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Simplicity and uncountable categoricity in excellent classes

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Abstract

We introduce Lascar strong types in excellent classes and prove that they coincide with the orbits of the group generated by automorphisms fixing a model. We define a new independence relation using Lascar strong types and show that it is well-behaved over models, as well as over finite sets. We then develop simplicity (when this independence relation has local character) and show that, under simplicity, the independence relation satisfies all the properties of nonforking in a stable first order theory. Further, simplicity for an excellent class, as well as the independence relation itself, is uniquely determined. Finally, we show that an excellent class is simple if and only if it has extensible U-rank (excellence does not imply simplicity in general). We deduce that any excellent class of finite U-rank is simple, and that any uncountably categorical excellent class has an expansion with countably many constants which is simple.

Introduction

Simplicity, in the context of first order model theory, is a very successful generalisation of stability; it is characterised by the existence of a well-behaved independence relation

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(given by nonforking) which gives rise to a good dimension theory. Simplicity has also been developed in contexts beyond first order model theory, with some compactness and homogeneity for example in Robinson theories [7], existentially closed models [20], and Compact Abstract Theories [3], as well as without compactness in homogeneous model theory [4]. By simplicity in a nonelementary context, we mean the existence of an independence relation satisfying *all* the properties of nonforking in a simple first order theory.

The goal of this paper is to develop simplicity in the context of excellent classes. Excellence is a model-theoretic property which was discovered by Shelah in his work around uncountable categoricity for classes of models of a sentence in $L_{\omega_1,\omega}$ [23,24]; it is the property that primary models exist over certain kinds of countable sets. Excellence implies the existence of full models, which can be used as universal domains, much in the way that saturated models are used in the first order case and homogeneous models in homogeneous model theory; full models exist and are unique up to isomorphism in each cardinality, and realise all the realisable types. Homogeneous uncountably categorical classes are excellent ([13] or [21]), but Shelah showed using an example of Marcus [18] that there are excellent uncountably categorical classes without uncountably homogeneous models. Excellent classes are still well-behaved: Shelah proved the parallel to Morley's theorem [24]; then Grossberg and Hart proved the Main Gap [5]; the second author proved a Baldwin–Lachlan theorem [16] introducing a U-rank for types over models with the usual additivity properties (the U-rank will be used in this paper); and finally, together with Shelah, the authors proved a generalisation of Hrushovksi's theorem [6] on group configuration [11]. In fact, some results of this paper are used in [11]. Excellence is also a key property in the classification of almost free algebras [19]. More recently, Zilber rediscovered excellence in his work around complex exponentiation and quasiminimality [25,26]. In this context, he showed that excellence is equivalent to natural arithmetic conditions [27].

The context of excellence assumes a nonelementary version of $\aleph_0$-stability, but without compactness, even $\aleph_0$-stability does not imply simplicity (see [10] for example). Typically, extension may fail, even local character, and the properties that hold over general sets in the first order case only hold over models. There are, however, interesting examples of simple excellent classes, e.g., Zilber's pseudo-analytic structures [25] (which are quasiminimal) and free groups [4] or free algebras (which are almost quasiminimal). More generally, we prove in this paper that *every* uncountably categorical excellent class is simple, once it is expanded with names for the prime model over the empty set. This shows that simplicity can play an important rôle in understanding uncountably categorical excellent classes, since simplicity has proved a convenient context in which to generalise the tools of geometric stability theory (see for example [2] and [9] in homogeneous model theory, and in a forthcoming paper we show that we can import canonical bases without destroying simplicity or excellence).

In this paper, we fix an excellent class $\mathcal{K}$ and work inside a large full model $\mathfrak{C}$ in $\mathcal{K}$. We introduce *Lascar strong types*; equality of two Lascar strong types is the finest invariant equivalence relation with a bounded number of classes. We prove all the first order results for Lascar strong types, in particular that Lascar strong types are the orbits of the group of strong automorphisms (those automorphisms fixing each Lascar strong type) and that the group of strong automorphisms is generated by the automorphisms fixing a model.

We then consider *Lascar-splitting*, which is a version of strong splitting, except that we have to work with special kinds of indiscernibles (essentially Morley sequences). This is due to the fact that indiscernible sequences are not so well-behaved at this level of generality (see below). We introduce an invariant and monotone independence relation with built-in extension property (an idea originating in [12] and also used in [10]). This independence relation satisfies all the usual properties over models as well as a Pairs Lemma over all sets, i.e., if a is independent from B over A and b is independent from Ba over Aa, then ab is independent from B over A. In addition, it satisfies natural restricted versions of extension and local character. Let us explain the use of the term *restricted*. In general, we do not have existence, i.e., it is possible that an element b may not be independent from a set A over the same set A. But, if b *is* independent from A over A, then (1) there is a finite $C \subseteq A$ such that b is independent from A over C (this is restricted local character); (2) for each B containing A there is b' realising $\mathrm{tp}(b/A)$ such that b' is independent from B over A (this is restricted extension). This independence relation is especially well-behaved over finite sets: If A is finite and b is independent from A over A, then (3) if a is independent from Ab over A, then b is independent from Aa over A (this is restricted symmetry); (4) for each B containing A there is b' realising the same Lascar strong type as b over A such that b' is independent from B over A (this is restricted strong extension). By monotonicity, the restriction that b is independent from A over A is necessary. We can also prove a good stationarity property for Lascar strong types.

As we pointed out, in a general excellent class, there may not be *any* independence relation satisfying all the first order properties of nonforking (see [10] for example), so the restrictions above are unavoidable. This is in spite of the fact that $\mathcal{K}$ is ω-stable. So we say that an excellent class is *simple* if the independence relation we introduced has local character. We are able to show that all the usual properties of nonforking in a stable first order theory hold for our independence relation when $\mathcal{K}$ is simple, namely, in addition to monotonicity, invariance, and local character, we also have finite character, extension, symmetry, transitivity, and stationarity of Lascar strong types. Furthermore, an excellent class $\mathcal{K}$ is simple if and only if *some* independence relation satisfies all these properties, and the properties themselves characterise the independence relation. Hence, the context of simplicity does not depend on our definition of the independence relation. We also show that the behaviour over finite sets entirely determines simplicity.

Finally, we revisit the U-rank. It was shown in [16] that in a general excellent class, the U-rank for types over models is well-behaved. We extend the U-rank to a complete type p over an arbitrary set by the supremum of the U-rank of types over models extending p. We say that $\mathcal{K}$ has extensible U-rank when the supremum is always achieved by a type over a model. We prove that an excellent class $\mathcal{K}$ is simple if and only if it has extensible U-rank (generalising [8]). We derive two interesting corollaries. Any excellent class of finite U-rank is simple. Any uncountably categorical excellent class has a countable expansion which is also simple.

Although many of the results are similar to the first order case, the technology used to prove them is not. In the first order case, it is possible to work inside a compact and homogeneous model. We do not have any compactness here, unlike Robinson theories [6], existentially closed models [20], or Compact Abstract Classes [3]. We do not even have much homogeneity left (in contrast to [4]), as full models are not even $\aleph_1$-homogeneous in general. Thus, two elements may have the same type over a subset A of $\mathfrak{C}$ without being automorphic over A. However, when A is finite or a model, then the natural semantic notion of a type over A (as an orbit of the group of automorphisms fixing A pointwise) has a good syntactic equivalent as a set of formulas with parameters over A. This dual aspect is used repeatedly, and is the chief reason why we can obtain good properties over finite sets or models in general. Another consequence of the failure of homogeneity is that indiscernible sequences are not so well-behaved; they cannot be extended in general, and a permutation of their elements does not necessarily extend to an automorphism.

0. Preliminaries

In this section, we remind the reader of a few facts about excellent classes due to Shelah, which can be found in [23,24], or in the expository paper [17] or in Baldwin's online book [1]. For simplicity, and without real loss of generality (see the remark below), we consider the case of the class $\mathcal{K}$ of *atomic* models of a countable first order theory T, i.e., $M \in \mathcal{K}$ if and only if $M \models T$ and each finite sequence $c \in M$ realises an isolated type over the empty set.

So we fix an atomic, excellent class $\mathcal{K}$ and consider a suitably large full model $\mathfrak{C}$ in $\mathcal{K}$. More precisely, we assume that $\mathfrak{C}$ satisfies the conditions (I)–(VIII) below. We denote by $Aut(\mathfrak{C}/C)$ the group of automorphisms of $\mathfrak{C}$ fixing C pointwise. The first two conditions concern *homogeneity*.

(I) $\mathfrak{C}$ is *strongly $\aleph_0$-homogeneous*, i.e., if $\text{tp}(a/C) = \text{tp}(b/C)$, where $a, b \in \mathfrak{C}$ are elements and $C \subseteq \mathfrak{C}$ is finite, then there exists $f \in Aut(\mathfrak{C}/C)$ such that $f(a) = b$.

(II) $\mathfrak{C}$ is *model homogeneous*, i.e., if $\text{tp}(a/M) = \text{tp}(b/M)$, where $a, b \in \mathfrak{C}$ are elements and $M \prec \mathfrak{C}$ with $\|M\| < \|\mathfrak{C}\|$, then there exists $f \in Aut(\mathfrak{C}/M)$ such that $f(a) = b$.

This gives us rather rich automorphism groups $Aut(\mathfrak{C}/C)$. Recall that *C-invariant* means preserved under automorphisms in $Aut(\mathfrak{C}/C)$. Note that, in general, we cannot assume that if $\text{tp}(a/C) = \text{tp}(b/C)$ there is $f \in Aut(\mathfrak{C}/C)$ such that $f(a) = b$. This is one of the main additional difficulties when dealing with excellent, rather than homogeneous classes.

We now consider the notion of *types*. In general, if $A \subseteq \mathfrak{C}$ and p is a complete type over A, there may not be $c \in \mathfrak{C}$ realising p. In fact, if $c \models p$, then Ac is atomic, i.e., $\text{tp}(ac/\emptyset)$ is isolated for any $a \in A$. This gives us a necessary condition for a type to be realised in $\mathfrak{C}$ and is the justification for the next definition.

Definition 0.1. Let $A \subseteq \mathfrak{C}$. We say that $p \in \text{S}_{\text{at}}(A)$ if p is a complete first order type (in $L(T)$) over A and for any c (in some elementary extension of $\mathfrak{C}$) realising p the set Ac is atomic.

The next fact about $\mathfrak{C}$ is a form of saturation for the appropriate notion of types; it gives a syntactic description of orbits over models. For uncountable models, the next definition is equivalent to fullness.

(III) $\mathfrak{C}$ *is full*, i.e., if $p \in \text{S}_{\text{at}}(M)$, with $M \prec \mathfrak{C}$ and $\|M\| < \|\mathfrak{C}\|$, then p is realised in $\mathfrak{C}$.

We now consider *stability* and *splitting*. Recall that $p \in S_{at}(A)$ *splits* over $C \subseteq A$ if there are $d, e \in A$ with $tp(d/C) = tp(e/C)$ and a formula $\phi(x, y)$ such that $\phi(x, d) \in p$ and $\neg\phi(x, e) \in p$.

