Sublinear Rigidity of Lattices in Semisimple Lie Groups

Ido Grayevsky
Pembroke College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2022

Acknowledgments

This work is dedicated to my late grandmother, Chava Friedman

I thank my supervisor Cornelia Druţu, for guiding me in my first steps in the world of Lie groups. I thank her for her support during the Covid-19 period. I thank my master thesis supervisor, Shahar Mozes, who introduced me to the field and who sparked me with the curiosity for symmetric spaces. I thank Dorit Aharonov for teaching me to approach big problems in small steps, rather than the other way round.

I thank Patrick Eberlein for his wonderful book on the geometry of nonpositively curved manifolds, which is the basis of my thesis and most of what I know about symmetric spaces.

I thank Or Landsberg, for numerous wonderful talks about symmetric spaces. His interest and ability to listen were essential to my mathematical development.

To my office mates Alice Kerr and Chris Weiss, thank you for being kind and making me feel at home in the department.

To my friends from the Weizmann Institute: Tal Cohen, Gil Goffer, Elyasheev Leibtag and Itamar Vigdorovich. You inspire me, as people as well as mathematicians. I probably would not have managed it without you.

I thank Uri Bader and Tsachik Gelander from the Weizmann Institute, for teaching me the basics of symmetric spaces and Lie groups, and for welcoming me into their research group during the last couple of years of my DPhil, when I was away from Oxford due to the pandemic.

I thank Gabriel Pallier, Elon Lindenstrauss and Omri Solan for insightful discussions on sublinear rigidity. I also thank the members of my DPhil committees: Dawid Kielak, Panos Papazoglou and Daniel Woodhouse, for their comments and ideas. I thank Yair Glasner for his careful review of this thesis. I especially thank my examiners, Emmanuel Breuillard and Yves Cornulier, for numerous comments and suggestions that have greatly improved this text and some of the results that appear in it.

I thank the University of Oxford, the maths institute and Pembroke College for the financial support and academic atmosphere. I also thank the Leo Baeck Institute, AJA Kartan and the Stuart Young Foundations for their financial support.

Finally, I thank my family. My mother Gitza, who takes care of me and is always teaching me to look after myself. My father Yoram who has been supporting and encouraging me in every step of my studies. To my sister Adar who is always on my side. To my aunt Pnina for her clever advice that always helps when I am mathematically (and otherwise) stuck. To my wife Maayan, for her love during these difficult years. To our daughter Ayala, who fills me with joy. And to my late grandmother Savta Chava, the best person ever.

Contents

I Introduction **1**

I.1 The Wide Context - Rigidity of Lattices . . . 3

I.1.1 Metric Rigidity . . . 4

I.2 About the Proof . . . 4

I.2.1 Sublinear Rigidity . . . 4

I.2.2 SBE Rigidity . . . 5

I.3 Hypotheses and Possible Improvements of Theorem I.0.1 . . . 6

I.4 Future Work . . . 7

I.4.1 Outline and Reading Suggestion . . . 9

II Lattices in semisimple Lie Groups Are Sublinearly Rigid **10**

II.1 Statements and Proof Outline . . . 10

II.1.1 Statements . . . 10

II.1.2 Proof Outline . . . 12

II.2 Preliminaries . . . 13

II.2.1 Symmetric Spaces . . . 15

II.2.2 Lie Groups and Lie Algebras . . . 18

II.2.2.1 Lie Algebras . . . 18

II.2.2.2 Lie groups . . . 21

II.2.3 Symmetric Spaces and Lie Groups Correspondence . . . 25

II.2.4 Algebraic Groups . . . 28

II.2.5 Generalities and Classification of Lattices . . . 31

II.2.5.1 Discrete Subgroups of Lie Groups . . . 31

II.2.5.2 $\mathbb{Q}$-rank 1 Lattices . . . 32

II.2.5.3 Kazhdan's Property (T) . . . 34

II.2.5.4 Separation Into Cases. . . . 35

II.3 Lattices with Property (T) . . . 36

II.3.1 Proof of Theorem II.3.1 . . . 37

II.4 Uniform Lattices . . . 41

II.4.1 Notations and Terminology . . . 42
II.4.2 Proof of Theorem II.4.2 . . . 44
II.5 $\mathbb{Q}$-rank 1 Lattices . . . 45
II.5.1 Strategy . . . 46
II.5.2 Geometry of Symmetric Spaces of Noncompact Type . . . 48
II.5.2.1 The Basic Geometry . . . 48
II.5.2.2 Cusps and the Rational Tits Building . . . 62
II.5.3 Translating Geometry into Algebra . . . 73
II.5.3.1 A Horospherical Lattice . . . 74
II.5.3.2 Indecomposable Horospherical Lattices . . . 81
II.5.3.3 Zariski Density . . . 81
II.5.4 A Λ-cocompact horosphere . . . 85
II.5.4.1 Λ-Free Horoballs . . . 86
II.5.4.2 A Γ-orbit point Lying Deep Inside a Λ-Free Horoball . . . 87
II.5.4.3 Intersection of Λ-Free Regions and the Existence of a Λ-Cocompact Horosphere . . . 93
II.5.5 The Bounded Case . . . 99
II.5.5.1 Schwartz's Argument . . . 102
II.5.6 Proof of Theorem II.5.1 . . . 105

III SBE Rigidity for Lattices of Higher $\mathbb{Q}$-Rank **106**
III.1 Introduction . . . 106
III.1.1 Sublinear Distortion and SBE Maps . . . 106
III.1.2 SBE Rigidity . . . 107
III.1.3 The Quasi-Isometry Case . . . 108
III.1.4 The SBE Case . . . 108
III.1.5 Outline . . . 109
III.2 SBE Maps are Close to Isometries . . . 110
III.2.1 Asymptotic Cones . . . 111
III.2.2 The quasi-isometry case . . . 112
III.2.2.1 A High-Level Description . . . 112
III.2.2.2 Flat Rigidity . . . 113
III.2.3 Generalization to SBE: Adapting Cone Arguments . . . 115
III.2.4 Some Remarks On $\mathbb{R}$-rank 1 Factors . . . 118
III.3 From SBE to Sublinearly Close Groups . . . 119

List of Figures

II.1 Basic setting of Λ-free balls 43
II.2 Bound on the distance of horoballs near x_0 89
II.3 A Γ-free horoball trapped between x_0 and the corresponding Λ-free horoball 96
II.4 A contradiction to almost cocompactness 98

I

Introduction

In this thesis I prove a metric rigidity result for lattices in real semisimple Lie groups without compact factors and with trivial centre. I prove that if a *sublinear neighbourhood* of a discrete subgroup Λ of such a Lie group G contains a lattice $\Gamma \leq G$, then Λ is itself a lattice.

Theorem I.0.1. *Let G be a real semisimple Lie group without compact factors and with trivial centre. Let $\Gamma \leq G$ be an irreducible lattice, and $\Lambda \leq G$ a discrete subgroup. Assume that $\Gamma \subset \mathcal{N}_u(\Lambda)$ for some sublinear function u. In case G is of higher rank but does not have Kazhdan's property (T), assume further that Λ is irreducible and that $\Lambda \subset \mathcal{N}_u(\Gamma)$. Then Λ is a lattice in G. Moreover, if Γ is uniform then so is Λ, and if G is higher rank and Γ is of $\mathbb{Q}$-rank 1 then so is Λ.*

I call this property *sublinear rigidity*. The hypothesis one should focus on is that $\Gamma \subset \mathcal{N}_u(\Lambda)$, which should be read as 'Λ sublinearly covers Γ'. Precise definitions (including the irreducibility of Λ) and a more refined statement of the theorem are given in Chapter II. For now I explain this hypothesis in the language of symmetric spaces, which is the one used in the proof. A function u is called *sublinear* if $\lim_{r\to\infty} \frac{u(r)}{r} = 0$. Let $X = G/K$ be the symmetric space corresponding to G, and x_0 some arbitrary base point. The hypothesis says that there is a sublinear function u for which $B\big(\gamma x_0, u(|\gamma x_0|)\big) \cap \Lambda \cdot x_0 \neq \emptyset$, where $|x| := d(x, x_0)$. In words, if one opens a ball of 'sublinear radius' about each Γ-orbit point, then inside that ball one can find a Λ-orbit point.

In Section I.3 I discuss the role of the other hypotheses and some improvements to this statement which I believe one can obtain without much difficulty.

The motivation to prove sublinear rigidity came from the problem of SBE-rigidity of lattices in Lie groups. Sublinear BiLipschitz Equivalence (SBE) maps are a generalization of quasi-isometries in which the 'additive' constant is replaced by a function that is allowed to grow sublinearly in the distance from a fixed base point of a space. Denote $a \vee b := \max\{a, b\}$ for any $a, b \in \mathbb{R}_{\geq 0}$.

Definition I.0.2. Given two pointed metric spaces $(X, d_X, x_0), (Y, d_Y, y_0)$, a map $f : X \to Y$ is an (L, u)-SBE (Sublinear BiLipschitz Equivalence) if there is a constant $L > 1$ and a sublinear function u such that:

1. For all $x_1, x_2 \in X$,

$$\frac{1}{L} d_X(x_1, x_2) - u(|x_1| \vee |x_2|) \leq d_Y\big(f(x_1), f(x_2)\big) \leq L d_X(x_1, x_2) + u(|x_1| \vee |x_2|)$$

2. For every $y \in Y$, $d_Y\big(y, f(X)\big) \leq u(|y|)$.

The question of SBE rigidity is the following: assume that Λ is a general, abstract, finitely generated group. In particular, it is not assumed to lie in G. Assume that Λ, considered as a metric space with any word metric, is SBE to a lattice in $\Gamma \leq G$. Is Λ a lattice? More precisely, the question is as follows.

Question 1. *Let G be a real semisimple Lie group without compact factors and with trivial centre, $\Gamma \leq G$ an irreducible lattice, and Λ an abstract finitely generated group. Assume that Λ is SBE to Γ. Is it true that Λ can be embedded (up to finite index subgroup and finite kernel) as a lattice in G?*

The class of lattices in G is called *SBE-complete* if the answer is affirmative. I prove that this is the case for a wide variety of groups G, namely those that do not admit a real rank-1 factor:

Theorem I.0.3 (Theorem III.1.6)**.** *Let G be a real semisimple Lie group of noncompact type, of higher $\mathbb{R}$-rank, with trivial centre and with no $\mathbb{R}$-rank 1 factors.*

1. *The class of uniform lattices of G is SBE complete.*

2. *The class of non-uniform lattices of G is SBE complete.*

The proof of Theorem I.0.3 uses Theorem I.0.1 in the final step of the argument. The other substantial step is proving the following result. In the statement, X_0 denotes the compact core of a lattice in $X = G/K$.

Theorem I.0.4. *Let G be a semisimple Lie group without compact factors, with trivial centre and without real rank-1 factors. For every constant $L \in \mathbb{R}_{>0}$ and sublinear function u there is another sublinear function $v = v(L, u)$ such that for every $\big(L, u\big)$-SBE $f : X_0 \to X_0$, there exists a unique isometry $g = g(f) \in \mathrm{Isom}(X)$ such that $d\big(f(x), g(x)\big) \leq v(|x|)$, for all $x \in X_0$.*

I.1 The Wide Context - Rigidity of Lattices

Metrically or otherwise, there are very few results in which one gets as a conclusion that a certain discrete subgroup is a lattice. Probably the most important of these is the Borel and Harish-Chandra criterion [12] (see Theorem II.2.72 below), which roughly states that every arithmetic group in a real algebraic group is a lattice. In fact, in the course of my research I could only find about five lattice criteria that involve geometry, and I use most of these in the course of the proofs. These criteria will be presented later on. The one criterion that I do not use in this thesis is still worth mentioning: this is a recent result by Fraczyk and Gelander [33], proving a conjecture of Margulis. Their result states that for a simple Lie group of higher rank and trivial centre G, a discrete subgroup $\Lambda \leq G$ is a lattice if and only if it has finite injectivity radius in its action on $X = G/K$.

In this context, my result supplies a metric criterion for a discrete subgroup to be a lattice, which is something that is not so easy to come across.

More generally, the notion of 'rigidity' in maths refers to the extent to which a certain structure is determined by some other property of the object in question, a property that is a priori weaker, or at least different, than the defining characterization of the structure.

The theme of rigidity has been extremely fruitful in the context of Lie groups and symmetric spaces. In particular, in various settings and under various assumptions, lattices in Lie groups have been shown to be very rigid. Two remarkable examples are Mostow's Strong Rigidity [56] and Margulis Super-rigidity [53], which I present here due to their immense significance to the theory of Lie groups and in particular to the subject of metric rigidity.

Strong rigidity. This theorem states that an isomorphism between irreducible lattices in two different semisimple Lie groups (without compact factors and that are not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$) must arise as the restriction of an isomorphism between the ambient groups. This has the implication that two uniform lattices in the same Lie group which are abstractly isomorphic are in fact conjugated in the ambient group. Viewing the non-uniform lattice as the fundamental group of the corresponding locally symmetric space $\mathcal{V} = \Gamma\backslash G/K$, strong rigidity can be viewed as a type of homotopy rigidity: the homotopy type of $\mathcal{V}$ determines its isometry type.

Super-rigidity. This theorem states that for higher rank real algebraic semisimple Lie groups without compact factors, essentially any embedding of an irreducible lattice $\Gamma \leq G$ in another such Lie group G' must arise as the restriction of a homomorphism of G into G'. The remarkable point to notice is that Γ is just a discrete finitely generated group,

and yet every embedding of it in another (continuous!) Lie group G' must come from a continuous homomorphism of G to G'.

I.1.1 Metric Rigidity

More closely related to this thesis is the notion of 'metric rigidity'. The *geometry* or *coarse geometry* of a finitely generated group is the quasi-isometry type of any chosen word metric on the group. It is a fundamental aspect of geometric group theory to study those features of a group that are invariant under quasi-isometries. A motivating question is: to what extent does the coarse geometry of a group determine other properties of the group?

In the case of lattices in Lie groups, this question has a very strong answer: the class of lattices in a semisimple Lie group G without compact factors and with finite centre is quasi-isometrically complete. Put differently, if an abstract finitely generated group has the quasi-isometry type of a lattice in such a Lie group, then the abstract group is actually isomorphic (up to finite kernel) to a lattice in G. This result is due to the accumulated work of many, in particular Druţu [24], Eskin [30], Farb [29] [31], Kleiner-Leeb [45], Pansu [62] and Schwartz [67] [68] (see [32] for a summary on this topic). In view of the Margulis Arithmeticity Theorem [52] (see Theorem II.2.88), in many cases this means that the coarse geometric structure of a group determines the arithmetic nature of that group. This is quite amazing: to begin with the group was completely abstract and only coarsely geometrically related to Lie groups.

Sublinear rigidity is a strong form of metric rigidity for discrete groups Λ that already lie in G. It basically means that a group arising as the coarse sublinear deformation of a lattice must be a lattice. SBE rigidity is a generalization of quasi-isometric rigidity, both dealing with the case where the group Λ is completely abstract and has no relation to G other than being SBE to a lattice in G. While quasi-isometries are bi-Lipschitz maps up to some uniformly bounded distortion, SBE maps are bi-Lipschitz maps that have an unbounded (but still sublinearly controlled) distortion, so they represent a more coarse geometric feature of a finitely generated group. From this point of view, results like Theorem I.0.1 of sublinear rigidity, or SBE rigidity, expand Gromov's program to understand groups by their coarse geometric structures.

I.2 About the Proof

I.2.1 Sublinear Rigidity

The proof of sublinear rigidity is divided into three non-mutually exclusive cases: uniform lattices, non-uniform lattices that have Kazhdan's property (T), and non-uniform lattices

of $\mathbb{Q}$-rank 1, that admit a very well understood geometric structure. These notions will become clear in Chapter II. The $\mathbb{Q}$-rank 1 lattices include all non-uniform lattices in groups of $\mathbb{R}$-rank 1 (including those locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$), but also lattices in higher rank Lie groups: see Definition II.2.91. The case of $\mathbb{Q}$-rank 1 lattices requires most of the work. In this case I give a proof that is entirely geometric, which I perform in the symmetric space $X = G/K$. I describe it here briefly.

In general, non-uniform lattices act cocompactly on a subspace $X_0 \subset X$, called the *compact core* of the lattice, which is the complement of a union of horoballs. In case Γ is a $\mathbb{Q}$-rank 1 lattice, these horoballs can be taken to be disjoint, and, more importantly, $\Gamma \cdot x_0$ intersects the corresponding horospheres in a cocompact (metric) lattice. These horospheres are sometimes called *parabolic horospheres*. There is a well developed theory of the visual boundary $X(\infty)$ and of the rational and real (spherical) Tits building structures that $X(\infty)$ accommodates. This theory allows me to use the sublinear constraint in order to prove that also Λ intersects a (parabolic) horosphere in a cocompact lattice. This is Proposition II.5.138.

This result is very powerful in its own, and it has very powerful geometric implications. I use it in order to conclude that Λ admits the conditions of two lattice criteria, rendering it a lattice. For groups of $\mathbb{R}$-rank 1 this is done using a result of Kapovich-Liu [43] on geometrically infinite groups (Theorem II.5.4). For higher rank groups I use a theorem of Benoist-Miquel [8] resolving a conjecture of Margulis. Their criterion is about groups that contain horospherical lattices (Theorem II.5.5).

Semisimple Lie groups without compact factors and with trivial centre that have property (T) admit the peculiar property, brought to my knowledge by Elon Lindenstrauss, that a discrete subgroup is a lattice if and only if it grows at the same exponential growth rate as a lattice. This is due to Leuzinger [49] (Theorem II.3.8), and allows for a straightforward proof of Theorem I.0.1 for these groups.

I.2.2 SBE Rigidity

The general argument in the proof of SBE rigidity closely follows that of quasi-isometric rigidity. There are two main steps. One must first embed the (abstract group) Λ in G. Then one has to prove that the embedded group is a lattice.

The first step in proving an embedding exists is to show that a self SBE of the compact core (associated to a the non-uniform lattice $\Gamma \leq G$) $f : X_0 \to X_0$ is sublinearly close to an isometry. This means that for any such SBE there exists a sublinear function v and an isometry $g \in G = \mathrm{Isom}(X)$ for which $d\big(f(x), g(x)\big) \leq v(|x|)$ for every $x \in X_0$. Call v the sublinear bound associated to f. I prove such a bound exists using the methods of Druţu from her work on quasi-isometric rigidity. She worked in the asymptotic cone, and

my proof is yet another demonstration of adapting cone arguments from quasi-isometries to SBE maps. This type of adaptation is part of the philosophy for studying SBE maps in general, and the reason for the name SBE given by Cornulier [21] who initiated the research on these maps in recent years: SBE stands for Sublinear bi-Lipschitz Equivalence, which reflects to the fact that SBE maps induce bi-Lipschitz maps between corresponding asymptotic cones.

A conjugation argument, again from the quasi-isometry setting, gives for each $\lambda \in \Lambda$ an SBE $f_\lambda : X_0 \to X_0$. Using the first step one gets a map $\Phi : \Lambda \to G = \mathrm{Isom}(X)$. Using sublinear rigidity, the proof would be complete as soon as one is able to prove that Φ is a group homomorphism with finite kernel and image that sublinearly covers Γ. All of these properties are easy to prove if one assumes some control on the sublinear bound v_λ associated to each f_λ. This control is the essence of Lemma III.2.2.

I.3 Hypotheses and Possible Improvements of Theorem I.0.1

To appreciate the strength of Theorem I.0.1, note that one cannot replace the arbitrary sublinear neighbourhood by an arbitrary linear neighbourhood. Indeed for $f(r) = r$ it is a matter of fact that G itself is contained in the f neighbourhood of the trivial subgroup $\{e_G\}$. Some relaxation of the sublinear constraint is applicable though, at least in the case of uniform lattices and of lattices with property (T). More precisely, in these cases I can prove that there is an $\epsilon \in (0, 1)$ for which one can replace the function u with the function $f(r) = \epsilon r$. I call this property ϵ-linear rigidity: see Theorem II.3.7 and Theorem II.4.2.

I believe that there are more improvements to Theorem I.0.1 that are within close reach. I list these ideas below.

Irreducibility of Λ. I believe it should be possible to remove the hypothesis that Λ is irreducible, which is needed for the Benoist-Miquel criterion. A much more natural assumption in the context of rigidity is to require Γ to be irreducible. I believe that in case Γ is irreducible, the geometric results I prove (which are independent of irreducibility of Λ and to some extent even independent of the irreducibility of Γ) are sufficient in order to deduce that Λ is irreducible. In fact I believe there should be a simple argument proving that a group which sublinearly covers an irreducible lattice is irreducible.

Sublinear Hausdorff distance. The extra assumption that $\Lambda \subset \mathcal{N}_u(\Gamma)$, which I require in the case of a $\mathbb{Q}$-rank 1 lattice without property (T), is in fact used only if $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$ (see Section II.5.5). It is probable that one can deduce that Λ is a lattice directly from the fact that $\Gamma \subset \mathcal{N}_D(\Lambda)$. See Section II.5.5 for an elaborate account of this bounded case.

Rational rank. If G is of higher rank, then the Margulis Arithmeticity Theorem assures Γ is arithmetic, and therefore one can associate a $\mathbb{Q}$-rank to Γ. Morally, groups that are SBE to a lattice should have the same $\mathbb{Q}$-rank as the lattice. For example, see Leuzinger's characterization of $\mathbb{Q}$-rank of a non-uniform lattice in terms of the asymptotic cone over the (quotient of the) rational Tits building associated to Γ. I prove this $\mathbb{Q}$-rank equality when Γ has $\mathbb{Q}$-rank 1. In order to add this conclusion in the general case, I suspect one would have to use the (quite natural, after all) assumption that also Λ is sublinearly covered by Γ.

Commensurability. In the case of quasi-isometries, non-uniform lattices admit a very strong form of rigidity, namely: if a finitely generated group is quasi-isometric to a non-uniform lattice in a semisimple Lie group without compact factors and with finite centre G (that is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$), then it can be embedded in G as a lattice *commensurable* to Γ. It may very well be true that this also holds in the sublinear setting, i.e. in the setting of Theorem II.1.5. At least for lattices of $\mathbb{Q}$-rank 1, I believe this might be easy to prove. Specifically, in these cases my proof relates the question of commensurability to the construction of a $\mathbb{Q}$-structure on G which is used by Benoist and Miquel in their lattice criterion. The construction itself is due to Raghunathan and, independently, Margulis. See Proposition 4.11 in [8].

I.4 Future Work

I list here some more topics for further research that arise from this thesis.

SBE Rigidity for Groups with $\mathbb{R}$-rank 1 Factors. The proof of Theorem I.0.4 for groups that admit $\mathbb{R}$-rank 1 factors must be essentially different from the one I aim for for groups without $\mathbb{R}$-rank 1 factors. There are some potential ways of tackling such groups. I briefly discuss my attempts in section III.2.4.

Geometric Proof for Lattices of Higher $\mathbb{Q}$-Rank. The proof of sublinear rigidity for groups G with property (T) goes by using an amazing consequence of Kazhdan's property (T), namely that a discrete subgroup of G is a lattice if and only if it has the 'right' growth rate. However this proof cannot, on the face of it, lead to stronger results like commensurability of Λ and Γ or even to some control on the $\mathbb{Q}$-rank of Λ. While it was much easier for me to complete the geometric proof for $\mathbb{Q}$-rank 1 lattices, I suspect that the geometric ideas I used could be generalized to any $\mathbb{Q}$-rank, at least to some extent. I remark that for lattices of higher $\mathbb{Q}$-rank the parabolic horospheres are not disjoint, but the lattice still intersects them in (non-uniform) metric lattices.

An Example. As of yet I could not construct or come up with an example of groups that are SBE to a lattice in a non-trivial manner. Of course, if being SBE to a lattice would also imply being commensurable to that lattice, then no such example exists. Even for sublinear rigidity, I still do not know whether a discrete subgroup $\Lambda \leq G$ that sublinearly covers a lattice Γ must actually admit $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some fixed $D > 0$.

A Geometric Analogue of the Margulis Conjecture. In the case of $\mathbb{Q}$-rank 1 lattices I use an algebraic criterion of Benoist and Miquel, resolving a conjecture of Margulis, in order to deduce Λ is a lattice. It is reasonable to believe that one could use the geometric implications of Section II.5 in order to give a completely geometric analogue of this criterion in the language of symmetric spaces and horospheres.

The ϵ-Linear Rigidity for $\mathbb{Q}$-rank 1 Lattices. The question of ϵ-linear rigidity for $\mathbb{Q}$-rank 1 lattices remains open. In the general context of ϵ-linearity it is interesting to check whether there exist groups that ϵ-linearly cover a lattice but that do not sublinearly cover it.

Generalizing to $\mathrm{Aut}(F_n), \mathrm{Out}(F_n)$, and Mapping Class Groups. It is a general theme in geometric group theory to generalize results about lattices in Lie groups to these three classes of groups. In view of some works regarding their large scale geometry, e.g. [6], it might be worth while to consider the notions of quasi-isometric rigidity, sublinear rigidity, sublinear distortions, and SBE maps for any of these groups. It seems like an interesting direction, but one that goes into a very unfamiliar territory for me and on which I can not even speculate the possible outcome.

Relation to Fraczyk-Gelander. In [33], Fraczyk and Gelander use *invariant random subgroups* and *stationary random subgroups* in order to come up with a correlation between the injectivity radius of a discrete subgroup of a higher rank simple Lie group G and its covolume. In other words, they use dynamics and ergodic theory on the Chabauty space of closed subgroups of G, and show that a subgroup that has finite injectivity radius must have finite covolume. When I first heard about this result I had a strong impression that I would be able to use it in the context of sublinear rigidity. However, the nature of their work does not give explicit bounds on the injectivity radius, and eventually I was not able to use it. More precisely, given $r \in \mathbb{R}_{>0}$, one cannot tell directly from the results [33] how 'far' one must wander in X in order to find a point with injectivity radius r. In yet another formulation: what is the relation between $|x|$ and InjRad(x)? It may be interesting to check if one can use the sublinear rigidity of Theorem I.0.1 in order to get some control on the quantities that appear in [33], or if one can produce explicit bounds for quantities in that work in order to prove Theorem I.0.1.

I.4.1 Outline and Reading Suggestion

The thesis is divided into two chapters, one devoted to sublinear rigidity and the other to SBE-completeness. The main objects of interest are symmetric spaces of noncompact type. Most of the original work is concentrated in the geometric arguments of Section II.5, specifically in Section II.5.4. These arguments generally concern horospheres and the properties of a compact core of non-uniform lattices. I think also Section II.5.3 contains parts that are not widely known even to experts.

I wrote chapter II with the goal that it would be completely self contained. For this reason, I included a lot of preliminaries which are definitely well known for those who work with symmetric spaces. The preliminaries may be used as references for facts that are used in the proofs. Section II.5.2 deals with the geometry of symmetric spaces of noncompact type. Due to the key role they play in this thesis, I give a rather detailed exposition which could serve as a condensed primer on horospheres and the structure of non-uniform lattices. It contains some interesting properties that are not so well known for non-experts.

The proofs in chapter III are based quite heavily on previous works, mainly of Druţu [24] and Schwartz [67]. I kept it much more concise, and in particular I chose not to present the preliminaries that go into these works.

II

Lattices in semisimple Lie Groups Are Sublinearly Rigid

II.1 Statements and Proof Outline

II.1.1 Statements

Definition II.1.1. Let (X, x_0) be a pointed metric space. Define the *size* of a point $x \in X$ to be $|x| := d(x, x_0)$, the distance from x to the base point x_0.

Definition II.1.2. Let (X, x_0) be a pointed metric space, $Y \subset X$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ some function. The *u-neighbourhood* of Y in X is defined to be

$$\mathcal{N}_u(Y) := \{x \in X \mid d(x, Y) \leq u(|x|)\}$$

The particular case of interest in this section is where the metric space is a Lie group (G, e_G) with a metric induced by some left invariant Riemannian metric. I consider two discrete subgroups, $\Lambda, \Gamma \leq G$, where Γ is a lattice and Λ is an arbitrary subgroup such that $\Gamma \subset \mathcal{N}_u(\Lambda)$ for a *sublinear* function u.

Definition II.1.3. A function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ is *sublinear* if $\lim_{r\to\infty} \frac{u(r)}{r} = 0$.

Remark II.1.4. Given a sublinear function u' one can define another sublinear function u such that $u' \leq u$ and u is monotonically non-decreasing: simply define

$$u(r) := \sup_{0\leq s\leq r} u'(s)$$

Throughout this thesis I assume that a sublinear function is monotonically non-decreasing. The use I make in sublinearity readily implies that there is no loss of generality in this assumption.

I can now formulate the main theorem. The definitions for the classic Lie theory and algebraic group theory notions are found in Section II.2.

Theorem II.1.5. *Let G be a semisimple Lie group without compact factors and with trivial centre. Let $\Gamma \leq G$ be a lattice, $\Lambda \leq G$ a discrete subgroup, and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ a sublinear function. Assume that $\Gamma \subset \mathcal{N}_u(\Lambda)$.*

If G is either of $\mathbb{R}$-rank 1 or has Kazhdan's property (T), or if Γ is uniform, then Λ is a lattice. Otherwise,

1. *If $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$, assume further that $\Lambda \subset \mathcal{N}_u(\Gamma)$.*
2. *If else $\Gamma \not\subset \mathcal{N}_D(\Lambda)$ for any $D > 0$, assume further that Λ is irreducible.*

Then Λ is a lattice. Moreover, Λ is uniform if Γ is uniform, and if G is of higher rank then Λ is $\mathbb{Q}$-rank 1 lattice if and only if Γ is a $\mathbb{Q}$-rank 1 lattice.

Remark II.1.6.

1. Trivial centre is a natural assumption that is used e.g. in order to ensure G may be considered as an algebraic group. The proof in Section II.4 of Theorem II.1.5 for uniform lattices is valid also for non-algebraic groups.
2. The extra assumption that Λ is irreducible (see Definition II.2.83) is imposed in order to use the Benoist-Miquel lattice criterion ([8], see Theorem II.5.5 and Lemma II.5.129 below). As mentioned in Section I.3, I believe the theorem is true under the more natural assumption that only Γ is irreducible.
3. The extra assumption that $\Lambda \subset \mathcal{N}_u(\Gamma)$ is only used in the case where $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$, i.e., when Γ lies in a bounded neighbourhood of Λ. Also here I suspect that it is true that $\Gamma \subset \mathcal{N}_D(\Lambda)$ alone should imply Λ is a lattice, but as of yet I need the extra assumption that $\Lambda \subset \mathcal{N}_u(\Gamma)$. See Section II.5.5.

In the case of uniform lattices or when G has property (T), I prove stronger results:

Theorem II.1.7 (see Section II.4)**.** *In the notation of Theorem II.1.5 and in the case Γ is uniform, the function u can be replaced with a function of the form $u(r) = \epsilon r$ for any $\epsilon < 1$.*

Theorem II.1.8 (see Section II.3)**.** *In the notation of Theorem II.1.5 and in the case G has Kazhdan's property (T), there is $\epsilon = \epsilon(G) > 0$ such that the function u can be replaced with any function of the form $u(r) = \epsilon' r$ for any $\epsilon' \leq \epsilon$.*

I call this property *ϵ-linear rigidity*.

II.1.2 Proof Outline

Four Lattice criteria and three cases to consider. Theorem II.1.5 adds a (metric) criterion for a discrete subgroup of a semisimple Lie group without compact factors and trivial centre to be a lattice. The proof uses four lattice criteria, which together capture all cases in question. Below is a rough sketch of these criteria. Precise formulations are given in each of the relevant sections.

1. If the function u in Theorem II.1.5 is bounded, then Λ is a lattice. This is an easy observation in the uniform case, and due to Schwartz [67] in the case of non-uniform $\mathbb{Q}$-rank 1 lattices. See Theorem II.5.2 and Theorem II.5.156 below.

2. If G has Kazhdan's property (T), then any discrete subgroup is a lattice if and only if its critical exponent equals that of a lattice. I.e., if and only if it grows 'as fast as a lattice grows'. This is due to Leuzinger [49]. I thank Elon Lindenstrauss for telling me about this criterion.

3. For groups of $\mathbb{R}$-rank 1, a discrete subgroup is geometrically infinite if and only if the set of non-conical limit points has cardinality continuum. This is a fairly recent result due to Kapovich-Liu [43].

4. If a discrete Zariski dense subgroup intersects a horospherical subgroup in an irreducible lattice, then it is a lattice. This is a conjecture of Margulis, recently solved in full generality by Benoist-Miquel [8], with major specific cases previously solved by Oh [57] (see discussion in Section 1 in [8]).

The proofs for the uniform case and for groups G with property (T) are quite straightforward. The essential part of this thesis is the proof of the case where the group G admits $\mathbb{R}$-rank 1 factors (including groups G of $\mathbb{R}$-rank 1). Along with the respective criteria, the cardinal step in the proof is the following proposition.

Proposition II.1.9 (Proposition II.5.138)**.** *Let (X, x_0) be a pointed symmetric space of noncompact type, $G := \mathrm{Isom}(X)^\circ$, $\Gamma \leq G$ a lattice and $\Lambda \leq G$ a discrete subgroup such that $\Gamma \cdot x_0 \subset \mathcal{N}_u(\Lambda \cdot x_0)$ for some sublinear function u. Assume that for every $D > 0$, $\Gamma \cdot x_0 \not\subset \mathcal{N}_D(\Lambda \cdot x_0)$. i.e. the Γ-orbit is not contained in a bounded neighbourhood of the Λ-orbit. Then there exists a horosphere $\mathcal{H} \subset X$, based in a Γ-parabolic limit point, such that $\big(\Lambda \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ intersects $\mathcal{H}$ in a cocompact metric lattice.*

Remark II.1.10. 1. Proposition II.5.138 is a strong geometric statement. I suspect that one might be able to use it in order to prove that Λ is a lattice without appealing to the powerful 'hammers' in the results of Benoist-Miquel or even that of Kapovich-Liu. I did not pursue this direction in my work.

2. Both the horospherical criterion (published 2020) and the geometric finiteness criterion (published 2019) are quite recent. As far as I am aware, Theorem II.1.5 is only the second use of the horospherical arithmeticity criterion of Benoist and Miquel or even of the previous partial results of Oh. The first use was in a recent work on Origamis [41]. I am not aware of any prior use of the result of Kapovich-Liu in the context of lattices, or in general.

The proof is separated into three (not mutually exclusive) cases:

1. Uniform lattices in any group G.
2. Non-uniform lattices in groups G with property (T).
3. Non-uniform lattices $\Gamma \leq G$ that are of $\mathbb{Q}$-rank 1. See Section II.2.5.2 for a definition.

Lemma II.2.103 shows that these cases indeed cover all real semisimple Lie groups G without compact factors and with trivial centre, and all lattices $\Gamma \leq G$ in such groups. I.e, all the cases that are considered in Theorem II.1.5.

Chapter Outline. Section II.2 contains only the background that is necessary to understand the statements made in the introduction, namely Theorem II.1.5 and the separation of the proof into three cases. More specific and proof related preliminaries appear separately in the relevant sections. The general preliminaries consist almost entirely of basic notions of the objects involved, and may be skipped by anyone who can read through the statements of Theorem II.1.5 and Lemma II.2.103.

The different cases of the proof are divided into 3 sections: groups with property (T) in Section II.3, uniform lattices in general Lie groups in Section II.4, and lattices of $\mathbb{Q}$-rank 1 in Section II.5. Each section begins with a precise statement of the lattice criterion which is used in that section, along with the strategy of proof and an explanation of where the sublinearity of u comes into play. Section II.5 is very involved so I include a detailed description of the argument before the formal proof. Furthermore the proof of that case requires many geometric facts about symmetric spaces as well as an algebraic interpretation of the geometric results obtained. The geometric background is in Section II.5.2 and the algebraic interpretation appears in Section II.5.3.

II.2 Preliminaries

In this section I define the notions that are needed in order to read through the main results, and the facts needed in order to understand the strategy of proof. For the most part I just give a direct citation and a reference, without getting into proofs.

Geometry vs. Algebra Theorem II.1.5 is stated in the algebraic language of Lie groups. The proofs however are performed for the most part in the geometric setting of symmetric spaces. There is a close relation between these settings that allows for the translation. Personally, I was always interested in Lie groups, but could not find a way through the immense preliminary mathematics that go into 'Lie theory'. What I did manage is to get myself more familiar with the related geometric objects - buildings and symmetric spaces. The geometry allowed me to eventually get a grip on the algebra of Lie groups and Lie algebras.

I learned that much of the subject indeed could be understood from a geometric point of view, which is the approach I take in this thesis. Namely I focus on the geometry of symmetric spaces, and present the algebraic side mainly (though not entirely) for completeness. Algebra could not be ignored completely when working with symmetric spaces, and one must say something about Lie groups, Lie algebras and algebraic groups. Still, I want to keep this 'something' as concise and restricted as possible. For those readers who are unfluent in or slightly deterred by the standard Lie-theoretic preliminaries sections that often appear in papers, I would like to stress explicitly that for much of this thesis no algebra is required. This is especially true for the original mathematical ideas in this thesis, that are entirely geometric. If one is willing to accept for granted some geometric facts that arise easily from the algebra - e.g. that the group of isometries acts transitively on the set of flats - then one could completely understand this thesis without knowing any Lie theory (including that which I present in Section II.2.2). My main goal in the algebraic sections of the preliminaries is actually to allow a reader who wishes to do so to relate the geometry of symmetric spaces to the algebra of Lie groups. From this perspective, the most important section of the preliminaries is Section II.2.3 where I explain some of the basic features of the Lie theory - symmetric spaces correspondence.

Preliminaries Outline. I begin with symmetric spaces in Section II.2.1, and continue with the Lie theoretic side in Section II.2.2. In Section II.2.3 I give an informal description of some aspects of the symmetric spaces-Lie groups correspondence, especially of those features which would be relevant in the sequel.

I then turn to make some necessary definitions and present basic concepts in algebraic groups. I finish with a section containing some generalities about lattices, with the goal of proving Lemma II.2.103. I include mainly the classification of groups G with Kazhdan's property (T) and some aspects of $\mathbb{Q}$-rank 1 lattices.

The presentation of the preliminaries is based mostly on the books of Eberlein [28] (symmetric spaces), Helgason [39] and Knapp [46] (Lie theory), and Borel [11] (algebraic groups).

II.2.1 Symmetric Spaces

Symmetric spaces are Riemannian manifolds with a special property, namely that the *local geodesic symmetry* at each point is a global isometry. In this thesis though I only need very basic Riemannian geometry, since many of the features can be expressed in much more basic geometric terms. In particular, the notion of *curvature* is mostly implicit and usually appears in the form of concrete metric properties.

Below are a couple of equivalent definitions for symmetric spaces (see Chapter 1 in [54] for a proof of equivalence). Recall that a *bi-infinite geodesic* in a metric space X is an isometric embedding $\eta : \mathbb{R} \to X$.

Definition II.2.1 (Section 1.3.1 in [28])**.** Let X be a connected Riemannian manifold. At any point $x \in X$, there exists a *geodesic symmetry* at x: a diffeomorphism $\sigma_x : X \to X$ that satisfies $\sigma\big(\eta(t)\big) = \eta(-t)$ for all bi-infinite geodesics $\eta : \mathbb{R} \to X$ with $\eta(0) = x$.

X is a *symmetric space* if for every $x \in X$, σ_x is an isometry of X.

Definition II.2.2 (Definition 1.1.5 in [54])**.** A Riemannian manifold X is a *symmetric space* if

1. X is connected.
2. $\mathrm{Isom}(X)$ acts transitively on X.
3. There is an isometry $\sigma : X \to X$ that is *involutive*, i.e., $\sigma^2 = id_X$, and σ admits at least one isolated fixed point in X.

The transitive action of $\mathrm{Isom}(X)$ assures that X is *geodesically complete*, meaning that any local geodesic $\eta : (-\epsilon, \epsilon) \to X$ can be prolonged indefinitely in both directions. The Hopf-Rinow theorem then implies that X is complete.

The object defined above is sometimes referred to as a *globally symmetric space*, to distinguish from:

Definition II.2.3. A complete Riemannian manifold M is called a *locally symmetric space* if its universal cover is a symmetric space.

One justification for the terminology is that in locally symmetric spaces, the geodesic symmetry is only a local isometry (i.e., an isometry of a small neighbourhood of the relevant point) which could not in general be extended to a global isometry.

I draw the reader's attention to the different flavours of the two definitions of (globally) symmetric spaces. The first emphasizes the local-to-global character of a symmetric space, namely that some local feature can be extended metrically to the entire space. The second definition emphasizes the homogeneity of symmetric spaces: all points in a symmetric spaces are the same from a geometric point of view.

Definition II.2.4. A symmetric space is *irreducible* if it is irreducible as a Riemannian manifold, i.e. if it has trivial de Rham decomposition.

Definition II.2.5 (Section 1.3 in [28])**.** A symmetric space X is *of noncompact type* if X has no compact or Euclidean de Rham factors.

The absence of compact factors is equivalent to the manifold X having nonpositive sectional curvature, and one has:

Proposition II.2.6 (Theorem 2.1.1 in [28])**.** *A symmetric space X is of noncompact type if and only if it has nonpositive sectional curvature and no Euclidean de Rham factors.*

The proof for this fact goes through the correspondence of symmetric spaces and Lie groups, specifically through the correspondence between compact de Rham factors and compact Lie algebra factors (see Section II.2.2 below).

Remark II.2.7. One can decompose a general nonpositively curved symmetric space (i.e., one that may contain a Euclidean factor) into a symmetric space of noncompact type and a maximal Euclidean factor, then study each component separately. In this way one can extend much of the theory of symmetric spaces of noncompact type to the more general setting of nonpositively curved symmetric spaces (see e.g. Theorem 7.3.3 in [28]).

Virtually everything I prove in this chapter relies on the nonpositive curvature of X and more concretely on the metric implications of nonpositive curvature. I depict the relevant metric implications as I go along.

From now on and unless otherwise stated, all symmetric spaces will be of noncompact type.

Definition II.2.8. A *lattice* of a symmetric space X is a discrete subgroup $\Gamma \leq \mathrm{Isom}(X)$ such that the quotient Riemannian orbifold $\mathcal{V} = \Gamma\backslash X$ has finite measure. If Γ is torsion-free then $\mathcal{V}$ is a Riemannian manifold. A lattice is *uniform* if $\mathcal{V}$ is compact, and *non-uniform* otherwise.

Notice that if Γ is torsion-free, then by definition the quotient manifold $\Gamma\backslash X$ is a locally symmetric space. An equivalent definition (see Section 5.6.4 in [25]) uses the notion of a *fundamental domain* for the Γ-action on X.

Definition II.2.9. Let X be a symmetric space. A *lattice* in X is a discrete subgroup $\Gamma \leq Isom(X)$ such that X admits a Γ *fundamental domain*: a set $U \subset X$ with finite measure, such that:

1. U is open, and $\gamma U \cap U = \emptyset$ for all $\gamma \in \Gamma$.

2. $X = \cup_{\gamma\in\Gamma}\gamma\overline{U}$.

3. ∂U has measure 0.

Definition II.2.10 (Section 3.11 in [28])**.** A lattice $\Gamma \leq \mathrm{Isom}(X)$ is *reducible* if it is virtually a product $\Gamma_1 \times \Gamma_2 \leq \mathrm{Isom}(X)$. This implies that the locally symmetric space $M = \Gamma\backslash X$ has a finite cover that splits as a Riemannian product $M_1 \times M_2$. A group is *irreducible* if it is not reducible.

Irreducibility is a notion that is appears usually in the context of lattices. In this thesis however a form of irreducibility is also required for the general discrete subgroup $\Lambda \leq G$, which is not a priori a lattice. See Definition II.2.83.

The last notion concerning symmetric spaces that appears in Theorem II.1.5, at least in its algebraic guise, is the *rank* of the space X.

Definition II.2.11. A *totally geodesic submanifold* $Y \subset X$ is a connected submanifold such that any geodesic in Y with the induced metric is also a geodesic of X.

Definition II.2.12. Let X be a symmetric space of non-compact type. A *flat* of X is a totally geodesic submanifold $F \subset X$ that is isometric to an Euclidean space $\mathbb{E}^m$. The number m is called the *dimension* of F.

Geodesics are isometric embeddings of $\mathbb{R}$ into X, so they are 1-dimensional flats. Much of the geometric structure of X is determined by the structure and mere presence of *maximal flats*.

Definition II.2.13. $F \subset X$ is a *maximal flat* if it is not strictly contained in a flat.

Proposition II.2.14. *All maximal flats of a symmetric space X have the same dimension. This dimension is called the* rank *of X, denoted* $\mathrm{rk}(X)$.

The proof for this fact usually goes through the correspondence with the Lie group structure (see Chapter 2 in [28] and Section II.2.3 below).

There is a substantial difference between spaces of rank 1 and spaces that admit flats of rank at least 2. These are usually referred to as *higher rank* spaces. One elementary and useful manifestation of this fact is that $\mathrm{Isom}(X)$ acts transitively on geodesics if and only if $\mathrm{rk}(X) = 1$, a fact which has many geometric implications. It is also illuminating to note that $\mathrm{Isom}(X)$ does act transitively on the set of maximal flats in X, regardless of its rank. This hints that it is fruitful to think about geodesics as maximal flats. See Section II.5.2) for a much more elaborate description of this and other geometric facts.

II.2.2 Lie Groups and Lie Algebras

Definition II.2.15. A group G is a *real Lie group* if it admits the structure of a smooth real manifold such that the multiplication and inverse maps are smooth.

This definition is relatively simple, but it hides more than it reveals. It hides the intimate relation between Lie groups and Lie algebras, which is the true power of Lie theory. The fundamental relation between these objects is that the tangent space T_eG has the structure of a Lie algebra (see Theorem II.2.37 below). Even a partial presentation of this relationship is beyond the scope of this thesis. I define and highlight only the most fundamental properties that are relevant. I begin with presenting Lie algebras because much of the Lie groups terminology originates from the associated Lie algebra.

II.2.2.1 Lie Algebras

It is often the case that the interest in Lie algebras arise in relation to an associated Lie group. However Lie algebras are defined and could be dealt with abstractly without any appeal to an underlying (Lie) group.

Definition II.2.16. A *(real) Lie algebra* is a (real) vector space $\mathfrak{g}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that:

1. $[x, x] = 0$ for all $x \in \mathfrak{g}$.
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The operation is called the *bracket* operation, and the equation of the second item is called the *Jacobi identity*.

Remark II.2.17. In this thesis I will only deal with real finite dimensional Lie algebras.

Remark II.2.18. Some important Lie algebras are subspaces of $M(n, \mathbb{R})$ with the bracket operation $[A, B] = AB - BA$, called the *commutator bracket*. Much of the terminology of Lie algebras is based on these 'matrix Lie algebras', e.g. a Lie algebra is called *abelian* if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, it is *nilpotent* (*solvable*) if the (bracket) lower central series (derived central series) terminates in $\{0\}$.

Definition II.2.19. A *Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace that is closed under the bracket operation, i.e. $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.

A *Lie ideal* $\mathfrak{i} \subset \mathfrak{g}$ is a vector subspace that satisfies $[x, y] \in \mathfrak{i}$ for all $x \in \mathfrak{i}$ and all $y \in \mathfrak{g}$.

Definition II.2.20. An *inner direct sum* of a Lie algebra $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of two *commuting* Lie subalgebras, i.e. $\mathfrak{h}_1, \mathfrak{h}_2$ are both Lie subalgebras such that $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$.

Definition II.2.21. A Lie algebra $\mathfrak{g}$ is *simple* if $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$ (i.e. if $\mathfrak{g}$ is non-abelian), and in addition it does not contain any non-trivial proper ideals. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras.

My next goal is to define the *rank*, or *Cartan rank*, of a Lie algebra. A finite dimensional real Lie algebra admits an inner product called the *Killing form*. This form is natural and is fundamental to the Lie structure theory. The definition of the Killing form requires the definition of another fundamental object in Lie theory, namely the adjoint representation. While these objects are fundamental for Lie theory, the specifications of the construction of the Killing form are not at all important for this thesis. I will just need some facts about the relation between the Killing form and the *Cartan decomposition* that a geodesic symmetry of X induces on the Lie algebra of the Lie group $G = \mathrm{Isom}(X)^\circ$ (see Section II.2.3).

Definition II.2.22. A vector space homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras is a *Lie algebra homomorphism* if it respects the bracket operation, i.e. $[\phi(x),\phi(y)]_{\mathfrak{h}} = \phi\big([x,y]_{\mathfrak{g}}\big)$

Definition II.2.23. The space of linear endomorphisms $GL(\mathfrak{g})$ is made into a Lie algebra with the bracket operation defined by $[\phi,\psi](x) = \phi\circ\psi(x) - \psi\circ\phi(x)$.

