Stratified hyperkähler spaces and Nahm's equations

Maxence Mayrand

St Hilda's College

University of Oxford

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To my wife, Anne-Elizabeth.

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Abstract

When a compact Lie group acts freely and in a Hamiltonian way on a symplectic manifold, the Marsden–Weinstein theorem says that the reduced space is a smooth symplectic manifold. If we drop the freeness assumption, the reduced space is usually fairly singular, but Sjamaar and Lerman showed that it can still be stratified into smooth symplectic manifolds which “fit together nicely”, in a precise sense. In this thesis, we prove analogues of Sjamaar–Lerman's results in hyperkähler geometry, yielding to the notion of stratified hyperkähler spaces.

We also study examples of stratified hyperkähler spaces coming from hyperkähler quotients of certain moduli spaces of solutions to the so-called *Nahm equations*. In particular, we prove a Kempf–Ness type theorem which realises these spaces as quasi-projective algebraic varieties. We then focus on an interesting family of examples whose stratification structure can be described explicitly by combinatorial data associated with the root system of a complex semisimple Lie algebra.

Leaving stratified spaces aside, we investigate how Nahm's equations, which are non-linear systems of ODEs, can generate groupoid structures by concatenation of paths, in a manner analogous to the fundamental groupoid of a topological space.

Finally, we study another moduli space of solutions to Nahm's equations, which is a smooth hyperkähler manifold diffeomorphic to a variety studied in geometric representation theory called the *universal centraliser*. We draw analogies between this space and the moduli space of Higgs bundles, and explain how mirror symmetry naturally enters into the picture.

Contents

Chapter 1

Introduction

A hyperkähler manifold is a Riemannian manifold (M, g) with three complex structures I, J, K that are Kähler with respect to g and satisfy the quaternionic relations $\mathsf{I}^2 = \mathsf{J}^2 = \mathsf{K}^2 = \mathsf{IJK} = -1$. Equivalently, a hyperkähler manifold is a Riemannian manifold of dimension $4n$ whose holonomy group is contained in $\mathrm{Sp}(n)$; hence, their existence was anticipated in the 1950s by Berger's list of possible holonomy groups [9]. The space $\mathbb{H}^n$ of n-tuples of quaternions is, of course, such a manifold, but the first non-trivial examples only came in the late 1970s. The earliest one appeared in physics through the work of Eguchi–Hanson [44, 45] who found such a metric on the cotangent bundle of a sphere. It was soon generalised by Calabi [26] who showed that $T^*\mathbb{CP}^n$ is hyperkähler for all n (and who coined the term *hyperkähler*).

But examples remained sparse and difficult to find until Hitchin–Karlhede–Lindström–Roček [77] introduced in 1987 the *hyperkähler quotient construction*, which is now the most widely used tool for constructing these special manifolds. It is the direct analogue of the Marsden–Weinstein symplectic reduction procedure [104]: given a compact Lie group K acting freely on a hyperkähler manifold M, this process produces (under certain conditions) a new hyperkähler manifold of dimension $\dim M - 4 \dim K$. More precisely, the manifold is the quotient $\mu^{-1}(0)/K$, where μ is a hyperkähler version of the moment map in symplectic geometry. The main feature of this construction is that one can start with a very simple hyperkähler manifold, such as $M = \mathbb{H}^n$, and still get an interesting result. For instance, Calabi's metric on $T^*\mathbb{CP}^n$ can be obtained from a hyperkähler quotient of $\mathbb{H}^{n+1}$. The study of hyperkähler quotients of $\mathbb{H}^n$ went into two far-reaching directions, Nakajima quiver varieties [115] and hypertoric varieties [13], both of which are still very active research areas.

Hyperkähler quotients also make sense in infinite dimensions, and this has led to highly non-trivial examples from gauge theory. Given a principal bundle over a hyperkähler manifold, one can see the space of connections as a flat hyperkähler manifold (analogous to $\mathbb{H}^n$) and the self-dual Yang–Mills equations as the components of a hyperkähler moment map for the action of the group of gauge transformations. Thus, by restricting to self-dual connections invariant under a special symmetry and imposing suitable boundary conditions, one can get a moduli space of connections which is a hyperkähler manifold. For example, moduli spaces of instantons, monopoles [80], or Higgs bundles [78] have been shown to be hyperkähler manifolds by this method. Another example is to take a compact Lie group K and an interval $I \subseteq \mathbb{R}$, and considering self-dual connections on the trivial principal K-bundle over $I \times \mathbb{R}^3$ which are $\mathbb{R}^3$-invariant. The self-duality equations reduce to the so-called *Nahm equations* and, after imposing some boundary conditions, the moduli space is a non-trivial hyperkähler manifold associated with K. This provides a beautiful bridge between hyperkähler geometry and Lie theory. For instance, many complex-symplectic varieties canonically associated with the complexification $G := K_{\mathbb{C}}$ of K have been shown to admit a hyperkähler structure by identifying them with such a moduli space. Examples include the cotangent bundle T^*G [95], coadjoint orbits in $\mathrm{Lie}(G)^*$ [97, 98, 14, 94], and $G \times \mathbb{C}^{\mathrm{rk}\, G}$ [11] (see also [22] and [33]).

Another direction in which the hyperkähler quotient construction is generalised is by relaxing the condition that K acts freely and thus considering singular spaces. For example, Nakajima quiver varieties [115] are often of this form. Although hyperkähler quotients by non-free actions are singular topological spaces, it is known that they can be partitioned into smooth manifolds, each carrying a hyperkähler structure [35]. This is proved by adapting the seminal work of Sjamaar–Lerman [134] on singular symplectic reduction.

However, not all results of Sjamaar–Lerman [134] have been adapted to the hyperkähler setting. For instance, they also showed that the partition of a singular symplectic reduction into symplectic manifolds is a *stratification*. This is a topological condition for a space X endowed with a partition $\mathcal{P}$ which is, in some sense, the notion of topological spaces that is closest to manifolds while still allowing singularities. The idea is, roughly speaking, that each piece S in the partition $\mathcal{P}$ of X is a topological manifold, and the way in which it embeds in X remains constant along S. This notion has been crucial for constructing cohomology theories for singular spaces: intersection cohomology, as developed by Goresky–MacPherson [53], is built on stratified spaces. However, the techniques used by Sjamaar–Lerman to show that singular symplectic reductions are stratified spaces cannot be directly applied to hyperkähler quotients, and hence the condition remained unknown in this case.

In this thesis, we show that hyperkähler quotients by non-free actions are indeed stratified spaces, provided that the action satisfies a mild integrability assumption, namely, that it extends to a holomorphic action of the complexification. We also explore more of the connections that Nahm's equations make between Lie theory and hyperkähler geometry. In particular, we study hyperkähler quotients of T^*G: we prove a Kempf–Ness type theorem about them, and describe an interesting family of examples with a non-trivial stratification structure. Leaving stratified spaces aside, we also investigate groupoid structures emerging from moduli spaces of solutions to Nahm's equations on compact intervals. Finally, we study another moduli space of solutions to Nahm's equations, which is diffeomorphic to an interesting variety studied in geometric representation theory. A more detailed summary of each chapter follows:

Chapter 2. Background on stratified spaces and reduction

This is a review chapter, where we introduce the necessary background on stratified spaces and their link with symplectic reduction. We also briefly review Geometric Invariant Theory (GIT) quotients and their complex-analytic counterpart.

Chapter 3. Singular hyperkähler quotients

We prove that the decomposition of a singular hyperkähler quotient into hyperkähler manifolds is a stratification if the action extends to a holomorphic action of the complexification. We also show that such a hyperkähler quotient carries a natural complex-analytic structure with a Poisson bracket and obtain local models which describe this structure canonically around any point. The main ingredient in the proof is a new local normal form for the underlying complex-Hamiltonian manifold of a tri-Hamiltonian hyperkähler manifold, which is analogous to Guillemin–Sternberg's local normal form for the moment map [60].

Chapter 4. A Kempf–Ness type theorem

The Kempf–Ness Theorem is a standard result in algebraic and symplectic geometry which identifies certain symplectic reductions with GIT quotients. In this chapter, we prove a new version of that theorem which applies to a broad class of (possibly transcendental) Kähler structures on complex affine varieties.

Chapter 5. Nahm's equations and GIT

For any compact Lie group K, the cotangent bundle T^*G, where $G := K_{\mathbb{C}}$, admits a hyperkähler structure by a diffeomorphism with the moduli space of solutions to Nahm's equations on a compact interval. Moreover, this structure is invariant under the action of $K \times K$, and there is a hyperkähler moment map. Hence, for any closed subgroup H of $K \times K$, we can consider the hyperkähler quotient of T^*G by H. In this chapter, we show that the Kempf–Ness type theorem proved in Chapter 4 can be used to identify these spaces with GIT quotients. This requires showing a particular relationship between the hyperkähler and algebraic structures of T^*G.

Chapter 6. Stratified hyperkähler spaces from semisimple Lie algebras

We study in detail a special case of the hyperkähler quotients of T^*G introduced in Chapter 5, namely, when the group acting is $T_K \times T_K$, where T_K is a maximal torus in K. The quotient is shown to depend only on the Lie algebra $\mathfrak{g}$ of G and is denoted by $\mathcal{D}(\mathfrak{g})$. It is a stratified space by the results of Chapter 3, and we show that the structure of the stratification can be determined explicitly by combinatorial data associated with the root system of $\mathfrak{g}$. More precisely, as for any partitioned space, there is a natural partial order on the set of strata given by $S_1 \leq S_2$ if S_1 is contained in the closure of S_2. We show that the poset of strata of $\mathcal{D}(\mathfrak{g})$ is isomorphic to the poset of root subsystems of $\mathfrak{g}$. We also compute many examples and draw the Hasse diagrams of their stratification posets.

Chapter 7. Groupoid structures from Nahm's equations

Coming back to the moduli space $\mathcal{M}_I$ of solutions to Nahm's equations on a compact interval I (which is diffeomorphic to T^*G) we study how these spaces interact for different intervals I. In particular, we show that if $a < b < c$ then elements of $\mathcal{M}_{[a,b]}$ and $\mathcal{M}_{[b,c]}$ which agree at b can be combined to form elements of $\mathcal{M}_{[a,c]}$ by concatenation of paths. This bring natural questions about possible groupoid structures arising in a manner analogous to the fundamental groupoid of a topological space, and the goal of this chapter is to answer some of these questions.

Chapter 8. Universal centralisers, Higgs bundles, and mirror symmetry

The universal centraliser $\mathfrak{Z}_G$ of a complex reductive group G is an interesting complex-symplectic variety which has been studied in geometric representation theory [119, 120, 10]. It has also been shown to be diffeomorphic to a moduli space of solutions to Nahm's equations [11, 22] and hence has a hyperkähler structure. If $\check{G}$ denotes the Langlands dual group of G, then $\mathfrak{Z}_G$ and $\mathfrak{Z}_{\check{G}}$ fibre onto a common base space $\mathbb{C}^{\operatorname{rk} G}$ and these fibrations are completely integrable systems with Lagrangian fibres. Moreover, the generic fibres are dual complex-algebraic tori, in a manner reminiscent to the SYZ mirror symmetry picture. We draw analogies between this and the mirror symmetry program of Hitchin systems [65, 39]. In particular, we show that $\mathfrak{Z}_G$ can be seen as a "moduli space of regular marked G-Higgs bundles over a point", in a precise sense. We also prove the existence of non-trivial branes inside $\mathfrak{Z}_G$ analogous to those constructed in the moduli space of G-Higgs bundles by Baraglia–Schaposnik [7]. However, we also investigate the Hodge numbers of $\mathfrak{Z}_G$ and $\mathfrak{Z}_{\check{G}}$ and show that they don't behave like those of a mirror pair, at least when $G = \mathrm{SL}(2, \mathbb{C})$.

