The Jacobian of Solutions to the Conductivity Equation and Problems arising from EIT



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This thesis is dedicated to my family, friends and Vicky for their utmost support, love, and sacrifices .

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Abstract

The first contribution of this thesis is an extension of the work originally done in [12]. We derive a general asymptotic expansion for the solution u_n to the conductivity equation in the presence of perturbative inclusions of very conductive or very insulating material. We study the perturbative regime, that is the case when this solution u_n converges to a limiting "background" potential u_0, in which, the convergence may be described at first order volume expansion by a limiting Radon measure μ and a polarisation tensor $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu\right)$ which describes the variations of ∇u_n when a material is present in the conducting domain. The convergence result here is general for any inclusion set Q_n which is Lebesgue measurable and has Lebesgue measure $|Q_n|$ converging to 0. From a numerical point of view, an asymptotic formula in terms of volume expansion is useful as it may be used to tackle a problem posed on very "small" sets which may be difficult to simulate by traditional finite elements methods due to the small mesh sizes.

The second focus of this thesis is on the behaviour of the solution to the conductivity system in $\mathbb{R}^d$ $(d \geq 3)$, where the solution satisfies a certain Jacobian determinant constraint. Namely, we demonstrate the failure of univalent $\gamma-$harmonic mappings in dimension $d \geq 3$ by constructing a class of materials Q , and boundary conditions $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ such that the Jacobian determinant of the $\gamma-$harmonic extension of ϕ in Ω changes sign in some open neighbourhoods of Ω. Our focus here is on quantifying the measure of the sets where the Jacobian determinant is positive and simillarly negative. We provide a general rule of thumb to obtain these estimates by a constructive approach.

The results obtained in this thesis would find itself useful in several problems in the field of elliptic inverse problems, homogenization and electrical impedance tomography. In particular, the sign changing Jacobian determinant property is applicable to the mathematical theory of cloaking whereas the asymptotic formula for the conductivity equation may be applied to a wide array of applications in the detection of inhomogeneity sets of small volume fraction such as cancer cells detection and mine detection for example.

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Chapter 1

Prologue

The study of inverse problems in mathematical and engineering sciences is integral in the recent developments of imaging technology. Finding causes from the knowledge of their effects, which actually constitutes the idea of solving inverse problems, is necessary in determining its applications in a wide array of critical areas such as medical imaging and mine detection. Imaging modalities, such as Magnetic Resonance Imaging (MRI), Ultrasound Imaging (UI), Optical Tomography (OT) and Electrical Impedance Tomography (EIT), are examples of this study that have proven to be successful in various medical applications such as the early stages of cancer cells detection. Much research work has been dedicated to the theoretical aspect and applications of these imaging techniques in the last decade to understand its underlying theory and current drawbacks. In particular, one of the aspects which is desired in the application of these imaging techniques is the ability to reconstruct an image with both high contrast, that is, distinguishing small and large variations in the values of the parameters, and high resolution, that is, a very accurate spatial resolution.

The focus of this thesis is several related questions pertaining to the conductivity equation for composite materials. This model appears very frequently in the aforementioned examples. We study the elliptic partial differential equation

$$\begin{aligned} -\nabla \cdot (\gamma \nabla U) &= 0 \,\text{in}\, \Omega \\ U &= \phi \,\text{on}\, \partial\Omega \end{aligned} \tag{1.1}$$

in a bounded domain Ω of $\mathbb{R}^d$, where γ is a bounded positive matrix valued function which takes the form $\gamma = \begin{cases} \gamma_0 & x \in \Omega \setminus \overline{Q} \\ \gamma_1 & x \in Q \end{cases}$ for some material Q lying within the domain . This equation, alongside with its variants such

as the Helmholtz's equation and Maxwell's equation, are core topics in the mathematical analysis of inverse problems. The conductivity equation models electrical current flowing through a conducting domain when two mediums of contrasting coefficients are present. For example, Ω may be taken to model the skin and Q may be taken to represent damaged tissue cells. In this scenario, the damaged tissue cells possess a different current resistivity from the healthy tissue surrounding it. The electrical impedance tomography problem for the conductivity equation seeks to reconstruct the electrical conductivity parameter γ by performing voltage-current measurements on the boundary $\partial\Omega$. In other words, given Dirichlet boundary condition $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ in which U solves (1.1), and measurements $\Phi = \big(\phi, \Lambda_\gamma(\phi)\big) \in X := H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ where

$$\Lambda_\gamma(\phi) := \gamma \nabla U \cdot \nu_\Omega,$$

the problem is then to determine γ by studying the inverse of the mapping

$$\gamma \mapsto \Lambda_\gamma.$$

When γ_0 and γ_1 are isotropic conductivity matrices, meaning they take the form $\gamma = \gamma I_d$, the electrical impedance tomography inverse problem leads to the well known Calderon's problem. The general impedance tomography problem considers matrix-valued conductivities corresponding to anisotropic mediums. In this generality, the mapping Λ_γ is not necessarily injective. Instead of asking for a full reconstruction of the conductivity parameter γ for this particular application of detecting inhomogeneity sets, we may instead ask for some partial information, in particular :

Problem 1. Can we determine the geometry, size and location of the material Q from performing certain measurements of the Dirichlet to Neumann map Λ_γ?

This problem is challenging in general due to the ill-posedness of the mathematical formulation in Calderon's problem. However, when Q possesses a small volume fraction, it is known that the problem becomes much more approachable by employing an asymptotic expansion method see Cedio-Fengya et al. [14], Friedman and Vogelius [22] and Capdeboscq and Vogelius [11].

In the first part of this thesis, we address this problem by developing an asymptotic formula for U in terms of a contrast-volume expansion of the material Q . We treat the solution U to (1.1) as a perturbation of its background potential U_0 (here U_0 is defined as the solution to (1.1) without the presence of Q, i.e, $Q = \emptyset$). The asymptotic formula that we derive here is

valid for any inclusion set Q, which is Lebesgue measurable and possesses a small volume fraction. Moreover, it is valid even when

$$\operatorname{esssup}(\gamma) \to \infty \quad \text{and} \quad \operatorname{essinf}(\gamma) \to 0$$

simultaneously under some suitable situations. Generally speaking, the solution U may not always converge to U_0 as the ellipticity bounds degenerate to infinity or zero, see example Dapogny and Vogelius [18] and Charnley and Vogelius [15] for instance. We study the perturbative regime in which U converges to U_0 in the presence of conductivities either non uniformly bounded from above, or non uniformly coercive, or both. We thus extend the validity of the asymptotic expansion derived in Capdeboscq and Vogelius [12] to include such inclusions.

> This work will be the subject of a forthcoming publication in collaboration with Yves Capdeboscq.

In the third chapter of this thesis, we study the elliptic system of the model equation

$$\begin{aligned} -\nabla \cdot (\gamma \nabla U) &= 0 \,\text{in}\, \Omega \\ U &= \phi \,\text{on}\, \partial\Omega \end{aligned} \tag{1.2}$$

that is, when $U = (u_1, \cdots, u_d)$ is now a vectorial solution for some prescribed boundary condition $\phi = (\phi_1, \cdots, \phi_d)$. We focus on the case when each u_i solve the scalar PDE 1.1 with prescribed boundary data ϕ_i and subsequently study the Jacobian determinant of DU. Motivated by the failure of the Radó-Choquet-Kneser (RCK) Theorem and its extensions in dimensions $d \geq 3$, we study the conditions imposed on γ , and ϕ so that a Jacobian constraint would be satisfied by $\det DU$. The goal here is to quantify the size of the set where the Jacobian determinant of DU is both positive and negative. If we are able to impose conditions such that the size of the set where the Jacobian determinant is negative (and positive) is comparable to the volume of Ω, or even understand the variations of the volume of this sign changing set when we perturb γ, it would then be applicable to a variety of applications in the field of hybrid inverse problems such as cloaking and numerical homogenisation. To give a precise statement of what we are investigating in this chapter

Problem 2. Given suitable boundary conditions ϕ to the PDE (1.1), is it possible to construct a class of topologically simple structures Q such that the Jacobian determinant of the solution $\det DU$, corresponding to a two phase conductivity coefficient matrix changes sign in some open neighbourhoods of Ω? If so, could we provide a quantitative estimate to the

Lebesgue measures of these sets?

Hybrid inverse problems relies on internal measurements or some functional formulation of the physical quantity to reconstruct the unknown parameters of the governing PDE rather than performing boundary measurements, in contrast with the mathematical formulation of Calderon's problem. In this context, a non-vanishing Jacobian allows us to construct the parameters of the PDE from the knowledge of its solution and sign changing Jacobian determinants typically render such approach unsuccessful. The same issues also arise in numerical homogenisation, where Cordes' conditions on the coefficients are often imposed to avoid sign changes in the Jacobian.

We exhibit a class of inclusions Q and boundary conditions ϕ such that the Jacobian determinant of the solution of the boundary value problem satisfies a sign-changing constraint. We show that it is possible to provide lower bounds for the measure of the sets where the Jacobian determinant is greater than a positive constant τ_0 (or lower than a negative constant) and prove quantitative estimates for this sign-changing set using the so-called $\alpha-$ admissible boundary conditions. Establishing bounds for $(\det DU)^{-1}(-\infty,-\tau_0)$ and $(\det DU)^{-1}(\tau_0,\infty)$ is the first step towards an optimisation scheme aimed at constructing topologically simple structures in which the sign of the Jacobian can be prescribed in most of the domain.

We provide some relevant sets of literature review in this section alongside the announcement of our main result in this thesis. A full and detailed illustration of the results may be found in Chapter 2 and 3 respectively although most of the content in Chapter 3 may also be found in the following preprint which is currently in the process of being reviewed

- Yves Capdeboscq, Shaun Chen Yang Ong. Quantitative Jacobian Bounds For The Conductivity Equation In High Contrast Composite Media. 2019. <hal-02168286>.

The summaries of the following chapters are as follows.

1.0.1 The asymptotic formula for the conductivity equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain in $\mathbb{R}^d$ where $d = 2$ or 3. Let $\gamma_0 \in W^{1,p}\left(\Omega;\mathbb{R}^{d\times d}\right) \cap W^{2,2}_{loc}\left(\Omega;\mathbb{R}^{d\times d}\right)$ for some $p > d$ and $\gamma_n \in L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)$ be symmetric matrix-valued functions which satisfies the ellipticity bounds

$$\lambda_0 |\zeta|^2 \leq \gamma_0 \zeta\cdot\zeta \leq \Lambda_0 |\zeta|^2 \qquad \text{and} \qquad \lambda_n |\zeta|^2 \leq \gamma_n \zeta\cdot\zeta \leq \Lambda_n |\zeta|^2, \quad (1.3)$$

for all $\zeta \in \mathbb{R}^d$, and where $0 < \lambda_n < \Lambda_n$ for each $n \in \mathbb{N}$. The $d \times d$ matrix valued functions γ_0 and γ_n here represent the conductivity contrast of the domain Ω.

Consider a conducting object $Q_n \cup R_n$ placed within Ω . Here $Q_n \cup R_n$ denotes any Lebesgue measurable subset which is compactly contained within Ω. Furthermore, let u_n and u_0 denote the family of solutions to the Dirichlet boundary value problems

$$\begin{aligned} -\nabla \cdot \left(\gamma_n \nabla u_n\right) &= 0 \quad \text{in} \quad \Omega \\ u_n &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

$$\begin{aligned} -\nabla \cdot \left(\gamma_0 \nabla u_0\right) &= 0 \quad \text{in} \quad \Omega \\ u_0 &= g \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.5}$$

for some $g \in H^{\frac{1}{2}}(\partial\Omega)$ where $\gamma_n = \gamma_0$ a.e. in $\Omega \setminus (R_n \cup Q_n)$. We are interested in deriving an asymptotic formula for u_n where the convergence is with respect to the background potential u_0. Adapting the method presented in Capdeboscq and Vogelius [12], we say that the inclusion sets Q_n and R_n are not intertwined if for each n, there exists a function $\chi_n \in W^{1,\infty}(\Omega)$ such that

$$\begin{gathered} 0 \leq \chi_n \leq 1, \\ Q_n \subset \chi_n^{-1}(\{1\}), R_n \subset \chi_n^{-1}(\{0\}) \\ \|\nabla \chi_n\| \leq C(\Omega), \end{gathered} \tag{1.6}$$

for some constant C independent of n. Furthermore, we say that the inclusion sets Q_n and R_n are mixed if this property does not hold. In particular, if there exists a $C^{1,\alpha}$ open set Ω_B such that

$$\overline{\bigcup_n Q_n} \Subset \Omega_B \text{ and } \overline{\Omega_B} \cap \overline{\bigcup_n R_n} = \emptyset,$$

then $(Q_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ are not intertwined.

We assume that the following statements hold true for our inclusions Q_n and R_n.

Assumption 3. *Given $(Q_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$, two sequences of measurable subsets of Ω whose Lebesgue measures tend to 0, the following hold.*

1. *The inclusions are well within the domain. There exists K a compact subset of Ω with C^∞ boundary, that is, $d(\partial K, \partial\Omega) > 0$, which contains*

both sequences. In other words,

$$\bigcup_{n\in\mathbb{N}} Q_n \cup R_n \subset K$$

2. *The inclusions are not low contrast. There exists* $\eta \in (0,1)$ *such that for each* $n \in \mathbb{N}\setminus\{0\}$, $Q_n = \left\{x \text{ in } \Omega : \gamma_n > \frac{1}{1-\sqrt{\eta}}\gamma_0\right\}$, *and* $R_n = \left\{x \text{ in } \Omega : \gamma_n < \frac{1}{1+\sqrt{\eta}}\gamma_0\right\}$, *the inequalities holding in the sense of quadratic forms. There holds*

$$\gamma_n = \gamma_0 \text{ a.e. in } \Omega\setminus(R_n\cup Q_n),$$

$$\frac{1}{1-\sqrt{\eta}}\gamma_0 < \gamma_n \text{ a.e. in } Q_n, \text{ and } \gamma_n < \frac{1}{1+\sqrt{\eta}}\gamma_0 \text{ a.e. in } R_n.$$

3. *We consider the perturbative regime, that is*

$$\lim_{n\to\infty} \left\|(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\right\|_{L^1(\Omega)} = 0.$$

4. *In addition, one of the following conditions hold*
   (a) *When the inclusions* Q_n *and* R_n *are not intertwined and the dimension of the ambient space is larger than 3, we assume additionaly that there exists* $p > \frac{d}{2}$ *such that*

$$\limsup_{n\to\infty} \left\|(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\right\|_{L^p(Q_n)} < \infty,$$

   (b) *When the inclusions* Q_n *and* R_n *are mixed , we assume that there exists* $p > 2d$ *such that*

$$\limsup_{n\to\infty} \left\|(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\right\|_{L^p(Q_n)} < \infty.$$

Remark 4. Note that for a matrix $A \in L^1\left(\Omega;\mathbb{R}^{d\times d}\right)$, we take

$$\|A\|_{L^p(\Omega)} := \||A|_F\|_{L^p(\Omega)}$$

where $|A|_F$ denotes the Frobenius norm of the matrix A , i.e $|A|_F := \sqrt{\sum_{i=1}^d \lambda_i^2}$ with λ_i being the i-th eigenvalue of A.

The assumption that $\left\|(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\right\|_{L^1(\Omega)} \to 0$ is assumed throughout this work. Note that this assumption is indeed a sufficient one for the convergence of $u_n \to u_0$ to take place in $L^2(\Omega)$ as we shall see from the improved L^2 bound 2.26 derived in the following chapter. Generally speaking, the solution u_n does not always converge to its background potential u_0 when this assumption is not met as seen in 16. However, there is a known exception in dimensions $d = 2$ or 3 in the case of diametrically small inhomogeneities as the authors in Nguyen and Vogelius [34] proved that the

convergence of $u_n \to u_0$ still holds valid uniformly whenever $\gamma_n \to \infty$ or $\gamma_n \to 0$. We characterise the convergence of u_n to its background potential u_0 by means of an integral convergence. This generalises the observation in [12], that a perturbative regime depending on the product of contrast and volume exist for the convergence of u_n to u_0 to take place. Notably, when the conductivity matrix is isotropic and piecewise scalar (i.e γ_n and γ_0 are two postive real numbers and $\gamma_n = \gamma_0$ outside $Q_n \cup R_n$), the L^1 norm of $(\gamma_n - \gamma_0)\,\gamma_n^{-1}\,(\gamma_n - \gamma_0)$ satisfies

$$\left\|(\gamma_n - \gamma_0)\,\gamma_n^{-1}\,(\gamma_n - \gamma_0)\right\|_{L^1(\Omega)} \approx \max\left\{(\text{contrast}), \frac{1}{(\text{contrast})}\right\} \times (\text{volume of } Q).$$

With the aid of the Green's function $G(x,y)$ for the operator $\nabla \cdot (\gamma_0 \nabla \cdot)$ in the domain Ω, our main theorem in Chapter 2 is the following,

Theorem 5. *Let u_n and u_0 be the solutions to* (2.2) *and* (2.1) *respectively. Let d_n be given by* (2.3)*. Suppose that Assumption 3 holds. Then, there exists a subsequence also denoted by d_n and a matrix valued function $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; d\mu\right)$, where μ is the Radon measure identified as the weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F$, such that for any compact set $K_1 \Subset \overline{\Omega}$ with $dist(K_1, K) > 0$, $y \in K_1$ and $x \in K$, the following representation formula holds*

$$\left(u_n - u_0\right)(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij} \frac{\partial u_0}{\partial x_i} \frac{\partial G(x,y)}{\partial x_j} d\mu(x) + Err_n(y)$$

in which, the function $Err_n(y) \in L^\infty\left(K_1\right)$ satisfies $\frac{\|Err_n\|_{L^\infty(K_1)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ uniformly in the unit ball $\left\{ g \in H^{\frac{1}{2}}(\partial\Omega) : \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1 \right\}$.

This asymptotic formula is an extension of the one originally derived in Capdeboscq and Vogelius [12, 13]. The situation considered there corresponds to the case when $\int_\Omega |d_n|_F \,\mathrm{d}x \to 0$ and d_n bounded almost everywhere by a bound independent of n. This implies in particular, by interpolation, that for any $p \in [1, \infty)$

$$\lim_{n\to\infty} \left(\int_\Omega |d_n|_F^p \,\mathrm{d}x \right)^{\frac{1}{p}} = 0.$$

We do not assume here that d_n is bounded uniformly for all n. In fact, we treat the case when the ellipticity bounds from (1.3) satisfies

$$\lambda_n \to 0 \quad \text{and} \quad \Lambda_n \to \infty$$

simultaneously as $n \to \infty$ by allowing the convergence of the term d_n to take place in $L^p(\Omega)$. The main difference in the argument of this theorem when

compared with the finite contrast case in Capdeboscq and Vogelius [12, 13] is the fact that Meyer's theorem fails to give us a satisfactory estimate for the gradient field $\nabla(u_n - u)$. In addition, the De–Giorgi–Nash–Moser's type of estimate also fails to describe the convergence of $\|u_n - u_0\|_{L^\infty(\Omega)}$ in the extreme conductivity regime. Both estimates come at the price of the ellipticity ratio $\frac{\Lambda_n}{\lambda_n}$ which is too costly for our analysis. Moreover, it does not capture the optimal rate of convergence. Due to this obstruction, a different approach is needed to address this issue.

This representation formula generalises and unifies the specific formulas already derived for a finite set of inhomogeneities of small diameter and for a finite set of inhomogeneities of small thickness. For some particular geometries, namely for diametrically small domains, it has been shown that this representation formula is valid uniformly and independently of the conductivity contrasts Nguyen and Vogelius [34]. On the other hand, for thin-closed shells an alternative expansion has been recently derived in Dapogny and Vogelius [18] and Charnley and Vogelius [15], which is also valid uniformly as the conductivity contrast approaches the extreme cases. Heuristically, this asymptotic formula shows that from a distance at first order, there is a separation of scales between the macroscopic behaviour of u_n and the small volume fraction perturbation presented through the matrix M i.e

$$\|d_n\|^{-1}_{L^1(\Omega)} ||\chi_{Q_n \cup R_n}(u_n - u_0)||^2_{L^2(\Omega)} \to 0$$

while,

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0) \nabla u_n \, \mathrm{d}x \to M \nabla u_0 \, \mathrm{d}\mu$$

in the weak-* topology of $C\left(\overline{\Omega}\right)$. The matrix valued function $M \in L^2\left(\Omega, \mathrm{d}\mu\right)$ is shown to be independent of the prescribed boundary condition g. Moreover, the probability measure μ generated by $\frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F$ is supported in the compact set K which compactly contains $\cup_n Q_n \cup R_n$. When the sets Q_n and R_n are not intertwined and non-empty, the matrix M may be further decomposed as the difference between two matrices which has disjoint supports. In particular, we may write

$$M \, \mathrm{d}\mu = E \, \mathrm{d}\mu_Q - F \mathrm{d}\mu_R$$

where the Radon measures μ_Q and μ_R are supported in $\cup_n Q_n$ and $\cup_n R_n$ respectively. The matrix-valued functions E and F are symmetric positive definite. Moreover, from the assumption that

$$\frac{1}{1-\sqrt{\eta}} \gamma_0 < \gamma_n \text{ a.e. in } Q_n, \text{ and } \gamma_n < \frac{1}{1+\sqrt{\eta}} \gamma_0 \text{ a.e. in } R_n,$$

for some $\eta \in (0,1)$, it can be shown that they satisfy the polarisation tensor bounds

$$0 \leq \quad \langle E(x)\zeta,\zeta\rangle, \langle F(x)\zeta,\zeta\rangle \quad \leq \left(\frac{1}{\sqrt{\eta}}-1\right)|\zeta|^2,$$

where the inequality holds for $\mu - a.e.x$ in Ω and for any $\zeta \in \mathbb{R}^d$. We present an example at the end of Chapter 2, discussing the optimality of this inequality.

1.0.2 Using composite materials to enforce sign-changing Jacobian determinant constrains in $\gamma-$Harmonic mappings

In the second part of this thesis, we assume that Ω is a bounded convex polygonal domain in $\mathbb{R}^d$ where $d \geq 3$. Given a material Q as before, we assume now that $Q \Subset \Omega$ is a C^2 simply connected inclusion compactly contained in Ω . Furthermore, given prescribed boundary conditions $\phi = (\phi_1, \cdots, \phi_d)$, with $\phi \in C^{1,\sigma}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ for some $\sigma \in (0,1)$, we study the elliptic system variant of our model equation (1.1). We are interested in finding a class of composite materials Q and boundary conditions ϕ such that the solution U corresponding to the elliptic system

$$\begin{aligned} -\text{Div}\,(\gamma DU) &= 0 \quad \text{in} \quad \Omega, \\ U &= \phi \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.7}$$

(i.e for $i \in 1,\ldots,d$, we have u_i satisfying the scalar elliiptic equation (1.1) with boundary condition ϕ_i) where γ takes the form

$$\gamma = (1+(k-1)\,\chi_Q)\,I_d$$

for some large positive scalar $k > 1$, satisfies a sign-changing Jacobian determinant in Ω. In this context, we say that $\det DU$ changes sign in Ω if

$$|\{x \in \Omega : \det DU > 0\}| \times |\{x \in \Omega : \det DU < 0\}| > \delta$$

for some $\delta > 0$ and $k > 1$. Note that since γ is not smooth, therefore the positivity (and negativity) of the Jacobian determinant must be understood in a measure theoretical sense. Given parameters (Q,k,ϕ), our aim here is to quantify lower bounds for the mapping δ

$$\delta : (Q,k,\phi) \mapsto [0,\infty).$$

The study of positive Jacobian determinant constraints which arises in elliptic systems constitutes a major topic in harmonic analysis. Proving pos-

itivity of the Jacobian corresponds to the absence of critical points of DU in Ω. The dimension of the ambient space plays an instrumental role in the analysis of these critical points. When $\gamma = I_d$, the theory of harmonic mappings provides a positive answer to this question through the Radó-Choquet-Kneser (RCK) Theorem in two dimensions. The usage of tools from complex analysis, namely the theory of quasi-conformal mappings proves to be extremely useful in tackling this problem owing to the deep connection between conformal and harmonic maps. The RCK Theorem has been extended by several authors to the so-called $\gamma-$harmonic mappings in Alessandrini and Nesi [3], Alessandrini and Magnanini [1], that is, when the identity matrix is replaced by any general symmetric positive definite matrix $\gamma \in L^{\infty}\left(\Omega;\mathbb{R}^{d\times d}\right)$ and ϕ is assumed to be any homeomorphism such that $\phi : \partial\Omega \to \Gamma$, a closed convex curve of $\mathbb{R}^2$. When γ is more regular, say, γ belongs to the space $C^{0,\sigma}$ for some $0 < \sigma < 1$, then a stronger result holds. In particular, it can be shown that $\det DU$ is bounded below by a positive constant which depends entirely on the a priori bounds of γ in Ω, see Bauman et al. [8].

In contrast, the situation is completely different in higher dimensions ($d \geq 3$) where the theory of univalent $\gamma-$ harmonic mappings fail (i.e the solution U to (1.2) is no longer a homeomorphism of Ω to itself). Counter-examples to the positivity of the Jacobian determinant for harmonic extensions date all the way back to the 90's when several authors including Melas, and Laugensen independently provided counter-examples to the RCK Theorem in dimension three and above. To mention an example, Laugensen provided his counter-example in Laugesen [31] by constructing a boundary homeomorphism of the unit ball in dimension $d = 3$ whose harmonic extension fails to be injective that is $\exists f = (f_1, f_2, f_3) \in C^1(\partial B)$, such that its harmonic extension F satisfies

$$\det DF > 0 \,\text{in}\, B^+ \quad , \quad \det DF < 0 \,\text{in}\, B^-$$

for some open balls B^+ and B^-. Laugensen's result plays an important role in showing the failure of univalent γ-harmonic mappings for dimensions $d > 2$. By using a classical change of variables argument for PDEs in divergence form, Laugensen's result may be used to construct examples of conductivity matrices $\gamma \in C^{\infty}\left(\Omega;\mathbb{R}^{3\times 3}\right)$ such that the $\gamma-$harmonic extension of the identity map

$$\begin{aligned} -\nabla\cdot\left(\gamma DU\right) &= 0 \quad \text{in} \quad B(0,1) \\ U &= I_d \quad \text{on} \quad \partial B(0,1) \end{aligned}$$

fails to be sense preserving (sign-changing). The fact that the Jacobian

determinant stays positive without further assumptions on the conductivity matrix γ besides the ellipticity and a priori bounds of γ is significant in numerical homogenisation. The observation that this property fails to hold in dimensions greater than 2 poses an obstruction for improved error estimates, hence, naturally one might wonder if the theory of univalent γ- harmonic mappings may be extended to a subclass of conductivity matrices in dimension $d \geq 3$ for a substantially large class of boundary data $\phi \in H^{\frac{1}{2}}\left(\partial\Omega\right)^3$. In particular, one might ask if it is possible under the assumptions of Theorem A in Alessandrini and Nesi [4] that a positive Jacobian determinant constraint is attainable for all isotropic conductivity matrices $\gamma = \gamma I_d$. If we are able to determine a priori choices of boundary data which enforces a strong positive determinant constraint on DU for all two phase isotropic γ, then it would be possible to design efficient numerical algorithms if that were the case. The answer to this problem was shown to be false in the paper Capdeboscq [10] that no choice of boundary data $\phi \in H^{\frac{1}{2}}\left(\partial\Omega\right)^3$ whose harmonic extension satisfies a strong positive determinant constraint could enforce a local positive determinant constraint for all two phase isotropic conductivity matrices.

Notably, consider a subdomain $\Omega' \Subset B(0,1)$, $\rho > 0$, and $x_0 \in \Omega'$ so that $B_\rho(x_0) \subset \Omega'$. Furthermore, consider $U_\phi$to be the solution of the PDE

$$\begin{aligned} -\Delta U_\phi &= 0 \quad \text{in} \quad B_1 \\ U_\phi &= \phi \quad \text{on} \quad \partial B_1. \end{aligned}$$

Denoting the set A to be

$$A(x_0,\lambda,\rho) := \{\phi \in H^{\frac{1}{2}}\left(\partial\Omega\right)^3 : \det DU_\phi > \lambda \left\|\phi\right\|^3_{H^{\frac{1}{2}}\left(\partial\Omega\right)} \quad \forall x \in B_\rho(x_0)\},$$

then the set $A(x_0,\lambda,\rho)$ contains all boundary data whose harmonic extension satisfy the strong positive Jacobian determinant bound in the ball of radius ρ centred at the point x_0. The set $A(x_0,\lambda,\rho)$ is non-empty for λ sufficiently small as the harmonic extension of the identity map $x = (x_1,x_2,x_3)$ is itself. The main result of the paper Capdeboscq [10] states the following.

Theorem 6. *Given $\rho > 0$,$x_0 \in \Omega^{'}$ such that $B_\rho(x_0) \subset \Omega^{'}$, and $\lambda > 0$, then there exist $n > 0$, depending on $\rho,\Omega,\Omega^{'}$ and λ only, a universal constant $\tau > 0$ and two open subsets $B_+$and B_- of $B_\rho(x_0)$ such that*

$$|B_+| > \tau|B_\rho(x_0)| \quad \textit{and} \quad |B_-| > \tau|B_\rho(x_0)|,$$

and for any $\phi \in A(x_0,\rho,\lambda)$,

$$\det(DU_n)(x) < -\tau\lambda||\phi||^3_{H^{\frac{1}{2}}(\partial\Omega)} \quad \textit{on} \quad B_-$$

and

$$\det(DU_n)(x) > \tau\lambda\,||\phi||^3_{H^{\frac{1}{2}}(\partial\Omega)} \quad \text{on} \quad B_+,$$

where U_n is the $\gamma(n\cdot)$-harmonic extension of ϕ.

The proof of (6) relied on the key result of the paper Briane et al. [9] which showed that for a specific inclusion Q and $\gamma = \big(1+(k-1)\chi_Q\big)I_3$ defined for values in the unit cube $Y = [-1,1]^3$, the periodic corrector matrix $P = D\zeta$ associated with the two phase homogenisation problem which is the solution ζ to

$$\begin{aligned} -\mathrm{div}(\gamma D\zeta) &= 0 \quad \text{in} \quad \mathbb{R}^3 \\ \zeta(y) - y &\in H^1_{\#}(Y), \end{aligned}$$

satisfies

$$\det(P)(y) > 2\tau \quad \text{in} \quad Y_+ \quad \text{and} \quad \det(P)(y) < -2\tau \quad \text{in} \quad Y_-$$

for some open neighbourhoods Y_+ and Y_-. The inclusion Q which was used for the proof in [10] is a slightly adapted version of the one constructed by Briane, Milton and Nesi in [9] which can be seen in the figure below

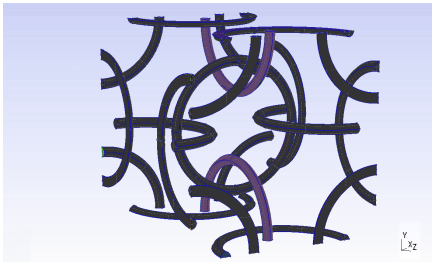


Figure 1.1: Sign changing microstructure

As observed from the above, the Radó-Choquet-Kneser Theorem and its extensions fails badly in three dimensions and more information on γ is needed to determine the sign changing phenomenon for the Jacobian determinant of U. Whilst a specific sign-changing microstructure was used in constructing the counterexample above, not many similar sign-changing microstructure have been exhibited so far and only a few examples are known in the literature. In particular, while it is known that some structures are sign changing, that property is not quantified, that is, the proportion of negative and positive Jacobian volumes are not known. The ultimate goal of this study is to obtain a topologically "simple" and "balanced" structure which we could provide quantitative bounds for the volume of the set where the Jacobian determinant is negative and similarly positive. The search for a topologically "simple" structure which exhibits this sign changing property is twofold, firstly, it is numerically cheaper to simulate a topologically

simple structure and observe its properties and secondly, a simple structure would render optimisation algorithms possible. It is unknown to us at this point if such a study has been done before and we hope that the results in this thesis would lead the way to a more systematic study of this phenomenon. The main result of our study here is that under the following assumptions imposed on the inclusion set Q, we are able to obtain quantifiable lower bounds for the parameter $\delta = \delta(Q,k,\phi)$ using the so-called $\alpha-$admissible boundary condition.

Assumption 7. *We assume that*

- *For any $x = (x_1,\dots,x_d) \in \Omega$ (resp. Q), $(\pm x_1,\dots,\pm x_d) \in \Omega$ (resp. Q).*
- *The inclusion $Q \subset \Omega$ is closed, its interior of is non empty, and Q connected.*
- *The (hyper)surface ∂Q is C^2 and orientable,*
- *The sets $\Omega \setminus Q$ and $(\Omega \setminus Q) \cap (0,\infty)^d$ are connected.*
- *For $i \in \{1,\dots,d\}$, $Q \cap (\mathbb{R}e_i) \neq \emptyset$.*
- *The origin isn't in Q, thus $d(0,Q) > 0$.*

An example of such a structure is the following

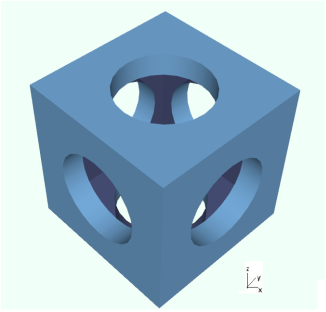


Figure 1.2: Sign changing structure

Definition 8. Given $\alpha > 1$, we say that $\phi \in C^{1,\sigma}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ for some $\sigma \in (0,1)$ is an α-admissible boundary condition if

1. For all $x \in \partial\Omega$, and every $i \in \{1,\dots,d\}$,

$$e_i \cdot \phi(x) = -e_i \cdot \phi(x - 2x \cdot e_i) \text{ and } e_i \cdot \phi(x) = e_i \cdot \phi\left(x - 2x \cdot e_j\right) \text{ for any } j \neq i. \tag{1.8}$$

2. For all $x \in \partial\Omega$, and every $i \in \{1,\dots,d\}$,

$$\alpha^{-1} \|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)} e_i \cdot x \leq e_i \cdot \phi(x) \leq \alpha \|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)} e_i \cdot x.$$

Roughly speaking, the $\alpha-$ admissible boundary condition can been seen as a perturbation of the identity map which preserves cubic symmetry. For illustration purposes, the figure below shows that any function ϕ in the unit ball of the space $C^{1,\sigma}\left(\Omega;\mathbb{R}^d\right)$, with each component ϕ_i lying within the region bounded by $\alpha \cdot x_i$ and $\frac{1}{\alpha} \cdot x_i$ is an $\alpha-$admissible boundary condition if they satisfy the condition (1.8).

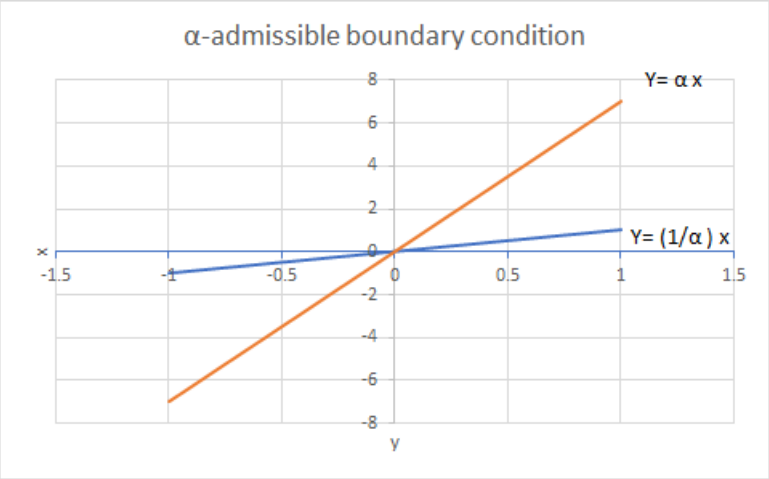


Figure 1.3: $C^{1,\sigma}$ perturbation of the identity

The main result of chapter 3 is the following theorem.

Theorem 9. *Suppose that assumption 7 holds. Then there exists two positive geometrical constants τ and C depending on Ω and Q only such that for any $\alpha > 0$, any $k \geq C \cdot \max\left\{1, \frac{1}{\alpha^d \tau}\right\}$ and any α-admissible $\phi \in C^{1,\sigma}\left(\mathbb{R}^d\right)$ the solution of (1.1) satisfies*

$$B\left(0, \left(\frac{\tau\alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(\tau\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)}, \infty\right)$$

and

$$\cup_{i=1}^{2d} B\left(p_i, \left(\frac{\tau\alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(-\infty, -\tau\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)}\right),$$

where the points p_i, $i \in \{1, \ldots, d\}$ may be explicitly given.

For a more detailed version of this result, we direct the reader to see Proposition 82. The main idea behind this theorem can be summarised as follows. The first step is to observe that constructing an inclusion which satisfies the sign changing Jacobian determinant constraint can be done by observing the limiting properties of the sequence of solutions U_k to (1.7) as $k \to \infty$.

1. The sequence of solutions U_k for $0 < k < \infty$ converges to a limiting solution U_∞ which is the solution to the perfect conductivity equation

$$\begin{aligned} -\Delta U_\infty &= 0 \text{ in } \Omega \setminus Q, \\ U_\infty &= \phi \text{ on } \partial\Omega, \\ U_\infty &= \text{constant in } Q. \end{aligned} \tag{1.9}$$

2. By imposing our assumptions on Q and the boundary condition ϕ, we utilise the symmetry of the solution U_k to deduce that this constant must be 0. Furthermore, using the method introduced in Briane et al. [9], we are able to show that the Jacobian determinant of DU_∞ (and similarly DU_k for all finite k) is diagonalisable along all (hyper)lines which enables us to study the sign-changing phenomenon. The sign-changing property of the Jacobian determinant follows from Hopf's Lemma and the classical maximum principle.

3. Using the quantitative version of Hopf's Lemma which holds for any arbitrary positive harmonic function H in the ball $B(x_0, R)$,

$$\frac{1-|x|}{(1+|x|)^{d-1}} H(x_0) \leq H(x) \text{ for all } x \in B(x_0, R) \text{ and } \lambda \in [0,1)$$

and the α-admissible boundary condition, we construct sub-solutions V with respect to U_∞ to obtain quantifiable estimates for the Jacobian determinant.