Definition II.2.24. Let $\mathfrak{g}$ be a Lie algebra. The *adjoint representation* is the Lie algebra homomorphism $ad : \mathfrak{g} \to GL(\mathfrak{g})$ defined by $x \mapsto ad_x$, where $ad_x : \mathfrak{g} \to \mathfrak{g}$ is defined by $y \mapsto [x,y]$.

Remark II.2.25.

1. It is routine to check ad is indeed a Lie algebra homomorphism. The map ad_x however is in general not a Lie algebra homomorphism.
2. The adjoint representation is also the derivative of the *adjoint representation* of a Lie group corresponding to $\mathfrak{g}$, see Definition II.2.59.

Definition II.2.26. Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. The *Killing form* on $\mathfrak{g}$ is the (symmetric, bilinear) form

$$B(x,y) := tr(ad_x \circ ad_y)$$

It is routine to check that the Killing form is invariant under any Lie algebra automorphism, i.e. $B\big(\phi(x),\phi(y)\big) = B(x,y)$ for every Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ and any $x, y \in \mathfrak{g}$.

Definition II.2.27 (Proposition 6.6, Chapter 2 in [39])**.** A Lie algebra $\mathfrak{g}$ is *compact* if the Killing form is strictly negative definite. Otherwise it is noncompact.

Remark II.2.28. Another definition states that a compact Lie algebra is any Lie algebra of a compact Lie group. While these definitions are not equivalent, the following is true:

1. If $\mathfrak{g}$ is a semisimple real Lie algebra that is compact according to Definition II.2.27, then there is a compact Lie group G with $\mathfrak{g}$ isomorphic to Lie(G) (Corollary 6.7, Chapter 2 in [39]).

2. Conversely, If $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of a compact Lie group G, then the Killing form is negative semidefinite. Moreover, the structure of $\mathfrak{g}$ in this case implies that if G has finite centre, then the Killing form is in fact negative definite (Corollaries 4.25 and 4.26, Chapter 4 in [46]).

Definition II.2.29. Let $\mathfrak{g}$ be a real semisimple Lie algebra. It is *of noncompact type* if every simple factor of it not compact.

Remark II.2.30. Definition II.2.29 is a slight modification of the general definition, see Section 1, Chapter 10 in [40].

Definition II.2.31. A Lie automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ is called a *Cartan involution* if it is an involution (i.e. $\theta^2 = id_{\mathfrak{g}}$) and the bilinear form $B_\theta(x, y) := -B\big(x, \theta(y)\big)$ is strictly positive definite.

Definition II.2.32. Let $\mathfrak{g}$ be a real finite dimensional Lie algebra and $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution. Since $\theta^2 = id_{\mathfrak{g}}$, it admits (at most) two eigenvalues $\{-1, 1\}$, to which there correspond two eigenspaces, $\mathfrak{p}$ for -1 and $\mathfrak{k}$ for 1. The direct sum decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is called the *Cartan decomposition associated to* θ.

Proposition II.2.33 (See Section 7, Chapter 3 in [39])**.** *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. Then $\mathfrak{g}$ admits a Cartan involution, hence it admits a Cartan decomposition. Moreover, every two Cartan decompositions are conjugate under an (inner) automorphism. That is, If $\mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{p}_i$ for $i \in \{1, 2\}$, then there exists an (inner) automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ for which $\phi(\mathfrak{p}_1) = \mathfrak{p}_2, \phi(\mathfrak{k}_1) = \mathfrak{k}_2$.*

Remark II.2.34. The fact that the isomorphism in Proposition II.2.33 is inner is not important for this presentation. For completeness, I mention that an *inner* automorphism of a Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is an automorphism of the form $\text{Ad}(G)$ for some $g \in G$, where $Ad : G \to GL(\mathfrak{g})$ is the adjoint representation (Definition II.2.59).

Definition II.2.35 (Section 1, Chapter 9 in [39])**.** Let θ be a Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. The *rank* of $\mathfrak{g}$ is defined to be the dimension of $\mathfrak{a}$. This is well defined as all maximal abelian subspaces have the same dimension.

Remark II.2.36. The subspace $\mathfrak{a}$ is the key object in the structure theory of the Weyl group and Weyl chambers, that play an important role in the proof of the $\mathbb{Q}$-rank 1 case, as well as in Druţu's work [24] on which I rely in Chapter III. I present these objects in Section II.5.2.

II.2.2.2 Lie groups

The relation between Lie groups and Lie algebras is given by the fact that the tangent space at the identity element admits the structure of a Lie algebra. The definition uses the notion of *vector fields* and *derivatives* from differential geometry, which I do not define.

Theorem II.2.37 (Section 1, Chapter 2 in [39])**.** *For a Lie group G, the tangent space T_eG at the identity element $e \in G$ has the structure of a Lie algebra. This structure is given via the identification of T_eG with the set of all left-invariant vector fields on G. These are vector fields X that admit $dL_gX = X$, where $L_g : G \to G$ is the smooth map $h \mapsto gh$, and dL_g is its derivative. The brackets in this Lie algebra are given by the vector field brackets $[X,Y](f) = X\big(Y(f)\big) - Y\big(X(f)\big)$.*

This Lie algebra is denoted $\mathfrak{g} = \mathrm{Lie}(G)$ and is called the Lie algebra of G.

The relation between a Lie group and its Lie algebra allows one to study many aspects of the 'wild' group structure of G by studying the more simple structure of its Lie algebra, which is after all a vector space with an additional bracket operation. 'Lie theory' could be viewed as the establishment of this relation. In this section I present some of the basics of this theory.

Definition II.2.38. Let G be a Lie group. A *Lie subgroup* $H \leq G$ is an abstract subgroup of G that is moreover a submanifold.

Theorem II.2.39 (Theorems 2.1 and 2.3 in [39])**.** *Let G be a Lie group.*

1. *Any abstract subgroup of G that is topologically closed in G admits a smooth structure that makes it into a Lie subgroup of G. Put differently, every closed abstract subgroup of G is a Lie subgroup.*

2. *Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. If $H \leq G$ is a Lie subgroup, then $\mathfrak{h} = \mathrm{Lie}(H)$ is a Lie subalgebra of $\mathfrak{g}$. Each Lie subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

Definition II.2.40. Let G, H be two Lie groups. A *Lie group homomorphism* is a smooth group homomorphism $\Phi : G \to H$.

Lemma II.2.41 (Lemma 1.12 in [39]). *Let $\Phi : G \to H$ be a Lie group homomorphism between the Lie groups G and H. Then $d\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

As for abstract groups, quotients by normal Lie subgroups have the structure of Lie groups (see Sections 3 and 4, Chapter 2 in [39]), and the quotient map is a Lie group homomorphism. This is the essential ingredient in the following correspondence.

Corollary II.2.42. *Let G be a Lie group, $N \lhd G$ a normal Lie subgroup. Then $\mathfrak{n} = \mathrm{Lie}(N)$ is an ideal in $\mathfrak{g} = \mathrm{Lie}(G)$.*

The Lie Exponential Map. The most fundamental relation between a Lie group and its Lie algebra is given by the *Lie exponential map*. As for general tangent spaces, picking $X \in \mathfrak{g} = T_{e_G}G$ gives rise to a parametrized curve through e_G in G. In Lie groups, this curve has the structure of a Lie subgroup, and gives rise to a map $\exp : \mathfrak{g} \to G$. I define this map explicitly below. A most significant aspect of the exponential map in the context of symmetric spaces is that it gives rise to the curves, and in particular to the geodesics, of the symmetric spaces. This is one of the corner stones of the relation between symmetric spaces, Lie groups and Lie algebras, and plays a role in virtually every geometric argument in a symmetric space.

Proposition II.2.43 (Proposition 1.13.1 in [28]). *For each $X \in \mathfrak{g} = \mathrm{Lie}(G) = T_eG$ there is a unique Lie group homomorphism $\eta_x : (\mathbb{R}, +) \to G$ with $\dot{\eta}_X(0) = \frac{d}{dt}\eta_X(0) = X$. The image $\eta(\mathbb{R})$ is called a* 1-parameter subgroup *of G.*

Definition II.2.44. For each $X \in \mathfrak{g} = \mathrm{Lie}(G)$, define $\exp(X) = \eta_X(1)$, where η is as in Proposition II.2.43. The map given by $\exp : \mathfrak{g} \to G$ is called the *exponential map of G*.

Due to the significant role geodesics play in this thesis, I give two points of view on the exponential map that help understand geodesics. One view point is that taken in Definition II.2.44: a 1-parameter subgroup is a curve in G. Fixing an element $X \in \mathfrak{g}$ is like fixing a 1-dimensional subspace of the vector space $\mathfrak{g}$ together with an $\mathbb{R}$-parameterization of this subspace (according to the vector X). The 1-parameter subgroup associated to this copy of $(\mathbb{R}, +)$ is indeed a 1-dimensional submanifold in G, i.e., a curve parameterized by $\mathbb{R}$, and the exponential map maps X to the point 'at time 1' on this curve.

From another point of view (see Section 10, Chapter 1 in [46]), if G, H are Lie groups with G simply connected, any Lie algebra homomorphism $\phi : \mathfrak{g} = \mathrm{Lie}(G) \to \mathfrak{h} = \mathrm{Lie}(H)$ induces a Lie group homomorphism $\Phi : G \to H$ such that $d\Phi = \phi$. Consider the space $\mathbb{R}$ as the Lie algebra of the group $(\mathbb{R}, +)$. The vector $1 \in \mathbb{R} = Lie\big((\mathbb{R}, +)\big)$ is in fact the tangent vector at $0 \in (\mathbb{R}, +)$ to the unit speed geodesic $t \mapsto t$. Thinking of $\mathbb{R}$ as a 1-dimensional Lie algebra, one can define a Lie algebra homomorphism from $\mathbb{R}$ to any

other Lie algebra $\mathfrak{g}$ by mapping $1 \in \mathbb{R} = Lie\big((\mathbb{R}, +)\big)$ to any choice of $X \in \mathfrak{g}$, and then extend linearly. Since $\mathbb{R}$ is simply connected, this map induces a map between the Lie groups, which turns out to be $t \mapsto \exp(tX)$.

The exponential map is the basis for the Lie groups-Lie algebras correspondence, but it will play an explicit role only in the discussion about horospherical subgroups in Section II.5.3.

Lie Groups of Noncompact Type. I now aim to explain the notion of compact factors in a Lie group and to present Lie groups of noncompact type. I start with a converse to the above statement about homomorphism between Lie algebras of Lie groups.

Definition II.2.45. Let G, H be two Lie groups. A *local homomorphism* between G and H is a smooth map $f : U \to V$ from an open neighbourhood of $e_G \in G$ to an open neighbourhood of $e_H \in H$, such that for every $a, b \in U$ with $ab \in U$, one has $f(ab) = f(a)f(b)$. A *local isomorphism* is a local homomorphism that is a bijection between the corresponding neighbourhoods.

Theorem II.2.46 (Theorem 1.11, Chapter 2 in [39])**.** *Two Lie groups are locally isomorphic if and only if their respective Lie algebras are isomorphic.*

Definition II.2.47. A connected real semisimple Lie group G is *of noncompact type* if its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is of noncompact type.

Definition II.2.48. A connected real Lie group G is *simple* (*semisimple*) if $\mathrm{Lie}(G)$ is simple (semisimple).

Theorem II.2.49 (Theorem 1.1, Section 1, Chapter 6 in [40])**.** *Let G be a connected real semisimple Lie group of noncompact type, and let $K \leq G$ be a maximal compact subgroup. Then G is diffeomorphic to the product manifold $K \times \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ is a vector subspace.*

Remark II.2.50. As the notation suggests, the subspace $\mathfrak{p}$ in Theorem II.2.49 is in fact the eigenspace corresponding to eigenvalue -1 in a suitable Cartan decomposition. See Section II.2.3 for some details.

Corollary II.2.51. *In the setting of Theorem II.2.49, $\pi_1(G) = \pi_1(K)$.*

Proposition II.2.52 (Proposition 27, Chapter 3 in [13])**.** *A connected non-trivial Lie group is simple if and only if every normal subgroup of G is either finite or of finite index.*

Remark II.2.53. In particular, a *simple Lie group* does not have to be *abstractly simple*, i.e. it may contain non-trivial normal subgroups.

Definition II.2.54. The *rank* of a real connected semisimple Lie group of noncompact type G is the rank of $\mathfrak{g} = \mathrm{Lie}(G)$. If G has rank greater than 1, G is said to be *of higher rank*.

Proposition II.2.55 (Proposition 28, Chapter 3 in [13])**.** *Let G be a connected, simply connected real Lie group. Then G is semisimple if and only if G is isomorphic to a product $G_1 \times G_2 \times \cdots \times G_n$, where each G_i is a simple Lie group. Each G_i is called a* factor *of G.*

If G is not simply connected, one has:

Theorem II.2.56 (Theorem 4 and Problem 21, Chapter 4, Section 1 in [58])**.** *Let G be a connected semisimple Lie group, and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ the decomposition of* $\mathrm{Lie}(G)$ *into simple Lie algebras. Let $G_i \leq G$ be the unique connected Lie subgroup with* $\mathrm{Lie}(G_i) = \mathfrak{g}_i$*. Then $G = G_1 \cdots G_n$, and for each $i \in \{1, \ldots, n\}$ the intersection $G_i \cap (G_1 \cdots G_{i-1} G_{i+1} \cdots G_n)$ is discrete. Every normal Lie subgroup of G is a product of some of the G_i.*

For groups with finite centre but not simply connected, Theorem II.2.56 gives rise to a decomposition analogous to the one given by Theorem II.2.55.

Corollary II.2.57. *Let G be a connected semisimple Lie group with finite centre, and $G_1, \ldots, G_n$ the subgroups given by Theorem II.2.56. Then there is a finite group $F \leq G_1 \times \cdots \times G_n$ such that*

$$1 \longrightarrow F \longrightarrow G_1 \times \cdots \times G_n \longrightarrow G \longrightarrow 1$$

By a slight abuse of terminology, each of the G_i is called a simple factor*, or* factor*, of G.*

Proof. Since G has finite centre and the $\{G_i\}_{i=1}^n$ pairwise commute, it is clear that the centre $Z(G_i)$ of each G_i is finite. Consider the multiplication map $m : G_1 \times \cdots \times G_n \to G$ given by $m\big((g_1, \ldots, g_n)\big) \mapsto g_1 \ldots g_n$. By Theorem II.2.56, this is a surjective Lie homomorphism. It is straightforward that the kernel of this map is contained in the product $Z(G_1) \times \cdots \times Z(G_n)$. Therefore $F = \mathrm{Ker}(m)$ is finite, proving the corollary. □

Corollary II.2.58. *A connected real semisimple Lie group with finite centre G is of noncompact type if and only if every simple factor of G is not compact.*

Proof. See Remark II.2.28. □

The Adjoint Representation. I end with the definition of the important *adjoint representation* of a Lie group on its Lie algebra.

Definition II.2.59. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. The *adjoint representation* of G is the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ constructed as follows.

For $g \in G$, let $i_g : G \to G$ be the conjugation map $x \mapsto gxg^{-1}$. This is a smooth map, so a Lie group homomorphism. The derivative at the identity element $di_g : T_eG \to T_eG$ is a Lie algebra automorphism of $\mathfrak{g}$. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is defined to be $g \mapsto di_g$.

Remark II.2.60. Assume G has trivial centre. The adjoint representation is then faithful, and G is naturally identified with $\mathrm{Ad}(G)$. An element $g \in G$ is called *semisimple* if it is diagonalizable, when viewed as the linear transformation $\mathrm{Ad}(g)$ of $\mathfrak{g}$. If $G = \mathrm{Isom}(X)$ for a symmetric space of noncompact type, then g is semisimple in the algebraic sense if and only if g acts on X as an *elliptic* or *hyperbolic* element - see Proposition II.5.57.

II.2.3 Symmetric Spaces and Lie Groups Correspondence

In this section I draw some of the relations between a symmetric space of noncompact type X, the Lie group $G = \mathrm{Isom}(X)^\circ$ and the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The basic correspondence between Lie groups and symmetric spaces is given by the following proposition.

Proposition II.2.61 (Proposition 1.2.6 in [54])**.** *Let G be a connected, simple Lie group of noncompact type and with finite centre. Then G has a maximal compact subgroup K (unique up to conjugacy), and G/K is a simply connected, irreducible symmetric space of noncompact type. Furthermore, G/K has non-positive sectional curvature and is not flat.*

Conversely, any noncompact, non-flat, irreducible symmetric space is of the form G/K, where G is connected, noncompact, simple Lie group with trivial centre, and K is a maximal compact subgroup of G.

The form of this correspondence for semisimple Lie groups is easily derived from the simple case above:

Proposition II.2.62 (See Section 3, Chapter 4 in [39])**.** *Let X be a symmetric space of noncompact type and fix $x_0 \in X$. Then $G = \mathrm{Isom}(X)^\circ$ has the structure of a real connected Lie group. The subgroup $K = \mathrm{Stab}_G(x_0)$ is compact, and G/K is analytically diffeomorphic to X by the map $g \mapsto gx_0$. On the other hand, any symmetric space is obtained this way, namely as the quotient of a connected real Lie group by a closed subgroup $K \leq G$ with $Ad(K)$ compact.*

More concretely, given a symmetric space X of noncompact type, the group $G := \mathrm{Isom}(X)^\circ$ is a semisimple Lie group with X diffeomorphic to the space G/K for $K \leq G$ a maximal compact subgroup. The irreducible de Rham factors of X correspond to the simple factors of G. Up to some minor adaptations, the metric on X is the metric induced by the Killing form on the corresponding Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ (see Proposition II.2.66).

In the previous sections I defined the notion of *noncompact type* separately for symmetric spaces and for Lie groups. The above correspondence implies that the notions are compatible.

Proposition II.2.63 (Theorem 2.1.1 in [28]). *Let X be a symmetric space and $G = \mathrm{Isom}(X)^\circ$ the corresponding Lie group. Then X is of noncompact type if and only if G is of noncompact type.*

In some places (e.g. Chapter 9 in [39]), this is taken as the definition of a noncompact type symmetric space.

Fix $x_0 \in X$ and corresponding $K \leq G$. The geodesic symmetry at x_0 is an isometry $s_0 \in G$. Its derivative ds_0 is a Cartan involution on $\mathfrak{g}$, giving rise to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $K = \exp(\mathfrak{k})$. This gives rise to the diffeomorphism of G and $K \times \mathfrak{p}$ depicted in Theorem II.2.49. This diffeomorphism is compatible with the diffeomorphism of X and G/K, and so X is diffeomorphic to $\mathfrak{p}$. This diffeomorphism is given by the Riemannian exponential map (which I do not define), and in particular, $\mathfrak{p}$ is identified with the tangent space $T_{x_0}X$.

From this viewpoint, geodesics of X are the orbits of 1-parameter subgroups of G coming from $X \in \mathfrak{p}$. This extends to flats. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, and set $A = \exp(\mathfrak{a})$. The maximal flats through gx_0 are orbits of x_0 by the cosets $\{gA \mid g \in G\}$. This allows to relate flats - geometric objects - to the algebraic structure of G, and in this way to get a grip on the action of G on X by isometries. For example, it implies that G acts transitively on the set of maximal flats of X. The group gAg^{-1} acts by Euclidean translations on the flat $F = gAx_0$. As geometric objects flats are preserved by isometries. An element $k \in K$ must fix x_0, thus it permutes the set of flats through x_0. I present one proposition that makes a part of the discussion above formal. Namely, that a flat is necessarily an orbit of an abelian subgroup. The proof is a result of the curvature tensor formula (Theorem 12.2, Section 12, Chapter 1 in [39]) and the definition of the Lie algebra of G in terms of vector fields (see Theorem II.2.37). I begin with an identification of the Riemannian inner product at a point x_0 and an inner product on $\mathfrak{p}$ that is essentially the restriction of the Killing form.

Definition II.2.64. Fix $x_0 \in X$. The map $x_0 \mapsto gx_0$ is a smooth map between G and X, which induces a map $dx_0 : \mathfrak{g} \to T_{x_0}X$. The kernel of the differential dx_0 is $\mathrm{Lie}(K)$

for $K = \mathrm{Stab}_G(x_0)$. The function $Q(Y,Z) = \langle d_{x_0}(Y), d_{x_0}(Z)\rangle_{x_0}$ is an inner product on $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition associated to the involutive isometry in x_0.

Lemma II.2.65 (Section 2.3 in [28], see also Theorem 8.2.9 in [72]). *In the notation of Definition II.2.64, $\mathfrak{p}$ decomposes into an orthogonal direct sum with respect to Q, $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$, where each $\mathfrak{p}_i$ is irreducible and $Ad(K)$-invariant subspace. Up to rescaling by a positive constant on each factor, Q is just the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{p}$.*

Proposition II.2.66 (Section 2.10 in [28]). *Let X be a symmetric space of noncompact type and of higher rank, $G = \mathrm{Isom}(X)^\circ, \mathfrak{g} = \mathrm{Lie}(G)$. Denote $r = \mathrm{rk}(X) = \mathrm{rk}(G)$.*

1. *If $x \in X$ is a point and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition corresponding to $K = \mathrm{Stab}_G(x)$, then every r-flat in X that contains x is of the form $F = \exp(\mathfrak{a})p$, where $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace.*

2. *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and define $f_p : \mathfrak{a} \to F = \exp(\mathfrak{a})p$ by $f_p(A) = \exp(A)p$. Then f_p is an isometry of $\mathfrak{a}$ onto F with respect to the inner product Q on $\mathfrak{p}$ defined in Definition II.2.64.*

3. *Let F_1 and F_2 be two arbitrary r-flats in X, and $p_1 \in F_1, p_2 \in F_2$ arbitrary points in them. There exists an element $g \in G$ such that $g(p_1) = p_2$ and $g(F_1) = F_2$.*

4. *If $\eta : \mathbb{R} \to F$ is any geodesic, then there is at least one r-flat containing η.*

I will use the first and third items of Proposition II.2.66 in the sequel, but all items are essential to any understanding of the geometry of X and its relation to the algebra of G and $\mathfrak{g}$. I list them here to further convey the idea that $\mathfrak{p}$ can in many aspects be identified with X. For more facts on this relation in the context of parabolic groups, see Section II.5.3.

Lastly, in the context of measured spaces one needs to relate the volume form on X to the Haar measure on G. This is the content of the following proposition, adapted from corollary B.1.7 in [7] using the fact that semisimple Lie groups of noncompact type are unimodular.

Proposition II.2.67 (see Corollary B.1.7 and Section B.1 in [7]). *Let G be a semisimple Lie group of noncompact type, $K \leq G$ a maximal compact subgroup. There is a G-invariant volume form on G/K, unique up to rescaling. On the other hand, let μ_X be a measure on the symmetric space $X = G/K$ that arises from a left-invariant Riemannian metric on X. Then up to rescaling, the Haar measure μ_G on G admits $\mu(UK) = \mu_X(U/K)$ for every measurable $U \subset G$.*

II.2.4 Algebraic Groups

I follow the concise and simple presentation given in [8], that contains most of what is needed for this thesis. Some intuition regarding nilpotent subgroups of algebraic groups would be useful in the context of Section II.5.3. For that part I consulted Borel's book [11] *Linear Algebraic Groups*.

Definition II.2.68. A *complex linear algebraic group* $\mathbf{G}$ is a subgroup of $\mathrm{GL}(V)$ for some finite dimensional complex vector space V, that is also the set of zeros of a family of polynomial functions $\mathcal{P}$ on V:

$$\mathbf{G} = \{g \in \mathrm{GL}(V) \mid P(g) = 0 \text{ for all } P \in \mathcal{P}\}$$

For a subfield $k \leq \mathbb{C}$, a group $\mathbf{G}$ is said to be *defined over* k if there exists a basis $\mathcal{B}$ of V and such that $\mathcal{P}$ can be chosen to have coefficients in k. I sometime use the notation $\mathbf{G}_k$ to emphasize the fact that the group $\mathbf{G}$ is defined over k. The group $\mathbf{G}_k$ together with a choice of a suitable basis $\mathcal{B}$ and polynomials $\mathcal{P}$ defined over k is called *k-form of* $\mathbf{G}$, or a k-structure on $\mathbf{G}$.

For any field $k \leq \mathbb{C}$ one can view V as a k-vector space V_k. Let $\mathbf{G}$ be defined over k for a basis $\mathcal{B}$. Under this basis, identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{C})$. The fact that the set of polynomials $\mathcal{P}$ is also defined over k for the basis $\mathcal{B}$ means that the group of k-rational points $G(k) = \mathbf{G} \cap \mathrm{GL}(n, k)$ is a closed non-trivial group. This group is called the *k-rational points* of $\mathbf{G}$.

Definition II.2.69. Let $\mathbf{G}, \mathbf{H}$ be two algebraic groups defined over a field $k \leq \mathbb{C}$. A *k-algebraic homomorphism* is a group homomorphism $\phi : \mathbf{G} \to \mathbf{H}$ which is also a morphism of algebraic varieties defined over k.

In this thesis k will be either $\mathbb{R}$ or $\mathbb{Q}$. If $k = \mathbb{Q}$ one is also interested in

$$G(\mathbb{Z}) = \mathbf{G} \cap GL(n, \mathbb{Z})$$

This group is called the *integer points* of $\mathbf{G}_{\mathbb{Q}}$. In this context it is worth to mention that if $k \leq k'$ then clearly every k-structure is a k'-structure. In particular for a $\mathbb{Q}$-group $\mathbf{G}$ one may consider its real points $G(\mathbb{R}) = \mathbf{G}_{\mathbb{Q}} \cap GL(n, \mathbb{R})$. Clearly $G(\mathbb{Z})$ is a discrete subgroup of $G(\mathbb{R})$ with the induced Euclidean topology.

For a Lie group G with trivial centre, the adjoint representation is a Lie group isomorphism of G into an algebraic group $\mathrm{GL}(\mathfrak{g})$. Thus such G can be assumed to be an algebraic group. Notice that the adjoint representation is only a Lie group homomorphism and not necessarily an *algebraic homomorphism*. But still the image group $\mathrm{Ad}(G)$

is an algebraic group as a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. Consequently one can think of the group G in Theorem II.1.5 as a real linear algebraic group.

In the theory of lattices in Lie groups, the great importance of $\mathbb{Q}$-structures comes from two celebrated theorems relating lattices to *arithmetic subgroups*: the Borel and Harish-Chandra Theorem and the Margulis Arithmeticity Theorem.

Up to minor technicalities, the definition of an arithmetic group is 'the $\mathbb{Z}$ points inside the $\mathbb{R}$ points of a $\mathbb{Q}$-algebraic group'.

Definition II.2.70. Let G be an abstract group. Two subgroups $H, K \leq G$ are called *commensurable* if $[H : H \cap K] < \infty$ and $[K : H \cap K] < \infty$.

Definition II.2.71. Let G be a real algebraic group. A subgroup $\Delta \leq G(R)$ is called an *arithmetic* group if there are a $\mathbb{Q}$-group H and an epimorphism $\phi : H \to \mathrm{Ad}(G)$ defined over $\mathbb{R}$ such that $\ker(\phi)$ is compact and the groups $H(\mathbb{Z})$ and $Ad(\Delta)$ are commensurable.

The Borel and Harish-Chandra Theorem below states that arithmetic groups are lattices. The Margulis arithmeticity (Theorem II.2.88) goes in the other direction, stating that in higher rank, every irreducible lattice in G can be viewed as an arithmetic subgroup for some $\mathbb{Q}$-structure on G.

Theorem II.2.72 (Borel Harish-Chandra, Theorem 7.8 in [12])**.** *Let* $\mathbf{G}$ *be a semisimple algebraic Lie group defined over* $\mathbb{Q}$*. Then* $G(\mathbb{R})/G(\mathbb{Z})$ *has finite Haar measure, i.e,* $G(\mathbb{Z})$ *is a lattice in* $G(\mathbb{R})$*.*

The Borel and Harish-Chandra Theorem gives a very useful way for finding lattices in real algebraic Lie groups. It is surely the most, and in some situations the only, practical way. One example among many in the context of this thesis is in the work of Benoist and Miquel [8] in the course of proving Theorem II.5.5. The basic definition for lattices, as well as the statement of the Margulis Arithmeticity Theorem, are found in Section II.2.5.

Another fundamental concept of algebraic groups is the Zariski topology.

Definition II.2.73. Let $\mathbf{G}_k$ be an algebraic k-group. An *algebraic set* is the set of zeros in G of some set of polynomials defined over k.

Definition II.2.74. Let $\mathbf{G}_k$ be a k-algebraic group. The *Zariski topology* on $\mathbf{G}_k$ is the topology that one gets by declaring algebraic sets to be closed. A subgroup of $\mathbf{G}_k$ is *Zariski dense* if it is dense in the Zariski topology.

A useful way to think about Zariski density is by the following property: A subgroup $H \leq G$ is Zariski dense if and only if every polynomial that vanishes on H must vanish on G. The Zariski topology is very different from the topology induced from $GL(V)$. In particular, the following theorem is important:

Theorem II.2.75 (Borel Density Theorem [10]). *Let G be a real algebraic Lie group without compact factors, $\Gamma \leq G$ a lattice. Then Γ is Zariski dense.*

Up to now I defined, in a limited setting, the most basic notions of algebraic groups. Next, I aim to present the definition of a *horospherical subgroup* of an algebraic group, which plays a key role in this thesis. The definition requires only elementary definitions from linear algebra.

Definition II.2.76. Let $\mathbf{G}$ be an algebraic group. An element $u \in \mathbf{G}$ is *unipotent* if $(u - Id)$ is nilpotent. A *unipotent* subgroup U of G is a subgroup whose elements are all unipotent. The *unipotent radical* of G is the maximal normal unipotent subgroup of G.

Remark II.2.77. The notions of a unipotent element and a unipotent subgroup are preserved under algebraic homomorphisms. See Theorem 4.4 in [11].

Definition II.2.78. A *minimal parabolic* subgroup $P \leq G$ is the normalizer of a maximal unipotent subgroup of G. A *parabolic* subgroup is any group containing a minimal parabolic subgroup. A parabolic group is *non-trivial* if it does not equal G.

Definition II.2.79. A *horospherical subgroup* U of G is the unipotent radical of a non-trivial parabolic subgroup.

The typical examples of the above definitions are presented and dealt with in Section II.5.3. Finally, I establish that there is a well defined notion of a *k-rank* for a k-algebraic group.

Definition II.2.80. Let $\mathbf{G}$ be an algebraic k-group. A *k-split torus* of $\mathbf{G}$ is a subgroup that is k-isomorphic to the group $(k^\times)^n$ for some $n \in \mathbb{N}$.

Theorem II.2.81 (Theorem 20.9 in [11]). *Let $\mathbf{G}$ be an algebraic k-group. Then all maximal k-split tori of $G(k) = \mathbf{G}_k \cap GL(n,k)$ are conjugated by an element of $G(k)$.*

From this one gets a well defined notion of the *k-rank* of a k-algebraic group.

Definition II.2.82. Let $\mathbf{G}_k$ be a k-algebraic group. The *k-rank* of $\mathbf{G}_k$ is the dimension of a maximal split tori in $G(k) = \mathbf{G}_k \cap GL(n,k)$.

Both the $\mathbb{R}$-rank and the $\mathbb{Q}$-rank of an algebraic Lie group play a role in the geometry of the corresponding symmetric space and in the geometry of its arithmetic quotients. This is described in Section II.5.2.

II.2.5 Generalities and Classification of Lattices

II.2.5.1 Discrete Subgroups of Lie Groups

I start with the most irregular notion in the statement of Theorem II.1.5: the irreducibility of a general discrete subgroup $\Lambda \leq G$.

Definition II.2.83 (Section 2.4.1 in [8])**.** Let G be a semisimple Lie group without compact factors and with finite centre. A discrete subgroup $\Delta \leq G$ is called *irreducible* if the intersection $\Delta \cap N$ is finite for every normal subgroup $N \leq G$.

It is known that for lattices many of the different notions for irreducibility coincide (see e.g. Section 2.B in [18]), while for general subgroups this is not the case. In this thesis however the irreducibility of Λ is needed only in order to satisfy the relevant hypothesis in the Benoist-Miquel criterion Theorem II.5.5. The reader should not be worried about other non-equivalent definitions.

I now turn to the more standard objects in Theorem II.1.5.

Definition II.2.84. Let G be a Lie group, μ_G a Haar measure. A discrete subgroup $\Gamma \leq G$ is called a *lattice* if $\Gamma\backslash G$ carries a finite G-invariant measure. A lattice is *uniform* if $\Gamma\backslash G$ is compact, and *non-uniform* otherwise. Equivalently, Γ is a lattice if and only if there exists a measurable set $U \subset G$ such that:

1. $\Gamma U = G$.
2. $\mu_G(U) < \infty$.

Remark II.2.85. The existence of a set $U \subset G$ as in the above definition assures the existence of a *fundamental domain* (recall Definition II.2.9) for Γ in G. See Chapter 1 in [65].

There is a basic correspondence between lattices in Lie groups and lattices in the corresponding symmetric spaces (see Definitions II.2.8, II.2.9).

Proposition II.2.86 (Exercise 5.65 in [25])**.** *Let G be a connected semisimple Lie group without compact factors, $K \leq G$ a maximal compact subgroup, μ_X the volume form on the symmetric space $X = G/K$ that is given by the left invariant metric. A discrete subgroup $\Gamma \leq G = Isom(X)^\circ$ is a lattice in X if and only if it is a lattice in G.*

I sketch a proof, see Section 5.6.4 in [25] for details.

Proof. Let μ_G be the Haar measure on G induced from μ_X as in Proposition II.2.67. Assume Γ is a lattice in X. Then by definition there is a fundamental domain $F = U/K \subset G/K$ for Γ. Since $\Gamma \cdot U/K = G/K$ one sees that $\Gamma \cdot UK = G$. Moreover, by Proposition II.2.67, $\mu_G(UK) = \mu_X(F) < \infty$, and so UK is the required set rendering Γ a lattice.

On the other hand, assume there is a fundamental domain U_G for Γ in G that is of finite measure. Since Γ is discrete, there exits a fundamental domain $F = U/K \subset X$ (U a set of representative for F) for the Γ-action on X (Lemma 5.62 in [25]). a priori, F may not have finite μ_X measure. But, as in the previous paragraph, $\mu_X(F) = \mu_G(UK)$, so it is enough to show that $\mu_G(UK) < \infty$. Every two fundamental domains for the Γ-action on G have the same measure (Exercise 5.63 in [25]), so it is enough to show that UK is a fundamental domain for Γ in G. Since F is a fundamental domain for Γ in X, there are $\gamma \in \Gamma, u \in U$ for which $\gamma gK = uK$, and in particular $\gamma g \in UK$. It is a simple exercise to see that such γ is unique, because there is a unique γ with $\gamma gK \in F = U/K$. Therefore UK is indeed a fundamental domain for Γ in G, hence $\mu_X(F) = \mu_G(UK) = \mu_G(U_G) < \infty$ and Γ is a lattice in X. □

As an immediate corollary of Proposition II.2.86 one has:

Corollary II.2.87. *Let X be a symmetric space, and $G = Isom(X)^\circ$ the corresponding Lie group. Then Γ is a lattice of X if and only if it is a lattice of G.*

II.2.5.2 $\mathbb{Q}$-rank 1 Lattices

The $\mathbb{Q}$-rank of the $\mathbb{Q}$-algebraic group G plays a role in the algebraic and geometric structure of an arithmetic lattice in G. As in the case of real rank, the major distinction is between groups with $\mathbb{Q}$-rank 1 and those with $\mathbb{Q}$-rank greater than 1. The case of $\mathbb{Q}$-rank 1 groups is much more simple, and these lattices admit a very well understood 'compact core' in their action on the corresponding symmetric space. In a nutshell, the structure of such lattices resembles that of non-uniform lattices in $\mathbb{R}$-rank 1 groups. This structure is described in detail in Section II.5.2. It is a major part of the proof of Proposition II.5.138 which is a key step in Theorem II.5.1.

At this point, I only wish to say what is needed in order to prove Lemma II.2.103. I begin with the celebrated Margulis Arithmeticity Theorem.

Theorem II.2.88 (Margulis Arithmeticity Theorem, Theorem 1 in [52])**.** *Let G be a connected semisimple real algebraic Lie group of noncompact type. If $\mathrm{rk}(G) > 1$ then every irreducible non-uniform lattice is arithmetic.*

As mentioned in Section II.2.4, a real algebraic group does not have a canonical $\mathbb{Q}$-structure. The Margulis Arithmeticity Theorem allows one to make the following definition.

Definition II.2.89. Let (G, Γ) be as in Theorem II.2.88. *The $\mathbb{Q}$-structure of* (G, Γ) (or the (G, Γ) $\mathbb{Q}$-*structure*) is the (unique) $\mathbb{Q}$-structure $\mathbf{G}_{\mathbb{Q}}$ for which Γ is commensurable to $\mathbf{G}_{\mathbb{Q}}(\mathbb{Z})$.

The notion of (G, Γ) $\mathbb{Q}$-structure is indeed well defined: there is such a $\mathbb{Q}$-structure by Theorem II.2.88, and it is unique when G is connected (see remark 2.1 in [48]). I will just be interested in the implied fact that the $\mathbb{Q}$-rank of the $\mathbb{Q}$-structure of (G, Γ) is an invariant of (G, Γ). This fact is a straightforward corollary of Theorem C and Corollary D of [48] (see Theorem II.5.95 below).

Remark II.2.90. While as of yet the uniqueness of the $\mathbb{Q}$-structure (G, Γ) is not used in the proofs, it should probably be relevant for establishing my suspicion that Λ is in fact commensurable to Γ, at least in the $\mathbb{Q}$-rank 1 case.

Definition II.2.91. Let $G = \mathbf{G}(\mathbb{R})$ be a connected semisimple real algebraic Lie group without compact factors and with trivial centre. An irreducible non-uniform lattice $\Gamma \leq G$ is said to be *a $\mathbb{Q}$-rank 1 lattice* (or simply of $\mathbb{Q}$-rank 1) if every nontrivial unipotent element of Γ is contained in a unique maximal unipotent subgroup of Γ.

Lemma II.2.92 (Lemma 9.2.5 in [54], Lemma 1.4 and Remark 1.5 in [63])**.** *Let $G = \mathbf{G}(\mathbb{R})$ be a connected semisimple real $\mathbb{Q}$-algebraic Lie group of higher rank and without compact factors, with finite centre. Let Γ an irreducible non-uniform arithmetic lattice of G. The following are equivalent:*

1. *Γ is of $\mathbb{Q}$-rank 1 .*
2. *Every nontrivial unipotent subgroup of Γ is contained in a unique maximal unipotent subgroup of Γ.*
3. *Every nontrivial unipotent $\mathbb{Q}$-subgroup of G is contained in a unique maximal unipotent $\mathbb{Q}$-subgroup of G.*
4. *$\mathbf{G}$ is of $\mathbb{Q}$-rank 1.*

Lemma II.2.92 gives the relation between $\mathbb{Q}$-rank 1 lattices and the $\mathbb{Q}$-rank of an algebraic $\mathbb{Q}$-group. The next statement draws a relation between $\mathbb{Q}$-rank 1 lattices and the $\mathbb{R}$-rank of such a group.

Lemma II.2.93 (Lemma 1.1 in [63]). *Let $G = \mathbf{G}(\mathbb{R})$ be a connected semisimple real algebraic Lie group without compact factors and with trivial centre, and let $\Gamma \leq G$ an irreducible non-uniform lattice. If G has a $\mathbb{R}$-rank 1 factor, then Γ is of $\mathbb{Q}$-rank 1.*

Remark II.2.94. Garland-Raghunathan proved that item 2 of Lemma II.2.92 holds for non-uniform lattices Γ in groups of $\mathbb{R}$-rank 1, proving Lemma II.2.93 in case G is of $\mathbb{R}$-rank 1. See Section 1 in [34].

In view of the Margulis Arithmeticity Theorem it is natural to think of the lattice Γ in its guise as the $\mathbb{Z}$-points of some real algebraic Lie group, and work with the associated $\mathbb{Q}$-structure. The next lemma by Prasad shows that the property of being $\mathbb{Q}$-rank 1 is not effected by this change of perspective.

Definition II.2.95. An element $g \in \mathrm{GL}_n(\mathbb{C})$ is called *net* if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion-free. A subgroup of a $\mathrm{GL}_n(\mathbb{C})$ is called *net* if every element is net.

Lemma II.2.96 (Lemma 1.8 in [63]). *Let G, G' be linear semisimple Lie groups without compact factors and with trivial centre. Let $\Gamma \leq G$ be an irreducible non-uniform $\mathbb{Q}$-rank 1 lattice in G. Let $\Gamma' \leq G'$ be a lattice in G' and $\theta : \Gamma \to \Gamma'$ an isomorphism. Assume Γ and Γ' are both net. Then Γ' is an irreducible non-uniform lattice of $\mathbb{Q}$-rank 1.*

Remark II.2.97. Every lattice in a real linear Lie group admits a finite index subgroup that is net (see Theorem 6.11 in [65]). Passing to a finite index subgroup does not effect any of the conditions I impose on Γ, and therefore throughout this thesis I will simply assume that Γ is net. In particular, it is also assumed to be torsion-free.

II.2.5.3 Kazhdan's Property (T)

Definition II.2.98. Let G be a topological group, (π, H) a unitary representation of G.

1. For a subset $Q \subset G$ and $\epsilon > 0$, a vector $v \in H$ is (Q, ϵ)-*invariant* if

$$\sup_{x \in Q} \|\pi(x)v - v\| < \epsilon \|v\|$$

2. The representation (π, H) *almost has invariant vectors* if it has (Q, ϵ)-invariant vectors for every compact subset $Q \subset G$ and every $\epsilon > 0$.

3. The representation (π, H) *has non-zero invariant vectors* if there exists $0 \neq v \in H$ such that $\pi(g)v = v$ for all $g \in G$.

Definition II.2.99. Let G be a topological group. A subset $Q \subset G$ is a *Kazhdan set* if there exists $\epsilon > 0$ with the following property: every unitary representation (π, H) of G which has a (Q, ϵ)-invariant vector has a non-zero invariant vector.

The group G has *Kazhdan's Property* (T) if G has a compact Kazhdan set.

Lemma II.2.103 requires the well known classification of groups with property (T) (see Theorem 5.6 in [53] for a more general statement).

Theorem II.2.100 (Theorem 3.5.4 in [7])**.** *Let G be a connected semisimple real Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Then G has property (T) if and only if $\mathfrak{g}$ does not admit a simple factor locally isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$.*

The following theorem is another basic fact that will be useful in the formulation of Lemma II.2.103.

Theorem II.2.101 (Theorem 1.7.1 in [7])**.** *Let G be a locally compact group, $\Gamma \leq G$ a lattice. The group G has property (T) if and only if Γ has property (T).*

Remark II.2.102. Property (T) does not depend on the algebraic structure of the group. This can be seen from the definition, since the family of unitary representations does not depend on the algebraic structure of G.

II.2.5.4 Separation Into Cases.

Lemma II.2.103. *Let G be a real semisimple Lie group without compact factors and with trivial centre. Assume that $\Gamma \leq G$ is an irreducible non-uniform lattice. Then either G has property (T) or Γ is of $\mathbb{Q}$-rank 1.*

Proof. If the group G does not have (T), then by Theorem II.2.100 G must admit an $SO(n,1)$ or $SU(n,1)$ factor. By Lemma II.2.93, this means that the $\mathbb{Q}$-structure of (G, Γ) is of $\mathbb{Q}$-rank 1 , and in particular that Γ is a $\mathbb{Q}$-rank 1 lattice. □

Together with the case where Γ is uniform, one has an immediate corollary:

Corollary II.2.104. *Theorem II.1.5 follows from Theorems II.3.1,II.4.1 and II.5.1.*

I fix the general setting for the rest of Chapter II:

Setting II.2.105. G is a semisimple Lie group without compact factors and with trivial centre, $X = G/K$ the corresponding symmetric space, $x_0 = eK$. The group $\Gamma \leq G$ is a lattice, and $\Lambda \leq G$ is a discrete subgroup such that $\Gamma \cdot x_0 \subset \mathcal{N}_u(\Lambda \cdot x_0)$, where u is a monotonically increasing continuous sublinear function.

II.3 Lattices with Property (T)

In this section I prove:

Theorem II.3.1. *In the Setting II.2.105, if $\Gamma \leq G$ is has property (T) then Λ is a lattice.*

The underlying assumption in this thesis is that the function u for which $\Gamma \subset \mathcal{N}_u(\Lambda)$ is sublinear. However in the case Γ has property (T) (and also for uniform lattices, see Section II.4), a slightly stronger result holds.

Definition II.3.2. Let $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be two monotonically increasing functions. Call f *asymptotically smaller* than g if $\limsup \frac{f}{g} \leq 1$. Denote this relation by $f \preceq_\infty g$.

Theorem II.3.3. *Under the assumptions of Theorem II.3.1, there exists $\epsilon = \epsilon(G)$ such that for every $\epsilon' < \epsilon$, the conclusion of Theorem II.3.1 holds also under the relaxed assumption that $u(r) \preceq_\infty \epsilon' r$.*

Clearly Theorem II.3.3 implies Theorem II.3.1. I thank Emmanuel Breuillard for suggesting this generalization. From now and until the end of this section, the standing assumptions are those of Theorem II.3.3.

Lattice Criterion. For groups with property (T) I use a criterion by Leuzinger, stating that being a lattice is determined by the exponential growth rate. The formulation requires a definition.

Definition II.3.4. Given a pointed metric space (X, d_X, x_0), denote:

1. $b_X(r) = |B(x_0, r)|$
2. $b^u_X(r) = \sup_{x \in X} |B(x, r)|$

When a group Δ acts on a pointed metric space X, the orbit $\Delta \cdot x_0$ together with the metric induced from X is a pointed metric space $(\Delta \cdot x_0, d_{X\restriction_{\Delta \cdot x_0}}, x_0)$. In this setting $b_{\Delta \cdot x_0}(r) = |B_X(x_o, r) \cap \Delta \cdot x_0|$. When the action is by isometries, i.e. $\Delta \leq \mathrm{Isom}(X)$, it is straightforward to observe that this quantity does not depend on the centre of the ball, and so $b^u_{\Delta \cdot x_0}(r) = b_{\Delta \cdot x_0}(r)$. The pointed metric spaces of interest in this section are the Γ and Λ orbits in the symmetric space $X = G/K$.

Definition II.3.5. Let X be a symmetric space and $\Delta \leq G = \mathrm{Isom}(X)^\circ$ a subgroup of isometries. The *critical exponent* of Δ is defined to be

$$\delta(\Delta) := \limsup_{r \to \infty} \frac{\log\left(b_{\Delta \cdot x_0}(r)\right)}{r}$$

Remark II.3.6. Throughout this section there is no risk of ambiguity, and I allow myself to ease notation and let $b_\Delta(r) = b_{\Delta\cdot x_0}(r)$, $b^u_\Delta(r) = b^u_{\Delta\cdot x_0}(r)$.

To a semisimple Lie group G one can associate a quantity $\|\rho\|$, where $\rho = \rho(G)$ is the half sum of positive roots in the root system of $(\mathfrak{g}, \mathfrak{a})$ (see Section 2 in [49]).

Theorem II.3.7 (Theorem 2 in [49]). *Let G be a semisimple Lie group without compact factors and with trivial centre. Let Δ be a discrete, torsion-free subgroup of G that is not a lattice. If G has Kazhdan's property (T), then there is a constant $c^*(G)$ (depending on G but not on Δ) such that $\delta(\Delta) \leq 2\|\rho\| - c^*(G)$.*

It is known that the critical exponent of a discrete subgroup $\Delta \leq G$ is bounded above by $2\|\rho\|$ (see Section 2.2 in [49]). Moreover, every lattice $\Gamma \leq G$ admits $\delta(\Gamma) = 2\|\rho\|$ (Example 2.3.5 in [49], Theorem C in [1]). Combining these facts with Theorem II.3.7 yields:

Theorem II.3.8 (Theorem B in [49]). *Let G be a semisimple Lie group without compact factors and with trivial centre. Let Δ be a discrete, torsion-free subgroup of G. If G has Kazhdan's property (T), then Δ is a lattice iff $\delta(\Delta) = 2\|\rho\|$.*

Line of Proof and the Use of ϵ-Linearity. The proof of Theorem II.3.1 goes by showing that ϵ-linear distortion cannot decrease the exponential growth rate by much. This fact is essentially manifested in a proposition by Cornulier [21], stated here in Proposition II.3.9. This is the only use I make of ϵ-linearity, and the computations involved are straightforward. Theorem II.3.3 is then an immediate consequence of Leuzinger's criterion Theorem II.3.7.