Chapter 2

Background on stratified spaces and reduction

This chapter gives background material on stratified spaces, symplectic reduction, geometric invariant theory, quotients of complex-analytic spaces, and the links between these notions. We start with a review of the basic theory of stratified spaces and then explain the work of Sjamaar–Lerman [134] on singular symplectic quotients. We then discuss links with complex-analytic geometry, reviewing results of Heinzner–Loose [71] and Sjamaar [133].

2.1 Stratified spaces

The idea behind stratified spaces is to describe singular topological spaces by decomposing them into manifolds which "fit together nicely". The underlying category for this theory is thus the following:

Definition 2.1.1. A **partitioned space** is a pair $(X, \mathcal{P})$ where X is a topological space and $\mathcal{P}$ a partition of X, i.e. a collection of non-empty disjoint subsets of X whose union is X. The elements of $\mathcal{P}$ are called the **pieces** of the partitioned space. A **morphism** between two partitioned spaces $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ is a continuous map $f : X \to Y$ such that for every piece $S \in \mathcal{P}$, the image $f(S)$ is contained in another piece $T \in \mathcal{Q}$. This defines a category **PTop** of partitioned spaces under composition.

Note that a morphism $f : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ of partitioned spaces determines a map $\hat{f} : \mathcal{P} \to \mathcal{Q}$. In fact, this gives a functor **PTop** $\to$ **Set**, $(X, \mathcal{P}) \mapsto \mathcal{P}$ to the category of sets. We note the following easy observation.

Proposition 2.1.2. *Let $f : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ be a morphism of partitioned spaces. Then, f is an isomorphism in* **PTop** *if and only if $f : X \to Y$ is a homeomorphism and $\hat{f}$ is a bijection.* □

Just like manifolds are topological spaces satisfying additional conditions (second countable, Hausdorff, and locally Euclidean), stratified spaces are partitioned spaces with additional conditions imposed. The first step is the following notion.

Definition 2.1.3 ([53, §1.1])**.** A **decomposed space** is a partitioned space $(X, \mathcal{P})$ such that X is second countable and Hausdorff, and the following conditions hold:

- **Manifold condition.** Each element of $\mathcal{P}$ is a topological manifold in the subspace topology.
- **Local condition.** $\mathcal{P}$ is locally finite and its elements are locally closed.
- **Frontier condition.** For all $S, T \in \mathcal{P}$, if $S \cap \overline{T} \neq \emptyset$ then $S \subseteq \overline{T}$.

In that case, we say that $\mathcal{P}$ is a **decomposition** of X.

If $(X, \mathcal{P})$ is a decomposed space, then there is a natural relation on $\mathcal{P}$ given by $S \leq T$ if $S \subseteq \overline{T}$. It follows from the local closedness of the strata that this relation is a partial order. Moreover, the frontier condition is equivalent to

$$\overline{S} = \bigcup_{T \leq S} T, \quad \text{for all } S \in \mathcal{P}.$$

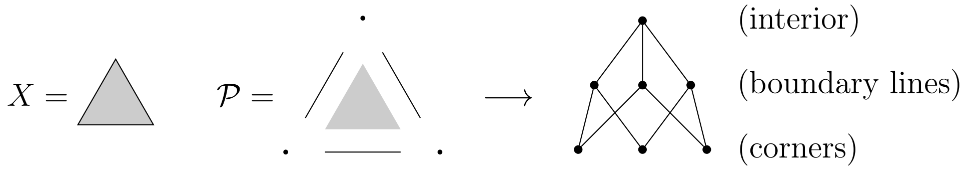
If $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are decomposed spaces and $f : (X, \mathcal{P}) \to (Y, \mathcal{Q})$ is a morphism of partitioned spaces, then $\hat{f} : \mathcal{P} \to \mathcal{Q}$ is a morphism of posets, and this gives a functor

$$\mathbf{DTop} \longrightarrow \mathbf{Poset}, \quad (X, \mathcal{P}) \longmapsto (\mathcal{P}, \leq)$$

from the category of decomposed spaces to the category of posets. We call $(\mathcal{P}, \leq)$ the **stratification poset** of $(X, \mathcal{P})$. It captures some key structural information about X.

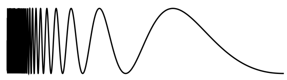
Remark 2.1.4. This notion is sometimes incorporated in the definition of decomposed spaces, namely, we fix a poset $\mathcal{I}$ and say that an $\mathcal{I}$-decomposed space is a decomposed space $(X, \mathcal{P})$ with an isomorphism $\mathcal{P} \cong \mathcal{I}$ of posets.

It is often useful to use a **Hasse diagram** to visualise the stratification poset. This is the diagram obtained by drawing a node for each $S \in \mathcal{P}$ and an edge upward from node S to node T if $S < T$ but there is no $Q \in \mathcal{P}$ such that $S < Q < T$. In particular, the highest dimensional strata are on the top and the lowest dimensional ones on the bottom. For example, the following figure represents a decomposition of the filled equilateral triangle and the corresponding stratification poset:



In Chapter 6, we will identify the stratification poset of a family of decomposed spaces $\mathcal{D}(\mathfrak{g})$ associated with the semisimple Lie algebras $\mathfrak{g}$ and draw examples of these diagrams (see §6.5).

This definition captures the intuitive idea of a space decomposed into manifolds, but it does not tell us *how* the pieces fit together. For example, the topologist's sine curve



is a perfectly valid decomposed space with two strata (the vertical segment on the left and the curve on the right). Roughly speaking, stratified spaces avoid such pathologies by requiring that every point has a neighbourhood which retracts continuously onto it. We also impose that this neighbourhood is compatible with the partition in some sense. To make this precise, we need a few extra notions. First, the **dimension** of a decomposed space $(X, \mathcal{P})$ is

$$\dim(X, \mathcal{P}) \coloneqq \sup\{\dim S : S \in \mathcal{P}\}.$$

Given two partitioned spaces $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$, their **cartesian product** is the partitioned space $(X \times Y, \mathcal{P} \times \mathcal{Q})$ where $\mathcal{P} \times \mathcal{Q} = \{S \times T : S \in \mathcal{P}, T \in \mathcal{Q}\}$. If $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are decomposed spaces, then so is $(X \times Y, \mathcal{P} \times \mathcal{Q})$, and $\dim(X \times Y, \mathcal{P} \times \mathcal{Q}) = \dim(X, \mathcal{P}) + \dim(Y, \mathcal{Q})$. Next, the **cone** over a partitioned space $(X, \mathcal{P})$ is the partitioned space $(CX, C\mathcal{P})$ where CX is the open cone over X, i.e.

$$CX \coloneqq (X \times [0, \infty))/\{(p, 0) \sim (q, 0), \text{for all } p, q \in X\}$$

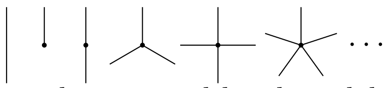
and $C\mathcal{P}$ is the natural partition of CX given by

$$C\mathcal{P} \coloneqq \{S \times (0, \infty) : S \in \mathcal{P}\} \cup \{\text{vertex}\}.$$

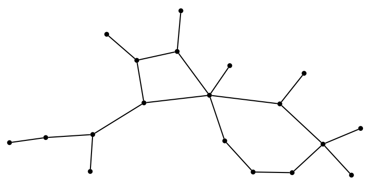
The cone over a decomposed space $(X, \mathcal{P})$ is itself a decomposed space and has dimension $\dim(CX, C\mathcal{P}) = \dim(X, \mathcal{P}) + 1$. A stratified space is defined inductively as a decomposed space $(X, \mathcal{P})$ which is locally isomorphic to $\mathbb{R}^n$ times a cone over a lower-dimensional stratified space:

Definition 2.1.5 ([53, 134])**.** A zero-dimensional stratified space is any countable set of points with the discrete topology and any partition. A **stratified space** is a finite-dimensional decomposed space $(X, \mathcal{P})$ such that every point $p \in X$ has a neighbourhood isomorphic as a partitioned space to $\mathbb{R}^n \times CL$ for some $n \geq 0$ and some compact stratified space L, by a map sending $p \mapsto \{0\} \times \{\text{vertex}\}$. In that case, we say that $\mathcal{P}$ is a **stratification** of X.

For example, one-dimensional stratified spaces are locally modelled on cones over finite sets of points, which means that they are the same thing as graphs:



one-dimensional local models



a one-dimensional stratified space

Then, two-dimensional stratified spaces are locally modelled on cones over graphs, etc. Also, all manifolds with corners are stratified spaces.

The compact stratified space L associated with a point p in Definition 2.1.5 is called the **link** at p and is unique up to homeomorphisms. Moreover, for a connected stratum $S \in \mathcal{P}$, every point of S has the same link, so we may speak of the link of the stratum. This is the closest notion of "locally Euclidean" that we can get for partitioned spaces, namely, the local structure along a stratum is constant. Note that for any link L, the space $\mathbb{R}^n \times CL$ is contractible. In particular, the topologist's sine curve above is not a stratified space.

A typical way of proving that a decomposed space $(X, \mathcal{P})$ is a stratified space is by the Whitney conditions:

Definition 2.1.6 (Whitney [145])**.** Let S and T be two disjoint smooth submanifolds of $\mathbb{R}^n$. We say that S is **regular** over T if the following two conditions hold:

- **Whitney Condition A.** If $x_i \in S$ is a sequence converging to $y \in T$ and the sequence of subspaces $T_{x_i}S \subseteq \mathbb{R}^n$ converges (in the Grassmannian) to $V \subseteq \mathbb{R}^n$, then $T_yT \subseteq V$.

- **Whitney Condition B.** If $x_i \in S$ and $y_i \in T$ are two sequences converging to $y \in T$ in such a way that that the sequence of lines $\mathbb{R}(x_i - y_i) \subseteq \mathbb{R}^n$ converges to $l \in \mathbb{RP}^{n-1}$ and the subspaces $T_{x_i}S$ to $V \subseteq \mathbb{R}^n$, then $l \subseteq V$.

A **Whitney stratification** of a subset X of $\mathbb{R}^n$ is a decomposition $\mathcal{P}$ of X into smooth submanifolds of $\mathbb{R}^n$ such that S is regular over T for all $S, T \in \mathcal{P}$.

We have (see, e.g., Goresky–MacPherson [53, Part I, Ch. 1, §1.4] or Mather [105]):

Proposition 2.1.7. *Whitney stratifications are stratifications in the sense of* Definition 2.1.5. □

For example, Whitney [145] showed that complex-algebraic varieties have Whitney stratifications, and this theorem is used to infer that they are stratified spaces (see [128]).

Although Whitney stratifications are initially defined in $\mathbb{R}^n$, the definition is purely local and is invariant under diffeomorphisms [105, §2]. In particular, it makes sense for complex-analytic spaces:

Definition 2.1.8. A **complex-analytic Whitney stratified space** is a complex-analytic space $(X, \mathcal{O}_X)$ together with a decomposition $\mathcal{P}$ of X into complex submanifolds satisfying Whitney conditions A and B.

By Proposition 2.1.7, complex-analytic Whitney stratified spaces are stratified spaces as in Definition 2.1.5.

2.2 Smooth manifold quotients

Many interesting examples of stratified spaces come from quotients of topological spaces by non-free group actions. The simplest case is the quotient of a smooth manifold M by a proper action of a Lie group K. We review this case here as it will introduce some important notation and concepts for later.

