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**MANAGING DEFAULT RISK FOR COMMODITY DEPENDENT
COUNTRIES: PRICE HEDGING IN AN OPTIMIZING MODEL**

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Managing Default Risk for Commodity Dependent Countries: Price Hedging in an Optimizing Model

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Abstract

Macroeconomic volatility, in particular from exposure to volatile terms of trade in the form of volatile commodity prices, is an important source of risk for emerging market countries. As a consequence of this exposure, it has been argued, their probability of facing solvency problems on payments of their foreign currency debt is high, as are the country risk premia they must pay in order to borrow from international capital markets. While the availability of derivative contracts on many major commodity prices makes it possible to hedge commodity price exposure, many emerging market sovereigns either do not hedge a significant amount of their fiscal exposure to their major export and import commodities or do not clearly report their hedging activities.

In this light of this phenomenon, and with the goal of crisis prevention in mind, we illustrate how a country exposed to shocks can execute its own insurance strategy against fluctuations in the prices of its major export commodities using futures and options markets. In the context of a model of sovereign default with endogenous sovereign spread and debt choice (Catao and Kapur (2004)), we demonstrate the resulting benefits of this insurance in terms of increased welfare for the country, a reduced sovereign spread, and a higher debt ceiling. Additionally, we highlight some political economy problems leaders might face that hinder them from hedging in practice, and describe a hedging strategy to overcome these problems.

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1 Introduction

There has been during the past several years a growing sense of recognition and concern about the role external shocks play as a destabilizing force for developing countries. As stated in the recent IMF discussion paper *Fund Assistance for Countries Facing Exogenous Shocks* (2003), “exogenous shocks, such as natural disasters, terms-of-trade shocks, and conflicts or crises in neighboring countries, can have a significant negative impact on developing countries’ growth, macroeconomic stability, debt sustainability, and poverty” (p. 3). The most prevalent types of shocks, those authors go on to note, are terms-of-trade shocks, that is, shocks to export and import prices, and natural disasters. In this paper, we will focus in on one major link in the above story: the connection between terms-of-trade shocks and debt sustainability for commodity dependent countries.

This link is very important for several reasons. As the International Task Force on Commodity Risk Management in Developing Countries points out in its discussion paper *Dealing with Commodity Price Volatility in Developing Countries: A Proposal for a Market-based Approach* (1999), “managing risks in highly volatile commodity markets remains one of the major challenges of development, especially for the poorest countries...yet more than 50 developing countries depend on three or fewer leading commodities for more than half of their export earnings”. The primary means that developing countries have with which to service their foreign debt are export earnings, foreign reserves, and net foreign transfers. They must also purchase their imports using foreign currency gained from one of these three sources. Thus, it is clear that for countries which depend on a small group of commodities with high price volatility for the bulk of their export revenue, trouble with servicing foreign debt payments is likely. To make the problem worse, it is precisely such countries which often also lack the resources to hold large stocks of foreign reserves as a buffer against shocks (*Fund Assistance for Countries Facing Exogenous Shocks* (2003)).

In what follows, we begin by reviewing the recent research linking macroeconomic volatility and sovereign default. We then focus in on a recent paper by Catao and Kapur (2004), which develops a model of sovereign default in which macroeconomic volatility plays a key role and can be used to explain the so-called “debt-intolerance” phenomenon, that some countries have a record of bad sovereign credit despite historically moderate debt-to-GDP ratios (à la Reinhart, Rogoff, and Savastano (2003)). Our primary contribution in this paper is a generalization of the model of Catao and Kapur (2004) which accomplishes three important goals:

- We make explicit the link between the prices faced by the country on its major export commodities (e.g. oil) and the probability that it will default on its external debt.
- We implement within the model hedging strategies for the country using commodity futures and options contracts that allow it to insure itself against this price volatility.
- We prove that, as a result of implementing these insurance strategies in a transparent way, both the country and international lenders benefit.

Our original contribution lies in demonstrating, in a default model with both endogenous sovereign spread and debt choice, the positive impact of hedging activity for both borrowers and lenders. The derivatives we consider for use, futures and vanilla options, are the most transparent and liquid available, with low transactions costs and relatively high accessibility. Whereas previous research has focused primarily on altering the debt contract itself to achieve benefits, for example by making the payoffs a function of commodity prices, our results apply directly to conventional debt, with which borrowers and lenders are already familiar. Furthermore, in addition to their applicability to the context of sovereign debt, our results provide a theoretical basis to explain the success of programs such as ANACAFÉ in Guatemala, which makes cheap loans to coffee growers on the condition that they appropriately hedge their coffee price exposure using futures and options.

We finish the paper with a discussion of how to operationalize such hedging strategies in practice, what role might be played by the international financial institutions, and directions for future research.

2 Literature Review

In response to the question, “why should developing countries hedge revenue volatility?”, the discussion paper *Commodity Risk Management and Development* (1998) by Larson, Varangis, and Yabuki offers the following representative justification:

The simplest reason is so that government revenues are not a function of commodity prices. Governments can increase the

probability that the expected revenues will actually be materialized. If governments borrow, lenders will see a less volatile source of revenue and they will lend at more attractive rates. Another reason for hedging is that the effect of an extreme move in commodity prices can be such as to create significant financial and budgetary problems for the government (p. 21).

In the model that we develop in this paper, we provide a theoretical justification for the claim that hedging revenue volatility will induce lenders to lend at more attractive rates, among other benefits. Of course, it is important to ask a more basic question, which is, why do developing countries borrow in the first place, and how does the fact that they face greater macroeconomic volatility impact this activity? Luis Catao and Sandeep Kapur, in their paper *Missing Link: Volatility and the Debt Intolerance Paradox* (2004) give a compelling answer:

While volatility increases the need for international borrowing to help smooth domestic consumption, the ability to borrow is constrained by the higher default risk that volatility engenders.

The model developed by Catao and Kapur (2004) does not consider the question of hedging macroeconomic volatility. Rather, their model, which our analysis employs, sheds light on the so-called “debt intolerance” phenomenon. This observation, put forward by Reinhart, Rogoff, and Savastano (2003), refers to the fact that many “emerging market” countries, which have moderate external debt-to-GDP ratios by international standards, are nevertheless perceived as riskier and unable to tolerate much debt. Consequently, the sovereign spread over the risk-free rate on their debt in international capital markets is quite high and the ceiling they face in contracting new (or rolling-over old) debt is sometimes binding. The latter phenomenon is associated with the constraint on external borrowing referred to as a sudden-stop, which has its own, growing literature (see e.g. Calvo (1998), Caballero and Krishnamurthy (2001), Mendoza (2004)).

While Reinhart, Rogoff, and Savastano (2003) invoke history to explain and identify “debt-intolerant” countries, with the basic premise that countries that have defaulted often in the past are likely to behave similarly in the future, their argument fails to offer a developed theoretical justification for why this might be the case. Catao and Kapur (2004) advance and support empirically a highly reasonable explanation, “that the high volatility of macroeconomic aggregates—in particular, of domestic output and external terms of trade—is a key factor in the sovereign risk of many developing

countries” (pp. 3-4). Further, they argue that “this greater volatility is associated with higher default probability and, as a result, these countries hit borrowing constraints at lower levels of indebtedness...to the extent that such volatility stems from structural and hence slowly evolving factors, the phenomenon can be fairly persistent even if there is scope for countries to gradually evolve out of this state” (p. 4).

Our paper builds on the idea that, whatever slowly-evolving structural factors may contribute to macroeconomic volatility, there is clear hope and the potential for success in devising ways to help developing countries insure themselves against exogenous shocks, in particular through the use of tools in existence or being developed in financial markets. Our particular focus here is on the effect undertaking such insurance would have on the ability of sovereign borrowers to service their foreign debt.

The literature linking commodity price volatility and debt service has focused primarily on two areas: (1) finding ways of improving the ability of the country to repay their debt by changing the nature of the bond contract or repayment schedule; and (2) taking independent measures to reduce the volatility of the country’s income stream from commodity exports, with the notion that other fiscal benefits will follow.

With regards to the first area, several strategies have been suggested. One of the most prominent is the commodity-linked or commodity-indexed bond, in which the principal and perhaps the interest payments are not guaranteed in nominal terms but are a function of the price of the underlying commodity on which the bond is written. Examples of papers proposing commodity-linked bonds as a potential solution to the debt problem of commodity dependent developing countries include Atta-Mensah (2004), Miura and Yamauchi (1998), O’Hara (1984), Budd (1983), and Schwartz (1982). Technically, the commodity-indexed bond has features of a forward contract on the underlying commodity, whereas the commodity-linked bond gives the buyer exposure to the commodity price in the form of an option (Atta-Mensah (2004)). With the latter type of contract, the terminal payoff is the lesser of the face value and the monetary value of a pre-specified unit of a commodity. This makes the payoff a piecewise linear function of the underlying commodity price. Commodity-linked bonds, on the other hand, have payoff functions that are linear. While the pricing problem has been solved for these bonds and their potential benefits are promising, there is a clear downside to forcing the lender to bear the commodity price risk.