II.3.1 Proof of Theorem II.3.1

In his study on SBE maps, Cornulier proves the following growth discrepancy result.

Proposition II.3.9 (Proposition 3.6 in [21]). *Let X, Y be two pointed metric spaces. Let u be a non-decreasing sublinear function and $p : X \to Y$ a map such that for some $L, R_0 > 0$:*

1. *$|p(x)| \leq \max(|x|, R_0)$, i.e. $p(|x|) \leq |x|$ for all large enough $x \in X$.*
2. *$d_Y\big(p(x), p(x')\big) \geq \frac{1}{L} d_X(x, x') - u(\max\{|x|, |x'|\})$*

Then for all $r > R_0$, $b_Y(r) \geq b_X(r)/b^u_X\big(L \cdot u(r)\big)$.

I need a slightly modified version of Proposition II.3.9:

Proposition II.3.10. *Let X, Y be two pointed metric spaces. Let u be a non-decreasing function that admits $u(r) \preceq_\infty \epsilon r$ for some $\epsilon < 1$, and $p : X \to Y$ a map such that for some $L, R_0 > 0$:*

1. $|p(x)| \leq \max(|x| + u(|x|), R_0)$, *i.e.* $|p(x)| \leq |x| + u(|x|)$ *for all large enough* $x \in X$.
2. $d_Y\big(p(x), p(x')\big) \geq \frac{1}{L} d_X(x, x') - u(\max\{|x|, |x'|\})$

Then for all $r > R_0$,

$$b_Y(r) \geq b_X\big(r - u(r)\big) / b_X^u\Big(L \cdot u\big(r - u(r)\big)\Big)$$

Proof. Repeat *verbatim* the proof for Proposition 3.6. in [21]. □

Corollary II.3.11. *Let $u(r) \preceq_\infty \epsilon \cdot r$ for some $\epsilon < \frac{1}{2}$. Assume that $\Gamma \subset \mathcal{N}_u(\Lambda)$. Then $\delta(\Lambda) \geq (1 - 4\epsilon) \cdot \delta(\Gamma)$.*

Corollary II.3.11 can be formulated in a slightly more general fashion. Using the notation $\delta(W) := \limsup_{r\to\infty} \frac{\log\left(b_W(r)\right)}{r}$ for a general subset W in a general metric space X, the following general version holds:

Corollary II.3.12. *Let (X, x_0) be a pointed metric space, $Y, Z \subset X$ two subsets, and $u(r) \preceq_\infty \epsilon r$ for some $\epsilon < \frac{1}{2}$. Assume that $b_Z^u(r) = b_Z(r)$. If $Z \subset \mathcal{N}_u(Y)$, then $\delta(Y) \geq (1 - 4\epsilon) \cdot \delta(Z)$. Moreover, if u is sublinear, then $\delta(Z) = \delta(Y)$.*

In particular, Corollary II.3.11 holds even when the group G does not have property (T). Corollary II.3.11 follows from Corollary II.3.12 because the fact that Γ is a group of isometries implies that $b_\Gamma^u(r) = b_\Gamma(r)$. Below I prove the more general version Corollary II.3.12.

Remark II.3.13. Restricting to $\epsilon < \frac{1}{2}$ stems from a 2 factor that appears in the proof and could possibly be dropped using a slightly more sophisticated approach. For my needs this is more than enough, since in any case I eventually restrict attention to a small interval around 0.

Proof. Observe that the closest point projection $p_Y : Z \to Y$ defined by $z \mapsto z_y$ for some point in the closed ball $z_y \in \overline{B}\big(z, u(|z|)\big)$ admits:

1. $|y_z| \leq |z| + u(|z|)$
2. $d\big(y_z, y_{z'}\big) \geq d(z, z') - 2u(\max\{|z|, |z'|\})$

The first item follows from triangle inequality: $|y_z| \leq d(y_z, z) + d(z, x_0)$. The second item follows from the quadrilateral inequality, i.e., using triangle inequality twice along the quadrilateral $[z, z', y_{z'}, y_z]$.

The above properties allow me to use Proposition II.3.10 with constant $L = 1$ and function $u' = 2u$ to get

$$b_Y(r) \geq b_Z\big(r - u'(r)\big)/b_Z^u\Big(u'\big(r - u'(r)\big)\Big)$$

Since I assume $b_Z^u = b_Z$, I can omit the superscript u in the last expression. Recalling the definition $\delta(W) = \limsup_{r\to\infty} \frac{b_W(r)}{r}$, it remains to prove:

$$\limsup_{r\to\infty} \frac{1}{r} \cdot \log\left(b_Z\big(r - u'(r)\big)/b_Z\Big(u'\big(r - u'(r)\big)\Big)\right) \geq (1 - 4\epsilon) \cdot \delta(Z)$$

The proof of this inequality involves nothing more than log rules and arithmetic of limits:

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{1}{r} \cdot \log\left(b_Z\big(r - u'(r)\big)/b_Z\Big(u'\big(r - u'(r)\big)\Big)\right) \\
&= \limsup_{r\to\infty} \frac{1}{r} \cdot \left(\log\Big(b_Z\big(r - u'(r)\big)\Big) - \log\left(b_Z\Big(u'\big(r - u'(r)\big)\Big)\right)\right) \\
&\geq \limsup_{r\to\infty} \left[\frac{1}{r} \cdot \log\Big(b_Z\big(r - u'(r)\big)\Big) - \limsup_{s\to\infty} \left[\frac{1}{s}\log\left(b_Z\Big(u'\big(s - u'(s)\big)\Big)\right)\right]\right] \qquad \text{(II.1)} \\
&= \limsup_{r\to\infty} \frac{1}{r} \cdot \log\Big(b_Z\big(r - u'(r)\big)\Big) - \limsup_{s\to\infty} \frac{1}{s}\log\left(b_Z\Big(u'\big(s - u'(s)\big)\Big)\right) \\
&\geq \limsup_{r\to\infty} \frac{1}{r} \cdot \log\Big(b_Z\big(r - 2\epsilon r\big)\Big) - \limsup_{s\to\infty} \frac{1}{s}\log\Big(b_Z\big(2\epsilon s\big)\Big) \\
&= (1 - 2\epsilon)\delta(Z) - 2\epsilon\delta(Z) = (1 - 4\epsilon)\delta(Z)
\end{aligned}$$

Below I justify the steps in the above inequalities:

1. First equality is by rules of log.
2. Second and third inequalities are by arithmetic of limits: let $(a_n)_n, (b_n)_n$ be two sequences of positive numbers, and $A = \limsup_n a_n$, $B = \limsup_n b_n$. Then $\limsup(a_n - b_n) \geq \limsup_n(a_n - B) = A - B$.
3. Fourth inequality: $u'(r) < 2\epsilon(r)$ for all large enough r.
4. Fifth equality: definition of δ.

This completes the proof in the general case, which is what is needed for the proof of Theorem II.3.3. For the more refined statement in the case u is sublinear, one has to show a bit more. From inequality II.1 (specifically from the fourth line of the inequality) it is clearly enough to prove:

1. $\limsup_{r\to\infty} \frac{1}{r} \cdot \log\left(b_Z\big(r - u'(r)\big)\right) = \delta(Z)$.
2. $\limsup_{s\to\infty} \frac{1}{s} \log\left(b_Z\Big(u'\big(s - u'(s)\big)\Big)\right) = 0$.

Starting from the second item, indeed it holds that

$$\frac{1}{s}\log\left(b_Z\Big(u'\big(s-u'(s)\big)\Big)\right) = \frac{u'\big(s-u'(s)\big)}{s} \cdot \frac{\log\left(b_Z\Big(u'\big(s-u'(s)\big)\Big)\right)}{u'\big(s-u'(s)\big)}$$

Clearly lim sup of the right factor in the above product is bounded by $\delta(Z)$, and in particular it is uniformly bounded. On the other hand sublinearity of u' implies that the left factor tends to 0. I conclude that this product tends to 0 as s tends to ∞.

It remains to prove $\limsup_{r\to\infty} \frac{1}{r} \cdot \log\left(b_Z\big(r - u'(r)\big)\right) = \delta(Z)$. In a similar fashion,

$$\frac{1}{r}\log\left(b_Z\big(r-u'(r)\big)\right) = \frac{r-u'(r)}{r} \cdot \frac{\log\left(b_Z\big(r-u'(r)\big)\right)}{r-u'(r)}$$

The left factor limits to 1 by sublinearity of u'. The right factor is nearly the expression in the definition of $\delta(Z)$, and I want to prove that indeed taking lim sup of it equals $\delta(Z)$. *A priori* $\{r - u'(r)\}_{r\in\mathbb{R}_{>0}}$ is just a subset of $\mathbb{R}_{>0}$, so changing variable and writing $t := r - u'(r)$ requires a justification. But there is no harm in assuming that u' is a non-decreasing continuous function, hence $\mathbb{R}_{\geq R} \subset \{r - u'(r)\}_{r\in\mathbb{R}_{>0}}$ for some $R \in \mathbb{R}_{>0}$. Therefore for any sequence $r_n \to \infty$ there is a sequence r'_n with $r_n = r'_n - u'(r'_n)$ for all large enough n (note that in particular $r'_n \to \infty$). In the other direction, for every sequence $r'_n \to \infty$ there is clearly a sequence $r_n \to \infty$ for which $r_n = r'_n - u'(r'_n)$. I conclude

$$\limsup_{r\to\infty} \frac{\log\left(b_Z\big(r-u'(r)\big)\right)}{r-u'(r)} = \limsup_{r\to\infty} \frac{1}{r} \cdot \log\big(b_Z(r)\big) = \delta(Z)$$

This completes the proof.

□

Proof of Theorem II.3.3. Define $\epsilon(G) = \frac{c^*(G)}{4\cdot 2\|\rho\|}$, and assume $u(r) \preceq_\infty \epsilon(G)\cdot r$. Notice that $\epsilon(G) < \frac{1}{2}$, and since $\delta(\Gamma) = 2\|\rho\|$ Corollary II.3.11 gives

$$\delta(\Lambda) \geq \big(1 - 4\epsilon(G)\big)\cdot 2\|\rho\| = 2\|\rho\| - 4\epsilon(G)\cdot 2\|\rho\| \geq 2\|\rho\| - c^*(G)$$

By Theorem II.3.7, Λ is a lattice. □

Remark II.3.14. The question of existence of interesting groups that coarsely cover a lattice is a key question that arises naturally in the context of this thesis. In view of the quasi-isometric rigidity results [24, 30, 67], the most basic question is whether there exists groups that are not commensurable to a lattice but that sublinearly, or even ϵ-linearly, cover one.

Assume the setting of Theorem II.3.3, so that $\Gamma \subset \mathcal{N}_u(\Lambda)$ for some function $u(r) \preceq_\infty \epsilon(G)r$. The theorem states that Λ is a lattice, and by Theorem II.3.8 one has $\delta(\Lambda) = 2\|\rho\|$.

The line of proof, and specifically that of Corollary II.3.12, suggest that this growth rate should allow one to prove the existence of another, more restrictive,'smaller' function u' for which $\Gamma \subset \mathcal{N}_{u'}(\Lambda)$.

The idea is simple: suppose for example that for every element $\gamma \in \Gamma$ one has that the closest point projection $\gamma \mapsto \lambda_\gamma$ has the property that $d(\gamma x_0, \lambda_\gamma x_0) > \epsilon'|\gamma|$. This fact, together with the growth rate of Γ, has an effect on the growth rate of Λ. It is plausible to suspect that this would force $\delta(\Lambda)$ to be strictly smaller than $2\|\rho\|$, contradicting the fact that Λ is a lattice. Similar valuations should hold if the bound $d(\gamma x_0, \lambda_\gamma x_0) > \epsilon'|\gamma|$ holds only for a substantial portion of the elements in γ (but not necessarily for all $\gamma \in \Gamma$). With the right definitions one might be able to show that u could always taken to be 'mostly sublinear', in the sense that most points in Γ are sublinearly covered by Λ.

II.4 Uniform Lattices

In this section I prove:

Theorem II.4.1. *In the Setting II.2.105, if Γ is a uniform lattice then Λ is a uniform lattice.*

Also for uniform lattices the stronger form of ϵ-linear rigidity holds, this time with arbitrary $\epsilon < 1$:

Theorem II.4.2. *The conclusion of Theorem II.4.1 holds also under the relaxed assumption that $u(r) \preceq_\infty \epsilon r$ for any $0 < \epsilon < 1$.*

Clearly Theorem II.4.2 implies Theorem II.4.1. From now and until the end of this section, the standing assumptions are those of Theorem II.4.2.

Lattice Criterion. A discrete group is a uniform lattice if and only if it admits a relatively compact fundamental domain (see Chapter 5.6.4 in [25] for basic (and non-basic) facts on fundamental domains). Hence it is immediate that if Γ is uniform and u is bounded, i.e. $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$, then Λ is a uniform lattice.

Line of Proof and Use of ϵ-Linearity. The goal is to show that Γ actually lies inside a bounded neighbourhood of Λ. The proof is by way of contradiction. Arbitrarily large $d(\gamma x_0, \Lambda \cdot x_0)$ readily imply the existence of arbitrarily large balls that do not intersect Λ, but are tangent to a Λ-orbit point (see Figure II.1). The proof goes by showing that certain large Λ-free balls (ones that are tangent to some fixed arbitrary point in X) imply the existence of large Γ-free balls. But the size of such balls is uniformly bounded because Γ is a uniform lattice.

The argument relies on a an extremely simple observation: that Λ-free balls are a Λ-invariant object. Thus if a Λ-free ball is tangent to a Λ-orbit point, this picture can be translated back to the base point of the space. Near the base point, the ϵ-linearity constraint imposed by u is meaningful and could easily be exploited to induce large Γ-free balls.

Remark II.4.3. While the proof is essentially no more than a few lines, its logic is used repeatedly and is the foundation of the much more involved case of $\mathbb{Q}$-rank 1 lattices. The simple idea is that the true power of a sublinear (or ϵ-linear) constraint is only visible 'near' the base point. This is combined with the observation that Λ-invariance allows one to pull a geometric situation back to the base point. The difficulty in the non-uniform case is that X does admit arbitrarily large Γ-free balls. This situation requires a different lattice criteria and some extra work.

II.4.1 Notations and Terminology

Definition II.4.4. Let X be a metric space, $Y, Z \subset X$ two closed subsets of X. The *closest point projection* of Y to Z is the set theoretic map $p_Z : Y \to Z$ defined by $p_Z(y) := z_y$, where $z_y \in Z$ is any point realizing the distance $d(y, Z) = d(y, z)$. If X is a proper metric space and Z discrete, then there are at most finitely many such points. In any case of multiple points, p_Z chooses one arbitrarily.

The particular case of interest from now on is where the metric space is the pointed symmetric space (X, x_0), the two subsets are the orbits $\Gamma \cdot x_0$ and $\Lambda \cdot x_0$, and the projection is $p_{\Lambda \cdot x_0} : \Gamma \cdot x_0 \to \Lambda \cdot x_0$. To ease notation I often denote this projection by p_Λ (there is no risk of ambiguity since the subgroups Λ and Γ are always considered in the context of their respective orbits in X and not in G).

Remark II.4.5. The distance of a point from a set is defined as the infimum $d(x, Y) := \inf_{y \in Y}\{d(x, y)\}$. In particular if Y is closed or discrete and X is proper, as are all cases I consider in this thesis, the distance is realized by a point $y \in Y$ for which $d(x, y) = d(x, Y)$. I use the fact that $d(x, Y)$ is a minimum in some of the proofs, but it is probably not essential.

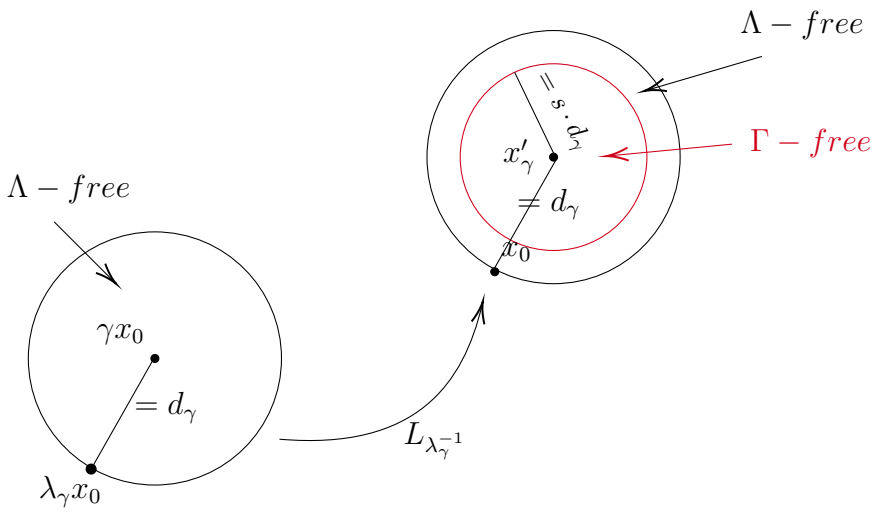


Figure II.1: Basic Setting and Lemma II.4.8. A Λ-free ball about γx_0 of radius d_γ, translated by λ_γ^{-1} to a ball tangent to x_0. The linear ratio between $|x'_\gamma| = d_\gamma$ and the Λ-free radius force the red ball to be Γ-free.

The following definitions will be used repeatedly in both this section and in Section II.5. It mainly fixes terminology and notation of the geometric situation illustrated in Figure II.1.

Definition II.4.6. Let $H \leq G = \mathrm{Isom}(X)^\circ$. A set $U \subset X$ is called *H-free* if $H \cdot x_0 \cap \mathrm{Int}(U) = \emptyset$, where $\mathrm{Int}(U)$ is the topological interior of U. That is, U is called H-free if its interior does not intersect the H-orbit $H \cdot x_0$.

Definition II.4.7. Denote $P_\Lambda(\gamma x_0) = P_\Lambda(\gamma) = \lambda_\gamma x_0$.

1. $d_\gamma := d(\gamma x_0, \lambda_\gamma x_0)$.
2. $B_\gamma := B(\gamma x_0, d_\gamma)$. It is a Λ-free ball centred at γx_0 and tangent to $\lambda_\gamma x_0$.
3. $x'_\gamma := \lambda_\gamma^{-1} \gamma x_0$. Denote $|x'_\gamma| = d_\gamma$.
4. $B'_\gamma := \lambda_\gamma^{-1} B_\gamma = B(x'_\gamma, d_\gamma)$. It is Λ-free as a Λ-translate of the Λ-free ball B_γ, and is tangent to x_0.
5. For $s \in \mathbb{R}_{>0}$ and a ball $B = B(x, r)$, denote $sB := B(x, sr)$, the rescaled ball with same centre and radius sr.
6. For a sequence γ_n, denote by $\lambda_n, d_n, B_n, B'_n, x'_n$ the respective $\lambda_{\gamma_n}, d_{\gamma_n}$, etc.

II.4.2 Proof of Theorem II.4.2

Lemma II.4.8. *Let $x \in X$. There exists $S = S(x,u) \in (0,1)$ such that for every $s \in (0,S)$ there is $R = R(s,S)$ such that if $r > R$ and $B = B(y,r)$ is a Λ-free ball tangent to x, then sB is Γ-free.*

In particular, the existence of arbitrarily large Λ-free balls that are all tangent to a fixed point $x \in X$ implies the existence of arbitrarily large Γ-free balls.

Proof. The proof is more easily read if one assumes $x = x_0$ and $u(r) = \epsilon r$ so I begin with this case. Assume $B = B(y,R)$ is Λ-free for some $y \in X$. The assumption $x = x_0$ means that $|y| = d(y,x_0) = r$. Assume that for a given $s \in (0,1)$, the ball sB intersects $\Gamma \cdot x_0$. This gives rise to an element $\gamma \in \Gamma$ such that:

1. $|\gamma| = d(\gamma x_0, x_0) \leq d(y,x_0) + d(\gamma x_0, y) = (1+s)r$ (triangle inequality). In particular, $d(\gamma x_0, \Lambda \cdot x_0) \leq \epsilon(1+s)r$
2. $B\big(\gamma x_0, (1-s)r\big) \subset B(y,r)$, so it is Λ-free.

I conclude that for s for which $sB \cap \Gamma \cdot x_0 \neq \emptyset$, one has the inequality $(1-s)r \leq \epsilon(1+s)r$, i.e. $\frac{1-s}{1+s} \leq \epsilon$. The number ϵ is fixed and smaller than 1, while $\frac{1-s}{1+s}$ limit to 1 monotonically from below as $s > 0$ tend to 0. I conclude that there is a segment $(0,S) \subset (0,1)$ such that for all $s \in (0,S)$, sB is Γ-free.

Assume now that $x \neq x_0$ and $u(r) \preceq_\infty \epsilon r$. As above, if $\gamma x_0 \in B(y,sr)$ then it is the centre of a Λ-free ball of radius $(1-s)r$, and so $(1-s)r \leq u(|\gamma|)$. I wish to use the ϵ-linear bound on u as I did before, only this time u is only asymptotically smaller than ϵr. To circumvent this I just need to show that $|\gamma|$ is large enough. Indeed since $B\big(\gamma x_0, (1-s)r\big)$ is Λ-free it does not contain $x_0 \in \Lambda \cdot x_0$ and in particular $(1-s)r \leq d(x_0, \gamma x_0) = |\gamma|$. For some $R_1(s) = R_1(s,u)$ one therefore has for all $r > R_1(s)$

$$(1-s)r \leq u(|\gamma|) \leq \epsilon|\gamma|$$

On the other hand $|y| \leq d(x,y) + d(x,x_0) = r + |x|$, and consequently $|\gamma| \leq (1+s)r + |x|$. For $r > R_1(s)$ one has

$$(1-s)r \leq u(|\gamma|) \leq \epsilon|\gamma| \leq \epsilon\big((1+s)r + |x|\big)$$

This means that s for which $\Gamma \cdot x_0 \cap B(y,sr) \neq \emptyset$ must admit, for all $r > R_1(s)$,

$$\frac{1-s}{1+s+\frac{|x|}{r}} = \frac{(1-s)r}{(1+s)r + |x|} \leq \epsilon < 1 \tag{II.2}$$

The rest of the proof is just Calculus 1, and concerns with finding $S = S(x,\epsilon,u) \in (0,1)$ so that for any $s \in (0,S)$ there is $R(s)$ such that all $r > R(s)$ satisfy

$$\epsilon < \frac{1-S}{1+S+\frac{|x|}{r}} \leq \frac{1-s}{1+s+\frac{|x|}{r}} \tag{II.3}$$

The lemma readily follows from inequalities II.2,II.3.

Explicitly, fix $\epsilon' > \epsilon$. As before, monotonic approach of $\frac{1-s}{1+s}$ to 1 allows to fix $S \in (0,1)$ for which $\epsilon < \epsilon' < \frac{1-s}{1+s}$ for all $s \in (0,S)$. Next note that for any fixed $s \in (0,S)$, $\lim_{r\to\infty} \frac{1-s}{1+s+\frac{|x|}{r}} = \frac{1-s}{1+s}$, and that the approach in monotonically increasing with r. Since $\epsilon < \epsilon'$, this limit implies that for some $R_2 > R_1(S)$, all $r > R_2$ admit $\epsilon < \frac{1-S}{1+S+\frac{|x|}{r}}$. Finally notice that for any fixed r the function $\frac{1-s}{1+s+\frac{|x|}{r}}$ is again monotonically increasing as s tends to 0 from above. Therefore inequality II.3 holds for every $s \in (0,S)$ and all $r > R_2(S)$ (capital S is intentional and important).

To conclude the proof, notice that if moreover $r > R_1(s)$ (again lowercase s is intentional and important) then inequalities II.2,II.3 both hold. This means that for any $s \in (0,S)$ there is $R(s) := \max\{R_1(s), R_2(S)\}$ such that $r > R(s) \Rightarrow B(y,sr)$ is Γ-free. The constants $R_1(s), R_2(S)$ have the desired dependencies, hence so does $R(s)$, proving the lemma. □

Corollary II.4.9 (Theorem II.4.2)**.** *There is a uniform bound on $\{d_\gamma\}_{\gamma\in\Gamma}$, i.e., $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$. In particular, Λ is a uniform lattice.*

II.5 $\mathbb{Q}$-rank 1 Lattices

In this section I prove:

Theorem II.5.1. *In the Setting II.2.105, assume further that Γ is an irreducible non-uniform $\mathbb{Q}$-rank 1 lattice. In case G is of higher rank, assume further one of the following:*

1. *If $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$, assume that $\Lambda \subset \mathcal{N}_u(\Gamma)$.*
2. *Else if $\Gamma \not\subset \mathcal{N}_D(\Lambda)$ for any $D > 0$, assume that Λ is irreducible.*

Then Λ is a $\mathbb{Q}$-rank 1 lattice.

The standing assumption for this section is that Γ is an irreducible $\mathbb{Q}$-rank 1 lattice.

II.5.1 Strategy

Lattice Criteria.

Theorem II.5.2 (Section 10.4 in [67]). *If u is bounded, i.e. $\Gamma \subset \mathcal{N}_D(\Lambda)$ and $\Lambda \subset \mathcal{N}_D(\Gamma)$, then Λ is a lattice commensurable to Γ.*

Remark II.5.3. The proof of Schwartz for Theorem II.5.2 is given in [67] for $\mathbb{R}$-rank 1 groups, but the same proof works for general $\mathbb{Q}$-rank 1 lattices. See Section II.5.5.1 for a discussion and proof sketch of this result.

It is not so hard to see that under the assumptions of Theorem II.5.1, $\Gamma \subset \mathcal{N}_D(\Lambda)$ implies $\Lambda \subset \mathcal{N}_{D'}(\Gamma)$ (see Proposition II.5.154). The proof of Theorem II.5.1 is therefore reduced to the case where $\Gamma \not\subset \mathcal{N}_D(\Lambda)$, for any $D > 0$. In that case, the lattice criteria I use for $\mathbb{Q}$-rank 1 lattices differ depending on the $\mathbb{R}$-rank of G. For $\mathbb{R}$-rank 1, I use a theorem by Kapovich-Liu, and in higher rank a theorem of Benoist-Miquel resolving a conjecture of Margulis.

Theorem II.5.4 (Theorem 1.5 in [43]). *Let X be a $\mathbb{R}$-rank 1 symmetric space. A discrete subgroup $\Delta \leq \mathrm{Isom}(X)$ is geometrically infinite if and only if the set of non-conical limit points has the cardinality of the continuum.*

Theorem II.5.5 (Theorem 2.16 in [8]). *Let G be a semisimple real algebraic Lie group of real rank at least 2 and U be a non-trivial horospherical subgroup of G. Let Δ be a discrete Zariski dense subgroup of G that contains an indecomposable lattice Δ_U of U. Then Δ is a non-uniform irreducible arithmetic lattice of G.*

Remark II.5.6. For the definition of an indecomposable horospherical lattice, see Definition II.5.127. See Definition II.5.73 for a definition of a geometrically infinite group. Theorem II.5.4 is originally formulated in [43] with the hypothesis that X has pinched negative curvature. The well known fact that every symmetric space of $\mathbb{R}$-rank 1 has pinched negative curvature is proved in [37], see corollary to Proposition 4 therein.

Outline of Proof and Use of Sublinearity. Lattices of $\mathbb{Q}$-rank 1 admit concrete geometric structure. This structure is manifested in the geometry of an orbit of such a lattice in the corresponding symmetric space $X = G/K$. One important geometric property is the existence of a set of horospheres which the orbit intersects in a (metric) cocompact lattice. Furthermore, the orbit does not intersect the interior of the horoballs bounded by these horospheres.

The core of the proof is in showing the existence of horospheres in X which a Λ-orbit intersects in a metric lattice. This is proved purely geometrically, using the geometric

structure of $\mathbb{Q}$-rank 1 lattices and the sublinear constraint. Together with some control on the location of these horospheres, I prove two statements:

1. $\Lambda \cdot x_0$ intersects a horosphere $\mathcal{H} \subset X$ in a cocompact lattice (Proposition II.5.138).

2. Every Γ-conical limit point is also a Λ-conical limit point (Corollary II.5.153).

The $\mathbb{R}$-rank 1 case of Theorem II.5.1 follows directly by using Theorem II.5.4 of Kapovich-Liu and the second item above. The higher rank case requires a bit more, namely to deduce the hypotheses of the Benoist-Miquel Theorem II.5.5. I use a well known geometric criterion (Lemma II.5.130) in order show that Λ is Zariski dense, and a lemma of Mostow (Lemma II.5.105) to show that Λ intersects a horospherical subgroup in a cocompact lattice.

Outline for Section II.5. The $\mathbb{Q}$-rank 1 case requires a considerable amount of background, both on the geometry of symmetric spaces and on the algebraic objects that come into play. Section II.5.2 contains a presentation of various basic geometric aspects of symmetric spaces. I include all the objects and properties that I use, with the hope that this presentation can serve as a comprehensive and self-contained view point on symmetric spaces. However I do not give proofs or elaborate explanations for well known facts. I do elaborate on more advanced properties and on the adaptations that are required specifically for this thesis.

Still on the geometric side, I then describe the fundamental features of $\mathbb{Q}$-rank 1 lattices. These features are more advanced and I present them at some length, even though they are not new. Section II.5.3 is still somewhat of a preliminary, and is devoted to the translation of the geometric results of Section II.5.4 to the algebraic language required in Theorem II.5.5. As it is, it contains some non-trivial translation work which I prove in full.

Section II.5.4 is the core of the original mathematics of this thesis. It is devoted to proving the fact that $\Lambda \cdot x_0$ intersects some horospheres in a cocompact lattice. The proof is quite delicate and somewhat involved. For this reason, I include a few figures and a detailed informal overview of the proof. The figures are quite detailed and may take a few moments to comprehend, but I believe they get the geometric message across.

Section II.5.5 deals with the case where $\Gamma \subset \mathcal{N}_D(\Lambda)$, and elaborates on Schwartz's proof for Theorem II.5.2.

Finally, in Section II.5.6 I put everything together for a complete proof of Theorem II.5.1.

II.5.2 Geometry of Symmetric Spaces of Noncompact Type

The proof for lattices of $\mathbb{Q}$-rank 1 is geometric, but it does not use much differential geometry. I list here some very basic Riemannian geometric facts that will be used throughout the presentation of the geometry of symmetric spaces.

From now on and unless explicitly stated otherwise, M denotes a complete smooth and simply connected Riemannian manifold of nonpositive sectional curvature.

Lemma II.5.7. *1. M is diffeomorphic to $\mathbb{R}^n$. This is the Cartan-Hadamard Theorem (see Section 2 in [4]).*

2. The unit tangent space at any point $p \in M$ is compact, that is the set of tangent vectors at p of size 1 is compact.

3. A bi-infinite geodesic η of M is completely and uniquely determined by its tangent vector at any point in time.

4. Let $f : M \to \mathbb{R}$ be a smooth map with no critical points. The sublevel set $f^{-1}(c) \subset M$ is a smooth embedded co-dimension 1 submanifold.

II.5.2.1 The Basic Geometry

In this section, X is a symmetric space of noncompact type (Definition II.2.5) and $G = \mathrm{Isom}(X)^\circ$. Some statements are formulated in the more general setting of complete simply connected Riemannian manifolds of nonpositive sectional curvature.

Since the geometry of $\mathbb{Q}$-rank 1 lattices resembles that of lattices in $\mathbb{R}$-rank 1, the reader could for the most part simply have the image of the hyperbolic plane in mind. If one wishes to see flats that are not geodesics, then a product of two hyperbolic planes is enough. Even the product of the hyperbolic plane and $\mathbb{R}$ is helpful, albeit this space has a Euclidean factor.

An important result in the theory of lattices that is most fundamental to this thesis is the geometric description of the quotient space $\Gamma\backslash X$ for a non-uniform lattice $\Gamma \leq G$ (see Theorem II.5.78 below). This result serves as a focal point for Section II.5.2. My presentation is mainly based on the books of Eberlein [28] and Ballmann-Gromov-Schroeder [4] on nonpositively curved manifolds.

Convex Distance Functions. I begin with the notion of convexity that lies at the the core of nonpositive curvature.

Lemma II.5.8 (Lemma 5.8, Chapter 2 in [17])**.** *For every $\epsilon > 0$, every geodesic segment $\eta : [-\epsilon, \epsilon] \to M$ can be uniquely prolonged indefinitely in both directions.*

Definition II.5.9. Given two points $x, y \in X$, denote by $[x, y]$ the unique geodesic segment joining them.

Definition II.5.10. A function $f : X \to \mathbb{R}$ is *convex* if the functions $f \circ \eta : \mathbb{R} \to \mathbb{R}$ are convex for every bi-infinite geodesic $\eta : \mathbb{R} \to X$.

Convex functions on X play a critical role in the geometry of symmetric spaces. Most importantly, the following characterization of nonpositive curvature allows one to work with symmetric spaces of noncompact type without appealing much to differential geometry.

Definition II.5.11. Let (Z, d_Z) be a metric space. The function d_Z is *locally convex* if every point $z \in Z$ admits a neighbourhood $W \subset Z$ such that for every two geodesics $\eta_1, \eta_2 : [0, \epsilon] \to W$ with $\eta_1(0) = z = \eta_1(0)$, the function $t \mapsto d\big(\eta_1(t), \eta_2(t)\big)$ is convex.

Theorem II.5.12 (Section 1.3 in [4])**.** *Let M be a Riemannian manifold. It is of nonpositive sectional curvature if and only if the distance function on M is locally convex.*

In particular, the distance function on X is locally convex. Other important convex functions on X are the distance functions to convex sets.

Lemma II.5.13. *Let $A \subset M$ be a closed convex set. Then $d_A : M \to \mathbb{R}$ defined by $d_A(p) = d(A, p)$ is a convex function.*

Lemma II.5.13 is an immediate corollary of the following two theorems:

Theorem II.5.14 (Theorem 1.6.1 in [28])**.** *If A is a closed convex subset of M then A is a totally geodesic submanifold of M. It is a complete submanifold if and only the boundary of A is empty.*

Theorem II.5.15 (Theorem 4.1 in [9])**.** *Let $A \subset M$ be a closed totally geodesic submanifold of M. Then the distance to A function $d_A : M \to \mathbb{R}$ is a convex function.*

Proposition II.5.16 (Proposition 1.6.3 in [28])**.** *Let $A \subset M$ be a closed convex subset. Then:*

1. *For each point $p \in M$ there exists a unique point $P_A(p) \in A$ such that $d\big(p, P_A(p)\big) \leq d(p, q)$ for all $q \in A$.*
2. *The* orthogonal projection map *$P_A : M \to A$ is distance non-increasing, i.e. $d\big(P_A(p), P_A(q)\big) \leq d(p, q)$.*

Definition II.5.17. Let $f : M \to \mathbb{R}$ be any function. A *sublevel set* of f is any set of the form $\{p \in M \mid f(p) \leq c\}$ for some $c \in \mathbb{R}$. An *open sublevel set* of f is any set of the form $\{p \in M \mid f(p) < c\}$. Level sets are sets of the form $\{p \in M \mid f(p) = c\}$.

Corollary II.5.18. *The following subsets of M are convex:*

1. *Sublevel sets of convex functions.*
2. *Metric balls.*
3. *Complete totally geodesic submanifolds.*
4. *Geodesic segments, rays and bi-infinite rays.*

Proof. The first item follows directly from the definition of convexity of real functions: for any $y \in [p,q]$ and a convex function f, $f(y) \leq \max\{f(p), f(q)\}$. The other items follows directly from item 1 and Lemma II.5.13. □

Horospheres, Horoballs, Visual Boundary. I turn to present some of the most basic geometric objects of symmetric spaces: the visual boundary $X(\infty)$ (and two topologies on it), horoballs and horosphere.

Definition II.5.19. A *Busemann function* on X is any function of the form

$$f_\eta(x) = \lim_{t\to\infty} d\big(x, \eta(t)\big) - t$$

for some geodesic η of X.

Lemma II.5.20 (Lemma 3.4 in [4])**.** *Busemann functions on X are convex.*

Definition II.5.21. A *horoball* $\mathcal{HB} \subset X$ is an open sublevel set of a Busemann function. A *horosphere* $\mathcal{H} \subset X$ is a level set of a Busemann function. Clearly the topological boundary $\partial\mathcal{HB}$ of each horoball $\mathcal{HB}$ is a horosphere $\mathcal{H}$, and each horosphere $\mathcal{H}$ bounds a unique horoball $\mathcal{HB}$ such that $\partial\mathcal{HB} = \mathcal{H}$. In this situation I say that $\mathcal{H}$ is the *bounding horosphere* of $\mathcal{HB}$.

Horoballs and horospheres play a crucial role in the proof. The origin of it is the role horoballs play in the geometric description of Leuzinger (see Theorem II.5.78 below). I also use one other characterization of horoballs, as a union of balls with centres 'tending to ∞'.

Lemma II.5.22 (Section 3 in [4])**.** *Every horoball is of the form $\cup_{t>0} B\big(\eta(t), t\big)$ for some geodesic η of X.*

This characterization motivates the description of horoballs as 'balls centred at ∞'. The definition of this 'centre', and more generally of 'points at infinity', is of great importance.

Definition II.5.23. Two geodesic rays η_1, η_2 of X are called *asymptotic* if the distance $d\big(\eta_1(t), \eta_2(t)\big)$ is bounded uniformly in $t > 0$. This is an equivalence relation. If η is a geodesic , I sometime write $\eta(\infty)$ (respectively $\eta(-\infty)$) to mean the equivalence class of the geodesic ray $\eta_{\restriction[0,\infty)}$ (respectively of $\eta_{\restriction(-\infty,0]}$).

Definition II.5.24. The *visual boundary* of X is the set of all equivalence classes of asymptotic rays, denoted $X(\infty)$.

Claim II.5.25. For every $x \in X, \xi \in X(\infty)$ there is a unique geodesic ray $\eta : [0, \infty)] \to X$ with $\eta(0) = x, \eta(\infty) = \xi$. This geodesic ray is denoted $[x, \xi)$.

Proof. Existence is by definition of $X(\infty)$. The distance function to a geodesic is convex, hence if two geodesics emanate from the same point then their distance is either uniformly 0 or else unbounded. □

Definition II.5.26. Let $Y \subset X$ be a submanifold. Denote by $Y(\infty) \subset X(\infty)$ all those points $\xi \in X(\infty)$ that admit a geodesic ray $\eta \subset X$ with $\eta(\infty) = \xi$ and so that $d(\eta, Y)$ is globally bounded.

Remark II.5.27. If Y is totally geodesic then the fact that projections do not increase distances imply that $P_Y(\eta)$ is a geodesic with $P_Y(\eta)(\infty) = \xi$. That is, for a totally geodesic submanifold $Y \subset X$ one has that $Y(\infty)$ is the subset of $X(\infty)$ comprised of all points ξ that admit a geodesic ray $\eta \subset Y$ with $\eta(\infty) = \xi$.

Definition II.5.28. Let $\eta = [x, \xi)$, where $\xi = \eta(\infty) \in X(\infty)$.

1. The Busemann function f_η, as well as the associated horoballs and horosphere (i.e. level and sublevel sets of f_η) are said to be *based* at ξ.

2. The horosphere containing x and based at ξ is denoted $\mathcal{H}(x, \xi)$, and the horoball bounded by $\mathcal{H}(x, \xi)$ is denoted $\mathcal{HB}(x, \xi)$.

Proposition II.5.29 (Proposition 1.10.2 in [28])**.** *Let f_{η_1}, f_{η_2} be two Busemann functions based at the same point at $X(\infty)$, i.e. $\eta_1(\infty) = \eta_2(\infty)$. Then $f_{\eta_1} - f_{\eta_2}$ is a constant function on X. In particular the set of horospheres based at ξ is independent on the choice of Busemann function.*

Proposition II.5.30 (Proposition 1.10.5 in [28])**.** *Let $x \in X, \xi \in X(\infty)$, and let $\mathcal{H} = \mathcal{H}(x, \xi)$, $\mathcal{HB}$ the horoball bounded by $\mathcal{H}$ and f the Busemann function based at ξ with $f(x) = 0$.*

1. *For any point $y \in X$, let η be the bi-infinite geodesic determined by the geodesic $[y, \xi)$. Then $P_{\mathcal{H}}(y) = \eta \cap \mathcal{H}$, where $P_{\mathcal{H}}(y)$ is the unique point closest to y on $\mathcal{H}$.*

2. *For any point* $y \in X$, $f(y) = \pm d\big(y, P_{\mathcal{H}}(y)\big)$. *Moreover,* $f(y)$ *is negative if and only if* $y \in \mathcal{HB}$.

3. *If* $x' \in X$, *then the horospheres* $\mathcal{H} = \mathcal{H}(x, \xi)$ *and* $\mathcal{H}' = \mathcal{H}(x', \xi)$ *are equidistant: if* $y \in \mathcal{H}, y' \in \mathcal{H}'$, *then* $d(y, \mathcal{H}') = d(y', \mathcal{H})$. *Such horospheres are called* parallel.

Remark II.5.31. Note that $P_{\mathcal{H}}$ is not a projection on a convex set since $\mathcal{H}$ is not convex. See Corollary II.5.121

Remark II.5.32. It holds that the stabilizer of a point $\xi \in X(\infty)$ acts transitively on the set of horospheres based at ξ (Proposition II.5.63). In particular, every two such horospheres are isometric. Moreover, there is a close relation between the induced metrics on horospheres with the same base point. Briefly, if $d_{\mathcal{H}}$ denotes the induced distance on a horosphere $\mathcal{H} \subset X$, then $d_{\mathcal{H}}\big(P_{\mathcal{H}}(x), P_{\mathcal{H}}(y)\big)$ for any two points $x, y \in \mathcal{H}'$ can be bounded uniformly below and above as a function of the distance $d_{\mathcal{H}'}(x, y)$ and the curvature bounds on X. See Heintze-Im hof [38] for precise statements.

Below are two immediate corollaries.

Corollary II.5.33. *Two horospheres are parallel if and only if they are based at the same point in* $X(\infty)$. *In particular,* $\mathrm{Stab}_G(\mathcal{H}) \subset \mathrm{Stab}_G(\xi)$.

Proof. One direction is the definition in item 3 of Proposition II.5.30. The other direction requires some differential geometry, in order to deduce that a Busemann function is uniquely determined by the base point ξ. The proof for this fact goes through the gradient characterization of Busemann functions, see Section 3 in [4].

In order to get more familiar with horospheres, I give a simple geometric proof that covers almost all cases. Let $\mathcal{H}, \mathcal{H}'$ be two parallel horospheres based at points $\xi, \xi' \in X(\infty)$ respectively. Being parallel is clearly an equivalence relation of horospheres, and but so every two horospheres based at ξ and ξ' would be parallel. In particular this is true for those horospheres containing some fixed arbitrary point $x \in X$. It is also clear that two parallel horospheres that contain the same point must be the same horosphere. It thus remains to show that if $\mathcal{H} = \mathcal{H}(x, \xi)$ is a horosphere based at some $\xi \in X(\infty)$, then $\mathcal{H}$ does not equal $\mathcal{H}(x, \xi')$ for any $x \in X, \xi' \in X(\infty)$.

To prove this easy fact, take for instance two bi-infinite geodesic determined by $\eta = [x, \xi)$ and $\eta' = [x, \xi')$. They define Busemann functions based at ξ, ξ' respectively. Consider $\mathcal{H}_t := \mathcal{H}\big(\eta'(t), \xi\big)$ and $\mathcal{H}'_t := \mathcal{H}\big(\eta'(t), \xi'\big)$. All these horospheres are parallel. By item 2 of Proposition II.5.30, $\mathcal{H}'_t$ is at distance $\pm t$ from $\mathcal{H}$. By item 1, $P_{\mathcal{H}}\big(\eta(t)\big) = \eta(0)$ if and only if $\xi' \in \{\eta(\infty), \eta(-\infty)\}$. Therefore unless $\eta = \eta'$ (possible with change of orientation), one has $d\big(\eta'(t), \mathcal{H}\big) < t$. On the other hand, $\mathcal{H}$ is also $\mathcal{H} = \mathcal{H}\big(\eta'(0), \xi'\big)$, so again by item 1 one has $d(\mathcal{H}, \mathcal{H}'_t) = t$. I obtain a contradiction.

It still remains to show that $\eta(-\infty) \neq \eta'(\infty)$. This means that for every point $x \in X$

$$\lim d\big(x, \eta(t)\big) - t = \lim d\big(x, \eta(-t)\big)$$

This situation occurs only if X is flat. See section 3 in [4].

□

The main step in the proof of Theorem II.5.1 is to find a horosphere that $\Lambda \cdot x_0$ intersects cocompactly, i.e., in a *cocompact metric lattice*.

Definition II.5.34. Let (Z, d_Z) be a metric space. A discrete subset $B \subset Z$ is a *cocompact metric lattice* if $Z = \mathcal{N}_D(B)$ for some $D > 0$.

Corollary II.5.35. *If $\mathcal{H}, \mathcal{H}'$ are two parallel horospheres based at the same $\xi \in X(\infty)$ and $A \subset \mathcal{H}$ is a cocompact metric lattice in $\mathcal{H}$, then $\pi_{\mathcal{H}'}(A)$ is a cocompact metric lattice in $\mathcal{H}'$.*

Proof. Let $D > 0$ be such that for every point $y \in \mathcal{H}$ there is $a_y \in A \cap B(y, D)$. Let $l := d(\mathcal{H}, \mathcal{H}')$, fix $x \in \mathcal{H}'$ and denote $y = p_{\mathcal{H}}(x)$. Triangle inequality gives

$$d\big(x, p'_{\mathcal{H}}(a_y)\big) \leq d(x, y) + d(y, a_y) \leq l + D$$

By Remark II.5.32, the projection $p_{\mathcal{H}'}(A)$ is a discrete set in $\mathcal{H}'$. Together with the above inequality, this renders $p_{\mathcal{H}'}(A)$ a cocompact metric lattice in $\mathcal{H}'$. □

Two Topologies on $X(\infty)$. I now turn to define two topologies on $X(\infty)$. Both topologies play an important role in the structure theory of X and specifically in this thesis. I start with a short description of the *cone* topology.

Proposition II.5.36 (See Section 8, Chapter 2 of [17])**.** *Denote $\overline{X} = X \cup X(\infty)$, and fix $x_0 \in X$. The* cone topology *on $\overline{X}$ is the one given by viewing $X(\infty)$ as the inverse system $\varprojlim \overline{B}(x_0, r)$ for $r \to \infty$, where $\overline{B}(x_0, r)$ is the closed metric ball of radius r about x_0. The limit is taken with respect to the projections p_r on the closed balls $\overline{B}(x_0, r)$, where $p_r(\xi)$ is defined to be the point at time r of the unique geodesic ray $[x_0, \xi)$.*

The induced topology on $X(\infty)$ is (also) called the cone topology*. The space $\overline{X}$ is a compactification of X, and $\partial\overline{X} = X(\infty)$. Together with the cone topology $X(\infty)$ is homeomorphic to the unit tangent space at x_0, in fact to the tangent space at any point $x \in X$.*

As a basis of open neighbourhoods for the cone topology on $X(\infty)$ one may take for all $r, \epsilon > 0$ and all geodesic rays η of X with $\eta(0) = x_0$, the sets:

$$U(\eta, r, \epsilon) := \{\xi \in X(\infty) \mid d\big(p_r(\eta), p_r(\xi)\big) < \epsilon\}$$

Remark II.5.37. In the cone topology, two points $\xi, \zeta \in X(\infty)$ are 'relatively close' if the geodesics $[x_0, \xi)$ and $[x_0, \zeta)$ stay relatively close to each other for a long time. A more concrete fact in this context is that two geodesics with small *comparison angle* stay close together. See Definition II.5.38 and Corollary II.5.46 below.

The next major goal is to define the Tits topology and metric on $X(\infty)$. This requires the notions of angles and the angular metric on $X(\infty)$. The angular metric on $X(\infty)$ will further help in understanding the cone metric on $X(\infty)$ (see Propositions II.5.45 and II.5.53).

Another important aspect which will play a major role in these definitions and in the proofs is the structure of flat regions in X. In the spirit of preferring metric notions over Riemannian notions, I give the metric interpretation of Riemannian angles. Consult Section 1, Chapter 1 in [17] for more details, in particular for a definition of a Riemannian angle.