Let K be a Lie group acting smoothly and properly on a smooth manifold M. Then, the quotient space M/K is a stratified space with respect to a natural partition by orbit-types. To define this partition, for each subgroup $H \subseteq K$, let (H) be the conjugacy class of H in K. We say that $p \in M$ has **orbit-type** (H) if its stabiliser subgroup K_p is in (H). Denote the set of points of orbit-type (H) by

$$M_{(H)} \coloneqq \{p \in M : K_p \in (H)\}.$$

Then, the connected components of the subspaces of the form $M_{(H)}/K$ for all $H \subseteq K$ form a stratification of M/K. The proof is an application of the slice theorem for proper group actions (see, e.g., [42, Theorem 2.7.4]). Recall that the slice theorem gives a local model for the K-manifold M near a point $p \in M$ in terms of K, K_p and the vector space $W \coloneqq T_pM/T_p(K\cdot p)$ (see, e.g., [42, Theorem 2.4.1]). More precisely, there is a K_p-invariant neighbourhood U of 0 in W and a K-equivariant diffeomorphism from $K \times_{K_p} U$ to a neighbourhood of p in M (where $K \times_{K_p} U$ is the quotient of $K \times U$ by the K_p-action $h \cdot (k, v) = (kh^{-1}, h \cdot v))$. This reduces K-equivariant local properties of M to the representation theory of compact Lie groups.

2.3 Stratified symplectic spaces

Symplectic geometry provides another important source of stratified spaces, as we now recall. We say that a **Hamiltonian manifold** is a triple (M, K, μ), where M is a symplectic manifold, K a compact Lie group acting on M by symplectomorphisms, and μ a moment map for this action, i.e. a K-equivariant smooth map $\mu : M \to \mathfrak{k}^*$ such that $d\langle \mu, x\rangle = x^\# \lrcorner\, \omega$ for all $x \in \mathfrak{k}$, where $\mathfrak{k} := \mathrm{Lie}(K)$, ω is the symplectic form of M, and $x^\#$ is the vector field on M generated by x. Recall that the Marsden–Weinstein Theorem [104] says that if the action of K on $\mu^{-1}(0)$ is free, then the quotient

$$M /\!\!/_{\mu} K := \mu^{-1}(0)/K$$

is a smooth symplectic manifold, called the **symplectic reduction** of M by K with respect to μ. If the action is not necessarily free, then $M /\!\!/_{\mu} K$ is usually fairly singular, but Sjamaar–Lerman [134] showed that it still has a natural stratification into smooth symplectic manifolds. The strata are the connected components of the subspaces of the form $\mu^{-1}(0)_{(H)}/K$ for all subgroups $H \subseteq K$. Moreover, the symplectic structures on these strata also fit together nicely, in the sense that they are compatible with a certain Poisson bracket on $M /\!\!/_{\mu} K$. Sjamaar–Lerman also gave local models for $M /\!\!/_{\mu} K$, namely, every point has a neighbourhood homeomorphic to a linear symplectic reduction with a homeomorphism respecting the natural stratifications and Poisson brackets on both sides.

We will review these statements more precisely, as it will introduce useful ideas and notation for the next chapter, which is to extend Sjamaar–Lerman's result to the hyperkähler setting.

The symplectic forms on the strata can be seen as follows (see [134, Theorem 3.5]). For a closed subgroup $H \subseteq K$, let M_H be the set of points $p \in M$ whose stabiliser is precisely H. Then, the connected components of M_H are smooth symplectic submanifolds of M (of possibly different dimensions) and the group $L := N_K(H)/H$ (where $N_K(H)$ is the normaliser of H in K) is compact and acts freely on M_H by preserving the symplectic forms. Now, $\mathfrak{l}^* := \mathrm{Lie}(L)^*$ can be identified with a subspace of $\mathfrak{k}^*$, namely, $\mathfrak{h}^\circ \cap (\mathfrak{k}^*)^H$, where $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h} := \mathrm{Lie}(H)$ and $(\mathfrak{k}^*)^H$ is the set of points fixed by H. Moreover, if M'_H denotes the union of the connected components of M_H which intersect $\mu^{-1}(0)$, then μ restricts to a moment map $\mu_H : M'_H \to \mathfrak{l}^*$ for the action of L on M'_H. Since this action is free, each connected component of $M_H /\!/_{\mu_H} L = \mu_H^{-1}(0)/L$ is a smooth symplectic manifold by the standard Marsden–Weinstein Theorem. Then, the inclusion $\mu_H^{-1}(0) \subseteq \mu^{-1}(0)_{(H)}$ descends to a homeomorphism $M_H /\!/_{\mu_H} L \cong \mu^{-1}(0)_{(H)}/K$, and this endows each connected component of $\mu^{-1}(0)_{(H)}/K$ with a symplectic structure. Furthermore, the pullback of each symplectic form to the corresponding connected component of $\mu^{-1}(0)_{(H)}$ (which is a smooth submanifold of M) is the restriction of the symplectic form of M.

The symplectic structures on the strata of $M /\!/_\mu K$ can also be viewed more globally as a Poisson structure on $M /\!/_\mu K$. Let $C^\infty(M /\!/_\mu K)$ be $\mathbb{R}$-algebra of continuous functions on $M /\!/_\mu K$ which descend from smooth K-invariant functions on M. Then, there is a natural Poisson bracket on $C^\infty(M /\!/_\mu K)$ such that the inclusion of each stratum into $M /\!/_\mu K$ is a Poisson map. This motivated Sjamaar–Lerman to make the following definition.

Definition 2.3.1. A **stratified symplectic space** is a stratified space $(X, \mathcal{P})$ with smooth and symplectic structures on each stratum, a subalgebra $C^\infty(X)$ of the $\mathbb{R}$-algebra of continuous functions on X, and a Poisson bracket on $C^\infty(X)$ such that for each stratum $S \in \mathcal{P}$ the embedding $S \hookrightarrow X$ is a Poisson map, i.e. for all $f, g \in C^\infty(X)$ the restrictions $f|_S, g|_S$ are smooth and $\{f|_S, g|_S\} = \{f, g\}|_S$.

Theorem 2.3.2 (Sjamaar–Lerman [134])**.** *For every Hamiltonian manifold* (M, K, μ)*, the quotient* $M /\!/_\mu K$ *is a stratified symplectic space.* □

In fact, they showed the stronger statement that $M /\!/_\mu K$ has an embedding in $\mathbb{R}^n$ such that the orbit-type partition is a Whitney stratification and used Proposition 2.1.7 to deduce that $M /\!/_\mu K$ is a stratified space.

An important class of examples of stratified symplectic spaces which we will use later are the **linear symplectic quotients**. Let (V, ω) be a symplectic vector space, i.e. a finite-dimensional real vector space V with a non-degenerate skew-symmetric bilinear form ω. Then, (V, ω) can be viewed as a symplectic manifold. A **symplectic representation** of a Lie group K on (V, ω) is a Lie group homomorphism $K \to \mathrm{Sp}(V, \omega)$, where $\mathrm{Sp}(V, \omega)$ is the group of $g \in \mathrm{GL}(V)$ such that $\omega(gu, gv) = \omega(u, v)$ for all $u, v \in V$. Such a representation gives rise to a Hamiltonian manifold:

Proposition 2.3.3. *Let* $K \to \mathrm{Sp}(V, \omega)$ *be a symplectic representation. Then, there is a canonical moment map for the* K*-action on* V*, namely,*

$$\phi_V : V \longrightarrow \mathfrak{k}^*, \quad \phi_V(v)(x) = \frac{1}{2}\omega(x \cdot v, v), \tag{2.3.1}$$

for all $v \in V$ *and* $x \in \mathfrak{k}$*.* □

Thus, each symplectic representation $K \to \mathrm{Sp}(V, \omega)$ of a compact Lie group K induces a canonical stratified symplectic space $V /\!\!/_{\phi_V} K$. In fact, Sjamaar–Lerman [134, Theorem 5.1] showed that these spaces are universal models for symplectic reductions:

Theorem 2.3.4 (Sjamaar–Lerman [134, Theorem 5.1]). *Let (M, K, μ) be a Hamiltonian manifold. Then every point $x \in M /\!\!/_\mu K$ has a neighbourhood isomorphic as a stratified symplectic space to a neighbourhood of the image of 0 in a linear symplectic reduction $V /\!\!/_{\phi_V} H$. More precisely, $V = T_p(K \cdot p)^\omega / T_p(K \cdot p)$ (where $(\cdot)^\omega$ is the symplectic complement), and $H = K_p$.* □

Just as for quotients of smooth manifolds (§2.2), the proof that symplectic reductions are stratified spaces uses an appropriate local model. This time, it is the local normal form for the moment map of Guillemin–Sternberg [60] and Marle [103], which is a generalisation of the Darboux Theorem to Hamiltonian manifolds. In the next chapter, we will adapt Sjamaar–Lerman's argument to the hyperkähler setting by proving a holomorphic version of this normal form. Thus, it will be useful to first review the symplectic local normal form here.

Recall that the Darboux Theorem can be interpreted as saying that every point p in a symplectic manifold (M, ω) has a neighbourhood symplectomorphic to a neighbourhood of 0 in the symplectic vector space $V = T_pM$, i.e. symplectic forms can be linearised, and V is the local model. Similarly, the local normal form for the moment map says that a Hamiltonian manifold (M, K, μ) is completely determined in a neighbourhood of a point $p \in \mu^{-1}(0)$ by the representation of $H = K_p$ on the symplectic slice $V \coloneqq (T_p(K \cdot p))^\omega / T_p(K \cdot p)$. In this case, the local model is the associated vector bundle $K \times_H (\mathfrak{h}^\circ \times V)$ over $K/H = K \cdot p$. This space is homeomorphic to a symplectic reduction of $T^*K \times V$ by H and hence has a canonical symplectic form. Moreover, the left K-action $k \cdot [g, \xi, v] = [kg, \xi, v]$ is Hamiltonian, and there is an explicit expression for the moment map. One shows that a neighbourhood of $K \cdot p$ in M is isomorphic as a Hamiltonian K-manifold to a neighbourhood of the zero-section in $K \times_H (\mathfrak{h}^\circ \times V)$. Setting $K = 1$ recovers the Darboux Theorem. Sjamaar–Lerman used this to prove Theorem 2.3.2 by reducing to the case of the Hamiltonian manifold $K \times_H (\mathfrak{h}^\circ \times V)$ near the zero-section. Our approach for the hyperkähler case will be similar, using a holomorphic-symplectic version of the local normal form.

2.4 Kähler quotients

A **Hamiltonian Kähler manifold** is a Hamiltonian manifold (M, K, μ) with a K-invariant Kähler structure compatible with the symplectic form. If the K-action is free, it is a standard result that $M/\!\!/_{\mu} K$ has a Kähler structure compatible with the reduced symplectic form (see, e.g., [77, Theorem 3.1]). More generally, when the action is not necessarily free, each symplectic stratum in Sjamaar–Lerman's stratification is Kähler. To see this, it suffices to note that for each closed subgroup $H \subseteq K$, the space M_H of points with stabiliser H is now a complex submanifold of M and hence is Kähler. Thus, the connected components of $M_H /\!\!/_{\mu_H} L$ (where μ_H and L are as in §2.3) are Kähler manifolds, and the homeomorphism $M_H /\!\!/_{\mu_H} L \cong \mu^{-1}(0)_{(H)}/K$ gives the desired Kähler structures.

But we can say much more about the holomorphic aspect of $M/\!\!/_{\mu} K$ if we assume that the action of K extends to a holomorphic action of the complexification $G := K_{\mathbb{C}}$. In that case, we say that the action is **integrable** and call (M, K, μ) an **integrable Hamiltonian Kähler manifold**. This terminology comes from the fact that the action is integrable if and only if for all $x \in \mathfrak{k} := \mathrm{Lie}(K)$, the vector field $\mathsf{I}x^{\#}$ is complete, where I is the complex structure on M. This holds, for example, if M is compact or is a smooth complex affine variety and the map $K \times M \to M$ is real algebraic.

We will review below how this integrability assumption implies that $M /\!\!/_{\mu} K$ is homeomorphic to a categorical quotient of complex-analytic space $M^{\mu\text{-ss}} /\!\!/ K_{\mathbb{C}}$, where $M^{\mu\text{-ss}}$ is an open subset of M. This quotient is, more precisely, an analytic Hilbert quotient: the complex-analytic analogue of Geometric Invariant Theory (GIT) quotients in algebraic geometry. Good expositions can be found in Heinzner–Huckleberry [68, 69] or Greb [55, §2–3]; we summarise the main points in this section. See also [139, §2.4] [54, §2] [56, §2] [57, §1] [67, §0] [72] [70] [71].