For the commodity indexed bond, lending countries may face lower coupon rates by sharing, on maturity, the appreciation of the commodity price with the bearer (Atta-Mensah (2004)), but conversely the depreci-

ation of the commodity price would lead to higher coupon rates faced by the country. This would lead, consequently, to a larger debt service burden that, depending on the structure of the contract, could be worse than if the country had issued conventional sovereign bonds. Similarly, in the case of commodity-linked bonds, Attah-Mensah (2004) and others have shown that the coupon rate would have to be larger than for a conventional bond because investors would have to be compensated for accepting the prescribed terminal payoff.

Another direction taken, besides commodity bonds, is to alter the schedule of debt repayment to better coincide with the country's ability to pay, as dictated by the current prices of its main export commodities. This is the focus of a paper, *Realignment of Debt Service Obligations and Ability to Pay in Concessional Lending: Feasibility and Modalities* by Gilbert and Tabova (2003). In their paper, the authors consider schemes which have the potential to increase the flexibility of heavily indebted primary producing countries in meeting their debt obligations by augmenting existing debt contracts with a set of floating for fixed swaps, where the swap prices are defined in terms of the world prices of the export commodities. Their results are mixed; while they find that the schemes do ease the debt service burden and can also offset the variability of export earnings, changes in export prices are not always related to changes in world commodity prices. The discrepancy comes from two factors, which are differences between world prices and the prices obtained by individual exporters (also known as basis risk) and quantity variation. They conclude by stating that their scheme may be an overelaborate means of achieving the end of easing concessional debt service, even for countries in which it is moderately successful. Furthermore, they note that if the objective "is that of offsetting variations in countries' export earnings, using concessional debt as a lever, there are probably superior means of achieving this".

This leads to the second point discussed in the literature, which is finding ways aimed specifically at reducing commodity price volatility. The most important of these are international commodity agreements, and direct hedging by producers or the government using futures, forwards, or options contracts. As Atta-Mensah (2004) notes, international commodity agreements involve a pact between producer and consumer countries to stabilize the world prices of commodities via export quotas or buffer stocks. The recent consensus among the development community is that buffer stock and similar schemes that have been attempted to interfere with commodity markets directly have failed or been abandoned (Gilbert and Tabova (2003)).

Direct hedging by producers and governments is a better way to deal with commodity price volatility (*Dealing with Commodity Price Volatility in Developing Countries: A Proposal for a Market-based Approach* (1999)), but there are several problems with implementing such strategies. End producers, such as farmers, may lack the knowledge of how to use derivatives markets for risk management or may face difficulties accessing these markets for a variety of reasons, the most simple of which being that they do not meet the minimum quantity or quality specifications for a standard futures or options contract. For a forward contract, the individual specifications can be tailored according to the needs of the counterparties, but the lack of an exchange creates the problem of counterparty risk and exacerbates the role of asymmetric information. And, as mentioned above, hedging revenue using derivative contracts written on international commodity prices faces the dual problems of basis risk and quantity variation. The latter can be dealt with, to some extent, directly via insurance against crop failure, adverse weather, and natural disasters, but such instruments are only now evolving (*Commodity Risk Management and Development* (1998)).

3 A Proposal, a Precedent, and Role for the IMF

Despite the problems with hedging commodity price risk (and export revenue volatility) using financial derivatives, and the complexities involved with formulating such strategies in a way that improves the ability of developing countries to service their debt, this approach seems to be the most promising of those available at present. There are several challenges, however, both theoretical and practical, which must be addressed. First, the hedging strategy should be welfare-improving for the country. Second, the strategy should be as complex as necessary to achieve an acceptable level of welfare improvement, but no more. Complex contracts carry with them hidden costs in terms of difficulty in pricing, and this is an issue both for borrowers and lenders. Finally, there are political economy problems that may hinder politicians from implementing a hedging strategy whose economic case is clear. A successful strategy will be one that gets around these problems.

Our paper aims to contribute to the literature by proposing a hedging strategy for commodity dependent developing countries that is relatively simple, flexible, and whose benefits can be clearly demonstrated in the context of a reasonable model of debt default.

3.1 The Proposal

The strategy we propose is as follows: the lending country issues conventional sovereign bonds, but those bonds are backed by the condition that the country is required to enter into a hedging strategy, for example using futures and/or options contracts, of its primary import and export commodity prices. This condition is verified by the International Monetary Fund (IMF), which also oversees the implementation of the hedging strategy, and may provide funds to cover start-up costs for the hedging operation when necessary.

In the default model considered, we focus on the case of a country which gains the majority of its export revenue from selling one commodity, (e.g. oil) on international markets. We illustrate how such a sovereign borrower would hedge its oil revenues using futures contracts and put options on the price of oil, and give theoretical results and illustrative numerical model solutions to demonstrate the benefits of hedging for the borrower and lenders.

3.2 ANACAFÉ: An Implementation in Progress on the Micro Level

While the model that we generalize, of Catao and Kapur (2004), was intended for the context of a sovereign borrower, it can also be applied to the context of borrowing and default on the micro level. In particular, the success of ANACAFÉ (Asociación Nacional de Café), an association of coffee producers in Guatemala, is explained by the model. ANACAFÉ facilitates loans to coffee farmers at lower interest rates on the condition that the farmers hedge the price risk of their product using futures and options contracts in the manner considered herein. The coffee sector in Guatemala accounts for about 30% of the country's total exports in a normal year and provides jobs directly or indirectly to about 30% of the population (International Task Force on Commodity Risk Management in Developing Countries (1999)). The hedged coffee loan program was introduced in 1994 and totaled US\$20 million in 1998; ANACAFÉ plans to expand the system to cover all the financial needs of the coffee sector, estimated at \$200 million.

As described by the International Task Force on Commodity Risk Management in Developing Countries (1999), the operational procedure of ANACAFÉ works as follows:

Farmers receive education in credit related issues, develop an understanding of their commercial operations, and determine their costs and break-even prices with assistance from ANACAFÉ.

Trained extension staff from ANACAFÉ verify the production potential of the farm and assist with the necessary paperwork for the loan. ... The bank approves the loan which is conditional on the farmer's obtaining a hedge (for example selling forward or purchasing options) from an exporter. The hedge provides protection against the drop in market prices thus guaranteeing that they will be able to cover their loan payments. ... To hedge prices, producers usually contact an exporter with whom they fix a price for future delivery of the crop purchased. Subsequently, exporters sell futures or purchase options in the New York Coffee, Sugar, and Cocoa Exchange (CSCE) to hedge their assumed exposure.

It is important, however, that the exporters hedge the right quantity. To address this problem of quantity risk, which is due to the uncertain yield on the farmers' crops, the International Task Force on Commodity Risk Management in Developing Countries (1999) notes:

If the producer fails to deliver and prices increase, exporters could occur significant financial losses. For this reason, ANACAFÉ assists in providing estimates of the expected crop so that producers will not over- or under-hedge their exposure. The majority of the hedging operations involve future/forward contracts; however, there is an increase in the use of options strategies such as the purchase of puts or construction of price fences (purchase of puts and sale of calls).

In the context of the default model, we consider two strategies: (1) shorting futures contracts and (2) purchasing put options. These two hedging strategies each have relative advantages and disadvantages. The upfront cost of futures and forward contracts is effectively zero, except in the case of large futures transactions that require the deposit of securities as margin (Hull (2002)). However, this problem would not be restrictive for sovereign borrowers, who could use their reserves and/or part of their foreign debt for this purpose. The cost of put options, in contrast, must be paid upfront. On the other hand, while purchasing put options offers the benefits of a limited downside and an unlimited upside for the sale price of the commodity, futures contracts fix the delivery price exactly and preclude the possible gains from a high commodity price in the future.

3.3 A Role for the IMF: Risk-Manager of First Resort

In the context of a hedging scheme for sovereigns, the IMF could aid with several aspects of implementation and oversight. Hedging schemes are likely to have start-up costs, in terms of setting up the necessary trading facility to dynamically purchase and sell options on derivative exchanges. The IMF and the World Bank could cover a significant portion of these costs, an expenditure that would clearly be in their benefit.

In terms of technical assistance, the IMF could help sovereign borrowers determine the right hedging amount for a given level of debt by estimating the production quantity of the underlying commodity and the relevant "impact factor" on government revenues. In particular, if the government is the primary owner and manager of the operation which produces and exports the commodity (such as the oil producer PEMEX, in the case of Mexico), its revenue stream from commodity exports can be hedged directly. If the government's primary source of revenue from exporting a particular commodity derives from taxation of private or semi-private company profits, the IMF could help it determine the appropriate number of hedging contracts to enter into in order to gain the appropriate exposure given the specifics of its shock distribution for the revenues from that commodity.

Additionally, the IMF could play a key role in facilitating transparency between sovereign borrowers and lenders. It would be important, for the successful execution of a scheme such as the one outlined here, that the parameters and design of the hedging strategy are clear to lenders and that in the event of a negative exogenous shock to commodity income, the positive income from the hedging strategy would be directly available and committed toward servicing the debt ².

As we discuss in section 7, more effective and politically feasible hedging strategies can be designed using combinations of options that reduce the initial cost of the hedging strategy while insuring downside risk and preserving significant upside exposure.