4. We proceed to use the Hölder regularity of DU_∞ to obtain quantifiable size estimates for the measure of the sets where $\det DU_\infty > 0$ and $\det DU_\infty < 0$.

5. Finally, we use layer potential estimates to establish our main result by means of an explicit convergence rate of the Jacobian determinant when the contrast k tends to infinity.

The method presented here is very general. The $\alpha-$admissible boundary conditions that we introduced here was done so that we could construct explicit sub-solutions by comparing it with the solution U_∞ when ϕ is given by the identity map. We do not claim that these bounds are optimal and there is certainly room for improvement. Loosely speaking, an "optimal" numerical bound is attainable if we are able to construct explicit sub-solutions V for a domain Ω' which possesses the largest volume fraction, however, this approach completely depends on the geometrical constraints of Q. To clarify the assumptions and results in the theorem above,

- We have assumed in (7) that the material Q touches the (hyper)lines $Q \cap (\mathbb{R}e_i) \neq \emptyset$ at least twice for all principal directions e_i. In fact, we

only need $Q \cap (\mathbb{R}e_i) \neq \emptyset$ for one particular direction to ensure that the Jacobian determinant of DU_∞ changes sign.

- The regularity assumption of Q may be relaxed. In general, as long as Q satisfy the cubic symmetry assumption above and consists only of regular points for the Laplace operator at the points $Q \cap (\mathbb{R}e_i)$, then by elliptic regularity, the continuity of the solution U_∞ is ensured along the (hyper)lines $(\Omega \setminus Q) \cap (0,\infty)^d$ and the same argument applies as well. We have assumed Q to be a C^2 material in this section of the thesis for the sake of utilising the global Hölder regularity estimates given by classical layer potential theory,
- The measure of the sets where the Jacobian determinant is positive or negative is not controlled by the Lebesgue measure of the inclusion Q; in particular it may happen when Q has zero measure. For example, consider the case $d = 3$ where Σ is a two dimensional annulus crack inclusion embedded in $\mathbb{R}^3$ (centred at 0).
- The points p_i in (9) are explicitly given and depends only on the Hölder norm of DU_∞. To obtain a numerical bound from here, it would require us to perform numerical estimations on $\|DU_\infty\|_{C^\sigma(\overline{\Omega \setminus Q})}$.
- The fact that γ is piecewise constant of the form $\gamma = \big(1 + (k-1)\chi_Q\big) I_d$ for $x \in \Omega$ is not required for the sign-changing phenomenon to hold. Using a perturbation argument, we may consider a regularised γ using the standard mollifier $\eta \in C^\infty(\mathbb{R})$, where $\eta_\varepsilon(x) = c_\eta \exp\left(\frac{-1}{1-|x/\varepsilon|^2}\right)$ for $|x| < \varepsilon$ and 0 otherwise, in which, the constant c_η is a normalising constant so that $\|\eta_\varepsilon\|_{L^1\left(\mathbb{R}^d\right)} = 1$. Observing that the regularised conductivity matrix $\tilde{\gamma} = \gamma \star \eta_\varepsilon$ is smooth and yet satisfies assumption 7, we observe the Jacobian determinant of the solution $D\tilde{U}$ with coefficient matrix $\tilde{\gamma}$ would still satisfy the sign changing phenomenon. The main result, however, makes use of layer potential techniques which are ill-suited for variable coefficients.

Chapter 2

On a general representation formula for the solution of elliptic boundary value problems of divergence form in the presence of asymptotically degenerate inclusions

In this chapter, we study the perturbative regime of the solution to the conductivity equation in the presence of defect sets with extreme contrasts. We show that a perturbative regime exists and depends solely on the size and contrast of the defect sets if we allow the sequence of conductivity matrices to approach the degenerate regime under a mild assumption on their L^p convergence. The main result of this chapter is the derivation of a general representation formula for the "far field" expansion of the solution in the presence of such defects with respect to its background solution where no defects are present, extending the results of [12]. The geometric assumptions on the defect sets are simple, notably, we only assume that they are Lebesgue measurable and they are uniformly separated away from the exterior boundary $\partial\Omega$. The asymptotic formula that we derive here describes the convergence of a first order volume expansion with the aid of

1. A limiting Radon measure μ which describes the geometry of the limiting set
2. A background fundamental solution $G(x,y)$
3. An "effective" polarisation tensor $M \in L^2\left(\Omega, \mathrm{d}\mu\right)$ which describes the variations of the field ∇u_n in the presence of inhomogeneity sets.

2.1 The general framework

Let Ω be an open, bounded regular (that is, with a C^2 boundary) domain of $\mathbb{R}^d$ with $d \geq 2$. Given $g \in H^{1/2}(\partial\Omega)$, we study the following family of Dirichlet boundary value problems for the conductivity equation

$$\begin{aligned} -\nabla \cdot (\gamma_0 \nabla u_0) &= 0 \quad \text{in} \quad \Omega, \\ u_0 &= g \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} -\nabla \cdot \left(\gamma_n \nabla u_n\right) &= 0 \quad \text{in} \quad \Omega \\ u_n &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

where the conductivity coefficients are assumed to be symmetric positive definite matrix-valued functions with $\gamma_0 \in W^{1,p}\left(\Omega;\mathbb{R}^{d\times d}\right) \cap W^{2,2}_{loc}\left(\Omega;\mathbb{R}^{d\times d}\right)$ for some $p > d$, $\gamma_n \in L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)$, and they satisfy the ellipticity condition

$$\lambda_0|\zeta|^2 \leq \gamma_0\zeta\cdot\zeta \leq \Lambda_0|\zeta|^2 \quad \text{and} \quad \lambda_n|\zeta|^2 \leq \gamma_n\zeta\cdot\zeta \leq \Lambda_n|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^d,$$

with $0 < \lambda_n < \Lambda_n$ for all $n \in \mathbb{N}$.

For $f \in L^p(\Omega)$, $1 \leq p \leq \infty$, $\|f\|_{L^p(\Omega)}$ is the canonical $L^P(\Omega)$ norm. For $U \in L^p\left(\Omega;\mathbb{R}^d\right)$, we use the notation $\|U\|_{L^p(\Omega)} = \||U|_d\|_{L^p(\Omega)}$ where $|\cdot|_d$ denotes the Euclidean norm in $\mathbb{R}^d$. For $A \in L^p\left(\Omega;\mathbb{R}^{d\times d}\right)$, $\|A\|_{L^p(\Omega)}$ means $\||A|_F\|_{L^p(\Omega)}$ where $|\cdot|_F$ is the Frobenius norm (i.e $|\cdot|_F = \sqrt{\sum_{i=1}^d \lambda_i^2(\cdot)}$) and λ_i denotes the i-th eigenvalue in increasing order. We remind the reader that $|AU|_d \leq |A|_F\,|U|_d$ a.e. in Ω, even though the Frobenius norm is not the subordinate matrix norm associated with the Euclidean distance in $\mathbb{R}^d$.

The key quantity in this chapter will be $d_n \in L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)$ a positive semi-definite matrix valued function given by

$$d_n = (\gamma_n - \gamma_0)\,\gamma_n^{-1}\,(\gamma_n - \gamma_0) \text{ a.e. in } \Omega \tag{2.3}$$

which is a scaled difference between the background conductivity and the inclusion conductivity.

We are interested in studying the perturbation regime of u_n with respect to its background solution u_0, in the presence of small inhomogeneity sets. In particular, given two measurable subsets of $\mathbb{R}^d$, Q_n and R_n whose Lebesgue measures tend to 0, we assume that $\gamma_n = \gamma_0$ a.e.x in $\Omega \setminus (Q_n \cup R_n)$ and subsequently study the perturbation regime given that γ_n differs from γ_0 by a "small" perturbation . Before underlying the main assumptions of

this chapter, we introduce the following definition regarding the seperation of the sets Q_n and R_n.

Definition 10. We say that the inclusion sets Q_n and R_n are not intertwined if for each n, there exists a function $\chi_n \in W^{1,\infty}(\Omega)$ such that

$$\begin{aligned} 0 \leq \chi_n \leq 1, \\ Q_n \subset \chi_n^{-1}(\{1\}), R_n \subset \chi_n^{-1}(\{0\}) \\ \|\nabla \chi_n\| \leq C(\Omega), \end{aligned} \tag{2.4}$$

for some constant C independent of n. Furthermore, we say that the inclusion sets Q_n and R_n are mixed if this property does not hold.

Remark 11. In particular, if there exists a $C^{1,\alpha}$ open set Ω_B such that

$$\overline{\bigcup_n Q_n} \Subset \Omega_B \text{ and } \overline{\Omega_B} \cap \overline{\bigcup_n R_n} = \emptyset,$$

then $(Q_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ are not intertwined.

We make the following assumptions on the inclusion sets.

Assumption 12. *Given $(Q_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$, two sequences of measurable subsets of Ω whose Lebesgue measures tend to 0, the following hold.*

1. *The inclusions are well within the domain. There exists K a compact subset of Ω with C^∞ boundary, that is, $d(\partial K, \partial\Omega) > 0$, which contains both sequences. In other words,*

$$\bigcup_{n\in\mathbb{N}} Q_n \cup R_n \subset K$$

2. *The inclusions are not low contrast. There exists $\eta \in (0,1)$ such that for each $n \in \mathbb{N}\setminus\{0\}$, $Q_n = \left\{x \text{ in } \Omega : \gamma_n > \frac{1}{1-\sqrt{\eta}}\gamma_0\right\}$, and $R_n = \left\{x \text{ in } \Omega : \gamma_n < \frac{1}{1+\sqrt{\eta}}\gamma_0\right\}$, the inequalities holding in the sense of quadratic forms. There holds*

$$\gamma_n = \gamma_0 \text{ a.e. in } \Omega \setminus (R_n \cup Q_n),$$

$$\frac{1}{1-\sqrt{\eta}}\gamma_0 < \gamma_n \text{ a.e. in } Q_n, \text{ and } \gamma_n < \frac{1}{1+\sqrt{\eta}}\gamma_0 \text{ a.e. in } R_n. \tag{2.5}$$

In particular[1]*,*

$$d_n > \eta\gamma_n \text{ on } Q_n \cup R_n. \tag{2.6}$$

[1]The condition 2 imply that on Q_n, $\gamma_n^{-1}\gamma_0 < (1-\sqrt{\eta})I_d$, therefore

$$d_n = \gamma_n\left(I_d - \gamma_n^{-1}\gamma_0\right)^2 > \eta\gamma_n \text{ a.e. on } Q_n.$$

3. *We assume that*

$$\lim_{n\to\infty} \|d_n\|_{L^1(\Omega)} = 0.$$

4. *In addition, we assume that either one of the following cases holds*

   (a) *When the inclusions Q_n and R_n are not intertwined and the dimension of the ambient space is larger than 3, we assume additionaly that there exists $p > \frac{d}{2}$ such that*

$$\limsup_{n\to\infty} \|d_n\|_{L^p(Q_n)} < \infty,$$

   (b) *When the inclusions Q_n and R_n are mixed , we assume that there exists $p > 2d$ such that*

$$\limsup_{n\to\infty} \|d_n\|_{L^p(Q_n)} < \infty.$$

Remark 13. We make a remark here that the condition

$$\lim_{n\to\infty} \|d_n\|_{L^1(\Omega)} = 0$$

is equivalent to the fact that

$$\|\gamma_n - \gamma_0\|_{L^1(Q_n)} \to 0 \quad \text{and} \quad \left\|\gamma_n^{-1} - \gamma_0^{-1}\right\|_{L^1(R_n)} \to 0.$$

Observe that since condition (2.5) entails

$$\begin{aligned} d_n &> \eta\,\gamma_n \\ &> \eta\,(\gamma_n - \gamma_0) \ \text{ on } \ Q_n, \end{aligned}$$

and

$$\begin{aligned} d_n &= (\gamma_n - \gamma_0)\left(I_d - \gamma_n^{-1}\gamma_0\right) \\ &< |\gamma_n - \gamma_0|_F \end{aligned}$$

we have by the triangle inequality

$$|d_n|_F \quad \leq |\gamma_n - \gamma_0|_F \leq \quad \frac{1}{\eta}\,|d_n|_F$$

which shows that $\|\gamma_n - \gamma_0\|_{L^1(Q_n)} \to 0 \Leftrightarrow \|d_n\|_{L^1(\Omega)} \to 0$.

Similarly, on R_n, $\gamma_n^{-1}\gamma_0 > 1 + \sqrt{\eta}$, therefore

$$d_n = \gamma_n\left(\gamma_n^{-1}\gamma_0 - I_d\right)^2 > \eta\,\gamma_n \text{ a.e. on } R_n.$$

Simillarly, we have from (2.5) that

$$\gamma_0^{-1} < (1+\sqrt{\eta})\,\gamma_0^{-1} < \gamma_n^{-1} \text{ on } R_n, \tag{2.7}$$

which implies

$$\begin{aligned}
|d_n|_F &= \left|(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\right|_F \\
&= \left|(\gamma_n-\gamma_0)^2\,\gamma_n^{-1}\right|_F \\
&\leq \left|(\gamma_n-\gamma_0)^2\right|_F \left|\gamma_n^{-1}\right|_F \\
&\leq \left|\gamma_0^2\right|_F \left|\gamma_n^{-1}\right|_F \text{ (since } \gamma_0 > \gamma_n \text{ here)} \\
&\leq \left|\gamma_0^2\right|_F \left(\left|\gamma_n^{-1}-\gamma_0^{-1}\right|_F + \left|\gamma_0^{-1}\right|_F\right) \\
&\leq \left\|\gamma_0^2\right\|_{L^\infty(\Omega)} \left(\left|\gamma_n^{-1}-\gamma_0^{-1}\right|_F + \frac{1}{\sqrt{\eta}}\left|\gamma_n^{-1}-\gamma_0^{-1}\right|_F\right) \text{ by (2.7)} \\
&\leq C\left(\left\|\gamma_0^2\right\|_{L^\infty(\Omega)}, \eta\right)\left|\gamma_n^{-1}-\gamma_0^{-1}\right|_F \text{ on } R_n
\end{aligned}$$

and since

$$(\gamma_0-\gamma_n)\left(\left(\gamma_n^{-1}-\gamma_0^{-1}\right)+\gamma_0^{-1}\right)(\gamma_0-\gamma_n) = d_n,$$

we have again from (2.5) that

$$\left(\frac{\sqrt{\eta}}{1+\sqrt{\eta}}\right)^2 \gamma_0\left(\gamma_n^{-1}-\gamma_0^{-1}\right)\gamma_0 \leq (\gamma_0-\gamma_n)\left(\left(\gamma_n^{-1}-\gamma_0^{-1}\right)\right)(\gamma_0-\gamma_n) \leq d_n$$

in the sense of quadratic forms. Thus,

$$\begin{aligned}
\left|\left(\gamma_n^{-1}-\gamma_0^{-1}\right)\right|_F &\leq \left(\left(\frac{\sqrt{\eta}}{1+\sqrt{\eta}}\right)^2 \operatorname{essinf}(\gamma_0)^2\right)^{-1} |d_n|_F \\
&\leq C\left(\left\|\gamma_0^{-2}\right\|_{L^\infty(\Omega)}, \eta\right)|d_n|_F \text{ on } R_n
\end{aligned}$$

and we deduce that $\left\|\gamma_n^{-1}-\gamma_0^{-1}\right\|_{L^1(R_n)} \to 0 \Leftrightarrow \|d_n\|_{L^1(\Omega)} \to 0$.

Remark 14. The various condition appearing in Assumption 12 are of different natures. Conditions 1, 2 and 3 are natural. Near the boundary of the domain, the behaviour of the solution is different, as the imposed boundary condition plays an increased role. Condition 2 excludes the small amplitude perturbation regime : in such a case, the perturbative expansion of (u_n-u_0) depends linearly $\gamma_n-\gamma_0$, not quadratically as d_n does. It is possible to remove this condition and it will be part of a forthcoming paper; it merely requires to modify the function d_n so that, when condition 2 is not satisfied, $d_n \propto \gamma_n-\gamma_0$. The focus of this chapter being the high contrast regime, the choice was made to exclude low contrast altogether. By this condition, we also impose that $Q_n \cap R_n = \emptyset$: there cannot be an anisotropic inclusion which is very large in one direction and very small in another.

Condition 3 is necessary in general, as example 16 shows, to stay in a perturbative regime.

On the other hand, condition 4 needs more explanation. It imposes condition on d_n only for highly conductive materials, not insulating ones, since when $Q_n = \emptyset$ it is automatically satisfied. In the two dimensional case, no higher integrability is imposed either, provided that the insulating and highly conductive parts are not intertwined. While theses conditins are sufficient, it is not clear if any of these conditions are necessary; these assumptions are, as far as the author is aware, much less restrictive than what is currently known in the litterature.

The main result of this chapter is the following theorem

Theorem 15. *Let u_n and u_0 be the solutions to* (2.2) *and* (2.1) *respectively. Let d_n be given by* (2.3)*. Suppose that Assumption 12 holds. Then, there exists a subsequence also denoted by d_n and a matrix valued function $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; d\mu\right)$, where μ is the Radon measure identified as the weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F$, such that for any compact set $K_1 \Subset \overline{\Omega}$ with $dist(K_1, K) > 0$, $y \in K_1$ and $x \in K$, the following representation formula holds*

$$\left(u_n - u_0\right)(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij} \frac{\partial u_0}{\partial x_i} \frac{\partial G(x,y)}{\partial x_j} d\mu(x) + Err_n(y),$$

in which, the function $Err_n(y) \in L^\infty\left(K_1\right)$ satisfies $\frac{\|Err_n\|_{L^\infty(K_1)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ uniformly in the unit ball $g \in H^{\frac{1}{2}}(\partial\Omega) : \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$.

Note that when the condition

$$\lim_{n\to\infty} \int_\Omega |d_n|_F \, \mathrm{d}x = 0 \tag{2.8}$$

is not met, it may happen that $u_n \not\to u$ for some sequence $(\gamma_n)_{n\in\mathbb{N}}$ as illustrated in the following example.

Example 16. Suppose that $\Omega = B(0,2) \subset \mathbb{R}^d$, choose $Q_n = B\left(0, 1+\frac{1}{n}\right) \setminus B\left(0, 1-\frac{1}{n}\right)$, and $g = x_1$. Then for $\gamma_0 = I_d$, the unperturbed solution of (2.1) corresponds to $u = x_1$. Suppose that γ_n is radial and constant on $(I_i)_{i\leq 1\leq 4}$, where $I_1 = \left(0, 1-\frac{1}{n}\right)$, $I_2 = \left(1-\frac{1}{n}, 1\right)$. $I_3 = \left(1, 1+\frac{1}{n}\right)$, $I_4 = \left(1+\frac{1}{n}, 2\right)$, with values

$$\gamma_n = \chi_{I_1 \cup I_4} + n^\alpha \chi_{I_2} + n^\beta \chi_{I_3},$$

and

$$\int_\Omega |d_n|_F \, \mathrm{d}x = \frac{(n^\alpha - 1)^2}{n^{\alpha+1}} + \frac{\left(n^\beta - 1\right)^2}{n^{\beta+1}}$$

where α, β are real parameters. Then the solution u_n of (2.2) takes the form

$$u_n = \sum_{i=1}^{4} A_i^n x_1 \mathbf{1}_{I_i}(|x|) + |x|^{-d} \sum_{i=2}^{4} B_i^n x_1 \mathbf{1}_{I_i}(|x|),$$

for some constants $(A_i^n)_{1\le i\le 4}$ and $(B_i^n)_{2\le i\le 4}$. As $n\to\infty$, then $u_n \to v$ pointwise where $v = \left(\lim_{n\to\infty} A_1^n\right) x_1$ for $x<1$ and $v = \left(\lim_{n\to\infty} A_4^n\right) x_1 + \left(\lim_{n\to\infty} B_4^n\right) |x|^{-d} x_1$ for $x>1$. Computing the value of the constants, we find that $\left(\lim_{n\to\infty} A_1^n\right) = \left(\lim_{n\to\infty} A_4^n\right) = 1$ and $\left(\lim_{n\to\infty} B_4^n\right) = 0$ if and only if $-1<\alpha<1$ and $-1<\beta<1$. We further note that if we write $\delta = \min(1+\alpha, 1+\beta, 1-\alpha, 1-\beta) > 0$, $u_n - x_1$ is of order $n^{-\delta}$. Subsequently, this computation shows that there exists a positive constant C depending on α,β and d but independent of n such that for all $n\ge 1$ there holds

$$C^{-1}\int_\Omega |d_n|_F \,\mathrm{d}x \le \|u_n - x_1\|_{L^1(\Omega)} \text{ and } \|u_n - x_1\|_{L^\infty(\Omega)} \le C\int_\Omega |d_n|_F \,\mathrm{d}x.$$

2.2 A priori estimates

Given $u_n, u_0 \in H^1(\Omega)$ as the above, we define $w_n = u_n - u_0 \in H_0^1(\Omega)$ as the weak solution of

$$\begin{aligned} -\nabla\cdot(\gamma_n \nabla w_n) &= \nabla\cdot(\gamma_n - \gamma_0)\nabla u_0 \text{ in } \Omega \\ w_n &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.9}$$

or equivalently,

$$\int_\Omega \gamma_n \nabla w_n \cdot \nabla\phi = -\int_\Omega (\gamma_n - \gamma_0)\nabla u_0 \cdot \nabla\phi \text{ for all } \phi \in H_0^1(\Omega). \tag{2.10}$$

Note that if u_0 is the solution of (2.1), then by the H^3_{loc} regularity estimate from Corollary 86 in the appendix, it follows from Sobolev's embedding inequality that $u_0 \in H^1(\Omega)\cap C^1(K)$ and $\|u_0\|_{C^1(K)} \le C(\Omega)\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$.

Lemma 17. *Let $d_n \in L^\infty\left(\Omega; \mathbb{R}^{d\times d}\right)$ denote the symmetric positive semi-definite matrix valued function given by*

$$d_n = (\gamma_n - \gamma_0)\gamma_n^{-1}(\gamma_n - \gamma_0) \text{ a.e. in } \Omega.$$

Then, the sequence $\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}$ converges up to the possible extraction of a subsequence, in the sense of measures to a positive radon measure μ, that is

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F \,\phi\, dx \to \int_\Omega \phi \, d\mu \text{ for all } \phi \in C^0(\overline{\Omega}), \tag{2.11}$$

where $C^0(\overline{\Omega})$ denotes the space of bounded continuous functions on $\overline{\Omega}$.

Proof. The proof is a direct consequence of Banach--Alaoglu's theorem and the continuous embedding between $L^1(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right)^*$, where we have identified the continuous dual space of $C^0\left(\overline{\Omega}\right)$ as the space of bounded Radon measures on Ω. □

Proposition 18. *The weak solution of* (2.10) $w_n \in H_0^1(\Omega)$ *satisfies*

$$E(w_n)^2 := \int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|^2_{L^\infty(K)}. \tag{2.12}$$

As a consequence, there holds

$$\|(\gamma_n - \gamma_0)\nabla w_n\|_{L^1(\Omega)} \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)}. \tag{2.13}$$

Furthermore, up to the possible extraction of a subsequence, $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n - \gamma_0)\nabla w_n$ *converges in the sense of measures to a limit given by*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n - \gamma_0)\nabla w_n \cdot \Psi\, dx \to \int_\Omega M \cdot \Psi\, d\mu \text{ for all } \Psi \in C^0(\overline{\Omega}; \mathbb{R}^d), \tag{2.14}$$

where $M \in \left[L^2(\Omega, d\mu)\right]^d$ *and* μ *is given by* (2.11).

Remark 19. The upper estimates (2.12) and (2.13) are sharp with respect to the order of dependence on $\|d_n\|_{L^1(\Omega)}$ as shown in Example 16.

Proof of Proposition 18. Note that w_n is the unique minimiser over $H_0^1(\Omega)$ of the functional

$$J(w) = \int_\Omega \gamma_n(x)\left(\nabla w + \gamma_n^{-1}(\gamma_n - \gamma_0)\nabla u_0\right) \cdot \left(\nabla w + \gamma_n^{-1}(\gamma_n - \gamma_0)\nabla u_0\right) \mathrm{d}x,$$

Clearly, $J(w_n) \geq 0$, thus

$$-\int_\Omega \gamma_n(x)\nabla w_n \cdot \nabla w_n \,\mathrm{d}x + 2\int_\Omega \gamma_n\left(\nabla w_n + \gamma_n^{-1}(\gamma_n - \gamma_0)\nabla u_0\right) \cdot \nabla w_n + \int_\Omega d_n \nabla u_0 \cdot \nabla u_0 \geq 0,$$

which shows

$$\int_\Omega \gamma_n(x)\nabla w_n \cdot \nabla w_n \,\mathrm{d}x \leq \int_\Omega d_n(x)\nabla u_0 \cdot \nabla u_0 \,\mathrm{d}x. \tag{2.15}$$

Thus, as $u_0 \in C^1(K)$ and d_n has support in K,

$$\int_\Omega \gamma_n(x)\nabla w_n \cdot \nabla w_n \,\mathrm{d}x \leq \left(\int_\Omega |d_n|_F \,\mathrm{d}x\right) \|\nabla u_0\|^2_{L^\infty(K)}. \tag{2.16}$$

We now turn to the second estimate. Using Cauchy–Schwartz and again

$|AU|_d \leq |A|_F \, |U|_d$ we find

$$\begin{aligned}
&\|(\gamma_n - \gamma_0)\nabla w_n\|_{L^1(\Omega)} \qquad (2.17)\\
&= \int_\Omega \left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\gamma_n^{\frac{1}{2}}\nabla w_n\right| \mathrm{d}x\\
&\leq \int_\Omega \left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\right|_F \left|\gamma_n^{\frac{1}{2}}\nabla w_n\right|_d \mathrm{d}x \qquad (2.18)\\
&\leq \left(\int_\Omega \left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\right|_F^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

Note that for any matrix $A \in \mathbb{R}^{d\times d}$, there holds

$$\begin{aligned}
|A|_F^2 &= \sum_{i=1}^d \lambda_i^2(A)\\
&\leq C(d)\left(\sum_{i=1}^d \lambda_i^4(A)\right)^{\frac{1}{2}}\\
&= C(d)\left(\sum_{i=1}^d \lambda_i^2\left(A^2\right)\right)^{\frac{1}{2}}\\
&= C(d)\left|A^2\right|_F,
\end{aligned}$$

where we have used the fact that Euclidean norms are equivalent for finite dimensional spaces in the second line, and the fact that the eigenvalues of A^2 are the squared of the eigenvalues of A in the third line. Furthermore, by using the fact that the eigenvalues of AB are the same as BA including multiplicities for any arbitrary square matrix in $\mathbb{R}^{d\times d}$, we infer that

$$\begin{aligned}
\left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\right|_F^2 &\leq C(d)\left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\right|_F\\
&\leq C(d)\left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\gamma_n^{-\frac{1}{2}}(\gamma_n - \gamma_0)\right|_F\\
&= C(d)\,|d_n|_F,
\end{aligned}$$

and we conclude from (2.16) that

$$\begin{aligned}
\|(\gamma_n - \gamma_0)\nabla w_n\|_{L^1(\Omega)} &\leq \left(\int_\Omega \left|(\gamma_n - \gamma_0)\,\gamma_n^{-\frac{1}{2}}\right|_F^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \mathrm{d}x\right)^{\frac{1}{2}}\\
&\leq C(d)\left(\int_\Omega |d_n|_F \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \mathrm{d}x\right)^{\frac{1}{2}}\\
&\leq C(d)\,\|d_n\|_{L^1(\Omega)}\,\|\nabla u_0\|_{L^\infty(K)}.
\end{aligned}$$

Finally, since $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n - \gamma_0)\nabla w_n$ is uniformly bounded in $L^1(\Omega)$, we

may extract a subsequence in which,

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n \overset{*}{\rightharpoonup} \mathrm{d}\mathscr{M}$$

in the space of bounded vector Radon measures. Moreover, for any $\Psi \in C^0(\overline{\Omega};\mathbb{R}^d)$,

$$\begin{aligned}\int_\Omega \Psi\cdot \mathrm{d}\mathscr{M} &= \lim_{n\to\infty}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n\cdot\Psi \mathrm{d}x\\ &\le \lim_n \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \gamma_n \nabla w_n\cdot\nabla w_n \mathrm{d}x\right)^{\frac12}\left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega d_n\Psi\cdot\Psi \mathrm{d}x\right)^{\frac12}\\ &\le C\left(\int_\Omega |\Psi|^2\,\mathrm{d}\mu\right)^{\frac12}\end{aligned}$$

thanks to the estimate above. As a consequence of this estimate, it follows that the functional

$$\Psi \to \int_\Omega \Psi\cdot \mathrm{d}\mathscr{M}$$

may be extended to a bounded linear functional on $\left[L^2(\Omega,\mathrm{d}\mu)\right]^d$. Hence, by Riesz's Representation Theorem, we may identify

$$\mathrm{d}\mathscr{M} = M\mathrm{d}\mu$$

for some function $M \in \left[L^2(\Omega,\mathrm{d}\mu)\right]^d$, which is our statement. □

Corollary. *There holds*

$$\|\gamma_n\nabla w_n\|_{L^1(Q_n)} \le \frac{1}{\sqrt{\eta}}\|d_n\|_{L^1(\Omega)}\|\nabla u_0\|_{L^\infty(K)}, \tag{2.19}$$

and for any $p>1$, there holds

$$\|\gamma_n\nabla w_n\|_{L^{\frac{2p}{p+1}}(Q_n)} \le \frac{d^{\frac14}}{\sqrt{\eta}}\|d_n\|^{\frac12}_{L^1(Q_n)}\|d_n\|^{\frac12}_{L^p(Q_n)}\|\nabla u_0\|_{L^\infty(K)} \tag{2.20}$$

Proof. Note that from (2.13) we have

$$\left\|\left(I_d-\gamma_0\gamma_n^{-1}\right)\gamma_n\nabla w_n\right\|_{L^1(Q_n)} \le \left\|\left(I_d-\gamma_0\gamma_n^{-1}\right)\gamma_n\nabla w_n\right\|_{L^1(\Omega)} \le \|d_n\|_{L^1(\Omega)}\|\nabla u_0\|_{L^\infty(K)}.$$

Assumption 12 shows that on Q_n

$$\gamma_n > \frac{1}{1-\sqrt{\eta}}\gamma_0 \Leftrightarrow -\gamma_n^{-1} > (\sqrt{\eta}-1)\,\gamma_0^{-1}$$

therefore

$$I_d > I_d-\gamma_0\gamma_n^{-1} > \sqrt{\eta}I_d \text{ a.e. on } Q_n. \tag{2.21}$$

As a result,

$$\begin{aligned}\|\gamma_n \nabla w_n\|_{L^1(Q_n)} &= \left\| \left(I_d - \gamma_0 \gamma_n^{-1}\right)^{-1} \left(I_d - \gamma_0 \gamma_n^{-1}\right) \gamma_n \nabla w_n \right\|_{L^1(Q_n)} \\ &\leq \frac{1}{\sqrt{\eta}} \left\| \left(I_d - \gamma_0 \gamma_n^{-1}\right) \gamma_n \nabla w_n \right\|_{L^1(Q_n)} \\ \text{and } \|\gamma_n \nabla w_n\|_{L^1(Q_n)} &\leq \frac{1}{\sqrt{\eta}} \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)}. \end{aligned} \tag{2.22}$$

On the other hand,

$$|\gamma_n \nabla w_n| \leq \left|\gamma_n^{\frac{1}{2}}\right|_F \left|\gamma_n^{\frac{1}{2}} \nabla w_n\right|,$$

and Hölder's inequality shows that for any $p > 1$

$$\|\gamma_n \nabla u_n\|_{L^{\frac{2p}{p+1}}(Q_n)} \leq \left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(Q_n)} E(w_n)^{\frac{1}{2}} \tag{2.23}$$

We have

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(Q_n)} = \left(\int_{Q_n} \left|\gamma_n^{\frac{1}{2}}\right|_F^{2p} \mathrm{d}x\right)^{\frac{1}{2p}},$$

and, using the fact that for $d \times d$ symmetric matrix A, there holds[2]

$$\left|A^2\right|_F \leq |A|_F^2 \leq \sqrt{d} \left|A^2\right|_F,$$

we find, using (2.6),

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(Q_n)} \leq d^{\frac{1}{4}} \left(\int_{Q_n} |\gamma_n|_F^p \,\mathrm{d}x\right)^{\frac{1}{2p}} = d^{\frac{1}{4}} \|\gamma_n\|_{L^p(Q_n)}^{\frac{1}{2}} \leq \frac{d^{\frac{1}{4}}}{\sqrt{\eta}} \|d_n\|_{L^p(Q_n)}^{\frac{1}{2}}. \tag{2.24}$$

Putting together (2.12), (2.23) and (2.24) the conclusion follows. □

Lemma 20. *Let K' be a compact such that $K \subset K' \subset \Omega$ with $3d(K', \partial\Omega) > d(K, \partial\Omega)$ and $3d(\partial K', \partial K) > d(K, \partial\Omega)$. There exists a constant $C > 0$ depending on Ω, K, K', the a priori bounds Λ_0 and λ_0, η defined in* (2) *and $\|\gamma_0\|_{W^{1,p}(\Omega)}$ only such that*

$$\|w_n\|_{L^\infty(K)} \leq C \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(K')} \right).$$

In particular, there exists $C > 0$ which depends on Ω, K the a priori bounds Λ_0 and λ_0, η defined in (2) *and $\|\gamma_0\|_{W^{1,p}(\Omega)}$ only such that*

$$\|w_n\|_{L^\infty(K)} \leq C \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{2.25}$$

[2]Indeed, since A is real and symmetric, naming $\lambda_1, \ldots, \lambda_d$ its eigenvalues, we have $|A|_F^2 = \lambda_1^2 + \ldots + \lambda_d^2$ and

$$\left|A^2\right|_F = \sqrt{\lambda_1^4 + \ldots + \lambda_d^4} \leq \lambda_1^2 + \ldots + \lambda_d^2 \leq \sqrt{d}\sqrt{\lambda_1^4 + \ldots + \lambda_d^4}.$$

Frobenius norm is the sum of its eigenvalues (as the trace is invariant under change of variable)

Proof. Given Ω', Ω'' two open domains such that $K \subset \Omega' \subset \Omega'' \subset K'$, with $d(K, \partial\Omega') > 0$, $d(\partial\Omega', \partial\Omega'') > 0$ and $d(\partial\Omega'', \partial K') > 0$. In what follows, we use the notation $a \lesssim b$ to mean $a \leq Cb$, where C is a constant, possibly changing from line to line, which may depend on $\Omega', \Omega'', \Omega, K$, Λ_0,λ_0,η and $\|\gamma_0\|_{W^{1,p}(\Omega)}$.

Since

$$-\nabla \cdot (\gamma_0 \nabla w_n) = 0 \quad \text{on} \quad \Omega'' \setminus \Omega'$$

and $\gamma_0 \in W^{1,p}(\Omega)$ with $p > d$, Proposition 84 from the Appendix shows that $w_n \in W^{1,p^*}_{loc}(\Omega)$ where $p^* = \frac{2d}{d-2}$ for $d > 2$, and p^* may be arbitrarily chosen between $\left(2, \infty\right)$ when $d = 2$. Subsequently, we infer from the embedding between $W^{1,p^*} \hookrightarrow C^{0,\alpha}$ that

$$\|w_n\|_{C\left(\overline{\Omega'' \setminus \Omega'}\right)} \lesssim \|w_n\|_{L^2(\Omega \setminus K)}.$$

By Poincaré's inequality on $\Omega \setminus K$, since w_n vanishes on $\partial\Omega$, there holds

$$\|w_n\|_{L^2(\Omega \setminus K)} \lesssim \|\nabla w_n\|_{L^2(\Omega \setminus K)}.$$

On the other hand,

$$\begin{aligned}
\|\nabla w_n\|_{L^2(\Omega \setminus K)} &\lesssim \left(\int_{\Omega \setminus K} \gamma_0 \nabla w_n \cdot \nabla w_n \right)^{\frac{1}{2}} \\
&= \left(\int_{\Omega \setminus K} \gamma_n \nabla w_n \cdot \nabla w_n \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\Omega} \gamma_n \nabla w_n \cdot \nabla w_n \right)^{\frac{1}{2}} \\
&\leq \|d_n\|^{\frac{1}{2}}_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \\
&\lesssim \|\nabla u_0\|_{L^\infty(K)},
\end{aligned}$$

where we used (2.12) for the penultimate inequality and the a priori bound on d_n for the last one. Now, note that $u_n = w_n + u_0$ satisfies

$$\begin{aligned}
-\nabla \cdot (\gamma_n \nabla u_n) &= 0 \text{ on } \Omega'' \\
u_n &= w_n + u_0 \text{ on } \partial\Omega'',
\end{aligned}$$

therefore by the maximum principle $\|u_n\|_{L^\infty(\Omega'')} \leq \|w_n\|_{C(\partial\Omega'')} + \|u_0\|_{C(\partial\Omega'')}$, and in turn, $\|w_n\|_{L^\infty(K)} \leq \|w_n\|_{L^\infty(\Omega'')} = \|u_n - u_0\|_{L^\infty(\Omega'')} \leq \|w_n\|_{C(\partial\Omega'')} + 2\|u_0\|_{L^\infty(K')}$. Note that our inequality depends on the choice of Ω' and Ω'' a priori : we minimise over all such subdomains to remove this dependence. The second estimate follows from the H^3_{loc} regularity estimate for u_0 given in Corollary 86 in the appendix . □

Following the strategy introduced in Capdeboscq and Vogelius [12], we now derive an improved L^2 estimate for the potential w_n. We prove this result by using a variant of the so-called Aubin-Nitsche (or Céa's) duality argument in numerical analysis. This result is new, as it depends on $\|d_n\|_{L^1(\Omega)}$, but not on a priori bound on γ_n.

