History Dependence in Stock Price Evolution

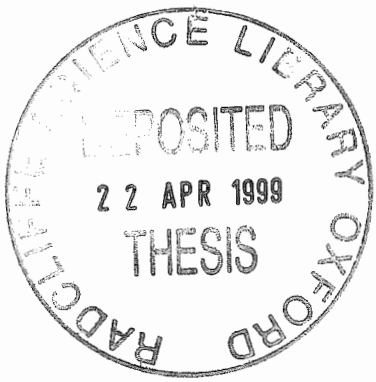
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ABSTRACT

This thesis describes the modelling of stock price volatility as a function of the stock price and an exponentially weighted moving average. The data used is the Dow Jones Industrial Average Index for the years 1901 until 1995.

The random walk of asset prices and the Black-Scholes model are modified to include a moving average as a third variable, the other two being stock price and time. The ideas behind technical analysis are introduced while no attempt is made to justify its use. Particular mention is made to the widely used technical indicator, the moving average. Numerous tests are performed on the Dow Jones data in order to determine whether or not the volatility, σ, can be a function of the ratio of the stock price and the corresponding exponentially weighted moving average, S/I.

Explicit finite differencing is carried out on the p.d.e. associated with the modified Black-Scholes model and various checks are made to study the agreement between the numerical scheme and the exact results. A 2-D option, which has a payoff dependent on the performance of an exponentially weighted moving average, is introduced and solved numerically.

Finally, the results detailed in the thesis are discussed along with possible future extensions. Also mentioned is an Asian option type payoff used by traders to trade the cross-overs of two different (time scale) moving averages.

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Chapter 1

Review of Classical Models and the Black Scholes Analysis

1.1 Introduction

The aim of this thesis is to investigate whether or not asset price evolution may be influenced not only by the current price, S, but also some history dependent functional of S such as a technical[1] indicator, I, an exponentially weighted moving average. If so, then what implications does this have on the pricing of derivatives securities which have as their underlying such assets ?

In particular, we would like to determine whether or not stock price volatility, σ, can be a function of S and of I. If it turns out that it is, then in effect we will be using past stock prices and moving averages to predict the future volatility.

The thesis is divided into seven chapters. In Chapter 1, this chapter, we introduce the Markov stochastic process used to model the behaviour of stock prices over time and the key concepts of the Efficient Market Hypothesis. We describe in detail the Black-Scholes model which uses this stochastic process in order to value non-dividend paying securities. We observe that the one variable in the Black-Scholes partial differential equation which cannot be evaluated directly is the volatility. The causes and the methods of estimation of this volatility are discussed. We end the chapter by comparing the Black-Scholes analysis to the real world.

Chapter 2 begins with an introduction to the main concepts and components of technical analysis - the use of past prices to predict future changes. There is a

[1]See Chapter 2 for a discussion on technical analysis and indicators

technical analysis literature review. All of the papers discussed in this review report on the evidence of predictability and question the validity of the postulates of the Efficient Market Hypothesis. There is a brief description of the different charts used by analysts in their attempts at forecasting stock price changes.

In Chapter 3 we compare the merits of some of the different moving averages - a important technical indicators, and derive the random walk followed by the exponentially weighted moving average. We discuss some of the popular methods used by technical analysts to trade using stock prices and moving averages. We also determine the s.d.e. and the steady-state p.d.f. followed by the ratio of the moving average to the stock price, I/S, a ratio often used by analysts to trade.

In Chapter 4 we discuss the history of the Dow Jones Industrial Average Index - one of the most famous stock market indices. We investigate the properties associated with stock price returns (as detailed in Chapter 1) - namely the frequency distribution and the dependence or independence of the Dow Jones returns. We end this chapter by estimating the historical volatility and the drift rate from 94 years worth of Dow Jones data.

Chapter 5 determines whether or not the Dow Jones index volatility (σ) can be modelled as a function of the ratio of the stock price (S) and the corresponding moving average (I). For various different time scale moving averages, we plot the σ versus S/I for the Dow Jones data. Various tests, carried out in order to explain the shape of these plots, are discussed. The key theoretical concepts of the pure jump and jump diffusion processes are also detailed. We finally describe the model which best fits the σ versus S/I plots. We end this chapter by attempting to simulate the Dow Jones price graph - a plot of the Dow Jones Index versus time.

Chapter 6 details the extension of the two dimensional Black-Scholes model to include an exponentially weighted moving average. We describe the main concepts of discretisation, stability, and convergence for the finite difference method. We use this explicit finite difference approach to change the modified Black-Scholes p.d.e. into a set of difference equations. We check the results from the resultant numerical scheme against analytical solutions. We conclude this chapter by devising and solving numerically a moving average dependent option.

Chapter 7 gives a brief summary of the thesis and the conclusions drawn from our investigations into the dependence of stock price volatility on the stock price

and the corresponding moving average. We also report on the σ dependence on S and I for other data sets - namely: the S&P 500 Index and the FT-SE 100 Index. Finally, possible extensions of the analysis detailed in this thesis are discussed.

1.2 The Random Walk Hypothesis and Efficient Markets

The **Random Walk Hypothesis** is a statement that price changes are in some way random and so prices wander ('walk') in an unpredictable manner. When prices follow such a process the only relevant information in the series of present and past prices, for traders, is the most recent price. Thus, people involved in the market have already made perfect use of the information in the past prices. Assuming that price are available frequently, then if only the recent price is relevant it follows that prices very quickly reflect the information in the historical record of prices. All of this is closely related to the concept of **Efficient Markets**.

The **Weak-form** of the **Efficient Market Hypothesis (EMH)** states that the present price of a stock[2] contains all the information available in a record of past prices. It is competition in the market which tends to ensure that the weak-form of market efficiency holds. Also, the very fact that numerous investors following the stock market closely and attempting to make a profit from it leads to a situation where a stock price at any given time contains information from past prices.

Not only is there a weak-form of the EMH but there are also **semi-strong** and **strong-forms** as well. In general they say:

- The past history is fully reflected in the present price, which does not hold any further information;
- Markets respond immediately to any new information about an asset.

Hence, the modelling of stock prices is in effect the modelling of the arrival of new information which affects the prices. Using these assumptions, the unexpected changes in stock prices follow a **Markov Process** [1]. This process embodies the fact that the predictions for the future should be unaffected by the results from the

[2]Asset and stock will be used interchangeably throughout this thesis

past and that only the present value is needed to predict the future. That is, a continuous time process X, taking values in $\Re$, is called a Markov process if

$$P(X(t_n) \leq x \,|\, X(t_n), \cdots, X(t_{n-1})) = P(X(t_n) \leq x \,|\, X(t_{n-1})), \tag{1.1}$$

for all x and all increasing sequences $t_1 < t_2 < \cdots < t_n$ of times.

Before we proceed to the 'classical' model describing stock price evolution it has to be noted that the **absolute** change in the stock price is not as 'useful' as the **relative** change. This is termed the **return** and defined as the change in the stock price divided by the original price.

Thus, if the current stock price (at time t) is S then at a infinitesimally later time $(t + \mathrm{d}t)$ it is $S + \mathrm{d}S$. We would like to model the corresponding return on the stock, $\mathrm{d}S/S$. The classical model comprises to parts: a predictable component and a random component. The predictable part is a deterministic return (cf. return on an investment in a bank). The other part models the random change in the stock price, for example the arrival of unanticipated news. The result is a **stochastic differential equation**

$$\frac{\mathrm{d}S}{S} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}X, \tag{1.2}$$

where :

- S and $\mathrm{d}S$ are the stock price and its change in a time interval $\mathrm{d}t$.
- μ is the expected return (per unit time) on the stock.
- $\mathrm{d}X$ models the random nature of stock price changes.

The $\mathrm{d}X$ term is known as the **Wiener process**. This is a particular type of Markov stochastic process.

Formally, if X is a variable which follows a Wiener process then the change in its value, ΔX, in a small time interval, Δt, is given by

$$\Delta X = \varepsilon \sqrt{\Delta t}, \tag{1.3}$$

where ε is a random drawing from a standardised normal distribution with a mean 0 and a standard deviation of 1. Note that equation 1.3 is true for any Δt. Equation 1.2 is a continuous-time s.d.e. The equivalent discrete time version is

$$\frac{\Delta S}{S} = \mu \,\Delta t + \sigma \,\Delta X. \tag{1.4}$$

The values of ΔX for any two different short intervals of time Δt are independent. Note that in the limit that $\Delta t \to 0$, equation 1.3 can be written as

$$\mathrm{d}X = \varepsilon \sqrt{\mathrm{d}t}. \tag{1.5}$$

The process represented by equations 1.2 and 1.4 is widely used in the valuation of derivative securities and is sometimes known as the **Geometric Brownian Motion**. Under this process, the proportional rate of return to the holder of the stock in any small time interval is normally distributed, with mean $\mu\,\mathrm{d}t$ or $\mu\,\Delta t$ and standard deviation $\sigma^2\,\mathrm{d}t$ or $\sigma^2\,\Delta t$, and the returns in any two small time intervals are independent. All in all, this model fits real time series data very well - at least for equities and indices. The are some discrepancies such as: real data seems to be more 'prone' to sudden sharp rises and falls but on the whole equation 1.2 has stood the test of time.

Equation 1.2 is a typical example of a random walk. It cannot be solved to give a deterministic path for the stock price but every time different parameter values are used to start a time series the result is a different path. Each path is called a **realisation** of the random walk.

If at time t the stock price is S then at a later time t' the stock price S' will be distributed about S with a lognormal probability density function. Hence, if S follows a random walk with a lognormal p.d.f., it is known as the **lognormal random walk**.

1.2.1 Monte Carlo Simulation

One way to gain an intuitive understanding of this stochastic process is to simulate the behaviour of the stock price, S. This involves dividing the time interval into many small (equal) time steps and randomly sampling possible paths for S.

Suppose that the expected return from the stock is 7 % per annum and the volatility is 20 % (that is $\sigma = 0.20$). Now suppose that Δt is 0.0040 so that we are considering changes in the stock price in time intervals of length 0.0040 year or 1 day (assuming a 250 day trading year). Thus, from equation 1.4 it follows that $\Delta S/S$ is normal with a mean 0.0028 (0.0040×0.07) and standard deviation 0.0126 ($0.20 \times \sqrt{0.0040}$), that is

$$\frac{\Delta S}{S} \sim N(0.0028, 0.0126), \tag{1.6}$$

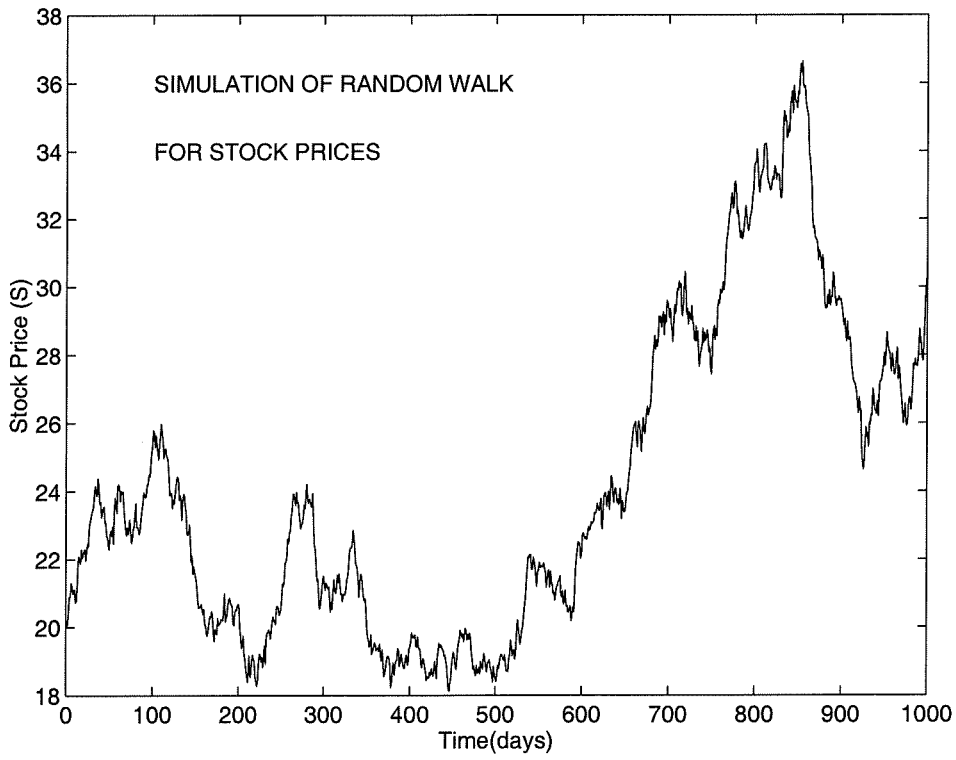


Figure 1.1: A simulation of stock price movements with a drift (μ) of 7 %, a volatility (σ) of 20 % and a time step (Δt) of 0.0040.

where $N(a, b)$ denotes a normal distribution with mean a and standard deviation b. A path for the stock price can be simulated by sampling repeatedly from $N(0.0028, 0.0126)$. One way to do this is to sample values, ν_1, from a standardised normal distribution (i.e $N(0, 1)$) and then convert these samples, ν_2, from $N(0.0028, 0.0126)$ using

$$\nu_2 = 0.0028 + 0.0126\,\nu_1. \tag{1.7}$$

This particular stock price simulation is shown in Figure 1.1 and the computer code used to generate it is given in Appendix B (Section B.1).

1.3 The Black-Scholes Analysis

In 1973 Fischer Black and Myron Scholes achieved a major breakthrough by deriving a partial differential equation that must be satisfied by the price of any derivative security on a non-dividend paying stock [2].

The now famous Black-Scholes partial differential equation is an equation that must be satisfied by the price, V, of any derivative security on a non-dividend paying

stock. We have all the necessary tools in place to detail its derivation. To start off with, we need to state the assumptions used:

- The stock price follows the lognormal random walk (equation 1.2) with constant volatility, σ, and rate of return, μ.
- The risk-free interest rate, r, is constant and the same for all maturities.
- The short selling of securities with full use of proceeds is permitted.
- There are no transaction costs or taxes and all securities are perfectly divisible.
- There are no dividends during the life of the derivative security.
- There are no riskless arbitrage opportunities.
- Security trading is continuous.

Now, let $V(S,t)$ be the value of an option which depends only on the asset price, S, and time, t. To proceed further we need to introduce **Ito's Lemma.**[3] This is to functions of stochastic variables what Taylor's theorem is to functions of deterministic variables. Expanding $V(S+\mathrm{d}S, t+\mathrm{d}t)$ in a Taylor series about (S,t) we get:

$$\mathrm{d}V = \frac{\partial V}{\partial S}\,\mathrm{d}S + \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{1}{2}\,\frac{\partial V^2}{\partial S^2}\,\mathrm{d}S^2 + \cdots, \tag{1.8}$$

where the dots denote the fact that we have ignored the higher-order terms which are smaller in magnitude compared to the terms retained. Recall that $\mathrm{d}S$ is given by equation 1.2. What about $\mathrm{d}X^2$? We can say that with probability one ([3] or [4])

$$\mathrm{d}X^2 = \mathrm{d}t. \tag{1.9}$$

That is the smaller the $\mathrm{d}t$ the closer $\mathrm{d}X^2$ becomes to $\mathrm{d}t$. Hence, to leading order in $\mathrm{d}t$, equation 1.8 becomes

$$\mathrm{d}V = \sigma\, S\,\frac{\partial V}{\partial S}\,\mathrm{d}X + \left(\mu\, S\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\, S^2\,\frac{\partial V^2}{\partial S^2} + \frac{\partial V}{\partial t}\right)\,\mathrm{d}t. \tag{1.10}$$

This equation gives us the random walk followed by the value of an option. Now we construct a portfolio comprising one option and a number, $-\,\Delta$, of the underlying asset. This portfolio is worth

$$\Pi = V - \Delta S. \tag{1.11}$$

[3] See Ito [3] or Øksendal [4].

The change in the value of the portfolio in time $\mathrm{d}t$ is given by

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta\,\mathrm{d}S. \tag{1.12}$$

Substituting equations 1.10 and 1.2 into equation 1.12 yields

$$\mathrm{d}\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)\mathrm{d}X + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right)\mathrm{d}t, \tag{1.13}$$

which is the random walk followed by Π. In order to eliminate the Wiener process, $\mathrm{d}X$, and hence the random component in this random walk we need to choose

$$\Delta = \frac{\partial V}{\partial S}. \tag{1.14}$$

Thus, equation 1.13 becomes

$$\mathrm{d}\Pi = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t. \tag{1.15}$$

This portfolio is riskless during the time $\mathrm{d}t$. The assumptions listed earlier imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. Thus,

$$\mathrm{d}\Pi = r\,\Pi\,\mathrm{d}t, \tag{1.16}$$

where r is the risk-free interest rate. Now, equating equation 1.16 with equation 1.15 gives

$$\left(\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)\mathrm{d}t = r\,(V - \Delta S)\,\mathrm{d}t, \tag{1.17}$$

and so the Black Scholes equation is:

$$\frac{\partial V}{\partial t} + r S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = r V. \tag{1.18}$$

The Black-Scholes equation has many solutions, corresponding to all the different derivative securities that can be defined with S as the underlying variable. The particular derivative security that is obtained when the equation is solved depends on the boundary conditions that are used. These conditions specify the values of the derivative security at the boundaries of possible values of S and t. We will discuss the Black-Scholes equation and its extension to include a non-constant volatility in Chapter 6.

The one parameter in equation 1.18 that cannot be observed directly is the volatility of the stock price. Before we go on to describe the various methods used by practitioners to determine volatility, the causes of volatility are discussed.

1.3.1 The Cause(s) of Volatility

Proponents of the Efficient Market Hypothesis claim that the volatility of an asset price is caused solely by the random arrival of new information about future returns from the asset. Others claim that it is caused largely by trading. Thus, an interesting question is whether or not volatility is the same when the exchange is open as when it is closed.

Fama [5] and French [6] carried out empirical tests of this question. They accumulated end-of-trading-day stock prices over a large period of time and then calculated:

- The variance of stock price returns between the close of trading on one day and the close of trading on the next trading day when there were no intervening non-trading days.
- The variance of the stock price returns between the close of trading on Fridays and the close of trading on Mondays.

Assuming the equivalence of trading and non-trading days, the variance in the second situation should have been three times as large as the variance in the first situation. Both Fama and French found that it was in fact much smaller.[4] This suggests that the volatility is much larger when the exchange is open than when it is closed. Hence, a reasonable conclusion is that volatility is caused, to some extent, by trading itself. For a detailed investigation of this point see French and Roll [7]. In the Black-Scholes model if daily data is used this implies that non-trading days can be ignored.

But how does one determine volatility ? The two popular methods are:

- Estimation from historical data.
- Implied volatility.

[4]Fama found that it was only 22 % higher and French found it to be 19 %.

1.3.2 Volatility Estimation from Historical Data

Given that we have values of the asset price, S, consisting of $n+1$ values at equal time steps (S_1, S_2, S_3, ..., S_{n+1}), and using the assumption that the logarithms of these prices are normally distributed then the unbiased variance estimate $\hat{\sigma}^2$ can be used. Letting

$$\mu = \frac{1}{n\,\mathrm{d}t} \sum_{i=1}^{n} \log\left(\frac{S_{i+1}}{S_i}\right), \tag{1.19}$$

the variance can be written as:

$$\hat{\sigma}^2 = \frac{1}{(n-1)\,\mathrm{d}t} \sum_{i=1}^{n} \left[\log\left(\frac{S_{i+1}}{S_i}\right) - \mu\,\mathrm{d}t\right]^2. \tag{1.20}$$

The time step, $\mathrm{d}t$, is assumed to be constant. Choosing an appropriate value for n is not easy but generally speaking more data leads to a better accuracy. However, the volatility does change over time and data that is too old may not be relevant for predicting the future.

1.3.3 Implied Volatility

This is the volatility implied by an option price in the market. Substituting the value of the interest rate, the underlying asset price, the expiry or strike price of the option, and the time to expiry into the Black-Scholes equation and using the quoted value of the option, one can calculate the volatility or σ of the option. More than one option price can be used. For example, using option prices for different expiry dates one can deduce the markets opinion of future values of volatility of the underlying. This is referred to as the **term structure of volatility**.

1.4 Black-Scholes Versus Real World

When calculating option values using the Black-Scholes model, and comparing them with 'real' option prices, one would find that there usually is a difference. This could be due to any one of the following reasons:

- The calculated value is correct but the option value is out of line. In this scenario it may be possible to trade profitably using the formula.
- Wrong inputs to the Black-Scholes formula have been used. A wrong value for the volatility or an out-of-date interest rate could be, for example, the problem.

- The Black-Scholes formula is incorrect.

We would like to concentrate on the last point, that the Black-Scholes could be wrong, and in particular we want to know how the values might change if we substituted more realistic assumptions (instead of the assumptions used in the Black-Scholes analysis). We consider these in turn.

Interest Rates : These are liable to change over a period of time. If this is the case, Merton [8] showed that one can sometimes simply substitute the interest rate on a bond with no coupons and a maturity equal to the option maturity for the short-term interest rate in the original formula. But this only works if the volatility of the stock is constant. In general, the effects of interest rate changes do not have a large effect on the option values.

Short Selling Penalties : An investor cannot sell a stock short on a **downtick**. This is defined as the sale of a security at a price below that of the preceding sale. He must go to the expense of borrowing stock if he wants to sell short. In addition, he may have to put up margin with his broker in cash, and he may not receive interest on cash balances with his broker. For options, the penalties are less severe. An investor who does not meet a suitability test may not be allowed to write naked options. Even the investor who meets the test may have to put up cash margin on which he receives no interest.

Transaction Costs : An outside investor must pay brokerage charges on his options and stock trades. An inside investor must pay floor brokerage charges, or execute the trade himself. He must pay clearing charges, and the costs of any exchange membership he may have. These transaction costs are often substantial compared to any potential profits.

Taxes : Some investor pay no taxes; some are taxed as individuals, paying taxes on dividends, interest, and capital gains; and some are taxed as corporations, also paying taxes on dividends, interest, and capital gains, but at different rates. The very existence of taxes will affect options values. The fact that investor tax rates differ will affect values too.

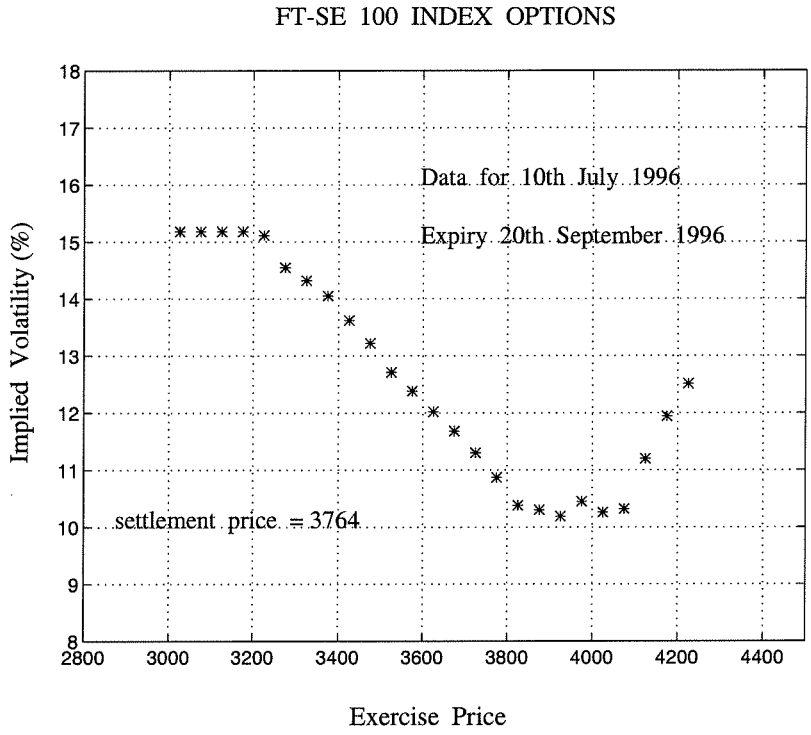


Figure 1.2: Implied volatilities plotted as a function of exercise price for FT-SE 100 Index Options. These volatilities were obtained from [10].

Dividends : The Black-Scholes formula does not take into account dividends. But dividends reduce the values of call options and the increase the value of put options. They make early exercise of a call options more likely; and the early exercise of put options less likely. There are various ways to modify options formulae to take into account dividends. See for example [9].

Takeovers : The formula assumes that the underlying stock will continue trading for the life of the option. Takeovers can make this assumption false.

Volatility Changes : The volatility of a stock is not constant and the fact that it can change may have a major impact on certain options values, especially far out-of-the-money options.

We have detailed the effects on option values when one uses realistic assumptions in the Black-Scholes formula. It has been noted that option values are indeed very sensitive to volatility changes. We would like to expand on this point.

In the previous section we described the two popular methods for estimating volatility: historical estimation and implied volatilities. Both methods of estimation reveal that the Black-Scholes model is not 'completely' accurate.

When using the Black-Scholes model one can the value of the volatility to be a constant measured from historic data. Volatility is not constant for long periods of time [11] nor is it obvious that volatility is independent of the time series from which it is evaluated.

Implied volatilities are in general not constant across exercise prices. That is, if the asset price, interest rate, and time to expiry are fixed then according to Black-Scholes the prices of options across exercise prices should reflect a uniform value for the volatility. But in practice this is not the case, see Figure 1.2. Note that the volatility for in-the-money options is larger than the volatility for at-the-money options. This particular shape is usually termed the volatility **smile**.

Whichever way is used for volatility estimation (historical or implied), we cannot obtain the future value of volatility. The value we obtain is either an estimate of the values in the past or the markets estimate of the future. Thus, both implied volatilities and historical volatility show that the use of the Black-Scholes model with a constant volatility is not an adequate choice.

But the model has withstood the test of time and is used by a large majority of practitioners. We do not want to suggest an alternative, but relaxing the Black-Scholes assumption of constant volatility we would like to model it using a different approach - different to historical estimation and implied volatilities.

Chapter 2

Technical Analysis

Chapter Abstract

Technical Analysis, the use of past price movements to predict future changes, is described in this chapter. There is a survey of the available literature on technical analysis. A brief introduction is also given to prices charts - diagrams used by technical analysts to formulate their trading strategies.

2.1 An Introduction to Technical Analysis

What determines the price of a stock, commodity or index ? What impact do past prices have on the current price and what impact does the current price have on the future price ? The answers to these questions depend on whom you ask: **Fundamental Analysts** or **Technical Analysts**.

- The **Fundamental Analysts** study a composite of supply-and-demand elements: statistical reports on production, expenditure, political ramifications, labour influences, price support programs, industrial development and other criteria. The result is a price forecast, a prediction of where prices will be at some time in the future.

- The **Technical Analysts** study patterns and movements. These include, but are not normally restricted, to price, volume, and open-interest. The result might be a short- or long-term forecast based on the statement that prices are moving up or down.

Many fundamental and economic analysts advocate that there is no sequential correlation between the direction of price movement from one day to the next. Their

position is that prices will seek a level that will balance the supply-and-demand factors, but that this level will be reached in an unpredictable manner. They firmly believe in the various forms of the Efficient Market Hypothesis (EMH).

Technical Analysis is the term used for trading techniques which attempt to use past prices and volume information to predict future price movements. It is the earliest investment analysis technique and is still used by analysts all over the world. Efficient market theory states (as mentioned in Chapter 1) that in a weak-form efficient market, technical analysts cannot consistently generate excess returns since this method uses past information that is already incorporated in the stock prices. Nonetheless, investors the world over employ technical analysis in their day-to-day trading strategies and so these technical analysis rules can be used to test market efficiency.

Technical analysis is essentially the search for recurrent and predictable patterns in stock prices. Although technicians recognise the value of information regarding future economic prospects of a firm, they believe that such information is not necessary for a successful trading strategy. This is because whatever the fundamental reason for a change in a stock price, if the price responds slowly enough, the analyst will be able to identify a trend that can be exploited during the adjustment period. The key to successful technical analysis is a sluggish response of stock prices to fundamental supply-and-demand factors. This prerequisite is diametrically opposed to the notion of an efficient market.

2.1.1 Literature Review

In recent years the EMH has come under serious threat and there are various papers[1] which have questioned its validity.