We now turn to an exposition of the basic default model of Catao and Kapur (2004), which is followed by an illustration of our proposal involving futures and put option hedging for a sovereign whose income risk stems from

²One possible way to ensure this would be to construct a variation on the financing mechanism known as asset-backed securitization of future flow receivables. This scheme, which has been used to raise external finance during liquidity crises for companies in developing countries that wish to escape their sovereign's credit ceiling, involves the sale by the borrowing entity of its future product directly or indirectly to an offshore Special Purpose Vehicle (SPV), which issues the debt instrument and, via a collection account, pays the lenders ((Ketar and Ratha (2001)).

the price uncertainty of a primary export commodity.

4 The Basic Default Model: Catao and Kapur (2004)

The basic model we build upon is that presented in the paper of Catao and Kapur (2004). Their model employs several standard assumptions from the sovereign default literature, which we will mention in due course. The novelty of their contribution is in modeling the optimal choice of debt directly, unlike in several previous studies (see Grossman and Van Huyck (1988), Grossman and Han (1999), and Alfaro and Kanczuk (2002)), which only consider fixed levels of debt, and in examining the impact of output shocks on optimal debt and sovereign spreads. They find that income volatility raises spreads and lowers the maximum debt threshold beyond which the sovereign is unable to borrow. Also, they find that the relationship between optimal borrowing levels is nonmonotonic. This is most likely due to the fact that, as they suggest, on one hand volatility makes the sovereign more eager to borrow to smooth consumption, but on the other hand, it deters borrowing through higher spreads.

4.1 The Country

Sovereign borrowing in this setting is driven by the desire to smooth consumption in the face of shocks to domestic income. The model has two periods. In the first, the sovereign chooses its level of borrowing in order to smooth its second period consumption.

The sovereign's autarkic real income in the absence of borrowing is $\bar{Y} + \theta$ in period 2, where $\bar{Y}$ is the terms of trade adjusted mean output and $\theta \in [-\theta_m, \theta_m]$ denotes a random shock with mean zero. We rule out the possibility of negative output. Although Catao and Kapur (2004) do not model the source of the income shocks, they can be thought of in their model as the result of exogenous changes in terms of trade, technology, weather, and so on. In our extension, we consider the case of a country with a primary export commodity (e.g. oil) that can be bought, sold, and hedged on the international markets.

Any borrowed funds of the sovereign can be invested until period 2 at the risk free rate; this feature can be thought of as either investing in domestic ventures or holding central bank reserves. The latter seems more reasonable in this case because domestic investment, especially in the countries under

consideration, certainly has a risky payoff. For the case of a country that does invest in domestic ventures, the risky payoff of that investment would have to be modelled explicitly. Thus let $R = 1 + r$ denote the gross risk-free interest rate. If the sovereign borrows $D > 0$ in the initial period, income in period 2 is:

$$Y_2(D) = \bar{Y} + RD + \theta.$$

4.2 The Shock and the Default Event

In this model the debt contract is accompanied by a contractual repayment obligation, but as in Catao and Kapur (2004), the sovereign may choose to default in some states. Contracting debt D in period 1 requires a repayment of $R_L D$ in period 2. The spread, denoted by $R_L - R$, arises because of the default risk for the country in question. In this model the level of D affects the likelihood of default and through that mechanism affects the spread R_L . The model follows Sachs and Cohen (1985) and assumes that lenders have an enforcement technology that allows them to capture a fraction $0 < \eta < 1$ of the defaulter's aggregate period-2 income, $Y_2(D)$. Thus it is rational for the sovereign to default when the amount that the lender could capture in the event of default would be exceeded by their contractual repayment obligation. This is a reasonable assumption; we would never expect that lenders could recapture all of the country's period-2 income, nor that they would fail to capture some of it, through a variety of means, from trade penalties to explicit renegotiation of the debt contract. In this setting the borrower will repay the debt given high realizations of θ . The repayment function is:

$$P(\theta, R_L, D) = \min[R_L D, \eta Y_2(D)].$$

This implies a critical value θ^* such that the borrower will repay if and only if the shock $\theta > \theta^*$. Assuming that $-\theta_m < \theta^* < \theta_m$, we have

$$P(\theta, R_L, D) = \begin{cases} \eta[\bar{Y} + RD + \theta] & \text{for } -\theta_m \leq \theta < \theta^* \\ R_L D & \text{for } \theta^* \leq \theta \leq \theta_m \end{cases} \quad (1)$$

where

$$\theta^*(R_L, D) = \frac{[R_L - \eta R]D}{\eta} - \bar{Y}$$

4.3 The Lenders

Given positive capture rates, as Catao and Kapur (2004) note, default is partial, and the size of the default is given by the difference between the

contractual repayment obligation and actual repayments:

$$S(\theta, D) = R_L D - P(\theta, R_L, D).$$

In addition to the above direct costs, they continue, default may also involve a negative externality, for example if default in one country increases the risk of default by other borrowers through contagion effects. In this model, the magnitude of such “spillover costs” is proportional to the size of the default. That is, a default of size S imposes a cost of $(1 + q)S$ on the lender, where the parameter $q \geq 0$ is a measure of the spillover costs. Thus the net return to lenders is given by contractual repayments less total default costs:

$$P^*(\theta, R_L, D) = R_L D - (1 + q)S(\theta, D).$$

As is standard in the literature, and following Catao and Kapur (2004), we assume that the capital market is competitive and that lenders are risk neutral. This implies that for a given level of borrowing D , lenders must choose R_L to break even:

$$\int_{-\theta_m}^{\theta_m} P^*(\theta, R_L, D) \pi(\theta) d\theta = RD,$$

where $\pi(\theta)$ is the density function of random shocks:

$$\int_{-\theta_m}^{\theta_m} \pi(\theta) d\theta = 1.$$

We assume, for comparison with Catao and Kapur (2004), and for analytical simplicity, that shocks follow a uniform distribution. That is, the probability density function of the shock variable takes the form

$$\pi(\theta) = \begin{cases} \frac{1}{2\theta_m} & \text{if } -\theta_m \leq \theta \leq \theta_m \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

4.4 The Spread Function in the No-hedging Case

As shown by Catao and Kapur (2004), the spread function in this case takes the form

$$R_L(D) = R \quad \text{for } D \leq \frac{\eta(\bar{Y} - \theta_m)}{(1 - \eta)R}$$

and

$$R_L(D) = R + \left[\sqrt{\frac{\eta\theta_m}{D(1+q)}} - \sqrt{\eta\frac{\bar{Y}}{D} - R(1-\eta) - \frac{\eta q\theta_m}{D(1+q)}} \right]^2$$

for $\frac{\eta(\bar{Y} - \theta_m)}{(1-\eta)R} < D < D_{\max},$

where

$$D_{\max} = \frac{\eta}{(1-\eta)R} \left[\bar{Y} - \frac{q\theta_m}{(1+q)} \right]$$

denotes the supply constraint on the amount of debt that the market is willing to supply to the country. At levels of debt which are sufficiently low, there is no scenario in which it will be optimal for the country to default, and in this case the sovereign spread is zero. Above this threshold and below $D_{\max}$, the spread is given above. The spread tends to a maximum but finite value as $D \rightarrow D_{\max}$. Note that maximum debt level is decreasing in the parameter θ_m . That is, the larger the variance of the macroeconomic shocks to which the country is exposed, the lower is their maximum debt ceiling.

It is worth making a comment on why exactly the sovereign lender faces a potential credit constraint in this model. In contrast to earlier papers, such as Stiglitz and Weiss (1981), which explain credit rationing in terms of asymmetric information between borrowers and lenders, the case considered here is one of full information. The debt ceiling exists because, given a certain output volatility, there is always a maximum value for R_L above which the lenders will never break even, since the high cost of the debt will always force the borrowers to default for rates above this value. The fact that the maximum value for R_L is finite occurs because the shock is assumed to be uniform and is itself bounded between finite maximum and minimum values. In contrast, if the shock was drawn from a normal distribution, for example, we would have a vertical asymptote for R_L as $D \rightarrow D_{\max}$.

4.5 The Optimal Choice of Debt in the Absence of Hedging

The sovereign is assumed to have a concave utility function and cares only about the expected utility of period-2 consumption, $E[U(C_2(\theta, D))]$. Consumption in period-2 is given by total output net of repayments:

$$C_2(\theta, D) = \bar{Y} + RD + \theta - P(\theta, R_L, D).$$

Consumption in the default and the non-default states is given by:

$$C_2(\theta, D) = \begin{cases} C_L = (\bar{Y} + RD + \theta)(1-\eta) & \text{if } \theta < \theta^* \\ C_H = \bar{Y} + \theta + (R - R_L)D & \text{if } \theta > \theta^* \end{cases} \quad (3)$$

The borrower's problem is to choose D to maximize expected utility,

$$\max_D \int_{-\theta_m}^{\theta^*} U(C_L)\pi(\theta)d\theta + \int_{\theta^*}^{\theta_m} U(C_H)\pi(\theta)d\theta$$

subject to the condition that

$$R_L = R_L(D).$$

A couple of points are worth noting. First, debt is a double-edged sword. On one hand, the cost of carrying debt is $D(R_L - R)$. Thus carrying more debt, in the absence of hedging, increases R_L and increases the total cost of each dollar of debt contracted. On the other hand, debt provides partial insurance against adverse shocks. As Catao and Kapur (2004) note, default on positive levels of debt yields results in consumption (net of what the lenders are able to capture) of $(1 - \eta)[\bar{Y} + RD + \theta]$, while consumption in the absence of debt would have been $\bar{Y} + \theta$. Thus debt provides partial insurance whenever

$$\theta < \frac{1 - \eta}{\eta}RD - \bar{Y}.$$

The second point is that, to solve the above optimization problem, we must specify a form for the utility function $U(C_2)$. In our implementation we follow Catao and Kapur (2004) in choosing a utility function in the constant elasticity of substitution class,

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}.$$

For purposes of comparison, we consider as do they values of γ in the set $\{1.3, 1.5, 2\}$ and values of the shock parameter θ_m in the set $\{30, 40, 50, 60\}$. A value of $\gamma = 2$ is closer to the values assumed in the real business cycle literature, they note, and in our results we find that the gains from hedging increase with increasing risk-aversion (in particular, they are greatest for the case of $\gamma = 2$). We solve the model using standard numerical optimization procedures for given values of the parameters. Since we are optimizing with respect to a single variable, D , it is possible to simply march through the state space for D and choose the value that produces the maximum utility.