Proposition 21. *For any $\varepsilon > 0$, there holds*

$$\|w_n\|_{L^2(\Omega)} \leq C \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}\left(1+\frac{1}{d-1+\varepsilon}\right)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2.26}$$

where the constant C may depend on ε, Ω, K, and the a priori bounds Λ_0 and λ_0 , η defined in (2) *and $\|\gamma_0\|_{W^{1,p}(\Omega)}$ only.*

Proof. In what follows, we use the notation $a \lesssim b$ to mean $a \leq Cb$, where C is a constant, possibly changing from line to line, which may depend on ε, Ω, K, Λ_0,λ_0 and $\|\gamma_0\|_{W^{1,p}(\Omega)}$.

Consider the following auxiliary equation

$$\begin{aligned} -\nabla \cdot (\gamma_0 \nabla \psi_n) &= w_n \quad \text{in} \quad \Omega \\ \psi_n &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.27}$$

Since $\gamma_0 \in W^{1,p}\left(\Omega; \mathbb{R}^{d\times d}\right)$ for $p > d$, we infer from elliptic regularity theory (see e.g. Di Fazio [19]) that for any $2 \leq q < \infty$, the solution ψ_n satisfies

$$\|\psi_n\|_{W^{2,q}(\Omega)} + \|\psi_n\|_{W^{1,q}(\Omega)} \lesssim \|w_n\|_{L^q(\Omega)}. \tag{2.28}$$

Testing (2.27) with w_n , and recalling that $\operatorname{supp}(\gamma_n - \gamma_0) \subset (Q_n \cup R_n) \Subset K$, a straightforward computation gives us

$$\begin{aligned} \|w_n\|_{L^2(\Omega)}^2 &= \int_\Omega \gamma_0 \nabla \psi_n \cdot \nabla w_n \, \mathrm{d}x \\ &= \int_\Omega (\gamma_0 - \gamma_n) \nabla (w_n + u_0) \cdot \nabla \psi_n \, \mathrm{d}x \\ &= \int_\Omega (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla \psi_n \, \mathrm{d}x + \int_\Omega (\gamma_0 - \gamma_n) \nabla u_0 \cdot \nabla \psi_n \, \mathrm{d}x \\ &\leq \left(\int_{Q_n \cup R_n} \gamma_n \nabla w_n \cdot \nabla w_n \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega d_n \nabla \psi_n \cdot \nabla \psi_n \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &+ \left(\int_{Q_n \cup R_n} \gamma_n \nabla u_0 \cdot \nabla u_0 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega d_n \nabla \psi_n \cdot \nabla \psi_n \, \mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Now, thanks to (2.12) on one hand, and (2.6) on the other,

$$\left(\int_{Q_n \cup R_n} \gamma_n \nabla w_n \cdot \nabla w_n \, \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{Q_n \cup R_n} \gamma_n \nabla u_0 \cdot \nabla u_0 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq \left(1+\frac{1}{\sqrt{\eta}}\right) \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}} \|\nabla u_0\|_{L^\infty(K)},$$

On the other hand, choosing $q = d+\varepsilon$ in (2.28) there holds

$$\|\nabla \psi_n\|_{L^\infty(\Omega)} \lesssim \|\psi_n\|_{W^{2,d+\varepsilon}(\Omega)} \lesssim \|w_n\|_{L^{d+\varepsilon}(\Omega)} \tag{2.29}$$

and

$$\left(\int_\Omega d_n \nabla \psi_n \cdot \nabla \psi_n \,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}} \|w_n\|_{L^{d+\varepsilon}(\Omega)}$$

By interpolation, and using the a priori bound on u_n given by (2.25) in Lemma 20 above, we find

$$\begin{aligned}\|w_n\|_{L^{d+\varepsilon}(\Omega)} &\leq \|w_n\|_{L^2(\Omega)}^{\frac{2}{d+\varepsilon}} \|w_n\|_{L^\infty(\Omega)}^{1-\frac{2}{d+\varepsilon}} \\ &\lesssim \|w_n\|_{L^2(\Omega)}^{\frac{2}{d+\varepsilon}} \left(\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)^{1-\frac{2}{d+\varepsilon}}.\end{aligned}$$

Altogether, we have obtained

$$\begin{aligned}\|w_n\|_{L^2(\Omega)}^{2\left(1-\frac{1}{d+\varepsilon}\right)} &\lesssim \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \left(\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)^{1-\frac{2}{d+\varepsilon}}, \\ &\lesssim \|d_n\|_{L^1(\Omega)} \left(\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)^{2\left(1-\frac{1}{d+\varepsilon}\right)}\end{aligned}$$

which can be rewritten as

$$\|w_n\|_{L^2(\Omega)} \lesssim \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}\left(1+\frac{1}{d-1+\varepsilon}\right)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

which is our statement. □

Remark 22. Note that the proposition above shows that when $d = 2$, for any ε we find

$$\|w_n\|_{L^2(\Omega)} \leq C(\varepsilon) \|d_n\|_{L^1(\Omega)}^{1-\varepsilon} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{2.30}$$

Corollary 23. *For any $s \geq 2$ and any $\varepsilon > 0$ there holds*

$$\|w_n\|_{L^s(\Omega)} \leq C(\varepsilon) \|d_n\|_{L^1(\Omega)}^{\frac{1}{s}\left(1+\frac{1}{d-1+\varepsilon}\right)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{2.31}$$

Proof. We write

$$\|w_n\|_{L^s(\Omega)} \leq \|w_n\|_{L^2(\Omega)}^{\frac{2}{s}} \|w_n\|_{L^\infty(\Omega)}^{s-2}$$

and the conclusion follows from (2.24) and (2.26). □

Definition 24. For each $i = 1, \ldots, d$, we define the *correctors* $w_n^i \in H_0^1(\Omega)$

as the weak solution of

$$\int_\Omega \gamma_n \nabla w_n^i \cdot \nabla \phi = \int_\Omega (\gamma_0 - \gamma_n)\, \mathbf{e}_i \cdot \nabla \phi \text{ for all } \phi \in H_0^1(\Omega). \tag{2.32}$$

In particular, we have that $w_n^i = u_n^i - x^i$ where u_n^i satisfies the perturbed problem

$$\begin{aligned} -\nabla \cdot \gamma_n \left(\nabla u_n^i \right) &= -\nabla \cdot (\gamma_0 \mathbf{e}_i) \quad \text{in} \quad \Omega \\ u_n^i &= x^i \quad \text{on} \quad \partial\Omega. \end{aligned}$$

We say that the gradient field of w_n is *polarisable* if there exists a matrix function $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu\right)$ where μ is the Radon measure generated by the sequence $\frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F$, such that

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0) \nabla w_n \quad \mathrm{d}x \quad \rightharpoonup^* \quad M \nabla u_0 \quad \mathrm{d}\mu \tag{2.33}$$

in the space of bounded Radon measures. Extending the notion introduced in [12], we call such a matrix M a polarisation tensor.

The following proposition, which is an original contribution of this thesis is key to our results. The proposition below establishes the constitutive relationship between $\frac{\chi_{Q_n \cup R_n}}{\|d_n\|_{L^1(\Omega)}} \frac{\partial w_n^i}{\partial x_j} \mathrm{d}x$ and the gradient of the background potential u_0. Its proof is based on a variation of the clever "integration by parts technique" originally developed by Murat and Tartar in the context of H-convergence and the Div-Curl Lemma.

Proposition 25. *For any $\phi \in C^1(\overline{\Omega})$, there holds*

$$\begin{aligned} &\int_\Omega \left(\left(\gamma_n - \gamma_0 \right) \nabla w_n \cdot \nabla x_i \right) \phi \\ &= \int_\Omega \left(\left(\gamma_n - \gamma_0 \right) \nabla w_n^i \cdot \nabla u_0 \right) \phi + \int_\Omega r_n \cdot \nabla \phi \end{aligned}$$

with $r_n \in L^1(\Omega)$. Furthermore for any $\varepsilon > 0$, the following estimate holds

$$\left| \int_\Omega r_n \cdot \nabla \phi \right| \leq C \|\nabla \phi\|_{L^\infty(\Omega)} \left(\|d_n\|_{L^1(\Omega)}^{1+\frac{1}{2(d-1+\varepsilon)}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \varepsilon_n \right)$$

where C is a constant, which depends on ε, Ω, K, $\Lambda_0, \lambda_0, \eta$ and $\|\gamma_0\|_{W^{1,p}(\Omega)}$ only, and ε_n satisfies the following two a priori estimates

$$\varepsilon_n \leq \|d_n\|_{L^1(\Omega)} \left(\|w_n\|_{L^\infty(Q_n)} + \left\|w_n^i\right\|_{L^\infty(Q_n)} \|\nabla u_0\|_{L^\infty(K)} \right) \tag{2.34}$$

and,

$$\varepsilon_n \leq \|d_n\|_{L^1(\Omega)}^{1+\tau} \|d_n\|_{L^{2d+4\varepsilon}(Q_n)}^{\frac{1}{2}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2.35}$$

with

$$\tau = \frac{\varepsilon}{2\,(d+\varepsilon-1)\,(d+2\varepsilon)} > 0.$$

Proof. We compute

$$\begin{aligned}
&\int_\Omega \left(\left(\gamma_n - \gamma_0\right)\nabla w_n \cdot \nabla x_i\right)\phi \\
&= \int_\Omega \nabla\left(w_n\phi\right)\cdot\left(\gamma_n-\gamma_0\right)\nabla x_i - \int_\Omega w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi \\
&= -\int_\Omega \gamma_n\nabla\left(w_n\phi\right)\cdot\nabla w_n^i + \int_\Omega r_n^1\cdot\nabla\phi,
\end{aligned}$$

where

$$r_n^1 = -w_n\left(\gamma_n-\gamma_0\right)\nabla x_i. \tag{2.36}$$

Passing the localisation ϕ on the other gradient, we find

$$\int_\Omega \gamma_n\nabla\left(w_n\phi\right)\cdot\nabla w_n^i = \int_\Omega \gamma_n\nabla w_n\cdot\nabla\left(w_n^i\phi\right) + \int_\Omega r_n^2\cdot\nabla\phi,$$

with

$$r_n^2 = w_n\gamma_n\nabla w_n^i - w_n^i\gamma_n\nabla w_n, \tag{2.37}$$

Finally, performing the reverse integration by parts, we find

$$\begin{aligned}
&\int_\Omega \gamma_n\nabla w_n\cdot\nabla\left(w_n^i\phi\right) \\
&= \int_\Omega \left(\gamma_0-\gamma_n\right)\nabla\left(w_n^i\phi\right)\cdot\nabla u_0 \\
&= \int_\Omega \left(\gamma_0-\gamma_n\right)\nabla w_n^i\cdot\nabla u_0\,\phi + \int_\Omega r_n^3\cdot\nabla\phi,
\end{aligned}$$

where

$$r_n^3 = w_n^i\left(\gamma_0-\gamma_n\right)\nabla u_0. \tag{2.38}$$

Thus collecting (2.36), (2.37) and (2.38), the remainder term $r_n \in L^1(\Omega)$ is given as

$$r_n = \left(\gamma_n-\gamma_0\right)\left(w_n^i\nabla u_0 - w_n\nabla x_i\right) + w_n^i\gamma_n\nabla w_n - w_n\gamma_n\nabla w_n^i.$$

Now, let us define $T_1 = \mathbf{1}_{\gamma_n\le\gamma_0}\left(r_n\cdot\nabla\phi\right)$ and $T_2 = r_n\cdot\nabla\phi - T_1$ we find.

$$\begin{aligned}\|T_1\|_{L^1(\Omega)} &\leq \int_{\Omega\cap\{\gamma_n\leq\gamma_0\}} \left|w_n\left(\gamma_n\nabla w_n^i\right)\cdot\nabla\phi\right| + \int_{\Omega\cap\{\gamma_n\leq\gamma_0\}} \left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right| \\ &+ \int_{\Omega\cap\{\gamma_n\leq\gamma_0\}} \left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right| + \int_{\Omega\cap\{\gamma_n\leq\gamma_0\}} \left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right| \\ &\leq \|\nabla\phi\|_{L^\infty(\Omega)}\|\gamma_0\|_{L^\infty(\Omega)}^{\frac{1}{2}}\left(\|w_n\|_{L^2(\Omega)}E\left(w_n^i\right)^{\frac{1}{2}} + \left\|w_n^i\right\|_{L^2(\Omega)}E\left(w_n\right)^{\frac{1}{2}}\right. \\ &\left. + \left\|w_n^i\right\|_{L^2(\Omega)}\|d_n\|_{L^1(\Omega)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(K)} + \|w_n\|_{L^2(\Omega)}\|d_n\|_{L^1(\Omega)}^{\frac{1}{2}}\right)\end{aligned}$$

Thanks to estimate (2.12) and (2.26) (applied to $u_0 = x_i$ for the corrector terms w_n^i) we find

$$\|T_1\|_{L^1(\Omega)} \lesssim \|d_n\|_{L^1(\Omega)}^{1+\frac{1}{2(d-1+\varepsilon)}}\|\nabla\phi\|_{L^\infty(\Omega)}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

where we use the notation $a \lesssim b$ to mean $a \leq Cb$, where C is a constant, possibly changing from line to line, which may depend on ε, Ω, K, Λ_0,λ_0,η and $\|\gamma_0\|_{W^{1,p}(\Omega)}$ only.

We now turn to the other term T_2. The triangle inequality gives

$$\begin{aligned}\|T_2\|_{L^1(\Omega)} &\leq \int_{Q_n} \left|w_n\left(\gamma_n\nabla w_n^i\right)\cdot\nabla\phi\right| + \int_{Q_n} \left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right| \\ &+ \int_{Q_n} \left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right| + \int_{Q_n} \left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|.\end{aligned} \tag{2.39}$$

Recalling from (2.21), we have on Q_n, $\sqrt{\eta}I_d < I_d - \gamma_n^{-1}\gamma_0 < I_d$, therefore

$$0 < \gamma_n - \gamma_0 < \frac{1}{\sqrt{\eta}}\left(\gamma_n-\gamma_0\right)\left(I_d-\gamma_n^{-1}\gamma_0\right) = \frac{1}{\sqrt{\eta}}d_n \text{ a.e. on } Q_n. \tag{2.40}$$

Thus using (2.19) and (2.40) we deduce from (2.39) that

$$\|T_2\|_{L^1(\Omega)} \lesssim \left(\|w_n\|_{L^\infty(Q_n)}\|d_n\|_{L^1(\Omega)} + \left\|w_n^i\right\|_{L^\infty(Q_n)}\|d_n\|_{L^1(\Omega)}\|\nabla u_0\|_{L^\infty(K)}\right)\|\nabla\phi\|_{L^\infty(K)},$$

which corresponds to estimate (2.34).

Alternatively, applying Hölder's inequality, and then the L^p flux bound (2.20) with $p = 2d+4\varepsilon$, and the L^s bound (2.31) with the conjugate ex-

ponent, we find

$$\begin{aligned}
&\int_{Q_n} \left| w_n \gamma_n \nabla w_n^i \cdot \nabla \phi \right| \\
&\leq \left\| \gamma_n \nabla w_n^i \right\|_{L^{\frac{2p}{p+1}}(Q_n)} \left\| w_n \right\|_{L^{\frac{2p}{p-2}}(Q_n)} \left\| \nabla \phi \right\|_{L^\infty(K)} \\
&\lesssim \left\| d_n \right\|_{L^1(Q_n)}^{\frac{1}{2}} \left\| d_n \right\|_{L^p(Q_n)}^{\frac{1}{2}} \left\| d_n \right\|_{L^1(\Omega)}^{\left(\frac{1}{2}-\frac{1}{p}\right)\left(1+\frac{1}{d-1+\varepsilon}\right)} \left(\left\| \nabla u_0 \right\|_{L^\infty(K)} + \left\| g \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \right) \left\| \nabla \phi \right\|_{L^\infty(K)} \\
&= \left\| d_n \right\|_{L^1(\Omega)}^{1+\tau} \left\| d_n \right\|_{L^{2d+4\varepsilon}(Q_n)}^{\frac{1}{2}} \left\| g \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \left\| \nabla \phi \right\|_{L^\infty(K)},
\end{aligned}$$

with

$$\tau = \frac{\varepsilon}{2\left(d+\varepsilon-1\right)\left(d+2\varepsilon\right)} > 0.$$

Similarly,

$$\int_{Q_n} \left| w_n^i \gamma_n \nabla w_n \cdot \nabla \phi \right| \lesssim \left\| d_n \right\|_{L^1(\Omega)}^{1+\tau} \left\| d_n \right\|_{L^{2d+4\varepsilon}(Q_n)}^{\frac{1}{2}} \left\| g \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \left\| \nabla \phi \right\|_{L^\infty(K)}.$$

On the other hand, using (2.40), Hölder's inequality with $p = 2d + 4\varepsilon$, and the L^s bound (2.31) with the appropriate exponent, we can write

$$\begin{aligned}
\int_{Q_n} \left| w_n^i \left(\gamma_n - \gamma_0\right) \nabla u_0 \cdot \nabla \phi \right| &\lesssim \left\| d_n^{\frac{1}{2}} \right\|_{L^2(Q_n)} \left\| d_n^{\frac{1}{2}} w_n^i \right\|_{L^2(Q_n)} \left\| \nabla u_0 \right\|_{L^\infty(K)} \left\| \nabla \phi \right\|_{L^\infty(K)} \\
&\lesssim \left\| d_n \right\|_{L^1(Q_n)}^{\frac{1}{2}} \left\| d_n \right\|_{L^p}^{\frac{1}{2}} \left\| w_n^i \right\|_{L^{\frac{2p}{p-2}}(Q_n)} \left\| \nabla u_0 \right\|_{L^\infty(K)} \left\| \nabla \phi \right\|_{L^\infty(K)} \\
&\lesssim \left\| d_n \right\|_{L^1(\Omega)}^{1+\tau} \left\| d_n \right\|_{L^{2d+4\varepsilon}(Q_n)}^{\frac{1}{2}} \left\| g \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \left\| \nabla \phi \right\|_{L^\infty(K)},
\end{aligned}$$

and finally, by the same argument,

$$\int_{Q_n} \left| w_n \left(\gamma_n - \gamma_0\right) \nabla x_i \cdot \nabla \phi \right| \lesssim \left\| d_n \right\|_{L^1(\Omega)}^{1+\tau} \left\| d_n \right\|_{L^{2d+4\varepsilon}(Q_n)}^{\frac{1}{2}} \left\| g \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \left\| \nabla \phi \right\|_{L^\infty(K)},$$

which concludes the proof of estimate (2.35). □

In what follows, we will work towards showing that the remainder term r_n from the proposition above tends to 0 at a rate faster than $\left\| d_n \right\|_{L^1(\Omega)}$. We shall treat several cases seperately. Notably, we first treat the case of an insulating inclusion, that is to say $Q_n = \emptyset$ or the case of a mixed inclusion i.e when the inclusion sets Q_n and R_n are non-empty and possibly intertwined.

The insulating inclusion case and the mixed case.

Proposition 26. *Suppose that either $Q_n = \emptyset$, or $\limsup_{n\to\infty} \left\| d_n \right\|_{L^p(\Omega)} < \infty$ for some $p > 2d$. Then, under Conditions 1,2, and 3 of Assumption 12,*

there holds

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n \cdot \left(\gamma_n - \gamma_0\right)_i dx \quad \overset{*}{\rightharpoonup} \quad M_i \cdot \nabla u_0 \, d\mu$$

in the weak topology of* $C(\overline{\Omega})^*$. *Here* $\left(\gamma_n - \gamma_0\right)_i$ *denotes the i-th column vector of the matrix* $\left(\gamma_n - \gamma_0\right)$. *By writing it in a different form, this shows that*

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n \, dx \quad \overset{*}{\rightharpoonup} \quad M\nabla u_0 \, d\mu$$

with entries M_{ij} *being the scalar weak* limit of* $\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n^i \cdot \left(\gamma_n - \gamma_0\right)_j$. *Furthermore, the convergence is uniform with respect to* $g \in H^{\frac{1}{2}}(\partial\Omega)$ *provided that* $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$.

Proof. If $Q_n = \emptyset$ choose $\varepsilon = 1$ and otherwise choose $\varepsilon = \frac{p-2d}{5}$. According to Proposition 25, the remainder term r_n satisfies in theses two cases ,

$$\left|\int_\Omega r_n \cdot \nabla\phi\right| \leq C\|d_n\|_{L^1(\Omega)}^{1+\tau}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)},$$

with

$$\tau \geq \frac{\varepsilon}{2(d+\varepsilon-1)(d+2\varepsilon)}.$$

Thus,

$$\lim_{n\to\infty}\left|\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega r_n \cdot \nabla\phi\right| = 0.$$

By passing the limit as $n \to \infty$ in Proposition 25, we find that

$$\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \phi\left(\gamma_n - \gamma_0\right)\nabla w_n \cdot \nabla x_i = \lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \phi\left(\gamma_n - \gamma_0\right)\nabla w_n^i \cdot \nabla u_0$$

which implies that

$$\int_\Omega \phi \, \mathrm{d}N_i = \int_\Omega \phi M_i \cdot \nabla u_0 \, \mathrm{d}\mu$$

for any $\phi \in C^1(\overline{\Omega})$, where M_i and N_i in the calculation above denotes the Radon measure generated by the sequence $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n^i$ and $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n$ respectively. Using the fact that the space $C^1(\overline{\Omega})$ is dense in $C^0(\overline{\Omega})$, and both of the bounded linear functionals agree on a dense subset, we may extend this equality by continuity so that they agree

on the whole of $C^0\left(\overline{\Omega}\right)$. Therfore, we obtain that

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n \cdot \left(\gamma_n - \gamma_0\right)_i \quad \overset{*}{\rightharpoonup} \quad M_i \cdot \nabla u_0$$

in $C^0\left(\overline{\Omega}\right)^*$. Finally, to see that this convergence holds uniformly for all g in the unit ball of $H^{\frac{1}{2}}(\partial\Omega)$,note that we have

$$\begin{aligned}\left|\int_\Omega r_n \cdot \nabla\phi\right| &\lesssim \|\nabla\phi\|_{L^\infty(\Omega)} \left(\|d_n\|_{L^1(\Omega)}^{1+\tau} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \\ &\lesssim o\left(\|d_n\|_{L^1(\Omega)}\right)\left(\|\nabla\phi\|_{L^\infty(\Omega)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\right),\end{aligned}$$

which subsequently implies that

$$\sup_{g\in H^{\frac{1}{2}}(\partial\Omega):\|g\|=1} \frac{|\int_\Omega r_n \cdot \nabla\phi|}{\|d_n\|_{L^1(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)}} \lesssim o(1),$$

and we arrive at our statement. □

In the next section, we treat the case when the inclusion is highly conducting, that is to say when $R_n = \emptyset$ in dimension 2. The highly conducting case in dimension 3 is treated at the last section as it requires a different method to arrive at our result. We shall use the result from Proposition 26 to establish the weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n$ in this case. It turns out that the highly conducting inclusion problem in dimension 2 is related to the insulating case via Helmholtz's decomposition theorem. In particular, we shall utilise the following theorem (see Girault and Raviart [25, 26])

Theorem 27. *(Helmholtz's Decomposition Theorem) Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with Lipschitz- continuous boundary, furthermore, let v be a vector field in $L^p\left(\Omega\right)^2$ such that*

$$\nabla \cdot v = 0 \quad in \quad \Omega$$

for any $p \geq 2$, then there exists a stream function $\phi \in W^{1,p}\left(\Omega\right)$ such that

$$v = \nabla\phi^\perp \quad in \quad \Omega, \tag{2.41}$$

in which, the stream function ϕ is unique up to an additive constant.

The novelty of the method presented here is that we are able to establish the convergence of the remainder term r_n in Proposition 26 without assuming that d_n converges to 0 in $L^p(\Omega)$ for any $p > 1$.

The high conductivity inclusion case in dimension two

Observe that the vector fields

$$F_n := \gamma_n \nabla u_n$$
$$F_0 := \gamma_0 \nabla u_0$$

in (2.9) belong to $L^2(\Omega)^2$and are divergence free, so we may choose an arbitrary smooth compact simply connected subset $K_1 \subset \Omega$ which contains the set K and deduce the existence of stream functions $\psi_n, \psi_0 \in H^1(K_1)$ such that

$$F_n = \nabla \psi_n^{\perp}$$
$$F_0 = \nabla \psi_0^{\perp}$$

in K_1 as a consequence of Helmholtz's decomposition theorem. As the stream functions ψ_n, ψ_0 may be chosen uniquely up to an additive constant, we may assume without loss of generality that they satisfy the constraint

$$\frac{1}{|K_1|}\int_{K_1} \psi_n = \frac{1}{|K_1|}\int_{K_1} \psi_0 = 0.$$

Writing J as the antisymmetric matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that $J(v) = v^{\perp}$ for any vector $v \in \mathbb{R}^2$,we observe that by taking the divergence of each terms $J^T \nabla u_n$ and $J^T \nabla u_0$ from the equations (2.9) and (2.1) respectively, the stream function $\phi_n = \psi_n - \psi_0$ satisfies

$$-\nabla \cdot (\sigma_n \nabla \phi_n) = \nabla \cdot ((\sigma_n - \sigma_0) \nabla \psi_0) = 0 \quad \text{in} \quad \mathscr{D}'(\Omega) \tag{2.42}$$

in the sense of distributions. The new conductivity matrices σ_n and σ_0 are defined as

$$\sigma_n := J^T \gamma_n^{-1} J \quad \text{and} \quad \sigma_0 := J^T \gamma_0^{-1} J.$$

Moreover, they are elliptic and the solution ϕ_n to (2.42) now satisfies a low conductivity inclusion problem .

Proposition 28. *Let K_1 be a smooth connected compact subset such that $K \Subset K_1 \Subset \Omega$. Furthermore, let ψ_n and ψ_0 be the stream functions which are chosen accordingly as the above, then for any $\varepsilon > 0$,*

$$\|\sigma_0 \nabla \phi_n \cdot \nu_{K_1}\|_{H^{-\frac{1}{2}}(\partial K_1)} \leq C \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}\left(1+\frac{1}{d-1+\varepsilon}\right)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2.43}$$

where the constant C may depend on ε, Ω, K, and the a priori bounds Λ_0

and λ_0 , η *defined in* (2) *and* $\|\gamma_0\|_{W^{1,p}(\Omega)}$ *only.*

Proof. Observe that the stream function $\phi_n = \psi_n - \psi_0$ satisfies

$$-\nabla \cdot (\sigma_0 \nabla \phi_n) \;=\; 0 \text{ in } K_1 \setminus K. \tag{2.44}$$

Note that since $\sigma_0 \nabla \phi_n$ is divergence free in $K_1 \setminus K$ and since ∂K_1 is smooth, the normal derivative $\sigma_0 \nabla \phi_n \cdot \nu_{K_1}$ is well defined in $H^{-\frac{1}{2}}(\partial K_1)$. Now, let $\phi \in H^1(K_1 \setminus K)$ be such that its trace on ∂K is 0. By the weak formulation of (2.44). we have

$$\begin{aligned}
\left| \int_{\partial K_1} \phi \, (\sigma_0 \nabla \phi_n \cdot \nu_{K_1}) \, \mathrm{d}x \right| &= \left| \int_{K_1 \setminus K} \sigma_0 \nabla \phi_n \cdot \nabla \phi \, \mathrm{d}x \right| \\
&= \left| \int_{K_1 \setminus K} J^T \nabla w_n \cdot \nabla \phi \, \mathrm{d}x \right| \\
&\leq \|\nabla w_n\|_{L^2(K_1 \setminus K)} \|\phi\|_{H^1(K_1 \setminus K)}.
\end{aligned}$$

Note that since $d\left(\bigcup_{n\in\mathbb{N}} Q_n \cup R_n, \partial K\right) > 0$,

$$-\nabla \cdot (\gamma_0 \nabla w_n) = 0 \text{ in } (K_1 \setminus K) + \lambda B(0,1)$$

for some $\lambda > 0$ sufficiently small, thus, we may choose a smooth cut-off function $\chi \in C_c^\infty(\Omega)$ such that $\chi = 1$ on $(K_1 \setminus K)$ and supported in $(K_1 \setminus K) + \lambda B(0,1)$, so that by testing the equation above with $w_n \chi^2$, Caccioppoli's (in)equality writes

$$\int_\Omega \gamma_0 \nabla (\chi w_n) \cdot \nabla (\chi w_n) = \int_\Omega (\gamma_0 \nabla \chi \cdot \nabla \chi) \, w_n^2,$$

and

$$\begin{aligned}
\int_{(K_1 \setminus K)} \gamma_0 \nabla (\chi w_n) \cdot \nabla (\chi w_n) &\leq \int_{(K_1 \setminus K) + \lambda B(0,1)} w_n^2 \\
&\leq C(\lambda, K) \|w_n\|_{L^2(\Omega)}.
\end{aligned}$$

Altogether, this implies that

$$\begin{aligned}
\|\sigma_0 \nabla \phi_n \cdot \nu_{K_1}\|_{H^{-\frac{1}{2}}(\partial K_1)} &\leq C(K_1 \setminus K) \|w_n\|_{L^2(\Omega)} \\
&\leq C(K_1 \setminus K) \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}\left(1 + \frac{1}{d-1+\varepsilon}\right)} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)},
\end{aligned}$$

where the estimate on the right-hand side follows from (2.26). □

Corollary 29. *Let* u_n *and* u_0 *be the solutions to* (2.2) *and* (2.1) *respectively. Furthermore, assuming that* γ_n *is highly conducting, that is to say* $R_n = \emptyset$ *and by denoting* $\phi_n = \psi_n - \psi_0$ *to be the weak solution to* (2.42)*, where* ψ_n *and* ψ_0 *are the stream functions associated to* ∇u_n *and* ∇u_0 *respectively, we*

have

$$\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\phi_n\,dx \quad \overset{*}{\rightharpoonup} \quad \tilde{M}\nabla\psi_0\,d\nu \tag{2.45}$$

in $C^0\left(\overline{\Omega}\right)^*$*, where* Σ_n *denotes the symmetric positive definite matrix*

$$\Sigma_n=\Big(\sigma_n-\sigma_0\Big)\sigma_n^{-1}\Big(\sigma_n-\sigma_0\Big)\,a.e.x\,in\,\Omega,$$

$\tilde{M}\in L^2\big(\Omega,\mathbb{R}^{d\times d};d\nu\big)$*, and* ν *is the Radon measure identified as the weak** *limit of the sequence* $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Sigma_n$.

Proof. The proof follows directly from proposition 36 and Theorem (38). □

Lemma 30. *Let* Σ_n *be the symmetric positive definite matrix as the above, then*

$$\lambda_1\left(\gamma_0^{-2}\right)|d_n|_F\leq|\Sigma_n|_F\leq\lambda_d\left(\gamma_0^{-2}\right)|d_n|_F \tag{2.46}$$

a.e.x in Ω *, where* $\lambda_k\left(\gamma_0^{-2}\right)$ *here denotes the k-th eigenvalue of the matrix* γ_0^{-2} *in increasing order.*

Proof. Note that

$$\begin{aligned}\Sigma_n &= J^T\left(\gamma_n^{-1}-\gamma_0^{-1}\right)\gamma_n\left(\gamma_n^{-1}-\gamma_0^{-1}\right)J\\ &= J^T\left(\gamma_0^{-1}\left(\gamma_0-\gamma_n\right)\gamma_n^{-1}\right)\gamma_n\left(\gamma_n^{-1}\left(\gamma_0-\gamma_n\right)\gamma_0^{-1}\right)J\\ &= J^T\gamma_0^{-1}\left(\gamma_0-\gamma_n\right)\gamma_n^{-1}\left(\gamma_0-\gamma_n\right)\gamma_0^{-1}J\\ &= J^T\gamma_0^{-1}d_n\gamma_0^{-1}J.\end{aligned}$$

Therefore, as $|AB|_F=|BA|_F$ for any two arbitrary square matrices A,B in $\mathbb{R}^{d\times d}$, we have

$$\begin{aligned}|\Sigma_n|_F &= \left|\gamma_0^{-1}d_n\gamma_0^{-1}\right|_F\\ &= \left|d_n\gamma_0^{-2}\right|_F\end{aligned}$$

and it follows from the eigenvalue comparison inequality that

$$\lambda_k\left(d_n\right)\lambda_1\left(\gamma_0^{-2}\right)\leq\lambda_k\left(d_n\gamma_0^{-2}\right)\leq\lambda_k\left(d_n\right)\lambda_d\left(\gamma_0^{-2}\right) \tag{2.47}$$

Thus, we deduce from the definition of the Frobenius norm that

$$\lambda_1\left(\gamma_0^{-2}\right)|d_n|_F\leq|\Sigma_n|_F=\sqrt{\sum_{i=1}^{d}\lambda_k^2\left(d_n\gamma_0^{-2}\right)}\leq\lambda_d\left(\gamma_0^{-2}\right)|d_n|_F\,.$$

□

Remark 31. In particular, the lemma above shows that

$$\|d_n\|_{L^p} \to 0 \Longleftrightarrow \|\Sigma_n\|_{L^p} \to 0.$$

Moreover, by denoting ν and μ to be the Radon measures generated by the sequences $\frac{\Sigma_n}{\|\Sigma_n\|_{L^1(\Omega)}}$ and $\frac{d_n}{\|d_n\|_{L^1(\Omega)}}$ respectively, the lemma above also shows that the Radon-Nikodym derivatives $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ and similarly $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ belongs to $L^\infty(\Omega;\mathrm{d}\mu)$ and $L^\infty(\Omega;\mathrm{d}\nu)$ respectively. As a consequence of this, the spaces $L^p(\Omega;\mathrm{d}\mu)$ are equivalent to $L^p(\Omega;\mathrm{d}\nu)$ for any $p>1$.

Proposition 32. *Suppose that $d=2$ and γ_n is highly conducting, that is to say $R_n=\emptyset$. Then, under Conditions 1,2 and 3 there holds*

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n \cdot \left(\gamma_n-\gamma_0\right)_i dx \quad \overset{*}{\rightharpoonup} \quad M_i \cdot \nabla u_0 \, d\mu$$

in the weak* topology of $C(\overline{\Omega})^*$. Here $\left(\gamma_n-\gamma_0\right)_i$ denotes the i-th column vector of the matrix $\left(\gamma_n-\gamma_0\right)$. By writing it in a different form, this shows that

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n-\gamma_0\right)\nabla w_n \, dx \quad \overset{*}{\rightharpoonup} \quad M\nabla u_0 \, d\mu$$

with entries M_{ij} being the scalar weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n^i \cdot \left(\gamma_n-\gamma_0\right)_j$. Furthermore, the convergence is uniform with respect to $g \in H^{\frac{1}{2}}(\partial\Omega)$ provided that $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$.

Proof. By the inequality in (2.46), we have

$$\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}} \quad \leq \quad \left\|\lambda_d\left(\gamma_0^{-2}\right)\right\|_{L^\infty(\Omega)},$$

therefore, by taking a convergent subsequence of $\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}}$ which converges to a positive real number a_0, and a possible further extraction of the subsequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\phi_n$, Corollary 2.45 implies that

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\phi_n \,\mathrm{d}x &= \lim_{n\to\infty}\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}}\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\phi_n \,\mathrm{d}x \\
&= a_0\lim_{n\to\infty}\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\phi_n \,\mathrm{d}x \\
&= a_0\tilde{M}\nabla\psi_0 \,\mathrm{d}\nu \\
&= a_0\tilde{M}\nabla\psi_0\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu
\end{aligned}$$

for some $\tilde{M}\in L^2\big(\Omega,\mathbb{R}^{d\times d};\mathrm{d}\nu\big)$, and ν is the Radon measure identified as the weak* limit of the sequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Sigma_n$. Note that by remark (31),

the term $a_0 \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\tilde{M}$ belongs to $L^2(\Omega;\mathrm{d}\mu)$. Thus, by rewriting the above in a slightly different form, we have

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\sigma_n - \sigma_0\Big) \nabla \psi_n \,\mathrm{d}x \\ = \quad & \lim_{n\to\infty} \frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\sigma_n - \sigma_0\Big) \nabla \psi_0 \,\mathrm{d}x + a_0 \tilde{M} \nabla \psi_0 \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu \\ = \quad & N \nabla \psi_0 \,\mathrm{d}\mu \end{aligned}$$

for some other matrix $N \in L^2(\Omega;\mathrm{d}\mu)$.

Let $\Xi \in C^0\big(\overline{\Omega},\mathbb{R}^2\big)$ be an arbitrary vector field, since $\big(J^T\gamma_0\big)\,\Xi \in C^0\big(\overline{\Omega},\mathbb{R}^2\big)$, we test the above with $\big(J^T\gamma_0\big)\,\Xi$ to find

$$\begin{aligned} & \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\sigma_n - \sigma_0\Big) \nabla \psi_n \cdot \big(J^T\gamma_0\big)\,\Xi \,\mathrm{d}x \\ = & \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(J^T \gamma_0^{-1} \Big(\gamma_0 - \gamma_n\Big) \gamma_n^{-1} J\Big) \big(J^T\gamma_n \nabla u_n\big) \cdot \big(J^T\gamma_0\big)\,\Xi \,\mathrm{d}x \\ = & \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} J^T \gamma_0^{-1} \Big(\gamma_0 - \gamma_n\Big) \nabla u_n \cdot J^T\gamma_0 \Xi \,\mathrm{d}x. \\ = & \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\gamma_0 - \gamma_n\Big) \nabla u_n \cdot \Xi \,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int_\Omega N \nabla \psi_0 \cdot \big(J^T\gamma_0\big)\,\Xi \,\mathrm{d}\mu \quad &= \quad \int_\Omega N J^T \gamma_0 \nabla u_0 \cdot \big(J^T\gamma_0\big)\,\Xi \,\mathrm{d}\mu \\ &= \quad \int_\Omega (\gamma_0 J)\, N\, (\gamma_0 J)^T \nabla u_0 \cdot \Xi \,\mathrm{d}\mu. \end{aligned}$$

Subsequently, by equating the terms above, we have

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\gamma_0 - \gamma_n\Big) \nabla u_n \,\mathrm{dx} \quad \overset{*}{\rightharpoonup} \quad (\gamma_0 J)\, N\, (\gamma_0 J)^T \nabla u_0 \,\mathrm{d}\mu$$

while

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\gamma_0 - \gamma_n\Big) \nabla u_0 \,\mathrm{dx} \quad \overset{*}{\rightharpoonup} \quad \tilde{N} \nabla u_0 \,\mathrm{d}\mu$$

for some matrix $\tilde{N} \in L^2(\Omega;\mathrm{d}\mu)$ trivially. Thus, we obtain

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \Big(\gamma_0 - \gamma_n\Big) \nabla w_n \,\mathrm{dx} \quad \overset{*}{\rightharpoonup} \quad M \nabla u_0 \,\mathrm{d}\mu.$$

Finally, the fact that this convergence is uniform for all g in the unit ball of $H^{\frac{1}{2}}(\partial\Omega)$ follows from the same argument in Proposition 26. □

The non-intertwined inclusions case.

