

Rates of convergence to equilibrium for collisionless kinetic equations in slab geometry

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Abstract

This work deals with free transport equations with partly diffuse stochastic boundary operators in slab geometry. Such equations are governed by stochastic semigroups in L^1 spaces. We prove convergence to equilibrium at the rate $O\left(t^{-\frac{k}{2(k+1)+1}}\right)$ ($t \rightarrow +\infty$) for L^1 initial data g in a suitable subspace of the domain of the generator T where $k \in \mathbb{N}$ depends on the properties of the boundary operators near the tangential velocities to the slab. This result is derived from a quantified version of Ingham's tauberian theorem by showing that $F_g(s) := \lim_{\varepsilon \rightarrow 0_+} (is + \varepsilon - T)^{-1} g$ exists as a C^k function on $\mathbb{R} \setminus \{0\}$ such that $\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C}{|s|^{2(j+1)}}$ near $s = 0$ and bounded as $|s| \rightarrow \infty$ ($0 \leq j \leq k$). Various preliminary results of independent interest are given and some related open problems are pointed out.

1 Introduction

This paper is devoted to rates of convergence to equilibrium for one-dimensional free (i.e. collisionless) transport equations with mass-preserving partly diffuse boundary operators. We provide a general L^1 theory relying on a *quantified tauberian theorem* [11]. In linear or non-linear kinetic theory, various non-local (combinations of specular and diffuse) boundary conditions are physically relevant, see e.g. [14][23] and the references therein. Furthermore, general free transport equations with smooth vector fields and positive contractive boundary operators are well posed, see e.g. [3][4]. On the other hand, the existence of an invariant density and the return to this equilibrium state for solutions to free

transport equations has not received much attention; see however [1][5][15][26] for the vector field $v \cdot \nabla_x$ with a Maxwell diffuse boundary operator with constant temperature; in this case, the invariant density is given by a maxwellian function. The L^1 convergence to this maxwellian equilibrium goes back to [5] while the analysis of rates of convergence was considered more recently in [1][15] after some numerical investigations in [26]; we will comment below on some results in [1][15]. We note that collisionless transport semigroups present a lack of spectral gap which make them akin to collisional linear kinetic equations with *soft* potentials. More recently, the authors of [21] provided a convergence theory to equilibrium for a general class of monoenergetic free transport equations in slab geometry with azimuthal symmetry and abstract boundary operators. In this abstract model, the existence of invariant density is characterized and shown for a general class of partly diffuse boundary operators. Our aim here is to derive a quantified version (with algebraic rates) of this convergence theory from a quantified version of Ingham's tauberian theorem [11]. We provide a general theory based on some natural structural conditions on the boundary operators in the vicinity of the tangential velocities to the slab. To keep the ideas of this work more transparent, we restrict ourselves to monoenergetic models; (non-monoenergetic free models in slab geometry could be treated similarly, see Remark 33). Besides the main result on the rates of convergence, our construction provides us with various new mathematical results of independent interest. Several open problems are also pointed out.

We note that a special quantified version of Ingham's theorem for "asymptotically analytic" C_0 -semigroups (see [11] Corollary 2.12) was already used for the first time in kinetic theory to deal with spatially homogeneous linear Boltzmann equations with soft potentials where the generators are bounded [18]. Finally, we point out that there exists a substantial literature on rates of convergence to equilibrium for collisional (linear or non-linear) kinetic equations relying mostly on *entropy methods*. In particular, collisional kinetic equations with soft potentials exhibit algebraic rates of convergence, see e.g. [6][7][12][18][25] and references therein.

We consider here the monoenergetic free transport equation in slab geometry with azimuthal symmetry

$$\frac{\partial f}{\partial t}(t, x, v) + v \frac{\partial f}{\partial x}(t, x, v) = 0, \quad (x, v) \in \Omega \quad (1)$$

$$f(0, x, v) = g(x, v) \quad (2)$$

where

$$\Omega = (-a, a) \times (-1, 1)$$

(with $a > 0$). The boundary conditions are

$$|v| f(t, -a, v) = \alpha_1 |v| f(t, -a, -v) + \beta_1 K_1(|\cdot| f_{-a}^-(t)) \quad (v > 0), \quad (3)$$

$$|v| f(t, a, v) = \alpha_2 |v| f(t, a, -v) + \beta_2 K_2(|\cdot| f_a^+(t)) \quad (v < 0) \quad (4)$$

where

$$\alpha_i \geq 0, \quad \beta_i \geq 0, \quad \alpha_i + \beta_i = 1 \quad (i = 1, 2); \quad (5)$$

here $f_{-a}^-(t)$ (resp. $f_a^+(t)$) denotes the restriction of $f(t, -a, \cdot)$ (resp. $f(t, a, \cdot)$) to $(-1, 0)$ (resp. to $(0, 1)$),

$$|\cdot| f_{-a}^-(t) : (-1, 0) \ni v \rightarrow |v| f(t, -a, v)$$

$$|\cdot| f_a^+(t) : (0, 1) \ni v \rightarrow f(t, a, v)$$

and K_i ($i = 1, 2$) are *stochastic* (i.e. positive and norm preserving on the positive cone) *weakly compact* operators

$$K_1 : L^1((-1, 0); dv) \rightarrow L^1((0, 1); dv),$$

$$K_2 : L^1((0, 1); dv) \rightarrow L^1((-1, 0); dv).$$

The weak compactness assumption implies that K_i has a *kernel* $k_i(\cdot, \cdot)$, ($i = 1, 2$) (see remark in [13], p. 508); it also plays a key role in several places of this work. Note that the boundary conditions are convex combinations of *specular* (deterministic) parts and *diffuse* (random) ones modelled by K_i ($i = 1, 2$); in particular, we can write (3)(4) as

$$h_{-a}^+ = O_1 h_{-a}^- \text{ and } h_a^- = O_2 h_a^+$$

where

$$h_a^\pm(v) = |v| f_a^\pm(v), \quad h_{-a}^\pm(v) = |v| f_{-a}^\pm(v)$$

$$O_1 = \alpha_1 R_1 + \beta_1 K_1 : L^1((-1, 0); dv) \rightarrow L^1((0, +1); dv)$$

$$O_2 = \alpha_2 R_2 + \beta_2 K_2 : L^1((0, +1); dv) \rightarrow L^1((-1, 0); dv)$$

and

$$R_1 : L^1((-1, 0); dv) \rightarrow L^1((0, +1); dv)$$

$$R_2 : L^1((0, +1); dv) \rightarrow L^1((-1, 0); dv)$$

denotes the specular reflection operators defined by

$$(R_i \varphi)(v) = \varphi(-v) \quad (i = 1, 2).$$

We point out that for the physical model in slab geometry with azimuthal symmetry,

$$v \in (-1, +1)$$

is not a "velocity" but rather the cosine of the angles of the monoenergetic velocities (of particles moving in the slab) with an oriented axis perpendicular to the slab. In particular, the tangential velocities to the slab correspond to

$$v = 0$$

i.e. to the degeneracy of the vector field $v \frac{\partial}{\partial x}$. These tangential velocities turn out to play a natural and fundamental role in our construction. Finally, we

note that the boundary conditions are local in space, i.e. we have two separated boundary conditions (one at $x = -a$ and another one at $x = a$) even if one can imagine much more complex models including a coupling of the fluxes at $-a$ and at a .

It is known that the problem (1)(2)(3)(4) is well-posed in $L^1(\Omega)$ in the sense of semigroup theory and the corresponding C_0 -semigroup $(e^{tT_O})_{t \geq 0}$ with generator T_O (indexed by $O := (O_1, O_2)$) is stochastic, i.e. norm preserving on the positive cone [21]. We deal here with the *partly diffuse* model

$$\beta_1 + \beta_2 > 0 \quad (6)$$

only, i.e. we assume that at least one boundary condition is at least partly diffuse. It is known that under condition (6) the semigroup admits an invariant density, [21]; (see below for the details). Furthermore, the C_0 -semigroup converges strongly to its ergodic projection as time goes to infinity provided that

$$\beta_1 \beta_2 > 0,$$

[21]. The lack of spectral gaps for such collisionless kinetic models means there are no obvious rates of convergence to equilibrium. Our aim here is to give a *quantified* version of the convergence theory given in [21]. The most important statement in this paper is:

MAIN THEOREM *Let the kernels $k_i(.,.)$ of K_i ($i = 1, 2$) be continuous and let $(e^{tT_O})_{t \geq 0}$ be irreducible. We assume that at least one of the boundary conditions is completely diffuse, i.e.*

$$\beta_1 = 1 \text{ or } \beta_2 = 1. \quad (7)$$

Let there exist an integer $k \geq 1$ such that the following operators

$$\begin{aligned} & O_1 \frac{1}{|v|^j} O_2 \frac{1}{|v|^{p-j}} \quad (0 \leq j \leq p \leq k) \\ & \frac{1}{|v|^{k+1}} O_1 O_2, \quad \frac{1}{|v|^k} O_1 \frac{1}{|v|} O_2, \quad \frac{1}{|v|^k} O_1 O_2 \frac{1}{|v|}, \quad \frac{1}{|v|^{k+1}} O_1 |v|^{k+1} \\ & |v|^{-(k+1-p)} O_2 |v|^{k+1-p} \quad (0 \leq p \leq k) \end{aligned}$$

are bounded. Then $(e^{tT_O})_{t \geq 0}$ has a unique invariant density ψ_0 and

$$\left\| e^{tT_O} g - \left(\int_{\Omega} g \right) \psi_0 \right\| = O \left(t^{-\frac{k}{2(k+1)+1}} \right) \quad (t \rightarrow +\infty) \quad (8)$$

for any initial data $g \in D(T_O)$ such that

$$\int_{\Omega} |g(x, v)| |v|^{-(k+1)} dx dv < +\infty.$$

Note that the operator $|v|^j$ ($j \in \mathbb{Z}$) refers to the multiplication operator by the function $|v|^j$. We will comment below on our assumptions. The rate of

convergence (8) is derived from a quantified version of Ingham's tauberian theorem [11]; (see Section 2 below). Its proof is quite involved and consists in showing that *the restriction of the resolvent $(\lambda - T_O)^{-1}$ to a suitable subspace extends continuously to $i\mathbb{R} \setminus \{0\}$ as a C^k function with suitable C^k estimates on $i\mathbb{R} \setminus \{0\}$* . The main object of this work is therefore to show how to obtain such estimates provided that one of the boundary conditions is completely diffuse. (For the obstruction to the treatment of the general case (6), see Remark 31.) We note that for the stochastic kinetic semigroups we consider here, 0 always belongs to the spectrum of the generator and it may happen (e.g. if $\beta_1 = 1$ or $\beta_2 = 1$) that the whole imaginary axis is included in the spectrum of the generator. Note also that (7) need not be the completely diffuse model which corresponds to $\beta_1 = \beta_2 = 1$; for instance the case of a diffuse boundary condition at $x = -a$ and a specular boundary condition at $x = a$ is covered by our statement. The proof of the rate of convergence (8) is given at the end of this article (see Theorem 29) as a consequence of various preliminary results of independent interest.

To our knowledge, the theorem above provides us with the first systematic quantitative result in collisionless kinetic theory for L^1 initial datum. Indeed, until now, the sole known quantified L^1 results in collisionless kinetic theory are much better rates obtained for *bounded initial datum in balls* with Maxwell diffuse boundary conditions and constant boundary temperature. More precisely, in dimension 3, the rate of convergence in L^1 norm is $O(t^{-1})$ if the initial data is radial (in space and in velocity) and is dominated by a maxwellian function (see [1] Theorem 4.1); this result was improved in ([15] Corollary 2) where the rate is shown to be $O(t^{-d})$ in dimension $d \leq 3$ for bounded initial datum; (see [1][15] for additional results which we do not comment on here). We point out that Maxwell diffuse boundary conditions refer to boundary operators which are (local in space and) *rank-one* in velocity. Finally, we mention that quantitative time asymptotics have never been dealt with for *partly* diffuse boundary operators.

We give now a more precise view on the mathematical construction behind the rate of convergence (8). Let

$$W_1(\Omega) = \left\{ f \in L^1(\Omega); v \frac{\partial f}{\partial x} \in L^1(\Omega) \right\}$$

($v \frac{\partial f}{\partial x}$ is understood in the sense of distributions) be endowed with the norm

$$\|f\|_{W_1} = \|f\| + \left\| v \frac{\partial f}{\partial x} \right\|$$

where

$$\|g\| = \int_{-a}^{+a} \int_{-1}^{+1} |g(x, v)| dx dv, \quad g \in L^1(\Omega).$$

According to classical trace theory (see [8][9]), the elements of $W_1(\Omega)$ admit a trace on

$$\{-a\} \times (-1, +1) \text{ and } \{a\} \times (-1, +1)$$

belonging to the weighted L^1 -space

$$L^1((-1, +1); |v| dv).$$

More precisely, the trace operator is surjective, continuous and admits a continuous lifting operator. For any $f \in W_1(\Omega)$, we denote by f_{-a}^- (resp. f_{-a}^+) the restriction of $f(-a, \cdot)$ to $(-1, 0)$ (resp. to $(0, 1)$), i.e.

$$f_{-a}^-: (-1, 0) \ni v \rightarrow f(-a, v); \quad f_{-a}^+: (0, 1) \ni v \rightarrow f(-a, v).$$

Similarly

$$f_a^-: (-1, 0) \ni v \rightarrow f(a, v); \quad f_a^+: (0, 1) \ni v \rightarrow f(a, v).$$

We keep in mind that

$$f_{-a}^-, f_a^- \in L^1((-1, 0); |v| dv) \quad \text{and} \quad f_{-a}^+, f_a^+ \in L^1((0, +1); |v| dv).$$

We define also

$$h_a^\pm(v) = |v| f_a^\pm(v), \quad h_{-a}^\pm(v) = |v| f_{-a}^\pm(v)$$

and keep in mind that

$$h_{-a}^-, h_a^- \in L^1((-1, 0); dv) \quad \text{and} \quad h_{-a}^+, h_a^+ \in L^1((0, +1); dv).$$

The transport operator

$$T_O : D(T_O) \subset L^1(\Omega) \rightarrow L^1(\Omega),$$

indexed by $O := (O_1, O_2)$, is defined by

$$T_O f = -v \frac{\partial f}{\partial x}$$

on the domain

$$D(T_O) = \{f \in W_1(\Omega); \quad h_{-a}^+ = O_1 h_{-a}^-, \quad h_a^- = O_2 h_a^+\}.$$