4.6 The Core Results of the Model without Hedging

Catao and Kapur (2004) state the following set of core results for their model of sovereign default, which we reprint below:

1. $R_L(D)$ is well-defined for levels of debt in some bounded interval $[0, D_{\max})$, where $D_{\max}$ depends, inter alia, on the probability distribution of shocks, $\pi(\theta)$.
2. $R_L(D) = R$ for $D \in [0, \frac{\eta(\bar{Y}-\theta_m)}{(1-\eta)R}]$. For higher values of D , $R_L(D)$ exceeds R and is strictly increasing in D .
3. $R_L(D)$ is increasing in the variance of shocks.
4. At the optimum, the equilibrium threshold below which the borrower defaults, denoted by $\theta^* = \theta^*(R_L(D^*), D^*)$, is increasing in the variance of shocks.
5. The effect of volatility on the level of optimal debt, D^* , is ambiguous and will depend on the borrower's degree of risk aversion.

The most important facts above, of course, are the results that increased output volatility leads to higher spreads for any given level of debt and a higher ex-ante probability of default. We turn now to our generalization of this model to the case of hedging.

5 Hedging Macroeconomic Volatility: The Case of an Oil-Exporting Country

Of the many external shocks faced by developing countries, one of the most important is a shock to their terms of trade. We will focus here on the case of an oil-exporting country whose oil income makes up the majority (approximately 70%) of its government revenue in hard currency terms. This figure is not unreasonable as a proxy for the export income share of government revenue for many commodity-dependent developing countries ³.

We imagine for simplicity that the government oil monopoly, or in general the oil producers, in the country are able to set the quantity Q of oil that they will produce for sale in (or by) period-2. The stochastic component of oil revenue, therefore, will be the stochastic return on the spot price of oil, which we will denote by R^{spot} . We denote by s_1 the spot price of oil at time $t = 1$ and by s_2 the spot price of oil at time $t = 2$. Thus

$$s_2 = s_1 R^{spot}$$

³In the case of a small coffee producer in Guatemala with a loan facilitated by ANACAFÉ, this percentage is probably an underestimate.

by definition. We assume by arbitrage arguments that the expected return on the price of oil between now and period-2 is the risk-free rate, $E[R^{spot}] = R$. The return on the spot price of oil is assumed to follow a uniform distribution, with $R^{spot} \in [R - R_m, R + R_m]$.

5.1 Hedging Strategy 1: Shorting Oil Futures

Due to the risk it faces because of changes in the international spot price of oil, the borrower potentially stands to gain by hedging its oil revenues. There are several strategies involving derivative instruments that the country might employ to achieve this goal. Following the example of ANACAFÉ, we consider separately two simple instruments: futures and put options. First, we outline the strategy of shorting oil futures. The basic principle here is that, by gaining a negative exposure to oil price increases using the futures market, the country can protect itself in case that the oil price falls.

In general, suppose that the IMF, in the case of a developing country, or perhaps a large investment bank which is brokering a debt issue for an emerging market, can enforce the commitment of the borrowing country to place a proportion α of its total debt D into a futures hedge on the period-2 price of oil, s_2 . Since the borrower has a revenue stream with a positive exposure to the oil price, the hedging strategy would require them to enter into a short futures position. This position could be undertaken by the government, and would make a windfall in the case that oil prices experience a negative shock and a loss in cases when oil prices are high. By undertaking such a strategy, the risk-averse country could both reduce its output volatility and, at the margin, for any given level of debt, lower the ex-ante probability of default on that debt via positive hedging revenues in bad states of the world.

The price of a futures contract on one unit of output in this setting should be, under no arbitrage, $F = s_1 R$. If the borrower hedges an amount equal to αD , then the number of unit-contracts it will short at time $t = 1$ is $\frac{\alpha D}{s_1 R}$. At time $t = 2$, the borrower can close out its futures positions in one of two ways: it can either make physical delivery on the futures contracts using a proportion of its oil output, or it can go long the same number of contracts at the prevailing futures price, which at time $t = 2$ will converge to the spot price s_2 . It can be shown that these strategies will result in the same net return, and we will assume that, in light of coordination costs in the case of physical delivery, the sovereign closes out its futures position and collects revenue from its normal channels of distribution at the prevailing market price.

With the futures contract, the sovereign can invest its entire debt, as before, in reserves at the risk free rate, and this debt can effectively be held as collateral by the futures exchange as the futures positions are marked to market each day (as they would be in the multi-period setting). This would result in an effective initial outlay of zero capital to enter into the futures position, since whatever the country does not invest in the position it would hold as reserves earning the risk free rate anyway. The strategy will generate a cash flow at time 2 of:

$$-\left[\frac{\alpha D}{s_1 R}\right] s_2 + \left[\frac{\alpha D}{s_1 R}\right] s_1 R + R\alpha D.$$

Thus the total gross return on the amount αD placed in the strategy will be

$$R + \left(1 - \frac{R^{spot}}{R}\right).$$

The total period-2 revenue for the foreign debt, D , all of which is placed in reserves and a proportion α of which is committed to the futures position, will be

$$RD + \alpha D \left[1 - \frac{R^{spot}}{R}\right].$$

Since the production quantity of oil, which we denote by Q , is certain, the oil revenues in period-2 will be $s_2 Q = s_1 R^{spot} Q$. We assume that the cost of producing this quantity of oil is fixed, or alternatively that there are oil reserves worth selling at nearly zero marginal cost in the event that s_2 is low. We subsume the fixed cost of producing the oil into the non-stochastic component of period-2 income. Then the total period-2 income of the country under hedging is:

$$Y_2 = \bar{Y}^{non-oil} + s_1 R^{spot} Q + RD + \alpha D \left[1 - \frac{R^{spot}}{R}\right],$$

where $\bar{Y}^{non-oil}$ is the expected net non-oil revenue. From the above, consider the stochastic component of period-2 income

$$\epsilon = s_1 R^{spot} Q + \alpha D \left[1 - \frac{R^{spot}}{R}\right].$$

Given that $E[\epsilon] = s_1 R Q$, write $\theta = \epsilon - E[\epsilon]$ to obtain

$$\theta = \left[s_1 Q - \frac{\alpha D}{R}\right] [R^{spot} - R].$$

We can now rewrite period-2 income as

$$Y_2 = \bar{Y} + RD + \theta,$$

where $E[\theta] = 0$ by construction and $\bar{Y} = \bar{Y}^{non-oil} + s_1 RQ$. In the analysis of the model that follows, we set $\bar{Y} = 100$ and $s_1 RQ = 70$. The distribution of the shock variable θ is uniform with $\theta \in [-\theta_m, \theta_m]$. With futures hedging, we have the result that

$$\theta_m = \left[s_1 Q - \frac{\alpha D}{R} \right] R_m^{spot}.$$

Thus higher oil price volatility translates into higher oil revenue volatility, and greater hedging for a given debt level reduces the volatility of national income. Also, in the case where the distribution of the return R^{spot} is assumed to be uniform, hedging reduced the magnitude of the worst-case-scenario shock to national income.

5.2 Optimal Debt Choice with Futures Hedging

As before, consumption in period-2 is given by total output net of repayments:

$$C_2(\theta, D) = \bar{Y} + RD + \theta - P(\theta, R_L, D).$$

With futures hedging, however, the shock variable θ has a different distribution with smaller variance, as shown above. For a given hedging proportion α , the borrower's problem is to choose D to maximize expected utility,

$$\max_D \int_{-\theta_m}^{\theta^*} U(C_L) \pi(\theta) d\theta + \int_{\theta^*}^{\theta_m} U(C_H) \pi(\theta) d\theta$$

subject to the condition that

$$R_L = R_L(D),$$

where C_L and C_H stand for consumption in default and non-default states, as before. Now, however, positive levels of hedging imply that the distribution of the shock, $\pi(\theta)$, is also a function of the debt choice D :

$$\pi(\theta) = \frac{1}{2\theta_m} = \frac{1}{2 \left[s_1 Q - \frac{\alpha D}{R} \right] R_m^{spot}}.$$

We solve the optimal debt choice problem via numerical optimization, taking into account the above changes in the objective function. The equilibrium sovereign spread is given by the formula for the no-hedging case, but with the modified value for θ_m and reinterpretation of $\bar{Y}$ stated in the previous section.