Definition II.5.38. Given 3 points $p_1, p_2, p_3 \in M$, their *comparison triangle* is the triangle in the Euclidean plane (p_1', p_2', p_3') such that $d_{\mathbb{E}^2}(p_i', p_j') = d_M(p_i, p_j)$. Consider the geodesic segments $[p_1, p_2]$ and $[p_1, p_3]$. The *comparison angle* at p_1 of these geodesics is defined to be the Euclidean angle at p_1' in the comparison triangle. Denote it by $\measuredangle_{p_1}(p_2, p_3)$.

Definition II.5.39. Let $\eta_1, \eta_2 : [0,1] \to M$ be two geodesic segments with $\eta_1(0) = \eta_2(0) = p$. The *Alexandrov upper angle* between η_1 and η_2 at p is defined to be

$$\limsup_{t,s \to 0} \measuredangle_p\big(\eta_1(t), \eta_2(s)\big)$$

Proposition II.5.40 (Corollary 1A.7, Appendix A, Chapter 1 in [17])**.** *Let M be a smooth Riemannian manifold. Let $\eta_1, \eta_2 : [0,1] \to M$ be two geodesic segments with $\eta_1(0) = \eta_2(0) = p$.*

$$\lim_{t,s \to 0} \measuredangle_p\big(\eta_1(t), \eta_2(s)\big)$$

exists, and equals the Riemannian angle. Denote this angle by $\measuredangle_p(\eta_1, \eta_2)$, and call it the angle between η_1 and η_2 at p.

The next step is to define an angle between two point at $X(\infty)$. Notice that the definition from Proposition II.5.40 depends only on a small neighbourhood of the starting points of η_1, η_2. This allows for the following definitions.

Definition II.5.41. Let $\xi, \zeta \in X(\infty)$ and $x \in X$. The *angle* between ξ and ζ at x is defined to be $\measuredangle_x(\xi, \zeta) := \measuredangle_x\big([x, \xi), [x, \zeta)\big)$. Define *the angle* between ξ and ζ to be

$$\measuredangle(\xi, \zeta) = \sup_{x \in X} \measuredangle_x(\xi, \zeta)$$

Remark II.5.42. Angles are defined via the comparison angles in Euclidean plane, so they are all bounded by π and the supremum in Definition II.5.41 exists.

Definition II.5.43. A *flat sector in* $\mathbb{E}^2$ is a triangular sector in the Euclidean plane that is bounded by two geodesics emanating from the same point with Euclidean angle strictly smaller than π. A *flat sector in* X is a smooth embedding of a flat sector in $\mathbb{E}^2$ into X, whose image is totally geodesic.

Proposition II.5.44 (Proposition 3.1.2 in [28])**.** *Let $\xi, \zeta \in X(\infty)$. The map $x \mapsto \measuredangle_x(\xi, \zeta)$ attains a global maximum at $x \in X$ if and only if the two geodesic rays $[x, \xi)$ and $[x, \zeta)$ bound a flat sector in X and $\measuredangle_x(\xi, \zeta) < \pi$, or else if x lies on a geodesic (ξ, ζ) in which case $\measuredangle_x(\xi, \zeta) = \pi$.*

Proposition II.5.45 (Section 3.1 in [28])**.** *The pair $\big(X(\infty), \measuredangle\big)$ is a metric space. The metric on it is called the angular metric, and one has:*

1. *Isometries of X induce isometries of $\big(X(\infty), \measuredangle\big)$.*
2. *Semicontinuity: for any $\xi, \zeta \in X(\infty)$ and every $\epsilon > 0$, there exists neighbourhoods of the cone topology $U, V \subset X(\infty)$ of ξ and ζ respectively such that for all $\xi' \in U, \zeta' \in V$ one has*

$$\measuredangle(\xi', \zeta') \geq \measuredangle(\xi, \zeta) - \epsilon$$

Corollary II.5.46. *Given time T and $\epsilon > 0$, there is an angle $\alpha = \alpha(T, \epsilon)$ so that if η_1, η_2 are two geodesic rays with $\eta_1(0) = \eta_2(0) = x$ for some $x \in X$ and $\measuredangle_x(\eta_1, \eta_2) \leq \alpha$ then $d_X\big(\eta_1(t), \eta_2(t)\big) < \epsilon$ for all $t \leq T$.*

Proof. This is immediate from the fact that Hadamard spaces are $CAT(0)$ spaces, which I do not define here. See Section 4, Chapter 2 in [17]. □

Definition II.5.47 (See Chapter 1.3 in [17])**.** Given a metric space (Z, d_Z), the associated *inner metric* is the function $\overline{d_Z} : Z \times Z \to \mathbb{R}_{>0} \cup \{\infty\}$ that assigns to a pair of points the minimal length of a rectifiable curve in Z joining them, or ∞ if there is no such curve. The space Z is called a *length space* If $d_Z = \overline{d_Z}$.

Definition II.5.48 (Section 3.4 in [28])**.** The *Tits metric* on $X(\infty)$ is the inner metric of the metric space $\big(X(\infty), \measuredangle\big)$. Denote the Tits metric by d_T.

The Tits metric is closely related to the structure of flats and horoballs in X. A significant fact about flats in X is the following:

Proposition II.5.49 (Proposition 2.21.14 in [28]). *Every two points $\xi, \zeta \in X(\infty)$ lie on the boundary of some flat $F \subset X$, i.e. there is a flat F and geodesic rays $\eta_1, \eta_2 : [0, \infty) \to F$ such that $\eta_1(\infty) = \xi, \eta_2(\infty) = \zeta$. In particular, Euclidean geometry implies that for any point $x \in F$ one can take η_1 and η_2 to emanate from x. The equality $\measuredangle(\xi, \zeta) = \pi$ holds if and only if there is a geodesic $\eta \subset F$ for which $\eta(-\infty) = \xi$ and $\eta(\infty) = \zeta$.*

Combined with Proposition II.5.44, one obtains a more straightforward definition of the Tits metric:

Corollary II.5.50. *Let $\xi, \zeta \in X(\infty)$. The Tits distance $d_T(\xi, \zeta)$ equals the angle $\measuredangle_x(\xi, \zeta)$ for any $x \in X$ that lies on a flat F such that $\xi, \zeta \in F(\infty)$. This also equals the angle between ξ and ζ in the sense of Definition II.5.41.*

Remark II.5.51. I could not produce a geometric proof for Proposition II.5.49. In [28] there seems to be a suggestion for a geometric proof using Proposition II.5.53 below. However as far as I can see this proof eventually relies on the fact that all flats in X have the same rank (Proposition II.2.14) which, as far as I know, goes through the algebra of the Lie group and Lie algebra.

Corollary II.5.52. *In $\mathbb{R}$-rank 1, every two distinct points $\xi, \zeta \in X(\infty)$ admit $d_T(\xi, \zeta) = \pi$.*

Below are some properties of the Tits metric, in relation to flats and otherwise.

Proposition II.5.53 (Propositions 3.4.3 and 3.6.1 in [28]). *The following properties hold:*

1. *The diameter of $\big(X(\infty), d_T\big)$ equals π.*
2. *For any flat $F \subset X$, the cone topology and the Tits topology coincide on $F(\infty)$.*
3. *The space $\big(X(\infty), d_T\big)$ is a geodesic metric space, i.e. every two points could be joined by a geodesic. The so called* Tits geodesic *is unique if and only if the distance is strictly less than π.*
4. *Let $\xi, \zeta \in X(\infty)$. The following are equivalent.*
   (a) $d_T(\xi, \zeta) \leq \frac{\pi}{2}$
   (b) *For some point $x \in X$ it holds that $[x, \xi) \subset \mathcal{HB}(x, \zeta)$*
   (c) *For every point $x \in X$ it holds that $(x, \xi) \subset \mathcal{HB}(x, \zeta)$.*

5. *Let $x, y \in X(\infty)$ and x_n, y_n sequences with $x_n \to x$, $y_n \to y$ in the cone topology. Then $d_T(x, y) \leq \liminf d_T(x_n, y_n)$.*

Below are two basic facts that help understand the relation between horoballs, flats and the Tits metric.

Lemma II.5.54. *Let $\mathcal{HB}$ be a horoball based at $\xi \in X(\infty)$, and F a flat with $\xi \in F(\infty)$. Then the intersection of F with $\mathcal{HB}$ is a half space of F.*

Proof. F is totally geodesic, so $\xi \in F(\infty)$ implies there is $\eta : [0, \infty) \to F$ with $\eta(\infty) = \xi$. The definition of a horoball via Busemann functions depend only on the distance between a point $x \in X$ and the points on the geodesic $\eta(t)$ for $t > 0$. Since F is totally geodesic, this picture in $F \subset X$ is exactly the picture of the Euclidean plane, in which case the lemma is easily seen to hold. □

Proposition II.5.55 (Proposition 3.6.2 in [28])**.** *Assume X is of higher rank, let $\xi, \zeta \in X(\infty)$ and $c : [0, \alpha] \to X(\infty)$ some Tits geodesic with $c(0) = \xi, c(\alpha) = \zeta$. There is a flat F such that $c \subset F$.*

Isometries and Stabilizers of Points at $X(\infty)$. I present the well known classification of isometries in X and some properties of groups that stabilize points at $X(\infty)$.

Definition II.5.56. Let $G = \mathrm{Isom}(X)$. The *displacement function* of $g \in G$ is the map $d_g : X \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto d(x, gx)$. Denote by C_g the set of points $x \in X$ on which d_g attains a minimum.

Proposition II.5.57 (Section 6 in [4] and Section 1.9 in [28])**.** *Let $g \in G$. Then g satisfies the trichotomy:*

1. *The element g fixes a point $x \in X$, and is called* elliptic*.*
2. *The element g stabilizes a bi-infinite geodesic η of X. It acts by translations along that geodesic, i.e. there exists $l = l(g) > 0$ for which $g\eta(t) = \eta(t + l)$. In this case g is called* hyperbolic *(or* loxodromic*). Such η is called an* axis *of g and l is called the* translation length *of g.*
3. *The element g is neither elliptic or hyperbolic, in which case it is called* parabolic*.*

Moreover, one has:

1. *g is parabolic if and only if $C_g = \emptyset$.*
2. *If g is elliptic, the set C_g is the fixed point set of g and it is a complete, totally geodesic submanifold of X.*

3. *For g hyperbolic, the set C_g is closed, convex and consists of the union of all the axes of g.*

Symmetric spaces admit as many hyperbolic isometries as possible, many of which come in the form of *transvections*.

Definition II.5.58. Let η be a bi-infinite geodesic of X, and let g_t, g_s be the two geodesic symmetries at $\eta(t)$ and $\eta(s)$, respectively. These can be viewed as isometries $g_t, g_s \in G = \mathrm{Isom}(X)$, since X is a symmetric space. The composition $g_t g_s$ is called a *transvection* of X.

Lemma II.5.59. *Let η be a geodesic and $l > 0$. Then there is a hyperbolic element $g \in G$ with axis η and translation length l.*

Proof. In the notation of Definition II.5.58, $g_{\frac{l}{2}} g_0$ is easily seen to be the desired hyperbolic element: it preserves $\eta(\infty)$ and $\eta(-\infty)$ and maps $\eta(0)$ to $\eta(l)$. □

An important aspect of isometries is their induced action on $X(\infty)$, especially for parabolic elements. One property in this context is the following.

Proposition II.5.60 (Propositions 4.1.1 and 4.1.2 in [28])**.** *Let $g \in \mathrm{Isom}(X)$ be a parabolic element, and let $X(\infty)_g$ be the set of fixed points of g in $X(\infty)$. Then $X(\infty)_g \neq \emptyset$, and one has:*

1. *$X(\infty)_g$ is closed with respect to the cone topology.*
2. *If $\xi, \zeta \in X(\infty)_g$ and $d_T(\xi, \zeta) < \pi$, then the (unique) geodesic joining them in $\big(X(\infty), d_T\big)$ lies in $X(\infty)_g$ as well.*
3. *There exists a unique $\xi \in X(\infty)_g$ such that $d_T(\xi, \zeta) \leq \frac{\pi}{2}$ for all $\zeta \in X(\infty)_g$. The point ξ is called the* centre of gravity *of $X(\infty)_g$.*

Definition II.5.61. A group $P \leq \mathrm{Isom}(X)$ is called *geometrically parabolic* if it fixes a point $\xi \in X(\infty)$.

Lemma II.5.62 (See Proposition 1.13.14 in [28])**.** *Let $G = \mathrm{Isom}(X)^\circ$ and $H \leq G$ a closed subgroup, $x \in X$ a point. Recall that $X = G/K$, for some maximal compact subgroup $K \leq G$.*

1. *H is compact if and only if it admits a bounded orbit.*
2. *If $H \cdot x$ is bounded, then H admits a fixed point $y \in X$.*

3. *The stabilizer of a point $x \in X$ (i.e. fixator of x in G) is a maximal compact subgroup of G.*

4. *All maximal compact subgroups of G are conjugated to $K = \mathrm{Stab}_G(x_0)$, where $x_0 = eK$.*

5. *Let $K' \leq G$ be a maximal compact subgroup. Then for any $\xi \in X(\infty)$, $K \cdot \xi = G \cdot \xi$.*

Proof. Consider the orbit map on H defined by $h \mapsto h \cdot x$. The fact that $X = G/K$ and K is compact implies that both the orbit map and its inverse are proper maps, proving the first item. The second item is the Bruhat-Tits Fixed Point Theorem for symmetric spaces of noncompact type (Theorem 13.5, Section 13, Chapter 1 in [39]). Item 3 is a corollary of item 2, and item 4 is due to the fact that G acts transitively on X. For the proof of the last item, see Proposition 4.4 in [19]. □

Corollary II.5.63. *Let $\xi \in X(\infty)$.*

1. *$G = \mathrm{Stab}_G(\xi) \cdot K$. In particular, $\mathrm{Stab}_G(\xi)$ acts transitively on X, and so does $\mathrm{Stab}_G(\xi)^\circ$.*

2. *Let $\mathcal{H} \subset X$ be a horosphere based at $\xi \in X(\infty)$. The subgroup $\mathrm{Stab}_G(\mathcal{H})$ acts transitively on $\mathcal{H}$.*

Proof. Let $x, y \in X$, and let $g \in G$ with $gx = y$. Consider $\eta_x = [x, \xi)$. Denote by $K_y := \mathrm{Stab}_G(y)$. It is a maximal compact subgroup, and by item 5 in Proposition II.5.62 one has that $g\eta_x(\infty) = g\xi \in G \cdot \xi$ implies that there is $k_y \in K_y$ with $k_y \cdot g\eta_x(\infty) = \xi$. Therefore $k_y \cdot g\eta_x = [y, \xi)$, and so $k_y g \in \mathrm{Stab}_G(\xi)$ maps x to y. If $x, y \in \mathcal{H}$, then by the definition of horospheres this forces $k_y g \in \mathrm{Stab}_G(\mathcal{H})$.

The identity component $\mathrm{Stab}_G(\xi)^\circ$ of $\mathrm{Stab}_G(\xi)$ is of finite index in $\mathrm{Stab}_G(\xi)$ (Proposition II.5.112). Therefore it is both closed and open in $\mathrm{Stab}_G(\xi)$. Since the action on X is by homeomorphisms, the orbits of $\mathrm{Stab}_G(\xi)^\circ$ are open and closed in X. On the other hand for any $x \in X$,

$$X = \mathrm{Stab}_G(\xi) \cdot x = \cup_{i=1}^n g_i \mathrm{Stab}_G(\xi)^\circ \cdot x$$

Since X is connected, it must hold that $\mathrm{Stab}_G(\xi)^\circ \cdot x = X$. □

As an immediate corollary, one has:

Corollary II.5.64. *Let $\xi \in X(\infty)$. Then $\mathrm{Stab}_G(\xi)$ acts transitively on the set of geodesics in the equivalence class of ξ, namely on the set of geodesics $\{\eta \mid \eta(\infty) = \xi\}$.*

Limit Set. The geometric classification of isometries is handy in trying to imagine how the different parts of orbits of general discrete subgroups $\Delta \leq G$ might look. An important related topic is the classification of limit points of a discrete subgroup of isometries.

Definition II.5.65. Let $\Delta \leq \mathrm{Isom}(X)$ be a discrete subgroup. The *limit set* of Δ is the intersection with $X(\infty)$ of the closure, in the compactification $\overline{X} = X \cup X(\infty)$ described above, of an orbit $\Delta \cdot x$. I denote the limit set of Δ

$$\mathcal{L}_\Delta := \overline{\Delta \cdot x} \cap X(\infty)$$

Note that for any two points $x_1, x_2 \in X$, the Hausdorff distance between $\Delta \cdot x_1$ and $\Delta \cdot x_2$ is finite, therefore $\mathcal{L}_\Delta$ does not depend on the choice of $x \in X$.

For lattices, the limit set is the whole visual boundary. In fact, much more is true:

Definition II.5.66. Let $\Delta \leq G = \mathrm{Isom}(X)$, and SX the unit tangent bundle. A vector $v \in SX$ is Δ-*periodic* if there is $\delta \in \Delta$ and $s > 0$ such that $\delta\eta(t) = \eta(t+s)$ for all $t \in \mathbb{R}$, where η is the bi-infinite geodesic determined by the vector v.

For a flat $F \subset X$ (including geodesics), denote $\Delta_F := \{\delta \mid \delta F = F\}$. The flat F is called a Δ-*periodic flat* if there exists a compact set $C \subset F$ such that $\Delta_F C = F$.

Proposition II.5.67 (Propositions 4.7.3., 4.7.5, 4.7.7 in [28], Lemma 8.3′ in [56])**.** *If $\Gamma \leq G = Isom(X)$ is a lattice, then:*

1. *The subset in SX of Γ-periodic vectors is dense.*
2. *Let $F \subset X$ be any flat, η any bi-infinite geodesic in F, and denote $v = \dot{\eta}(0) \in SX$ the initial velocity vector. There is a sequence $v_n \in SX$ of regular vectors such that*
   (a) $\lim_{n\to\infty} v_n = v$.
   (b) *The bi-infinite geodesics η_n determined by v_n are all Γ-periodic.*
   (c) *Denote by F_n the (unique) flat containing η_n. Each F_n is Γ-periodic.*

   Put differently, the set of Γ-periodic flats is dense in the set of flats of X.

Remark II.5.68. See Definition II.5.86 for the notion of *regular* tangent vectors.

Remark II.5.69. Eberlein formulates Proposition II.5.67 using the important notion of the *duality condition*. Every lattice satisfies this condition - see Definitions 1.9.10, 1.9.15 and Proposition 1.9.32 in [28]. Moreover, the statements are formulated for higher rank groups only, and the density of periodic flats only for uniform lattices:

1. In $\mathbb{R}$-rank 1, the proof is just that the action of a lattice on the Furstenberg boundary is minimal. In this case the Furstenberg boundary equals the visual boundary, and a lattice acts minimally on it (see Section 3.8 in [28] and Lemma 8.5 in [56]). Therefore it is enough to prove that there is one Γ-periodic vector in SX, a fact which follows from easy geometric considerations in Gromov-hyperbolic spaces.

2. The proof for density of periodic flats for non-uniform lattices is due to Prasad-Raghunathan [64]. Density of orbits in the action on the Furstenberg boundary plays a similar role as in the $\mathbb{R}$-rank 1 case.

In [36], Hattori describes areas in the *rational* Tits building by the different types of limit points. I define these types now, and explain his results in the subsequent section.

Definition II.5.70. Let $\Delta \leq G$ be a discrete subgroup, $\xi \in \mathcal{L}_\Delta$.

1. The point ξ is called *conical* if for some (hence any) x and some (hence every) geodesic ray η with $\eta(\infty) = \xi$ there is a number $D = D(x, \eta)$ such that for every $T \in \mathbb{R}_{>0}$ there is $t > T$ for which $B\big(\eta(t), D\big) \cap \Delta \cdot x \neq \emptyset$. Since Δ is discrete, this is equivalent to $\Delta \cdot x \cap \mathcal{N}_D(\eta)$ being infinite.

2. The point ξ is called *horospherical* if for every horoball $\mathcal{HB}$ based at ξ and every $x \in X$, $\Delta \cdot x \cap \mathcal{HB}$ is non-empty. In particular, a conical limit point is horospherical.

3. The point ξ is *non-horospherical* if it is not horospherical.

As a corollary of Proposition II.5.67, one has:

Corollary II.5.71. *If $\Gamma \leq \mathrm{Isom}(X)$ is a lattice, then*

1. *The set of Γ-conical limit points is dense in $X(\infty)$ with the cone topology.*

2. *$\mathcal{L}_\Gamma = X(\infty)$.*

Proof. This is clear from the definitions of limit set, conical limit points, Γ-periodic orbits, and the relation between the topology on SX and the cone topology on $X(\infty)$. □

At this point I define a *geometrically finite* subgroup of G for $\mathbb{R}$-rank 1 symmetric spaces, which is another basic notion in the study of the visual boundary and in studying discrete subgroups of G that are not lattices. It is closely related to conical limit points.

Definition II.5.72. Let X be a $\mathbb{R}$-rank 1 symmetric space and $\Delta \leq Isom(X)$ a discrete subgroup. Denote by $\overline{\mathrm{Hull}(\Delta)}$ the closed convex hull in $\overline{X} = X \cup X(\infty)$ of the limit set $\mathcal{L}_\Delta$, and $\mathrm{Hull}(\Delta) = X \cap \overline{\mathrm{Hull}(\Delta)}$. By virtue of negative curvature, $\mathrm{Hull}(\Delta)$ is the union of all geodesics η such that $\eta(\infty), \eta(-\infty) \in \mathcal{L}_\Delta$. The *convex core* of Δ is defined to be $\Delta \backslash \mathrm{Hull}(\Delta) \subset \Delta \backslash X$, i.e., the quotient of the Δ-action on $\mathrm{Hull}(\Delta)$.

Definition II.5.73 (Bowditch [15], see Theorem 1.4 in [43])**.** Let X be a $\mathbb{R}$-rank 1 symmetric space, i.e. a symmetric space of pinched negative curvature. A discrete group $\Delta \leq G = \mathrm{Isom}(X)$ is *geometrically finite* if for some $\delta > 0$, the uniform δ-neighbourhood in $\Delta\backslash X$ of the convex core $\mathcal{N}_\delta\big(\Delta\backslash \mathrm{Hull}(\Delta)\big)$, has finite volume and there is a bound on the orders of finite subgroups of Δ. A group is *geometrically infinite* if it is not geometrically finite.

Immediately from the definition of geometrical finiteness, one has:

Corollary II.5.74. *Let X be a $\mathbb{R}$-rank 1 symmetric space. If $\Delta \leq \mathrm{Isom}(X)$ is geometrically finite and admits $\mathcal{L}_\Delta = X(\infty)$, then Δ is a lattice in $\mathrm{Isom}(X)$.*

II.5.2.2 Cusps and the Rational Tits Building

In this section I present some of the structure theory of non-compact quotients of X. The focus is on the structure of 'cusps' in noncompact finite volume quotients of symmetric spaces, also called *locally* symmetric spaces of finite volume, and their 'location' on the visual boundary.

Cusps and Compact Core. I give a short informal description of the geometric situation, and then formally present the results that play an active role in the proofs.

Consider $\mathcal{V} = \Gamma\backslash X$, for $\Gamma \leq G$ a non-uniform lattice. This is a locally symmetric space of finite volume. The term 'cusps' is an informal name given to those areas in a locally symmetric space through which one can 'escape to infinity'. Another description is that cusps are the ends of the complement of a large enough compact set in $\mathcal{V}$. In strictly negative curvature, i.e. in $\mathbb{R}$-rank 1 locally symmetric spaces, these cusps have a precise description as submanifolds of the form $C \times \mathbb{R}_{\geq 0}$ for a compact manifold C, and metrically (C, t) gets narrower as $t \to \infty$. There are finitely many cusps, each corresponding to a point at $X(\infty)$ called a 'parabolic point'. See e.g. Introduction in [4] or [27]. A fundamental feature of the cusps is that one can 'chop' them out of the quotient manifold V and get a compact manifold. This could be done in such a way so that:

1. The lifts of the chopped parts to the universal cover X would be disjoint.
2. Each cusp is covered in X by the Γ-orbit of a horoball, whose base point is called a *parabolic point*. That is, the lift of a cusp is the Γ-orbit of a horoball, and all the respective base points are called parabolic points of Γ in $X(\infty)$.
3. Γ acts on $X\setminus\big(\cup_{i\in I}\mathcal{HB}_i\big)$ cocompactly, where $\{\mathcal{HB}_i\}_{i\in I}$ is the set of horoballs coming from the lifts of cusps.

Since there are only finitely many cusps and Γ discrete, there are exactly countably many such horoballs. See for example Section 12.6 in [25] where this is illustrated in the case of the real hyperbolic spaces $\mathbb{H}^n$. In general, one has the following theorem, that establishes the above properties for $\mathbb{R}$-rank 1 spaces.

Theorem II.5.75 (Theorem 3.1 in [27]). *Assume X is of $\mathbb{R}$-rank 1, and $\Gamma \leq G$ a non-uniform lattice. The space $\mathcal{V} = \Gamma\backslash X$ has only finitely many (topological) ends and each end is parabolic and Riemannian collared, and in particular a quotient of a horoball.*

Remark II.5.76. For the precise terminology of the theorem, see Introduction in [27]. Since it is important for the sequel I explicitly mention that *Riemannian collared* implies that for the relevant horoball $\mathcal{HB}$ with bounding horosphere $\mathcal{H}$ and base point ξ, it holds $\Gamma \cap \mathrm{Stab}_G(\xi)$ acts cocompactly on $\mathcal{H}$.

For symmetric spaces of higher rank, a similar construction is available (see [50]). By removing a countable family of horoballs from X, one obtains a subspace on which Γ acts cocompactly. There are two main differences from the situation in $\mathbb{R}$-rank 1. One is that an orbit map $\gamma \mapsto \gamma x$ is a quasi-isometric embedding of Γ (with the word metric) into X. This result is due to Lubotzki-Mozes-Raghunathan [51]. It plays a significant preliminary role in the proofs of quasi-isometric rigidity for non-uniform lattices in higher rank symmetric spaces in both [24] and [30].

Theorem II.5.77 (Theorem A in [51]). *Let G be a semisimple Lie group of higher $\mathbb{R}$-rank, d_G a left invariant metric induced from some Riemannian metric on G. Let Γ an irreducible lattice, d_Γ the corresponding word metric on Γ. Then $d_{G|\Gamma\times\Gamma}$ and d_Γ are Lipschitz equivalent.*

The other major difference is that the horoballs could not in general be taken to be disjoint. However, in the special case of $\mathbb{Q}$-rank 1 lattices (recall Definition II.2.91), the horoballs can be taken to be disjoint. Recall that the $\mathbb{Q}$-structure of (G, Γ) is a $\mathbb{Q}$-structure on $G = \mathbf{G}(\mathbb{R})$ in which Γ is an arithmetic lattice (Definition II.2.89) of G.

Theorem II.5.78 (Theorem 4.2 in [47]). *Assume X is of higher rank, and $\Gamma \leq G$ an irreducible torsion-free non-uniform lattice. On the locally symmetric space $\mathcal{V} = \Gamma\backslash X$ there exists a continuous and piece-wise real analytic exhaustion function $h : \mathcal{V} \to [0, \infty)$ such that, for any $s > 0$, the sublevel set $\mathcal{V}(s) := \{h < s\}$ is a compact submanifold with corners of $\mathcal{V}$. Moreover the boundary of $\mathcal{V}(s)$, which is a level set of h, consists of projections of subsets of horospheres in X.*

The Γ-action on the above set of horospheres has finitely many orbits, and the following conditions are equivalent:

1. *The corresponding horoballs bounded by these horospheres can be taken to be disjoint.*

2. *The action of* Γ *on each bounding horosphere of these horoballs is cocompact.*

3. *The* $\mathbb{Q}$*-structure of* (G, Γ) *is of* $\mathbb{Q}$*-rank* 1.

4. *The lattice* Γ *is of* $\mathbb{Q}$*-rank* 1.

Remark II.5.79. Theorem II.5.78 is a thick version of the original statement in [47]. The fact that the action of Γ on the set of horospheres is with finitely many orbits is a conclusion of the reduction theory presented in proposition 2.1 in [47], due to Borel and Harish-Chandra. The first three equivalent conditions for $\mathbb{Q}$-rank 1 lattices appear in the introduction of [47] as well as remarks 3, 4 in [50], where they appear in the language of the $\mathbb{Q}$-structure of (G, Γ). The equivalence of the third and fourth items is by Lemma II.2.92 (see Section II.2.5).

Definition II.5.80. In the setting of Theorem II.5.75 and Theorem II.5.78, the horoballs and horospheres that appear in the statement are called (global) *horoballs (horospheres) of* Γ, or *bounding horospheres of* Γ. Base points of these are called *parabolic limit points of* Γ.

Other than the fact that horoballs of Γ can be taken to be disjoint, I will use the intersection pattern of a Γ orbit and the horospheres of Γ, given by the following theorem by Raghunathan [65]. I state it here in a geometric language.

Theorem II.5.81 (Section 13 in [65], see also Proposition 2.1 in [63])**.** *Assume that* Γ *is a* $\mathbb{Q}$*-rank* 1 *lattice, and* $\mathcal{H}$ *a global horosphere of* Γ*. Then* $\Gamma \cap \mathrm{Stab}_G(\mathcal{H})$ *acts cocompactly on* $\mathcal{H}$*.*

The proof of the key Proposition II.5.138 only requires that Γ has the properties given by Theorems II.5.75 and II.5.81, i.e. that the horoballs of Γ are disjoint and that $\Gamma \cdot x_0$ intersects horospheres of parabolic points in a cocompact metric lattice. I prefer to make this geometric assumption explicit rather than it being implicit in the algebraic hypotheses on Γ. Namely, for the purposes of Section II.5.4, one could define a $\mathbb{Q}$-rank 1 lattice like so:

Definition II.5.82 ($\mathbb{Q}$-rank 1 lattices, a geometric definition)**.** Let $\Gamma \leq G$ be an irreducible torsion-free non-uniform lattice. It is said to be a $\mathbb{Q}$*-rank* 1 *lattice* if:

1. The horoballs of Γ are disjoint.

2. For every $\mathcal{HB}$ a horoball of Γ with bounding horosphere $\mathcal{H} = \partial\mathcal{HB}$, and every $x \in \mathcal{H}$, the orbit $\Gamma \cdot x \cap \mathcal{H}$ is a cocompact metric lattice in $\mathcal{H}$.

One defines the *compact core* of Γ as the complement in X of the horoballs of Γ. The group Γ acts on it cocompactly.

In light of Theorems II.5.75 and II.5.81, it is a matter of terminology to get:

Corollary II.5.83. *Every lattice considered in Theorem II.5.1 is a $\mathbb{Q}$-rank 1 lattice according to the geometric definition.*

The horospheres that appear in item 2 of Definition II.5.82 is the most important object in the proof of Proposition II.5.138. To further ease the notation and terminology, I make another definition in the following corollary.

Corollary II.5.84. *Let $\Gamma \leq G$ be a $\mathbb{Q}$-rank 1 lattice, $x \in X$, and ξ a parabolic limit point. There is a unique horosphere $\mathcal{H}$ based at ξ such that both following conditions hold:*

1. *$\Gamma \cdot x \cap \mathcal{H}$ is a cocompact metric lattice in $\mathcal{H}$.*
2. *$\Gamma \cdot x \cap \mathcal{HB} = \emptyset$, where $\mathcal{HB}$ is the horoball bounded by $\mathcal{H}$.*

$\mathcal{H}$ is called an x-horosphere of Γ at ξ, *and the corresponding bounded horoball an* x-horoball of Γ. *Moreover, one has:*

1. *For every C there exists a bound $K = K(x, C)$ so that $B(x, C)$ intersects at most K x-horospheres of Γ.*
2. *There is $D = D(x) > 0$ such that $\mathcal{H} \subset \mathcal{N}_D(\Gamma \cdot x \cap \mathcal{H})$ for all x-horospheres of Γ. The constant D is called* the compactness number of (Γ, x).
3. *There is a number N such that every point x admits exactly N horospheres of (Γ, x) that pass through x. These are called* the horospheres of (Γ, x).

Proof. Let $x \in X$. Since Γ is a $\mathbb{Q}$-rank 1 lattice, the stabilizer in Γ of a parabolic limit point $\xi \in X(\infty)$ acts cocompactly on each horosphere based at ξ, and in particular on $\mathcal{H}_x := \mathcal{H}(x, \xi)$. Let $D(x, \mathcal{H})$ be such that $\mathcal{H}_x \subset \mathcal{N}_D(\Gamma \cdot x \cap \mathcal{H}_x)$. A priori $D(x)$ depends on $\mathcal{H}$, but the fact that Γ acts by isometries implies that for every horosphere of the form $\mathcal{H}' := \gamma\mathcal{H} = \mathcal{H}(\gamma x, \gamma\xi)$ for some $\gamma \in \Gamma$, one has $\mathcal{H}' \subset \mathcal{N}_D(\Gamma \cdot x \cap \mathcal{H}')$. So $D(x)$ depends only on the Γ-orbit of $\mathcal{H}$. Since Γ acts on the set of parabolic limit points with finitely many orbits (that is to say $\Gamma\backslash X$ has finitely many cusps) one may take $D = D(x)$ to be the maximum of the respective bounds on each orbit. This gives the required 'compactness number'.

Thinking of horoballs of γ as lifts of ends of the complement of some compact subset of $\mathcal{V} = \Gamma\backslash X$, one sees that there is some horoball $\mathcal{HB}$ based at ξ so that $\Gamma \cdot x \notin \mathcal{HB}$.

Denote $\mathcal{H} = \partial\mathcal{H}\mathcal{B}$, and $y = P_{\overline{\mathcal{H}\mathcal{B}}}(x) \in \mathcal{H}$ be the projection on the closed convex set that is the closure of the horoball $\mathcal{H}\mathcal{B}$. Finally, Let $\eta = [x, y]$ and denote $l = d(x, y)$. The existence of the required horosphere is equivalent to the fact that the following non-empty set admits a maximum:

$$\mathcal{H}_{x,\xi} := \{t \in [0, l] \mid \Gamma \cdot x \cap \mathcal{H}\big(\eta(t), \xi\big) \neq \emptyset\}$$

Indeed $0 \in \mathcal{H}_{x,\xi}$ and it is a bounded set, so it admits a supremum T. Moreover, this set is discrete. If t were an accumulation point, then for any small $\epsilon > 0$ $\eta_{\restriction(t-\epsilon,t+\epsilon)}$ intersects infinitely many horospheres which the orbit $\Gamma \cdot x$ intersects cocompactly, and each with cocompactness number D. Orbit points on different horospheres are in particular different points, and therefore this suggests $B\big(\eta(t), D+\epsilon\big) \cap \Gamma \cdot x$ is infinite, contradicting discreteness of Γ. I conclude that T is a maximum, and that $\mathcal{H}\big(\eta(T), \xi\big)$ is the unique desired horosphere.

The argument above generally shows that there cannot be an accumulation point in X of x-horospheres of Γ. In particular, for every C, the ball $B(x, C)$ intersects only finitely many x-horospheres of Γ, say $K(C)$, proving item 1 in the 'moreover' part.

For the last statement, simply note that the horoballs of Γ are the Γ-translates of finitely many horoballs. In the terminology of the statement, infinitely many horospheres of (Γ, x) imply that infinitely many of them are in the same Γ-orbit. Suppose these are $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, with base points ξ_n that are evidently pairwise different. Finally let $\gamma_n \in \Gamma$ for which $\gamma_n\mathcal{H}_1 = \mathcal{H}_n$. Since $\Gamma \cap \mathrm{Stab}_G(\mathcal{H}_n)$ acts cocompactly on $\mathcal{H}_n$, there is $\gamma'_n \in \Gamma \cap \mathrm{Stab}_G\mathcal{H}_n$ that maps $\gamma_n x$ to $B(x, D)$. Discreteness of Γ implies that the set $\gamma'_n\gamma_n x$ is finite, hence infinitely many of these points are the same point. However $\gamma'_n\gamma_n\xi_1 = \xi_n$ and therefore $\gamma'_n\gamma_n \neq Id_X$, contradicting the fact that Γ is torsion-free. □

Remark II.5.85. For the most part, I am interested in a fixed base point x_0 and the x_0-horospheres and horoballs. By a slight abuse of terminology I refer to these objects as the '*horospheres of* Γ' and 'horoballs of Γ', respectively, and denote the associated cocompactness number D_Γ.

Real and Rational Tits Buildings. In case Γ is an arithmetic lattice, the location of the parabolic points in $X(\infty)$ also plays an important role in the geometry of X and in its relation to the algebra of $G = \mathrm{Isom}(X)^\circ$. The natural way to work with this set of parabolic points is by considering the so called *rational Tits building*. This is a building structure on the subset of parabolic points at $X(\infty)$, that are also known in this case as 'rational points'. The name 'rational' comes from the fact that these points are exactly those that correspond to (algebraic) parabolic groups of G that are defined over $\mathbb{Q}$ (see

Section II.5.3 for more details). I present this object, which I denote $\mathcal{W}_{\mathbb{Q}}(\Gamma)$, along with the more familiar *real Tits building* structure on $X(\infty)$ with the Tits topology. I present the results of Hattori [36], that give a good description of the rational Tits building in terms of conical and horospherical limit points. In case where G is of $\mathbb{R}$-rank 1, by $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ I mean the (countable) set of parabolic limit points of Γ (so that $X(\infty) \setminus \mathcal{W}_{\mathbb{Q}}(\Gamma)$ is comprised of conical limit points only, see Theorem II.5.98).

I stress that while buildings have very rich geometric structure, virtually none of that structure is relevant to the proof of Theorem II.5.1. Therefore I do not give a general exposition to buildings. A reader who is a bit familiar with buildings should easily identify the building structure from the geometric description I give. Further details about these buildings can be found in [42].

I begin with a geometric presentation of Weyl chambers and the Tits buildings at $X(\infty)$, then present the some of Leuzinger and Hattori that are important for the sequel.

Definition II.5.86. A geodesic η is said to be *regular* if it is contained in a unique maximal flat $F \subset X$. The point $\eta(\infty) \in X(\infty)$ is called a *regular* point of $X(\infty)$. A point $\xi \in X(\infty)$ is singular if it is not regular.

Remark II.5.87. By Corollary II.5.64, the notion of a regular or singular point at $X(\infty)$ is well defined, since a regular geodesic is mapped by isometries to a regular geodesic.

Definition II.5.88. Fix $x \in X$, and let $\mathcal{S}_x$ denote the set of singular geodesics emanating from x. *Weyl chambers* of X at x are the connected components of $X \setminus \mathcal{S}_x$.

Let $\mathcal{S} \subset X(\infty)$ be the subset of singular points at $X(\infty)$. A *Weyl chamber* at $X(\infty)$, or an open *spherical chamber*, is any connected component in the Tits topology of $X(\infty) \setminus \mathcal{S}$.

Proposition II.5.89 (Propositions 2.2 and 3.2 in [42] and Section 8 in [4])**.** *The Weyl chambers induce a simplicial complex structure on $X(\infty)$ that is a* spherical Tits building*. The apartments of the building are exactly the sets of the form $F(\infty) \subset X(\infty)$ for all flats $F \subset X$, and the chambers are exactly the Weyl chambers at $X(\infty)$. Moreover, the Tits metric completely determines the building structure, and vice versa, and $\left(X(\infty), d_T\right)$ is a metric realization of the Tits building at $X(\infty)$.*

Remark II.5.90. The existence of both regular and singular points at $X(\infty)$ highlights the fact that not all geodesics are the same in terms of their geometry. Along a singular 'direction', there is in fact a continuum of flats all 'glued' along a geodesic, while for a regular direction there is only one such flat. This is just an explicit (and quantitative) restatement of the definition, but it nonetheless highlights a very basic and important geometric aspect of X.

Many properties of the action of G on $X(\infty)$ can be formulated in terms of the Weyl chambers. A fundamental example is that maximal compact subgroups act simply transitively on the Weyl chambers of $X(\infty)$.

Proposition II.5.91 (Proposition 2.17.24 in [28]). *Let $\xi \in X(\infty)$ be a regular point, and consider the Weyl chamber $\mathcal{C}(\xi)$ it is contained in. For any point $\zeta \in X(\infty)$, $G\cdot\zeta = K\cdot\zeta$ intersects $\overline{\mathcal{C}(\xi)}$ at exactly one point. If ζ is regular then this point lies in $\mathcal{C}(\xi)$ (and not on its boundary). In particular, $K = \mathrm{Stab}_G(x_0)$ acts simply transitively on the set of Weyl chambers at x_0.*

Apart from the Tits building that was described in Proposition II.5.89, there is another building structure on a subset of $X(\infty)$ that is of great importance in the context of irreducible arithmetic lattices, i.e. for non-uniform lattices in higher rank. This is the *rational* Tits building, which can be defined if $G = \mathrm{Isom}(X)$ is an algebraic group defined over $\mathbb{Q}$.

I now describe the rational Tits building in analogy with the (real) Tits building. Recall the latter is described by the simplicial structure of the Weyl chambers. These, in turn, lie on the boundary of maximal flats (this is implicit in the statement of Proposition II.5.89). On the Lie groups side, flats correspond to real maximal split tori (see Section II.2.4). Since G is an algebraic group defined over $\mathbb{Q}$, one can consider the maximal $\mathbb{Q}$-split tori. The *rational flats* of X are then the $G(\mathbb{Q})$-orbits of maximal $\mathbb{Q}$-split tori of G, and the *rational boundary* are all points $\xi \in X(\infty)$ such that $\xi \in F_{\mathbb{Q}}(\infty)$ for some rational flat $F_{\mathbb{Q}}(\infty)$.

One defines *regular rational directions* and *rational Weyl chambers* in an analogous way to the real case, this time taking only rational flats into account.

Proposition II.5.92 (See [42], and Section 2 in [36]). *Assume that $G = \mathrm{Isom}(X)^\circ$ is an algebraic group defined over $\mathbb{Q}$. The* rational Tits building *at $X(\infty)$ is the set $X_{\mathbb{Q}}(\infty)$ together with the simplicial structure given by the rational Weyl chambers.*

There is a 1-1 correspondence between Weyl chambers and stabilizers of regular points at $X(\infty)$. For real Weyl chambers, this is given in the following proposition.

Proposition II.5.93 (Proposition 2.17.15 in [28]). *Let $\xi, \zeta \in X(\infty)$ be two regular points, and let $\mathcal{C}(\xi), \mathcal{C}(\zeta)$ be the Weyl chambers containing ξ, ζ respectively. Then $\mathrm{Stab}_G(\xi)^\circ = \mathrm{Stab}_G(\zeta)^\circ$ if and only if $\mathcal{C}(\xi) = \mathcal{C}(\zeta)$.*

The more standard approach to these Tits buildings is given in terms of (algebraic) parabolic groups and a partial order on them that gives a building structure. See e.g. [36] for a concise presentation, or the notes by Lizhen Ji [42]. The correspondence between

the geometric and the algebraic constructions is essentially due to Proposition II.5.93, together with the correspondence between stabilizers of points at $X(\infty)$ (i.e. parabolic subgroups in the geometric sense) and parabolic subgroups in the algebraic sense. See Proposition II.5.114, and consult Leuzinger's work in Section 2 of [50] and Section 1 in [47].

This correspondence goes even further, and one can give a geometric interpretation to the $\mathbb{Q}$-rank of an algebraic group in terms of the rational Tits building. This is similar to the interpretation of the real rank as the dimension of maximal flats in X. For a definition of an asymptotic cone, see Chapter III.

Definition II.5.94. Denote by $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ the rational Tits building of $\mathbf{G}_{\mathbb{Q}}$, where $\mathbf{G}_{\mathbb{Q}}$ is the canonical $\mathbb{Q}$-structure of (G, Γ).

Theorem II.5.95 (Theorem C and Corollary D in [48])**.** *If X is of higher rank and Γ an irreducible lattice, then the asymptotic cone of the locally symmetric space $\mathcal{V} = \Gamma\backslash X$ is quasi-isometric to $\mathcal{V}$, and is isometric to the Euclidean cone over the finite simplicial complex $\Gamma\backslash\mathcal{W}_{\mathbb{Q}}(\Gamma)$.*

Moreover, the rational rank of the $\mathbb{Q}$-structure of (G, Γ) equals the dimension of the asymptotic cone of $\Gamma\backslash\mathcal{W}_{\mathbb{Q}}(\Gamma)$.

I now present Hattori's characterization.

Theorem II.5.96 (Theorem A in [36])**.** *Let $X = G/K$ be a symmetric space of non-compact type and of higher rank, and let Γ be an irreducible non-uniform lattice. Then $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ does not include horospherical limit points. The $\frac{\pi}{2}$-neighbourhood*

$$\mathcal{N}_{\frac{\pi}{2}}\big(\mathcal{W}_{\mathbb{Q}}(\Gamma)\big) := \{\xi \in X(\infty) \mid d_T\big(\xi, \mathcal{W}_{\mathbb{Q}}(\Gamma)\big) < \frac{\pi}{2}\}$$

does not include conical limit points.

In $\mathbb{Q}$-rank 1 , the converse statement also holds:

Theorem II.5.97 (Theorem B in [36])**.** *Let $X = G/K$ be a symmetric space of non-compact type of higher rank and Γ be an irreducible non-uniform lattice. Let*

$$\mathcal{V} = \{\xi \in X(\infty) \mid d_T\big(\xi, \mathcal{W}_{\mathbb{Q}}(\Gamma)\big) \geq \frac{\pi}{2}\}$$

Suppose that Γ is of $\mathbb{Q}$-rank 1 . Then $\mathcal{V}$ consists of conical points only.

In groups of $\mathbb{R}$-rank 1 one has the following well known fact:

Theorem II.5.98 (Proposition 5.4.2 and Theorem 6.1 in [16], see also Theorem 12.29 in [25]). *Let $X = G/K$ be a symmetric space of noncompact type and of $\mathbb{R}$-rank 1, $\Gamma \leq Isom(X)$ a lattice. Then every $\xi \in X(\infty)$ is either conical or non-horospherical.*

Corollary II.5.99. *When Γ is of $\mathbb{Q}$-rank 1 , the following holds:*

1. $\mathcal{W}_{\mathbb{Q}}(\Gamma) = \{\xi \in X(\infty) \mid \mathcal{N}_{\frac{\pi}{2}}(\xi)$ *does not contain conical limit points*$\}$
2. *Any two points $\xi, \xi' \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ are at Tits distance $= \pi$.*

Proof. In view of Theorem II.5.98, for $\mathbb{R}$-rank 1 both statements hold trivially. In higher rank, both follow from the following observation: for any point $\xi' \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ and any point $\zeta \in X(\infty)$ with $d(\zeta, \xi') = \frac{\pi}{2}$, ζ is conical. To see this notice that ζ lies on the boundary of a horosphere based at ξ': take a flat $F \subset X$ with $\xi', \zeta \in F(\infty)$. Any geodesic with limit ζ is contained in (an Euclidean) horosphere based at ξ'. The fact that Γ is cocompact on the horospheres based at $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ implies that ζ is conical.

The second item of the corollary follows: let $\xi, \xi' \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ and $c : [0, \alpha]$ a Tits geodesic joining them. By Proposition II.5.55, there is a flat $F \subset X$ containing both ξ, ξ' as well as $c \subset F(\infty)$. Every point ζ that is at Tits distance $\frac{\pi}{2}$ from either ξ or ξ' is conical, and no point inside the $\frac{\pi}{2}$ neighbourhood of either ξ or ξ' is conical. On the flat F, the definition of the Tits metric and Proposition II.5.44 readily imply that the Tits metric of $F(\infty)$ is the same as the Tits metric on the Euclidean space of the same rank. Therefore one may prolong the geodesic c so that $c(0) = \xi, c(\alpha) = \xi'$ and $c(\pi) = \xi''$. If $d_T(\xi, \xi') < \pi$, then there is a point along this prolonged geodesic that is at Tits distance exactly $\frac{\pi}{2}$ from ξ (so it is conical by the first paragraph), but at Tits distance strictly less than $\frac{\pi}{2}$ from ξ' (so it cannot be conical by Theorem II.5.96). Therefore $d_T(\xi, \xi') = \pi$.

For the first item, one containment is just Hattori's Theorem II.5.96. For the other containment, pretty much the same argument from above works. Assume for some $\xi \in X(\infty)$ that $\mathcal{N}_{\frac{\pi}{2}}(\xi)$ consists of non-conical limit point. In particular it is not conical, and by Theorem II.5.97 it holds that $d(\xi, \mathcal{W}_{\mathbb{Q}}(\Gamma)) < \frac{\pi}{2}$. Let $\xi' \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ be a point realizing this distance. As above, this gives rise to a flat F containing ξ, ξ' and another point ζ that is at Tits distance $\frac{\pi}{2}$ from ξ' but at Tits distance strictly less than $\frac{\pi}{2}$ from ξ. The first forces ζ to be conical, and the latter forces it to be non-conical, a contradiction. □

Another useful statement that appears in [36] is the following lemma, that gives an estimate on the rate of penetration of a geodesic η to a given horoball. The estimates essentially depend only on the Tits distance between $\eta(\infty)$ and the base point of the horoball. The main takeaway is that this rate of penetration (or escape) is bounded by a linear function.

