

ANALYSIS OF A DILUTE POLYMER MODEL WITH A TIME-FRACTIONAL DERIVATIVE*

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Abstract. We investigate the well-posedness of a coupled Navier–Stokes–Fokker–Planck system with a time-fractional derivative. Such systems arise in the kinetic theory of dilute solutions of polymeric liquids, where the motion of noninteracting polymer chains in a Newtonian solvent is modelled by a stochastic process exhibiting power-law waiting time, in order to capture subdiffusive processes associated with non-Fickian diffusion. We outline the derivation of the model from a subordinated Langevin equation. The elastic properties of the polymer molecules immersed in the solvent are modelled by a finitely extensible nonlinear elastic (FENE) dumbbell model, and the drag term in the Fokker–Planck equation is assumed to be corotational. We prove the global-in-time existence of large-data weak solutions to this time-fractional model of order $\alpha \in (\frac{1}{2}, 1)$, and derive an energy inequality satisfied by weak solutions.

Key words. Existence of weak solutions; Navier–Stokes–Fokker–Planck system; dilute polymer model; Hookean and FENE-type bead-spring-chain model; Riemann–Liouville fractional derivative; time-fractional PDE

MSC codes. 35Q30, 35Q84, 35R11, 60G22, 82C31, 82D60

1. Introduction. This paper is concerned with the existence of weak solutions to a system of nonlinear partial differential equations that arises in the kinetic theory of dilute solutions of polymeric fluids. Within this class of models we focus on finitely-extensible nonlinear elastic, FENE-type, dumbbell models with a corotational drag term. In contrast to previous literature on the analysis of these models we assume power-law waiting times in the derivation of the system, which results in the appearance of a time-fractional derivative in the Fokker–Planck equation describing the evolution of the probability density function. This raises new questions about the study of well-posedness, and we provide rigorous results concerning the existence of global-in-time weak solutions to the system of partial differential equations featuring in the model.

Dilute polymer models are derived and extensively described in the monograph [16] and in the book by Öttinger [57]; see also [66] for a mathematically rigorous derivation of the Hookean bead-spring-chain model from Brownian dynamics. We also refer to the papers [30, 41] for a comparison of several FENE-type dumbbell models. Such systems are of microscopic-macroscopic type since they involve a coupling of the (macroscopic) Navier–Stokes equations for the description of incompressible fluid flow and the Fokker–Planck equation for the microscopic processes associated with the statistical properties of polymer molecules immersed in the fluid. Concerning the weak and strong well-posedness of FENE-type models, we refer to the works [32, 34, 50, 62, 72]. More general dilute polymer models are analyzed in [6–8, 10, 11]. Further, we mention the papers [6, 9, 18, 21, 43–45, 49, 63], which, similarly to the discussion herein, are concerned with dumbbell models that assume a corotational

*Submitted to the editors March 4, 2024.

Funding: MF is supported by the State of Upper Austria.

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drag term in the Fokker–Planck equation. In such models it is supposed that polymer molecules are not stretched by the surrounding solvent, although they are allowed to rotate without stretching; see, for example, [35].

Time-fractional differential equations have been the focus of considerable attention in the mathematical and engineering literature in recent years. Such equations are nonlocal in time and have an innate history effect. They are of relevance in applications where memory effects are present and hereditary properties of materials are studied; see, for example, the textbooks on viscoelasticity [48, 70], hydrology [65], financial economics [23], and mechanical processes [2, 60]. The time-fractional Fokker–Planck equation, in particular, allows subdiffusive behaviour and has been previously studied in [4, 5, 27–29, 36, 53–55] with regards to its derivation and applicability. The articles [37–40, 61] have investigated the numerical analysis and the simulation of solutions to the time-fractional Fokker–Planck equation. The time-fractional model considered herein has been explored computationally in [15], albeit in the simpler setting of a linear (Hookean) elastic spring force instead of the FENE spring model that we study here.

We employ a spatial Galerkin approximation in conjunction with a compactness argument to prove the existence of weak solutions to the time-fractional Navier–Stokes–Fokker–Planck system under consideration. More specifically, we discretize the system in space and derive appropriate energy bounds, which then enable us to pass to the limit in the discretized system. Spatial discretizations of dilute polymer models were previously considered in [9, 13, 14]. In addition, weak solutions to time-fractional PDEs have been investigated using the Galerkin approach in the publications [24–26]. There have also been initial steps in the analysis of a decoupled time-fractional Fokker–Planck equation with time-dependent forces; see the papers [39, 40, 51, 52]. However, only mild, strong, and classical solutions have been studied. The coupling of the time-fractional Fokker–Planck equation to the Navier–Stokes system gives rise to new technical complications, which have not been addressed previously.

In Section 2 we derive the model from the Langevin equation assuming power-law waiting time. In this way time-fractional derivatives in the sense of Riemann–Liouville appear in the associated Fokker–Planck equation. By mimicking the technique for the derivation of the standard dumbbell model, a time-fractional Navier–Stokes–Fokker–Planck system is obtained. In Section 3 we introduce several function spaces of Sobolev-type and recall some important results from the theory of fractional derivatives, including chain inequalities and embedding theorems. In Section 4 we transform the model in order to make it amenable to the subsequent analysis. We then equip the model with suitable initial and boundary conditions and we make use of the associated Maxwellian to rescale the Navier–Stokes–Fokker–Planck system. In Section 5 we finally state and prove a theorem asserting the existence of large-data global-in-time weak solutions to the model with a time-fractional derivative of order $\alpha \in (\frac{1}{2}, 1)$.

2. Derivation of the time-fractional FENE-type system. In this section we derive the time-fractional Navier–Stokes–Fokker–Planck model, admitting both Hookean and FENE-type bead-spring-chains. The classical Hookean bead-spring-chain model is derived from a system of stochastic differential equations; see, for example, the articles [7, 66] and the thesis [71, Section 1.3]. The time-fractional Fokker–Planck equation is derived from a Langevin equation in [46]. We shall emphasize the differences in the derivation of the time-fractional Navier–Stokes–Fokker–Planck model and examine the steps where the time-fractional derivative is introduced.

In this work, the time-fractional derivative of order $\alpha \in (0, 1)$ is understood in the sense of Riemann–Liouville and is given by

$$(2.1) \quad \partial_t^\alpha w(t) := \partial_t \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} w(s) ds,$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$ is Euler’s Gamma function.

2.1. Derivation. We shall idealize each polymer molecule as a pair of massless beads connected by a massless elastic spring. It is assumed that the resulting, so called, *dumbbell* is suspended in a Newtonian solvent, whose motion is governed by the incompressible Navier–Stokes equations for the velocity u and the pressure p of the fluid. Let us denote the position vectors (with respect to an arbitrary, but fixed, reference point in \mathbb{R}^3) of the centers of mass of the two beads at time t by $x_i(t) \in \mathbb{R}^3$ for $i \in \{1, 2\}$. At time t , the center of mass of the dumbbell is at $x_c(t) := \frac{1}{2}(x_1(t) + x_2(t))$ and the elongation (or conformation) vector pointing from $x_1(t)$ to $x_2(t)$ is $q_1(t) := x_2(t) - x_1(t)$. We denote the vector pointing in the opposite direction by $q_2(t) := -q_1(t)$ and assume that $q_1(t)$ and $q_2(t)$ are contained, for all $t \geq 0$, within a given convex open set $D \subset \mathbb{R}^3$ that satisfies $0 \in D$, and $-q \in D$ whenever $q \in D$.

Three kinds of force act on the i -th bead in the dumbbell suspended in the fluid: a drag force (Stokes drag) arising from the motion of the bead through the solvent, an elastic spring force, and a random force, which is assumed to be Brownian, modelling the random collisions of the bead with the molecules of the surrounding Newtonian solvent. As each of the two beads is assumed to be massless, Newton’s second law implies that

$$(2.2) \quad \text{Drag Force}_i + \text{Spring Force}_i + \text{Brownian Force}_i = 0, \quad i = 1, 2.$$

To define the drag force acting on the i -th bead of the dumbbell suspended in the fluid, we apply Stokes’ law and get

$$-\zeta \left(\frac{d}{dt} x_i(t) - u(x_i(t), t) \right), \quad i \in \{1, 2\}.$$

Here ζ is the friction coefficient and $u(\cdot, \cdot)$ stands for the fluid velocity. We note that ζ carries the SI unit $[\text{kg/s}]$ and is linear in the dynamic viscosity η . For a sphere of radius R it reads $\zeta = 6\pi R\eta$.

The elastic spring force $F : D \rightarrow \mathbb{R}^3$ of the spring connecting the two beads is

$$F(q) := HU'(\tfrac{1}{2}|q|^2)q,$$

where U is a nonnegative continuously differentiable potential and H is the spring constant having the SI unit $[\text{kg/s}^2]$. In the case of the Hookean dumbbell model the spring force is linear and is given by $F(q) = Hq$ with $q \in D = \mathbb{R}^3$ and the corresponding potential is $U(s) = s$ for $s \in [0, \infty)$, while in the case of a classical FENE model one has, instead,

$$D = B_{|q_{\max}|}(0), \quad F(q) = \frac{Hq}{1 - |q|^2/|q_{\max}|^2}, \quad U(s) = -\frac{|q_{\max}|^2}{2} \ln \left(1 - \frac{2s}{|q_{\max}|^2} \right),$$

for $q \in D$ and $s \in [0, \frac{1}{2}|q_{\max}|^2)$, where $B_{|q_{\max}|}(0)$ is an open ball in \mathbb{R}^3 with radius $|q_{\max}|$ centered at the origin and $|q_{\max}| > 0$ is a strict upper bound on the maximal extension to which a dumbbell can be stretched.

The Brownian force acting on the i -th bead at time t is denoted by $B_i(t)$ and is formally defined by

$$B_i(t) := \sqrt{2k_B\mu_T\zeta} \frac{dW_i(t)}{dt}, \quad i \in \{1, 2\},$$

where k_B is the Boltzmann constant in $[\text{kg m}^2/(\text{s}^2\text{K})]$, μ_T denotes the temperature in $[\text{K}]$, and $W(t) = (W_1(t)^T, W_2(t)^T)^T \in \mathbb{R}^6$ is a (column) vector of two independent Wiener processes. As each Wiener process is distributed according to $\mathcal{N}(0, t)$, we find that the SI unit for $\frac{dW_i(t)}{dt}$ is $[1/\sqrt{\text{s}}]$.

By introducing the notation

$$(2.3) \quad X(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad b(X(t), t) := \begin{pmatrix} u(x_1(t), t) + \zeta^{-1}F(q_1(t)) \\ u(x_2(t), t) + \zeta^{-1}F(q_2(t)) \end{pmatrix},$$

and dividing (2.2) by ζ , we get as impulse balance the so-called Langevin equation, an Itô stochastic differential equation of the form

$$dX(t) = b(X(t), t) dt + \sqrt{\frac{2k_B\mu_T}{\zeta}} dW(t), \quad t \geq 0.$$

The partial differential equation describing the evolution of the probability density function $\tilde{\psi} := \tilde{\psi}(x_1, x_2, t)$ of the random variable $X(t)$ is the standard Fokker–Planck equation [59]:

$$(2.4) \quad \partial_t \tilde{\psi} = \sum_{i=1}^2 \left(-\text{div}_{x_i} (b_i((x_1^T, x_2^T)^T, t) \tilde{\psi}) + \frac{k_B\mu_T}{\zeta} \Delta_{x_i} \tilde{\psi} \right),$$

with x_1 and x_2 considered to be column-vectors in \mathbb{R}^3 .

At this point, we deviate from the usual derivation of the Fokker–Planck equation for the evolution of the probability density function of the stochastic process X and introduce a subordination of the Langevin equation. This allows us to model trapping events to the motion of the particles. For $\alpha \in (0, 1)$, let $U_\alpha(\cdot)$ be an α -dependent subordinator with expectation $\mathbb{E}[e^{-\lambda U_\alpha(\tau)}] = \exp(-\tau \Phi_\alpha(\lambda))$, where

$$\Phi_\alpha(\lambda) := \tau_0^{\alpha-1} \lambda^\alpha$$

is the so-called Laplace exponent and τ_0 is a characteristic time-scale (to be fixed). We note that the limiting value of $\alpha = 1$ results in the standard integer-order case. The inverse subordinator S_α^t , defined as the first-passage time of U_α , is then given by

$$S_\alpha^t := \inf_{\tau > 0} \{ \tau : U_\alpha(\tau) > t \}.$$

Suppose that $Y_\alpha(t)$ is a solution of the stochastic differential equation

$$dY_\alpha(t) = b(Y_\alpha(t), U_\alpha(t)) dt + \sqrt{\frac{2k_B\mu_T}{\zeta}} dW(t), \quad t \geq 0,$$

and define $X(t) := Y_\alpha(S_\alpha^t)$. It then follows from Theorem 1 in [47] that, if b is twice continuously differentiable with respect to its variables and satisfies the usual Lipschitz condition, then the probability density function of the process X is a solution of the

time-fractional Fokker–Planck equation resulting from the replacement of $\tilde{\psi}$ on the right-hand side of (2.4) by $\tau_0^{1-\alpha} \partial_t^{1-\alpha} \tilde{\psi}$, i.e.,

$$\partial_t \tilde{\psi} = \sum_{i=1}^2 \left(-\operatorname{div}_{x_i} (b_i((x_1^T, x_2^T)^T, t) \tau_0^{1-\alpha} \partial_t^{1-\alpha} \tilde{\psi}) + \frac{k_B \mu_T}{\zeta} \Delta_{x_i} \tau_0^{1-\alpha} \partial_t^{1-\alpha} \tilde{\psi} \right).$$

Next, we perform the linear change of variables $(x_1, x_2) \mapsto (\frac{1}{2}(x_1 + x_2), x_2 - x_1) =: (x, q)$, whereby we have identified a point $x \in \Omega \subset \mathbb{R}^3$ in the (macroscopic) flow domain Ω with the center of mass of the dumbbell, and have, as before, denoted by q the vector pointing from x_1 to x_2 . By recalling the definition (2.3) of $b(\cdot, \cdot)$ and setting $\psi(x, q, t) := \psi(x - \frac{1}{2}q, x + \frac{1}{2}q, t)$, we find that

$$\begin{aligned} (2.5) \quad & \partial_t \psi + \tau_0^{1-\alpha} \operatorname{div}_x \left(\frac{u(x - \frac{1}{2}q, t) + u(x + \frac{1}{2}q, t)}{2} \partial_t^{1-\alpha} \psi \right) \\ & + \tau_0^{1-\alpha} \operatorname{div}_q \left((u(x + \frac{1}{2}q, t) - u(x - \frac{1}{2}q, t)) \partial_t^{1-\alpha} \psi - \frac{2F(q)}{\zeta} \partial_t^{1-\alpha} \psi \right) \\ & = \frac{k_B \mu_T \tau_0^{1-\alpha}}{2\zeta} \Delta_x \partial_t^{1-\alpha} \psi + \frac{2k_B \mu_T \tau_0^{1-\alpha}}{\zeta} \Delta_q \partial_t^{1-\alpha} \psi. \end{aligned}$$