It is known (see [21]) that T_O generates a stochastic (i.e. mass preserving on the positive cone) C_0 -semigroup $(e^{tT_O})_{t \geq 0}$ and, for $g \in L^1(\Omega)$,

$$f := (\lambda - T_O)^{-1} g, \quad (\operatorname{Re} \lambda > 0)$$

is given by

$$f(x, v) = \frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+ + \int_{-a}^x e^{-\frac{\lambda}{v}(x-y)} \frac{1}{v} g(y, v) dy \quad (v > 0) \quad (9)$$

$$f(x, v) = \frac{1}{|v|} e^{-\frac{\lambda}{|v|}(a-x)} h_a^- + \int_x^a e^{-\frac{\lambda}{|v|}(y-x)} \frac{1}{|v|} g(y, v) dy \quad (v < 0) \quad (10)$$

with

$$\begin{aligned} h_{-a}^+ &= (1 - G_\lambda)^{-1} O_1 e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) \\ &\quad + (1 - G_\lambda)^{-1} O_1 \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \end{aligned} \quad (11)$$

$$h_a^- = O_2 \left(e^{-\frac{2\lambda a}{|v|}} h_{-a}^+ + \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) \quad (12)$$

and

$$G_\lambda f = O_1 \left(e^{-\frac{2\lambda a}{|v|}} O_2 \left(e^{-\frac{2\lambda a}{|v|}} f \right) \right) \quad (13)$$

where the operator " $e^{-\frac{2\lambda a}{|v|}}$ " refers to the *multiplication* operator by the function $e^{-\frac{2\lambda a}{|v|}}$. For the sake of simplicity, if no ambiguity may occur, the different (natural) L^1 norms as well as their corresponding operator norms are denoted by the symbol $|||$. Note that $\|G_\lambda\| \leq e^{-4a \operatorname{Re} \lambda}$ ($\operatorname{Re} \lambda \geq 0$) and $\|G_0\| = 1$. Under the general assumption (6), the *essential* spectral radii of the stochastic operators

$$G_0 = O_1 O_2 : L^1((0, +1); dv) \rightarrow L^1((0, +1); dv)$$

$$\tilde{G}_0 = O_2 O_1 : L^1((-1, 0); dv) \rightarrow L^1((-1, 0); dv)$$

are *strictly* less than 1; in particular, G_0 and \tilde{G}_0 admit 1 as an isolated eigenvalue associated respectively to the nonnegative eigenfunctions

$$h_0 \in L_+^1((0, +1); dv) \quad (14)$$

and

$$\tilde{h}_0 \in L_+^1((-1, 0); dv)$$

with

$$O_2 h_0 = \tilde{h}_0 \text{ and } O_1 \tilde{h}_0 = h_0. \quad (15)$$

Furthermore, T_O admits 0 as eigenvalue (i.e. $(e^{tT_O})_{t \geq 0}$ has an invariant density) *if and only if*

$$\int_0^1 \frac{h_0(v)}{v} dv + \int_{-1}^0 \frac{\tilde{h}_0(v)}{|v|} dv < \infty; \quad (16)$$

in this case, a space homogeneous invariant density is given by

$$\psi_0(v) = \begin{cases} \frac{1}{v} h_0(v) & (v > 0) \\ \frac{1}{|v|} \tilde{h}_0(v) & (v < 0); \end{cases} \quad (17)$$

see [21] for all these results. We note that (16) requires that the kernels of the diffuse parts K_i vanish (in an appropriate sense) at $v = 0$. For example, (16) is *not* satisfied in the purely diffuse case (i.e. $\beta_1 = \beta_2 = 1$) if

$$\inf_{(v, v')} k_i(v, v') > 0 \quad (i = 1, 2). \quad (18)$$

Of course, the object of this paper is meaningful *only if* $(e^{tT_O})_{t \geq 0}$ has an invariant density. A sufficient condition ensuring (16) is given in Theorem 5 below, (see also Remark 6). Actually, the present paper is built on much stronger structural assumptions (see below) so that the existence of the invariant density is guaranteed.

If $(e^{tT_O})_{t \geq 0}$ is irreducible (a criterion is given in Theorem 7) then, under the normalization $\int_{\Omega} \psi_0 = 1$, the invariant density ψ_0 is unique and the C_0 -semigroup $(e^{tT_O})_{t \geq 0}$ is mean ergodic with ergodic projection

$$P : g \rightarrow \left(\int_{\Omega} g \right) \psi_0, \quad (19)$$

i.e.

$$L^1(\Omega) = \text{Ker}(T_O) \oplus \overline{\text{Ran}(T_O)}$$

and the mean ergodic convergence

$$s \lim_{t \rightarrow +\infty} t^{-1} \int_0^t e^{sT_O} ds = P$$

($s \lim_{t \rightarrow +\infty}$ refers to *strong* limit) holds where P is the projection on $\text{Ker}(T_O)$ along $\overline{\text{Ran}(T_O)}$.

The convergence

$$s \lim_{t \rightarrow +\infty} e^{tT_O} = P$$

is proved in [21] under the condition $\beta_1 \beta_2 > 0$ by using a result (from [24]) on *partially* integral semigroups; (a new approach of this result is considered in [22]).

We point out that if (16) were *not* satisfied then $(e^{tT_O})_{t \geq 0}$ would be *sweeping* with respect to the sets

$$(-a, a) \times [(-1, -\varepsilon) \cup (\varepsilon, +1)] \quad (\varepsilon > 0)$$

in the sense that the total mass of $e^{tT_O} g$ concentrates in the vicinity of $v = 0$ (i.e. around the tangential velocities) as $t \rightarrow +\infty$, i.e.

$$\int_{-1}^{-\varepsilon} \int_{-a}^{+a} |(e^{tT_O} g)(x, v)| dx dv + \int_{\varepsilon}^1 \int_{-a}^{+a} |(e^{tT_O} g)(x, v)| dx dv \rightarrow 0 \quad (20)$$

as $t \rightarrow +\infty$, [21]. Actually, the following alternative holds: $(e^{tT_O})_{t \geq 0}$ is either strongly convergent if an invariant density exists or is sweeping in the sense (20) otherwise; (i.e. a *Foguel-like* alternative holds, see [16], Theorem 5. 10. 1, p. 130).

Thus, we are concerned here with quantitative time asymptotics of strongly convergent kinetic models; (the relevant open question for the non-convergent kinetic models is whether we can *quantify their sweeping behaviour* (20), see Remark 32 (ii)). To this end, a key preliminary result is that

$$r_{\sigma}(G_{is}) < 1 \quad (s \in \mathbb{R} \setminus \{0\})$$

(r_σ refers to spectral radius) and

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \ni \lambda \rightarrow (1 - G_\lambda)^{-1} \in \mathcal{L}(L^1(\Omega))$$

extends continuously (in the strong operator topology) to $i\mathbb{R} \setminus \{0\}$. Various technical estimates are given in this paper. We can summarize them in *two key statements*. Let $k \in \mathbb{N}$, $k \neq 0$; (the integer k comes from the structural assumptions).

The first statement is: if

$$\int_{-a}^a \int_{-1}^1 \frac{|g(x, v)|}{|v|^{k+1}} dx dv < +\infty$$

then

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \ni \lambda \rightarrow (\lambda - T_O)^{-1} g \in L^1(\Omega)$$

extends continuously to $i\mathbb{R} \setminus \{0\}$ and, with

$$F_g(s) := \lim_{\substack{\lambda \rightarrow is \\ \operatorname{Re} \lambda > 0}} (\lambda - T_O)^{-1} g, \quad (21)$$

the map

$$\mathbb{R} \setminus \{0\} \ni s \rightarrow F_g(s) \in L^1(\Omega)$$

lies in $C^k(\mathbb{R} \setminus \{0\}; L^1(\Omega))$ with the uniform C^k estimates

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq C \left(\sum_{p=0}^{j+1} \|(1 - G_{is})^{-1}\|^p \right) \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k, s \neq 0)$$

where $C > 0$ is a constant, see Theorem 20.

The second statement is:

$$\sup_{|s| \geq \eta} \|(1 - G_{is})^{-1}\| < \infty, \quad (\eta > 0)$$

and there exists a constant $C > 0$ such that

$$\|(1 - G_{is})^{-1}\| \leq \frac{C}{s^2} \quad (\text{for small } s \in \mathbb{R} \setminus \{0\}),$$

see Theorem 23.

It follows that

$$\sup_{|s| \geq \eta} \left\| \frac{d^j}{ds^j} F_g(s) \right\| < +\infty \quad (\eta > 0, 0 \leq j \leq k)$$

and there exists a constant $C > 0$ such that

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C}{|s|^{2(j+1)}} \quad (0 \leq j \leq k) \quad (\text{for small } s \in \mathbb{R} \setminus \{0\})$$

and consequently a quantified version of Ingham's theorem (see Corollary 3 below) implies

$$\left\| e^{tT_O} g - \left(\int_{\Omega} g \right) \psi_0 \right\| = O \left(t^{-\frac{k}{2(k+1)+1}} \right), \quad (t \rightarrow +\infty) \quad (22)$$

for any initial data $g \in D(T_O)$ such that $\left\| \frac{g}{|v|^{k+1}} \right\| < +\infty$, see Theorem 29.

Apart from Theorem 21 and Theorem 23 (which hold under the general condition $\beta_1 + \beta_2 > 0$), the paper is based upon a set of structural assumptions (37)(38)(43)(44). A priori, Assumptions (43)(44), which say that

$$|v|^{-(k+1)} O_1 |v|^{k+1} \quad \text{and} \quad |v|^{-(k+1-p)} O_2 |v|^{k+1-p} \quad \text{are bounded} \quad (0 \leq p \leq k),$$

are *checkable*. Indeed $|v|^{-j} R_i |v|^j$ ($i = 1, 2$) are always bounded while the boundedness of $|v|^{-j} K_i |v|^j$ ($i = 1, 2$) is a condition on the kernel of K_i ($i = 1, 2$) in the neighborhood of $v = 0$. On the other hand, (37)(38) are checkable only if O_1 is a kernel operator (or if O_2 is a kernel operator); this explains why the condition " $\beta_1 = 1$ or $\beta_2 = 1$ " appears in some statements. We point out that the need for conditions on the kernels of K_i ($i = 1, 2$) near the tangential velocities (i.e. $v = 0$) is not fortuitous since the existence of an invariant density already *requires* a condition in the same spirit, see (16) and (18).

The fact that $\frac{k}{2(k+1)+1} \rightarrow \frac{1}{2}$ ($k \rightarrow \infty$) shows that if there exists $C_j > 0$ such that

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C_j}{|s|^{2(j+1)}} \quad (0 < |s| \leq 1, \quad j \in \mathbb{N})$$

(this occurs if the structural assumptions are satisfied for *all* $k \in \mathbb{N}$) then the quantified version of Ingham's tauberian theorem provides us with the rate

$$O \left(\frac{1}{t^{\frac{1}{2+\varepsilon}}} \right), \quad (\varepsilon > 0). \quad (23)$$

It is a priori unclear whether we can reach the limit rate $O \left(\frac{1}{\sqrt{t}} \right)$ or can go beyond this rate for the kinetic semigroups $(e^{tT_O})_{t \geq 0}$ (note that much better rates of convergence occur for *bounded* initial datum in balls, see [1][15]). We refer to Remark 31 and Remark 32 for different open problems suggested by our construction.

Our paper is organized as follows:

In Section 2, we give a corollary of a quantified version of Ingham's theorem [11] which implies the rates of convergence

$$\|e^{tT} g - P g\| = O \left(t^{-\frac{k}{\alpha(k+1)+1}} \right), \quad t \rightarrow +\infty$$

for bounded mean ergodic C_0 -semigroups $(e^{tT})_{t \geq 0}$ on a Banach space X with ergodic projection P (and generator T) for initial data

$$g \in D(T) \cap (Ker(T) + Ran(T)) \quad (24)$$

provided that $F_g(s) := \lim_{\varepsilon \rightarrow 0_+} (\varepsilon + is - T)^{-1}g$ ($s \neq 0$) exists, lies in $C^k(\mathbb{R} \setminus \{0\}; X)$ for some $k \in \mathbb{N}$ and satisfies the estimates

$$\sup_{|s| \geq 1} \|F_g^{(j)}(s)\| < +\infty \text{ and } \|F_g^{(j)}(s)\| \leq C |s|^{-\alpha(j+1)}, \quad (0 \leq j \leq k, \quad 0 < |s| \leq 1).$$

In Section 3, we give a sufficient criterion for the existence of an invariant density of $(e^{tT_O})_{t \geq 0}$. A sufficient criterion of irreducibility of $(e^{tT_O})_{t \geq 0}$ is given in Section 4. The combination of the last two results implies that the C_0 -semigroup $(e^{tT_O})_{t \geq 0}$ is mean ergodic. Because of the importance of (24), a sufficient criterion for a given $g \in L^1(\Omega)$ to belong to the range of T_O is given in Section 5. Section 6 is devoted to $\sigma(T_O) \cap i\mathbb{R}$, the boundary spectrum of the generator; while $0 \in \sigma(T_O)$ is always true, we show that the imaginary axis is equal to the boundary spectrum at least when $\beta_1 = 1$ or $\beta_2 = 1$. In Section 7, we explain why

$$F_g(s) := \lim_{\varepsilon \rightarrow 0_+} (\varepsilon + is - T_O)^{-1}g \quad (s \neq 0)$$

exists and lies in $C^k(\mathbb{R} \setminus \{0\}; L^1(\Omega))$ if $\int_{\Omega} |g(x, v)| |v|^{-(k+1)} dx dv < +\infty$ and if the boundary fluxes h_a^+ and h_a^- given by (11)(12) with $\lambda = is$ ($s \neq 0$) are C^k functions of $s \in \mathbb{R} \setminus \{0\}$ and their j th derivatives belong to suitable *weighted* spaces depending on j ($1 \leq j \leq k$). Such conditions depends heavily on the existence of $(1 - G_{is})^{-1}$ ($s \neq 0$) and its derivatives in s (in suitable spaces) which are thus the cornerstone of this work. The existence and estimate of $(1 - G_{is})^{-1}$ ($s \neq 0$) are postponed until Section 11. Under the general condition

$$\beta_1 + \beta_2 > 0, \quad (25)$$

we show that $r_{\sigma}(|G_{is}|) < 1$ ($s \neq 0$) where $|G_{is}|$ is the *linear modulus* of G_{is} (see [10]). The proof relies on strict comparison of spectral radii of positive operators in a context of domination [19]. We show also the key estimates

$$\sup_{|\lambda| \geq \eta} \|(1 - G_{\lambda})^{-1}\| < +\infty \quad (\eta > 0, \quad \operatorname{Re} \lambda \geq 0)$$

$$\|(1 - G_{\lambda})^{-1}\| = O\left(\frac{1}{|\operatorname{Im} \lambda|^2}\right) \quad (\lambda \rightarrow 0, \quad \operatorname{Re} \lambda \geq 0, \quad \lambda \neq 0). \quad (26)$$

The proof of (26) is quite involved and relies on a second order expansion about $s = 0$ (uniformly in $\varepsilon \geq 0$) of a suitable function related to $\mathbb{R} \ni s \rightarrow \|G_{\varepsilon + is}\|$. In Section 8, we show by induction the key estimate of the *derivatives*

$$\left\| \frac{d^j}{ds^j} (1 - G_{is})^{-1} \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+j} dv))} \leq C \sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \quad (1 \leq j \leq k)$$

by exploiting a differential equation satisfied by $\mathbb{R} \setminus \{0\} \ni s \rightarrow (1 - G_{is})^{-1}$. It is at this place that we need that at least one of the boundary conditions must be

completely diffuse. In Section 9, we deduce the estimate on the left flux

$$\left\| \frac{d^j h_{-a}^+}{d\lambda^j} \right\|_{L^1(|v|^{-k-1+p} dv)} \leq C \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k)$$

and a similar estimate on the right flux h_a^- . In Section 10, we sum up the previous estimates in the statement

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq C \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k, s \neq 0).$$

Finally, in Section 12, we deduce the algebraic estimates of (21) on $i\mathbb{R} \setminus \{0\}$

$$\sup_{|s| \geq \eta} \left\| \frac{d^j}{ds^j} F_g(s) \right\| < +\infty, \quad (\eta > 0, 0 \leq j \leq k)$$

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C}{s^{2(j+1)}} \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k, s \rightarrow 0)$$

and derive, from the quantified version of Ingham's theorem, the rate of convergence

$$\left\| e^{tT_O} g - \left(\int_{\Omega} g \right) \psi_0 \right\| = O \left(t^{-\frac{k}{2(k+1)+1}} \right), \quad (t \rightarrow +\infty)$$

for any initial data $g \in D(T_O) \cap L^1(\Omega; \frac{dv}{|v|^{k+1}})$.