Lemma II.5.100 (Lemma 3.4 in [36]). *Let X be a symmetric space of higher rank and of noncompact type. Let $\eta_1, \eta_2 : [0, \infty) \to X$ be two geodesic rays, $\alpha := d_T\big(\eta_1(\infty), \eta_2(\infty)\big)$ and b_2 the Busemann function corresponding to η_2. Then there exists a positive constant C_1, depending only on η_1 and η_2, such that:*

1. *If $\alpha > \frac{\pi}{2}$, then for all $t \geq 0$*

$$b_2\big(\eta_1(t)\big) \geq -t \cdot \cos\alpha - C_1$$

2. *If $\alpha = \frac{\pi}{2}$, then $b_2\big(\eta_1(t)\big)$ is monotone non-increasing in t and $-C_1 \leq b_2\big(\eta_1(t)\big)$.*

3. *If $\alpha < \frac{\pi}{2}$, then for all $t \geq 0$*

$$b_2\big(\eta_1(t)\big) \leq -t \cdot \cos\alpha - C_1$$

Remark II.5.101. If X is a symmetric spaces of $\mathbb{R}$-rank 1, maximal flats are geodesics. Therefore every two points $\xi, \zeta \in X(\infty)$ admit $d_T(\xi, \zeta) = \pi$, and it is clear from the strict negative curvature that Lemma II.5.100 is true also in this case.

In the context of sublinear distance, one has the following lemma about the limit set of two sublinearly close subgroups.

Lemma II.5.102. *Let $\Gamma, \Lambda \leq G$ be discrete subgroups and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ a sublinear function. If $\Gamma \subset \mathcal{N}_u(\Lambda)$, then $\mathcal{L}_\Gamma \subset \mathcal{L}_\Lambda$, i.e. every Γ-limit point is a Λ-limit point. In particular, if Γ is a lattice then the limit set of Λ is the entire $X(\infty)$.*

Remark II.5.103. In Section II.5.4 I prove that in the setting of Proposition II.5.138, the set of Λ-conical limit points contains the set of Γ-conical limit points (Corollary II.5.153). It holds that the conical limit points of Γ are dense in $X(\infty)$ (in the cone topology, see Corollary II.5.71) and therefore $\mathcal{L}_\Lambda = \mathcal{L}_\Gamma = X(\infty)$. In particular, every Γ-limit point is a Λ-limit point. The strength of Lemma II.5.102 is that it does not assume anything on Γ other than that it lies in the u-neighbourhood of Λ. In particular, in Lemma II.5.102 Γ is not assumed to be a lattice.

Proof. By definition, one has to show that given a point $\xi \in X(\infty)$ and a sequence $\gamma_n \in \Gamma$ such that $\gamma_n x_0 \to \xi$ (in the cone topology on $\overline{X}$), there is a corresponding sequence $\lambda_n \in \Lambda$ with $\lambda_n x_0 \to \xi$. Define $\lambda_n := \lambda_{\gamma_n}$ to be the closest point to γ_n in Λ, and to ease notation denote $x_n := \gamma_n x_0, x_n' = \lambda_n x_0$. Let also $\eta_n := [x_0, x_n]$ and $\eta_n' := [x_0, x_n']$ be unit speed geodesics. Finally, let T_n denote the time in which η_n terminates, i.e. $\eta_n(T_n) = x_n$.

Convergence in the cone topology $x_n \to \xi$ is equivalent to the fact that the geodesics η_n converge to $\eta := [x_0, \xi)$ uniformly on compact sets. This means in particular that $T_n \to$

∞. Non-positive curvature guarantees that the functions $F_n(t) := d\big(\eta(t), \eta_n(t)\big), F_n'(t) := d\big(\eta(t), \eta_n'(t)\big)$ and $G_n := d\big(\eta_n(t), \eta_n'(t)\big)$ are convex (F_n' is just a notation, completely unrelated to the derivative of F_n). Since $G_n(0) = F_n(0) = F_n'(0) = 0$ any of these functions is either constant 0 or monotonically increasing, so proving uniform convergence of η_n' to η amounts to proving $\lim_n F_n'(T) = 0$ for every $T \in \mathbb{R}_{\geq 0}$.

From triangle inequality $F_n'(T) \leq F_n(T) + G_n(T)$. I know that $\lim_n F_n(T) = 0$ by assumption. Notice that $G_n(T_n) = d(x_n, x_n') \leq u(|\gamma_n|) = u(T_n)$. Writing $T = \frac{T}{T_n} \cdot T_n$, convexity of G_n implies

$$G_n(T) \leq (1 - \frac{T}{T_n})G_n(0) + \frac{T}{T_n}G_n(T_n) \leq 0 + \frac{T}{T_n}u(T_n) = T \cdot \frac{u(T_n)}{T_n}$$

As $\lim_n T_n = \infty$ it follows from sublinearity that $\lim_n G_n(T) = 0$. I conclude that η_n' converge to η uniformly on compact sets, therefore ξ lies in the limit set of Λ. $\square$

I end this section with an easy fact that will be of use.

Lemma II.5.104. *Assume that a torsion-free discrete group $\Delta \leq \mathrm{Isom}(X)^\circ =: G$ admits the following:*

1. *There is a bound N such that at each point $x \in \Delta \cdot x_0$ there are at most N horoballs that are tangent to x and do not intersect $\Delta \cdot x_0$.*

2. *There is a horosphere $\mathcal{H} \subset X$ such that*

   (a) *The set $\Delta \cdot x_0 \cap \mathcal{H}$ is a cocompact metric lattice in $\mathcal{H}$.*

   (b) *The horoball $\mathcal{HB}$ bounded by $\mathcal{H}$ is Δ-free, i.e. $\Delta \cdot x_0 \cap \mathcal{HB} \subset \mathcal{H}$.*

Then $\big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ is also a cocompact metric lattice in $\mathcal{H}$.

Proof. This is the Pigeonhole Principle. Fix $x \in \Delta \cdot x_0 \cap \mathcal{H}$, and let $\{\mathcal{HB}_i\}_{i \in \{1,\dots,N\}}$ be the finite set of horoballs tangent to x that do not intersect $\Delta \cdot x_0$. Assume w.l.o.g that $\mathcal{H}$ is the bounding horosphere of $\mathcal{HB}_1$. Fix a Δ-orbit point $\delta_0 x_0 \in \Delta \cdot x_0 \cap \mathcal{H}$. Up to translating by some element of Δ, I may assume $x = x_0$. For any other Δ-orbit point $\delta x_0 \in \Delta \cdot x_0 \cap \mathcal{H}$ let $i(\delta) \in \{1, \dots, N\}$ be the index of the horoball $\delta^{-1}\mathcal{HB}_1$. Notice that from hypothesis 1 it indeed follows that $\delta^{-1}\mathcal{HB}_1 \in \{\mathcal{HB}_i\}_{i \in \{1,\dots,N\}}$, because the action of Δ is by isometries. Define δ_i to be the element in Δ for which:

1. $i(\delta_i) = i$, i.e. $\delta_i^{-1}(\mathcal{HB}_1) = \mathcal{HB}_i$

2. $d(\delta_i x_0, x_0)$ is minimal among all such $\delta \in \Delta$.

For some $i \in \{1, \dots, N\}$ there is a $\delta \in \Delta$ with $i = i(\delta)$, while for others there might not be. I will only care about those i for which there is such δ. Assume w.l.o.g that these are $i \in \{1, \dots M\}$ for $M \leq N$, and let $L := \max_{1 \leq i \leq M}\{d(x_0, \delta_i x_0)\} < \infty$. For such an index i_0, the Δ-orbit points that share the same $i(\delta) = i_0$ are in the same $\Delta \cap \mathrm{Stab}_G(\mathcal{H})$ orbit, namely

$$\{\delta x_0 \mid i(\delta) = i_0\} \subset \big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot \delta_{i_0} x_0$$

Indeed, if $\delta^{-1}\mathcal{H}\mathcal{B}_1 = \mathcal{H}\mathcal{B}_{i_0}$ then by definition $\delta\delta_{i_0}^{-1}\mathcal{H}\mathcal{B}_1 = \delta\mathcal{H}\mathcal{B}_{i_0} = \mathcal{H}\mathcal{B}_1$, hence $\delta\delta_{i_0}^{-1} \in \mathrm{Stab}_G(\mathcal{H})$ is an element mapping $\delta_{i_0}x_0$ to δx_0. Let now $\delta x_0 \in \mathcal{H}$. Its distance from the orbit $\big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ is $\big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big)$-invariant, therefore

$$d\Big(\delta x_0, \big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big)x_0\Big) = d\Big(\delta_{i(\delta)} x_0, \big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big)x_0\Big) \leq d(\delta_{i(\delta)}, x_0)$$

The right-hand side is uniformly bounded by L, proving that

$$(\Delta \cdot x_0 \cap \mathcal{H}) \subset \mathcal{N}_L\Big(\big(\Delta \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0\Big)$$

The fact that $\Delta \cdot x_0 \cap \mathcal{H}$ is a cocompact metric lattice in $\mathcal{H}$ renders $(\Delta \cap \mathrm{Stab}_G(\mathcal{H})) \cdot x_0$ a cocompact metric lattice in h as well, as claimed.

□

II.5.3 Translating Geometry into Algebra

The goal of this section is to prove that the results of Section II.5.4 imply that Λ satisfies the two hypotheses of the Benoist-Miquel criterion Theorem II.5.5. Namely, that Λ is Zariski dense, and that it intersects a horospherical subgroup in a cocompact indecomposable lattice. These are algebraic properties, and the proof that Λ satisfies them is in essence just a translation of the geometric results of Section II.5.4 to an algebraic language. The geometric data given by Section II.5.4 is that for some horosphere $\mathcal{H}$ bounding a Λ-free horoball, $\Lambda \cdot x_0$ intersects $\mathcal{H}$ in a cocompact metric lattice (Proposition II.5.138), and that the set of Λ-conical limit points contains the set of Γ-conical limit points (Corollary II.5.153).

Before I start, I remark that by Proposition II.2.86, the fact that $\Lambda \cdot x_0 \cap \mathcal{H}$ is a cocompact metric lattice implies $\Lambda \cap \mathrm{Stab}_G(\mathcal{H})$ is a cocompact lattice in $\mathrm{Stab}_G(\mathcal{H})$.

II.5.3.1 A Horospherical Lattice

From now on I assume that $\Lambda \cap \mathrm{Stab}_G(\mathcal{H})$ is a lattice in $\mathrm{Stab}_G\mathcal{H}$, and I want to show that Λ intersects a horospherical group U of G in a lattice. This step requires quite a bit of algebraic background, which I give below in full. In short, the first goal is to show that $\mathrm{Stab}_G(\mathcal{H})$ admits a subgroup $U \leq \mathrm{Stab}_G(\mathcal{H})$ that is a horospherical subgroup of G. A lemma of Mostow (Lemma II.5.105 below) allows to conclude that Λ intersects U in a lattice.

A horospherical subgroup U is the unipotent radical of a parabolic group (Definition II.2.79). While parabolic groups are usually viewed as algebraic objects (Definition II.2.78), they could equivalently be defined in terms of the geometry of X. It is a matter of fact that parabolic groups of G are exactly the stabilizers of points in the visual boundary of X, i.e. groups of the form $\mathrm{Stab}_G(\xi)$ for $\xi \in X(\infty)$ (Proposition II.5.114). The Langlands decomposition of a parabolic group is described in a geometric context by Eberlein ([28], see Proposition II.5.112 below). This description readily yields the 'Langlands' decomposition of $\mathrm{Stab}_G(\mathcal{H}) \leq \mathrm{Stab}_G(\xi)$ (where ξ is the base point of $\mathcal{H}$). From this structure I am able to deduce that $\mathrm{Stab}_G(\mathcal{H})$ and U satisfy the hypotheses of Mostow's Lemma II.5.105 (Corollary II.5.126).

I begin with Mostow's lemma. It appears in [55] as part of the elaborate research on the structure theory of lattices of nilpotent and solvable Lie groups, having the question of arithmeticity of lattices in mind. Much of the theory for nilpotent groups is due to Malcev, which lends this theory its name as *Malcev theory* (see Shapter 2 in [65]). For a more detailed account on lattices of solvable Lie groups, see Auslander's notes on solvable manifolds [2].

Lemma II.5.105 (Lemma 3.9 in [55])**.** *Let H be a Lie group having no compact connected normal semisimple non-trivial Lie subgroups, and let N be the maximal connected nilpotent normal Lie subgroup of H. Let $\Gamma \leq H$ be a lattice. Then $N/N \cap \Gamma$ is compact.*

Remark II.5.106. In the original statement Mostow uses the term 'analytic group', which I replaced here with 'connected Lie subgroup'. This appears to be Mostow's definition of an analytic group. See e.g. Section 10, Chapter 1 in [46]. In Chevalley's *Theory of Lie Groups*, he defines a *Lie group* as a locally connected topological group whose identity component is an analytic group (Definition 1, Section 8, Chapter 4 in [20]), and proves (Theorem 1, Section 4, Chapter 4 therein) a 1-1 correspondence between analytic subgroups of an analytic group and Lie subalgebras of the corresponding Lie algebra.

Lemma II.5.105 lays the rationale for the rest of this section. Explicitly, I prove that $\mathrm{Stab}_G(\mathcal{H})$ admits a subgroup that is a horospherical subgroup U of G (Corollary II.5.119),

and that U is maximal connected nilpotent normal Lie subgroup of $\mathrm{Stab}_G(\mathcal{H})$ (Corollary II.5.125).

I now recall the algebraic definitions for horospherical and parabolic subgroups of G. There are several equivalent definitions for these objects. Since I use them in the context of Theorem II.5.5, I chose the ones given in [8].

Definition II.5.107. A *minimal parabolic* subgroup $P \leq G$ is the normalizer of a maximal unipotent subgroup of G. A *parabolic* subgroup is any group containing a minimal parabolic subgroup. A parabolic group is *non-trivial* if does not equal G.

Definition II.5.108. A *horospherical subgroup* U of G is the unipotent radical of a non-trivial parabolic subgroup.

In fact, [8] provides two equivalent definitions. Here is the second:

Definition II.5.109. Let $g \in G$. A *horospherical subgroup* of G is a subgroup of the form $U_g := \{u \in G \mid \lim_{n\to\infty} g^{-n}ug^n = id_G\}$. This group is called the *stable group of* g^{-1}.

Remark II.5.110. I do not use nor prove the equivalence of the two definitions. A proof goes through the Langlands decomposition, which I present below. See also Remark II.5.116.

Definition II.5.109 can be related naturally to the definition of geometric parabolic groups, i.e. to stabilizers of points at $X(\infty)$. This is done via the structure theorem for such groups, given below in Proposition II.5.112. Recall (Definition II.2.44) that the exponential map $\exp : \mathfrak{g} \to G$ gives rise to a family of 1-parameter subgroups of the form $\exp(tX)$ for each $X \in \mathfrak{p}$.

Proposition II.5.111 (Proposition 2.17.3 in [28])**.** *Let $x \in X(\infty)$, and let $X \in \mathfrak{p}$ be the tangent vector of the unit speed geodesic $[x_0, \xi)$. Let h^t_ξ be the 1-parameter subgroup defined by $t \mapsto \exp(tX)$. Then an element $g \in G$ fixes ξ if and only if $\lim_{t\to\infty} h^{-t}_\xi g h^t_\xi$ exists.*

Proposition II.5.112 (Propositions 2.17.5 and 2.17.25 in [28])**.** *Let $\xi \in X(\infty)$ and h^t_ξ as in Proposition II.5.111. Let F be a flat containing $[x_0, \xi)$ and $A \leq G$ the maximal abelian subgroup such that $Ax_0 = F$. Denote $G_\xi := \mathrm{Stab}_G(\xi)$, and define $T_\xi : G_\xi \to G$ by $g \mapsto \lim_{n\to\infty} h^{-n}_\xi g h^n_\xi$. Then T_ξ is a homomorphism, and there are subgroups $N_\xi, A_\xi, K_\xi \leq G_\xi$ such that:*

1. *$A_\xi = exp\big(Z(X) \cap \mathfrak{p}\big)$, where $Z(X)$ is the centralizer of X in $\mathfrak{g}$. Moreover, every element $a \in A_\xi$ lies in some conjugate $A^g = gAg^{-1}$ with the property that $[x_0, \xi) \subset F^g := A^g x_0$.*

2. $K_\xi \leq K = \mathrm{Stab}_G(x_0)$ *fixes the bi-infinite geodesic determined by* $[x_0, \xi)$. *In particular, it is a compact subgroup.*

3. $K_\xi A_\xi = A_\xi K_\xi$.

4. $N_\xi = Ker(T_\xi)$. *It is a connected normal subgroup of* G_ξ.

5. $G_\xi = N_\xi A_\xi K_\xi$, *and the indicated decomposition of an element is unique.*

6. $G = N_\xi A_\xi K$, *and the indicated decomposition of an element is unique. In case* ξ *is a regular point at* $X(\infty)$, *this decomposition is the Iwasawa decomposition.*

7. G_ξ *has finitely many connected components, and* $G_\xi^\circ = (K_\xi A_\xi)^\circ N_\xi$.

8. G_ξ *is self normalizing.*

Remark II.5.113. The decomposition of G_ξ is the called *Langlands decomposition*, see e.g. Section 7 of Chapter 7 in [46], where it is described in purely Lie algebraic terms. I give here an informal geometric interpretation of this decomposition. First, consider the case where ξ is a regular point. The geodesic $[x_0, \xi)$ is contained in a unique flat $F = Ax_0$ of X. Since every maximal flat of X has the same dimension as F, the fact that ξ is regular implies that $\exp(X)$ does not commute with any element in $\exp(\mathfrak{p} \setminus \mathfrak{a})$, and therefore $A_\xi = A$. Thinking about A as a copy of $\mathbb{R}^n$, the action of A on itself by left multiplication is an action by translation. Therefore the action of A on $F = Ax_0$ is by translations. In particular one readily concludes that A fixes every point in $F(\infty)$.

The elements in K that fix ξ fix $[x_0, \xi)$, thus map F to a flat that contains $[x_0, \xi)$. The only possibility for such a flat is F. Therefore the isometries of K_ξ are those that map F to itself and that fix $[x_0, \xi)$. From Euclidean geometry one can classify exactly what these isometries could be (e.g, a reflection about some hyperplane of F that contains $[x_0, \xi)$). On the other hand, not any possible isometry of this form in $\mathbb{R}^n$ can actually be realized. The complete list of those isometries that do arise in K_ξ are closely related to the Weyl group and the Weyl chamber structure on X with respect to x_0, see e.g. Section 2.17 in [28].

N_ξ does not lie in K so it does not fix the point x_0. Necessarily $n \in N_\xi$ maps $[x_0, \xi)$ to an asymptotic geodesic $[nx_0, \xi)$. It therefore maps F to another flat F' of X. By Corollary II.5.64, $\mathrm{Stab}_G(\xi)$ acts transitively on the set of geodesics asymptotic to $[x_0, \xi)$. Therefore N_ξ acts transitively on those geodesics. Moreover, the fact that ξ is regular implies that each such geodesic $[nx_0, \xi)$ lies on a unique flat F_n for which $\xi \in F(\infty)$. Therefore the set of such flats is contained in the set $\{nF \mid n \in N_\xi\}$. Moreover, the set of such flats is actually parameterized by N_ξ, i.e. $nF = F$ if and only if $n = e_G$. The reason

is that $nF = F$ implies $n[x_0, \xi)$ is a geodesic that is parallel (in the Euclidean sense) to $[x_0, \xi)$. This will eventually contradict the definition of N_ξ as the kernel of T_ξ.

The decomposition $K_\xi A_\xi N_\xi$ for ξ regular turns into $K_\xi AN$, where N is the same N from the more famous *Iwasawa decomposition* of G (see e.g. Section 4, Chapter 6 in [46]).

If ξ is a singular direction, the geometric ideas from the regular case are still morally true, with some necessary adaptations. The set A_ξ now contains multiple elements that are not in A but that commute with h^t_ξ. These elements correspond to other maximal abelian subgroups A' for which $A'x_0 = F'$ is a flat containing the geodesic ray $[x_0, \xi)$. Elements of such A' act just like the elements of A act in the regular case, only not on F but rather on another flat $F' = A'x_0$. Since $[x_0, \xi)$ is contained in all these flats, the action by an element of A_ξ is by a translation on a flat F', such that the 'branching axis' $[x_0, \xi)$ is mapped to some other branching axis in F'. The elements of K_ξ can permute these flats that branch of $[x_0, \xi)$ while keeping $[x_0, \xi)$ in place.

Back to the proof, the main focus is on N_ξ, which is readily seen to be a horospherical subgroup if one uses Definition II.5.109. However I use the definition of a horospherical subgroup as a unipotent radical of a parabolic subgroup. I therefore address to the statement of the important identification between algebraic parabolic and geometric parabolic subgroups.

Proposition II.5.114 (Proposition 3.8 in [35])**.** *Every non-trivial parabolic subgroup of G is a stabilizer $\mathrm{Stab}_G(\xi)$ for some $\xi \in X(\infty)$, and all such stabilizers are non-trivial parabolic subgroups of G.*

Moreover, one has the following property.

Lemma II.5.115 (See Proposition 7.78 in [46])**.** *Let $P = \mathrm{Stab}_G(\xi) \leq G$ be a parabolic group, with Langlands decomposition $P = K_\xi A_\xi N_\xi$. Then N_ξ is the unipotent radical of P. In particular, it is a horospherical subgroup of G.*

Remark II.5.116. 1. The proof for Proposition II.5.114 is carried via the identification of the Langlands decomposition of $\mathrm{Stab}_G(\xi)$ and $\mathfrak{g}_\xi = Lie\big(\mathrm{Stab}_G(\xi)\big)$ given in Propositions 2.17.5 and 2.17.13, and on the other hand the Langlands decomposition for (algebraic) parabolic subgroups, given e.g. in Section 7, Chapter 7 in [46]. Another view point of this fact stems from Proposition 2.19.16 in [28], which states that an element of G is unipotent if and only if it is contained in some N_ξ. This hints in the direction of Lemma II.5.115. See also Corollary 11.3 and Section 11.21 in [11].

2. Lemma II.5.115 states that the quotient group $K_\xi A_\xi = K_\xi A_\xi N_\xi / N_\xi$ is *reductive*, namely it has trivial unipotent radical. This is formulated in [46] in terms of the corresponding Lie algebra, i.e. it is proved there that $\mathrm{Lie}(K_\xi A_\xi)$ is a reductive Lie algebra.

In order to use Lemma II.5.105, I show that N_ξ is a maximal normal nilpotent connected Lie subgroup of $\mathrm{Stab}_G(\mathcal{H})^\circ$, and that $\mathrm{Stab}_G(\mathcal{H})^\circ$ admits no compact normal factors. This requires to establish the structure of $\mathrm{Stab}_G(\mathcal{H})^\circ$.

Definition II.5.117. In the notation $h_\xi^t = \exp(tX)$ and $A_\xi = \exp\big(Z(X) \cap \mathfrak{p}\big)$ of Proposition II.5.112, define $A_\xi^\perp$ to be the codimension-1 submanifold of A_ξ that is orthogonal to $\{h_\xi(t)\}_{t\in\mathbb{R}}$ (with respect to the Killing form in the Lie algebra).

Claim II.5.118. Every element $a \in A_\xi^\perp$ stabilizes $\mathcal{H} = \mathcal{H}(x_0, \xi)$.

Proof. An element in A_ξ is an element that maps x_0 to a point on a flat $F \subset X$ that contains the geodesic ray $[x_0, \xi)$. If $a \in A_\xi^\perp$, then the geodesic $[x_0, ax_0]$ is orthogonal to $[x_0, \xi)$, and lies in F. From Euclidean geometry and structure of horospheres in Euclidean spaces, it is clear that $ax_0 \in \mathcal{H}(x,\xi)$. Since $a \in G_\xi$, this means $a\mathcal{H} = \mathcal{H}(ax_0, \xi) = \mathcal{H}(x, \xi) = \mathcal{H}$. □

Corollary II.5.119. *Let $\mathcal{H}$ be a horosphere based at ξ. Then $\mathrm{Stab}_G(\mathcal{H})^\circ = (K_\xi A_\xi^\perp)^\circ N_\xi$, and in particular it contains a horospherical subgroup of G. Moreover, $\mathrm{Stab}_G(\mathcal{H})^\circ$ is normal in $\mathrm{Stab}_G(\xi)^\circ$ and acts transitively on $\mathcal{H}$.*

Proof. Clearly $(K_\xi A_\xi^\perp)^\circ N_\xi$ is a codimension-1 subgroup of $\mathrm{Stab}_G(\xi)^\circ$. Since $\mathrm{Stab}_G(\mathcal{H}) \neq \mathrm{Stab}_G(\xi)$ (e.g. $h_\xi^t \notin \mathrm{Stab}_G(\mathcal{H})$ for $t \neq 0$), it is enough to show that $(K_\xi A_\xi^\perp)^\circ N_\xi \leq \mathrm{Stab}_G(\mathcal{H})$. Let $kan \in (K_\xi A_\xi^\perp)^\circ N_\xi$. It fixes ξ, so it is enough to show that $kanx_0 \in \mathcal{H}$. Since $k \in K_\xi$ and $kx_0 = x_0$, it stabilizes $\mathcal{H}$. From Claim II.5.118 $a \in \mathrm{Stab}_G(\mathcal{H})$. So it remains to check that N_ξ stabilizes $\mathcal{H}$, but this is more or less the definition: fixing a base point x_0, the horospheres based at ξ are parameterized by $\mathbb{R}$. Denote them by $\{\mathcal{H}_t\}_{t\in\mathbb{R}}$, where $\mathcal{H} = \mathcal{H}_0$. In this parameterization, any element $g \in G_\xi$ acts on $\{\mathcal{H}_t\}_{t\in\mathbb{R}}$ by translation. I can thus define for $g \in \mathrm{Stab}_G(\xi)$ the real number $l(g)$ to be that number for which $g\mathcal{H}_t = \mathcal{H}_{t+l(g)}$. Clearly $l\big(h_\xi(t)\big) = t$. The element n fixes ξ, so one has

$$h_\xi^{-t} n h_\xi^t \mathcal{H}_0 = h_\xi^{-t} \mathcal{H}_{t+l(n)} = \mathcal{H}_{t+l(n)-t} = \mathcal{H}_{l(n)}$$

The fact that $n \in Ker(T_\xi)$, i.e. that $\lim_{t\to\infty} h_\xi^{-t} n h_\xi^t = e_G$ readily implies that necessarily $l(n) = 0$. I conclude that $(K_\xi A_\xi^\perp)^\circ N_\xi = \mathrm{Stab}_G(\mathcal{H})^\circ$, as wanted.

To see that $\mathrm{Stab}_G(\mathcal{H})^\circ$ acts transitively, recall that $\mathrm{Stab}_G(\xi)^\circ$ acts transitively on X (Corollary II.5.63). Let $x, y \in \mathcal{H}$, and consider $g \in \mathrm{Stab}_G(\xi)^\circ$ with $gx = y$. Writing an

element $g \in G_\xi$ as $ka_t a_\perp n \in K_\xi h_\xi^t A_\xi^\perp N_\xi$, the argument above shows that $kh_\xi^t a_\perp n\mathcal{H}_0 = \mathcal{H}_0$ if and only if $t = 0$, i.e., if and only if $g \in \mathrm{Stab}_G(\mathcal{H})^\circ$.

Finally, let $g \in \mathrm{Stab}_G(\xi)$ and $h \in \mathrm{Stab}_G(\mathcal{H})$. By the discussion above $h \cdot \mathcal{H}_t = \mathcal{H}_t$ for all $t \in \mathbb{R}$. Clearly $-l(g) = l(g^{-1})$, and therefore

$$ghg^{-1}\mathcal{H}_0 = gh\mathcal{H}_{-l(g)} = g \cdot \mathcal{H}_{-l(g)} = \mathcal{H}_0$$

Therefore $\mathrm{Stab}_G(\mathcal{H})$ is normal in $\mathrm{Stab}_G\xi$, and the same is true for the respective identity components. □

Corollary II.5.120. $\mathrm{Stab}_G(\mathcal{H})^\circ$ *is a connected Lie group with no connected compact normal semisimple non-trivial Lie subgroups.*

Proof. Every compact subgroup of G fixes a point. Let $H \leq G$ be some closed subgroup. It is standard to note that a normal $N \leq H$ that fixes a point $x \in X$ must fix every point in the orbit $H \cdot x$: $hnh^{-1}hx = hx$. Since $H = \mathrm{Stab}_G(\mathcal{H})^\circ$ acts transitively on $\mathcal{H}$, it shows that a normal compact subgroup of $\mathrm{Stab}_G(\mathcal{H})^\circ$ fixes every point in $\mathcal{H}$. An isometry fixing a horosphere pointwise while fixing its base point is clearly the identity, proving the claim. □

Corollary II.5.121. *A horosphere in X is not convex.*

Proof. Let $\mathcal{H}'$ be some horosphere in X, with base point $\zeta \in X(\infty)$, and assume towards contradiction that it is convex. Fix $x \in \mathcal{H}'$ and a'_t the one parameter subgroup with $\eta'(\infty) = a'_t x$, and denote $\mathcal{H}'_t = \mathcal{H}(a'_t x, \zeta)$. Let $e_G \neq n \in N_\zeta$ (N_ζ defined with respect to a'_t in a corresponding Langlands decomposition), and consider the curve $\eta'_n(t) := a'_t nx$. I claim that this is a geodesic. On the one hand, the fact that $\mathcal{H}'$ is convex implies that the geodesic segment $[x, nx]$ is contained in $\mathcal{H}'$. Therefore $a'_t[x, nx] = [a'_t x, a'_t nx] \subset \mathcal{H}'_t$. More generally it is clear that because $a'_t\mathcal{H}'_s = \mathcal{H}'_{s+t}$ it holds that $\mathcal{H}'_t$ is convex for every t as soon as it is convex for some t.

On the other hand, for every point $y \in [x, nx]$, $d(y, \mathcal{H}'_t) = t$, and more generally for any $y \in [a'_s nx, a'_s x]$ it holds that $d(y, \mathcal{H}'_t) = |s - t|$. In particular this is true for $\eta'_n(t) = y_t := a'_t nx$. I get that $d\big(\eta'_n(t), \eta'_n(s)\big) = |s - t|$. Therefore η'_n is a geodesic (to be pedantic, by my definition of a geodesic one has to show that η'_n is a continuous curve, which is a result of the fact that a'_t is a one parameter subgroup of isometries). Clearly

$$d\big(\eta'_n(t), \eta'(t)\big) = d(a'_t nx, a'_t x) = d(nx, x)$$

and therefore η'_n is at uniformly bounded distance to η'. This bounds $d(\eta_n, \eta'_n)$ as bi-infinite geodesics, i.e. for all $t \in \mathbb{R}$, not just as infinite rays. The Flat Strip Theorem

(Theorem 2.13, Chapter 2.2 in [17]), then implies that the geodesics η_n, η'_n bound a flat strip: an isometric copy of $\mathbb{R} \times [0, l]$ (where $l = d(x, nx)$).

Up to now I did not use the fact that $n \in N_\xi$, only that the point nx lies on a geodesic that is contained in $\mathcal{H}' = \mathcal{H}'_0$. Therefore the entire bi-infinite geodesic that is determined by $[x, nx]$ lies on a 2-dimensional flat F that contains η'. The two elements n, a'_t therefore admit $nx, a'_t x \in F$. It is a fact that two such elements must commute (see Proposition II.2.66). I can conclude that $[n, a'_t] = e_G$, i.e. n and a'_t commute. This contradicts the fact that $n \in N_\zeta = Ker(T_\zeta)$, thus proving that an arbitrary horosphere $\mathcal{H}'$ of X is not convex. □

Lemma II.5.122 (Theorem 11.13 in [66]). *Let N be a connected real Lie group. Then* $\mathrm{Lie}(N)$ *is a nilpotent Lie algebra if and only if N is a nilpotent Lie group.*

Proposition II.5.123 (Proposition 13, Section 4, Chapter 1 in [13]). *In the notation of Proposition II.5.112,* $\mathfrak{n}_\xi = \mathrm{Lie}(N_\xi)$ *is a maximal nilpotent ideal in* $\mathfrak{g}_\xi = \mathrm{Lie}(G_\xi)$.

Remark II.5.124.

1. The presentation of $\mathfrak{n}_\xi$ in [14] is given by means of the root space decomposition of $\mathrm{Stab}_G(\xi)$, that appears in Proposition 2.17.13 in [28].

2. There are two main objects in the literature that are referred to as the *nilpotent radical* or the *nilradical* of a Lie algebra. These are: (a) the maximal nilpotent ideal of the Lie algebra, and (b) the intersection of the kernels of all irreducible finite-dimensional representations. Proposition 13 in Section 4 of Chapter 9 in [13] shows that in the case of Lie algebras of parabolic Lie groups, these notions coincide.

Corollary II.5.125. *N_ξ is a maximal connected nilpotent normal Lie subgroup of the identity connected component* $\mathrm{Stab}_G(\mathcal{H})^\circ$.

Proof. Lemma II.5.122 implies N_ξ is nilpotent. Since $\mathrm{Stab}_G\mathcal{H} \lhd \mathrm{Stab}_G(\xi)$, every normal subgroup of $\mathrm{Stab}_G(\mathcal{H})$ containing N_ξ is in fact a normal subgroup of $\mathrm{Stab}_G(\xi)$, still containing N_ξ. It remains to prove maximality of N_ξ among all connected nilpotent normal Lie subgroups of $\mathrm{Stab}_G(\xi)$. Any such subgroup $N' \lhd \mathrm{Stab}_G(\xi)$ gives rise to an ideal $\mathfrak{n}'$ of $\mathfrak{g}_\xi = \mathrm{Lie}\big(\mathrm{Stab}_G(\xi)\big)$, and by Lemma II.5.122 it is a nilpotent ideal. Therefore by Proposition II.5.123 it is contained in $\mathfrak{n}_\xi = \mathrm{Lie}(N_\xi)$, implying that $N' \leq N_\xi$.

□

Corollary II.5.126. *A lattice in* $\mathrm{Stab}_G(\mathcal{H})$ *intersects the horospherical subgroup N_ξ in a lattice.*

Proof. Corollaries II.5.120 and II.5.125 imply that the pair $N_\xi \lhd \mathrm{Stab}_G(\mathcal{H})$ satisfy the hypotheses of Mostow's Lemma II.5.105. □

II.5.3.2 Indecomposable Horospherical Lattices

The Benoist-Miquel criterion requires the horospherical lattice to be indecomposable. It is shown in [8] that if this lattice is contained in a Zariski dense discrete subgroup, then the indecomposability condition is equivalent to irreducibility of the ambient group. The precise definitions and statements are as follows.

Definition II.5.127 (Definition 2.14 in [8])**.** For a semisimple real algebraic Lie group G and U a horospherical subgroup of G, let Δ_U be a lattice in U.

1. Δ_U is *irreducible* if for any proper normal subgroup N of G°, one has $\Delta_U \cap N = \{e\}$.
2. Δ_U is *indecomposable* if one cannot write G° as a product $G^\circ = N'N''$ of two proper normal subgroups $N', N'' \lhd G$ with finite intersection such that the group

$$\Delta'_U := (\Delta_U \cap N')(\Delta_U \cap N'')$$

has finite index in Δ_U.

I recall the definition for the irreducibility of a general discrete subgroup:

Definition II.5.128 (Definition II.2.83 above, see Section 2.4.1 in [8])**.** Let G be a semisimple real algebraic Lie group. A discrete subgroup $\Lambda \leq G$ is said to be *irreducible* if, for all proper normal subgroups $N \lhd G$, the intersection $\Lambda \cap N$ is finite.

Lemma II.5.129 (Lemma 4.3 in [8])**.** *Let G be a semisimple real algebraic Lie group, $U \subset G$ a non-trivial horospherical subgroup, and $\Delta_U \leq U$ a lattice of U which is contained in a discrete Zariski dense subgroup Δ of G. Then the following are equivalent:*

1. *Δ is irreducible.*
2. *Δ_U is irreducible.*
3. *Δ_U is indecomposable.*

II.5.3.3 Zariski Density

The last assumption in the Benoist-Miquel criterion is that the discrete subgroup would be Zariski dense. In order to prove that the group Λ is Zariski dense, I use a geometric criterion which is well known to experts.

Lemma II.5.130 (Proposition 2 in [44])**.** *Let X be a symmetric space of noncompact type, $G = \mathrm{Isom}(X)^\circ$. A subgroup $\Delta \leq G$ is Zariski dense if and only if:*

1. *Δ does not globally fix a point in $X(\infty)$, i.e. $\Delta \not\leq \mathrm{Stab}_G(\zeta)$ for any $\zeta \in X(\infty)$.*

2. *The identity component of the Zariski closure of Δ does not leave invariant any proper totally geodesic submanifold in X.*

In the proof I use several facts - mostly algebraic, and two geometric. I warmly thank Elyasheev Leibtag for his help and erudition in algebraic groups. The first property I need is very basic.

Lemma II.5.131. *Let $\Delta \leq G$ be a discrete subgroup, and let $H \leq G$ be the Zariski closure of Δ. Then $\Delta \cap H^\circ$ is of finite index in Δ.*

Proof. H° is normal and of finite index in H. □

The following fact is probably known to experts. It appears in a recent work by Bader and Leibtag.

Lemma II.5.132 (Lemma 3.9 in [3])**.** *Let k be a field, $\mathbf{G}$ a connected k algebraic group, $P \leq G = \mathbf{G}(\mathbb{R})$ a parabolic subgroup. Then the centre of G contains the centre of P.*

Still on the algebraic side, I need a Theorem of Dani, generalizing the Borel Density Theorem for solvable groups.

Theorem II.5.133 (See [23])**.** *Let $\mathbf{S}$ be a real solvable algebraic group. If $S = \mathbf{S}(\mathbb{R})$ is $\mathbb{R}$-split, then every lattice $\Gamma_S \leq S$ is Zariski dense.*

Remark II.5.134. It is a fact (see Theorem 15.4 and Section 18 in [11]) that:

1. Every unipotent group over $\mathbb{R}$ is $\mathbb{R}$-split.

2. For a field k of characteristic 0, a solvable linear algebraic k-group is k-split if and only if its maximal torus is k-split.

Finally I need two geometric facts. The first is a characterization determining when does a unipotent element belongs to N_ζ for some $\zeta \in X(\infty)$.

Proposition II.5.135 (Proposition 4.1.8 in [28])**.** *Let X be a symmetric space of non-compact type and of higher rank, $n \in G = \mathrm{Isom}(X)^\circ$ a unipotent element, and $\zeta \in X(\infty)$. The following are equivalent:*

1. *For N_ζ as in Proposition II.5.112, $n \in N_\zeta$.*

2. *For some geodesic ray η with $\eta(\infty) = \zeta$ it holds that $\lim_{t\to\infty} d\big(n\eta(t), \eta(t)\big) = 0$.*

3. *For every geodesic ray η with $\eta(\infty) = \zeta$ it holds that $\lim_{t\to\infty} d\big(n\eta(t), \eta(t)\big) = 0$.*

The last property I need is a characterization of the displacement function for unipotent elements.

Proposition II.5.136 (See proof of Proposition 3.4 in [5])**.** *Let X be a symmetric space of noncompact type, $\zeta \in X(\infty)$ some point and $n \in N_\zeta$ an element of the unipotent radical of $\mathrm{Stab}_G(\zeta)$. The displacement function $x \mapsto d(nx, x)$ is constant on horospheres based at ζ, and for every $\epsilon > 0$ there is a horoball $\mathcal{HB}_\epsilon$ based at ζ such that $d(nx, x) < \epsilon$ for every $x \in \mathcal{HB}_\epsilon$.*

I can now show that the geometric results of Section II.5.4 imply the Zariski density of Λ.

Corollary II.5.137. *Assume that:*

1. *$\big(\Lambda \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ is a cocompact metric lattice in a horosphere $\mathcal{H} \subset X$ bounding a Λ-free horoball.*

2. *Every Γ-conical limit point is a Λ-conical limit point.*

Then Λ is Zariski dense.

Proof. I show the criteria of Lemma II.5.130 are met. I start by proving that $\Lambda \not\leq \mathrm{Stab}_G(\zeta)$ for any $\zeta \in X(\infty)$. To this end, I first prove that $\Lambda \cdot x_0$ is not contained in any bounded neighbourhood of any horosphere $\mathcal{H}'$. Let ξ' be the base point of $\mathcal{H}'$. By Hattori's Lemma II.5.100 (and Remark II.5.101), it is enough to find a Λ-conical limit point ζ' with $d_T(\xi', \zeta') \neq \frac{\pi}{2}$. Take some $\zeta'' \in X(\infty)$ at Tits distance π of ξ', i.e. take a flat F on which ξ' lies and let ζ'' be the antipodal point to ξ' in F. Fix $\epsilon = \frac{\pi}{4}$. By Corollary II.5.50 and Proposition II.5.45, there are neighbourhoods of the cone topology $U, V \subset X(\infty)$ of ξ', ζ'' (respectively) so that every point $\zeta' \in V$ admits $d_T(\xi', \zeta') \geq d_T(\xi', \zeta'') - \frac{\pi}{4} = \frac{3}{4}\pi$. Recall that the set of Γ-conical limit points is dense (in the cone topology), so the second hypothesis implies there is indeed a Λ-conical limit point in V and therefore at Tits distance different (in this case larger) than $\frac{\pi}{2}$ from ξ'. I conclude that $\Lambda \cdot x_0$ is not contained in any bounded metric neighbourhood of any horosphere of X.

Assume towards contradiction that $\Lambda \leq \mathrm{Stab}_G(\zeta)$. I show that this forces $\Lambda \cap N_\zeta \neq \emptyset$. By Proposition II.5.135 it is enough to find a unipotent element $\lambda \in \Lambda$ and a geodesic η with $\eta(\infty) = \zeta$ such that $\lim_{t\to\infty} d\big(\lambda\eta(t), \eta(t)\big) = 0$. Let F be a maximal flat with $\xi, \zeta \in F(\infty)$, $x \in F$ some point and $X, Y \in \mathfrak{a} \leq \mathfrak{p}$ two vectors such that $\exp(tY) = \eta(t)$ for the unit speed geodesic $\eta = [x, \zeta)$, and $\exp(tX) = \eta'(t)$ for the unit speed geodesic $\eta' = [x, \xi)$ (where $\mathfrak{a} \leq \mathfrak{p}$ a maximal abelian subalgebra in a suitable Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$). Let $\mathrm{Stab}_G(\xi) = K_\xi A_\xi N_\xi$ be the decomposition described in

Proposition II.5.112 with respect to Y (notice that N_ξ does not depend on choice of Y, see item 3 of Proposition 2.17.7 in [28]).

The assumption that $\Lambda \leq \mathrm{Stab}_G(\zeta)$ implies that for any $\lambda \in \Lambda$ the distance $d\big(\lambda\eta(t), \eta(t)\big)$ either tends to 0 as $t \to \infty$ or is uniformly bounded for $t \in \mathbb{R}$. In the latter case there is some constant $c > 0$ for which $d\big(\lambda\eta(t), \eta(t)\big) = c$ for all $t \in \mathbb{R}$. As in the proof of Corollary II.5.121, the Flat Strip Theorem (Theorem 2.13, Chapter 2.2 in [17]) and Proposition II.2.66 imply that λ and $a_t := \exp(tY)$ commute.

From the first hypothesis of the statement and Mostow's result (Corollary II.5.126) I know that $\Lambda \cap N_\xi$ is a cocompact lattice in N_ξ (attention to subscripts). Therefore $\Lambda \cap N_\xi$ is Zariski dense in N_ξ (Theorem II.5.133). Moreover, since commuting with an element is an algebraic property, an element $g \in G$ that commutes with $\Lambda \cap N_\xi$ must also commute with its Zariski closure, namely with N_ξ. This means that if a_t commutes with all $\Lambda \cap N_\xi$ then it commutes with N_ξ, i.e. $a_t n = n a_t$ for all $t \in \mathbb{R}$ and all $n \in N_\xi$. I know that $a_t \in A_\xi$ commutes with both K_ξ and A_ξ (Proposition II.5.112) therefore if a_t also commutes with N_ξ then a_t lies in the centre of $\mathrm{Stab}_G(\xi)$. This means that a_t is central in G (Lemma II.5.132). For a group G with compact centre this cannot happen, so there is indeed some unipotent element $\lambda \in \Lambda \cap N_\xi$ for which $\lim_{t\to\infty} d\big(\lambda\eta(t), \eta(t)\big) = 0$. I conclude from Proposition II.5.135 that $\Lambda \cap N_\zeta \neq \emptyset$.

Recall from the first paragraph that $\Lambda \cdot x_0$ does not lie in any bounded neighbourhood of any horosphere, and in particular in any bounded neighbourhood of a horosphere $\mathcal{H}'$ based at ζ. The assumption that $\Lambda \subset \mathrm{Stab}_G(\zeta)$ implies that every $\lambda \in \Lambda$ acts by translation on the filtration of horospheres $\{\mathcal{H}'_t\}_{t\in\mathbb{R}}$ based at ζ. Therefore as soon as $\Lambda \cdot x_0 \not\subset \mathcal{H}_t$ for some $t \in \mathbb{R}$ one concludes that ζ is a horospherical limit point of Λ, i.e. that every horoball based at ζ intersects the orbit $\Lambda \cdot x_0$.

By Proposition II.5.136 it holds that for a unipotent element $g \in N_\zeta$ the displacement function $x \mapsto d(gx, x)$ depends only on the horosphere $\mathcal{H}'_t$ in which x lies and that, for $x_t \in \mathcal{H}'_t$ it holds that $\lim_{t\to\infty} d(gx_t, x_t) = 0$ (up to reorienting the filtration $t \in \mathbb{R}$ so that $\eta(t) \in \mathcal{H}'_t$). For a non-trivial element $\lambda_\zeta \in \Lambda \cap N_\zeta$ the previous paragraph therefore yields a sequence of elements $\lambda_n \in \Lambda$ such that $\lim_{n\to\infty} d(\lambda_\zeta \lambda_n x_0, \lambda_n x_0) = 0$, contradicting the discreteness of Λ. I conclude that $\Lambda \not\leq \mathrm{Stab}_G(\zeta)$ for every $\zeta \in X(\infty)$.

Assume that $H := \big(\overline{\Lambda}^Z\big)^\circ$, the identity connected component of the Zariski closure of Λ, stabilizes a totally geodesic submanifold $Y \subset X$. By Lemma II.5.131, $\Lambda_0 := \Lambda \cap H$ is of finite index in Λ, therefore $\Lambda_0 \cap \mathrm{Stab}_G(\mathcal{H})$ is also a cocompact lattice in $\mathrm{Stab}_G(\mathcal{H})$. The fact that $\big(\Lambda_0 \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ is a cocompact metric lattice in $\mathcal{H}$ readily implies that $\big(\Lambda_0 \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot y$ is a cocompact metric lattice in $\mathcal{H}_y = \mathcal{H}(y, \xi)$. This goes to show that there is no loss of generality in assuming $x_0 \in \mathcal{H} \cap Y$. It follows that $\Lambda_0 \cap \mathrm{Stab}_G(\mathcal{H}) \cdot x_0 \subset Y \cap \mathcal{H}$, and therefore $\mathcal{H} \subset \mathcal{N}_D(Y)$ for some $D > 0$. A horosphere

is a codimension-1 submanifold, implying that Y is either all of X or of codimension-1. The latter forces $Y = \mathcal{H}$, which is impossible since $\mathcal{H}$ is not totally geodesic ($\mathcal{H}$ is not convex, see Corollary II.5.121). I conclude that H does not stabilize any totally geodesic proper submanifold, and hence that Λ is Zariski dense. □

II.5.4 A Λ-cocompact horosphere

I retain the notation from Section II.4. In particular $d_\gamma = d(\gamma x_0, \lambda_\gamma x_0)$. In this section I prove:

Proposition II.5.138. *If $\{d_\gamma\}_{\gamma\in\Gamma}$ is unbounded, then there exists a horosphere $\mathcal{H}$ based at $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ such that $\big(\Lambda \cap \mathrm{Stab}_G(\mathcal{H})\big) \cdot x_0$ intersects $\mathcal{H}$ in a cocompact metric lattice. Moreover, the bounded horoball $\mathcal{HB}$ is Λ-free.*

Throughout Section II.5.4 the standing assumption is that $\{d_\gamma\}_\gamma \in \Gamma$ is unbounded (as well as Γ being a $\mathbb{Q}$-rank 1 lattice).