5.3 Hedging Strategy 2: Buying Put Options on the Oil Price

The second hedging strategy we consider is purchasing put options on the period-2 price of oil. If we denote by p the price of a put option in period-1, by β the fraction of total debt D used to purchase the put options, and by N the total number of unit-contracts purchased, then we have the identity

$$Np = D\beta.$$

For simplicity we consider only the case where the country purchases put options of one strike price, which we denote by K . As before, we assume that the distribution of the spot return on the oil price is uniform on the interval $R^{spot} \in [R - R_m, R + R_m]$. Then the period-2 income of the country under put option hedging, Y_2 , is given by

$$Y_2 = \bar{Y}^{non-oil} + s_1 R^{spot} Q + (1 - \beta)DR + N \max[0, K - s_1 R^{spot}],$$

where $(1 - \beta)DR$ is the income from the portion of the debt placed in securities earning the risk-free rate R and $N \max[0, K - s_1 R^{spot}]$ is the total payout from the portfolio consisting of N put options with strike price K . We substitute the identity $N = D\beta/p$ into the above to obtain

$$Y_2 = \bar{Y}^{non-oil} + s_1 R^{spot} Q + (1 - \beta)DR + \frac{D\beta}{p} \max[0, K - s_1 R^{spot}].$$

Under no-arbitrage and risk-neutral pricing, we obtain the result that the price of the put option is given by

$$p = \frac{1}{R} E[\max[0, K - s_1 R^{spot}]].$$

Taking expectations of the period-2 income and applying this pricing result, we obtain

$$E[Y_2] = \bar{Y}^{non-oil} + s_1 RQ + DR = \bar{Y} + DR,$$

where $\bar{Y} = \bar{Y}^{non-oil} + s_1 RQ$ as before. We can then write

$$Y_2 = \bar{Y} + DR + \theta$$

as in the futures hedging case, where θ is the stochastic component of period-2 income and $E[\theta] = 0$. For the put options hedging case, the income shock θ is given by

$$\theta = s_1 Q(R^{spot} - R) + \beta DR \left[\frac{\max[0, K - s_1 R^{spot}]}{E[\max[0, K - s_1 R^{spot}]]} - 1 \right].$$

Note that unlike in the futures hedging case, the distribution of the shock to income is no longer uniform. In this case, the shock function θ is piecewise linear in the shock variable R^{spot} .

The country will still default when the required repayment exceeds the amount foreign lenders can recapture. Thus the critical shock value below which the country will default takes the same form:

$$\theta^* = \frac{[R_L - \eta R]D}{\eta} - \bar{Y}.$$

Given that the country borrows debt D , the sovereign spread is determined by the requirement that the lenders break even,

$$E[P^*(\theta, R_L, D)] = RD.$$

However, in this case, since the distribution of the shock is no longer uniform, we take the expectation with respect to the underlying distribution of R^{spot} , and the requirement becomes

$$\int_{R-R_m}^{R+R_m} P^*(\theta(R^{spot}, \beta), R_L, D) \pi(R^{spot}) dR^{spot} = RD,$$

where $\pi(R^{spot})$ is the density function for the spot price of oil:

$$\int_{R-R_m}^{R+R_m} \pi(R^{spot}) dR^{spot} = 1.$$

The closed-form solution for R_L is cumbersome to write down, so we omit it here. However, it can be computed given values for the model parameters and D , and we provide sample numerical solutions to the model in the following section for a range of different parameter assumptions. It is worth noting that, in deriving the solution, we must split the expectation integral into two parts, for values of R^{spot} before and after the critical default value, respectively. This requires solving for the value of R^{spot} for which $\theta(R^{spot}, \beta) = \theta^*$.

The Strategic Strike Price K

Up until this point, we have delayed discussion of one crucial factor in the strategy: how to choose the strike price K . Adapting the strategy of ANACAFÉ to the sovereign context, we choose our strike price at the

default threshold price level for s_2 in equilibrium in the case of no hedging. In other words, we choose K to solve the equation

$$\theta = s_1 Q \left(\frac{K}{s_1} - R \right) = \frac{[R_L - \eta R]D}{\eta} - \bar{Y} = \theta^*.$$

This yields a strategic strike price of

$$K = \frac{[R_L - \eta R]D}{\eta Q} - \frac{\bar{Y}}{Q} + s_1 R,$$

where R_L and D denote specifically the interest rate on sovereign debt and the level of debt contracted in equilibrium in the case of no hedging.

Note that this is only a rule of thumb. We have reasoned simply that a sensible price to insure the country at is the price below which it would default in the absence of insurance. This is not exactly the optimal price, since of course the put option insurance has a cost to the borrowing country, but testing of different values indicates that the performance of the strategy using this rule of thumb is generally quite good in comparison to strike prices close to the K computed above.

With K set at this value, the threshold spot return R_*^{spot} on oil below which the country defaults satisfies the equation

$$s_1 Q (R_*^{spot} - R) + \beta D R \left[\frac{\max[0, K - s_1 R_*^{spot}]}{E[\max[0, K - s_1 R^{spot}]]} - 1 \right] = \frac{[R_L - \eta R]D}{\eta} - \bar{Y},$$

given that the sovereign interest rate and debt are R_L and D , respectively. A couple of additional points are worth noting. First, the country will never have an incentive to default for debt levels D that satisfy the equation

$$R_L D \leq \eta Y_2^{\min} = \eta(\bar{Y} + R D + \theta^{\min}),$$

where $\theta^{\min}$ is given by

$$\theta^{\min} = -s_1 Q R_m + \beta D R \left[\frac{K - s_1(R - R_m)}{E[\max[0, K - s_1 R^{spot}]]} - 1 \right]$$

and in this case we must have $R_L = R$. Thus the country will never default for levels of debt below

$$D^{\min} = \frac{\eta(\bar{Y} - s_1 Q R_m)}{(1 - \eta)R - \eta \beta R \left[\frac{(K - s_1(R - R_m))}{E[\max[0, K - s_1 R^{spot}]]} - 1 \right]}$$

The second point is that the above expression gives a natural upper bound on the hedging proportion β . By imposing the requirement $D^{\min} < \underline{D}$, we necessitate that

$$\beta < \frac{[(1 - \eta)R\underline{D} - \eta(\bar{Y} - s_1QR_m)]E[\max[0, K - s_1R^{spot}]]}{\eta R\underline{D}(K - s_1(R - R_m))}.$$

Whereas in the case of futures hedging we had the upper bound of $\alpha \leq 1$, put option hedging gives the country a lot more leverage to reduce its revenue volatility and we must be aware of this when choosing the proportion of debt to commit to the hedging portfolio, so that the country does not over-hedge.

5.4 Optimal Debt Choice with Put Option Hedging

The optimal debt in the case of put hedging is determined as the solution of the maximization problem

$$\max_D \int_{R-R_m}^{R_*^{spot}} U(C_L)\pi(R^{spot})dR^{spot} + \int_{R_*^{spot}}^{R+R_m} U(C_H)\pi(R^{spot})dR^{spot},$$

subject to the constraint

$$R_L = R_L(D),$$

and where R_*^{spot} is obtain by solving the equation in the previous section. As in the case of futures hedging, we solve the optimization problem above numerically for a given value of β .

6 Gains From Hedging

6.1 Futures Hedging

We present here summary tables of equilibrium debt, sovereign spread, ex-ante default probabilities, expected size of defaults, and worst case shocks with and without hedging. We solved the model for three values of the risk aversion parameter $\gamma \in \{1.3, 1.5, 2\}$ and four values for the shock parameter $\theta_m \in \{30, 40, 50, 60\}$. These are the same combinations of scenarios considered by Catao and Kapur (2004). Given that several different combination of parameters in the new model can lead to the same values for θ_m , we fixed the values for s_1 and Q so that $s_1QR = 70\%$ of total income. Then variation in the initial shock size parameter $\theta_m = s_1QR_m^{spot}$ was achieved by choosing the appropriate value of R_m^{spot} . We consider the cases of no hedging, partial hedging with $\alpha = 0.5$, and full hedging with $\alpha = 1$. The

results presented here are the more representative case of the country with risk-aversion $\gamma = 2$. The corresponding tables of results for the countries with risk-aversions $\gamma = 1.3$ and $\gamma = 1.5$ are located in Appendix 1 for easy reference.

Several trends are apparent from the model results:

1. Optimal debt size has a nonmonotonic relationship to shock size, which seems to change with greater hedging or greater risk aversion.
2. Greater shock variance is associated with higher sovereign spreads, with and without hedging.
3. Great hedging increases the optimal level of sovereign debt for every combination of parameters.
4. For sufficiently large shocks ($\theta \geq 40$) at each level of risk aversion, and for all shock sizes in the case of the more risk averse country ($\gamma = 2$), greater hedging achieves greater reduction in the equilibrium sovereign spread.
5. The expected utility evaluated at the optimal level of debt and the corresponding sovereign spread improves uniformly from increased hedging in all cases.

The first two of these observations are consistent with the findings of Catao and Kapur (2004). With regards to hedging, clear gains are apparent for the sovereign borrower: it can choose a higher level of debt and usually is able to do this at a lower cost per dollar of debt contracted.

