said to be *compatible* if they have a common refinement.

We shall be concerned almost entirely with the case $G = F_N$.

Each vertex stabilizer of a free splitting of F_N is a free factor. We work with the standard left action of $\mathrm{Aut}(F_N)$ on F_N. There is then a right action of $\mathrm{Aut}(F_N)$ on the set of all free splittings of F_N: the action of $\phi \in \mathrm{Aut}(F_N)$ sends $f : F_N \to \mathrm{Isom}(T)$ to $f \circ \phi$. This action respects equivalence classes of F_N-trees, and the inner automorphisms leave each equivalence class invariant. Thus there is an induced action of $\mathrm{Out}(F_N)$ on the set of equivalence classes of free splittings of F_N. Stabilizers under this action have been studied extensively in the literature; the most general results (replacing F_N with an arbitrary group and allowing more general splittings) appear in work of Bass–Jiang and Levitt [BJ96, Lev05].

We write $\mathcal{FS}$ for the set of equivalence classes of free splittings of F_N. When there is no danger of ambiguity, we shall not distinguish between a free splitting T and its equivalence class $[T]$.

Unpacking the definitions, we see that $[T] \in \mathcal{FS}$ is fixed by an outer automorphism $\Phi \in \mathrm{Out}(F_N)$ if and only if for each representative $\phi \in \Phi$ there is a homeomorphism $f_\phi : T \to T$ such that

$$f_\phi(gx) = \phi(g) f_\phi(x) \tag{1}$$

for all $x \in T$ and $g \in F_N$; in other words, $[T]$ is fixed by $\phi \in \mathrm{Aut}(F_N)$ if and only if there is an isomorphism from T to itself that is 'ϕ-twistedly equivariant'. The map f_ϕ is unique. (This is true, more generally, for stabilizers of minimal irreducible G-trees.) If ϕ is conjugation by g, then $f_\phi(x) = gx$.

We use $\mathrm{Stab}(T)$ to denote the stabilizer of $[T]$ in $\mathrm{Out}(F_N)$. There is a homomorphism

$$\mathrm{Stab}(T) \to \mathrm{Aut}(T/F_N)$$

given by the left action of each outer automorphism on the F_N-orbits of edges and vertices in T. We call the kernel of this map $\mathrm{Stab}^0(T)$. (Here, T/F_N is the quotient graph, not the quotient graph of groups.)

We use $\mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ to denote the kernel of the map

$$\mathrm{Out}(F_N) \to \mathrm{GL}_N(\mathbb{Z}/3\mathbb{Z})$$

given by the action of $\mathrm{Out}(F_N)$ on $H_1(F_N, \mathbb{Z}/3\mathbb{Z})$. The analogous subgroup of the mapping class group consists of *pure* mapping classes [Iva02, Theorem 7.1.E] and behaves similarly; both groups are torsion-free and passing to them avoids a good deal of troublesome periodic behaviour. In this vein, we will require the following consequence of [HM20, Theorem 3.1].

Proposition 2.5 ([HW20], Lemma 2.6). *Suppose that $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$. If the G-orbit of $T \in \mathcal{FS}$ is finite, then G fixes T; moreover $G < \mathrm{Stab}^0(T)$ and G fixes every collapse of T.*

If T is a free splitting with one F_N-orbit of edges, we say that T is a *one-edge splitting*. A one-edge splitting is *nonseparating* if the quotient graph T/F_N is a loop, and *separating* otherwise. The link between free splittings and our study of direct products of free groups is the following extract from [HW20, Theorem 6.1]. Its original proof was in the context of direct products of free groups; we sketch below how the proof extends to commuting families of full-sized groups.

Theorem 2.6 ([HW20], Theorem 6.1). *Suppose $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ is generated by a set $G_1, \ldots, G_{2N-4}$ of pairwise-commuting full-sized subgroups. Then G fixes a one-edge nonseparating free splitting of F_N.*

Sketch proof. Let $\mathcal{F}$ be a maximal, proper, G-periodic free factor system. Then by [HM20, Theorem 3.1], as G is contained in $\mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$, the free factor system $\mathcal{F}$ is fixed by G. A free factor system is *sporadic* if either $\mathcal{F} = \{[A]\}$ is given by a single factor of rank $N-1$ or $\mathcal{F} = \{[A], [B]\}$, where $\mathrm{Rank}(A) + \mathrm{Rank}(B) = N$. A system is *nonspordadic* if it is not sporadic.

If $\mathcal{F}$ is nonsporadic, then the relative free factor complex $\mathrm{FF}(F_N, \mathcal{F})$ is hyperbolic and has infinite diameter, and the considerations in Sect. 4 of [HW20] apply equally well to commuting families as they do to direct products: either G has bounded orbits in $\mathrm{FF}(F_N, \mathcal{F})$, or after reindexing the G_i, the subgroup generated by $G_1, \ldots, G_{2N-5}$ has a finite orbit in the Gromov boundary of $\mathrm{FF}(F_N, \mathcal{F})$ (this happens if the last subgroup G_{2N-4} contains a loxodromic element Φ: as the rest of the G_i commute with Φ they preserve the endpoints of its axis). The group G cannot have bounded orbits as by [GH19, Proposition 5.1], G would have a finite orbit in $\mathrm{FF}(F_N, \mathcal{F})$, contradicting maximality of $\mathcal{F}$. Furthermore, the subgroup generated by $G_1,\dots,G_{2N-5}$ cannot have a finite orbit in the boundary of $\mathrm{FF}(F_N,\mathcal{F})$, as by [HW20, Lemma 6.5], the product rank of the stabilizer of any point on the Gromov boundary of $\mathrm{FF}(F_N,\mathcal{F})$ is less than $2N-5$.

We may therefore assume $\mathcal{F}$ is sporadic. In this case, $\mathcal{F}$ determines a G-invariant one-edge splitting T (there is a unique one-edge splitting where $\mathcal{F}$ is elliptic). It is shown in the proof of [HW20, Theorem 6.1] that the stabilizer of a separating one-edge splitting has product rank at most $2N-5$, therefore T is nonseparating. □

2.4 Automorphic lifts. Subgroups that stabilize free splittings in $\mathrm{Out}(F_N)$ have the striking feature that they virtually lift to $\mathrm{Aut}(F_N)$ (see, for instance [HM23]). To describe this lifting, we need some further notation. Given $G<\mathrm{Out}(F_N)$, we let $\tilde{G}$ denote the preimage of G in $\mathrm{Aut}(F_N)$. We view F_N as a subgroup of $\tilde{G}$ via the identification $g\mapsto \mathrm{ad}_g$ of F_N with the inner automorphisms. The following lemma is well known (see, for example, Lemma 6.7 of [BGH22]).

Lemma 2.7. *Let T be a splitting of F_N such that the action of F_N on T is irreducible. A subgroup $G<\mathrm{Out}(F_N)$ fixes T if and only if the action of F_N on T extends to an action of $\tilde{G}$ on T.*

Proof. As in Sect. 2.3, G fixes T if and only if for each $\phi\in\tilde{G}$ there exists a unique isomorphism $f_\phi: T\to T$ such that $f_\phi(gx)=\phi(g)f_\phi(x)$. If the action $f: F_N\to \mathrm{Isom(T)}$ extends to an action $\tilde{f}:\tilde{G}\to\mathrm{Isom(T)}$ then $f_\phi:=\tilde{f}(\phi)$ suffices. Conversely, if G fixes T and $\phi,\psi\in\tilde{G}$ then the isomorphisms f_ϕ and f_ψ satisfy

$$\begin{aligned} f_\phi f_\psi(gx) &= f_\phi(\psi(g)f_\psi(x)) \\ &= \phi\psi(g)f_\phi f_\psi(x), \end{aligned}$$

so by uniqueness $f_\phi f_\psi=f_{\phi\psi}$, and $\phi\mapsto f_\phi$ gives the required action. □

Understanding this extended action allows one to construct automorphic lifts.

Proposition 2.8 (Existence of automorphic lifts). *Let G be a subgroup of $\mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ that fixes a free splitting T and let $\tilde{G}<\mathrm{Aut}(F_N)$ be its preimage, which acts on T. For each edge e in T, the stabilizer $\tilde{G}_e<\tilde{G}$ of e is isomorphic to G. The isomorphism is given by the restriction to $\tilde{G}_e$ of the quotient map $\pi:\mathrm{Aut}(F_N)\to\mathrm{Out}(F_N)$.*

Proof. As T is a free splitting, no inner automorphism fixes e and the map $\tilde{G}_e\to G$ is injective. To see that it is surjective, suppose $\Phi\in G$ and let $\phi\in\Phi$ be a representative of Φ. Proposition 2.5 tells us that $G<\mathrm{Stab}^0(T)$ and hence $\tilde{G}$ preserves the F_N-orbits of edges in T. Therefore $f_\phi(e)=ge$ for some $g\in F_N$ and

$$f_{\mathrm{ad}_g^{-1}\phi}(e)=f_{\mathrm{ad}_g^{-1}}f_\phi(e)=g^{-1}\cdot ge=e.$$

It follows that $\mathrm{ad}_g^{-1}\phi$ is a representative of Φ in $\tilde{G}_e$. □

3 Direct products of free groups in $M_k(F_2)$ with maximal product rank

We consider semidirect products of the form $M_k(A) = (A \times \cdots \times A) \rtimes \mathrm{Aut}(A)$, where there are k copies of the arbitrary group A and the action of $\mathrm{Aut}(A)$ is diagonal. Writing elements of $M_k(A)$ in the form $(g_1, \ldots, g_k; \phi)$, the group operation is

$$(g_1, \ldots, g_k; \phi).(h_1, \ldots, h_k; \psi) = (g_1\phi(h_1), \ldots, g_k\phi(h_k); \phi \circ \psi).$$

These groups arise naturally in many settings. For example, if $X = K(A, 1)$ then the group of homotopy classes of homotopy equivalences $X \to X$ fixing $(k+1)$ marked points is isomorphic to $M_k(A)$. When A is free, these groups play an important role in the study of graph cohomology [C+16] and homology stability results for automorphism groups of free groups [HV04]. We shall be concerned almost entirely with the case $A = F_2$.

Our interest in $M_k(F_2)$ stems from the fact that if T is the Bass–Serre tree of a collapsed rose with $N-2$ petals in the boundary of Outer space then, as we will see later, $\mathrm{Stab}^0(T) \cong M_{2N-5}(F_2)$. Our purpose in this section is to give a detailed description of the ways in which a direct product of $(k+1)$ nonabelian free groups can be embedded in $M_k(F_2)$.