As far as we know, all these functional analytic results on collisionless kinetic theory appear here for the first time. Some open problems suggested by our construction in slab geometry are pointed out in Remark 31 and Remark 32 below. We note that this work could be extended to *non*-monoenergetic free transport equations in slab geometry with more general reflection operators R_i ($i = 1, 2$), see Remark 33. However, its extension to multidimensional-space geometries is an open problem, see Remark 34. For the sake of simplicity, in all the paper, we will denote by the same symbol C various positive constants occuring in our different proofs and statements. The authors thank the referee for constructive comments.

2 A quantified version of Ingham's theorem

Let X be a complex Banach space. For any $f \in L^\infty(\mathbb{R}_+, X)$, we define its Laplace transform by

$$\widehat{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt \quad (\operatorname{Re} \lambda > 0).$$

Let $\eta \in \mathbb{R}$. We say that $i\eta$ is a weakly regular point for \widehat{f} if there exist $\varepsilon > 0$ and $h \in L^1((\eta - \varepsilon, \eta + \varepsilon), X)$ such that

$$\widehat{f}(\alpha + i.) \rightarrow h(.) \text{ in the distributional sense on } (\eta - \varepsilon, \eta + \varepsilon) \text{ as } \alpha \rightarrow 0_+.$$

The weak half-line spectrum $sp_w(f)$ of f is defined as the set of all real numbers η such that $i\eta$ is not weakly regular for \hat{f} . Then $sp_w(f)$ is a closed subset of \mathbb{R} and there exists $F \in L^1_{loc}(\mathbb{R} \setminus sp_w(f), X)$ such that

$$\hat{f}(\alpha + i.) \rightarrow F(.) \text{ in the distributional sense on } \mathbb{R} \setminus sp_w(f) \text{ as } \alpha \rightarrow 0_+, \quad (27)$$

see e.g. [2] Lemma 4.9.9, p. 326. We give now a quantified version of the classical Ingham's tauberian theorem (see e.g. [2] Theorem 4.9.5, p. 327). This version is a special case of ([11] Theorem 2.13 (a)).

Theorem 1 *Let X be a complex Banach space and suppose that f belongs to $L^\infty(\mathbb{R}_+, X)$, is Lipschitz continuous and $\sup_{t \geq 0} \left\| \int_0^t f(s) ds \right\| < +\infty$. Suppose furthermore that $sp_w(f) \subset \{0\}$ and F (given by (27)) lies in $C^k(\mathbb{R} \setminus \{0\}; X)$ for some $k \in \mathbb{N}$. If $\sup_{|s| \geq 1} \|F^{(j)}(s)\| < +\infty$ ($0 \leq j \leq k$) and if*

$$\|F^{(j)}(s)\| \leq C |s|^{-\alpha(j+1)}, \quad (0 \leq j \leq k, 0 < s \leq 1)$$

for some constants $C > 0$, $\alpha \geq 1$ then

$$\|f(t)\| = O\left(t^{-\frac{k}{\alpha(k+1)+1}}\right), \quad (t \rightarrow +\infty).$$

Now suppose that $(e^{tT})_{t \geq 0}$ is a bounded C_0 -semigroup with generator T on X , and that $f(t) = e^{tT}g$ ($t \geq 0$) for some $g \in X$. Then f is a bounded continuous function. It is Lipschitz continuous if $g \in D(T)$ and has uniformly bounded primitive if $g \in \text{Ran}(T)$. Recall that the Laplace of f is given by $(\mathcal{L}f)(\lambda) = R(\lambda, T)g$ for $\text{Re } \lambda > 0$. Hence, by a calculation similar to that in ([11], Eq. (1.2)) we see that f satisfies the assumptions of Theorem 1 provided that $g \in D(T) \cap \text{Ran}(T)$ and $R(\lambda, T)g$ extends continuously to a sufficiently smooth function on $i\mathbb{R} \setminus \{0\}$. Note that, crucially for us, this is possible for particular initial values $g \in X$ even if $i\mathbb{R} \subset \sigma(T)$. We obtain the following corollary of Theorem 1.

Corollary 2 *Let $(e^{tT})_{t \geq 0}$ be a bounded C_0 -semigroup with generator T on a complex Banach space X and let $g \in D(T) \cap \text{Ran}(T)$. Suppose that $R(\lambda, T)g$ ($\text{Re } \lambda > 0$) extends continuously to $i\mathbb{R} \setminus \{0\}$ and that*

$$F_g(s) := \lim_{\varepsilon \rightarrow 0_+} R(is + \varepsilon, T)g$$

lies in $C^k(\mathbb{R} \setminus \{0\}; X)$ for some $k \in \mathbb{N}$. If $\sup_{|s| \geq 1} \|F_g^{(j)}(s)\| < +\infty$ ($0 \leq j \leq k$) and if

$$\|F_g^{(j)}(s)\| \leq C |s|^{-\alpha(j+1)}, \quad (0 \leq j \leq k, 0 < s \leq 1)$$

for some constants $C > 0$, $\alpha \geq 1$ then

$$\|e^{tT}g\| = O\left(t^{-\frac{k}{\alpha(k+1)+1}}\right), \quad (t \rightarrow +\infty).$$

In this paper, we need the following simple consequence of Corollary 2.

Corollary 3 *Let $(e^{tT})_{t \geq 0}$ be a bounded mean ergodic C_0 -semigroup with generator T on a complex Banach space X with ergodic projection P . Let*

$$g \in D(T) \cap (Ker(T) + Ran(T)).$$

Suppose that $R(\lambda, T)g$ ($\operatorname{Re} \lambda > 0$) extends continuously to $i\mathbb{R} \setminus \{0\}$ and that

$$F_g(s) := \lim_{\varepsilon \rightarrow 0_+} R(is + \varepsilon, T)g$$

lies in $C^k(\mathbb{R} \setminus \{0\}; X)$ for some $k \in \mathbb{N}$. If $\sup_{|s| \geq 1} \|F_g^{(j)}(s)\| < +\infty$ ($0 \leq j \leq k$) and if

$$\|F_g^{(j)}(s)\| \leq C |s|^{-\alpha(j+1)}, \quad (0 \leq j \leq k, \quad 0 < s \leq 1)$$

for some constants $C > 0$, $\alpha \geq 1$ then

$$\|e^{tT}g - Pg\| = O\left(t^{-\frac{k}{\alpha(k+1)+1}}\right), \quad (t \rightarrow +\infty).$$

Remark 4 *Theorem 1 can be complemented by the statement that if $F \in C^\infty(\mathbb{R} \setminus \{0\}; X)$ satisfies $\sup_{|s| \geq 1} \|F^{(j)}(s)\| < \infty$ ($j \in \mathbb{Z}_+$) and if there exists a constant $C > 0$ such that*

$$\|F^{(j)}(s)\| \leq C j! |s|^{-\alpha(j+1)+1}, \quad (j \in \mathbb{Z}_+, \quad 0 < s \leq 1)$$

then $\|f(t)\| = O\left(\left(\frac{\ln(t)}{t}\right)^{\frac{1}{\alpha}}\right)$, ($t \rightarrow +\infty$), (see [11] Theorem 2.13 (b)).

3 On existence of invariant density

We complement a result from [21].

Theorem 5 *We assume that either $O_1 = K_1$ and both $|v|^{-1} K_1$ and $|v|^{-1} K_2 |v|$ are bounded or $O_2 = K_2$ and both $|v|^{-1} K_2$ and $|v|^{-1} K_1 |v|$ are bounded. Then (16) is satisfied and consequently $(e^{tT_O})_{t \geq 0}$ has an invariant density.*

Proof. Without loss of generality, we may suppose that $\beta_1 = 1$ and therefore $\alpha_1 = 0$. We know that $G_0 h_0 = h_0$ and $\tilde{G}_0 h_0 = \tilde{h}_0$. Thus

$$K_1 (\alpha_2 R_2 + \beta_2 K_2) h_0 = h_0$$

so $\int_0^1 \frac{h_0(v)}{v} dv < \infty$. By (15)

$$\tilde{h}_0 = O_2 h_0 = \alpha_2 R_2 h_0 + \beta_2 K_2 h_0.$$

By assumption $K_2 h_0 \in L^1\left(\frac{dv}{|v|}\right)$ if $h_0 \in L^1\left(\frac{dv}{|v|}\right)$. Since $L^1\left(\frac{dv}{|v|}\right)$ is invariant under R_2 we have $\tilde{h}_0 \in L^1\left(\frac{dv}{|v|}\right)$. ■

Remark 6 *The assumptions in Theorem 5 can be weakened. For instance, we can replace the boundedness of $|v|^{-1} K_1$ by the assumption that*

$$|v|^{-1} K_1 R_2 K_1 \text{ and } |v|^{-1} K_1 K_2 K_1 \text{ are bounded.}$$

Indeed, since $[K_1 (\alpha_2 R_2 + \beta_2 K_2)]^2 h_0 = h_0$ then $h_0 \in L^1 \left(\frac{dv}{|v|} \right)$ provided that $|v|^{-1} K_1 (\alpha_2 R_2 + \beta_2 K_2) K_1$ is bounded.

4 On irreducibility of $(e^{tT_O})_{t \geq 0}$

We give two complementary irreducibility criteria.

Theorem 7 *We assume that either $O_1 = K_1$ or $O_2 = K_2$. If*

$$G_0 = O_1 O_2 : L^1((0, +1); dv) \rightarrow L^1((0, +1); dv)$$

is irreducible and if O_2 is strict positivity preserving in the sense that

$$h(v) > 0 \text{ a.e.} \implies (O_2 h)(v) > 0 \text{ a.e.}$$

then $(e^{tT_O})_{t \geq 0}$ is irreducible.

Proof. Note that

$$(1 - G_\lambda)^{-1} = \sum_{j=0}^{\infty} G_\lambda^j \quad (\lambda > 0)$$

so that for any nonnegative h and h^*

$$\langle (1 - G_\lambda)^{-1} h, h^* \rangle \geq \langle G_\lambda^j h, h^* \rangle, \quad (j \in \mathbb{N}, \lambda > 0).$$

Since G_0 is irreducible then for any non trivial nonnegative h and h^* there exists an integer j (depending a priori on h and h^*) such that $\langle G_0^j h, h^* \rangle > 0$. Since

$$\lim_{\lambda \rightarrow 0_+} \langle G_\lambda^j h, h^* \rangle = \langle G_0^j h, h^* \rangle$$

then $\langle G_\lambda^j h, h^* \rangle > 0$ for λ small enough. Since $\lambda \rightarrow \langle G_\lambda^j h, h^* \rangle \in \mathbb{R}_+$ is nonincreasing, an analyticity argument shows that

$$\langle G_\lambda^j h, h^* \rangle > 0, \quad (\lambda > 0)$$

and finally $\langle (1 - G_\lambda)^{-1} h, h^* \rangle > 0$ so $(1 - G_\lambda)^{-1} h > 0$ a.e. Thus (11) gives $h_{-a}^+ > 0$ a.e. for any non trivial nonnegative g and (12) gives $h_a^- > 0$ a.e. since O_2 is strict positivity preserving. Finally $(1 - T_O)^{-1}$ is positivity improving or equivalently $(e^{tT_O})_{t \geq 0}$ is irreducible. ■

Remark 8 Note that G_0 is an integral operator with kernel $q(v, v')$. The irreducibility of G_0 amounts to

$$\int_{[0,1] \setminus S} \left(\int_S q(v, v') dv' \right) dv > 0$$

for any measurable $S \subset [0, 1]$ such that S and $[0, 1] \setminus S$ have positive measure. In particular, this is the case if $q(v, v') > 0$ a.e. Note that $O_2 = \alpha_2 R_2 + \beta_2 K_2$ is automatically strict positivity preserving if $\alpha_2 > 0$.

Remark 9 Another irreducibility criterion is a "dual" version of Theorem 7: Assume that either $O_1 = K_1$ or $O_2 = K_2$. If

$$\tilde{G}_0 = O_2 O_1 : L^1((-1, 0); dv) \rightarrow L^1((-1, 0); dv)$$

is irreducible and if O_1 is strict positivity preserving then $(e^{tT_O})_{t \geq 0}$ is irreducible. Indeed, it is easy to see that

$$h_a^- = \left(I - \tilde{G}_\lambda \right)^{-1} O_2 \left[e^{-\frac{\lambda}{v} 2a} O_1 \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) \right]$$

$$\tilde{G}_\lambda := O_2 e^{-\frac{\lambda}{v} 2a} O_1 e^{-\frac{\lambda}{|v|} 2a}$$

$$h_{-a}^+ = O_1 h_{-a}^- = O_1 \left(e^{-\frac{\lambda}{|v|} 2a} h_a^- + \int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right)$$

and then it suffices to exchange the roles of h_{-a}^+ and h_a^- and to argue as previously.