2.4.1 GIT quotients

We now briefly recall the notion of GIT quotients. Let G be a complex reductive group (not necessarily connected) acting algebraically on a complex-algebraic variety $(X, \mathcal{O}_X)$. A **good quotient** of X by G is a complex-algebraic variety $(Y, \mathcal{O}_Y)$ together with a G-invariant surjective affine morphism $X \to Y$ such that $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$. Such a quotient, if it exists, is a categorical quotient in the category of complex-algebraic varieties, i.e. every G-invariant morphism $X \to Z$ factors uniquely through a morphism $Y \to Z$. In particular, $(Y, \mathcal{O}_Y)$ is unique up to isomorphisms. Mumford's work [111] shows that for every line bundle $\mathcal{L}$ on X with a lift $G \to \mathrm{Aut}(\mathcal{L})$ of the

G-action (a **linearisation**), there is a G-invariant Zariski-open subset $X^{\mathcal{L}\text{-ss}}$ of X (the set of **$\mathcal{L}$-semistable points**) on which a good quotient exists. It is called the **GIT quotient** of X by G with respect to $\mathcal{L}$ and is denoted by $X/\!\!/_{\mathcal{L}} G$. More precisely, $X^{\mathcal{L}\text{-ss}}$ is the union of the sets $X_\sigma \coloneqq \{x \in X : \sigma(x) \neq 0\}$ for all G-invariant sections $\sigma \in \bigoplus_{n=1}^{\infty} H^0(X, \mathcal{L}^{\otimes n})$ such that X_σ is affine. For example, if X is affine and $\mathcal{L}$ is the trivial linearisation, then $X^{\mathcal{L}\text{-ss}} = X$ and the GIT quotient is $X/\!\!/G \coloneqq \operatorname{Spec}(\mathbb{C}[X]^G)$ with the map $X \to X/\!\!/G$ induced by the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$.

2.4.2 Analytic Hilbert quotients

There is an analytic version of GIT quotients in the category of complex-analytic spaces, which we now recall (see, e.g., [68, 69]). We refer to Gunning–Rossi [63] for background on complex-analytic spaces.

Definition 2.4.1. Let $(X, \mathcal{O}_X)$ be a complex-analytic space and G a complex reductive group acting holomorphically on X. An **analytic Hilbert quotient** of X by G is a complex-analytic space $(Y, \mathcal{O}_Y)$ together with a G-invariant surjective holomorphic map $\pi : X \to Y$ such that:

(i) the map $\pi : X \to Y$ is **locally Stein**, i.e. Y has a cover by Stein open sets whose preimages are Stein;

(ii) $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$.

Analytic Hilbert quotients are also sometimes called *semistable quotients* [72]. An important consequence of this definition is that, if it exists, an analytic Hilbert quotient is a categorical quotient in the category of complex-analytic spaces. In particular, it is unique up to biholomorphisms. We denote it

$$X/\!\!/G \coloneqq \text{the analytic Hilbert quotient of } X \text{ by } G \text{ (if it exists).}$$

Topologically, $X/\!\!/G$ is the quotient of X by the equivalence relation $x \sim y$ if $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$ and $\pi : X \to X/\!\!/G$ is the corresponding quotient map. The space $X/\!\!/G$ can also be viewed as the set of closed G-orbits, i.e. by defining the set of **polystable** points

$$X^{\text{ps}} \coloneqq \{x \in X : \text{the orbit } G \cdot x \text{ is closed in } X\},$$

the inclusion $X^{\text{ps}} \subseteq X$ descends to a homeomorphism $X^{\text{ps}}/G \to X/\!\!/G$. In particular, for every $p \in X/\!\!/G$, there is a unique closed G-orbit in the fibre $\pi^{-1}(p) \subseteq X$.

Two other properties of analytic Hilbert quotients that we will use later are as follows.

Definition 2.4.2. Let X be a topological space and G a group acting on X. A subset A of X is called ***G*-saturated** if $\overline{G \cdot x} \subseteq A$ for all $x \in A$.

Proposition 2.4.3. *Let $\pi : X \to X/\!\!/G$ be an analytic Hilbert quotient.*

(i) *An open set $U \subseteq X$ is G-saturated if and only if it is saturated with respect to π, i.e. $\pi^{-1}(\pi(U)) = U$. In that case, $U/\!\!/G := \pi(U)$ is open in $X/\!\!/G$, and the restriction $U \to U/\!\!/G$ is an analytic Hilbert quotient.*

(ii) *If $Y \subseteq X$ is a G-invariant closed complex-analytic subspace, then $Y/\!\!/G := \pi(Y)$ is a closed complex-analytic subspace of $X/\!\!/G$, and the restriction $Y \to Y/\!\!/G$ is an analytic Hilbert quotient.* □

For (i) see [72, §2 Remark and §1 Corollary] and for (ii) see [72, §1(ii)].

Recall that every complex-algebraic variety has an underlying complex-analytic space and this gives a functor from the category of complex-algebraic varieties to the category complex-analytic spaces called **analytification** (Serre [130]). The following will be useful:

Proposition 2.4.4. *The analytification of a GIT quotient is an analytic Hilbert quotient.*

Proof. The affine case is proved in [66, §6.4] and the general case follows from the fact that a GIT quotient is constructed by glueing affine quotients. □

2.4.3 The Heinzner–Loose Theorem

Just as for GIT quotients, the question of the existence of an analytic Hilbert quotient is a subtle one. In complete analogy with GIT, for an action of a complex reductive group G on a complex-analytic space X, there does not always exist an analytic Hilbert quotient, but in good cases, one can find a large open subset of X on which the quotient exists. For GIT, this set depends on a choice of a linearisation, and for analytic Hilbert quotients, it depends on a choice of a moment map for the action of a maximal compact subgroup $K \subseteq G$, as we now explain.

Let (M, K, μ) be an integrable Hamiltonian Kähler manifold and let $G := K_{\mathbb{C}}$. Define the set of **$\boldsymbol{\mu}$-semistable** points by

$$M^{\mu\text{-ss}} := \{p \in M : \overline{G \cdot p} \cap \mu^{-1}(0) \neq \emptyset\}$$

and the set of **$\boldsymbol{\mu}$-polystable** points by

$$M^{\mu\text{-ps}} := (M^{\mu\text{-ss}})^{\text{ps}} = \{p \in M : G \cdot p \text{ is closed in } M^{\mu\text{-ss}}\}.$$

Theorem 2.4.5 (Heinzner–Loose [71]). *The set $M^{\mu\text{-ss}}$ is a G-invariant open subset of M, and the analytic Hilbert quotient $M^{\mu\text{-ss}}/\!\!/G$ exists. We have*

$$p \in M^{\mu\text{-ps}} \iff G \cdot p \cap \mu^{-1}(0) \neq \emptyset. \tag{2.4.1}$$

Moreover, the inclusion $\mu^{-1}(0) \hookrightarrow M^{\mu\text{-ss}}$ descends to a homeomorphism $M/\!\!/_\mu K \to M^{\mu\text{-ss}}/\!\!/G$. Also, for every $p \in M^{\mu\text{-ps}}$ we have $G_p = (K_p)_{\mathbb{C}}$, so G_p is a complex reductive group. □

Remark 2.4.6.

(1) Special cases of Theorem 2.4.5 were known long before [71]. See, for example, Guillemin–Sternberg [61, §4] and Kirwan [88, §7.5]. It was also obtained independently by Sjamaar [133] under an additional assumption on the moment map.

(2) This result can be thought of as an "analytic" version of the Kempf–Ness Theorem. The latter will be discussed in detail in Chapter 4.

(3) Heinzner–Loose [71] do not mention analytic Hilbert quotients directly, but the above theorem can be deduced from their proofs. The reformulation which we gave is found in Heinzner–Huckleberry [67, §0]. To translate from [71] and [67, §0] to Theorem 2.4.5, note the following: the statement that the analytic Hilbert quotient $M^{\mu\text{-ss}}/\!\!/G$ exists is [67, §0(i)]; the equivalence (2.4.1) is in [71, (1.2)(a) and (1.3)]; the homeomorphism $M/\!\!/_\mu K \to M^{\mu\text{-ss}}/\!\!/G$ is [67, §0(iv)] or [71, (1.3)]; by [67, §0(iii)] or [71, (2.2) and (2.7)] we have that $G_p = (K_{\mathbb{C}})_p$ for all $p \in \mu^{-1}(0)$ and hence also for all $p \in M^{\mu\text{-ps}}$ since $M^{\mu\text{-ps}} = G \cdot \mu^{-1}(0)$ by (2.4.1).

The main ingredient in the proof of the Heinzner–Loose Theorem is the Holomorphic Slice Theorem. We briefly review it here, since we will use it later. If H is a complex Lie subgroup of a complex Lie group G and S is a complex H-manifold, we denote by $G \times_H S$ the quotient of $G \times S$ by the H-action $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. Since the H-action is free and proper, there is a unique complex manifold structure on $G \times_H S$ such that $G \times S \to G \times_H S$ is a holomorphic submersion.

Definition 2.4.7. Let G be a complex reductive group acting holomorphically on a complex manifold M. A **slice** at a point p in M is a G_p-invariant complex submanifold $S \subseteq M$ containing p such that $G \cdot S$ is open in M and the map

$$G \times_{G_p} S \longrightarrow G \cdot S, \quad [g, x] \longmapsto g \cdot x$$

is a G-equivariant biholomorphism.

Theorem 2.4.8 (Holomorphic Slice Theorem [71, §2.7] [133, Theorem 1.12])**.** *Let (M, K, μ) be an integrable Hamiltonian Kähler manifold. Then, there exists a slice at every point of $M^{\mu\text{-ps}}$.* □

Remark 2.4.9. In [71], this is stated only for points $p \in M$ such that $\mu(p)$ is fixed by the coadjoint action, but since $M^{\mu\text{-ps}} = G \cdot \mu^{-1}(0)$ we deduce the above version.

This enables us to study G-equivariant local properties of the complex manifold M near a closed orbit of $M^{\mu\text{-ss}}$ by the local model $G \times_{G_p} S$. This was used by Heinzner–Loose to prove the existence of the analytic Hilbert quotient.

2.4.4 Stratification of analytic Hilbert quotients

Let $\pi : X \to X/\!\!/G$ be an analytic Hilbert quotient (e.g. a GIT quotient). Then, as in §2.2, the orbit space X^{ps}/G has a natural partition by G-orbit-types, i.e. the pieces are the connected components of the sets $(X^{\text{ps}})_{(H)}/G$ for $H \subseteq G$. Then, the bijection $X^{\text{ps}}/G \to X/\!\!/G$ defines a natural partition on $X/\!\!/G$ which we call the ***G*-orbit-type partition**. Equivalently, the orbit-type of a point $p \in X/\!\!/G$ is defined to be the orbit-type of the unique closed orbit in $\pi^{-1}(p)$.

If (M, K, μ) is an integrable Hamiltonian Kähler manifold, then $M/\!\!/_{\mu}K \cong M^{\mu\text{-ss}}/\!\!/G$ is an analytic Hilbert quotient and hence has a G-orbit-type partition. But it also has the K-orbit-type partition of Sjamaar–Lerman. Moreover, each stratum in the K-orbit-type partition is a Kähler manifold and hence has a complex structure. The next result shows that these partitions and complex structures are the same:

Theorem 2.4.10 (Sjamaar [133, Theorem 2.10])**.**

(i) *The homeomorphism $M/\!\!/_{\mu}K \to M^{\mu\text{-ss}}/\!\!/G$ is an isomorphism of partitioned spaces.*

(ii) *The G-orbit-type strata of $M^{\mu\text{-ss}}/\!\!/G$ are complex submanifolds.*

(iii) *Let S be a K-orbit-type stratum in $M/\!\!/_{\mu}K$ and S' the corresponding G-orbit-type stratum in $M^{\mu\text{-ss}}/\!\!/G$. Then, the restriction $S \to S'$ is a biholomorphism with respect to the Kähler structure on S and the complex structure on S' obtained from* (ii)*.* □

Remark 2.4.11.