3.1 Generalities. We distinguish between the k visible copies of A in $M_k(A)$ by writing $M_k(A) = (A_1 \times \cdots \times A_k) \rtimes \mathrm{Aut}(A)$. If A has trivial centre, then $g \mapsto \mathrm{ad}_g$ defines an isomorphism from A to the group of inner automorphisms $\mathrm{Inn}(A) < \mathrm{Aut}(A)$. We claim that this gives rise to a natural embedding of the direct product $A^{k+1} \hookrightarrow M_k(A)$ with image

$$M_k^0(A) := (A_1 \times \cdots \times A_k) \rtimes \mathrm{Inn}(A),$$

where the last summand in A^{k+1} maps to

$$J := \{(g^{-1}, \ldots, g^{-1}; \mathrm{ad}_g) \mid g \in A\} < M_k(A).$$

The existence of this embedding points to the fact $A_1 \times \cdots \times A_k < M_k(A)$ is not a characteristic subgroup, and the semidirect product decomposition defining $M_k(A)$ is less canonical than the short exact sequence

$$1 \to M_k^0(A) \to M_k(A) \to \mathrm{Out}(A) \to 1.$$

Lemma 3.1. *With the notation established above,*

1. $(g_1, \ldots, g_k, x) \mapsto (g_1x^{-1}, \ldots, g_kx^{-1}; \mathrm{ad}_x)$ *defines an isomorphism* $A^{k+1} \xrightarrow{\cong} M_k^0(A)$ *with inverse* $(g_1, \ldots, g_k; \mathrm{ad}_x) \mapsto (g_1x, \ldots, g_kx, x)$.
2. *For* $i = 1, \ldots, k$, *there exists an involution* $\alpha_i \in \mathrm{Aut}(M_k(A))$ *that exchanges* A_i *and* J *while restricting to the identity on* $\mathrm{Aut}(A)$ *and* $A_{\ell \neq i}$;
3. *these* α_i *generate a copy of the symmetric group* $\mathrm{Sym}(k+1) \hookrightarrow \mathrm{Aut}(M_k(A))$, *and the embedding* $A^{k+1} \to M_k^0(A) < M_k(A)$ *is* $\mathrm{Sym}(k+1)$*-equivariant.*

Proof. For (1): it is clear that these maps are mutually inverse and a straightforward calculation establishes that they are homomorphisms.

To prove (2) and (3), one verifies that the formula

$$\alpha_i : (g_1, \ldots, g_k; \phi) \mapsto (g_1 g_i^{-1}, \ldots, g_{i-1} g_i^{-1}, g_i^{-1}, g_{i+1} g_i^{-1}, \ldots, g_k g_i^{-1}; \mathrm{ad}_{g_i}\phi)$$

defines an automorphism with the desired properties. □

3.2 Fixed subgroups, centralisers, and Nielsen transformations. We remind the reader that an automorphism $\tau \in \mathrm{Aut}(F_2)$ is called a *Nielsen transformation* if there is a basis $\{x_1, x_2\}$ of F_2 such that

$$\tau(x_1) = x_1 x_2 \text{ and } \tau(x_2) = x_2. \tag{2}$$

Any two Nielsen transformations are conjugate in $\mathrm{Aut}(F_2)$ and there are various ways to distinguish them from other automorphisms. In $\mathrm{Out}(F_2) \cong \mathrm{GL}(2,\mathbb{Z})$, an automorphism represents a power of a Nielsen transformation if and only if its associated matrix has trace 2 and determinant 1.

Lemma 3.2. *If a subgroup $H < \mathrm{Aut}(F_2)$ consists entirely of powers of Nielsen transformations, then it is cyclic.*

Proof. H intersects $\mathrm{Inn}(F_2)$ trivially and therefore injects into $\mathrm{Out}(F_2) \cong \mathrm{GL}(2,\mathbb{Z})$, where its image consists entirely of elements that have trace 2 and determinant 1; let M be such an element. We choose a basis so that the image of H contains $E_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some non-zero integer n. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then ME_n has trace 2 only if $c = 0$ and $a = d = 1$. Hence the image of H is contained in the cyclic subgroup of $\mathrm{GL}(2,\mathbb{Z})$ generated by E_1. □

A characterisation of Nielsen transformations that will be useful to us here concerns the rank of fixed subgroups; this is due to Collins and Turner [CT96]. Bestvina and Handel's solution to the Scott Conjecture [BH92] shows that the subgroup of $\mathrm{Fix}(\phi) < F_N$ fixed by an automorphism ϕ has rank at most N. Theorem A of [CT96] gives a complete description of the automorphisms with $\mathrm{rk}(\mathrm{Fix}(\phi)) = N$; in the case $N = 2$, these automorphisms are the powers of Nielsen transformations.

Proposition 3.3 ([CT96], Theorem A). *If $\phi \in \mathrm{Aut}(F_2)$ fixes a non-cyclic subgroup of F_2, then ϕ is a power of a Nielsen transformation. If τ is the Nielsen automorphism given in Equation (2) then the fixed subgroup of every nontrivial power of τ is $\langle x_1 x_2 x_1^{-1}, x_2 \rangle$.*

Corollary 3.4.
1. *If $\Lambda < \mathrm{Inn}(F_2)$ is not cyclic, then its centraliser $C(\Lambda) < \mathrm{Aut}(F_2)$ is either trivial or else the cyclic subgroup generated by a Nielsen transformation.*
2. *Let $\tau_0, \tau_1 \in \mathrm{Aut}(F_2)$ be Nielsen transformations. If $\langle \tau_0 \rangle \cap \langle \tau_1 \rangle \neq 1$, then $\tau_0 = \tau_1^{\pm 1}$.*

3. *Let* $\phi, \psi \in \mathrm{Aut}(F_2)$, *not both trivial. If* $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi)$ *is not cyclic, then there is a unique Nielsen transformation* $\tau^{\pm 1}$ *such that* $\phi, \psi \in \langle \tau \rangle$.
4. $\mathrm{rk}_F(\mathrm{Aut}(F_2)) = 1$.

Proof. This proof is a variation on the argument used by Gordon [Gor04, Theorem 3.2] to prove that $\mathrm{rk}_F(\mathrm{Aut}(F_2)) = 1$. If $\phi \in \mathrm{Aut}(F_2)$ centralizes Λ, then $\phi(w) = w$ for all $\mathrm{ad}_w \in \Lambda$, so ϕ is a power of a Nielsen transformation. Thus $C(\Lambda)$ consists entirely of powers of Nielsen transformations, so it is cyclic, by Lemma 3.2. If τ is a Nielsen transformation then $\mathrm{Fix}(\tau) = \mathrm{Fix}(\tau^p)$ for $p \neq 0$, so $\tau^p \in C(\Lambda)$ implies $\tau \in C(\Lambda)$. This proves (1).

For (2), using again the fact that $\mathrm{Fix}(\tau) = \mathrm{Fix}(\tau^p)$ for all $p \neq 0$ we see that $\mathrm{Fix}(\tau_0) = \mathrm{Fix}(\tau_1)$. Then apply (1) with $\Lambda = \{\mathrm{ad}_w : w \in \mathrm{Fix}(\tau_0)\}$, noting that a Nielsen transformation generates any cyclic subgroup to which it belongs. The existence of τ in (3) is proved by applying (1) to $\{\mathrm{ad}_x \mid x \in \mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi)\}$. Uniqueness follows immediately from (2).

For (4), since $\mathrm{Out}(F_2)$ is virtually free, the centraliser of any nonabelian free subgroup is finite. So if there were a copy of $F_2 \times F_2$ in $\mathrm{Aut}(F_2)$ then at least one of the factors, say $F_2 \times 1$, would have to intersect $\mathrm{Inn}(F_2)$ non-trivially. The intersection would be normal in $F_2 \times 1$, hence non-cyclic. On the other hand, the centraliser of this intersection would contain $1 \times F_2$, contradicting (1). ☐

Lemma 3.5. *Let* $\tau_0, \tau_1 \in \mathrm{Aut}(F_2)$ *be Nielsen transformations. If some non-zero powers of* τ_0 *and* τ_1 *commute, then* τ_0 *and* τ_1 *commute, and* $\tau_1 = \mathrm{ad}_w \tau_0^{\pm 1}$ *for some* $w \in \mathrm{Fix}(\tau_0)$.

Proof. By hypothesis, $[\tau_0^p, \tau_1^q] = 1$ for some $p, q \neq 0$, that is $(\tau_0^{-p}\tau_1\tau_0^p)^q = \tau_1^q$. From Corollary 3.4(2) we deduce $\tau_0^{-p}\tau_1\tau_0^p = \tau_1$. Similarly, $(\tau_1^{-1}\tau_0\tau_1)^p = \tau_0^p$ implies $\tau_1^{-1}\tau_0\tau_1 = \tau_0$.

In the linear action of $\mathrm{GL}(2,\mathbb{Z}) \cong \mathrm{Out}(F_2)$ on $\mathbb{R}^2$, the image of each Nielsen transformation τ has a unique eigenspace V and τ generates the pointwise stabiliser of V in $\mathrm{SL}(2,\mathbb{Z})$. As τ_0 and τ_1 commute, their eigenspaces coincide. Thus, replacing τ_0 by its inverse if necessary, $\tau_1 = \mathrm{ad}_w \tau_0$. Comparing $\tau_0\tau_1 = \mathrm{ad}_{\tau_0(w)}\tau_0^2$ to $\tau_1\tau_0 = \mathrm{ad}_w \tau_0^2$, we have $\tau_0(w) = w$. ☐

We remind the reader that two elements $\phi, \psi \in \mathrm{Aut}(F_N)$ are defined to be *similar* if there is an inner automorphism ad_y such that $\phi = \mathrm{ad}_y \circ \psi \circ \mathrm{ad}_y^{-1} = \mathrm{ad}_{y\psi(y)^{-1}} \circ \psi$. As conjugacy preserves the set of (powers of) Nielsen transformations, so does similarity.

Lemma 3.6. *Let* $\tau \in \mathrm{Aut}(F_2)$ *be a Nielsen transformation and suppose* $p \neq 0$ *and* $w \in F_2$. *Then* $\phi = \mathrm{ad}_w \tau^p$ *is a power of a Nielsen transformation if and only if* $w = y\tau^p(y)^{-1}$ *for some* $y \in F_2$. *Equivalently,* $\phi = \mathrm{ad}_w \tau^p$ *is a power of a Nielsen transformation if and only if there exists* $y \in F_2$ *such that* $\phi = (\mathrm{ad}_y \circ \tau \circ \mathrm{ad}_y^{-1})^p$.

Proof. In the light of Proposition 3.3, it suffices to argue that $\mathrm{Fix}(\phi)$ is noncyclic if and only if ϕ is similar to τ^p. This follows from the strengthened form of the Scott

Conjecture proved in [BH92, Corollary 6.4], which says that if $\phi \in \mathrm{Aut}(F_N)$ and if S is a set of representatives of the similarity classes of ϕ in its outer class $[\phi]$, then

$$\sum_{\psi \in S} \max\{(\mathrm{Rank}(\mathrm{Fix}(\psi)) - 1), 0\} \le N - 1.$$

In particular, there are only finitely many similarity classes in any outer class that have noncyclic fixed subgroups, and when $N = 2$ there can be at most one. Hence any automorphism in $[\tau]$ with a noncyclic fixed subgroup is similar to τ. $\square$

3.3 A description of subgroups in $M_k(F_2)$ with maximal product rank. In the remainder of this section, we focus on the case $A = F_2$. To lighten the notation, we make the abbreviations $M_k = M_k(F_2)$ and $F^k := (A_1 \times \cdots \times A_k)$. We have a fixed identification of each A_i with F_2, with respect to which the action defining the semidirect product $M_k = F^k \rtimes \mathrm{Aut}(F_2)$ is diagonal. Note that, for each index i, the subgroup $A_i^\tau < A_i$ fixed by $\tau \in \mathrm{Aut}(A_i) = \mathrm{Aut}(F_2)$ is the intersection of $A_i < F^k$ with the centraliser in M_k of $\underline{\tau} = (1, \ldots, 1; \tau)$. Likewise, we define $J^\tau < J$ to be the tuples of the form $(g^{-1}, \ldots, g^{-1}; \mathrm{ad}_g)$ with $g \in \mathrm{Fix}(\tau)$. Then J^τ is defined to be the intersection of J (as defined in Sect. 3.1) with the centraliser of $\underline{\tau}$.