5 On the range of T_O

According to Corollary 3, the knowledge of the range of the generator is a key point. To this end, we describe now a useful subspace of the range of T_O .

Theorem 10 We assume that either $O_1 = K_1$ and both $|v|^{-1} K_1$ and $|v|^{-1} K_2 |v|$ are bounded or $O_2 = K_2$ and both $|v|^{-1} K_2$ and $|v|^{-1} K_1 |v|$ are bounded. We assume additionally, in the first case, that $G_0 = O_1 O_2$ is irreducible or, in the second case, that $\tilde{G}_0 = O_2 O_1$ is irreducible. Let $g \in L^1(\Omega)$. If

$$\frac{1}{|v|} g \in L^1(\Omega) \tag{28}$$

and if

$$\int_\Omega g = 0 \tag{29}$$

then $g \in \text{Ran}(T_O)$.

Proof. Note that $(e^{tT_O})_{t \geq 0}$ is a stochastic semigroup so that

$$\int_{\Omega} T_O \varphi = 0, \quad \varphi \in D(T_O)$$

and consequently (29) is a necessary condition for $g \in L^1(\Omega)$ to belong to $Ran(T_O)$. We consider the case $\beta_1 = 1$. Let

$$g_- : (y, v) \in (-a, a) \times (-1, 0) \rightarrow g(y, v),$$

$$g_+ : (y, v) \in (-a, a) \times (0, +1) \rightarrow g(y, v)$$

and

$$\widehat{g}_-(v) := \int_{-a}^a g_-(y, v) dy, \quad \widehat{g}_+(v) := \int_{-a}^a g_+(y, v) dy.$$

Note that by assumption

$$\widehat{g}_- \in L^1\left((-1, 0); \frac{dv}{|v|}\right) \text{ and } \widehat{g}_+ \in L^1\left((0, +1); \frac{dv}{|v|}\right).$$

By inspection of (9)(10)(11)(12), for solving $(\lambda - T_O)f = g$ with $\lambda = 0$, it suffices that (28) is satisfied,

$$O_1 O_2 (\widehat{g}_+) + O_1 (\widehat{g}_-) \in Ran(1 - G_0), \quad (30)$$

that h_{-a}^+ , given by (11), is such that

$$\frac{1}{|v|} h_{-a}^+ \in L^1((0, +1); dv) \quad (31)$$

and that

$$\frac{1}{|v|} h_a^- := \frac{1}{|v|} O_2 (h_{-a}^+ + \widehat{g}_+) \in L^1((-1, 0); dv). \quad (32)$$

Note that 1 is an isolated algebraically simple eigenvalue of $G_0 = O_1 O_2$ associated with the eigenfunction h_0 (see (14)) so that

$$Ran(1 - G_0) \text{ is closed in } L^1((0, +1); dv).$$

By the Fredholm alternative

$$G_0 (\widehat{g}_+) + O_1 (\widehat{g}_-) \in Ran(1 - G_0)$$

if and only if $G_0 (\widehat{g}_+) + O_1 (\widehat{g}_-)$ is orthogonal (for the duality pairing) to the dual eigenfunction $h_0^* \in L^\infty((0, +1); dv)$ i.e.

$$\langle G_0 (\widehat{g}_+) + O_1 (\widehat{g}_-), h_0^* \rangle = 0$$

or

$$\langle G_0 (\widehat{g}_+), h_0^* \rangle + \langle O_1 (\widehat{g}_-), h_0^* \rangle = 0.$$

Since

$$\langle G_0(\widehat{g}_+), h_0^* \rangle = \langle \widehat{g}_+, G_0^* h_0^* \rangle = \langle \widehat{g}_+, h_0^* \rangle$$

we have

$$\langle \widehat{g}_+ + O_1(\widehat{g}_-), h_0^* \rangle = 0.$$

On the other hand, $G_0 : L^1((0, +1); dv) \rightarrow L^1((0, +1); dv)$ is integral preserving, i.e.

$$\int_0^1 G_0 \varphi = \int_0^1 \varphi \quad \forall \varphi \in L^1((0, +1); dv)$$

so $G_0^* 1 = 1$ and $h_0^* = 1$. Thus

$$\int_0^1 \widehat{g}_+ + \int_0^1 O_1(\widehat{g}_-) = 0.$$

Since O_1 is also integral preserving we have

$$\int_0^1 O_1(\widehat{g}_-) = \int_{-1}^0 \widehat{g}_-$$

and finally (30) amounts to

$$\int_0^1 \widehat{g}_+ + \int_{-1}^0 \widehat{g}_- = 0$$

which is nothing but (29). Hence (30) is satisfied. Note that

$$L^1((0, +1); dv) = \text{Ker}(I - G_0) \oplus \text{Ran}(1 - G_0)$$

and $\text{Ran}(1 - G_0)$ is invariant under G_0 . It follows that on $\text{Ran}(1 - G_0)$

$$(1 - G_0)^{-1} = I + G_0(1 - G_0)^{-1}$$

so

$$\begin{aligned} h_{-a}^+ &= (1 - G_0)^{-1} [O_1 O_2(\widehat{g}_+) + O_1(\widehat{g}_-)] \\ &= O_1 O_2(\widehat{g}_+) + O_1(\widehat{g}_-) + G_0(1 - G_0)^{-1} [O_1 O_2(\widehat{g}_+) + O_1(\widehat{g}_-)] \end{aligned}$$

shows that $h_{-a}^+ \in L^1\left((0, +1); \frac{dv}{|v|}\right)$ since $\frac{1}{|v|} O_1$ is bounded. Note that

$$\begin{aligned} \frac{1}{|v|} h_a^- &= \frac{1}{|v|} O_2(h_{-a}^+ + \widehat{g}_+) \\ &= \frac{1}{|v|} R_2(h_{-a}^+ + \widehat{g}_+) + \frac{1}{|v|} K_2(h_{-a}^+ + \widehat{g}_+) \\ &= \frac{1}{|v|} R_2 |v| \left(\frac{h_{-a}^+}{|v|} + \frac{\widehat{g}_+}{|v|} \right) + \frac{1}{|v|} K_2 |v| \left(\frac{h_{-a}^+}{|v|} + \frac{\widehat{g}_+}{|v|} \right). \end{aligned}$$

Since

$$h_{-a}^+ + \widehat{g}_+ \in L^1 \left((0, +1); \frac{dv}{|v|} \right)$$

we have, using our assumption,

$$\frac{1}{|v|} K_2 |v| \left(\frac{h_{-a}^+}{|v|} + \frac{\widehat{g}_+}{|v|} \right) \in L^1 \left((-1, 0); \frac{dv}{|v|} \right).$$

We always have

$$\frac{1}{|v|} R_2 |v| \left(\frac{h_{-a}^+}{|v|} + \frac{\widehat{g}_+}{|v|} \right) \in L^1 \left((-1, 0); \frac{dv}{|v|} \right).$$

This shows (32). The case $\beta_2 = 1$ can be treated similarly. ■

Remark 11 *We do not know whether (28) is a necessary condition for g to belong to $\text{Ran}(T_O)$.*

6 On the boundary spectrum of T_O

This section is devoted to the analysis of $\sigma(T_O) \cap i\mathbb{R}$. Note first that the type of $(e^{tT_O})_{t \geq 0}$ is equal to 0 since $(e^{tT_O})_{t \geq 0}$ is a stochastic semigroup. Thus $0 \in \sigma(T_O)$ since the type of $(e^{tT_O})_{t \geq 0}$ coincides with the spectral bound of its generator, see e.g. [20].

Theorem 12 *Suppose that $O_1 = K_1$ and that $|v|^{-1} K_1$ is bounded (or $O_2 = K_2$ and $|v|^{-1} K_2$ is bounded). Then $i\mathbb{R} \subset \sigma(T_O)$.*

Proof. A simple inspection of $(\lambda - T_O)^{-1}g$ shows that it consists of two parts, the *first* one being

$$H_\lambda g := \chi_{\{v > 0\}} \int_{-a}^x e^{-\frac{\lambda}{v}(x-y)} \frac{1}{v} g(y, v) dy + \chi_{\{v < 0\}} \int_x^a e^{-\frac{\lambda}{|v|}(y-x)} \frac{1}{|v|} g(y, v) dy$$

which is nothing but $(\lambda - T_0)^{-1}g$ where T_0 is the classical free transport operator with the "zero incoming" boundary condition. It is well known (see [17]) that

$$\sigma(T_0) = \{\lambda \in \mathbb{C}; \text{Re } \lambda \leq 0\}; \quad (33)$$

(the proof is given there in $L^2(\Omega)$ but is the same in all L^p spaces ($p \geq 1$)). Let us regard this result in a slightly different way. Indeed, let

$$\Omega_+ = (-a, a) \times (0, 1) \text{ and } \Omega_- = (-a, a) \times (-1, 0).$$

We note that $L^1(\Omega_+)$ and $L^1(\Omega_-)$ are invariant under $(\lambda - T_0)^{-1}$ (or equivalently under $(e^{tT_0})_{t \geq 0}$) and therefore T_0 splits as $T_0 = T_0^- \oplus T_0^+$ where T_0^\pm are the generators of the restrictions of $(e^{tT_0})_{t \geq 0}$ to the subspaces $L^1(\Omega_\pm)$. Thus

$$(\lambda - T_0^+)^{-1}g_+ = \int_{-a}^x e^{-\frac{\lambda}{v}(x-y)} \frac{1}{v} g_+(y, v) dy$$

and

$$(\lambda - T_0^-)^{-1}g_- = \int_x^a e^{-\frac{\lambda}{|v|}(y-x)} \frac{1}{|v|} g_-(y, v) dy$$

where g_{\pm} are the restrictions of g to $L^1(\Omega_{\pm})$. As in [17], we can show that

$$\sigma(T_0^-) = \sigma(T_0^+) = \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \leq 0\}.$$

In particular

$$\lim_{\varepsilon \rightarrow 0_+} \|(\varepsilon + is - T_0^+)^{-1}\| = +\infty \quad (s \in \mathbb{R}) \quad (34)$$

and

$$\lim_{\varepsilon \rightarrow 0_+} \|(\varepsilon + is - T_0^-)^{-1}\| = +\infty \quad (s \in \mathbb{R}). \quad (35)$$

(i) Suppose first that $\beta_1 = 1$ and that $|v|^{-1}K_1$ is bounded. We know that $(\lambda - T_O)^{-1}g$ is given for *positive* v by

$$\frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+ + (\lambda - T_0^+)^{-1}g_+$$

where

$$h_{-a}^+ = (1 - G_{\lambda})^{-1}O_1 \left[e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \right].$$

Note that

$$(1 - G_{\lambda})^{-1}O_1 = O_1 + G_{\lambda}(1 - G_{\lambda})^{-1}O_1$$

and $G_{\lambda} = O_1 e^{-\frac{2\lambda a}{|v|}} O_2 e^{-\frac{2\lambda a}{|v|}}$. According to Corollary 25 $(1 - G_{\lambda})^{-1}$ ($\operatorname{Re} \lambda > 0$) extends continuously to $i\mathbb{R} \setminus \{0\}$ (in the strong operator topology). It follows that the norm of the operator (depending on $\lambda = \varepsilon + is$; $\varepsilon > 0$)

$$L^1(\Omega) \ni g \rightarrow \frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+ \in L^1(\Omega_+)$$

remains *uniformly bounded* when $\varepsilon \rightarrow 0_+$ ($\forall s \neq 0$). Finally (34) implies that

$$\lim_{\varepsilon \rightarrow 0_+} \sup_{\|g\| \leq 1} \|(\varepsilon + is - T_O)^{-1}g\|_{L^1(\Omega_+)} = +\infty \quad (s \neq 0)$$

whence $is \in \sigma(T_O)$ ($\forall s \neq 0$).

(ii) Suppose now that $\beta_2 = 1$ and that $|v|^{-1}K_2$ is bounded. It is easy to see that $(\lambda - T_O)^{-1}g$ can also be given by

$$f(x, v) = \frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+ + \int_{-a}^x e^{-\frac{\lambda}{v}(x-y)} \frac{1}{v} g(y, v) dy \quad (v > 0)$$

$$f(x, v) = \frac{1}{|v|} e^{-\frac{\lambda}{|v|}(a-x)} h_a^- + \int_x^a e^{-\frac{\lambda}{|v|}(y-x)} \frac{1}{|v|} g(y, v) dy \quad (v < 0)$$

where

$$h_a^- = \left(I - \tilde{G}_\lambda\right)^{-1} O_2 \left[e^{-\frac{\lambda}{v} 2a} O_1 \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) \right]$$

$$\tilde{G}_\lambda := O_2 e^{-\frac{\lambda}{v} 2a} O_1 e^{-\frac{\lambda}{|v|} 2a}$$

and

$$h_{-a}^+ = O_1 h_{-a}^- = O_1 \left(e^{-\frac{\lambda}{|v|} 2a} h_a^- + \int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right).$$

In particular, $(\lambda - T_O)^{-1}g$ is given for *negative* v by

$$\frac{1}{|v|} e^{-\frac{\lambda}{|v|}(a-x)} h_a^- + (\lambda - T_0^-)^{-1} g_-.$$

By noting that

$$\left(I - \tilde{G}_\lambda\right)^{-1} O_2 = O_2 + \tilde{G}_\lambda (1 - \tilde{G}_\lambda)^{-1} O_2,$$

and using the fact that $(1 - \tilde{G}_\lambda)^{-1}$ ($\operatorname{Re} \lambda > 0$) extends continuously to $i\mathbb{R} \setminus \{0\}$ in the strong operator topology (see Remark 26), we see as before that (35) implies

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\|g\| \leq 1} \|(\varepsilon + is - T_O)^{-1} g\|_{L^1(\Omega_-)} = +\infty \quad (s \neq 0)$$

and $is \in \sigma(T_O)$ ($\forall s \neq 0$). ■

Remark 13 *A priori, it is not clear whether $i\mathbb{R} \subset \sigma(T_O)$ for more general partly diffuse models.*

7 The objects to be estimated

Note that $H_\lambda g = (\lambda - T_0)^{-1}g$ does not extend to $i\mathbb{R}$ for *all* g because of (33). On the other hand, we can extend it on a suitable subspace. Indeed, let $k \in \mathbb{N}$, ($k \neq 0$). It is easy to see that $H_\lambda g$ extends to the whole closed half space $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0\}$ with the C^k norm estimates

$$\left\| \frac{\partial^j}{\partial \lambda^j} H_\lambda g \right\| \leq (2a)^j \left\| \frac{g}{|v|^{j+1}} \right\| \quad (0 \leq j \leq k, \operatorname{Re} \lambda \geq 0) \quad (36)$$

provided that $\left\| \frac{g}{|v|^{k+1}} \right\| < +\infty$. Actually, to estimate $(\lambda - T_O)^{-1}g$ up to the imaginary axis, the *key* point is to estimate in C^k norm the *boundary* terms

$$\frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+, \quad \frac{1}{|v|} e^{-\frac{\lambda}{|v|}(a-x)} h_a^-.$$