To proceed, we assume ‘local homogeneity’, i.e., that the spatial variation of the velocity field over the microscopic length-scale of a single dumbbell is negligibly small. Consequently, the arithmetic mean $(u(x - \frac{1}{2}q, t) + u(x + \frac{1}{2}q, t))/2$ can be approximated by $u(x, t)$ in the second term on the left-hand side of (2.5). In the case of the third term, we use Taylor expansion of u about the point x to obtain

$$(2.6) \quad u(x + \frac{1}{2}q, t) - u(x - \frac{1}{2}q, t) = \nabla_x u(x, t) q + \mathcal{O}(|q|^3).$$

We omit the $\mathcal{O}(|q|^3)$ term, which can be justified on the grounds that $|q| \ll 1$. Thus for our analysis, we consider the following Fokker–Planck equation with a time-fractional derivative:

$$\begin{aligned} (2.7) \quad & \partial_t \psi + \tau_0^{1-\alpha} \operatorname{div}_x (u \partial_t^{1-\alpha} \psi) + \tau_0^{1-\alpha} \operatorname{div}_q ((\nabla_x u) q \partial_t^{1-\alpha} \psi) \\ & = \frac{k_B \mu_T \tau_0^{1-\alpha}}{2\zeta} \Delta_x \partial_t^{1-\alpha} \psi + \frac{2k_B \mu_T \tau_0^{1-\alpha}}{\zeta} \Delta_q \partial_t^{1-\alpha} \psi + \tau_0^{1-\alpha} \operatorname{div}_q \left(\frac{2F(q)}{\zeta} \partial_t^{1-\alpha} \psi \right). \end{aligned}$$

Let $\psi(x, q, t)$ denote from now on the probability density function for a collection of $N \gg 1$ dumbbells. As we are dealing with a dilute polymeric fluid, the polymer molecules suspended in the fluid are assumed not to interact with each other and they move without self-interaction. The function ψ therefore satisfies the same partial differential equation, (2.7), as in the case of a single dumbbell. The only difference is in the choice of the initial datum $\psi^0 \geq 0$ for the Fokker–Planck equation. A typical choice of ψ^0 in the present context is

$$\psi^0(x, q) = \frac{1}{N} \sum_{j=1}^N \alpha_j(q) \Psi_j(x),$$

where $\alpha_j(q) \geq 0$ for all $q \in D$ and $\int_D \alpha_j(q) dq = 1$, $j = 1, \dots, N$, and $\Psi_j \geq 0$, for all $x \in \Omega$ and $\int_\Omega \Psi_j(x) dx = 1$, $j = 1, \dots, N$. For example, one may choose Ψ_j as a mollifier (i.e. a nonnegative C_0^∞ approximation to the Dirac measure) concentrated

at a point $z_j \in \Omega$, $j = 1, \dots, N$, with $\{\Psi_j\}_{j=1}^N$ forming a scaled partition of unity; here z_j can be thought of as the initial location of the center of mass of the j -th dumbbell.

For the sake of simplicity we shall confine our attention to the case when the boundary $\partial\Omega$ of the macroscopic flow domain $\Omega \subset \mathbb{R}^3$ has no inflow or outflow parts, i.e., the velocity u will be supposed to satisfy the no-slip boundary condition $u(x, t) = 0$ for all $(x, t) \in \partial\Omega \times (0, T)$. We shall supplement the Fokker–Planck equation (2.7) with no-flux (homogeneous Neumann) boundary conditions on $\partial\Omega \times D \times (0, T)$ and on $\Omega \times \partial D \times (0, T)$, which will ensure that

$$\int_{\Omega} \int_D \psi(x, q, t) \, dq \, dx = \int_{\Omega} \int_D \psi^0(x, q) \, dq \, dx = 1.$$

Moreover the total number $N \gg 1$ of polymer molecules contained in Ω remains constant in time. The number density $\varrho_P := \varrho_P(x, t)$, in $[m^{-3}]$, of the polymer molecules contained in Ω is called the *polymer number density* and it is related to N by $N = \int_{\Omega} \varrho_P(x, t) \, dx$. The polymer number density is further related to the probability density function ψ satisfying the Fokker–Planck equation (2.7) by

$$(2.8) \quad \varrho_P(x, t) = N \int_D \psi(x, q, t) \, dq,$$

with the understanding that ψ has the SI unit of $[m^{-6}]$.

Having established the “microscopic” equation that describes the statistical properties of polymer molecules in the continuum, we turn our attention to the continuum mechanical “macroscopic” equations of motion of the incompressible fluid in which the polymer molecules are suspended. We note that polymeric fluids are non-Newtonian fluids and the presence of the $N \gg 1$ polymer molecules contributes an additional term in the balance of linear momentum equation in the Navier–Stokes system: a symmetric polymeric extra stress tensor, which we shall define below. We assume that the evolution of the velocity field u and the pressure p_1 is governed by the incompressible Navier–Stokes system

$$(2.9) \quad \begin{aligned} \varrho(\partial_t u + (u \cdot \nabla_x)u) - \eta \Delta_x u + \nabla_x p_1 &= \operatorname{div}_x \tau_1(\psi) && \text{in } \Omega \times (0, T), \\ \operatorname{div}_x u &= 0 && \text{in } \Omega \times (0, T), \end{aligned}$$

supplemented with the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega \times (0, T)$ and the initial condition $u(0) = u^0$ in Ω at $t = 0$. As before, η denotes the dynamic viscosity and ϱ stands for the macroscopic density, which is assumed to be constant in space and time. The polymeric extra-stress tensor is defined by the so-called *Kramers expression*, see, e.g., [16] for the integer-order case (i.e., when $\alpha = 1$),

$$(2.10) \quad \tau_1(\psi) := \varrho_P (\mathcal{C}(\tau_0^{1-\alpha} \partial_t^{1-\alpha} \psi) - k_B \mu_T I_3),$$

where, as before, ϱ_P is the polymer number density, k_B and μ_T are the Boltzmann constant and the absolute temperature, respectively, and I_3 is the 3×3 identity matrix. In the time-fractional case the usual argument ψ in $\mathcal{C}(\cdot)$ is replaced by $\tau_0^{1-\alpha} \partial_t^{1-\alpha} \psi$. In the following, we do not work with p_1 and τ_1 but introduce the pressure p and the redefined extra-stress tensor τ :

$$(2.11) \quad \tau(\psi) := \varrho_P \mathcal{C}(\tau_0^{1-\alpha} \partial_t^{1-\alpha} \psi) := N \int_D F(q) q^T \tau_0^{1-\alpha} \partial_t^{1-\alpha} \psi \, dq,$$

$$(2.12) \quad p := p_1 + k_B \mu_T \varrho_P.$$

Recall that ψ is assumed to have the SI unit $[m^{-6}]$ and thus the definition of $\tau(\psi)$ results in the SI unit $[kg/(s^2m^2)]$ for the polymeric extra stress tensor. The tensor τ is responsible for coupling the velocity u and the redefined pressure p to the probability density function ψ . Dividing the Navier–Stokes momentum equation by the macroscopic density ϱ and introducing the kinematic viscosity as $\nu := \eta/\varrho$, we arrive at the Navier–Stokes system in its dimensional form:

$$(2.13) \quad \begin{aligned} \partial_t u + (u \cdot \nabla_x)u - \nu \Delta_x u + \frac{1}{\varrho} \nabla_x p &= \frac{1}{\varrho} \operatorname{div}_x \tau(\psi) & \text{in } \Omega \times (0, T), \\ \operatorname{div}_x u &= 0 & \text{in } \Omega \times (0, T). \end{aligned}$$

2.2. Nondimensionalization. Next, we transform the Navier–Stokes–Fokker–Planck system, see (2.13) and (2.7), into its nondimensionalized form. To this end, we define the quantities \hat{x} , \hat{t} and \hat{u} by setting

$$x = L_0 \hat{x}, \quad t = T_0 \hat{t}, \quad u(x, t) = U_0 \hat{u}(\hat{x}, \hat{t}),$$

where L_0 and T_0 stand for the characteristic macroscopic length-scale and time-scale, respectively, and U_0 denotes the characteristic velocity of the macroscopic flow. In a similar manner, we introduce \hat{q} by letting $q = \ell_0 \hat{q}$, where ℓ_0 is a characteristic microscopic length-scale; recall again that ψ has been assumed to have the SI unit $[m^{-6}]$, and define the dimensionless quantity $\hat{\psi}$ by

$$(2.14) \quad \hat{\psi}(\hat{x}, \hat{q}, \hat{t}) := (L_0 \ell_0)^3 \psi(x, q, t).$$

Consequently, we obtain from the time-fractional Fokker–Planck equation (2.7) the following partial differential equation:

$$(2.15) \quad \begin{aligned} \frac{1}{T_0} \partial_{\hat{t}} \hat{\psi} + \frac{\tau_0^{1-\alpha} U_0}{T_0^{1-\alpha} L_0} \operatorname{div}_{\hat{x}} (\hat{u}(\psi) \partial_{\hat{t}}^{1-\alpha} \hat{\psi}) + \frac{\tau_0^{1-\alpha} U_0}{T_0^{1-\alpha} L_0} \operatorname{div}_{\hat{q}} ((\nabla_{\hat{x}} \hat{u}(\psi)) \hat{q} \partial_{\hat{t}}^{1-\alpha} \hat{\psi}) \\ = \frac{k_B \mu_T \tau_0^{1-\alpha}}{2 \zeta T_0^{1-\alpha} L_0^2} \Delta_{\hat{x}} \partial_{\hat{t}}^{1-\alpha} \hat{\psi} + \frac{2 k_B \mu_T \tau_0^{1-\alpha}}{\zeta \ell_0^2 T_0^{1-\alpha}} \Delta_{\hat{q}} \partial_{\hat{t}}^{1-\alpha} \hat{\psi} + \frac{2 H \tau_0^{1-\alpha}}{\zeta T_0^{1-\alpha}} \operatorname{div}_{\hat{q}} (\hat{F}(\hat{q}) \partial_{\hat{t}}^{1-\alpha} \hat{\psi}) \end{aligned}$$

defined on the nondimensionalized domain $\hat{\Omega} \times \hat{D} \times (0, \hat{T})$. Here we have used the notation $\hat{u}(\psi)$ to indicate that the velocity field is understood to depend on the original (dimensional) probability density function ψ , and will only be expressed as a function depending on the nondimensionalized probability density function $\hat{\psi}$ in the next step. We set the characteristic macroscopic and microscopic time-scales to, respectively, $T_0 := L_0/U_0$, $\tau_0 := T_0$ and the nondimensionalized force in case of the standard FENE type model to

$$(2.16) \quad \hat{F}(\hat{q}) := \frac{\hat{q}}{1 - |\hat{q}|^2 / |\hat{q}_{\max}|^2}, \quad \hat{q}_{\max} := q_{\max} / \ell_0.$$

The prefactors of the last two terms on the right-hand side of (2.15) are equal if the microscopic length-scale ℓ_0 is defined as follows:

$$(2.17) \quad \ell_0^2 := \frac{k_B \mu_T}{H}.$$

Next we introduce the Deborah number λ , defined as the ratio of the time it takes for the material to adjust to applied stresses/deformations and the characteristic time scale, T_0 , by

$$\lambda := \frac{\zeta/(4H)}{T_0} = \frac{\zeta}{4HT_0} = \frac{\zeta\ell_0^2}{4k_B\mu_T T_0}.$$

Finally, we define the nondimensional parameter

$$\varepsilon := \frac{k_B\mu_T}{2\zeta U_0 L_0},$$

whose numerator and denominator both carry the SI unit of $J = [kg\,m^2/s^2]$ corresponding to energy. Thus, we obtain from (2.15) by multiplying it with T_0 the following nondimensional equation:

$$\begin{aligned} (2.18) \quad & \partial_t \hat{\psi} + (\hat{u}(\psi) \cdot \nabla_{\hat{x}}) \partial_t^{1-\alpha} \hat{\psi} + \operatorname{div}_{\hat{q}}((\nabla_{\hat{x}} \hat{u}(\psi)) \hat{q} \partial_t^{1-\alpha} \hat{\psi}) \\ & = \varepsilon \Delta_{\hat{x}} \partial_t^{1-\alpha} \hat{\psi} + \frac{1}{2\lambda} \operatorname{div}_{\hat{q}}(\nabla_{\hat{q}} \partial_t^{1-\alpha} \hat{\psi} + \hat{F}(\hat{q}) \partial_t^{1-\alpha} \hat{\psi}). \end{aligned}$$

We note in passing that the ratio of the diffusion coefficients ε and $1/(2\lambda)$ featuring in the equation (2.18) is equal to $(\ell_0/(2L_0))^2$, which is $\ll 1$, so the first-term on the right-hand side of (2.18), called the center-of-mass diffusion term, is frequently neglected in practical considerations. Crucially, we shall retain this term in the equation and will continue to work with a strictly positive center-of-mass diffusion coefficient ε , as is implied by the derivation of the Fokker–Planck equation (2.18) performed above.