(IV) $\mathfrak{C}$ is $\aleph_0$-*stable*, i.e., $\mathfrak{C}$ realises only countably many types over countable subsets.
(V) If $p \in S_{at}(M)$, for $M \prec \mathfrak{C}$ and $\|M\| < \|\mathfrak{C}\|$, there exists a finite $C \subseteq M$ such that p does not split over C.
(VI) The independence relation

$$A \underset{C}{\overset{ns}{\downarrow}} B,$$

defined by $tp(a/CB)$ does not split over a finite subset of C for each $a \in A$, satisfies Invariance and Monotonicity, as well as Local Character, Extension, Symmetry, and Stationarity provided C is a model.

Finally, we have two conditions on primary models. Recall that a model M is *primary over* A, if $M = A \cup \{a_i : i < \alpha\}$ with $tp(a_i/A \cup \{a_j : j < i\})$ is isolated for each $i < \alpha$.

(VII) For each $M \prec \mathfrak{C}$ and finite sequence $a \in \mathfrak{C}$, there is a primary model $M(a) \prec \mathfrak{C}$ over $M \cup a$.
(VIII) Let M_ℓ for $\ell = 0, 1, 2 \prec \mathfrak{C}$ with $M_0 \prec M_1, M_2$. If $M_1 \underset{M_0}{\overset{ns}{\downarrow}} M_2$, then there exists a primary model M^* over $M_1 \cup M_2$.

We finish this list with a couple of remarks. We first point out some of the important connections between the conditions (I)–(VIII).

Remark 0.2. First, we work with an atomic $\mathcal{K}$ so all models are $\aleph_0$-homogeneous. Thus, in each cardinality where $\mathcal{K}$ has a model, $\mathcal{K}$ has a strongly $\aleph_0$-homogeneous model (this almost gives (I)). In order to develop excellence, we work in an $\aleph_0$-stable class, i.e., each model $M \in \mathcal{K}$ is $\aleph_0$-stable. This gives (IV), and modulo some additional assumptions (for example the amalgamation property, but (III) is enough), $\aleph_0$-stability implies that the independence relation built from nonsplitting has good properties over models (V, VI). This independence is then used to define excellence, which is a condition on the existence of a primary model over countable independent systems of models; for example, in (VIII) the system (M_0, M_1, M_2) is independent. It is not clear that these conditions imply excellence, which requires the existence of primary models over independent systems of larger dimensions. Excellence implies (VII) and (VIII), i.e., the existence of primary models in higher cardinalities can be deduced. Excellence also implies (III). Finally (I) and (II) follows from (III) and (VII).

We now say a word on how to deal with nonatomic classes.

Remark 0.3. For the problem of uncountable categoricity, Shelah [22] showed that it is enough to consider the case of an atomic class; what this means is that every uncountably categorical class of models of a sentence $\psi \in L_{\omega_1,\omega}$, or a class of D-model for some set of types D, can be naturally expanded to form an atomic class of models with the same uncountable spectrum. In practice, however, given a class of D-models, for example, we may not want to expand it. It is still possible to develop excellence in this setting (see [16] for details); in fact, throughout all the proofs and the statements of this paper, it suffices to replace 'models' with '$(D, \aleph_0)$-homogeneous models'. For example, types over $(D, \aleph_0)$-homogeneous models are stationary for nonsplitting, $(D, \aleph_0)$-homogeneous models are a-saturated below. Full models realise types $p \in S_D(M)$, for M a $(D, \aleph_0)$-homogeneous model; where $p \in S_D(M)$ if $p \in S(M)$ and for each $c \models p$, the set Mc realises only types in D. For primary models, we consider instead $\mathbb{F}^s_{\aleph_0}$-primary $(D, \aleph_0)$-homogeneous models (which are prime in the class of $(D, \aleph_0)$-homogeneous models), and so forth.

From now on, we work inside the full model $\mathfrak{C}$, which we use as a universal domain. All sets, sequences, and models will be assumed to be inside $\mathfrak{C}$ of size less than $\|\mathfrak{C}\|$ (though we may repeat this for emphasis). Uppercase letters A, B, C, denote sets; letters like M, N denote models; and lowercase letters a, b, c denote finite sequences.

The next fact follows easily from (III) and (VII) and will be used several times in this paper.

Fact 0.4. *Let M^* be full. Let $f : M \to N$ be an isomorphism, with $M, N \prec M^*$ of size less than $\|M^*\|$. Then there exists $g \in Aut(M^*)$ extending f.*

We now consider the problem of indiscernible sequences. Indiscernible sequences in excellent classes do not necessarily behave as well as in the homogeneous case. For example, some infinite indiscernible sequences cannot be extended in $\mathfrak{C}$. A nice example is given by $\mathbb{Z}$ in the quasi-minimal excellent class worked out by Zilber (see [25] for details) to model the behaviour of

$$0 \to \mathbb{Z} \to^{i} \mathbb{C}_+ \to^{exp} \mathbb{C}^* \to 0.$$

Further, it is not clear that a permutation of the elements always extends to an automorphism of $\mathfrak{C}$. This is the reason why we introduce the following definition:

Definition 0.5. We say that I is *strongly indiscernible over C* (or *strongly C-indiscernible*) if for any λ, there is a C-indiscernible sequence I' extending I of size λ with the property that any injective map from I' into I' extends to an automorphism of $\mathfrak{C}$ fixing C pointwise.

We will spend the rest of this section showing that this notion of indiscernibles is appropriate for our purposes, i.e., it satisfies, at least over finite sets, the sort of properties that indiscernibles satisfy in the first order case and in the homogeneous case (see [12] and [4] where such results are proved for all indiscernibles).

Note that being strongly C-indiscernible is a C-invariant notion. Also, if I is strongly indiscernible over C, and $I' \subseteq I$ and $C' \subseteq C$, then I' is strongly indiscernible over C'.

Recall that $(a_i : i < \alpha)$ is a *Morley sequence* for $p \in \mathrm{S_{at}}(M)$, where M is a model, if each $a_i \models p$ and

$$a_i \underset{M}{\overset{ns}{\downarrow}} \{a_j : j < i\}, \qquad \text{for each } i < \alpha.$$

It follows from (VI) that I is indiscernible over M and *independent over M*, i.e.,

$$a_i \underset{M}{\overset{ns}{\perp}} I \setminus \{a_i\}, \qquad \text{for each } i < \alpha.$$

The definition of a Morley sequence does not really depend on the particular independence relation that we use, provided it satisfies monotonicity, stationarity over models and invariance (see for example Kolesnikov [15] or Proposition 2.7 below). We will show that strongly indiscernible sequences coincide with Morley sequences. To see this, we need to recall a fact also due to Shelah [23]. We sketch the proof to illustrate the rôle of (I)–(VIII).

Fact 0.6. *Let $I \cup C \subseteq \mathfrak{C}$ with $|I| > \lambda \geq |C| + \aleph_0$. Then there is $M \prec \mathfrak{C}$ containing C of size λ and $J \subseteq I$ of size λ^+ such that J is a Morley sequence for some $p \in \mathrm{S}_{\mathrm{at}}(M)$. It follows that J is strongly indiscernible over M.*

Proof. Let $M \prec \mathfrak{C}$ containing C be of size λ. Construct an increasing and continuous sequences $(M_i : i < \lambda^+)$ of models of size λ with $M_0 = M$ and $a_i \in I$ such that $a_i \in M_{i+1} \setminus M_i$. This is possible since I has size λ^+. Each type $\mathrm{tp}(a_i/M_i)$ is stationary and does not split over a finite subset of M_i by (V) and (VI). Hence, by Fodor's lemma, we may assume that none of them split over some finite subset of M_0, and by the pigeonhole principle that they do not split over the same finite subset of M_0. Another application of the pigeonhole principle, using $\aleph_0$-stability (IV), shows that we may assume that all the a_i's satisfy the same type over M_0. Then, $(a_i : i < \lambda^+)$ is a Morley sequence for $\mathrm{tp}(a_0/M_0)$.

A standard argument now shows that I is M_0-indiscernible. Thus, $(a_i : i < \lambda^+)$ can be extended to an M_0-indiscernible sequence $(a_i : i < \mu)$ of any desired size (by simply extending this Morley sequence, using (VI) and (III)).

Now by (VII), we can choose $(N_i : i < \mu)$ increasing and continuous, such that $N_0 = M_0$, $N_i \prec M_i$, and N_{i+1} is primary over $N_i \cup a_i$. Then by pasting together all the constructions of N_{i+1} over $N_i \cup a_i$ and using orthogonality calculus we obtain that $N := \bigcup_{i<\mu} N_i$ is primary over $M_0 \cup \{a_i : i < \mu\}$ (see [5] for details). Thus, any permutation of $(a_i : i < \mu)$ extends to an automorphism of N fixing M_0. By Fact 0.4, this automorphism extends further to an automorphism of $\mathfrak{C}$ fixing M_0. This shows that $(a_i : i < \lambda^+)$ is strongly indiscernible over M_0. $\square$

We can now prove the desired characterisation.

Proposition 0.7. *I is strongly C-indiscernible if and only if I is the Morley sequence of a type $p \in \mathrm{S}_{\mathrm{at}}(M)$ for some model M containing C.*

Proof. The last two paragraphs of the previous proof show that if I is the Morley sequence of a type $p \in \mathrm{S}_{\mathrm{at}}(M)$, with M containing C, then I is strongly C-indiscernible.

Let us prove the converse. Suppose that I is strongly C-indiscernible. By extending I if necessary, we may assume that $I = (a_i : i < \lambda^+)$, for some $\lambda \geq |C| + \aleph_0$. Choose a model M containing C such that some $J \subseteq I$ of size λ^+ is a Morley sequence for a complete type over M (this is possible by the previous fact). Write $J = (a_{i_\ell} : \ell < \lambda^+)$. Let $f \in Aut(\mathfrak{C}/C)$ such that $f(a_{i_\ell}) = a_\ell$, for each $\ell < \lambda^+$, which exists since I is strongly C-indiscernible. Then I is a Morley sequence for $\mathrm{tp}(a_0/f(M))$, and $f(M)$ is a model containing C. $\square$

We now prove a couple of lemmas about strongly indiscernible sequences which will be used later. The next lemma will be used in the proof of the Pairs Lemma (Proposition 2.8).

Lemma 0.8. *Let $C \subseteq B$ with B finite. Let $(a_i : i < \omega)$ be B-indiscernible and strongly C-indiscernible. Then, for each $n < \omega$, there exists a strongly B-indiscernible sequence $(a'_i : i < \omega)$ such that $a'_i = a_i$ for each $i < n$.*

Proof. Let $n < \omega$ be given. Let $(a_i : i < \omega_1)$ be strongly C-indiscernible extending $(a_i : i < \omega)$. By the pigeonhole principle and $\aleph_0$-stability (IV), we may assume that

$$\mathrm{tp}(a_0 \dots a_{n-1}/B) = \mathrm{tp}(a_{i_0} \dots a_{i_n}/B), \qquad \text{for each } i_0 < \dots < i_{n-1} < \omega_1.$$

By the Fact 0.6, there exists $S \subseteq \omega_1$ of size ω_1 such that $(a_i : i \in S)$ is strongly B-indiscernible. The result now follows by strong ω-homogeneity of $\mathfrak{C}$ (I) by sending the n first elements of $(a_i : i \in S)$ to $a_0, \dots, a_{n-1}$ fixing B. □

This next lemma is a technical result used in the proof of Theorem 4.7.