Early papers found evidence that daily, weekly and monthly returns were predictable from past returns. For example, Fama [5] found that the first-order autocorrelations of daily returns were positive for 23 of the 30 Dow Jones Industrials. Fisher's [13] results suggested that autocorrelations of monthly returns on diversified portfolios were positive and larger than those for individual stocks. As surveyed by Fama [14] and [15], the evidence of predictability in these earlier works often lacked

[1]For an overview of the different papers see Gencay [12].

statistical power and the portion of the variance of returns explained by variations in expected returns was typically accepted as a good working model.

More recently there has been extensive work on temporal dynamics of security returns. Lo and Mackinlay [16] found that weekly returns on portfolios of NYSE stocks grouped according to size show positive autocorrelation. Conrad and Kaul [17] examined the autocorrelations of Wednesday-to-Wednesday returns for size-grouped portfolios of stocks that trade on both Wednesdays. Their findings were similar to those of Lo and Mackinlay - that weekly returns are positively autocorrelated. Lo and Mackinlay [18] reported positive serial correlation in weekly returns for indices and portfolios and negative serial correlation for individual stocks. Chopra, Lakonishok and Ritter [19], Fama and French [20] and Poterba and Summers [21] all found negative serial correlation in returns of individual stocks and various portfolios over three to ten year intervals. Overall, the findings of the recent literature confirm the earlier findings that daily and weekly returns are at least partly predictable from past returns and other economic and financial variables.

Another highly quoted article is the study by Brock, Lakonishok and LeBaron [22] which showed that technical trading rules have predictive power for the Dow Jones Index. Their results documented two important facts. The first was that buy signals consistently generated higher returns than sell signals. The second was that second moments of the distributions of the buy and sell signals behaved quite differently because returns following buy signals were less volatile than returns following sell signals. Overall, their findings showed that linear conditional mean estimators (such as GARCH) failed to characterise the temporal dynamics of security returns and hence suggested possible nonlinearities.

Thus, the results of these studies are in sharp contrast with majority of the earlier studies which supported the random walk hypothesis and concluded that the predictable variation in equity returns was economically and statistically insignificant.

Note that in Chapter 1 (Section 1.4) we compared the Black-Scholes analysis to the "real world". It was shown that some of the assumptions used in the derivation of the Black-Scholes p.d.e. were incorrect, none more so than the assumption of constant volatility. Technical analysts, though ridiculed by many, also attempt (in a way) to compare the real world with the Black-Scholes one. They try to prove that

the assumption of efficient markets, which leads to the lognormal random walk, can be wrong (not all the time) but sometimes. In doing so, they also attempt to make a profit from this disagreement (between the real and the Black-Scholes worlds).

2.2 Chartists

It has been mentioned that technical analysts use past prices to predict future movements. In effect, they are hoping to find patterns which can be exploited to make a profit. As these past prices are recorded in the form of charts,[2] and technical analysts study such charts, they are sometimes referred to as **chartists**.

The chartist may draw a **trend-line** across various prices. This is made by connecting the highest or lowest prices to which a security or commodity has risen or fallen within a particular period. The angle of the resulting line will indicate if the security or commodity is in a **downtrend** or an **uptrend**. If the price rises above a downward sloping trend-line or drops below a rising uptrend line, they say that a new direction may be emerging. When drawing trend-lines chartists are searching for **momentum** or the direction in which security or commodity prices might be moving. An example of a momentum search is shown in Figure 2.2.

More complex patterns, such as the **breakaway** and the **head and shoulders** (Figure 2.1), are also believed to convey clear buy or sell signals. The breakaway indicators can be one of the strongest indicators of a reversal in trend. By definition the breakaway or break-out is a rise in security's price above a previous high price or a drop below a former lowest price. The head and shoulders is named for its rough resemblance to a portrait of a head with surrounding shoulders. Once the right shoulder is penetrated (known as **piercing the neckline**) chartists believe the stock is on the verge of a major decline in price. A reverse head and shoulders pattern has the head at the bottom of the chart, meaning that prices should be rising.

To complete this short introduction on charts it should also be mentioned that one of the most commonly heard components of technical analysis is the notion of **support** and **resistance** levels. These values are said to be the price levels below which it is unlikely for stock prices to fall, or above which it is difficult for them

[2]See Jobman [23], Bodie, Kane and Marcus [24] and Charters [25].

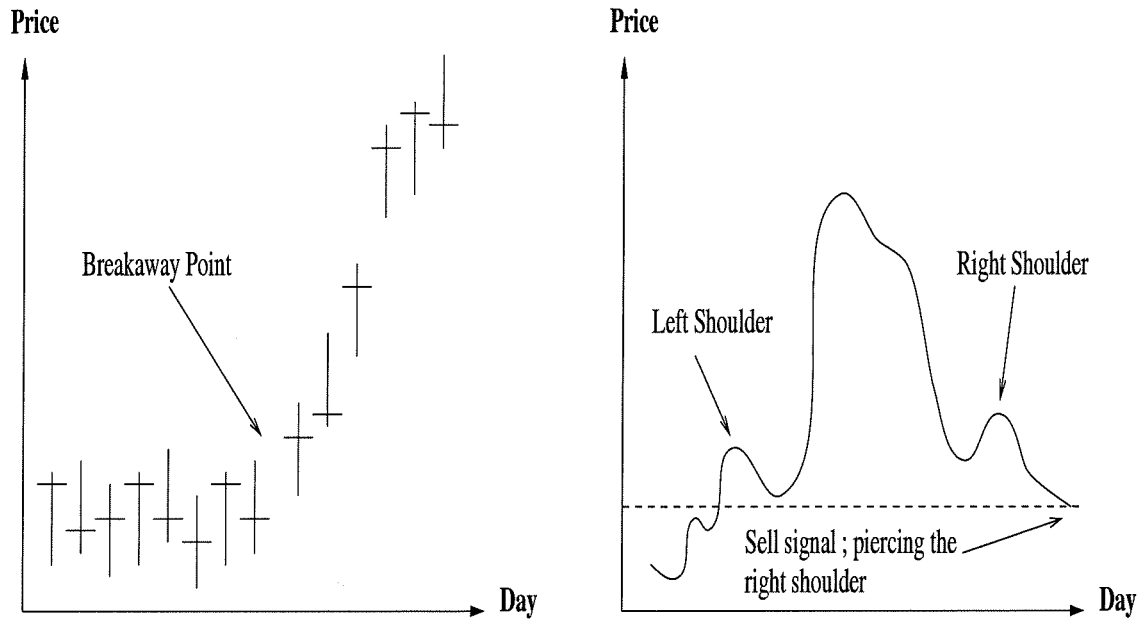


Figure 2.1: Some of the typical patterns chartists use to make buy/sell decisions. The left plot shows the breakaway pattern and right one the head and shoulders pattern. Note that these are 'ideal' plots and in reality chartists would be looking for plots which closely resemble these ones.

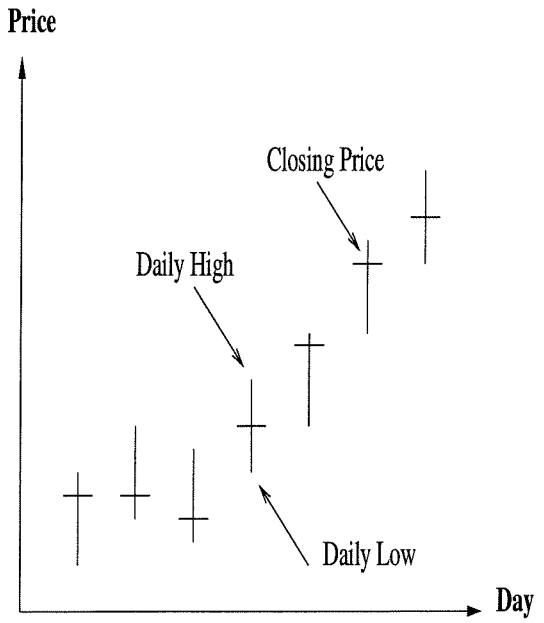


Figure 2.2: An upward trend formation. Chartists use such plots for the purposes of a momentum search.

to rise. Note that all these formations (the breakaway, head and shoulders, and support and resistance levels) are said to be models of psychological effects.

A lot of trading is based on the buy/sell signals generated from such charts. Analysing the movement and quality of price as much as price itself could lead to signals for future prices. How fast are prices changing ? What does this say about the strength or weakness of a move ? How much momentum does a market have ? **Technical Indicators** are the analysis tools used to answer these questions and hence predict future price movements.

Chapter 3

The Moving Average Technical Indicator

Chapter Abstract

This chapter contains a discussion and comparisons of various moving average systems. Some of the popular trading techniques (based on moving averages), used by traders, are described. A detailed derivation is given for the random walk followed by the ratio of the moving average and the stock price along with the p.d.f. associated with this ratio.

3.1 Moving Averages

One of the most common trend-following technical indicator is a **moving average**. It is an average of past stock prices and moves ("runs") with time to include the most recent price. This has the crucial effect of smoothing out short term price fluctuations and can reveal the underlying characteristics or trends. It is probably the single most commonly used technical indicator (certainly all the papers described in the technical analysis literature review use it) and we shall be using it throughout this thesis. There is a plethora of moving average systems, some of which are discussed below.

Simple Moving Average: This places equal weight on every price for the time period in question. That is to say, if you have n prices then

$$I_n = \frac{\sum_{i=1}^{n} S_i}{n}, \tag{3.1}$$

where S_i is the i'th price. Assuming that these prices are daily prices then I_n is known as the n-day moving average. When the next price becomes available

then the same moving average can be expressed as

$$I_n = \frac{\sum_{i=2}^{n+1} S_i}{n}. \tag{3.2}$$

This is still a n-day moving average but the difference is that the latest value $(n+1)$ has been included at the expense of the earliest one $(i = 1)$.

Weighted Moving Average: This places more emphasis on the latest price than the earliest one, because recent prices may be more important than older ones. Using the same framework as above we can write a weighted moving average as

$$I_n^{weight} = \frac{\sum_{i=1}^{n} W_i S_i}{\sum_{i=1}^{n} W_i}, \tag{3.3}$$

where W_n is the weight associated with the nth price and the denominator is the sum of all the weights. Once again, when the next price becomes available the moving average would be the same as equation 3.3 but with the same difference (as above) that the earliest value $(i = 1)$ is discarded to include the latest one $(n+1)$.

Exponentially Weighted Moving Average EWMA: Both the simple and the weighted moving average exclude earlier prices and this could lead to the elimination of a valuable piece(s) of information. Not only does the EWMA include all the prices, assigning higher weights to newer prices, but it is analytically 'very' tractable. We define the EWMA to be

$$I = \lambda_n \int_{-\infty}^{t} e^{-\lambda(t-\tau)} S(\tau)\, \mathrm{d}\tau, \tag{3.4}$$

where λ_n has dimensions of inverse time and is called the **smoothing constant**. Equation 3.4 makes the EWMA a continuous time average. Observe one important property of λ_n, that the larger λ_n is the more the moving average is weighted towards the spot or present value of the stock price, S(t). Thus, $1/\lambda_n$ is the "half-life" or the number of days (or time scale in question) used in the exponential smoothing process.

Irrespective of the moving average type, the basic principle of any moving average system is that, if there is more buying pressure than selling pressure, the prices will be above the average and the market will be in an uptrend; if

there is more selling pressure, prices fall below the moving average, indicating a downtrend. Trading signals are generated when prices cross (moving up or down) the moving average.

What we actually require is the value of I for successive time steps, that is, we want to calculate the rate of change of I with respect to the change in time, t. As long as S is a continuous function of t then evaluating the time derivative of equation 3.4 gives the following result:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lambda_n \left(S - I\right). \tag{3.5}$$

This is the random walk followed by I; note that there is no random component and that it is independent of the random walk followed by S.

The time derivative, equation 3.5, can be approximated as a **finite-difference** equation. For $\mathrm{d}t$ small, using a forward time difference we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{I(t+\mathrm{d}t) - I(t)}{\mathrm{d}t} + \mathcal{O}(\mathrm{d}t). \tag{3.6}$$

Discretising the right-hand-side of this equation and ignoring the $\mathcal{O}(\mathrm{d}t)$ term, equation 3.5 can be re-written as

$$I^{t+1} = I^t + \lambda_n \left(S^t - I^t\right) \mathrm{d}t. \tag{3.7}$$

where S^t and I^t represent the stock price and the moving average at the current time t and I^{t+1} represents the moving average at the next time step. This (equation 3.7 can be approximated to be

$$I^{t+1} = e^{-\lambda_n \, dt} \, I^t + \left(1 - e^{-\lambda_n \, dt}\right) S^t. \tag{3.8}$$

This equation is known as the updating equation and using the known values of S^t, S^{t+dt} and I^t, the EWMA can be evaluated at successive time steps.

Figure 3.1 shows the effect of using exponential smoothing on a randomly generated set (equation 1.4 was used) of stock prices.[1]

[1] For the computer code used to generate this figure see Appendix B (Section B.2).

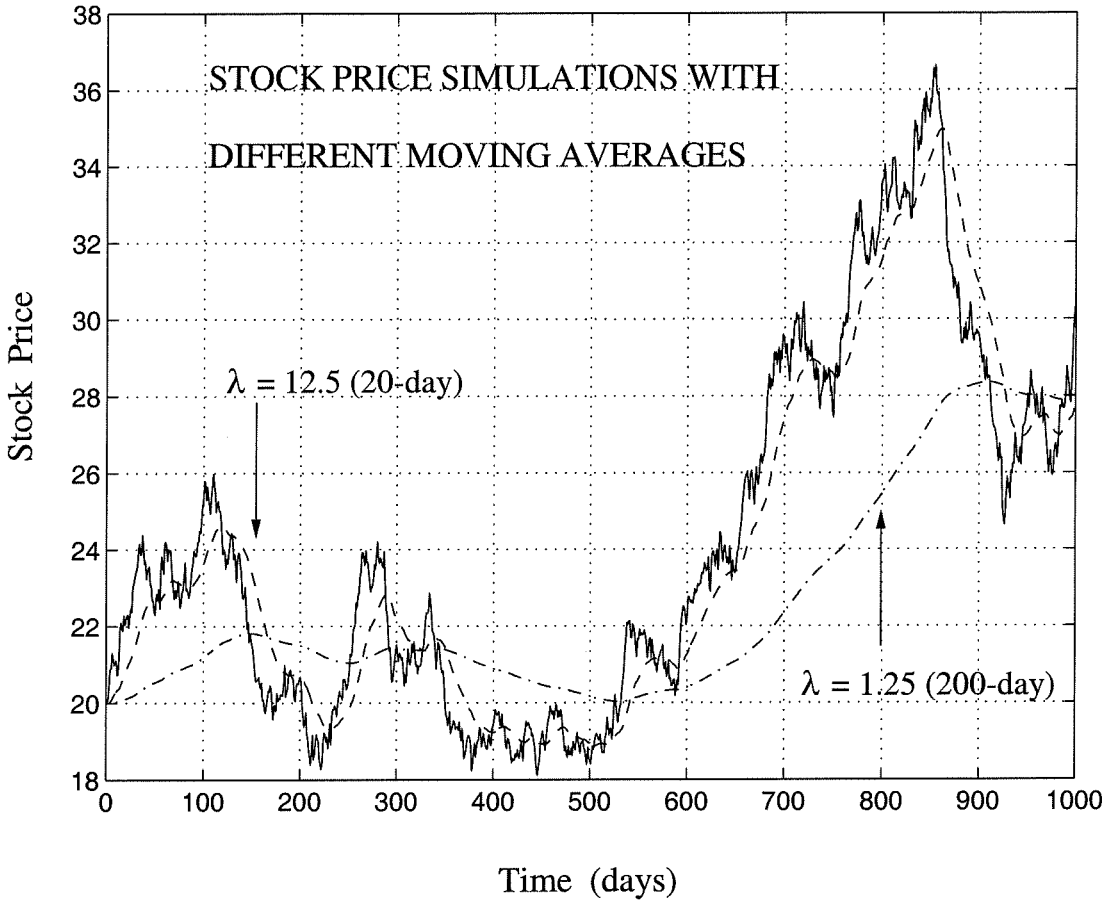


Figure 3.1: A random walk simulation for stock prices with EWMAs of scale: 20-day (λ_n = 12.5) and 200-day (λ_n) = 1.25). Note that the other parameters are the same as Figure 1.1.

3.2 How are Trading Signals Produced ?

It has already been mentioned that traders trade the cross-over of moving averages and stock prices. The basic rule is that: Buy when prices rise from below the moving average line to above the line; sell when prices go from above the line to below the line. It is very important to decide the length or time scale of the moving average used. When the market is moving rapidly in one direction, one wants a fast moving average (one with fewer days in it) to 'catch' the trend. When the market is trending erratically, one wants a slower moving average (one with more days in it) to avoid the generation of spurious trading signals.

A simple trading system might use the relationship of prices to just one moving average to generate its signals. More complicated ones use multiple moving average, usually ones which are short-term, medium-term and long-term, to improve the odds that a signal really does indicate a new trend. The basic signal comes when a shorter moving average moves above or below a longer moving average.

Ideally, in the case of an uptrend, one wants the prices to be on top with the short-term, medium-term and long-term moving averages (in that order) below the prices. If the short-term moving average falls below the medium-term one, then one could 'get out of' some contracts rather than reversing ones entire position. If the short-term moving average continues to fall through the long-term one, then one could close out all long positions and go short. But note, in the Black-Scholes world this will not work in the long-term.

3.3 Random Walk for the Moving Average and the Stock Price in the Black-Scholes World

It has been mentioned that a large portion of the volume of shares etc. are traded in reaction to technical indicators such as moving averages. A trade is triggered, for example, when a moving average and the underlying share cross each other. Thus, the ratio I/S (or S/I) is an important quantity in determining whether or not a trade takes place. Our goal in this section is to evaluate, within the Black-Scholes world, the p.d.f. for I/S. But first we determine the s.d.e. governing I/S.

Now, in order to derive[2] this s.d.e. we use the definitions of the random walk followed by the stock price S (equation 1.2) and the definition of the EWMA (equation 3.4):

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}X, \tag{3.9}$$

and

$$I = \lambda \int_{-\infty}^{t} e^{-\lambda(t-\tau)}\, S(\tau)\,\mathrm{d}\tau. \tag{3.10}$$

We then define a scaled moving average M as

$$M = \frac{\lambda}{S(t)} \int_{-\infty}^{t} e^{-\lambda(t-\tau)}\, S(\tau)\,\mathrm{d}\tau, \tag{3.11}$$

which is just the ratio of equation 3.10 to $S(t)$. Now, in a small time step $\mathrm{d}t$, M changes to $M + \mathrm{d}M$ where

$$M + \mathrm{d}M = \frac{\lambda}{S + \mathrm{d}S} \int_{-\infty}^{t+dt} e^{-\lambda(t+dt-\tau)}\, S(\tau)\,\mathrm{d}\tau. \tag{3.12}$$

This can be approximated by

$$\frac{\lambda}{S}\left(1 - \frac{\mathrm{d}S}{S} + \frac{\mathrm{d}S^2}{S^2} + \cdots\right)\left(\int_{-\infty}^{t} e^{-\lambda(t-\tau)}\,(1 - \lambda\,\mathrm{d}t)\,S(\tau)\,\mathrm{d}\tau + S(t)\,\mathrm{d}t + \cdots\right). \tag{3.13}$$

[2]See Atkinson and Wilmott [26].

Now, using equations 3.9 and 3.11 we obtain the s.d.e. for M

$$\mathrm{d}M = (\lambda + a\,M)\mathrm{d}t - \sigma\,M\mathrm{d}X, \tag{3.14}$$

where

$$a = -\lambda + \sigma^2 - \mu. \tag{3.15}$$

This is the random walk followed by M or I/S. Note that the s.d.e. for M does not depend on S, where as other definitions for a moving average would not necessarily have this property. This greatly simplifies any problem that is only dependent upon the underlying asset price via its moving average.

3.4 Probability Density Function for I/S

Now that we have the s.d.e. for I/S we are in a position to determine its steady-state probability density function. In order to do so we use the fact that, if G is a random variable which follows the following stochastic differential equation:

$$\mathrm{d}G = A(G,t)\,\mathrm{d}X + B(G,t)\mathrm{d}t, \tag{3.16}$$

where A and B are any functions of G and t, then we can define [27] a **transition probability density function** as $P(G,t;G',t')$ which satisfies to the following forward parabolic equation:

$$\frac{1}{2}\frac{\partial^2(A^2(G',t')\,P)}{\partial G'^2} - \frac{\partial(B(G',t')\,P)}{\partial G'} - \frac{\partial P}{\partial t'} = 0. \tag{3.17}$$

This is equation known as the **Fokker-Planck equation**. Note that the current time and value of the random variable are denoted by G and t and the future ones by G' and t'.

Comparing the random walk (equation 3.16) and the random walk followed by M (equation 3.14) we can write the forward equation for M as

$$\frac{1}{2}\frac{\partial^2(\sigma^2\,M^2\,P)}{\partial M^2} - \frac{\partial((\lambda + aM)\,P)}{\partial M} - \frac{\partial P}{\partial t} = 0, \tag{3.18}$$

or

$$\frac{1}{2}(\sigma^2\,M^2\,P)_{MM} - [(\lambda + a\,M)\,P]_M = P_t, \tag{3.19}$$

where the subscripts represent partial derivatives. Now, in the steady-state case, $P_t = 0$, and integrating equation 3.19 with respect to M gives

$$\frac{P_M}{P} = \frac{\lambda + M\,(a - \sigma^2)}{\frac{1}{2}\,\sigma^2\,M^2}. \tag{3.20}$$

Using the boundary condition

$$P \rightarrow 0 \text{ at } 0\,,\,\infty \tag{3.21}$$

and the fact that

$$\int_0^{+\infty} P\,\mathrm{d}M = 1, \tag{3.22}$$

we obtain the final steady-state p.d.f. for M:

$$P(M) = \frac{\left(\frac{2\lambda}{\sigma^2}\right)^{1\,-\,2a/\sigma^2}}{\Gamma\,[1\,-\,2a/\sigma^2]}\,e^{-\,2\,\lambda/\sigma^2\,M}\,M^{-\,2\,+\,2a/\sigma^2}, \tag{3.23}$$

where Γ is the Gamma function.

The mean and the variance of a continuous distribution function are defined as:

$$\begin{aligned} \hat{\mu} &= \int_{-\infty}^{+\infty} y\,f(y)\,\mathrm{d}y \\ \hat{\sigma^2} &= \int_{-\infty}^{+\infty} (y - \hat{\mu})^2\,f(y)\,\mathrm{d}y, \end{aligned} \tag{3.24}$$

respectively. In our case using equation 3.23 as $f(y)$ for the mean we get

$$\hat{\mu} = \int_0^{+\infty} M\,P(M)\,\mathrm{d}M = 1, \tag{3.25}$$

and for the variance

$$\begin{aligned} \hat{\sigma^2} &= \int_0^{+\infty} \hat{M}^2\,P(M)\,\mathrm{d}M - \left[\int_0^{+\infty} M\,P(M)\,\mathrm{d}M\right]^2 \\ &= \frac{2\,\lambda^2}{2\,a^2 + a\,\sigma^2} - \frac{1}{a^2}. \end{aligned} \tag{3.26}$$

Probability density functions for different values of a are evaluated using equation 3.23 and compared to the ones obtained from real data in Chapter 5 (Section 5.3).

Chapter 4

Analysis of Dow Jones Returns

Chapter Abstract

In this chapter we analyse some real data which was available in the form of the Dow Jones Industrial Average. In particular, the independence of Dow Jones returns is checked along with a test of whether or not these returns conform to a normal distribution. The Dow Jones historical volatility and the drift rate are also estimated.

4.1 The Dow Jones Industrial Average

The data used in this thesis was in the form of a financial time series for the **Dow Jones Industrial Average**[1] from the years 1901 until 1995.[2] The format of this time series is shown in Table 4.1. The DJIA is published by Dow-Jones & Co.,

Date	Index Value
01-01-02	51.60
01-01-03	49.79
01-01-04	50.79
01-01-05	49.58
01-01-07	49.17
01-01-08	49.56
01-01-09	49.47

Table 4.1: The format of the Dow Jones Industrial Average data file used in the data analysis. The dates are of the form: yy-mm-dd.

New York and is the 'grand-daddy' of all share price indices, first calculated on 12 stocks in January 1897. It now contains 30 stocks, chosen by Dow-Jones and the

[1] From now on referred to as DJIA.

[2] The Dow Jones data was obtained from [28].

Wall Street Journal to represent a balanced selection of blue-chip corporations. In recent years, the constituents have been gradually altered to reflect the shift in the US economy from old-line manufacturing towards computers and service industries.

It is a **simple aggregate price index** (price weighted average), which means that it is computed[3] by adding the stock prices of the 30 companies and dividing by a *divisor*. Thus, in general, at time t the index is

$$IX_t = \frac{K \sum_{s=1}^{m} P_{st}}{D_t}, \tag{4.1}$$

where

- m represents the number of shares.
- K is the index number at the base date, in this case 100.
- P_{st} is the share price of constituent s at time t.
- D_t is the value of the divisor at time t after adjusting for past capital and constituent changes to ensure continuity.

The divisor at the base date, D_0, is set equal to

$$\sum_{s=1}^{m} P_{s0}, \tag{4.2}$$

thus giving a value of K for the index at the base date. If there is a capital or constituent change at time t, the new divisor, D'_t, is calculated from the former divisor, D_t, using:

$$D'_t = \frac{\sum_{s=1}^{m} P'_{st}}{\sum_{s=1}^{m} P_{st}} D_t, \tag{4.3}$$

where P'_{st} is the share price of constituent s immediately after the change and P_{st} is the price before the change. This ensures the continuity of the index.

This type of index performs like an investment portfolio in which the price of a stock determines its weighting in the portfolio. Thus, a move in the share price of a company with a relatively modest market capitalisation may have as great an influence on the day's price movements as a comparable move in the price of a large one. But, in the case of the Dow, however, this factor is mitigated by the selection of very large companies, and by the tendency for US companies to manage the nominal

[3]The frequency of calculations is one minute.

value of their shares so as to ensure market prices that fall into roughly comparable ranges, between $ 30 and $ 100.

What happens in the case of a stock split ? Let us assume that three stocks are selling at $ 5, $ 10 and $ 15 each. Their average price is $ 10. Now, if there is a three-for-one split for the $ 15 share, it subsequently sells for $ 5. The average becomes $ 6.67 and not $ 10. An adjustment has to be so that the average remain as $ 10. This is achieved by changing the divisor to 2 (note, the original divisor was 3).

4.2 The Dynamics of Stock Price Returns

Using the Dow Jones (returns) data set, we tested some of the properties of stock price returns discussed in Chapter 1. We wanted to check whether or not the Dow Jones returns were independent of one another and also whether or not they conformed to a normal distribution.

There are different ways of calculating stock price returns (see for example Taylor [29]) but the one we used was

$$\frac{\mathrm{d}S}{S} = \frac{S_{t+1} - S_t}{S_t}, \tag{4.4}$$

where S_t andS_{t+1} are the stock prices at times t and $t+1$ respectively.