The Argument. The proof is entirely geometric, and is essentially composed of chasing down the geometric implications of unbounded d_γ in the geometric structure of $\mathbb{Q}$-rank 1 lattices. This proof is much more involved than the proof for uniform lattices, but the basic mind set and ideas are the same. I encourage the reader to have a look at Section II.4 before reading this one.

While the geometric arguments are quite elementary, some implications are more delicate than they may seem at first glance, and but so the reader might have doubts regarding the necessity of some lemmas or arguments in the proof. For this reason, I now give a detailed sketch of the argument. I also highlight the delicate issues as they appear along the course of this section. The proof consists of the following steps:

1. Unbounded d_γ results in Λ-free horoballs $\mathcal{HB}^\Lambda$ tangent to Λ-orbit points. Each such horoball is based at $\mathcal{W}_{\mathbb{Q}}(\Gamma)$, giving rise to corresponding Γ-free horoballs $\mathcal{HB}^\Gamma$.

2. If d_γ is large, then γx_0 lies deep inside a unique Λ-free horoball tangent to $\lambda_\gamma x_0$. This is achieved using two bounds:

   (a) A bound on the distance $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma)$.

   (b) A bound on the angle $\angle_{\lambda_\gamma x_0}([\lambda_\gamma x_0, \gamma x_0], [\lambda_\gamma x_0, \xi))$, where $\xi \in X(\infty)$ is the base point of a suitable Λ-free horoball tangent to $\lambda_\gamma x_0$.

3. There exist horospheres of Γ, say $\mathcal{H}^\Gamma$, such that if $\gamma x_0 \in \mathcal{H}^\Gamma$ then large d_γ implies large Λ-free areas along the bounding horosphere of some $\mathcal{HB}^\Lambda$.

4. If $\mathcal{HB}^\Lambda$ is boundedly close to some Λ-orbit point, then $\mathcal{H}^\Lambda$ is almost Λ-cocompact, that is $\mathcal{H}^\Lambda \subset \mathcal{N}_D(\Lambda \cdot x_0)$ for some universal D. Together with the previous step, this yields a uniform bound on d_γ along certain horospheres of Γ.

5. Finally, I elevate the almost cocompactness to actual cocompactness, i.e., $\mathcal{H}^\Lambda \subset \mathcal{N}_D(\Lambda \cdot x_0 \cap \mathcal{H}^\Lambda)$ for some $D > 0$. This is immediately elevated to $\mathcal{H}^\Lambda \subset (\Lambda \cap \mathrm{Stab}_G(\mathcal{H}^\Lambda)) \cdot x_0$, proving the proposition.

The Properties of Γ. Here are the geometric properties of Γ which I use:

1. In higher rank: characterization of $\mathcal{W}_{\mathbb{Q}}(\Gamma)$, using conical / non-horospherical limit points (Corollary II.5.99). For $\mathbb{R}$-rank 1 groups: If a point is horospherical, it is conical (see Theorem II.5.98 above).

2. Γ-cocompactness along the horospheres of Γ.

3. For every point $x \in X$ and $C > 0$ there is a bound $K(C)$ on the number of horospheres of Γ that intersect $B(x, C)$ (Corollary II.5.84).

II.5.4.1 Λ-Free Horoballs

I retain the notations and objects defined in Section II.4.1.

Lemma II.5.139. *There exists a Λ-free horoball tangent to x_0.*

Proof. Since $\{d_\gamma\}_{\gamma\in\Gamma}$ is unbounded, there are $\gamma_n \in \Gamma$ be a sequence with $d_n = d_{\gamma_n} = d(\gamma_n, \lambda_n) \to \infty$ monotonically, where $\lambda_n \in \Lambda$ is a Λ-orbit point closest to γ_n. Denote $x'_n = \lambda_n^{-1}\gamma_n x_0$, $\eta_n := [x_0, x'_n]$, and $v_n \in S_{x_0}X$ the initial velocity vectors $v_n := \dot{\eta}_n(0)$. The tangent space $S_{x_0}X$ is compact, so up to a subsequence, v_n converge monotonically in angle to a direction $v \in S_{x_0}X$. Let η be the unit speed geodesic ray emanating from x_0 with initial velocity $\dot{\eta}(0) = v$. Denote $\xi := \eta(\infty)$ the limit point of η in $X(\infty)$.

I claim that the horoball $\mathcal{HB} := \cup_{t>0} B\big(\eta(t), t\big)$, based at ξ and tangent to x_0, is Λ-free. Let $t > 0$ and consider $\eta(t)$. For every $\epsilon > 0$, there is some angle $\alpha = \alpha(t, \epsilon)$ such that any geodesic η' with $\angle_{x_0}(\eta, \eta') < \alpha$ admits $d\big(\eta(t), \eta'(t)\big) < \epsilon/2$. The convergence $v_n \to v$ implies $d\big(\eta(t), \eta_n(t)\big) < \epsilon/2$ for all but finitely many $n \in \mathbb{N}$. In particular, $B\big(\eta(t), t\big) \subset \mathcal{N}_\epsilon\Big(B\big(\eta_n(t), t\big)\Big)$ for all such $n \in \mathbb{N}$.

For a fixed $t \le d_n$, it is clear from the definitions that $B\big(\eta_n(t), t\big) \subset B'_n = B(x'_n, d_n)$. One has $d_n \to \infty$, and so for a fixed $t > 0$ it holds that $t < d_n$ for all but finitely many $n \in \mathbb{N}$. I conclude that for any fixed $t > 0$ there is n large enough such that

$$B\big(\eta(t), t\big) \subset \mathcal{N}_\epsilon\Big(B\big(\eta_n(t), t\big)\Big) \subset \mathcal{N}_\epsilon B'_n$$

I conclude that for every $\epsilon > 0$, $\mathcal{HB} \subset \bigcup_n \mathcal{N}_\epsilon(B'_n) = \mathcal{N}_\epsilon\big(\bigcup_n B'_n\big)$. This fact clearly implies that any point in the interior of $\mathcal{HB}$ must be contained in the interior of one of the Λ-free balls B'_n. This proves $\mathcal{HB}$ is Λ-free.

□

Lemma II.5.140. *Suppose $\mathcal{HB}$ is a Λ-free horoball, based at some point $\xi \in X(\infty)$. Then $\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$.*

Proof. For any geodesic η with limit ξ, the size $d(x_0, \gamma x_0)$ of the Γ-orbit points γx_0 that lie boundedly close to η grows linearly in the distance to any fixed horosphere based at ξ, and in particular to $\mathcal{H} = \partial\mathcal{HB}$. The sublinear constraint $d(\gamma x_0, \lambda_\gamma x_0) \leq u(|\gamma|)$ together with the fact that $\mathcal{HB}$ is Λ-free imply that the size of such γ is bounded. In $\mathbb{R}$-rank 1 every limit point is either conical or in $\mathcal{W}_{\mathbb{Q}}(\Gamma)$, proving the lemma in this case. For higher rank, the above argument actually shows more: it shows that a point $\xi' \in \mathcal{N}_{\frac{\pi}{2}}(\xi)$ is not conical, because every geodesic with limit $\xi' \in \mathcal{N}_{\frac{\pi}{2}}(\xi)$ entres $\mathcal{HB}$ at a linear rate (Lemma II.5.100). Hattori's characterization of $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ (Corollary II.5.99) implies $\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$.

□

Definition II.5.141. Given a Λ-free horoball $\mathcal{HB}^\Lambda$, Lemma II.5.140 gives rise to a horoball of Γ based at the same point at $X(\infty)$. Call this the *horoball corresponding to* $\mathcal{HB}^\Lambda$, and denote it by $\mathcal{HB}^\Gamma$. The corresponding horosphere is denoted $\mathcal{H}^\Gamma$.

Remark II.5.142. During the work on this part I had had a few conversations with Omri Solan regarding the penetration of geodesics into Λ-free horoballs. Assuming $\Lambda \subset \mathcal{N}_u(\Gamma)$ implies that Λ preserves $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ (see Lemma II.5.155). This is the case in the motivating setting where Λ is an abstract finitely generated group that is SBE to Γ, see Claim III.3.12 in Chapter III. Omri suggested, in the case of $SL_2(\mathbb{R})$, to use the action of Λ on the Bruhat-Tits tree of $Sl_2(\mathbb{Q}_p)$ (for all primes p) and the classification of these elements into elliptic and hyperbolic elements (separately for each p) in order to deduce that Λ actually lies in $SL_2(\mathbb{Z})$. We did not pursue that path nor its possible generalization to the SL_n case and general Bruhat-Tits buildings.

II.5.4.2 A Γ-orbit point Lying Deep Inside a Λ-Free Horoball

I established the existence of Λ-free horoballs. It is somewhat odd that the first step in proving $\Lambda \cdot x_0$ is 'almost everywhere' is to prove that there are Λ-free regions. But this fits perfectly well with the algebraic statement that non-uniform lattices must admit unipotent elements (see Proposition 5.3.1 in [54]). The goal of this section is to obtain some control on the location of the Λ-free horoballs, in order to conclude that some γx_0 lies deep inside $\mathcal{HB}^\Lambda$. This allows one to find yet more Λ-free regions.

I will need one property of sublinear functions. I thank Panos Papazoglou for noticing a mistake in the original formulation of this property.

Lemma II.5.143. *Let u be a sublinear function, $f, g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{>0}$ two positive monotone functions with $\lim_{x\to\infty} f(x) + g(x) = \infty$. If for all large enough x it holds that $f(x) \leq u\big(f(x) + g(x)\big)$, then for every $1 < s$ and all large enough x it holds that $f(x) \leq u\big(s \cdot g(x)\big)$. In particular $f(x) \leq u'\big(g(x)\big)$ for some sublinear function u'.*

Proof. Assume as always that u is non-decreasing. By definition of sublinearity $\lim_{x\to\infty} \frac{u\left(f(x)+g(x)\right)}{f(x)+g(x)} = 0$, so by hypothesis $\lim_{x\to\infty} \frac{f(x)}{f(x)+g(x)} = 0$. This means that for every $\epsilon > 0$ one has $f(x) \leq \epsilon \cdot g(x)$ for all large enough x, resulting in

$$f(x) \leq u\big(f(x) + g(x)\big) \leq u\big((1+\epsilon) \cdot g(x)\big)$$

Notice that for a fixed $s > 0$, the function $u'(x) = u(sx)$ is sublinear, as

$$\lim_{x\to\infty} \frac{u(sx)}{x} = \lim_{x\to\infty} s \cdot \frac{u(sx)}{sx} = 0$$

□

Lemma II.5.144. *Let $C > 0$. There is $L = L(C)$ such that if $\mathcal{HB}^\Lambda$ is any Λ-free horoball tangent to a point $x \in B(x_0, C)$ then $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma) \leq L$. Moreover, there is a sublinear function u' such that:*

$$L(C) \leq \begin{cases} u'(C) & \text{if } \mathcal{HB}^\Gamma \subset \mathcal{HB}^\Lambda \\ C & \text{if } \mathcal{HB}^\Lambda \subset \mathcal{HB}^\Gamma \end{cases}$$

Proof. If $\mathcal{HB}^\Lambda \subset \mathcal{HB}^\Gamma$, then clearly $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma) \leq C$, simply because $\mathcal{HB}^\Gamma$ is Γ-free and in particular cannot contain x_0. Therefore $\mathcal{H}^\Gamma$ must separate $\mathcal{H}^\Lambda$ from x_0 and in particular $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma) \leq C$.

Assume that $\mathcal{HB}^\Gamma \subset \mathcal{HB}^\Lambda$, and denote $l = d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma)$. The horoball $\mathcal{HB}^\Gamma$ is a horoball of Γ, hence $\Gamma \cdot x_0$ is D_Γ-cocompact along $\mathcal{H}^\Gamma$ and there is an element $\gamma \in \Gamma$ with $|\gamma| \leq C + l + D_\Gamma$ and $\gamma x_0 \in \mathcal{H}^\Gamma$. Since $\mathcal{HB}^\Lambda$ is Λ-free one has

$$l \leq d(\gamma x_0, \lambda_\gamma x_0) \leq u(|\gamma|) \leq u(C + l + D_\Gamma)$$

C, D_Γ are fixed, so this inequality can only occur for boundedly small l, say $l < L'(C)$ (D_Γ is a universal constant and may be ignored). Consult figure II.2 for a geometric visualization of this situation.

It remains to show that $L'(C)$ is indeed sublinear in C. First define $L(C)$ to be the minimal L that bounds the distance $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma)$ for all possible $\mathcal{HB}^\Lambda$ tangent to a

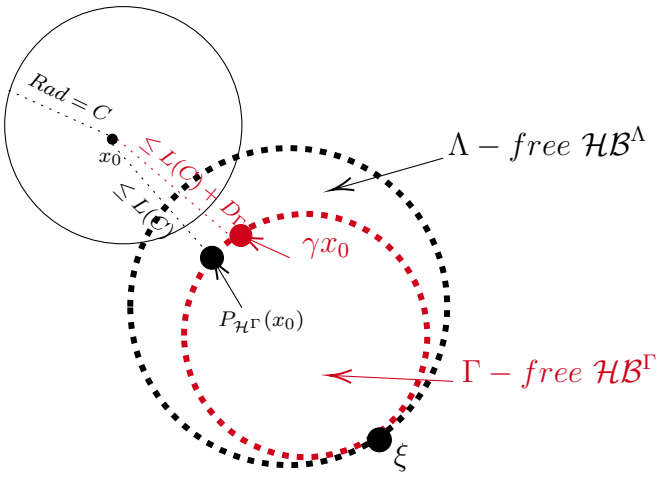


Figure II.2: Lemma II.5.144. A Λ-free horoball $\mathcal{HB}^\Lambda$ intersects a ball of radius C about x_0. The associated Γ-free horoball $\mathcal{HB}^\Gamma$ is boundedly close, essentially due to the uniform cocompactness of $\Gamma \cdot x_0$ along the Γ horospheres.

point $x \in B(x_0, C)$. This is indeed a minimum, since by Corollary II.5.84 there are only finitely many horoballs of Γ intersecting $B\big(x_0, C + L'(C)\big)$. For every C there is thus a horoball $\mathcal{HB}^\Gamma_C$ and an element $\gamma \in \Gamma$ such that $\gamma x_0 \in \mathcal{H}^\Gamma$, $d(\mathcal{H}^\Lambda_C, \mathcal{H}^\Gamma_C) = L(C)$ and $|\gamma| \leq C + L(C) + D_\Gamma$. The fact that $\mathcal{HB}^\Lambda_C$ is Λ-free implies

$$L(C) = d(\mathcal{H}^\Lambda_C, \mathcal{H}^\Gamma_C) \leq u(|\gamma|) \leq u\big(C + L(C) + D_\Gamma\big)$$

Lemma II.5.143 implies that $L(C) \leq u'(C)$ for some sublinear function u'. □

The following is an immediate corollary, apparent already in the above proof.

Corollary II.5.145. *For every $C > 0$ there is a bound $K = K(C)$ and a fixed set $\xi_1, \xi_2, \ldots \xi_K \in \mathcal{W}_\mathbb{Q}(\Gamma) \subset X(\infty)$ so that every Λ-free horoball $\mathcal{HB}^\Lambda$ which is tangent to some point $x \in B(x_0, C)$ is based at ξ_i for some $i \in \{1, \ldots, K\}$.*

In particular, for any specific point $x \in \mathcal{N}_C(\Lambda \cdot x_0)$ there are at most K Λ-free horoballs tangent to x.

Proof. Let $\mathcal{HB}^\Lambda$ be a horoball tangent to a point $x \in B(x_0, C)$. Lemma II.5.144 bounds $d(\mathcal{HB}^\Lambda, \mathcal{HB}^\Gamma)$ by $L(C)$, hence $\mathcal{HB}^\Gamma$ is tangent to a point $x' \in B\big(x_0, C + L(C)\big)$. By Corollary II.5.84 there are only finitely many possibilities for such $\mathcal{HB}^\Gamma$. In particular there are finitely many base-points for these horoballs, say $\xi_1, \xi_2, \ldots, \xi_{K(C)} \in \mathcal{W}_\mathbb{Q}(\Gamma)$. Finally, recall that a horoball is determined by a base point and a point $x \in X$ tangent to it, so the last statement of the corollary holds for any $x \in B(x_0, C)$. But the property in question is Λ-invariant so the same holds for any point $x \in \Lambda \cdot B(x_0, C) = \mathcal{N}_C(\Lambda \cdot x_0)$. □

The bound on $d(\mathcal{HB}^\Lambda, \mathcal{HB}^\Gamma)$ given by Lemma II.5.144 further strengthen the relationship between $\mathcal{HB}^\Lambda$ and $\mathcal{HB}^\Gamma$. The ultimate goal is to show that the $\mathcal{HB}^\Lambda$-s play the role of the Γ-horoballs in the geometric structure of $\mathbb{Q}$-rank 1 lattices, namely to show that $\Lambda \cdot x_0$ is cocompact on the $\mathcal{H}^\Lambda$-s. This requires to actually find Λ-orbit points somewhere in X, and not just Λ-free regions as was done up to now. As one might suspect, these points arise as $\lambda_\gamma x_0$ corresponding to points $\gamma x_0 \in \mathcal{H}^\Gamma$, which exist in abundance since $\Gamma \cdot x_0 \cap \mathcal{H}^\Gamma$ is a cocompact lattice in $\mathcal{H}^\Gamma$.

The hope is that a Λ-free horoball $\mathcal{HB}^\Lambda$ tangent to $\lambda_\gamma x_0$ would correspond to a horoball of Γ tangent to γx_0. This would have forced all the λ_γ to actually lie on the same bounding horosphere, and therefore $\{\lambda_\gamma x_0 \mid \gamma x_0 \in \mathcal{H}^\Gamma\}$ would be a cocompact lattice in $\mathcal{H}^\Lambda$. This hope turns out to be more or less true, but it requires some work. The goal of the rest of this section is to establish a relation between a Λ-free horoball $\mathcal{HB}^\Lambda$ tangent to $\lambda_\gamma x_0$ and γx_0. I start with some notations.

Definition II.5.146. In light of Corollary II.5.145, there is some finite number N of Λ-free horoballs tangent to x_0. I fix the following notations:

1. $\{\mathcal{HB}_i^\Lambda\}_{1=1}^N$ are the Λ-free horoballs tangent to x_0.
2. $\xi^i \in \mathcal{W}_\mathbb{Q}(\Gamma)$ is the base point of $\mathcal{HB}_i^\Lambda$.
3. $v^i \in S_{x_0}X$ is the unit tangent vector in the direction ξ^i.
4. $\eta^i := [x_0, \xi_i)$ is the unit speed geodesic ray emanating from x_0 with limit ξ^i. In particular $v^i = \frac{d}{dt}\eta^i(0)$.
5. $\mathcal{HB}_i^\Gamma$ is the horoball of Γ that corresponds to $\mathcal{HB}_i^\Lambda$. It is based at ξ^i.
6. $\mathcal{HB}_{\lambda,i}^\Lambda, \xi_\lambda^i, \eta_\lambda^i$ are the respective λ-translates of the objects above. For example, $\mathcal{HB}_{\lambda,i}^\Lambda := \lambda \cdot \mathcal{HB}_i^\Lambda$.
7. The notation $\mathcal{H}$ decorated by the proper indices denotes the horosphere bounding $\mathcal{HB}$ with the respective indices, e.g. $\mathcal{H}_i^\Lambda := \partial\mathcal{HB}_i^\Lambda$.
8. For an angle $\alpha > 0$ and a tangent vector $v_0 \in S_xX$, define
   (a) The *α-sector of v in S_xX* is the set $\{v \in S_xX \mid v \in \mathcal{N}_\alpha(v_0)\}$. Recall that the metric on S_xX is the angular metric.
   (b) The *α-sector of v in X* are all points $y \in X$ for which the tangent vector at 0 of the unit speed geodesic $[x, y]$ lies in the α-sector of v in SxX.

Lemma II.5.147. *For every angle $\alpha \in (0, \frac{\pi}{2})$ there exists $D = D(\alpha)$ such that if $d_\gamma > D$ then for some $i \in \{1, \dots, N\}$, γx_0 lies inside the α-sector of $v^i_{\lambda_\gamma}$ at $\lambda_\gamma x_0$. Furthermore whenever α is uniformly small enough, there is a unique such $i = i(\gamma)$, independent of α.*

Proof. Translation by the isometry λ_γ^{-1} preserves angles and distances, so it is enough to prove that there is an i for which $x'_\gamma := \lambda^{-1}\gamma x_0$ lies inside the α-sector of v^i, and that this i is unique if α is uniformly small. Assume towards contradiction that there is $\alpha \in (0, \frac{\pi}{2})$ and a sequence $\gamma_n \in \Gamma$, $\lambda_n := \lambda_{\gamma_n} \in \Lambda$ with d_{γ_n} unbounded, and $x'_n := \lambda_n^{-1}\gamma_n x_0$ not lying in the union of the α-sectors of v^i. By perhaps taking smaller α I may assume all the α-sectors of the v^i in $S_{x_0}X$ are pairwise disjoint. This can be done because there are only finitely many v^i.

Compactness of $S_{x_0}X$ allows me to take a converging subsequence $v'_n := \dot{[x_0, x'_n]}$, with limit direction v'. Denote by η' the geodesic ray emanating from x_0 with initial velocity v'. The exact same argument of Lemma II.5.139 proves that $\eta'(\infty)$ is the base point of a Λ-free horoball tangent to x_0. But this means $v' = v^i$ for some $i \in \{1, \dots, N\}$, contradicting the fact that all v'_n lie outside the α-sectors of the v^i. This proves that there is a bound $D = D(\alpha)$ such that if $d_\gamma > D$ then x'_γ lies within the α-sector of some v^i.

The proof clearly shows that whenever α is small enough so that the α-sectors of the v^i are disjoint, x'_γ lies in the α-sector of a unique v^i as soon as $d_\gamma > D(\alpha)$.

□

Remark II.5.148. In the proof of Lemma II.5.139 I used compactness of $S_{x_0}X$ to induce a converging subsequence of directions. Lemma II.5.147 actually shows that the fact there are finitely many Λ-free horoballs tangent to x_0 implies *a posteriori* that there was not much choice in the process - all directions $[x_0, x'_\gamma]$ must fall into one of the finitely many directions v^i.

Next, I want to control the actual location of certain points with respect to the horoballs of interest, and not just the angles. This turns out to be a more difficult of a task than one might suspect, since control on angles does not immediately give control on distances.

Recall that large Λ-free balls near x_0 imply large concentric Γ-free balls. The precise quantities and bounds are given by Lemma II.4.8.

Proposition II.5.149. *Let $S \in (0, 1)$ be the constant given by Lemma II.4.8, and let $s \in (0, S)$. There is a bound $D = D(s)$ such that $d_\gamma > D$ implies that γx_0 lies sd_γ deep in $\mathcal{HB}^\Lambda_{\lambda_\gamma}$.*

Proof. The proof is a bit delicate but very similar to that of Lemma II.4.8. In essence, I use the Γ-free balls near x_0 to produce a Γ-free cylinder, which would force a certain geodesic not to cross a horosphere of Γ, i.e. force it to stay inside a Γ-free horoball.

As in Lemma II.5.147 it is only required to show that $x' = \lambda_\gamma^{-1}\gamma x_0$ is sd_γ deep inside $\mathcal{HB}^\Lambda_{i(\gamma)}$. I start with proving that $x' \in \mathcal{HB}^\Gamma_{i(\gamma)}$. I learned the hard way that even this is not a triviality.

Recall the notation $B_\gamma = B(\gamma x_0, d_\gamma)$. The ball $B'_\gamma = \lambda_\gamma^{-1}B_\gamma$ is a Λ-free ball of radius d_γ about $x' = \lambda_\gamma^{-1}\gamma x_0$. Denote by x'_t the point at time t along the unit speed geodesic $\eta' := [x_0, x']$. It holds that $|x'_t| = t$ and, for $t \le d_\gamma$, x'_t is the centre of a Λ-free ball of radius t tangent to x_0. The constant s is fixed and by Lemma II.4.8 there is $T' = T'(s)$ such that if $t > T'$, the ball $sB\big(x'_t, t\big)$ is Γ-free.

The next goal is to show that $x'_T \in \mathcal{HB}^\Gamma$ for some adequate T. For any time $T > 0$, let $\alpha = \alpha(\epsilon, T)$ be the angle for which $d\big(\eta(T), \eta^{i(\gamma)}(T)\big) < \epsilon$ for every η in the α-sector of $v^{i(\gamma)}$. By perhaps taking smaller α I may assume that α is uniformly small as stated in Lemma II.5.147. Let $D(\alpha)$ be the bound given by Lemma II.5.147 guaranteeing

$$D(\alpha) < d_\gamma \Rightarrow d\big(x'_T, \eta^{i(\gamma)}(T)\big) < \epsilon$$

For my needs in this lemma ϵ may as well be chosen to be 1. I now choose a specific time T for which I want x'_T and $\eta^{i(\gamma)(T)}$ to be close. There are only finitely many Λ-free horoballs $\{\mathcal{HB}^\Lambda_i\}_{i\in\{1,\dots,N\}}$ tangent to x_0, giving rise to a uniform bound $L = \max_{i\in\{1,\dots,N\}}\{d(\mathcal{H}^\Lambda_i, \mathcal{H}^\Gamma_i)\}$ on the distance $d(\mathcal{H}^\Lambda_{i(\gamma)}, \mathcal{H}^\Gamma_{i(\gamma)})$. Fix T to be any time in the open interval $(T' + L + \epsilon, d_\gamma)$. The fact that $L + \epsilon < T$ implies that $\eta^{i(\gamma)}(T)$ lies at least ϵ-deep inside $\mathcal{HB}^\Gamma_{i(\gamma)}$, and therefore $\eta'(T) \in \mathcal{HB}^\Gamma_{i(\gamma)}$. Recall that any point on $\mathcal{H}^\Gamma$ is D_Γ-close to a point $\gamma x_0 \in \mathcal{H}^\Gamma$. By perhaps enlarging T and shrinking α if necessary, I may assume that $D_\Gamma < sT$. Thus for all $T < t \le d_\gamma$, x'_t is the centre of a Γ-free ball of radius $st > sT > D_\Gamma$, hence $\{x'_t\}_{T\le t\le d_\gamma}$ does not cross a horosphere of Γ. Since $x'_T \in \mathcal{HB}^\Gamma_{i(\gamma)}$, this implies that x'_t stays in $\mathcal{HB}^\Gamma_{i(\gamma)}$ for all $T < t \le d_\gamma$. In particular $x'_{d_\gamma} = x' \in \mathcal{HB}^\Gamma_{i(\gamma)}$.

To get the result of the proposition, recall that $sB'_\gamma = B(x', sd_\gamma)$ is Γ-free, so x' must be at distance at least $sd_\gamma - D_\Gamma$ from any horosphere of Γ, and in particular from $\mathcal{H}^\Gamma_{i(\gamma)}$. In terms of Busemann functions, this means that $b_{\eta^{i(\gamma)}}(x') \le -sd_\gamma + D_\Gamma$ whenever I can find $T' + L + \epsilon < T < d_\gamma$. Since $\mathcal{HB}^\Lambda_{i(\gamma)}$ is tangent to x_0, the corresponding horoball $\mathcal{HB}^\Gamma_{i(\gamma)}$ lies inside it, and so x' lies $(sd_\gamma - D_\Gamma)$-deep inside $\mathcal{HB}^\Lambda_{i(\gamma)}$. A close look at the argument yields the desired bound $D = D(s)$ such that the above holds whenever $d_\gamma > D(s)$. To help the reader take this closer look, I reiterate the choice of constants and their dependencies as they appear in the proof:

1. Fix $\epsilon = 1$.

2. Let $T' = T'(s)$ the constant from Lemma II.4.8 and $L = \max_{i\in\{1,\dots,N\}}\{d(\mathcal{HB}_i^\Lambda, \mathcal{HB}_i^\Gamma)\}$ be the constants given by previous results. The constant L is universal and does not depend on anything.

3. Fix $T > T' + L + 1$.

4. Fix $\alpha = \alpha(1, T)$.

5. Fix $D(s) = \max\{D(\alpha), T + 1\}$.

I remark, for the reader worried about the D_Γ which appears in the final bound but not in the statement, that (a) D_Γ is a fixed universal constant and may as well be ignored, and (b) the discrepancy can be formally corrected by taking a slightly larger $s < s'$ to begin with and as a result perhaps enlarging the bound D for d_γ).

□

II.5.4.3 Intersection of Λ-Free Regions and the Existence of a Λ-Cocompact Horosphere

In this section I find Λ-orbit points that lie close to the bounding horosphere of a Λ-free horoball $\mathcal{HB}^\Lambda$. In order to find such points I need to make sure $\mathcal{HB}^\Lambda$ is not contained inside a much larger Λ-free horoball. I introduce the following definition.

Definition II.5.150. A Λ-free horoball $\mathcal{HB}^\Lambda$ is called *maximal* if it is tangent to a point $x = \lambda x_0 \in \Lambda \cdot x_0$. It is called *$\epsilon$-almost maximal* if $d(\Lambda \cdot x_0, \mathcal{H}^\Lambda) < \epsilon$.

Remark II.5.151. It may happen that a discrete group has no *maximally free* horoballs - see e.g. discussion in section 4 of [70]. In any case it is clear that any Λ-free horoball can be 'blown-up' to an ϵ-almost maximal Λ-free horoball, for every $\epsilon > 0$. Moreover, every two ϵ-almost maximal horoballs based at the same point $\xi \in \mathcal{W}_\mathbb{Q}(\Gamma)$ lie at distance at most ϵ of one another.

For my needs any fixed ϵ would suffice, and I simply take $\epsilon = 1$.

Lemma II.5.152. *There is $D_\Lambda > 0$ such that if $\mathcal{HB}^\Lambda$ is 1-almost maximal Λ-free horoball, then for all $x \in \mathcal{H}^\Lambda$, $d(x, \Lambda \cdot x_0) \le D_\Lambda$, i.e, $\mathcal{H}^\Lambda \subset \mathcal{N}_{D_\Lambda}(\Lambda \cdot x_0)$.*

Notice that Lemma II.5.152 does not state $\Lambda \cdot x_0$ intersects $\mathcal{H}^\Lambda$ at all, and in particular $\Lambda \cdot x_0$ is not, as of yet, a metric lattice in $\mathcal{H}^\Lambda$.

Proof. I start with a short sketch of the proof. Consider a 1-maximal horoball with distance D to $\Lambda \cdot x_0$ at some point of $\mathcal{H}^\Lambda$. One may translate this situation to x_0, which results in a Λ-free horoball $\mathcal{HB}^\Lambda$ intersecting the (closed) D-ball about x_0 at a point w with $B(w, D)$ Λ-free.

The proof then differs depending on whether the $\mathcal{HB}^\Gamma \subset \mathcal{HB}^\Lambda$ or the other way round, since I use the bounds from II.5.144:

1. If $\mathcal{HB}^\Gamma \subset \mathcal{HB}^\Lambda$, there is a sublinear bound on $d(\mathcal{HB}^\Lambda, \mathcal{HB}^\Gamma)$, which readily yields a bound on D.

2. if $\mathcal{HB}^\Lambda \subset \mathcal{HB}^\Gamma$, there are two steps: first see that there is a bound (independent of D) on $d(x_0, \mathcal{HB}^\Gamma)$. So there are only finitely many possibilities for $\mathcal{HB}^\Gamma$, independent of D. Hence there are only finitely many possible base points for $\mathcal{HB}^\Gamma$. These in turn correspond to possible base points for such $\mathcal{HB}^\Lambda$, and this finiteness yields a bound on the distance $d(\mathcal{HB}^\Gamma, \mathcal{HB}^\Lambda) < L$ that is independent of D. The rest of the proof is quite routine.

Let $\mathcal{HB}^\Lambda$ be a 1-almost maximal Λ-free horoball. By definition there is $\lambda \in \Lambda$ and $z \in \mathcal{H}^\Lambda$ such that $d(\lambda x_0, z) < 1$. Fix $D > 0$. I show that if there is some $z' \in \mathcal{H}^\Lambda$ for which $d(z', \Lambda \cdot x_0) \geq D$, then D must be uniformly small. Exactly how small will be set in the course of the proof.

Fix $D > 1$ and assume that there is $z' \in \mathcal{H}^\Lambda$ with $d(z, \Lambda \cdot x_0) \geq D$. Up to sliding z' along $\mathcal{H}^\Lambda$, the continuity of the function $x \mapsto d(x, \Lambda \cdot x_0)$ together with Intermediate Value Theorem allows to assume that $d(z', \Lambda \cdot x_0) = D$. Let $\lambda' \in \Lambda$ be the element for which $d(z', \lambda' x_0) = D$. Translating by λ'^{-1} yields

1. A Λ-free horoball $\mathcal{HB}_0^\Lambda := \lambda'^{-1}\mathcal{HB}^\Lambda$.

2. A point $w := \lambda'^{-1} z' \in \mathcal{H}_0^\Lambda$ for which $|w| = d(w, x_0) = d(w, \Lambda \cdot x_0) = D$.

I aim to use the sublinear constraint for a Γ-orbit point in $\mathcal{H}_0^\Gamma$ in order to show that D must be uniformly bounded.

Assume first that $\mathcal{HB}_0^\Gamma \subset \mathcal{HB}_0^\Lambda$. By Lemma II.5.144 there is a sublinear function u' such that $d(\mathcal{H}_0^\Gamma, \mathcal{H}_0^\Lambda) \leq u'(D)$. This yields a point $\gamma x_0 \in \mathcal{H}_0^\Gamma$ for which $d(w, \gamma x_0) \leq u'(D) + D_\Gamma$. Thus $|\gamma x_0| \leq D + u'(D) + D_\Gamma$ and the reverse triangle inequality gives

$$D - \big(u'(D) + D_\Gamma\big) \leq d(w, \lambda_\gamma x_0) - d(w, \gamma x_0) < d(\gamma x_0, \lambda_\gamma x_0)$$

Together with the bound $d(\gamma x_0, \lambda_\gamma x_0) \leq u(|\gamma x_0|)$ and rearranging, one obtains

$$D \leq u\big(D + u'(D) + D_\Gamma\big) + u'(D) + D_\Gamma$$

The right hand side is clearly a sublinear function in D, hence this inequality may hold only for boundedly small D, say $D < D_1$. I conclude that $\mathcal{HB}_0^\Gamma \subset \mathcal{HB}_0^\Lambda$ may occur only when $D < D_1$. Notice that D_1 depends only on u and u', but not on $\mathcal{HB}^\Lambda$.

Assume next that $\mathcal{HB}_0^\Lambda \subset \mathcal{HB}_0^\Gamma$, and that the containment is strict. This situation requires a bit more work, but the proof is essentially the same as in the first case. Since $x_0 \in \Gamma \cdot x_0$, the geodesic $\tau := [x_0, w]$ is of length D and intersects $\mathcal{H}_0^\Gamma$. Denote by $t_0 \in [0, D)$ the time in which τ intersects $\mathcal{H}_0^\Gamma$, and let $w' := \tau(t_0) \in \mathcal{H}_0^\Gamma$ be the intersection point. In particular $|w'| = t_0$. It is clear that $B(w', t_0)$ is Λ-free, as a subset of the ball $B(w, D)$. Similarly to the first case, there is $\gamma x_0 \in B(w', D_\Gamma) \cap \mathcal{H}_0^\Gamma$ and so $|\gamma x_0| \leq t_0 + D_\Gamma$. By reverse triangle inequality

$$t_0 - D_\Gamma \leq d(w', \lambda_\gamma x_0) - d(w', \gamma x_0) \leq d(\gamma x_0, \lambda_\gamma x_0)$$

and the sublinear constraint gives $t_0 - D_\Gamma \leq u(t_0 + D_\Gamma)$. This can only happen for boundedly small t_0, say $t_0 < T$. I conclude that if $\mathcal{HB}_0^\Lambda \subset \mathcal{HB}_0^\Gamma$, then $\mathcal{HB}_0^\Gamma$ is a horoball of Γ tangent to some point $y \in B(x_0, T)$.

By Corollary II.5.84 there are finitely many horoballs of Γ tangent to points in $B(x_0, T)$. In particular there is a finite set $\{\xi_1', \dots, \xi_K'\} \in \mathcal{W}_\mathbb{Q}(\Gamma)$ of possible base points for $\mathcal{HB}_0^\Gamma$. This set depends only on T, and since the choice of T was completely independent of D, the set of possible base points is independent of D as well. Let $\widetilde{\mathcal{HB}_i^\Gamma}$ be the horoball of Γ based at ξ_i'.

I can now bound the distance $d(\mathcal{HB}_0^\Gamma, \mathcal{HB}_0^\Lambda)$. Let $1 \leq i \leq K$ be an index for which there is a Λ-free horoball based at ξ_i' that is contained in $\widetilde{\mathcal{HB}_i^\Gamma}$. There is thus some 1-almost-maximal Λ-free horoball based at ξ_i'. Fix an arbitrary such 1-almost-maximal Λ-free horoball $\widetilde{\mathcal{HB}_i^\Lambda}$ for each such i, and let $L_i := d(\widetilde{\mathcal{HB}_i^\Lambda}, \widetilde{\mathcal{HB}_i^\Gamma})$. Finally, define $L := \max\{L_i\} + 1$ among such i. As stated in Remark II.5.151, $d(\mathcal{HB}_0^\Lambda, \widetilde{\mathcal{HB}_i^\Lambda}) \leq 1$ for some i, therefore $d(\mathcal{HB}_0^\Gamma, \mathcal{HB}_0^\Lambda) \leq L$.

Recall $|w| = D$ and $B(w, D)$ is Λ-free. It holds that $d(w, \mathcal{H}_0^\Gamma) \leq L$, and so there is $\gamma x_0 \in \mathcal{H}_0^\Gamma$ for which $d(w, \gamma x_0) \leq L + D_\Gamma$. In particular $|\gamma x_0| \leq D + L + D_\Gamma$ (in fact it is clear that $|\gamma x_0| \leq T + D_\Gamma$, but this won't be necessary). Reverse triangle inequality gives

$$D - (L + D_\Gamma) \leq d(w, \lambda_\gamma x_0) - d(w, \gamma x_0) \leq d(\gamma x_0, \lambda_\gamma x_0)$$

and from the sublinear constraint I conclude $D - (L + D_\Gamma) \leq u(D + L + D_\Gamma)$. Since L, D_Γ are fixed constants independent of D, this can only hold for boundedly small D, say $D < D_2$. In particular, one gets a uniform bound $D_\Lambda = \max\{D_1, D_2\}$ such that $x \in \mathcal{H}^\Lambda \Rightarrow d(x, \Lambda \cdot x_0) < D_\Lambda$.

□

With Lemma II.5.152 I can already prove the missing piece for the $\mathbb{R}$-rank 1 case:

Corollary II.5.153. *Every Γ-conical limit point is a Λ-conical limit point.*

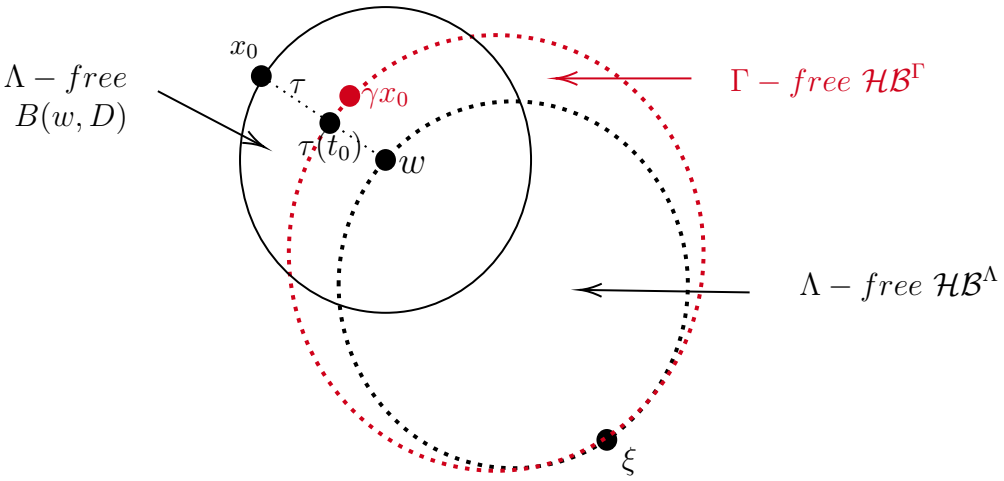


Figure II.3: Lemma II.5.152, case $\mathcal{HB}^\Lambda \subset \mathcal{HB}^\Gamma$. The red horosphere of Γ is trapped between x_0 and $\mathcal{H}^\Lambda$, and is at distance t_0 from x_0. A Γ-orbit point on the red horosphere close to x_0 allows to use sublinearity to get a bound on t_0.

Proof. Let $\xi \in X(\infty)$ be a Γ-conical limit point. Let $\eta : \mathbb{R}_{\geq 0} \to X$ be a geodesic with $\eta(\infty) = \xi$. By definition there is a bound $D > 0$ and sequences $t_n \to \infty, \gamma_n \in \Gamma$ such that $d\big(\gamma_n x_0, \eta(t_n)\big) < D$. Consider the corresponding $\lambda_n := \lambda_{\gamma_n}$ and $\lambda_n x_0$. If d_n is uniformly bounded, then ξ is Λ-conical by definition. Otherwise, assume d_n is monotonically increasing to ∞. For some fixed $s \in (0,1)$ it holds that for all but finitely many $n \in \mathbb{N}$, $\gamma_n x_0$ is sd_n deep inside $\mathcal{HB}^\Lambda_n := \mathcal{HB}^\Lambda_{\lambda_n, i(\gamma_n)}$. I assume d_n is large enough so that $sd_n > D$, and in particular $\eta(t_n) \in \mathcal{HB}^\Lambda_n$. Let $\xi_n = \mathcal{W}_{\mathbb{Q}}(\Gamma)$ be the respective base points of $\mathcal{HB}^\Lambda_n$. The point ξ is Γ-conical, and by Theorem II.5.97 $\frac{\pi}{2} \leq d(\xi, \mathcal{W}_{\mathbb{Q}}(\Gamma)) \leq d(\xi, \xi_n)$.

The proof differs depending on whether the above inequality is strict or not, for a specific $n \in \mathbb{N}$. Assume first that for some $m \in \mathbb{N}$, $d(\xi, \xi_m) = \frac{\pi}{2}$. By item 2 of Lemma II.5.100, $d\big(\mathcal{H}^\Lambda_m, \eta(t)\big)$ is uniformly bounded, i.e., there is $C > 0$ such that for every $t > 0$ there is $x_t \in \mathcal{H}^\Lambda_m$ for which $d\big(x_t, \eta(t)\big) < C$. By Lemma II.5.152, $d(x_t, \Lambda \cdot x_0) < D_\Lambda$, hence $d\big(\eta(t_n), x_{t_n}\big) \leq C + D_\Lambda$. This means that ξ is Λ-conical.

Otherwise, for all $n \in \mathbb{N}$ it holds that $\frac{\pi}{2} < d(\xi, \xi_n)$. The fact that $\eta(t_n) \in \mathcal{HB}^\Lambda_n$ together with Lemma II.5.100 implies that at some later time the geodesic ray η leaves $\mathcal{HB}^\Lambda_n$. Thus there is $s_n > t_n$ for which $\eta(s_n) \in \mathcal{H}^\Lambda_n$. Since $\mathcal{HB}^\Lambda_n$ are maximal Λ-free horoballs, Lemma II.5.152 gives rise to points $\lambda_n x_0$ such that $d\big(\lambda_n x_0, \eta(s_n)\big) \leq D_\Lambda$. This renders ξ as a Λ-conical limit point, as wanted.

□

I now prove Proposition II.5.138.

Proof of Proposition II.5.138. The strategy for finding a horosphere which $\Lambda \cdot x_0$ intersects in a cocompact metric lattice is as follows. First recall that there are maximal Λ-free horoballs in X - see Lemma II.5.139. For $\mathcal{HB}^\Lambda = \mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$, one uses Proposition II.5.149 to get that $\mathcal{HB}^\Gamma \subset \mathcal{HB}^\Lambda$ and that the distance $d(\mathcal{H}^\Lambda, \mathcal{H}^\Gamma)$ is large with d_γ. The horosphere $\mathcal{H}^\Gamma$ admits a Γ-cocompact metric lattice, and so the projections of these metric lattice points onto $\mathcal{H}^\Lambda$ form a cocompact metric lattice in $\mathcal{H}^\Lambda$ (Lemma II.5.35). It remains to show that for each $\gamma' x_0 \in \Gamma \cdot x_0 \cap \mathcal{H}^\Gamma$, the corresponding $\lambda' = \lambda_{\gamma'}$ indeed lies on the same $\mathcal{H}^\Lambda$ and boundedly close to the projection $P_{\mathcal{H}^\Lambda}(\gamma' x_0)$. This is done by putting together all the geometric facts obtained up to this point, specifically Lemma II.5.152. One delicate fact that will be of use is that two maximal Λ-free horoballs that are based at the same point must be equal, because none of them can contain a Λ-orbit point while on the other hand both bounding horospheres intersect $\Lambda \cdot x_0$.

Fix $s > 0$ for which Proposition II.5.149 yields a corresponding bound $D(s)$, and let $\gamma \in \Gamma$ such that $sd_\gamma > 2 \cdot \big(D_\Lambda + D(s)\big)$. Consider the (maximal) Λ-free horoball $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ based at $\xi^{i(\gamma)}_\lambda$. I show that $\Lambda \cdot x_0 \cap \mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$ is a cocompact metric lattice in $\mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$. I keep the subscript notation because the proof is a game between $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ and another Λ-free horoball.

Let $\mathcal{HB}^\Gamma_{\lambda_\gamma, i(\gamma)}$ be the Γ-horoball corresponding to $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$. I can conclude that $\mathcal{HB}^\Gamma_{\lambda_\gamma, i(\gamma)} \subset \mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$, because the choice of $d_\gamma > D(s)$ guarantees γx_0 is sd_γ deep inside $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$. In particular $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ is not Γ-free. Moreover, it holds that $L := d(\mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}, \mathcal{H}^\Gamma_{\lambda_\gamma, i(\gamma)}) \geq sd_\gamma$. Let $\gamma' \in \Gamma$ be any element in the cocompact metric lattice $\Gamma \cdot x_0 \cap \mathcal{H}^\Gamma_{\lambda_\gamma, i(\gamma)}$, and consider two associated points: (a) $\lambda' x_0 = \lambda_{\gamma'} x_0$ and (b) the projection of $\gamma' x_0$ on $\mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$, denoted $p'_\gamma := P_{\mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}}(\gamma' x_0) \in \mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$. The horoball $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ is a maximal Λ-free horoball so it is also 1-almost maximal, hence $d(p'_\gamma, \Lambda \cdot x_0) \leq D_\Lambda$ and the following holds:

$$sd_\gamma \leq L \leq d_{\gamma'} \leq d(\gamma' x_0, p'_\gamma) + d(p'_\gamma, \Lambda \cdot x_0) \leq L + D_\Lambda \tag{II.4}$$

Consider $\xi^{i(\gamma')}_{\lambda'}$, and assume towards contradiction that $\xi^{i(\gamma')}_{\lambda_{\gamma'}} \neq \xi^{i(\gamma)}_{\lambda_\gamma}$. Both points lie in $\mathcal{W}_\mathbb{Q}(\Gamma)$ and therefore must be at Tits distance π of each other. Therefore the fact that $\gamma' x_0$ lies in $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ implies that the geodesic $[\gamma' x_0, \xi^{i(\gamma')}]$ leaves $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ at some point $z \in \mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$.

The fact that $D(s) \leq sd_\gamma \leq d_{\gamma'}$ implies that $\gamma' x_0$ lies $s^2 d_\gamma$ deep inside $\mathcal{HB}^\Lambda_{\lambda_{\gamma'}, i(\gamma')}$. Therefore the point z also lies at least $s^2 d_\gamma$ deep inside $\mathcal{HB}^\Lambda_{\lambda_{\gamma'}, i(\gamma')}$, and therefore z is the centre of a Λ-free horoball of radius at least $s^2 d_\gamma$.