Lemma 0.9. *Let C be finite. There exists a model N of size less than $\|\mathfrak{C}\|$ with the following properties:*

(1) *Whenever $(a_i : i < \omega) \subseteq \mathfrak{C}$ is strongly C-indiscernible with $a_0, a_1 \in N$, there is $(a'_i : i < \omega_1) \subseteq N$ a strongly C-indiscernible sequence with $a_0 = a'_0$ and $a_1 = a'_1$.*
(2) *Whenever $(a_i : i < \omega_1) \subseteq N$ is strongly C-indiscernible and σ is a permutation of $(a_i : i < \omega_1)$, there exists $f \in Aut(N/C)$ extending σ.*

Proof. Let $\lambda = 2^{\aleph_1}$. We prove that there exists a full model N of size λ^+ satisfying (1) and (2). We construct an increasing and continuous sequence of full models $(M_i : i < \lambda^+)$ such that $C \subseteq M_0$, each M_i has size λ, and whenever $(a_i : i < \omega) \subseteq \mathfrak{C}$ with $a_0, a_1 \in M_i$ is strongly C-indiscernible, there exists $(a'_i : i < \omega_1) \subseteq M_{i+1}$ strongly C-indiscernible with $a'_0 = a_0$ and $a'_1 = a_1$. Further, we choose M_{i+1} such that whenever $(b_i : i < \omega_1) \subseteq M_i$ is strongly C-indiscernible, there is $M \prec M_{i+1}$ countable such that $(b_i : i < \omega_1)$ is a Morley sequence for $\mathrm{tp}(b_0/M)$.

This is possible: Choose M_0 arbitrary of size λ containing C. Having constructed M_i, there are at most λ choices for (a_0, a_1) in M_i, so only λ many sequences $(a'_i : i < \omega_1)$ to add in M_{i+1} (we only need to choose *one* strongly indiscernible sequence for each pair (a_0, a_1)). By Proposition 0.7, each strongly indiscernible sequence is the Morley sequence of a type over some model. But, the number of strongly C-indiscernible sequences $(b_i : i < \omega_1)$ in M_i is at most $\lambda^{\aleph_1} = \lambda$, so we can add all the necessary countable models in M_{i+1} without violating the cardinality requirement. Finally, making M_{i+1} into a full model is easy by (III) and (IV). At limit, choose M_i full containing $\bigcup_{j<i} M_j$ (but actually, $\bigcup_{j<i} M_j$ is already full).

This is enough: Let $N = \bigcup_{i<\lambda^+} M_i$. Then N is full since λ^+ is regular. Now, certainly (1) is satisfied. To see that (2) holds, let $(b_i : i < \omega_1) \subseteq N$ be C-strongly indiscernible. Then, by regularity, there exists $i < \lambda^+$ such that $(b_i : i < \omega_1) \subseteq M_i$. Thus, there is $M \prec M_{i+1}$ countable such that $(b_i : i < \omega_1)$ is a Morley sequence for $\mathrm{tp}(b_0/M)$. Then, any permutation of $(b_i : i < \omega_1)$ extends to an automorphisms of N fixing M just like in the last paragraph of the proof of Fact 0.6 using Fact 0.4. □

1. Lascar strong types and strong automorphisms

In this section, we introduce Lascar strong types for the context of excellent classes. In the first order case without stability, many of these results were obtained by Kim and Pillay [14]. For the homogeneous context, this was done first in the stable context by Hyttinen and Shelah [12] and by Buechler and Lessmann [4] without stability.

We say that a set is *bounded* if it has size less than $\|\mathfrak{C}\|$. Fact 0.6 shows that a C-invariant set X is bounded if and only if X has at most $|C| + \aleph_0$ elements.

Definition 1.1. We say that a and b have the same *Lascar strong type over* C, written $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, if $E(a, b)$ holds for any C-invariant equivalence relation E with a bounded number of classes.

Equality between Lascar strong types over C is clearly a C-invariant equivalence relation; we will show in the next few lemmas that it is the *finest* C-invariant equivalence relation with a bounded number of classes.

Lemma 1.2. *Let E be a C-invariant equivalence relation with a bounded number of classes. Let J be strongly indiscernible over C. Then $E(a, b)$ for any $a, b \in J$.*

Proof. If not, then $\neg E(a, b)$ holds for some $a \neq b$ in J. By strong indiscernibility, we may assume that J has size $\|\mathfrak{C}\|$. But now, by C-invariance and strong indiscernibility again, we have $\neg E(c, d)$ for any $c \neq d \in J$, so E has an unbounded number of equivalence classes, a contradiction. □

Lemma 1.3. *Let E be a C-invariant equivalence relation with a bounded number of classes, then E has at most $|C| + \aleph_0$ many classes.*

Proof. Let $\lambda = |C| + \aleph_0$. Suppose that $\{a_i : i < \lambda^+\}$ are E-inequivalent. By Fact 0.6, there is an infinite $J \subseteq \{a_i : i < \lambda^+\}$, which is strongly indiscernible over C. But for any $a, b \in J$, we have $E(a, b)$, by the previous lemma, a contradiction. □

Before we prove the next corollary, we point out that there are at most $2^{|C|+\aleph_0}$ distinct C-invariant subsets of $\mathfrak{C}$. In the homogeneous case, this is obvious; here, we notice that if Z is C-invariant, then Z is a union of orbits of $Aut(\mathfrak{C}/C)$. Now, by Fact 0.6, there are at most $|C| + \aleph_0$ distinct orbits (in the given arity) over C, which implies the conclusion.

Corollary 1.4. *The relation $E(a, b)$ given by* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *is the finest C-invariant equivalence relation with a bounded number of classes.*

Proof. It is enough to show that equality of Lascar strong types itself has a bounded number of classes. As we pointed out, there are at most $2^{|C|+\aleph_0}$ equivalence relations which are C-invariant. Hence, if $\{a_i : i < \lambda\}$ realise distinct Lascar strong types over C, and λ is suitably large (given by Erdős–Rado), there is a subset of size $(|C| + \aleph_0)^+$ which would be E-inequivalent for a specific C-invariant equivalence relation with a bounded number of classes. This contradicts the previous lemma. □

We will refer to the distinct classes under Lascar strong type equality as *Lascar strong types*. Since the number of complete types over a bounded set C is bounded, equality of types is a C-invariant equivalence relation with a bounded number of classes, so if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, then $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$. Further, for the same reason, there is $f \in Aut(\mathfrak{C}/C)$ such that $f(a) = b$. We prove a better result in Proposition 1.9 below.

We now introduce the bounded closure of a set.

Definition 1.5. Let C be a set. We say that b is in the *bounded closure of* C, written $b \in \mathrm{bcl}(C)$, if $\mathrm{tp}(b/C)$ has a bounded number of realisations in $\mathfrak{C}$.

Again, Fact 0.6 shows that $|\mathrm{bcl}(C)| \le |C| + \aleph_0$.

Lemma 1.6. *If $p \in \mathrm{S}_{\mathrm{at}}(C)$ has a bounded number of realisations and $a, b \models p$. Then,* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *if and only if $a = b$.*

Proof. Define $E(a, b)$ if whenever $a, b \models p$ then $a = b$. Then E is a C-invariant equivalence relation with a bounded number of classes, so the result follows from Corollary 1.4. □

We can in fact prove more:

Lemma 1.7. *If* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *then* $\mathrm{tp}(a/\mathrm{bcl}(C)) = \mathrm{tp}(b/\mathrm{bcl}(C))$.

Proof. Define $E(a, b)$ if $\mathrm{tp}(a/\mathrm{bcl}(C)) = \mathrm{tp}(b/\mathrm{bcl}(C))$. This is a C-invariant equivalence relation, for if $f \in Aut(\mathfrak{C}/C)$ then f fixes $\mathrm{bcl}(C)$ setwise. Further, E has a bounded number of classes since $|\mathrm{bcl}(C)| \le |C| + \aleph_0$. Hence, by Corollary 1.4 if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ we must have $E(a, b)$. □

We now introduce *strong automorphisms*.

Definition 1.8. We say that $f \in Aut(\mathfrak{C}/C)$ is a *strong automorphism over* C if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(f(a)/C)$, for each $a \in \mathfrak{C}$.

We denote $Saut(\mathfrak{C}/C)$ the group of strong automorphisms over C. It is easy to check that $Saut(\mathfrak{C}/C)$ is a normal subgroup of $Aut(\mathfrak{C}/C)$. The next proposition shows that Lascar strong types over C are the orbits of $Saut(\mathfrak{C}/C)$, and that this group is generated by the automorphisms of $\mathfrak{C}$ fixing a model containing C.

Proposition 1.9. *Let a, b be finite sequences. The following conditions are equivalent:*

(1) $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$.
(2) *There exists $f \in Saut(\mathfrak{C}/C)$ such that $f(a) = b$.*
(3) *There is $n < \omega$, M_i for $i < n$, with $C \subseteq M_i \prec \mathfrak{C}$, and $f_i \in Aut(\mathfrak{C}/M_i)$ such that $a = f_{n-1} \circ \cdots \circ f_0(b)$.*

Proof. (2) implies (1) is by definition. To see that (1) implies (2), let $E(a, b)$ hold if there is $f \in Saut(\mathfrak{C}/C)$ such that $f(a) = b$. This is a C-invariant equivalence relation. Hence by Corollary 1.4 it is enough to show that E has a bounded number of classes.

There are only $|C| + \aleph_0$ many distinct Lascar strong types over C, so we can choose M bounded containing a realisation for each Lascar strong type over C. Suppose that $\{a_i : i < (|C| + \aleph_0)^+\}$ are E-inequivalent. By $\aleph_0$-stability (IV), there are $i \neq j < (|C| + \aleph_0)^+$ such that $\mathrm{tp}(a_i/M) = \mathrm{tp}(a_j/M)$. By model homogeneity, there exists $f \in Aut(\mathfrak{C}/M)$ such that

$f(a_i) = a_j$. We claim that $f \in Saut(\mathfrak{C}/C)$: Let $a \in \mathfrak{C}$, and choose $a' \in M$ such that $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(a'/C)$. Then $\mathrm{Lstp}(f(a)/C) = \mathrm{Lstp}(f(a')/C)$ by C-invariance. But $f(a') = a'$, and hence $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(f(a)/C)$.