In the next section, we treat the case when either γ_n is highly conducting in dimension 3 (i.e $R_n = \emptyset$) or when the inclusion sets $Q_n \cup R_n$ are non-empty and non-intertwined. The main objective here is to address the L^∞ estimate which arise from the remainder term r_n in (2.34).

Proposition 33. *Let $w_n \in H_0^1(\Omega)$ be the weak solution to* (2.10)*, then for any ball $B_\rho \Subset \overline{\Omega \setminus (Q_n \cup R_n)}$ and for any $0 < \rho' < \rho$, we have $w_n \in C^{1,\alpha}(B_{\rho'})$, moreover, the following estimate holds*

$$\|w_n\|_{C^{1,\alpha}(B_{\rho'})} \leq C\left(\alpha, \rho, \rho'\right) \|w_n\|_{L^2(B_\rho)}, \tag{2.48}$$

where $\alpha \in (0,1)$ depends only on d.

Proof. Note that by our assumption $\gamma_0 \in W^{2,2}_{loc}(\Omega)$ and that w_n satisfies

$$-\nabla \cdot (\gamma_0 \nabla w_n) \quad = \quad 0 \text{ in } \mathscr{D}'\left(B_\rho\right),$$

for any ball $B_\rho \Subset \overline{\Omega \setminus (Q_n \cup R_n)}$, thus, we may use the H^3_{loc} estimate derived in appendix 86 to obtain

$$\|w_n\|_{H^3(B_{\rho'})} \leq C\left(\rho, \rho'\right) \|w_n\|_{L^2(\Omega)}$$

for any $0 < \rho' < \rho$. Thanks to Sobolev's embedding inequality, we have

$$\|w_n\|_{C^{1,\alpha}(B_{\rho'})} \leq C\left(\alpha, \rho'\right) \|w_n\|_{H^3(B_{\rho'})} \leq C\left(\alpha, \rho, \rho'\right) \|w_n\|_{L^2(\Omega)}$$

where $\alpha \in (0,1)$ depends only on the dimension d. □

We require the following lemma by Stampacchia in the following Proposition.

Lemma 34. *(Stampacchia) Let $\phi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a nonincreasing function such that*

$$\phi(h) \leq \frac{M\phi(k)^\delta}{|h-k|^\gamma} \quad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$, and $\gamma > 0$. Then we have $\phi(d) = 0$, where

$$d^\gamma = M\phi(0)^{\delta-1} 2^{\frac{\delta\gamma}{\delta-1}}.$$

Proof. We refer the reader to Lemma 7.1 in the paper by Stampacchia [35] . □

Proposition 35. *Suppose that the sets Q_n and R_n are not intertwined. Let $w_n \in H_0^1(\Omega)$ be the weak solution to* (2.10)*. When $d = 2$, p^* may be ar-*

bitrarily chosen between $\left(2,\infty\right)$*, and when* $d>2$*,* $p^*=\frac{2d}{d-2}$*. Then, for any* $\tau\in\left(0,\frac{p^*}{2}-1\right)$ *there holds*

$$\|w_n\|_{L^\infty(Q_n)}\leq 2^{\frac{p^*}{2\tau}}C\left(d,\Omega,\|\gamma_0\|_{L^\infty},\tau\right)\left(\|d_n\|_{L^{\frac{p^*}{p^*-2(1+\tau)}}(\Omega)}\right)^{\frac{1}{2}}\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \tag{2.49}$$

Proof. Let V_n and U_n be smooth open sets such that $Q_n\Subset V_n\Subset U_n\Subset(\Omega\setminus R_n)$. Since $d\left(Q_n,R_n\right)\geq C_0$ for all $n\geq 1$, we may select V_n and U_n such that $d\left(U_n,V_n\right)=R$ for some sufficiently small $R>0$. Let χ_n be a smooth cut off function such that $\chi_n=1$ on V_n , and is supported in U_n. Since $d\left(V_n,U_n\right)=R$, we may choose the cut off function χ_n so that it satisfies

$$|\nabla\chi_n|\leq C\left(\frac{1}{R}\right).$$

Note that the constant $C\left(\frac{1}{R}\right)$ is independent of n. Moreover, it depends only on the dimension of the ambient space and the parameter R. Observe that $w_n\chi_n\in H_0^1\left(\Omega\right)$ and satisfies

$$\begin{aligned}-\nabla\cdot\left(\gamma_n\nabla\left(w_n\chi_n\right)\right) &= -\nabla\cdot\left(\chi_n\gamma_n\nabla\left(w_n\right)\right)-\nabla\cdot\left(w_n\gamma_n\nabla\left(\chi_n\right)\right)\\ &= -\nabla\cdot\left(\chi_n\gamma_n\nabla\left(w_n\right)\right)-\nabla\cdot\left(w_n\gamma_0\nabla\left(\chi_n\right)\right)\text{ in }\mathscr{D}'\left(\Omega\right).\end{aligned}$$

On the other hand, from testing (2.10) with $\phi\chi_n$ for any $\phi\in H_0^1\left(\Omega\right)$, we find that

$$\begin{aligned}-\nabla\cdot\left(\chi_n\gamma_n\nabla\left(w_n\right)\right) &= -\nabla\cdot\left(\chi_n\left(\gamma_0-\gamma_n\right)\nabla u_0\right)+\left(\gamma_0-\gamma_n\right)\nabla u_0\cdot\nabla\chi_n\\ &\quad -\gamma_n\nabla w_n\cdot\nabla\chi_n\\ &= -\nabla\cdot\left(\chi_n\left(\gamma_0-\gamma_n\right)\nabla u_0\right)-\gamma_0\nabla w_n\cdot\nabla\chi_n\,\text{in }\mathscr{D}'\left(\Omega\right)\end{aligned}$$

as $\operatorname{supp}(\nabla\chi_n)\subset\left(\Omega\setminus\left(Q_n\cup R_n\right)\right)$. Altogether, we have

$$\begin{aligned}-\nabla\cdot\left(\gamma_n\nabla\left(w_n\chi_n\right)\right) &= -\nabla\cdot\left(\chi_n\left(\gamma_0-\gamma_n\right)\nabla u_0\right)-\gamma_0\nabla w_n\cdot\nabla\chi_n\\ &\quad -\nabla\cdot\left(w_n\gamma_0\nabla\chi_n\right)\text{ in }\mathscr{D}'\left(\Omega\right).\end{aligned} \tag{2.50}$$

For $k>0$, set

$$A_k=\left\{x\in\Omega:|w_n\chi_n|>k\right\}.$$

We follow Stampacchia's method of proof [35]. Denoting $G_k(u)$ to be the truncation operator, i.e $G_k(u)=\begin{cases}u & |u|\leq k\\ k & u>k\\ -k & u<-k\end{cases}$ with $k>0$, we test the

equation (2.50) against $v_n = w_n\chi_n - G_k(w_n\chi_n)$ to obtain

$$\begin{aligned}\int_\Omega \gamma_n \nabla v_n \cdot \nabla v_n &= \int_\Omega \gamma_n \nabla(w_n\chi_n)\cdot\nabla v_n \\ &= \int_\Omega \chi_n(\gamma_0-\gamma_n)\nabla u_0\cdot\nabla v_n - \int_\Omega \gamma_0(\nabla w_n\cdot\nabla\chi_n)v_n \\ &+ \int_\Omega w_n\gamma_0\nabla\chi_n\cdot\nabla v_n\end{aligned}$$

Applying Cauchy–Schwarz on the right-hand side, we obtain

$$\begin{aligned}\int_\Omega \gamma_n\nabla v_n\cdot\nabla v_n \le \Big(\int_{A_k}\chi_n d_n\nabla u_0\cdot\nabla u_0\Big)^{\frac12}\Big(\int_\Omega \gamma_n\nabla v_n\cdot\nabla v_n\Big)^{\frac12} \\ + \|\nabla\chi_n\|_{L^\infty(\Omega)}\|\gamma_0\|_{L^\infty(\Omega)}\left(\int_\Omega v_n^2\right)^{\frac12}\left(\int_{(U_n\setminus V_n)\cap A_k}\nabla w_n\cdot\nabla w_n\right)^{\frac12} \\ + \|\nabla\chi_n\|_{L^\infty(\Omega)}\|\gamma_0\|_{L^\infty(\Omega)}\left(\int_{(U_n\setminus V_n)\cap A_k} w_n^2\right)^{\frac12}\left(\int_\Omega\nabla v_n\cdot\nabla v_n\right)^{\frac12}.\end{aligned}$$

Note that $\operatorname{supp}(v_n)\subset U_n\subset(\Omega\setminus R_n)$, thus

$$\int_\Omega \gamma_n\nabla v_n\cdot\nabla v_n \ge \int_\Omega \gamma_0\nabla v_n\cdot\nabla v_n.$$

Consequently, we have by Poincaré's inequality,

$$\|v_n\|_{L^2(\Omega)} + \|\nabla v_n\|_{L^2(\Omega)} \le C(\Omega)\left(\int_\Omega \gamma_n\nabla v_n\cdot\nabla v_n\right)^{\frac12},$$

which in turn shows that

$$\begin{aligned}\left(\int_\Omega \gamma_n\nabla v_n\cdot\nabla v_n\right)^{\frac12} &\le C\left(\Omega,\frac{1}{R},\|\gamma_0\|_{L^\infty}\right)\left(\int_{A_k}\chi_n d_n\nabla u_0\cdot\nabla u_0\right)^{\frac12} \\ &+ \left(\int_{(U_n\setminus V_n)\cap A_k}\nabla w_n\cdot\nabla w_n\right)^{\frac12} + \left(\int_{(U_n\setminus V_n)\cap A_k} w_n^2\right)^{\frac12}\Bigg)\end{aligned}$$

Applying Sobolev's inequality to v_n we find

$$\begin{aligned}\left(\int_{A_k}|v_n|^{p^*}\right)^{\frac{1}{p^*}} &= \left(\int_\Omega |v_n|^{p^*}\right)^{\frac{1}{p^*}} \\ &\le C_d\left(\int_\Omega \nabla v_n\cdot\nabla v_n\right)^{\frac12} \\ &= C_d\left(\int_{A_k}\gamma_n^{-1}\left(\gamma_n^{\frac12}\nabla v_n\right)\cdot\left(\gamma_n^{\frac12}\nabla v_n\right)\right)^{\frac12}\end{aligned}$$

where C_d depends only on the dimension of the ambient space. This in turn

yields

$$\begin{aligned}\left(\int_{A_k}|v_n|^{p^*}\right)^{\frac{1}{p^*}} &\leq C_d\left\|\left(\gamma_n^{-1}\right)\right\|_{L^\infty(A_k)}^{\frac{1}{2}}\left(\int_{A_k}\gamma_n\nabla v_n\cdot\nabla v_n\right)^{\frac{1}{2}}\\ &\leq C_d\left(\Omega,\frac{1}{R},\|\gamma_0\|_{L^\infty}\right)\left(\left(\int_{A_k}\chi_n d_n\nabla u_0\cdot\nabla u_0\right)^{\frac{1}{2}}\right.\\ &\left.+\left(\int_{(U_n\setminus V_n)\cap A_k}\nabla w_n\cdot\nabla w_n\right)^{\frac{1}{2}}+\left(\int_{(U_n\setminus V_n)\cap A_k}w_n^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

For any $h>k\geq 0$ we have

$$\begin{aligned}\left(\int_{A_h}|v_n|^{p^*}\right)^{\frac{1}{p^*}} &\leq \left(\int_{A_k}|v_n|^{p^*}\right)^{\frac{1}{p^*}}\\ &\lesssim\left(\|d_n\|_{L^r(\Omega)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(\Omega)}+\|w_n\|_{W^{1,2r}((U_n\setminus V_n)\cap A_k)}\right)|A_k|^{\frac{1}{2s}}\\ &\lesssim\left(\|d_n\|_{L^r(\Omega)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(\Omega)}+\|w_n\|_{C^1((U_n\setminus V_n))}\right)|A_k|^{\frac{1}{2s}}\\ &\lesssim\left(\|d_n\|_{L^r(\Omega)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(\Omega)}+\|w_n\|_{L^2(\Omega)}\right)|A_k|^{\frac{1}{2s}}\end{aligned}$$

where $r,s\in[1,\infty)$ satisfy $r^{-1}+s^{-1}=1$. Note that the last line in the inequality above follows from the estimate in (2.48) and a topological covering argument. Consequently, this gives

$$|A_h|\leq M^{p^*}\frac{|A_k|^{\frac{p^*}{2s}}}{|h-k|^{p^*}},\tag{2.51}$$

with

$$M=C_d\left(\Omega,\frac{1}{R},\|\gamma_0\|_{L^\infty}\right)\left(\|d_n\|_{L^r(\Omega)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(\Omega)}+\|w_n\|_{L^2(\Omega)}\right).$$

Choose $\frac{1}{s}=\frac{2}{p^*}\left(1+\tau\right)$ for some $0<\tau<\frac{p^*}{2}-1$. Then the map $\phi:h\to|A_h|$ satisfies, for all $h>k\geq 0$

$$\phi(h)\leq M^{p^*}\frac{\phi(k)^{1+\tau}}{|h-k|^{p_*}},$$

which implies

$$\phi(h)=0\text{ for }h\geq M\phi(0)^{\frac{\tau-1}{p^*}}2^{1+\frac{1}{\tau}},$$

in other words for any $\tau \in \left(0, \frac{p^*}{2} - 1\right)$,

$$\begin{aligned}
&\|w_n\|_{L^\infty(Q_n)} \\
\leq &\|w_n \chi_n\|_{L^\infty(\Omega)} \\
\leq &2^{\frac{p^*}{2\tau}} C_d\left(\Omega, \frac{1}{R}, \|\gamma_0\|_{L^\infty}\right)\left(\|d_n\|^{\frac{1}{2}}_{L^{\frac{p^*}{p^*-2(1+\tau)}}(\Omega)} \|\nabla u_0\|_{L^\infty(\Omega)} + \|w_n\|_{L^2(\Omega)}\right)
\end{aligned}$$

and by the improved L^2 bound (2.26), this implies that

$$\|w_n\|_{L^\infty(Q_n)} \leq 2^{\frac{p^*}{2\tau}} C_d\left(\Omega, \frac{1}{R}, \|\gamma_0\|_{L^\infty}\right)\left(\|d_n\|_{L^{\frac{p^*}{p^*-2(1+\tau)}}(\Omega)}\right)^{\frac{1}{2}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

which is our statement. □

Proposition 36. *Suppose that Q_n and R_n are not intertwined and*

$$\left(\int_\Omega |d_n|_F^p \, dx\right)^{\frac{1}{p}} \to 0$$

some $p > d/2$, then

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \nabla w_n \cdot \left(\gamma_n - \gamma_0\right)_i dx \quad \overset{*}{\rightharpoonup} \quad M_i \cdot \nabla u_0 \, d\mu$$

in the weak* topology of $C(\overline{\Omega})^*$. By writing it in a different form, this shows that

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\gamma_n - \gamma_0\right) \nabla w_n \, dx \quad \overset{*}{\rightharpoonup} \quad M \nabla u_0 \, d\mu$$

with entries M_{ij} being the scalar weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}} \nabla w_n^i \cdot \left(\gamma_n - \gamma_0\right)_j$. Furthermore, the convergence is uniform with respect to $g \in H^{\frac{1}{2}}(\partial\Omega)$ provided that $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$.

Proof. We use the estimate for the remainder term r_n given by Proposition (25). From (2.34) there holds

$$\varepsilon_n \lesssim \|\nabla \phi\|_{L^\infty(\Omega)} \|d_n\|_{L^1(\Omega)} \left(\|w_n\|_{L^\infty(Q_n)} + \left\|w_n^i\right\|_{L^\infty(Q_n)} \|\nabla u_0\|_{L^\infty(\Omega)}\right).$$

According to (2.49), given any $p > d/2$, we may choose p^* and τ so that

$$\frac{p^*}{p^* - 2(1+\tau)} = p$$

and consequently,

$$\|w_n\|_{L^\infty(Q_n)} \lesssim \left(\|d_n\|_{L^p(\Omega)}\right)^{\frac{1}{2}} \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

As a consequence ,

$$\lim_{n\to\infty}\left|\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega r_n\cdot\nabla\phi\right| = \lim_{n\to\infty}\left|\frac{1}{\|d_n\|_{L^1(\Omega)}}\varepsilon_n\right| = 0.$$

Thus, we have

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\nabla w_n\cdot\left(\gamma_n-\gamma_0\right)_i \quad \overset{*}{\rightharpoonup} \quad M_i\cdot\nabla u_0$$

in $C^0\left(\overline{\Omega}\right)^*$, and by a simillar argument in Proposition 26 , we obtain that the convergence here is uniform for all g in the unit ball of $H^{\frac{1}{2}}(\partial\Omega)$. □

2.3 The representation formula

As a consequence of proposition (36), we are now able to derive the representation formula for $u_n - u_0$ which characterises all the possible limit points of $\left(u_n-u_0\right)(y)$, as $\|d_n\|_{L^1(\Omega)}$ goes to 0 for any $y \in \overline{\Omega\setminus K}$. To derive this representation formula, we introduce the Green's function $G(x,y)$ for the operator $\nabla\cdot\left(\gamma_0\nabla\cdot\right)$. For any $y\in\Omega$, the Green's function $G(x,y)$ is defined as the unique solution to the boundary value problem given by

$$\begin{aligned}\nabla_x\cdot\left(\gamma_0\nabla G(x,y)\right) &= \delta_y \text{ in } \Omega \\ G(x,y) &= 0 \text{ on } \partial\Omega\end{aligned} \tag{2.52}$$

where δ_y denotes the Dirac measure at the point y. Since Ω is assumed to be a C^2 domain and the background conductivity matrix γ_0 belongs to $W^{1,p}(\Omega)$ for $p>d$, the Green's function $G(x,y)$ is continuous $\forall x,y\in(\Omega\times\Omega)\setminus\{x=y\}$ by a result of Widman and Grüter in [28]. In particular, it is shown in Theorem 3.5 in the aforementioned paper that $\nabla G(x,y)\in C^{0,\alpha}\left((\Omega\times\Omega)\setminus\{x=y\}\right)$ for any $\alpha\in(0,1)$. Moreover, there exists a positive constant $C(\Omega)$ such that

$$|\nabla_x G(x,y)| \le C(\Omega)\,|x-y|^{1-d}$$

for $d\ge 3$. A simillar result holds true as well in dimension two as mentioned in [28], although a different approach was used in arriving at that result.

By invoking the continuity of $\nabla_x G(x,y)$ for the operator $\nabla\cdot(\gamma_0\nabla\cdot)$ in $(\Omega\times\Omega)\setminus\{x=y\}$, our main theorem for this chapter is read as follows.

Theorem 37. *Let u_n and u_0 be the solutions to* (2.2) *and* (2.1) *respectively. Let d_n be given by* (2.3)*. Suppose that Assumption 12 holds. Then, there exists a subsequence also denoted by d_n and a matrix valued function $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; d\mu\right)$, where μ is the Radon measure identified as the weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}}|d_n|_F$, such that for any compact set $K_1 \Subset \overline{\Omega}$ with $dist(K_1,K) > 0$, $y \in K_1$ and $x \in K$, the following representation formula holds*

$$\left(u_n - u_0\right)(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij} \frac{\partial u_0}{\partial x_i} \frac{\partial G(x,y)}{\partial x_j} d\mu(x) + Err_n(y)$$

in which, the function $Err_n(y) \in L^\infty\left(K_1\right)$ satisfies $\frac{\|Err_n\|_{L^\infty(K_1)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ uniformly in the unit ball $\left\{ g \in H^{\frac{1}{2}}(\partial\Omega) : \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1 \right\}$.

Proof. Observe that the weak formulation for the solution $w_n = u_n - u_0$ reads

$$-\int_\Omega \gamma_0 \nabla w_n \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega \left(\gamma_n - \gamma_0\right) \nabla u_n \cdot \nabla \phi \, \mathrm{d}x \tag{2.53}$$

for any $\phi \in H_0^1(\Omega)$. Let $y \in K_1$ be a fixed point and let $\phi_m \in C^1\left(\overline{\Omega}\right)$ be a sequence such that $\phi_m \to G(\cdot, y)$ in $W^{1,1}\left(\Omega\right)$ and $\phi_m \to G(\cdot, y)$ in $C^1\left(K\right)$. Using the fact that w_n is smooth away from the set K and the fact that $\gamma_n - \gamma_0$ is supported in K, we may insert ϕ_m into (2.53) and pass to the limit to conclude that

$$-\int_\Omega \gamma_0 \nabla w_n \cdot \nabla_x G(x,y) \mathrm{d}x = \int_\Omega \left(\gamma_n - \gamma_0\right) \nabla u_n \cdot \nabla_x G(x,y) \mathrm{d}x.$$

Thus, by integration by parts with (2.52), this implies that

$$\begin{aligned} \left(u_n - u_0\right)(y) &= \int_\Omega \left(\gamma_n - \gamma_0\right) \nabla u_n \cdot \nabla_x G(x,y) \mathrm{d}x \qquad (2.54) \\ &= \|d_n\|_{L^1(\Omega)} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\gamma_n - \gamma_0\right) \nabla u_n \cdot \nabla_x G(x,y) \mathrm{d}x. \end{aligned}$$

Since we have

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\gamma_n - \gamma_0\right) \nabla u_n \, \mathrm{d}x \overset{*}{\rightharpoonup} M \nabla u_0 \, \mathrm{d}\mu \tag{2.55}$$

in $C^0\left(\overline{\Omega}\right)^*$ for any fixed $g \in H^{\frac{1}{2}}(\partial\Omega)$ and uniformly for all g in the unit ball of $H^{\frac{1}{2}}(\partial\Omega)$, we use the fact that $\forall y \in K_1$, we may find a C^1 function Φ_y such that

$$\begin{aligned} \Phi_y(x) &= G(x,y) \\ \nabla_x \Phi_y(x) &= \nabla_x G(x,y) \quad \forall x \in K, \end{aligned}$$

and subsequently insert $\nabla_x \Phi_y(x)$ into (2.55) to obtain

$$\begin{aligned}\left(u_n - u_0\right)(y) &= \|d_n\|_{L^1(\Omega)} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\gamma_n - \gamma_0\right) \nabla u_n \cdot \nabla_x \Phi_y \mathrm{d}x \\ &= \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij} \frac{\partial u_0}{\partial x_i} \frac{\partial G(x,y)}{\partial x_j} \mathrm{d}\mu(x) + Err_n(y)\end{aligned}$$

where $M \in L^2\left(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu\right)$ denotes the weak * limit of the sequence $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla u_n^i$ and

$$Err_n(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega r_n \cdot \nabla_x G(x,y)\, \mathrm{d}x$$

where r_n is the remainder term from Proposition 25. Recalling from the estimate done in Proposition 25, we have

$$\begin{aligned}\sup_{y\in K_1} \left| \frac{1}{\|d_n\|_{L^1(\Omega)}} Err_n(y) \right| &\leq \left| \int_\Omega r_n \cdot \nabla_x G(x,y)\, \mathrm{d}x \right| \\ &\leq \sup_{x\in K, y\in K_1} |\nabla_x G(x,y)| \, \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \, o(1).\end{aligned}$$

Thus, we deduce that $Err_n(y) \in L^\infty\left(K_1\right)$ and satisfies $\frac{\|Err_n\|_{L^\infty(K_1)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ uniformly in the unit ball $\left\{ g \in H^{\frac{1}{2}}(\partial\Omega) : \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1 \right\}$ which is our statement. □

2.4 Properties of the polarisation tensor M

Polarisation tensors heuristically speaking are the basic building blocks for the asymptotic expansion of the boundary voltage perturbations due to the presence of an inclusion with small volume fraction. The term was coined by Polya, Schiffer and Szego and it is known in the theory of homogenisation as the low volume fraction limit of the effective properties of the diluted two phase composites. It captures the geometrical properties of the inclusion and provides information on the variations of the electrical field ∇u when a material is present in the conducting domain. In some particular cases of small inclusions such as diametrically small domains, the polarisation tensors can be explicitly derived. They have been used successfully in determining the conductivity coefficient γ of the electric potential u_n caused by low volume fraction internal conductivity inhomogeneities, which is essential to answering Calderon's problem, see [6], [29] for instance. From a numerical point of view, it is important for us to understand a few properties regarding the polarisation tensor M . While traditional numerical methods such as the finite elements and finite difference schemes are useful in

approximating solutions to elliptic boundary problems, it is however not feasible to study small inclusions due to the requirement of a very refined mesh around the neighbourhoods of these small objects. Showing the basic properties of M allows us to derive a more stable and accurate algorithm for the numerical computation of the steady-state voltage in the presence of small conductivity inclusions while circumventing the difficulties of traditional numerical schemes. In what follows, we reserve the notation M to denote the polarisation tensor with entries M_{ij} generated by the scalar weak* limit of $\nabla w_n^i \cdot \left(\gamma_n - \gamma_0\right)_j$.

2.4.1 Boundary independence

In this section, we address the boundary independence of the polarisation tensor M. We demonstrate that the result from the section before is indeed a result of gradient fields which are supported in the compact set K . In other words, we show that the matrix M is invariant under a perturbation of its prescribed Dirichlet data g and the exterior domain Ω which compactly contains all the inhomogeneity sets $\cup_n \left(Q_n \cup R_n\right)$.

Theorem 38. *Suppose that Assumption 12 holds. Let $\tilde{\Omega}$ be any bounded regular open set such that $K \subset\subset \tilde{\Omega}$ with $dist(K, \partial\tilde{\Omega}) > 0$. Suppose further that w_n^X belongs to the space X where X is either*

$$H_0^1(\tilde{\Omega}), \quad \tilde{H}^1(\tilde{\Omega}) := \left\{ \phi \in H^1\left(Q\right) : \int_Q \phi = 0 \right\}$$

or

$$H_{\#}^1(\tilde{\Omega}) := \left\{ \phi \in H_{loc}^1\left(\mathbb{R}^d\right) \, : \, \int_Q \phi dx = 0 \text{ and } \phi \quad \tilde{\Omega} - \text{periodic} \right\}.$$

When $X = H_{\#}^1(\tilde{\Omega})$, we take $\tilde{\Omega}$ to be any cube such that $\overline{\Omega} \subseteq \tilde{\Omega}$.

Setting $w_n^X \in X$ to be the unique weak solution to

$$\int_{\tilde{\Omega}} \gamma_n \nabla w_n^X \cdot \nabla \phi = \int_{\tilde{\Omega}} \left(\gamma_n - \gamma_0\right) \nabla u_0 \cdot \nabla \phi \text{ for all } \phi \in X, \tag{2.56}$$

then there exists $C > 0$ such that for any $\varepsilon > 0$

$$\frac{1}{\|d_n\|_{L^1(\tilde{\Omega})}} \| \left(\gamma_n - \gamma_0\right) \nabla \left(w_n - w_n^X\right) \|_{L^1(\tilde{\Omega})} \leq C\left(\varepsilon, d\right) \|d_n\|_{L^1(\tilde{\Omega})}^{\frac{1}{2(d-1+\varepsilon)}}.$$

Proof. Testing (2.56) by w_n^X as before, we have

$$E\left(w_n^X\right) := \int_{\tilde{\Omega}} \gamma_n \nabla w_n^X \cdot \nabla w_n^X \leq C \|d_n\|_{L^1(\tilde{\Omega})} \|\nabla u_0\|_{L^\infty(K)}^2,$$

and by using a Poincaré's inequality adapted to each situation,

$$\|w_n^X\|_{H^1(\tilde{\Omega})} \leq C\|d_n\|_{L^1(\tilde{\Omega})} \|\nabla u_0\|_{L^\infty(K)}.$$

Taking $z_n^X \in X$ to solve the adjoint problem

$$-\nabla \cdot \left(\gamma_0 \nabla z_n^X\right) \quad = \quad w_n^X \text{ in } \tilde{\Omega}$$

with boundary conditions, i.e

B.C	X
$\gamma_0 \nabla z_n^X = 0$	$\tilde{H}^1(\tilde{\Omega})$
$z_n^X = 0$	$H_0^1(\tilde{\Omega})$
z_n^X $\tilde{\Omega}-$periodic	$H_\#^1(\tilde{\Omega})$

, we employ the same argument as Proposition 21 for z_n^X to find

$$\left\|z_n^X\right\|_{W^{2,q}_{loc}(\tilde{\Omega})} + \left\|z_n^X\right\|_{W^{1,q}(\tilde{\Omega})} \lesssim \|w_n\|_{L^q(\tilde{\Omega})}$$

for any $2 \leq q < \infty$. Subsequently, by a similar computation as (2.26), we have

$$\left\|w_n^X\right\|_{L^2(\tilde{\Omega})} \leq C(\varepsilon, d) \|d_n\|_{L^1(\tilde{\Omega})}^{\frac{1}{2}\left(1+\frac{1}{d-1+\varepsilon}\right)} \|\nabla u_0\|_{L^\infty(\tilde{\Omega})}. \tag{2.57}$$

Now, we choose a smooth cut-off function $\chi \in C_c^\infty\left(\tilde{\Omega}\right)$ such that $\chi = 1$ on K, Caccioppoli's inequality writes (2.52)

$$\int_{\tilde{\Omega}} \gamma_n \nabla\left(\chi\left(w_n - w_n^X\right)\right) \cdot \nabla\left(\chi\left(w_n - w_n^X\right)\right) = \int_{\tilde{\Omega}\setminus K} \left(\gamma_0 \nabla\chi \cdot \nabla\chi\right)\left(w_n - w_n^X\right)^2,$$
$$\int_K \gamma_n \nabla\left(w_n - w_n^X\right) \cdot \nabla\left(w_n - w_n^X\right) \leq C\left(\tilde{\Omega}, K\right) \int_{\tilde{\Omega}\setminus K} \left(w_n - w_n^X\right)^2,$$

hence we find that

$$\|\sqrt{\gamma_n}\nabla\left(w_n - w_n^X\right)\|_{L^2(\tilde{\Omega})} \leq C\left(\tilde{\Omega}, K\right) \left\|w_n - w_n^X\right\|_{L^2(\tilde{\Omega}\setminus K)}.$$

As a result of this estimate, we have by Cauchy-Schwarz's inequality,

$$\begin{aligned}
\|\left(\gamma_n - \gamma_0\right)\nabla\left(w_n - w_n^X\right)\|_{L^1(\tilde{\Omega})} &\leq \|d_n\|_{L^1(\tilde{\Omega})}^{\frac{1}{2}} \left(\int_K \gamma_n \nabla\left(w_n - w_n^X\right) \cdot \nabla\left(w_n - w_n^X\right)\right)^{\frac{1}{2}} \\
&\leq C\left(\tilde{\Omega}, K\right) \|d_n\|_{L^1(\tilde{\Omega})}^{\frac{1}{2}} \left\|w_n - w_n^X\right\|_{L^2(\tilde{\Omega}\setminus K)} \\
&\leq C\left(\tilde{\Omega}, K, \varepsilon, d\right) \|d_n\|_{L^1(\tilde{\Omega})}^{1+\frac{1}{2(d-1+\varepsilon)}}.
\end{aligned}$$

where the last line follows from the triangle inequality, (2.57), and Proposition 21. In other words, this estimate shows that the term $\frac{1}{\|d_n\|_{L^1(\tilde{\Omega})}} \left\|\left(\gamma_n - \gamma_0\right)\nabla\left(w_n - w_n^X\right)\right\|_{L^1(\tilde{\Omega})}$ converges to 0, in which, it effect-

ively implies the equivalence of the limiting measure

$$\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\tilde{\Omega})}}\big(\gamma_n-\gamma_0\big)\nabla w_n^X\,\mathrm{dx}=\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\tilde{\Omega})}}\big(\gamma_n-\gamma_0\big)\nabla w_n\,\mathrm{dx}=M\nabla u_0\,\mathrm{d}\mu$$

which is our statement. □

2.4.2 Symmetry

Lemma 39. *The entries of the polarisation tensor M satisfies $M_{ij}=M_{ji}$ μ-almost everywhere in Ω.*

Proof. Using the definition of M_{ij} and revisiting the identity of the remainder term from Proposition 25, we have

$$\begin{aligned}\int_\Omega\left(\Big(\gamma_n-\gamma_0\Big)\nabla w_n\cdot\nabla x_i\right)\phi &= \int_\Omega\left(\Big(\gamma_n-\gamma_0\Big)\nabla w_n^i\cdot\nabla u\right)\phi \quad (2.58)\\ &+ \int_\Omega r_n\cdot\nabla\phi\end{aligned}$$

for any $\phi\in C^1\left(\overline{\Omega}\right)$. Note that by taking $w_n=w_n^j$ where $i\neq j$ in the equation above, we find that

$$\begin{aligned}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega\left(\Big(\gamma_n-\gamma_0\Big)\nabla w_n^j\cdot\nabla x_i\right)\phi=\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega\Big(\Big(\gamma_n-\gamma_0\Big)\nabla w_n^i\cdot\nabla x_j\Big)\,\phi\\ +o(1)\end{aligned}$$

By definition, this implies for any $\phi\in C^1\left(\overline{\Omega}\right)$ that

$$\int_\Omega\phi M_{ji}\,\mathrm{d}\mu=\int_\Omega\phi M_{ij}\,\mathrm{d}\mu,$$

thus, $M_{ij}=M_{ji}$ μ-almost everywhere in Ω. □

2.4.3 Positivity

Lemma 40. *Let u_n^i denote the weak solution to* (2.2) *with $g=x_i$. Furthermore, let E_{ij} and F_{ij} be the weak * limit generated by the following sequence*

$$\begin{aligned}E_{ij}&:=\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\big(\gamma_n-\gamma_0\big)\nabla u_n^i\cdot\nabla x_j\right)\chi_{Q_n}\\ F_{ij}&:=\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\big(\gamma_0-\gamma_n\big)\nabla u_n^i\cdot\nabla x_j\right)\chi_{R_n},\end{aligned}$$

where χ_{Q_n} and χ_{R_n} are the characteristic functions of Q_n and R_n respectively. Fix $\zeta = (\zeta)_{i=1}^d$ to be an arbitrary vector in $\mathbb{R}^d$ and denote

$$U = \sum_{i=1}^{d} x_i \zeta_i \text{ and } U_n = \sum_{i=1}^{d} u_n^i \zeta_i.$$

Then, for any $\phi \in C^1\left(\overline{\Omega}\right)$, and open sets V_n, S_n such that $Q_n \subset V_n \subset (\Omega \setminus R_n)$, $R_n \subset S_n \subset (\Omega \setminus Q_n)$, we may choose smooth cut-off functions η_{V_n} and η_{S_n} which takes the value 1 on V_n and S_n respectively such that the following inequalities hold

$$\begin{aligned} \int_\Omega \sum_{i,j} E_{i,j}\zeta_i\zeta_j \phi \, d\mu &= \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0\right) \nabla U \cdot \nabla U \right) \phi \qquad (2.59) \\ &- \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \eta_{V_n} \phi \left(\gamma_n \nabla \left(U_n - U\right) \cdot \nabla \left(U_n - U\right)\right) + o(1) \end{aligned}$$

$$\begin{aligned} \int_\Omega \sum_{i,j} F_{i,j}\zeta_i\zeta_j \phi \, d\mu &= \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{R_n} \left(\left(\gamma_0 - \gamma_n\right) \nabla U \cdot \nabla U \right) \phi \qquad (2.60) \\ &+ \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \eta_{S_n} \phi \left(\gamma_n \nabla \left(U_n - U\right) \cdot \nabla \left(U_n - U\right)\right) + o(1). \end{aligned}$$

Proof. Let $\zeta = (\zeta)_{i=1}^d$ be an arbitrary vector in $\mathbb{R}^d$. Observe that

$$\begin{aligned} \zeta_i\zeta_j \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0\right) \nabla u_n^i \cdot \nabla x_j \right) \phi &= \zeta_i\zeta_j \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0\right) \nabla x_i \cdot \nabla x_j \right) \phi \\ &+ \zeta_i\zeta_j \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0\right) \nabla w_n^i \cdot \nabla x_j \right) \phi. \end{aligned}$$

Let $Q_n \subset V_n \subset O_n$ be open sets as in the proof of Proposition 35. Take η_{V_n} to be the smooth cut-off function such that $\eta_{V_n} = 1$ on V_n with support contained in $O_n \subset (\Omega \setminus R_n)$ and satisfies $|\nabla \eta_{V_n}| \leq C(\Omega)$. We test the equation satisfied by w_n^j with $\zeta_i\zeta_j w_n^i \eta_{V_n} \phi \in H_0^1(\Omega)$ to find

$$\begin{aligned} & \zeta_i\zeta_j \int_\Omega \gamma_n \nabla w_n^j \cdot \nabla \left(w_n^i \eta_{V_n} \phi \right) \\ = \; & \zeta_i\zeta_j \int_\Omega \eta_{V_n} \phi \left(\gamma_n \nabla w_n^j \cdot \nabla w_n^i \right) + \zeta_i\zeta_j \int_\Omega w_n^i \left(\gamma_n \nabla w_n^j \cdot \nabla \left(\eta_{V_n} \phi \right) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned}\zeta_i\zeta_j\int_\Omega \gamma_n\nabla w_n^j\cdot\nabla\left(w_n^i\eta_{V_n}\phi\right) &= \zeta_i\zeta_j\int_\Omega(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla\left(w_n^i\eta_{V_n}\phi\right)\\ &= \zeta_i\zeta_j\int_\Omega w_n^i(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla\left(\eta_{V_n}\phi\right)\\ &+ \zeta_i\zeta_j\int_\Omega \eta_{V_n}\phi(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla w_n^i.\end{aligned}$$

Rewriting in a slightly different form, we have

$$\begin{aligned}\zeta_i\zeta_j\int_\Omega \chi_{Q_n}\phi(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla w_n^i &= \zeta_i\zeta_j\int_\Omega \eta_{V_n}\phi(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla w_n^i\\ &= \zeta_i\zeta_j\int_\Omega \eta_{V_n}\phi\left(\gamma_n\nabla w_n^j\cdot\nabla w_n^i\right)+e_{n_{ij}}\end{aligned}$$

where

$$e_{n_{ij}}=\zeta_i\zeta_j\int_\Omega w_n^i\left(\gamma_n\nabla w_n^j\cdot\nabla\left(\eta_{V_n}\phi\right)\right)-\zeta_i\zeta_j\int_\Omega w_n^i(\gamma_0-\gamma_n)\nabla x_j\cdot\nabla\left(\eta_{V_n}\phi\right). \tag{2.61}$$

Thus,

$$\begin{aligned}\zeta_i\zeta_j\int_\Omega\chi_{Q_n}\left((\gamma_n-\gamma_0)\nabla u_n^i\cdot\nabla x_j\right)\phi &= \zeta_i\zeta_j\int_\Omega\chi_{Q_n}\left((\gamma_n-\gamma_0)\nabla x_i\cdot\nabla x_j\right)\phi\\ &- \zeta_i\zeta_j\int_\Omega\eta_{V_n}\phi\left(\gamma_n\nabla w_n^j\cdot\nabla w_n^i\right)-e_{n_{i,j}},\end{aligned}$$

and consequently,

$$\begin{aligned}\int_\Omega\sum_{i,j}E_{i,j}\zeta_i\zeta_j\phi\,\mathrm{d}\mu &= \frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega\chi_{Q_n}\left((\gamma_n-\gamma_0)\nabla U\cdot\nabla U\right)\phi\\ &- \frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega\eta_{V_n}\phi\left(\gamma_n\nabla(U_n-U)\cdot\nabla(U_n-U)\right)+o(1)\end{aligned}$$

as the term $e_{n_{i,j}}$ satisfies

$$\begin{aligned}\left|e_{n_{i,j}}\right| &\le |\zeta|^2\left(\left(\int_\Omega\gamma_n\nabla w_n^j\cdot\nabla w_n^j\right)^{\frac12}\left(\int_\Omega\left(w_n^i\right)^2 d_n\nabla\left(\eta_{V_n}\phi\right)\cdot\left(\eta_{V_n}\phi\right)\right)^{\frac12}\right.\\ &\left.+\left(\int_\Omega\gamma_n\nabla x_j\cdot\nabla x_j\right)^{\frac12}\left(\int_\Omega\left(w_n^i\right)^2 d_n\nabla\left(\eta_{V_n}\phi\right)\cdot\left(\eta_{V_n}\phi\right)\right)^{\frac12}\right)\\ &\le C(\Omega,V_n)|\zeta|^2\|\phi\|_{C^1(\overline{\Omega})}\|d_n\|_{L^1(\Omega)}^{\frac12}\left\|w_n^i\right\|_{L^\infty(Q_n)}\\ &\le C(\Omega,V_n)|\zeta|^2\|\phi\|_{C^1(\overline{\Omega})}\|g\|_{H^{\frac12}(\partial\Omega)}\,o\left(\|d_n\|_{L^1(\Omega)}\right)\end{aligned}$$

where we have used the L^∞ estimate from Proposition 35in the last line. Note that since the sets Q_n and R_n are non-intertwined, the constant which appears in the above is uniformly bounded for all n. Using the same strategy,

we may take a simillar cut-off function η_{S_n} such that $\eta_{S_n} = 1$ on S_n and $|\nabla \eta_{S_n}| \leq C(\Omega)$ to obtain

$$\begin{aligned}\zeta_i\zeta_j \int_\Omega \chi_{R_n}\left((\gamma_0-\gamma_n)\nabla u_n^i \cdot \nabla x_j\right)\phi &= \zeta_i\zeta_j \int_\Omega \chi_{R_n}\left((\gamma_0-\gamma_n)\nabla x_i \cdot \nabla x_j\right)\phi \\ &+ \zeta_i\zeta_j \int_\Omega \chi_{R_n}\left((\gamma_0-\gamma_n)\nabla w_n^i \cdot \nabla x_j\right)\phi.\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \sum_{i,j} F_{i,j}\zeta_i\zeta_j\,\phi\,\mathrm{d}\mu &= \frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \chi_{R_n}\left((\gamma_0-\gamma_n)\nabla U\cdot\nabla U\right)\phi \\ &+ \frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \eta_{S_n}\phi\left(\gamma_n\nabla(U_n-U)\cdot\nabla(U_n-U)\right)+o(1)\end{aligned}$$

for any open set S_n such that $R_n \subset S_n \subset (\Omega \setminus Q_n)$. □

Proposition 41. *Let $\eta > 0$ be such that*

$$\frac{1}{1-\sqrt{\eta}}\gamma_0 < \gamma_n \text{ a.e. in } Q_n, \text{ and } \gamma_n < \frac{1}{1+\sqrt{\eta}}\gamma_0 \text{ a.e. in } R_n.$$

the inequalites being understood in the sense of quadratic forms. Then, the symmetric matrices $(E)_{ij}$, $(F)_{ij}$ defined as the above are positive definite and

$$0 \leq \langle E(x)\zeta,\zeta\rangle, \langle F(x)\zeta,\zeta\rangle \leq \left(\frac{1}{\sqrt{\eta}}-1\right)|\zeta|^2,$$

where the inequalities holds for $\mu - a.e.x$ in Ω and for any $\zeta \in \mathbb{R}^d$.