(1) Point (iii) is not stated in this way in [133, Theorem 2.10], but is nonetheless part of the proof.

(2) As explained earlier, Sjamaar [133] obtained Heinzner–Loose's Theorem 2.4.5 independently of them, but under an additional assumption on the moment map which he calls *admissibility*. He then stated Theorem 2.4.10 under the same assumption, but his proof relies only on the validity of Theorem 2.4.5 but not on the admissibility of the moment map.

Chapter 3

Singular hyperkähler quotients

3.1 Introduction

3.1.1 Overview

In Chapter 2, we reviewed how Sjamaar–Lerman generalised the Marsden–Weinstein Theorem [104] by showing that if (M, K, μ) is a Hamiltonian manifold whose K-action is not necessarily free, then $M /\!\!/_{\mu} K := \mu^{-1}(0)/K$ is stratified into symplectic manifolds. Moreover, the symplectic structures are compatible with a Poisson bracket on an appropriate substitute for the algebra of smooth functions. Sjamaar–Lerman also showed that linear symplectic reductions are local models for the stratified symplectic structure on $M /\!\!/_{\mu} K$.

In hyperkähler geometry, there is an analogue of symplectic reduction due to Hitchin–Karlhede–Lindström–Roček [77] which has been a very important tool for constructing new examples of these special manifolds. The goal of this chapter is to extend Sjamaar–Lerman's results to this setting.

It is already known [35] that hyperkähler quotients by non-free actions of compact Lie groups are partitioned into smooth hyperkähler manifolds. The main contribution of this chapter is to show that this partition is a stratification, that the hyperkähler structures are compatible with complex-analytic Poisson structures, and to obtain a complex-analytic version of Sjamaar–Lerman's local models. This chapter is based on the author's paper [107].

3.1.2 Statement of results

Let $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ be a **hyperkähler manifold**, i.e. a smooth manifold M with a Riemannian metric g and three complex structures I, J, K that are Kähler with respect to g and satisfy $\mathsf{IJ} = \mathsf{K}$. This implies that for all $(a, b, c) \in \mathbb{R}^3$ on the unit two-sphere, the endomorphism $a\mathsf{I} + b\mathsf{J} + c\mathsf{K}$ is another complex structure which is

Kähler with respect to g. Thus, M has a **two-sphere of complex structures**. Let $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$ be the Kähler forms of $\mathsf{I}, \mathsf{J}, \mathsf{K}$, respectively. If K is a compact Lie group acting on M by preserving the hyperkähler structure, a **hyperkähler moment map** is a map $\mu = (\mu_{\mathsf{I}}, \mu_{\mathsf{J}}, \mu_{\mathsf{K}}) : M \to \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$, where $\mathfrak{k} := \mathrm{Lie}(K)$ and $\mu_{\mathsf{I}}, \mu_{\mathsf{J}}, \mu_{\mathsf{K}}$ are moment maps for $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$, respectively. If such a map μ exists, we say that the K-action is **tri-Hamiltonian** and call the triple (M, K, μ) a **tri-Hamiltonian hyperkähler manifold**. The group K in such a triple will always be assumed to be compact. The **hyperkähler quotient** of M by K with respect to μ is the quotient space

$$M /\!/\!/_{\mu} K := \mu^{-1}(0)/K.$$

This construction was introduced in [77, §3(D)], where it is shown that if K acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ and $M /\!/\!/_{\mu} K$ are smooth manifolds and $M /\!/\!/_{\mu} K$ has a canonical hyperkähler structure descending from M. If the K-action is not necessarily free, then $M /\!/\!/_{\mu} K$ can be partitioned by orbit-types as in the symplectic case. That is, we partition $M /\!/\!/_{\mu} K$ into the connected components of the spaces $\mu^{-1}(0)_{(H)}/K$ for all subgroups $H \subseteq K$. We call this the **orbit-type partition** of $M /\!/\!/_{\mu} K$. By adapting Sjamaar–Lerman's arguments in [134, Theorem 3.5], Dancer–Swann [35, §2] showed that each piece in the orbit-type partition is a hyperkähler manifold. We state this result in the following form (see §3.2 for details).

Theorem 3.1.1 (Dancer–Swann [35])**.** *Let $((M, g, \mathsf{I}, \mathsf{J}, \mathsf{K}), K, \mu)$ be a tri-Hamiltonian hyperkähler manifold, let $\pi : \mu^{-1}(0) \to M /\!/\!/_{\mu} K$ be the quotient map, and let $S \subseteq M /\!/\!/_{\mu} K$ be a piece of the orbit-type partition. Then, S is a topological manifold, $\pi^{-1}(S)$ is a smooth submanifold of M, there is a unique smooth structure on S such that $\pi^{-1}(S) \to S$ is a smooth submersion, and there is a unique hyperkähler structure $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ on S such that the pullbacks of the Kähler forms $\omega_{\mathsf{I}_S}, \omega_{\mathsf{J}_S}, \omega_{\mathsf{K}_S}$ to $\pi^{-1}(S)$ are the restrictions of $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$.*

However, the question of whether the orbit-type partition of $M /\!/\!/_{\mu} K$ is a stratification as in the symplectic case was left open in Dancer–Swann's work. The main issue is that the arguments used by Sjamaar–Lerman [134] are based on the local normal form for the moment map [60, 103], but there is no hyperkähler equivalent. Indeed, the local normal form implies the Darboux Theorem, so we would have a canonical form describing all three symplectic forms simultaneously and hence they could not carry any local information. But the symplectic forms on a hyperkähler manifold determine the Riemannian metric which does carry local information: the curvature.

Nevertheless, we will show that if the K-action is integrable with respect to some element of the two-sphere of complex structures, then we do get a stratification. Without loss of generality, we may assume that the action is integrable with respect to I.

The way around the problem is to use the close relationship between hyperkähler and complex-symplectic geometry. Recall that a **complex-symplectic manifold** is a complex manifold (M, I) together with a non-degenerate holomorphic closed 2-form $\omega_{\mathbb{C}}$. A **complex-Hamiltonian manifold** is a complex-symplectic manifold $(M, \mathsf{I}, \omega_{\mathbb{C}})$ along with a holomorphic action of a complex Lie group G preserving $\omega_{\mathbb{C}}$ and a **complex moment map**, i.e. a G-equivariant holomorphic map $\mu_{\mathbb{C}} : M \to \mathfrak{g}^*$ such that $d\langle \mu_{\mathbb{C}}, x\rangle = x^{\#} \lrcorner\, \omega_{\mathbb{C}}$ for all $x \in \mathfrak{g} := \mathrm{Lie}(G)$.

Now, if $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ is a hyperkähler manifold, then $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$ is a complex-symplectic form on (M, I). Moreover, if K acts on M with hyperkähler moment map μ and this action extends to an I-holomorphic action of $G := K_{\mathbb{C}}$, then $\mu_{\mathbb{C}} := \mu_{\mathsf{J}} + i\mu_{\mathsf{K}} : M \to \mathfrak{g}^*$ is a complex moment map for the action of G on $(M, \mathsf{I}, \omega_{\mathbb{C}})$ (see [77, §3(D)]). We call $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$ the **underlying complex-Hamiltonian manifold** of (M, K, μ).

To prove that $M /\!\!/\!\!/_{\mu} K$ is a stratified space as in the symplectic case, the idea is to use the Heinzner–Loose Theorem to identify $M /\!\!/\!\!/_{\mu} K$ with the analytic Hilbert quotient $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$, where $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} := \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}\text{-ss}}$. Then, $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$ can be viewed as a symplectic reduction in the category of complex-analytic spaces, so it suffices to prove a holomorphic version of the local normal form for the moment map and adapt Sjamaar–Lerman's argument to this setting.

To state this normal form, we first introduce some terminology. We say that a **complex-symplectic representation** is a complex-symplectic vector space (V, Ω) (i.e. Ω is a non-degenerate skew-symmetric complex-bilinear form on V) together with a complex-linear action of a complex reductive group H on V preserving Ω. We denote this by $H \to \mathrm{Sp}(V, \Omega)$. In particular, (V, Ω) can be viewed as a complex-symplectic manifold. The following is well-known:

Proposition 3.1.2. *Let $H \to \mathrm{Sp}(V, \Omega)$ be a complex-symplectic representation. Then,*

$$\Phi_V : V \longrightarrow \mathfrak{h}^*, \quad \Phi_V(v)(x) = \frac{1}{2}\Omega(x \cdot v, v), \quad (v \in V, x \in \mathfrak{h})$$

is a complex moment map for the action of H on V. □

Now, let $p \in \mu^{-1}(0)$ and let $V := (T_p(G\cdot p))^{\omega_{\mathbb{C}}}/T_p(G\cdot p)$, where $(\cdot)^{\omega_{\mathbb{C}}}$ is the complex-symplectic complement with respect to $\omega_{\mathbb{C}}$. Then, V is a complex-symplectic vector space, and the action of the stabiliser $H := G_p$ on V gives a complex-symplectic representation. Roughly speaking, the local normal form says that the complex-Hamiltonian manifold $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$ is completely determined in a neighbourhood of p by this representation. More precisely, let E be the complex-symplectic reduction of $T^*G \times V$ by H, where H acts by right translations on T^*G and linearly on V (see §3.3 for details). Then, E is a complex-Hamiltonian G-manifold. As a complex G-manifold, E can be identified with the associated vector bundle $G \times_H (\mathfrak{h}^\circ \times V)$, where $\mathfrak{h}^\circ \subseteq \mathfrak{g}^*$ is the annihilator of $\mathfrak{h} \coloneqq \mathrm{Lie}(H)$ and G acts by left multiplication on the G-factor. Moreover, there is an explicit expression for the moment map (see (3.3.5)). We will show:

Theorem 3.1.3 (Complex-Hamiltonian Local Normal Form)**.** *Let $((M, g, \mathsf{I}, \mathsf{J}, \mathsf{K}), K, \mu)$ be a tri-Hamiltonian hyperkähler manifold whose K-action is I-integrable. Let $G \coloneqq K_{\mathbb{C}}$, $\mu_{\mathbb{R}} \coloneqq \mu_{\mathsf{I}}$, $\mu_{\mathbb{C}} \coloneqq \mu_{\mathsf{J}} + i\mu_{\mathsf{K}}$, and $\omega_{\mathbb{C}} \coloneqq \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$. Let $p \in \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}}$ and let $E \coloneqq G \times_H (\mathfrak{h}^\circ \times V)$, where $H \coloneqq G_p$, $\mathfrak{h} \coloneqq \mathrm{Lie}(H)$, and $V \coloneqq (T_p(G \cdot p))^{\omega_{\mathbb{C}}}/T_p(G \cdot p)$. Endow E with its canonical complex-symplectic form η and complex moment map κ* (see §3.3.2)*. Then, there is a G-saturated neighbourhood U of $G \cdot p$ in $M^{\mu_{\mathbb{R}}\text{-ss}}$, a G-saturated neighbourhood U' of the zero-section in E, and a G-equivariant biholomorphism $f : U \to U'$ such that $f(p) = [1, 0, 0]$, $f^*\eta = \omega_{\mathbb{C}}$, and $\kappa \circ f = \mu_{\mathbb{C}}$. Moreover, U' can be chosen to be of the form $G \times_H (H \cdot B)$, where B is an open ball around zero in $\mathfrak{h}^\circ \times V$.*

Remark 3.1.4. Losev [101] proved a closely related statement in the algebraic setting.