In the same manner, we multiply the Navier–Stokes system (2.13) by T_0/U_0 and arrive at the following nondimensionalized system posed on $\hat{\Omega} \times (0, \hat{T})$:

$$\begin{aligned} (2.19) \quad & \partial_t \hat{u} + (\hat{u} \cdot \nabla_{\hat{x}}) \hat{u} - \frac{1}{\operatorname{Re}} \Delta_{\hat{x}} \hat{u} + \nabla_{\hat{x}} \hat{p} = \operatorname{div}_{\hat{x}} \hat{\tau}(\hat{\psi}), \\ & \operatorname{div}_{\hat{x}} \hat{u} = 0, \end{aligned}$$

where Re stands for the Reynolds number and \hat{p} for a scaled pressure. The definition (2.11) of $\tau(\psi)$ in combination with the nondimensionalization steps, the definition of \hat{F} and $\hat{\psi}$, see (2.16) and (2.14), respectively, and the fact that $T_0 = \tau_0$ yield

$$\begin{aligned} & \frac{1}{\varrho} \frac{T_0}{U_0} \frac{1}{L_0} H \ell_0^2 \ell_0^3 \int_{\hat{D}} \hat{F}(\hat{q}) \hat{q}^T \partial_t^{1-\alpha} \hat{\psi} \, d\hat{q} (L_0 \ell_0)^{-3} N \\ & = \frac{N}{\varrho L_0^3} \frac{k_B \mu_T}{U_0^2} \int_{\hat{D}} \hat{F}(\hat{q}) \hat{q}^T \partial_t^{1-\alpha} \hat{\psi} \, d\hat{q} = \gamma \hat{\mathcal{C}}(\partial_t^{1-\alpha} \hat{\psi}) =: \hat{\tau}(\hat{\psi}), \end{aligned}$$

with the dimensionless quantities γ and $\hat{\mathcal{C}}(\cdot)$ being defined by, respectively,

$$(2.20) \quad \gamma := \frac{k_B \mu_T N}{\varrho U_0^2 L_0^3} \quad \text{and} \quad \hat{\mathcal{C}}(\partial_t^{1-\alpha} \hat{\psi}) := \int_{\hat{D}} \hat{F}(\hat{q}) \hat{q}^T \partial_t^{1-\alpha} \hat{\psi} \, d\hat{q}.$$

For the sake of notational simplicity we drop from now on all $\hat{\cdot}$ symbols. We close this section by recalling from (2.19) and (2.18) our nondimensionalized model problem:

$$\begin{aligned} (2.21) \quad & \partial_t \psi + (u \cdot \nabla_x) \partial_t^{1-\alpha} \psi + \operatorname{div}_q((\nabla_x u) q \partial_t^{1-\alpha} \psi) \\ & - \frac{1}{2\lambda} \operatorname{div}_q(\nabla_q \partial_t^{1-\alpha} \psi + F(q) \partial_t^{1-\alpha} \psi) = \varepsilon \partial_t^{1-\alpha} \Delta_x \psi \quad \text{in } \Omega \times D \times (0, T), \\ & \partial_t u + (u \cdot \nabla_x) u - \frac{1}{\operatorname{Re}} \Delta_x u + \nabla_x p = \operatorname{div}_x \tau(\psi) \quad \text{in } \Omega \times (0, T), \\ & \operatorname{div}_x u = 0 \quad \text{in } \Omega \times (0, T), \end{aligned}$$

and we supplement this system of equations with the following boundary conditions:

$$\begin{aligned} \left(\frac{1}{2\lambda}(\nabla_q \partial_t^{1-\alpha} \psi + U' q \partial_t^{1-\alpha} \psi) - (\nabla_x u) q \partial_t^{1-\alpha} \psi\right) \cdot n_{\partial D} &= 0 & \text{on } \Omega \times \partial D \times (0, T), \\ \varepsilon \nabla_x \partial_t^{1-\alpha} \psi \cdot n_{\partial \Omega} &= 0 & \text{on } \partial \Omega \times D \times (0, T), \\ u &= 0 & \text{on } \partial \Omega \times (0, T). \end{aligned}$$

The nondimensional initial data $u^0(x)$ and $\psi^0(x, q)$ for the velocity field and the probability density function, respectively, are obtained from their dimensional counterparts. We note that the scaling (2.14) ensures that $\int_{\Omega} \int_D \psi^0(x, q) dx dq = 1$; this then guarantees, thanks to the homogeneous Neumann boundary conditions on ψ , that $\int_{\Omega} \int_D \psi(x, q, t) dx dq = 1$ for all $t \geq 0$.

At this point, we introduce a simplification to this, so called, *general noncorotational Navier–Stokes–Fokker–Planck system*. We split the spatial gradient $\nabla_x u$ of u , which appears in the third term in the time-fractional Fokker–Planck equation and in the first of the associated boundary conditions stated above, into its symmetric and antisymmetric parts, which are, respectively, defined as follows:

$$(2.22) \quad \sigma(u) := \frac{\nabla_x u + (\nabla_x u)^T}{2}, \quad \omega(u) := \frac{\nabla_x u - (\nabla_x u)^T}{2}.$$

In our analysis of the existence of weak solutions, we shall omit the symmetric part of the gradient of u , i.e., we consider the, so called, *corotational model*. Our omission in the analysis of the term $\sigma(u(x, t))$ from the additive decomposition $\nabla_x u(x, t) = \sigma(u(x, t)) + \omega(u(x, t))$ is for purely technical reasons. In a corotational model, polymer molecules are allowed to rotate, but they are forced to do so without stretching. This modelling assumption weakens the coupling between the Navier–Stokes equation and the (time-fractional) Fokker–Planck equation; for example, if the initial datum for ψ happens to be spherically symmetric with respect to q and independent of the spatial variable x , then the Fokker–Planck equation is decoupled from the Navier–Stokes equation. In this respect, the model that we study here theoretically is no different from the corotational model considered (in the case of $\alpha = 1$) in the works [6, 9, 18, 21, 43–45, 49, 63].

We shall rely in our analysis on the fact that $q^T \omega(u) q \equiv 0$, thanks to the skew-symmetry of $\omega(u)$. In the general noncorotational case $q^T (\nabla_x u) q$ is of course not identically zero in the flow domain, which complicates the analysis. For $\alpha = 1$ at least, the proof of the existence of large-data global-in-time weak solutions to the general noncorotational Navier–Stokes–Fokker–Planck system relies, instead, on a logarithmic entropy estimate (cf. [12]). It is this entropy estimate that needs to be replicated in the time-fractional case, for $\alpha \in (0, 1)$, as a key ingredient of the proof of the existence of global-in-time large-data weak solutions. The analysis of the general noncorotational time-fractional model is deferred to future work.

3. Mathematical preliminaries. In this section, we introduce some useful definitions and results regarding the fractional derivative in the sense of Riemann–Liouville and recall the Aubin–Lions lemma, which is a key result featuring in proofs of existence of weak solutions to nonlinear PDEs based on compactness arguments.

For a Hilbert space H with inner product $(\cdot, \cdot)_H$ and norm $\|\cdot\|_H$, we shall denote the duality pairing between H and its dual space H' by $\langle \cdot, \cdot \rangle_H$. We shall denote the inner product on the Bochner space $L^2(0, T; H)$ by $(\cdot, \cdot)_{L^2 H}$, and we shall write $(\cdot, \cdot)_{L_t^2 H}$ when in this inner product the temporal interval of integration is $(0, t)$ for

some $t \in (0, T)$ rather than the complete time interval $(0, T)$, i.e.,

$$(u, v)_{L_t^2 H} := \int_0^t (u(s), v(s))_H \, ds \quad \forall u, v \in L^2(0, T; H).$$

The norm induced by this inner product will be denoted by $\|\cdot\|_{L_t^2 H}$.

3.1. Riemann–Liouville kernels. The Riemann–Liouville kernel function g_α of order α is defined by $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t \in (0, T)$, for $\alpha > 0$ and $g_0(t) := \delta_0(t)$ (the Dirac distribution concentrated at $t = 0$) for $\alpha = 0$. We observe that $g_\alpha \in L^p(0, T)$ for any $\alpha \in (1 - 1/p, 1)$ and $p \in [1, \infty)$, and the kernel function satisfies the following semigroup property; see [22, Theorem 2.4]:

$$(3.1) \quad g_\alpha * g_\beta = g_{\alpha+\beta} \quad \forall \alpha, \beta \geq 0.$$

We note that when $\alpha \in (0, 1)$, one can bound the $L^p(0, t)$ -norm of a function $u : (0, T) \rightarrow \mathbb{R}$ by its convolution with g_α as follows: for any $t \in (0, T]$, we have that

$$(3.2) \quad \begin{aligned} \|u\|_{L^p(0, t)}^p &:= \int_0^t |u(s)|^p \, ds \leq t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} |u(s)|^p \, ds \\ &\leq T^{1-\alpha} \Gamma(\alpha) (g_\alpha * |u|^p)(t). \end{aligned}$$

This implies that the space

$$L_\alpha^p(0, T) := \left\{ u : (0, T) \rightarrow \mathbb{R} : \sup_{t \in (0, T)} (g_\alpha * |u|^p)(t) < \infty \right\},$$

is a subspace of $L^p(0, T)$. If the order α of the kernel function g_α is larger than 1, then one can exploit the semigroup property of the kernel and apply Young's convolution inequality (cf. Lemma 3.2 in [56]) as follows:

$$(g_{1+\alpha} * u)(t) = (g_1 * g_\alpha * u)(t) = \int_0^t (g_\alpha * u)(s) \, ds \leq \|g_\alpha\|_{L^1(0, t)} \|u\|_{L^1(0, t)},$$

for any $u \in L^1(0, T)$ and any $t \in (0, T]$.

3.2. Time-fractional derivative. We can rewrite the definition of the Riemann–Liouville derivative stated in (2.1) in a compact form by using the convolution operator $*$ as $\partial_t^\alpha w = \partial_t(g_{1-\alpha} * w)$. We refer to the classical textbooks [3, 22] and the newer monographs [20, 31] regarding fractional calculus and fractional differential equations.

We define the fractional Riemann–Liouville–Bochner space for $\alpha \in (0, 1)$ and $p \in [1, \infty)$ on $(0, T)$ with values in H by

$$\mathcal{W}^{\alpha, p}(0, T; H) := \left\{ u \in L^p(0, T; H) : g_{1-\alpha} * u \in W^{1, p}(0, T; H) \right\}.$$

Here, the convolution $*$ is of course understood to be with respect to the temporal variable $t \in (0, T)$. In the limit, when $\alpha = 1$, we have that $g_{1-\alpha} = g_0 = \delta$, and then

$$\mathcal{W}^{1, p}(0, T; H) := W^{1, p}(0, T; H) := \left\{ u \in L^p(0, T; H) : \partial_t u \in L^p(0, T; H) \right\}.$$

However for $0 < \alpha < 1$, the Riemann–Liouville–Bochner space $\mathcal{W}^{\alpha, p}(0, T; H)$ differs from the fractional-order Sobolev–Bochner space

$$W^{\alpha, p}(0, T; H) := \left\{ u \in L^p(0, T; H) : (s, t) \mapsto \frac{\|u(t) - u(s)\|_H}{|t-s|^{\alpha+1/p}} \in L^p((0, T) \times (0, T)) \right\},$$

which can be confirmed by noting that the function g_α is an element of $\mathcal{W}^{\alpha, p}(0, T) := \mathcal{W}^{\alpha, p}(0, T; \mathbb{R})$ for $\alpha \in (1 - \frac{1}{p}, 1)$ but not of $W^{\alpha, p}(0, T)$; see [19, Proposition 3.13].

Remark 3.1. Even though the space $\mathcal{W}^{\alpha,p}(0, T)$ is not a subspace of the Sobolev–Slobodeckii space $W^{\alpha,p}(0, T)$, it is nevertheless continuously embedded into $C([0, T])$, the space of uniformly continuous functions defined on $[0, T]$, for $\alpha \in (1 - \frac{1}{p}, 1]$ and $p \in [1, \infty)$; see, [19, Remark 6.2].

We also introduce the following Riemann–Liouville–Bochner space incorporating a homogeneous initial condition at $t = 0$, albeit in a somewhat nonstandard manner:

$$\mathcal{W}_0^{\alpha,p}(0, T; H) := \{u \in \mathcal{W}^{\alpha,p}(0, T; H) : (g_{1-\alpha} * u)(0) = 0\}.$$

We note that the function $g_{1-\alpha} * u : [0, T] \rightarrow H$ has a well-defined trace at $t = 0$ (even when the function u itself might not have one) thanks to the continuous embedding

$$g_{1-\alpha} * u \in W^{1,p}(0, T; H) \hookrightarrow AC([0, T]; H).$$

For a given element $z \in H$, the convolution $g_{1-\alpha} * z$ should be understood to mean the function $t \mapsto (g_{1-\alpha} * g_1)(t)z \in H$; recall that $g_1(t) \equiv 1$ for all $t \geq 0$. Thus, $z \in H$ is in this context now thought of as the mapping $t \mapsto g_1(t)z \in \mathcal{W}^{\alpha,p}(0, T; H)$, for $\alpha \in (0, 1)$, $p \in [1, \infty)$ and $0 < \alpha p < 1$, or if $\alpha = 1$ and $p \in [1, \infty)$. Thanks to the semigroup property (3.1) we then have that

$$t \mapsto (g_{1-\alpha} * z)(t) = z g_{2-\alpha}(t) = \frac{z}{\Gamma(2-\alpha)} t^{1-\alpha} \in C([0, T]; H)$$

for any $\alpha \in [0, 1]$. Thus, for $\alpha \in (0, 1]$, $p \in [1, \infty)$ and $z \in H$ we define the following ‘translated’ Riemann–Liouville–Bochner space:

$$(3.3) \quad \mathcal{W}_z^{\alpha,p}(0, T; H) := \{u \in L^p(0, T; H) : u - z \in \mathcal{W}^{\alpha,p}(0, T; H), (g_{1-\alpha} * u)(0) = 0\}.$$

Note that if $\alpha \in (0, 1)$, $p \in [1, \infty)$ and $0 < \alpha p < 1$, or if $\alpha = 1$ and $p \in [1, \infty)$, then $u - z \in \mathcal{W}^{\alpha,p}(0, T; H)$ if, and only if $u \in \mathcal{W}_0^{\alpha,p}(0, T; H)$, and therefore, for such α and p we have that $\mathcal{W}_z^{\alpha,p}(0, T; H) = \mathcal{W}_0^{\alpha,p}(0, T; H)$ irrespective of the choice of $z \in H$.

Next, we state an inverse convolution (or deconvolution) property. Its name stems from the fact that convolution with the kernel g_α acts as an inverse mapping on the operator of taking α -th fractional derivative, up to a term that involves the initial value at $t = 0$.