For (1) implies (3), it is enough to check that the relation between a and b defined by (3) is a C-invariant equivalence relation with a bounded number of classes, which is easy. For (3) implies (1), it is enough to see that if $f \in Aut(\mathfrak{C}/M)$, where $C \subseteq M \prec \mathfrak{C}$, then $\mathrm{Lstp}(b/C) = \mathrm{Lstp}(f(b)/C)$. If $b \in M$, then $b = f(b)$ and there is nothing to show. Otherwise, let $M_0 = M$. Let M_1 be primary over $M_0bf(b)$. Let $a_1 \models \mathrm{tp}(b/M) = \mathrm{tp}(f(b)/M)$ be such that $\mathrm{tp}(a_1/M_1)$ does not split over M. Continue inductively to obtain a Morley sequence $(a_i : 0 < i < \omega)$. Notice that both $(b, a_i : 0 < i < \omega)$ and $(f(b), a_i : 0 < i < \omega)$ are Morley sequences for $\mathrm{tp}(b/M)$, hence strongly indiscernible over C with elements in common. This implies that $\mathrm{Lstp}(b/C) = \mathrm{Lstp}(f(b)/C)$. □

It follows that if $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ and $c \in \mathfrak{C}$ then there is $d \in \mathfrak{C}$ such that $\mathrm{Lstp}(ac/C) = \mathrm{Lstp}(bd/C)$.

Proposition 1.10. *Suppose that $a \neq b$. Then the following conditions are equivalent:*

(1) $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$
(2) *There exists $n < \omega$, a_i and strongly C-indiscernible sequences J_i for $i \leq n$ such that $a_0 = a$, $a_n = b$, and $a_i, a_{i+1} \in J_i$ for $i < n$.*

Proof. (2) implies (1) is clear by Lemma 1.2. To see that (1) implies (2), let $E(a, b)$ be the equivalence relation defined by (2). It is easily seen to be C-invariant. Hence, by Corollary 1.4, it is enough to show that it has a bounded number of classes. This is clear: If $\{a_i : i < \|\mathfrak{C}\|\}$ were E-inequivalent, then by Fact 0.6, there is $J \subseteq \{a_i : i < \|\mathfrak{C}\|\}$ a Morley sequence for some stationary $p \in S_{\mathrm{at}}(M)$, where M contains C. Thus, any $a_i, a_j \in J$ are E-equivalent, a contradiction. □

The next concept will be useful in understanding the stationarity of the independence relations we introduce in the next section.

Definition 1.11. We say that a model is *a-saturated* if it realises every Lascar strong type over a finite subset.

Proposition 1.12. *Every model is a-saturated.*

Proof. Let M be given. Let $C \subseteq M$ be finite. We want to show that M realises every Lascar strong type over C. Without loss of generality, we may assume that M is countable. Since there are only countably many distinct Lascar strong types over C, we can find a countable $N \prec \mathfrak{C}$ containing C realising all the Lascar strong types over C, that is, containing a complete set of representatives over C. But N and M are isomorphic over C, and moreover by Fact 0.4, there is an automorphism f of $\mathfrak{C}$ sending N into M fixing C pointwise. If M did not contain a complete set of representatives, and $b \in M$ such that $\mathrm{Lstp}(b/C) \neq \mathrm{Lstp}(a/C)$ for each $a \in M$, then $f^{-1}(b)$ shows that N does not contain a complete set of representatives over C either. □

The previous proposition also shows that the Lascar strong types do not depend on $\mathfrak{C}$.

2. Lascar-splitting and independence

We first introduce a new independence relation based on strong splitting and strong indiscernibles.

Definition 2.1. We say that $p \in S_{at}(A)$ *Lascar-splits over* C if there is a strongly C-indiscernible sequence $(a_i : i < \omega)$ and a formula $\phi(x, y)$ such that $\phi(x, a_0) \in p$, and $\neg\phi(x, a_1) \in p$.

Clearly, the type $tp(a/A)$ Lascar-splits over C if there is a strongly C-indiscernible sequence $(a_i : i < \omega)$, with $a_0, a_1 \in A$, such that $tp(aa_0/C) \neq tp(aa_1/C)$. It is obvious that if p does not split over C then p does not Lascar-split over C. Hence, by (V) for any $p \in S_{at}(M)$ there exists a finite $C \subseteq M$ such that p does not Lascar-split over C. The next proposition justifies the name.

Proposition 2.2. *Let $p \in S_{at}(M)$ and let $C \subseteq M$ be finite. Then p Lascar-splits over C if and only if there are $a, b \in M$ realising the same Lascar strong type over C and ϕ such that $\phi(x, a) \in p$ and $\neg\phi(x, b) \in p$.*

Proof. The 'if' part follows from Lemma 1.2, since if $(a_i : i < \omega)$ is strongly C-indiscernible with $\phi(x, a_0), \neg\phi(x, a_1) \in p$, then $Lstp(a_0/C) = Lstp(a_1/C)$. For the 'only if' part, suppose that p does not Lascar-split over C. Suppose that a_i and J_i, for $i \leq n$, witness the fact that $Lstp(a/C) = Lstp(b/C)$ as Proposition 1.10(2). We may assume that the J_i's are countable. By ω-homogeneity of M we may also assume that all the J_i are in M. Suppose that $\phi(x, a) \in p$. Since each J_i is strongly C-indiscernible, then $\phi(x, a_i) \in p$, for each $i \leq n$, so $\phi(x, b) \in p$. □

The next proposition shows that types over models are stationary for non-Lascar-splitting.

Proposition 2.3. *Suppose that $p \in S_{at}(M)$ does not Lascar-split over the finite $C \subseteq M$. For any B containing M there is $a \in \mathfrak{C}$ realising p such that* $tp(a/B)$ *does not Lascar-split over C. Furthermore, the type* $tp(a/B)$ *is unique.*

Proof. Without loss of generality, we may assume that B is a model N. Define $q \in S_{at}(N)$ as follows. For each $b' \in N$, choose $b \in M$ such that $Lstp(b/C) = Lstp(b'/C)$. This is possible since M is a-saturated. Then for each $\phi(x, y)$, let $\phi(x, b') \in q$ if and only if $\phi(x, b) \in p$. It is easy to check that $q \in S(N)$, extends p, and does not Lascar-split over C using Proposition 2.2. By fullness of $\mathfrak{C}$ (III), we can find $a \in \mathfrak{C}$ realising q. Uniqueness of q follows from the a-saturation of M (Proposition 1.12) and Proposition 2.2. □

We point out the following connection of Lascar-splitting with bounded closure:

Lemma 2.4. *If $a \in bcl(C)$ then* $tp(a/B)$ *does not Lascar-split over C for any B.*

Proof. Suppose, for a contradiction, that $tp(a/B)$ Lascar-splits over C. Let λ be suitably large and let $(b_i : i < \lambda)$ be strongly C-indiscernible such that $b_0, b_1 \in B$ and $tp(ab_0/C) \neq tp(ab_1/C)$. We may assume that $tp(ab_0/C) \neq tp(ab_i/C)$ for each $i < \lambda$. By strong indiscernibility, for each $i < \lambda$, there is $f_i \in Aut(\mathfrak{C}/C)$ permuting $(b_j : j < \lambda)$ such that $f_i(b_0) = b_i$. Let $a_i = f_i(a)$. Then $a_i \neq a_j$ for $i \neq j$, so $\mathrm{tp}(a/C)$ has an unbounded number of realisations, a contradiction. □

We now introduce a new independence relation. It is based on non-Lascar-splitting and has a built-in extension property (as in [12] and [10]). We let

$$a \underset{C}{\downarrow} B$$

if there is a finite $C' \subseteq C$ such that for all D containing $C \cup B$ there is a' realising the type $\mathrm{tp}(a/BC)$ such that $\mathrm{tp}(a'/D)$ does not Lascar-split over C'. We then write

$$A \underset{C}{\downarrow} B$$

if $a \underset{C}{\downarrow} B$ for any finite $a \in A$.

We will show that it satisfies all the usual nonelementary properties of an independence relation over a model, as well as restricted versions of the expected properties over finite (and sometimes infinite) sets. We first list a few obvious ones. Notice that Finite Character is not one of them (we will only be able to prove this result in the next section).

Proposition 2.5. (1) *(Invariance)* $\downarrow$ *is invariant under automorphisms of* $\mathfrak{C}$.
(2) *(Restricted Local Character) If* $a \underset{C}{\downarrow} B$ *then there exists a finite* $C' \subseteq C$ *such that* $a \underset{C'}{\downarrow} B$.
(3) *(Monotonicity) Suppose that* $C \subseteq B \subseteq D$. *If* $a \underset{C}{\downarrow} D$ *then* $a \underset{C}{\downarrow} B$ *and* $a \underset{B}{\downarrow} D$.
(4) *(Restricted Extension) Let* $C \subseteq B$. *If* $a \underset{C}{\downarrow} B$ *and* D *contains* B *then there is* $a' \models \mathrm{tp}(a/B)$ *such that* $a' \underset{C}{\downarrow} D$.

Proof. (1), (2), and (3) are clear. For (4), using Monotonicity (3), it is enough to show the result for $D = M$, a model. Since $a \underset{C}{\downarrow} B$, there exist $C' \subseteq C$ finite and $a' \models \mathrm{tp}(a/BC)$ such that $\mathrm{tp}(a'/M)$ does not Lascar-split over C'. Suppose that D' contains M. By Proposition 2.3 there exists a unique $a'' \models \mathrm{tp}(a'/M)$ such that $\mathrm{tp}(a''/D)$ does not Lascar-split over C'. Hence $a'' \underset{C}{\downarrow} M$ by Proposition 2.3 and Monotonicity. □

Not only is $\downarrow$ is Invariant, but more is true: If $\mathrm{tp}(a/CB) = \mathrm{tp}(a'/CB)$ then $a \underset{C}{\downarrow} B$ if and only if $a' \underset{C}{\downarrow} B$. The next lemma explores a few simple properties of the bounded closure.

Lemma 2.6. *Let* C *be finite.*

(1) *If* $a \in \mathrm{bcl}(C)$ *then* $a \underset{C}{\downarrow} B$ *for any* B.

(2) *If* $a \notin \mathrm{bcl}(C)$ *then* $a \underset{C}{\not\downarrow} a$.

(3) *If* $a \in \mathrm{bcl}(B) \setminus \mathrm{bcl}(C)$ *then* $a \underset{C}{\not\downarrow} B$.

Proof. (1) is immediate by Lemma 2.4. For (2), since C is finite, the orbit of a under $Aut(\mathfrak{C}/C)$ is unbounded. So by Fact 0.6, there exists a strongly C-indiscernible sequence $(a_i : i < \omega)$ such that each a_i is C-automorphic to a. We may assume that $a = a_0$. Suppose that $a \underset{C}{\downarrow} a$. Let $a' \models \mathrm{tp}(a/Ca)$ such that $a' \underset{C}{\downarrow} \{a_i : i < \omega\}$. Then $a' = a$, so $a \underset{C}{\downarrow} \{a_i : i < \omega\}$ and this is a contradiction since $\mathrm{tp}(a/C\{a_i : i < \omega\})$ Lascar-splits over C, since $x = a_0$ and $x \neq a_1$ belong to $\mathrm{tp}(a/C\{a_i : i < \omega\})$ and $(a_i : i < \omega)$ is strongly C-indiscernible.

For (3), suppose, for a contradiction, that $a \underset{C}{\downarrow} B$. Let $a' \models \mathrm{tp}(a/CB)$ such that $a' \underset{C}{\downarrow} \mathrm{bcl}(B)$. But $a' \in \mathrm{bcl}(B)$, so $a' \underset{C}{\downarrow} a'$, contradicting (2). □

We now look at the behaviour over models. The main point here is that, over models, this independence relation coincides with the familiar one $\overset{ns}{\downarrow}$ defined via nonsplitting in (VI). We have stationarity over models (1), symmetry over models (3), and transitivity when the middle set is a model (4).