4.2.1 Autocorrelations for Dow Jones Returns

In order to demonstrate that the Dow Jones returns were independent, we needed to evaluate the **sample autocorrelation coefficient**. For two return observations with a time lag of k, R_t and R_{t+k}, it is defined as

$$r_k = \frac{\sum_{t=1}^{N-1}(R_t - \hat{R})(R_{t+k} - \hat{R})}{\sum_{t=1}^{N}(R_t - \hat{R})^2}, \tag{4.5}$$

where $\hat{R}$ is the average

$$\hat{R} = \frac{1}{N}\sum_{t=1}^{N} R_t. \tag{4.6}$$

In other words it is a measure of the correlation between R_t and R_{t+k}. We calculated and plotted a **correlogram** - a plot of r_k against k. One such plots is shown in Figure 4.1. Looking at Figure 4.1 we cannot see any predictive or repetitive patterns

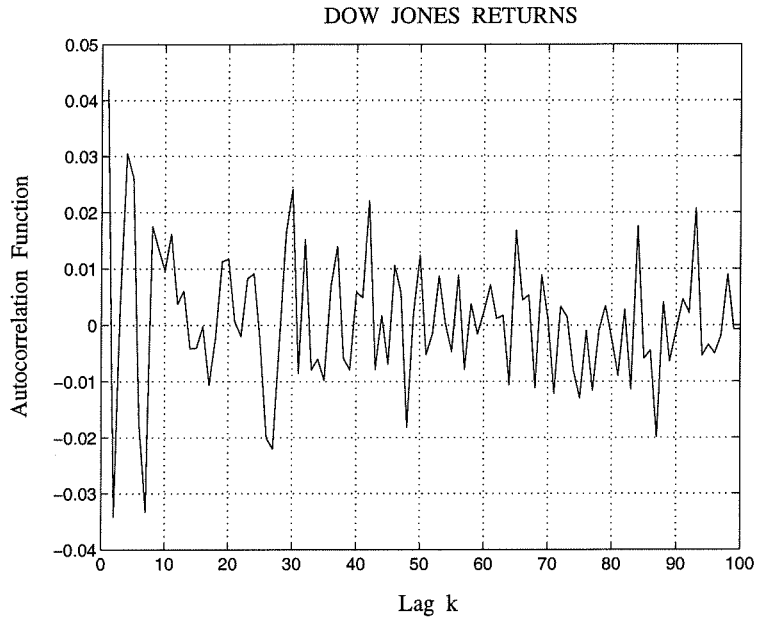


Figure 4.1: The autocorrelation function for the Dow Jones returns.

and also that the correlation coefficients are very small. The coefficients at small k are greater in magnitude than the ones at large values of k. This is a well known fact and a lot of research has been carried out in this area. See for example [30] and [31].

On the above evidence, it is satisfactory to assume that the Dow Jones returns are independent of one another.

4.2.2 Dow Jones Returns Frequency Distribution

To confirm that the Dow Jones returns were distributed normally we had to evaluate various statistical quantities. Dow Jones returns were evaluated using equation 4.4 and the corresponding frequency distribution is shown in Figure 4.2. Defining the return given by equation 4.4 as η, the mean daily return (first moment) can be written as

$$\hat{\eta} = \frac{1}{N}\sum_{t=1}^{N}\eta_t, \tag{4.7}$$

where N represents the total number of returns used in the calculation. This number is 0.000230 which is very small. An easier way to interpret this would be to consider the returns over one year, that is an annualised return. Suppose that the relevant market trades for N days every year. The \$ 1 invested at time 0 would have been worth

$$z_n/z_0 = e^{n\hat{\eta}}, \tag{4.8}$$

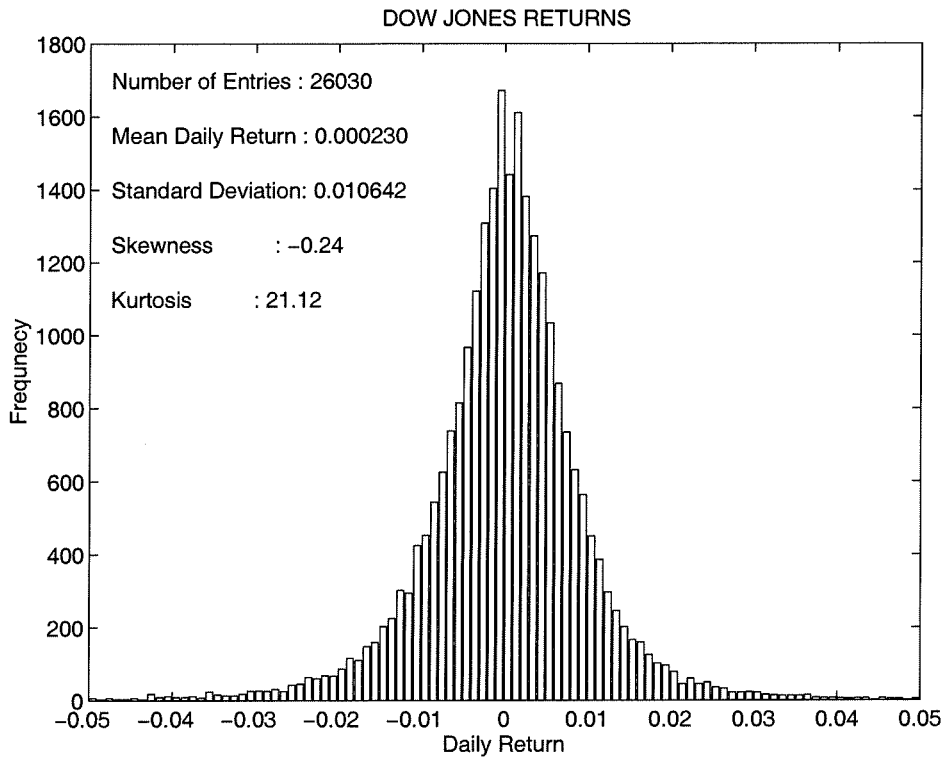


Figure 4.2: The frequency distribution for the Dow Jones Index returns. For a description of the terms in this figure see the text below.

n days or n/N years later - ignoring transaction costs. The historic annual compound rate, G, giving the same return after n/N is defined by

$$(1+G)^{n/N} = e^{n\hat{\eta}}$$

$$G = e^{M\hat{\eta}} - 1. \tag{4.9}$$

Using $N = 250$ the mean daily return of 0.000230 can be converted to an annualised return of 5.92 % - a plausible number.

The standard deviation, sd, is defined as

$$sd = \left[\frac{1}{M-1}\sum_{t=1}^{M}(\eta_t - \hat{\eta})^2\right]^{1/2}. \tag{4.10}$$

Standard deviations (second moment) measure the speed with which prices are changing. A low (high) standard deviation implies that the chances of a substantial price fall/increase are relatively low (high). Thus, risks from two alternative investments can be compared by looking at their standard deviations. The DJIA standard deviation is shown, for comparison, in Table 4.2.2 with other standard deviations obtained from Taylor [29]. Note that this is a standard deviation of returns and so is a measure of the volatility.

Investment	Standard Deviation of Returns
Currencies	0.006
Standard & Poor's 500	0.007
DJIA	0.011
FT-SE 30	0.013
US Stocks (large)	0.015
Gold and Metals	0.018
London Agricultural Futures	0.022
US Stocks (random)	0.027

Table 4.2: The standard deviation of returns calculated from the DJIA as compared to standard deviations from other investments.

The skewness (third moment) of a distribution is used to study the symmetry of the distribution. It is calculated as

$$sk = \frac{1}{M-1}\sum_{t=1}^{M}\frac{(\eta_t - \hat{\eta})^3}{sd^3}, \tag{4.11}$$

where sd is given by equation 4.10. The value we obtained, -0.24, is small and negative implying that the distribution is relatively symmetric. The standard error associated with an estimate of sk calculated using N returns depends on N and the population distribution. For a random sample drawn from a normal distribution it is $\sqrt{6/N}$. But this does not apply to stock price returns as they have a high kurtosis.

The kurtosis (fourth moment) is defined as

$$Ku = \frac{1}{M-1}\sum_{t=1}^{M}\frac{(\eta_t - \hat{\eta})^4}{sd^4}, \tag{4.12}$$

and takes the value of 3 for normal distributions. Even though Figure 4.2 closely resembles a normal distribution, the value of the kurtosis we obtained, 21.12, is much larger. A large skew and/or a high kurtosis value is usually attributed to what are known as **outliers**. These are observations several standard deviations from the mean predicted by normal distributions.

When Fama [5] investigated the empirical distribution of daily returns on New York Stock exchange securities he discovered that they were distributed symmetrically but that the empirical distribution had "fat tails" and no finite variance. The distribution suggested as an alternative to the normal distribution is the **Pareto-Levy** distribution. For a description see Hull [32] or Adams, Bloomfield, Booth and England [33] or Taylor [29].

Dates	dt	σ_{hist}
2/1/1901 - 9/8/95	1/250	16.9 %
2/1/1901 - 9/8/95	1/300	18.5 %
2/1/1901 - 31/5/51	1/300	20.7 %
1/6/1951 - 9/8/95	1/250	13.7 %

Table 4.3: The historical volatility (column 3) for the Dow Jones Industrial Average calculated using different combinations (column 2) of the number of trading days in a year.

In summary, the DJIA daily returns are characterised by firstly low autocorrelation and secondly approximately symmetric distributions having long tails and high kurtosis.

4.3 Dow Jones: Historical Volatility and Drift Rate

We also evaluated the historical σ and the drift rate, μ, for the DJIA. In order to calculate the historical volatility, σ_{hist}, equation 1.20 was used. That is

$$\sigma_{hist} = \left[\frac{1}{(n-1)\,\mathrm{d}t}\sum_{i=1}^{n}\left[\log\left(\frac{S_{i+1}}{S_i}\right) - \hat{m}\,\mathrm{d}t\right]^2\right]^{\frac{1}{2}}, \tag{4.13}$$

where n is the total number of stock prices used in the calculation, $\mathrm{d}t$ is the time step (in our case one day) and $\hat{m}$ is given by

$$\hat{m} = \frac{1}{n\,\mathrm{d}t}\sum_{i=1}^{n}\log\left(\frac{S_{i+1}}{S_i}\right). \tag{4.14}$$

It was observed that the DJIA time series up to the end of May 1951 included closing prices for Saturday, which meant a six day trading week. Thus, the historical volatility was evaluated using various combinations of the number of trading days in a year. The results are shown in Table 4.3. These figures only serve as a rough estimate of the historical volatility. Using a 250 day trading year we obtained a σ_{hist} of 16.9 % and for a 300 day trading year we obtained 18.5 %. Hence, a good estimate for σ_{hist} would be approximately 17.7 %. Note that all these figures represent an annualised quantity.

Now, in order to calculate the drift rate, μ, or rather estimate it we have to use equation 1.10 and let $V = \log S$. This gives

$$\mathrm{d}(\log S) = (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}X. \tag{4.15}$$

Dates	dt	b	Drift Rate (μ)
2/1/1901 - 9/8/95	1/250	0.0398	5.4 %
2/1/1901 - 9/8/95	1/300	0.0477	6.5 %
2/1/1901 - 31/5/51	1/300	0.0287	5.0 %
1/6/1951 - 9/8/95	1/250	0.0484	5.8 %

Table 4.4: The drift rates for the different periods for the DJIA. Column 3 represents the gradients from the least squares fit to Figure 4.3. The drift rates have been calculated using equation 4.20 and the historical σ's from Table 4.3.

And taking expectations we obtain

$$E\left[\mathrm{d}(\log S)\right] = (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t. \tag{4.16}$$

Thus, if we plot $\log S$ versus t and fit a straight line across the data, the gradient of this line would give us the estimate for the drift rate. That is, if the straight line is

$$\log S = a + b\,t, \tag{4.17}$$

where a is the intercept on the y-axis and b is the gradient, then

$$\mathrm{d}(\log S) = b\,\mathrm{d}t, \tag{4.18}$$

and

$$E\left[\mathrm{d}(\log S)\right] = b\,\mathrm{d}t. \tag{4.19}$$

And so, equating equations 4.16 and 4.19 we obtain the drift rate:

$$\mu = \frac{1}{2}\sigma^2 + b. \tag{4.20}$$

Straight line least squares fits were carried out to Figure 4.3 corresponding to the different periods in Table 4.3. The resulting drift rates are shown in Table 4.4. Once again, these number are only approximate.

For completeness, the annualised returns and the standard deviations of the returns (see section 4.2) were also re-evaluated for the different periods. See Table 4.5. The difference between the standard deviation calculation (equation 4.10) and the historical volatility calculation (equation 4.13) is that one uses simple returns (equation 4.4) and the other uses log returns (equation 4.14). In principle the two should give the same estimate of the volatility. Comparing Tables 4.3 and 4.5, it can be seen that there is satisfactory agreement between the two volatility estimation methods.

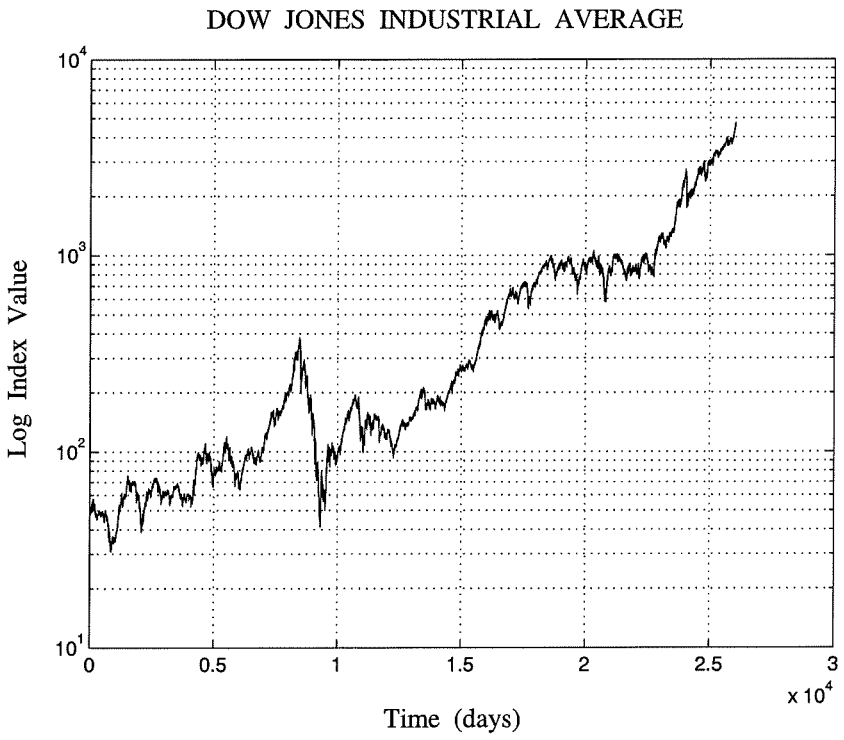


Figure 4.3: The log of the index value versus time for the DJIA from 1901 until 1995.

Dates	dt	Annualised Return (%)	Standard Deviation (%)
2/1/1901 - 9/8/95	1/250	5.9	16.8
2/1/1901 - 9/8/95	1/300	7.1	18.4
2/1/1901 - 31/5/51	1/300	5.5	20.7
1/6/1951 - 9/8/95	1/250	7.8	13.6

Table 4.5: The DJIA annualised returns (equation 4.9) for the different time periods. The annualised standard deviations were calculated by multiplying the original standard deviations by the square root of the number of trading days.

Chapter 5

Data Analysis and Suggested Models

Chapter Abstract

This chapter details the investigations carried out to determine whether or not stock price volatility (σ) can be a function of the ratio of the stock price (S) and the corresponding moving average (I). The Dow Jones σ is evaluated and plotted versus S/I for different time scale moving averages. Various tests are performed in order to obtain an explanation of these plots. The theory associated with the jump and jump diffusion processes is discussed. A model which best fits these plots is derived. Finally, an attempt is made to simulate the Dow Jones price graph (index value plotted against time).

5.1 Is Volatility a Function of S and I ?

In Chapter 1 we mentioned the fact that empirical work had shown that one of the main causes of volatility was trading itself. Even though finance academics discard the use of technical analysis many practitioners use it. In Chapter 2 we showed that technical analysts trade the cross-over of stock prices and their corresponding moving averages.

Thus, the ratio S/I or I/S, which in effect is a measure of the separation between S and I, could serve as an indication as to what traders perceive to be the strength or weakness of a stock. Hence, if trading decisions are based on such ratios (representing cross-overs) then volatility 'could' be expressed as a function of S/I.

Now, in order to test the dependence, if any, of σ on S/I we wanted to plot σ as $\mathrm{d}S/S\sqrt{\mathrm{d}t}$ versus S/I (note that if we use $\mathrm{d}t$ as 1 over the number of trading days in a year, then we are calculating σ as the annualised standard deviation of returns). Before these plots were made we had to decide what length moving averages were

Time Scale (n)	Λ ($1/n$)
1-Day	250
10-Day	25
20-Day	12.5
100-Day	2.5
200-Day	1.25

Table 5.1: The different time scale used for the exponentially weighted moving average.

to be used. A mixture of short-, medium-, and long-term moving averages were employed. These are given in Table 5.1. Note, that the n's in Table 5.1, representing the number of days, have been annualised using

$$\text{Annualised days} = \frac{n}{250}, \tag{5.1}$$

where 250 represents the number of trading days in a calendar year. The five plots obtained using these λ's are shown in Figures 5.1 - 5.5. A **bucket** approach was used in these plots. The horizontal axis (S/I) was divided in to buckets of fixed width, in our case 0.0001, and an average was taken of the different $\mathrm{d}S/S\sqrt{\mathrm{d}t}$ in each bucket. More formally

$$\sigma_{ave}^{S/I} = \frac{1}{ntot}\sum_{i=1}^{ntot}\sigma_i^{S/I}, \tag{5.2}$$

where $\sigma_i^{S/I}$ represents the i'th $\mathrm{d}S/S\sqrt{\mathrm{d}t}$ in each bucket of S/I and $ntot$ is the total number of $\mathrm{d}S/S\sqrt{\mathrm{d}t}$ in that particular bucket.

A quadratic shape is clearly visible in all the plots and it is more apparent for the long-term moving averages. Note that the loss of clarity in the LTMA[1] plots was due to the handling of the small S/I bin size by the **MATLAB** graphics package.

A Monte Carlo check was performed where data was simulated for a fixed value of the volatility, σ. Carrying out the S/I analysis (as for the data) various plots were made. One such plot is shown in Figure 5.6. As expected, the plot shows a constant σ for all values of S/I.

[1]From now on, short-, medium- and long-term moving averages are referred to as STMA, MTMA and LTMA respectively.

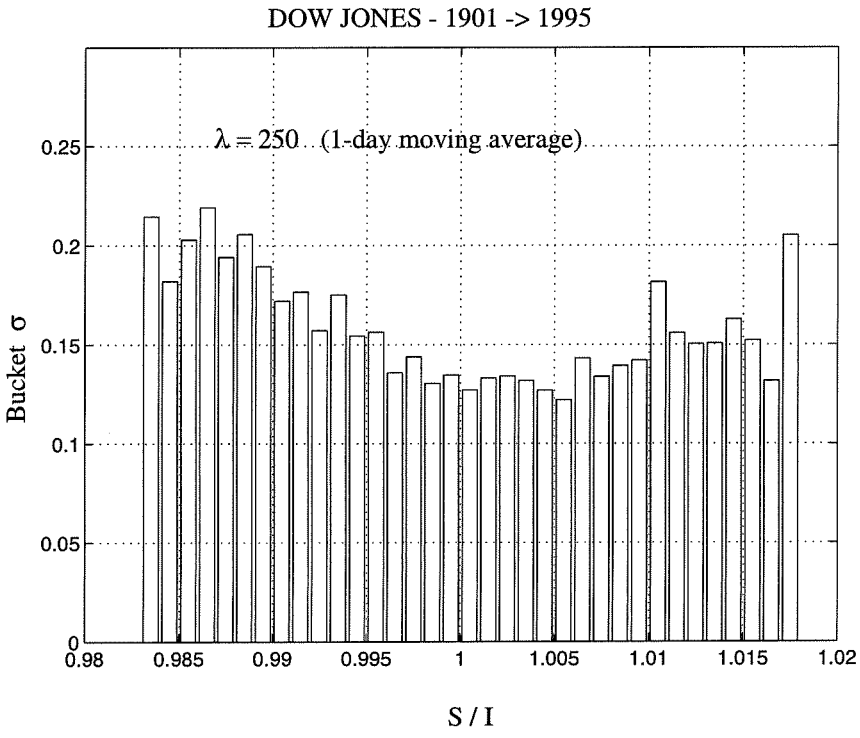


Figure 5.1: Volatility as a function of S/I for a 1-day moving average.

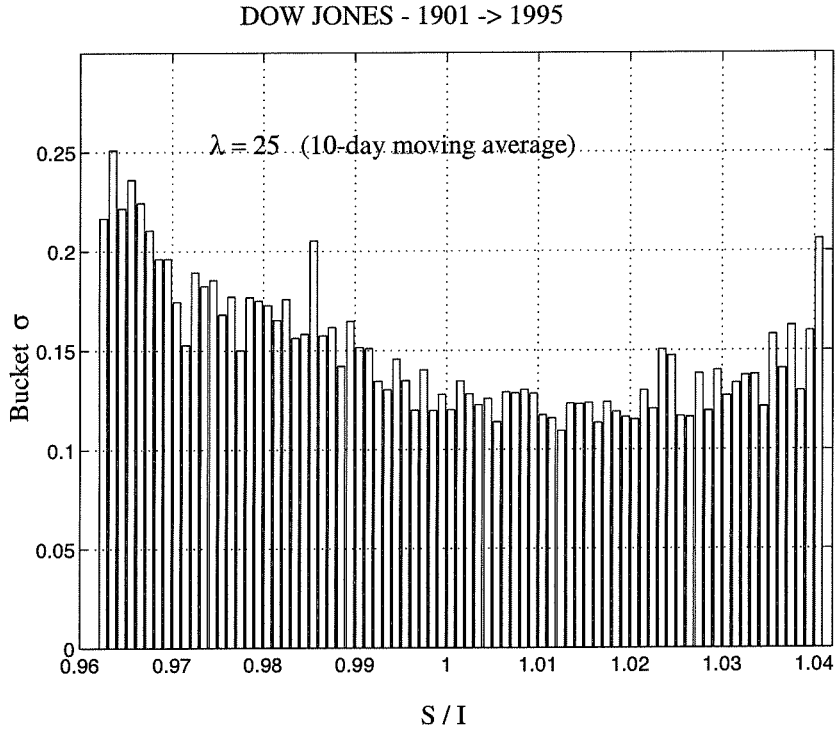


Figure 5.2: σ vs. S/I : 10-day moving average.

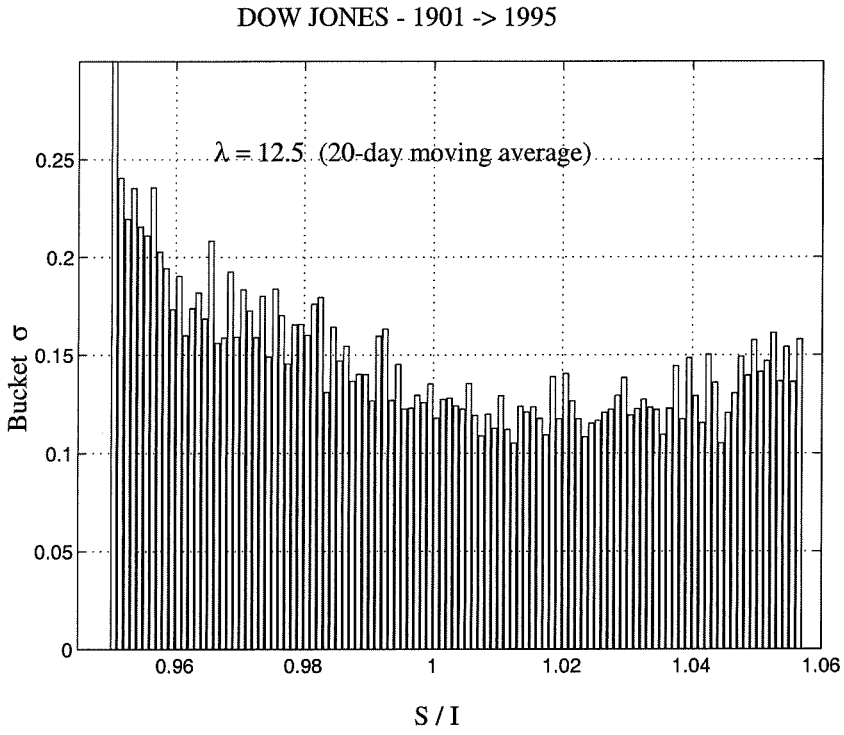


Figure 5.3: σ vs. S/I : 20-day moving average.

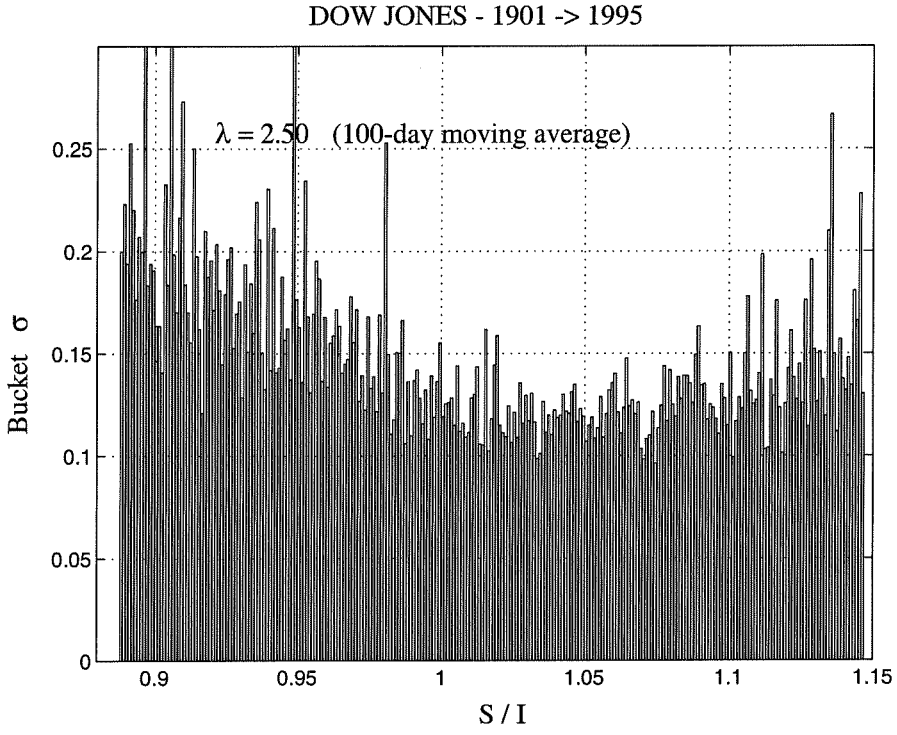


Figure 5.4: σ vs. S/I : 100-day moving average.

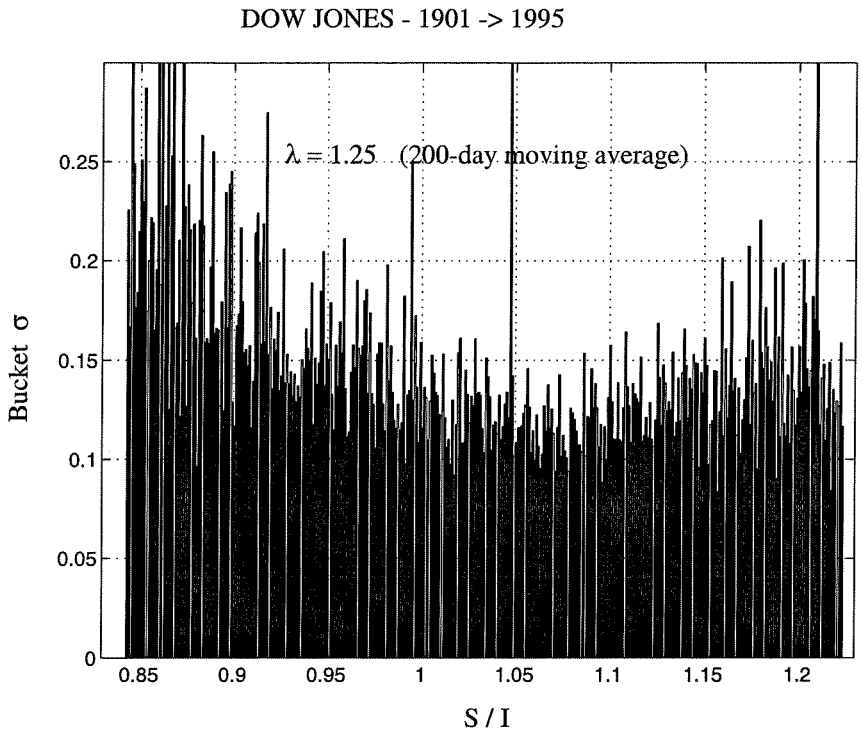


Figure 5.5: σ vs. S/I : 200-day moving average.