The structure of the proof is as follows. We first use the Holomorphic Slice Theorem to show that a neighbourhood of p is biholomorphic to a neighbourhood of the zero-section of E. We then use some basic results of complex-symplectic representations to construct a biholomorphism $E \to E$ which will make the complex-symplectic form $\omega_{\mathbb{C}}$ of the hyperkähler structure match the canonical one η on the zero-section of E. Then, we use a holomorphic version of the Darboux–Weinstein Theorem (which we prove in §3.3.3) to deform E further so that $\omega_{\mathbb{C}}$ and η match on a full neighbourhood of the zero-section.

We then use this local normal form to describe the local structure of singular hyperkähler quotients:

Theorem 3.1.5. *Let $((M, g, \mathsf{I}, \mathsf{J}, \mathsf{K}), K, \mu)$ be a tri-Hamiltonian hyperkähler manifold whose K-action is I-integrable. Let $G \coloneqq K_{\mathbb{C}}$, let $\mu_{\mathbb{R}} \coloneqq \mu_{\mathsf{I}}$, and let $\mu_{\mathbb{C}} \coloneqq \mu_{\mathsf{J}} + i\mu_{\mathsf{K}}$.*

(i) **Complex Structure.** *The inclusion $\mu^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ descends to a homeomorphism $M /\!/\!/_{\mu} K \cong \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!/ G$ and hence $M /\!/\!/_{\mu} K$ inherits the* *structure $\mathcal{O}_{\mathsf{I}}$ of a complex-analytic space. For each piece $S \subseteq M /\!\!/\!\!/_{\mu} K$ in the orbit-type partition, we have:*

- *S is a non-singular complex-analytic subspace of $(M /\!\!/\!\!/_{\mu} K, \mathcal{O}_{\mathsf{I}})$.*
- *Let $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ be the hyperkähler structure of S as in* Theorem 3.1.1. *Then, the inclusion $S \hookrightarrow M /\!\!/\!\!/_{\mu} K$ is holomorphic with respect to I_S and $\mathcal{O}_{\mathsf{I}}$.*

(ii) **Stratification Structure.** *The orbit-type partition of $M /\!\!/\!\!/_{\mu} K$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}_{\mathsf{I}}$.*

(iii) **Poisson Structure.** *There is a unique Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ such that for each S in the orbit-type partition, the inclusion $S \hookrightarrow M /\!\!/\!\!/_{\mu} K$ is a Poisson map with respect to the complex-symplectic form $\omega_{\mathsf{J}_S} + i\omega_{\mathsf{K}_S}$ on (S, I_S).*

(iv) **Local Model.** *Let $q \in M /\!\!/\!\!/_{\mu} K$. Take a point $p \in \mu^{-1}(0)$ above q, let $H := G_p$, let $V := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p)$, where $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$, and let $\Phi_V : V \to \mathfrak{h}^*$ be the canonical complex moment map* (Proposition 3.1.2). *Then, H is a complex reductive group and q has a neighbourhood biholomorphic with respect to $\mathcal{O}_{\mathsf{I}}$ to a neighbourhood of* 0 *in the affine GIT quotient $\Phi_V^{-1}(0) /\!\!/ H = \operatorname{Spec} \mathbb{C}[\Phi_V^{-1}(0)]^H$. Moreover, this biholomorphism preserves the natural partitions and holomorphic Poisson brackets on both sides.*

Using Kempf–Ness type theorems (see Chapter 4), there are many situations where $M /\!\!/\!\!/_{\mu} K$ is isomorphic to a GIT quotient $\mu_{\mathbb{C}}^{-1}(0) /\!\!/_{\mathcal{L}} G$ for some linearisation $\mathcal{L}$, i.e. when $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ coincides with the set of $\mathcal{L}$-semistable points. In that case, the sheaf $\mathcal{O}_{\mathsf{I}}$ is simply the underlying complex-analytic structure. In Chapter 5, we will show that this is the case for hyperkähler quotients of T^*G.

The complex-symplectic affine GIT quotients $\Phi_V^{-1}(0) /\!\!/ H$ which appear in (iv) have been well studied in the literature. For example, it has recently been shown [24] that if H is a complex torus, then $\Phi_V^{-1}(0) /\!\!/ H$ is normal. In particular, we deduce the following result (see §3.4.10 for details).

Corollary 3.1.6. *Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold whose K-action is I-integrable. If K is abelian, then $(M /\!\!/\!\!/_{\mu} K, \mathcal{O}_{\mathsf{I}})$ is normal.*

3.1.3 Stratified hyperkähler spaces

Let $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ be a hyperkähler manifold. Then, for every positively oriented orthonormal frame $\mathbf{f} = (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in (S^2)^3$, we get another hyperkähler structure on M which we denote by $(g, \mathsf{I}^{\mathbf{f}}, \mathsf{J}^{\mathbf{f}}, \mathsf{K}^{\mathbf{f}})$. More precisely, if $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\mathsf{I}^{\mathbf{f}} := u_1\mathsf{I} + u_2\mathsf{J} + u_3\mathsf{K}, \quad \mathsf{J}^{\mathbf{f}} := v_1\mathsf{I} + v_2\mathsf{J} + v_3\mathsf{K}, \quad \mathsf{K}^{\mathbf{f}} := w_1\mathsf{I} + w_2\mathsf{J} + w_3\mathsf{K}.$$

The following notion will be useful later:

Definition 3.1.7. A **hyperkähler rotation** is a smooth map $\psi : M \to M$ which is an isomorphism of hyperkähler manifolds with respect to $(g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ on the left and $(g, \mathsf{I}^{\mathbf{f}}, \mathsf{J}^{\mathbf{f}}, \mathsf{K}^{\mathbf{f}})$ on the right for some positively oriented orthonormal frame $\mathbf{f}$, i.e. ψ is an isometry such that

$$d\psi \circ \mathsf{I} = \mathsf{I}^{\mathbf{f}} \circ d\psi, \quad d\psi \circ \mathsf{J} = \mathsf{J}^{\mathbf{f}} \circ d\psi, \quad d\psi \circ \mathsf{K} = \mathsf{K}^{\mathbf{f}} \circ d\psi.$$

An **SO(3)-hyperkähler rotation** is an action of SO(3) on M such that for all $a \in \mathrm{SO}(3)$, the action map $p \mapsto a \cdot p$ is a hyperkähler rotation with respect to the frame consisting of the columns of a.

If (M, K, μ) is a tri-Hamiltonian hyperkähler manifold, it is often the case that the action is integrable with respect to all elements $a\mathsf{I} + b\mathsf{J} + c\mathsf{K}$ of the two-sphere of complex structures. For example, this happens when there is an SO(3)-hyperkähler rotation commuting with the K-action; we will see an important example in Chapter 5. We introduce a terminology for this:

Definition 3.1.8. A tri-Hamiltonian hyperkähler manifold is called **fully integrable** if the action is integrable with respect to all elements of the two-sphere of complex structures.

By applying our results to $(M, g, \mathsf{I}^{\mathbf{f}}, \mathsf{J}^{\mathbf{f}}, \mathsf{K}^{\mathbf{f}})$ for all frames $\mathbf{f}$, we naturally obtain the following definition.

Definition 3.1.9. A **stratified hyperkähler space** is a stratified space $(X, \mathcal{P})$ with the following data:

(a) A hyperkähler structure $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ on each stratum $S \in \mathcal{P}$.

(b) For each $\mathbf{u} \in S^2$, a complex-analytic structure $\mathcal{O}^{\mathbf{u}}$ on X.

(c) For each positively oriented orthonormal frame $\mathbf{f} = (\mathbf{u}, \mathbf{v}, \mathbf{w})$, a Poisson bracket $\{\cdot, \cdot\}^{\mathbf{f}}$ on $\mathcal{O}^{\mathbf{u}}$.

And with the following properties:

(i) For each $\mathbf{u} \in S^2$, the partition $\mathcal{P}$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}^{\mathbf{u}}$.

(ii) For each positively oriented orthonormal frame $\mathbf{f}$ and each $S \in \mathcal{P}$, the inclusion $S \hookrightarrow X$ is a holomorphic Poisson map with respect to the complex-symplectic structure $(\mathsf{I}^{\mathbf{f}}_S, \omega_{\mathsf{J}^{\mathbf{f}}_S} + i\omega_{\mathsf{K}^{\mathbf{f}}_S})$ on S and the Poisson structure $(\mathcal{O}^{\mathbf{u}}, \{\cdot,\cdot\}^{\mathbf{f}})$ on X.

Note that since a hyperkähler structure is completely determined by the three Kähler forms, we can recover the hyperkähler structures on all strata by only two of the holomorphic Poisson sheaves.

Theorem 3.1.5 has the following evident corollary:

Corollary 3.1.10. *Let (M, K, μ) be a fully integrable tri-Hamiltonian hyperkähler manifold. Then, $M /\!\!/\!\!/_{\mu} K$ is a stratified hyperkähler space.*

There is an obvious category of stratified hyperkähler spaces, which we define here since we will use its notion of isomorphisms in Chapter 6.

Definition 3.1.11. A **morphism** between stratified hyperkähler spaces X and Y is a morphism $f : X \to Y$ of partitioned spaces such that for any positively oriented orthonormal frame $\mathbf{f} = (\mathbf{u}, \mathbf{v}, \mathbf{w})$, the map f is a holomorphic Poisson map with respect to $(\mathcal{O}^{\mathbf{u}}_X, \{\cdot,\cdot\}^{\mathbf{f}}_X)$ and $(\mathcal{O}^{\mathbf{u}}_Y, \{\cdot,\cdot\}^{\mathbf{f}}_Y)$. This defines a category under composition.

In particular, a morphism of stratified hyperkähler spaces restricts to hyperkähler maps on the strata, i.e. tri-holomorphic isometries. Hence, an isomorphism in this category is, in particular, an isomorphism of partitioned spaces which restricts to isomorphisms of hyperkähler manifolds on the strata.

Remark 3.1.12. It is natural to expect that the one-to-one correspondence between smooth hyperkähler manifolds and twistor spaces (see [77, §3(F)]) extends to a one-to-one correspondence between stratified hyperkähler spaces and some notion of singular twistor spaces. Using the above definition of a stratified hyperkähler space X, one can construct a sheaf $\mathcal{O}_Z$ on $Z := X \times \mathbb{CP}^1$ together with a Lie bracket of degree -2 on the graded sheaf $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_Z(n)$, where $\mathcal{O}_Z(n)$ is the sheaf functions on Z with values in the line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$. One can hope that $\mathcal{O}_Z$ is a complex-analytic structure and that the stratified hyperkähler structure on X can be recovered from this Lie bracket.

3.2 Hyperkähler structures on the orbit-type pieces

Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold. In this section, we recall, following Dancer–Swann [35], the construction of a hyperkähler structure on each piece of the orbit-type partition of $M /\!/\!/_\mu K$, arriving at Theorem 3.1.1 above. The latter is slightly stronger than the original Dancer–Swann Theorem [35, Theorem 2.1] since it characterises these structures uniquely. The purpose of this section is thus to give a complete proof of this stronger version.

The proof is very similar to the construction of the Kähler structures on the orbit-type strata of a Kähler quotient explained in §2.4. For this section, it will be convenient to slightly relax the definition of a manifold so that different connected components may have different dimensions. Also, a **smooth submanifold** will always mean an embedded one.