Proposition 2.7. (1) *(Stationarity over models) Let* M *be a model. If* $a \underset{M}{\downarrow} B$ *and* $b \underset{M}{\downarrow} B$ *and* $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, *then* $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$.

(2) *Let* M *be a model. Then,* $A \underset{M}{\downarrow} B$ *if and only if* $A \underset{M}{\overset{ns}{\downarrow}} B$.

(3) *If* $A \underset{M}{\downarrow} B$, *then* $B \underset{M}{\downarrow} A$.

(4) *Let* M *be a model and let* $C \subseteq M \subseteq D$. *If* $a \underset{C}{\downarrow} M$ *and* $a \underset{M}{\downarrow} D$, *then* $a \underset{C}{\downarrow} D$.

(5) *If* $a \underset{M}{\not\downarrow} B$, *then there is a finite* $b \in B$ *such that* $a \underset{M}{\not\downarrow} b$.

Proof. (1) follows immediately from Proposition 2.3.

For (2), the right to left direction is easy: certainly, if $\mathrm{tp}(a/MB)$ does not split over a finite subset C of M, then $\mathrm{tp}(a/MB)$ does not Lascar-split over C. Further, for any D containing MB there is $a' \models \mathrm{tp}(a/M)$ such that $\mathrm{tp}(a'/D)$ does not split over C, hence does not Lascar-split over C. But, $\mathrm{tp}(a'/MB) = \mathrm{tp}(a/MB)$ by stationarity (for nonsplitting). This shows $a \underset{M}{\downarrow} B$. Now suppose that $a \underset{M}{\downarrow} B$. Choose $C \subseteq M$ finite such that $\mathrm{tp}(a/M)$ does not split over C and such that $a \underset{C}{\downarrow} MB$ (this is possible by Restricted Local Character and Monotonicity from Proposition 2.5). Let $a' \models \mathrm{tp}(a/M)$ be such that $\mathrm{tp}(a'/MB)$ does not split over C (this is possible since types over models are stationary for nonsplitting). Then $\mathrm{tp}(a'/MB)$ does not Lascar-split over C, and hence $\mathrm{tp}(a'/MB) = \mathrm{tp}(a/MB)$ Proposition 2.3. This shows that $\mathrm{tp}(a/MB)$ does not split over the finite subset C of M.

(3) is true by (2) since Symmetry holds for $\overset{ns}{\mathop{\smile\!\!\!\!\!\!\mid}}$. (4) follows easily from Proposition 2.3 (or (1)). (5) holds again by (2) and is, in fact, easily seen to follow directly from (1). $\square$

We can prove the Pairs Lemma (sometimes called Left Transitivity) when the sets are finite.

Proposition 2.8 (*Pairs Lemma*). *Let $C \subseteq B$ with B finite. Assume that $a \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} B$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{Ca} Ba$. Then $ab \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} B$.*

Proof. Assume, for a contradiction, that $ab \not\mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} B$. Hence, there is a D containing B such that whenever $a'b' \models \mathrm{tp}(ab/B)$ then $\mathrm{tp}(a'b'/D)$ Lascar-splits over C. By Monotonicity, we may assume that $D = M$ is a model. By definition, we can find $a'b' \models \mathrm{tp}(ab/B)$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} M$ and $b' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{Ca'} Ma'$. Let $(c_i : i < \omega)$ be strongly C-indiscernible with $\mathrm{tp}(a'b'c_0/C) \neq \mathrm{tp}(a'b'c_1/C)$. We may assume that $(c_i : i < \omega)$ in M.

We now claim that $(c_i : i < \omega)$ is indiscernible over Ca': Otherwise, for some $n < \omega$ and some $i_0 < \dots i_{n-1} < \omega$ we have

$$\models \phi(a', c_0, \dots, c_{n-1}) \quad \text{and} \quad \models \neg\phi(a', c_{i_0}, \dots, c_{i_{n-1}}).$$

Then, the sequence I' consisting of n-element subsequences from I is clearly strongly indiscernible over C, which implies that $\mathrm{tp}(a'/CI)$ Lascar-splits over C, hence $a' \not\mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} M$, a contradiction.

Since $(c_i : i < \omega)$ is strongly indiscernible over C and indiscernible over Ca', by Lemma 0.8, we can find a strongly Ca'-indiscernible sequence $(c'_i : i < \omega)$ with $c_0 = c'_0$ and $c_1 = c'_1$. But this implies that $\mathrm{tp}(b'/Ma')$ Lascar-splits over Ca', which contradicts the fact that $b' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{Ca'} Ma'$. $\square$

We can now prove a restricted form of Symmetry over finite sets.

Proposition 2.9 (*Restricted Symmetry*). *Let C be finite and suppose that $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} C$. If $a \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} b$ then $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} a$.*

Proof. Let N be a model containing C. Since $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} C$, there exists $b' \models \mathrm{tp}(b/C)$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} N$. Choose an automorphism $f \in Aut(\mathfrak{C}/C)$ such that $f(b') = b$. Then, $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} f(N)$ by Invariance. This shows that we can choose a model M containing C such that $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} M$.

Now since $a \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} b$, there exists $a' \models \mathrm{tp}(a/Cb)$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} Mb$. Then $a' \mathop{\smile\!\!\!\!\!\!\mid}\limits_{M} b$ by Monotonicity, so by Symmetry over models (Proposition 2.7(3)), we have $b \mathop{\smile\!\!\!\!\!\!\mid}\limits_{M} a'$, so that by Transitivity when the middle set is a model (Proposition 2.7(4)), we have $b \underset{C}{\downarrow} a'$. Thus $b \underset{C}{\downarrow} a$ since $\mathrm{tp}(ab/C) = \mathrm{tp}(a'b/C)$. □

We now consider the problem of existence and uniqueness of extensions of Lascar strong types. We first prove Strong Extension when C is finite (note that, without the finiteness of $C \subseteq B$ below, we do not even know whether a B-automorphic image of a is independent from D over C).

Proposition 2.10. *Let $C \subseteq B$ with B finite. If $a \underset{C}{\downarrow} B$ and D contains B, there exists a strong automorphism $f \in Saut(\mathfrak{C}/B)$ such that $f(a) \underset{C}{\downarrow} D$.*

Proof. If $a \in \mathrm{bcl}(B)$, then $a \in \mathrm{bcl}(C)$, and so $a \underset{C}{\downarrow} D$ by Lemma 2.6, and there is nothing to prove. Suppose that $a \notin \mathrm{bcl}(B)$.

We first claim that there is $a' \models \mathrm{Lstp}(a/B)$ such that $a' \underset{C}{\downarrow} Ba$: Choose $(a_i : i < \omega_1)$ such that $a_i \models \mathrm{tp}(a/B)$, $a_i \underset{C}{\downarrow} B\{a_j : j < i\}$ (this is possible by Restricted Extension). By Lemma 2.6, we have that $a_i \neq a_j$ if $i \neq j$. Hence by Fact 0.6, we may assume that $(a_i : i < \omega)$ is strongly B-indiscernible. Since B is finite, we can choose $f \in Aut(\mathfrak{C}/B)$ with $f(a) = a_0$. Hence, without loss of generality we may assume that $a = a_0$. Now $\mathrm{Lstp}(a_1/B) = \mathrm{Lstp}(a_0/B)$ by Lemma 1.2 and $a_1 \underset{C}{\downarrow} Ba_0$. This shows the claim.

For the general case, let D be given containing B. Fix $a' \underset{C}{\downarrow} Ba$ with $a' \models \mathrm{Lstp}(a/B)$. By Restricted Extension, there is $a'' \models \mathrm{tp}(a'/Ba)$ such that $a'' \underset{C}{\downarrow} D$. Then, by Proposition 1.9 there exists $g \in Saut(\mathfrak{C}/B)$ such that $g(a) = a'$. Further, since B is finite, there is $f \in Aut(\mathfrak{C}/Ba)$ such that $f(a') = a''$. Then $f^{-1} \circ g \circ f \in Saut(\mathfrak{C}/B)$ by normality of $Saut(\mathfrak{C}/B)$ in $Aut(\mathfrak{C}/B)$. Then,

$$a'' = f(a') = f \circ g(a) = f \circ g \circ f^{-1}(a) \quad \text{and} \quad a'' \underset{C}{\downarrow} D,$$

so we are done. □

We finish this section with the problem of uniqueness of the extension, i.e., the stationarity of Lascar strong types. We first prove a lemma.

Lemma 2.11. *Let C be finite. Suppose that $c \underset{C}{\downarrow} ab$. If* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *then* $\mathrm{tp}(a/Cc) = \mathrm{tp}(b/Cc)$.

Proof. Let M be a model containing Cab. Let $c' \models \mathrm{tp}(c/Cab)$ such that $c' \underset{C}{\downarrow} M$. Choose $f \in Aut(\mathfrak{C}/Cab)$ such that $f(c') = c$. Hence, $c \underset{C}{\downarrow} f(M)$ by Invariance, so in particular, $\mathrm{tp}(c/f(M))$ does not Lascar-split over C. Since $f(M)$ is a model containing Cab, and $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$, we must have $\mathrm{tp}(ac/C) = \mathrm{tp}(bc/C)$ by Proposition 2.2. □

The next proposition is the form of stationarity for Lascar strong types which we can prove. We call it *restricted* stationarity for suggestiveness, but here, in contrast to our previous usage of the term restricted, the restriction does not follow from the conclusion. In fact, in the homogeneous case, the restriction can be bypassed (see [12] or [10]) due to the good behaviour of indiscernibles. Notice, however, that in applications (for example [11]), this form of stationarity is enough for many purposes.

Proposition 2.12 (*Restricted Stationarity*). *Suppose* $c \underset{C}{\downarrow} C$. *If* $a \underset{C}{\downarrow} c$, $b \underset{C}{\downarrow} c$ *and* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *then* $\mathrm{tp}(a/Cc) = \mathrm{tp}(b/Cc)$.

Proof. Suppose, for a contradiction, that $\mathrm{tp}(a/Cc) \neq \mathrm{tp}(b/Cc)$. Choose $C' \subseteq C$ finite such that $c \underset{C'}{\downarrow} C$, $a \underset{C'}{\downarrow} c$, $b \underset{C'}{\downarrow} c$, and $\mathrm{tp}(a/C'c) \neq \mathrm{tp}(b/C'c)$. By Proposition 2.10 and $a \underset{C'}{\downarrow} c$ we can choose $a' \models \mathrm{Lstp}(a/C'c)$ such that $a' \underset{C'}{\downarrow} abc$. By the Pairs Lemma (Proposition 2.8), since $b \underset{C'}{\downarrow} c$, we must have $a'b \underset{C'}{\downarrow} c$. Then, by restricted symmetry over a finite set (Proposition 2.9) we have that $c \underset{C'}{\downarrow} ba'$, since $c \underset{C'}{\downarrow} C'$. But $\mathrm{Lstp}(a'/C') = \mathrm{Lstp}(a/C') = \mathrm{Lstp}(b/C')$, so $\mathrm{tp}(a'c/C') = \mathrm{tp}(bc/C')$ by the previous lemma. Thus, $\mathrm{tp}(ac/C') = \mathrm{tp}(bc/C')$ by choice of a'. This is a contradiction. □

We finish with two lemmas about independence over finite sets. They will allow us to characterise simplicity in the next section.