LEMMA 3.2 (Inverse convolution). *Let $\alpha \in (0, 1]$ and $p \in [1, \infty)$. Suppose further that H is a Hilbert space and $z \in H$. Then, for any $t \in (0, T)$, we have the following equalities:*

$$(3.4) \quad (g_\alpha * \partial_t^\alpha u)(t) = u(t) - (g_{1-\alpha} * u)(0)g_\alpha(t) \quad \forall u \in \mathcal{W}^{\alpha,p}(0, T; H),$$

$$(3.5) \quad (g_\alpha * \partial_t^\alpha u)(t) = u(t) \quad \forall u \in \mathcal{W}_z^{\alpha,p}(0, T; H).$$

Proof. We start with the proof of the equality (3.4). Recall that for any function $u \in \mathcal{W}^{\alpha,p}(0, T; H)$ we have $g_{1-\alpha} * u \in AC([0, T]; H)$, and the fundamental theorem of calculus for absolutely continuous functions therefore yields, for any $t \in [0, T]$,

$$(g_{1-\alpha} * u)(t) - (g_{1-\alpha} * u)(0) = \int_0^t \partial_s(g_{1-\alpha} * u)(s) ds = (g_1 * \partial_t^\alpha u)(t).$$

We convolve this equality with the kernel g_α and make use of the semigroup property (3.1) to obtain

$$(g_1 * u)(t) - (g_{1-\alpha} * u)(0)g_{1+\alpha}(t) = (g_{1+\alpha} * \partial_t^\alpha u)(t),$$

where we have used that $g_\alpha * 1 = g_\alpha * g_1 = g_{1+\alpha}$, because $\alpha\Gamma(\alpha) = \Gamma(1+\alpha)$. Next, we differentiate this equality in t and observe that $\partial_t(g_1 * u) = u$, $\partial_t g_{1+\alpha} = g_\alpha$, and $\partial_t(g_{1+\alpha} * v) = g_\alpha * v$, which yields (3.4). We finally note that (3.5) follows trivially from (3.4) and (3.3). \square

The following result is a direct consequence of the interaction between fractional derivatives and kernel functions.

COROLLARY 3.3. *The following identities hold:*

$$(3.6) \quad \begin{aligned} \partial_t^\alpha(g_\alpha * u) &= \partial_t(g_{1-\alpha} * g_\alpha * u) = \partial_t(1 * u) = u & \forall u \in L^1(0, T; H), \\ \partial_t^{1-\alpha} \partial_t^\alpha u &= \partial_t(g_\alpha * \partial_t^\alpha u) = \partial_t u & \forall u \in W_0^{1,1}(0, T; H). \end{aligned}$$

We shall require the following special case of the classical Aubin–Lions lemma; see [64]. Suppose that the Hilbert spaces V, H, Z form a Gelfand triple $V \hookrightarrow H \hookrightarrow Z$; then, the following compact embeddings hold:

$$(3.7) \quad \begin{aligned} W^{1,1}(0, T; Z) \cap L^p(0, T; V) &\hookrightarrow L^p(0, T; H), \quad p \in [1, \infty), \\ W^{1,r}(0, T; Z) \cap L^\infty(0, T; V) &\hookrightarrow C([0, T]; H), \quad r \in (1, \infty); \end{aligned}$$

see [64]. Several fractional counterparts of the Aubin–Lions lemma have been proposed; see [42, 58, 69]. We make use of the following result; see [58, Corollary 3.2]:

$$\mathcal{W}^{\alpha,1}(0, T; Z) \cap L^p(0, T; V) \hookrightarrow L^r(0, T; H), \quad p \in (1, \infty), \quad r \in [1, p), \quad \alpha \in (0, 1).$$

The proof can be easily adapted to the limit case $r = p$ if the α -th fractional derivative is in a better space than $L^1(0, T; Z)$. This is done for Caputo derivatives in [42]. In fact, we obtain

$$(3.8) \quad \mathcal{W}^{\alpha,r}(0, T; Z) \cap L^p(0, T; V) \hookrightarrow L^p(0, T; H), \quad r \in (1, \infty), \quad \alpha \in (0, 1).$$

The classical chain rule does not hold for fractional derivatives, but one can use instead the following inequality as a remedy; see [67, Theorem 2.1]:

$$(3.9) \quad \frac{1}{2} \partial_t^\alpha \|u\|_H^2 + \frac{1}{2} g_{1-\alpha}(t) \|u\|_H^2 \leq (u, \partial_t^\alpha u)_H \quad \forall u \in \mathcal{W}_z^{\alpha,2}(0, T; H),$$

for $z \in H$ and almost all $t \in (0, T)$.

4. Model revisited. Having summarised the required results from fractional calculus, we revisit the mathematical model that we have derived in Section 2. Let us assume for the moment that the solution ψ to the Fokker–Planck equation belongs to $\mathcal{W}^{1-\alpha,p}(0, T; H) \cap C([0, T]; H)$ for some $\alpha \in (0, 1)$ and a suitable Hilbert space H , to be chosen. As $\psi \in C([0, T]; H)$, it follows that $\|(g_\alpha * \psi)(t)\|_H \leq \frac{t^\alpha}{\Gamma(1+\alpha)} \|\psi\|_{C([0,T];H)}$, and therefore $(g_\alpha * \psi)(0) = 0$. Hence, $\psi \in \mathcal{W}_0^{1-\alpha,p}(0, T; H)$. It then follows from (3.4), with α replaced by $1 - \alpha$ and $u = \psi$ there, that $(g_{1-\alpha} * \partial_t^{1-\alpha} \psi)(t) = \psi(t)$ for $t \in (0, T)$. Motivated by these properties, we introduce the auxiliary function ϕ by

$$(4.1) \quad \phi := \partial_t^{1-\alpha} \psi = \partial_t(g_\alpha * \psi),$$

whereby $\psi = g_{1-\alpha} * \phi$. We then have that $\partial_t \psi = \partial_t(g_{1-\alpha} * \phi) = \partial_t^\alpha \phi$; and, thanks to the assumed continuity of ψ (i.e. $\psi \in C([0, T]; H)$) it makes sense to require attainment of the initial condition $\psi(0) = \psi^0$, i.e. $(g_{1-\alpha} * \phi)(0) = \psi^0$. We shall therefore

introduce the substitution $\phi := \partial_t^{1-\alpha}\psi$ in (2.21), which results in the following system of equations:

$$(4.2) \quad \begin{aligned} \partial_t u + (u \cdot \nabla_x)u - \nu \Delta_x u + \nabla_x p - \gamma \operatorname{div}_x \mathcal{C}(\phi) &= 0, \\ \operatorname{div}_x u &= 0, \\ \partial_t^\alpha \phi + (u \cdot \nabla_x)\phi + \operatorname{div}_q(\omega(u)q\phi) - \frac{1}{2\lambda} \operatorname{div}_q(\nabla_q \phi + U'q\phi) - \varepsilon \Delta_x \phi &= 0, \end{aligned}$$

subject to the initial conditions $u(0) = u^0$ and $(g_{1-\alpha} * \phi)(0) = \psi^0$ for a given non-negative ψ^0 that fulfils $\int_D \psi^0 dq = 1$. Furthermore, we equip the system with the following boundary conditions:

$$(4.3) \quad \begin{aligned} u &= 0 && \text{on } \partial\Omega \times (0, T), \\ \left(\frac{1}{2\lambda}(\nabla_q \phi + U'q\phi) - \omega(u)q\phi\right) \cdot n_{\partial D} &= 0 && \text{on } \Omega \times \partial D \times (0, T), \\ \varepsilon \nabla_x \phi \cdot n_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times D \times (0, T). \end{aligned}$$

4.1. The Maxwellian and Maxwellian-weighted function spaces. We introduce the normalized Maxwellian by

$$(4.4) \quad M(q) := \frac{e^{-U(\frac{1}{2}|q|^2)}}{\int_D e^{-U(\frac{1}{2}|s|^2)} ds}.$$

Moreover, we define the (Maxwellian-weighted) Hilbert spaces

$$\begin{aligned} \mathcal{H} &:= \{h \in L^2(\Omega; \mathbb{R}^d) : \operatorname{div} h = 0\}, & \mathcal{H}_0 &:= \{h \in \mathcal{H} : h \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\}, \\ \mathcal{V} &:= \{v \in H^1(\Omega; \mathbb{R}^d) : \operatorname{div} v = 0\}, & \mathcal{V}_0 &:= \{v \in \mathcal{V} : v|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}, \\ \mathcal{Y} &:= L^2(\Omega \times D), & \widehat{\mathcal{Y}} &:= L_M^2(\Omega \times D) := \{y \in L_{\text{loc}}^2(\Omega \times D) : \|y\|_{\widehat{\mathcal{Y}}} < \infty\}, \\ \mathcal{X} &:= H^1(\Omega \times D), & \widehat{\mathcal{X}} &:= H_M^1(\Omega \times D) := \{\phi \in H_{\text{loc}}^1(\Omega \times D) : \|\phi\|_{\widehat{\mathcal{X}}} < \infty\}, \\ \mathcal{Z} &:= H^1(D; H^1(\Omega)), & \widehat{\mathcal{Z}} &:= H_M^1(D; H^1(\Omega)) := \{\zeta \in H_{\text{loc}}^1(D; H^1(\Omega)) : \|\zeta\|_{\widehat{\mathcal{Z}}} < \infty\}, \end{aligned}$$

where the norms on $\widehat{\mathcal{Y}}$, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Z}}$ are defined by, respectively,

$$\|y\|_{\widehat{\mathcal{Y}}} := \|M^{1/2}y\|_{\mathcal{Y}}, \quad \|\phi\|_{\widehat{\mathcal{X}}}^2 := \|\phi\|_{\mathcal{Y}}^2 + \|\nabla_q \phi\|_{\mathcal{Y}}^2 + \|\nabla_x \phi\|_{\mathcal{Y}}^2, \quad \|\zeta\|_{\widehat{\mathcal{Z}}}^2 := \|\zeta\|_{\mathcal{X}}^2 + \|\nabla_x \nabla_q \zeta\|_{\mathcal{Y}}^2.$$

Obviously, $H_M^2(\Omega \times D) \subseteq \widehat{\mathcal{Z}}$, where $H_M^2(\Omega \times D)$ is the subspace of $H_M^1(\Omega \times D)$ consisting of all functions defined on $\Omega \times D$ whose second (weak) partial derivatives belong to $\widehat{\mathcal{Y}} = L_M^2(\Omega \times D)$. We refer to [6] regarding theoretical results on these weighted Hilbert spaces. In particular, we have the Gelfand triples

$$\begin{aligned} \mathcal{V} &\hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}', & \mathcal{V}_0 &\hookrightarrow \mathcal{H}_0 \hookrightarrow \mathcal{V}_0', \\ \mathcal{X} &\hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X}', & \widehat{\mathcal{X}} &\hookrightarrow \widehat{\mathcal{Y}} \hookrightarrow \widehat{\mathcal{X}}', \end{aligned}$$

where \mathcal{V}' , \mathcal{V}_0' , \mathcal{X}' and $\widehat{\mathcal{X}}'$ denote the dual space of, respectively, \mathcal{V} , \mathcal{V}_0 , \mathcal{X} and $\widehat{\mathcal{X}}$.

Using the definition of the normalized Maxwellian M , see (4.4), we have that

$$M(q) \nabla_q M(q)^{-1} = -M(q)^{-1} \nabla_q M(q) = \nabla_q U(\tfrac{1}{2}|q|^2) = U'(\tfrac{1}{2}|q|^2)q.$$

We introduce the scaled variable $\widehat{\phi} := \phi/M$ and with the formula

$$M \nabla_q \widehat{\phi} = \nabla_q \phi + M \nabla_q M^{-1} \phi = \nabla_q \phi + U'q\phi$$

we can rewrite the fractional Fokker–Planck equation in (4.2) as

$$\partial_t^\alpha \phi + (u \cdot \nabla_x) \phi + \operatorname{div}_q(\omega(u)q\phi) = \frac{1}{2\lambda} \operatorname{div}_q(M\nabla_q \widehat{\phi}) + \varepsilon \Delta_x \phi.$$

As was indicated earlier, we shall confine ourselves here to considering the corotational model, i.e., $\omega(v) = -\omega(v)^T$, $q^T \omega(v)q = 0$; if $\operatorname{div} v = 0$ it then follows that

$$(4.5) \quad (M\widehat{\phi}\omega(v)q, \nabla_q \widehat{\phi})_Y = \frac{1}{2} (M\omega(v)q, \nabla_q \widehat{\phi}^2)_Y = -\frac{1}{2} (\operatorname{div}_q(M\omega(v)q), \widehat{\phi}^2)_Y = 0;$$

see [6, 9]. We note in passing that partial integration yields the following equalities:

$$(4.6) \quad \begin{aligned} -2(M\omega(u)q\widehat{\phi}, \nabla_q \widehat{\phi})_Y &= (\nabla_x(M\widehat{\phi}\nabla_q \widehat{\phi})q, u)_Y + (u \cdot q, \operatorname{div}_x(M\widehat{\phi}\nabla_q \widehat{\phi}))_Y \\ &= (M\nabla_x \widehat{\phi}(\nabla_q \widehat{\phi})^T q, u)_Y + (M\widehat{\phi}\nabla_x \nabla_q \widehat{\phi} q, u)_Y \\ &\quad + (u \cdot q, M\nabla_x \widehat{\phi} \cdot \nabla_q \widehat{\phi})_Y + (u \cdot q, M\widehat{\phi} \operatorname{div}_x \nabla_q \widehat{\phi})_Y. \end{aligned}$$

For $\mathcal{C}(M\widehat{\phi})$, we find the following bound; see also [9, Eq. (3.7)]:

$$(4.7) \quad \begin{aligned} \int_\Omega |\mathcal{C}(M\widehat{\phi})|^2 dx &= \int_\Omega \left| \int_D F(q)q^T M\widehat{\phi} dq \right|^2 dx \\ &\leq \int_D M|F(q)q^T|^2 dq \int_{\Omega \times D} M|\widehat{\phi}|^2 d(x, q) \leq C\|\widehat{\phi}\|_{\widehat{\mathcal{Y}}}^2 \quad \forall \widehat{\phi} \in \widehat{\mathcal{Y}}, \end{aligned}$$

provided that $|q_{\max}|^2 > 2\ell_0^2$ (i.e. $|\widehat{q}_{\max}|^2 > 2$), which we shall assume to hold in what follows, in order to ensure that $\int_D M|F(q)q^T|^2 dq < \infty$. This assumption is typical in FENE-type models and has been considered, e.g., in [12].