Remark 3.7. As $\mathrm{rk}_F(\mathrm{Aut}(F_2)) = 1$ and product rank is subadditive with respect to exact sequences (Lemma 2.2), the exact sequence $1 \to F_2^k \to M_k(F_2) \to \mathrm{Aut}(F_2) \to 1$ implies that $\mathrm{rk}_F(M_k(F_2)) \le k+1$. The discussion in Sect. 3.1 gives an embedding of F_2^{k+1} in $M_k(F_2)$, showing that the product rank is exactly $k+1$.

We want to classify embeddings $D \hookrightarrow M_k$, where

$$D = \langle D_1, \ldots, D_{k+1} \rangle$$

is generated by a collection of pairwise commuting subgroups D_i such that each D_i is full-sized. (The D_i need not be finitely generated.) We will be led to consider the images of the subgroups D_i under the retraction $\pi : M_k \to \mathrm{Aut}(F_2)$ and the associated projection $\overline{\pi} : M_k \to \mathrm{Out}(F_2)$. Let $K_i := \ker \pi|_{D_i}$.

Theorem 3.8. *With the notation established above*:

1. *After permuting the indices, $[D_i, D_i] < A_i$ for $i = 1, \ldots, k$ and $[D_{k+1}, D_{k+1}] < J$;*
2. *if the centraliser $C_{M_k}(D) < M_k$ is non-trivial then there is a Nielsen transformation τ such that $C_{M_k}(D) = \langle \underline{\tau} \rangle$, and after conjugating by an element of $F^k \times J$,*

   $$D < A_1^\tau \times \cdots \times A_k^\tau \times J^\tau \times \langle \underline{\tau} \rangle,$$

   with $D_i < A_i^\tau \times \langle \underline{\tau} \rangle$ for $i = 1, \ldots, k$ and $D_{k+1} < J^\tau \times \langle \underline{\tau} \rangle$;
3. *if $C_{M_k}(D)$ is trivial, then D satisfies one of the following conclusions, after conjugating by an element of $F^k \times J$:*

i. $\overline{\pi}(D) = 1$ *and* $D_i < A_i$ *for* $i = 1, \dots, k$*, while* $D_{k+1} < J$ *– in particular,*

$$D < A_1 \times \cdots \times A_k \times J;$$

ii. $\overline{\pi}(D_i) = 1$ *for* $i = 1, \dots, k$ *but* $\overline{\pi}(D_{k+1}) \neq 1$*: in this case there is a Nielsen transformation* τ *such that* $D_i < A_i^{\tau}$ *for* $i = 1, \dots, k$ *and* $D_{k+1} < \langle J, \underline{\tau} \rangle$*, so*

$$D < A_1^{\tau} \times \cdots \times A_k^{\tau} \times \langle J, \underline{\tau} \rangle;$$

iii. $\overline{\pi}(D_{k+1}) = 1$ *but there is a unique* $j \leq k$ *such that* $\overline{\pi}(D_j) \neq 1$*: in this case* $D_j < \langle A_j, \underline{\tau} \rangle$ *while* $D_{k+1} < J^{\tau}$ *and* $D_i < A_i^{\tau}$ *for* $j \neq i \leq k$*, whence*

$$D < A_1^{\tau} \times \cdots \times A_{j-1}^{\tau} \times \langle A_j, \underline{\tau} \rangle \times A_{j+1}^{\tau} \cdots \times A_k^{\tau} \times J^{\tau}.$$

The following implicit feature of the theorem warrants explicit mention.

Addendum 3.9. *If* $\overline{\pi}(D_i)$ *is non-trivial for at least two of the subgroups* $D_i < D$*, then* $C_{M_k}(D) = \langle \underline{\tau} \rangle$ *for some Nielsen transformation* τ*.*

Remarks 3.10.

1. The action of $\alpha_j \in \mathrm{Aut}(M_k)$ interchanges cases 3(**ii**) and 3(**iii**).
2. If $C_{M_k}(D)$ is non-trivial, D might still conform to one of the descriptions in 3(**i-iii**).
3. In cases 3(**ii**) and 3(**iii**), the subgroups $\langle A_j, \underline{\tau} \rangle$ and $\langle J, \underline{\tau} \rangle$ are isomorphic to $F_2 \rtimes_{\tau} \mathbb{Z}$, a virtually special 3-manifold group in which free groups abound.

3.4 The proof of Theorem 3.8.

Lemma 3.11. *Consider* $D = \langle D_1, \dots, D_{k+1} \rangle \hookrightarrow M_k = F^k \rtimes \mathrm{Aut}(F_2)$ *with each* D_i *full-sized and* $[D_i, D_j] = 1$ *for all* $1 \leq i < j \leq k+1$*.*

1. *There is a unique index* i_0 *such that* $\pi|_{D_{i_0}}$ *is injective.*
2. *There is a Nielsen transformation* τ *such that* $\pi(D_j) < \langle \tau \rangle$ *for all* $j \neq i_0$*, and*
3. $\overline{\pi}(D_{i_0}) < \langle [\tau] \rangle$*.*
4. $\pi|_{D_{i_0}}^{-1}(\mathrm{Inn}(F_2)) < J$*.*

Proof. Let $D_j' < D_j$ be a free nonabelian subgroup. As in the proof of Proposition 2.3, the direct product $D_1' \times D_2' \times \cdots \times D_k'$ embeds into D. Let $K_j = \ker \pi|_{D_j}$ and let $K_j' = \ker \pi|_{D_j'}$. If K_j' is nontrivial then it is a nonabelian free group. Since $K_1' \times \cdots \times K_{k+1}' < F^k$ and $\mathrm{rk}_F(F^k) = k$, it follows that $\pi|_{D_j'}$ is injective for at least one index i_0. As $\mathrm{rk}_F(\mathrm{Aut}(F_2)) = 1$, this index i_0 is unique. We claim that $\pi|_{D_{i_0}}$ is also injective. Indeed, the remaining K_j generate a subgroup of F^k isomorphic to a direct product of k nonabelian free groups, which has a trivial centralizer in F^k. As K_{i_0} is contained in this centralizer it is also trivial.

To lighten the notation, we assume $i_0 = k+1$. Then $K = K_1 \times K_2 \times \cdots \times K_k < F^k$ is a direct product of k nonabelian free groups, and we can relabel the indices to assume $K_j < A_j$ for $j = 1, \dots, k$. We fix a pair of non-commuting elements $\underline{u}_j, \underline{v}_j \in K_j$ for $j = 1, \dots, k$ and denote their non-trivial coordinates by u_j and v_j respectively; for example, $\underline{u}_1 = (u_1, 1 \dots, 1; 1)$ and $\underline{v}_1 = (v_1, 1 \dots, 1; 1)$.

The remaining parts of the lemma require an analysis of the elements of D_{k+1} – let $g=(\omega_1,\dots,\omega_k;\phi)$ be such. Since D_{k+1} commutes with K_1, we have $g\underline{u}_1=\underline{u}_1 g$, hence

$$(\omega_1\phi(u_1),\omega_2,\dots,\omega_k;\phi)=(u_1\omega_1,\omega_2,\dots,\omega_k;\phi),$$

whence $\omega_1\phi(u_1)\omega_1^{-1}=u_1$. Similarly, $\omega_1\phi(v_1)\omega_1^{-1}=v_1$. Since u_1 and v_1 do not commute, $\mathrm{Fix}(\mathrm{ad}_{\omega_1}\circ\phi)$ is not cyclic. Therefore, by Proposition 3.3, either $\phi=\mathrm{ad}_{\omega_1}^{-1}$ or else $\mathrm{ad}_{\omega_1}\circ\phi$ is a non-zero power of a Nielsen transformation. In the latter case, Lemma 3.6 tells us that ϕ itself must be a power of a Nielsen transformation, say $\phi=\tau_0^p$, and $\omega_1=y_1\tau_0^p(y_1)^{-1}$ for some $y_1\in F_2$.

Repeating this argument with $\{u_j,v_j\}$ in place of $\{u_1,v_1\}$, we see that either $\phi=\mathrm{ad}_{\omega_j}^{-1}$ for $j=1,\dots,k$ or else $\phi=\tau_0^p$ and ω_j has the form $y_j\tau_0^p(y_j)^{-1}$ for $j=1,\dots,k$. From the former case we deduce that

$$\pi|_{D_{k+1}}^{-1}(\mathrm{Inn}(F_2))<J, \tag{3}$$

as claimed in (4). From the latter case we deduce that $\overline{\pi}(D_{k+1})<\mathrm{Out}(F_2)$ consists entirely of elements of trace 2 and determinant one. As in Lemma 3.2, this implies that $\overline{\pi}(D_{k+1})<\mathrm{Out}(F_2)$ is cyclic, generated by $[\tau_0^r]$, say, where $\tau_0^r\in\pi(D_{k+1})$.

At this stage, we know that $\pi(D_{k+1})<\mathrm{Aut}(F_2)$ is full-sized and $\overline{\pi}(D_{k+1})<\mathrm{Out}(F_2)$ is cyclic. Thus $\Lambda:=\pi(D_{k+1})\cap\mathrm{Inn}(F_2)$ is not abelian, and Corollary 3.4 provides a Nielsen transformation τ that generates the centralizer of Λ in $\mathrm{Aut}(F_2)$. As $\pi(D_j)$ commutes with Λ when $j\le k$, part (2) of the lemma is proved. (If $\pi(D_j)=1$ for all $j\le k$, then we take $\tau=\tau_0$.)

If $\pi(D_j)=\langle\tau^p\rangle$ for some $j\le k$ and $p\neq 0$, then $\tau^p\in\pi(D_j)$ commutes with $\tau_0^r\in\pi(D_{k+1})$. Hence $\tau_0=\mathrm{ad}_w\circ\tau^{\pm1}$, by Lemma 3.5, and (3) is proved. $\square$

Proof of Theorem 3.8. With the lemma in hand, we may assume that for $j\le k$ we have $D_j=K_j<A_j$ or $D_j=K_j\rtimes\langle T_j\rangle$ with $K_j<A_j$ nonabelian and $T_j=(t_{j1},\dots,t_{jk};\tau^{p_j})$, some $p_j\neq 0$. Also, $D_{k+1}<J$ or else $D_{k+1}=K_0\rtimes\langle T_0\rangle$ with $K_0<J$ and $T_0=(t_{01},\dots,t_{0k};\tau_0^{p_0})$. Part (1) of the theorem follows. Observe that $C(D)$ is contained in the centralizer of $K_1\times K_2\times\cdots\times K_k$, so intersects $A_1\times A_2\times\cdots\times A_k$ trivially. Hence $C(D)<1\rtimes\mathrm{Aut}(F_2)$. This subgroup of $\mathrm{Aut}(F_2)$ will fix the non-abelian subgroup $K_1\subset A_1\cong F_2$ in the natural action, so is trivial or else contained in the subgroup generated by a Nielsen automorphism, as in Corollary 3.4(1).

Let us first assume that there are at least two indices with $D_j=K_j\rtimes\langle T_j\rangle$ and prove that this forces D to be as described in part (2) of the theorem. For clarity of exposition, we assume that this set of indices includes $\{1,2\}$. (The superficially-exceptional case where one of the indices is $(k+1)$ is reduced to this case by applying one of the automorphisms α_i.)