Proof. We show that this inequality holds for $(E)_{ij}$. Let η_{V_n} be the cut-off function as in the proof of Lemma 40.

Take $\phi \in C^1\left(\overline{\Omega}\right)$ such that $\phi > 0$. Thanks to (2.59),

$$\begin{aligned}&\int_\Omega \eta_{V_n}\gamma_n\nabla(U-U_n)\cdot\nabla(U-U_n)\,\phi \\ &= \int_\Omega \chi_{Q_n}(\gamma_n-\gamma_0)\nabla U\cdot\nabla U\,\phi\,\mathrm{d}x \\ &- \int_\Omega \chi_{Q_n}\left((\gamma_n-\gamma_0)\nabla U_n\cdot\nabla U\right)\phi + e_n \\ &= \int_\Omega \chi_{Q_n}\left((\gamma_n-\gamma_0)\nabla(U-U_n)\cdot\nabla U\right)\phi + e_n \\ &\leq \left(\int_\Omega \chi_{Q_n} d_n\nabla U\cdot\nabla U\,\phi\right)^{\frac{1}{2}}\left(\int_\Omega \eta_{V_n}\gamma_n\nabla(U-U_n)\cdot\nabla(U-U_n)\,\phi\right)^{\frac{1}{2}} + |e_n|,\end{aligned}$$

where

$$
\begin{aligned}
|e_n| &= \left| \sum_{i,j} e_{n_{i,j}} \right| \\
&= \left| \sum_{i,j} \zeta_i \zeta_j \int_\Omega w_n^i \left(\gamma_n \nabla w_n^j \cdot \nabla \left(\eta_{V_n} \phi \right) \right) - \zeta_i \zeta_j \int_\Omega w_n^i \left(\gamma_0 - \gamma_n \right) \nabla x_j \cdot \nabla \left(\eta_{V_n} \phi \right) \right|.
\end{aligned}
$$

Thus, using the fact that $a^2 < ab + c \Longrightarrow a^2 < b^2 + 2c$ for positive a, b, c,we find that

$$
\begin{aligned}
\int_\Omega \eta_{V_n} \gamma_n \nabla (U - U_n) \cdot \nabla (U - U_n) \, \phi \quad &\leq \quad \int_\Omega \chi_{Q_n} d_n \nabla U \cdot \nabla U \, \phi + 2 \left| e_n \right| \\
&= \quad \int_\Omega \chi_{Q_n} \left(\gamma_n - \gamma_0 \right) \nabla U \cdot \nabla U \, \phi \\
&- \quad \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0 \right) \gamma_n^{-1} \gamma_0 \nabla U \cdot \nabla U \right) \phi \\
&+ \quad 2 \left| e_n \right|.
\end{aligned}
$$

Consequently, as $n \to \infty$ we have

$$
\begin{aligned}
&\frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \eta_{V_n} \gamma_n \nabla (U - U_n) \cdot \nabla (U - U_n) \, \phi \\
+ &\frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \left(\chi_{Q_n} \left(\gamma_n - \gamma_0 \right) \gamma_n^{-1} \gamma_0 \nabla U \cdot \nabla U \right) \phi \\
\leq &\frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(\gamma_n - \gamma_0 \right) \nabla U \cdot \nabla U \, \phi \mathrm{d}x
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\int_\Omega \sum_{i,j} E_{i,j} \zeta_i \zeta_j \, \phi \, \mathrm{d}\mu \quad &\geq \quad \lim_{n \to \infty} \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(d_n \left(\gamma_n - \gamma_0 \right)^{-1} \gamma_0 \nabla U \cdot \nabla U \right) \phi \\
&\geq \quad 0. \qquad (2.62)
\end{aligned}
$$

For the upper bound, it is straightforward from (2.59) that

$$
\begin{aligned}
&\int_\Omega \sum_{i,j} E_{i,j} \zeta_i \zeta_j \, \phi \, \mathrm{d}\mu \\
\leq &\frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0 \right) \nabla U \cdot \nabla U \right) \phi \\
\leq &\frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\int_\Omega \chi_{Q_n} \left(d_n \nabla U \cdot \nabla U \right) \phi + \int_\Omega \left(\chi_{Q_n} \left(\gamma_n - \gamma_0 \right) \gamma_n^{-1} \gamma_0 \nabla U \cdot \nabla U \right) \phi \right).
\end{aligned}
$$

Note that from the assumption of $\left(1 + \frac{\sqrt{\eta}}{1 - \sqrt{\eta}} \right) \gamma_0 < \gamma_n$ a.e. in Q_n, the latter

term satisfies

$$\begin{aligned} & \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(\left(\gamma_n - \gamma_0\right) \gamma_n^{-1} \gamma_0 \nabla U \cdot \nabla U \right) \phi \\ = \; & \frac{1}{\|d_n\|_{L^1(\Omega)}} \int_\Omega \chi_{Q_n} \left(d_n \left(\gamma_n - \gamma_0\right)^{-1} \gamma_0 \right) \nabla U \cdot \nabla U \, \phi \\ \leq \; & \left(\frac{1}{\sqrt{\eta}} - 1 \right) \int_\Omega |\zeta|^2 \, \phi \, \mathrm{d}\mu \end{aligned} \tag{2.63}$$

Altogether, we combine estimates (2.62) and (2.63) to obtain

$$0 \leq \quad E_{ij} \zeta_i \zeta_j \quad \leq \left(\frac{1}{\sqrt{\eta}} - 1 \right) |\zeta|^2$$

where the inequality holds for any $\zeta \in \mathbb{R}^d$, and $\mu - a, e, x$ in Ω. We omit the proof for $(F)_{ij}$ as it follows directly from the same method. □

Corollary 42. *Let M denote the polarisation tensor with entries M_{ij} generated by the scalar weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}} \nabla w_n^i \cdot \left(\gamma_n - \gamma_0\right)_j$. Then, if the sets Q_n and R_n are non-intertwined and non-empty, we may write M as the difference between two symmetric positive definite matrices which have disjoint supports in Ω. In particular, we may write*

$$M \, d\mu = E \, d\mu_Q - F \, d\mu_R,$$

in which, the Radon measures μ_Q and μ_R are supported in $\cup_n Q_n$ and $\cup_n R_n$ respectively.

Proof. The proof follows directly from the definition of M and the Proposition 41. □

In the following example, we shall demonstrate that the lower bound obtained in Proposition 41 is indeed optimal.

Example 43. Consider the inclusion set

$$Q_n := \left\{ (x,y) \subset \mathbb{R}^2 : \frac{x^2}{\cosh^2(\varepsilon_n)} + \frac{y^2}{\sinh^2(\varepsilon_n)} \leq 1 \right\}$$

consisting of collapsing ellipses as $\varepsilon_n \to 0$. Take

$$\gamma_n(x) = \begin{cases} 1 & x \in Q_1 \setminus Q_n \\ \lambda_n & x \in Q_n, \end{cases}$$

where $\lambda_n > 0$ is a positive real number. The solution u_n^i to the equation

$$\begin{aligned} -\nabla\cdot\left(\gamma_n \nabla u_n^i\right) &= 0 \quad \text{in} \quad Q_1 \\ u_n^i &= x_i \quad \text{on} \quad \partial Q_1 \end{aligned} \tag{2.64}$$

is computed explicitly by elliptic coordinates as

$$\begin{pmatrix} u_n^1 \\ u_n^2 \end{pmatrix} = \begin{pmatrix} (A_{11}\cosh r + A_{12}\sinh r)\cos\theta \\ (A_{21}\cosh r + A_{22}\sinh r)\sin\theta \end{pmatrix} \chi_{\Omega\setminus Q_n} + \begin{pmatrix} A_{13}\cosh r\cos\theta \\ A_{23}\sinh r\sin\theta \end{pmatrix} \chi_{Q_n}$$

with

$$\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} = \begin{bmatrix} -\frac{K}{\tanh 1 - K} \\ \frac{1}{\tanh 1 - K} \\ \frac{1}{\lambda_n - 1}\frac{\coth \varepsilon_n}{\tanh 1 - K} - \frac{1}{\lambda_n - 1}\frac{1}{\tanh 1 - K}\tanh \varepsilon_n \end{bmatrix}$$

$$\begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} = \begin{bmatrix} \frac{1}{\coth 1 - L} \\ -\frac{L}{\coth 1 - L} \\ -\frac{1}{\lambda_n - 1}\frac{\coth \varepsilon_n}{\coth 1 - L} + \frac{1}{\lambda_n - 1}\frac{1}{\coth 1 - L}\tanh \varepsilon_n \end{bmatrix}$$

where K and L are defined as

$$K := \frac{\lambda_n}{\lambda_n - 1}\left(\tanh \varepsilon_n - \frac{1}{\lambda_n}\coth \varepsilon_n\right) \quad , \quad L := \frac{\lambda_n}{\lambda_n - 1}\left(\coth \varepsilon_n - \frac{1}{\lambda_n}\tanh \varepsilon_n\right).$$

Note that $u_n^i = A_{i,3} x_i$ for $x \in Q_n$.

Take $\phi \in C\left(\overline{\Omega}\right)$, we compute

$$\begin{aligned} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - I_d\right)\nabla u_n^i \cdot \nabla x_j \, \phi \, \mathrm{d}x &= \left(\lambda_n - 1\right)\int_{Q_n} A_{i,3}\frac{\nabla x_i \cdot \nabla x_j}{\|d_n\|_{L^1(\Omega)}}\,\phi \,\mathrm{d}x \\ &= \frac{\operatorname{sgn}(\lambda_n - 1)\lambda_n}{\left|\left(\lambda_n - 1\right)\right|} A_{i,3}\delta_{ij}. \end{aligned}$$

Noticing that

$$A_{1,3} \longrightarrow 1 \quad \& \quad A_{2,3} \longrightarrow 0$$

as $\lambda_n \to \infty$, moreover,

$$A_{1,3} \longrightarrow 1 \quad \& \quad A_{2,3} \longrightarrow \frac{1}{\lambda_n}$$

as $\lambda_n \to 0$, we compute the polarisation tensors $(E)_{ij}$ and $(F)_{ij}$ associated to the high contrast and low contrast ellipse inclusion respectively as

$$(E)_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \& \quad (F)_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Remark 44. The lower bounds for $(E)_{ij}$ and $(F)_{ij}$ derived in (41) are indeed optimal as shown in the example above. Moreover, it is worth noting that $(E)_{ij}$ approaches the lower bound 0 at a scale which is independent of the rate which the volume of the inclusion set $|Q_n| \to 0$. Denote by $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ to be the smallest and largest eigenvalues of a matrix in $\mathbb{R}^{d\times d}$ respectively. We observe the following,

$$\begin{aligned}
\int_\Omega \sum_{i,j} E_{i,j}\zeta_i\zeta_j\,\phi\,\mathrm{d}\mu &\geq \lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \chi_{Q_n}\left(d_n(\gamma_n-\gamma_0)^{-1}\gamma_0\nabla U\cdot\nabla U\right)\phi \\
&\geq \lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \chi_{Q_n}\frac{\lambda_{min}(\gamma_0)}{\lambda_{max}(\gamma_n-\gamma_0)}d_n\nabla U\cdot\nabla U\,\phi \\
&\geq \left(\int_\Omega \lambda_{min}(\gamma_0)\,\phi\,|\zeta|^2\,\mathrm{d}\mu\right)\lim_{n\to\infty}\left(\frac{1}{\|\lambda_{max}(\gamma_n-\gamma_0)\|_{L^\infty(Q_n)}}\right)
\end{aligned}$$

Thus, at a scale of $\|\lambda_{max}(\gamma_n-\gamma_0)\|_{L^\infty(Q_n)}$,

$$\|\lambda_{max}(\gamma_n-\gamma_0)\|_{L^\infty(Q_n)}\left(\int_\Omega \sum_{i,j} E_{i,j}\zeta_i\zeta_j\,\phi\,\mathrm{d}\mu\right) \geq \left(\int_\Omega \lambda_{min}(\gamma_0)\,\phi\,|\zeta|^2\,\mathrm{d}\mu\right).$$

This fact is unsurprising. Note that when $\gamma_n = \gamma_1$, a fixed isotropic conductivity matrix, the authors of the paper [12] showed that the polarisation tensor M_{ij} associated to γ_0 and γ_1 indeed satisfies

$$\min\left\{1,\frac{\gamma_0(x)}{\gamma_1(x)}\right\}|\zeta|^2 \leq M_{ij}(x)\zeta_i\zeta_j \leq \max\left\{1,\frac{\gamma_0(x)}{\gamma_1(x)}\right\}|\zeta|^2$$

$\mu - a.e.x$ in Ω. Therefore, by formally sending $\gamma_1 \to \infty$ in this inequality, we effectively recover the bounds for $(E)_{ij}$ in the high contrast conductivity setting.

Chapter 3

Quantitative Jacobian bounds for the conductivity equation in high contrast composite media

3.1 Preliminaries

Given Ω a convex polygonal domain in $\mathbb{R}^d$ where $d \geq 3$, consider the following elliptic boundary value problem

$$\begin{aligned} -\mathrm{Div}(\gamma DU) &= 0 \quad \text{in} \quad \Omega, \\ U &= \phi \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

with $U = (u_1, \cdots, u_d)$ and $\phi \in C^{1,\sigma}\left(\mathbb{R}^d; \mathbb{R}^d\right)$, for some $\sigma \in (0,1)$. In this chapter, we assume that the conductivity matrix γ is a real-valued function such that $\gamma(\Omega) = \{1, k\}$ with $k > 0$. More precisely,

$$\gamma = \Big(1 + (k-1)\chi_Q\Big) I_d \tag{3.2}$$

where χ_Q denotes the characteristic function of a sub domain $Q \subset \Omega$, which we assume to be a C^2 open connected inclusion, with $d(\partial\Omega, Q) > 0$. We are interested in studying the Jacobian determinant of U, that is,

$$J = \det DU \text{ in } \Omega \setminus Q.$$

The goal of this chapter is to address the following problem:

Problem 45. Given suitable boundary conditions ϕ to the PDE (3.1), is it possible to construct a class of topologically simple structures Q such that the Jacobian determinant of the solution $\det DU$, corresponding to a two phase conductivity coefficient matrix changes sign in some open neigh-

bourhoods of Ω? If so, could we provide a quantitative estimate to the Lebesgue measures of these sets?

Note that since γ is not smooth, the positivity (and negativity) of the Jacobian determinant must be understood in a measure theoretical sense. In particular, we say that $\det DU$ changes sign in Ω if

$$\left|J^{-1}\left((-\infty,0)\right)\right| \times \left|J^{-1}\left((0,\infty)\right)\right| > 0. \tag{3.3}$$

Specifically, we want to construct a class of inclusions Q for which J changes sign, that is, in which, we want to establish lower bounds for the measures of

$$J_- = J^{-1}\left((-\infty,0)\right) \text{ and } J_+ = J^{-1}\left((0,\infty)\right), \tag{3.4}$$

by exhibiting open subsets included in both sets. It is known for example in [32], that in this setting J is piecewise continuous, and thus the sets appearing in (3.3) are indeed measurable. Unlike the two dimensional case where such inclusions do not exist [4, 2], the situation is completely different in higher dimensions ($d \geq 3$). As mentioned in the introduction, counterexamples to the positivity of the Jacobian determinant for harmonic extensions have been known for several decades see [31, 37, 33] and [9]. The dimension of the ambient space plays an important role in this problem. In two dimensions, The Radó–Choquet–Kneser Theorem, and its extension to γ-harmonic maps [3, 1] states that given a bi-continuous map from the boundary $\partial\Omega$ of a two dimensional domain Ω onto the boundary $\partial\Sigma$ of a convex two dimensional domain Σ generates a harmonic extension of that map within the domain which is a diffeomorphism from Ω to Σ. In particular, for any $\gamma \in L^\infty(\Omega)$ bounded below by a positive constant, the Jacobian determinant belongs to the space of functions of bounded mean oscillation

$$\log\det DU \in BMO\big(\Omega\big),$$

therefore either J_+ or J_- is null.

This chapter is structured as follows. In Section 3.1.1 we show that assumption 46 is sufficient to guarantee that both J_+ and J_- are non empty provided k is large enough. We adapt here a method introduced in [9], and show that it is sufficient to focus on the study of the Jacobian determinant on particular lines. The next two sections are devoted to quantitative estimates. In section 3.2 we turn to the case $k = \infty$, and establish a counterpart to the main theorem Theorem (9) in that case. In the last section 3.3, we use layer potential estimates to establish our main result by means of an explicit convergence rate of Jacobian determinant when the contrast $k \to \infty$

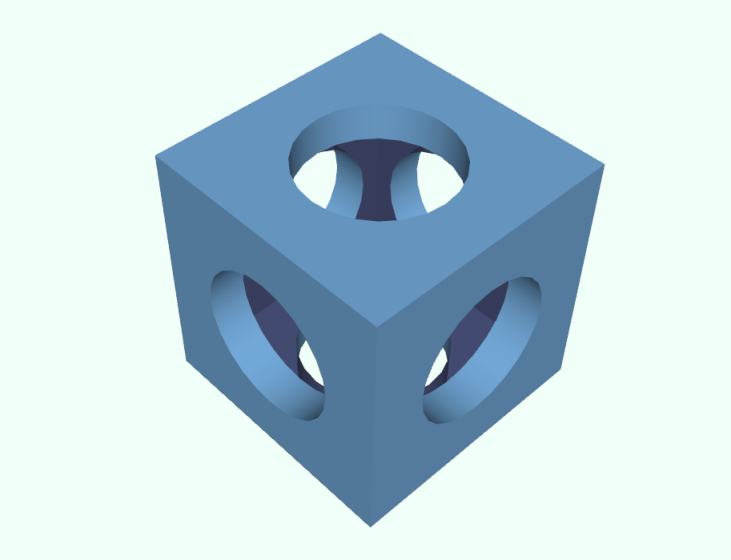


Figure 3.1: Sign changing structure

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3.1.1 Symmetrical inclusions with sign-changing Jacobian determinants

In this section, we derive a simplified form for the Jacobian determinant $\det DU$. The key observation here which was originally shown in the work by Briane and Milton [9], is that the Jacobian matrix DU is in fact diagonalisable along its principal coordinate axes $\Omega \cap (\mathbb{R}e_i)$ under approriate assumptions on the symmetries of the prescribed boundary data ϕ , the exterior domain Ω and the underlying inclusion Q . We summarise a class of inclusions Q here which is sufficient for us to obtain the simplification of $\det DU$. In particular, we assume that the following holds:

Assumption 46. *We assume*

- *For any $x = (x_1, \dots, x_d) \in \Omega$ (resp. Q), $(\pm x_1, \dots, \pm x_d) \in \Omega$ (resp. Q).*
- *The inclusion $Q \subset \Omega$ is closed and the interior of is non empty and Q connected.*
- *The (hyper)surface ∂Q is C^2 and orientable,*
- *The sets $\Omega \setminus Q$ and $(\Omega \setminus Q) \cap (0,\infty)^d$ are connected.*
- *For $i \in \{1, \dots, d\}$, $Q \cap (\mathbb{R}e_i) \neq \emptyset$.*
- *The origin isn't in Q, and $d(0, Q) > 0$.*

An example of such a structure includes

We introduce the following class of boundary conditions which are sufficient for obtaining a sign-changing Jacobian determinant.

Definition 47. Given $\alpha > 1$, we say that $\phi \in C^{1,\sigma}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ for some $\sigma \in (0,1)$ is an α-admissible boundary condition if

1. For all $x \in \partial\Omega$, and every $i \in \{1,\dots,d\}$,

$$e_i \cdot \phi(x) = -e_i \cdot \phi(x - 2x \cdot e_i) \text{ and } e_i \cdot \phi(x) = e_i \cdot \phi\left(x - 2x \cdot e_j\right) \text{ for any } j \neq i.$$

2. For all $x \in \partial\Omega$, and every $i \in \{1,\dots,d\}$,

$$\alpha^{-1} \|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)} e_i \cdot x \leq e_i \cdot \phi(x) \leq \alpha \|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)} e_i \cdot x.$$

Remark 48. Note that if $\phi = (\phi_1, \cdots, \phi_d)$ is an α-admissible boundary condition then ϕ_i is odd with respect to the variable x_i.

Remark 49. Moreover, observe that not all boundary conditions ϕ leads to a sign-changing Jacobian determinant. When ϕ is a constant and $\gamma = I_d$, the Jacobian determinant is trivially 0 everywhere.

The following Lemma which shows the simplification of the Jacobian determinant $\det DU$ is key to showing the sign-changing constrain when the Assumption (46) holds as we shall see.

Lemma 50. *Let $\gamma = \left(1 + (k-1)\chi_Q\right) I_d$ be as the above, and suppose assumption* (46) *holds and that ϕ is α-admissible. If U satisfies the Dirichlet problem*

$$\begin{aligned} -\nabla \cdot (\gamma DU) &= 0 \quad \text{in} \quad \Omega \\ U &= \phi \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.5}$$

then the Jacobian matrix of the solution U is diagonalisable along the line $L := \Omega \cap \left(\mathbb{R} \times \{0\}^{d-1}\right)$ and the Jacobian determinant is given by

$$\det DU(s) = \partial_1 U_1(s) \times \ldots \times \partial_d U_d(s) \text{ for all } s \in L.$$

Proof. We first show that the solution U_i for $i = 1,..,d$ corresponding to the scalar equation

$$\begin{aligned} -\nabla \cdot (\gamma \nabla U_i) &= 0 \\ U_i &= \phi_i \end{aligned}$$

is odd with respect to x_i and even with respect to x_j for $i \neq j$. Letting $x = (x_i, z)$ where $z \in \mathbb{R}^{d-1}$ denotes the other remaining $d-1$ variables, we denote by $V_i(x) = -U_i(-x_i, z)$. Notice that from the symmetry of Ω ,

Q and ϕ, we have $V_i \in H^1_{\phi_i}(\Omega)$. Moreover,

$$\begin{aligned} \partial_i V_i(x_i,z) &= \partial_i U_i(-x_i,z) \\ \partial_j V_i(x_i,z) &= -\partial_j U_i(-x_i,z) \quad \text{for} \quad i \neq j \end{aligned}$$

Using the symmetry of γ and denoting by $T_i : \Omega \mapsto \Omega$ the change of variables mapping given by $T_i(x) = (x_1, \cdots, -x_i, \cdots, x_d)$, a straightforward computation yields

$$\begin{aligned} \int_\Omega \gamma DV_i \cdot DV_i \mathrm{d}x &= \int_\Omega \gamma DU_i(-x_i,z) \cdot DU_i(-x_i,z)\ \mathrm{d}x_i\ \mathrm{d}z \\ &= \int_{T_i(\Omega)} \gamma DU_i(\tau,z) \cdot DU_i(\tau,z)\,|\det T_i|\ \mathrm{d}\tau\ \mathrm{d}z \\ &= \int_\Omega \gamma DU_i \cdot DU_i\ \mathrm{d}x. \end{aligned} \tag{3.6}$$

Considering the variational formulation for U_i, we note that U_i is the unique minimizer of the Dirichlet energy, namely U_i satisfies

$$\int_\Omega \gamma DU_i \cdot DU_i\ \mathrm{d}x = \min_{V \in H^1_{\phi_i}(\Omega)} \int_\Omega \gamma DV \cdot DV\ \mathrm{d}x.$$

Thus, by the uniqueness of minimisers, we deduce that $U(x_i,z) = V_i(x_i,z) = -U_i(-x_i,z)$. The evenness with respect to x_j is established by similar arguments. In particular, we may apply the same argument to the function $W_i(x) = U_i(x_i,-z)$ to show that the function U_i is indeed even with respect to $x_j\ \forall i \neq j$. Since $U_i(x_i,z) = U_i(x_i,-z)$ for any $(x_i,z) \in \Omega$, a straightforward computation shows that for $i \neq j$,

$$\partial_i U_j(x_j,z) = -\partial_i U_j(x_j,-z)$$

and for $i = j$,

$$\partial_i U_i(x_i,z) = \partial_i U_i(-x_i,z)$$

altogether, this implies that for $y = (x_1, -x_2, \ldots, -x_d)$,

$$DU(y) = \begin{bmatrix} \partial_1 U_1 & 0 & \ldots & 0 \\ 0 & \partial_2 U_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \partial_d U_d \end{bmatrix}(x) - \begin{bmatrix} 0 & \partial_1 U_2 & \ldots & \partial_1 U_d \\ \partial_2 U_1 & 0 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \partial_{d-1} U_d \\ \partial_d U_1 & \partial_d U_2 & \ldots & 0 \end{bmatrix}(x)$$

In particular, for $s = (x_1, 0)$,

$$DU(s) = \begin{bmatrix} \partial_1 U_1 & 0 & \dots & 0 \\ 0 & \partial_2 U_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \partial_d U_d \end{bmatrix} (s) \quad \forall s \in L.$$

Subsequently,

$$\det DU\,(s) = \partial_1 U_1 \times \cdots \times \partial_d U_d(s) \tag{3.7}$$

□

Definition 51. Given $k > 0$, we denote by $U_k \in H^1\left(\Omega; \mathbb{R}^d\right)$ the solution to the Dirichlet problem

$$\begin{aligned} -\nabla \cdot \left(\left(1 + (k-1)\chi_Q\right) DU_k\right) &= 0 \quad \text{in} \quad \Omega \\ U_k &= \phi \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.8}$$

We referred to this solution as U previously; the k dependence is underlined because we will compare U_k to its perfectly conducting limit. We write

$$J_k = \det DU_k, \tag{3.9}$$

and

$$J_\infty = \det DU_\infty, \tag{3.10}$$

where U_∞ is defined in (3.11).

Lemma 52. *Suppose that assumption 46 holds and that ϕ is α-admissible. Let U_k denote the solution to* (3.8) *with conductivity $\gamma_k = \left(1 + \left(k-1\right)\chi_Q\right) I_d$. Then U_k strongly converges in $H^1(\Omega; \mathbb{R}^d)$ to U_∞ as $k \to \infty$ where U_∞ satisfies*

$$\begin{aligned} -\Delta U_\infty &= 0 \text{ in } \Omega \setminus Q, \\ U_\infty &= \phi \text{ on } \partial\Omega, \\ U_\infty &= 0 \text{ in } Q. \end{aligned} \tag{3.11}$$

Proof. Consider the variational formulation for U_k given by

$$\int_\Omega \gamma_k DU_k : DU_k \, \mathrm{d}x = \min_{v \in H^1_\phi\left(\Omega; \mathbb{R}^d\right)} \int_\Omega \gamma_k DV : DV \, \mathrm{d}x,$$

with $\gamma_k = \left(1 + (k-1)\chi_Q\right) I_d$. Multiplying by a cut-off function, we may construct a function $f \in H^1_\phi(\Omega; \mathbb{R}^d)$ such that f is a constant vector on Q

and

$$\begin{aligned}\int_\Omega \gamma_k DU_k : DU_k \, \mathrm{d}x &= k\int_Q DU_k : DU_k \, \mathrm{d}x + \int_{\Omega\setminus Q} DU_k : DU_k \, \mathrm{d}x \\ &\leq \int_{\Omega\setminus Q} Df : Df \, \mathrm{d}x.\end{aligned}$$

This implies that

$$\|U_k\|_{H^1(\Omega;\mathbb{R}^d)} \leq M$$

for some positive constant M. Thus, by taking a subsequence which we still denote it by U_k , we have $U_k \rightharpoonup U_\infty$ for some function U_∞ in $H^1(\Omega)$. By the uniform H^1 bound of U, we may infer that

$$\|DU_k\|_{L^2(Q)} \to 0 \quad \text{in} \quad Q$$

which implies that the limiting function U_∞ is a constant vector in Q. Now, consider the functional

$$I : \mathscr{M} \to \mathbb{R}$$

defined as

$$I(U) := \int_{\Omega\setminus Q} DU : DU \, \mathrm{d}x$$

with arguments taken from the constraint set

$\mathscr{M} := \Big\{ U \in H^1_\phi(\Omega;\mathbb{R}^d) : U = \text{constant on } Q \Big\}$. Since $\mathscr{M}$ is a closed subspace of $H^1_\phi(\Omega;\mathbb{R}^d)$, the functional I is convex continuous, and therefore it admits a unique minimiser in the space $\mathscr{M}$. We have

$$\min_{V\in\mathscr{M}} \int_\Omega DV : DV \, \mathrm{d}x \leq \int_{\Omega\setminus Q} DU_\infty : DU_\infty \, \mathrm{d}x,$$

and, on the other hand,

$$\int_\Omega \gamma_k DU_k : DU_k \, \mathrm{d}x \leq \inf_{V\in\mathscr{M}} \int_\Omega DV : DV \, \mathrm{d}x \tag{3.12}$$

as $\mathscr{M} \subset H^1_{I_d}(\Omega;\mathbb{R}^d)$. Using the fact that norms are weakly lower semicontinuous, we deduce

$$\begin{aligned}\int_\Omega DU_\infty : DU_\infty \, \mathrm{d}x &\leq \liminf_{k\to\infty} \int_\Omega DU_k : DU_k \, \mathrm{d}x \\ &\leq \liminf_{k\to\infty}(k) \int_Q DU_k : DU_k \, \mathrm{d}x \\ &+ \int_{\Omega\setminus Q} DU_k : DU_k \, \mathrm{d}x \\ &\leq \inf_{V\in\mathscr{M}} \int_\Omega DV : DV \, \mathrm{d}x,\end{aligned} \tag{3.13}$$

hence, the weak limit U_∞ minimises the functional I in $\mathscr{M}$. Finally, by

(3.12), we have

$$\limsup_{k\to\infty}\left((k)\int_Q DU_k : DU_k \,\mathrm{d}x + \int_{\Omega\setminus Q} DU_k : DU_k \,\mathrm{d}x\right) \;\leq\; \int_\Omega DU_\infty : DU_\infty \,\mathrm{d}x.$$

Combining this fact with (3.13), we deduce that the sequence of energies converges $I_k(U_k) \to I(U_\infty)$. Hence, by the uniform convexity of L^2 norms, the sequence DU_k converges strongly in L^2 to the limit DU_∞ where U_∞ satisfies the equation (3.11) as claimed. Finally, for any $i \in \{1,\dots,d\}$, we have that $(U_k)_i$ is odd with respect to the variable x_i as in Lemma 50, and $U_k \to U_\infty$ in $H^1(\Omega)$. Thus, the limit $(U_\infty)_i$ is also odd in x_i, and we obtain that $(U_\infty)_i(\Pi_i \cap \Omega) = 0$. Since Q is a connected component of Ω, we infer that $(U_\infty)_i$ must be identically 0 on the whole of Q. Note that since U_∞ is the unique solution to the PDE (3.11), we may repeat the same argument as before for any subsequence of U_k to show that the full sequence U_k converges to U_∞ as $k \to \infty$, which is our statement □

We now proceed to show that Assumption 46 and α-admissible boundary conditions 47 are sufficient to obtain sign changing structures.

Proposition 53. *Under the assumptions of Lemma 52, there holds*

$$\left|J_\infty^{-1}(0,\infty)\right| \times \left|J_\infty^{-1}(-\infty,0)\right| > 0. \tag{3.14}$$

Proof. In this proof, we write $U_\infty = (U_i)_{1\leq i\leq d}$. Given $i = 1,\dots,d$, consider the connected region $D_i := \{x \in \Omega\setminus Q : x_i > 0\} \subset \Omega\setminus Q$ in the upper half plane $x_i \geq 0$. From (3.11), we have that U_i is harmonic in D_i, $U_i|_{\partial D_i} \geq 0$ and $U_i \geq \alpha^{-1}(\phi)x_i$ on $\partial\Omega \cap D_i$. Thus, thanks to weak maximum principle, this implies that $U_i > 0$ in the interior of D_i.

Observe that since U_1 is odd with respect to x_1 and even with respect to the other variables, we have $U_1(0) = 0$. By assumption, $B(0,\varepsilon)\cap Q = \emptyset$ and $\mathbb{R}e_1 \cap Q \neq \emptyset$, therefore there exists $s_1 > 0$ such that $x^* = s_1e_1 \in \left(\mathbb{R}^+ \times \{0\}^{d-1}\right)\cap\partial Q$, and $U_1(se_1) > 0$ for any $0 < s < s_1$. In particular, we have $F : s \to U_1(se_1)$ satisfies

$$F(0) = F(s_1) = 0 \text{ and } F(s) > 0 \text{ on } (0,s_1).$$

Since Q is a C^2 inclusion, elliptic regularity asserts that the solution U is smooth in the interior of $\Omega\setminus Q$ and is continuous up to the boundary of Q, hence, we infer that $F'(s) = \frac{\partial U_1}{\partial x_1}$ changes signs for some $s \in (0,s_1)$.