Lemma 2.13. *Let C be finite. Suppose that $c \underset{C}{\downarrow} C$ for each finite sequence c. If $a \underset{C}{\not\downarrow} B$, then there exists a finite $b \in B$ such that $a \underset{C}{\not\downarrow} b$.*

Proof. Assume that $a \underset{C}{\downarrow} C$. By Proposition 2.10, we can find $a' \models \mathrm{Lstp}(a/C)$ such that $a' \underset{C}{\downarrow} B$. Then $\mathrm{tp}(a'/CB) \neq \mathrm{tp}(a/CB)$ since $a' \underset{C}{\not\downarrow} B$. Hence, there is $b \in B$ finite such that $\mathrm{tp}(a'/Cb) \neq \mathrm{tp}(a/Cb)$. But $b \underset{C}{\downarrow} C$, so $a \underset{C}{\not\downarrow} b$ by Restricted Stationarity (Proposition 2.12) since $a' \underset{C}{\downarrow} b$ by Monotonicity. □

Lemma 2.14. *Suppose that $c \underset{C}{\downarrow} C$ for each finite C and each finite c. If $(C_i : i < \omega)$ is an increasing sequence of finite sets and a is given, there exists $i < \omega$ such that $a \underset{C_i}{\downarrow} C_{i+1}$.*

Proof. Suppose, for a contradiction, that for each $i < \omega$ we have $a \underset{C_i}{\not\downarrow} C_{i+1}$. We will construct an increasing sequence of models $(M_i : i < \omega)$ and c such that

$$c \underset{M_i}{\not\downarrow} M_{i+1}, \quad \text{for each } i < \omega.$$

This is a contradiction since $\mathrm{tp}(c/\bigcup_{i<\omega} M_i)$ does not Lascar-split over a finite subset of $\bigcup_{i<\omega} M_i$.

We define an increasing sequence of models M_i and accessory sets A_i^j, for $i < \omega$ and $j \leq i$ such that:

(1) $A_i^i \subseteq M_i \prec M_{i+1}$.
(2) $A_{j+1}^i \underset{A_j^i}{\downarrow} M_i$, for $i \leq j$.
(3) For each $j < i$, there is an automorphism $f_i^j \in Aut(\mathfrak{C}/M_j)$ such that $f_i^j(A_k^j) = A_k^{j+1}$, where $j < k \leq i$.

Let $A_0^0 = C_0$ and choose M_0 any countable model containing C_0. Now choose A_i^0 according to (2) and (3), which is possible by independence and strong ω-homogeneity. Now assume that A_k^j has been constructed for $k < j \leq n$. We let $A_j^j = A_j^{j-1}$ and M_j be any countable model containing $M_{j-1} \cup A_j^j$. Again, choose A_k^j for $k > j$ such that (2) and (3) hold (this only uses strong ω-homogeneity and the properties of independence proved in Section 2).

Now, let $b_i = f_i^i \circ f_{i-1}^{i-1} \dots (a)$. Clearly $\mathrm{tp}(b_i/M_j) = \mathrm{tp}(b_j/M_j)$ when $j < i$. Thus, choose c such that $\mathrm{tp}(c/M_i) = \mathrm{tp}(b_i/M_i)$ for each $i < \omega$. Then, $c \underset{A_i^i}{\not\downarrow} A_{i+1}^{i+1}$ and $c \underset{A_{i+1}^{i+1}}{\downarrow} M_i$, so that $c \underset{M_i}{\not\downarrow} M_{i+1}$, the desired contradiction. □

3. Simplicity

In the first order case, simplicity is equivalent to the local character of nonforking. This is the motivation for the next definition.

Definition 3.1. We say that the excellent class $\mathcal{K}$ is *simple* if for each a and B there is $C \subseteq B$ finite such that $a \underset{C}{\downarrow} B$.

There are excellent classes which fail to be simple (recall Shelah's example in [10]). We prove a first characterisation of simplicity. Not only is simplicity equivalent to Existence, but it is enough to check Existence over finite sets.

Theorem 3.2. *$\mathcal{K}$ is simple if and only if $a \underset{C}{\downarrow} C$, for each a and each finite C.*

Proof. The left to right direction is clear by Monotonicity. So assume that $a \underset{C}{\downarrow} C$ for any finite C. By Restricted Local Character, it is enough to show that $a \underset{C}{\downarrow} C$ for any, possibly infinite, set C.

Suppose, for a contradiction, that $a \underset{C}{\not\downarrow} C$ for some a and (infinite) C. Let $C_0 = \emptyset$. By Monotonicity, we have $a \underset{C_0}{\not\downarrow} C$, so by Lemma 2.13 there exists a finite $C_1 \subseteq C$ such that $a \mathop{\not\downarrow}_{C_0} C_1$. But, by Monotonicity again, we have $a \mathop{\not\downarrow}_{C_1} C$. Continuing inductively, we construct an increasing and continuous sequence of finite sets $(C_i : i < \omega)$ such that

$$a \mathop{\not\downarrow}_{C_i} C_{i+1}, \quad \text{for each } i < \omega.$$

This contradicts Lemma 2.14. □

An immediate consequence of simplicity is that bounded closure has finite character.

Proposition 3.3. *Assume that $\mathcal{K}$ is simple. If $a \in \mathrm{bcl}(B)$ then there exists a finite $C \subseteq B$ such that $a \in \mathrm{bcl}(C)$.*

Proof. By simplicity, we can choose $C \subseteq B$ finite such that $a \mathop{\downarrow}_{C} B$. Then, $c \in \mathrm{bcl}(C)$ by Lemma 2.6. □

Another immediate consequence is the stationarity of Lascar strong types.

Proposition 3.4 (*Stationarity of Lascar Strong Types*). *Suppose that $\mathcal{K}$ is simple. Suppose that $a \mathop{\downarrow}_{C} B$ and $b \mathop{\downarrow}_{C} B$ and* $\mathrm{Lstp}(a/C) = \mathrm{Lstp}(b/C)$ *then* $\mathrm{tp}(a/BC) = \mathrm{tp}(b/BC)$.

Proof. If not, there is $c \in B$ such that $\mathrm{tp}(a/Cc) \neq \mathrm{tp}(b/Cc)$. But $c \mathop{\downarrow}_{C} C$ by simplicity of $\mathcal{K}$, which contradicts Proposition 2.12. □

We now show that our independence notion satisfies all the properties of nonforking in a simple first order theory. We start with Symmetry.

Proposition 3.5 (*Symmetry*). *Assume that $\mathcal{K}$ is simple. If $a \mathop{\downarrow}_{C} b$ then $b \mathop{\downarrow}_{C} a$.*

Proof. Let $C' \subseteq C$ be finite such that $a \mathop{\downarrow}_{C'} Cb$ and $ab \mathop{\downarrow}_{C'} C$ (this latter uses simplicity). Then since C' is finite and $b \mathop{\downarrow}_{C'} C'$, we have $b \mathop{\downarrow}_{C'} a$ by Proposition 2.9. By the Pairs Lemma (Proposition 2.8), we have that $ca \mathop{\downarrow}_{C'} b$ for each $c \in C$. Hence by Proposition 2.9 again, we have $b \mathop{\downarrow}_{C'} ca$ for each $c \in C$. This implies $b \mathop{\downarrow}_{C'} Ca$ by simplicity and Lemma 2.13. Thus, $b \mathop{\downarrow}_{C} a$ by Monotonicity. □

Transitivity over all sets follows from Symmetry and the Pairs Lemma using Lemma 2.13.

Proposition 3.6 (*Transitivity*). *Assume that $\mathcal{K}$ is simple. Let $A \subseteq B \subseteq C$. If $a \mathop{\downarrow}_{A} B$ and $a \mathop{\downarrow}_{B} C$, then $a \mathop{\downarrow}_{A} C$.*

Proof. Choose $A' \subseteq A$ finite and $B' \subseteq B$ finite such that $a \underset{A'}{\perp} A$ and $a \underset{A'B'}{\perp} B$. By Monotonicity, it is enough to show that $a \underset{A'}{\perp} C$. By Lemma 2.13, it is enough to show that $a \underset{A'}{\perp} c$ for each finite $c \in C$. By Symmetry, it is enough to show that $c \underset{A'}{\perp} a$ for each finite $c \in C$. Let $c \in C$ be finite. Write $c = bc_0$, with $b \in B$ and $c_0 \in C \setminus B$. We may assume that b contains B'. Then by assumption and Monotonicity, we have $a \underset{A'}{\perp} b$ as well as $a \underset{A'b}{\perp} c_0$. So $b \underset{A'}{\perp} a$ and $c_0 \underset{A'b}{\perp} a$ by Symmetry. Hence, $c \underset{A'}{\perp} a$ by the Pairs Lemma (Proposition 2.8). This completes the proof. □

We can finally prove Finite Character.

Proposition 3.7 (*Finite Character*). *Assume that $\mathcal{K}$ is simple. Let A, B, C be sets such that $a \underset{C}{\perp} b$ for each finite $a \in A$ and $b \in B$. Then $A \underset{C}{\perp} B$.*

Proof. Suppose that $A \underset{C}{\not\perp} B$. By definition, there is $a \in A$ such that $a \underset{C}{\not\perp} B$. By simplicity, we can choose $C' \subseteq C$ finite such that $a \underset{C'}{\perp} C$. Now, for each $b \in CB$, we have $a \underset{C'}{\perp} b$ by Transitivity (Proposition 3.6) since by assumption $a \underset{C}{\perp} b$. Hence, by Lemma 2.13 we have $a \underset{C'}{\perp} B$. This contradicts Monotonicity. □

We summarise our results about the main properties of $\perp$ in the next theorem. By Local Character we really mean $\aleph_0$-Local Character.

Theorem 3.8. *Assume that $\mathcal{K}$ is simple. Then, $\perp$ satisfies the following properties:*

(1) *(Invariance) If $A \underset{C}{\perp} B$ then $f(A) \underset{f(C)}{\perp} f(B)$ for an $f \in Aut(\mathfrak{C})$.*
(2) *(Finite Character) $A \underset{C}{\perp} B$ if and only if $a \underset{C}{\perp} b$ for any finite $a \in A$ and finite $b \in B$.*
(3) *(Monotonicity) If $A \underset{C}{\perp} B$ and $C \subseteq D \subseteq C \cup B$ then $A \underset{C}{\perp} D$ and $A \underset{D}{\perp} B$.*
(4) *(Local Character) For any finite a and any B there exists a finite $C \subseteq B$ such that $a \underset{C}{\perp} B$.*
(5) *(Extension) For any a, C and B containing C there $a' \models \mathrm{tp}(a/C)$ such that $a \underset{C}{\perp} B$.*
(6) *(Symmetry) $A \underset{C}{\perp} B$ if and only if $B \underset{C}{\perp} A$.*
(7) *(Transitivity) Let $B \subseteq C \subseteq D$. If $A \underset{B}{\perp} C$ and $A \underset{C}{\perp} D$, then $A \underset{B}{\perp} D$.*
(8) *(Stationarity over models) Let M be a model. If $\mathrm{tp}(a_1/M) = \mathrm{tp}(a_2/M)$ and $a_\ell \underset{M}{\perp} B$, for $\ell = 1, 2$, then $\mathrm{tp}(a_1/B) = \mathrm{tp}(a_2/B)$.*

(9) *(Stationarity of Lascar strong types) If* $\mathrm{Lstp}(a_1/C) = \mathrm{Lstp}(a_2/C)$ *and* $a_\ell \underset{C}{\downarrow} B$, *for* $\ell = 1, 2$, *then* $\mathrm{tp}(a_1/B) = \mathrm{tp}(a_2/B)$.