	D^*			$R_L^* - R$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	33.29	35.09	36.85	2.21%	1.48%	0.582%
$\theta_m = 40$	33.09	34.75	36.5	7.10%	4.48%	1.91%
$\theta_m = 50$	32.49	34.28	36.2	13.63%	8.77%	4.03%
$\theta_m = 60$	31.64	33.65	35.84	21.39%	14.08%	6.84%

Table 1: More Risk Averse Borrower ($\gamma = 2.0$): Optimal Debt and Sovereign Spread

In addition, for any given level of debt, the country would prefer to hedge a greater portion of it at the margin; that is, the optimal level of α is $\alpha^* = 1$ unless the debt exceeds the level $D = s_1 QR$ which corresponds to zero

output volatility. However, this level is never approached in our case because it is set far above the highest levels of the debt ceiling $D^{\max}$. Furthermore, in reality an important problem in the implementation of hedging strategies requiring large positions in derivative markets are the exchange-dictated upper bounds on position sizes taken by any one market participant (see Hull (2002) for details). While such limits are put in place to combat the effects of speculative behavior, for example the attempt to influence or “corner” the spot market by taking large futures positions, it is important to increase the depth of futures and options markets to improve the effectiveness of hedging strategies such as those outlined here. In practice, the value a country could choose for α may be limited by the depth of the futures markets for its primary export (or potentially import) commodities.

As another measure of the gains from hedging, we computed the maximum size of the shock magnitude θ_m for output after the country has undertaken their hedge and contracted their chosen level of debt at the applicable sovereign spread. We find significant reductions in the volatility of oil export income as a result of futures hedging.

	$\gamma = 1.3$		$\gamma = 1.5$		$\gamma = 2.0$	
	$\alpha = .5$	$\alpha = 1$	$\alpha = .5$	$\alpha = 1$	$\alpha = .5$	$\alpha = 1$
$\theta_m^* = 30$	22.79	14.49	22.69	14.41	22.48	14.21
$\theta_m^* = 40$	30.57	19.71	30.39	19.52	30.07	19.14
$\theta_m^* = 50$	38.36	24.90	38.13	24.62	37.76	24.14
$\theta_m^* = 60$	46.20	30.15	45.96	29.83	45.58	29.28

Table 2: Worst Case Shocks: Before and After Futures Hedging

Here the set of values $\theta_m \in \{30, 40, 50, 60\}$ represent the magnitudes of the maximum negative shock before hedging for the four shock distributions considered. This reduction in θ_m translates directly in the model into an increase in the debt ceiling applicable to the country, as is apparent from the relation

$$D_{\max} = \frac{\eta}{(1-\eta)R}[\bar{Y} - \frac{q\theta_m}{(1+q)}].$$

Finally, we examine the benefits of hedging from the point of view of the creditors. In particular, we record the ex-ante default probability and expected default size for different shock distributions and levels of hedging. The following results come to light:

1. The ex-ante default probability has a nonmonotonic relationship with the hedging parameter α . For lower or medium risk aversion and lower

levels of shock variance, the default probability is actually greater than in the no-hedging case.

2. The above generalization also holds for the expected size of default.
3. For the more realistic case of $\gamma = 2$, both ex-ante default probability and expected size of default are decreasing as hedging increases.

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	23.36%	22.64%	18.32%	2.10	1.53	0.781
$\theta_m = 40$	36.13%	33.91%	28.45%	4.34	3.06	1.63
$\theta_m = 50$	44.36%	42.06%	36.65%	6.65	4.76	2.65
$\theta_m = 60$	50.06%	48.07%	43.14%	9.01	6.57	3.79

Table 3: Ex-ante Default Probability and Expected Size of Default ($\gamma = 2$)

While the reduction in ex-ante default probability and expected size of default occurs in many cases, and the net result in nearly all cases translates into a lower sovereign spread for the borrower, how can we explain the cases where this does not hold? In particular, in the case of the less risk averse country, it seems to suggest a paradox that greater hedging of their income could result in having to face a higher sovereign spread, while still leading to an increase in their expected utility. This paradox is resolved by the observation that, in this model of sovereign default, hedging, first, allows the optimizing borrower to contract a higher level of debt in all cases, and second, increases the insurance value of that debt to the country. In the seminal article of Eaton and Gersovitz (1981), as well as Catao and Kapur (2004), which is a modified and analytically tractable version of that framework, the welfare function is increasing in debt for values of debt D less than or equal to the optimal debt D^* . This is a consequence of the second order condition for obtaining a maximum.

Recall that debt serves as consumption insurance whenever the realized income shock is below a certain value:

$$\theta < \theta' \equiv \frac{1 - \eta}{\eta} RD - \bar{Y}.$$

When futures hedging is allowed, we obtain the result that for a given level of debt D^* , increasing the hedging proportion α at the margin increases the insurance value the country derives from the debt. This is stated formally in

Theorem 2. It is more direct to see that for any given (fixed) level of debt, greater hedging always leads to lower spreads and a lower default probability, so long as the quantity (notional amount) hedged is less than or equal to the total output quantity Q . As a precursor to Theorem 2, Theorem 1 states the result that futures hedging is welfare improving.

Theorem 1. *From the optimum level of debt for a given hedging level, greater hedging increases total welfare at the margin as well as increasing total optimal debt. That is, for the welfare function $V(D, \alpha) = E[U(C_2(D, \alpha))]$,*

$$\frac{\partial V(D^*, \alpha)}{\partial \alpha} > 0$$

and

$$\frac{\partial D^*}{\partial \alpha} > 0.$$

Proof. See Appendix 2. □

Thus, when a sovereign borrower can hedge its hard currency income using futures contracts, it can improve its welfare and, for the parameter values considered here, will want to borrow more debt. Having established this, we obtain directly the result the greater hedging leads to greater consumption insurance in states of the world that require sovereign default:

Theorem 2. *Let $G(D, \theta) = (1 - \eta)(\bar{Y} + RD + \theta) - (\bar{Y} + \theta)$ denote the surplus income consumed by the borrowing country in the event of default, given a debt level D , over the amount that they would have consumed for the same shock realization but without contracting debt. Then at the optimum, the insurance value of debt increases with greater hedging. That is, we have*

$$\frac{\partial}{\partial \alpha} E[G(D^*, \theta)] > 0$$

and

$$\frac{\partial}{\partial \alpha} \text{Var}[G(D^*, \theta)] < 0.$$

Proof. See Appendix 2. □

This result implies that, as long as the country does not over-hedge, it will increase the insurance value of its debt by hedging a higher percentage of that debt. This is highly intuitive: hedging is a much more effective insurance mechanism than simply defaulting in bad states of the world.

6.2 Put Options Hedging

To evaluate the put options hedging strategy, we used the same parameter combinations as in the futures hedging case. The key difference here is that, instead of using the hedging parameter α which has a natural upper bound of 1 in the case of futures hedging, we denote the hedging proportion of debt by β , whose natural upper bound turns out to be a function of the underlying parameters in the model. As indicated previously, if we assume that the minimum level of debt that creates nonzero default risk, $D^{\min}$, is less than some upper bound, $\underline{D}$, then this implies an upper bound for the hedging parameter:

$$\beta \leq \beta^{\max} = \frac{[(1 - \eta)R\underline{D} - \eta(\bar{Y} - s_1QR_m)]E[\max[0, K - s_1R^{spot}]]}{\eta R\underline{D}(K - s_1(R - R_m))}.$$

In the following results we set $\underline{D} = \bar{Y} = 100$. That is, the country cannot contract more riskless debt than its total GDP (or more accurately, its total hard currency income). Naturally this would seem to be a loose upper bound, but in practice this calibration seems to generate reasonable results, because we consider only hedging parameters β in the set $\frac{\beta}{\beta^{\max}} \in \{0, .25, .50\}$.

First we examine the benefits of hedging from the perspective of the sovereign borrower. The following results come to light:

1. Greater shock variance is associated with higher sovereign spreads, as before, with and without hedging.
2. Put option hedging can significantly reduce sovereign spreads, and in sufficiently large quantities, can sometimes eliminate sovereign default risk entirely.
3. If a large enough proportion of the debt is used to purchase the put option insurance, then optimal debt may actually decrease in cases where the default risk is eliminated.

Overall, the results indicate that put option hedging is very effective. In particular, we note that in contrast to the case of futures hedging, put option hedging uniformly decreases the sovereign spread in all instances. Why might this be the case? Essentially, the put option hedging strategy is a more directed and versatile means of insuring against the default event. Whereas the futures contract has a unique price, it is possible to select among several different put option contracts with different strike prices and purchase the one that is best suited to the default threshold faced by the country. To

	D^*			$R_L^* - R$		
	$\beta = 0$	$\frac{\beta}{\beta^{\max}} = .25$	$\frac{\beta}{\beta^{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta^{\max}} = .25$	$\frac{\beta}{\beta^{\max}} = .50$
$\theta_m = 30$	33.29	32.02	16	2.21%	0%	0%
$\theta_m = 40$	33.09	33.46	31.23	7.10%	2.78%	0%
$\theta_m = 50$	32.49	33.06	33.75	13.63%	7.86%	2.56%
$\theta_m = 60$	31.64	32.38	33.16	21.39%	14.42%	7.61%

Table 4: More Risk Averse Borrower ($\gamma = 2$): Optimal Debt and Sovereign Spread

further illustrate the benefits of put option hedging, from the perspective of the lenders, the following tables summarize the ex-ante default probability and the expected size of default for the various parameter combinations.