If $j\neq 1$, then for every $\underline{w}\in K_j$ we have $T_1\underline{w}=\underline{w}T_1$. The j-coordinate of $T_1\underline{w}$ is $t_{1j}\tau^{p_1}(w)$, whereas the j-coordinate of $\underline{w}T_1$ is wt_{1j}. Thus $\tau^{p_1}(w)=t_{1j}^{-1}wt_{1j}$ for all $w\in K_j$; in other words $f_{1j}:=\mathrm{ad}_{t_{1j}}\tau^{p_1}$ fixes K_j.

Lemma 3.6 provides $y_{1j} \in F_2$ such that

$$t_{1j} = y_{1j}\,\tau^{p_1}(y_{1j})^{-1},$$

so $f_{1j} = \mathrm{ad}_{y_{1j}} \circ \tau^{p_1} \circ \mathrm{ad}_{y_{1j}}^{-1}$. Similarly, for $j \neq 2$ we obtain $f_{2j} = \mathrm{ad}_{y_{2j}} \circ \tau^{p_2} \circ \mathrm{ad}_{y_{2j}}^{-1}$ fixing K_j. Corollary 3.4(3) tells us that f_{1j} and f_{2j} are powers of a common Nielsen transformation, hence

$$\mathrm{ad}_{y_{1j}} \circ \tau \circ \mathrm{ad}_{y_{1j}}^{-1} = \mathrm{ad}_{y_{2j}} \circ \tau \circ \mathrm{ad}_{y_{2j}}^{-1},$$

with $y_{1j}y_{2j}^{-1} \in \mathrm{Fix}(\tau)$ (note that by examining the images of these automorphisms in $\mathrm{Out}(F_N)$, we cannot have τ on one side and τ^{-1} on the other side of this equation). Taking powers and simplifying, this implies that for all $m \in \mathbb{Z}$,

$$y_{1j}\tau^m(y_{1j})^{-1} = y_{2j}\tau^m(y_{2j})^{-1}. \tag{4}$$

We now conjugate D by $\gamma = (y_{21}, y_{12}, y_{13}, \ldots, y_{1k}; 1)^{-1}$, noting that

$$(y_{21}, y_{12}, y_{13}, \ldots, y_{1k}; 1)^{-1}\ T_1\ (y_{21}, y_{12}, y_{13}, \ldots, y_{1k}; 1) = (*, z_{12}, z_{13}, \ldots, z_{1k}; \tau^{p_1}),$$

where for $j \geq 2$ we have

$$z_{1j} = y_{1j}^{-1}t_{1j}\tau^{p_1}(y_{1j}) = y_{1j}^{-1}y_{1j}\tau^{p_1}(y_{1j})^{-1}\tau^{p_1}(y_{1j}) = 1.$$

Likewise,

$$(y_{21}, y_{12}, y_{13}, \ldots, y_{1k}; 1)^{-1}\ T_2\ (y_{21}, y_{12}, y_{13}, \ldots, y_{1k}; 1) = (z_{21}, *, z_{23}, \ldots, z_{2k}; \tau^{p_2}),$$

where, using equation (4) to replace $y_{2j}\tau^{p_2}(y_{2j})^{-1}$ by $y_{1j}\tau^{p_2}(y_{1j})^{-1}$, for $j \neq 1, 2$ we have

$$z_{2j} = y_{1j}^{-1}t_{2j}\tau^{p_2}(y_{1j}) = y_{1j}^{-1}y_{2j}\tau^{p_2}(y_{2j})^{-1}\tau^{p_2}(y_{1j}) = y_{1j}^{-1}y_{1j}\tau^{p_2}(y_{1j})^{-1}\tau^{p_2}(y_{1j}) = 1.$$

An entirely similar argument applies to each index j with $D_j = K_j \rtimes \langle T_j \rangle$.

Thus, after this conjugation (and abusing notation by identifying D with D^γ), the T_j have the form

$$T_1 = (t_1, 1, \ldots, 1; \tau^{p_1}),\quad T_2 = (1, t_2, 1, \ldots, 1; \tau^{p_2}),\quad T_j = (1, \ldots, t_j, \ldots, 1; \tau^{p_j}).$$

The commutation $[T_1, K_j] = 1$ now forces $K_j < \mathrm{Fix}(\tau^{p_1}) = \mathrm{Fix}(\tau)$ for $j > 1$ (including the case $K_j = D_j$). And $[T_2, K_1] = 1$ forces $K_1 < \mathrm{Fix}(\tau)$. Finally, the relations $[T_1, T_j] = 1$ imply $t_j \in \mathrm{Fix}(\tau)$ for $j > 1$. In particular, $T_j \in A_j^\tau \times \langle \underline{\tau} \rangle$, hence $D_j < A_j^\tau \times \langle \underline{\tau} \rangle$ for $j \leq k$. (The superficially-exceptional case $j = k + 1$ can again be handled by exchanging D_{k+1} and some D_j using $\alpha_j \in \mathrm{Aut}(M_k)$.) This shows that when at least two of the T_j are nontrivial, the D_j are as described in (2). In particular, $\underline{\tau}$ centralizes D. As $C(D)$ is contained in a cyclic subgroup as described at the start of this proof, we conclude that $C(D) = \langle \underline{\tau} \rangle$.

It remains to consider what happens when $\overline{\pi}(D_j) \neq 1$ for at most one index j. Case 3(**i**) is covered by Lemma 3.11 and Case 3(**ii**) can be reduced to 3(**iii**) by applying the automorphism α_j, so we address Case 3(**iii**), taking $j = 1$ for clarity. The proof in this case is a simplified version of the proof of (2): we have $D_j < A_j$ for $j = 2, \ldots, k$ and $D_{k+1} < J$, and after conjugating we may assume that $D_1 < A_1 \rtimes \langle T_1 \rangle$ where $T_1 = (t_1, 1, \ldots, 1; \tau^p)$ with $p \neq 0$. Again, the commutation $[T_1, D_j] = 1$ forces $D_j < A_j^\tau$ for $2 \leq j \leq k$ and $D_{k+1} < J^\tau$.

In order to complete the proof, we have to argue that if $C(D)$ is nontrivial, then D is as described in (2). Suppose $C(D)$ contains a nontrivial element ζ. Let $D_j^+ = \langle D_j, \zeta \rangle$, and let D^+ be the subgroup of M_k generated by the pairwise-commuting family $D_1^+, D_2^+, \ldots, D_{k+1}^+$. The centre of each D_j^+ is nontrivial, so D_j^+ is of the form $D_j^+ = K_j^+ \rtimes \langle T_j^+ \rangle$ and hence projects nontrivially to $\mathrm{Aut}(F_2)$. We have proved that in this case, D^+ is as described in (2), so $D_j < D_j^+ < A_j^\tau \times \langle \underline{\tau} \rangle$ and $D_{k+1} < D_{k+1}^+ < J^\tau \times \langle \underline{\tau} \rangle$. Finally, since $\underline{\tau}$ visibly centralizes D, and the centralizer of D is contained in a cyclic subgroup generated by a Nielsen transformation, $C(D) = \langle \underline{\tau} \rangle$. □

3.5 Extending by $\mathrm{Sym}(k+1)$.

Theorem 3.8 describes embeddings into M_k of groups generated by a commuting family of $(k+1)$ full-sized groups. The following proposition shows that one gets no extra embeddings when the target is enlarged to $M_k \rtimes \mathrm{Sym}(k+1)$, where the action in the semidirect product is the same as Lemma 3.1: the transposition $(i\ k+1) \in \mathrm{Sym}(k+1)$ acts as α_i.

Proposition 3.12. *If D is generated by a set of $(k+1)$ pairwise-commuting full-sized subgroups, then the image of every embedding $D \hookrightarrow M_k \rtimes \mathrm{Sym}(k+1)$ lies in $M_k \times 1$. Furthermore, if $\phi \in M_k \rtimes \mathrm{Sym}(k+1)$ centralizes D, then ϕ lies in $M_k \times 1$.*

Proof. We identify D with its image in $M_k \rtimes \mathrm{Sym}(k+1)$. There is no loss of generality in assuming that D is finitely generated. Let $D^* < D$ be the subgroup obtained by replacing each subgroup $D_i < D$ with the intersection of the kernels of all non-trivial homomorphisms from D_i to (the abstract group) $\mathrm{Sym}(k+1)$. Then $D^* < M_k \times 1$ and as in Theorem 3.8(1) we may assume that $[D_i^*, D_i^*] < A_i$ for $i \leq k$ and $[D_{k+1}^*, D_{k+1}^*] < J$. The action of D by conjugation on D^* preserves each of the subgroups $[D_i^*, D_i^*]$. In contrast, conjugation in $M_k \rtimes \mathrm{Sym}(k+1)$ by any element of the form $(m; \sigma)$ will (if we write $J = A_{k+1}$) send A_i to $A_{\sigma(i)}$; in particular it will not leave $[D_i^*, D_i^*]$ invariant if $\sigma(i) \neq i$. If ϕ centralizes D then ϕ will also leave the groups $[D_i^*, D_i^*]$ invariant, so must also lie in $M_k \times 1$. □

4 Stabilizers of collapsed roses and cages

In this paper we restrict our attention to two simple examples of free splittings, which are *collapsed roses* and *cages*. We will make use of Bass–Serre theory, for which the standard references are Serre's book [Ser80] and the topological approach given by Scott and Wall [SW79].

4.1 The Bass–Serre tree of a collapsed rose. A *collapsed rose with k petals* is a graph of groups decomposition of F_N with a single vertex group and k loops with trivial edge groups. The vertex group is necessarily isomorphic to a free factor $A \cong F_{N-k}$. We abuse notation slightly and refer to a free splitting T of F_N as a collapsed rose if the corresponding graph of groups is a collapsed rose. In the above notation, the vertex stabilizers of T are the conjugates of the free factor A. A rose with one loop is simply a nonseparating free splitting.

Let b_A be the vertex of T with stabilizer A and pick representatives $e_1, \dots, e_k$ of each orbit of edges that have initial vertex b_A. A *stable letter* for e_i is a choice of element x_i such that $x_i b_A$ is the terminal vertex of e_i. Changing either the representative e_i or the translating element gives possible stable letters of the form $ux_i^{\pm 1}v$, where $u, v \in A$. The free group F_N is generated by A and the stable letters $x_1, \dots, x_k$.

4.2 Stabilizers of collapsed roses in $\mathbf{Out}(F_N)$. Let T be a splitting of F_N corresponding to a collapsed rose with k petals. As in Sect. 2.3, we let $\mathrm{Stab}(T)$ denote the stabilizer of T in $\mathrm{Out}(F_N)$. Letting $x_1, \dots, x_k$ be a choice of stable letters for the rose, there is a finite subgroup W_k generated by automorphisms σ such that σ is the identity when restricted to A and for every stable letter $\sigma(x_i) = x_j^{\epsilon}$ for some j and $\epsilon \in \{1, -1\}$. One can think of W_k as the subgroup of $\mathrm{Out}(F_N)$ given by permuting and inverting the petals of the rose, and W_k is isomorphic to the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^k$ and the symmetric group $\mathrm{Sym}(k)$. The homomorphism

$$F : \mathrm{Stab}(T) \to \mathrm{Aut}(T/F_N)$$

to the automorphism group of the rose is split surjective, and W_k is a set of coset representatives of $\mathrm{Stab}^0(T)$ in $\mathrm{Stab}(T)$.