Consider first

$$\frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+.$$

Note that a priori $h_{-a}^+ \in L^1((0, +1]; dv)$. Since

$$\begin{aligned} \frac{\partial^k \left(\frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} h_{-a}^+ \right)}{\partial \lambda^k} &= \sum_{j=0}^k \binom{k}{j} \frac{\partial^j}{\partial \lambda^j} \left(\frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} \right) \frac{\partial^{k-j}}{\partial \lambda^{k-j}} (h_{-a}^+) \\ &= \sum_{j=0}^k \binom{k}{j} \left(-\frac{x+a}{v} \right)^j \frac{1}{v} e^{-\frac{\lambda}{v}(x+a)} \frac{\partial^{k-j}}{\partial \lambda^{k-j}} h_{-a}^+ \end{aligned}$$

our main concern is to estimate the norms

$$\left\| \frac{1}{|v|} \frac{\partial^k}{\partial \lambda^k} h_{-a}^+ \right\|, \left\| \frac{1}{|v|^2} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} h_{-a}^+ \right\|, \dots, \left\| \frac{1}{|v|^k} \frac{\partial}{\partial \lambda} h_{-a}^+ \right\|, \left\| \frac{1}{|v|^{k+1}} h_{-a}^+ \right\|$$

in

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0, \lambda \neq 0\}.$$

8 Operator estimates up to the imaginary axis

Since

$$h_{-a}^+ = (1 - G_\lambda)^{-1} O_1 \left[e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \right]$$

then the key object to deal with is the resolvent $(1 - G_\lambda)^{-1}$ where

$$G_\lambda = O_1 e^{-\frac{2\lambda a}{|v|}} O_2 e^{-\frac{2\lambda a}{|v|}}.$$

Note that G_λ is defined on the closed half space $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0\}$ and $\|G_\lambda\| \leq e^{-4a \operatorname{Re} \lambda}$ ($\operatorname{Re} \lambda \geq 0$). The derivatives of G_λ

$$\begin{aligned} \frac{\partial^p G_\lambda}{\partial \lambda^p} &= \sum_{j=0}^p \binom{p}{j} O_1 \frac{\partial^j}{\partial \lambda^j} \left(e^{-\frac{2\lambda a}{|v|}} \right) O_2 \frac{\partial^{p-j}}{\partial \lambda^{p-j}} \left(e^{-\frac{2\lambda a}{|v|}} \right) \\ &= (-2a)^p \sum_{j=0}^p \binom{p}{j} O_1 \left(\frac{1}{|v|^j} e^{-\frac{2\lambda a}{|v|}} \right) O_2 \left(\frac{1}{|v|^{p-j}} e^{-\frac{2\lambda a}{|v|}} \right) \quad (0 \leq p \leq k) \end{aligned}$$

are *uniformly bounded* on $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0\}$ (for the usual operator norms) provided that

$$O_1 \frac{1}{|v|^j} O_2 \frac{1}{|v|^{p-j}} \text{ are bounded operators } (0 \leq j \leq p \leq k). \quad (37)$$

We need also the additional conditions

$$G_\lambda : L^1((0, +1]; dv) \rightarrow L^1((0, +1]; \frac{dv}{|v|^{k+1}})$$

and

$$\frac{d}{d\lambda}G_\lambda : L^1((0, +1]; dv) \rightarrow L^1((0, +1]; \frac{dv}{|v|^k})$$

or more precisely

$$\frac{1}{|v|^{k+1}}O_1O_2, \frac{1}{|v|^k}O_1\frac{1}{|v|}O_2 \text{ and } \frac{1}{|v|^k}O_1O_2\frac{1}{|v|} \text{ are bounded operators.} \quad (38)$$

Remark 14 *If O_1 or O_2 is weakly compact then at least one of the two is an integral operator and consequently Assumptions (37)(38) are checkable in principle.*

We will show, under the condition $\beta_1 + \beta_2 > 0$, that $r_\sigma(G_\lambda) < 1$ ($\text{Re } \lambda \geq 0, \lambda \neq 0$) and $(1 - G_\lambda)^{-1}$ extends continuously (in the strong operator topology) to $i\mathbb{R} \setminus \{0\}$, (see Corollary 25). We are ready to give our key estimates of the derivatives of $(1 - G_\lambda)^{-1}$ in terms of $\|(1 - G_\lambda)^{-1}\|$.

Lemma 15 *Suppose that (37)(38) are satisfied. Then there exists a constant $C > 0$ such that for all $s \in \mathbb{R}, s \neq 0$*

$$\left\| \frac{d^j}{ds^j} (1 - G_{is})^{-1} \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+j}dv))} \leq C \sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \quad (1 \leq j \leq k).$$

Proof. Note that

$$(1 - G_\lambda)^{-1} = I + G_\lambda(1 - G_\lambda)^{-1} \quad (39)$$

and

$$\frac{d}{d\lambda}(1 - G_\lambda)^{-1} = (1 - G_\lambda)^{-1}G'_\lambda(1 - G_\lambda)^{-1} \quad (40)$$

so

$$\begin{aligned} \frac{d}{d\lambda}(1 - G_\lambda)^{-1} &= (I + G_\lambda(1 - G_\lambda)^{-1})G'_\lambda(I + G_\lambda(1 - G_\lambda)^{-1}) \\ &= G'_\lambda(I + G_\lambda(1 - G_\lambda)^{-1}) + G_\lambda(1 - G_\lambda)^{-1}G'_\lambda(I + G_\lambda(1 - G_\lambda)^{-1}) \end{aligned}$$

and (37)(38) show that

$$\frac{d}{d\lambda}(1 - G_\lambda)^{-1} : L^1((0, +1]; dv) \rightarrow L^1((0, +1]; \frac{dv}{|v|^k})$$

and that there exists a constant $C > 0$ such that

$$\left\| \frac{d}{d\lambda}(1 - G_\lambda)^{-1} \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k}dv))} \leq C \left(1 + \|(1 - G_\lambda)^{-1}\| + \|(1 - G_\lambda)^{-1}\|^2 \right).$$

Let us show by induction that

$$\frac{d^j}{d\lambda^j}(1 - G_\lambda)^{-1} : L^1((0, +1]; dv) \rightarrow L^1((0, +1]; \frac{dv}{|v|^{k+1-j}}) \quad (1 \leq j \leq k)$$

and there exists a constant $C > 0$ such that

$$\left\| \frac{d^j}{d\lambda^j} (1 - G_\lambda)^{-1} \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+j} dv))} \leq C \left(1 + \sum_{l=1}^{j+1} \|(1 - G_\lambda)^{-1}\|^l \right). \quad (41)$$

We already know that this statement is true for $j = 1$. It suffices to show that if $1 \leq p < k$ and that if

$$\left\| \frac{d^j}{d\lambda^j} (1 - G_\lambda)^{-1} \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+j} dv))} \leq C \left(1 + \sum_{l=1}^{j+1} \|(1 - G_\lambda)^{-1}\|^l \right) \quad (1 \leq j \leq p)$$

then estimate (41) is true for $j = p + 1$. Let

$$f(\lambda) = (1 - G_\lambda)^{-1}.$$

According to (40), $f(\lambda)$ satisfies the differential equation

$$f'(\lambda) = f(\lambda) G'(\lambda) f(\lambda). \quad (42)$$

Differentiating (42) p times we get

$$\begin{aligned} \frac{d^{p+1}}{d\lambda^{p+1}} f &= \sum_{q=0}^p \binom{p}{q} \left(\frac{d^q}{d\lambda^q} f \right) \frac{d^{p-q}}{d\lambda^{p-q}} (G'(\lambda) f(\lambda)) \\ &= \sum_{q=0}^p \binom{p}{q} \left(\frac{d^q}{d\lambda^q} f \right) \sum_{m=0}^{p-q} \binom{p-q}{m} \left(\frac{d^{p-q-m}}{d\lambda^{p-q-m}} G'(\lambda) \right) \left(\frac{d^m}{d\lambda^m} f \right) \\ &= \sum_{q=0}^p \binom{p}{q} \left(\frac{d^q}{d\lambda^q} f \right) \sum_{m=0}^{p-q} \binom{p-q}{m} \left(\frac{d^{p-q-m+1}}{d\lambda^{p-q-m+1}} G(\lambda) \right) \left(\frac{d^m}{d\lambda^m} f \right) \\ &= \sum_{q=0}^p \sum_{m=0}^{p-q} \binom{p}{q} \binom{p-q}{m} \left(\frac{d^q}{d\lambda^q} f \right) \left(\frac{d^{p-q-m+1}}{d\lambda^{p-q-m+1}} G(\lambda) \right) \left(\frac{d^m}{d\lambda^m} f \right). \end{aligned}$$

Note that $|v|^{-k-1+j} \geq |v|^{-k-1+j'}$ ($j \leq j'$) shows that

$$L^1(|v|^{-k-1+j} dv) \subset L^1(|v|^{-k-1+j'} dv)$$

and

$$\|\varphi\|_{L^1(|v|^{-k-1+j'} dv)} \leq \|\varphi\|_{L^1(|v|^{-k-1+j} dv)} \quad \forall \varphi \in L^1(|v|^{-k-1+j} dv).$$

Thus

$$\left\| \frac{d^m}{d\lambda^m} f(\lambda) \right\|_{\mathcal{L}(L^1(dv))} \leq \left\| \frac{d^m}{d\lambda^m} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+m} dv))}$$

and (by assumption)

$$\left\| \frac{d^m}{d\lambda^m} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+m} dv))} \leq C \left(1 + \sum_{l=1}^{m+1} \|f(\lambda)\|^l \right) \quad (m \leq p - q)$$

so

$$\left\| \frac{d^m}{d\lambda^m} f(\lambda) \right\|_{\mathcal{L}(L^1(dv))} \leq C \left(1 + \sum_{l=1}^{m+1} \|f(\lambda)\|^l \right) \quad (m \leq p - q).$$

By (37) the derivatives $\frac{d^{p-q-m+1}}{d\lambda^{p-q-m+1}} G(\lambda)$ are uniformly bounded for the natural operator norms. Similarly,

$$\frac{d^q}{d\lambda^q} f(\lambda) : L^1(dv) \rightarrow L^1(|v|^{-k-1+q} dv) \subset L^1(|v|^{-k-1+p} dv) \quad (q \leq p)$$

and

$$\left\| \frac{d^q}{d\lambda^q} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+p} dv))} \leq \left\| \frac{d^q}{d\lambda^q} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+q} dv))}$$

so (using the assumption)

$$\begin{aligned} \left\| \frac{d^q}{d\lambda^q} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+p} dv))} &\leq \left\| \frac{d^q}{d\lambda^q} f(\lambda) \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+q} dv))} \\ &\leq C \left(1 + \sum_{r=1}^{q+1} \|f(\lambda)\|^r \right) \quad (q \leq p). \end{aligned}$$

On the other hand

$$\left(1 + \sum_{l=1}^{m+1} \|f(\lambda)\|^l \right) \left(1 + \sum_{r=1}^{q+1} \|f(\lambda)\|^r \right) = 1 + \sum_{l=1}^{m+1} \|f(\lambda)\|^l + \sum_{r=1}^{q+1} \|f(\lambda)\|^r + \sum_{l=1}^{m+1} \sum_{r=1}^{q+1} \|f(\lambda)\|^{l+r}.$$

Since $m \leq p - q$ we have

$$l + r \leq m + 1 + q + 1 \leq p + 2$$

and there exists $C > 0$ such that

$$\left\| \frac{d^{p+1}}{d\lambda^{p+1}} f \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+p} dv))} \leq C \left(1 + \sum_{l=1}^{p+2} \|f(\lambda)\|^l \right).$$

Finally

$$\left\| \frac{d^{p+1}}{d\lambda^{p+1}} f \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+p+1} dv))} \leq \left\| \frac{d^{p+1}}{d\lambda^{p+1}} f \right\|_{\mathcal{L}(L^1(dv); L^1(|v|^{-k-1+p} dv))} \leq C \left(1 + \sum_{l=1}^{(p+1)+1} \|f(\lambda)\|^l \right)$$

and hence we are done. ■

9 Estimates of boundary fluxes

We note that if

$$O_1 : L^1((0, +1]; \frac{dv}{|v|^{k+1}}) \rightarrow L^1((0, +1]; \frac{dv}{|v|^{k+1}}) \text{ is bounded}$$

i.e. if

$$\frac{1}{|v|^{k+1}} O_1 |v|^{k+1} \text{ is bounded} \quad (43)$$

then (39) gives

$$\begin{aligned} h_{-a}^+ &= (1 - G_\lambda)^{-1} O_1 \left[e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \right] \\ &= G_\lambda (1 - G_\lambda)^{-1} O_1 \left[e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) + \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \right] \\ &\quad + O_1 e^{-\frac{2\lambda a}{|v|}} O_2 \left(\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right) + O_1 \left(\int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right) \end{aligned}$$

and

$$\begin{aligned} \|h_{-a}^+\|_{L^1(|v|^{-(k+1)} dv)} &\leq 2 \left\| \frac{1}{|v|^{k+1}} O_1 O_2 \right\| \|(1 - G_\lambda)^{-1}\| \|g\| \\ &\quad + \left\| \frac{1}{|v|^{k+1}} O_1 O_2 \right\| \|g\| + \left\| \frac{1}{|v|^{k+1}} O_1 |v|^{k+1} \right\| \left\| \frac{g}{|v|^{k+1}} \right\|. \end{aligned}$$

Leibnitz's rule shows that $\frac{d^p h_{-a}^+}{d\lambda^p}$ is given by

$$\sum_{j=0}^p \binom{p}{j} \left(\frac{d^j}{d\lambda^j} (1 - G_\lambda)^{-1} \right) \frac{d^{p-j}}{d\lambda^{p-j}} \left[O_1 e^{-\frac{2\lambda a}{|v|}} O_2 \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy + O_1 \int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} g(y, v) dy \right]$$

or indeed by

$$\begin{aligned} &\sum_{j=0}^p (-1)^{p-j} \binom{p}{j} \left(\frac{d^j}{d\lambda^j} (1 - G_\lambda)^{-1} \right) \times \\ &\sum_{m=0}^{p-j} \binom{p-j}{m} (2a)^m O_1 \frac{1}{|v|^m} e^{-\frac{2\lambda a}{|v|}} O_2 \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} (a-y)^{p-j-m} \frac{g(y, v)}{|v|^{p-j-m}} dy \\ &+ \sum_{j=0}^p (-1)^{p-j} \binom{p}{j} \left(\frac{d^j}{d\lambda^j} (1 - G_\lambda)^{-1} \right) \left(O_1 \int_{-a}^a e^{-\frac{\lambda}{|v|}(y+a)} (y+a)^{p-j} \frac{g(y, v)}{|v|^{p-j}} dy \right). \end{aligned}$$

Finally, Lemma 15 implies:

Lemma 16 *Suppose that (37)(38)(43) are satisfied. There exists a constant $C > 0$ such that*

$$\left\| \frac{d^j h_{-a}^+}{d\lambda^j} \right\|_{L^1(|v|^{-k-1+p} dv)} \leq C \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k).$$

We deal now with h_a^- .