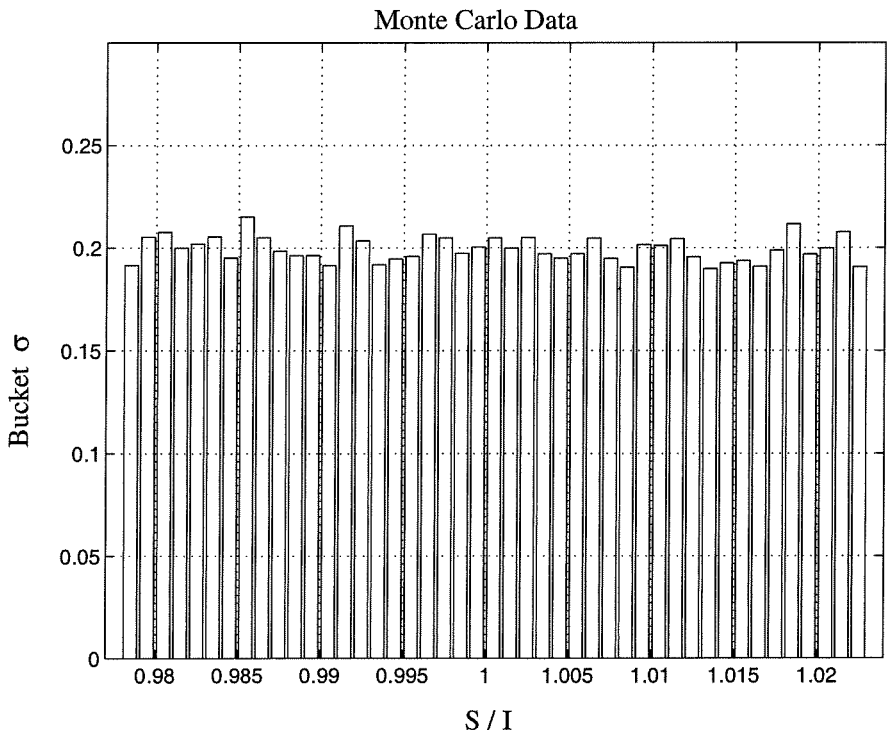


Figure 5.6: σ vs. S/I : 1-day moving average. Data generated with a constant volatility, σ, of 20 %.

5.2 Modified Random Walk

The evidence from Figures 5.1 - 5.5 pointed to the fact that σ for the Dow Jones Industrial Average 'seemed' to be a function of S/I. In this non-constant volatility scenario the standard random walk equation 1.2 would be re-written as:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\left(\frac{S}{I}\right)\mathrm{d}X. \tag{5.3}$$

Calculating the variance of the return, $(\mathrm{d}S/S)$, we obtain:

$$\begin{aligned}\mathrm{Var}\left[\frac{\mathrm{d}S}{S}\right] &= E\left[\mathrm{d}S^2/S^2\right] - E^2\left[\mathrm{d}S/S\right] \\ &= E\left[\mu^2\,\mathrm{d}t + \sigma^2\,\mathrm{d}X^2 + 2\,\mu\,\sigma\,\mathrm{d}t\,\mathrm{d}X\right] - E^2\left[\mu\,\mathrm{d}t + \sigma\,\mathrm{d}X\right] \\ &= \sigma^2\left(\frac{S}{I}\right)\mathrm{d}t,\end{aligned} \tag{5.4}$$

where we have used the fact that $E\left[\mathrm{d}X\right] = 0$. Squaring equation 5.3 we get

$$\mathrm{d}S^2 = \sigma^2\,S^2\,\mathrm{d}X^2 + 2\,\sigma\,\mu\,S^2\,\mathrm{d}t\,\mathrm{d}X + \mu^2\,S^2\,\mathrm{d}t^2. \tag{5.5}$$

Looking at the magnitude at each term shows us that the first term is the largest for small $\mathrm{d}t$ and dominates the other two. Hence, to leading order,

$$\mathrm{d}S^2 = \sigma^2\,S^2\,\mathrm{d}X^2 + \cdots, \tag{5.6}$$

which, using equation 1.9, can be written as

$$\sigma^2\left(\frac{S}{I}\right) = \frac{\mathrm{d}S^2}{S^2\mathrm{d}t}. \tag{5.7}$$

And so, one way to study the dependence of σ on S/I would be to examine the plots of $\mathrm{d}S/S\sqrt{\mathrm{d}t}$ versus S/I. This is what was done for the plots in Figures 5.1 - 5.5.

Before investigating the shape of these plots, we compared the real data I/S frequency distributions with the ones obtained using equation 3.23 (Chapter 3, Section 3.4).

5.3 Theoretical Versus Real $\sigma(S/I)$ P.D.F.

In Chapter 3 we had derived the p.d.f. for the ratio of the moving average and the stock price, I/S, as being

$$P(M) = \frac{\left(\frac{2\lambda}{\sigma^2}\right)^{1\,-\,2a/\sigma^2}}{\Gamma\left[1\,-\,2a/\sigma^2\right]}\,e^{-\,2\,\lambda/\sigma^2\,M}\,M^{-\,2\,+\,2a/\sigma^2}, \tag{5.8}$$

where Γ is the Gamma function, M is I/S and $a = -\lambda + \sigma^2 - \mu$. λ, σ and μ are the time scale of the moving average, the stock price volatility and the drift rate respectively. The λ's that we are dealing with (Table 5.1) when substituted into $-2a/\sigma^2$ translate into large arguments of the Gamma function. Using **Stirling's formula** [34] for large positive α

$$\Gamma[1+\alpha] \approx \sqrt{2\pi\,\alpha}\left(\frac{\alpha}{e}\right)^{\alpha}. \tag{5.9}$$

Now

$$\begin{aligned}\frac{-2\,a}{\sigma^2} &= \frac{2\,\lambda - 2\,\sigma^2 + 2\,\mu}{\sigma^2} \\ &\approx \frac{2\,\lambda}{\sigma^2},\end{aligned} \tag{5.10}$$

where we have used the fact that for large values the λ term in a dominates the σ^2 and μ terms. Thus, using $\alpha = 2\,\lambda/\sigma^2$ we can express $P(M)$ as

$$P(M) \approx \frac{\left(\frac{2\,\lambda}{\sigma^2}\right)^{1+\alpha}}{\Gamma[1+\alpha]}\, e^{-2\,\lambda/\sigma^2\, M}\, M^{-2-\alpha}. \tag{5.11}$$

Hence, using equation 5.9

$$P(M) \approx \frac{e^{-\alpha\,(1/M-1)}\, M^{-2-\alpha}}{\sqrt{2\,\pi\,/\alpha}}, \tag{5.12}$$

where $\alpha = 2\,\lambda/\sigma^2$. Figures 5.7 and 5.8 compare this theoretical p.d.f. result with the frequency distribution for the Dow Jones I/S for different values of λ. Note that the Dow Jones distributions were evaluated by dividing the I/S into fixed length bins and determining the frequency of each bin.

It can be seen from these plots that even though the Dow Jones frequency distributions have 'slightly' fatter tails, there is satisfactory agreement between the real (Dow Jones) frequency distribution and the simulated ($P(M)$) p.d.f's.

5.4 Monte Carlo Simulation of $\sigma(S/I)$

We wanted to check whether all the five plots (Figures 5.1 - 5.5) could be generated by one $\sigma(S/I)$ dependence. The first step in this process was to let σ be a function of S/I in the quadratic form

$$\sigma(S/I) = a\,(S/I)^2 + b\,(S/I) + c, \tag{5.13}$$

where a, b and c are constants. Now, for each of the data plots, Figure 5.1 - 5.5, a least squares quadratic fit was made and these coefficients determined.

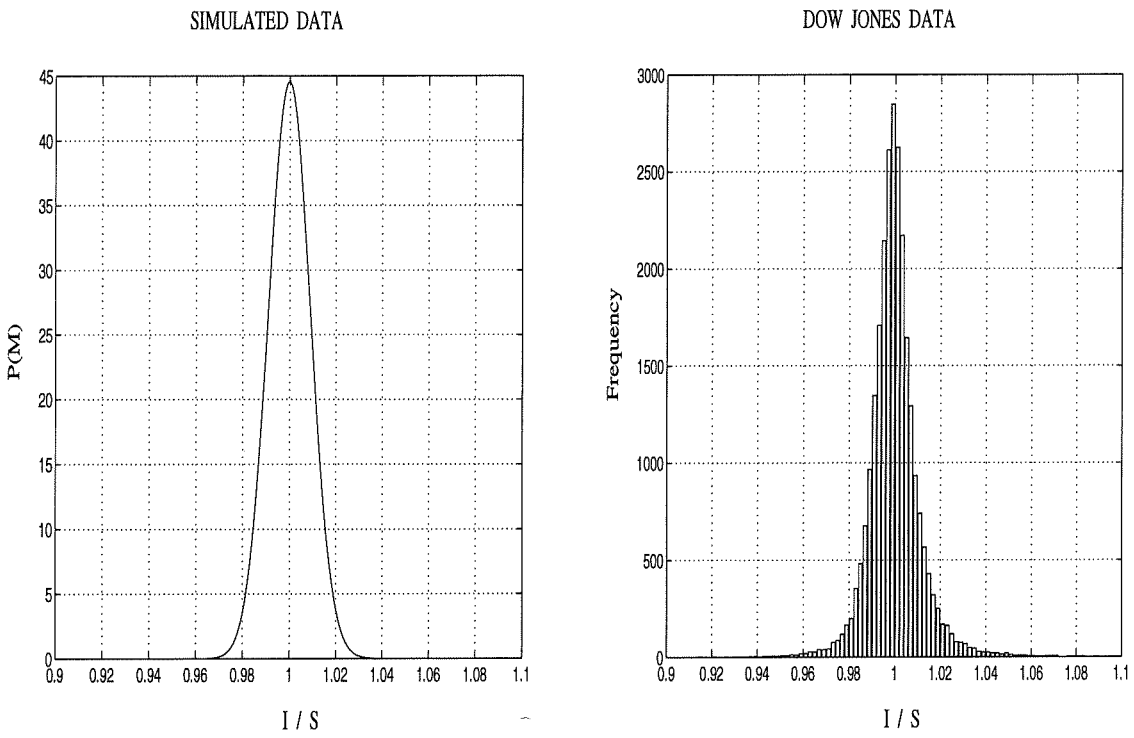


Figure 5.7: The theoretical p.d.f. for I/S for a 1-day ($\lambda = 250$) moving average (left plot). The right plot shows the equivalent frequency distribution obtained from the Dow Jones data.

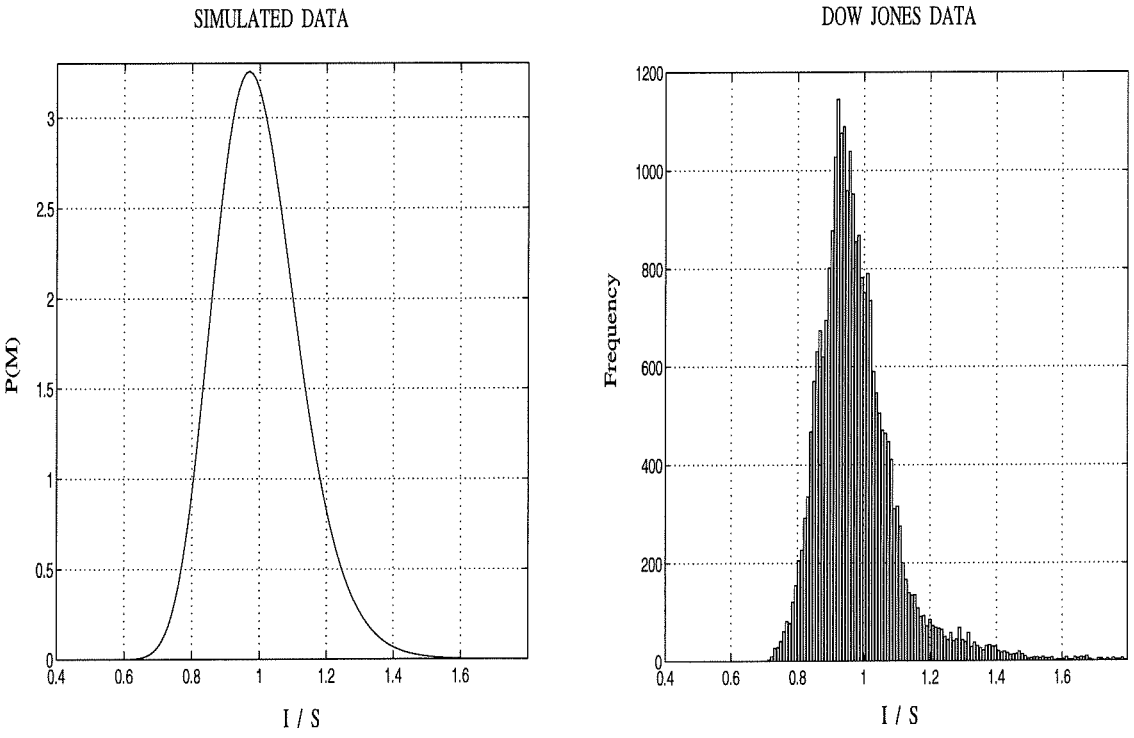


Figure 5.8: Same as the above plot, but for a 200-day ($\lambda = 1.25$) moving average.

Time scale (days)	a	b	c
1	208.53	-418.68	210.29
10	43.51	-88.08	44.70
20	26.31	-53.59	27.41
100	3.97	-8.34	4.50
200	1.97	-4.28	2.43

Table 5.2: The coefficients of the equation $\sigma(S/I) = a(S/I)^2 + b(S/I) + c$ for the different moving averages.

Least Squares Fitting : The basic method is that if

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \tag{5.14}$$

are a sample of n observations with a regression curve

$$Y_i = \alpha\, x_i^2 + \beta\, x_i + \gamma, \tag{5.15}$$

the coefficients α, β and γ are those values of a, b and c for which the sum

$$SS = \sum_1^n (y_i - a\, x_i^2 - b\, x_i - c)^2, \tag{5.16}$$

is a minimum. Differentiating SS with respect to a, b and c we get the **normal equations**, which for a summation from 1 to n are

$$\begin{aligned} c\,n + b \sum x_i + a \sum x_i^2 &= 0 \\ c \sum x_i + b \sum x_i^2 + a \sum x_i^3 &= 0 \\ c \sum x_i^2 + b \sum x_i^3 + a \sum x_i^4 &= 0. \end{aligned} \tag{5.17}$$

This system is symmetric and we could use various methods, for example Gauss elimination, to solve for the coefficients.

The value of the coefficients (for the different moving averages) determined using this method are shown in Table 5.4. Note, these values were checked against the values calculated using the mathematics package, **Mathematica**. For further discussion of the least squares method or the Gauss elimination method see for example Newbold [35] or Kreyszig [34].

We now had five equations of the volatility as a function of the stock price and the EWMA. The next step was to simulate data using equation 5.3, that is

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\left(\frac{S}{I}\right)\mathrm{d}X. \tag{5.18}$$

where σ is function of S/I, as given in equation 5.13. Once we had the simulated data, it was analysed using various combinations of short-, medium- and long-term moving averages. That is, if we were using data generated using a MTMA it was analysed with different STMAs and LTMAs. Other iterations were also performed, with the end result that no matter with what time scale moving average the data was generated, once it was analysed with a different time scale MA a quadratic shape (like in the real data plots) was not seen in the analysed plots. This implied that the process had 'no memory' of the generation stage and hence the Dow Jones volatility did not appear to be a function of S and one particular I.

5.5 Effect of Jumps in Volatility Estimation

At the end of Section 4.2.2 it was noted that the high kurtosis value (for returns) was usually attributed to a few outliers - large jumps in the stock price. It could well be that the quadratic shape we observed was due to such jumps and so a detailed investigation was carried out into this possibility.

As the returns distribution, Figure 4.2, is relatively symmetric with a small skew it was decided to redo the σ versus S/I analysis with the exclusion of

$$\left|\frac{\mathrm{d}S}{S}\right| > 0.02. \tag{5.19}$$

That is, we excluded all returns that were > 2 % or $<$ - 2 %. Had the distribution not been symmetric, we would have had to determine what values other than $\pm$ 2 % were the ones that would have allowed us to remove the same number of entries from both the positive and the negative sides of the distribution.

Two plots (taking into account equation 5.19) for the STMA of 1-day and the LTMA of 200-day are shown in Figures 5.9 and 5.10 respectively. Though there is still a residual quadratic shape, these plots are less pronounced than the corresponding Dow Jones plots, Figures 5.1 and 5.5. That is, the plots with the 2 % returns removed possess, on average, a smaller bucket σ compared to the original Dow Jones plots.

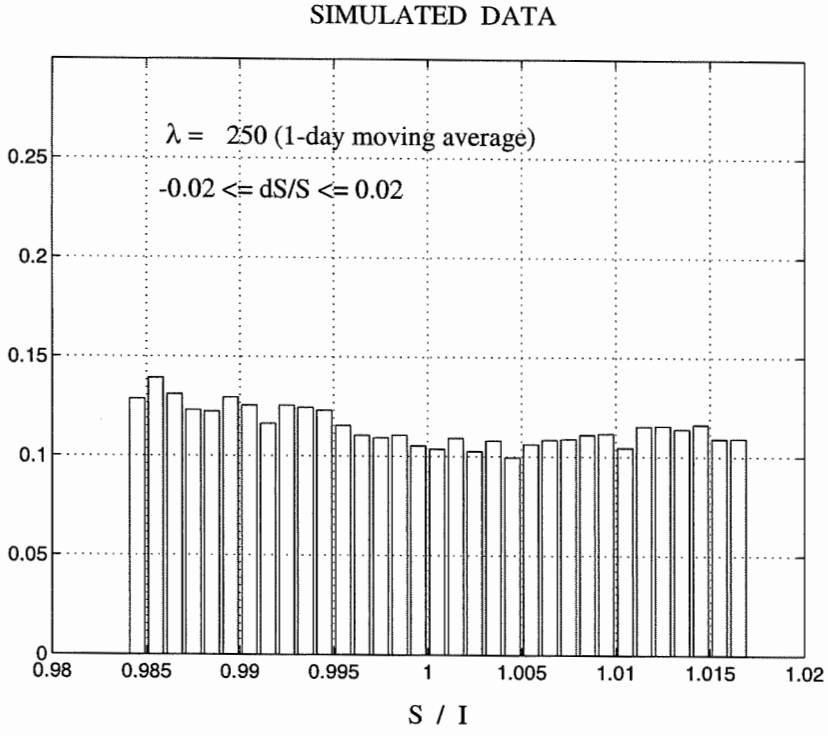


Figure 5.9: σ vs. S/I : 1-day moving average for the Dow Jones with high returns excluded.

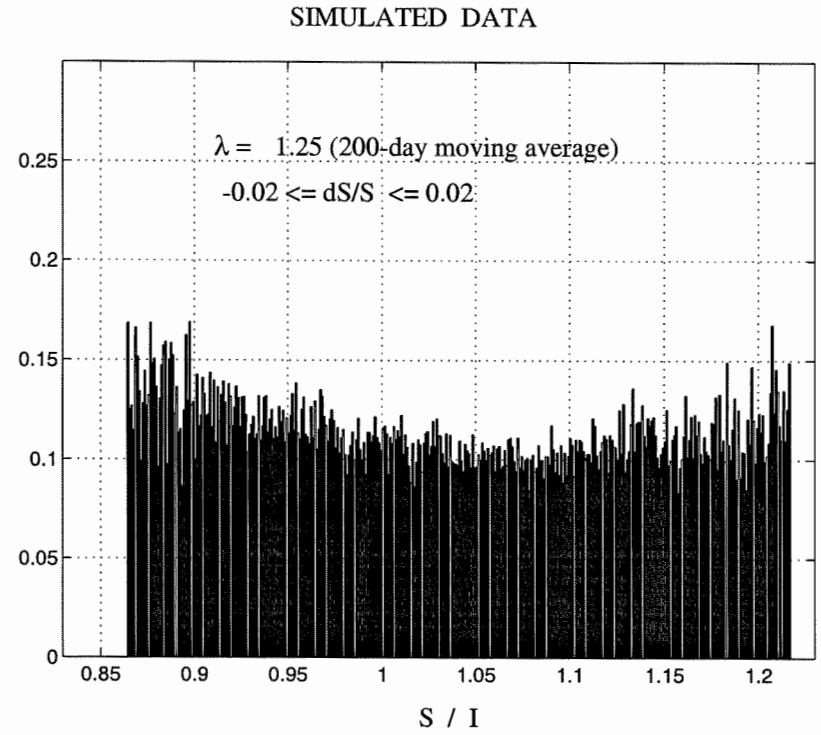


Figure 5.10: σ vs. S/I : 200-day moving average for the Dow Jones with high returns excluded.

Jump Probability : We also wanted to compare the probability of jumps evaluated from real data with the probability predicted by theory. The theoretical probability for a jump of F % or more is given by

$$N(-(\mathrm{F}/100)/\sigma\sqrt{\mathrm{d}t}), \tag{5.20}$$

which for a dt of $1/300$, σ of 0.16 and F of 5 % is 3.2×10^{-8}. For the DJIA data from 1901 until 1995 the frequency of a jump of greater than 5 % was approximately 2.1×10^{-3}. Thus, the real data contained more jumps than predicted by the theory.

Using this and the fact that the bucket σ's were reduced, after the removal of large returns, we could say that jumps in the Dow Jones index value were the 'probable' cause of the shape of the plots.

5.6 Isolation of Jump Point(s)

We needed to identify whether or not there was a certain value(s) of the return(s) where jumps were more likely to take place. Least squared fits were made to the data plots of σ vs. S/I and the coefficients of a linear or a quadratic equation were determined. The predicted σ (as a function of S/I) and the observed σ were compared to determine the **residual** from the fit. Formally, if the equation of a straight line is

$$y_i = a + b\,x_i + e_i, \tag{5.21}$$

and if $\hat{y}_i$ is the value predicted by the regression line (least squares fit) then

$$y_i = \hat{y}_i + e_i, \tag{5.22}$$

where e_i is the residual. Now, if the sample mean is defined as $\overline{y}$ then equation 5.22 can be written as:

$$(y_i - \overline{y}) = (\hat{y}_i - \overline{y}) + e_i, \tag{5.23}$$

where the left-hand-side is the observed deviation from the sample mean, the first term on the right-hand-side is the predicted deviation from the sample mean. The sum of the square of the residuals is then defined as

$$\sum_i^n (y_i - \overline{y})^2 = \sum_i^n (\hat{y}_i - \overline{y})^2 + \sum_i^n e_i^2, \tag{5.24}$$

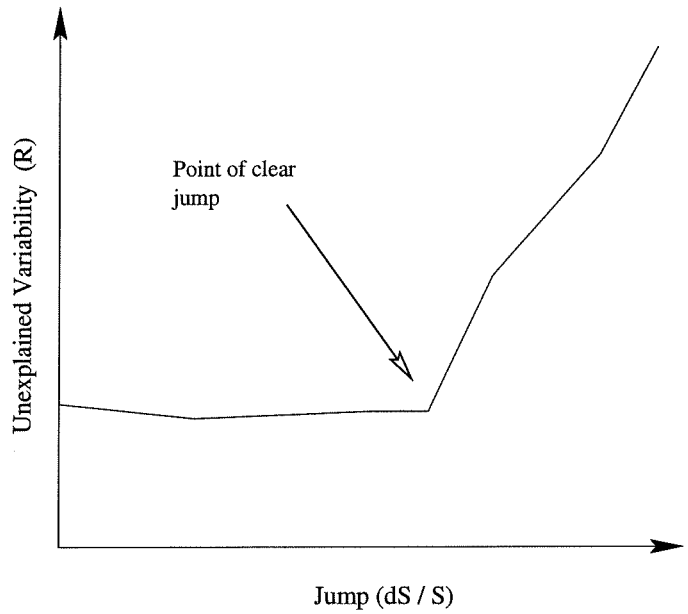


Figure 5.11: The expected behaviour of the residuals from the least squares fit to the data showing the fact that we anticipate a clear jump point.

where the left-hand-side is the total sample variability and the first term on the right-hand-side is the explained variable variability and the sum of the squares of the residuals is the unexplained variability. If there were to be a clear jump in the index value then if we plotted this quantity, defined as R, versus the return $\mathrm{d}S/S$ we would have expected the plot shown in Figure 5.11, but, instead we obtained the plots shown in Figures 5.12 and 5.13. Note that the returns in these plots are threshold returns. That is, a return value of 0.01 (in the plots) means that the residual was evaluated with returns of > 0.01 or < -0.01 excluded.

It is clear that irrespective of the time scale of the moving average the results were the same - the plots are 'approximately' linear, in that there is no clear point or rather value of the return where jumps were more likely to take place.

5.7 The Jump and Jump Diffusion Processes

Even though we were not able to find a particular point (or points) where a jump was most likely to occur, the evidence in the previous sections showed that jumps in the Dow Jones Index value were 'somehow' responsible for the shape of the data plots, Figures 5.1 - 5.5. The first step is to ask the question: "How are jumps in stock prices modelled ?"

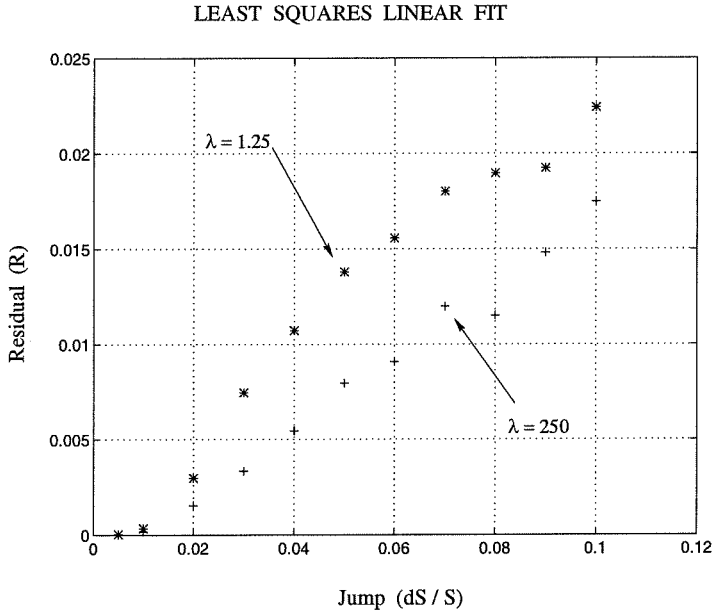


Figure 5.12: Residuals for a linear least squares fit to the data using 1-day ($\lambda = 250$) and 200-day ($\lambda = 1.25$) moving averages.

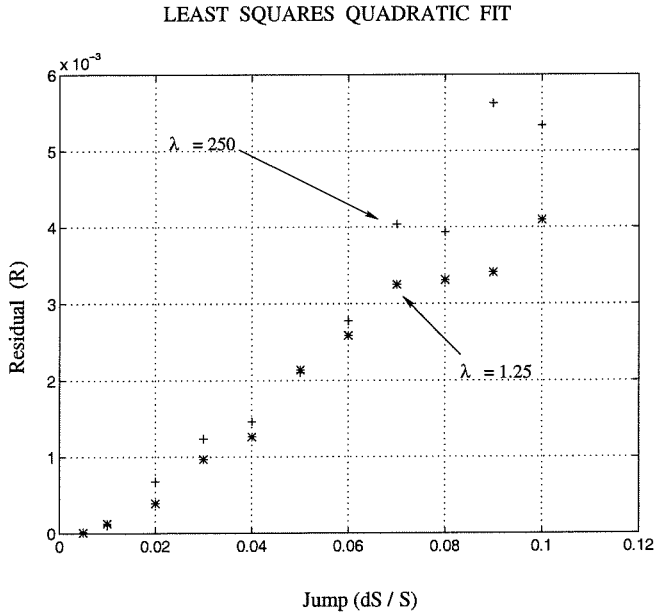


Figure 5.13: Residuals for a quadratic least squares fit to the data where the time scales of the moving averages are the same as the ones in the figure above.

The Black Scholes model (Section 1.3) assumes that stock prices follow a continuous sample path and in particular a continuous time diffusion process. But real markets do not possess continuous sample paths: prices are recorded at discrete times and specified to the nearest penny. Although such models are theoretically useful, they do not always agree with observation. Note that it was shown (Section 5.5) that the Dow Jones data contained more jumps than theory predicted. Hence, we would like a model which not only allows stock prices to follow a diffusion process but also allows for occasional jumps (in the stock price) to occur.