Next, we show that for $i \neq 1$, $\frac{\partial U_i}{\partial x_i}$ preserves a constant sign along the line $\ell = (0,s_1)\times\{0\}^{d-1}$. To see this, we note that $\ell \subset \left(\Omega\setminus Q\right)$. For any point $x \in \ell$, we may find an open ball centred at x with radius r_x so that

$B(x, r_x) \subset \Omega \setminus Q$. In particular, $B_{i,x} = B\left(x + \frac{1}{2} r_x e_i, \frac{1}{2} r_x\right) \subset D_i$ and

$$\overline{B_{i,x}} \cap \mathbb{R} e_i = x.$$

Since U_i is strictly positive in the open ball $B_{i,x}$ and $U_i(x) = 0$, Hopf's lemma (see Lemma 96) ensures that

$$\frac{\partial U_i}{\partial x_i}(x) > 0.$$

Since x is chosen arbitrary on the line ℓ, together with the continuity of DU, we can conclude

$$|\{x \in \Omega \setminus Q : J_\infty > 0\}| \times |\{x \in \Omega \setminus Q : J_\infty < 0\}| > 0.$$

□

Remark 54. Note that we have assumed in (46) that the material Q touches the (hyper)lines $Q \cap (\mathbb{R} e_i) \neq \emptyset$ at least twice for all principal directions e_i. Strictly speaking, this fact is not necessary for $\det DU_\infty$ to change sign in $\Omega \setminus Q$. In fact, we only need Q to touch the (hyper)lines twice for one particular direction to ensure that the Jacobian determinant of DU_∞ changes sign along that direction. We made this assumption here for the purpose of obtaining the largest possible volume estimates for the sign-changing sets of $\det DU_\infty$ as we shall see later on in the next section.

As a consequence of the Proposition above, it follows from a perturbation argument that the sign-changing phenomenon holds uniformly for all $k \geq k_0$, for some large positive k_0 contained in the following proposition.

Proposition 55. *Suppose that assumption 46 holds and that ϕ is α-admissible. There exist $k_0 > 0$ and $\delta_0 > 0$ such that for $k \geq k_0$, the Jacobian determinant J_k given by* (3.9) *satisfies*

$$|\{x \in \Omega \setminus Q : J_k > 0\}| \times |\{x \in \Omega \setminus Q : J_k < 0\}| \geq \delta_0.$$

Proof. Observing that the solution U_k is harmonic and hence smooth away from the inclusion Q, we may infer from standard regularity estimates that in any compact subset $K \subset \Omega \setminus Q$, the sequence of functions $\{U_k\}_{k \geq k_0}$ is uniformly bounded in C^2 norm. By Ascoli-Arzela's theorem, and the limiting argument of Lemma 52, U_k converges to U_∞ in C^1 norm as $k \to \infty$ in any fixed set K. Thus, in some compact subset $K_+ \subset \{x \in \Omega \setminus Q : J_\infty > 0\}$ and $K_- \subset \{x \in \Omega \setminus Q : J_\infty < 0\}$, we have

$$\det DU_k \to \det DU_\infty$$

converging uniformly as $k \to \infty$ in $K_+ \cup K_-$. Hence, $\exists k_0(Q) > 1$, and $\delta_0 > 0$ s.t .$\forall k \geq k_0$

$$|\,|\{x \in \Omega \setminus Q : J_k > 0\}| \times |\{x \in \Omega \setminus Q : J_k < 0\}| > \delta_0,$$

which is our thesis. □

Considering the fact that realisable materials correspond to a finite value contrast conductivity, it is important for practical purposes to quantify the magnitude of the parameter k_0 in which the sign-changing phenomenon occurs. We address this issue in the last section of this chapter, where we provide a stronger, quantitative version of Theorem 72.

Remark 56. The assumption that Q is a C^2 inclusion may be suitably relaxed. Observe that since U is harmonic in the open set $\Omega \setminus Q$, thus smooth in the interior of $\Omega \setminus Q$ by elliptic regularity theory, we note that the only argument which was needed in the proof above was the fact that U belongs to $C^0\left(\overline{\Omega \setminus Q}\right)$. This property for U is achievable as long as points on the topological boundary of $\Omega \setminus Q$ consists of regular points to the Laplace operator see Theorem 2.14 and Section 2.9 in [24]. Loosely speaking, a regular point for the Laplace operator is a point on ∂Q such that a "barrier" function exists, in which, it may be determined by Wiener's criterion, which is to say that the sum

$$\sum_{j=0}^{\infty} \frac{\mathrm{cap}_2\left\{x \in (\Omega \setminus Q)^c : |x - x_0| \leq \lambda^j\right\}}{\lambda^{j(d-2)}}$$

diverges for any fixed $\lambda \in (0,1)$ depending on x_0 , where cap_2 here denotes the H^1 capacity measure of a set in $\mathbb{R}^d$. Thus, the proof in the proposition above allows for a more general set of inclusion sets. In particular, it can be shown for example in Lemma 94 that Wiener's criterion is satisfied whenever $Q \subset \mathbb{R}^3$ is a surface of co-dimension 1 with smooth edges.

Remark 57. The fact that γ is piecewise constant of the form $\gamma = \big(1 + (k-1)\chi_Q\big)I_d$ for $x \in \Omega$ is not required for the sign-changing phenomenon to hold. Using a perturbation argument, we may consider a regularised γ using the standard mollifier $\eta \in C^\infty(\mathbb{R})$, where $\eta_\varepsilon(x) = c_\eta \exp\left(\frac{-1}{1-|x/\varepsilon|^2}\right)$ for $|x| < \varepsilon$ and 0 otherwise, in which, the constant c_η is a normalising constant so that $\|\eta_\varepsilon\|_{L^1\left(\mathbb{R}^d\right)} = 1$. Observing that the regularised conductivity matrix $\tilde{\gamma} = \gamma \star \eta_\varepsilon$ is smooth and yet satisfies assumption 46, we observe the Jacobian determinant of the solution $D\tilde{U}$ with coefficient matrix $\tilde{\gamma}$ would still satisfy the sign changing phenomenon.

Definition 58. Let ϕ be an $\alpha-$admissible boundary condition. We say that an inclusion set $Q \Subset \Omega \subset \mathbb{R}^d$ is a sign-changing inclusion if the solution U_∞

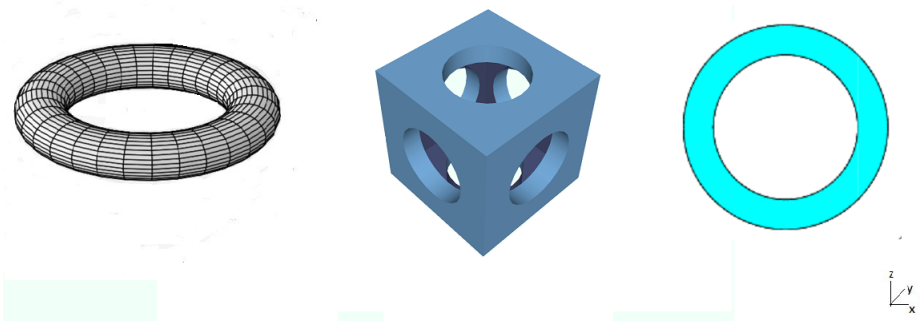


Figure 3.2: Sign-changing inclusions

corresponding to the perfect conductivity equation (3.11) changes signs in the measure theoretical sense

$$|\{x \in \Omega \setminus Q : J_\infty > 0\}| \times |\{x \in \Omega \setminus Q : J_\infty < 0\}| > 0.$$

Example 59. Some examples of a sign-changing inclusion Q in $\mathbb{R}^3$ include

1. A proper torus centred at the origin
2. A cube with circular faces removed and oriented diagonally
3. A 2 two dimensional annular region embedded in $\mathbb{R}^3$. (See figure above)

Example 60. Some non-examples in $\mathbb{R}^3$ includes

1. A solid sphere with $\Omega = B(0,R)$ and $Q = \overline{B(0,R_0)}$, where $0 < R_0 < R$, and
2. Any inclusion set of the form $\overline{B(0,R_0)} \setminus S$, where S is an arbitrary open subset compactly contained within $B(0,R_0)$.

Note that both of these inclusion give rise to the same solution U_∞ to the perfect conductivity equation (3.11). Thanks to the explicit computation done in (95), it is shown that

$$\det DU_\infty(x) \geq \frac{1}{R^2}\left(\frac{2R_0^3 + R^3}{R^3 - R_0^3}\right)|U_\infty(x)|^2 \quad \text{in} \quad B(0,R) \setminus \overline{B(0,R_0)}.$$

3.2 Volume estimates for the sign changing set of $\det DU_\infty$

In this section, we present a method which enables us to derive quantitative bounds for the Jacobian determinant of DU_∞ in the perfectly conducting

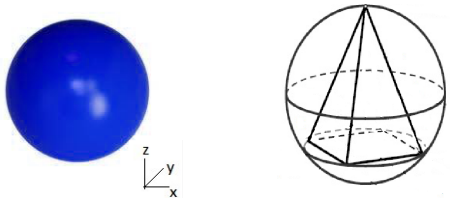


Figure 3.3: Positive Jacobian inclusion

case. We will make use of the result of Lemma 50, namely that the Jacobian determinant may be diagonalised along the line

$$L := (\Omega \setminus Q) \cap \left(\mathbb{R}e_1\right) \tag{3.15}$$

as

$$\det DU(s) = \partial_1 U_1(s) \times \ldots \times \partial_d U_d(s) \text{ for all } s \in L.$$

Obtaining a pointwise bound for $\det DU$ along this line corresponds to estimating each of the partial derivatives $\partial_i U_i$ along the line L. There are several methods to available to do so. One approach is to use the following quantitative version of Hopf's lemma.

Lemma 61. *Let H be a positive harmonic function in some ball $B(x_{0,},R) \subset \mathbb{R}^d$. Assuming that H is continuous in $\overline{B(x_{0,},R)}$ and vanishes at some point $y_0 \in \partial B(x_0,R)$, then*

$$\frac{\partial H}{\partial \nu}(y_0) \leq -\frac{1}{2^{d-1}} H(x_0)$$

where ν here is the unit normal vector $\frac{y_0-x_0}{|y_0-x_0|}$.

Proof. By Harnack's inequality, we have for any $x \in B(x_{0,},R)$, such that $|x-x_0| = \lambda R$ where $\lambda \in (0,1)$,

$$\frac{1-\lambda}{(1+\lambda)^{d-1}} H(x_0) \leq H(x).$$

Consequently,

$$\frac{H(x)-H(y_0)}{1-\lambda} \geq \frac{H(x_0)-H(y_0)}{(1+\lambda)^{d-1}},$$

and by sending $\lambda \to 1$, we have

$$\frac{\partial H}{\partial \nu}(y_0) \leq -\frac{1}{2^{d-1}} H(x_0).$$

□

To derive a quantitative result, we will use the fact that the domain is (for

lack of a better word) tube connected, which roughly means that $\Omega \setminus Q$ is path connected with 'thick' paths.

Definition 62. The set $\Omega \setminus Q$ is tube connected to the exterior boundary $\partial\Omega$ if for all $a \in (\Omega \setminus Q) \cap [0,\infty)^d$, and for each direction $i \in \{1,\dots,d\}$, there exists $N \in \mathbb{N}$, and cylinders $C_1^i,\dots,C_N^i$ of the form
$C_m^i = S_m^i \times \left\{x_m^k + t v_m^k, t \in (0,1)\right\}$, with $\left(x_m^k, v_m^k\right) \in \mathbb{R}^d$ and $S_m^k \subset \mathbb{R}^{d-1}$ a non-empty bounded open hypersurface for each $m = 1,\dots,N$, such that

- The last cylinder contains a: $a \in C_N^i$,
- The intersection of the first cylinder with the exterior boundary is such that $\partial\Omega \cap C_1^i \subset \{x \cdot e_i > \max(Q \cap (\mathbb{R}_+ e_i))\}$.
- For each $2 \le m \le N$ there holds $\overline{S_m^i} \times \left\{x_m^i\right\} \subset C_{m-1}^i$
- If $a \cdot e_i > 0$, then $\cup_{m=1}^N C_m^i \subset (0,\infty)^d$. If $a \cdot e_i = 0$, then $\cup_{m=1}^{N-1} C_m^i \subset (0,\infty)^d$.

Remark 63. For explicit constructions, one can consider cylinders whose cross sections are rectangles or discs.

Proposition 64. *If Ω and Q satisfy assumption 46, then $\Omega \setminus Q$ is tube connected.*

Proof. Consider first the case when $a \in (\Omega \setminus Q) \cap (0,\infty)^d$. Pick $b \in \Omega \cap (0,\infty)^d$ such that $b \cdot e_i = \frac{1}{2}\max(\Omega \cap (\mathbb{R}_+ e_i)) + \frac{1}{2}\max(Q \cap (\mathbb{R}_+ e_i))$. Because $\Omega \setminus Q \cap (0,\infty)^d \subset \mathbb{R}^d$ is connected, there exists a finite chain of open balls $B_2,\dots,B_M$ in $\Omega \setminus Q \cap (0,\infty)^d$ such that $B_i \cap B_{i+1} \neq \emptyset$ for all $i \in 1, M-1$. with $a \in B_M$ and $b \in B_2$. Choose $S_1^i = B_1 \cap \{x \cdot e_i = b \cdot e_i\} - b$, and $C_1^i = S_1^i \times \{b + 2(b \cdot e_i) e_i\}$, so that the initial constraint is satisfied. The construction is straightforward afterwards. Finally, when $a \notin (\Omega \setminus Q) \cap (0,\infty)^d$, for some n with $1 \le n \le d$, consider instead the point

$$\tilde{a} = a + \sum_{\{j : a \cdot e_j = 0\}} \frac{1}{2} \min\left(\mathrm{d}\left(a, \partial Q \cap (0,\infty)^d\right), \mathrm{d}\left(a, \partial\Omega \cap (0,\infty)^d\right)\right) e_j,$$

and add a final cylinder to connect to a. □

Lemma 65. *Suppose that Assumption 46 holds and that ϕ is α-admissible are satisfied. Then, there exists $p_1 \in L$, where L is the set given by* (3.15)*, and two explicit constants M_1 and M_0 such that*

$$J_\infty(p_1) < -\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)} M_1, \text{ and } J_\infty(0) > \alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)} M_0.$$

Remark 66. The lower bounds for M_0 and M_1 can be computed depending on geometric invariants of the surface ∂Q , the tubes connecting p_1, and an intermediate point $p_{1/2} \in L$. Possible locations for $p_1, p_{1/2} \in L$ can be

determined based on the exterior ball condition satisfied at y_0,the point of contact between L and Q closest to the origin.

The following quantitative lower bound for the measure of the domains where Jacobian is positive or negative follows.

Corollary 67. *Under the assumptions of Assumption 46, there holds*

$$B\left(p_1, \left(\frac{M_1}{2dC(\Omega,Q)^d}\alpha^d\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset J_\infty^{-1}\left(\left(-\infty, -\frac{1}{2}\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)} M_1\right)\right),$$

and

$$B\left(0, \left(\frac{M_0}{2dC(\Omega,Q)^d}\alpha^d\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset J_\infty^{-1}\left(\left(\frac{1}{2}\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)} M_0, \infty\right)\right).$$

Proof. Thanks to a variant of Hadamard's inequality (see Friedland [21]) there holds for any matrix A ,B in $\mathbb{R}^{d\times d}$,

$$|\det A - \det B| \leq d\,|A-B| \max\left(|A|, |B|\right)^{d-1}.$$

Thus, for any $x, y \in \Omega \setminus Q$, we have

$$\begin{aligned} |\det DU_\infty(x) - \det DU_\infty(y)| &\leq d\,|(DU_\infty(x) - DU_\infty(y))| \max\left(|DU_\infty(x)|, |DU_\infty(y)|\right)^{d-1} \\ &\leq d\,|x-y|^\sigma \|U_\infty\|_{C^{1,\sigma}(\Omega\setminus Q)} \|U_\infty\|^{d-1}_{C^1\left(\overline{\Omega\setminus Q}\right)}, \end{aligned} \tag{3.16}$$

Furthermore, thanks to Lemma 80,

$$\begin{aligned} \|U_\infty\|_{C^1\left(\overline{\Omega\setminus Q}\right)} &\leq \|U_\infty\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \\ &\leq C(\Omega, Q) \|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)}, \end{aligned}$$

where C_Q depends on Q and Ω only, and the result follows. □

Proof of lemma 65. First step : construction of the point p_1. Since ∂Q is C^2, it satisfies the exterior ball condition, and we may place a ball $B\left(p_{1/2}, r_0\right)$ which is completely outside of Q and centred at some (intermediate) point $p_{1/2} \in L$, tangential to the boundary of the inclusion ∂Q at the point of contact $y_0 \in \partial L$. Since $\Omega \setminus Q$ is tube connected, there exists N tubes $C_1^1, \ldots, C_N^1$ connecting $p_{1/2}$ to the exterior facet $C_1^1 \cap \partial\Omega$. Consider the following Di-

richlet problems

$$\begin{aligned}
-\Delta V_m^1 &= 0 \quad \text{in} \quad C_m^1 \\
V_m^1 &= 0 \text{ on } \partial S_m^1 \times \left\{x_m^1 + t v_m^1, t \in (0,1)\right\}, \\
V_m^1 &= 0 \text{ on } S_m^1 \times \{x_m^1 + v_m^1\}, \\
V_m^1 &= 1 \text{ on } S_m^1 \times \{x_m^1\},
\end{aligned}$$

and set

$$M_2^1 = \max\left(Q \cap (\mathbb{R}_+ e_1)\right) \inf_{S_2^1} V_1^1 \times \inf_{S_3^1} V_2^1 \times \ldots \inf_{S_N^1} V_{N-1}^1 \times \inf_{L} V_N^1. \qquad (3.17)$$

Since

$$\partial\Omega \cap C_1^1 \subset \left\{x \cdot e_1 > \inf\left(Q \cap (\mathbb{R}_+ e_1)\right)\right\},$$

There holds for all $x \in \partial\Omega \cap C_1^1$,

$$U_1(x) = \phi(x) \cdot e_1 \geq \alpha(\phi)\, x \cdot e_1 > \alpha(\phi) \max\left(Q \cap (\mathbb{R}_+ e_1)\right)$$

Thus $U_1 > \alpha(\phi) \max\left(Q \cap (\mathbb{R}_+ e_1)\right) V_1^1$ in C_1^1 by the maximum principle. Then

$$U_1 > \alpha(\phi) \max\left(Q \cap (\mathbb{R}_+ e_1)\right) \left(\inf_{S_2} V_1^1\right) V_2^1$$

on C_2^1, and iterating this argument we obtain

$$U_1 > \alpha(\phi) \max\left(Q \cap (\mathbb{R}_+ e_1)\right) \inf_{S_2^1} V_1^1 \times \inf_{S_3^1} V_2^1 \times \ldots \inf_{S_N^1} V_{N-1}^1 \times V_N^1 \text{ on } C_N^1,$$

and in turn

$$U_1 \geq \alpha(\phi) M_2^1 \text{ on } B\left(p_{1/2}, r_0\right) \cap L.$$

Thanks to the quantitative Hopf's Lemma (61) we obtain that

$$\frac{\partial U_1}{\partial \nu}(y_0) = \frac{\partial U_1}{\partial x_1}(y_0) \leq -\frac{1}{2^{d-1}} \alpha(\phi) M_2^1.$$

Since $U_1 \in C^{1,\alpha}\left(\overline{\Omega \setminus Q}\right)$, see Lemma 80 below, we can choose p_1 between $p_{1/2}$ and y_0 such that

$$\frac{\partial U_1}{\partial \nu}(p_1) \leq -\frac{1}{2^d} \alpha(\phi) M_2^1.$$

Indeed, for every $\tau \in (0,1)$

$$\left|\frac{\partial U_1}{\partial \nu}\left(\tau y_0 + (1-\tau) p_{1/2}\right) - \frac{\partial U_1}{\partial x_1}(y_0)\right| \leq (1-\tau)^\sigma \left|p_{1/2} - y_0\right|^\sigma \|U\|_{C^{1,\alpha}\left(\overline{\Omega \setminus Q}\right)}$$

Second step : lower bound for $\partial_i U_i(p_1)$ for $i = 2, \ldots, d$. At the point p_1,

for each $i=2,\dots,d$ let $C_1^i,\dots,C_N^i$ be the tubes connecting x_1 the exterior facets $\partial\Omega\cap\{x\cdot e_i>\inf(Q\cap(\mathbb{R}_+e_i))\}$ (for a possibly different N from the one above). We note that due to the evenness of χ_Q in the direction e_i, $U_i=0$ on L. Consider the following Dirichlet problems for $m=2,\dots N$

$$\begin{aligned}
-\Delta V_m^i &= 0 \quad \text{in} \quad C_m^i \\
V_m^i &= 0 \text{ on } \partial S_m^i\times\left\{x_m^i+tv_m^i, t\in(0,1)\right\}, \\
V_m^i &= 0 \text{ on } S_m^i\times\{x_m^i+v_m^i\}, \\
V_m^i &= 1 \text{ on } S_m^i\times\{x_m^i\},
\end{aligned}$$

and, for the last tube, consider the Dirichlet problem posed the positive half plane. Defining -

$$\begin{aligned}
-\Delta V_N^i &= 0 \quad \text{in} \quad C_N^i\cap\left\{x\in\mathbb{R}^d : x\cdot e_i>0\right\}, \\
V_N^i &= 1 \text{ on } S_N^i\times\{x_N^i\}, \\
V_N^i &= 0 \text{ on the rest of the boundary.}
\end{aligned}$$

Introducing

$$M_2^i=\max(Q\cap(\mathbb{R}_+e_i))\inf_{S_2^i}V_1^i\times\inf_{S_3^i}V_2^i\times\dots\inf_{S_N^i}V_{N-1}^i\times V_N^i, \tag{3.18}$$

and arguing as above we have $U_i\geq\alpha(\phi)M_2^iV_N^i$ in $C_N^i\cap\left\{x\in\mathbb{R}^d : x\cdot e_i>0\right\}$. In particular, since $V_N^i=U_i=0$ on $L\cap C_N^i$, we have

$$\partial_iU_i(p_1)\geq\alpha(\phi)M_2^i\partial_iV_N^i(p_1)>0.$$

Third step: upper bound for $J_\infty(p_1)$ and lower bound for$J_\infty(0)$. In the first two steps, we have obtained that

$$\partial_1U_1(p_1)\dots\partial_dU_d(p_1)\leq-\frac{\alpha(\phi)^d}{2^d}\Pi_{i=1}^dM_2^i\Pi_{j=2}^d\partial_iV_N^i(p_1),$$

which is the announced upper bound. By the same argument as in the second step, applied at the origin, we have

$$\Pi_{i=2}^d\partial_iU_i(0)\geq\alpha(\phi)^{d-1}\Pi_{i=2}^d\tilde{M}_2^i\partial_iV_N^i(0)>0. \tag{3.19}$$

Turning to $\partial_1U_1(0)$, we notice that because of the symmetry of χ_Q, $U_1(0)=0$. We thus repeat once again the second step method, to obtain

$$\partial_1U_1(0)\geq\alpha(\phi)\tilde{M}_2^1\partial_1V_N^1(0)>0, \tag{3.20}$$

and the conclusion follows from (3.19) and (3.20). □

Remark 68. Using explicitly cylinders to connect the points is certainly not necessary. A more streamlined proof would simply use auxiliary problems defined in $(\Omega \setminus Q) \cap (0,\infty)^d$. The reason why this hands-on approach is employed here is that it allows explicit bounds (as closed form solutions can be written down) in practical cases, see Proposition (71).

Remark 69. We make a remark here that the solutions V_m^i that we considered above should not be regarded as a H^1 energy solution. Strictly speaking, the functions V_m^i which satisfies

$$\begin{aligned} -\Delta V_m^i &= 0 \quad \text{in} \quad C_m^i \\ V_m^i &= 0 \text{ on } \partial S_m^i \times \left\{x_m^i + t v_m^i, t \in (0,1)\right\} \\ V_m^i &= 0 \text{ on } S_m^i \times \{x_m^i + v_m^i\}, \\ V_m^i &= 1 \text{ on } S_m^i \times \{x_m^i\}, \end{aligned}$$

in each cylinder should be understood within the framework of harmonic functions with L^p boundary data. Following the pioneering work of Cimmino in Cimmino [16], it is understood that such a solution V_m^i exists, is unique, and it satisfies the weak maximum principle

$$\max_{x\in\Omega} V_m^i = \operatorname{ess\,sup}_{x\in\partial\Omega} V_m^i$$

in each C_m^i.

Example 70. Consider the case $d = 3$ and Q is a toroidal inclusion centred at the origin with minor radii a and major radii c. We exhibit below p_1, and provide upper bounds for M_0 and M_1 as defined Lemma 65.

Proposition 71. *Let U_∞ be as the above, suppose that Q is a proper torus centred at the origin with minor radii a, and major radii c, with $0 < a < c < 1$. Then we may chose $p_{1/2} = (c-a,0,0) - \frac{C_1^2}{4L^2} e_1$, where*

$$C_1 := \left(1 - \frac{1}{\sqrt{2}}\right) \frac{2(c-a)}{\pi^2 \cosh\left(\frac{2\pi}{c-a}\right)}, \text{ and } L := \|U\|_{C^{1,\frac{1}{2}}\left(\overline{\Omega\setminus Q}\right)},$$

and there holds

$$M_0 \geq \left(\left(2 - \sqrt{2}\right) \frac{c-a}{\pi^2 \cosh\left(\frac{\pi}{2}(c-a)\right)} \right)^2 \cdot \frac{16}{\pi(c-a)} \frac{1}{\sinh\left(\frac{\pi}{c-a}\right)}$$

and

$$M_1 \geq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{8\sqrt{5}}{\pi^5} \frac{1}{\cosh\left(\frac{\pi}{2}(c-a)\right) \cosh\left(\frac{\pi}{\tau}\sqrt{\frac{1}{4}+4}\right) \sinh\left(\frac{\pi}{\tau}\sqrt{\frac{5}{4}}\right)},$$

with

$$\tau(r_2) = \frac{r_2^{\frac{3}{2}}\sqrt{2(c-a)-r_2}}{r_2+\sqrt{2(c-a)\,r_2}}$$
$$r_2(L) = \frac{4(c-a)^2}{L^2}\left(\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\pi^2\cosh(c-a)}\right)^2$$

Proof. The straightforward but tedious computation is provided in the Appendix. □

3.3 Continuous dependence of gradient field in conductivity contrast k

In this section, we derive a stronger quantitative version of Corollary 67. In particular, our main result of this chapter is the following theorem.

Theorem 72. *Suppose that assumption 46 holds. Then there exist two positive constants τ and C depending on Ω and Q only such that for any $\alpha > 0$, any $k \geq C \cdot \max\left\{1, \frac{1}{\alpha^d \tau}\right\}$ and any α-admissible $\phi \in C^{1,\sigma}\left(\mathbb{R}^d\right)$ the solution of* (3.1) *satisfies*

$$B\left(0, \left(\frac{\tau\alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(\tau\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)}, \infty\right)$$

and

$$\cup_{i=1}^{2d} B\left(p_i, \left(\frac{\tau\alpha^d}{C}\right)^{1/\sigma}\right) \cap (\Omega \setminus Q) \subset (\det DU)^{-1}\left(-\infty, -\tau\alpha^d \|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)}\right),$$

where the points $p_i \in \Omega \setminus Q, i = 1, \ldots, 2d$ are all located on different coordinate half-axes.

The method that we use here is by an integral equation approach. We investigate the continuous dependency of DU_k, where U_k is the solution of (3.8), when k varies. Several authors have investigated this question including Li and Vogelius [32], Ammari et al. [7], Kang et al. [30]. While it is known that the Jacobian changes sign when the contrast $k = \infty$, it is however not useful in most practical applications as materials which are perfectly conducting does not exist. We will use the fact that U_k admits a representation formula similar to that in [5], and we use the classical theory of weakly singular boundary integral operators to show the continuous dependence of DU_k with respect to k as $k \to \infty$. Finally, we will use a perturbation argu-

ment combined with our first result 67, to obtain the quantitative version of our main result given as the above.

We remind the reader of the definitions of the classical boundary layer potentials.

Definition 73. Given $\psi \in C^{0,\sigma}(\partial Q)$, where Q is a connected closed orientable (hyper)surface of $\mathbb{R}^d$ and $d \geq 3$, the single layer potential $S_Q(\psi)$ and double layer potential $D_Q(\psi)$ are defined as

$$\begin{aligned} [S_Q(\psi)](x) &:= \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-y|^{d-2}} \psi(y) \mathrm{d}\sigma(y) \quad x \in \mathbb{R}^d \setminus \partial Q \\ [D_Q(\psi)](x) &:= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu(y) \rangle}{|x-y|^d} \psi(y) \mathrm{d}\sigma(y) \quad x \in \mathbb{R}^d \setminus \partial Q \end{aligned}$$

where ω_d denotes the $d-1$ dimensional surface area of the unit ball.

The layer potential operators maps functions ψ which are defined on the boundary of the surface Q to the space of harmonic functions in $\mathbb{R}^d \setminus \partial Q$.

For a function u defined on $\mathbb{R}^d \setminus \partial Q$, we denote:

$$\begin{aligned} u\,|_{\pm}(x) &:= \lim_{t \to 0^+} u(x \pm t\nu_x), \quad x \in \partial Q \\ \frac{\partial}{\partial \nu_x} u\,|_{\pm}(x) &:= \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \partial Q \end{aligned}$$

where ν_x is the outward unit normal to ∂Q at the point x. Note that by assumption Q is orientable and the outer and inner limits $\frac{\partial}{\partial \nu_x} u\,|_{\pm}(x)$ are well-defined.

The single layer and double layer potential also satisfy the well known jump formulas given by

$$\begin{aligned} [S_Q(\psi)\,|_+](x) &= [S_Q\psi\,|_-](x) \quad a.e. x \in \partial Q \\ \left[(I_d - n \otimes n) \nabla S_Q(\psi)\,|_+ \right](x) &= \left[(I_d - n \otimes n) \nabla S_Q(\psi)\,|_- \right](x) \quad a.e. x \in \partial Q \\ \left[\nabla S_Q(\psi) \cdot n\,|_{\pm} \right](x) &= \left[\left(\pm \frac{1}{2} I + K_Q^* \right) \psi \right](x) \quad a.e. x \in \partial Q \quad (3.21) \\ \left[D_Q(\psi)\,|_{\pm} \right](x) &= \left[\left(\mp \frac{1}{2} I + K_Q \right) \psi \right](x) \quad a.e. x \in \partial Q \end{aligned}$$

where

$$K_Q^*(\psi) := \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \psi(y) \mathrm{d}\sigma(y) \quad x \in \partial Q \qquad (3.22)$$

denotes the Poincaré-Neumann operator and $K_Q(\psi)$ is the L^2-adjoint of K_Q^* given by

$$K_Q(\psi) := \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} \psi(y) \mathrm{d}\sigma(y) \quad x \in \partial Q.$$

We refer the reader to [17] for proofs of the jump formulas.

Proposition 74. *(See theorem 2.17 in [5]) Let u_k be the solution to the following scalar Dirichlet boundary value problem*

$$\begin{aligned} -\nabla\cdot\big(1+(k-1)\chi(Q)\big)\nabla u_k &= 0 \quad in \quad \Omega, \\ u_k|_{\partial\Omega} &= f \quad on \quad \partial\Omega, \end{aligned} \tag{3.23}$$

with the Dirichlet to Neumann map $\Lambda : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined by

$$\Lambda(f) = \frac{\partial u_k}{\partial \nu}|_{\partial\Omega}.$$

Then, the solution u_k admits a layer potential representation given by

$$u_k(x) = H_k(x) + \left[S_Q(\psi_k)\right](x), \quad x \in \Omega$$

for $k \in (0,\infty]$, where the function H is harmonic of the form

$$H_k(x) = \left(-S_\Omega \circ \Lambda + D_\Omega\right)(f), \quad x \in \Omega, \tag{3.24}$$

and ψ_k satisfies the integral equation:

$$\left(\frac{k+1}{2(k-1)}I - K_Q^*\right)\psi_k = \frac{\partial H_k}{\partial \nu}\,|_{\partial Q} \quad on \quad \partial Q. \tag{3.25}$$

Lemma 75. *Let u_k denote the solution to* (3.23)*. Assuming that $f \in H^{\frac{1}{2}}\big(\partial\Omega\big)$ is fixed, then the Dirichlet to Neumann map $\Lambda_k\big(f\big) = \nabla u_k \cdot \nu_\Omega$ is uniformly bounded for all $k \geq 1$ in $H^{-\frac{1}{2}}\big(\partial\Omega\big)$. In particular, the following estimate holds*

$$\|\nabla u_k \cdot \nu_\Omega\|_{H^{-\frac{1}{2}}\left(\partial\Omega\right)} \leq C_{2,T}\,\|f\|_{H^{\frac{1}{2}}\left(\partial\Omega\right)}. \tag{3.26}$$

for some positive constant $C_{2,T}$ depending only on the domain Ω.

Proof. Let Ω_0 be a smooth subdomain of Ω such that $dist\big(\partial\Omega_0, \partial\Omega\big) > 0$ and contains the closed domain $\overline{Q}$. In this proof, we write $A \lesssim B$ when $A \leq CB$ where C depends on Ω and Ω_0 only. Let $\zeta \in H^{\frac{1}{2}}\big(\partial\Omega\big)$ be arbitrarily chosen. Since ζ belongs to $H^{\frac{1}{2}}(\partial\Omega)$, there exists an extension of ζ which we shall denote by $\tilde{\zeta}$, such that

$$\left\|\tilde{\zeta}\right\|_{H^1\left(\Omega\right)} \lesssim \|\zeta\|_{H^{\frac{1}{2}}\left(\partial\Omega\right)}.$$

Furthermore, we may extend ζ in such a way that it satisfies $\text{trace}(\tilde{\zeta}) = \begin{cases} \zeta & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega_0 \end{cases}$ and $\tilde{\zeta} = 0$ in Ω_0. By testing the equation (3.23)

with $\tilde{\zeta}$, we have

$$\int_{\partial\Omega} \tilde{\zeta} \nabla u_k \cdot \nu_\Omega \mathrm{d}\sigma \;=\; \int_{\Omega\setminus\Omega_0} \nabla u_k \cdot \nabla \tilde{\zeta}\, \mathrm{d}x.$$

Note that from standard elliptic regularity, the solution u_k to (3.23) satisfies

$$\|\nabla u_k\|_{H^1(\Omega)} \lesssim \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Consequently, we infer that

$$\begin{aligned}
\left|\langle \tilde{\zeta}, \nabla u_k \cdot \nu_\Omega \rangle_{H^{\frac{1}{2}}, H^{-\frac{1}{2}}}\right| &= \left|\int_{\partial\Omega} \tilde{\zeta} \nabla u_k \cdot \nu_\Omega \mathrm{d}\sigma\right| \\
&= \left|\int_{\Omega\setminus\Omega_0} \nabla u_k \cdot \nabla \tilde{\zeta}\, \mathrm{d}x\right| \\
&\leq \|\nabla u_k\|_{L^2(\Omega\setminus\Omega_0)} \left\|\nabla \tilde{\zeta}\right\|_{L^2(\Omega\setminus\Omega_0)} \\
&\lesssim \|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \left\|\tilde{\zeta}\right\|_{H^1(\Omega)} \\
&\lesssim \|f\|_{H^{\frac{1}{2}}(\partial\Omega)} \|\zeta\|_{H^{\frac{1}{2}}(\Omega)}
\end{aligned}$$

which further implies

$$\|\nabla u_k \cdot \nu_\Omega\|_{H^{-\frac{1}{2}}(\partial\Omega)} \lesssim \|f\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

□

Lemma 76. *Let u_k denote the solution to* (3.23)*. Given $f \in H^{\frac{1}{2}}(\partial\Omega)$, the function H_k is bounded in $C^2(\overline{Q})$ independently of $k \geq 1$. Furthermore, there holds*

$$\|H_k\|_{C^2(\overline{Q})} \leq \frac{(d+1)(d-2)}{\omega_d} \frac{|\partial\Omega|^{\frac{1}{2}}}{C_0} (1 + C_{2,T}) \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

where $C_{2,T}$ is defined in Lemma 75 and $C_0 = \min\left\{dist(\partial\Omega, \partial Q)^d, dist(\partial\Omega, \partial Q)^{d-2}\right\}$.

Proof. Let x be an arbitrary point in $\overline{Q}$. A straightforward computation of

$\partial_{\alpha\beta} H_k(x)$ gives

$$\begin{aligned}
\partial_{\alpha\beta} H_k(x) &= \partial_{\alpha\beta}\left(\frac{1}{(d-2)\omega_d}\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}\nabla u_k\cdot\nu_\Omega(y)\mathrm{d}\sigma(y)\right.\\
&\quad\left.+\frac{1}{\omega_d}\int_{\partial\Omega}\frac{\langle x-y,\nu_y\rangle}{|x-y|^d}f(y)\mathrm{d}\sigma(y)\right)\\
&= \frac{1}{(d-2)\omega_d}\int_{\partial\Omega}\partial_{\alpha\beta}\frac{1}{|x-y|^{d-2}}\nabla u_k\cdot\nu_\Omega(y)\,\mathrm{d}\sigma(y)\\
&\quad+\frac{1}{\omega_d}\int_{\partial\Omega}\partial_{\alpha\beta}\frac{\langle x-y,\nu_y\rangle}{|x-y|^d}f(y)\mathrm{d}\sigma(y)\\
&= \frac{1}{\omega_d}\int_{\partial\Omega}\left(\frac{(-d)(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^{d+2}}+\frac{\delta_{\alpha\beta}}{|x-y|^d}\right)\nabla u_k\cdot\nu_\Omega(y)\,\mathrm{d}\sigma(y)\\
&+ \frac{1}{\omega_d}\int_{\partial\Omega}(2-d)\langle x-y,\nu_y\rangle\left(\frac{(-d)(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^{d+2}}\right.\\
&\quad\left.+\frac{\delta_{\alpha\beta}}{|x-y|^d}\right)f(y)\,\mathrm{d}\sigma(y)\\
&+ \frac{1}{\omega_d}\int_{\partial\Omega}\left((2-d)\frac{(x_\alpha-y_\alpha)(x_\beta-y_\beta)}{|x-y|^d}\left(\nu_y^\beta+\nu_y^\alpha\right)\right)f(y)\,\mathrm{d}\sigma(y).
\end{aligned}$$

Hence, by (3.26), we have

$$\begin{aligned}
\sup_{x\in\overline{Q}}\left|\partial_{\alpha\beta}H_k(x)\right| &\le \frac{d+1}{\omega_d}\left|\int_{\partial\Omega}\frac{1}{|x-y|^d}\nabla u_k\cdot\nu_\Omega(y)\,\mathrm{d}\sigma(y)\right|\\
&+ \frac{(d+1)(d-2)}{\omega_d}\left|\int_{\partial\Omega}\frac{1}{|x-y|^{d-1}}f(y)\,\mathrm{d}\sigma(y)\right|\\
&+ \frac{2(d-2)}{\omega_d}\left|\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}f(y)\,\mathrm{d}\sigma(y)\right|\\
&\le \frac{(d+1)(d-2)}{\omega_d}\frac{|\partial\Omega|^{\frac{1}{2}}}{C_0}\left(\|\nabla u_k\cdot\nu_\Omega\|_{H^{-\frac{1}{2}}(\partial\Omega)}+\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}\right)\\
&\le \frac{(d+1)(d-2)}{\omega_d}\frac{|\partial\Omega|^{\frac{1}{2}}}{C_0}\left(1+C_{2,T}\right)\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

□

Remark. Since there is some distance between $\partial\Omega$ and ∂Q, the function $H_k=(-S_\Omega\circ\Lambda+D_\Omega)(f)$ which is defined as a boundary layer potential on $\partial\Omega$ is trivially smooth on $\overline{Q}$. However, in the spirit of tracking down constants, the purpose of the lemma above is to provide a quantitative version for the second order derivatives of H_k on the set $\overline{Q}$.