5. Existence of weak solutions. In this section, we prove the existence of a weak solution to the time-fractional Navier–Stokes–Fokker–Planck system. We use a Galerkin procedure and discretize the partial differential equations in space and derive suitable energy estimates. We will emphasize the places where the time-fractional derivative comes into play. We shall then pass to the limit in the sequence of Galerkin approximations to deduce the existence of a weak solution. We shall proceed step by step and prove this result through several lemmas. While, for physical reasons, the derivation of the system was in the previous sections discussed in the case of $d = 3$ space dimensions, the analysis below applies to both two and three space dimensions. We begin by introducing the concept of a weak solution to the time-fractional corotational Navier–Stokes–Fokker–Planck system under consideration.

DEFINITION 5.1. *Suppose that $d \in \{2, 3\}$. We call the pair $(u, \widehat{\phi})$ a weak solution to the system (4.2), (4.3) provided that*

$$\begin{aligned} u &\in L^\infty(0, T; \mathcal{H}_0) \cap L^2(0, T; \mathcal{V}_0) \cap W^{1, \frac{8}{4+d}}(0, T; \mathcal{V}'_0), \\ \widehat{\phi} &\in L^2(0, T; \widehat{\mathcal{X}}), \quad \partial_t^\alpha \widehat{\phi} \in L^{\frac{8}{4+d}}(0, T; \widehat{\mathcal{Z}}'), \end{aligned}$$

*the functions u and $\widehat{\phi}$ satisfy the initial conditions $u(0) = u^0$, $(g_{1-\alpha} * \widehat{\phi})(0) = \widehat{\psi}^0 := \psi^0/M$ and the variational problems*

$$(5.1) \quad \begin{aligned} \langle \partial_t u, v \rangle_{L^{8/(4-d)}\mathcal{V}} + ((u \cdot \nabla_x)u, v)_{L^2\mathcal{H}} + \nu(\nabla_x u, \nabla_x v)_{L^2\mathcal{H}} \\ + \gamma(\mathcal{C}(M\widehat{\phi}), \nabla_x v)_{L^2\mathcal{H}} = 0, \end{aligned}$$

$$(5.2) \quad \langle \partial_t^\alpha \widehat{\phi}, \widehat{\zeta} \rangle_{L^{8/(4-d)}(\widehat{\mathcal{Z}})} - (u \widehat{\phi}, \nabla_x \widehat{\zeta})_{L^2 \widehat{\mathcal{Y}}} + \frac{1}{2\lambda} (\nabla_q \widehat{\phi}, \nabla_q \widehat{\zeta})_{L^2 \widehat{\mathcal{Y}}} + \varepsilon (\nabla_x \widehat{\phi}, \nabla_x \widehat{\zeta})_{L^2 \widehat{\mathcal{Y}}} \\ + \frac{1}{2} (\nabla_x (\widehat{\phi} \nabla_q \widehat{\zeta}) q, u)_{L^2 \widehat{\mathcal{Y}}} - \frac{1}{2} (u \cdot q, \operatorname{div}_x (\widehat{\phi} \nabla_q \widehat{\zeta}))_{L^2 \widehat{\mathcal{Y}}} = 0,$$

for all $v \in L^{8/(4-d)}(0, T; \mathcal{V}_0)$ and $\widehat{\zeta} \in L^{8/(4-d)}(0, T; \widehat{\mathcal{Z}})$, respectively. In these variational problems and hereafter, for a Hilbert space H and $p \in [1, \infty]$, subscripts of the form $L^p H$ and $L_t^p H$ appearing in the various inner products, norms, and duality pairings, signify $L^p(0, T; H)$ and $L^p(0, t; H)$, respectively.

We summarize the assumptions that we require for proving the existence of a weak solution in the sense of [Definition 5.1](#) in [Assumption 5.2](#) below.

ASSUMPTION 5.2. *Let the following assumptions hold:*

- $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded open ball centered at the origin, $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain (i.e., bounded, open, connected set in \mathbb{R}^d , with a Lipschitz-continuous boundary $\partial\Omega$), and $T \in (0, \infty)$ is a fixed final time;
- $u^0 \in \mathcal{H}_0$, $\widehat{\psi}^0 \in \widehat{\mathcal{X}}$ with $\widehat{\psi}^0 \in H_M^1(D; H^{1+d/2+\delta}(\Omega))$ for some $\delta \in (0, 1)$;
- $\mathcal{C}(\cdot)$ is given by [\(2.20\)](#);
- $\alpha \in (1/2, 1)$ and $|\widehat{q}_{\max}|^2 > 2$;
- $\gamma, \nu, \lambda, \varepsilon$ are dimensionless positive constants.

The assumption $\alpha \in (1/2, 1)$ in the proof of the existence theorem below is in line with the results in [\[58, Theorem 4.1\]](#) where it is shown that the initial-value problem even for the simpler time-fractional heat equation $\partial_t^\alpha \phi - \Delta \phi + f(\phi) = 0$ with the initial condition $(g_{1-\alpha} * \phi)(0) = v$ is ill-posed in the case of $v \neq 0$ and $\alpha \in (0, 1/2]$.

The main result of this paper is the following theorem asserting the existence of global-in-time large-data weak solutions to the time-fractional Navier–Stokes–Fokker–Planck system under consideration.

THEOREM 5.3. *Let [Assumption 5.2](#) hold; then, there exists a weak solution $(u, \widehat{\phi})$ to the system [\(4.2\)](#), [\(4.3\)](#) in the sense of [Definition 5.1](#).*

In order to prove this theorem, we state several lemmas, which will eventually imply [Theorem 5.3](#). We begin by constructing a sequence of Galerkin approximations $\{(u_k, \widehat{\phi}_k)\}_{k=1}^\infty$ to the system of partial differential equations under consideration, resulting in a system of (fractional-order) ordinary differential equations, which admits a local-in-time solution $(u_k, \widehat{\phi}_k)$ for each $k \geq 1$ thanks to standard theory. We then derive an energy estimate for the sequence of Galerkin approximations, which is uniform with respect to k ; this then implies that, for each $k \geq 1$, the local-in-time solution of the fractional-order system of ordinary differential equations can be extended to the entire time-interval $[0, T]$; it also implies the existence of a weakly/weakly-* convergent subsequence $\{(u_{k_j}, \widehat{\phi}_{k_j})\}_{j=1}^\infty$. Finally, we pass to the limit $j \rightarrow \infty$ and apply a compactness argument to deduce that the limiting pair of functions $(u, \widehat{\phi})$ is in fact a weak solution to the system of partial differential equations in the sense of [Definition 5.1](#). The Galerkin method has been applied to various time-fractional PDEs; see, e.g., [\[24–26, 68\]](#); it has also been applied to Navier–Stokes–Fokker–Planck systems in [\[9, 13, 14, 33\]](#), with an integer-order Fokker–Planck equation.

5.1. Galerkin discretization. We follow the construction of [\[17, Section 2.1\]](#) and conclude by the Hilbert–Schmidt theorem [\[17, Lemma A.4\]](#) the existence of a countable set $\{h_j\}_{j=1}^\infty$ of eigenfunctions in $\mathcal{V}_0 \cap H^{1+\frac{d}{2}+\delta}(\Omega)^d$, with $\delta \in (0, 1)$ (fixed), whose linear span is dense in \mathcal{H}_0 such that the h_j , $j \in \{1, 2, \dots\}$, are orthonormal in \mathcal{H} and orthogonal in $H^{1+\frac{d}{2}+\delta}(\Omega)^d$ in the sense that $(h_j, h_i)_{H^{1+\frac{d}{2}+\delta}(\Omega)} = \lambda_j \delta_{i,j}$ for

any $i, j \in \{1, 2, \dots\}$ and $\lambda_j > 0$ for all $j = 1, 2, \dots$. Similarly, we fix a countable set $\{y_j\}_{j=1}^\infty$ in $H_M^2(\Omega \times D)$ that forms an orthogonal system in $H_M^2(\Omega \times D)$ and an orthonormal system in $\widehat{\mathcal{Y}}$. We then define the k -dimensional linear spaces

$$\mathcal{H}_k := \text{span}\{h_1, \dots, h_k\}, \quad \widehat{\mathcal{Y}}_k := \text{span}\{y_1, \dots, y_k\},$$

and we consider the Galerkin approximations

$$(5.3) \quad u_k(t) = \sum_{j=1}^k u_k^j(t) h_j, \quad \widehat{\phi}_k(t) = \sum_{j=1}^k \widehat{\phi}_k^j(t) y_j,$$

where u_k^j and $\widehat{\phi}_k^j$ are real-valued coefficient functions for all $j \in \{1, \dots, k\}$. The canonical orthogonal projection onto the finite-dimensional space \mathcal{H}_k is defined by $\Pi_{\mathcal{H}_k} : \mathcal{H} \rightarrow \mathcal{H}_k$, $h \mapsto \sum_{j=1}^k (h, h_j)_{\mathcal{H}} h_j$, and in the same way for $\Pi_{\widehat{\mathcal{Y}}_k} : \widehat{\mathcal{Y}} \rightarrow \widehat{\mathcal{Y}}_k$. For $h = \sum_{j=1}^\infty (h, h_j)_{\mathcal{H}} h_j$ we have that

$$\|h\|_{H^{1+\frac{d}{2}+\delta}(\Omega)}^2 = \sum_{j=1}^\infty \lambda_j |(h, h_j)_{\mathcal{H}}|^2, \quad \|\Pi_{\mathcal{H}_k} h\|_{H^{1+\frac{d}{2}+\delta}(\Omega)}^2 = \sum_{j=1}^k \lambda_j |(h, h_j)_{\mathcal{H}}|^2,$$

from which we conclude via the Sobolev embedding theorem that, for each $k \geq 1$,

$$\|\Pi_{\mathcal{H}_k} h\|_{W^{1,\infty}(\Omega)} \leq C \|\Pi_{\mathcal{H}_k} h\|_{H^{1+\frac{d}{2}+\delta}(\Omega)} \leq C \|h\|_{H^{1+\frac{d}{2}+\delta}(\Omega)}.$$

The Galerkin equations read as follows: we wish to find a tuple $(u_k, \widehat{\phi}_k) \in \mathcal{H}_k \times \widehat{\mathcal{Y}}_k$ for each $k \geq 1$ such that $u_k(0) = u_k^0 := \Pi_{\mathcal{H}_k} u^0$, $(g_{1-\alpha} * \widehat{\phi}_k)(0) = \widehat{\psi}_k^0 := \Pi_{\widehat{\mathcal{Y}}_k} \widehat{\psi}^0$, and

$$(5.4) \quad (\partial_t u_k, v_k)_{\mathcal{H}} + ((u_k \cdot \nabla_x) u_k, v_k)_{\mathcal{H}} + \nu (\nabla_x u_k, \nabla_x v_k)_{\mathcal{H}} + \gamma (\mathcal{C}(M \widehat{\phi}_k), \nabla_x v_k)_{\mathcal{H}} = 0,$$

$$(5.5) \quad (\partial_t (g_{1-\alpha} * \widehat{\phi}_k), \widehat{\zeta}_k)_{\widehat{\mathcal{Y}}} + ((u_k \cdot \nabla_x) \widehat{\phi}_k, \widehat{\zeta}_k)_{\widehat{\mathcal{Y}}} + \frac{1}{2\lambda} (\nabla_q \widehat{\phi}_k, \nabla_q \widehat{\zeta}_k)_{\widehat{\mathcal{Y}}} + \varepsilon (\nabla_x \widehat{\phi}_k, \nabla_x \widehat{\zeta}_k)_{\widehat{\mathcal{Y}}} - (\omega(u_k) q \widehat{\phi}_k, \nabla_q \widehat{\zeta}_k)_{\widehat{\mathcal{Y}}} = 0,$$

for all $v_k \in \mathcal{H}_k$ and $\widehat{\zeta}_k \in \widehat{\mathcal{Y}}_k$.

LEMMA 5.4. *Suppose that [Assumption 5.2](#) holds; then, for each $k \geq 1$, there exists a local-in-time solution $(u_k, \widehat{\phi}_k)$ to the Galerkin system (5.4), (5.5).*

Proof. Let $U_k(t) := (u_k^1(t), \dots, u_k^k(t))^T$ and $\widehat{\Phi}_k(t) = (\widehat{\phi}_k^1, \dots, \widehat{\phi}_k^k)^T$. With this notation the Galerkin subsystem (5.4) becomes an initial-value problem for a system of ordinary differential equations of the form $\frac{d}{dt} U_k = F(t, U_k, \widehat{\Phi}_k)$, while, by noting that $\frac{d}{dt} (g_{1-\alpha} * \widehat{\phi}_k) = \partial_t^\alpha \widehat{\phi}_k$, the Galerkin subsystem (5.5) takes the form of an initial-value problem for a system of fractional-order ordinary differential equations $(\frac{d}{dt})^\alpha \widehat{\Phi}_k = G(t, U_k, \widehat{\Phi}_k)$. As the functions F and G are continuous with respect to their arguments and locally Lipschitz continuous with respect to their second and third arguments, we can appeal to the generalization of the Cauchy–Lipschitz theorem stated in Theorem 5.1 of [22] to deduce the existence of a unique continuous solution, defined on a time interval $[0, T_k]$ where $0 < T_k \leq T$, where U_k is, in fact, a continuously differentiable function of t by the classical Cauchy–Lipschitz theorem. \square

5.1.1. Energy estimates. Next, we derive a k -uniform energy estimate, which will allow us to extend, for each $k \geq 1$, the corresponding local-in-time Galerkin solution, whose existence is guaranteed by Lemma 5.4, to the entire time interval $[0, T]$; it will also enable us to extract weakly converging subsequences of Galerkin approximations. We begin by deriving a bound on the solution to the Galerkin approximation of the Navier–Stokes equation; we shall then derive a bound on the solution to the Galerkin approximation of the Fokker–Planck equation. At the end, we will add the two bounds and apply Gronwall’s lemma to obtain a k -uniform energy estimate.