By choice of d_γ the point z therefore admits a $2D_\Lambda$ neighbourhood that is Λ-free. But z lies on $\mathcal{H}^\Lambda_{\lambda_\gamma, i(\gamma)}$, a maximal horosphere of Λ, contradicting Lemma II.5.152. I conclude that $\xi^{i(\gamma')}_{\lambda_{\gamma'}} = \xi^{i(\gamma)}_{\lambda_\gamma}$, so $\mathcal{HB}^\Lambda_{\lambda_\gamma, i(\gamma)}$ and $\mathcal{HB}^\Lambda_{\lambda_{\gamma'}, i(\gamma')}$ are two Λ-free horoballs that are tangent

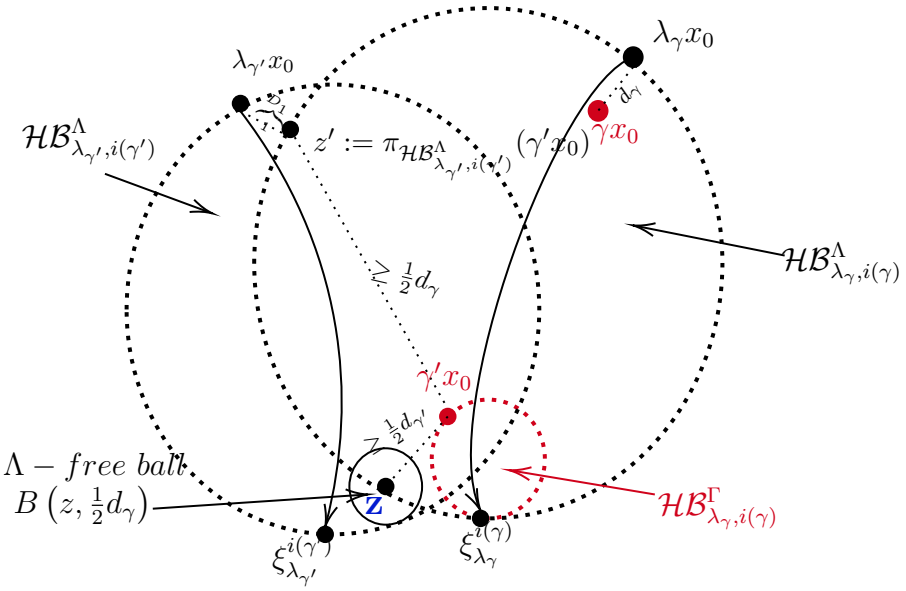


Figure II.4: Proposition II.5.138. Assuming towards contradiction that $\xi^{i(\gamma)}_{\lambda_\gamma} \neq \xi^{i(\gamma')}_{\lambda_{\gamma'}}$ results in a point $z \in \mathcal{HB}^\Lambda_{\lambda_\gamma,i(\gamma)}$ (blue coloured and bold faced in the bottom part of the figure) admitting a large Λ-free neighbourhood, contradicting almost cocompactness.

to a $\Lambda \cdot x_0$ point and based at the same point at ∞. This implies $\mathcal{HB}^\Lambda_{\lambda_\gamma,i(\gamma)} = \mathcal{HB}^\Lambda_{\lambda_{\gamma'},i(\gamma')}$, and in particular $\lambda_{\gamma'} x_0 \in \mathcal{H}^\Lambda$. Finally, it is clearly seen from Inequality II.4 that

$$d(\lambda' x_0, p'_\gamma) \leq d(\lambda' x_0, \gamma' x_0) + d(\gamma' x_0, p'_\gamma) \leq d_{\gamma'} + L \leq L + D_\Lambda + L$$

The element $\gamma' x_0 \in \Gamma \cdot x_0 \cap \mathcal{H}^\Gamma_{\lambda_\gamma,i(\gamma)}$ was as arbitrary element, and the above argument shows that the corresponding Λ-orbit points satisfy:

1. $\lambda' x_0$ all lie on $\mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$.
2. Each p'_γ is $2L + D_\Lambda$ close to the point $\lambda' x_0$.

This shows that the cocompact metric lattice $\{p'_\gamma \mid \gamma' x_0 \in \mathcal{H}^\Gamma_{\lambda_\gamma,i(\gamma)}\}$ lies in a bounded neighbourhood of the set of points $\Lambda \cdot x_0 \cap \mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$, proving that $\Lambda \cdot x_0 \cap \mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$ is a cocompact metric lattice in $\mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$. Lemma II.5.104 elevates this to

$$\left(\Lambda \cap \mathrm{Stab}_G(\mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)})\right) \cdot x_0 \cap \mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$$

being a cocompact metric lattice in $\mathcal{H}^\Lambda_{\lambda_\gamma,i(\gamma)}$, completing the proof. □

II.5.5 The Bounded Case

Proposition II.5.138 is enough in order to prove Theorem II.5.1 in the case $\Gamma \not\subset \mathcal{N}_D(\Lambda)$ for any $D > 0$, i.e. in case Γ does not lie in a bounded neighbourhood of Λ. In this section I deal with the case where $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$. While this seems like a stronger condition, my proof for this case requires the extra assumption that $\Lambda \subset \mathcal{N}_u(\Gamma)$. It very well may be that this assumption is not necessary. I prove the following proposition:

Proposition II.5.154. *Let G be a real Lie group without compact factors and with finite centre, $\Gamma \leq G$ a lattice of $\mathbb{Q}$-rank 1 , $\Lambda \leq G$ a discrete subgroup, and u a sublinear function. Assume that $\Gamma \subset \mathcal{N}_D(\Lambda)$ for some $D > 0$. If $\Lambda \subset \mathcal{N}_u(\Gamma)$, then actually $\Lambda \subset \mathcal{N}_{D'}(\Gamma)$ for some $D' > 0$.*

Moreover, if G is of $\mathbb{R}$-rank 1, the conclusion holds under the relaxed assumption that $u(r) \preceq_\infty \epsilon r$ for some $\epsilon < 1$.

Proposition II.5.154 allows one to reduce Theorem II.5.1 to Schwartz's Theorem II.5.2 in the case $\Gamma \subset \mathcal{N}_D(\Lambda)$.

In Section II.5.5.1 I present Schwartz's argument for Theorem II.5.2 in full. I do this for three reasons. First, Theorem II.5.2 is a slight modification of the original statement and I want to explain this modification. Second, the full argument does not appear in the literature and with the encouragement of my examiners I decided to take the opportunity and make it public. Lastly, it is a beautiful argument that uses a different viewpoint that is somewhat complementary to the one I take in this thesis. I thank Rich Schwartz for explaining it to me in detail.

I start with the proof of Proposition II.5.154. The arguments are geometric and similar to those used in Section II.5.4. The first step is to establish the fact that Λ must preserve $\mathcal{W}_{\mathbb{Q}}(\Gamma)$.

Lemma II.5.155. *Let G be a real Lie group without compact factors and with finite centre, $\Gamma \leq G$ a lattice of $\mathbb{Q}$-rank 1 , $\Lambda \leq G$ a discrete subgroup. Assume that $\Gamma \subset \mathcal{N}_u(\Lambda)$ and that $\Lambda \subset \mathcal{N}_{u'}(\Gamma)$ for a sublinear functions u, u'. Then $\Lambda \cdot \mathcal{W}_{\mathbb{Q}}(\Gamma) \subset \mathcal{W}_{\mathbb{Q}}(\Gamma)$.*

Moreover, if G is of $\mathbb{R}$-rank 1, the conclusion holds under the relaxed assumption that $u'(r) \preceq_\infty \epsilon r$ for some $\epsilon < 1$.

Proof. The proof is similar to the argument of Lemma II.5.140, and uses the linear penetration rate of a geodesic into a horoball. Let $\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$, and let $\mathcal{H}^\Gamma$ be a horosphere bounding a Γ-free horoball $\mathcal{HB}^\Gamma$ with $\mathcal{H}^\Gamma \cap \Gamma \cdot x_0$ a metric lattice in $\mathcal{H}^\Gamma$. Assume first that u' is sublinear. Since $\mathcal{HB}^\Gamma$ is Γ-free and $\Lambda \subset \mathcal{N}_{u'}(\Gamma)$, I can conclude that $\Lambda \cdot x_0 \cap \mathcal{HB}^\Gamma \subset \mathcal{N}_{u'}(\mathcal{H}^\Gamma)$. Recall (Lemma II.5.100) that every geodesic ray η with limit point $\xi' \in \mathcal{N}_{\frac{\pi}{2}}(\xi)$

penetrates $\mathcal{HB}^\Gamma$ at linear rate. Therefore for every such geodesic ray η and every sublinear function v there is $R = R(\eta, v) > 0$ for which $\mathcal{N}_v(\eta_{|r>R})$ is Λ-free.

On the other hand, let $\lambda \in \Lambda$, and assume towards contradiction that $\lambda\xi \notin \mathcal{W}_{\mathbb{Q}}(\Gamma)$. Then by Proposition II.5.99 there is a Γ-conical limit point $\xi' \in \mathcal{N}_{\frac{\pi}{2}}(\lambda\xi)$. The hypothesis that $\Gamma \subset \mathcal{N}_u(\Lambda)$ then implies that for every $R > 0$, $\mathcal{N}_u(\eta_{|r>R}) \cap \Lambda \cdot x_0 \neq \emptyset$. Translating by λ^{-1} yields a contradiction to the previous paragraph. I conclude that $\lambda\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$.

I now modify the argument to include $u'(r) \preceq_\infty \epsilon r$ when G is of $\mathbb{R}$-rank 1. In this case, the only point $\xi' \in \mathcal{N}_{\frac{\pi}{2}}(\lambda\xi)$ is $\lambda\xi$ itself. Therefore by the same argument as above, the assumption that $\lambda\xi \notin \mathcal{W}_{\mathbb{Q}}(\Gamma)$ implies that the u-sublinear neighbourhood of every geodesic ray with limit point ξ intersects $\Lambda \cdot x_0$. I.e., for every η with limit point ξ and every $R > 0$ it holds that $\mathcal{N}_u(\eta_{|r>R}) \cap \Lambda \cdot x_0 \neq \emptyset$.

On the other hand, every such geodesic penetrates $\mathcal{HB}^\Gamma$ at 1-linear rate. This amounts to the following fact: if $v'(r) = \epsilon r$ for some $\epsilon \in (0, 1)$, then for some $R > 0$, the set $\mathcal{N}_u(\eta_{|r>R}) \cap \mathcal{N}_{v'}(\mathcal{H}^\Gamma) = \emptyset$. This is a contradiction to $\Lambda \subset \mathcal{N}_{u'}(\Gamma)$. □

Proof of Proposition II.5.154. The proof is geometric. In short, I obtain a characterization of certain horoballs of Γ in terms of their intersection pattern with $\Lambda \cdot x_0$, and use the Λ-invariance of this pattern in order to view this situation near x_0, where I have some control on how the picture should look like. Assume towards contradiction that there is a sequence λ_n such that $d(\lambda_n x_0, \Gamma \cdot x_0) > n$. Recall that $\Gamma \cdot x_0$ is a cocompact metric lattice in the compact core of Γ. This implies that there is a number $D' > 0$ such that any $\lambda \in \Lambda$ for which $\lambda x_0 \notin \mathcal{N}_{D'}(\Gamma \cdot x_0)$ must lie at least $\frac{1}{2}D'$-deep inside a horoball of Γ. Without loss of generality I can assume that for all $n \in \mathbb{N}$ there are corresponding horoballs of Γ, which I denote $\mathcal{HB}^\Gamma_n$, for which $\lambda_n \cdot x_0 \in \mathcal{HB}^\Gamma_n$. The fact that $\Gamma \subset \mathcal{N}_D(\Lambda)$ then implies that $\mathcal{N}_D(\Lambda \cdot x_0)$ covers a cocompact metric lattice in $\mathcal{H}^\Gamma_n$, namely the metric lattice $\Gamma \cdot x_0 \cap \mathcal{H}^\Gamma_n$. In the terminology of Section II.5.4, $\mathcal{H}^\Gamma_n$ is almost Λ-cocompact, or D-almost Λ-cocompact.

I first prove that every horoball of Γ contains a Λ-free horoball (this is of course immediate if $\Lambda \subset \mathcal{N}_C(\Gamma)$ for some $C > 0$). Assume towards contradiction that there is a horoball $\mathcal{HB}^\Gamma$ of Γ that does not contain a Λ-free horoball. Denote $\mathcal{H}^\Gamma := \partial\mathcal{HB}^\Gamma$. In the notations of the previous paragraph, I can assume without loss of generality that $\mathcal{HB}^\Gamma = \mathcal{HB}^\Gamma_n$ for all $n \in \mathbb{N}$. Denote by ξ the base point of $\mathcal{HB}^\Gamma$, fix some arbitrary $x \in \mathcal{H}^\Gamma$ and consider the geodesic ray $\eta := [x, \xi)$. The constraint that $\Lambda \subset \mathcal{N}_u(\Gamma)$ implies that for every $R > 0$ there is some $L > 0$ for which the ball $B\big(\eta(L + t), R\big)$ is Λ-free, for all $t \geq 0$. In particular, for all large enough $n \in \mathbb{N}$ (depending on R), the horosphere that is parallel to $\mathcal{H}^\Gamma$ and that passes through $\lambda_n x_0$ contains a point that is the centre of Λ-free ball of radius R. This property is Λ-invariant, as well as the fact that $\mathcal{HB}^\Gamma$ is based at

$\mathcal{W}_{\mathbb{Q}}(\Gamma)$. In particular, these two properties hold for the horoballs $\mathcal{HB}_n := \lambda_n^{-1} \cdot \mathcal{HB}^\Gamma$, whose respective base points I denote $\xi_n := \lambda_n^{-1}\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$.

Fix $R = D + 2D_\Gamma$ (recall that D_Γ is such that every horosphere $\mathcal{H}$ of Γ admits $\mathcal{H} \subset \mathcal{N}_{D_\Gamma}(\Gamma \cdot x_0 \cap \mathcal{H})$). Let $L = L(D + D_\Gamma)$ be the corresponding bound from the previous paragraph. For every $n > L$ the horoball $\mathcal{HB}_n$ has bounding horosphere $\mathcal{H}_n$ that admits a point $z_n \in \mathcal{H}_n$ for which $B(z_n, D + 2D_\Gamma)$ is Λ-free. Moreover, the same is true for every horosphere that is parallel to $\mathcal{H}_n$ which lies inside $\mathcal{HB}_n$. Since $\Gamma \subset \mathcal{N}_D(\Lambda)$, this means that every horosphere that lies inside $\mathcal{HB}_n$ admits a point that is the centre of a Γ-free ball of radius $2D_\Gamma$. I conclude that none of those horospheres could be the horosphere of Γ corresponding to the parabolic limit point $\xi_n \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$. Since $x_0 \in \mathcal{H}_n$ it must therefore be that $\mathcal{H}_n$ is a horosphere of Γ. But this contradicts the fact that $z_n \in \mathcal{H}_n$ and $B(z_n, 2D_\Gamma)$ is Γ-free. This shows that no horoball of Γ contains a sequence of Λ-orbit points that lie deeper and deeper in that horoball. Put differently, it shows that every horoball of Γ contains a Λ-free horoball.

I remark that the above argument shows something a bit stronger, which I will not use but which I find illuminating. It proves that as soon as $d(\lambda x_0, \Gamma \cdot x_0)$ is uniformly large enough, say more than M, then λx_0 must lie on a $(D + D_\Gamma)$-almost Λ-cocompact horosphere parallel to $\mathcal{H}^\Gamma$, where $\mathcal{H}^\Gamma$ is the bounding horosphere of any horoball of Γ in which λx_0 lies (recall that it must lie in at least one such horoball). On the other hand if $d(\lambda x_0, \Gamma \cdot x_0) < M$, then since every point in the Γ-orbit lies on a horosphere of Γ one concludes that λx_0 lies on a horosphere $\mathcal{H}$ based at $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ that is $(M + D)$-almost Λ-cocompact.

I can now assume that every $\mathcal{HB}_n^\Gamma$ contains a Λ-free horoball. In particular it contains a 1-almost maximal Λ-free horoball $\mathcal{HB}_n^\Lambda$ (see Definition II.5.150). By definition, there is a point $\lambda'_n x_0$ that is at distance at most 1 from $\mathcal{H}_n^\Lambda = \partial\mathcal{HB}_n^\Lambda$. Up to enlarging $d(\lambda_n x_0, \Gamma \cdot x_0)$ or decreasing it by at most 1, I can assume $\lambda_n = \lambda'_n$ to begin with. Consider $\mathcal{HB}_n := \lambda_n^{-1} \cdot \mathcal{HB}_n^\Lambda$ with $\mathcal{H}_n = \partial\mathcal{HB}_n$. This is a sequence of horoballs, each of which contains a Λ-free horoball at depth at most 1, based at corresponding parabolic limit points $\xi_n \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$, and tangent to points that are at distance at most 1 from x_0, i.e., $\mathcal{H}_n \cap B(x_0, 1) \neq \emptyset$.

Since $\Gamma \subset \mathcal{N}_D(\Lambda)$ I conclude that each of the $\mathcal{HB}_n$ contain a horoball of depth at most $D + 1$ that is Γ-free. Therefore the horoball of Γ that is based at ξ_n must have its bounding horosphere intersecting $B(x_0, D + 2)$. By Corollary II.5.84 there are only finitely such horoballs. I conclude that there are finitely many points $\xi'_1, \dots, \xi'_K \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ such that for every $n \in \mathbb{N}$ there is $i(n) \in \{1, \dots, K\}$ with $\xi_n = \xi'_{i(n)}$. From the Pigeonhole Principle there is some $\xi' \in \{\xi'_1 \dots, \xi'_K\}$ for which $\xi_n = \xi'$ for infinitely many $n \in \mathbb{N}$. Passing to a subsequence I assume that this is the case for all $n \in \mathbb{N}$.

To begin with the $\mathcal{HB}_n^\Gamma$ are horoballs of Γ, and therefore as in the first case the bounding horospheres $\mathcal{H}_n^\Gamma$ are $D+D_\Gamma$-almost Λ-cocompact. This is a Λ-invariant property and therefore the same thing holds for the λ_n^{-1} translate of it. These are the horospheres which are based at ξ' and lie outside $\mathcal{HB}_n$ at distance $d(\lambda_n x_0, \Gamma \cdot x_0) > n-1$ from $\mathcal{H}_n$. They form a sequence of outer and outer horospheres based at the same point at $\mathcal{W}_\mathbb{Q}(\Gamma)$, all of which are $D + D_\Gamma$-almost Λ-cocompact. This is a contradiction, since the union of such horospheres intersect every horoball of X, contradicting the existence of Λ-free horoballs. Formally, take some $\zeta \in \mathcal{W}_\mathbb{Q}(\Gamma)$ different from ξ'. Since both ξ' and ζ lie in $\mathcal{W}_\mathbb{Q}(\Gamma)$, they admit $d_T(\zeta, \xi') = \pi$ and there is a geodesic η with $\eta(-\infty) = \xi'$ and $\eta(\infty) = \zeta$. Let $\mathcal{HB}_\zeta^\Gamma$ be the horoball of Γ that is based at ζ. By the first step of this proof, every such horoball must contain a Λ-free horoball $\mathcal{HB}_\zeta^\Lambda$. Therefore there is some $T > 0$ such that for all $t > T$ the point $\eta(t)$ lies $2(D + D_\Gamma)$ deep in $\mathcal{HB}_\zeta^\Lambda$. I conclude that for all $t > T$, $B\big(\eta(t), 2D\big)$ is Λ-free. On the other hand, for arbitrarily large t it holds that the horosphere based at ξ' and tangent to $\eta(t)$ is $D + D_\Gamma$-almost Λ-cocompact, and in particular $d\big(\eta(t), \Lambda \cdot x_0\big) < D + D_\Gamma$, a contradiction.

I conclude that $\Lambda \subset \mathcal{N}_{D'}(\Gamma)$ for some $D' > 0$, as claimed. □

II.5.5.1 Schwartz's Argument

In his work, Schwartz proves the quasi-isometric rigidity of lattices in semisimple Lie groups of $\mathbb{R}$-rank 1, i.e., he proves that the class of non-uniform is quasi-isometrically complete. The key step in his argument is showing that the quasi-isometric image of horospheres of Γ must lie uniformly boundedly close to horospheres of Γ. That is, if $X_0 \subset X$ is the compact core of Γ, $\mathcal{H}_1^\Gamma \subset X$ a horosphere of Γ and q an (L, c)-quasi-isometry, then $q(\mathcal{H}_1^\Gamma)$ is contained within the $D = D(L, c)$ neighbourhood of some other horosphere of Γ. This is a form of 'flat rigidity', a fundamental geometric notion which I discuss in Chapter III (mostly in the context of higher rank symmetric spaces).

From the flat rigidity Schwartz can deduce that every quasi-isometry is close to an isometry. This means that for every pair of constants (L, c) there is $D = D(L, c)$ such that if q is an (L, c)-quasi-isometry of X_0, then there is an isometry f of X such that $d\big(f(x), q(x)\big) < D$ for all $x \in X_0$. This allows him to take an abstract group Δ that is quasi-isometric to the lattice $\Gamma \leq G$ and embed it (with finite kernel) as a discrete subgroup in G. By construction (see the definition of Φ in Section III.3), this embedding of Δ has two properties:

1. Every element of Δ (viewed as a subgroup of G) *quasi-preserves* the compact core of the lattice Γ. Namely, each element of Δ is an isometry of X that preserves $\mathcal{W}_\mathbb{Q}(\Gamma)$ and that maps every horosphere of Γ to within the $D = D(L, c)$ neighbourhood of some other horosphere of Γ.

2. For the same $D = D(L, c)$ it holds that $\Gamma \subset \mathcal{N}_D(\Delta)$.

From these two properties Schwartz is able to deduce that Δ has finite covolume, i.e. that Δ is a lattice in G. Here is a sketch of his argument:

Theorem II.5.156. *In the setting described above, Δ is a lattice in G.*

Proof sketch. Consider $X_0' := \bigcup_{g\in\Delta} g \cdot X_0$, where X_0 is the compact core of Γ. This space serves as a 'compact core' for Δ: the fact that Δ quasi-preserves X_0 implies that $X_0' \subset \mathcal{N}_D(X_0)$. It is a Δ-invariant space, and therefore one gets an isometric action of Δ on X_0'. This action is cocompact: the reason is that Γ acts cocompactly on X_0, and $\Gamma \subset \mathcal{N}_D(\Delta)$. Formally, every point in X_0' is D-close to a point in X_0. Every point in X_0 is D_Γ-close to a point in $\Gamma \cdot x_0$. Every point in $\Gamma \cdot x_0$ is D-close to a point in $\Delta \cdot x_0$. Therefore the ball of radius $2D + D_\Gamma$ contains a fundamental domain for the action of Δ on X_0'.

It remains to see that the action of Δ on $X \setminus X_0'$ is of finite covolume. As a result of the cocompact action of Δ on X_0', there is $B := B(x_0, R)$ so that $X_0' \subset \Delta \cdot B$. X_0' is the complement of a union of horoballs, which one may call *horoballs of* Δ, with bounding horospheres called *horospheres of* Λ. The fact that Γ is of $\mathbb{Q}$-rank 1 means that the horoballs of Γ are disjoint, and therefore those of Λ are almost disjoint: there is some $C > 0$ such that for every horosphere $\mathcal{H}$ of Λ and every point $x \in \mathcal{H}$, $d(x, X_0') < C$. Up to enlarging the radius of B by C, I can assume that $\mathcal{H} \subset \Delta \cdot B$ for every horosphere $\mathcal{H}$ of Λ. I thank Alex Eskin for noticing this fact.

Each horoball of Λ is based at $\mathcal{W}_{\mathbb{Q}}(\Gamma)$, and each lies uniformly boundedly close to the corresponding horoballs of Γ. From Corollary II.5.84 one therefore sees that there are finitely many horoballs of Δ that intersect B. Denote them by $\mathcal{HB}_1, \dots, \mathcal{HB}_N$, their bounding horospheres by $\mathcal{H}_i = \partial\mathcal{HB}_i$, and their intersection with B by $B_i := B \cap \mathcal{HB}_i$. Let also $\xi_i \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$ denote the base point of each $\mathcal{HB}_i$. Each B_i is pre-compact and therefore the projection of each B_i on $\mathcal{H}_i$ is pre-compact as well (this is a consequence e.g. of the results of Heintze-Im hof recalled in Remark II.5.32). Let $D_i \subset \mathcal{H}_i$ be a compact set that contains this projection, i.e. $P_{\mathcal{H}_i}(B_i) \subset D_i \subset \mathcal{H}_i$. In particular $B \cap \mathcal{H}_i \subset D_i$.

Observe now that for every horoball $\mathcal{HB}$ of Δ, with bounding horosphere $\mathcal{H} = \partial\mathcal{HB}$, the Δ-orbit of every point $x \in \mathcal{H}$ intersects some D_i. First notice that for $x \in \mathcal{H}$ the choice of B implies that the Δ-orbit of x must intersect B, say $gx \in B$. In particular $g\mathcal{H} \cap B \neq \emptyset$, and since $g\mathcal{HB}$ is a horoball of Δ then by definition $g\mathcal{H} = \mathcal{H}_i$ for some $i \in \{1, \dots, N\}$. One concludes that indeed $gx \subset D_i$.

Moreover, let $y \in X$ is any point that lies inside a horoball $\mathcal{HB}$ of Δ, and $x = P_{\mathcal{H}}(y)$ its projection on the bounding horosphere $\mathcal{H} = \partial\mathcal{HB}$. By the previous paragraph there

is some $g \in \Delta$ and $i \in \{1, \dots, N\}$ for which $gx \in D_i$, and therefore it is clear that gy lies on a geodesic emanating from D_i to ξ_i.

Finally, define $\mathrm{Cone}(D_i)$ to be the set of all geodesic rays that emanate from D_i and with limit point ξ_i. The previous paragraph proves that $\bigcup_{i=1}^{N} \mathrm{Cone}(D_i)$ contains a fundamental domain for the action of Δ on $X \setminus X_0'$. Moreover, the fact that $D_i \subset \mathcal{H}_i$ is compact readily implies that each $\mathrm{Cone}(D_i)$ has finite volume, and so this fundamental domain is of finite volume.

To conclude, $B \cup \left(\bigcup_{i=1}^{N} \mathrm{Cone}(D_i) \right)$ is a set of finite volume and it contains a fundamental domain for the Δ-action on X, as wanted.

□

There is one essential difference between Theorem II.5.2 and Theorem II.5.156, and that is the assumption that $\Lambda \subset \mathcal{N}_D(\Gamma)$ rather than quasi-preserving the compact core of Γ. In Schwartz's work, the fact that $\Delta \subset \mathcal{N}_D(\Gamma)$ is not relevant (even though it easily follows from the construction of his embedding of Δ in G). He only uses the two properties described above, namely the quasi-preservation of X_0 and $\Gamma \subset \mathcal{N}_D(\Delta)$.

The assumption that Λ quasi-preserves $\mathcal{W}_{\mathbb{Q}}(\Gamma)$ does not feel appropriate in the context of my thesis, while the metric condition $\Lambda \subset \mathcal{N}_D(\Gamma)$ seems much more natural. It is a stronger condition as I now show. By Lemma II.5.155, $\Lambda \cdot \mathcal{W}_{\mathbb{Q}}(\Gamma) \subset \mathcal{W}_{\mathbb{Q}}(\Gamma)$. Let $\mathcal{H}_1^{\Gamma}$ be a horosphere of Γ, based at $\xi \in \mathcal{W}_{\mathbb{Q}}(\Gamma)$, and let $\gamma x_0 \in \mathcal{H}_1^{\Gamma}$ be some point on the metric lattice of $\Gamma \cdot x_0$ on $\mathcal{H}_1^{\Gamma}$. There is an element $\lambda \in \Lambda$ such that $d(\lambda x_0, \gamma x_0) < D$. Moreover, since $\Lambda \subset \mathcal{N}_D(\Gamma)$ one knows that the parallel horoball that lies D-deep inside $\mathcal{HB}_1^{\Gamma}$ is Λ-free.

Let $\lambda' \in \Lambda$ be an arbitrary element of Λ, and consider $\lambda' \cdot \mathcal{H}_1^{\Gamma}$. The last statement in the previous paragraph is Λ-invariant, and so the horoball that lies D-deep inside $\lambda' \cdot \mathcal{HB}_1^{\Gamma}$ is Λ-free. The fact that $\Gamma \subset \mathcal{N}_D(\Lambda)$ then implies that the parallel horoball that lies $2D$ deep inside $\lambda' \mathcal{HB}_1^{\Gamma}$ is Γ-free. Let $\mathcal{H}_2^{\Gamma}$ be the horosphere of Γ that is based at $\lambda' \xi$. The last statement amounts to saying that $\mathcal{H}_2^{\Gamma}$ lies at most $2D$-deep inside $\lambda' \mathcal{HB}_1^{\Gamma}$. On the other hand, one has $d(\lambda' \lambda x_0, \lambda' \mathcal{H}_1^{\Gamma}) = d(\lambda x_0, \mathcal{H}_1^{\Gamma}) \leq D$, so there is a Λ-orbit point that lies within D of $\lambda' \mathcal{H}_1^{\Gamma}$. The parallel horoball that lies D-deep inside $\mathcal{HB}_2^{\Gamma}$ must also be Λ-free, so I conclude that $\mathcal{H}_2^{\Gamma}$ must be contained in the parallel horoball to $\lambda' \mathcal{HB}_1^{\Gamma}$ which contains it and that is at distance D from it. I conclude that $d(\lambda' \mathcal{H}_1^{\Gamma}, \mathcal{H}_2^{\Gamma}) \leq 2D$, and so that Λ quasi-preserves X_0.

Remark II.5.157. One could also prove Theorem II.5.2 using the same type of arguments that appear repeatedly in this thesis. The argument is very similar to that of the proof of Proposition II.5.154, and I do not present it here.

Remark II.5.158. It is not clear how to make the geometric argument for the proof of Theorem II.5.2 work for higher $\mathbb{Q}$-rank lattices. On the other hand for such lattices Theorem II.5.2 easily follows from Leuzinger's criterion for groups with property (T) (Theorem II.3.8), since bounded Hausdorff distance readily implies the equality of the critical exponents. While this is mathematically sound, Theorem II.5.2 was used several years prior to Leuzinger's work (already in [24], [30]). It would thus be more plausible to find an argument not involving Leuzinger's result.

II.5.6 Proof of Theorem II.5.1

I can now complete the proof of the main theorem for $\mathbb{Q}$-rank 1 lattices.

Proof of Theorem II.5.1. If $\{d_\gamma\}_{\gamma\in\Gamma}$ is unbounded, then the conclusions of both Proposition II.5.138 and Corollary II.5.153 hold. I separate into cases depending on the $\mathbb{R}$-rank of G.

In $\mathbb{R}$-rank 1, the fact that the set of Λ-non-conical limit points is contained in the set of Γ-non-conical limit points implies that the cardinality of that set is strictly smaller than the continuum, because Γ is a lattice and in particular geometrically finite. Therefore Λ is geometrically finite by Theorem II.5.4. Lemma II.5.102 then implies that Λ is a geometrically finite group with a full limit set, rendering it a lattice.

Assume G is of higher rank. Proposition II.5.138 and Corollary II.5.153 imply that Λ is Zariski dense (this is Corollary II.5.137). Moreover, $\Lambda\cap\mathrm{Stab}_G(\mathcal{H})$ is a lattice in $\mathrm{Stab}_G(\mathcal{H})$ and so Λ intersects a horospherical group of G in a lattice (this is Corollary II.5.126). Since Λ is irreducible by hypothesis, Lemma II.5.129 implies that this horospherical lattice is indecomposable. The group Λ therefore meets the hypotheses of Theorem II.5.5, rendering it an arithmetic non-uniform lattice. As a matter of fact, $\Lambda\cdot x_0$ intersects a horosphere based at a non-horospherical limit point of Λ in a cocompact metric lattice: this is Proposition II.5.138. Theorem II.5.78 states that this means Λ is of $\mathbb{Q}$-rank 1 .

Assume now that $\{d_\gamma\}_{\gamma\in\Gamma}$ is bounded. If $\Lambda\subset\mathcal{N}_u(\Gamma)$, then by Proposition II.5.154 it must be that also $\Lambda\subset\mathcal{N}_{D'}(\Gamma)$ for some D'. From Theorem II.5.2 I deduce that Λ is a lattice commensurable to Γ.

Notice that for groups of $\mathbb{R}$-rank 1, if d_γ is bounded I can right away claim as above that every Γ-conical limit point is also Λ-conical, thus Λ is a lattice by Theorem II.5.4. I stress that in this case I indeed do not invoke Proposition II.5.154 or the assumption that $\Lambda\subset\mathcal{N}_u(\Gamma)$. □

III

SBE Rigidity for Lattices of Higher $\mathbb{Q}$-Rank

III.1 Introduction

III.1.1 Sublinear Distortion and SBE Maps

Denote $a \vee b := \max(a, b)$ for $a, b \in \mathbb{R}$. For a pointed metric space (X, x_0, d_X) and $x, x_1, x_2 \in X$, denote $|x|_X := d_X(x, x_0)$ and $|x_1 - x_2|_X := d_X(x_1, x_2)$ (or simply $|x|$ and $|x_1 - x_2|$ when there is no ambiguity about the space X).

Following Cornulier [22], Pallier [59] makes the following definition:

Definition III.1.1. A function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *admissible* if it satisfies the following conditions:

- u is non-decreasing
- u grows sublinearly: $\limsup_{r \mapsto \infty} \frac{u(r)}{r} = 0$.
- u is doubling: $\frac{u(tr)}{u(r)}$ is bounded above for all $t > 0$.
- $u \geq 1$.

The focus in this paper is condition 2, namely that the function u is strictly sublinear. I moreover require it to be subadditive, resulting in the following terminology:

Definition III.1.2. A function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *sublinear* if it is admissible and subadditive, i.e. $u(t + s) \leq u(t) + u(s)$ for all $t, s > 0$.

Definition III.1.3. Let $(X, d_X, x_0), (Y, d_Y, y_0)$ be two pointed metric spaces. A map $q : X \to Y$ is an SBE if there exists $L \in \mathbb{R}_{>0}$ and a *sublinear function* $u : \mathbb{R} \to \mathbb{R}_{\geq 1}$ such that:

1. $q(x_0) = y_0$

2. $\forall x_1, x_2 \in X$,

$$\frac{1}{L}|x_1 - x_2|_X - u\big(|x_1|_X \vee |x_2|_X\big) \leq |(q(x_1) - q(x_2)|_Y \leq L|x_1 - x_2|_X + u\big(|x_1|_X \vee |x_2|_X\big)$$

3. $\forall y \in Y\ \exists x \in X$ such that $|y - q(x)|_Y \leq u(|y|)$

In analogy with quasi-isometries, the constant L and function u are called the Lipschitz sublinear *constants* of q (in the case of u this is clearly an abuse of language). The map q is said to be as an (L, u)-SBE. The spaces X and Y are said to be *SBE* if there is an (L, u)-SBE between them for some L, u as above.

Proposition III.1.4 (Proposition 2.4 in [21])**.** *For pointed metric spaces, being SBE is an equivalence relation.*

SBE maps generalize quasi-isometries, in that the additive error term is replaced by a sublinear error term, i.e., the farther you are in the space, the more the additive term may grow. The class of SBE maps form the largest class of maps which induce bi-Lipschitz maps between the respective *asymptotic cones*. I refer to [22] for a general review of the topic by Cornulier, who initiated the research of these maps.

III.1.2 SBE Rigidity

Two finitely generated groups Γ and Λ are said to be SBE if they are SBE when viewed as metric spaces with some word metrics and base points e_Γ, e_Λ. Observe that every quasi-isometry is an SBE, and in particular the word metric is an SBE-invariant.

Definition III.1.5. A class of groups $\mathcal{A}$ is said to be *SBE complete* if, for every finitely generated group Λ that is SBE with some group $\Gamma \in \mathcal{A}$, there is a short exact sequence

$$1 \to F \to \Lambda \to \Lambda_1 \to 1$$

for a finite group $F \leq \Lambda$ and some $\Lambda_1 \in \mathcal{A}$.

In this chapter I prove:

Theorem III.1.6. *Let G be a real semisimple Lie group of noncompact type, of higher $\mathbb{R}$-rank, with trivial centre and with no $\mathbb{R}$-rank 1 factors.*

1. *The class of uniform lattices of G is SBE complete.*

2. *The class of non-uniform lattices of G is SBE complete.*

Remark III.1.7. The proof I present is quite indifferent to whether the lattice Γ is uniform or not. In order to have a unified proof and to ease notation, I fix the convention that for both uniform and non-uniform lattices, X_0 denotes the compact core of the lattice. This just means that $X = X_0$ in case Γ is uniform. My works heavily relies on that of Druţu in [24], where the theorems are stated for non-uniform lattices. Nonetheless one readily sees that her proofs work perfectly well for uniform lattices. Indeed the arguments of [24] are only much simpler in the uniform case.

Remark III.1.8. In case of quasi-isometries, quasi-isometric completeness of non-uniform lattices holds also for groups that admit $\mathbb{R}$-rank 1 factors. Indeed, it holds also for groups of $\mathbb{R}$-rank 1, see [32]. In the SBE setting, I could not adapt any strategy in order to tackle $\mathbb{R}$-rank 1 factors or groups of $\mathbb{R}$-rank 1. See Section III.2.4 for a more detailed discussion.

III.1.3 The Quasi-Isometry Case

My argument for proving Theorem III.1.6 follows closely the proof in the quasi-isometry setting, specifically that of Druţu in [24]. I sketch this argument now. Recall that a non-uniform lattice $\Gamma \leq G$ admits a 'compact core' $X_0 \subset X$ (see Theorem II.5.78). The main step is to show that for any quasi-isometry $f : X_0 \to X_0$ there exists an isometry $g : X \to X$ such that f, g are boundedly close, i.e. there is some $D > 0$ for which $d\big(f(x), g(x)\big) < D$ for all $x \in X_0$.

Let Λ be an abstract group with an quasi-isometry $q : \Lambda \to \Gamma$. Using the result of Lubotzki-Mozes-Raghunathan ([51], see Theorem II.5.77), Γ is quasi-isometrically embedded in X as the compact core X_0. One can thus extend q to a quasi-isometry $q_0 : \Lambda \to X_0$. A conjugation trick allows to associate to each $\lambda \in \Lambda$ a quasi-isometry $f_\lambda : X_0 \to X_0$ defined by $f_\lambda := q \circ L_\lambda \circ q^{-1}$ ($L_\lambda : \Lambda \to \Lambda$ is the left multiplication by λ in Λ).

By the first paragraph, there exists $g_\lambda \in \mathrm{Isom}(X)$ that is boundedly close to f_λ. Moreover, the proof also shows that the bound D depends only on the quasi-isometry constants of f_λ. These could be seen to depend only on q and not on any specific λ. From this one concludes that the map $\lambda \mapsto g_\lambda$ is a group homomorphism $\Phi : \Lambda \to G$. It is then straightforward to show that Φ has finite kernel and that $\Gamma \subset \mathcal{N}_D\big(\mathrm{Im}(\Phi)\big)$. One then uses Schwartz's argument (see Section II.5.5 in Chapter II) to deduce that $\mathrm{Im}(\Phi)$ is a non-uniform lattice in G. Moreover $\mathrm{Im}(\Phi)$ is commensurable to Γ by a result of Shah [69].

III.1.4 The SBE Case

The first step is to find an isometry of X that is close to an SBE of X_0. Druţu's proof is preformed in the asymptotic cone of X, which allows for a smooth transition to the SBE

setting. Indeed, given an SBE $f : X_0 \to X_0$, one can find an isometry $g : X \to X$ that is close to it. The difference is that in the SBE setting, these maps are only *sublinearly close*.

Definition III.1.9. Let (X, x_0) be a pointed metric space, and $f, g : X \to X$ be two maps. Maps f, g are said to be *sublinearly close* maps on X if there is a sublinear function u such that $d\big(f(x), g(x)\big) \leq u(|x|)$.

Theorem III.1.10 (Theorem III.2.1)**.** *Let $f : X_0 \to X_0$ be an SBE. Then there exists a unique isometry $g \in \mathrm{Isom}(X)$ that is sublinearly close to f.*

From this point on, one more or less continues as in the quasi-isometry case: define the map $\Phi : \Lambda \to G$ in a similar fashion and show that $\Gamma \subset \mathcal{N}_u\big(\mathrm{Im}(\Phi)\big)$ for some sublinear function u. That $\mathrm{Im}(\Phi)$ is a lattice would then be a result of Theorem II.1.5, proving Theorem III.1.6.

There is one additional requirement in the SBE case which does not appear in the quasi-isometry case. Unlike the quasi-isometry case, the SBE constants of f_λ do depend on λ, and the resulting sublinear bound from Theorem III.1.10 is not enough to define Φ properly. As far as I can see, one needs to get some control on the sublinear bound of $d\big(f_\lambda(x), g_\lambda(x)\big)$. The following statement is enough:

Lemma III.1.11 (Lemma III.2.2)**.** *Let $\{f_r\}_{r\in\mathbb{R}_{>0}}$ be a family of (L', v_r)-SBE maps $f_r : X_0 \to X_0$, where $v_r(s) = L' \cdot v(s) + v(r)$ for some sublinear function $v \in O(u)$ and a constant L'. Let g_r be the associated isometry given by Theorem III.2.1. Then for any $x \in X_0$, there is a sublinear function $u^x \in O(u)$ such that $d\big(f_r(x), g_r(x)\big) \leq u^x(r)$.*

This type of uniform control is often needed when working with SBE maps, see e.g. Section I.3 in Pallier's thesis [60]. Using it, I am able to complete the argument as in the quasi-isometry case and prove:

Theorem III.1.12. *Let G be as in Theorem III.1.6. In the notations described above, the map $\Phi : \Lambda \to G$ is a group homomorphism with $\mathrm{Ker}(\Phi)$ finite, and there is a sublinear function u such that for $\Lambda_1 := \mathrm{Im}(\Phi)$ it holds that $\Gamma \subset \mathcal{N}_u(\Lambda_1)$ and $\Lambda_1 \subset \mathcal{N}_u(\Gamma)$.*

Theorem III.1.6 is an immediate corollary of Theorem III.1.12 and Theorem II.1.5.

III.1.5 Outline

This section is divided into two parts that correspond to the steps of the proof. These are Section III.2, dealing with the task of finding an isometry that is sublinearly close to an SBE, and Section III.3, where the properties of the map $\Phi : \Lambda \to G$ are established.

I keep Section III.2 slim and concise. The main reason for this choice is that the proof of Theorem III.2.1 is merely a mimic of Druţu's argument in [24], or an adaptation of it to the SBE setting. While these adaptations are somewhat delicate, giving a complete detailed proof would require reproducing Druţu's argument more or less in full. I felt that this is not desirable, and instead I only indicate the required adaptations. I believe that a reader who is familiar with Druţu's argument and with asymptotic cones could easily produce a complete proof using these indications.

There is therefore is no preliminary section. I do not present buildings or dynamical results that go into Druţu's argument. I only shortly present asymptotic cones and some ideas from Druţu's proof of the quasi-isometry version of Theorem III.2.1. Section III.3 is elementary and requires no preliminaries.

III.2 SBE Maps are Close to Isometries

In this section I indicate how to adapt Druţu's arguments in [24] in order to prove:

Theorem III.2.1. *There is a sublinear function $v = v(L,u)$ such that for every $\big(L,u\big)$-SBE $f : X_0 \to X_0$, there exists a unique isometry $g = g(f) \in \mathrm{Isom}(X)$ such that $d\big(f(x),g(x)\big) \leq v(|x|)$.*

As mentioned in the introduction, the proof of Theorem III.1.6 requires some control on the sublinear distance between f and g, in terms of the sublinear constants of f. This is the meaning of the following lemma.

Lemma III.2.2. *Let $\{f_r\}_{r\in\mathbb{R}_{>0}}$ be a family of (L',v_r)-SBE maps $f_r : X_0 \to X_0$, where $v_r = L'\cdot v + v(r)$ for some sublinear function $v \in O(u)$ and a constant L'. Let g_r be the associated isometry given by Theorem III.2.1. Then for any $x \in X_0$, there is a sublinear function $u^x \in O(u)$ such that $d\big(f_r(x),g_r(x)\big) \leq u^x(r)$.*

Remark III.2.3. Combined with Theorem III.2.1, a different way to phrase the above statement is to say that, in the notations and hypothesis of Lemma III.2.2, the function $D : \Lambda \times X_0 \to \mathbb{R}_{\geq 0}$ defined by $D(\lambda,x) = d\big(f_\lambda(x),g_\lambda(x)\big)$ is sublinear in each variable. I.e., there is a function $u : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ such that u is sublinear in each variable and $D(\lambda,x) \leq u(|\lambda|,|x|)$.

Outline. The proof for Theorem III.2.1 and Lemma III.2.2 is a sublinear adaptation to the work of Druţu [24]. For a reader familiar with Druţu's argument, the proof is really nothing more than a bookkeeping procedure. And but so, I allow myself to present only a detailed sketch of the argument. I believe and hope that this presentation would allow the interested reader who is familiar with [24] to easily obtain a complete proof.

I begin with a short presentation of asymptotic cones. I then give an account of the original proof of Theorem III.2.1 when $f : X_0 \to X_0$ is a quasi-isometry. I then present the routines that are required in order to modify the proof for the SBE setting. I exemplify this procedure in a specific representative example. I end the section with a road map for proving Theorem III.2.1 and Lemma III.2.2 using the aforementioned routines.

III.2.1 Asymptotic Cones

The definition of an asymptotic cone requires ultrafilters and ultralimits. I refer to Section 2.4 in [24] for these definitions.

Definition III.2.4. Let (X, d) be a metric space. Fix an ultrafilter ω, a sequence of points $x_n \in X$ and a sequence of scaling factors $\imath_n \underset{\omega}{\to} 0$. The *asymptotic cone of X w.r.t. $x_n, \imath_n$*, denoted $C(X)$, is the metric ω-ultralimit of the sequence of pointed metric spaces $(X, \frac{1}{\imath_n} \cdot d, x_n)$. The metric on $C(X)$ is denoted d_ω.

The strength of SBE maps is that they induce bi-Lipschitz maps between the respective asymptotic cones.

Lemma III.2.5. *Let $f : X \to Y$ be an (L, u)-SBE. Then f induces an L-bi-Lipschitz map $C(f) : C(X) \to C(Y)$ between the corresponding asymptotic cones with the same scaling factors $C(X) = (X, \frac{1}{\imath_n} d_X, x_n^0)$ and $C(Y) = (Y, \frac{1}{\imath_n} d_Y, y_n^0)$.*

Proof. Let $x_\omega \in C(X)$, and let $x_n \in X$ be points so that $x_\omega = \lim_\omega x_n$. Define $C(f)(x_\omega) = \big(f(x_n)\big)$. The proof that it is well defined is similar to the proof that it is bi-Lipschitz, which I now show. For $x_\omega^1, x_\omega^2 \in C(X)$, there are again corresponding sequences $x_n^i \in X_n$ such that $x_\omega^i = (x_n^i)$.

The fact f is an (L, u)-SBE gives

$$d_\omega\big(C(f)(x_\omega^1), C(f)(x_\omega^2)\big) = \lim_\omega \frac{1}{\imath_n} d\big(f(x_n^1), f(x_n^2)\big) \leq \lim_\omega \frac{1}{\imath_n} L d(x_n^1, x_n^2) + u(|x_n^1| \vee |x_n^2|)$$

Recall that $x_n^0 \in X$ is the sequence of base points defining $C(X)$. By definition of a metric ultralimit, for $i \in \{1, 2\}$ it holds that $\lim_\omega \frac{1}{\imath_n} |x_n^i| = \lim_\omega \frac{1}{\imath_n} d(x_n^i, x_n^0) < \infty$. This in turn implies $\lim_\omega \frac{1}{\imath_n} u(|x_n^i|) = 0$, by sublinearity of u. This implies that $d_\omega\big(C(f)(x_\omega^1), C(f)(x_\omega^2)\big) \leq \lim_\omega \frac{1}{\imath_n} d(x_n^1, x_n^2) = L \lim_\omega \frac{1}{\jmath_n} d(x_n^1, x_n^2) = L d_\omega(x_\omega^1, x_\omega^2)$. The lower bound Lipschitz inequality is symmetric, concluding the proof. □

The other main fact is that an asymptotic cone of a symmetric space of noncompact type and of higher rank admits the structure of a Euclidean building. See [45].

III.2.2 The quasi-isometry case

III.2.2.1 A High-Level Description

In this section I elaborate on the high-level description given in the introduction. A quasi-isometry $q : \Lambda \to \Gamma$ can be extended into a quasi-isometry $q_0 : \Lambda \to X_0$ using the quasi-isometry $q_\Gamma : \Gamma \to X_0$ given by Theorem II.5.77. A conjugation trick gives a representation of Λ into $\mathrm{QI}(X_0)$, the group of quasi-isometry maps from X_0 to itself up to bounded distortion (easily verified to be a group). This whole discussion is naturally generalized to the case where q is an SBE and the representation is into the group of SBE maps up to sublinear distortion $\mathrm{SBE}(X_0)$ (again easily seen to be a group).