The total change in the stock price is said to comprise two types of changes:

- The first is the "normal" variations in price due, for example, to a temporary imbalance between supply and demand, changes in the economic outlook or new information that causes marginal changes in the stock's value. The impact of such information per unit time is to produce a marginal change in the price. This component is modelled by a standard **Geometric Brownian Motion** with constant variance per unit time and it has a continuous sample path.

- The second is the "abnormal" variations in price caused by the arrival of important new information about the stock that has more than a marginal effect on the price. Most of the time such information would be specific to the firm or industry. But such information arrives only at discrete points in time. This component has to be modelled by a **jump process**[2] which reflects the non-marginal effect of the information.

Just as the natural prototype process for the continuous component of the stock price change is the Wiener process, so the prototype for the jump component is a **Poisson process** [39].

For a time interval of $(t, t + \mathrm{d}t)$, the Poisson process (usually written as $\mathrm{d}\pi$) is defined as

$$\mathrm{d}\pi = \begin{cases} 1 & \text{with probability } \alpha\,\mathrm{d}t \\ 0 & \text{with probability } 1 - \alpha\,\mathrm{d}t, \end{cases} \tag{5.25}$$

where the successive increments (or arrivals of information) are independent and identically distributed and that no more than one jump can occur in any interval. The number α is the frequency, or better known as the **rate** or **intensity**, of the

[2]See for example Cox and Ross [36] and Merton [37] or [38].

jumps. Note that the mean and variance of $d\pi$ are both $\alpha\,dt$ (ignoring $\mathcal{O}((dt)^2)$ terms).

Given that the Poisson event occurs, that is an important piece of information arrives, then a random number is drawn from a distribution which determines the impact of this information on the stock price. Thus, if at time t the stock price is $S(t)$ and J is the random number, then the stock price at time $t+dt$ is $S(t+dt) = J\,S(t)$. And so, for the pure jump process the stock price return can be written as

$$\frac{dS}{S} = (J-1)\,d\pi, \tag{5.26}$$

where J is the jump amplitude and $d\pi$ is the Poisson process with the properties given in equation 5.25. Note, that both J and $d\pi$ are independent. J is drawn from a normal distribution mean 0 and standard deviation σ_J, N(0,σ_J), and successive jump amplitudes are independent drawings from the same distribution. A sample path generated by equation 5.26 is shown in Figure 5.14. This is certainly unrealistic for an equity such as the Dow Jones Index.[3] For a formal definition and an "interesting" example of a jump (Poisson) process, see Appendix A.

This pure jump process can be extended to model stock prices that vary in the usual (diffusive) way but also have the occasional unpredictable jumps. In order to do so, we simply combine the pure diffusion model (equation 1.2) and the pure jump model (equation 5.26), to give the **jump diffusion** process where the stock price return is defined as

$$\frac{dS}{S} = \mu\,dt + \sigma\,dX + (J-1)\,d\pi, \tag{5.27}$$

where dX (Wiener process), J (jump amplitude) and $d\pi$ are all independent. A Monte Carlo simulation using this equation is shown in Figure 5.15. Using the jump diffusion recipe, we generated stock prices that were 'passed' through the σ versus S/I analysis. But this was to no avail, as we did not see the quadratic shape observed in the Dow Jones plots (Figures 5.1 - 5.5).

5.8 σ vs. S/I: A Final Word

What became obvious during our investigations of the question: "Is Volatility a Function of S/I ?" is that jumps in the Dow Jones Index values were certainly

[3]Note that such a model could be used for modelling base rates.

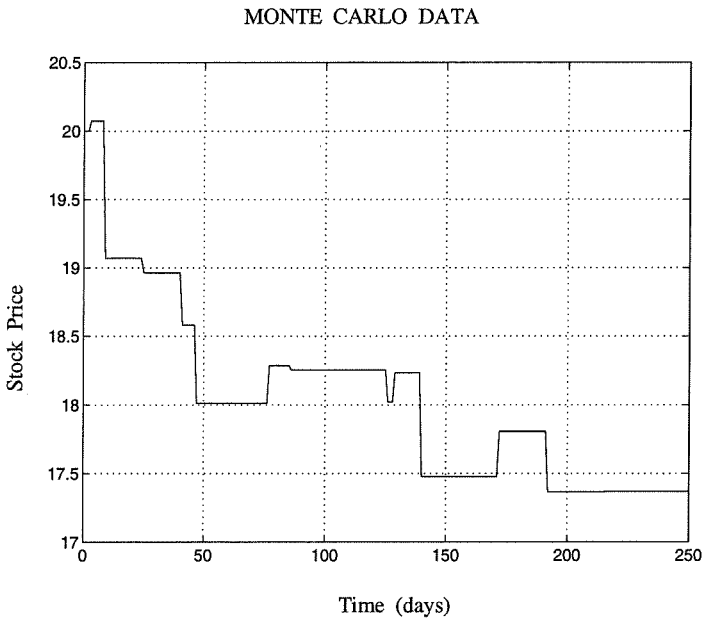


Figure 5.14: A simulation of the pure jump process (equation 5.26). A dt of 1/250 and an α of 12 were used giving the effect of 12 jumps in 250 days. The jump amplitudes, J, were determined from drawings from a normal distribution of mean 0 and a standard deviation of 0.03. This figure and the figures below were generated using the computer codes given in Appendix B (Section B.3).

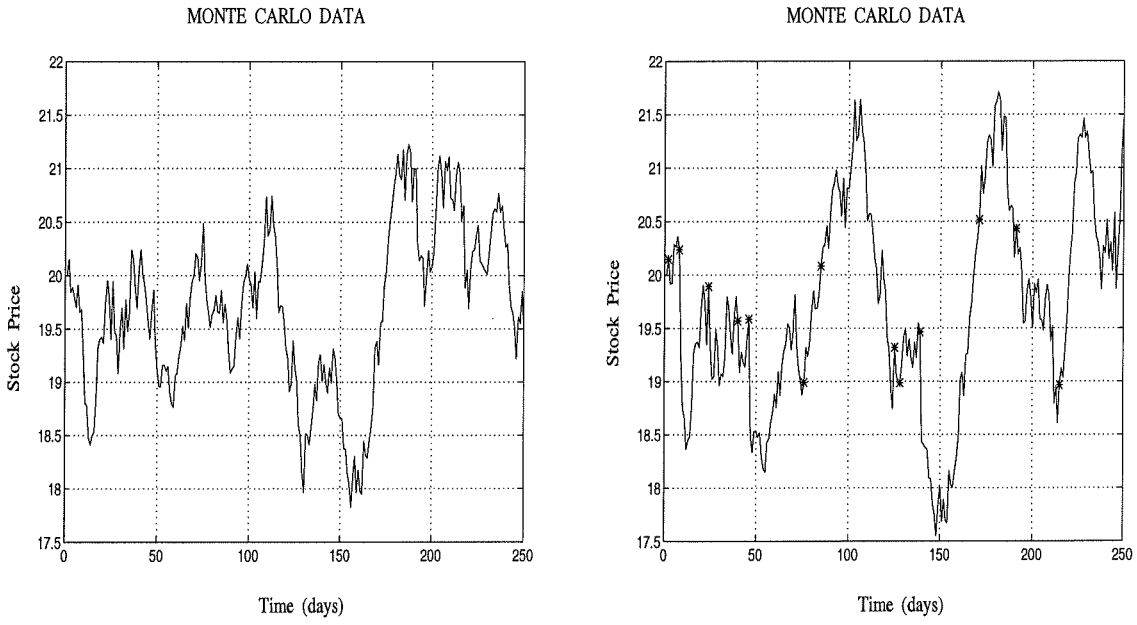


Figure 5.15: A simulation of the jump diffusion process (right plot) (equation 5.27). For comparison the same plot with no jumps is shown (left plot). The α and dt were the same as the figure above and a σ of 20 % and μ of 15 % were used. The stars in the right plot represent jump points.

responsible for the quadratic shapes of the σ versus S/I plots. The question then was: "How ?" A possible scenario is as follows:

- If a jump of a certain magnitude takes place the corresponding jump in the moving average is not of the same size. This effect is particularly enhanced in the long term moving averages which are 'far away' from the spot price (DJIA).

- Once a significant jump has taken place the next contribution to a bucket σ would be on the extremities of the σ versus S/I plots. That is, the particular data point (the one associated with the big jump) would be either on the extreme right or the extreme left of $S/I = 1$ line. The reason for this is that, when a sizeable jump in S takes place the corresponding jump in I would not be of the same magnitude - due to the effect of exponential smoothing associated with I. Hence, the ratio S/I would be either be $\gg 1.0$ (if S jumps up) or $\ll 1.0$ (if S jumps down).

- Finally, if a large jump were immediately followed by a jump of a similar magnitude, the effect would be an increase of the bucket σ. After a large jump, S/I is far away from 1. Thus, price movements after such a jump are at the extremities. Hence, if a jump is immediately followed by a jump of a similar magnitude then it would be placed in the same S/I bin as the first jump. The net effect is an increase in the average σ of that bin - that is, an increase in the bucket σ.

These possibilities were tested out on the DJIA data. A σ versus S/I plot was made for an I of 25 (10-day moving average) with returns > 2 % immediately followed by returns > 2 % excluded. That is 728 returns were not used out of a total of 26030. The resulting plot is shown in Figure 5.16. Comparing this figure with Figure 5.2 it is obvious that it is some what less pronounced, in that the bucket σ's at the left and right edges (in S/I) are smaller in magnitude. Note that both the plots are on exactly the same scale. A similar reduction in the values of the bucket σ's was also observed for the other time scales. That is, for the moving averages with λ's of 250 (1-day), 12.5 (20-day), 2.5 (100-day) and 1.25 (200-day).

To test the hypothesis that a jump followed by a jump affects the σ versus S/I plots, we carried out two Monte Carlo checks in which data was simulated with:

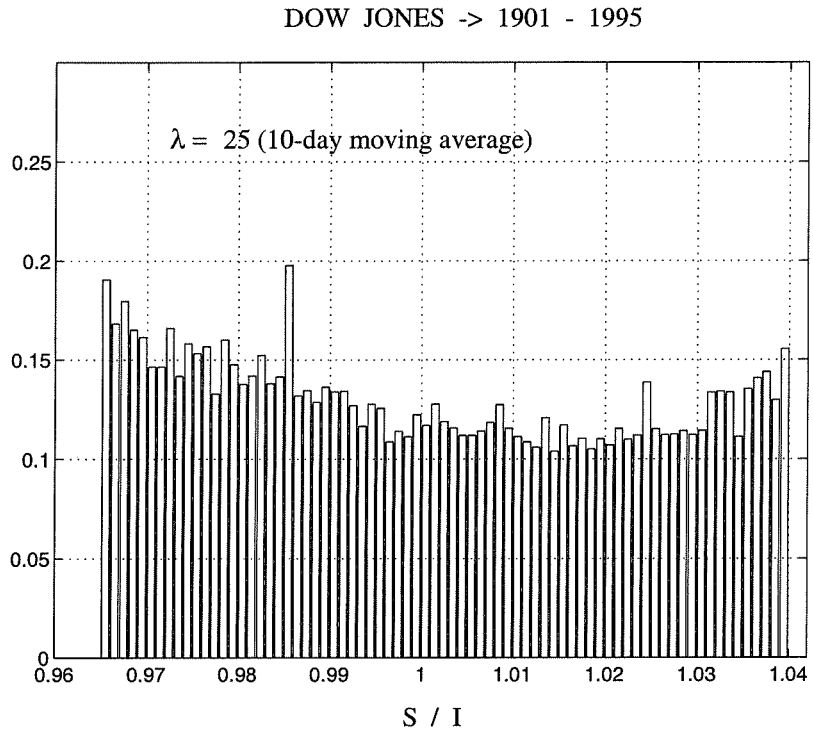


Figure 5.16: σ vs. S/I : 10-day moving average. Exactly the same as Figure 5.2 but with the vital difference that all returns > 2 % immediately followed by returns > 2 % have been removed.

- the jump frequency (α), in the first case, being made a function of S/I;
- and in the second case, being increased immediately after a jump had occurred.

5.8.1 Monte Carlo Check: $\alpha(S/I)$

Data was simulated, using equation 5.27, with the difference that the frequency of the jumps, α, in the jump process (Poisson process) was made to be a function of S/I. That is, the $\mathrm{d}\pi$ in equation 5.27 became

$$\mathrm{d}\pi = \begin{cases} 1 & \text{with probability } \alpha(S/I)\,\mathrm{d}t \\ 0 & \text{with probability } 1-\alpha(S/I)\,\mathrm{d}t, \end{cases} \tag{5.28}$$

A simple form used for S/I was

$$\alpha(S/I) = a_1\,(S/I)^2 + a_2\,(S/I) + a_3, \tag{5.29}$$

where a_1, a_2 and a_3 are constants. The constants were evaluated once the maximum and minimum values of $\alpha(S/I)$ were determined. It was difficult to estimate α from real data, as it is not clear whether to classify a given large change as a jump or a rare large change in the diffusive process. We applied a cut-off at a day-on-day return of $\pm$ 3.0 standard deviations - that is, a change of more than this was deemed

to be a jump. Thus, the minimum and maximum values of $\sigma(S/I)$ were estimated to be

$$\begin{aligned} \alpha_{min}\,\mathrm{d}t &= \frac{\#\,\text{double jumps}}{\text{total}} \\ &= \frac{269}{26030} \\ \alpha_{max}\,\mathrm{d}t &= \frac{\#\,\text{double jumps}}{\#\,\text{single jumps}} \\ &= \frac{269}{505}, \end{aligned} \tag{5.30}$$

where

- total is the total number of returns.
- double jumps is the number of jumps ≥ 3 % immediately followed by jumps ≥ 3 %.
- single jumps is the number of jumps ≥ 3 %.
- $\mathrm{d}t$ is $1/250$.

The resulting plot of $\alpha(S/I)$ versus S/I is shown in Figure 5.17. Monte Carlo data was simulated using this function and passed through the σ versus S/I analysis. The plots generated are shown in Figure 5.18. The two plots in Figure 5.18 were generated with the same parameters except for the value of the jumps amplitudes. The jump amplitudes were determined by random drawings from the normal distribution with mean 0 and a standard deviation σ_J. The σ_J for the two plots were 0.03 for Figure 5.18 (left) and 0.05 for Figure 5.18 (right). These figures do not resemble exactly the σ versus S/I plot for the 10-day moving average (Figure 5.2), but the presence of the quadratic shape in Figure 5.18 (left and right plots) should be noted.

Thus, it does appear that a jump of a large magnitude followed immediately by a jump of a similar magnitude causes the quadratic shapes that we observed in the σ versus S/I plots.

To present this argument from a different angle, the maximum and minimum values of $\alpha(S/I)$ were changed to be 250 and 25. This is shown in Figure 5.19. This

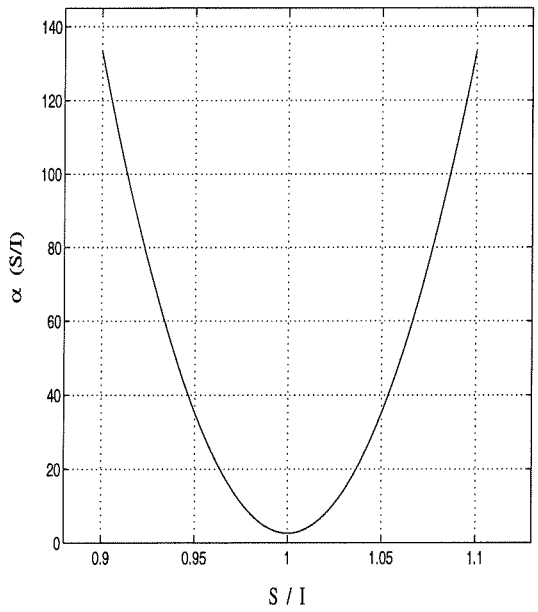


Figure 5.17: α as a function of S/I where the minimum and maximum values have been determined using equation 5.30.

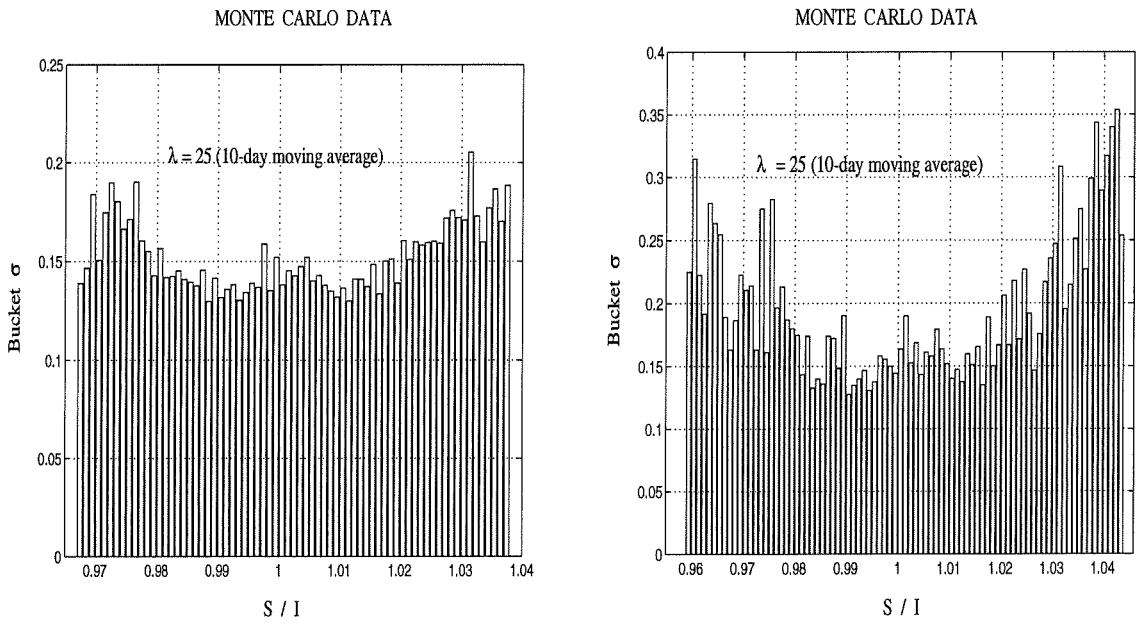


Figure 5.18: Simulations of jump-diffusion (equation 5.27) where the jump frequency is dependent on S/I (equation 5.28). The parameters are: $\mu = 6$ %, $\sigma = 13$ % (starting), $dt = 1/250$. For the left plot the standard deviation of the drawing for the jump amplitude is 0.03 and for the right plot it is 0.05.

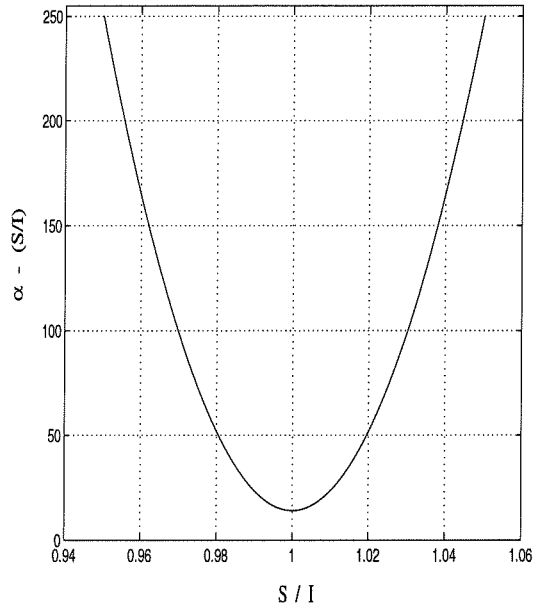


Figure 5.19: α as a function of S/I where the maximum values have been chosen in such a way that the probability of a jump followed by a jump is 1.

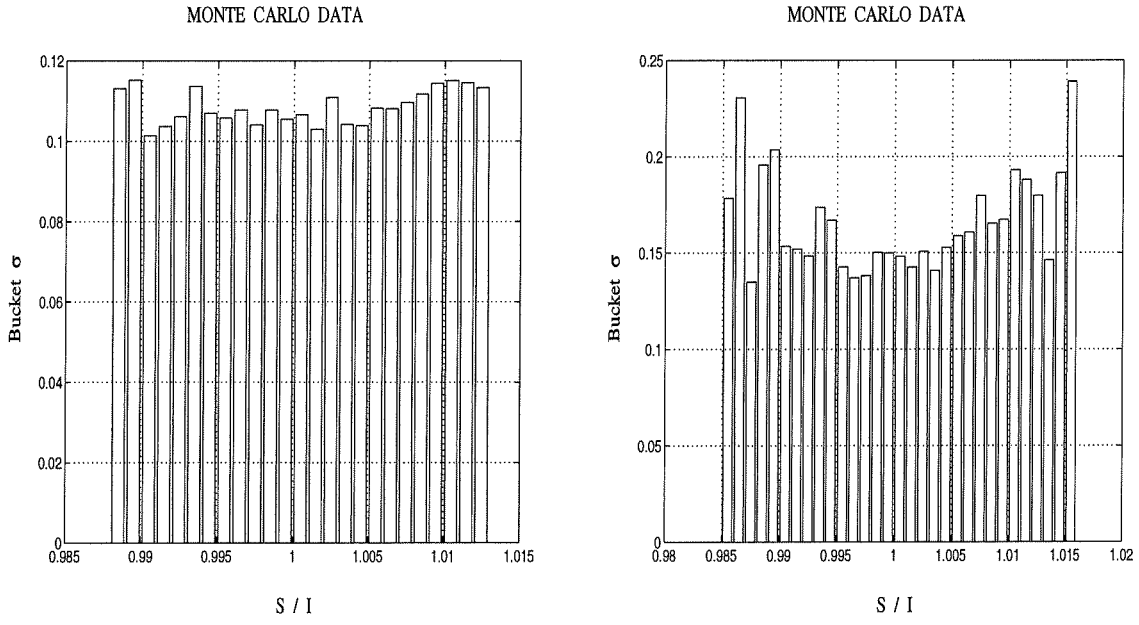


Figure 5.20: Simulations of jump-diffusion (equation 5.27) where the jump frequency is dependent on S/I for a 1-day ($\lambda = 250$) moving average. The parameters are: $\mu = 6$ %, $\sigma = 10$ % (starting), $dt = 1/250$. For the left plot the standard deviation of the drawing for the jump amplitude is 0.01 and for the right plot it is 0.03.

meant that at the left and right extremities of the plot the maximum values became

$$\begin{aligned} \alpha_{max}\, \mathrm{d}t &= 250 \times \frac{1}{250} \\ &= 1. \end{aligned} \tag{5.31}$$

That is, the probability of a very large jump immediately followed by another very large jump (previously determined from the Dow Jones data) was forced to be 1. In effect, if a large jump placed the data point in one of the left or right edge bins then this was immediately followed, with probability 1, by a jump of a similar magnitude (being placed in the same bin). The net result being that the σ in that particular bin or bucket increased. For a 1-day ($\lambda = 250$) moving average and σ_J's of 0.01 and 0.03, the plots of the bucket σ versus S/I are shown in Figure 5.20.

It is obvious from these plots that jumps followed by jumps certainly influence the volatility, σ. The relevant computer code is given in Appendix B (Section B.4).

5.8.2 Monte Carlo Check: Increased Jump Frequency

In this check, data was simulated with the jump frequency, α, being increased immediately after a jump had taken place. In effect, once a jump had occurred we 'forced' another to take place.

Firstly, we generated data using a volatility of 10 %, a drift rate of 7 % and a jump frequency of 12 (per 250 day trading year). If a jump had occurred, this frequency was increased to 150, otherwise it remained at its original value. The jumps were determined from drawings from a normal distribution with a standard deviation of 3 %. Finally, the σ versus S/I analysis was performed on this simulated data. One such plot, for the 1-day moving average ($\lambda = 250$) is shown in Figure 5.21. Comparing this with Figure 5.1 for the Dow Jones data, it can be seen that the agreement between the two is satisfactory. Thus, on this evidence and from the results given in the previous section, it can be said (with some certainty) that the scenario of a jump followed by a jump affecting the σ, is a satisfactory one.

5.9 Mirroring the DJIA and Stochastic $d\mu$

If we carefully observe the price graph of the DJIA (Figure 4.3), it appears that there are times when the index rises and falls in a partly predictable manner. For

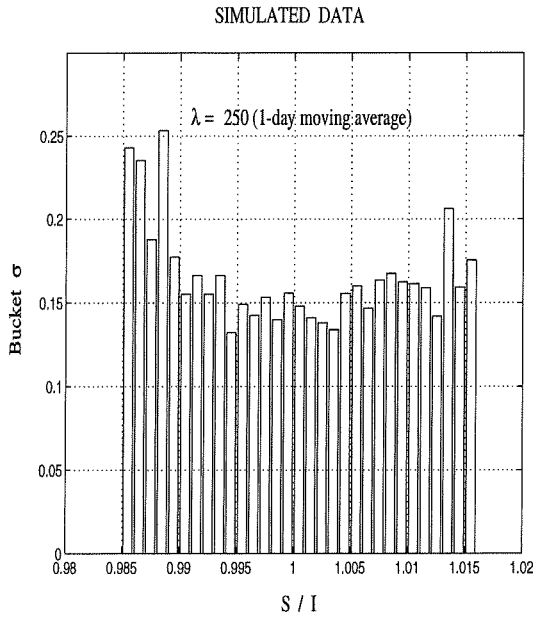


Figure 5.21: σ vs. S/I: 1-day ($\lambda = 250$) moving average. The jump frequency, in the simulated data, immediately following a jump was increased from 12 (per 250 days) to 150. There is satisfactory agreement between this plot and Figure 5.1.

a magnification of this plot see Figure 5.22. From within this region, if we plot the change in the index value, that is dS, versus time, this periodic rise and fall is more apparent (see Figure 5.22). Note that on average the magnitude of the rise is smaller than that of the fall. In order to simulate the behaviour shown in these two plots we first have to take a brief re-look at the lognormal random walk, equation 1.2:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}X. \tag{5.32}$$

If we were dealing with a stock that had no volatility (an unlikely situation), that is σ is 0, then this equation would reduce to an ordinary differential equation which could be solved to give

$$S = S_0\, e^{\mu(t-t_0)}, \tag{5.33}$$

where S_0 is the value of the stock at $t = t_0$. Thus, the stock price, which is totally deterministic in such a case, would rise exponentially. Now, if the volatility was not 0 then there would be randomness in the plot but in the long run there would be an ultimate rise in the prices (as shown in Chapter 1, Figure 1.1).

It is the drift, μ, which to some extent controls this ultimate rise in the prices. Hence, to obtain repeatability (as shown by the Dow Jones data) in simulated stock prices, we wanted μ to vary in a manner similar to Figure 5.22 (right plot). This

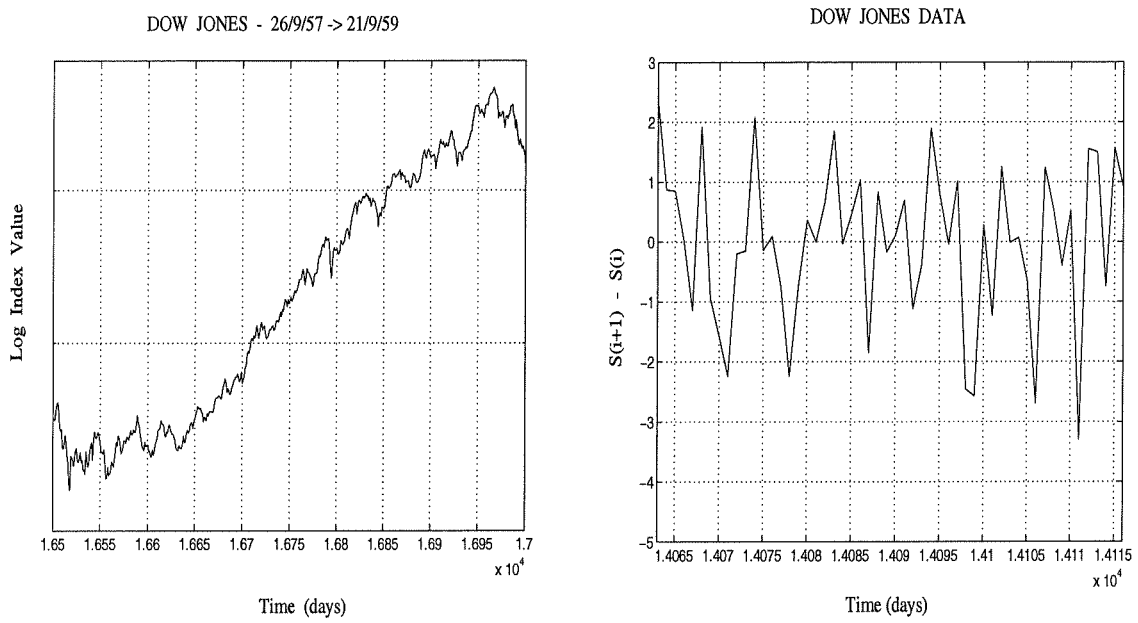


Figure 5.22: The Dow Jones rising and falling in a repeatable pattern (left plot). The right plots shows the change in the index value versus time. Note the periodic rise and fall in the index.

was attempted by letting μ follow the s.d.e.:

$$\mathrm{d}\mu = a\left[(1-\frac{\mu}{a})\,\mathrm{d}\pi - b\,\alpha\,\mathrm{d}t\right], \tag{5.34}$$

where

- a and b are constants.
- $\mathrm{d}\pi$ is Poisson process with the properties given in equation 5.25.
- α is the frequency of the jumps.
- $\mathrm{d}t$ is the time step.