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\beta = 0$	$\frac{\beta}{\beta^{\max}} = .25$	$\frac{\beta}{\beta^{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta^{\max}} = .25$	$\frac{\beta}{\beta^{\max}} = .50$
$\theta_m = 30$	23.36%	0%	0%	2.10	0	0
$\theta_m = 40$	36.13%	33.37%	0%	4.34	1.86	0
$\theta_m = 50$	44.36%	42.59%	38.60%	6.65	4.07	1.49
$\theta_m = 60$	50.06%	48.88%	46.39%	9.01	6.37	3.63

Table 5: Ex-ante Default Probability and Expected Size of Default ($\gamma = 2$)

Several observations are apparent from these results (see Appendix 1 for the full set of tables for the borrowers with $\gamma = 1.3$ and $\gamma = 1.5$).

1. The put option hedging strategy works primarily through the channel of reduced default sizes to achieve lower spreads, although there is in nearly all cases some reduction in the probability of default as well.
2. Increasing risk-aversion is associated with a higher propensity to default and a larger expected default size.
3. In contrast to the case of futures hedging, put option hedging appears to work particularly well in the case of the less risk-averse country.
4. Greater hedging (as a proportion of the applicable $\beta^{\max}$) uniformly achieves a reduction in expected default size.

In the cases where the country is able to completely eliminate default risk, this is achieved by hedging a very high (often over 90%) percentage of their output quantity Q . That is, the number of put option contracts purchased is

a fairly high proportion of their total number of units of output. Although the strategic strike prices are not reprinted here, they lie within a reasonable range, from approximately \$35 to \$50. While the put option hedging strategy requires a sunk cost upfront of the option premium, it allows the country a higher degree of leverage to insure itself against falls in the oil price without having to give up potential upside.

7 Using Multiple Options to Create Politically Feasible Hedging Strategies

So far we have focused on two of the most simple strategies using derivatives to insure against downside risk: purchasing futures and purchasing put options. Although the theoretical gains of risk management using these strategies is apparent in terms of lower default probabilities and lower borrowing costs, implementing them in practice is likely to face several obstacles for a sovereign borrower. Two of the most important are:

1. Initial cash outflows to pay for the insurance may have high opportunity costs for politicians. Why not use the funds to build a much-needed hospital, for instance?
2. Ex-post losses on the hedging strategy may create politically difficult situations for officials who, ex-ante, will decide not to bear this risk and thus will not initiate the hedge.

In light of these two problems, a realistic hedging strategy for a sovereign borrower needs to be low-cost, in the sense of requiring little or no initial cash outflow, should preserve significant exposure to a rising commodity price (in the case of an exporter), and should still limit significantly the downside risk of a large fall in the commodity price.

We can satisfy all of these requirements by expanding the set of options-based strategies available for use. Consider the example of the oil-exporting borrower subject to income shocks of up to $\theta_m = 30$ percent of total income, valued in dollars. The initial price of oil in our paper is assumed to be \$50.

Then let us construct a self-financing strategy where for a given number of output units (barrels of oil) we:

1. Sell β_1 in-the-money call options with a strike price of $K_1 \leq \$50$.
2. Buy β_2 out-of-the-money put options with a strike price $K_1 \leq \$50$.
3. Buy β_3 call options with a strike price $K_2 > K_1$

Here the sale of the in-the-money call options will finance the purchase of the out-of-the-money put options and the call options with a higher strike price, so that it does not cost anything for the country to initiate the strategy at time zero ⁴. That is, if $P(K)$ is the price of a put option with strike price K and $C(K)$ is the price of a call option with strike price K , all other things held equal, then we will choose the set $\{\beta_1, \beta_2, \beta_3, K_1, K_2\}$ to satisfy the zero initial cost condition

$$\beta_1 C(K_1) - \beta_2 P(K_1) - \beta_3 C(K_2) = 0.$$

In the above framework many payoff combinations are possible. For simplicity, consider the case where $\beta_1 = \beta_2 = \beta_3 = \beta$. In the general problem, the risk-averse sovereign will choose the two strike prices, β , and the level of debt to maximize utility subject to the zero cost condition and the spread equation. In fact, the general problem of choosing optimal hedging portfolios, given the ability to buy and sell options and perhaps other derivatives, is an interesting area for future research.

For now, we simply wish to illustrate the potential of hedging strategies such as those described for overcoming some of the political economy problems that face politicians and debt managers in emerging markets. In this example, the “price” of guaranteeing that the oil price received does not fall below $K_1 < \$50$ is agreeing to forego an amount $K_2 - K_1 > 0$ of future profits per barrel when prices are above the strike price K_2 . Figure 1 illustrates the price received per barrel of oil with and without hedging under the choice of strike prices K_1 and K_2 considered in what follows.

If we let K_1 equal the strategic strike price from the put option strategy we examined in this paper for a country with risk-aversion $\gamma = 2$ and the oil price shock $\theta_m = 30$, we get $K_1 = \$40.51$. Given this strategic strike price, we have $C(K_1) = \$12.59$ and $P(K_1) = \$1.17$. Then the strike price for the second (purchased) call option that satisfies the zero-initial cost relation is $K_2 = \$42.15$, which has cost $C(K_2) = \$11.42$. Thus our initial cash flow is zero, because

$$C(K_1) - P(K_1) - C(K_2) = \$12.59 - \$1.17 - \$11.42 = 0.$$

In this example, where the oil price shock is drawn from a uniform distribution, the sovereign would have to give up only $\$42.15 - \$40.51 = \$1.64$ of the period-2 oil price per barrel. This is a relatively modest price for insuring away a significant part of the country’s default risk.

⁴An organization called the CPM Group has proposed a somewhat similar approach, which is intended for use by various emerging market producers and consumers of commodities (CPM Group (2000)).

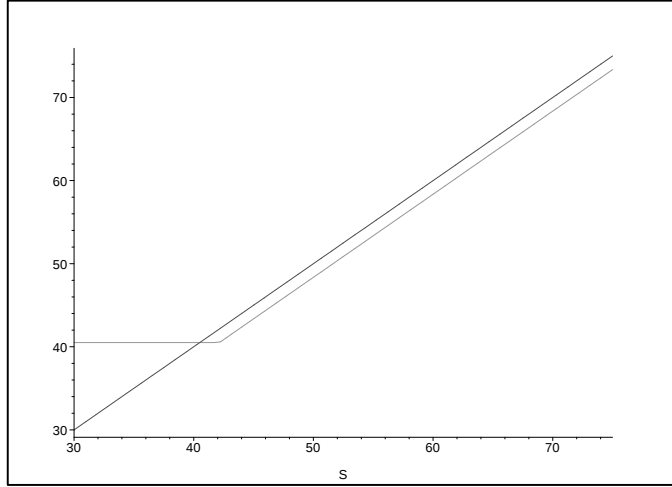


Figure 1: Hedged vs. Unhedged Price per Barrel

8 Conclusion

We have examined the benefits to be gained by commodity exporting countries from hedging their export revenues and thereby reducing their risk of defaulting on their sovereign debt. Towards this end, we extended the theoretical model of debt default of Catao and Kapur (2004) to the case of an oil exporting country that can hedge its oil revenues using futures or put options contracts on the international price of oil. This framework also can be applied to the case of a private borrower, such as the clients of ANACAFÉ.

The policy implications of our work are several. On the part of the IMF and regional development banks, there is a clear potential for acting as a facilitator between developing countries and capital markets, in a way analogous to the role played by ANACAFÉ between coffee producers and domestic banks in Guatemala. In this role, the multilateral institutions might provide the following services:

- Aid developing countries to determine reliable estimates of their production quantities and price exposure for major export commodities, in order to hedge appropriately.
- Provide advice on the derivative products available for hedging and how best to use them, as well as aiding with start-up costs for hedging operations.

- Provide advocacy on the behalf of developing country hedgers that leads to the deepening and expansion of existing futures, options and derivative markets for insurance purposes.
- Facilitate transparency between borrowers and capital markets to ensure that lenders are fully aware of the risk-reducing hedging activity being undertaken by the borrowers.

Avenues for Future Research

While the basic theoretical argument advanced here should carry over, with second-order modifications, to the general multi-period case, and where the currency composition and debt structure are modeled explicitly, implementing this approach to determine the fair spreads for borrowers who hedge their price exposure is probably best done in the context of a structural bond pricing model in the tradition of Merton (1974). Much work has been done on structural bond pricing models for corporate borrowers (see e.g. KMV Corporation (1999)), and recent work has begun to focus on adapting the structural approach to the case of the sovereign (Gapen et al. (2005)). In addition, our assumption of risk-neutral lenders is likely to understate the true gains that result from reducing downside risks from the perspective of actual lenders in capital markets, who are risk-averse.

Finally, two practical issues in conducting a viable hedging strategy are quantity variation and basis risk. In developing models for implementation, it would be necessary to take into account the uncertainty of output volume and model the imperfect correlation between the price of the actual commodity sold and the instrument used to hedge it.