Proposition 17 *Suppose that (37)(38)(43) are satisfied. If*

$$|v|^{-(k+1-p)} O_2 |v|^{k+1-p} \text{ is bounded } (0 \leq p \leq k) \quad (44)$$

then there exists a constant $C > 0$ such that

$$\begin{aligned} \|h_a^-\|_{L^1(|v|^{-(k+1)} dv)} &\leq C \left[\|h_{-a}^+\|_{L^1(|v|^{-(k+1)} dv)} + \left\| \frac{g}{|v|^{k+1}} \right\| \right] \\ \left\| \frac{d^p h_a^-}{d\lambda^p} \right\|_{L^1(|v|^{-k-1+p} dv)} &\leq C \left[\sum_{j=0}^p \left\| \frac{d^{p-j} h_{-a}^+}{d\lambda^{p-j}} \right\|_{L^1(|v|^{-k-1+p-j} dv)} + \left\| \frac{g}{|v|^{k+1}} \right\| \right] \quad (1 \leq p \leq k). \end{aligned}$$

Proof. Note that

$$h_a^- = O_2 \left(e^{-\frac{2\lambda a}{|v|}} h_{-a}^+ + \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} g(y, v) dy \right)$$

or

$$h_a^- = O_2 \left[|v|^{k+1} \left(e^{-\frac{2\lambda a}{|v|}} \frac{h_{-a}^+}{|v|^{k+1}} + \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} \frac{g(y, v)}{|v|^{k+1}} dy \right) \right]$$

shows that

$$\|h_a^-\|_{L^1(|v|^{-(k+1)} dv)} \leq \left\| |v|^{-(k+1)} O_2 |v|^{k+1} \right\| \left[\|h_{-a}^+\|_{L^1(|v|^{-(k+1)} dv)} + \left\| \frac{g}{|v|^{k+1}} \right\| \right].$$

Leibnitz's rule gives

$$\begin{aligned} \frac{d^p h_a^-}{d\lambda^p} &= \sum_{j=0}^p \binom{p}{j} O_2 \left[\left(\frac{d^j}{d\lambda^j} (e^{-\frac{2\lambda a}{|v|}}) \right) \frac{d^{p-j} h_{-a}^+}{d\lambda^{p-j}} \right] \\ &\quad + (-1)^p O_2 \left[\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} (a-y)^p \frac{g(y, v)}{v^p} dy \right] \\ &= \sum_{j=0}^p \binom{p}{j} (-2a)^j O_2 \left[\frac{1}{|v|^j} e^{-\frac{2\lambda a}{|v|}} \frac{d^{p-j} h_{-a}^+}{d\lambda^{p-j}} \right] \\ &\quad + (-1)^p O_2 \left[\int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} (a-y)^p \frac{g(y, v)}{v^p} dy \right] \end{aligned}$$

or

$$\begin{aligned} \frac{d^p h_a^-}{d\lambda^p} &= \sum_{j=0}^p \binom{p}{j} (-2a)^j O_2 \left[|v|^{k+1-p} \frac{1}{|v|^{k+1-p+j}} e^{-\frac{2\lambda a}{|v|}} \frac{d^{p-j} h_{-a}^+}{d\lambda^{p-j}} \right] \\ &\quad + (-1)^p O_2 \left[v^{k+1-p} \int_{-a}^a e^{-\frac{\lambda}{v}(a-y)} (a-y)^p \frac{g(y, v)}{v^{k+1}} dy \right] \end{aligned}$$

so there exists a constant $C' > 0$ such that

$$\left\| \frac{d^p h_a^-}{d\lambda^p} \right\|_{L^1(|v|^{-k-1+p} dv)} \leq C' \left\| |v|^{-(k+1-p)} O_2 |v|^{k+1-p} \right\| \left[\sum_{j=0}^p \left\| \frac{d^{p-j} h_{-a}^+}{d\lambda^{p-j}} \right\|_{L^1(|v|^{-k-1+p-j} dv)} + \left\| \frac{g}{|v|^{k+1}} \right\| \right].$$

This ends the proof. ■

Finally, Proposition 17 and Lemma 16 imply:

Corollary 18 *Suppose that (37)(38)(43)(44) are satisfied. There exists a constant $C > 0$ such that*

$$\left\| \frac{d^j h_a^-}{d\lambda^j} \right\|_{L^1(|v|^{-k-1+p} dv)} \leq C \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \left\| \frac{g}{|v|^{k+1}} \right\| \quad (0 \leq j \leq k).$$

Remark 19 *Note that (43) and (44) are checkable since they are always satisfied by the specular parts of the boundary operators O_i ($i = 1, 2$) and checkable for the diffuse parts.*

10 On the resolvent on the imaginary axis

Combining Lemma 15, Lemma 16, Corollary 18, Corollary 25, (36) and using the limit $F_g(s)$ defined by (21) we get:

Theorem 20 *Let $k \in \mathbb{N}$ and let (37)(38)(43)(44) be satisfied. Let*

$$Z := \left\{ g \in L^1(\Omega); \quad \frac{g}{|v|^{k+1}} \in L^1(\Omega) \right\}$$

be endowed with the norm $\|g\|_Z = \left\| \frac{g}{|v|^{k+1}} \right\|$. For any $g \in Z$,

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \ni \lambda \rightarrow (\lambda - T_O)^{-1} g \in L^1(\Omega)$$

extends continuously to $i\mathbb{R} \setminus \{0\}$ as a C^k function $F_g(\cdot)$ and there exists a constant $C > 0$ such that

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq C \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \|g\|_Z \quad (0 \leq j \leq k, \ s \neq 0).$$

11 Existence and estimates of $(1 - G_\lambda)^{-1}$

The preceeding sections show that the existence and estimate of $(1 - G_\lambda)^{-1}$ for $\lambda = is$ ($s \neq 0$) are the cornerstone of this work. We start with a general result.

Theorem 21 *If $\beta_1 + \beta_2 > 0$ then $r_\sigma(G_\lambda) < 1$ ($\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$).*

Proof. We have $G_\lambda = O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}}$ and $G_0 = O_1 O_2$. Note that $O_1 O_2$ is stochastic so $r_\sigma(G_0) = 1$. According to [21], $r_{ess}(G_0) < 1$ if $\beta_1 + \beta_2 > 0$ so $r_\sigma(G_0)$ is an isolated eigenvalue of G_0 with finite algebraic multiplicity. We know that

$$\|G_\lambda\| = \left\| O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}} \right\| \leq e^{-4a \operatorname{Re} \lambda} < 1 \text{ if } \operatorname{Re} \lambda > 0$$

since $\left| e^{-\frac{2\lambda a}{|\nu|}} \right| \leq e^{-2a \operatorname{Re} \lambda}$. Let $\lambda = i\alpha$ ($\alpha \in \mathbb{R}$). Note that the (operator) modulus $|G_\lambda|$ of G_λ (see [10]) is such that

$$\left| O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}} \right| \leq O_1 O_2 = G_0$$

and

$$\left| O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}} \right| \neq G_0 \quad (\alpha \neq 0)$$

so by [19]

$$r_\sigma\left(\left| O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}} \right|\right) < r_\sigma(G_0) = 1$$

whence

$$r_\sigma(O_1 e^{-\frac{2\lambda a}{|\nu|}} O_2 e^{-\frac{2\lambda a}{|\nu|}}) < 1 \quad (\alpha \neq 0)$$

and $r_\sigma(G_\lambda) < 1$ ($\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$). ■

Remark 22 *We can show similarly that $r_\sigma(\tilde{G}_\lambda) < 1$ ($\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$) where $\tilde{G}_\lambda := O_2 e^{-\frac{\lambda}{\nu} 2a} O_1 e^{-\frac{\lambda}{|\nu|} 2a}$.*

We complement now Theorem 21 in different directions by adding suitable assumptions.

Theorem 23 *Let K_i ($i = 1, 2$) be compact and let $\beta_1 + \beta_2 > 0$. Then:*

(i) *If $\beta_1 > 0$, $\beta_2 > 0$ and, for almost all v'' , $k_1(v'', \cdot) \in L^\infty(-1, 0)$ then $c_\eta := \sup_{|\lambda| \geq \eta} \|G_\lambda\| < 1$ ($\eta > 0$). If the kernels $k_i(\cdot, \cdot)$ of K_i ($i = 1, 2$) are continuous and $K_1 |v|^{-2} K_2$ is bounded then there exists $\hat{c} > 0$ such that*

$$\|G_\lambda\| \leq 1 - \hat{c} |\operatorname{Im} \lambda|^2 \quad (\lambda \rightarrow 0).$$

(ii) *If $\beta_1 > 0$, $\beta_2 = 0$ and, for almost all v'' , $k_1(v'', \cdot) \in L^\infty(-1, 0)$ then $c_\eta := \sup_{|\lambda| \geq \eta} \|G_\lambda^2\| < 1$ ($\eta > 0$). If the kernel $k_1(\cdot, \cdot)$ of K_1 is continuous and $K_1 |v|^{-2} R_2 K_1$ is bounded then there exists $\hat{c} > 0$ such that*

$$\|G_\lambda^2\| \leq 1 - \hat{c} |\operatorname{Im} \lambda|^2 \quad (\lambda \rightarrow 0).$$

(A similar statement holds if $\beta_1 = 0$ and $\beta_2 > 0$).

(iii) In particular, in both cases (i) and (ii) we have

$$\sup_{|\lambda| \geq \eta} \|(1 - G_\lambda)^{-1}\| < +\infty \quad (\eta > 0) \quad \text{and} \quad \|(1 - G_\lambda)^{-1}\| = O(|\operatorname{Im} \lambda|^{-2}) \quad (\lambda \rightarrow 0).$$

Proof. Note that $\|G_\lambda\| \leq e^{-4a \operatorname{Re} \lambda}$ ($\operatorname{Re} \lambda \geq 0$) so we may restrict ourselves to the strip $\{\lambda; 0 \leq \operatorname{Re} \lambda \leq 1\}$. Let $\lambda = \varepsilon + is$, $\varepsilon \in [0, 1]$. Without loss of generality, we may restrict ourselves to the case $\beta_1 > 0$. This case subdivides into two subcases:

$$\beta_1 > 0 \text{ and } \beta_2 > 0 \tag{45}$$

or

$$\beta_1 > 0 \text{ and } \beta_2 = 0. \tag{46}$$

Consider first the case (45).

$$\begin{aligned} G_\lambda &= O_1 e^{-\frac{2\lambda a}{|v|}} O_2 e^{-\frac{2\lambda a}{|v|}} = (\alpha_1 R_1 + \beta_1 K_1) e^{-\frac{2\lambda a}{|v|}} (\alpha_2 R_2 + \beta_2 K_2) e^{-\frac{2\lambda a}{|v|}} \\ &= \beta_1 \beta_2 K_1 e^{-\frac{2\lambda a}{|v|}} K_2 e^{-\frac{2\lambda a}{|v|}} + H_\lambda \end{aligned}$$

where

$$H_\lambda = \alpha_1 \alpha_2 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} + \alpha_1 \beta_2 R_1 e^{-\frac{2\lambda a}{|v|}} K_2 e^{-\frac{2\lambda a}{|v|}} + \beta_1 \alpha_2 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}}.$$

We have

$$\begin{aligned} \|G_\lambda\| &\leq \beta_1 \beta_2 \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| + \alpha_1 \alpha_2 + \alpha_1 \beta_2 + \beta_1 \alpha_2 \\ &= \beta_1 \beta_2 \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| + (1 - \beta_1)(1 - \beta_2) + (1 - \beta_1)\beta_2 + \beta_1(1 - \beta_2) \\ &= \beta_1 \beta_2 \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| + 1 - \beta_1 \beta_2 \end{aligned}$$

so

$$\|G_\lambda\| \leq 1 - \beta_1 \beta_2 \left(1 - \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| \right). \tag{47}$$

Note that $K_1 e^{-\frac{2\lambda a}{|v|}} K_2 = K_1 e^{-\frac{2i s a}{|v|}} \widehat{K}_2$ where \widehat{K}_2 has the kernel

$$\widehat{k}_2(v, v') = e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v') \leq k_2(v, v').$$

We have

$$\int_{-1}^0 k_1(v, v') dv = 1, \quad \int_0^1 \widehat{k}_2(v, v') dv \leq \int_0^1 k_2(v, v') dv = 1.$$

Since

$$\begin{aligned} K_1 e^{-\frac{2i s a}{|v|}} \widehat{K}_2 f &= \int_{-1}^0 dv k_1(v'', v) e^{-\frac{2i s a}{|v|}} \int_0^1 \widehat{k}_2(v, v') f(v') dv' \\ &= \int_0^1 \left[\int_{-1}^0 k_1(v'', v) e^{-\frac{2i s a}{|v|}} \widehat{k}_2(v, v') dv \right] f(v') dv' \end{aligned} \tag{48}$$

then

$$\left\| K_1 e^{-\frac{2isa}{|v|}} \widehat{K}_2 \right\| \leq \sup_{v' \in (0,1)} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| dv''.$$

We recall (see [27], Thm 1.39, p. 30) that for any complex function $h \in L^1(\mu)$,

$$\left| \int h d\mu \right| = \int |h| d\mu$$

if and only if there exists a constant α such that $\alpha h = |h|$. It follows that

$$\left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| < \int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv$$

otherwise there exists a constant α such that

$$\alpha k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') = k_1(v'', v) \widehat{k}_2(v, v')$$

so $\alpha e^{-\frac{2isa}{|v|}} = 1$ and $\alpha = e^{\frac{2isa}{|v|}}$ is *not* a constant. Thus, for $\operatorname{Re} \lambda \geq 0$ and $\lambda \neq 0$,

$$\begin{aligned} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| dv'' &< \int_0^1 \left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right) dv'' \\ &\leq \int_0^1 \left(\int_{-1}^0 k_1(v'', v) k_2(v, v') dv \right) dv'' = 1. \end{aligned}$$