Proposition 4.1. *Let T be a collapsed rose with k petals. Let $\widetilde{\mathrm{Stab}}(T)$ and $\widetilde{\mathrm{Stab}}^0(T)$ be the respective preimages of $\mathrm{Stab}(T)$ and $\mathrm{Stab}^0(T)$ in $\mathrm{Aut}(F_N)$. Let b_A be a vertex of the tree with F_N-stabilizer A and let $\widetilde{\mathrm{Stab}}(T)_A$ and $\widetilde{\mathrm{Stab}}^0(T)_A$ be the respective subgroups of $\widetilde{\mathrm{Stab}}(T)$ and $\widetilde{\mathrm{Stab}}^0(T)$ that fix b_A. An automorphism ϕ is an element of $\widetilde{\mathrm{Stab}}(T)_A$ if and only if there exists $w \in W_k$ such that:*

1. *ϕ restricts to an automorphism of A and,*
2. *for each stable letter x_i there exist $u_i, v_i \in A$ such that $\phi(x_i) = u_i w(x_i) v_i$.*

The group $\widetilde{\mathrm{Stab}}(T)_A$ is isomorphic to

$$M_{2k}(A) \rtimes W_k = (A^{2k} \rtimes \mathrm{Aut}(A)) \rtimes W_k,$$

via the isomorphism

$$\phi \mapsto (u_1^{-1}, \dots, u_k^{-1}, v_1, \dots, v_k; \phi|_A; w).$$

The group W_k acts on $M_{2k}(A)$ as a subgroup of the group $\mathrm{Sym}(2k+1)$ defined in Lemma 3.1, and $\phi \in \widetilde{\mathrm{Stab}}^0(T)_A$ if and only if $w = 1$ in the above.

If e_j is the edge joining b_A to x_jb_A then the subgroup $\widetilde{\mathrm{Stab}}^0(T)_{e_j} < \widetilde{\mathrm{Stab}}^0(T)_A$ *fixing e_j consists of automorphisms* $\phi \in \widetilde{\mathrm{Stab}}^0(T)_A$ *such that* $\phi(x_j) = x_jv_j$ (*i.e.* $u_j = 1$); *it is isomorphic to* $M_{2k-1}(A) = A^{2k-1} \rtimes \mathrm{Aut}(A)$.

Proof. By the work on automorphic lifts in Sect. 2.4, $\phi \in \widetilde{\mathrm{Stab}}(T)$ if and only if there exists a ϕ-twistedly equivariant map $f_\phi \colon T \to T$ that preserves the F_N-orbits of edges and their orientations. Suppose that ϕ satisfies conditions (1) and (2) of the proposition. We define f_ϕ on the vertex set of T by the map of cosets $gA \mapsto \phi(g)A$ (as the vertices of T correspond to cosets of A via Bass–Serre theory). Let $X = \{x_1, \dots, x_k, x_1^{-1}, \dots, x_k^{-1}\}$. The cosets gA adjacent to $1A$ are of the form axA, for $a \in A$ and $x \in X$. Therefore, under this correspondence between vertices and cosets of A, two vertices gA and hA span an edge in T if and only if $g^{-1}h \in AXA$. As $\phi(AXA) = AXA$, the map f_ϕ preserves edges and so determines an isomorphism of T which is clearly ϕ-twistedly equivariant. Conversely, if $\phi \in \widetilde{\mathrm{Stab}}(T)_A$ then let $f_\phi : T \to T$ be a ϕ-twistedly equivariant map fixing b_A. As $f_\phi(b_A) = b_A$, it follows that ϕ preserves $\mathrm{Stab}(b_A) = A$, so restricts to an automorphism of A. As f_ϕ preserves edges, using the same reasoning as above, we must have $\phi(AXA) = AXA$, from which it follows that ϕ must also satisfy (2).

The action of ϕ on the quotient graph T/F_N is given by the last coordinate $w \in W_k$ in its decomposition, so $\phi \in \widetilde{\mathrm{Stab}}^0(T)_A$ if and only if $w = 1$. This gives an identification of $\widetilde{\mathrm{Stab}}^0(T)_A$ with $M_{2k}(A)$ via the map

$$\phi \mapsto (u_1^{-1}, \dots, u_k^{-1}, v_1, \dots, v_k; \phi|_A).$$

Conjugation by an element of W_k acts on this group by a signed permutation of the k pairs $\{u_i^{-1}, v_i\}$, so W_k is a subgroup of the group $\mathrm{Sym}(2k+1)$ defined in Lemma 3.1.

An element $\phi \in \widetilde{\mathrm{Stab}}^0(T)_A$ fixes the edge between $1A$ and x_jA if and only if $\phi(x_j)A = x_jA$ (equivalently, f_ϕ fixes this terminal vertex). As $\phi(x_j)A = u_jx_jA$, this happens if and only if $u_j = 1$ and gives our description of $\widetilde{\mathrm{Stab}}^0(T)_{e_j}$. $\square$

Following Proposition 2.8, each edge stabilizer $\widetilde{\mathrm{Stab}}^0(T)_e$ is an automorphic lift of $\mathrm{Stab}^0(T) < \mathrm{Out}(F_N)$ to $\mathrm{Aut}(F_N)$.

4.3 Arc stabilizers in the Bass–Serre tree of a collapsed rose with $N-2$ petals.

In the proof of the Theorem A, we will need to understand arc stabilizers for the action of $\widetilde{\mathrm{Stab}}(T)$ on the Bass–Serre tree of a collapsed rose with $N-2$ petals (with the notation of the previous section). In this case, each vertex stabilizer A is isomorphic to F_2, so that $\widetilde{\mathrm{Stab}}^0(T)_A \cong M_{2N-4}(F_2)$. Proposition 4.1 and Remark 3.7 together imply that the product rank of a vertex stabilizer of T (in $\mathrm{Aut}(F_N)$) is $2N-3$, and the product rank of an edge stabilizer is $2N-4$. For later arguments, we need to show that the product rank drops further when one takes stabilizers of longer arcs in T.

LEMMA 4.2. *Let T be a collapsed rose with $N-2$ petals, and as above let $\widetilde{\mathrm{Stab}}(T)$ be the preimage of* $\mathrm{Stab}(T)$ *in* $\mathrm{Aut}(F_N)$. *If α is any edge path of length ≥ 2 in T then the pointwise stabilizer of α with respect to the $\widetilde{\mathrm{Stab}}(T)$–action on T has product rank at most $2N-5$.*

Proof. It is enough to prove the result for a path $\alpha = \{e, e'\}$ of length two, and as product rank is preserved under passing to finite-index subgroups, we may pass to $G := \widetilde{\mathrm{Stab}}^0(T)$. We want to show that the intersection $G_e \cap G_{e'}$ of the edge stabilizers in G has product rank at most $2N-5$. As above, suppose that $A \cong F_2$ is the subgroup of F_N which stabilizes the vertex adjacent to e and e'. Without loss of generality we may assume that e is the edge between vertices with stabilizers A and $x_1Ax_1^{-1}$ respectively. Then G_e is the subgroup of $\mathrm{Aut}(F_N)$ that preserves A, sends x_1 to x_1v_1, and sends each x_i to a word of the form $u_ix_iv_i$ for $2 \leq i \leq N-2$ (where as above each $u_i, v_i \in A$). Note that we can replace a stable letter x_i with $ux_i^{\pm 1}v$ if $2 \leq i \leq N-2$ and $u, v \in A$. We may also replace x_1 with x_1v for some $v \in A$ without changing the description of G_e. Up to the above replacements, there are three possibilities for the stabilizer of the second vertex of e': it is either of the form $x_jAx_j^{-1}$ for some j satisfying $2 \leq j \leq N-2$, of the form $x_1^{-1}Ax_1$, or of the form $(wx_1)A(wx_1)^{-1}$ for some nontrivial $w \in A$. In the first case, we see that $G_e \cap G_{e'}$ consists of automorphisms ϕ preserving A with the added restriction that both $u_j = 1$ and $u_1 = 1$ in the above notation. Hence the intersection decomposes as an exact sequence

$$1 \to A^{2k-2} \to G_e \cap G_{e'} \to \mathrm{Aut}(A) \to 1,$$

where $k = N-2$. The product rank of the kernel is then $2N-6$ and the product rank of $\mathrm{Aut}(A) = \mathrm{Aut}(F_2)$ is one, so the product rank of the intersection is at most $2N-5$.

In the second case, where the terminal vertex of e' has stabilizer $x_1^{-1}Ax_1$, a similar argument applies. One sees that $G_e \cap G_{e'}$ is the subgroup of the stabilizer of v_A which fixes x_1 (hence $u_1 = v_1 = 1$), and we have the same exact sequence as above.

In the final case, the intersection $G_e \cap G_{e'}$ is given by the automorphisms in G_e which also fix the subgroup $(wx_1)A(wx_1)^{-1}$. Note that

$$\phi((wx_1)A(wx_1)^{-1}) = \phi(w)\phi(x_1Ax_1^{-1})\phi(w)^{-1} = \phi(w)x_1Ax_1^{-1}\phi(w)^{-1}$$

as elements of G_e preserve $x_1Ax_1^{-1}$. If ϕ is also an element of $G_{e'}$ then

$$(wx_1)A(wx_1)^{-1} = \phi(w)x_1Ax_1^{-1}\phi(w)^{-1}.$$

This implies that $\phi(w) = w$ as $w \in A$. Therefore in the exact sequence

$$1 \to A^{2N-5} \to G_e \to \mathrm{Aut}(A) \to 1,$$

the intersection $G_e \cap G_{e'}$ projects to a subgroup of $\mathrm{Aut}(A)$ fixing the element w. However, parts 1 and 3 of Lemma 3.11 imply that a subgroup of $G_e \cong A^{2N-5} \rtimes \mathrm{Aut}(A)$ of maximal product rank projects to a subgroup of $\mathrm{Aut}(A)$ containing a nonabelian subgroup of inner automorphisms, and therefore cannot fix any $w \in A$. It follows that $G_e \cap G_{e'}$ has product rank at most $2N-5$. $\square$

PROPOSITION 4.3. *Let T be a collapsed rose with $N-2$ petals, and let D be a subgroup of $\widetilde{\mathrm{Stab}}(T) < \mathrm{Aut}(F_N)$ generated by $2N-3$ pairwise-commuting full-sized subgroups. Then D has a unique global fixed point v in T. The normalizer of D in $\widetilde{\mathrm{Stab}}(T)$ also fixes v.*

Proof. If a fixed vertex of D exists then it is unique as stabilizers of edges in T have product rank $2N-4$. By uniqueness, such a fixed point will also be invariant under the normalizer of D in $\widetilde{\mathrm{Stab}}(T)$. We are left with the matter of proving such a fixed point exists.