Proof. Invariance is Proposition 2.5(1), Finite Character is Proposition 3.7, and Monotonicity is given by Proposition 2.5(3). For Local Character, it is the definition of simplicity. As for Extension, if $a, A \subseteq B$ are given, since $a \underset{A}{\downarrow} A$, there exists $a' \models \mathrm{tp}(a/A)$ such that $a' \underset{A}{\downarrow} B$ by Proposition 2.5(4). Symmetry is Proposition 3.5 using Finite Character, Transitivity is Proposition 3.6, and Stationarity over models is Proposition 2.7(1). Stationarity of Lascar strong types is given Proposition 3.4. □

We finish this section by showing that, like in the first order case [14], the definition of the independence relation is uniquely determined by its properties, and that simplicity is equivalent to the existence of some independence relation with those properties.

Theorem 3.9. *Let $\mathcal{K}$ be an excellent class. $\mathcal{K}$ is simple if and only if there exists an independence relation on the subsets of $\mathfrak{C}$ satisfying Invariance, Finite Character, Monotonicity, Local Character, Extension, Symmetry, Transitivity, and Stationarity over models. Moreover, the independence relation coincides with $\downarrow$.*

Proof. Theorem 3.8 shows that $\downarrow$ satisfies all the desired properties when $\mathcal{K}$ is simple.

Let us prove the converse. Assume that there exists an independence relation, denoted $\overset{(A)}{\downarrow}$, on the subsets of $\mathfrak{C}$ satisfying the listed properties. Recall that $\downarrow$ satisfies Invariance, Restricted Local Character, Restricted Extension, and Monotonicity (Proposition 2.5). We will show that

$$a \overset{(A)}{\underset{C}{\downarrow}} B \quad \text{if and only if} \quad a \underset{C}{\downarrow} B.$$

This implies that $\mathcal{K}$ is simple, as well as the last sentence. We prove a few claims towards the full result.

Claim. *Let C be finite. If $a \overset{(A)}{\underset{C}{\downarrow}} B$ then $\mathrm{tp}(a/CB)$ does not Lascar-split over C.*

Proof. Suppose, for a contradiction, that $\mathrm{tp}(a/CB)$ Lascar-splits over C and let $(b_i : i < \omega)$ be a strongly C-indiscernible sequence, with $b_0, b_1 \in BC$ and $\mathrm{tp}(ab_0/C) \neq \mathrm{tp}(ab_1/C)$. By Monotonicity, we have that $a \overset{(A)}{\underset{C}{\downarrow}} b_0 b_1$. Now, let $p(x, b_0, b_1) = \mathrm{tp}(a/Cb_0b_1)$. By Lascar-splitting, $p(x, b_0, b_1) \cup p(x, b_1, b_2)$ is contradictory.

Let $\lambda = (2^{\aleph_0})^+$, and choose $(b_i : i < \lambda)$ a strong C-indiscernible sequence extending $(b_i : i < \omega)$. Thus, for any $i < j < k$, the type $p(x, b_i, b_j) \cup p(x, b_j, b_k)$ is contradictory (use an automorphism fixing C sending b_i, b_j, b_k to b_0, b_1, b_2 fixing C). Now choose a countable model M containing C. By Invariance (since C is finite) and Extension, for each $i, j < \lambda$ there is $a'_{i,j}$ realising $p(x, b_i, b_j)$ such that $a'_{i,j} \overset{(A)}{\underset{C}{\downarrow}} M \cup \{b_\ell : \ell < \lambda\}$. By ω-stability and Erdős–Rado, there exists an infinite set $S \subseteq \lambda$ such that $\mathrm{tp}(a'_{i,j}/M) = \mathrm{tp}(a'_{k,\ell}/M)$ for any $i < j$ and $k < \ell$ from S. Choose $i < j < k$ from S. Then, by stationarity over models for $\overset{(A)}{\downarrow}$, we have $\mathrm{tp}(a'_{i,j}/Cb_ib_jb_k) = \mathrm{tp}(a'_{j,k}/Cb_ib_jb_k)$. But this implies that $a'_{i,j}$ realises $p(x, b_i, b_j) \cup p(x, b_j, b_k)$, a contradiction. $\square$

We can now prove the left to right direction.

Claim. *If $A \underset{C}{\overset{(A)}{\downarrow}} B$ then $A \underset{C}{\downarrow} B$.*

Proof. By definition of $A \underset{C}{\downarrow} B$, it is enough to show that $a \underset{C}{\downarrow} B$ for each finite $a \in A$. Fix $a \in A$ and choose $C' \subseteq C$ finite such that $a \underset{C'}{\overset{(A)}{\downarrow}} B$. Let D contain $C \cup B$. By Extension and Transitivity there exists $a' \models \mathrm{tp}(a/C)$ such that $a' \underset{C'}{\overset{(A)}{\downarrow}} D$. By the previous claim $\mathrm{tp}(a'/D)$ does not Lascar-split over C'. Hence, $a \underset{C'}{\downarrow} B$ by definition. $\square$

Finally, we prove the converse:

Claim. *If $A \underset{C}{\overset{(A)}{\not\downarrow}} B$ then $A \underset{C}{\not\downarrow} B$.*

Proof. By Finite Character of $\overset{(A)}{\downarrow}$ and definition of $\downarrow$, it is enough to show this for A finite. So, assume that $A = a$ is finite and that $a \underset{C}{\overset{(A)}{\not\downarrow}} B$. Suppose, for a contradiction, that $a \underset{C}{\downarrow} B$. Then $a \underset{C}{\overset{(A)}{\not\downarrow}} b$ for some finite $b \in B$ by Finite Character. Then, $a \underset{C}{\downarrow} b$ by Monotonicity of $\downarrow$ and there is $C' \subseteq C$ finite such that $a \underset{C'}{\downarrow} b$ by Restricted Local Character. Then, by Monotonicity of $\overset{(A)}{\downarrow}$ we have that $a \underset{C'}{\overset{(A)}{\not\downarrow}} b$.

Notice that $\mathrm{tp}(b/C')$ must be unbounded: By Extension, there is $a' \models \mathrm{tp}(a/C')$ such that $a' \underset{C'}{\overset{(A)}{\downarrow}} \mathrm{bcl}(C')$. Since C' is finite, there exists $f \in Aut(\mathfrak{C}/C')$ such that $f(a') = a$. Hence $a \underset{C'}{\overset{(A)}{\downarrow}} \mathrm{bcl}(C')$ by Invariance, since f fixes $\mathrm{bcl}(C')$ setwise. This implies that $b \notin \mathrm{bcl}(C')$ by Monotonicity.

Now choose $(b_i : i < \omega)$ a strongly C-indiscernible sequence with $b_0 = b$, which is $\overset{(A)}{\downarrow}$-independent. By definition of $\downarrow$, we can find $a' \models \mathrm{tp}(a/C'b_0)$ such that $a' \underset{C'}{\downarrow} \{b_i : i < \omega\}$.

Then, $\mathrm{tp}(a'/C' \cup \{b_i : i < \omega\})$ does not Lascar-split over C' and hence $\mathrm{tp}(a'b_0/C') = \mathrm{tp}(a'b_i/C')$ for each $i < \omega$. By Invariance, this implies that $a' \overset{(A)}{\underset{C'}{\not\downarrow}} b_i$, for each $i < \omega$. But then, by Symmetry and Transitivity we have $a' \overset{(A)}{\underset{C' \cup \{b_j : j < i\}}{\not\downarrow}} b_i$, for each $i < \omega$. This contradicts Local Character of $\overset{(A)}{\downarrow}$. □

This finishes the proof. □

4. *U*-rank and simplicity

In this section, we establish another characterisation of simplicity and deduce from it (and [16]) that uncountably categorical excellent classes are essentially simple.

We recall the definition of the U-rank from [16] (notice that defining the U-rank with splitting or Lascar splitting is immaterial because of Proposition 2.5(2)): For $p \in S_{\mathrm{at}}(M)$, we define $U(p) \geq \alpha$ by induction on the ordinal α.

- $U(p) \geq 0$ for any $p \in S_{\mathrm{at}}(M)$, for any model M.
- $U(p) \geq \alpha + 1$ if there exists N, with $M \prec N$, and $a \models p$ such that $U(\mathrm{tp}(a/N)) \geq \alpha$ and $a \underset{M}{\not\downarrow} N$.
- For a limit α, we set that $U(p) \geq \alpha$ if $U(p) \geq \beta$ for each $\beta < \alpha$.

We say that $U(p) = \alpha$ if $U(p) \geq \alpha$ but $U(p) \not\geq \alpha + 1$, and $U(p) = \infty$ if $U(p) \geq \alpha$ for each ordinal α.

We abbreviate $U(a/M)$ for $U(\mathrm{tp}(a/M))$. The next two facts are proved in [16]:

Fact 4.1. *Let $\mathcal{K}$ be an excellent class and $M \in \mathcal{K}$.*

(1) $U(a/M) < \infty$.
(2) *If $\mathcal{K}$ is uncountably categorical, then $U(a/M) < \omega$.*

Fact 4.2. *Let $M \prec N$.*

(1) $U(a/N) \leq U(a/M)$.
(2) $U(a/M) = U(a/N)$ *if and only if* $a \underset{M}{\downarrow} N$.

We now extend the U-rank to types over arbitrary sets. For a set C and a finite sequence a, we let

$$U(\mathrm{tp}(a/C)) = \sup\{U(b/M) : b \text{ realises } \mathrm{tp}(a/C),\ M \in \mathcal{K} \text{ contains } C\}.$$

As before, we write $U(a/C)$ for $U(\mathrm{tp}(a/C))$. The next lemma shows that we can either fix the model, or fix the realisation:

Lemma 4.3. *Let C be finite.*

(1) *Let M be a model containing C. Then*

$$U(a/C) = \sup\{U(b/M) : b \models \mathrm{tp}(a/C)\}.$$

(2) $U(a/C) = \sup\{U(a/M) : M \in \mathcal{K}$ *contains* $C\}$.