A simulation using equation 5.34 is shown in Figure 5.23. Note that the pattern in this figure is similar to the one in Figure 5.22 (right plot).

Stock prices were generated using this equation and equation 5.32 with a volatility of 0 and 15 %. It can be seen that we managed to generate a repeatable pattern for the 0 volatility case, Figure 5.24 (left), but the introduction of volatility totally masked this effect.

This masking was eradicated by increasing the absolute magnitude of the simulated μ, but at the same time keeping the average (μ) low: see Figure 5.25. This did

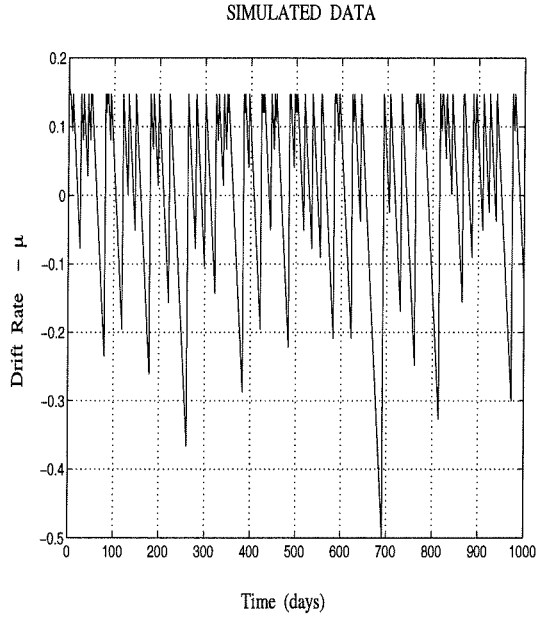


Figure 5.23: The drift rate used to induce repeatability in S.

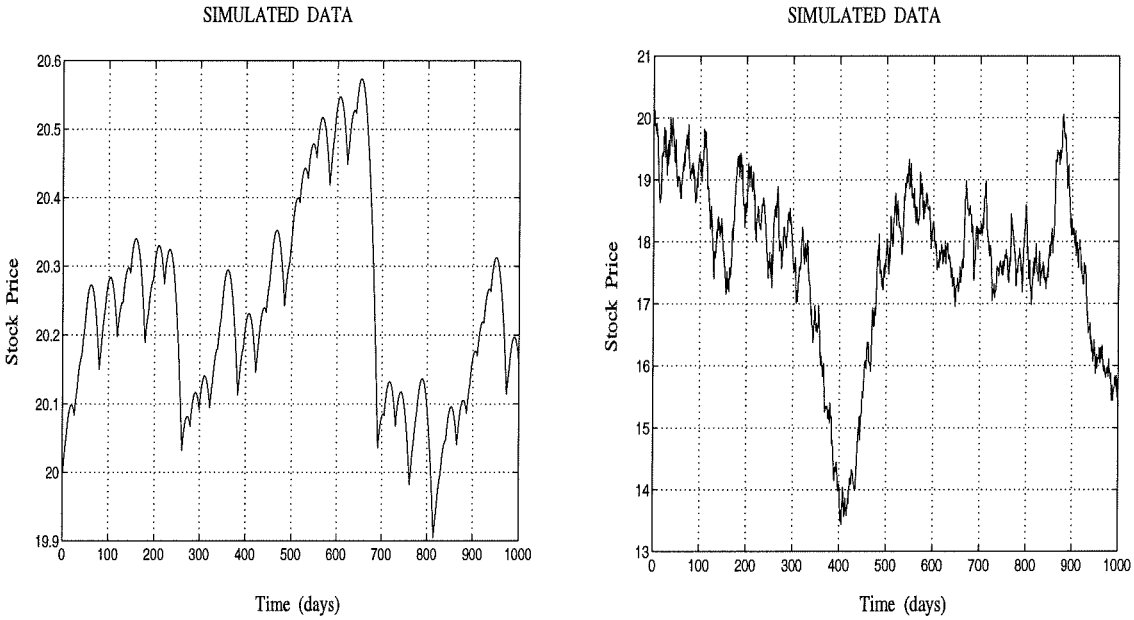


Figure 5.24: The simulated S's using no volatility (left) and a volatility of 20 % (right). The other parameters are $dt = 1/365$, $\alpha = 30$, $a = 0.15$, $b = 1$ and a starting μ of 15 %. Note that no repeatable patterns are observed in the right plot.

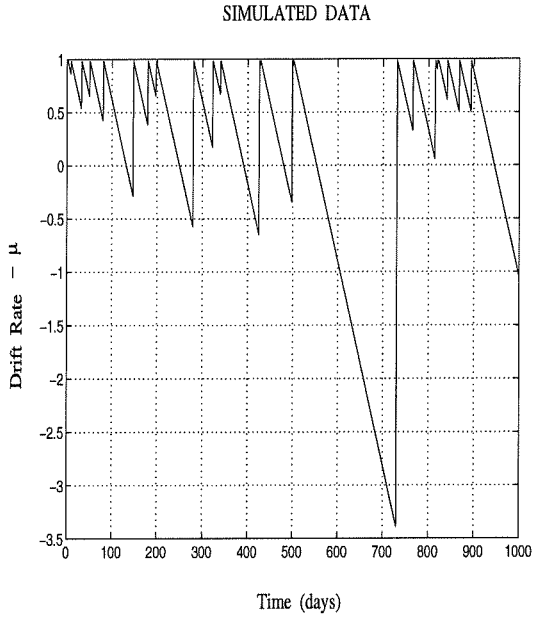


Figure 5.25: Much larger (compared to Figure 5.23) positive and negative values for the drift. But, the absolute average is only 0.7 %.

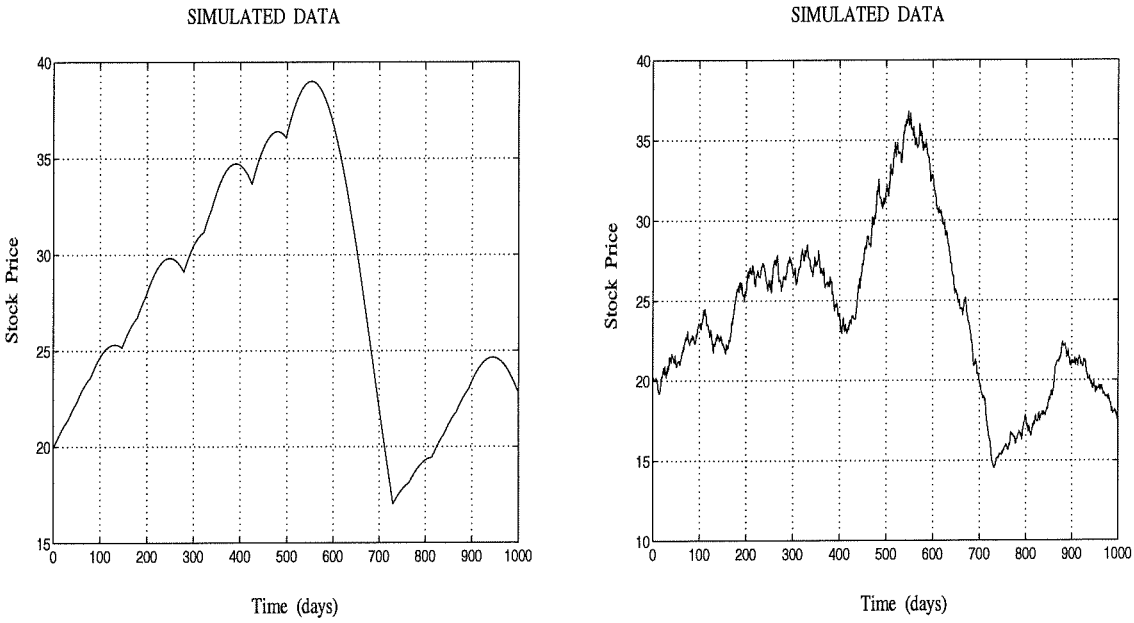


Figure 5.26: Simulated stock prices using the same parameters as Figure 5.24. Note the resemblance of the right and left plots.

induce the repeatable effect that we were trying to generate in S (see Figure 5.26). For a listing of the relevant computer program, see Appendix B (Section B.5).

Thus, we found that by allowing the drift to exhibit the stochastic behaviour given in equation 5.34, we were able to replicate the periodic rise and fall which was observed in the Dow Jones price graph.

If μ alone varies in the manner discussed above, the Black-Scholes equation would include terms involving μ derivatives. But, as the payoff never depends on μ (and μ is not in the 'usual' Black-Scholes equation), we would look for a solution independent of μ which satisfies the p.d.e.

Chapter 6

Numerical Analysis

Chapter Abstract

The 2-D Black-Scholes model is extended to include not only the stock price and time but also a moving average term. The essentials of the finite difference method for solving p.d.e.'s are discussed. An explicit finite difference method is used to transform (to a set of finite difference equations) and solve numerically the extended Black-Scholes p.d.e. The numerical solutions (of this p.d.e.), for different boundary conditions, are checked against the corresponding analytical results. Also, a moving average dependent option is devised and solved numerically.

6.1 The Black-Scholes Equation and the Moving Average

In Chapter 1 (Section 1.3) we derived the Black-Scholes partial differential equation. In order to extend the Black-Scholes framework to include not only the asset price, S, and time, t, but also the moving average, I, we have to apply Itô's lemma to the function $V(S, I, t)$.

$$V(S + \mathrm{d}S, I + \mathrm{d}I, t + \mathrm{d}t) = V + \mathrm{d}S\, V_S + \mathrm{d}t\, V_t + \mathrm{d}I\, V_I + \frac{1}{2}\mathrm{d}S^2\, V_{SS} + \ldots, \quad (6.1)$$

where the subscripts represent partial derivatives. Using equation 1.2 and 1.9 we obtain

$$\mathrm{d}V = \sigma\, S\, V_S\, \mathrm{d}X + \left(\frac{1}{2}\sigma^2\, S^2\, V_{SS} + \mu\, S\, V_S + V_t\right) \mathrm{d}t + \mathrm{d}I\, V_I. \quad (6.2)$$

This equation has been derived in the same manner as equation 1.10. It should be noted the new term, V_I, does not give rise to any new stochastic terms in the random walk followed by V.

Just as before, create a risk-free portfolio, comprising a long position in one option and a short position of Δ (equal to V_S) in the underlying. Using arbitrage arguments and equation 3.5 we arrive at the equation

$$V_t + \lambda (S - I) V_I + \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_S - r V = 0. \tag{6.3}$$

As it stands this is an equation backward in time and it is convenient to change it into a forward one. In order to do so we use a change of variable

$$t = -\tau. \tag{6.4}$$

Thus, equation 6.3 can be re-written as

$$-V_\tau + \lambda (S - I) V_I + \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_S - r V = 0. \tag{6.5}$$

This transformation of the governing p.d.e. into a forward p.d.e. allows us to impose an initial condition at $\tau = 0$ rather than a final condition had we used equation 6.3.

6.2 Finite Difference Methodology

In order to evaluate our governing p.d.e. numerically, we have to take into consideration the following:

- We need to construct a finite approximation to the p.d.e. which can be solved using a computer. This is termed the **discretisation** of the problem.
- We have to check for **stability**. That is, is the discretised model sensitive to small errors introduced through the use of finite precision computer mathematics ?
- We have to test whether the solutions to the discretised models approximate the solutions of the p.d.e. This is the problem of **convergence**.

Finite difference methods value a derivative security by solving the governing p.d.e. numerically. The differential or partial differential equations are converted into a set of difference equations and these are solved numerically.

The first step is to divide S into equally spaced nodes a distance δS apart, I into equally spaced nodes a distance δI apart, and τ into equally equally spaced

nodes a distance $\delta\tau$ apart. Envisaging this as a 3-D plot, the division has the effect of dividing the (S, I, τ) plane into a mesh where the mesh points have the form $(S\,\delta S, I\,\delta I, \tau\,\delta\tau)$. Thus, the exact solution of the p.d.e. can be written as

$$V(S, I, \tau) = U^n_{l,m} = U(l\delta S, m\delta I, n\delta\tau). \tag{6.6}$$

6.2.1 Explicit Finite Difference Method

All the partial derivatives in equation 6.5 can be reduced to corresponding difference equations. We start off with the time derivative, V_τ.

Time Derivative, V_τ : Taylor expanding $V(S, I, \tau + \delta\tau)$

$$V(S, I, \tau + \delta\tau) = V(S, I, \tau) + V_\tau\,\delta\tau + \mathcal{O}((\delta\tau)^2), \tag{6.7}$$

or

$$V_\tau = \frac{V(S, I, \tau + \delta\tau) - V(S, I, \tau)}{\delta\tau} + \mathcal{O}((\delta\tau)). \tag{6.8}$$

Using equation 6.6 this can be written as

$$V_\tau \approx \frac{U^{n+1}_{l,m} - U^n_{l,m}}{\delta\tau}. \tag{6.9}$$

This particular approximation is termed a **forward difference** as the differencing is in the forward τ direction.

Diffusion Term, V_{SS} : Repeating the same process as above we obtain (using a forward difference in S):

$$V(S + \delta S, I, \tau) = V(S, I, \tau) + V_S\,\delta S + \frac{1}{2}V_{SS}\,(\delta S)^2 + \frac{1}{3!}V_{SSS}\,(\delta S)^3 + \mathcal{O}((\delta S)^4), \tag{6.10}$$

and using a **backward difference** in S

$$V(S - \delta S, I, \tau) = V(S, I, \tau) - V_S\,\delta S + \frac{1}{2}V_{SS}\,(\delta S)^2 - \frac{1}{3!}V_{SSS}\,(\delta S)^3 + \mathcal{O}((\delta S)^4). \tag{6.11}$$

Adding these equations and re-arranging them gives

$$V_{SS} = \frac{V(S + \delta S, I, \tau) - 2\,V(S, I, \tau) + V(S - \delta S, I, \tau)}{(\delta S)^2} + \mathcal{O}((\delta S)^2). \tag{6.12}$$

This can be re-written as:

$$V_{SS} \approx \frac{U^n_{l+1,m} - 2\,U^n_{l,m} + U^n_{l-1,m}}{(\delta S)^2}. \tag{6.13}$$

Convection Term, V_S : As before (for V_τ) we can use a forward difference and write V_S as:

$$V_S \approx \frac{U^n_{l+1,m} - U^n_{l,m}}{\delta S}. \tag{6.14}$$

But this approximation is only accurate to $O(\delta S)$, whereas the approximation for V_{SS} is accurate to $O((\delta S)^2)$. This could reduce the accuracy of the scheme and effect the stability as well. Thus, we want to approximate V_S to order $((\delta S)^2)$. We see that

$$V(S+\delta S, I, \tau) = V(S, I, \tau) + V_S\,\delta S + \frac{1}{2} V_{SS}\,(\delta S)^2 + \mathcal{O}((\delta S)^3), \tag{6.15}$$

and

$$V(S+2\delta S, I, \tau) = V(S, I, \tau) + 2\,V_S\,\delta S + 2\,V_{SS}\,(\delta S)^2 + \mathcal{O}((\delta S)^3), \tag{6.16}$$

where the partial derivatives are evaluated at (S, I, τ). Eliminating V_{SS}, we have

$$V_S = \frac{4\,V(S+\delta S, I, \tau) - 3\,V(S, I, \tau) - V(S+2\delta S, \tau)}{2\delta S} + \mathcal{O}((\delta S)^2), \tag{6.17}$$

which can be approximated by

$$V_S \approx \frac{4\,U^n_{l+1,m} - 3\,U^n_{l,m} - U^n_{l+2,m}}{2\delta S}. \tag{6.18}$$

We know that

$$\delta S = \frac{S}{l_{max}}. \tag{6.19}$$

Now, if there is a boundary at $S = l_{max}\,\delta S$, the calculation of $U^n_{l,m}$ at $S = (l_{max} - 1)\,\delta S$ would involve the use of data from outside this boundary. In such cases, that is $S = (l_{max} - 1)\,\delta S$, we use equation 6.14.

Note that the problem arising at $S = (l_{max} - 1)\,\delta S$ can be also eliminated using a **central difference** in V_S. Subtracting equation 6.11 from equation 6.10 and dividing through by $2\,\delta S$ gives

$$V_S = \frac{V(S+\delta S, I, \tau) - V(S-\delta S, I, \tau)}{2\delta S} + \mathcal{O}((\delta S)^2), \tag{6.20}$$

which can be written as:

$$V_S \approx \frac{U^n_{l+1,m} - U^n_{l-1,m}}{2\delta S}. \tag{6.21}$$

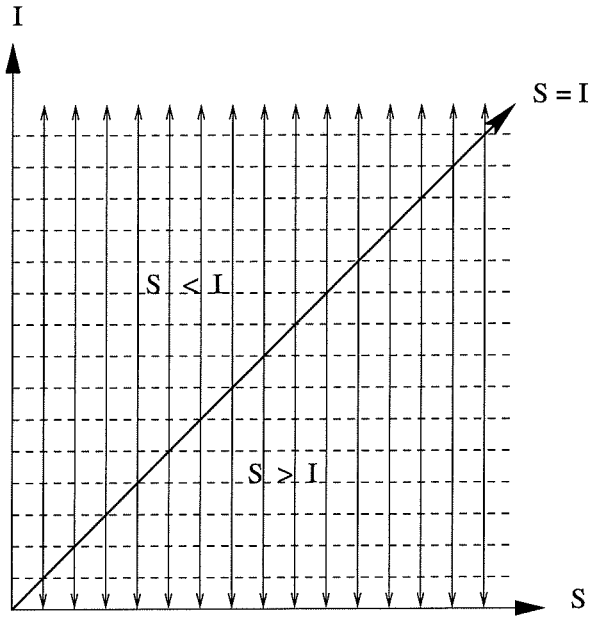


Figure 6.1: Plot showing the flow of information away from the line $S = I$ in the S, I plane.

Convection Term, V_I : Looking at equation 6.5 we see that for the term involving V_I there are two cases that we have to consider. Firstly, we have to difference V_I when $S > I$ and secondly, when $S < I$. This is because of the fact that at any time information flows away from the line $S = I$ as shown in Figure 6.1.

Using the same scheme as for V_S (equation 6.18), when $S > I$

$$V_I \approx \frac{4\,U^n_{l,m+1} - 3\,U^n_{l,m} - U^n_{l,m+2}}{2\delta I}, \tag{6.22}$$

and for $S < I$

$$V_I \approx \frac{4\,U^n_{l,m-1} - 3\,U^n_{l,m} - U^n_{l,m-2}}{-2\delta I}. \tag{6.23}$$

Collecting all the terms together we can now write down, in full, the explicit finite difference scheme which approximates our governing p.d.e. (equation 6.5):

$$\begin{aligned} \frac{U^{n+1}_{l,m} - U^n_{l,m}}{\delta\tau} &= \pm\lambda(S-I)\left(\frac{4\,U^n_{l,m\pm1} - 3\,U^n_{l,m} - U^n_{l,m\pm2}}{2\delta I}\right) \\ &+ \frac{1}{2}\sigma^2 S^2\left(\frac{U^n_{l+1,m} - 2\,U^n_{l,m} + U^n_{l-1,m}}{(\delta S)^2}\right) \\ &+ r\,S\left(\frac{4\,U^n_{l+1,m} - 3\,U^n_{l,m} - U^n_{l+2,m}}{2\delta S}\right) - r\,U^n_{l,m}, \end{aligned} \tag{6.24}$$

where the $\pm$ represent the two cases: $S > I$ and $S < I$. In order to solve this equation iteratively we only need to specify the initial condition at $\tau = 0$ and the two boundary conditions at $S = 0$ and $S = l_{max}\,\delta S$.

If, at time step n, we know $U_{l,m}^n$ for all l and m we can explicitly evaluate $U_{l,m}^{n+1}$. Hence, the name : explicit finite difference.

Numerical v.s Analytical Solution

We have to compare our numerical solution with an exact solution of equation 6.5. Firstly, in order to check the accuracy of our explicit finite difference scheme we look at the case when the volatility, σ, is constant. This has the effect of reducing equation 6.5 to the forward form of the Black-Scholes equation (as $V_I = 0$). The exact solutions of European call and put options, when the interest rate and volatility are constant, are (see for example Wilmott et al. [27]):

$$C(S,t) = S\,N(d_1) - E\,e^{-r(T-t)}\,N(d_2), \tag{6.25}$$

and

$$P(S,t) = E\,e^{-r(T-t)}\,N(-d_2) - S\,N(-d_1), \tag{6.26}$$

where $C(S,t)$ and $P(S,t)$ are the exact call and put solutions to the Black-Scholes equation and $N(.)$ is the cumulative distribution function for a standardised normal random variable given by

$$N(x) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,\mathrm{d}y. \tag{6.27}$$

Where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\,\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}}, \tag{6.28}$$

and

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\,\sigma^2)(T-t)}{\sigma\,\sqrt{T-t}}. \tag{6.29}$$

In order to solve equations 6.25 and 6.26, and compare with equation 6.5 we need to specify boundary conditions at $S = 0$ and $S = l_{max}\,\delta S$ and an initial condition at $\tau = 0$. For European calls and puts, these conditions are given in Table 6.1. Various combinations of S, E, σ, τ, and r were used to represent call and put options which were *out-of-the-money*, *at-the-money*, and *in-the-money*. Table 6.2 shows a comparison of the analytical solution with the numerical solution of equation 6.5 with σ not a function of S and I. Using the same S, E, σ, τ, and r as shown in Table 6.2 a comparison for various put options is shown in Table 6.3. Looking at Tables 6.2 and 6.3 it can be seen that the agreement between the numerical solution and the exact solution of equation 6.5, when σ is a constant, can be considered to be satisfactory. The computer code for this particular numerical scheme is given in Appendix B (Section B.6).

Option	$S = 0$	$S \to \infty$	$\tau = 0$
Call	$V(0,t) = 0$	$V(S,t) = S - Ee^{-r(T-t)}$	$\text{Max}(S-E,0)$
Put	$V(0,t) = Ee^{-r(T-t)}$	$V(S,t) = 0$	$\text{Max}(E-S,0)$

Table 6.1: The boundary conditions, columns 2 and 3, and the initial condition, column 4, for European call and put options. Where, in our finite difference formulation $S \to \infty$ is approximated by $S = l_{max}\,\delta S$.

Stock Price (S)	Strike Price (E)	Time to Expiry (τ)	Volatility (σ)	Interest Rate (r)	Analytical Solution	Numerical Solution
In-the-money						
100	80	0.50	0.20	0.06	22.5464	22.5495
110	100	0.33	0.15	0.10	13.5131	13.5208
90	50	0.25	0.35	0.14	41.7203	41.7204
At-the-money						
100	100	0.40	0.30	0.05	8.5147	8.5057
90	90	0.80	0.25	0.20	15.8830	15.8898
110	110	0.60	0.10	0.10	7.4447	7.4608
Out-of-the-money						
96	115	1.00	0.40	0.15	14.0080	14.0028
110	130	0.50	0.11	0.12	0.3484	0.3211
70	110	0.70	0.25	0.07	0.1839	0.1820

Table 6.2: Comparison of the numerical solution of equation 6.5 and the exact solution of the European call option, equation 6.25. The values of δS and $\delta\tau$ were 2 and 1/10000 respectively.

Numerical v.s Linear Solution

It should be noted that equation 6.5 also has a solution which is linear in S, I and τ. Let

$$V(S,I,\tau) = a\,S + b\,I + c, \tag{6.30}$$

where a, b and c are functions of τ. Substituting equation 6.30 into equation 6.5 we get

$$\begin{aligned} a_\tau - \lambda\, b &= 0, \\ b_\tau + b(r+\lambda) &= 0, \\ c_\tau + r\,c &= 0. \end{aligned} \tag{6.31}$$

These are three ordinary differential equations with the subscripts representing derivatives with respect to time, τ. Solving these equations we obtain:

$$a(\tau) = -\frac{b_0\,\lambda}{(r+\lambda)\,e^{(r+\lambda)\,\tau}} + a_0,$$

Analytical Solution	Numerical Solution
Out-of-the-money	
0.1821	0.1851
0.2669	0.2747
0.0005	0.0006
At-the-money	
6.5346	6.5256
2.5760	2.5826
1.0388	1.0554
In–the-money	
16.9894	16.9841
12.7778	12.7505
34.9238	34.9226

Table 6.3: Same variables as Table 6.2 but a comparison of the numerical solution with the exact solution for various put options. Same values as Table 6.2 were used for δS and $\delta\tau$.

$$\begin{aligned} b(\tau) &= \frac{b_0}{e^{(r+\lambda)\,\tau}}, \\ c(\tau) &= \frac{c_0}{e^{r\,\tau}}, \end{aligned} \tag{6.32}$$

where a_0, b_0 and c_0 are arbitrary constants. Thus, a linear solution for the governing p.d.e. is of the form

$$V(S,I,\tau) = \left(-\frac{b_0\,\lambda}{(r+\lambda)\,e^{(r+\lambda)\,\tau}} + a_0\right) S + \left(\frac{b_0}{e^{(r+\lambda)\,\tau}}\right) I + \frac{c_0}{e^{r\,\tau}}. \tag{6.33}$$

In order to compare this solution against the numerical one we need to specify an initial condition and two boundary conditions for equation 6.24. The initial condition is:

$$V(S,I,0) = \left(-\frac{b_0\,\lambda}{(r+\lambda)} + a_0\right) S + b_0\,I + c_0, \tag{6.34}$$

and the two boundary conditions at $S = 0$ and $S = l_{max}\,\delta S$ (or $S = S_{max}$):

$$V(0,I,\tau) = \left(\frac{b_0}{e^{(r+\lambda)\,\tau}}\right) I + c_0\,e^{-r\,\tau}, \tag{6.35}$$

and

$$V(S_{max},I,\tau) = \left(-\frac{b_0\,\lambda}{(r+\lambda)\,e^{(r+\lambda)\,\tau}} + a_0\right) S_{max} + \left(\frac{b_0}{e^{(r+\lambda)\,\tau}}\right) I + \frac{c_0}{e^{r\,\tau}} \tag{6.36}$$

are specified for each I. Now that we have the initial and boundary conditions (for equation 6.24) we can just substitute into equation 6.33 various values of S, I, τ, r

Interest Rate (r)	Time to Expiry (τ)	λ	S	I	Exact Result	Numerical Result
0.06	0.25	1.25	100	80	260.4718	260.4727
0.06	0.25	1.25	100	120	375.7875	375.7860
0.06	0.25	1.25	130	70	239.1173	239.1206
0.15	0.40	2.50	180	170	544.9703	544.9703
0.15	0.40	2.50	60	120	272.5648	272.5524
0.15	0.40	2.50	100	130	354.1278	354.1209
0.08	0.75	12.5	160	140	484.7028	484.7028
0.08	0.75	12.5	80	120	244.7218	244.7217
0.08	0.75	12.5	100	140	304.7218	304.7217

Table 6.4: For a linear solution of the governing p.d.e., a comparison of the exact results with ones obtained using our numerical scheme. The values of δS, δI and $\delta\tau$ were 2.0, 2.0 and 1/10000 respectively.

and λ and compare the results with the numerical results. It has to be noted that as this is a linear solution we can ignore the V_{SS} term in the governing p.d.e.. In other words, V_{SS} is 0 for all values of σ. A comparison of the results obtained using the numerical scheme and the corresponding results from the exact solution are shown in Table 6.4. For $r = 0.06, \tau = 0.25$ and $\lambda = 1.25$ a 3D surface was also plotted for both the exact solution and the numerical solution, see Figure 6.2.