Suppose D is generated by $D_1, D_2, \ldots, D_{2N-3}$. Firstly, suppose that some subgroup, for instance D_{2N-3}, contains a hyperbolic element g with respect to the action on T. As each subgroup D_i for $i < 2N-3$ commutes with g, the subgroup $\langle D_1, D_2, \ldots, D_{2N-4}\rangle$ acts on the axis A_g of g preserving the orientation. The commutator subgroup D_i' of each D_i acts trivially on this axis, and as these groups are full-sized we obtain a direct product of $2N-4$ nonabelian free groups fixing a line in T. This contradicts Lemma 4.2. We may therefore assume that each subgroup D_i consists of elliptic elements. Any product of commuting elliptic elements is also elliptic, so every element of D is elliptic. If D does not have a global fixed point, then as in [Ser80, I.6.5, Exercise 2] the group D is not finitely generated and fixes an end of T. Every element of D fixes a half-line towards this end, so that we can find a finitely generated subgroup $\langle D_1', D_2', \ldots, D_{2N-3}'\rangle$ with each D_i' a nonabelian free group fixing a half-line in T, which again contradicts Lemma 4.2. Hence D has a global fixed point in T. $\square$

4.4 Stabilizers of cages. A splitting T of F_N is a *cage* if T is a free splitting and the quotient graph T/F_N is isomorphic to a cage (a connected graph with two vertices and no loop edges). Suppose that T/F_N is a cage with k edges. Pick adjacent vertices v and w in T with stabilizers A and B. Let e be the edge from v to w and let $e_1, \ldots, e_{k-1}$ be edges based at v representing the other $k-1$ edge orbits in T. If x_i is an element that takes the terminal vertex of e_i to w, then F_N is generated by A, B, and $x_1, \ldots, x_{k-1}$.

As above, take $G = \mathrm{Stab}^0(T)$ and $\tilde{G}$ its preimage in $\mathrm{Aut}(F_N)$. The stabilizer $\tilde{G}_e$ of e gives an automorphic lift of G. As in the proof of Proposition 4.1, every element $\Phi \in \mathrm{Stab}^0(T)$ has a unique representative $\phi \in \tilde{G}_e$ such that

- ϕ restricts to an automorphism of A and B.
- For each x_i we have $\phi(x_i) = a_i x_i b_i$ for some $a_i \in A$ and $b_i \in B$.

It follows that $\mathrm{Stab}^0(T)$ fits in the exact sequence

$$1 \to A^{k-1} \times B^{k-1} \to \mathrm{Stab}^0(T) \to \mathrm{Aut}(A) \times \mathrm{Aut}(B) \to 1.$$

Furthermore, this sequence splits so that $\mathrm{Stab}^0(T)$ is a semidirect product of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ with $A^{k-1} \times B^{k-1}$, where $\mathrm{Aut}(A)$ acts diagonally on A^{k-1} and trivially on B^{k-1}, and the action of $\mathrm{Aut}(B)$ on $A^{k-1} \times B^{k-1}$ acts trivially on A^{k-1} and diagonally on B^{k-1}. In the proof of Proposition 5.1, we will use this decomposition to calculate $\mathrm{rk}_F(\mathrm{Stab}^0(T))$.

5 Fixed splittings of subgroups with maximal product rank

In this section, we prove the following proposition:

Proposition 5.1. *Suppose that $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ and $\mathrm{rk}_F(G) = 2N-4$. Any splitting fixed by G has one F_N-orbit of vertices (i.e. is a collapsed rose). Any two free splittings fixed by G are compatible.*

Proposition 5.1 follows from two lemmas that will be proved below. Lemma 5.4 asserts that if a subgroup $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ fixes two incompatible free splittings, then G fixes a free splitting with at least two F_N-orbits of vertices. Lemma 5.5 then shows that such a splitting cannot be fixed by a group with maximal product rank (in other words, subgroups with maximal product rank can only fix collapsed roses). We delay the full proof for the time being to first state two consequences of this proposition.

We say that a free splitting T is G-unrefinable if it admits no G-invariant refinement that is a free splitting. We have the following useful corollary:

Corollary 5.2. *Suppose $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ and $\mathrm{rk}_F(G) = 2N-4$.*

- *The group G fixes a unique G-unrefinable free splitting. If this splitting is non-trivial then it is a collapsed rose.*
- *If T is the unique G-unrefinable collapsed rose fixed by G, then T is also fixed by the normalizer $N_{\mathrm{Out}(F_N)}(G)$ of G in $\mathrm{Out}(F_N)$.*

Proof. Take X to be the set of all one-edge free splittings preserved by G. These are all compatible by Proposition 5.1, so X is finite and there is a common refinement T collapsing to every element of X ([SS03, Theorem 5.16] or [GL17, Proposition A.17]). Also by Proposition 5.1, the tree T is a collapsed rose. Any other G-invariant free splitting is a common refinement of a subset of X (this follows from the fact that G is contained in $\mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ and Proposition 2.5: if G fixes a free splitting T' then G also fixes all of the one-edge splittings to which T' collapses). It follows that T is the unique G-unrefinable, G-invariant free splitting. By uniqueness, T is invariant under the normalizer of G in $\mathrm{Out}(F_N)$. □

Corollary 5.3. *Let $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ be such that $\mathrm{rk}_F(G) = 2N-4$. If G fixes the Bass–Serre tree T of a collapsed rose with $N-2$ petals, then this rose is unique and the normalizer $N(G)$ of G in $\mathrm{Out}(F_N)$ also fixes T.*

Proof. A G-invariant refinement of such a rose would have to have $N-1$ or N petals. However it follows from Proposition 4.1 that the stabilizer of a collapsed rose with

$N-1$ petals is virtually abelian, while a rose with N petals has finite stabilizer. Hence if G fixes a collapsed rose with $N-2$ petals then this rose is G-unrefinable and invariant under the normalizer of G in $\mathrm{Out}(F_N)$. $\square$

We move on to the required lemmas.

LEMMA 5.4. *If $G < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$ fixes incompatible free splittings T and T' then G fixes a free splitting which contains at least two orbits of vertices.*

Proof. As refinements of incompatible trees are also incompatible, we may assume that the two splittings T and T' are incompatible and G-unrefinable. Let $\tilde{G}$ be the preimage of G in $\mathrm{Aut}(F_N)$. By Proposition 6.2 of [GH21], any two G-unrefinable free splittings belong to the same deformation space when viewed as $\tilde{G}$-trees (equivalently, T and T' have the same set of elliptic subgroups with respect to the $\tilde{G}$-action). However, if there are incompatible trees in a deformation space then there must be trees with at least two orbits of vertices. We give a brief explanation via folding. As T and T' belong to the same deformation space, there exists a $\tilde{G}$ equivariant map $f: T \to T'$ taking edges in T to (possibly trivial) edge paths in T' (see, e.g., [GL07]). The map f decomposes as a collapse $f_0 : T \to T_0$ followed by a morphism $f_1 : T_0 \to T_1$ that does not collapse edges. The map f_1 is nontrivial as T and T' are incompatible, which implies there exist edges $e \neq e'$ based at the same vertex in T_0 such that the edge paths $f(e)$ and $f(e')$ have a common initial egde (if f_1 were locally injective, then f_1 would be an isomorphism). Partially folding e and e' along a small initial segment gives a new $\tilde{G}$ tree T_1 with an extra $\tilde{G}$-orbit of vertices at the fold. There is an induced morphism $f_2 : T_1 \to T'$, which implies that any edge stabilizer of T_1 fixes a nondegenerate arc in T'. Thus $F_N < \tilde{G}$ has no nontrivial edge stabilizer in T_1, and T_1 is a G-invariant free splitting with at least two orbits of edges. $\square$

We now show that the stabilizer of any splitting with more than one orbit of vertices does not have maximal product rank. Proposition 5.1 follows immediately.

LEMMA 5.5. *Let T be a splitting of F_N which contains at least two F_N-orbits of vertices. Then $\mathrm{rk}_F(\mathrm{Stab}_{\mathrm{Out}(F_N)}(T)) \leq 2N-5$.*

Proof. Suppose for a contradiction that $\mathrm{rk}_F(\mathrm{Stab}(T)) = 2N-4$. Then the intersection of $\mathrm{Stab}(T)$ with the finite index subgroup $\mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z}) < \mathrm{Out}(\mathrm{F_N})$ also contains a direct product D of $2N-4$ nonabelian free groups, and D preserves every collapse of T (Proposition 2.5). We may therefore replace T with a collapse T' that has exactly two orbits of vertices v and w. Collapsing all loop edges in the quotient graph then gives us a cage S with $k \geq 1$ edges (allowing the degenerate case of a one-edge separating splitting). As $D < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$, we have $D < \mathrm{Stab}^0(S)$. If A and B are free factor representatives of the vertex stabilizers in S then, as in Sect. 4.4, the group $\mathrm{Stab}^0(S)$ decomposes as a short exact sequence

$$1 \to A^{k-1} \times B^{k-1} \to \mathrm{Stab}^0(S) \to \mathrm{Aut}(A) \times \mathrm{Aut}(B) \to 1.$$

Suppose A and B are of rank n and m respectively, so that $N = n + m + (k-1)$. If A and B are both noncyclic, then

$$\mathrm{rk}_F(\mathrm{Aut}(A)) + \mathrm{rk}_F(\mathrm{Aut}(B)) = (2n-3) + (2m-3) = 2N - 2k - 4$$

and

$$\mathrm{rk}_F(A^{k-1} \times B^{k-1}) = 2k - 2.$$

Hence $\mathrm{rk}_F(\mathrm{Stab}^0(S)) \leq (2N - 2k - 4) + 2k - 2 = 2N - 6$. If both A and B are cyclic (possibly trivial), then $\mathrm{Stab}^0(S)$ is virtually abelian, so $\mathrm{rk}_F(\mathrm{Stab}^0(S)) = 0$. Suppose A is nonabelian and B is cyclic. Then $\mathrm{rk}_F(A^{k-1} \times B^{k-1}) = k - 1$. If B is trivial then A is rank $N - k + 1$ and the vertex w has valence at least 3, so that $k \geq 3$. Then $\mathrm{rk}_F(\mathrm{Aut}(A) \times \mathrm{Aut}(B)) = \mathrm{rk}_F(\mathrm{Aut}(A)) = 2(N - k + 1) - 3$. Hence

$$\mathrm{rk}_F(\mathrm{Stab}^0(S)) \leq 2(N - k + 1) - 3 + k - 1 = 2N - 2 - k \leq 2N - 5.$$

Similarly, if B is infinite cyclic then A is rank $N - k$ and the same computation gives

$$\mathrm{rk}_F(\mathrm{Stab}^0(S)) \leq 2N - 4 - k \leq 2N - 5. \qquad \square$$

6 Completing the proofs of theorems A and B

We are now armed with enough knowledge to complete the proof of Theorem A.

Theorem 6.1 (Theorem A). *Let $N \geq 3$ and suppose*

$$D = \langle D_1, D_2 \ldots, D_{2N-4} \rangle < \mathrm{Out}(F_N)$$

is generated by $2N-4$ pairwise-commuting full-sized subgroups $D_1, \ldots, D_{2N-4}$. Then there is a unique collapsed rose with $N-2$ petals that is fixed by D.

Remark 6.2. Once we have established the existence of the collapsed rose in this theorem, its uniqueness is assured by Corollary 5.3, from which it follows that the collapsed rose is also fixed by the normalizer of D. This observation about normalizers will be useful in the inductive proof that follows.

Proof. We first reduce to the case where $D < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$. Let $D_i' = D_i \cap \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$, note that D_i' is full-sized and let

$$D' = \langle D_1', D_2', \ldots, D_{2N-4}' \rangle.$$

As D' is normal in D, if D' fixes a collapsed rose with $N-2$ petals, then so does D, by Remark 6.2. Thus we may assume without loss of generality that $D < \mathrm{IA_N}(\mathbb{Z}/3\mathbb{Z})$.

When $N = 3$, a rose with $N-2$ petals is simply a nonseparating free splitting. By Theorem 2.6, a group $D = \langle D_1, D_2 \rangle$ in $\mathrm{IA}_3(\mathbb{Z}/3\mathbb{Z})$ fixes such a splitting.