Proof. (2) is clear by strong ω-homogeneity of $\mathfrak{C}$ (I). The key point of (1) is to show that for any $b \models \mathrm{tp}(a/C)$ and N containing C, there exists $c \models \mathrm{tp}(a/C)$ such that

$$U(b/N) = U(c/M).$$

So, let b and N be given as above. First, there exists $N' \prec N$ countable containing C such that $b \mathop{\smile\!\!\!\!\!\!\mid}_{N'} N$. By ω-homogeneity of M and strong ω-homogeneity of $\mathfrak{C}$, we may assume that $bN' \subseteq M$. By extension over models, we can find $c \models \mathrm{tp}(b/N')$ such that $c \mathop{\smile\!\!\!\!\!\!\mid}_{N'} M$. Now $c \models \mathrm{tp}(a/C)$, and by Fact 4.2, we have:

$$U(b/N) = U(b/N') = U(c/N') = U(c/M). \quad \square$$

The U-rank is clearly invariant, and notice also that $U(a/C) \leq U(a'/B)$ if $B \subseteq C$ and $a \models \mathrm{tp}(a'/B)$. We make the following definition.

Definition 4.4. We say that $\mathcal{K}$ has *extensible U-rank* if for each finite C and each a and $M \in \mathcal{K}$ containing C, there is b realising $\mathrm{tp}(a/C)$ such that

$$U(a/C) = U(b/M).$$

We first show if $\mathcal{K}$ is simple, then $\mathcal{K}$ has extensible U-rank. The converse will be the direction of interest. Grossberg and Hart [5] proved the next fact for the independence notion $\overset{ns}{\mathop{\smile\!\!\!\!\!\!\mid}}$. By Proposition 2.7(2), this holds for $\mathop{\smile\!\!\!\!\!\!\mid}$ as well.

Fact 4.5 (*Dominance*). *If $a \mathop{\smile\!\!\!\!\!\!\mid}_{M} B$ and M' is primary over $M \cup B$ then $a \mathop{\smile\!\!\!\!\!\!\mid}_{M} M'$.*

Proposition 4.6. *Assume that $\mathcal{K}$ is simple. If $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} M$, with $C \subseteq M \in \mathcal{K}$, then $U(a/C) = U(a/M)$. It follows that $\mathcal{K}$ has extensible U-rank.*

Proof. The second sentence for C finite implies that $\mathcal{K}$ has extensible U-rank just like in Lemma 4.3.

Now assume $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} M$. We want to show that $U(a/C) = U(a/M)$. First, it is enough to show this for C finite: Let a and C be given. By simplicity $a \mathop{\smile\!\!\!\!\!\!\mid}_{C'} C$ for some finite $C' \subseteq C$. By Transitivity, we have $a \mathop{\smile\!\!\!\!\!\!\mid}_{C'} M$. Then

$$U(a/C') \geq U(a/C) \geq U(a/M),$$

hence if $U(a/C') = U(a/M)$, we have $U(a/C) = U(a/M)$.

So assume, in addition, that C is finite. Since $U(a/C) \geq U(a/M)$, it is enough to show by induction on α that $U(a/C) \geq \alpha$ implies $U(a/M) \geq \alpha$. For $\alpha = 0$ or a limit ordinal, there is nothing to prove. Suppose that $U(a/C) \geq \alpha + 1$. Then, $U(a/N) \geq \alpha + 1$ for some N containing C. By definition, there exists N', such that $N \prec N'$ and

$$a \not\downarrow_{N} N'.$$

Since $a \downarrow_C M$, we have $M \downarrow_C Ca$ by Symmetry, so that by Extension and finiteness of C there is $f \in Aut(\mathfrak{C}/Ca)$ such that $f(M) \downarrow_C N'a$. Thus, without loss of generality, we may assume that

$$M \downarrow_C N'a.$$

Then, $a \downarrow_{N'} MN'$ by Symmetry and Monotonicity. And by excellence (VIII) since $M \downarrow_N N'$ by Monotonicity (and thus $M \overset{ns}{\downarrow}_N N'$ by Proposition 2.7(2)), there exists a primary model M' over $M \cup N'$. By Dominance and $a \downarrow_{N'} MN'$, we have that

$$a \downarrow_{N'} M'.$$

Thus, $U(a/M') = U(a/N') \geq \alpha$ by Fact 4.2. But, we also have $a \not\downarrow_M M'$ (for otherwise $a \downarrow_M M'$, so $a \downarrow_C M'$ by Transitivity and so $a \downarrow_N N'$ by Monotonicity, which contradicts the choice of N'). Hence $U(a/M) \geq \alpha + 1$ by definition of the U-rank. This finishes the proof. □

The example of the class free groups $F(X)$ on infinitely many generators X is an atomic uncountably categorical, excellent class (even homogeneous) which is simple, uncountably categorical and therefore has extensible U-rank. However, the U-rank is not continuous in the first order sense since $U(x = x) := \sup_{a \in F(X)} U(a/\emptyset) = \omega$, and by Fact 4.1 the ordinal ω cannot be achieved (see [16] for details).

We now show the converse.

Theorem 4.7. *$\mathcal{K}$ is simple if and only if $\mathcal{K}$ has extensible U-rank.*

Proof. We have already shown that if $\mathcal{K}$ is simple, then $\mathcal{K}$ has extensible U-rank. We show the converse. Assume that $\mathcal{K}$ has extensible U-rank. To show that $\mathcal{K}$ is simple, it is enough to show that $a \downarrow_C C$ for any a and finite C by Theorem 3.2.

Let C be finite and a be given. Fix N a bounded model containing C satisfying the conditions (1) and (2) of Lemma 0.9. Since $\mathcal{K}$ has extensible U-rank, we can choose $b \models \text{tp}(a/C)$ such that

$$U(a/C) = U(b/N).$$

It is enough to show that $\text{tp}(b/N)$ does not Lascar-split over C, as this implies that $b \mathop{\downarrow}\limits_C N$ by Proposition 2.3 (so $b \mathop{\downarrow}\limits_C C$ and thus $a \mathop{\downarrow}\limits_C C$ since $\text{tp}(a/C) = \text{tp}(b/C)$).

Suppose, for a contradiction, that $(a_i : i < \omega)$ is strongly C-indiscernible, with $a_0, a_1 \in N$, and $\text{tp}(a_0/Cb) \neq \text{tp}(a_1/Cb)$. By assumption (1) of Lemma 0.9 on N, we may assume that $(a_i : i < \omega)$ is the beginning of a strongly C-indiscernible sequence $(a_i : i < \omega_1)$ which is contained in N. We may also assume that

$$\text{tp}(a_i/Cb) \neq \text{tp}(a_0/Cb), \quad \text{for each } i > 0.$$

By assumption (2) of Lemma 0.9 on N, for each $i < \omega_1$, we can choose $f_i \in Aut(N/C)$ permuting $(a_i : i < \omega_1)$ such that $f_i(a_0) = a_i$. By Fact 0.4, for each $i < \omega_1$, we can choose $g_i \in Aut(\mathfrak{C}/C)$ extending f_i. Then, letting $b_i = g_i(b)$, we have

$$\text{tp}(b_i/N) \neq \text{tp}(b_j/N), \quad \text{for } i < j < \omega_1.$$

Now let $M \prec N$ be a countable model containing C. Then since

$$U(b_i/C) = U(b_i/N) \quad \text{and} \quad U(b_i/N) \leq U(b_i/M),$$

we have $U(b_i/N) = U(b_i/M)$ so that $b_i \mathop{\downarrow}\limits_M N$, for each $i < \omega_1$ by Fact 4.2. By stationarity of types over models (Proposition 2.3), we must have

$$\text{tp}(b_i/M) \neq \text{tp}(b_j/M), \quad \text{for } i < j < \omega_1,$$

but this contradicts the $\aleph_0$-stability of $\mathfrak{C}$ (IV). □

So, when the excellent class $\mathcal{K}$ is simple, the U-rank over complete types behaves in a similar way to the first order case. The property of extensibility of the U-rank extends to types over all sets (by Proposition 4.6), and we can easily derive the usual additivity properties of the U-rank for types over arbitrary sets from this using the additivity properties over models (see [16] and [8]). Our proofs also show:

Remark 4.8. Let $\mathcal{K}$ be excellent. Suppose that $a \mathop{\downarrow}\limits_C C$ for each finite C. Then $\mathcal{K}$ is simple.

We finish this paper with two promised corollaries.

Definition 4.9. We say that an excellent, atomic class $\mathcal{K}$ has *finite U-rank* if

$$\sup\{U(a/M) : M \in \mathcal{K}, a \text{ a singleton}\} < \omega.$$

The next corollary was obtained in [8] for the homogeneous case.

Corollary 4.10. *Let $\mathcal{K}$ be an excellent atomic class with finite U-rank. Then $\mathcal{K}$ is simple.*

Proof. Recall the following U-rank equality from [16] (Theorem 3.9):

$$U(a/M(b)) + U(b/M) \leq U(ab/M) \leq U(a/M(b)) \oplus U(b/M), \tag{*}$$

where $M(b)$ is a primary model over Mb, and $\oplus$ is the natural sum of ordinals, which agrees with usual addition when the ordinals involved are finite. Assume that

$\sup\{U(a/M) : M \in \mathcal{K}, a$ a singleton$\} = k < \omega$. Then it is easy to show, by induction on the arity of a, Lemma 4.3 and (*) that

$$\sup\{U(a/M) : M \in \mathcal{K}, a \in \mathfrak{C}^n\} = nk < \omega.$$

It follows immediately that the supremum is achieved and so that $\mathcal{K}$ has extensible U-rank, so is simple by the previous theorem. □

As we pointed out, not all uncountably categorical excellent classes have finite U-rank; nevertheless, all such classes are still essentially simple:

Corollary 4.11. *Let T be a countable theory and let D be a diagram. Assume that the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models is excellent (or homogeneous). If $\mathcal{K}$ is uncountably categorical, then an expansion of $\mathcal{K}$ with countably many constants is uncountably categorical, excellent (resp. homogeneous), and simple.*

Proof. Consider $T' = Th(\mathfrak{C}, M)$, where M is the unique countable $(D, \aleph_0)$-homogeneous model and $\mathfrak{C}$ is a large full model (or a large homogeneous model in the homogeneous case). Let D' be the diagram of $\mathfrak{C}$ is the expanded language, and let $\mathcal{K}'$ be the class of $(D', \aleph_0)$-homogenous models. Observe that if $\mathfrak{C}$ is homogeneous, then any expansion $\mathfrak{C}'$ with names for M is still homogeneous. So if $\mathcal{K}$ was homogeneous (i.e., if D is good), then so is $\mathcal{K}'$ (resp. so is D'). Similarly, if $\mathfrak{C}$ is full, then $\mathfrak{C}'$ is full also (as types over $(D, \aleph_0)$-homogeneous models are stationary), and it is not difficult to see that if $\mathcal{K}$ is excellent, then so is $\mathcal{K}'$. Furthermore, uncountable categoricity is preserved since M is countable. Now, by Theorem 3.10 of [16], in $\mathcal{K}$ we have that $U(\text{tp}(a/M)) < \omega$ for any $a \in \mathfrak{C}$. It follows that $U(\text{tp}(a/MC)) \leq U(\text{tp}(a/M)) \leq n$, for any set C, as in the previous proof. This implies that $\mathcal{K}'$ has extensible U-rank, so $\mathcal{K}'$ is simple by Theorem 4.7. □

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