6.3 Moving Average Dependent Option

Let us a define an option with the payoff, in our case as an initial condition at $\tau = 0$, of

$$V(S,I)\,|_{\tau=0} = \begin{cases} 0 & \text{for } 0 < S < \infty,\ 0 < I < B \\ 1 & \text{for } 0 < S < \infty,\ I = B. \end{cases} \tag{6.37}$$

That is, the option pays \$ 1 if the moving average reaches B before expiry and 0 otherwise. The boundary conditions are

$$V(S,I,\tau) = \begin{cases} 0 & \text{at } S = 0 \\ 1 & \text{as } S \to \infty, \end{cases} \tag{6.38}$$

which means that if $S = 0$ then V is bound to be 0 (the option is not exercised) and as $S \to \infty$ it is very likely that $I \geq B$. The 3-D plot for the initial condition is shown in Figure 6.3. The numerical solutions for these set of conditions and with an interest rate of 10 %, time to expiry of 1 year, B of 100 and λ's of 250 (1-day) and 1.25 (200-day) are shown in Figure 6.4. Note that as λ tends to ∞ the moving average converges to the spot price S. In this scenario the V_I term in

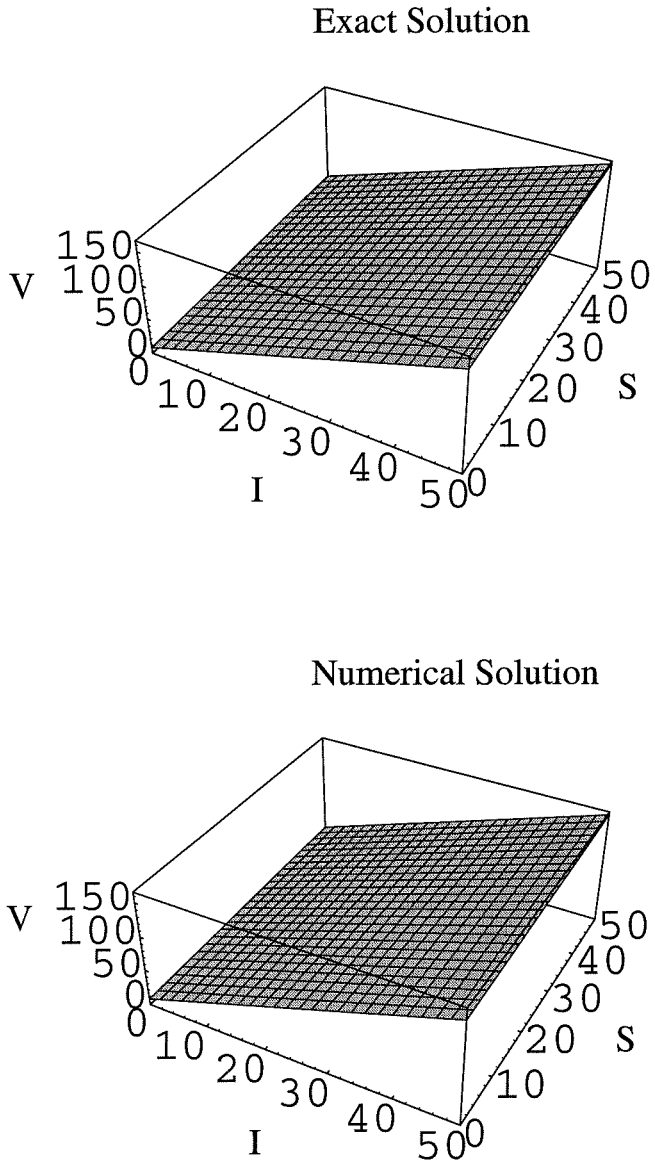


Figure 6.2: 3D surfaces, plotted using Mathematica, for $r = 0.06, \tau = 0.25$ and $\lambda = 1.25$.

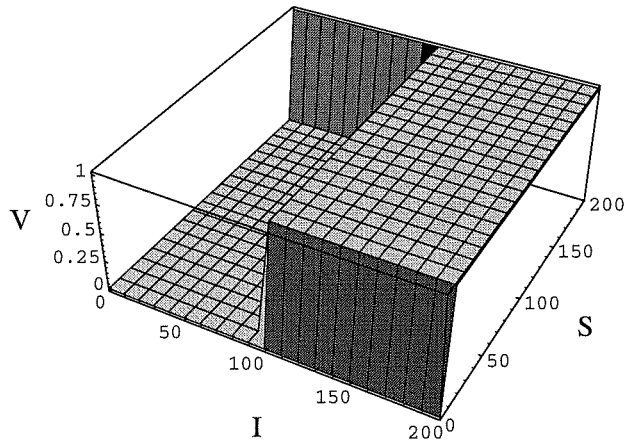


Figure 6.3: The payoff for the moving average dependent option with the initial conditions given by equation 6.37.

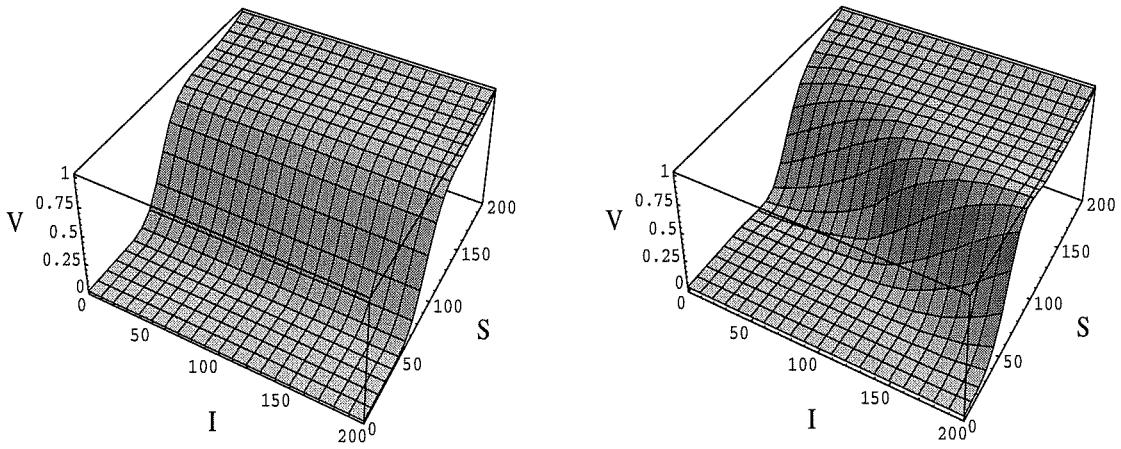


Figure 6.4: Two numerical solutions of the moving average dependent option with $r = 10\%$, $\tau = 1$ year, $B = 100$ and $\lambda = 250$ (left plot) and $\lambda = 1.25$ (right plot). The values of δS, δI and $\delta\tau$ were 10.0, 10.0 and 1/10000 respectively.

equation 6.5 drops out and with the conditions given in equations 6.37 and 6.38, the moving average dependent option becomes a simple cash-or-nothing call. This is apparent in Figure 6.4 (left) if we look 'into' the S-axis. Note that the payoff for a cash-or-nothing call can be written as:

$$B\mathcal{H}(S-E), \tag{6.39}$$

where $\mathcal{H}$ is the Heaviside function, which has the value 0 when its argument is negative and 1 otherwise. Essentially, it is a straight bet on the asset price.

Chapter 7

Conclusions

7.1 Thesis Overview

In this thesis we have attempted to:

- Model volatility, σ, as a function of a stock price, S, and an exponentially weighted moving average, I.
- Extend the Black-Scholes analysis to include not just the two variables, S and t, but also I.

In Chapter 1 we addressed the theory of the lognormal random walk, followed by asset prices, and described the important role it plays in the famous Black-Scholes model for valuing non-dividend paying securities. We described the Efficient Market Hypothesis, which states (in general) that todays stock price contains all available information about past prices - but no information about the future. We discussed the question of volatility in the Black-Scholes partial differential equation, its causes and popular methods of estimation. We concluded by saying that there is ample evidence of volatility being non-constant and that we would like to find a new measure (of volatility).

Chapter 2 gave an introduction to Technical Analysis, the use of past price movements to predict future changes. A review of the available literature on the subject was given. All the papers in this review reported results that question the validity of the EMH. A brief description was given of the popular charts, records of prices, used by technical analysts to make trading decisions. We discussed the

various methods of calculating the moving average, one of the most famous trend following technical indicators. The random walk followed by I was determined along with the probability density function for the ratio of the moving average to the stock price. Finally, we talked about how technical analysts use stock prices and moving averages to make trading decisions.

Chapter 4 detailed the different tests carried out on the Dow Jones Industrial Average Index. We began with an introduction to the history of the index itself and then verified that not only are the successive Dow Jones Returns values independent but that they also conform to a normal distribution. The results were in accordance with the random walk theory. We concluded the chapter by estimating the Dow Jones historical volatility and the drift rate.

Chapter 5 was all about the investigations carried out in establishing whether or not the volatility, σ, could be a function of S/I - the ratio of the stock price to the corresponding moving average. The motivation for choosing S/I was the fact that technical analysts use combinations of stock prices and moving averages to trade and one of the causes of volatility, as discussed in Chapter 1, is trading itself. Hence, the ratio S/I which in effect is a measure of the degree of separation between S and I, is a good choice to model volatility. We modified the standard random walk to include volatility as function of S/I, that is $\sigma(S/I)$.

Evaluating the volatility and plotting it against S/I, we found that it was not a constant. That is, it appeared that the Dow Jones σ was a function of S and of I. Various Monte Carlo tests confirmed that the quadratic shape of the plots was due to jumps of extraordinary magnitude in the index value. Further investigations showed that an increase in the σ was most likely to take place when a jump of a 'large' magnitude was immediately followed by another jump of a similar magnitude.

On close observation, it was discovered that the index rose and fell periodically in a predictable manner. We simulated this particular behaviour through the use of a stochastic drift rate, $\mathrm{d}\mu$.

In Chapter 6 we extended the 2-D Black-Scholes model to encompass not only S and t but also I. We solved the resulting 3-D p.d.e. using an explicit finite difference scheme. This involved the transformation of the p.d.e. into a set of difference equations. We checked our numerical scheme against two exact solutions, one for the case when σ is not a function of S/I and the other for an exact solution

linear in S, I and t. The results were satisfactory. We also formulated a payoff for a moving average dependent option and solved, numerically, the resulting p.d.e. The holder of this particular option would receive a predetermined sum of money if the moving average was above a certain value of the stock price.

7.2 The Dow Jones and Other Data Sets

Two other data sets were obtained: the S&P 500 Index and the FT-SE 100 Index. The σ versus S/I analysis was repeated to check for the quadratic behaviour that was found in the Dow Jones data.

The Standard and Poor's 500 Index data was obtained from [40]. For a description see Appendix A, Section A.1. The σ versus S/I plots for the S&P 500, Figures 7.1 - 7.2, are similar to the Dow Jones ones, Figures 5.1 - 5.4. This is not surprising, as there is a high degree of correlation between the two data sets. The correlation coefficients for the Dow Jones and the S & P 500 indices were determined, using MATLAB for the years 1992-95, and are shown in Table 7.1. The coefficients for the other years in this table were provided by [41]. The correlation coefficients determined using MATLAB were checked against the ones evaluated using the *Population Correlation Coefficient* equation [33]. This defines the correlation between two random variables, X and Y, as:

$$\begin{aligned} \rho_{xy} &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \\ &= \frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\overline{x})^2\sum_{i=1}^{n}(y_i-\overline{y})^2}}, \end{aligned} \tag{7.1}$$

where x and y are the two variables whose correlation is being determined, n is the total number of observables and $\overline{x}$ and $\overline{y}$ are the means of x and y respectively. Note that this equation is simply the sample covariance of X and Y divided by the square root of the variances of X and Y. It can be see from the table, that apart from the year 1992 the two indices are highly correlated (positively) and as we obtained a quadratic behaviour for the Dow Jones, we expected the same for the S&P 500.

The FT-SE 100 Index data was obtained from [42]. The σ versus S/I plots, Figures 7.3 - 7.4, do not exhibit a quadratic behaviour unlike the S&P 500 and the Dow Jones, though, for the longer term moving average plots, it can be seen

Year	Correlation
1931	1.00
1940	1.00
1950	0.99
1960	0.90
1970	0.93
1980	0.96
1990	0.98
1992	0.24
1993	0.93
1994	0.95
1995	0.99

Table 7.1: Table showing the correlation coefficients for the S&P 500 and the Dow Jones Industrial Average Indices.

that the σ's are large at the left and right extremities of S/I. It should be noted that the statistical error associated with the FT-SE data is much larger than the errors for the Dow Jones and the S&P 500. For the FT-SE we used 3083 data points compared to 16127 for the S&P 500 and 26031 for the Dow Jones. This draws us to the conclusion that the σ versus S/I plots are data set oriented. In other words, different plots for different data sets.

7.3 Extensions of Present Work

Numerical Analysis : The numerical scheme presented in Chapter 6 was an explicit one and as such has some drawbacks. As mentioned before, when checking the numerical results against the analytical ones we obtained satisfactory results. But, when the grid sizes (that is δS, δI and $\delta\tau$) were changed, the numerical scheme became unstable and divergent. The obvious way around these problems would be the use of the **Implicit Finite Difference Scheme**. The differencing in S and in I would be the same as in the explicit scheme but the time derivative would be replaced by a **backward difference** rather than a forward difference. This scheme might be more complicated to code but it 'would' be the answer to the convergence and stability problems associated with the explicit scheme.

Option Payoff Dependent on Two Crossing Moving Averages : Our governing p.d.e. (equation 6.5) can be further extended to take into account two

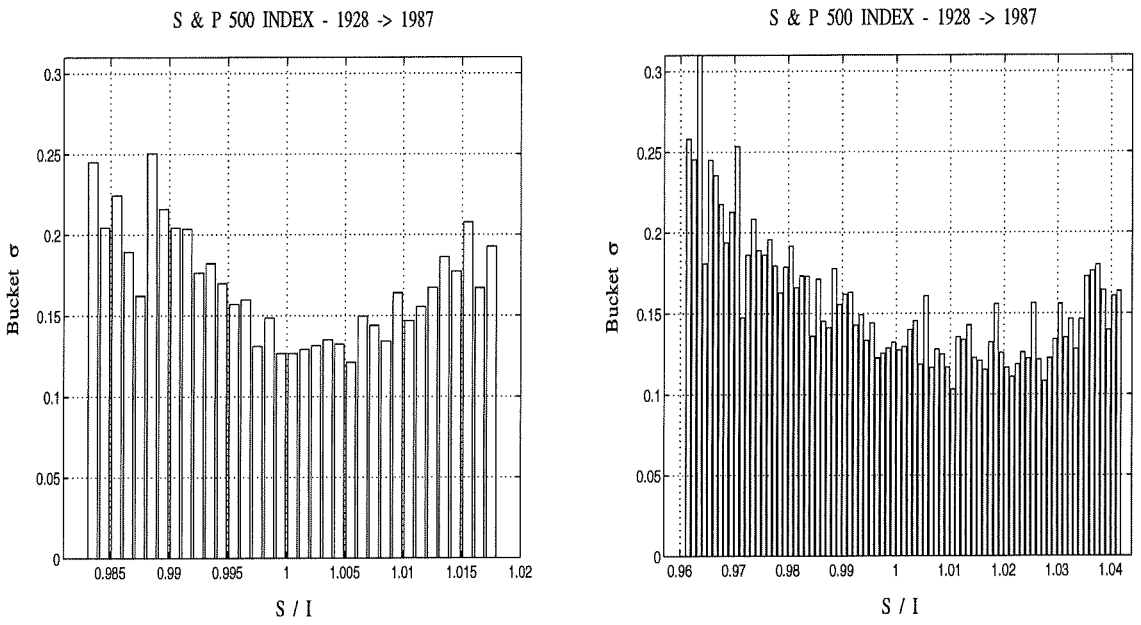


Figure 7.1: σ versus S/I : 1-day moving average (left) and 10-day moving average (right).

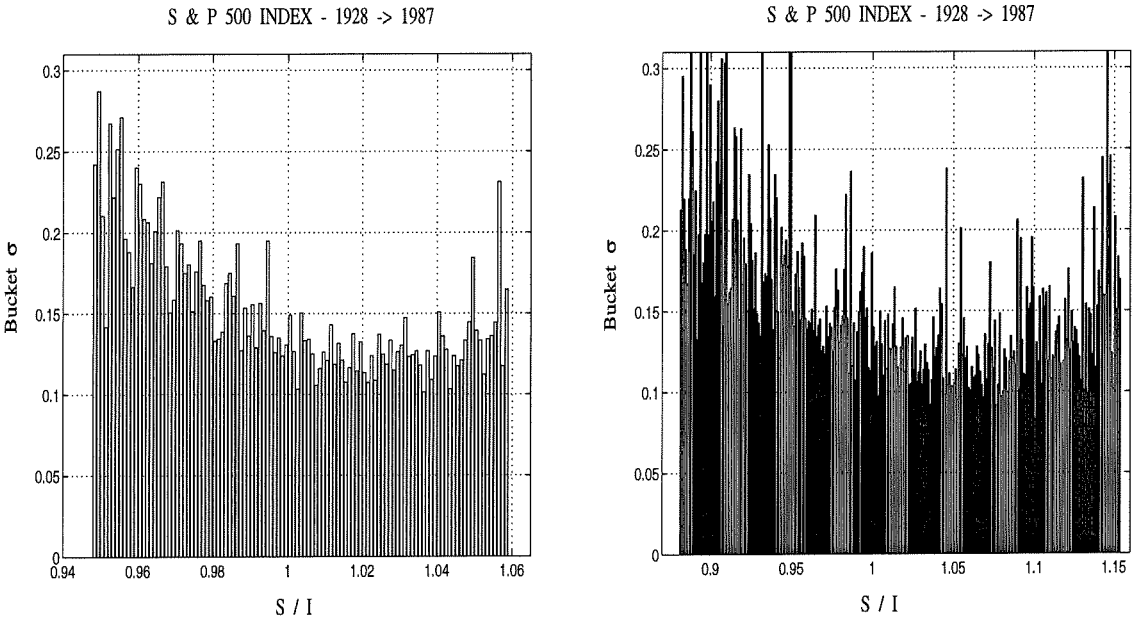


Figure 7.2: σ versus S/I : 20-day moving average (left) and 100-day moving average (right). Like the Dow Jones plots, these S & P 500 plots also exhibit a quadratic behaviour.

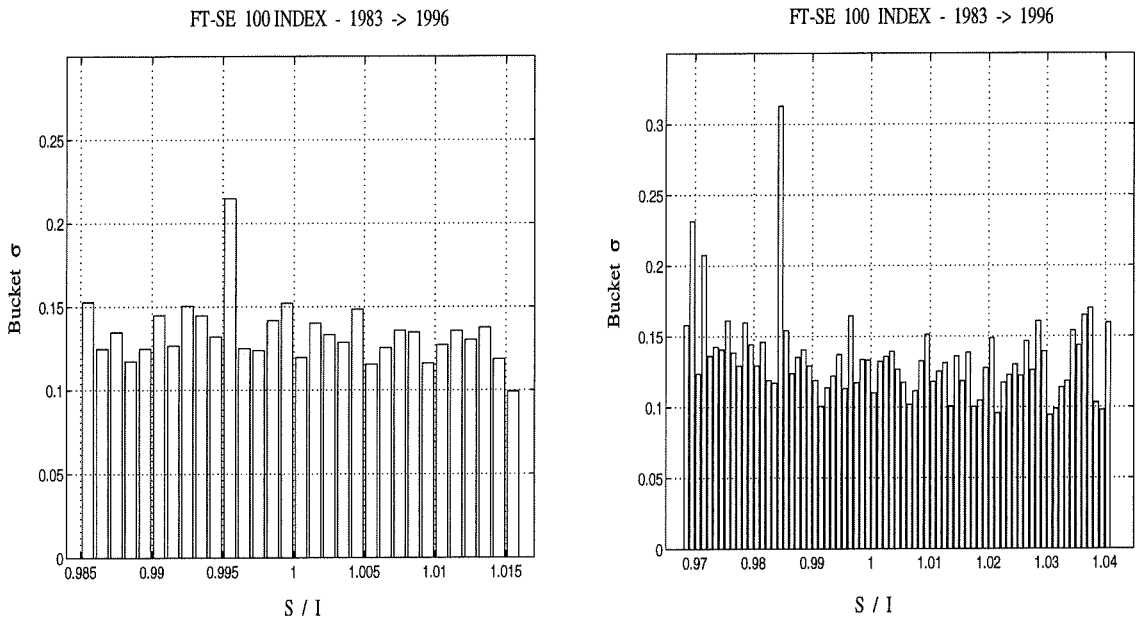


Figure 7.3: σ versus S/I : 1-day moving average (left) and 10-day moving average (right).

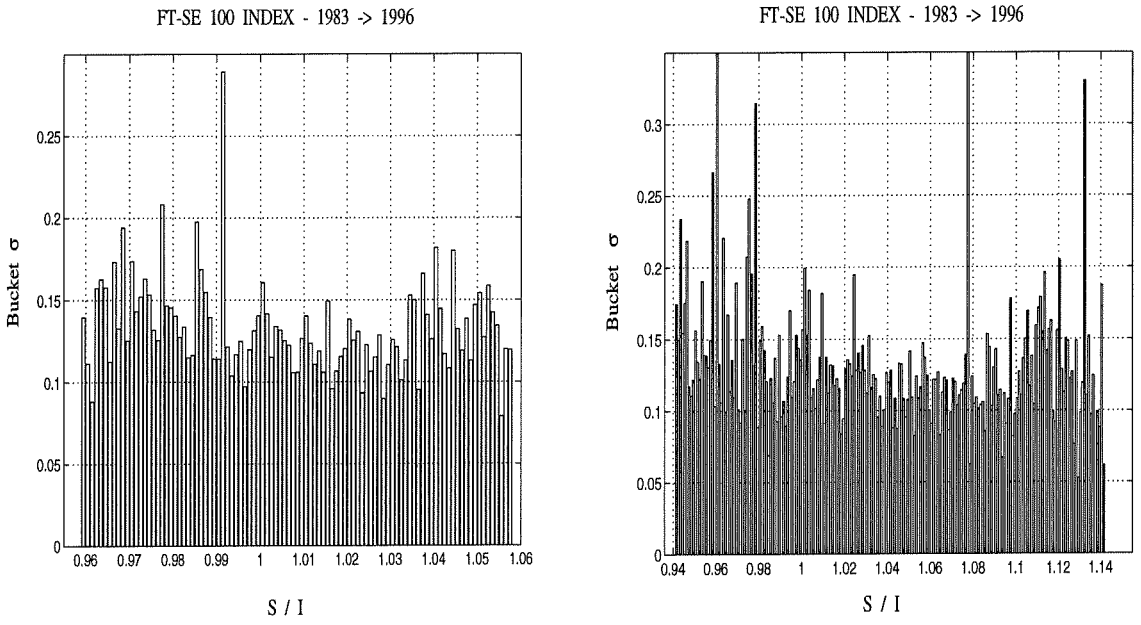


Figure 7.4: σ versus S/I : 20-day moving average (left) and 100-day moving average (right). Unlike the Dow Jones and the S&P 500 data, these plots do not have a quadratic shape. Though, note the high σ's at the left and right edges.

moving averages. The motivation behind this is the use by traders of a particular technical trading system where by they trade the crossing of two different time scale moving averages; see Chapter 2, Section 3.2. The governing p.d.e. can be written as

$$-V_\tau + \lambda_1 (S - I_1) V_{I_1} + \lambda_2 (S - I_2) V_{I_2} + \frac{1}{2}\sigma^2 S^2 V_{SS} + r S V_S - r V = 0, \quad (7.2)$$

where λ_1 and λ_2 are the time scales of the two moving average I_1 and I_2 respectively. A typical payoff could be one where the option holder receives a nominal sum when I_1 and I_2 cross before expiry. Equation 7.2 allows us greater flexibility when modelling volatility. σ as a function of I_1/I_2 could be a better measure of volatility. This is because, if a sizeable number of traders were using I_1/I_2 to trade, then I_1 and I_2 alone would affect the volatility. The only disadvantage with equation 7.2 is that it is a 4-D equation and so solving it numerically (to obtain accurate results) could take considerably longer than the 3-D equation 6.5.

Appendix A

Jump (Poisson) Process

A.1 Jump Process : An Example

The formal definition of a Poisson process is (like the Wiener process) in terms of the global process rather than the infinitesimal local version given in Chapter 5, Section 5.7. A random variable $\pi(t)$ is a Poisson process if

- $\pi(0)$ is 0, with a probability of 1.
- For each $0 \leq t_1 < t_2 < \infty$, $\pi(t_1) - \pi(t_2)$ has the Poisson distribution with a mean of $\alpha\,(t_2 - t_1)$. Thus, $\pi(t)$ can only take on integer values, and that the probability of x occurrences in the time interval between t_1 and t_2 is

$$P[\pi(t_2) - \pi(t_1) = x] = e^{-\alpha\,(t_2 - t_1)}\,\frac{(\alpha\,(t_2 - t_1))^x}{x\,!}. \tag{A.1}$$

- For each $0 \leq t_1 < t_2 < \infty$, $\pi(t_1)$ and $\pi(t_2)$ are independent.

Hence, $\pi(t)$ counts the number of jumps that have taken place, and that

$$\pi(t) = \int_0^t \mathrm{d}\pi. \tag{A.2}$$

It can be shown [39] that the interval from 0 up to the first event, and thereafter the intervals between successive events, are not only independent but independently distributed with an exponential p.d.f.:

$$\alpha\, e^{\alpha\, x} \;\; (x > 0). \tag{A.3}$$

Note that α is the frequency of jumps. Now, in order to illustrate that the time between successive jumps has such a p.d.f., use was made of **S&P 500 index futures**.

Time (hh:mm:ss)	Futures Price
08:30:01	356.35
08:30:03	356.40
08:30:07	356.35
08:30:07	356.30
08:30:07	356.25
08:30:24	356.30
08:30:26	356.35
08:30:28	356.40

Table A.1: An example of S&P 500 Index futures price data to study the frequency distribution of a typical jump process.

S&P 500 : This index trades on the **Chicago Mercantile Exchange (CME)** and is based on a portfolio of 500 different stocks: 400 industrials, 40 utilities, 20 transportation companies, and 40 financial institutions. This is a **market value-weighted index** unlike the Dow Jones Industrial average which has been mentioned to be a price-weighted average. It is computed by calculating the total market value of the 500 firms in the index and the total market value of those firms on the previous day of trading.

A futures contract on an index, unlike a futures contract on a commodity, is settled by a cash amount equal to the value of the stock index in question on the contract maturity date times a multiplier ($ 500 in the case of the S&P 500) that scales the size of the contract. For example a S&P 500 contract with a futures price of 250 and a final index value of 255 would result in a profit for the long side of $ 500 $\times$ (255 - 250) = $ 2500.

Note that a financial future is an agreement to buy or sell, on an organised exchange, a standard quantity of a specific financial instrument (or currency e.t.c) at a future date and at a price agreed between two parties.

Deals are transacted in a **futures pit** where price are quoted on an **open outcry** basis, i.e. all bids and offers are available for dealing by any other trader in the pit. Once, a deal has been finalised the price is then placed on a computer system (this is a simplified version of a complex chain of events). The result of this is 'tick-data', an example of which is shown in Table A.1. This table shows the times at which these particular S&P 500 contracts were traded.

Data in this format was available for the month of January (with 2 months until expiry) for the years 1987 through till 1990. That is, January S&P 500 futures prices for contracts that expire in March.