For the inductive step, we will take the preimage $\tilde{D}$ of D in $\mathrm{Aut}(F_N)$ and build an action of $\tilde{D}$ on a tree T such that the restriction to the inner automorphisms

$F_N < \mathrm{Aut}(F_N)$ is a free splitting with $N-2$ F_N-orbits of edges. This splitting is then D-invariant (Lemma 2.7) and is necessarily an $N-2$ rose by Corollary 5.2. We build the tree as a *graph of actions* (see [Lev94]). A graph of actions for a group H consists of the following data:

- A graph of groups decomposition of H.
- For every vertex u in the graph of groups, a tree T_u equipped with an action of the vertex group H_u.
- For every oriented edge e with terminus u, a fixed point x_e of $H_e < H_u$ in T_u.

Given a graph of actions where the vertex trees are all simplicial, if S is the Bass–Serre tree of the underlying graph of groups then one obtains a refinement T of S from the graph of actions. There is a natural collapse map $T \to S$, where the preimage of a vertex $\tilde{u} \in S$ is a copy of T_u. The points x_e are used as gluing instructions for the endpoints of the edges of S into the trees T_u. We will take S to be any one-edge nonseparating free splitting invariant under D, which exists by Theorem 2.6. Let $A \cong F_{N-1}$ be a free factor whose conjugates form the vertex stabilizers of S, and let b_A (hereafter shortened to b) be the vertex fixed by A. Let D_A be the image of D in $\mathrm{Out}(A)$. By considering product rank in the short exact sequence

$$1 \to A \times A \to \mathrm{Stab}^0(S) \to \mathrm{Out}(A) \to 1,$$

the kernel K of the homomorphism $D \to D_A$ has $\mathrm{rk}_F(K) \leq 2$. By Proposition 2.3, after reordering the D_i, the groups $D_3/K, \dots, D_{2N-4}/K$ are full-sized in D_A. In particular, D_A is in the normalizer of the group generated by a commuting family of $2N-6$ full-sized groups in $\mathrm{Out}(A)$. By induction and Remark 6.2, we know that D_A fixes a collapsed rose T_A with $N-3$ petals. Let $\tilde{D}_A$ be the preimage of D_A in $\mathrm{Aut}(A)$. As $\tilde{D}_b$ preserves A, the vertex group $\tilde{D}_b$ acts on T_A via the projection $\tilde{D}_b \to \mathrm{Aut}(A)$. In order to show that we can define a graph of actions, we need to take an edge e adjacent to b and check that $\tilde{D}_e$ has a fixed point with respect to its action on T_A. We let $f : \tilde{D}_e \to \tilde{D}_A$ be the map factoring through $\tilde{D}_b$ that determines this action. We have the following commutative diagram.

$$\begin{tikzcd}
\tilde{D}_e \arrow[dr, hook] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\cong"'] & & & \\
& \tilde{D}_b \arrow[r] \arrow[d] & \tilde{D}_A \arrow[d] & < \mathrm{Aut}(A) \\
& D \arrow[r] & D_A & < \mathrm{Out}(A)
\end{tikzcd}$$

The map from $\tilde{D}_e$ to D is an isomorphism as S is a free splitting, so that $\tilde{D}_e$ is an automorphic lift as described in Sect. 2.4. Without loss of generality, we can take e corresponding to the stable letter x_1, so that the stabilizers of its endpoints are A and

$x_1Ax_1^{-1}$. As in Sect. 4.2, every automorphism $\phi \in \tilde{D}_e$ restricts to an automorphism ϕ_A of A and satisfies $\phi(x_1) = x_1a$ for some $a \in A$. It follows that the kernel of f is a free group generated by these right transvections. Applying Proposition 2.3 to $\tilde{D}_e \cong D$, after permuting the D_i, the images of $D_2, \ldots, D_{2N-4}$ are full-sized in $\mathrm{Im}(f)$. Therefore $\mathrm{Im}(f)$ is contained in the normalizer of a commuting family of $2N-5 = 2(N-1)-3$ full-sized groups in $\mathrm{Aut}(A)$, so by Proposition 4.3, the group $\mathrm{Im}(f)$ has a fixed point x_e in T_A.

It follows that $\tilde{D}$ admits a graph of actions with the Bass–Serre tree S and vertex tree T_A, so that the refinement of S determined by this graph of actions is a free splitting of F_N with $N-2$ orbits of edges. □

Theorem 6.3 (Theorem B). *Let $N \geq 3$ and suppose $D < \mathrm{Aut}(F_N)$ is generated by a commuting family of $2N-3$ full-sized subgroups. Then, the image of D in $\mathrm{Out}(F_N)$ fixes a unique collapsed rose with $N-2$ petals, and D acts on the Bass–Serre tree of this collapsed rose with a unique global fixed point.*

Proof. The image $\bar{D}$ of D in $\mathrm{Out}(F_N)$ is contained in the normalizer of a commuting family of $2N-4$ nonabelian free groups (the kernel of the map $D \to \bar{D}$ is free and normal, so we may apply Proposition 2.3 with $l = 1$). Hence $\bar{D}$ fixes a collapsed rose T with $N-2$ petals in the boundary of Outer space, and D acts on this tree. Proposition 4.3 then states that D has a unique global fixed point with respect to the action on T. □

7 Algebraic descriptions of the direct products and their centralizers

In this section we prove Theorems C, D and F from the introduction, where the relevant notation was established. We prove an $\mathrm{Out}(F_N)$-version of Theorem C in Theorem 7.2.

Theorem 7.1 (Theorem C). *Let $N \geq 3$ and let $D < \mathrm{Aut}(F_N)$ be a direct product of $2N-3$ nonabelian free groups. Then a conjugate of D is contained in one of the following subgroups.*

- $L_1 \times \cdots \times L_{N-2} \times R_1 \times \cdots \times R_{N-2} \times I(A)$
- $L_1^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times I(A)^\tau \times \langle \tau \rangle$
- $\langle L_1, \tau \rangle \times L_2^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times I(A)^\tau$
- $L_1^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times \langle I(A), \tau \rangle$

In the second case, if $D = D_1 \times \cdots \times D_{2N-3}$, then each summand D_i is contained in a unique subgroup of the form $\langle L_i^\tau, \tau \rangle$, $\langle R_i^\tau, \tau \rangle$, or $\langle I(A)^\tau, \tau \rangle$. In the remaining cases, the inclusion respects the visible direct product decompositions, up to permuting the summands.

Proof. By Theorem B, there exists an action of D on the Bass–Serre tree T of a collapsed rose with $N-2$ petals such that D has a (unique) global fixed point. Up to conjugation, we may assume that the collapsed rose has vertex group $A = \langle a_1, a_2 \rangle$ and stable letters $x_1, \ldots, x_{N-2}$, and $D < \widetilde{\mathrm{Stab}}(T)_A$ (i.e. the vertex fixed by D in T has F_N–stabilizer equal to A). By Proposition 4.1, we have

$$\widetilde{\mathrm{Stab}}(T)_A = M_{2N-4}(A) \rtimes W_{N-2},$$

where W_{N-2} acts as a subgroup of the group $\mathrm{Sym}(2N-3)$ of automorphisms of $M_{2N-4}(A)$ defined in Lemma 3.1. By Proposition 3.12, the projection of D to W_{N-2} is trivial, so that D is contained in $\widetilde{\mathrm{Stab}}^0(T)_A = M_{2N-4}(A)$. Under the isomorphism between $\widetilde{\mathrm{Stab}}^0(T)_A$ and $M_{2N-4}(A)$ given in Proposition 4.1, the groups L_i and R_i are taken to factors in the A^{2N-4} subgroup of $M_{2N-4}(A)$, the inner automorphisms by elements of A are taken to J, and τ is taken to $\underline{\tau}$ (using the notation of Sect. 3). The possibilities for D are then given by Theorem 3.8. $\square$

Theorem D from the introduction is proved in exactly the same way as above and the proof is omitted. In the following theorem we blur the distinction between L_i and R_i and their (isomorphic) images in $\mathrm{Out}(F_N)$.

Theorem 7.2. *Let $N \geq 3$ and let $D < \mathrm{Out}(F_N)$ be a direct product of $2N-4$ non-abelian free groups. Then a conjugate of D is contained in one of the following subgroups.*

- $L_1 \times \cdots \times L_{N-2} \times R_1 \times \cdots \times R_{N-2}$
- $L_1^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau \times \langle [\tau] \rangle$
- $\langle [\tau], L_1 \rangle \times L_2^\tau \times \cdots \times L_{N-2}^\tau \times R_1^\tau \times \cdots \times R_{N-2}^\tau$

In the second case, if $D = D_1 \times \cdots \times D_{2N-4}$, then each summand D_i is contained in a unique subgroup of the form $\langle L_i^\tau, [\tau] \rangle$ or $\langle R_i^\tau, [\tau] \rangle$. In the remaining cases, the inclusion respects the visible direct product decompositions, up to permuting the summands.

Proof. We apply Theorem A to find a D-invariant collapsed rose with $N-2$ petals in the boundary of Outer space. We then conjugate D so that this rose T is the one given by A and $x_1, \ldots, x_{N-2}$. In this case it is more natural to see $\mathrm{Stab}^0(T) \cong M_{2N-5}(A)$ decomposing as the exact sequence:

$$1 \to L_1 \times \cdots \times L_{N-2} \times R_1 \times \cdots \times R_{N-2} \to \mathrm{Stab}^0(T) \to \mathrm{Out}(A) \to 1,$$

so that $\mathrm{Stab}(T) = \mathrm{Stab}^0(T) \rtimes W_{N-2}$ and W_{N-2} acts by signed permutations of the (R_i, L_i) pairs of factors in the kernel of this exact sequence. Automorphic lifting (see the proof of Proposition 2.8) tells us that $\mathrm{Stab}^0(T) \cong \widetilde{\mathrm{Stab}}^0(T)_e$ for any edge e in the tree. If e is the edge between the vertices corresponding to the cosets $1A$ and x_1A in the Bass–Serre tree, then in Proposition 4.1 we have seen that each $\phi \in \widetilde{\mathrm{Stab}}^0(T)_e$ preserves A, maps $x_1 \mapsto x_1 v_1$ and for $2 \leq i \leq N-2$ maps $x_i \mapsto u_i x_i v_i$ with the $u_i, v_i \in A$. This gives an isomorphism between $\mathrm{Stab}^0(T) = \widetilde{\mathrm{Stab}}^0(T)_e$ and

$M_{2N-5}(A)$ via

$$\Theta(\phi) = (u_2^{-1}, \ldots, u_{N-2}^{-1}, v_1, \ldots v_{N-2}; \phi|_A).$$

Although the images of $L_2, \ldots, L_{N-2}$ and $R_1, \ldots, R_{N-2}$ are easy to see under Θ, the 'natural' representatives of the elements of L_1 are not in $\widetilde{\mathrm{Stab}}^0(T)_e$. However the transvection ϕ mapping $x_1 \mapsto ax_1$ is equivalent in $\mathrm{Out}(F_N)$ to the automorphism ϕ' sending x_1 to x_1a and conjugating every other basis element by a^{-1}, so that $\Theta(\phi') = (a, \ldots, a; \mathrm{ad}_a^{-1})$. It follows that L_1 is mapped to the subgroup J of $M_{2N-5}(A)$ (using the notation of Sect. 3). With some care one can then check that, as in the Aut proof, W_{N-2} acts on $M_{2N-5}(A)$ through Θ as a subgroup of the group $\mathrm{Sym}(2N-4)$ defined in Sect. 3. In particular, Proposition 3.12 then tells us that the projection of D to W_{N-2} is trivial and $D < \mathrm{Stab}^0(T)$. The result then follows by combining Theorem 3.8 with the above isomorphism and the observation that $\Theta(L_1) = J$. $\square$