Let $S \subseteq M /\!/\!/_\mu K$ be an orbit-type piece. Then, S is a connected component of a set of the form $\mu^{-1}(0)_{(H)}/K$ for some closed subgroup $H \subseteq K$. The set M_H of points with stabiliser H is now a hyperkähler submanifold of M and μ restricts to a hyperkähler moment map $\mu_H : M_H' \to \mathfrak{l}^* \otimes \mathbb{R}^3 \subseteq \mathfrak{k}^* \otimes \mathbb{R}^3$ for the free action of $L := N_K(H)/H$ on the union M_H' of the connected components of M_H intersecting $\mu^{-1}(0)$. Hence, the connected components of $M_H /\!/\!/_{\mu_H} L = \mu_H^{-1}(0)/L$ are smooth hyperkähler manifolds by the usual hyperkähler quotient construction [77, Theorem 3.2]. Moreover, the inclusion $\mu_H^{-1}(0) \subseteq \mu^{-1}(0)_{(H)}$ descends to a homeomorphism $M_H /\!/\!/_{\mu_H} L \to \mu^{-1}(0)_{(H)}/K$ and hence endows each connected component of $\mu^{-1}(0)_{(H)}/K$ with a hyperkähler structure. To show that this map is indeed a homeomorphism and also to characterise the hyperkähler structure as in Theorem 3.1.1, we will need the following lemma. This result is implicit in Sjamaar–Lerman [134] and Dancer–Swann [35], but we give a short proof for completeness.

Lemma 3.2.1. *Let K be a compact Lie group acting smoothly on a smooth manifold M, let H be a closed subgroup of K, and let $L = N_K(H)/H$. Then, M_H and $M_{(H)}$ are smooth submanifolds of M, and the quotients M_H/L and $M_{(H)}/K$ are topological manifolds with unique smooth structures such that the quotients maps $M_H \to M_H/L$ and $M_{(H)} \to M_{(H)}/K$ are smooth submersions. Moreover, the inclusion $M_H \hookrightarrow M_{(H)}$ descends to a diffeomorphism $M_H/L \to M_{(H)}/K$.*

Proof. This follows easily from the slice theorem for proper group actions. The map $M_H/L \to M_{(H)}/K$ is bijective, so everything reduces to local statements, and hence we may assume (by the slice theorem) that $M = K \times_H W$ for some representation W of H. Then, $M_H = L \times W_H$, $M_{(H)} = K/H \times W_H$, and W_H is a linear subspace of W (the set of fixed points of H), so M_H and $M_{(H)}$ are smooth submanifolds of

M. Moreover, $M_H/L = W_H$ and the quotient map $M_H \to M_H/L$ is the projection $L \times W_H \to W_H$ and hence is a smooth submersion. Similarly, the quotient map $M_{(H)} \to M_{(H)}/K$ is the projection $K/H \times W_H \to W_H$. Under these identifications, the map $M_H/L \to M_{(H)}/K$ is the identity map $W_H \to W_H$. $\square$

Let

$$\pi : \mu^{-1}(0) \longrightarrow M /\!\!/\!\!/_\mu K$$

be the quotient map and let $S \subseteq M /\!\!/\!\!/_\mu K$ be an orbit-type piece as above. We can now prove Theorem 3.1.1:

Proposition 3.2.2. *The space S is a topological manifold, $\pi^{-1}(S)$ is a smooth submanifold of M (of pure dimension), there is a unique smooth structure on S such that $\pi^{-1}(S) \to S$ is a smooth submersion, and there is a unique hyperkähler structure $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ on S such that the pullbacks of the Kähler forms $\omega_{\mathsf{I}_S}, \omega_{\mathsf{J}_S}, \omega_{\mathsf{K}_S}$ to $\pi^{-1}(S)$ are the restrictions of $\omega_\mathsf{I}, \omega_\mathsf{J}, \omega_\mathsf{K}$.*

Proof. Let $Z := \mu^{-1}(0)$ so that S is a connected component of $Z_{(H)}/K$ for some $H \subseteq K$. As explained above, Z_H is a smooth submanifold of M_H and Z_H/L is a hyperkähler manifold, where $L := N_K(H)/K$. Now, Z_H/L is a smooth submanifold of M_H/L and its image under the diffeomorphism $M_H/L \to M_{(H)}/K$ is $Z_{(H)}/K$, so the latter is also a smooth submanifold. Recall that if $f : X \to Y$ is a smooth submersion between smooth manifolds and $Y' \subseteq Y$ is a smooth submanifold, then $f^{-1}(Y')$ is a smooth submanifold, and the restriction $f^{-1}(Y') \to Y'$ is a smooth submersion (this follows from the rank theorem). Thus, $Z_{(H)}$ is a smooth submanifold of $M_{(H)}$, and $Z_{(H)} \to Z_{(H)}/K$ is a smooth submersion. Note that $\pi^{-1}(S)$ is open in $Z_{(H)}$, so it is also a smooth submanifold and the restriction $\pi^{-1}(S) \to S$ is a smooth submersion. Moreover, $\pi^{-1}(S)$ has pure dimension since S is connected and all fibres are diffeomorphic to K/H.

To prove the claim about the hyperkähler structure, let $\eta_\mathsf{I}, \eta_\mathsf{J}, \eta_\mathsf{K}$ be the Kähler forms on $Z_{(H)}/K$ induced by the diffeomorphism $Z_H/L \to Z_{(H)}/K$ and consider the commutative diagram

$$\begin{array}{ccccc} Z_H & \xhookrightarrow{\;i\;} & Z_{(H)} & \xhookrightarrow{\;j\;} & M \\ \big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle\pi} & & \\ Z_H/L & \xrightarrow{\;\varphi\;} & Z_{(H)}/K. & & \end{array}$$

We want to show that $\pi^*\eta_\mathsf{I} = j^*\omega_\mathsf{I}$ and similarly for J and K. By the construction of the hyperkähler structure on Z_H/L we have $\rho^*\varphi^*\eta_\mathsf{I} = (ji)^*\omega_\mathsf{I}$ and hence $i^*(\pi^*\eta_\mathsf{I}) = i^*(j^*\omega_\mathsf{I})$. Hence, $\pi^*\eta_\mathsf{I}$ and $j^*\omega_\mathsf{I}$ agree on T_pZ_H for all $p \in Z_H$. Note that since $d\varphi_p$ and $d\rho_p$

are surjective we have $T_pZ_{(H)} = T_pZ_H + \ker d\pi_p$. Thus, to prove that $\pi^*\eta_{\mathsf{I}}$ and $j^*\omega_{\mathsf{I}}$ agree on $T_pZ_{(H)}$ it suffices to show that if $u \in \ker d\pi_p$ and $v \in T_pZ_{(H)}$ then $\pi^*\eta_{\mathsf{I}}(u,v) = j^*\omega_{\mathsf{I}}(u,v)$. Clearly, $\pi^*\eta_{\mathsf{I}}(u,v) = 0$ since $d\pi_p(u) = 0$. To show that also $j^*\omega_{\mathsf{I}}(u,v) = 0$, note that $\ker d\pi_p = T_p(K \cdot p)$ so $u = x_p^\#$ for some $x \in \mathfrak{k}$ and hence $\omega_{\mathsf{I}}(u,v) = x^\# \lrcorner \omega_{\mathsf{I}}(v) = d\langle \mu_{\mathsf{I}}, x\rangle(v) = 0$ since $v \in T_pZ_{(H)} \subseteq \ker(d\mu_{\mathsf{I}})_p$. Hence, $\pi^*\eta_{\mathsf{I}}$ and $j^*\omega_{\mathsf{I}}$ agree on $T_pZ_{(H)}$ for all $p \in Z_H$ and since they are K-invariant and $K \cdot Z_H = Z_{(H)}$ we conclude that $\pi^*\eta_{\mathsf{I}} = j^*\omega_{\mathsf{I}}$. The same argument also shows that $\pi^*\eta_{\mathsf{J}} = j^*\omega_{\mathsf{J}}$ and $\pi^*\eta_{\mathsf{K}} = j^*\omega_{\mathsf{K}}$. Since a hyperkähler structure is completely determined by its three symplectic forms (e.g. $\mathsf{I} = \omega_{\mathsf{J}}^{-1}\omega_{\mathsf{K}}$; see [80, bottom of p. 63]), this proves the proposition. □

3.3 The complex-Hamiltonian local normal form

3.3.1 Overview

The goal of this section is to prove the Complex-Hamiltonian Local Normal Form (Theorem 3.1.3), which establishes a local normal form for the underlying complex-Hamiltonian manifold of a tri-Hamiltonian hyperkähler manifold analogous to the local normal form of Guillemin–Sternberg [60] outlined in §2.3. It will be used in §3.4 to show that singular hyperkähler quotients are stratified spaces, in a proof similar to Sjamaar–Lerman's one for symplectic reductions.

Throughout this section, $((M, g, \mathsf{I}, \mathsf{J}, \mathsf{K}), K, \mu)$ is a tri-Hamiltonian hyperkähler manifold whose K-action is I-integrable, and $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$ is its underlying complex-Hamiltonian manifold, where $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$, $G := K_{\mathbb{C}}$, and $\mu_{\mathbb{C}} := \mu_{\mathsf{J}} + i\mu_{\mathsf{K}}$. Let $\mu_{\mathbb{R}} := \mu_{\mathsf{I}}$ so that $(M, K, \mu_{\mathbb{R}})$ is an integrable Hamiltonian Kähler manifold as in §2.4. In particular, we have the sets $M^{\mu_{\mathbb{R}}\text{-ss}}$ and $M^{\mu_{\mathbb{R}}\text{-ps}}$ as in §2.4.3, and we will use the notations

$$\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} := \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}\text{-ss}} \quad \text{and} \quad \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}} := \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}\text{-ps}}.$$

In analogy with the local normal form in symplectic geometry, the idea is to show that in a neighbourhood of a point $p \in \mu^{-1}(0)$, the underlying complex-Hamiltonian manifold $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$ is completely determined by the representation of $H := G_p$ on the **complex-symplectic slice**

$$V := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p).$$

By the Holomorphic Slice Theorem, the orbit $G \cdot p$ is embedded in M, so the tangent space $T_p(G \cdot p)$ is well-defined. Just as in the real case, the definition of a moment map implies that $T_p(G \cdot p) \subseteq \ker(d\mu_{\mathbb{C}})_p = (T_p(G \cdot p))^{\omega_{\mathbb{C}}}$ and hence V

is a well-defined complex-symplectic vector space. We have $H = G_p = (K_p)_{\mathbb{C}}$ (by Theorem 2.4.5), so H is a complex reductive group acting linearly on V and preserving its complex-symplectic form. In other words, p determines a complex-symplectic representation $\rho : H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$. The goal of the Complex-Hamiltonian Local Normal Form (Theorem 3.1.3) is to construct a complex-Hamiltonian manifold E from G and ρ which is isomorphic to a neighbourhood of p in $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$. The construction of E is the same as the one used by Guillemin–Sternberg [60], but in a complex-symplectic setting; see also [134, §2].

This section is organised as follows. In §3.3.2 we give a more precise construction of the local model E. In §3.3.3, we prove a holomorphic version of the Darboux–Weinstein Theorem which we will need for the proof of the Complex-Hamiltonian Local Normal Form. In §3.3.4, we reformulate the Holomorphic Slice Theorem in a way that is more suitable for our purpose. Finally, we prove the Complex-Hamiltonian Local Normal Form in §3.3.5.