Now, using the data (as in Table A.1) plots were made for the frequency distribution of the time between trades. It was observed that all the distributions (for the years 1987 - 1990) were negative exponential (c.f equation A.1). This implies that when a trade takes place, once a broker (or some such person) has decided to buy or sell, the random nature of the time of the trade is approximated by a jump (Poisson) process. Although the tick data is well modelled by a pure jump process, on longer time scales the diffusion process is better in comparison.

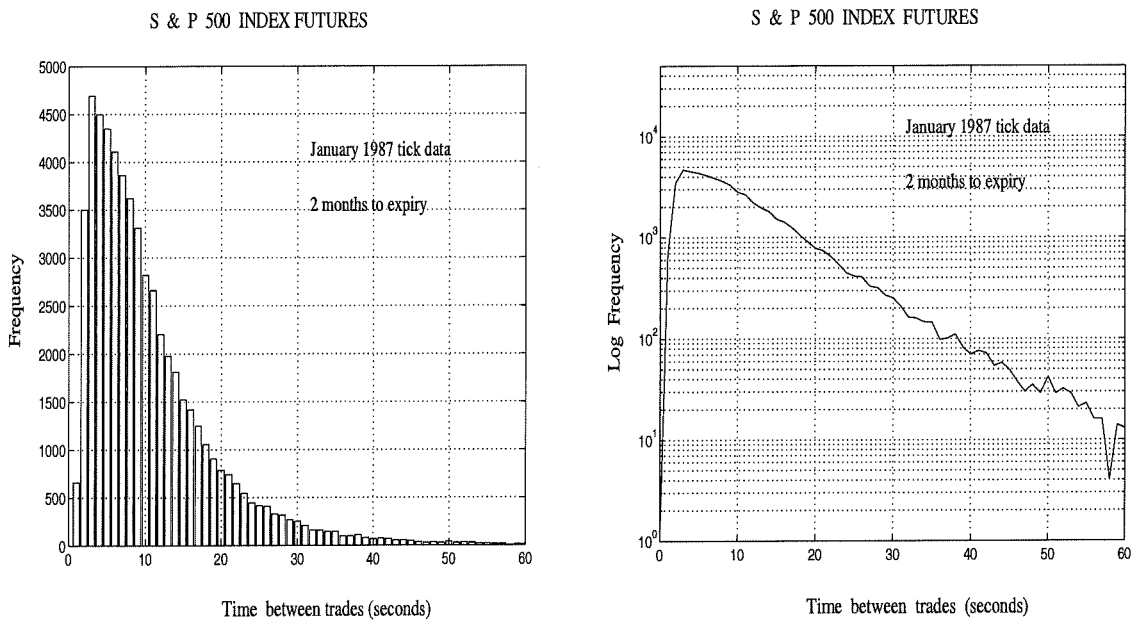


Figure A.1: The frequency and the log frequency distributions of the time between trades of the January 1987 S&P 500 futures contracts.

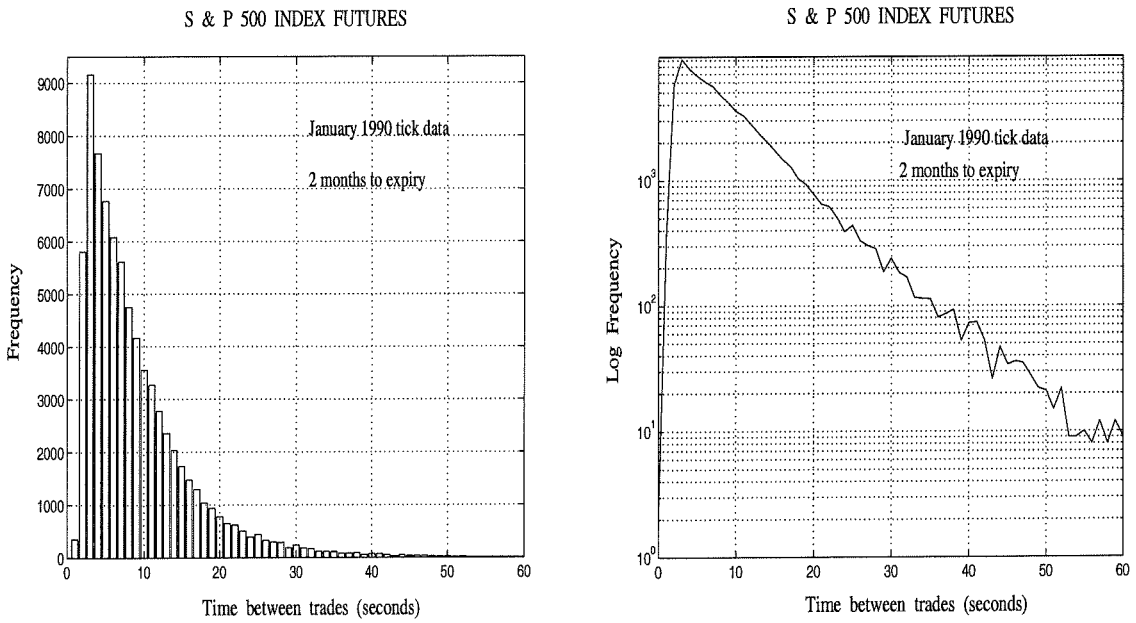


Figure A.2: Same as Figure A.1 but for the 1990 data set.

Appendix B

Fortran Programs

B.1 Random Walk Simulation

Fortran code used for the simulation of random walk for stock prices. See Section 1.2.1 and Figure 1.1.

```
************************************
            program random_walk
************************************

* number of data points to generate
parameter (ncount=1000)

* dimension the real numbers
real*8 sold,s,dt,amu,sigma,x,g05cbf,g05ddf,return(ncount)

* dimension the integers
integer i

* NAGLIB routines for random number generation
external g05cbf
external g05ddf

* open output data file
open(unit=23,file='random_walk_orig.dat',status='new')

* declare constants

* the expected rate of return
amu = 0.070d0

* the time step
dt = 1.0d0/250.0d0

* the volatility
sigma = 0.20d0
```

```
* the starting stock price
sold = 20.0d0

* initialise the random number generator
call g05cbf(0)

* generate the values
do i = 1 , ncount

* generate a random number drawn from a standard normal
* distribution
      x = g05ddf(0.0d0,1.0d0)

* calculate the new stock price
      s = sold + amu*sold*dt + sigma*sold*x*dsqrt(dt)

* write out generated stock prices into output file
      write(23,111) i,sold

* update old stock price
      sold = s

enddo

111 format(1x,i5,1x,f12.6)

stop

end
```

B.2 Random Walk and Moving Average Simulation

The computer code used for the simulation of random walk for stock prices and the calculation of moving averages with differing time scale - as done in Section 3.1 and shown in Figure 3.1.

```
************************************
        program moving_average
************************************

* number of data points to generate
parameter (ncount=1000)

* dimension the real numbers
real*8 sold,s,dt,amu,sigma,x,g05cbf,g05ddf,
       aiold,ai,alambda,sigdep,return(ncount),
       blambda,biold,bi

* dimension the integers
integer i

* NAGLIB routines for random number generation
external g05cbf
external g05ddf

* open output data file
open(unit=23,file='random_walk_ma.dat',status='new')

* declare constants

* the expected rate of return
amu = 0.070d0

* the time step
dt = 1.0d0/250.0d0

* the volatility
sigma = 0.20d0

* lambda for one moving average
alambda = 12.50d0

* lambda for the other moving average
blambda = 1.250d0

* the starting stock price
sold = 20.0d0

* the starting moving average
```

```
aiold = 20.0d0

* the starting moving average (2)
biold = 20.0d0

* initialise the random number generator
call g05cbf(0)

* generate the values
do i = 1 , ncount

* generate a random number drawn from a standard normal
* distribution
      x = g05ddf(0.0d0,1.0d0)

* calculate the new stock price
      s = sold + amu*sold*dt + sigma*sold*x*dsqrt(dt)

* calculate the moving average
      ai = (dexp(-(alambda*dt)) * aiold) +
           ((1-dexp(-(alambda*dt))) * sold)

* calculate the moving average(2)
      bi = (dexp(-(blambda*dt)) * biold) +
           ((1-dexp(-(blambda*dt))) * sold)

* write out generated stock prices into output file
      write(23,111) i,sold,aiold,biold

* update old stock price and old moving average
      sold = s
      aiold = ai
      biold = bi
enddo

111 format(1x,i5,1x,f12.6,1x,f12.6,1x,f12.6)

stop

end
```

B.3 Simulation of Jump and Jump Diffusion Processes

The computer codes used for the simulation of the pure jump process and the jump diffusion process. See Section 5.7 and figures 5.14 and 5.15.

```
************************************
         program pure_jump
************************************

parameter (ncount=250)

* declare the integers
integer injump,jup,jdown,inran

* declare the reals
real*8 time,dt,x,xa,freqjump,alpha,aj,g05def,s,
       g05ddf,g05cbf,sold,g05caf,ajump

* external functions
external g05ddf
external g05caf
external g05def

* external subroutine
external g05cbf

* open results file
open(unit=55,file='jump_pure.dat',status='new')

* constants

* time step (annualised)
dt = 1.0d0/250.0d0
alpha = 12.0d0

* frequency of jumps (annualised)
* once a month
freqjump = alpha*dt

* counter for elapsed time
time = 0.0d0

* starting S
sold = 20.0d0

* counter for # of jumps
injump = 0

* counter for # of no jumps
inran = 0
```

```
* up jumps
jup = 0

* down jumps
jdown = 0

* initialise the NAGLIB routine to generate random numbers
call g05cbf(0)

* generate values
do i = 1 , ncount

* elapsed time
      time = time + dt

* generate the random numbers drawn from a normal distribution
* mean 0 variance 1
      x = g05ddf(0.0d0,1.0d0)

* generate random numbers drawn from a uniform distribution
* between 0 and 1
      xa = g05caf(xa)

       if (xa.le.freqjump) then

        injump = injump + 1

* normal distn.  mean 0 and variance (0.03)^2
        aj = g05ddf(0.0d0,0.030d0)
        ajump = aj

        if (aj.eq.x) then
         write(6,*) 'Two norm.  distn.s same !'
        endif

* count the up and down jumps
        if (ajump.gt.0.0d0) then
         jup = jup + 1
        elseif(ajump.lt.0.0d0) then
         jdown = jdown + 1
        endif

* the new s
        s = sold + ajump*sold

       else

        inran = inran + 1

        s = sold

       endif
```

```
* write out results
      write(55,69) i,sold

* update old value of s
      sold = s

enddo

69 format(1x,i5,1x,f12.6)

write(6,*)
write(6,*)'# of Jumps :  ',injump
write(6,*)'# of Ran.  Walks :  ',inran
write(6,*)
write(6,*)'Jump up :  ',jup
write(6,*)'Jump down :  ',jdown
write(6,*)

stop

end



**********************************
        program jump_diffusion
**********************************

parameter (ncount=250)

* declare the integers
integer injump,inran,jup,jdown

* declare the reals
real*8 time,dt,x,xa,freqjump,alpha,aj,g05def,s,
      g05ddf,g05cbf,sold,g05caf,sigma,amu,ajump

* external functions
external g05ddf
external g05caf
external g05def

* external subroutine
external g05cbf
* open results file
open(unit=55,file='jump_old.dat',status='new')

* constants

* drift rate (annualised)
amu = 0.150d0
```

```
* time step (annualised)
dt = 1.0d0/250.0d0
alpha = 12.0d0

* frequency of jumps (annualised)
* once a month
freqjump = alpha*dt

* counter for elapsed time
time = 0.0d0

* volatility (annualised)
sigma = 0.20d0

* starting S
sold = 20.0d0

* counter for # of jumps
injump = 0

* counter for # of random walks
inran = 0

* up jumps
jup = 0

* down jumps
jdown = 0

* initialise the NAGLIB routine to generate random numbers
call g05cbf(0)

* generate values
do i = 1 , ncount

* elapsed time
      time = time + dt

* generate the random numbers drawn from a normal distribution
* mean 0 variance 1
      x = g05ddf(0.0d0,1.0d0)

* generate random numbers drawn from a uniform distribution
* between 0 and 1

      xa = g05caf(xa)

       if (xa.le.freqjump) then

        injump = injump + 1

* normal distn.  mean 0 and variance (0.03)^2
        aj = g05ddf(0.0d0,0.030d0)
```

```
          ajump = aj

          if (aj.eq.x) then
           write(6,*) 'Two norm.  distn.s same !'
          endif

 * count the up and down jumps
          if (ajump.gt.0.0d0) then
           jup = jup + 1
          elseif(ajump.lt.0.0d0) then
           jdown = jdown + 1
          endif

 * the new s
          s = sold + ajump*sold + amu*sold*dt +
              sigma*sold*x*dsqrt(dt)

         else

          inran = inran + 1

          s = sold + amu*sold*dt +
              sigma*sold*x*dsqrt(dt)

         endif

 * write out results
        write(55,69) i,sold

 * update old value of s
        sold = s

 enddo

 69 format(1x,i5,1x,f12.6)

 write(6,*)
 write(6,*)'# of Jumps :  ',injump
 write(6,*)'# of Random Walks :  ',inran
 write(6,*)
 write(6,*)'Jump up :  ',jup
 write(6,*)'Jump down :  ',jdown
 write(6,*)

 stop

 end
```

B.4 Jump Frequency as a Function of S and I.

Code for the simulation of the jump diffusion process where the jump frequency is a function of the stock price and the moving average. See Section 5.8.

```
************************************
         program jump_frequency
************************************

parameter (ncount=26031)

* declare the integers
integer injump,inran,jup,jdown,l1

* declare the reals
real*8 time,dt,x,xa,freqjump,alpha,aj,g05def,s,
       g05ddf,g05cbf,sold,g05caf,sigma,amu,ajump,
       amove,ambda,beta,w

* external functions
external g05ddf
external g05caf
external g05def

* external subroutine
external g05cbf

* open results file
open(unit=55,file='jump.dat',status='new')

* constants

* drift rate (annualised)
amu = 0.060d0

* time step (annualised)
dt = 1.0d0/250.0d0

* counter for elapsed time
time = 0.0d0

* volatility (annualised)
sigma = 0.10d0

* starting S
sold = 51.60d0

* starting I
amold = 51.60d0

* time scale of moving average (1-day)
ambda = 250.0d0
```

```
* counter for # of jumps
injump = 0

* counter for # of random walks
inran = 0

* up jumps
jup = 0

* down jumps
jdown = 0

* initialise the NAGLIB routine to generate random numbers
call g05cbf(0)

* generate values
do i = 1 , ncount

* elapsed time
      time = time + dt

* generate the random numbers drawn from a normal distribution
* mean 0 variance 1
      x = g05ddf(0.0d0,1.0d0)

* generate random numbers drawn from a uniform distribution
* between 0 and 1
      xa = g05caf(xa)

* jump frequency
      freqjump = dt*beta(sold/amold)

* the moving average
      amove = (dexp(-ambda*dt)*amold) +
           ((1 - dexp(-ambda*dt))*sold)

       if (xa.le.freqjump) then

        injump = injump + 1

* normal distn.  mean 0 and variance (0.01)^2
        aj = g05ddf(0.0d0,0.030d0)
        ajump = aj

        if (aj.eq.x) then
         write(6,*) 'Two norm.  distn.s same !'
        endif

* count the up and down jumps
        if (ajump.gt.0.0d0) then
         jup = jup + 1
        elseif(ajump.lt.0.0d0) then
```

```
            jdown = jdown + 1
           endif
* the new s (jump-diffusion)
           s = sold + ajump*sold + amu*sold*dt +
               sigma*sold*x*dsqrt(dt)
          else
           inran = inran + 1
* the old s (only diffusion)
           s = sold + amu*sold*dt +
               sigma*sold*x*dsqrt(dt)
          endif
* write out results
         write(55,69) i,sold
* update old value of s and am
         sold = s
         amold = amove
enddo
69 format(1x,i5,1x,f12.6)
write(6,*)
write(6,*)'# of Jump-Diffusions :  ',injump
write(6,*)'# of Diffusions :  ',inran
write(6,*)
write(6,*)'Jump up :  ',jup
write(6,*)'Jump down :  ',jdown
write(6,*)
enddo
stop
end
*****************
real*8 function beta(v)
*****************
* a function to force beta to be function of S/I.
* at S/I = 1 (=25), at 0.95 and 1.05 (=250) - a quadratic
* shape.
real*8 acon,bcon,ccon,v
acon = 94400.0d0
bcon = - 188800.0d0
ccon = 94414.0d0
         beta = acon*(v*v) + bcon*(v) + ccon
end
```

B.5 Stochastic Drift Rate

This section list the fortran code used to simulate a stochastic drift rate which when used in turn to generate stock prices, produced a repeatable pattern in the prices. See Chapter 5, Section 5.9.

```
************************************
         program drift_rate
************************************

parameter (ncount=1000)

* declare the integers
integer injump,inran,nsum

* declare the reals
real*8 time,dt,x,xa,freqjump,alpha,g05def,s,g05ddf,
g05cbf,sold,g05caf,sigma,amu,amuold,a,b

* external functions
external g05ddf
external g05caf
external g05def

* external subroutine
external g05cbf

* open results file
open(unit=55,file='jump_mu.dat',status='new')

* constants
* drift rate (annualised)
amuold = 1.0d0

* time step (annualised)
dt = 1.0d0/365.0d0
alpha = 10.0d0

* frequency of jumps (annualised)
freqjump = alpha*dt

* counter for elapsed time
time = 0.0d0

* volatility (annualised)
sigma = 0.20d0

* starting S
sold = 20.0d0

* the coefficient a
```

```
a = 1.0d0

* the coefficient b
b = 0.70d0

* counter for ran.  walks
inran = 0

* counter for jumps walks
injump = 0

* write constants in file
write(6,*)
write(6,*)'dmu = a*[(1-mu/a)*dp - b*alpha*dt]'
write(6,*)'**********************************'
write(6,*)
write(6,*)'starting mu :',amuold
write(6,*)'starting S :',sold
write(6,*)'a :',a
write(6,*)'b :',b
write(6,*)'sigma :',sigma
write(6,*)'alpha :',alpha
write(6,*)'dt :',dt
write(6,*)

* initialise the NAGLIB routine to generate random numbers
call g05cbf(0)

nsum = 0

* generate values
do i = 1 , ncount

* elapsed time
      time = time + dt

* generate the random numbers drawn from a normal distribution
* mean 0 variance 1
      x = g05ddf(0.0d0,1.0d0)

* generate random numbers drawn from a uniform distribution
* between 0 and 1
      xa = g05caf(xa)

       if (xa.le.freqjump) then

        injump = injump + 1

* the new mu
        amu = amuold + a*((1.0d0-(amuold/a)) - b*alpha*dt)

* the new s
```

```
          s = sold + amu*sold*dt + sigma*sold*x*dsqrt(dt)

* if no jump
         else

          inran = inran + 1

* the new mu
          amu = amuold + a*b*(- alpha)*dt

          s = sold + amu*sold*dt + sigma*sold*x*dsqrt(dt)

         endif

* write out results
        write(55,69) i,sold,amuold

        nsum = nsum + amuold

* update old value of s and amu
        sold = s
        amuold = amu

enddo

69 format(1x,i5,1x,f12.6,1x,f10.6)

write(6,*)
write(6,*)'Average mu:',((nsum)/dble(i))
write(6,*)
write(6,*)
write(6,*)'# of Jumps :  ',injump
write(6,*)'# of Random Walks :  ',inran
write(6,*)

stop

end
```

B.6 Explicit Finite Difference Codes

Fortran codes for the numerical schemes used when the volatility, σ, is a constant and when the solution linear in S, I, and τ. See Section 6.2.1.

```
************************************
         program explicit_no_sigma
************************************

parameter(nmax=200)

* declare the integers
integer lmax,n,ntime

* declare the reals
real*8 ds,dt,smax,s,e,r,t,sigma,uold(0:nmax),
       unew(0:nmax),a1,a2,a3

* maximum value for S
smax = 200.0d0

* S step size
ds = 2.0d0

* time step size
dt = 0.00010d0

* number of S steps
lmax = int(smax/ds)

* time to expiry
t = 0.70d0

* number of time steps
ntime = int(t/dt)

* interest rate
r = 0.070d0

* volatility
sigma = 0.250d0

* exercise price
e = 110.0d0

************************
* The initial conditions
************************

* for a European Call Option -> Max(S-E,0)
* for a Put -> Max(E-S,0)
```

```
do l = 0 , lmax

       s = l*ds

        if (s.gt.e) then
         uold(l) = s - e
        elseif (s.le.e) then
         uold(l) = 0.0d0
        endif

enddo

************************
* Boundary conditions and the
* finite difference scheme.
************************

* loop over the time steps
do n = 1 , ntime

* do the S values excluding the boundary values
       do l = 1 , lmax - 1

        s = l*ds

* the V_SS term
        a1 = (0.50d0*sigma*sigma*s*s*dt)*((uold(l+1)-
             2*uold(l)+uold(l-1))/(ds*ds))

* the V_S term - making sure that
* no eroneous values are introuced
        if(l.eq.(lmax-1)) then
         a2 = (r*s*dt)*((uold(l+1)-uold(l))/(ds))
        else
         a2 = (r*s*dt)*((4*uold(l+1)-3*uold(l)-
              uold(l+2))/(2*ds))
        endif

        a3 = - r*dt*uold(l)

* the new values after the scheme
        unew(l) = uold(l) + (a1+a2+a3)

       enddo

* updating the old values in the array
       do ii = 0 , lmax
        uold(ii) = unew(ii)
       enddo

* the boundary conditions
       uold(0) = 0.0d0
```

```
* for the Call Option
       uold(lmax) = smax - e*dexp(-r*n*dt)

* write out results
        if(n.eq.ntime) then
         write(67,*)
         write(67,*)'r =',r
         write(67,*)'sigma =',sigma
         write(67,*)'dt =',dt
         write(67,*)

         do kk = 0 , lmax
          write(67,*) kk*ds,e,uold(kk)
         enddo

         write(67,*)
        endif

enddo

stop

end



**********************************
         program explicit_linear
**********************************

parameter(nmax=200)

* declare the integers
integer lmax,n,ntime,mmax

* declare the reals
real*8 ds,dt,smax,s,e,r,t,sigma,uold(0:nmax,0:nmax),
       unew(0:nmax,0:nmax),a1,a2,a3,aimax,di,ai,a4,
       alambda,a0,b0,c0,x05baf,cptime,cp1,cp2

* for the NAGLIB cpu timer
cp1 = x05baf()

* maximum value for S
smax = 200.0d0

* S step size
ds = 10.0d0

* number of S steps
lmax = int(smax/ds)
```

```
* maximum value for I
aimax = 200.0d0

* I step size
di = 10.0d0

* number of I steps
mmax = int(aimax/di)

* time step size
dt = 0.00010d0

* time to expiry-annualised
t = 1.0d0

* number of time steps
ntime = int(t/dt)

* interest rate
r = 0.10d0

* volatility
* for the linear solution Vss is 0 - hence sigma = 0
sigma = 0.20d0

* lambda
alambda = 250.0d0

* constants in linear solution
a0 = 3.0d0
b0 = 4.0d0
c0 = 5.0d0

*************************
* The initial condition
*************************
* loop over the I steps
do m = 0 , mmax

       ai = m*di

* loop over the S steps
        do l = 0 , lmax

         s = l*ds

* for the moving average dependent option
         if(ai.gt.0.0d0.and.ai.lt.100.0d0) then
          uold(l,m) = 0.0d0
         elseif(ai.ge.100.0d0) then
          uold(l,m) = 1.0d0
         endif
```

```
          uold(0,m) = 0.0d0
          uold(lmax,m) = 1.0d0

cc initial condition for a linear solution
cc uold(l,m) = (-((alambda*b0)/(r + alambda)) +
cc a0)*s + b0*ai + c0

         enddo

enddo

*****************************
* Boundary conditions and the finite
* difference scheme.
*****************************
* loop over the time steps - beginning of time do-loop
do n = 1 , ntime

* loop over S values excluding the boundary ones
       do l = 1 , (lmax - 1)

         s = l*ds

****************
* for S <= I
***************

* doing the I
         do m = l , mmax

          ai = m*di

* for the V_SS term
          a1 = (0.50d0*sigma*sigma*s*s*dt)*((uold(l+1,m)-
               2.0d0*uold(l,m)+uold(l-1,m))/(ds*ds))

* the V_S term
          if(l.eq.(lmax-1)) then
           a2 = (r*s*dt)*((uold(l+1,m)-uold(l,m))/(ds))
          else
           a2 = (r*s*dt)*((4.0d0*uold(l+1,m)-3.0d0*uold(l,m)-
                uold(l+2,m))/(2.0d0*ds))
          endif

          a3 = - r*dt*uold(l,m)

* for the V_I term - making sure no values
* are taken from the S > I side
          if(m.eq.l.or.m.eq.(l+1)) then
           a4 = (alambda*(s - ai)*dt)*((uold(l,m)-
                uold(l,m-1))/di)
```

```
          else
           a4 = -(alambda*(s - ai)*dt)*((4.0d0*uold(l,m-1)-
                3.0d0*uold(l,m)-uold(l,m-2))/(2.0d0*di))
          endif

* the values after the scheme
          unew(l,m) = uold(l,m) + (a1+a2+a3+a4)

         enddo

**************
* for S > I
**************

         do m = (l - 1) , 0 , -1

          ai = m*di

* the V_SS term
          a1 = (0.50d0*sigma*sigma*s*s*dt)*((uold(l+1,m)-
               2.0d0*uold(l,m)+uold(l-1,m))/(ds*ds))

* the V_S term
          if(l.eq.(lmax-1)) then
           a2 = (r*s*dt)*((uold(l+1,m)-uold(l,m))/(ds))
          else
           a2 = (r*s*dt)*((4.0d0*uold(l+1,m)-3.0d0*uold(l,m)-
                uold(l+2,m))/(2.0d0*ds))
          endif

          a3 = - r*dt*uold(l,m)
* for the V_I term
          if(m.eq.(l-1)) then
           a4 = (alambda*(s - ai)*dt)*((uold(l,m+1)-
                uold(l,m))/di)
          else
           a4 = (alambda*(s - ai)*dt)*((4.0d0*uold(l,m+1)-
                3.0d0*uold(l,m)-uold(l,m+2))/(2.0d0*di))
          endif

          unew(l,m) = uold(l,m) + (a1+a2+a3+a4)

         enddo

        enddo

* update the old value at each time step
        do m = 0 , mmax
         do l = 0 , lmax
          uold(l,m) = unew(l,m)
         enddo
```

```
        enddo

* the boundary conditions at each I step
        do m = 0 , mmax

         ai = m*di

cc in S = 0 plane - for the linear solution
cc uold(0,m) = (b0*dexp(-(r+alambda)*n*dt))*ai +
cc c0*dexp(-r*n*dt)

         uold(0,m) = 0.0d0

cc in S = Smax plane - for the linear solution
cc uold(lmax,m) = (-((alambda*b0)/(r+alambda))*
cc dexp(-(r+alambda)*n*dt) + a0)*smax +
cc (b0*dexp(-(r+alambda)*n*dt))*ai +
cc c0*dexp(-r*n*dt)

         uold(lmax,m) = 1.0d0

        enddo

* end of time do-loop enddo

write(67,*)
write(67,*)'Interest rate =',r
write(67,*)'Expiry =',t
write(67,*)'Volatility =',sigma
write(67,*)'Time Step =',dt
write(67,*)'S-step =',ds
write(67,*)'I-step =',di
write(67,*)'Lambda =',alambda
write(67,*)

write(67,*)'I','S','V(S,I,t)'
write(67,*)'*','*','********'
write(67,*)

* write out the results
       do m = 0 , mmax

        ai = m*di

        do l = 0 , lmax

         s = l*ds
         write(67,222) ai,s,uold(l,m)

        enddo

       enddo
```

```
      cp2 = x05baf()

      cptime = cp2 - cp1

      write(6,*)
      write(6,300)'It took',cptime, ' seconds to complete'
      write(6,*)
      write(67,*)
      write(67,300)'It took',cptime, ' seconds to complete'
      write(67,*)

  222 format(1x,f6.1,1x,f6.1,1x,f6.2)

  300 format(1x,a,1p,d10.2,a)

      stop

      end
```

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