Proposition 77. *Let H_k be the harmonic function $H_k=-S_\Omega(\nabla u_k\cdot\nu_\Omega)+D_\Omega(f)$ given in* (3.24)*. Suppose that*
$\psi,\psi_k\in C_0^{0,\sigma}(\partial Q):=\left\{f\in C^{0,\sigma}(\partial Q):\int_{\partial Q}f\quad d\sigma(y)=0\right\}$ *solve the integral*

equations

$$\left(\frac{k+1}{2(k-1)}I-K_Q^*\right)\psi_k=\frac{\partial H_k}{\partial \nu}\mid_{\partial Q}$$

and

$$\left(\frac{1}{2}I-K_Q^*\right)\psi=\frac{\partial H_k}{\partial \nu}\mid_{\partial Q}$$

respectively. Then for any $k>1$, there holds

$$\|\psi_k-\psi\|_{C^{0,\sigma}\left(\partial Q\right)}\leq C_4\frac{1}{k-1}\|f\|_{H^{\frac{1}{2}}\left(\partial\Omega\right)}$$

in which, the constant C_4 is a geometrical constant which depends on the spectral radius of the operator K_Q^*.

Proof. Thanks to Proposition 90 from the appendix, a straightforward computation shows that for any $\phi\in C_0^{0,\sigma}(\partial Q)$, $\lambda_n,\lambda\in\mathbb{R}$ such that $\lambda_n>\lambda$ and λ is greater than the spectral radius of K_Q^* we have

$$\begin{aligned}\left(\left(\lambda_n I-K_Q^*\right)^{-1}-\left(\lambda I-K_Q^*\right)^{-1}\right)(\phi) &= \left(\frac{1}{\lambda_n}\sum_{i=0}^{\infty}\left(\frac{K_Q^*}{\lambda_n}\right)^i-\frac{1}{\lambda}\sum_{i=0}^{\infty}\left(\frac{K_Q^*}{\lambda}\right)^i\right)(\phi)\\ &= \sum_{i=0}^{\infty}\left|\frac{1}{\lambda_n^{i+1}}-\frac{1}{\lambda^{i+1}}\right|\left|[K_Q^*]^i(\phi)\right|\end{aligned}$$

$$\begin{aligned}&\left|\left(\left(\lambda_n I-K_Q^*\right)^{-1}-\left(\lambda I-K_Q^*\right)^{-1}\right)(\phi)\right|\\ \leq&|\lambda-\lambda_n|\left(\sum_{i=0}^{\infty}\sum_{k=0}^{k=i}(\frac{1}{\lambda_n})^{i-k}\cdot(\frac{1}{\lambda})^k\left\|\left(K_Q^*\right)^i\right\|\right)\|\phi\|_{C^{0,\sigma}(\partial Q)}\\ \leq&|\lambda-\lambda_n|\left(\sum_{i=0}^{\infty}(i+1)\cdot\frac{1}{\lambda^i}\left\|\left(K_Q^*\right)^i\right\|\right)\|\phi\|_{C^{0,\sigma}(\partial Q)}\\ =&M(\rho,|\lambda|)\,|\lambda-\lambda_n|\,\|\phi\|_{C^{0,\sigma}(\partial Q)}.\end{aligned}$$

where

$$\rho=\limsup_n\left\|\left(K_Q^*\right)^n\right\|^{\frac{1}{n}}$$

and

$$M(\rho,\lambda)\leq\frac{C(Q)}{|\lambda|-\rho}$$

where $C(Q)$ is independent of λ. Using Proposition 90, we know that the spectrum of K_Q^* on the set $C_0^{0,\sigma}\left(\partial Q\right):=\left\{f\in C^{0,\sigma}\left(\partial Q\right):\int_{\partial Q}f=0\right\}$ is less

than $\frac{1}{2}$. Applied the above argument to $\lambda = \frac{1}{2}$ and $\lambda_n = \frac{k+1}{2(k-1)}$, we obtain

$$\begin{aligned}
\|\psi_k - \psi\|_{C^{0,\sigma}(\partial Q)} &= \left\| \left(\left(\frac{k+1}{2(k-1)} I - K_Q^* \right)^{-1} - \left(\frac{1}{2} I - K_Q^* \right)^{-1} \right) \left(\frac{\partial H_k}{\partial \nu} \mid_{\partial Q} \right) \right\|_{C^{0,\sigma}(\partial Q)} \\
&\leq M\left(\rho, \frac{1}{2}\right) \left| \frac{1}{k-1} \right| \left\| \frac{\partial H_k}{\partial \nu} \mid_{\partial Q} \right\|_{C^{0,\sigma}(\partial Q)} \\
&\leq C_4 \frac{1}{k-1} \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},
\end{aligned}$$

as announced. □

Lemma 78. *Let Ω be an open connected subset of $\mathbb{R}^d$. Suppose further that Ω has cubic symmetry, then the single layer potential S_Ω and double layer potential D_Ω maps odd functions to odd functions.*

Proof. Let ψ be an arbitrary odd function belonging to $H^{\frac{1}{2}}(\partial\Omega)$. Denoting the map $z(y) = -y$ for $y \in \mathbb{R}^d$, observe that if ψ is odd and Ω has cubic symmetry, a straightforward computation shows that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
S_\Omega(\psi)(-x) &:= \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x+y|^{d-2}} \psi(y) \mathrm{d}\sigma(y) \\
&= \frac{1}{(2-d)\omega_d} \int_{z(\partial Q)} \frac{1}{|x-z|^{d-2}} \psi(-z) \, |\det z| \, \mathrm{d}\sigma(z) \\
&= \frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-z|^{d-2}} \psi(-z) \mathrm{d}\sigma(z) \\
&= -\frac{1}{(2-d)\omega_d} \int_{\partial Q} \frac{1}{|x-z|^{d-2}} \psi(z) \mathrm{d}\sigma(z) \\
&= -S_\Omega(\psi)(x).
\end{aligned}$$

and similarly,

$$\begin{aligned}
D_\Omega(\psi)(-x) &:= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle -x-y, \nu(y) \rangle}{|x+y|^d} \psi(y) \mathrm{d}\sigma(y) \\
&= \frac{1}{\omega_d} \int_{\partial Q} \frac{\langle -x+z, \nu(-z) \rangle}{|x-z|^d} \psi(-z) \mathrm{d}\sigma(z) \\
&= -\frac{1}{\omega_d} \int_{\partial Q} \frac{\langle x-z, \nu(z)) \rangle}{|x-z|^d} \psi(z) \mathrm{d}\sigma(z) \\
&= -D_\Omega(\psi)(x)
\end{aligned}$$

as $\nu(-y) = -\nu(y)$ for all $y \in \partial\Omega$, thus establishing our thesis. □

Corollary 79. *Let Ω be as the above and let K_Ω^* be the Poincaré-Neumann operator given in* (3.22)*. Denoting $C_{odd}^{0,\sigma}(\partial\Omega)$ to be the set*
$C_{odd}^{0,\sigma}(\partial\Omega) := \{ f \in C^{0,\sigma}(\partial\Omega) : f(-x) = -f(x) \text{ for } x \in \partial\Omega \}$*, the integral*

equation given by

$$\left(\frac{k+1}{2(k-1)}I - K_{\Omega}^{*}\right)\psi_k = f \quad \forall x \in \partial\Omega$$

is uniquely solvable for any $f \in C_{odd}^{0,\sigma}(\partial\Omega)$*, moreover, the solution* ψ_k *belongs to* $C_{odd}^{0,\sigma}(\partial\Omega)$.

Proof. Since $\partial\Omega$ has cubic symmetry, it is clear that the set $C_{\text{odd}}^{0,\sigma}(\partial\Omega)$ is non-empty. Thanks to Proposition 78, the operator $\left(\frac{k+1}{2(k-1)}I - K_{\Omega}^{*}\right)$ maps

$$\left(\frac{k+1}{2(k-1)}I - K_{\Omega}^{*}\right) : C_{\text{odd}}^{0,\sigma}(\partial\Omega) \longrightarrow C_{\text{odd}}^{0,\sigma}(\partial\Omega),$$

moreover, as $C_{\text{odd}}^{0,\sigma}$ is a closed subspace of
$C_0^{0,\sigma}(\partial\Omega) := \left\{f \in C^{0,\sigma}(\partial\Omega) : \int_{\partial\Omega} f \quad \text{d}\sigma(\text{y})=0\right\}$, Proposition 89 from the appendix ensures that the spectral radius of K_{Ω}^{*} is strictly less than $\frac{1}{2}$, thus, by Theorem 88, the operator $\left(\frac{k+1}{2(k-1)}I - K_{\Omega}^{*}\right)$ is invertible on the closed subspace $C_{\text{odd}}^{0,\sigma}(\partial\Omega)$ in which, $\left(\frac{k+1}{2(k-1)}I - K_{\Omega}^{*}\right)^{-1}$ may be given by its Neumann series. □

Lemma 80. *Let* $Q \Subset \Omega$ *be a bounded, connected* C^2 *subset so that* $\Omega \setminus Q$ *remains connected. Denoting* $U \in H^1\left(\Omega\right)$ *to be the weak solution to the conductivity equation in* $\Omega \setminus Q$

$$\begin{aligned} -\Delta U &= 0 \quad \text{in} \quad \Omega \setminus Q \\ U &= \phi \quad \text{on} \quad \partial\left(\Omega \setminus Q\right) \end{aligned} \tag{3.27}$$

where $\phi \in C^{1,\sigma}\left(\mathbb{R}^d\right)$*, the following estimate holds*

$$\|U\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \leq C\left(\Omega \setminus Q\right)\|\phi\|_{C^{1,\sigma}\left(\partial\left(\Omega\setminus Q\right)\right)}, \tag{3.28}$$

where the constant $C\left(\Omega \setminus Q\right)$ *depends on* $\Omega \setminus Q$ *only.*

Proof. See Widman [36] and Grisvard [27] (for the case when Ω is a convex polygon). □

Proposition 81. *Under the various assumptions of Assumption 46, the solution* U_k *to* (3.8) *converges to* U_∞*, the solution of the perfect conductivity equation* (3.11) *in the space* $C^{1,\sigma}\left(\overline{\Omega \setminus Q}\right)$*. In addition to that, there exists a constant* K_{SL} *which is independent of the conductivity parameter* k *such that*

$$\|U_k - U\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \leq K_{SL}\frac{1}{k-1}\|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)}.$$

Proof. Let $U_{i,k}$ denote the i-th component of the vector U_k. By the representation formula (3.24),

$$U_{i,k} = H_{i.k} + S_Q\big(\psi_{i,k}\big)$$

where $\psi_{i,k}$ satisfies the integral equation:

$$\left(\frac{k+1}{2(k-1)}I - K_Q^*\right)\psi_{i,k} = \frac{\partial H_{i.k}}{\partial \nu}\mid_{\partial Q} \quad \text{on} \quad \partial Q \tag{3.29}$$

and $H_{i,k}$ is defined from the boundary data $\phi = (\phi_i)_{1\le i\le d}$. Since the function $H_{i,k} \in C^2\big(\overline{Q}\big)$ by (76) , we have $\frac{\partial H_{i,k}}{\partial \nu}\mid_{\partial Q} \in C^{0,\sigma}_{\text{odd}}$. Invoking Corollary 79 here and the fact that the integral equation (3.29) is uniquely solvable (see Theorem 2.30 in Colton and Kress [17]), we further deduce that the solution $\psi_{i.k}$ belongs to $C^{0,\sigma}_{\text{odd}}$. Observe from the proof of Lemma 50 that $U_{i,k}$ is an odd function in Ω and therefore its gradient with respect to x, $\nabla U_{i,k}$ is even for all $x \in \Omega$. Subsequently, by the symmetry of $\partial\Omega$, we infer that $\nabla U_{i,k}\cdot \nu_\Omega$ is an odd function on $\partial\Omega$. Applying Lemma 78 to the function $\nabla U_{i,k}\cdot \nu_\Omega$ here, we deduce again that the function

$$H_{i,k} = -S_\Omega\big(\nabla U_{i,k}\cdot \nu_\Omega\big) + D_\Omega\big(\phi_i\big)$$

is also odd. Since Q shares the same symmetry with $\partial\Omega$, we may reiterate this argument to show that the function $\frac{\partial H_{i,k}}{\partial \nu}\mid_{\partial Q}$ is also odd. Now, define the function $V_{i,k} := H_{i,k} + S_Q\big(\psi_i\big)$ where $\psi_i \in C^{0,\sigma}_{\text{odd}}$ solves the integral equation

$$\left(\frac{1}{2}I - K_Q^*\right)\psi_i = \frac{\partial H_{i,k}}{\partial \nu}\mid_{\partial Q} \quad \text{on} \quad \partial Q.$$

Note that $V_{i,k}$ is piecewise harmonic in in $(\Omega\setminus Q)\cup Q$. Moreover, by the jump formula (3.21), we have

$$\begin{aligned}
\nabla V_{i,k}\cdot \nu_Q\mid_- &= \big(\nabla\left(H_{i,k} + S_Q\big(\psi_i\big)\right)\cdot \nu_Q\big)\mid_- \\
&= \nabla H_{i,k}\cdot \nu_Q\mid_- + \left(-\frac{1}{2}I + K_Q^*\right)\psi_i \\
&= 0.
\end{aligned}$$

Therefore, the function $V_{i,k}$ satisfies

$$\begin{aligned}
-\Delta V_{i,k} &= 0 \text{ in } \Omega\setminus Q \\
V_{i,k} &= \text{constant on } \partial Q \\
V_{i,k} &= H_k + S_Q\big(\psi_i\big) \text{ for } x\in\partial\Omega.
\end{aligned}$$

Applying Lemma 78 again here, we deduce that $V_{i,k}$ is and odd function in Ω. Subsequently, we have $V_{i,k} = 0$ for $x \in \overline{Q}$.

Next, consider the function $W_{i,k} = U_{i,k} - V_{i,k}$. By construction, we have

$$W_{i,k}(x) \quad = \quad S_Q\big(\psi_{i,k} - \psi_i\big) \quad \forall x \in \mathbb{R}^d \setminus \partial Q,$$

moreover, as ∂Q is C^2 and $\psi_{i,k}, \psi_i \in C^{0,\sigma}\big(\partial Q\big)$, we use the fact that the single layer potential S_Q maps

$$S_Q : C^{0,\sigma}\left(\partial Q\right) \mapsto C^{1,\sigma}\left(\overline{\Omega \setminus Q}\right)$$

boundedly to obtain the inequality

$$\big\|S_Q\big(\psi_{i,k} - \psi_i\big)\big\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \leq C_S \big\|\psi_{i,k} - \psi_i\big\|_{C^{0,\sigma}\left(\partial Q\right)}.$$

Now, observe that the function $F_{i,k} := V_{i,k} - U_i$ where U_i denotes the i-th component of the solution U to the perfect conductivity equation 3.27 satisfies

$$\begin{aligned} -\Delta F_{i,k} &= 0 \text{ in } \Omega \setminus Q \\ F_{i,k} &= 0 \text{ on } \partial Q \\ F_{i,k} &= -S_Q\big(\psi_{i,k} - \psi_i\big) \text{ on } \partial\Omega \end{aligned}$$

as $V_{i,k} - U_i|_{\partial\Omega} = V_{i,k} - U_{i,k}|_{\partial\Omega}$. Altogether, Lemma 80 implies that

$$\begin{aligned} \big\|F_{i,k}\big\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} &\leq C\left(\Omega, Q\right) \big\|S_Q\big(\psi_k - \psi_i\big)\big\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)} \\ &\leq C\left(\Omega, Q\right) C_S \big\|\psi_k - \psi_i\big\|_{C^{0,\sigma}\left(\partial Q\right)}. \end{aligned}$$

Thus, we have for any $i \in \{1, \ldots, d\}$ and $0 < \alpha < 1$,

$$\begin{aligned} \big\|U_{i,k} - U_i\big\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} &\leq \big\|W_{i,k}\big\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} + \big\|F_{i,k}\big\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \\ &\leq \big(C_S + C\left(\Omega, Q\right) C_S\big) \big\|\psi_k - \psi_i\big\|_{C^{0,\sigma}\left(\partial Q\right)}. \\ &\leq K_{SL} \frac{1}{k-1} \big\|\phi_i\big\|_{C^{1,\sigma}\left(\partial\Omega\right)} \end{aligned}$$

where we have used Proposition 77 in the penultimate line to arrive at our statement. □

We are now ready to prove a stronger version of our main result (3.1), which follows from the following proposition

Proposition 82. *Assume that assumption 46 holds and that ϕ is α-admissible. Let M_0 and M_1 denote the values given by Proposition 65, and K_{SL} be the*

constant given by Proposition 81. Write

$$k_0 := 1+4\cdot K_{SL}\cdot\max\left\{1,\frac{d2^{d-1}}{\alpha^d\min\{M_0,M_1\}}\right\} \text{ and } \tau_0 := \frac{1}{4}\alpha^d\min\{M_0,M_1\}.$$

Then for all $k > k_0$, for all α-admissible ϕ, the solution of (3.1) *satisfies*

$$\cup_{i=1}^d A_-^i \subset (\det DU)^{-1}\left(-\infty,-\tau_0\|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}\right)}\right) \text{ and } A_+ \subset (\det DU)^{-1}\left(\tau_0\|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}\right)},\infty\right).$$

The sets $A_-^1,\dots,A_-^d$ and A_+ are non-empty. Furthermore, we have

$$A_+ \supseteq B\left(0,\left(\frac{M_0}{2dC(\Omega,Q)}\alpha^d\right)^{1/\sigma}\right),$$

where the constant $C(\Omega,Q)$ depends on the regularity of Ω and Q only, and

$$A_-^i \supseteq \left(B\left(p_i,\left(\frac{M_1}{2dC(\Omega,Q)}\alpha^d\right)^{1/\sigma}\right)\cup B\left(-p_i,\left(\frac{M_1}{2dC(\Omega,Q)}\alpha^d\right)^{1/\sigma}\right)\right)\cap(\Omega\setminus Q),$$

where point p_i is constructed on $\mathbb{R}^+e_i\cap(\Omega\setminus Q)$.

Proof. Using Corollary 67, we observe that, writing

$$B_1 := B\left(p_1,\left(\frac{M_1}{2dC(\Omega,Q)}\alpha^d\right)^{1/\sigma}\right)\cap(\Omega\setminus Q)$$

and

$$B_0 := B\left(0,\left(\frac{M_0}{2dC(\Omega,Q)}\alpha^d\right)^{1/\sigma}\right)\cap(\Omega\setminus Q),$$

$$B_1 \subset J_\infty^{-1}\left(\left(-\infty,-\frac{1}{2}\alpha^d\|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}\right)}M_1\right)\right) \,\&\, B_0 \subset J_\infty^{-1}\left(\left(\frac{1}{2}\alpha^d\|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}\right)}M_1,\infty\right)\right). \tag{3.30}$$

Thanks to Proposition 81, we have

$$\|U_k-U\|_{C^{1,\sigma}\left(\overline{\Omega\setminus Q}\right)} \le K_{SL}\frac{1}{k-1}\|\phi\|_{C^{1,\sigma}\left(\mathbb{R}^d\right)},$$

therefore arguing as in the proof of Corollary (67) ,

$$\begin{aligned}\sup_{x\in\overline{\Omega\setminus Q}}|\det DU_k(x)-\det DU_\infty(x)| &\le d\sup_{x\in\overline{\Omega\setminus Q}}|(DU_k(x)-DU_\infty(x))|\max\left(|DU_k|,|DU_\infty|\right)^{d-1}\\ &\le K_{SL}\frac{1}{k-1}\cdot d\left(1+\frac{(K_{SL})}{k-1}\right)^{d-1}\|\phi\|^d_{C^{1,\sigma}\left(\mathbb{R}^d\right)},\end{aligned} \tag{3.31}$$

and and for any positive k such that

$$K_{SL}\frac{1}{k-1}\cdot d\left(1+\frac{(K_{SL})}{k-1}\right)^{d-1}\leq\frac{1}{4}\alpha^{d}\min\left(M_{0},M_{1}\right),$$

the triangle inequality and (3.30) shows for all $x\in B_1$,

$$\begin{aligned}\det DU_{k}(x)&\leq\det DU_{\infty}(x)+\frac{K_{SL}}{k-1}d2^{d-1}\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}\\&\leq-\frac{1}{4}\alpha^{d}\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}M_{1},\end{aligned}$$

and for all $x\in B_0$ there holds

$$\begin{aligned}\det DU_{k}(x)&\geq\det DU_{\infty}(x)-K_{SL}\frac{1}{k-1}\cdot d\left(1+\frac{K_{SL}}{k-1}\right)^{d-1}\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}\\&\geq\frac{1}{4}\alpha^{d}\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}M_{0}.\end{aligned}$$

Thus, by combining these two statements and by writing $\tau_0=\frac{1}{4}\alpha^{d}\min\left\{M_{0},M_{1}\right\}$,we have the set containment

$$B_{1}\subset J_{k}^{-1}\left(\left(-\infty,-\tau_{0}\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}\right)\right)$$

and

$$B_{0}\subset J_{k}^{-1}\left(\left(\left\|\phi\right\|_{C^{1,\sigma}\left(\mathbb{R}^{d}\right)}^{d}\tau_{0},\infty\right)\right),$$

which hold uniformly for all $k>k_0:=1+4\cdot K_{SL}\cdot\max\left\{1,\frac{d2^{d-1}}{\min\{M_{0},M_{1}\}}\alpha^{d}\right\}$. Finally, by the symmetry assumptions 46 and because ϕ is α-admissible, all of the above developments are valid for every canonical direction $e_1,\ldots,e_d$ and both in the positive and negative half lines, thus,

$$A_{-}^{i}=\left(B\left(p_{i},\left(\frac{M_{1}}{2dC\left(\Omega,Q\right)}\alpha^{d}\right)^{1/\sigma}\right)\cup B\left(-p_{i},\left(\frac{M_{1}}{2dC\left(\Omega,Q\right)}\alpha^{d}\right)^{1/\sigma}\right)\right)\cap\left(\Omega\setminus Q\right)$$

where p_i is constructed similarly to p_1 on the half-axes $\mathbb{R}_{\pm}e_i$. □

Chapter 4

Epilogue

In this section the main contributions of this thesis are summarised and some future problems are discussed.

4.1 Conclusions

4.1.1 The asymptotic formula for the conductivity equation

Previous results concerning the asymptotic formula for the conductivity equation in the presence of a general, measurable inclusion set of small volume fraction require the conductivity coefficient matrix γ_0 and γ_n to be smooth, symmetric positive definite matrices which are bounded uniformly away from 0 and ∞. In this thesis, we have shown that this result is extendable to perturbative inclusions of small volume with high contrast. In particular, we allow for the cases when either essinf$(\gamma_n) \to 0$, esssup$(\gamma_n) \to \infty$ or even when γ_n approaches these two extreme regions simultaneously. Moreover, we have shown in this work that the assumption that γ_0 and γ_n are smooth is unnecessary. We may simply take γ_0 to be in $W^{1,p} \cap W^{2,2}_{loc}$ for some $p > d$ and $\gamma_n \in L^\infty$ here. The main argument which was needed in the proof of the main result where improved L^2 and L^∞ convergence estimate, which we derived by direct estimations. Note that the L^∞ estimate does not follow directly from the De Giorgi–Nash–Moser Theorem. Finally, we derive the polarisation tensor bounds corresponding to a very contrasted medium in the last section of chapter 2 . Unlike the finite contrast case, we show that the polarisation tensor M no longer remains strongly positive definite.

The assumption that d_n converges to 0 in L^p norm for $1 \leq p \leq \frac{d}{2}$, for the non-intertwined case and similarly the mixed case for $1 \leq p \leq 2d$, cannot

be dealt with the same method. The approach we used here fails for this regime. In dimension two, the convergence result is indeed optimal for the insulated and the high conductivity case, as the result holds true for any $p \geq 1$. However, this is not the case for $d \geq 3$. In dimension three, one possible way to obtain the optimal polarisation tensor result for the high conductivity inclusion setting is to utilise the Helmholtz's decomposition argument simillar to 26. In this case, the resulting 'insulated' PDE is no longer an elliptic equation in divergence form. The PDE which arises from this argument may be written as a symmetric-curl-equation which takes the form

$$\begin{aligned} \nabla \times \left(\gamma_n^{-1} \nabla \times \Psi_n\right) &= \nabla \times \left(\left(\gamma_0^{-1} - \gamma_n^{-1}\right) \nabla \times \Phi_0\right) \quad \text{in} \quad \Omega \quad (4.1) \\ \Psi_n \times \nu_\Omega &= \mathbf{0} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

in which, an 'optimal' L^2 bound simillar to 21 is required to obtain the polarisation tensor result for $1 \leq p \leq \frac{3}{2}$. A natural question arises here: What is the optimal mode of convergence for d_n , in which, our main result remains true for dimensions $d \geq 3$? A possible answer to this question may be in determining the neccessary and sufficient conditions for $u_n \to u_0$ in L^∞ under the assumption that $d_n \to 0$ in L^p . If this is not possible, a counter-example would be helpful in understanding the drawbacks and obstructions of this analysis.

4.1.2 Sign-changing Jacobian constraints and quantitative Jacobian bounds in PDEs with high contrast composite media

Motivated from problems arising from hybrid imaging inverse problems, we studied the sufficient conditions on the geometry of a very contrasted composite material, which allows us to enforce a sign-changing Jacobian constraint on the solutions of a two-phase elliptic system in a bounded domain of $\mathbb{R}^d$ where $d \geq 3$. Inspired by (mathematical) elastic cavitation from the theory of homogenization, sign-changing inclusions were initially studied in the periodic cell [9] where exotic materials have been constructed to satisfy this Jacobian constraint. These materials introduces unconventional behaviour, as their microscopic properties are very different from the ones of their limit effective approximation. Moreover, only a few examples are known in the literature, and no systematic studies appears to have been conducted. While it is known that some structures are sign changing, that property is not quantified, that is, the proportion of negative and positive Jacobian volumes are not known.

In this thesis, we characterised a class of inclusions Q and boundary conditions ϕ such that the determinant of the solution of the boundary value problem satisfies this sign-changing constraint. We provide lower bounds for the measure of the sets where Jacobian determinant is greater than a positive constant (or lower than a negative constant). Although different sign changing structures were introduced in [9] where the existence of such media was first established, the quantitative estimates that are provided here are new. We proved a new contrast dependence result for the gradient DU in this setting, allowing us to give a precise estimate on the contrast parameter $k \gg 1$ for the sign-changing phenomenon to occur. The contrast and volume estimates that we provided here depends only on the geometrical characteristics of the exterior domain Ω and the material Q.

The ad-hoc approach that we used in deriving the volume estimates relies fundamentally on the harmonic comparison principles. The tube-connected assumption was certainly unnecessary. A more streamlined proof would simply use auxilliary problems defined in $\Omega \setminus Q \cap (0,\infty)^d$. The reason detailed this hands-on approach is that it allows explicit bounds (as closed form solutions can be written down) in practical cases, see Proposition. We note that the method presented here does not provide optimal bounds for, say,$J_k^{-1}((-\infty,0))$ and $J_k^{-1}((0,\infty))$, however it does give a rule of thumb to obtain larger bounds: the largest the cross section of the connecting tubes, the better, by comparison. In the proof of the sign-changing material , we discussed and showed that the measure of the sets where the Jacobian determinant is positive or negative is not controlled by the Lebesgue measure of the inclusion Q; in particular it may happen when Q has zero measure, for example, we showed that any surface of co-dimension 1 with smooth boundary edges embedded in dimension d which satisfy the symmetrical assumptions in is in fact a sign-changing inclusion. A natural question which arise here is that if it is possible to obtain simillar volume estimates for the Jacobian determinant when the inclusion Q is now a crack domain such as the annular region in $\mathbb{R}^3$. A possible way to study these inclusions it to understand the gradient regularity of harmonic functions in these crack domains. Harmonic functions in domains with cracks is known to have gradient discontinuities as the solution approaches the boundary edges, and this calls for a careful analysis on the discontinuity behaviour around these regions. If this study is succesful, it will pave the way towards constructing a topoligical optimization scheme, where the Jacobian of such simple structures can be prescribed in most of the domain, leading to a variety of applications.

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Appendix A

Appendix

Proposition 83. (*The Schauder perturbation theorem*) . *Let* Ω *be a bounded regular domain in* $\mathbb{R}^d$ *where* $d = 2$ *or* 3. *Let* $1 < p < d$ *and denote* $p^* = \frac{dp}{d-p}$. *For any* $1 \leq q \leq p^*$, *there exists* $\eta > 0$ *depending on* Ω, p, q *such that if*

$$\|A - I_d\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \left(\sum_{i,j} \left| A_{ij}(x) - \delta_{ij} \right| \right) \leq \delta,$$

then for any $f \in L^p(\Omega)$ *there exists a unique* $u \in W_0^{1,q}(\Omega)$ *weak solution of*

$$-\nabla \cdot (A \nabla u) \quad = \quad f \, in D'(\Omega).$$

Furhtermore, there exists constant C depending only on Ω, p, q *such that*

$$\|u\|_{W_0^{1,q}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Proposition 84. *Let* Ω *be a bounded regular domain and* $\gamma_0 \in W^{1,p}(\Omega)$ *where* $p > d$. *Suppose that* $u \in H^1(\Omega)$ *weakly solves the elliptic equation*

$$-\nabla \cdot (\gamma_0 \nabla u) = f \, in D'(\Omega), \tag{A.1}$$

for $f \in L^p(\Omega)$, *then for* $d = 2$, p^* *may be arbitrarily chosen between* $(2, \infty)$, *and when* $d > 2$, $p^* = \frac{2d}{d-2}$, *we have* $u \in W_{loc}^{1,p^*}(\Omega)$ *and the following estimate holds*

$$\|u\|_{W^{1,p^*}(V)} \leq C(V) \|f\|_{L^p(\Omega)} \tag{A.2}$$

where $V \Subset \Omega$.

Proof. Assume first that γ_0 is an isotropic conductivity matrix, i.e $\gamma_0 = \gamma_0 I_d$. Since $\gamma_0 \in C^{0,\alpha}(\Omega)$ for some $\alpha > 0$ depending only p. We have for $B_\rho(x_0)$

$$\gamma_0(x) = \gamma_0(x_0) + b(x)$$

with

$$|b(x)| = |\gamma_0(x_0) - \gamma_0(x)| \leq C(d,\rho)\,\|\gamma_0\|_{W^{1,p}(\Omega)}\,|x_0 - x|^{\alpha}.$$

Consider χ a cut-off function which is 1 in $B_\rho(x_0)$ and 0 outside $B_{2\rho}(x_0)$. We write

$$\begin{aligned} \nabla\cdot(\gamma_0\nabla u) &= \nabla\cdot(\chi\gamma_0\nabla u) - \gamma_0\nabla u\cdot\nabla\chi \\ -\nabla\cdot(\gamma_0\nabla(u\chi)) &= f\chi - \nabla\cdot(u\gamma_0\nabla\chi) - \gamma_0\nabla u\cdot\nabla\chi \text{ in } D'(\Omega). \end{aligned}$$

Since χu is supported in $B_{2\rho}(x_0)$, we may write

$$-\nabla\cdot(BI_d\nabla(\chi u)) = \frac{1}{\gamma_0(x_0)}\left(f\chi - \nabla\cdot(u\gamma_0\nabla\chi) - \gamma_0\nabla u\cdot\nabla\chi\right) \text{ in } D'(\Omega)$$

where

$$\begin{aligned} B &= 1 + \frac{\gamma_0(x) - \gamma_0(x_0)}{\gamma_0(x_0)} \text{ in } B_{2\rho}(x_0) \\ &= 1 \text{ in } \Omega\setminus B_{2\rho}(x_0). \end{aligned}$$

From Schauder's perturbation theorem (83), we have for $1 < p,q \leq p^*$, there exists a constant $\eta > 0$ depending on p,q,Ω such that the solution to

$$-\nabla\cdot(B\nabla v) = g \text{ in } D'(\Omega)$$

has a unique solution $v \in W_0^{1,q}(\Omega)$ such that for $g \in L^p(\Omega)$

$$\|v\|_{W_0^{1,q}(\Omega)} \leq C\eta\,\|g\|_{L^p(\Omega)}$$

provided

$$\|B - I_d\|_{L^\infty(\Omega)} \leq \eta.$$

Thus, choose ρsuch that $|\gamma_0(x_0) - \gamma_0(x)| \leq \eta$ on $B_{2\rho}(x_0)$ and conclude that

$$\|\chi u\|_{W_0^{1,2^*}(\Omega)} \leq \left\|\frac{1}{\gamma_0(x_0)}\left(f\chi - \nabla\cdot(u\gamma_0\nabla\chi) - \gamma_0\nabla u\cdot\nabla\chi\right)\right\|_{L^2(\Omega)}.$$

Note that

$$\begin{aligned} \left\|\frac{1}{\gamma_0(x_0)}\left(f\chi - \nabla\cdot(u\gamma_0\nabla\chi) - \gamma_0\nabla u\cdot\nabla\chi\right)\right\|_{L^2(\Omega)} &\leq C(\rho,\|\gamma_0\|_{L^\infty})\left(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(B_{2\rho})}\right) \\ &\leq C(\rho,\|\gamma_0\|_{L^\infty})\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \end{aligned}$$

by Cacciopoli's inequality.

By a usual topological covering argument, this shows that

$$\|u\|_{W_0^{1,2^*}(K)} \leq C(\rho,\|\gamma_0\|_{L^\infty})\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

for any compact subset $K \Subset \Omega$. When γ_0 is a symmetric matrix, we can come back to the isotropic case by a rotation and a dilation. □

Corollary 85. *Let $u \in W^{1,2^*}_{loc}(\Omega)$ be the weak solution to* (A.1)*. Then for any compact set $K \Subset \Omega$, we have $u \in H^2(K)$, and*

$$\|u\|_{H^2(K)} \leq C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant $C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)$ depeds only on the bound from $|\nabla\chi|$ and the $W^{1,p}$ norm of γ_0.