The core of the argument lies in elevating such a quasi-isometry q_0 (or an SBE) to an isometry $g_0 \in G = \mathrm{Isom}(X)$. There are two gaps to fill: first, Γ is non-uniform and so q_0 is not even defined on the whole space X. And obviously, q_0 is just an SBE.

Assume for a moment that Γ is uniform and that q_0 is defined on the whole space X. Elevating $q_0 : X \to X$ to an isometry is done by considering the map $C(q) : C(X) \to C(X)$ that q induces on an asymptotic cone $C(X)$. This map is bi-Lipschitz, and the work of Kleiner and Leeb [45] allows one to conclude that $C(q)$ is, up to a scalar, an isometry. In turn, this isometry induces an isometry ∂g on the spherical building structure of $X(\infty)$. This is done by the relation between the Euclidean building structure of $C(X)$ and the spherical building structure of $\partial_\infty X$. A theorem of Tits [71] associates to ∂q a unique isometry $g \in \mathrm{Isom(X)}$ that induces ∂q as its boundary map. By construction, it is then not too difficult to see that $d\big(g(x), q(x)\big) \leq D$ for some fixed D that depends on the quasi-isometry constants. In case Γ is non-uniform, an SBE $q : \Gamma \to \Gamma$ does not readily yield a cone map on $C(X)$, but only on $C(X_0)$. Overcoming this difficulty requires substantial work and is the heart of Druţu's proof. In short, she uses dynamical results stating that the vast majority of flats in X are close enough to X_0. As mentioned in Section II.5.2, the building structure on $X(\infty)$ is determined by the boundaries of flats $F \subset X$. The same holds for the (Euclidean) building structure of $C(X)$. Therefore the fact that the majority of flats in X are 'close enough' to X_0 results in the fact that $C(X_0)$ composes the majority of $C(X)$. This is a very rough sketch of the logic behind Druţu's argument.

The procedure described above results in an isometry $g_0 \in \mathrm{Isom}(X)$ associated to q_0. To complete the argument one needs to verify that the map $q_0 \mapsto g_0$ is a group homomorphism between $\mathrm{QI}(\Gamma) = \mathrm{QI}(X_0)$ and $\mathrm{Isom}(X)$. Composing this map with the representation of Λ into $\mathrm{QI}(\Gamma)$ yields a map $\Lambda \to G$ by $\lambda \mapsto q_\lambda \mapsto g_\lambda := g_{q_\lambda}$. A computation then shows that this map has finite kernel and that Γ lies in a bounded neighbourhood of the image.

III.2.2.2 Flat Rigidity

The adaptations that are required for the SBE setting lie mainly in the part of Druţu's work that concerns *flat rigidity*. That is, the proof that the quasi-isometry q_0 maps a flat $F \subset X_0$ to within a uniformly bounded neighbourhood of another flat $F' \subset X_0$. This is proved by passing to the cone map, using an analogous result for bi-Lipschitz maps between Euclidean buildings, and then translate this back down to the space X_0.

In fact, flat rigidity is also about showing that the quasi-isometry preserves the intersection pattern of maximal flats. Below is a concise description of the steps leading to flat rigidity. I then explain in detail how the proofs should be modified for the SBE setting.

Remark III.2.6. Druţu's argument requires many geometric and combinatorial definitions - some classical and widely known (e.g., Weyl chambers of a symmetric space X) some less known (e.g. an asymptotic cone with respect to an ultrafilter ω) and some new (e.g., the *horizon* of a set $A \subset X$). I use her definitions, terminology and notations freely without giving the proper preliminaries or even the definitions. I assume most readers who are interested in the question of SBE rigidity are familiar to some extent with at least the classical objects. For the new definitions, I try to say as little as needed to allow the reader to follow the argument.

The proof consists of 6 steps:

1. The *horizon* of an image of a Weyl chamber is contained in the *horizon* of a finite union of Weyl chambers, and the number of chambers in this union depend only on the Lipschitz constant. (Lemmas 3.3.5, 3.3.6 in [24], consult Remark III.2.7 below for a sketchy definition).

2. The horizon of an image of a flat coincides with the horizon of a finite union of Weyl chambers, and the number of chambers depends only on the Lipschitz constant of the quasi-isometry.

3. The union of Weyl chambers in the previous step limits to an apartment in the Tits building at $X(\infty)$. Such a union is called a *fan over an apartment*.

4. For each Weyl chamber W there corresponds a unique chamber W' such that $q_0(W)$ and W' have the same horizon. This amounts to an induced map on the Weyl chambers of the Tits building at $X(\infty)$ (Lemma 4.2.1 in [24]).

5. Given a flat F through a point x, the unique flat F' asymptotic to the union of Weyl chambers obtained in step 3 is at uniform bounded distance from $q_0(x)$. The

bound depends only on the quasi-isometry constants. The flat F' is called the flat *associated* to F.

6. If F_1 and F_2 are two flats through x which intersect along a hyperplane H, then the boundary at $X(\infty)$ of the associated flats F_1' and F_2' intersect along a hyperplane of the same codimension as H.

Remark III.2.7. For a precise definition of horizon see [24], section 3. For now, it suffices to say the following. The horizon of a set $A \subset X$ is contained in the horizon of a set $B \subset X$ if, looking far away at A from some point $x \in X$, A appears to be contained in an ϵ-neighbourhood of B. This intuition is made precise by considering the angle at x that a point $a \in A$ makes with the set B. Two sets have the same horizon if each set's horizon is contained in the other. In the case A and B have the same horizon, an important aspect is the distance R starting from which A and B seem to be ϵ-contained in one another. Call this distance the *horizon radius*. It depends on x and ϵ.

The proofs for most of these steps have similar flavour: in any asymptotic cone $C(X)$, q_0 induces a bi-Lipschitz map. Kleiner and Leeb [45] proved many results about such maps between cones of higher rank symmetric spaces. One assumes towards contradiction that some assertion fails (say, in step 5, assume that there is no bound on the distance between $q_0(x_n)$ and the associated flat F_n'). This gives an unbounded sequence of scalars (say, $d\big(q_0(x_n), F_n'\big) = \imath_n \to \infty$). These scalars are used to define a cone in which one obtains a contradiction to some fact about bi-Lipschitz maps (say, that the point $[q_0(x_n)]_\omega$ is at d_ω-distance 1 from $[F_n']_\omega$, while it should lie in $[F_n']_\omega$).

Typically, the bounds obtained this way depend on the quasi-isometry constants. *A priori*, they also depend on the specific point $x \in X$ or flat $F \subset X$ in which you work (e.g. the horizon radius for the chambers in step 3, the bound on $d\big(q(x), F'\big)$ in step 5). However, it is easy to see that in the quasi-isometry setting, the bounds are actually independent of the choice of point/flat/chamber. This independence stems from the fact that one can pre-compose q_0 with an isometry translating any given point/flat/chamber to a fixed point/flat/chamber (resp.), *without changing the quasi-isometry constants* (see e.g. Remark 3.3.11 in [24]). The fact that these bounds depend only on the quasi-isometry constants is essential for the proof that the map $\Lambda \to \mathrm{Isom}(X)$ has the desired properties.

Moving to the SBE setting, the essential difference is exactly this point, namely that the bounds one obtains depend on the specific point, Weyl chamber or flat. Indeed it is clear that these bounds should depend on the size $|x|$, since the sublinear function u depends on it. It is sensible to guess though that the bounds only grow sublinearly in $|x|$, which is enough in order to push the argument forward. In the next section I show how to elevate a typical cone argument from the quasi-isometry setting to the SBE setting. I

focus on showing that the bound one obtains depend only on the SBE constants (L, u) and sublinearly $|x|$.

III.2.3 Generalization to SBE: Adapting Cone Arguments

To adapt for the SBE setting, split each step into three sub-steps: the first two amount to proving Theorem III.2.1, and the third step amounts to proving Lemma III.2.2.

Sub-Step 1. Repeat the argument of the quasi-isometry setting *verbatim*, to obtain a bound $c = c(x)$ which depend on the point x.

Sub-Step 2. Assume towards contradiction that there is a sequence of points x_n for which $\lim_\omega \frac{c(x_n)}{|x_n|} \neq 0$. This means $\lim_\omega \frac{|x_n|}{c(x_n)} \neq \infty$, and so the point $(x_n)_n$ lies in the cone $C(X) = Cone\big(X, x_0, c(x_n)\big)$, and one may proceed as in the corresponding quasi-isometry setting to obtain a contradiction.

Sub-step 3. Fix $x \in X$ and a sequence of SBE maps as in Lemma III.2.2, i.e. $\{f_n\}_{n\in\mathbb{N}}$ with the same Lipschitz constant and with sublinear constants $v_n(s) = v(s) + u(n)$, for some sublinear functions u, v. Denote by $c_n(x)$ the constants that were achieved in the previous steps for x and the SBE map f_n, and assume towards contradiction that $|c_n(x)|$ is not bounded above by any function sublinear in n. This means in particular that $\lim_\omega \frac{u(n)}{c_n(x)} = 0$. One concludes that the cone map $C(f_{r_n})$ is bi-Lipschitz, and gets a contradiction in the same manner as in the first step.

Example III.2.8. *In order to give the reader a sense of what is actually required, I now demonstrate this procedure in full in a specific claim. I chose to do this for proposition* 4.2.7 *of* [24]*, which is complicated enough to require some attention to details, but not too much. The statement is as follows:*

Proposition III.2.9 (SBE version of Proposition 4.2.7 in [24])**.** *Let $f : X \to X$ be an (L, u)-SBE, and $F \subset X$ a flat through x to which q associates a fan over an apartment, $\cup_{i=0}^{p} W_i$. If F' is the maximal flat asymptotic to the fan, then $d\big(q(x), F'\big) \leq c(x)$ where $c(x) = c(|x|)$ is sublinear.*

Moreover, let $f_n : X \to X$ be a sequence of (L, v_n) SBE maps for $v_n = v + u'(n)$ for v, u' some sublinear functions. The constant $c_n(x)$ associated to x and the SBE f_n achieved in the first part of the proposition admits $c_n(x) \leq u^x(n)$ for some sublinear function u^x.

F' is then said to be the associated flat to F by q.

Proof. Proceed in two (sub-)steps.

Step 1. I show that for a given x, there exists such a constant $c(x)$ independent of the flat F. This is done exactly as in [24], but I repeat the proof here because it contains the terminology and necessary preparation for the second step.

Fix $x \in X$ and assume towards contradiction that there exists a sequence F_n of flats through x and a sequence $f_n : X \to X$ of (L,u)-SBE maps such that $c_n := d\big(f_n(x), F_n'\big) \to \infty$. In $\mathrm{Cone}(X, x, c_n^{-1})$ one can show that $[\cup_{i=0}^p W_i^n]$, the union of Weyl chambers associated to $f_n(F_n)$, is a maximal flat (see Proposition 4.2.6 in [24]). Denote $F_\omega := [\cup_{i=0}^p W_i^n]$. Furthermore, since the bi-Lipschitz flat $[f_n(F_n)] \subset \mathrm{Cone}(X, x, c_n^{-1})$ is contained in it, it coincides with it: $[f_n(F_n)] = F_\omega$. On the other hand, since the Hausdorff distance between $\cup_{i=0}^p W_i^n$ and F_n' is by assumption $c_n = d(x, F_n')$, in the cone the maximal flats F_ω and $F_\omega' := [F_n']$ are at Hausdorff distance 1.

This implies that $F_\omega = F_\omega'$ (see Corollary 4.6.4 in [45]). But since $d\big(q(x), F_n'\big) = c_n$ the limit point $y_\omega := Q(x_\omega) = [q(x)]$, which is contained in F_ω, is at distance 1 from F_ω' - a contradiction.

Step 2. Assume $c(x)$ is taken to be the smallest possible for each x, and then modify the function c so that $c(x) = \max_{y:|y|=|x|} c(y)$. The function $c : X \to \mathbb{R}$ now only depends on $|x|$. I wish to show that $c(|x|) = O(u(|x|))$. Assume towards contradiction that there exists a sequence x_n with $|x_n| \to \infty$ such that $\lim_\omega \frac{c(x_n)}{u(|x_n|)} = \infty$. Denote $c_n = c(x_n)$ and consider the cone $\mathrm{Cone}(X, x_n, c_n^{-1})$. The assumption $\lim_\omega \frac{c(x_n)}{u(|x_n|)} = \infty$ implies $(x_0)_\omega = (x_n)_\omega$ hence $\mathrm{Cone}(X, x_n, c_n) = \mathrm{Cone}(X, x_0, c_n)$. By the definition of $c(x)$ this means that there is a sequence of flats F_n through x_n such that $d\big(q(x_n), F_n'\big) = c_n$, so one may proceed as in step 1 for a cone with a fixed base point $(x_0)_\omega$. In this cone the flat $F_\omega' := [F_n']$ is at distance 1 from the point $[q(x_n)]$, which lies on the maximal flat $[\cup_{i=0}^p W_i^n]$. The latter flat is, on the one hand, at Hausdorff distance 1 from F_ω' (by the definition of the scaling factors c_n), so they actually coincide. On the other hand, $[\cup_{i=0}^p W_i^n]$ coincides with $F_\omega := Q(F_\omega) = [q(F_n)]$, so $F_\omega = F_\omega'$, contradicting the fact that $d_\omega([q(x_n)], F_\omega') = 1$. Thus $c(|x|) = O(u(|x|))$, as wanted. □

Step 3. *For the moreover part (uniform control on the growth of $c(x)$ as a function of the sublinear constants), the proof is identical to Step* 1*. This time, consider a sequence f_n as in the statement, and denote by $c_n = c_n(x)$ the constant obtained in step* 1 *w.r.t. the SBE constants (L, v_n). Assume towards contradiction that $\lim_\omega \frac{u(n)}{c_n(x)} = 0$. The proof goes exactly as in step* 1*, with the sole difference that now one might need convincing in the fact that in $C(X)$, the cone with $\frac{1}{c_n}$ as scaling factors, the cone map $C(f_n)$ is bi-Lipschitz. But indeed for any two cone points $(x_n), (y_n) \in C(X)$ it holds:*

$$d_\omega\big(C(f_n)(x_n), C(f_n)(y_n)\big) = \lim_\omega \frac{1}{c_n} d\big(f_n(x_n), f_n(y_n)\big) \leq \lim_\omega \frac{1}{c_n} L \cdot d(x_n, y_n) + v_n(|x_n| \vee |y_n|)$$

By definition of the cone metric, $\lim_\omega \frac{1}{c_n} L \cdot d(x_n, y_n) = L \cdot d_\omega\big((x_n), (y_n)\big)$. It thus remains to show $\lim_\omega \frac{1}{c_n} v_n(|x_n| \vee |y_n|) = 0$. By definition of v_n it amounts to proving $\lim_\omega \frac{1}{c_n} v(|x_n|) = 0 = \lim_\omega \frac{1}{c_n} v(|y_n|)$ and $\lim_\omega \frac{1}{c_n} u(n) = 0$. The former follows from the fact that $(x_n), (y_n) \in C(X)$ and therefore both $\lim_\omega \frac{1}{c_n} |x_n|$ and $\lim_\omega \frac{1}{c_n} |x_n|$ are finite. The latter follows from the assumption on the c_n. One obtains a contradiction identical to the one in Step 1.

Following the claims of Sections $3, 4, 5$ in [24] carefully, and making the SBE adaptations as depicted in the above example, one obtains flat rigidity in the SBE setting, that is Theorem III.2.1, as well as the uniform control described in Lemma III.2.2. Here is the complete list of claims involving cone arguments in [24] which should be modified.

- **Section 3.** All claims starting from Lemma 3.3.5 through Corollary 3.3.10. All statements should consider, instead of a quasi-isometry, a general (L, u)-SBE f and, when relevant, a general point $x \in X$ with $f(x) = y$ (i.e., $f(x)$ does not necessarily equal x). Also when relevant one should consider a family f_n of (L, v_n) SBE maps as depicted in Lemma III.2.2 above.

- **Section 4.** Propositions $4.2.6, 4.2.7, 4.2.9$. When relevant, statements should be modified so that the distance between $f(x_0)$ and an associated flat of it should be uniformly sublinear in $|x|$. Also when relevant one should consider a family f_n of (L, v_n) SBE maps as above.

- **Section 5.** Lemma 5.4.1 ($D = D(x)$ should be uniformly linear in $c = c(x)$). Proposition 5.4.2 (the constant $D = D(x)$ should be replaced be a sublinear function $D(|x|)$). When relevant one should consider a family f_n of (L, v_n) SBE maps as above.

This yields a proof for uniform flat rigidity in the SBE setting. The other major part of Druţu's argument concerns the fact that q_0 is defined only on X_0 and not on X. The considerations for this aspect are intertwined in the proof, but they all involve only Γ and the quasi-isometry between Γ and X_0. For this reason, the fact that q_0 is an SBE to begin with does not effect any of these arguments. Moreover, this argument is indifferent to whether or not Γ is uniform or not. If Γ is uniform all that changes is that that part of Druţu's argument dealing with extending the cone map from $C(X_0)$ to $C(X)$ is not necessary since $X = X_0$. Her proof still works perfectly well also for uniform lattices. Therefore the argument above proves Theorem III.2.1 and Lemma III.2.2.

III.2.4 Some Remarks On $\mathbb{R}$-rank 1 Factors

Quasi-isometric rigidity holds for groups of $\mathbb{R}$-rank 1 (except for those locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$). Indeed, the proof of Schwartz [67] for the $\mathbb{R}$-rank 1 case was the first in the series of works on quasi-isometric rigidity for non-uniform lattices. It is worth mentioning that his proof also relies on 'flat rigidity' - but in this case the flats are the horospheres of the space X. While these are not isometrically embedded flats, the induced metric on horospheres is flat and Schwartz uses that in order to construct the boundary map and find the associated isometry.

The same phenomena occurs in the SBE setting. Considering the compact core $X_0 \subset X$ of Γ, one can use Proposition 5.6 in Druţu-Sapir [26], in order to show that horospheres are mapped boundedly close to horospheres. In general, this work characterizes and explores a certain class of spaces they call *asymptotically tree graded*, a class that is very suitable for the setting of the compact core of a non-uniform lattice in $\mathbb{R}$-rank 1.

A key ingredient in the proof is the fact that the boundary map ∂q induced by the quasi-isometry is *quasi-conformal*. This in particular implies that it is almost everywhere differentiable. I spent some time trying to generalize the proof of Schwartz to the SBE setting. One obstacle is that it is not clear that the boundary map is going to be differentiable almost everywhere. Gabriel Pallier found that there are SBE maps of the hyperbolic space whose boundary maps are not quasi-conformal. For this reason Pallier develops the notion of quasi-conformality [61]. While these maps may be differentiable, he told me of examples he constructed where the differential is almost everywhere 0 - a property which also nullifies Schwartz's argument.

In the context of SBE rigidity, the maps I consider seem to indeed have 'flat rigidity', i.e. to map a horosphere to within bounded distance of a unique horosphere. As in the higher rank flat rigidity, this bound is not uniform but rather grows sublinearly with the distance of the horosphere to a fixed base point. These are very specific maps, that coarsely preserve the compact core of that lattice $X_0 \subset X$ and basically map horospheres to horospheres. This means there might still be hope for these specific maps to induce boundary maps that admit the required analytic properties.

In a subsequent paper [68], Schwartz proves quasi-isometric rigidity for lattices in products of $\mathbb{R}$-rank 1 groups, i.e. in Hilbert modular groups. His proof there is different, but it also makes use of the fact that horospheres are mapped to within uniformly bounded distance of horospheres. The fact that in the SBE setting this bound is not uniform seems like a real obstruction to any attempt of generalizing his proof in that case.

The quasi-isometric rigidity for general groups with $\mathbb{R}$-rank 1 factors is patched from the $\mathbb{R}$-rank 1 case of Schwartz and the higher rank (without $\mathbb{R}$-rank 1 factors) by either Druţu or Eskin. In his survey about the subject, Farb [32] explains how to combine these

proofs using a decomposition argument of Eskin (section 10 in [30]) in order to get the full result.

III.3 From SBE to Sublinearly Close Groups

In this section I prove Theorem III.1.12. I restate it here for convenience

Theorem III.3.1. *Let G be a real semisimple Lie group of noncompact type, of higher $\mathbb{R}$-rank, with trivial centre and with no $\mathbb{R}$-rank 1 factors. Let $\Gamma \leq G$ be a lattice, and Λ an abstract finitely generated group that is SBE to Γ. Then there is a group homomorphism $\Phi : \Lambda \to G$ with finite kernel such that $\Gamma \subset \mathcal{N}_u\big(\Phi(\Lambda)\big)$ and $\Phi(\Lambda) \subset \mathcal{N}_u(\Gamma)$ for a sublinear function u.*

The proof is a sublinear adaptation of the classical arguments by Schwartz. The only difference is that some calculations are in order, but there is no essential difference from Section 10.4 in [67].

Before I start, I need one well known preliminary fact, namely that sublinearly close isometries are equal.

Lemma III.3.2. *Let X be a symmetric space of noncompact type and with no $\mathbb{R}$-rank 1 factors. Let $\Gamma \leq \mathrm{Isom}(X)$ be a non-uniform lattice, X_0 its compact core with respect to $x_0 \in X$. Let $g, h \in G = \mathrm{Isom}(X)$ and u a sublinear function such that for every $x \in X_0$, $d\big(g(x), h(x)\big) \leq u(|x|)$. Then $g = h$.*

Proof. The proof is essentially just the fact that a sublinearly bounded convex function is uniformly bounded. Up to multiplying by h^{-1}, one may assume $h = id_X$. First I show that the continuous map $\partial g : X(\infty) \to X(\infty)$ is the identity map. Recall that the space $X(\infty)$ can be represented by all geodesics emanating from the fixed point x_0. Let $\eta : [x_0, \xi)$ be a Γ-periodic geodesic. By definition, there is some $T > 0$ and and a sequence $\gamma_n \in G$ for which $\eta(nT) = \gamma_n x_0$. In particular, $x_n := \gamma_n x_0 \in X_0$ hence $d\big(g(x_n), x_n\big) \leq u(|x_n|) = u(nT)$.

On the other hand, the distance function $d\big(\eta(t), g\cdot\eta(t)\big)$ is convex. A convex sublinear function is bounded, and so by definition in $X(\infty)$ one has $[\eta] = [g \cdot \eta]$, for all Γ-periodic geodesics η. The manifold X is of non-positive curvature, hence ∂g is a homeomorphism of $X(\infty)$, and the density of Γ-periodic geodesics implies $\partial g = id_{X(\infty)}$. This implies that $g = id_X$ (see Section 3.10 in [28] for a proof of this last implication). □

Remark III.3.3. The proof for the fact that $\partial g = id_{X(\infty)}$ implies $g = id_X$ appears in [28] (section 3.10) as part of the proof of the following important theorem of Tits:

Theorem III.3.4 ([71], see Theorem 3.10.1 in [28]). *Let X, X' be symmetric spaces of noncompact type and of higher $\mathbb{R}$-rank. Assume X has no $\mathbb{R}$-rank 1 factors, and let $\phi : X(\infty) \to X'(\infty)$ be a bijection that is a homeomorphism with respect to the cone topology and an isometry with respect to the Tits metric. Then, after multiplying the metric of X by positive constants on de Rham factors, there exists a unique isometry $g : X \to X'$ such that $\phi = \partial g$.*

This theorem is actually a key ingredient in Druţu's argument. Much of her work is directed towards showing that the cone map $C(q)$ will correspond to a map on $X(\infty)$ satisfying the above hypothesis. The restriction to X with no $\mathbb{R}$-rank 1 factors in Theorem III.1.6 comes from this restriction in Tits' Theorem III.3.4

Remark III.3.5. While the hypothesis requires that X has no $\mathbb{R}$-rank 1 factors, the proof that $\partial g = id_{X(\infty)} \Rightarrow g = id_X$ only uses the fact that X has no Euclidean de Rham factors (see pg. 251 in [28]). Since I only use it in the setting of no $\mathbb{R}$-rank 1 factors, I added this assumption to Lemma III.3.2.

The Map $\Phi : \Lambda \to G$. The orbit map $q_0 : \Gamma \to X_0$ defined by $\gamma \mapsto \gamma x_0$ is a quasi-isometric embedding: this is Švarc-Milnor in case Γ is uniform and $X_0 = X$, and Lubotzki-Mozes-Raghunathan (Theorem II.5.77) if Γ is non-uniform. An SBE $f : \Lambda \to \Gamma$ thus gives rise to an SBE $\Lambda \to X_0$, which I also denote by f.

For each $\lambda \in \Lambda$ let $f_\lambda := f \circ L_\lambda \circ f^{-1} : X_0 \to X_0$, where L_λ is the left multiplication by λ. The left translation L_λ is an isometry, hence f_λ is a self SBE of X_0. By Theorem III.2.1, there exists a unique isometry $g_\lambda \in \mathrm{Isom}(X)$ that is sublinearly close to f_λ. Define the map $\Phi : \Lambda \to G$ by $\lambda \mapsto g_\lambda$. The goal in this section is to prove Φ is a homomorphism with finite kernel, and that Γ is contained in a sublinear neighbourhood of $\Phi(\Lambda)$.

I begin by controlling the SBE constants of the f_λ.

Lemma III.3.6. *For each $\lambda \in \Lambda$, f_λ is an (L^2, v_λ)-SBE, for*

$$v_\lambda(|x|) := (L+1)u(|x|) + u(|\lambda|)$$

In particular $v_\lambda \in O(u)$.

Before the proof I state the immediate corollary, which follows immediately by combining Lemma III.3.6 with Lemma III.2.2.

Corollary III.3.7. *Assume Lemma III.2.2 holds. Then for any $x \in X$ there is a sublinear function u^x such that*

$$d\big(f_\lambda(x), g_\lambda(x)\big) \leq u^x(|\lambda|)$$

.

Proof of Lemma III.3.6. The proof is a straightforward computation. Up to an additive constant I may assume f^{-1} is an (L,u)-SBE with $q^{-1}(e_\Gamma) = e_\Lambda$. Let $x_1, x_2 \in X_0$, and assume w.l.o.g $|x_2| \leq |x_1|$. By the properties of an SBE, this also means that for $i \in \{1,2\}$:

$$|f^{-1}(x_i)| \leq L|x_i - x_0| + u(|x_i|) \leq L|x_1| + u(|x_1|) \tag{III.1}$$

Finally notice that $f_\lambda(x) = f\big(\lambda \cdot f^{-1}(x)\big)$, and f is an (L,u)-SBE. The following inequalities, justified below, give the required upper bounds:

$$\begin{aligned}\big|f_\lambda(x_1) - f_\lambda(x_2)\big| &\leq L \cdot \big|\lambda f^{-1}(x_1) - \lambda f^{-1}(x_2)\big| + u\big(|\lambda f^{-1}(x_1)| \vee |\lambda f^{-1}(x_2)|\big) \\ &\leq L^2|x_1 - x_2| + Lu\big(|x_1|\big) + u\Big(|\lambda|) + L|x_1| + u(|x_1|)\Big) \\ &\leq L^2|x_1 - x_2| + (L+1)u\big(|x_1|\big) + u(|\lambda|).\end{aligned} \tag{III.2}$$

From the first line to the second line I used:

1. For the first term: left multiplication in Λ is an isometry, and f^{-1} is an (L,u)-SBE.
2. For the second term: triangle inequality, left multiplication in Λ is an isometry, and Equation III.1.

From the second to the third line I used the properties of u as an admissible function, namely that it is sub-additive and doubling, so $u\big((L+1)|x_1|\big) \leq (L+1)u(|x_1|)$ for all large enough x_1.

□

Remark III.3.8. I remark that the proof of Lemma III.3.6 is the only place where I use the properties of an admissible function and not just the sublinearity of u.

Claim III.3.9. $\Phi : \Lambda \to G$ is a group homomorphism.

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$. I begin with some notations:

1. $f_1 = f_{\lambda_1}, f_2 = f_{\lambda_2}, f_{12} = f_{\lambda_1\lambda_2}$. By Lemma III.3.6, these are all $O(u)$ SBE maps with the same Lipschitz constant $L' := L^2$ and sublinear constants $v_1, v_2, v_{12} \in O(u)$.
2. $g_1 = \Phi(\lambda_1), g_2 = \Phi(\lambda_2), g_{12} = \Phi(\lambda_1\lambda_2)$
3. u_1, u_2, u_{12} the sublinear functions that bound the respective distances between any g and f, e.g. $|g_1(x) - f_1(x)| \leq u_1(|x|)$.

One has to prove that $g_{\lambda_2} \circ g_{\lambda_1} = g_{\lambda_1\lambda_2}$. In view of Lemma III.3.2, it is enough to find a sublinear function v such that for all $x \in X_0$ $|g_1g_2(x) - g_{12}(x)| \leq v(|x|)$. By triangle inequality and the above definitions and notation, it is enough to show that each of the following four terms are bounded by a function sublinear in x:

1. $|g_1g_2(x) - g_1f_2(x)| = |g_2(x) - f_2(x)| \leq u_2(|x|)$ (g_1 is an isometry).
2. $|g_1f_2(x) - f_1f_2(x)| \leq u_1\big(|f_2(x)|\big) \leq u_1\big(L^2|x| + v_2(|x|)\big)$
3. $|f_1f_2(x) - f_{12}(x)|$
4. $|f_{12}(x) - g_{12}(x)| \leq u_{12}(|x|)$

Clearly items $1, 2, 4$ are bounded by a sublinear function in $|x|$. It remains to bound $|f_1f_2(x) - f_{12}(x)|$. The map $\Lambda \to Aut(\Lambda)$ given by $\lambda \mapsto L_\lambda$ is a group homomorphism, i.e. $L_{\lambda_1\lambda_2} = L_{\lambda_1}L_{\lambda_2}$, so it remains to bound:

$$|f_1f_2(x) - f_{12}(x)| = |fL_{\lambda_1}f^{-1}fL_{\lambda_2}f^{-1}(x) - fL_{\lambda_1}L_{\lambda_2}f^{-1}(x)|$$

$f \circ L_\lambda$ is a composition of an isometry with an SBE, so it is still an SBE. Denote the SBE constants of fL_{λ_1} by L', v (clearly one can take $L' = L$ and $v \in O(u)$, but this is not needed). Writing $y := L_{\lambda_2}f^{-1}(x)$, this shows

$$|fL_{\lambda_1}f^{-1}f(y) - fL_{\lambda_1}(y)| \leq L|f^{-1}fy - y| + v(|f^{-1}fy| \vee |y|)$$

By definition of an SBE inverse it holds that $|f^{-1}f(y) - y| \leq u(|y|)$ and in particular also $|f^{-1}f(y)| \leq |y| + u(|y|)$. I conclude that

$$|f_1f_2(x) - f_{12}(x)| \leq L \cdot u(|y|) + v\big(|y| + u(|y|)\big)$$

The right-hand side is a sublinear function in $|y|$, hence it only remains to show that $|y|$ is bounded by a linear function in x. Indeed

$$|y| = |L_{\lambda_2}f^{-1}(x)| \leq |\lambda_2| + |f^{-1}(x)| \leq |\lambda_2| + L|x| + u(|x|)$$

This completes the proof, rendering Φ a group homomorphism. □

Claim III.3.10. Φ has discrete image and finite kernel.

Proof. I show that for any radius $R > 0$, there are finitely many $\lambda \in \Lambda$ for which $g_\lambda x_0 \in B(x_0, R)$. I.e., that there is a finite number of $\Phi(\Lambda)$-orbit points, with multiplicities, inside an R ball in X. In particular the set $\{\lambda \in \Lambda \mid g_\lambda x_0 = x_0\}$ is finite, and clearly contains $\mathrm{Ker}(\Phi)$. In addition, the actual number of $\Phi(\Lambda)$-orbit points inside that R ball is finite, so $\Phi(\Lambda)$ is discrete.

Let $R > 0$, and $\lambda \in \Lambda$. By the defining property of g_λ and the definition of f_λ, reverse triangle inequality gives

$$d\big(x_0, g_\lambda(x_0)\big) \geq d\big(x_0, f_\lambda(x_0)\big) - d\big(g_\lambda(x_0), f_\lambda(x_0)\big) \geq |f(\lambda)| - u_\lambda(|x_0|)$$

Corollary III.3.7 gives $d\big(g_\lambda(x_0), f_\lambda(x_0)\big) \leq u^{x_0}(|\lambda|)$ for some sublinear function $u^{x_0} \in O(u)$. On the other hand f is an SBE, and so $|f(\lambda)|$ grows close to linearly in λ. Formally,

$$|f(\lambda)| = d\big(f(\lambda), x_0\big) = d\big(f(\lambda), f(e_\Lambda)\big) \geq \frac{1}{L} d(\lambda, x_0) - u(|\lambda| \vee |e_\Lambda|) \geq \frac{1}{L}|\lambda| - u(|\lambda|)$$

To conclude, one has

$$d\big(x_0, g_\lambda(x_0)\big) \geq \frac{1}{L}|\lambda| - u(|\lambda|) - u^{x_0}(|\lambda|)$$

and both u, u^{x_0} are sublinear in $|\lambda|$. Therefore there is a bound $S \in \mathbb{R}_{>0}$ such that $|\lambda| > S \Rightarrow \frac{1}{L}|\lambda| - u(|\lambda|) - u^{x_0}(|\lambda|) > R$. The group Λ is finitely generated and so only finitely many $\lambda \in \Lambda$ admit $|\lambda| \leq S$, hence $g_\lambda(x_0) \in B(x_0, R)$ only for finitely many $\lambda \in \Lambda$. □

Up to this point I have established that Λ can be embedded into G as a discrete subgroup with finite kernel. The following lemma gives a sublinear bound that completes the proof of Theorem III.1.12.

Claim III.3.11. There exists a sublinear function $u' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ such that

$$\Gamma \cdot x_0 \subset \mathcal{N}_{u'}\big(\Phi(\Lambda) \cdot x_0\big)$$

Proof. I claim that there is a sublinear function u_0, depending only on f and q_0, such that for all $\gamma \in G$, $d\big(g_\lambda(x_0), \gamma(x_0)\big) \leq u_0(|\gamma|)$.

As before, I only have control on g_λ via f_λ, and so I use triangle inequality to get:

$$d\big(g_\lambda(x_0), \gamma(x_0)\big) \leq d\big(g_\lambda(x_0), f_\lambda(x_0)\big) + d\big(f_\lambda(x_0), \gamma(x_0)\big)$$

By Corollary III.3.7, $d\big(g_\lambda(x_0), f_\lambda(x_0)\big) \leq u_0^x(|\lambda|)$ for a sublinear function u^{x_0}.

To prove a bound on the second term it is beneficial to distinguish between the SBE $f_\Gamma : \Lambda \to \Gamma$ and the same SBE composed with the orbit quasi-isometry $q_0 : \Gamma \to X_0$. From now on I keep the notation $f_\Gamma : \Lambda \to \Gamma$ for the SBE of the groups and f_0 for the same SBE composed with the orbit quasi-isometry $q : \Gamma \to X_0$, so $f_0 = q_0 \circ f_\Gamma$.

Define $\lambda_\gamma := f_\Gamma^{-1}(\gamma)$. I show that $d\big(f_{\lambda_\gamma}(x_0), \gamma(x_0)\big)$ is bounded by a function sublinear in γ. Indeed, recall that I assumed w.l.o.g. that $q_0 \circ f_\Gamma^{-1}(x_0) = e_\Lambda$. Moreover, Γ is assumed to be torsion-free, and so there is no ambiguity or trouble in defining the restriction of the map q^{-1} to the orbit $\Gamma \cdot x_0$ to be of the form $q^{-1}(\gamma x_0) = \gamma$. All together, this gives

$d\big(f_{\lambda_\gamma}(x_0), \gamma(x_0)\big) = d\big(f_0(\lambda_\gamma), \gamma(x_0)\big) = d\big(q_0 \circ f_\Gamma f_\Gamma^{-1}(\gamma), q_0(\gamma)\big)$

f_Γ is an SBE, so $d\big(f_\Gamma f_\Gamma^{-1}(\gamma), \gamma\big) \leq u(\gamma)$. The fact that q_0 is an (L', C)-quasi-isometry implies that

$$d\big(q_0 \circ f_\Gamma f_\Gamma^{-1}(\gamma), q(\gamma)\big) \leq Ld\big(f_\Gamma f_\Gamma^{-1}(\gamma), \gamma\big) + C \leq L'u(|\gamma|) + C$$

Combining everything, one has

$$d\big(g_{\lambda_\gamma}(x_0), \gamma(x_0)\big) \leq u^{x_0}(|\lambda_\gamma|) + L'u(|\gamma|) + C$$

As before, $|\lambda_\gamma| = |f^{-1}(\gamma)| \leq |\gamma| + u(|\gamma|) \leq 2|\gamma|$, where the last inequality holds for all large enough γ. what matters is that $|\lambda_\gamma|$ is linear in $|\gamma|$. I conclude that indeed $\Gamma \cdot x_0 \subset \mathcal{N}_{u'}\big(\Phi(\Lambda) \cdot x_0\big)$ for the sublinear function $u' = u^{x_0} + L'u + C \in O(u)$, as wanted. (To be pedantic, $u' = u^{x_0} \circ 2 + L'u + C \in O(u)$ where 2 is the 'multiplication by 2' function, $r \mapsto 2r$).

□

Claim III.3.12. There exists a sublinear function $u' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ such that

$$\Phi(\Lambda) \cdot x_0 \subset \mathcal{N}_{u'}(\Gamma \cdot x_0)$$

Proof. Let $\lambda \in \Lambda$. Let $\gamma = f(\lambda)$ and consider the distance $d(g_\lambda x_0, \gamma x_0)$. From triangle inequality and Corollary III.3.7 one has

$$d(g_\lambda x_0, \gamma x_0) \leq d(g_\lambda x_0, f_\lambda x_0) + d(f_\lambda x_0, \gamma x_0) \leq u'(|\lambda|) + d(f_\lambda x_0, \gamma x_0)$$

To conclude, notice that by definition of f_λ it holds that $f_\lambda(x_0) = f(\lambda) \cdot x_0 = \gamma \cdot x_0$ and so $d\big(f_\lambda(x_0), \gamma x_0\big) = 0$. □

Claims III.3.9, III.3.10, III.3.11 and III.3.12 result in Theorem III.1.12. Theorem III.1.6 then follows from Theorem II.1.5.

Bibliography

[1] P Albuquerque, *Patterson-Sullivan theory in higher rank symmetric spaces*, GAFA Geom. Funct. Anal. **9** (March 1999), no. 1, 1–28.

[2] Louis Auslander, *An exposition of the structure of solvmanifolds. part I: Algebraic theory*, Bulletin of the American Mathematical Society **79** (1973), no. 2, 227–261.

[3] Uri Bader and Elyasheev Leibtag, *Homomorphic images of algebraic groups* (2022). Manuscript submitted for publication.

[4] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser Boston, 1985.

[5] Werner Ballmann, *Lectures on spaces of nonpositive curvature*, Vol. 25, Springer Science & Business Media, 1995.

[6] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, *Geometry and rigidity of mapping class groups*, Geometry Topology **16** (2008).

[7] B. Bekka, P. de La Harpe, and A. Valette, *Kazhdan's property (T)*, Cambridge university press, 2008.

[8] Yves Benoist and Sébastien Miquel, *Arithmeticity of discrete subgroups containing horospherical lattices*, Duke Mathematical Journal **169** (2020), no. 8, 1485 –1539.

[9] Richard Bishop and B. O'Neill, *Manifolds of negative curvature*, Transactions of the American Mathematical Society **145** (196911), 1–49.

[10] Armand Borel, *Density properties for certain subgroups of semi-simple groups without compact components*, Annals of Mathematics **72** (1960), no. 1, 179–188.

[11] ———, *Linear algebraic groups*, Springer New York, 1991.

[12] Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Annals of Mathematics **75** (1962), no. 3, 485–535.

[13] N. Bourbaki, *Lie groups and lie algebras: Chapters 1-3*, Bourbaki, Nicolas: Elements of mathematics, Springer-Verlag, 1989.

[14] ———, *Lie groups and lie algebras: Chapters 7-9*, Elements of mathematics, Springer Berlin Heidelberg, 2004.

[15] Brian H. Bowditch, *Geometrical finiteness for hyperbolic groups*, Journal of Functional Analysis **113** (1993), no. 2, 245–317.

[16] Brian H Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Mathematical Journal **77** (1995), no. 1, 229–274.

[17] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Vol. 319, 2009.

[18] Pierre-Emmanuel Caprace and Nicolas Monod, *A lattice in more than two kac-moody groups is arithmetic*, Israel Journal of Mathematics **190** (2012), 413–444.

[19] S. S. Chen and P. Eberlein, *Isometry groups of simply connected manifolds of nonpositive curvature*, Illinois Journal of Mathematics **24** (1980), no. 1, 73 –103.

[20] Claude Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

[21] Yves Cornulier, *On sublinear bilipschitz equivalence of groups*, Annales scientifiques de l'École normale supérieure **52** (201702).

[22] ———, *On sublinear bilipschitz equivalence of groups* (2017), available at arXiv:1702.06618.

[23] SG Dani, *On ergodic quasi-invariant measures of group automorphism*, Israel Journal of Mathematics **43** (1982), no. 1, 62–74.

[24] Cornelia Druţu, *Quasi-isometric classification of non-uniform lattices in semisimple groups of higher rank*, Geometric and Functional Analysis - GAFA **10** (200006).

[25] Cornelia Druţu and Michael Kapovich, *Geometric group theory*, American Mathematical Society, 2017.

[26] Cornelia Druţu and Mark Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058.

[27] Patrick Eberlein, *Lattices in spaces of nonpositive curvature*, Annals of Mathematics **111** (1980), no. 3, 435–476.

[28] ———, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, 1996.

[29] A. Eskin and B. Farb, *Quasi-flats and rigidity in higher rank symmetric spaces*, Journal of the American Mathematical Society **10** (1997), 653–692.

[30] Alex Eskin, *Quasi-isometric rigidity of nonuniform lattices in higher rank symmetric spaces*, Journal of the American Mathematical Society **11** (1998), no. 2, 321–361.

[31] B. Farb and R. E. Schwartz, *The large-scale geometry of Hilbert modular groups*, Journal of Differential Geometry **44** (January 1996), no. 3.

[32] Benson Farb, *The quasi-isometry classification of lattices in semisimple lie groups*, Mathematical Research Letters **4** (1997), 705–717.

[33] Mikolaj Fraczyk and Tsachik Gelander, *Infinite volume and infinite injectivity radius* (2021). preprint.

[34] H. Garland and M. S. Raghunathan, *Fundamental domains for lattices in (ℝ-)rank 1 semisimple lie groups*, Annals of Mathematics **92** (1970), no. 2, 279–326.

[35] Y. Guivarc'h, L. Ji, and J.C. Taylor, *Compactifications of symmetric spaces*, Progress in Mathematics, Birkhäuser Boston, 1998.

[36] Toshiaki Hattori, *Geometric Limit Sets of Higher Rank Lattices*, Proceedings of the London Mathematical Society **90** (200505), no. 3, 689–710, available at https://academic.oup.com/plms/article-pdf/90/3/689/4441923/90-3-689.pdf.

[37] Ernst Heintze, *On homogeneous manifolds of negative curvature*, Mathematische Annalen **211** (March 1974), no. 1, 23–34.

[38] Ernst Heintze and Hans-Christoph Im Hof, *Geometry of horospheres*, Journal of Differential Geometry **12** (1977), no. 4, 481 –491.

[39] Sigurdur Helgason, *Differential geometry, lie groups, and symmetric spaces*, ISSN, Elsevier Science, 1979.

[40] ———, *Differential geometry and symmetric spaces*, Vol. 341, American Mathematical Soc., 2001.

[41] P. Hubert and C. M. Santos, *An origami of genus 3 with arithmetic kontsevich–zorich monodromy*, Mathematical Proceedings of the Cambridge Philosophical Society **169** (2020), no. 1, 19–30.

[42] Lizhen Ji, *From symmetric spaces to buildings, curve complexes and outer spaces*, Innovations in Incidence Geometry **10** (2009), no. none, 33 –80.

[43] M. Kapovich and B. Liu, *Geometric finiteness in negatively pinched hadamard manifolds*, Annales Academiae Scientiarum Fennicae Mathematica (2019).

[44] Inkang Kim, *Rigidity on symmetric spaces*, Topology **43** (2004), no. 2, 393–405.

[45] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and euclidean buildings*, Publ. Math. IHES **86** (1997), 115–197 (en). MR98m:53068

[46] Anthony W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics, Birkhäuser Boston, 2002.

[47] Enrico Leuzinger, *An exhaustion of locally symmetric spaces by compact submanifolds with corners*, Inventiones Mathematicae **121** (December 1995), no. 1, 389–410.

[48] ———, *Tits geometry, arithmetic groups and the proof of a conjecture of siegel*, Journal of Lie Theory **14** (2002).

[49] ———, *Critical exponents of discrete groups and ^{2}–spectrum*, Proceedings of the American Mathematical Society **132** (2004), no. 3, 919–927.

[50] ———, *On polyhedral retracts and compactifications of locally symmetric spaces*, Differential Geometry and its Applications **20** (2004), no. 3, 293–318.

[51] A. Lubotzky, S. Mozes, and M. S. Raghunathan, *The word and riemannian metrics on lattices of semisimple groups*, Publ. Math. IHES **91** (2000), no. 1, 5–53.

[52] G. A. Margulis, *Arithmeticity of the irreducible lattices in the semi-simple groups of rank greater than 1*, Inventiones Mathematicae **76** (February 1984), no. 1, 93–120.

[53] ———, *Discrete subgroups of semisimple lie groups*, 3. Folge, Springer, 1991.

[54] Dave Witte Morris, *Introduction to arithmetic groups*, 2001.

[55] George D. Mostow, *Arithmetic subgroups of groups with radical*, Annals of Mathematics **93** (1971), no. 3, 409–438.

[56] ———, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, Princeton University Press, 1973.

[57] Hee Oh, *Discrete subgroups generated by lattices in opposite horospherical subgroups*, Journal of Algebra **203** (1998), 621–676.

[58] A.L. Onishchik, D.A. Leites, and E.B. Vinberg, *Lie groups and algebraic groups*, Springer Series in Soviet Mathematics, Springer Berlin Heidelberg, 2012.

[59] Gabriel Pallier, *Large-scale sublinearly lipschitz geometry of hyperbolic spaces* (2018), available at arXiv:1801.05163.

[60] ———, *Géométrie asymptotique sous-linéaire : hyperbolicité, autosimilarité, invariants*, Ph.D. Thesis, 2019. Thèse de doctorat dirigée par Pansu, Pierre Mathématiques fondamentales Université Paris-Saclay (ComUE) 2019.

[61] ———, *Sublinear quasiconformality and the large-scale geometry of Heintze groups*, Conform. Geom. Dyn. **24** (2020), 131–163. MR4127909

[62] Pierre Pansu, *Metriques de Carnot-Caratheodory et quasiisometries des espaces symetriques de rang un*, Annals of Mathematics **129** (1989), no. 1, 1–60.

[63] Gopal Prasad, *Strong rigidity of q-rank 1 lattices.*, Inventiones mathematicae **21** (1973), 255–286.

[64] Gopal Prasad and M. S. Raghunathan, *Cartan subgroups and lattices in semi-simple groups*, Annals of Mathematics **96** (1972), no. 2, 296–317.

[65] M.S. Raghunathan, *Discrete subgroups of lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, 1972.

[66] A.A. Sagle, R.E. Walde, and R. Walde, *Introduction to Lie groups and Lie algebras*, Pure and Applied Mathematics; A Series of Monographs and Tex, Academic Press, 1973.

[67] Richard Evan Schwartz, *The quasi-isometry classification of rank one lattices*, Publ. Math. IHES **82** (1995), no. 1, 133–168.

[68] ———, *Quasi-isometric rigidity and diophantine approximation*, Acta Mathematica **177** (1996), no. 1, 75–112.

[69] Nimish Shah, *Invariant measures and orbit closures on homogeneous spaces*, Proceedings of the international colloquium on lie groups and ergodic theory, mumbai, 1996, 1996, pp. 229–272.

[70] Dennis Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions* (Irwin Kra and Bernard Maskit, eds.), Princeton University Press, 2016.

[71] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Springer-Verlag, Berlin,New York, 1974.

[72] Joseph Albert Wolf, *Spaces of constant curvature*, New York: McGraw-Hill, 1967.