LEMMA 5.5. *Let Assumption 5.2 hold; then, the following bound on the Galerkin solution u_k , in terms of u_k^0 and $\hat{\phi}_k$, holds for all $t \in (0, T_k]$:*

$$(5.6) \quad \frac{1}{2} \|u_k(t)\|_{\mathcal{H}}^2 + \frac{\nu}{2} \|\nabla_x u_k\|_{L_t^2 \mathcal{H}}^2 \leq \frac{1}{2} \|u_k^0\|_{\mathcal{H}}^2 + \frac{CT\gamma^2}{2\nu} \|\hat{\phi}_k\|_{L_t^2 \mathcal{Y}}^2.$$

Proof. We take the test function $v_k = u_k(t)$ in the equation (5.1), which gives

$$\frac{1}{2} \frac{d}{dt} \|u_k\|_{\mathcal{H}}^2 + \nu \|\nabla_x u_k\|_{\mathcal{H}}^2 = -\gamma (\mathcal{C}(M\hat{\phi}), \nabla_x u_k)_{\mathcal{H}},$$

and we can further bound the right-hand side from above to deduce that

$$\frac{1}{2} \frac{d}{dt} \|u_k\|_{\mathcal{H}}^2 + \nu \|\nabla_x u_k\|_{\mathcal{H}}^2 \leq \frac{\nu}{2} \|\nabla_x u_k\|_{\mathcal{H}}^2 + \frac{\gamma^2}{2\nu} \|\mathcal{C}(M\hat{\phi}_k)\|_{\mathcal{H}}^2.$$

By bounding the term $\mathcal{C}(M\hat{\phi}_k)$ as in (4.7) we arrive at the inequality

$$(5.7) \quad \frac{1}{2} \frac{d}{dt} \|u_k\|_{\mathcal{H}}^2 + \frac{\nu}{2} \|\nabla_x u_k\|_{\mathcal{H}}^2 \leq \frac{C\gamma^2}{2\nu} \|\hat{\phi}_k\|_{\mathcal{Y}}^2.$$

We then integrate (5.7) over the interval $[0, t]$, where $t \in (0, T_k)$ and $T_k \leq T$, and we get the desired inequality (5.6). \square

Having derived a bound on u_k , we move on to the derivation of a bound on $\hat{\phi}^k$ by testing the Galerkin system (5.5).

LEMMA 5.6. *Let Assumption 5.2 hold and let $\tilde{\gamma} > 0$ be arbitrary but fixed; then, the following bound on the Galerkin solution $\hat{\phi}_k$, in terms of $\hat{\psi}_k^0$ and u_k , holds for all $t \in (0, T_k]$:*

$$(5.8) \quad \begin{aligned} & \frac{\tilde{\gamma}}{2} (g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_{\alpha}\|_{\mathcal{Y}}^2)(t) + \frac{\tilde{\gamma} T^{-\alpha}}{16 \Gamma(1-\alpha)} \|\hat{\phi}_k\|_{L_t^2 \mathcal{Y}}^2 + \frac{\tilde{\gamma}}{2\lambda} \|\nabla_q \hat{\phi}_k\|_{L_t^2 \mathcal{Y}}^2 + \tilde{\gamma} \varepsilon \|\nabla_x \hat{\phi}_k\|_{L_t^2 \mathcal{Y}}^2 \\ & \leq C(\alpha, \tilde{\gamma}) \|M^{1/2} \hat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \int_0^t g_{2\alpha-1}(s) \|u_k(s)\|_{\mathcal{H}}^2 ds + C(\alpha, \tilde{\gamma}, T) \|\hat{\psi}_k^0\|_{\mathcal{X}}^2. \end{aligned}$$

Proof. We note again that, thanks to the inverse convolution property, see (3.5), $\hat{\phi}_k - \hat{\psi}_k^0 g_{\alpha} = g_{\alpha} * \partial_t^{\alpha} \hat{\phi}_k$. We take this function as the test function in the variational Fokker–Planck equation (5.2), i.e., $\hat{\zeta} = \hat{\phi}_k - \hat{\psi}_k^0 g_{\alpha} = g_{\alpha} * \partial_t^{\alpha} \hat{\phi}_k$, which gives

$$(5.9) \quad \begin{aligned} & (\partial_t^{\alpha} \hat{\phi}_k, g_{\alpha} * \partial_t^{\alpha} \hat{\phi}_k)_{\mathcal{Y}} + \frac{1}{2\lambda} \|\nabla_q \hat{\phi}_k\|_{\mathcal{Y}}^2 + \varepsilon \|\nabla_x \hat{\phi}_k\|_{\mathcal{Y}}^2 \\ & = g_{\alpha}(t) \cdot \left((u_k \cdot \nabla_x) \hat{\phi}_k, \hat{\psi}_k^0 \right)_{\mathcal{Y}} + \frac{1}{2\lambda} (\nabla_q \hat{\phi}_k, \nabla_q \hat{\psi}_k^0)_{\mathcal{Y}} \\ & \quad + \varepsilon (\nabla_x \hat{\phi}_k, \nabla_x \hat{\psi}_k^0)_{\mathcal{Y}} - (\omega(u_k) q \hat{\phi}_k, \nabla_q \hat{\psi}_k^0)_{\mathcal{Y}} =: R. \end{aligned}$$

We then use the fractional chain inequality (3.9) to bound the left-hand side of (5.9) from below, which yields

$$\frac{1}{2} \partial_t^\alpha \|\widehat{\phi}_k - g_\alpha \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 \leq (\partial_t^\alpha \widehat{\phi}_k, \widehat{\phi}_k - g_\alpha \widehat{\psi}_k^0)_{\widehat{\mathcal{Y}}} = (\partial_t^\alpha \widehat{\phi}_k, g_\alpha * \partial_t^\alpha \widehat{\phi}_k)_{\widehat{\mathcal{Y}}}.$$

Regarding the right-hand side of (5.9), we integrate the last term containing $\omega(u_k)$ by parts, see (4.6), and get

$$\begin{aligned} -(M\omega(u_k)q\widehat{\phi}, \nabla_q \widehat{\psi}_k^0)_Y &= (M\nabla_x \widehat{\phi}_k (\nabla_q \widehat{\psi}_k^0)^T q, u_k)_Y + (M\widehat{\phi}_k \nabla_x \nabla_q \widehat{\psi}_k^0 q, u_k)_Y \\ &\quad + (u_k \cdot q, M\nabla_x \widehat{\phi}_k \cdot \nabla_q \widehat{\psi}_k^0)_Y + (u_k \cdot q, M\widehat{\phi}_k \operatorname{div}_x \nabla_q \widehat{\psi}_k^0)_Y. \end{aligned}$$

We apply Hölder's inequality to obtain the following bound on the right-hand side, R , of the equality (5.9):

$$\begin{aligned} R &\leq g_\alpha(t) \cdot \left(\|u_k\|_{\mathcal{H}} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}} \|M^{1/2} \widehat{\psi}_k^0\|_{L^2(D; L^\infty(\Omega))} \right. \\ &\quad \left. + \frac{1}{2\lambda} \|\nabla_q \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}} \|\nabla_q \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}} + \varepsilon \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}} \|\nabla_x \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}} \right. \\ &\quad \left. + C \|u_k\|_{\mathcal{H}} \|q\|_{L^\infty(D)} \|M^{1/2} \nabla_q \widehat{\psi}_k^0\|_{L^2(D; W^{1,\infty}(\Omega))} (\|\widehat{\phi}_k\|_{\widehat{\mathcal{Y}}} + \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}) \right). \end{aligned}$$

Hence, thanks to Young's inequality, we arrive at the following bound on R :

$$\begin{aligned} R &\leq \frac{1}{\varepsilon} g_\alpha(t)^2 \|M^{1/2} \widehat{\psi}_k^0\|_{L^2(D; W^{1,\infty}(\Omega))}^2 \|u_k\|_{\mathcal{H}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 \\ &\quad + \frac{1}{4\lambda} \|\nabla_q \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \frac{g_\alpha(t)^2}{4\lambda} \|\nabla_q \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \varepsilon g_\alpha(t)^2 \|\nabla_x \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 \\ &\quad + \varepsilon_1 \|\widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + C(\varepsilon_1) g_\alpha(t)^2 \|M^{1/2} \nabla_q \widehat{\psi}_k^0\|_{L^2(D; W^{1,\infty}(\Omega))}^2 \|u_k\|_{\mathcal{H}}^2, \end{aligned}$$

where $\varepsilon_1 > 0$ is sufficiently small, to be chosen appropriately later on. After combining the lower bound on the left-hand side of (5.9) and the upper bound on the right-hand side we have that

$$\begin{aligned} &\frac{1}{2} \partial_t^\alpha \|\widehat{\phi}_k - g_\alpha \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 + \frac{1}{2\lambda} \|\nabla_q \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \varepsilon \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 \\ &\leq \frac{1}{\varepsilon} g_\alpha(t)^2 \|M^{1/2} \widehat{\psi}_k^0\|_{L^2(D; W^{1,\infty}(\Omega))}^2 \|u_k\|_{\mathcal{H}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 \\ &\quad + \frac{1}{4\lambda} \|\nabla_q \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \frac{g_\alpha(t)^2}{4\lambda} \|\nabla_q \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \varepsilon g_\alpha(t)^2 \|\nabla_x \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 \\ &\quad + \varepsilon_1 \|\widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \frac{\varepsilon}{4} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + C(\varepsilon_1) g_\alpha(t)^2 \|M^{1/2} \nabla_q \widehat{\psi}_k^0\|_{L^2(D; W^{1,\infty}(\Omega))}^2 \|u_k\|_{\mathcal{H}}^2, \end{aligned}$$

and absorbing terms on the right-hand side into the left-hand side gives

$$\begin{aligned} &\frac{1}{2} \partial_t^\alpha \|\widehat{\phi}_k - g_\alpha \widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}}^2 + \frac{1}{4\lambda} \|\nabla_q \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + \frac{\varepsilon}{2} \|\nabla_x \widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 \\ &\leq C(\varepsilon_1) g_\alpha(t)^2 \|M^{1/2} \widehat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \|u_k\|_{\mathcal{H}}^2 + \varepsilon_1 \|\widehat{\phi}_k\|_{\widehat{\mathcal{Y}}}^2 + C(\varepsilon_1) g_\alpha(t)^2 \|\widehat{\psi}_k^0\|_{\widehat{\mathcal{X}}}^2. \end{aligned}$$

We note that $g_\alpha^2 = \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)^2} g_{2\alpha-1}$ is integrable for $\alpha \in (\frac{1}{2}, 1)$ and $g_1 * g_{2\alpha-1} = g_{2\alpha}$, which is continuous, bounded, and monotonically increasing on $[0, T]$ for $\alpha \in (\frac{1}{2}, 1)$.

We integrate the inequality over $(0, t)$ and exploit the representation $\partial_t^\alpha v = \partial_t(g_{1-\alpha} * v)$ of the Riemann–Liouville derivative, which gives

$$(5.10) \quad \begin{aligned} & \frac{1}{2}(g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_\alpha\|_{\hat{\mathcal{Y}}}^2)(t) + \frac{1}{4\lambda} \|\nabla_q \hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 + \frac{\varepsilon}{2} \|\nabla_x \hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 - \varepsilon_1 \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 \\ & \leq C(\varepsilon_1, \alpha) \|M^{1/2} \hat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \int_0^t g_{2\alpha-1}(s) \|u_k(s)\|_{\mathcal{H}}^2 ds + C(\varepsilon_1) \|\hat{\psi}_k^0\|_{\hat{\mathcal{X}}}^2 g_{2\alpha}(T). \end{aligned}$$

Further, we derive a lower bound on the first term of the left-hand side by noting that $(g_1 * v)(t) \leq T^\alpha \Gamma(1-\alpha)(g_{1-\alpha} * v)(t)$, see (3.2), and therefore we have that

$$\begin{aligned} & \frac{1}{2}(g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_\alpha\|_{\hat{\mathcal{Y}}}^2)(t) \\ & \geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \int_0^t \|\hat{\phi}_k(s) - \hat{\psi}_k^0 g_\alpha(s)\|_{\hat{\mathcal{Y}}}^2 ds \\ & \geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \int_0^t \|\hat{\phi}_k(s)\|_{\hat{\mathcal{Y}}} - g_\alpha(s) \|\hat{\psi}_k^0\|_{\hat{\mathcal{Y}}} \|^2 ds \\ & = \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \int_0^t \|\hat{\phi}_k(s)\|_{\hat{\mathcal{Y}}}^2 - 2g_\alpha(s) \|\hat{\phi}_k(s)\|_{\hat{\mathcal{Y}}} \|\hat{\psi}_k^0\|_{\hat{\mathcal{Y}}} + g_\alpha^2(s) \|\hat{\psi}_k^0\|_{\hat{\mathcal{Y}}}^2 ds, \end{aligned}$$

where we applied the reverse triangle inequality in the second bound. The function g_α belongs to $L^2(0, t)$ for any $\alpha \in (\frac{1}{2}, 1)$ and the integral of g_α^2 is positive. We apply Hölder's inequality in the second term of the integrand and note that the $L^2(0, t)$ -norm of g_α has the upper bound $C(\alpha)T^{\alpha-1/2}$. We thus have that

$$\begin{aligned} \frac{1}{2}(g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_\alpha\|_{\hat{\mathcal{Y}}}^2)(t) & \geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \left[\frac{1}{2} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 - 2C(\alpha)T^{\alpha-1/2} \|\hat{\psi}_k^0\|_{\hat{\mathcal{Y}}} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}} \right] \\ & \geq \frac{T^{-\alpha}}{2\Gamma(1-\alpha)} \left[\frac{1}{4} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 - C(T, \alpha) \|\hat{\psi}_k^0\|_{\hat{\mathcal{Y}}}^2 \right], \end{aligned}$$

where we have applied Young's inequality in the last step. We multiply the inequality (5.10) by $\tilde{\gamma} > 0$ and obtain for $\varepsilon_1 = \frac{T^{-\alpha}}{16\Gamma(1-\alpha)}$ the inequality (5.8). \square

Next, we combine the bounds on u_k and $\hat{\phi}_k$, see Lemma 5.5 and Lemma 5.6, to obtain a k -uniform bound.