Proposition 7.3. *Let $N \geq 3$ and let $D < \mathrm{Aut}(F_N)$ be a direct product of $2N-3$ nonabelian free groups in $\mathrm{Aut}(F_N)$. Then the centralizer of D is either trivial or generated by a Nielsen automorphism.*

Proof. Let $C = C(D)$ be the centralizer of D, and let $\overline{C}$, $\overline{D}$ be the respective images of C and D in $\mathrm{Out}(F_N)$. As in the proof of Theorem B, the group $\overline{D}$ is contained in the normalizer of a direct product of $2N-4$ nonabelian free groups, so by Theorem A it fixes a unique collapsed rose with $N-2$ petals in the boundary of Outer space. By uniqueness $\overline{C}$ also fixes this rose, so C acts on the associated Bass–Serre tree T. The group D acts on T with a unique fixed point (Proposition 4.3), therefore C also fixes this point. Hence C is in the subgroup $H = M_{2N-4}(A) \rtimes W_{N-2}$ described in Proposition 4.1. The second part of Proposition 3.12 tells us that the projections of C and D to the $W_{N-2} < \mathrm{Sym}(2N-3)$ factor are trivial, so $C < M_{2N-4}(A)$. We can therefore apply Part 2 of Theorem 3.8: if the centralizer of D in $M_{2N-4}(A)$ is nontrivial it is generated by the element $\underline{\tau}$, and this is mapped to a Nielsen transformation τ under the isomorphism between the group $M_{2N-4}(A)$ and the point stabilizer in T given in Proposition 4.1. $\square$

Corollary 7.4 (Theorem F). *Let $N \geq 3$ and suppose Γ is a finite-index subgroup of $\mathrm{Aut}(F_N)$. If $f\colon \Gamma \to \mathrm{Aut}(F_N)$ is an injective homomorphism then every power of a Nielsen automorphism is mapped to a power of a Nielsen automorphism under f.*

Proof. If ϕ is the power of a Nielsen automorphism then ϕ centralizes a direct product $D < \mathrm{Aut}(F_N)$ of $2N-4$ nonabelian free groups (see the second case of Theorem C). By taking finite-index subgroups of the factors, we can assume that $D < \Gamma$, so that $f(\phi)$ centralizes $f(D)$. Hence $f(\phi)$ is also a power of a Nielsen automorphism by Proposition 7.3. $\square$

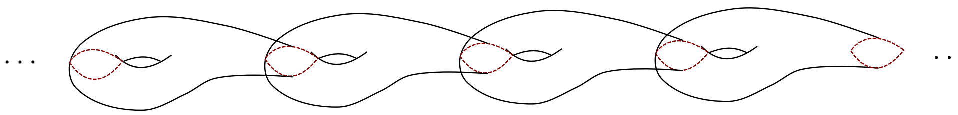

Figure 2: Kurosh's example of a locally free group that is not free is the fundamental group of the above infinite chain of tori.

8 Ascending chains of direct products

As well as taking direct products of free groups in $\mathrm{Aut}(F_N)$ or $\mathrm{Out}(F_N)$ with a maximal number of direct factors, one might also ask about maximality with respect to inclusion. For arbitrary groups, one must be very careful: if G is a group with product rank k and $\mathcal{D}$ is the poset of direct products of k nonabelian free groups in G (ordered by containment), it is not generally true that $\mathcal{D}$ contains maximal elements. One reason for this is the existence of *locally free groups* that are not free.

Recall that a group G is called *locally free* if and only if every finitely generated subgroup of G is free. If G is countable, this is equivalent to the condition that G is the direct limit of its free subgroups. The simplest example of a locally free group that is not free is $\mathbb{Q}$. The free product $\mathbb{Q} * \mathbb{Q}$ is also locally free and clearly contains nonabelian free groups. However, divisibility is not the only reason why a locally free group can fail to be free. We are grateful to Henry Wilton for directing us to the following example of Kurosh: take the one-relator group $G = \langle a, b, t \,|\, t[a,b]t^{-1} = a\rangle$, and let H be the kernel of the map to $\mathbb{Z}$ given by $a, b \mapsto 0$, $t \mapsto 1$. The group G is the fundamental group of the space X_G obtained from a one-holed torus T by gluing the boundary curve of T to a simple closed curve on T. The group H is then the fundamental group of an infinite chain of surfaces (shown in Fig. 2).

The group H has an infinite presentation with generating set $\{a_i, b_i : i \in \mathbb{Z}\}$ and relations $[a_i, b_i] = a_{i+1}$ for all $i \in \mathbb{Z}$. The subgroup generated by $a_{-n}, b_{-n}, \ldots, a_n, b_n$ is freely generated by a_{-n} and $b_{-n}, b_{-n+1}, \ldots, b_n$, which implies that H is locally free. However, H is not free as H is not residually nilpotent: the relations let us write each a_i as an arbitrarily long iterated commutator, so a_i is trivial in every nilpotent quotient of H.

The group G containing H is well-behaved—it is hyperbolic and the fundamental group of a 3-manifold with boundary (this follows by constructing X_G in $\mathbb{R}^3$ and thickening). Following the classification results in Sect. 7, we would like to rule out this behaviour in the mapping torus $M_\tau = F_2 \rtimes_\tau \mathbb{Z}$, where we will take $F_2 = \langle a, b\rangle$ and τ the automorphism taking $a \mapsto ab$ and fixing b. In the spirit of the rest of the paper, we embed M_τ in $\mathrm{Aut}(F_2)$ by identifying $F_2 \rtimes 1$ with the inner automorphisms

Proposition 8.1. *A subgroup of M_τ is free if and only if it does not contain a subgroup isomorphic to $\mathbb{Z}^2$. Any free subgroup of M_τ is contained in a maximal one.*

Proof. The second statement follows from the first via Zorn's Lemma. In more detail, if we have an ascending chain $H_1 < H_2 < H_3 < \cdots$ of free subgroups of M_τ, then the union $H = \cup H_i$ does not contain a subgroup isomorphic to $\mathbb{Z}^2$ (as it would be contained in one of the H_i). Therefore H is free, and we can apply Zorn's Lemma.

We are left with the trickier task of proving the first assertion. To do this, we look at the limiting tree T of the automorphism τ in the boundary of Outer space (see [CL95]). This is a cyclic splitting with vertex stabilizers conjugate to $\mathrm{Fix}(\tau) = \langle aba^{-1}, b\rangle$ and edge stabilizers conjugate to $\langle b\rangle$. As T is invariant under $[\tau]$, there is an action of M_τ on T with vertex stabilizers conjugate to $\langle aba^{-1}, b\rangle \times \langle \tau\rangle$ and edge stabilizers conjugate to $\mathbb{Z}^2 = \langle b, \tau\rangle$. At a vertex v, there are two $\mathrm{Stab}(v)$–orbits of adjacent edges. If $\mathrm{Stab}(v) = \langle aba^{-1}, b\rangle$, these are the conjugacy classes of $\langle aba^{-1}, \tau\rangle$ and $\langle b, \tau\rangle$ in $\mathrm{Stab}(v)$.

Recall that a *cylinder* in T is a subtree C_g that is fixed pointwise by a nontrivial element g of M_τ. In F_2 edge stabilizers are malnormal, so if a cylinder C_g contains more than one edge then $g \notin F_2$. Hence $g = \mathrm{ad}_x \tau^k$ for some $x \in F_2$ and $k \neq 0$. The subgroup of F_2 fixed by g is nonabelian, as g commutes with any inner automorphism fixing an edge in its cylinder. Hence g is similar to τ^k by Collins–Turner (Proposition 3.3). Hence the cylinder is a star of radius one (if τ fixes an edge then the edge is adjacent to the vertex with stabilizer $\mathrm{Fix}(\tau)$). This shows that every cylinder in T is either a point, a single edge, or a star of radius one.

Let H be a subgroup of M_τ that does not contain $\mathbb{Z}^2$. Then for every vertex $v \in T$, the stabilizer H_v is free. As above we may assume that $\mathrm{Stab}(v) = \langle aba^{-1}, b\rangle \times \langle \tau\rangle$. The 'exceptional' case is where $H_v \cap \langle \tau\rangle$ is nontrivial. Then $H_v < \langle \tau\rangle$ as H contains no $\mathbb{Z}^2$ subgroups. Then the H–stabilizer of every edge adjacent to v is also equal to H_v. The 'generic' case is where $H_v \cap \langle \tau\rangle$ is trivial. Then H_v embeds into $\langle aba^{-1}, b\rangle$ via the projection to this factor, and under this projection each adjacent edge stabilizer is contained in a $\mathrm{Stab}(v)$–conjugate of $\langle aba^{-1}\rangle$ or $\langle b\rangle$. Hence H_v splits relative to its adjacent edge groups via the free splitting $S_v := \langle aba^{-1}\rangle * \langle b\rangle$.

One can therefore blow up each generic vertex group via the splitting S_v. In the new H–tree T', the nontrivial edge stabilizers are equal to their adjacent vertex stabilizers, so the tree is a union of cylinders with identical cyclic edge and vertex groups, necessarily separated by edges with trivial stabilizers. As all new edges in the blow-up have trivial stabilizers, each cylinder in T' is unchanged from T and is either a point, a single edge, or a star of radius one. It follows that any setwise stabilizer of a cylinder fixes a point in that cylinder, and therefore fixes the cylinder pointwise (as the edge and vertex groups in cylinders are all identical). This means we can collapse each cylinder to a point, giving a tree T'' on which H acts with trivial edge stabilizers and vertex stabilizers that are either $\mathbb{Z}$ or trivial. Hence H is free. □

Corollary 8.2 (Corollary E). *Let $N \geq 3$ and let $\mathcal{D}$ be the family of subgroups of either* $\mathrm{Aut}(F_N)$ *or* $\mathrm{Out}(F_N)$ *that are direct products of* $2N-3$ *or* $2N-4$ *nonabelian free groups, respectively. Then every $D \in \mathcal{D}$ is contained in a maximal element* (*with respect to inclusion*).

Proof. We limit the proof to the $\mathrm{Aut}(F_N)$ case. Let $D \in \mathcal{D}$. We say a direct summand D_i of D is *twisted* if it acts nontrivially on the homology of $A = \langle a_1, a_2\rangle$, and is *untwisted* otherwise (roughly speaking, this detects if τ is seen in D_i). If at least two summands are twisted, then the proof of Theorem 3.8 implies that each summand of D is contained in a unique $F_2 \times \mathbb{Z}$ given by $\langle L_i^\tau, \tau\rangle$, $\langle R_i^\tau, \tau\rangle$, or $\langle I(A)^\tau, \tau\rangle$, and any $D' \in \mathcal{D}$ containing D also satisfies this conclusion. One then chooses maximal free subgroups of each $F_2 \times \mathbb{Z}$ containing each D_i. Otherwise, we are in one of the remaining cases of Theorem C, where each summand is embedded in an F_2 or a subgroup isomorphic to M_τ, and this enveloping group is a maximal direct product of full-sized subgroups of $\mathrm{Aut}(F_N)$ by Theorem D. Hence taking each F_2 and a maximal free subgroup of the M_τ summand (which exists by Proposition 8.1) will suffice. $\square$

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