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9 Appendix 1: Summary Tables of Model Solutions for Low and Moderate Risk-aversion

9.1 Futures Hedging

Optimal Debt and Sovereign Spreads

	D^*			$R_L^* - R$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	29.78	33.64	36.18	0.119%	.486%	0.25%
$\theta_m = 40$	29.78	32.99	35.52	2.15%	2.09%	1.01%
$\theta_m = 50$	29.76	32.6	35.14	6.53%	5.07%	2.49%
$\theta_m = 60$	29.49	32.19	34.82	12.8%	9.28%	4.68%

Table 6: Less Risk Averse Borrower ($\gamma = 1.3$): Optimal Debt and Sovereign Spread

	D^*			$R_L^* - R$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	30.97	34.11	36.37	0.494%	0.731%	0.330%
$\theta_m = 40$	31.1	33.62	35.84	3.59%	2.77%	1.25%
$\theta_m = 50$	30.91	33.23	35.53	8.83%	6.19%	2.97%
$\theta_m = 60$	30.4	32.76	35.2	15.59%	10.80%	5.35%

Table 7: Moderately Risk Averse Borrower ($\gamma = 1.5$): Optimal Debt and Sovereign Spread

Ex-ante Default Probabilities and Expected Sizes of Default

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	5.13%	12.62%	11.87%	0.462	0.863	0.516
$\theta_m = 40$	18.87%	22.41%	20.10%	2.27	2.06	1.19
$\theta_m = 50$	29.39%	30.94%	27.92%	4.41	3.56	2.09
$\theta_m = 60$	37.32%	37.91%	34.64%	6.72	5.25	3.13

Table 8: Ex-ante Default Probability and Expected Size of Default ($\gamma = 1.3$)

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$	$\alpha = 0$	$\alpha = .5$	$\alpha = 1$
$\theta_m = 30$	10.64%	15.63%	13.60%	0.958	1.06	0.588
$\theta_m = 40$	24.89%	26.11%	22.60%	2.99	2.38	1.32
$\theta_m = 50$	34.83%	34.62%	30.83%	5.22	3.96	2.28
$\theta_m = 60$	41.90%	41.35%	37.46%	7.54	5.70	3.35

Table 9: Ex-ante Default Probability and Expected Size of Default ($\gamma = 1.5$)

9.2 Put Option Hedging

Optimal Debt and Sovereign Spreads

	D^*			$R_L^* - R$		
	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$
$\theta_m = 30$	29.78	7.02	3.50	0.119%	0%	0%
$\theta_m = 40$	29.78	30.06	16.32	2.15%	0.181%	0%
$\theta_m = 50$	29.76	30.25	30.14	6.53%	3.29%	0%
$\theta_m = 60$	29.49	30.13	30.66	12.75%	8.34%	3.58%

Table 10: Less Risk Averse Borrower ($\gamma = 1.3$): Optimal Debt and Sovereign Spread

	D^*			$R_L^* - R$		
	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$
$\theta_m = 30$	30.97	14.59	7.28	0.494%	0%	0%
$\theta_m = 40$	31.10	31.43	21.52	3.59%	0.904%	0%
$\theta_m = 50$	30.91	31.41	31.95	8.83%	4.76%	0.748%
$\theta_m = 60$	30.40	31.05	31.67	15.59%	10.31%	4.87%

Table 11: Moderately Risk Averse Borrower ($\gamma = 1.5$): Optimal Debt and Sovereign Spread

Ex-ante Default Probabilities and Expected Sizes of Default

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$
$\theta_m = 30$	5.13%	0%	0%	0.461932	0	0
$\theta_m = 40$	18.87%	18.93%	0%	2.26	0.191	0
$\theta_m = 50$	29.39%	29.60%	0%	4.41	2.24	0
$\theta_m = 60$	37.32%	37.57%	36.91%	6.72	4.46	1.98

Table 12: Ex-ante Default Probability and Expected Size of Default ($\gamma = 1.3$)

	$Pr\{\theta < \theta^*\}$			$E[S(\theta, D) \theta < \theta^*]$		
	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$	$\beta = 0$	$\frac{\beta}{\beta_{\max}} = .25$	$\frac{\beta}{\beta_{\max}} = .50$
$\theta_m = 30$	10.64%	0%	0%	0.958	0	0
$\theta_m = 40$	24.89%	24.19%	0%	2.99	0.784	0
$\theta_m = 50$	34.83%	34.42%	31.72%	5.22	2.89	0.502
$\theta_m = 60$	41.90%	41.66%	40.14%	7.54	5.125	2.56

Table 13: Ex-ante Default Probability and Expected Size of Default ($\gamma = 1.5$)

10 Appendix 2: Theorem Proofs

10.1 Proof of Theorem 1

The first partial derivative is derived analytically, with the help of the envelope theorem, and evaluated for the combinations of parameters examined in this paper. In particular, we use the fact that

$$\frac{\partial V}{\partial \alpha} = \frac{\partial V}{\partial D} \Big|_{D=D^*} \frac{\partial D^*}{\partial \alpha} + \frac{\partial V}{\partial \alpha} \Big|_{D=D^*}$$

reduces to

$$\frac{\partial V}{\partial \alpha} = \frac{\partial V}{\partial \alpha} \Big|_{D=D^*}$$

because $\frac{\partial V}{\partial D} \Big|_{D=D^*} = 0$ at the optimum.

Under standard assumptions the second result is equivalent to showing that $\frac{\partial^2 V}{\partial \alpha \partial D} > 0$. In particular, we know that at the optimum we have

$$\frac{\partial^2 V}{\partial D^2}(D^*, \alpha_0) < 0,$$

where α_0 is the initial value for the hedging parameter. By the continuity of $V(D, \alpha)$, we know that for $\alpha' > \alpha_0$ with $\Delta\alpha = \alpha' - \alpha_0$ sufficiently small, we must also have

$$\frac{\partial^2 V}{\partial D^2}(D^*, \alpha') < 0.$$

Now keeping $D = D^*$ fixed, the change in the partial derivative of the value function with respect to debt is given by

$$\Delta V_D = V_{D\alpha}(\Delta\alpha) + O((\Delta\alpha)^2).$$

Given a sufficiently small positive change in α , this implies that $\Delta V_D > 0$ if and only if $V_{D\alpha} > 0$. Now, consider changing the debt to maximize the value function V starting from the new point (D^*, α') . From the approximation

$$\Delta V \approx V_D(D^*, \alpha')(\Delta D) + \frac{1}{2}V_{DD}(\Delta D)^2,$$

the value for ΔD that maximizes ΔV is given by

$$\Delta D \approx \frac{-V_D(D^*, \alpha')}{V_{DD}(D^*, \alpha')}.$$

Because we know that $V_{DD}(D^*, \alpha') < 0$, the sign of ΔD is the same as the sign of $V_D(D^*, \alpha')$. For $\Delta\alpha$ sufficiently small, this implies that the sign of ΔD is the same as the sign of $V_{D\alpha}(D^*, \alpha_0)$. Also, we have the result that

$$V_\alpha(D^*, \alpha_0) > 0,$$

which implies that $V(D^*, \alpha') - V(D^*, \alpha_0) > 0$ for $\Delta\alpha$ sufficiently small. Let $D' = D^* + \Delta D$ be the new (approximate) optimal debt. Then from the identity

$$\begin{aligned} V(D', \alpha') - V(D^*, \alpha_0) &= [V(D', \alpha') - V(D^*, \alpha')] \\ &\quad + [V(D^*, \alpha') - V(D^*, \alpha_0)] > 0, \end{aligned}$$

we see that for $\Delta\alpha$ sufficiently small and positive, the optimal debt must increase, and $\frac{\partial D^*}{\partial \alpha} > 0$ is equivalent to the condition $\frac{\partial^2 V}{\partial D \partial \alpha} > 0$.

Thus using this equivalence result we derive $V_{\alpha D}$ explicitly and evaluate at the relevant combinations of parameters considered in this paper.

10.2 Proof of Theorem 2

First take the expectation of $G(D^*, \theta)$ to obtain

$$E[G(D^*, \theta)] = (1 - \eta)RD^* - \eta\bar{Y}.$$

Then

$$\frac{\partial}{\partial \alpha} E[G(D^*, \theta)] = (1 - \eta)R \frac{\partial D^*}{\partial \alpha} > 0$$

follows from the previous result that $\frac{\partial D^*}{\partial \alpha} > 0$. The variance of $G(D, \theta)$ is given by

$$\begin{aligned} \text{Var}[G(D^*, \theta)] &= \eta^2 \text{Var}[\theta] \\ &= \frac{\eta^2}{3} \theta_m^2 \\ &= \frac{\eta^2}{3} [s_1 Q R_m - \frac{D^* \alpha}{R}]^2. \end{aligned}$$

Differentiating this with respect to α , we obtain

$$\frac{\partial}{\partial \alpha} \text{Var}[G(D^*, \theta)] = -\frac{2\eta^2}{3R} [s_1 Q R_m - \frac{D^* \alpha}{R}] [D^* + \alpha \frac{\partial D^*}{\partial \alpha}] < 0$$

from the result that $\frac{\partial D^}{\partial \alpha} > 0$ and the requirement $s_1 Q R_m - \frac{D^* \alpha}{R} > 0$.*