Let us show that for any constant $c > 0$

$$\sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0,1]} \sup_{v'} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| dv'' < 1. \quad (49)$$

Let us argue by contradiction by supposing that this supremum is equal to 1. Note that $\widehat{k}_2(v, v') = e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v')$ and K_2 is weakly compact, i.e.

$$\{k_2(., v'); v' \in (0, 1)\}$$

is a relatively *weakly compact* subset of the unit sphere of $L^1(-1, 0)$ (at this stage we do not need the compactness of K_2). There exist $\varepsilon_j \rightarrow \varepsilon$, $v'_j \rightarrow \omega$, $s_j \rightarrow s \in [c, c^{-1}]$ and $g \in L^1(-1, 0)$ such that

$$k_2(., v'_j) \rightarrow g \text{ weakly in } L^1(-1, 0). \quad (50)$$

and

$$\begin{aligned} &\int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv \right| dv'' \\ &\rightarrow \sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0,1]} \sup_{v'} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| dv'' = 1. \end{aligned}$$

Since the sequence $\{k_2(., v'_j)\}_j$ is equiintegrable and since, for almost all v'' , $k_1(v'', .) \in L^\infty(-1, 0)$ we have

$$\int_{\tau}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv \rightarrow 0 \quad (\tau \rightarrow 0_-)$$

uniformly in j and (50) implies

$$\begin{aligned} \int_{-1}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv &\rightarrow \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} e^{-\frac{2\varepsilon a}{|v|}} g(v) dv \\ \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv \right| &\rightarrow \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} e^{-\frac{2\varepsilon a}{|v|}} g(v) dv \right| \end{aligned}$$

and similarly

$$\int_{-1}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} k_2(v, v'_j) dv \rightarrow \int_{-1}^0 k_1(v'', v) e^{-\frac{2\varepsilon a}{|v|}} g(v) dv.$$

Since

$$\int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2is_j a}{|v|}} e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv \right| dv'' \leq \int_0^1 \left(\int_{-1}^0 k_1(v'', v) e^{-\frac{2\varepsilon_j a}{|v|}} k_2(v, v'_j) dv \right) dv'' \leq 1$$

then

$$\left(\int_{-1}^0 \left(\int_0^1 k_1(v'', v) dv'' \right) e^{-\frac{2is_j a}{|v|}} k_2(v, v'_j) dv \right) \rightarrow 1.$$

This last limit shows that $\varepsilon_j \rightarrow 0$ and

$$\int_{-1}^0 \left(\int_0^1 k_1(v'', v) dv'' \right) g(v) dv = 1$$

or indeed

$$\int_0^1 \left(\int_{-1}^0 k_1(v'', v) g(v) dv \right) dv'' = 1.$$

Hence

$$\int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} g(v) dv \right| dv'' = \int_0^1 \left(\int_{-1}^0 k_1(v'', v) g(v) dv \right) dv'' = 1$$

and the inequality

$$\left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} g(v) dv \right| \leq \int_{-1}^0 k_1(v'', v) g(v) dv$$

implies the *equality*

$$\left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} g(v) dv \right| = \int_{-1}^0 k_1(v'', v) g(v) dv$$

which is *not* possible since $s \neq 0$. This ends the proof of (49). Hence

$$\sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0,1]} \left\| K_1 e^{-\frac{2isa}{|v|}} \widehat{K}_2 \right\| < 1$$

and (47) gives

$$\sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0,1]} \|G_{\varepsilon+is}\| < 1. \quad (51)$$

We have

$$\left\| K_1 e^{-\frac{2isa}{|v|}} \widehat{K}_2 \right\| \leq \sup_{v' \in (0,1)} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v') dv \right| dv''.$$

Let us show that $\lim_{|s| \rightarrow \infty} \left\| K_1 e^{-\frac{2isa}{|v|}} \widehat{K}_2 \right\| = 0$ uniformly in $\varepsilon \in [0, 1]$. By weak compactness of K_i ($i = 1, 2$) (and an equiintegrability argument) it suffices to show that for any $\delta > 0$

$$\lim_{|s| \rightarrow \infty} \sup_{v' \in (0,1)} \int_{\delta}^1 \left| \int_{-1}^{-\delta} k_1(v'', v) e^{-\frac{2isa}{|v|}} e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v') dv \right| dv'' = 0 \quad (52)$$

uniformly in $\varepsilon \in [0, 1]$. If K_2 is *compact* then $\{k_2(., v'); v' \in (0, 1)\}$ is a *relatively compact* subset of $L^1(-1, 0)$ and consequently, for almost all $v'' \in (0, 1)$,

$$\left\{ k_1(v'', .) e^{-\frac{2\varepsilon a}{|v|}} k_2(., v'); v' \in (0, 1), \varepsilon \in [0, 1] \right\}$$

is a *relatively compact* subset of $L^1(-1, -\delta)$. A Riemann-Lebesgue argument gives

$$\lim_{|s| \rightarrow \infty} \int_{-1}^{-\delta} k_1(v'', v) e^{-\frac{2isa}{|v|}} e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v') dv = 0$$

uniformly in $v' \in (0, 1)$ and $\varepsilon \in [0, 1]$. Finally, (52) holds by the dominated convergence theorem. Hence

$$c_\eta := \sup_{|\lambda| \geq \eta} \|G_\lambda\| < 1, \quad (\eta > 0)$$

and

$$\sup_{|\lambda| \geq \eta} \|(1 - G_\lambda)^{-1}\| \leq (1 - c_\eta)^{-1}, \quad (\eta > 0).$$

Let us analyze the function

$$\mathbb{R} \ni s \rightarrow \sup_{v' \in (0,1)} \int_0^1 \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| dv''$$

(depending on $\varepsilon \in [0, 1]$) in the *vicinity* of $s = 0$. Consider first

$$\begin{aligned} & \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| \\ &= \sqrt{\left(\int_{-1}^0 k_1(v'', v) \cos\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right)^2 + \left(\int_{-1}^0 k_1(v'', v) \sin\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right)^2} \end{aligned}$$

and let

$$u_\varepsilon(s, v', v'') := \left(\int_{-1}^0 k_1(v'', v) \cos\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right)^2 + \left(\int_{-1}^0 k_1(v'', v) \sin\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right)^2$$

(ε comes from $\widehat{k}_2(v, v') = e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v')$). We may write $u_\varepsilon(s)$ or $u(s)$ for simplicity. We note that

$$u_\varepsilon(0, v', v'') = \left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right)^2 \leq \left(\int_{-1}^0 k_1(v'', v) k_2(v, v') dv \right)^2 = u_0(0, v', v'').$$

We have

$$\begin{aligned} \frac{\partial u}{\partial s} &= -4a \left(\int_{-1}^0 k_1(v'', v) \cos\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right) \int_{-1}^0 \frac{k_1(v'', v)}{|v|} \sin\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \\ &\quad + 4a \left(\int_{-1}^0 k_1(v'', v) \sin\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \cos\left(\frac{2sa}{|v|}\right) \widehat{k}_2(v, v') dv \right) \end{aligned}$$

so $\frac{\partial u}{\partial s}(0, v', v'') = 0$. We have

$$\left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| = \sqrt{u(\alpha, v', v'')}$$

so

$$\frac{\partial}{\partial s} \left(\sqrt{u(s, v', v'')} \right) = \frac{\frac{\partial u}{\partial s}}{2\sqrt{u(s, v', v'')}}.$$

is such that

$$\frac{\partial}{\partial s} \left(\sqrt{u(s, v', v'')} \right)_{s=0} = 0$$

and

$$\begin{aligned} \frac{\partial^2}{\partial s^2} \left(\sqrt{u(s, v', v'')} \right) &= \frac{1}{2} \frac{\frac{\partial^2 u}{\partial s^2} \sqrt{u} - \left(\frac{\partial u}{\partial s} \right)^2}{u} \\ &= \frac{1}{2} \frac{2 \frac{\partial^2 u}{\partial s^2} u - \left(\frac{\partial u}{\partial s} \right)^2}{2\sqrt{uu}} \end{aligned}$$

so

$$\frac{\partial^2}{\partial s^2} \left(\sqrt{u(s, v', v'')} \right)_{\alpha=0} = \frac{1}{2} \frac{\frac{\partial^2 u}{\partial s^2}(0, v', v'')}{\sqrt{u(0, v', v'')}}.$$

On the other hand

$$\begin{aligned} \left(\frac{\partial^2 u}{\partial s^2} \right)_{s=0} &= -8a^2 \left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|^2} \widehat{k}_2(v, v') dv \right) \\ &\quad + 8a^2 \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \widehat{k}_2(v, v') dv \right) \end{aligned}$$

so $-\frac{1}{8a^2} \left(\frac{\partial^2 u}{\partial s^2} \right)_{s=0}$ is given by

$$\left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|^2} \widehat{k}_2(v, v') dv \right) - \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \widehat{k}_2(v, v') dv \right)^2.$$

Since we have strict inequality in the Cauchy-Schwarz inequality which is to say

$$\left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \widehat{k}_2(v, v') dv \right)^2 < \left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|^2} \widehat{k}_2(v, v') dv \right)$$

we see that

$$c_\varepsilon(v', v'') := \left(\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv \right) \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|^2} \widehat{k}_2(v, v') dv \right) - \left(\int_{-1}^0 \frac{k_1(v'', v)}{|v|} \widehat{k}_2(v, v') dv \right)^2 > 0$$

and is *continuous* for smooth (say continuous) functions k_1 and k_2 (ε comes again from $\widehat{k}_2(v, v') = e^{-\frac{2\varepsilon a}{|v|}} k_2(v, v')$). Now

$$\sqrt{u(s, v', v'')} = \sqrt{u(0, v', v'')} + \frac{s^2}{2} \frac{\partial^2}{\partial s^2} \left(\sqrt{u(\zeta, v', v'')} \right)$$

where $\zeta \in (0, s)$ or $\zeta \in (s, 0)$ according as $s > 0$ or $s < 0$. Write it as

$$\sqrt{u(0, v', v'')} - \sqrt{u(s, v', v'')} = \frac{s^2}{2} \left(-\frac{\partial^2}{\partial s^2} \left(\sqrt{u(\zeta, v', v'')} \right) \right).$$

For smooth (say continuous) functions k_1 and k_2

$$-\frac{\partial^2}{\partial s^2} \left(\sqrt{u(s, v', v'')} \right) \rightarrow -\frac{\partial^2}{\partial s^2} \left(\sqrt{u(s, v', v'')} \right)_{s=0} = -\frac{1}{2} \frac{\frac{\partial^2 u}{\partial s^2}(0, v', v'')}{\sqrt{u(0, v', v'')}}$$

(as $s \rightarrow 0$) *uniformly* in (v', v'') and $\varepsilon \in [0, 1]$. On the other hand

$$-\frac{1}{8a^2} \left(\frac{\partial^2 u}{\partial s^2}(0, v', v'') \right) = c_\varepsilon(v', v'')$$

so

$$-\frac{1}{2} \frac{\frac{\partial^2 u}{\partial s^2}(0, v', v'')}{\sqrt{u(0, v', v'')}} = \widehat{c}_\varepsilon(v', v'') := \frac{4a^2 c_\varepsilon(v', v'')}{\sqrt{u_\varepsilon(0, v', v'')}}.$$

is (say) continuous and *bounded away from zero uniformly* in $\varepsilon \in [0, 1]$. Hence

$$\sqrt{u_\varepsilon(0, v', v'')} - \sqrt{u_\varepsilon(s, v', v'')} \geq \frac{s^2}{2} \frac{\widehat{c}_\varepsilon(v', v'')}{2}$$

for s small enough. Let

$$\widehat{\beta} := \inf_{\varepsilon \in [0, 1]} \inf_{v' \in (0, 1)} \int_0^1 \frac{\widehat{c}_\varepsilon(v', v'')}{2} dv'' > 0.$$

Thus,

$$\int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv - \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| \geq \frac{s^2}{2} \frac{\widehat{c}_\varepsilon(v', v'')}{2} \quad (\varepsilon \in [0, 1])$$

for s small enough. Thus

$$\int_0^1 dv'' \int_{-1}^0 k_1(v'', v) \widehat{k}_2(v, v') dv - \int_0^1 dv'' \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| \geq \frac{s^2}{2} \int_0^1 \frac{\widehat{c}_\varepsilon(v', v'')}{2} dv''$$

and

$$1 - \int_0^1 dv'' \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| \geq \frac{s^2}{2} \int_0^1 \frac{\widehat{c}_\varepsilon(v', v'')}{2} dv''$$

so that taking the infimum in $v' \in (0, 1)$ and $\varepsilon \in [0, 1]$ on both sides

$$\begin{aligned} & 1 - \sup_{\varepsilon \in [0, 1]} \sup_{v' \in (0, 1)} \int_0^1 dv'' \left| \int_{-1}^0 k_1(v'', v) e^{-\frac{2isa}{|v|}} \widehat{k}_2(v, v') dv \right| \\ & \geq \frac{s^2}{2} \inf_{\varepsilon \in [0, 1]} \inf_{v' \in (0, 1)} \int_0^1 \frac{\widehat{c}_\varepsilon(v', v'')}{2} dv'' \geq \frac{s^2}{2} \widehat{\beta} \end{aligned}$$

i.e.

$$1 - \sup_{\varepsilon \in [0, 1]} \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| \geq \frac{s^2}{2} \widehat{\beta}.$$

Hence

$$\sup_{\varepsilon \in [0, 1]} \|G_{\varepsilon+is}\| \leq 1 - \beta_1 \beta_2 \left(1 - \sup_{\varepsilon \in [0, 1]} \left\| K_1 e^{-\frac{2\lambda a}{|v|}} K_2 \right\| \right) \leq 1 - \beta_1 \beta_2 \frac{s^2}{2} \widehat{\beta}$$

This ends the proof in the case (45).