LEMMA 5.7. *Let Assumption 5.2 hold; then, the following k -uniform estimate on the Galerkin solution $(u_k, \hat{\phi}_k)$ holds, for all $t \in (0, T_k]$:*

$$(5.11) \quad \begin{aligned} & (g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_\alpha\|_{\hat{\mathcal{Y}}}^2)(t) + \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{X}}}^2 + \|u_k(t)\|_{\mathcal{H}}^2 + \|\nabla u_k\|_{L_t^2 \mathcal{H}}^2 \\ & \leq C(\alpha, T, \|u^0\|_{\mathcal{H}}^2, \|M^{1/2} \hat{\psi}^0\|_{H^1(D; H^{1+d/2+\delta}(\Omega))}^2). \end{aligned}$$

Proof. We add the integrated velocity inequality (5.6) to the bound (5.8) and obtain the following combined bound:

$$\begin{aligned} & \frac{\tilde{\gamma}}{2}(g_{1-\alpha} * \|\hat{\phi}_k - \hat{\psi}_k^0 g_\alpha\|_{\hat{\mathcal{Y}}}^2)(t) + \frac{\tilde{\gamma}T^{-\alpha}}{16\Gamma(1-\alpha)} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 + \frac{\tilde{\gamma}}{2\lambda} \|\nabla_q \hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 + \tilde{\gamma}\varepsilon \|\nabla_x \hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2 \\ & + \frac{1}{2} \|u_k(t)\|_{\mathcal{H}}^2 + \frac{\nu}{2} \|\nabla u_k\|_{L_t^2 \mathcal{H}}^2 \\ & \leq C(\alpha, \tilde{\gamma}) \|M^{1/2} \hat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \int_0^t g_{2\alpha-1}(s) \|u_k(s)\|_{\mathcal{H}}^2 ds + C(\alpha, \tilde{\gamma}, T) \|\hat{\psi}_k^0\|_{\hat{\mathcal{X}}}^2 \\ & + \frac{1}{2} \|u_k^0\|_{\mathcal{H}}^2 + \frac{C\gamma^2}{2\nu} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{Y}}}^2. \end{aligned}$$

We now choose $\tilde{\gamma}$ such that $\frac{\tilde{\gamma} T^{-\alpha}}{16\Gamma(1-\alpha)} \geq \frac{C\gamma^2}{\nu}$. Hence, we can absorb the last term on the right-hand side into the second term on the left-hand side, and the combined energy inequality thus becomes

$$(5.12) \quad \begin{aligned} & (g_{1-\alpha} * \|\widehat{\phi}_k - \widehat{\psi}_k^0 g_\alpha\|_{\widehat{\mathcal{Y}}}^2)(t) + \|\widehat{\phi}_k\|_{L_t^2 \widehat{\mathcal{X}}}^2 + \|u_k(t)\|_{\mathcal{H}}^2 + \|\nabla u_k\|_{L_t^2 \mathcal{H}}^2 \\ & \leq C(\alpha, T) \|M^{1/2} \widehat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \int_0^t g_{2\alpha-1}(s) \|u_k(s)\|_{\mathcal{H}}^2 ds \\ & \quad + C(\alpha, T) (\|\widehat{\psi}_k^0\|_{\widehat{\mathcal{X}}}^2 + \|u_k^0\|_{\mathcal{H}}^2), \end{aligned}$$

where we took the minimum of each prefactor of the norms on the left-hand side and divided the inequality by this value. Gronwall's lemma then implies that

$$(5.13) \quad \begin{aligned} & (g_{1-\alpha} * \|\widehat{\phi}_k - \widehat{\psi}_k^0 g_\alpha\|_{\widehat{\mathcal{Y}}}^2)(t) + \|\widehat{\phi}_k\|_{L_t^2 \widehat{\mathcal{X}}}^2 + \|u_k(t)\|_{\mathcal{H}}^2 + \|\nabla u_k\|_{L_t^2 \mathcal{H}}^2 \\ & \leq C(\alpha, T) \cdot (\|\widehat{\psi}_k^0\|_{\widehat{\mathcal{X}}}^2 + \|u_k^0\|_{\mathcal{H}}^2) \cdot \exp\left(\frac{T^{2\alpha-1}}{2\alpha-1} \|M^{1/2} \widehat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2\right). \end{aligned}$$

We note that the initial conditions of the Galerkin system are defined by $u_k^0 = \Pi_{\mathcal{H}_k} u^0$ and $\widehat{\psi}_k^0 = \Pi_{\widehat{\mathcal{Y}}} \widehat{\psi}^0$. Therefore, we have that $\|u_k^0\|_{\mathcal{H}}^2 \leq \|u^0\|_{\mathcal{H}}^2$ and

$$\|M^{1/2} \widehat{\psi}_k^0\|_{H^1(D; W^{1,\infty}(\Omega))}^2 \leq C \|M^{1/2} \widehat{\psi}^0\|_{H^1(D; H^{1+d/2+\delta}(\Omega))}^2.$$

We insert these bounds into the right-hand side of the inequality (5.13) and we thus arrive at the desired energy estimate (5.11). \square

As the right-hand side of the inequality (5.11) is independent of T_k , the Galerkin solution $(u_k, \widehat{\phi}_k)$ can be extended from $[0, T_k]$ to the whole of $[0, T]$ for each $k \geq 1$.

5.2. Convergence of subsequences. Having derived the k -uniform energy estimate (5.11) stated in Lemma 5.7, we shall extract weakly/weakly-* converging subsequences of Galerkin solutions $(u_k, \widehat{\phi}_k)$. We shall also prove strong convergence of a subsequence $\{u_{k_j}\}_{j=1}^\infty$ in $L^2(0, T; \mathcal{H}_0)$ in order to pass to the limit $j \rightarrow \infty$ in the nonlinear terms in the variational Navier–Stokes–Fokker–Planck system.

LEMMA 5.8. *Let Assumption 5.2 hold and assume that $r \in [1, \infty)$ for $d = 2$ and $r \in [1, 6)$ for $d = 3$; then, the sequence of Galerkin solutions $(u_k, \widehat{\phi}_k)$ from Lemma 5.4 contains a subsequence $(u_{k_j}, \widehat{\phi}_{k_j})$ that admits the following convergences as $j \rightarrow \infty$:*

$$(5.14) \quad \begin{aligned} u_{k_j} &\rightharpoonup u && \text{weakly-* in } L^\infty(0, T; \mathcal{H}_0), \\ u_{k_j} &\rightharpoonup u && \text{weakly in } L^2(0, T; \mathcal{V}_0) \cap L^{8/d}(0, T; L^4(\Omega)^d), \\ \widehat{\phi}_{k_j} &\rightharpoonup \widehat{\phi} && \text{weakly in } L^2(0, T; \widehat{\mathcal{X}}), \\ \partial_t u_{k_j} &\rightharpoonup \partial_t u && \text{weakly in } L^{8/(4+d)}(0, T; \mathcal{V}_0'), \\ u_{k_j} &\rightarrow u && \text{strongly in } L^2(0, T; L^r(\Omega; \mathbb{R}^d)), \\ u_{k_j} &\rightarrow u && \text{strongly in } C([0, T]; \mathcal{V}_0'), \\ \partial_t^\alpha \widehat{\phi}_{k_j} &\rightharpoonup \partial_t^\alpha \widehat{\phi} && \text{weakly in } L^{8/(4+d)}(0, T; \widehat{\mathcal{Z}}'), \\ \widehat{\phi}_{k_j} &\rightarrow \widehat{\phi} && \text{strongly in } L^2(0, T; \widehat{\mathcal{Y}}), \\ \mathcal{C}(M \widehat{\phi}_{k_j}) &\rightarrow \mathcal{C}(M \widehat{\phi}) && \text{strongly in } L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d})), \\ g_{1-\alpha} * \widehat{\phi}_{k_j} &\rightharpoonup g_{1-\alpha} * \widehat{\phi} && \text{weakly-* in } L^\infty(0, T; \widehat{\mathcal{Y}}) \cap L^2(0, T; \widehat{\mathcal{X}}), \\ g_{1-\alpha} * \widehat{\phi}_{k_j} &\rightarrow g_{1-\alpha} * \widehat{\phi} && \text{strongly in } C([0, T]; \widehat{\mathcal{X}}') \cap L^2(0, T; \widehat{\mathcal{Y}}). \end{aligned}$$

Proof. In Lemma 5.7 we stated various k -uniform bounds on u_k and $\widehat{\phi}_k$. Thanks to the Banach–Alaoglu and Eberlein–Šmulian theorems, see [1, Theorem 8.10], there are weakly/weakly-* converging subsequences $\{u_{k_j}\}_{j=1}^\infty$ and $\{\widehat{\phi}_{k_j}\}_{j=1}^\infty$. In particular, we obtain the following weak/weak-* convergences:

$$(5.15) \quad \begin{aligned} u_{k_j} &\rightharpoonup u && \text{weakly-* in } L^\infty(0, T; \mathcal{H}_0), \\ u_{k_j} &\rightharpoonup u && \text{weakly in } L^2(0, T; \mathcal{V}_0), \\ \widehat{\phi}_{k_j} &\rightharpoonup \widehat{\phi} && \text{weakly in } L^2(0, T; \widehat{\mathcal{X}}). \end{aligned}$$

We shall establish the strong convergence of u_{k_j} in $L^2(0, T; \mathcal{H}_0)$ by applying the Aubin–Lions compactness lemma; see (3.7). To this end, we need to bound the time derivative of u_k in a suitable dual space. Let us therefore consider an arbitrary element $v \in L^{8/(4-d)}(0, T; \mathcal{V}_0)$, move the second, third and fourth terms from the left-hand side of (5.4) to the right-hand side, and bound them by means of Hölder’s inequality:

$$\begin{aligned} \int_0^T |\langle \partial_t u_k, v \rangle_{\mathcal{V}_0}| dt &= \int_0^T \left| -((u_k \cdot \nabla_x) u_k, \Pi_{\mathcal{H}_k} v)_{\mathcal{H}} \right. \\ &\quad \left. - \nu(\nabla_x u_k, \nabla_x \Pi_{\mathcal{H}_k} v)_{\mathcal{H}} - \gamma(\mathcal{C}(M\widehat{\phi}_k), \nabla_x \Pi_{\mathcal{H}_k} v)_{\mathcal{H}} \right| dt \\ &\leq C \int_0^T \left(\|u_k\|_{L^4(\Omega)} \|u_k\|_{\mathcal{V}} \|\Pi_{\mathcal{H}_k} v\|_{L^4(\Omega)} \right. \\ &\quad \left. + \|u_k\|_{\mathcal{V}} \|\Pi_{\mathcal{H}_k} v\|_{\mathcal{V}} + \|\mathcal{C}(M\widehat{\phi}_k)\|_{\mathcal{H}} \|\Pi_{\mathcal{H}_k} v\|_{\mathcal{V}} \right) dt. \end{aligned}$$

Hence, using Ladyzhenskaya’s inequality, we have that

$$\begin{aligned} &\int_0^T |\langle \partial_t u_k, v \rangle_{\mathcal{V}_0}| dt \\ &\leq C \int_0^T \left(\|u_k\|_{\mathcal{H}}^{1-d/4} \|u_k\|_{\mathcal{V}}^{1+d/4} \|\Pi_{\mathcal{H}_k} v\|_{L^4(\Omega)} + \|u_k\|_{\mathcal{V}} \|\Pi_{\mathcal{H}_k} v\|_{\mathcal{V}} + \|\widehat{\phi}_k\|_{\widehat{\mathcal{Y}}} \|\Pi_{\mathcal{H}_k} v\|_{\mathcal{V}} \right) dt \\ &\leq C \left(\|u_k\|_{L^\infty \mathcal{H}}^{1-d/4} \|u_k\|_{L^2 \mathcal{V}}^{1+d/4} \|v\|_{L^{8/(4-d)} \mathcal{V}} + \|u_k\|_{L^2 \mathcal{V}} \|v\|_{L^2 \mathcal{V}} + \|\widehat{\phi}_k\|_{L^2 \widehat{\mathcal{Y}}} \|v\|_{L^2 \mathcal{V}} \right) \\ &\leq C \|v\|_{L^{8/(4-d)} \mathcal{V}}. \end{aligned}$$

This then implies that $\partial_t u_k$ is bounded in $L^{8/(4+d)}(0, T; \mathcal{V}_0')$. It follows by the Aubin–Lions lemma (3.7) that

$$(5.16) \quad \begin{aligned} \partial_t u_{k_j} &\rightharpoonup \partial_t u && \text{weakly in } L^{8/(4+d)}(0, T; \mathcal{V}_0'), \\ u_{k_j} &\rightarrow u && \text{strongly in } L^2(0, T; L^r(\Omega; \mathbb{R}^d)), \\ u_{k_j} &\rightarrow u && \text{strongly in } C([0, T]; \mathcal{V}_0'), \end{aligned}$$

where $r \in [1, \infty)$ for $d = 2$ and $r \in [1, 6)$ for $d = 3$.