Proof. Let K be a compact subset of Ω and let χ be a cut off function such that $\chi = 1$ on K and supported in Ω. Note that for $i = 1, \ldots, d$

$$-\partial_i\left(\nabla\cdot\left(\gamma_0\nabla\left(u\chi\right)\right)\right) \;=\; \partial_i\left(f\chi - \nabla\cdot\left(u\gamma_0\nabla\chi\right) - \gamma_0\nabla u\cdot\nabla\chi\right) \text{ in } D'(\Omega).$$

By a direct computation , we have

$$-\nabla\cdot\left(\gamma_0\nabla\left(\partial_i\left(u\chi\right)\right)\right) \;=\; \nabla\cdot\left(\partial_i\gamma_0\nabla\left(u\chi\right)\right) + \partial_i\left(f\chi - \nabla\cdot\left(u\gamma_0\nabla\chi\right) - \gamma_0\nabla u\cdot\nabla\chi\right) \quad \text{(A.3)}$$

Since for $p > d$,

$$\frac{1}{p} + \frac{1}{2^*} \leq \frac{1}{2}$$

it is clear from (A.2) that

$$\begin{aligned}\left\|\left(\partial_i\gamma_0\right)\nabla\left(u\chi\right)\right\|_{L^2(\Omega)} &\leq \left\|\left(\partial_i\gamma_0\right)\right\|_{L^p}\left\|\nabla\left(u\chi\right)\right\|_{L^{2^*}(\Omega)} \\ &\leq C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)\end{aligned}$$

and

$$\begin{aligned}\left\|\nabla\cdot\left(u\gamma_0\nabla\chi\right)\right\|_{L^2(\Omega)} &= \left\|u\left(\nabla\cdot\left(\gamma_0\nabla\chi\right)\right) + \nabla u\cdot\gamma_0\nabla\chi\right\|_{L^2(\Omega)} \\ &\leq C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)\left(\|u\|_{L^2(K)} + \|\nabla u\|_{L^2(K)}\right)\end{aligned}$$

thus, the terms on the R.H.S in (A.3) belongs to $H^{-1}(\Omega)$, therefore, we deduce from standard elliptic regularity estimates that $\partial_i u \in H^1_{loc}(\Omega)$ and the following estimate holds

$$\|\partial_i u\|_{H^1(K)} \leq C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant $C\left(K, \|\gamma_0\|_{W^{1,p}(\Omega)}\right)$ depeds only on the bound from $|\nabla\chi|$ and the $W^{1,p}$ norm of γ_0. Altogether, by combining the statement from (A.2), we arrive at our statement. □

Corollary 86. *Let $u \in H^2_{loc}(\Omega)$ be the weak solution to A.1. Suppose additionally that*

$\gamma_0 \in H^2_{loc}(\Omega)$, and $f \in W^{1,2}(\Omega)$ then $u \in H^3_{loc}(\Omega)$ and the following estimate holds for every compact set $K \Subset \Omega$

$$\|u\|_{H^3(K)} \leq C\left(K, \|\gamma_0\|_{W^{2,2}(\tilde{K})}\right)\left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

Proof. We focus the case when $d = 3$, the proof for $d = 2$ follows simillarly. Observe that

$$\begin{aligned} -\nabla \cdot (\gamma_0 \nabla (\partial_k u)) &= \partial_k f - \nabla \cdot ((\partial_i \gamma_0) \nabla u) \\ &= \partial_k f - \sum_{i,j} \partial_i \left(\partial_k \gamma_{0_{i,j}} \partial_j u\right) \\ &= \partial_k f - \sum_{i,j} \partial_k \gamma_{0_{i,j}} \partial^2_{i,j} u - \sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \partial_j u \text{ in } D'(\Omega) \end{aligned} \tag{A.4}$$

Note that

$$\begin{aligned} \sum_{i,j} \partial_k \gamma_{0_{i,j}} \partial^2_{i,j} u &\in L^6 \times L^2 = L^{\frac{3}{2}} \\ \sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \partial_j u &\in L^2 \times L^6 = L^{\frac{3}{2}} \\ \partial_k f &\in L^2 \subset L^{\frac{3}{2}} \end{aligned}$$

thus, by Schauder's perturbation theorem 83 we have

$$\partial_k u \in W^{1,3}_{loc}(\Omega)$$

and

$$\begin{aligned} \|\partial_k u\|_{W^{1.3}(K)} &\leq C(K) \left(\|\partial_k f\|_{L^{\frac{3}{2}}(\Omega)} + \left\| \sum_{i,j} \partial_k \gamma_{0_{i,j}} \right\|_{L^6(\tilde{K})} \left\| \partial^2_{i,j} u \right\|_{L^2(\tilde{K})} \right. \\ &\quad \left. + \left\| \sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \right\|_{L^2(\tilde{K})} \left\| \partial_j u \right\|_{L^6(\tilde{K})} \right) \\ &\leq C(K) \left(\|\partial_k f\|_{L^{\frac{3}{2}}(\Omega)} + \|\gamma_0\|_{W^{2,2}(\tilde{K})} \|u\|_{H^2(\tilde{K})} \right) \\ &\leq C\left(K, \|\gamma_0\|_{W^{2,2}(\tilde{K})}\right)\left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)}\right) \end{aligned}$$

Using this fact, we may bootstrap the argument in (A.4) to obtain

$$\begin{aligned} \sum_{i,j} \partial_k \gamma_{0_{i,j}} \partial^2_{i,j} u &\in L^6 \times L^3 \\ \sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \partial_j u &\in L^2 \times L^q \end{aligned}$$

for any $q < \infty$ and . Altogether, this shows that the R.H.S of (A.4) belongs to $L^{2-\varepsilon}$

for any $\varepsilon > 0$ (since $q < \infty$). Hence by reiterating Schauder's argument , we have $\partial_k u \in W^{1,6-\varepsilon}$. Going back to the line (A.4),

$$\begin{aligned}
\sum_{i,j} \partial_k \gamma_{0_{i,j}} \partial^2_{i,j} u &\in L^6 \times L^{6-\varepsilon} \\
\sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \partial_j u &\in L^2 \times L^\infty \\
\partial_k f &\in L^2.
\end{aligned}$$

Note that now the terms on the R.H.S of (A.4) belongs to L^2. We use Schauder's method to obtain $\partial_k u \in W^{1,6}_{loc}$ and

$$\begin{aligned}
\|\partial_k u\|_{W^{1.6}(K)} &\leq C(K) \left(\|\partial_k f\|_{L^2(\Omega)} + \left\| \sum_{i,j} \partial_k \gamma_{0_{i,j}} \right\|_{L^6(\tilde{K})} \left\| \partial^2_{i,j} u \right\|_{L^{6-\varepsilon}(\tilde{K})} \right. \\
&\quad \left. + \left\| \sum_{i,j} \partial^2_{i,k} \gamma_{0_{i,j}} \right\|_{L^2(\tilde{K})} \left\| \partial_j u \right\|_{L^\infty(\tilde{K})} \right) \\
&\leq C\left(K, \|\gamma_0\|_{W^{2,2}(\tilde{K})}\right) \left(\|\partial_k f\|_{L^2(\Omega)} + \|\partial_k u\|_{W^{1.3}(\tilde{K})} \right)
\end{aligned}$$

for some compact set $K \subset \tilde{K}$. Now, we are done, as

$$-\nabla \cdot \left(\gamma_0 \nabla \partial_{k,l} u \right) = \nabla \cdot \left(\partial_l \gamma_0 \nabla \partial_k u \right) + \nabla \cdot \left(\partial_{k,l} \gamma_0 \nabla u \right) \tag{A.5}$$

and

$$\begin{aligned}
\partial_l \gamma_0 \nabla \partial_k u &\in L^6 \times L^6 \subset L^2 \\
\partial_{k,l} \gamma_0 \nabla u &\in L^2 \times L^\infty \subset L^2.
\end{aligned}$$

In particular, the terms on the R.H.S of (A.5) now belongs to $H^{-1}(\Omega)$. We deduce by standard regularity result that $\partial_{k,l} u \in H^1_{loc}(\Omega)$ and

$$\begin{aligned}
\left\| \partial_{k,l} u \right\|_{H^1(K)} &\leq C(K) \left(\|\partial_l \gamma_0 \nabla \partial_k u\|_{L^2(K)} + \left\| \partial_{k,l} \gamma_0 \nabla u \right\|_{L^2(K)} + \left\| \partial_{k,l} u \right\|_{L^2(\tilde{K})} \right) \\
&\leq C(K) \left(\|\partial_l \gamma_0\|_{L^6(K)} \|\nabla \partial_k u\|_{L^6(K)} + \left\| \partial_{k,l} \gamma_0 \right\|_{L^2(K)} \|\nabla u\|_{L^\infty(K)} + \|\partial_k u\|_{W^{1,6}(\tilde{K})} \right) \\
&\leq C(K) \|\gamma_0\|_{W^{2,2}(K)} \|\partial_k u\|_{W^{1,6}(\tilde{K})}
\end{aligned}$$

for some compact set $K \subset \tilde{K}$.Altogether,

$$\begin{aligned}
\|u\|_{H^3(K)} &\leq C(K) \|\gamma_0\|_{W^{2,2}(K)} \|\partial_k u\|_{W^{1,6}(\tilde{K})} \\
&\leq C\left(K, \|\gamma_0\|_{W^{2,2}(\tilde{K})}\right) \left(\|f\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right)
\end{aligned}$$

□

Theorem 87. *(Girault-Raviart) Girault and Raviart [25, 26] Suppose that $\Omega \subset \mathbb{R}^2$ is*

a bounded simply connected domain with Lipschitz- continuous boundary, furthermore, let v be a vector field in $L^p(\Omega)^2$ *such that*

$$\nabla \cdot v = 0 \quad in \quad \Omega$$

for any $p \geq 2$*, then there exists a stream function* $\phi \in W^{1,p}(\Omega)$ *such that*

$$v = \nabla \phi^{\perp} \quad in \quad \Omega, \tag{A.6}$$

in which, the stream function ϕ *is unique up to an additive constant.*

Theorem 88. *(Colton & Kress) Let* $A : X \to X$ *be a bounded linear operator mapping the Banach space X onto itself. Then the Neumann series*

$$\left(\lambda I - K_Q^*\right)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n$$

converges in the uniform operator norm for all $\lambda > r(A)$ *where* $r(A)$ *is the spectral radius of A in X.*

Proposition 89. *Let* $\partial\Omega$ *be a* $C^{1,\alpha}$ *domain, consider the space*

$$\mathscr{H}_0 := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega} f = 0 \right\}$$

endowed with the inner product

$$\langle \phi, \psi \rangle_{\mathscr{H}} := -\int_{\partial\Omega} \phi \cdot S_{\Omega}(\psi) d\sigma(y),$$

then $\mathscr{H}_0$ *is a Hilbert space in which its norm is equivalent to the usual space* $H^{\frac{1}{2}}(\partial\Omega)$*. Moreover, the operator*

$$K_{\Omega}^* : \mathscr{H}_0 \to \mathscr{H}_0$$

is compact, self-adjoint, and only has real discrete spectrum accumulating to 0.

Proposition 90. *Let Q be a* C^2 *domain and let* $C_0^{0,\alpha}(\partial Q) := \left\{ f \in C^{0,\alpha}(\partial Q) \, : \, \int_{\partial Q} f \, d\sigma(y) = 0 \right\}$*. Then the spectrum of the operator*

$$K_Q^* : C_0^{0,\alpha}(\partial Q) \to C_0^{0,\alpha}(\partial Q)$$

consists only of discrete eigenvalues σ_n *where* $\sigma_n \in (-\frac{1}{2}, \frac{1}{2})$*, moreover, the Neumann series*

$$(\lambda I - K_Q^*)^{-1} = \frac{1}{\lambda} \sum_{i=0}^{\infty} \left(\frac{K_Q^*}{\lambda} \right)^i$$

converges in operator norm $\forall |\lambda| > \sup_n \sigma_n$*.*

Proof. Suppose that $\phi \in C_0^{0,\alpha}(\partial Q)$ and ϕ satisfies $\left(\sigma_n I - K_Q^*\right)(\phi) = 0$, where $\phi \neq 0$.

Consider the following quantities

$$A = \int_Q |\nabla S_Q(\phi)|^2 \, \mathrm{d}x; B = \int_{\mathbb{R}^3 \setminus Q} |\nabla S_Q(\phi)|^2 \, \mathrm{d}x, \tag{A.7}$$

since

$$\Delta S_Q(\phi) \quad = \quad 0 \quad \text{in} \quad \mathbb{R}^d \setminus \partial Q$$

we may apply the divergence theorem and the jump formula of $S_Q(\phi)$ to find

$$A = \int_{\partial Q} \left(-\frac{1}{2} I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \mathrm{d}x; B = -\int_{\partial Q} \left(\frac{1}{2} I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \mathrm{d}x. \tag{A.8}$$

Since $K_Q^*(\phi) = \sigma_n \phi$, we have

$$\sigma_n = \frac{1}{2} \frac{B-A}{B+A}.$$

Thus, if A and B are not identically 0, this shows that $|\sigma_n| < \frac{1}{2}$ and we are done.

Now consider the case that $\sigma_n = -\frac{1}{2}$ which corresponds to the case $B = 0$. Since $B = 0$, we deduce from (A.7) that $S_Q(\phi)$ is constant on $\mathbb{R}^d \setminus Q$. Moreover, by the usual decay estimates

$$|S_Q(\phi)|(x) \leq O\left(\frac{1}{|x|^{d-1}} \right) \text{ for } |x| \text{ large},$$

so, we deduce that

$$S_Q(\phi) \equiv 0 \text{ on } \mathbb{R}^d \setminus Q$$

which further implies that ϕ is identically 0 by the injectivity of S_Q. Since we have assumed that $\phi \neq 0$, this implies a contradiction and thanks to (A.8), this also implies that $\sigma_n = -\frac{1}{2}$ is not an eigenvalue.

When $\sigma_n = \frac{1}{2}$, we have that $A = 0$, so we deduce again from (A.7) that $S_Q(\phi) = m$, a constant in $\overline{Q}$. Now,

$$\begin{aligned} B &= -\int_{\partial Q} \left(\frac{1}{2} I + K_Q^* \right) (\phi) \cdot S_Q(\phi) \mathrm{d}x \\ &= -m \int_{\partial Q} \phi \mathrm{d}x. \end{aligned}$$

Since by our assumption $\phi \in C_0^0(\partial Q)$, this implies that $B = 0$, which further implies that $S_Q(\phi)$ is a constant in $\mathbb{R}^3 \setminus Q$. By the same argument as before $S_Q(\phi)$ must be 0 in $\mathbb{R}^3 \setminus Q$, which implies $m = 0$ by the continuity of $S_Q(\phi)$, therefore implying $\phi = 0$ and hence a contradiction. Combining these two facts, we deduce that the spectrum of K_Q^* in $C_0(\partial Q)$ lies in the open interval $\sigma_n \in (-\frac{1}{2}, \frac{1}{2})$.

Lastly, consider the space

$$\mathscr{H}_0 := \left\{ f \in L^2(\partial\Omega) : \int_{\partial\Omega} f = 0 \right\}$$

endowed with the inner product

$$\langle \phi, \psi \rangle_{\mathscr{H}} := -\int_{\partial\Omega} \phi \cdot S_\Omega(\psi) \mathrm{d}\sigma(y).$$

By Proposition 89 in the appendix, the spectrum of K_Q^* in $\mathscr{H}_0$ consists only of real discrete eigenvalues converging to 0 with spectral radius $\sup_n \sigma_n < \frac{1}{2}$. Since

$$\sigma(K^*_{\partial Q}|_{C_0^0}) \subseteq \sigma\left(K^*_{\partial Q}|_{\mathscr{H}_0}\right),$$

we deduce that the spectrum of $K^*_{\partial Q}$ in the space $C_0^0(\partial Q)$ also consists only of discrete eigenvalues with spectral radius $\sup_n \sigma_n < \frac{1}{2}$. Thus, we may apply Theorem 88 □

Proposition 91. *Let $d = 3$, $\Omega = [-1,1]^3$,and U_∞ be given by* (1.9) *and $\phi = I_3$. Suppose that Q is a proper torus centred at the origin with minor radii a, and major radii c, with $0 < a < c < R$. Then $y_0 = \inf(\Omega \setminus Q \cap (\mathbb{R}^+ e_1)) = \big(c-a, 0, 0\big)$ and the following inequalities holds*

$$\frac{\partial U_1}{\partial x_1}(0) \geq \Big(1 - \frac{1}{\sqrt{2}}\Big) \frac{2\,(c-a)}{\pi^2 \cosh\left(\frac{2\pi}{c-a}\right)} \text{ and } \frac{\partial U_1}{\partial x_1}(y_0) \leq -\Big(1 - \frac{1}{\sqrt{2}}\Big) \frac{2\,(c-a)}{\pi^2 \cosh\left(\frac{2\pi}{c-a}\right)}.$$

Proof. We parametrize Q as follows

$$Q\big(a,c,\theta,\phi\big) := \begin{bmatrix} (c + a\cos\theta)\cos\phi \\ (c + a\cos\theta)\sin\phi \\ a\sin\theta \end{bmatrix}; \qquad 0 < a < c < 1, \text{ and } \phi \in [0, 2\pi].$$

Set $p_{1/2} = \big(\frac{c-a}{2}, 0, 0\big)$. The ball $B\big(p_{1/2}, \frac{c-a}{2}\big)$ is tangential to the boundary of the torus Q at y_0. Consider the cuboid $D = p_{1/2} + (-r,r)^2 \times (-1,1)$ for some $r = \frac{c-a}{2\sqrt{2}}$, sufficiently small so that $D \cap Q = \emptyset$ and $D \subset \Big(\Omega \cap x_1 > 0\Big)$. Take V_1 to be the solution to

$$\begin{aligned} \Delta V_1 &= 0 \text{ in } D \\ V_1 &= \frac{c-a}{2} - r \text{ on } \partial D \cap \{x_3 = \pm 1\}. \\ V_1 &= 0 \text{ on } \partial D \setminus \{x_3 = \pm 1\}. \end{aligned}$$

By the weak maximum principle, we have $U_1|_{\partial D} \geq V_1|_{\partial D}$ so that V_1 is a sub-solution with respect to U_1. We construct the solution to V_1 by the standard method of separation of variables in D and obtain

$$V_1(x_1,x_2,x_3) = \sum_{m=n=\text{odd}}^{\infty} C_{m.n} \sin\Big(\frac{m\pi}{2r}\big(x_1 - (\frac{c-a}{2} - r)\big)\Big) \sin\Big(\frac{n\pi}{2r}\big(x_2 + r\big)\Big) \cosh\Big(k_{m,n} x_3\Big)$$

where

$$\begin{aligned} k_{m,n} &= \frac{\pi}{2r}\sqrt{m^2+n^2} \\ C_{m.n} &= \big(x^*-r\big)\frac{16}{mn\pi^2}\frac{1}{\cosh(k_{m,n})} \end{aligned}$$

and

$$\begin{aligned} V_1(p_{1/2}) &= \sum_{m=n=\text{odd}}^{\infty} C_{m,n}\cosh(0) \\ &\geq \frac{16}{\pi^2\cosh(k_{1,1})}\big(\frac{c-a}{2}-r\big). \end{aligned}$$

Therefore

$$U_1(z) \geq V_1(z) > \Big(1-\frac{1}{\sqrt{2}}\Big)\frac{8\,(c-a)}{\pi^2\cosh\big(\frac{2\pi}{c-a}\big)}$$

so that by (61),

$$\frac{\partial U_1}{\partial x_1}(0) \geq \Big(1-\frac{1}{\sqrt{2}}\Big)\frac{2\,(c-a)}{\pi^2\cosh\big(\frac{2\pi}{c-a}\big)} \text{ and } \frac{\partial U_1}{\partial x_1}(y_0) \leq -\Big(1-\frac{1}{\sqrt{2}}\Big)\frac{2\,(c-a)}{\pi^2\cosh\big(\frac{2\pi}{c-a}\big)} \quad \text{(A.9)}$$

which is our statement. □

Proposition 92. *At $p_{1/2} = y_0 - \frac{C_1^2}{4L^2}e_1$, where*

$$C_1 := \Big(1-\frac{1}{\sqrt{2}}\Big)\frac{2\,(c-a)}{\pi^2\cosh\big(\frac{2\pi}{c-a}\big)}, \text{ and } L := \|U\|_{C^{1,\frac{1}{2}}\left(\overline{\Omega\setminus Q}\right)},$$

there holds

$$\det DU\left(p_{1/2}\right) \leq -M_0,$$

where M_0 is given in Proposition 4.1.2.

Proof. Thanks to Lemma 80 there holds

$$\begin{aligned} \frac{\partial U_1}{\partial x_1}(p_{1/2}) - \frac{\partial U_1}{\partial x_1}(y_0) &\leq L\left|p_{1/2}-y_0\right|^{\frac{1}{2}} \\ \frac{\partial U_1}{\partial x_1}(p_{1/2}) &\leq -C_1 + L\left|p_{1/2}-y_0\right|^{\frac{1}{2}} \\ \frac{\partial U_1}{\partial x_1}(p_{1/2}) &\leq -\frac{1}{2}C_1. \end{aligned} \quad \text{(A.10)}$$

Now that $p_{1/2}$ lies on the line $(s,0,0)$ and at a distance $r_2 = \frac{1}{4L^2}\Big(C_1\Big)^2$ away from the boundary of ∂Q, we observe that in the x_1x_2 plane, we may consider the circle formed by the triangle with vertices E

$$E := p_{1/2} + \{(-r_2,0,0),(r_2,0,0),(0,h,0)\}$$

with $h = \sqrt{|x_0|^2 - |p_{1/2}|^2}$ so that the maximal circle contained in E is of radius

$$\begin{aligned}\tau &= \frac{hr_2}{\sqrt{r_2^2+h^2}+r_2} \\ &= \frac{r_2^{\frac{3}{2}}\sqrt{2(c-a)-r_2}}{r_2+\sqrt{r_2\Big(2(c-a)\Big)}}\end{aligned}$$

and centred at $x_\tau = x_0^* + \frac{1}{2}\tau\mathbf{e_2}$. Thus, by considering the domain

$$D_2 := p_{1/2} + (-\tau,+\tau)\times(0,\frac{1}{2}\tau)\times(-1,1)$$

and taking V_2 to be the solution to the Dirichlet problem

$$\begin{aligned}\Delta V_2 &= 0 \text{ in } D_2 \\ V_2 &= x_2 \text{ on } \partial D_2 \cap \{x_3 = \pm 1\} \\ V_2 &= 0 \text{ on } \partial D_2 \setminus \{x_3 = \pm 1\},\end{aligned}$$

we have $0 \le V_2|_{\partial D_2} \le U_2|_{\partial D_2}$ and V_2 is a sub-solution with respect to U_2. We compute the solution V_2 as

$$V_2(x_1,x_2,x_3) = \sum_{m=\text{odd},n\in\mathbb{Z}^+}^{\infty} D_{m.n}\sin\Big(\frac{m\pi}{2\tau}\big(x_1-(\big|p_{1/2}\big|-\tau)\big)\Big)\sin\Big(\frac{2n\pi}{\tau}\big(x_2\big)\Big)\cosh\Big(k_{m,n}x_3\Big)$$

$$\begin{aligned}k_{m,n} &= \frac{\pi}{\tau}\sqrt{\frac{m^2}{4}+4n^2} \\ D_{m.n} &= \frac{4\tau}{mn\pi^2}\frac{(-1)^{n+1}}{\cosh\Big(k_{m,n}\Big)}\end{aligned}$$

and subsequently, at the point $(\big|p_{1/2}\big|,\frac{1}{4}\tau,0)$, only the terms with m,n odd survives, so we obtain

$$\begin{aligned}V_2(\big|p_{1/2}\big|,\frac{1}{4}\tau,0) &= \sum_{m,n=\text{odd},}^{\infty} D_{m,n}\cosh(0) \\ &\ge \frac{4\tau}{\pi^2}\frac{1}{\cosh\Big(\frac{\pi}{\tau}\sqrt{\frac{1}{4}+4}\Big)}\end{aligned}$$

which implies that

$$\frac{\partial U_2}{\partial x_2}(p_{1/2}) > \frac{\tau}{\pi^2}\frac{1}{\cosh\Big(\frac{\pi}{\tau}\sqrt{\frac{1}{4}+4}\Big)} =: C_2. \tag{A.11}$$

by Harnack's inequality. From our choice of D_3 that we may place a ball centred at $(|x_0^*|, \frac{1}{4}\tau, 0)$, tangent to the point at x_0^*, and yet lies in the upper half x_2 plane.

Finally, choose

$$D_3 := p_{1/2} + (-\tau, +\tau) \times (-\frac{\tau}{2}, \frac{\tau}{2}) \times (0,1),$$

and V_3 to be the solution to

$$\begin{aligned} \Delta V_3 &= 0 \text{ in } D_3 \\ V_3 &= 1 \text{ on } \partial D_3 \cap \{x_3 = 1\} \\ V_3 &= 0 \text{ on } \partial D_3 \setminus \{x_3 = 1\}. \end{aligned}$$

We have

$$V_3(x_1,x_2,x_3) = \sum_{m,n=\text{odd}}^{\infty} E_{m.n} \sin\Big(\frac{m\pi}{2\tau}\big(x_1 - (|x_0^*| + \tau)\big)\Big) \sin\Big(\frac{n\pi}{\tau}\big(x_2 + \frac{\tau}{2}\big)\Big) \sinh\Big(k_{m,n}x_3\Big)$$

$$\begin{aligned} k_{m,n} &= \frac{\pi}{\tau}\sqrt{\frac{m^2}{4} + n^2} \\ E_{m.n} &= \frac{16}{mn\pi^2}\frac{1}{\sinh(k_{m,n})} \end{aligned}$$

and

$$\frac{\partial V_3}{\partial x_3}(x) = \sum_{m,n=\text{odd}}^{\infty} k_{m,n}E_{m.n} \sin\Big(\frac{m\pi}{2\tau}\big(x_1 - (|x_0^*| + \tau)\big)\Big) \sin\Big(\frac{n\pi}{h}\big(x_2 + \frac{\tau}{2}\Big) \cosh\Big(k_{m,n}x_3\Big)$$

as the Fourier coefficients converges uniformly for all $x \in D_3$. Thus,

$$\begin{aligned} \frac{\partial V_3}{\partial x_3}(x_0^*) &= \sum_{m,n=\text{odd}}^{\infty} k_{m,n}E_{m.n} \cosh\Big(0\Big) \\ &> \frac{8\sqrt{5}}{\pi\tau}\frac{1}{\sinh(\frac{\pi}{\tau}\sqrt{\frac{5}{4}})} = C_3. \end{aligned} \tag{A.12}$$

Altogether, we combine the estimates A.10,A.11 and A.12 to obtain our statement. □

Proposition 93. *There holds*

$$\det DU(0) \geq M_1$$

where M_1 is given in Proposition 4.1.2.

Proof. By the symmetry of U_1 and U_2 we have $U_1(s,0,0) = U_2(0,s,0)\ \forall s \in [-1,1]$, thus

$$\frac{\partial U_1}{\partial x_1}(s,0,0) = \frac{\partial U_2}{\partial x_2}(0,s,0)$$

so we may use Proposition 91 to provide the same quantitative lower bounds for $\frac{\partial U_2}{\partial x_2}(0)$ as for $\frac{\partial U_1}{\partial x_1}(0)$. To estimate $\frac{\partial U_3}{\partial x_3}(0)$, we simply consider the domain

$$D_4 := (-\frac{d}{2\sqrt{2}}, \frac{d}{2\sqrt{2}}) \times (-\frac{d}{2\sqrt{2}}, \frac{d}{2\sqrt{2}}) \times (0,1)$$

and take W to satisfy

$$\begin{aligned} -\Delta W &= 0 \quad \text{in} \quad D_3 \\ W(-\frac{d}{2\sqrt{2}}, x_2, x_3) &= W(\frac{d}{2\sqrt{2}}, x_2, x_3) = 0 \\ W(x_1, -\frac{d}{2\sqrt{2}}, x_3) &= W(x_1, \frac{d}{2\sqrt{2}}, x_3) = 0 \\ W(x_1, x_2, 1) &= 1 \\ W(x_1, x_2, 0) &= 0 \end{aligned}$$

we have that $W|_{\partial D_4} \leq U_3|_{\partial D_4}$ so that W is a sub-solution with respect to U_3. We compute W as

$$W(x_1,x_2,x_3) = \sum_{m,n=\text{odd}}^{\infty} F_{m.n} \sin\Big(\frac{\sqrt{2}m\pi}{d}\big(x_1 + \frac{d}{2\sqrt{2}}\big)\Big) \sin\Big(\frac{\sqrt{2}n\pi}{d}\big(x_2 + \frac{d}{2\sqrt{2}}\big)\Big) \sinh\Big(k_{m,n}x_3\Big)$$

$$\begin{aligned} k_{m,n} &= \frac{\sqrt{2}\pi}{d}\sqrt{m^2+n^2} \\ F_{m.n} &= \frac{16}{mn\pi^2}\frac{1}{\sinh(k_{m,n})} \end{aligned}$$

$$\frac{\partial W}{\partial x_3}(x) = \sum_{m,n=\text{odd}}^{\infty} k_{m,n}F_{m.n} \sin\Big(\frac{\sqrt{2}m\pi}{d}\big(x_1 + \frac{d}{2\sqrt{2}}\big)\Big) \sin\Big(\frac{\sqrt{2}n\pi}{d}\big(x_2 + \frac{d}{2\sqrt{2}}\big)\Big) \cosh\Big(k_{m,n}x_3\Big),$$

hence,

$$\begin{aligned} \frac{\partial W}{\partial x_3}(0) &= \sum_{m,n=\text{odd}}^{\infty} k_{m,n}F_{m.n} \\ &> \frac{32}{\pi d}\frac{1}{\sinh\left(\frac{2\pi}{d}\right)} \\ &= C_4. \end{aligned}$$

Altogether,

$$\begin{aligned} \det DU(0) &= \partial_1 U_1 \partial_2 U_2 \partial_3 U_3(0) \\ &\geq C_1^2 C_4 \\ &\geq \left(\left(2-\sqrt{2}\right)\frac{c-a}{\pi^2 \cosh\left(\frac{\pi}{2}(c-a)\right)}\right)^2 \cdot \frac{16}{\pi(c-a)}\frac{1}{\sinh\left(\frac{\pi}{c-a}\right)}. \end{aligned}$$

□

Lemma 94. *Suppose that*

$\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain*

$Q_\infty \subset\subset \Omega$ *is a connected planar domain*

Q_∞ *has smooth boundary edges.*

Then, $\forall \phi \in C^0(\overline{\Omega})$*, the Dirichlet problem*

$$\begin{aligned} -\Delta u &= 0 \quad in \quad \Omega \setminus \overline{Q}_\infty \\ u &= \phi \quad on \quad \partial(\Omega \setminus \overline{Q}_\infty) \end{aligned}$$

is well posed, moreover $u \in C^2(\Omega \setminus \overline{Q}_\infty) \cap C^0(\partial(\Omega \setminus \overline{Q}_\infty))$.

Proof. By theorem 2.14 in Gilbarg-Trudinger [23], the classical Dirichlet problem in any bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular with respect to the Laplace operator. Since by our assumption $\partial\Omega$ is regular, we just check that $\overline{Q}_\infty$ consists only of regular points and the proof follows from classical arguments. Consider $x_0 \in \overline{Q}_\infty$, by Wiener's criterion, x_0 is a regular boundary point of $\Omega \setminus \overline{Q}_\infty$if and only if the series

$$\sum_{j=0}^{\infty} \frac{C_j}{\lambda^j}$$

diverges for any fixed $\lambda \in (0,1)$ depending on x_0 , where

$$C_j = \mathrm{cap}_2 \left\{ x \in (\Omega \setminus \overline{Q}_\infty)^c : |x - x_0| \leq \lambda^j \right\}.$$

Consider first the case $x_0 \in Q_\infty$. Since Q_∞ is an open set in $\mathbb{R}^2$, $\exists r > 0 : B_2(x_0, r) \subset Q_\infty$, where B_2 denotes the two dimensional ball of radius r. In particular, taking $\lambda = r$, we have

$$C_j = \mathrm{Cap}_2 \left(B_2(x_0, r^j) \right)$$

$\forall j \geq 1$. Using the computation done by Evan's in [20], the capacity of a disk of radius r may be computed exactly to be

$$\mathrm{Cap}_2(D_r) = \frac{2}{\pi} r,$$

moreover, as the capacity measure is invariant under linear tranformations, we have $C_j = \frac{2}{\pi} r^j$, hence

$$\sum_{j=0}^{\infty} \frac{C_j}{r^j}$$

diverges, implying that $x_0 \in Q_\infty$ is a regular point. Now, take any point $x_0 \in \partial Q_\infty$, by our assumption ∂Q_∞ is twice continuously differentiable, thus from elementary geometry, ∂Q_∞ satisfies the interior disk condition. In particular $\forall x_0 \in \partial Q_\infty$, $\exists r > 0 :$ $B_2(x, r) \subset \overline{Q}_\infty$, and $B_2(x, r) \cap \partial Q_\infty = \{x_0\}$ for some $x \in Q_\infty$. Denoting the set $S_j =$

$\left\{x \in (\Omega \setminus \overline{Q}_\infty)^c : |x-x_0| \leq r^j\right\} \cap B_2(x_j, r^j)$ where x_j denotes the points in Q_∞ such that $B_2(x_j, r^j) \cap \partial Q_\infty = \{x_0\}$, we have

$$C_j \geq \mathrm{Cap}_2(S_j).$$

From elementary geometry again, we may place a disk of radius

$$R = \frac{\sin(\frac{\pi}{3})}{1+\sin(\frac{\pi}{3})} r^j$$

in the sector of the circle with opening angle $\frac{2}{3}\pi$ centered at x_0 with radius r^j and contained in S_j. This implies

$$C_j \geq \frac{\sin(\frac{\pi}{3})}{1+\sin(\frac{\pi}{3})} r^j,$$

hence the sum $\sum_{j=0}^{\infty} \frac{C_j}{r^j}$ diverges implying $x_0 \in \partial Q_\infty$is a regular point. □

Example 95. Let $U : B(0,R) \setminus \overline{B(0,R_0)} \to \mathbb{R}^3$ be the solution of the harmonic system given by

$$\begin{aligned} -\Delta U &= 0 \quad \text{in} \quad B(0,R) \setminus \overline{B(0,R_0)} \\ U &= 0 \quad \text{on} \quad \partial B(0,R_0) \\ U &= I_d \quad \text{on} \quad \partial B(0,R) \end{aligned} \tag{A.13}$$

with $0 < R_0 < R$, then the Jacobian determinant of the solution U satisfies

$$\det DU(x) \geq \frac{1}{R^2}\left(\frac{2R_0^3+R^3}{R^3-R_0^3}\right)|U(x)|^2 \quad \text{in} \quad B(0,R) \setminus \overline{B(0,R_0)}.$$

Proof. We look for solutions of the form $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ where

$u_1 = f_1(r)\sin\theta\cos\phi$
$u_2 = f_2(r)\sin\theta\sin\phi$
$u_3 = f_3(r)\cos\theta$
in terms of spherical coordinates.
Computing

$$\Delta u_i = \Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)u_i + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)u_i + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}u_i$$

for each ansatz above, we find that in all cases,

$$\Delta u_i = 0 \Longrightarrow f_i'' + \frac{2}{r}f_i' - \frac{2}{r^2}f_i = 0.$$

Thus, by solving this ODE, we find that f_i is of the form

$$f_i(r) = \left(\frac{a_1}{r^2} + a_2 r\right) \tag{A.14}$$

for some constants a_1 and a_2. Since $f_i(r)$ is of this form, and the boundary conditions that we impose on f_i are the same, namely

$$\begin{aligned} f_i(R) &= R \\ f_i(R_0) &= 0 \end{aligned} \tag{A.15}$$

for each i, we may restrict our computation for these undertermined constants to the first case only. Plugging A.14 into A.15, and solving the following linear system,

$$\begin{pmatrix} \frac{1}{R^2} & R \\ \frac{1}{R_0^2} & R_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

we find

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \frac{1}{\left(\frac{R_0}{R^2} - \frac{R}{R_0^2}\right)} \begin{pmatrix} R_0 & -R \\ -\frac{1}{R_0^2} & \frac{1}{R^2} \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix} \\ &= \frac{1}{\left(\frac{R_0}{R^2} - \frac{R}{R_0^2}\right)} \begin{pmatrix} RR_0 \\ -\frac{R}{R_0^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{R^3 R_0^3}{R^3 - R_0^3} \\ \frac{R^3}{R^3 - R_0^3} \end{pmatrix}. \end{aligned} \tag{A.16}$$

Thus, altogether, we have

$$U = \left(\left(-\frac{R^3 R_0^3}{R^3 - R_0^3}\right)\frac{1}{r^2} + \left(\frac{R^3}{R^3 - R_0^3}\right) r\right) \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}.$$

Computing the Jacobian matrix for DU with respect to (r, θ, ϕ), we have

$$J^U_{r,\theta,\phi} = \begin{bmatrix} A(r)\sin\theta\cos\phi & B(r)\cos\theta\cos\phi & -B(r)\sin\theta\sin\phi \\ A(r)\sin\theta\sin\phi & B(r)\cos\theta\sin\phi & B(r)\sin\theta\cos\phi \\ A(r)\cos\theta & -B(r)\sin\theta & 0 \end{bmatrix},$$

where $A(r) = \left(-2\frac{a_1}{r^3} + a_2\right)$ and $B(r) = \left(\frac{a_1}{r^2} + a_2 r\right)$. Furthermore, by computing the

determinant of $J^U_{r,\theta,\phi}$, we find

$$\begin{aligned}
\det J^U_{r,\theta,\phi} &= B(r)\sin\theta\cos\phi\left(A(r)\sin\theta\cos\phi(B(r)\sin\theta)+A(r)\cos\theta B(r)\cos\theta\cos\phi\right)\\
&\quad +B(r)\sin\theta\sin\phi\left(A(r)\sin\theta\sin\phi(B(r)\sin\theta)+A(r)\cos\theta B(r)\cos\theta\sin\phi\right)\\
&= A(r)B(r)^2\sin^3\theta\cos^2\phi+A(r)B(r)^2\sin\theta\cos^2\theta\cos^2\phi\\
&\quad +A(r)B(r)^2\sin^3\theta\sin^2\phi+A(r)B(r)^2\sin\theta\sin^2\phi\cos^2\theta\\
&= A(r)B(r)^2\left(\sin^3\theta\right)+A(r)B(r)^2\sin\theta\cos^2\theta\\
&= A(r)B(r)^2\sin\theta.
\end{aligned}$$

Now, by applying the chain rule given by

$$J^U_{x_1,x_2,x_3} = J^U_{r,\theta,\phi}J^{r,\theta\phi}_{x_1,x_2,x_3},$$

where the latter term denotes the Jacobian matrix for the change of variables between (r,θ,ϕ) and (x_1,x_2,x_3), altogether, we may compute

$$\begin{aligned}
\det DU &= \det J^U_{r,\theta,\phi}\det J^{r,\theta\phi}_{x_1,x_2,x_3}\\
&= A(r)B(r)^2\sin\theta\left(\frac{1}{r^2\sin\theta}\right)\\
&= \frac{A(r)B(r)^2}{r^2}.
\end{aligned}$$

Since $R_0\le r\le R$, and noting that from A.16, $a_1<0$, $a_2>0$, we find

$$\begin{aligned}
\det DU &= \frac{A(r)B(r)^2}{r^2}\\
&= \frac{1}{r^2}\left(-2\frac{a_1}{r^3}+a_2\right)\left(\frac{a_1}{r^2}+a_2r\right)^2\\
&\ge \frac{1}{R^2}\left(-2\frac{a_1}{R^3}+a_2\right)|U|^2\\
&= \frac{|U|^2}{R^2}\left(\frac{2}{R^3}\frac{R^3R_0^3}{R^3-R_0^3}+\frac{R^3}{R^3-R_0^3}\right)\\
&= \frac{|U|^2}{R^2}\left(\frac{2R_0^3+R^3}{R^3-R_0^3}\right)
\end{aligned}$$

□

Lemma 96. *(Hopf's Lemma) [23]Let $B(0,R)$ be a ball in $\mathbb{R}^d$. Supposed that $u\in C^2(B)\cap C^0(\partial B)$ and $-\Delta u=0$ in $B(0,R)$ with u attaining its maximum at $x_0\in\partial B$ and that $u(x_0)>u(x)$ for $x\in B(0,R)$. Then,*

$$\frac{\partial u}{\partial \nu}(x_0)>0.$$