Consider now the case (46). In this case $\beta_2 = 0$ and $\alpha_2 = 1$ so

$$\begin{aligned} G_\lambda &= (\alpha_1 R_1 + \beta_1 K_1) e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \\ &= \alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} + \beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}}. \end{aligned}$$

It follows that

$$\begin{aligned} G_\lambda^2 &= \left(\alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} + \beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right)^2 \\ &= \left(\alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \left(\alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \\ &\quad + \left(\beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \left(\beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \\ &\quad + \left(\alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \left(\beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \\ &\quad + \left(\beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \left(\alpha_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \right) \\ &= \beta_1^2 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} + H_\lambda \end{aligned}$$

where

$$\begin{aligned} H_\lambda &= \alpha_1^2 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \\ &\quad + \alpha_1 \beta_1 R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} \\ &\quad + \alpha_1 \beta_1 K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} R_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}}. \end{aligned}$$

Hence

$$\begin{aligned} \|G_\lambda^2\| &\leq \beta_1^2 \left\| K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 \right\| + (1 - \beta_1)^2 + 2(1 - \beta_1)\beta_1 \\ &= \beta_1^2 \left\| K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 \right\| + [1 - \beta_1 + \beta_1]^2 - \beta_1^2 \\ &= 1 - \beta_1^2 \left(1 - \left\| K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 \right\| \right). \end{aligned}$$

It is easy to see that

$$K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 f = \int_{-1}^0 k_1(v'', v) \left(e^{-\frac{4is a}{|v|}} \int_{-1}^0 e^{-\frac{4\varepsilon a}{|v|}} k_1(-v, v') f(v') dv' \right) dv$$

so that $K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1$ has the *same structure* as the operator $K_1 e^{-\frac{2\lambda a}{|v|}} K_2$ considered previously (see (48)). In particular, arguing as previously, one sees that for any $0 < c < c'$

$$\sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0, 1]} \left\| K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 \right\| < 1$$

so

$$\sup_{c \leq |s| \leq c^{-1}} \sup_{\varepsilon \in [0, 1]} \|G_{\varepsilon + is}^2\| < 1 \quad (53)$$

and

$$\lim_{|s| \rightarrow \infty} \left\| K_1 e^{-\frac{2\lambda a}{|v|}} R_2 e^{-\frac{2\lambda a}{|v|}} K_1 \right\| = 0$$

uniformly in $\varepsilon \in [0, 1]$. Finally, as previously, $c_\eta := \sup_{|\lambda| \geq \eta} \|G_\lambda^2\| < 1$ ($\eta > 0$) and there exists $\hat{c} > 0$ such that

$$\|G_\lambda^2\| \leq 1 - \hat{c} |\operatorname{Im} \lambda|^2 \quad (\lambda \rightarrow 0).$$

Since $r_\sigma(G_\lambda) < 1$ for $\operatorname{Re} \lambda \geq 0$ and $\lambda \neq 0$ (see Theorem 21) then for $\lambda \neq 0$

$$\begin{aligned} (1 - G_\lambda)^{-1} &= \sum_{j=0}^{\infty} G_\lambda^j = \sum_{j=0}^{\infty} G_\lambda^{2j} + \sum_{j=0}^{\infty} G_\lambda^{2j+1} = \sum_{j=0}^{\infty} G_\lambda^{2j} + G_\lambda \sum_{j=0}^{\infty} G_\lambda^{2j} \\ &= (1 - G_\lambda^2)^{-1} + G_\lambda (1 - G_\lambda^2)^{-1} \end{aligned}$$

and

$$\|(1 - G_\lambda)^{-1}\| \leq \frac{1}{1 - \|G_\lambda^2\|} + \frac{\|G_\lambda\|}{1 - \|G_\lambda^2\|} \leq \frac{2}{1 - \|G_\lambda^2\|}.$$

Finally $\sup_{|\lambda| \geq \eta} \|(1 - G_\lambda)^{-1}\| \leq \frac{2}{1 - c_\eta}$ and

$$\|(1 - G_\lambda)^{-1}\| \leq \frac{2}{1 - \|G_\lambda^2\|} \leq 2\hat{c}^{-1} |\operatorname{Im} \lambda|^{-2} \quad (\lambda \rightarrow 0).$$

■

Remark 24 (i) We have also a similar statement with $\tilde{G}_\lambda := O_2 e^{-\frac{\lambda}{v} 2a} O_1 e^{-\frac{\lambda}{|v|} 2a}$ instead of G_λ .

(ii) The compactness assumption on K_i ($i = 1, 2$) (which is used in the study of the norm of $\|G_\lambda\|$ or $\|G_\lambda^2\|$ as $|s| \rightarrow \infty$ only) could be avoided by analyzing G_λ^2 in the case (i) and G_λ^3 in the case (ii) (and using Dunford-Pettis arguments). Such a proof is however too cumbersome to be presented. Note that K_i ($i = 1, 2$) are compact if the kernels $k_i(., .)$ of K_i ($i = 1, 2$) are continuous.

Corollary 25 Let $\beta_1 + \beta_2 > 0$. We assume that for almost all $v'' \in (0, 1)$, $k_1(v'', .) \in L^\infty(-1, 0)$ (and for almost all $v'' \in (-1, 0)$, $k_2(v'', .) \in L^\infty((0, 1))$). Then

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \ni \lambda \rightarrow (1 - G_\lambda)^{-1}$$

extends continuously (in the strong operator topology) to $i\mathbb{R} \setminus \{0\}$.

Proof. Let $\hat{\lambda} = i\hat{s}$ ($\hat{s} \neq 0$) and $\lambda_l = \varepsilon_l + is_l \rightarrow \hat{\lambda}$ ($l \rightarrow \infty$). By the part of Theorem 23 which does *not* rely on the compactness of K_i ($i = 1, 2$) (see (51) and (53)) there exists $c < 1$ such that $\|G_{\lambda_l}^2\| \leq c \quad \forall l$ and then we can pass to the limit in

$$(1 - G_{\lambda_l})^{-1} f = (I + G_{\lambda_l}) (1 - G_{\lambda_l}^2)^{-1} f = (I + G_{\lambda_l}) \sum_{j=0}^{\infty} (G_{\lambda_l}^2)^j f$$

as $\varepsilon_l \rightarrow 0_+$ and $s_l \rightarrow \hat{s}$ to show that $(1 - G_{\lambda_l})^{-1} f \rightarrow (1 - G_{\hat{\lambda}})^{-1} f \quad (l \rightarrow \infty)$.

■

Remark 26 We have also a similar statement with $(1 - \tilde{G}_\lambda)^{-1}$ instead of $(1 - G_\lambda)^{-1}$.

Remark 27 In Theorem 23, the continuity assumption on the kernels k_1 and k_2 could probably be replaced by a piecewise continuity assumption; we have not tried to elaborate on this point here.

12 Rates of convergence to equilibrium

We give first algebraic estimates of the resolvent on the imaginary axis.

Theorem 28 *We assume that $O_1 = K_1$ or $O_2 = K_2$. Let the kernels $k_i(.,.)$ of K_i ($i = 1, 2$) be continuous. Let $k \in \mathbb{N}$ and let (37)(38)(43)(44) be satisfied. Then, for any $g \in Z$,*

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\} \ni \lambda \rightarrow (\lambda - T_O)^{-1}g \in L^1(\Omega)$$

extends continuously to $i\mathbb{R} \setminus \{0\}$ as a C^k function

$$\mathbb{R} \setminus \{0\} \ni s \rightarrow F_g(s) \in L^1(\Omega)$$

such that

$$\sup_{|s| \geq 1} \left\| \frac{d^j}{ds^j} F_g(s) \right\| < +\infty \quad (0 \leq j \leq k) \quad (54)$$

and there exists a constant $C > 0$ such that

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq \frac{C}{s^{2(j+1)}} \|g\|_Z \quad (0 \leq j \leq k, \quad 0 < |s| \leq 1). \quad (55)$$

Proof. By Theorem 20

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq C' \left(\sum_{l=0}^{j+1} \|(1 - G_{is})^{-1}\|^l \right) \|g\|_Z \quad (0 \leq j \leq k, \quad s \neq 0).$$

The fact that $s \rightarrow \|(1 - G_{is})^{-1}\|$ is uniformly bounded outside any neighborhood of 0 shows (54). It suffices to prove (55) for *small* s . By using Theorem 23

$$\begin{aligned} \sum_{l=0}^{p+1} \|(1 - G_{is})^{-1}\|^l &\leq \sum_{l=0}^{p+1} \left(\frac{C}{s^2} \right)^l = \frac{1 - \left(\frac{C}{s^2}\right)^{p+2}}{1 - \frac{C}{s^2}} = \frac{\left(\frac{C}{s^2}\right)^{p+2} - 1}{\frac{C}{s^2} - 1} \\ &= \frac{(C)^{p+2} \frac{s^2}{s^{2(p+2)}} - s^2}{C - s^2} = O\left(\frac{1}{|s|^{2(p+1)}}\right) \quad (s \rightarrow 0). \end{aligned}$$

This ends the proof. ■

We are now ready to prove the main result of this paper.

Theorem 29 *We assume that $O_1 = K_1$ or $O_2 = K_2$. Let the kernels $k_i(.,.)$ of K_i ($i = 1, 2$) be continuous and let $(e^{tT_O})_{t \geq 0}$ be irreducible. Let there exist an integer $k \geq 1$ such that (37)(38)(43)(44) are satisfied. If*

$$g \in D(T_O) \text{ and } \int_{\Omega} |g(x, v)| |v|^{-(k+1)} dx dv < +\infty$$

then

$$\left\| e^{tT_O} g - \left(\int_{\Omega} g \right) \psi_0 \right\| = O\left(t^{-\frac{k}{2(k+1)+1}}\right), \quad (t \rightarrow +\infty).$$

Proof. The ergodic projection of $(e^{tT_O})_{t \geq 0}$ is given by $Pg = \left(\int_{\Omega} g\right) \psi_0$ where ψ_0 is given by (17) and is normalized in $L^1(\Omega)$. Then

$$\int_{\Omega} (g - Pg) = \int_{\Omega} g - \int_{\Omega} Pg = \int_{\Omega} g - \left(\int_{\Omega} g\right) \left(\int_{\Omega} \psi_0\right) = 0.$$

Since $h_0 = O_1 O_2 h_0$ then the first part of (38) implies that $\frac{1}{|v|^{k+1}} h_0 \in L^1$. On the other hand, since $\tilde{h}_0 = O_2 h_0$ and

$$\frac{1}{|v|^{k+1}} \tilde{h}_0 = \frac{1}{|v|^{k+1}} O_2 h_0 = \frac{1}{|v|^{k+1}} O_2 v^{k+1} \left(\frac{1}{v^{k+1}} h_0\right)$$

then (44) implies that $\frac{1}{|v|^{k+1}} \tilde{h}_0 \in L^1$ too. Hence

$$\frac{1}{|v|^k} \psi_0 \in L^1(\Omega)$$

by (17) and

$$\frac{1}{|v|} (g - Pg) = \frac{1}{|v|} g - \left(\int_{\Omega} g\right) \frac{1}{|v|} \psi_0 \in L^1(\Omega)$$

since $k \geq 1$. Thus the assumptions in Theorem 10 are satisfied and $g - Pg \in \text{Ran}(T_O)$. Since $g - Pg \in D(T_O)$ then Theorem 28 and Corollary 3 end the proof. ■

Remark 30 A sufficient criterion of irreducibility of $(e^{tT_O})_{t \geq 0}$ is given in Theorem 7. The continuity of the kernels $k_i(.,.)$ ($i = 1, 2$) could probably be relaxed, see Remark 27.

Remark 31 A priori, the rates of convergence given in this paper depend on the condition $\beta_1 = 1$ or $\beta_2 = 1$. Two kinds of assumptions appear in this work: The "kernel" assumptions (37) (38) which can be checked only if $\beta_1 = 1$ or $\beta_2 = 1$ (see Remark 14) and the "non-kernel" assumptions (43)(44) which are satisfied even by the reflections conditions (see Remark 19). (Note that Theorem 21 and Theorem 23 hold under the very general condition $\beta_1 + \beta_2 > 0$.) The extension of the theory to the general case $\beta_1 + \beta_2 > 0$ (or at least to the case $\beta_1 \beta_2 > 0$) should depend on a weakening of the "kernel" assumptions which are used essentially in the proof of the key Lemma 15.

Remark 32 Three additional open problems are worth mentioning.

(i) We have seen that the quantified Ingham's theorem provides us with the rate of convergence $O\left(\frac{1}{t^{\frac{1}{2}+\varepsilon}}\right)$ for any $\varepsilon > 0$ if the structural assumptions are satisfied for all $k \in \mathbb{N}$, see (23). Whether one can reach the limit rate $O\left(\frac{1}{\sqrt{t}}\right)$ (or can go beyond this rate) in the context of kinetic theory is an open problem. Note also that if there exists a constant $C > 0$ such that

$$\left\| \frac{d^j}{ds^j} F_g(s) \right\| \leq C j! |s|^{-2(j+1)+1} \quad (j \in \mathbb{N}, 0 < |s| \leq 1) \quad (56)$$

then another quantified version of Ingham's theorem (see Remark 4) gives the rate $O(\sqrt{\frac{\ln(t)}{t}})$. However, in practice, the verification of (56) seems to be out of reach.

(ii) A completely open problem is to quantify the sweeping phenomenon (20) in case of lack of invariant densities.

(iii) We know (see Theorem 12) that the imaginary axis is the boundary spectrum of the generator if $\beta_1 = 1$ or $\beta_2 = 1$. The extension of this result to more general partly diffuse models is an open problem.

Remark 33 This work could be extended to non-monoenergetic free models in slab geometry

$$\frac{\partial f}{\partial t}(t, x, v, \rho) + \rho v \frac{\partial f}{\partial x}(t, x, v, \rho) = 0, \quad (x, v, \rho) \in \Omega$$

where $\Omega = (-a, a) \times (-1, 1) \times (0, +\infty)$ with stochastic partly diffuse boundary conditions

$$|v| f(t, -a, v, \rho) = \alpha_1 |v| f(t, -a, -v, \rho) + \beta_1 \int_0^{+\infty} d\rho' \int_{-1}^0 k_1(v, v', \rho, \rho') f(t, -a, v', \rho') dv'$$

for $v \in (0, 1)$ and

$$|v| f(t, a, v, \rho) = \alpha_1 |v| f(t, a, -v, \rho) + \beta_1 \int_0^{+\infty} d\rho' \int_0^1 k_2(v, v', \rho, \rho') f(t, a, v', \rho') dv'$$

for $v \in (-1, 0)$ under the convexity condition (5). Indeed, the approach taken in [21] could be extended to this model and the arguments of the present paper could be adapted accordingly. We have not tried to elaborate on this point here.

The specular reflection $R_1 : L^1((-1, 0); dv) \rightarrow L^1((0, +1); dv)$ defined by $(R_1\varphi)(v) = \varphi(-v)$ could also be replaced by a more general deterministic boundary operator of the form $(R_1\varphi)(v) = \varphi(\zeta(v))$ where $\zeta : (0, +1) \rightarrow (-1, 0)$ is a smooth measure-preserving function. A similar remark applies to R_2 .

Remark 34 We are confident that our formalism could extend to multidimensional (in space) geometries with partly diffuse boundary operators. However, such an extension is not straightforward and faces serious additional problems we hope to be able to deal with in the near future.

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