Similarly, we consider an arbitrary element $\widehat{\zeta} \in L^{\frac{8}{4-d}}(0, T; \widehat{\mathcal{Z}})$, with $\widehat{\mathcal{Z}}$ as defined in the beginning of subsection 4.1. We test the Galerkin equation satisfied by $\widehat{\phi}_k$ with $\Pi_{\mathcal{H}_k} \widehat{\zeta}$ to deduce that

$$(5.17) \quad \int_0^T |\langle \partial_t^\alpha \widehat{\phi}_k, \Pi_{\mathcal{H}_k} \widehat{\zeta} \rangle_{\widehat{\mathcal{X}}}| dt = \int_0^T \left| -((u_k \cdot \nabla_x) \widehat{\phi}_k, \Pi_{\mathcal{H}_k} \widehat{\zeta})_{\widehat{\mathcal{Y}}} - \frac{1}{2\lambda} (\nabla_q \widehat{\phi}_k, \nabla_q \Pi_{\mathcal{H}_k} \widehat{\zeta})_{\widehat{\mathcal{Y}}} \right. \\ \left. - \varepsilon (\nabla_x \widehat{\phi}_k, \nabla_x \Pi_{\mathcal{H}_k} \widehat{\zeta})_{\widehat{\mathcal{Y}}} + (\omega(u_k) q \widehat{\phi}_k, \nabla_q \Pi_{\mathcal{H}_k} \widehat{\zeta})_{\widehat{\mathcal{Y}}} \right| dt.$$

We note that u_k is bounded in $L^{8/d}(0, T; L^4(\Omega)^d)$ by the following interpolation result:

$$\int_0^T \|u_k\|_{L^4}^{8/d} dt \leq \int_0^T \|u_k\|_{\mathcal{H}}^{8/d-2} \|u_k\|_{\mathcal{V}}^2 dt \leq \|u_k\|_{L^\infty \mathcal{H}}^{8/d-2} \|u_k\|_{L^2 \mathcal{V}}^2.$$

Regarding the last term in (5.17), we integrate by parts and estimate by Hölder's inequality to deduce that

$$\begin{aligned} & -(\omega(u_k)q\hat{\phi}, \nabla_q \hat{\zeta})_{\hat{\mathcal{Y}}} \\ &= (\nabla_x \hat{\phi}_k (\nabla_q \hat{\zeta})^T q, u_k)_{\hat{\mathcal{Y}}} + (\hat{\phi}_k \nabla_x \nabla_q \hat{\zeta} q, u_k)_{\hat{\mathcal{Y}}} + (u_k \cdot q, \nabla_x \hat{\phi}_k \cdot \nabla_q \hat{\zeta})_{\hat{\mathcal{Y}}} \\ & \quad + (u_k \cdot q, \hat{\phi}_k \operatorname{div}_x \nabla_q \hat{\zeta})_{\hat{\mathcal{Y}}} \\ & \leq C \|\nabla_x \hat{\phi}_k\|_{L^2 \hat{\mathcal{Y}}} (\|\nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}} + \|\nabla_x \nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}}) \|q\|_{L^\infty} \|u_k\|_{L^{8/d} L^4} \\ & \quad + \|\hat{\phi}_k\|_{L^2 \hat{\mathcal{X}}} \|\nabla_x \nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}} \|q\|_{L^\infty} \|u_k\|_{L^{8/d} L^4} \\ & \quad + C \|u_k\|_{L^{8/d} L^4} \|q\|_{L^\infty} \|\nabla_x \hat{\phi}_k\|_{L^2 \hat{\mathcal{Y}}} (\|\nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}} + \|\nabla_x \nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}}) \\ & \quad + \|u_k\|_{L^{8/d} L^4} \|q\|_{L^\infty} \|\hat{\phi}_k\|_{L^2 \hat{\mathcal{X}}} \|\operatorname{div}_x \nabla_q \hat{\zeta}\|_{L^{8/(4-d)} \hat{\mathcal{Y}}} \\ & \leq C \|u_k\|_{L^{8/d} L^4} \|q\|_{L^\infty} \|\hat{\phi}_k\|_{L^2 \hat{\mathcal{X}}} \|\hat{\zeta}\|_{L^{8/(4-d)}(0, T; \hat{\mathcal{Z}})}. \end{aligned}$$

Using this estimate, we can bound (5.17) as follows:

$$\begin{aligned} & \int_0^T |\langle \partial_t^\alpha \hat{\phi}_k, \Pi_{\mathcal{H}_k} \hat{\zeta} \rangle_{\hat{\mathcal{X}}} | dt \\ (5.18) \quad & \leq C \left(\|u_k\|_{L^\infty \mathcal{H}_0} \|\hat{\phi}_k\|_{L^2 \hat{\mathcal{X}}} \|\hat{\zeta}\|_{L^2 \hat{\mathcal{X}}} + \|\nabla_q \hat{\phi}_k\|_{L^2 \hat{\mathcal{Y}}} \|\nabla_q \hat{\zeta}\|_{L^2 \hat{\mathcal{Y}}} \right. \\ & \quad \left. + \|\nabla_x \hat{\phi}_k\|_{L^2 \hat{\mathcal{Y}}} \|\nabla_x \hat{\zeta}\|_{L^2 \hat{\mathcal{Y}}} + \|u_k\|_{L^{8/d} L^4} \|q\|_{L^\infty} \|\hat{\phi}_k\|_{L^2 \hat{\mathcal{X}}} \|\hat{\zeta}\|_{L^{8/(4-d)}(0, T; \hat{\mathcal{Z}})} \right) \\ & \leq C \|\hat{\zeta}\|_{L^{8/(4-d)}(0, T; \hat{\mathcal{Z}})}. \end{aligned}$$

Hence, we obtain the k -uniform boundedness of $\partial_t(g_{1-\alpha} * \hat{\phi}_k) = \partial_t^\alpha \hat{\phi}_k$ in the space $L^{8/(4+d)}(0, T; \hat{\mathcal{Z}}')$, which is continuously embedded in $L^{8/(4+d)}(0, T; (H_M^2(\Omega \times D)))'$. Therefore, we are in the setting of the Gelfand triple

$$\hat{\mathcal{X}} \hookrightarrow \hat{\mathcal{Y}} \hookrightarrow (H_M^2(\Omega \times D))'.$$

We thus obtain from the fractional Aubin–Lions lemma, see (3.8), that

$$(5.19) \quad \begin{aligned} \partial_t^\alpha \hat{\phi}_{k_j} &\rightharpoonup \partial_t^\alpha \hat{\phi} \quad \text{weakly in } L^{8/(4+d)}(0, T; \hat{\mathcal{Z}}'), \\ \hat{\phi}_{k_j} &\longrightarrow \hat{\phi} \quad \text{strongly in } L^2(0, T; \hat{\mathcal{Y}}). \end{aligned}$$

We note that the mapping $M\varphi \mapsto \mathcal{C}(M\varphi)$ is linear and continuous thanks to (4.7), and therefore we have from (5.19)₂ that

$$(5.20) \quad \mathcal{C}(M\hat{\phi}_{k_j}) \longrightarrow \mathcal{C}(M\hat{\phi}) \quad \text{strongly in } L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d})).$$

The convolution $g_{1-\alpha} * \hat{\phi}_k$ is bounded in $L^2(0, T; \hat{\mathcal{X}})$ thanks to Young's convolution inequality; indeed, we have for all $t \in (0, T]$ that

$$\|g_{1-\alpha} * \hat{\phi}_k\|_{L_t^2 \hat{\mathcal{X}}} \leq \|g_{1-\alpha}\|_{L_t^1} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{X}}} \leq CT^{1-\alpha} \|\hat{\phi}_k\|_{L_t^2 \hat{\mathcal{X}}}.$$

Moreover, $g_{1-\alpha} * \widehat{\phi}_k$ is bounded in $L^\infty(0, T; \widehat{\mathcal{Y}})$ by the following chain of estimates:

$$\begin{aligned}
& \|g_{1-\alpha} * \widehat{\phi}_k\|_{L^\infty \widehat{\mathcal{Y}}} \\
& \leq \sup_{t \in (0, T)} \int_0^t g_{1-\alpha}(t-s) \|\widehat{\phi}_k(s)\|_{\widehat{\mathcal{Y}}} ds \\
& \leq \sup_{t \in (0, T)} \int_0^t g_{1-\alpha}(t-s) \|\widehat{\phi}_k(s) - \widehat{\psi}_k^0 g_\alpha(s)\|_{\widehat{\mathcal{Y}}} ds + (g_{1-\alpha} * g_\alpha)(t) \|\widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}} \\
& \leq \sup_{t \in (0, T)} \int_0^t g_{1-\alpha}(t-s) \|\widehat{\phi}_k(s) - \widehat{\psi}_k^0 g_\alpha(s)\|_{\widehat{\mathcal{Y}}}^2 ds + \frac{1}{4} \int_0^t g_{1-\alpha}(t-s) ds + \|\widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}} \\
& = \sup_{t \in (0, T)} (g_{1-\alpha} * \|\widehat{\phi}_k - \widehat{\psi}_k^0 g_\alpha\|_{\widehat{\mathcal{Y}}}^2)(t) + \frac{1}{4} g_{2-\alpha}(T) + \|\widehat{\psi}_k^0\|_{\widehat{\mathcal{Y}}},
\end{aligned}$$

and the first term on the right-hand side is bounded by [Lemma 5.7](#). Since we have already proved a bound on $\partial_t(g_{1-\alpha} * \widehat{\phi}_k) = \partial_t^\alpha \widehat{\phi}_k$, see (5.18), we may use the Aubin–Lions lemma, see (3.7), to obtain the following strong convergence results:

$$\begin{aligned}
(5.21) \quad & g_{1-\alpha} * \widehat{\phi}_{k_j} \longrightarrow g_{1-\alpha} * \widehat{\phi} \quad \text{strongly in } L^2(0, T; \widehat{\mathcal{Y}}), \\
& g_{1-\alpha} * \widehat{\phi}_{k_j} \longrightarrow g_{1-\alpha} * \widehat{\phi} \quad \text{strongly in } C([0, T]; \widehat{\mathcal{X}}').
\end{aligned} \quad \square$$

5.3. Passage to the limit. Next, we pass to the limit $j \rightarrow \infty$ in the time-integrated k_j -th Galerkin system (5.4), (5.5). Specifically, we shall use the convergence results stated in the preceding lemma to show that the weak limits, u and $\widehat{\phi}$, satisfy the variational Navier–Stokes–Fokker–Planck system in the sense of [Definition 5.1](#).

Proof of Theorem 5.3. We consider the time-integrated Galerkin system

$$\begin{aligned}
(5.22) \quad & \int_0^T \left(\langle \partial_t u_{k_j}, v \rangle_V + ((u_{k_j} \cdot \nabla_x) u_{k_j}, v)_{\mathcal{H}} \right. \\
& \quad \left. + \nu (\nabla_x u_{k_j}, \nabla_x v)_{\mathcal{H}} + \gamma (\mathcal{C}(M \widehat{\phi}_{k_j}), \nabla_x v)_{\mathcal{H}} \right) \eta(t) dt = 0 \\
(5.23) \quad & \int_0^T - (g_{1-\alpha} * \widehat{\phi}_{k_j}, \widehat{\zeta})_{\widehat{\mathcal{Y}}} \eta'(t) + \left(((u_{k_j} \cdot \nabla_x) \widehat{\phi}_{k_j}, \widehat{\zeta})_{\widehat{\mathcal{Y}}} + \frac{1}{2\lambda} (\nabla_q \widehat{\phi}_{k_j}, \nabla_q \widehat{\zeta})_{\widehat{\mathcal{Y}}} \right. \\
& \quad \left. + \varepsilon (\nabla_x \widehat{\phi}_{k_j}, \nabla_q \widehat{\zeta})_{\widehat{\mathcal{Y}}} - (\omega(u_{k_j}) q \widehat{\phi}_{k_j}, \nabla_q \widehat{\zeta})_{\widehat{\mathcal{Y}}} \right) \eta(t) dt = 0,
\end{aligned}$$

for all $v \in \mathcal{H}_{k_j}$, $\eta \in C_0^\infty(0, T)$ and $\widehat{\zeta} \in \widehat{\mathcal{Y}}_{k_j}$. Passing to the limit $j \rightarrow \infty$ in (5.22) using (5.15)–(5.20) is standard, and results in (5.1). It therefore remains to pass to the limit $j \rightarrow \infty$ in (5.23). In particular, the convergence of the linear terms follow immediately by weak convergence, and we only consider the two nonlinear terms. We note that $\widehat{\phi}_{k_j} \rightarrow \widehat{\phi}$ strongly in $L^2(0, T; \widehat{\mathcal{Y}})$ and $\omega(u_{k_j}) \rightarrow \omega(u)$ weakly in $L^2(0, T; \mathcal{H}_0)$, from which we deduce that

$$\int_0^T (\omega(u_{k_j}) q \widehat{\phi}_{k_j}, \nabla_q \widehat{\zeta})_{\widehat{\mathcal{Y}}} \eta(t) dt \longrightarrow \int_0^T (\omega(u) q \widehat{\phi}, \nabla_q \widehat{\zeta})_{\widehat{\mathcal{Y}}} \eta(t) dt \quad \text{as } j \rightarrow \infty.$$

With the same reasoning, it follows that

$$\int_0^T ((u_{k_j} \cdot \nabla_x) \widehat{\phi}_{k_j}, \widehat{\zeta})_{\widehat{\mathcal{Y}}} \eta(t) dt \longrightarrow \int_0^T ((u \cdot \nabla_x) \widehat{\phi}, \widehat{\zeta})_{\widehat{\mathcal{Y}}} \eta(t) dt \quad \text{as } j \rightarrow \infty.$$

We then use the density of $\cup_{k=1}^{\infty} \mathcal{H}_k$ in \mathcal{V} and of $\cup_{k=1}^{\infty} \widehat{\mathcal{Y}}_k$ in $H_M^2(\Omega \times D)$, which completes the proof by observing that the tuple $(u, \widehat{\phi})$ satisfies the variational form of the time-fractional Navier–Stokes–Fokker–Planck system as stated in [Definition 5.1](#).

It remains to check that the initial conditions are satisfied. First, we obtain the convergence $u_{k_j}(0) \rightarrow u(0)$ in \mathcal{V}'_0 as $j \rightarrow \infty$; see again [\(5.14\)](#). However, by definition, $u_{k_j}(0) = \Pi_{\mathcal{H}_{k_j}} u^0$, which converges to u^0 in \mathcal{H}_0 as $j \rightarrow \infty$. By the uniqueness of the limit it follows that $u(0) = u^0$. Regarding the solution of the Fokker–Planck equation, we use again the strong convergence [\(5.14\)](#) to conclude $(g_{1-\alpha} * \widehat{\phi})(0) = \widehat{\psi}^0$. \square

Having proved that a weak solution tuple $(u, \widehat{\phi})$ to the time-fractional system in the sense of [Definition 5.1](#) exists, we return to the original variable $\psi := g_{1-\alpha} * (M\widehat{\phi})$, whose evolution is governed by the time-fractional Fokker–Planck equation [\(2.18\)](#). Indeed, $g_{1-\alpha} * \psi = g_{2-2\alpha} * \phi$, which is continuous for $\alpha > 1/2$ and $(g_{1-\alpha} * \psi)(0) = 0$ as we have originally assumed in the model transformation. In this sense, we have also shown the existence of a variational solution tuple (u, ψ) to the original time-fractional model.

Conclusions and outlook. In this paper, we investigated the well-posedness of a coupled Navier–Stokes–Fokker–Planck system with a time-fractional derivative. Such systems arise in the kinetic theory of polymeric liquid solutions with noninteracting polymer chains. We outlined the derivation of the model from a subordinated Langevin equation and considered the case of a finitely extensible nonlinear elastic (FENE-type) dumbbell model with a corotational drag term. We proved the existence of large-data global-in-time weak solutions to the corotational time-fractional model of order $\alpha \in (\frac{1}{2}, 1)$ and derived a uniform energy inequality by considering a nonstandard and novel testing procedure. The existence of weak solutions to the general noncorotational time-fractional FENE model is an open problem, which will be studied in a forthcoming paper by using a different testing procedure; see [\[12\]](#) for the integer-order setting (corresponding to $\alpha = 1$). Concerning the numerical approximation of the time-fractional system considered here we refer the reader to the recent paper [\[15\]](#).

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