3.3.2 The local model

We now build the local model E. Let G and $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ be as above, so that G is a complex reductive group, H a reductive subgroup of G, and $(V, \omega_{\mathbb{C}})$ a complex-symplectic representation of H. Since G is a complex manifold, the cotangent bundle T^*G has a canonical complex-symplectic form $-d\theta$, where θ is the tautological holomorphic 1-form. We identify T^*G with $G \times \mathfrak{g}^*$ via left translation, i.e. via the biholomorphism

$$G \times \mathfrak{g}^* \longrightarrow T^*G, \quad (g, \xi) \longmapsto (dL_{g^{-1}})^*(\xi), \tag{3.3.1}$$

where $L_{g^{-1}} : G \to G$ is left multiplication by g^{-1}. Recall that a Lie group action on any manifold lifts to a Hamiltonian action on the cotangent bundle. By considering the action of $G \times G$ on G by left and right multiplications (i.e. $(a, b) \cdot g \coloneqq agb^{-1}$) its lift to $T^*G = G \times \mathfrak{g}^*$ is

$$(a, b) \cdot (g, \xi) = (agb^{-1}, \mathrm{Ad}_b^* \xi),$$

and the moment map is

$$T^*G \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*, \quad (g, \xi) \longmapsto (\mathrm{Ad}_g^* \xi, -\xi) \tag{3.3.2}$$

(see, e.g., [1, §4.4]). The representation $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ can also be viewed as a complex-Hamiltonian H-manifold with complex moment map Φ_V as in Proposition 3.1.2. Thus, there is a Hamiltonian action of H on $T^*G \times V$, where H acts on T^*G as the subgroup $1 \times H \subseteq G \times G$ and on V via the given representation. Let E be the complex-symplectic reduction of $T^*G \times V$ by H. Since the action of H on $T^*G \times V$ is free and proper, E is a complex-symplectic manifold. Moreover, the Hamiltonian action of the left factor of $G \times G$ on T^*G descends to a Hamiltonian action of G on E, making E into a complex-Hamiltonian G-manifold.

We can also rewrite E in a more convenient form where the complex moment map for the G-action is explicit. First, note that the complex moment map for the H-action on $T^*G \times V$ is

$$\lambda : T^*G \times V \longrightarrow \mathfrak{h}^*, \quad \lambda(g, \xi, v) = \Phi_V(v) - \xi|_{\mathfrak{h}}.$$

Take a Hermitian inner-product on $\mathfrak{g}$ invariant under the maximal compact subgroup $K \subseteq G$ and let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. This defines an H-equivariant isomorphism $\mathfrak{h}^* \cong \mathfrak{m}^\circ \subseteq \mathfrak{g}^*$ so we can view Φ_V as taking values in $\mathfrak{g}^*$. Then, the map

$$G \times \mathfrak{h}^\circ \times V \longrightarrow \lambda^{-1}(0), \quad (g, \xi, v) \longmapsto (g, \xi + \Phi_V(v), v)$$

is a biholomorphism. The H-action on $\lambda^{-1}(0)$ corresponds to the H-action on $G \times \mathfrak{h}^\circ \times V$ given by $h \cdot (g, \xi, v) = (gh^{-1}, \mathrm{Ad}_h^* \xi, h \cdot v)$, so E is the holomorphic vector bundle

$$E = G \times_H (\mathfrak{h}^\circ \times V) \tag{3.3.3}$$

over G/H. In this setup, the Hamiltonian G-action is

$$G \times E \longrightarrow E, \quad a \cdot [g, \xi, v] = [ag, \xi, v] \tag{3.3.4}$$

and the complex moment map is

$$\kappa : G \times_H (\mathfrak{h}^\circ \times V) \longrightarrow \mathfrak{g}^*, \quad [g, \xi, v] \longmapsto \mathrm{Ad}_g^*(\xi + \Phi_V(v)). \tag{3.3.5}$$

We summarise this discussion in the following proposition.

Proposition 3.3.1. *Let G be a complex reductive group, H a reductive subgroup of G, and V a complex-symplectic representation of H. Then, the complex-symplectic manifold* (3.3.3) *with the action* (3.3.4) *and moment map* (3.3.5) *is a complex-Hamiltonian manifold.* □

Remark 3.3.2. Dancer–Swann [34] showed that E is a tri-Hamiltonian hyperkähler manifold whose underlying complex-Hamiltonian manifold is the one described above.

3.3.3 Holomorphic Darboux–Weinstein Theorem

The Darboux–Weinstein Theorem [143] is a standard result in symplectic geometry which says that if two symplectic forms ω_0 and ω_1 on a manifold M agree on a submanifold $N \subseteq M$, then we can find a diffeomorphism f on a neighbourhood of N such that $f^*\omega_1 = \omega_0$. There is also an equivariant version of the theorem, where if ω_0, ω_1 and N are invariant under the action of a compact Lie group, then f can be taken to be equivariant. By the tubular neighbourhood theorem, it suffices to prove the result when M is a vector bundle and N the zero-section, and this is indeed how Weinstein's original proof goes [143]. In the holomorphic setting, there is no tubular neighbourhood theorem, but we can still adapt Weinstein's proof to formulate a similar statement on holomorphic vector bundles:

Theorem 3.3.3. *Let G be a group acting on a holomorphic vector bundle E by bundle automorphisms (not necessarily fixing the base). Let ω_0 and ω_1 be two G-invariant complex-symplectic forms on a G-invariant neighbourhood U of the zero-section $Z \subseteq E$ such that $\omega_0|_Z = \omega_1|_Z$. Then, there are G-invariant neighbourhoods U_0 and U_1 of Z in U and a G-equivariant biholomorphism $f : U_0 \to U_1$ such that $f^*\omega_1 = \omega_0$ and $f|_Z = \mathrm{Id}_Z$.*

Remark 3.3.4. Here $\omega_i|_Z$ is the restriction of ω_i to $(\Lambda^2 T^*E)|_Z$ (this is not the same as the pullback to Z).

The rest of this subsection is devoted to the proof of this theorem, which is an adaptation of Weinstein's proof [143] to the holomorphic setting. Let us first briefly sketch how we will proceed. The first step is to get a "Poincaré lemma" for the retraction of U onto Z, i.e. to construct an explicit homotopy operator $I : \Omega^k(U) \to \Omega^{k-1}(U)$ between the identity map and π^*, where $\pi : U \to U, v \mapsto 0 \cdot v$. Then, $\alpha = I(\omega_0 - \omega_1)$ is a 1-form on U and, for t small enough, $\omega_t := \omega_0 + t(\omega_1 - \omega_0)$ is non-degenerate, so we get a time-dependent holomorphic vector field $X_t = \omega_t^{-1}(\alpha)$ on a neighbourhood of Z. The proof concludes by showing that the time-dependent flow of X gives a biholomorphism with the desired properties.

Let us now construct the homotopy operator. Let $\overline{\mathbb{D}}$ be the closed unit disc centred at 0 in $\mathbb{C}$ and let $U \subseteq E$ be as in Theorem 3.3.3. By shrinking U if necessary, we may assume that it is preserved by $\overline{\mathbb{D}}$, i.e. $zu \in U$ for all $z \in \overline{\mathbb{D}}$ and $u \in U$. Let $\Omega^k(U)$ be the space of holomorphic k-forms on U and let

$$W := \{(z, u) \in \mathbb{C} \times U : zu \in U\}.$$

Then, W is open in $\mathbb{C} \times U$ and we have $\overline{\mathbb{D}} \times U \subseteq W$. Let ξ be the holomorphic vector field on W given by $\xi_{(z,p)} = (\frac{\partial}{\partial z}\big|_z, 0)$ under the identification $T_{(z,p)}W = T_z\mathbb{C} \times T_pU$. Let

$$\lambda : W \longrightarrow U, \quad (z, u) \longmapsto zu$$

be the scaling map, and for each $z \in \overline{\mathbb{D}}$ let

$$i_z : U \longrightarrow W, \quad p \longmapsto (z, p).$$

Then, for all $\omega \in \Omega^k(U)$, we have a holomorphic family of k-forms

$$W \longrightarrow \Lambda^k T^*U, \quad (z, u) \longmapsto (i_z^*(\xi \lrcorner \lambda^*\omega))_u.$$

Hence,

$$I\omega := \int_0^1 i_z^*(\xi \lrcorner \lambda^*\omega)\, dz$$

is a holomorphic $(k-1)$-form on U. Let $\pi : U \to U$ be the projection onto the zero-section, i.e. $u \mapsto 0 \cdot u$.

Proposition 3.3.5. *The map*

$$I : \Omega^k(U) \longrightarrow \Omega^{k-1}(U)$$

is a homotopy operator between the identity map and π^*, *i.e.*

$$d(I\omega) + I(d\omega) = \omega - \pi^*\omega$$

for all $\omega \in \Omega^k(U)$.

Proof. We have

$$d(I\omega) + I(d\omega) = \int_0^1 i_z^*(d(\xi \lrcorner \lambda^*\omega) + \xi \lrcorner d(\lambda^*\omega)) = \int_0^1 i_z^*(\mathcal{L}_\xi \lambda^*\omega) dz.$$

Moreover, the flow θ of the vector field ξ is $\theta_t(z, u) = (z + t, u) = i_{z+t}(u)$. In particular, $\theta_t \circ i_0 = i_t$ for all $t \in [0, 1]$. Thus, $i_t^*(\mathcal{L}_\xi \lambda^*\omega) = i_0^*\theta_t^*(\mathcal{L}_\xi \lambda^*\omega)$ and since $\theta_t^*(\mathcal{L}_\xi \lambda^*\omega) = \frac{d}{dt}\theta_t^*\lambda^*\omega$ we get

$$\begin{aligned} d(I\omega) + I(d\omega) &= \int_0^1 i_0^* \frac{d}{dt}\theta_t^*\lambda^*\omega\, dt = \int_0^1 \frac{d}{dt} i_0^*\theta_t^*\lambda^*\omega\, dt \\ &= i_0^*\theta_1^*\lambda^*\omega - i_0^*\theta_0^*\lambda^*\omega = \omega - \pi^*\omega. \end{aligned}$$

□

We will also need the following easy consequence of the definition of I.

Lemma 3.3.6. *Let* $\omega \in \Omega^k(U)$ *and let* $p \in Z$. *If* $\omega_p = 0$ *then* $(I\omega)_p = 0$.

Proof. For all $v \in T_pU$, we have

$$[i_t^*(\xi \lrcorner \lambda^*\omega)]_p(v) = [\xi \lrcorner \lambda^*\omega]_{(t,p)}(di_t(v)) = (\lambda^*\omega)_{(t,p)}(\xi_{(t,p)}, di_t(v))$$
$$= \omega_{tp}(d\lambda(\xi_{(t,p)}), d\lambda(di_t(v))) = 0$$

since $tp = p$ as p is in the zero-section. Thus, $[i_t^*(\xi \lrcorner \lambda^*\omega)]_p = 0$ for all $t \in [0,1]$ and hence $(I\omega)_p = 0$. ☐

Proof of Theorem 3.3.3. Let $\eta = \omega_1 - \omega_0$ and let $\alpha = -I\eta$, where I is the homotopy operator of Proposition 3.3.5. Then, $\eta = -d\alpha$. Since η is G-invariant, it follows easily from the definition of I that α is also G-invariant. Moreover, since $\eta|_Z = 0$ we have $\alpha|_Z = 0$ by Lemma 3.3.6.

For each $z \in \mathbb{C}$, define a G-invariant holomorphic 2-form on U by $\omega_z = \omega_0 + z\eta$. We have $\omega_z|_Z = \omega_0|_Z$, so in particular, $\omega_z|_p$ is non-degenerate for all $(z,p) \in \mathbb{C} \times Z$. Let $\mathbb{D}_r$ be the open disc of radius r centred at 0 in $\mathbb{C}$. By compactness of $\overline{\mathbb{D}}_2$, we can find a neighbourhood $U' \subseteq U$ of Z such that $\omega_z|_p$ is non-degenerate for all $(z,p) \in \overline{\mathbb{D}}_2 \times U'$. Moreover, by G-invariance of ω_z, we can take U' to be G-invariant. Thus, we may assume that $\omega_z|_p$ is non-degenerate for all $(z,p) \in \overline{\mathbb{D}}_2 \times U$. In particular, the maps

$$\hat{\omega}_z : TU \longrightarrow T^*U, \quad v \longmapsto \omega_z(v, \cdot)$$

are vector bundle isomorphisms for all $z \in \overline{\mathbb{D}}_2$. Define a holomorphic family of vectors fields on U by

$$X : \mathbb{D}_2 \times U \longrightarrow TU, \quad (z,p) \longmapsto (\hat{\omega}_z)^{-1}(\alpha_p).$$

Let $J = \mathbb{D}_2 \cap \mathbb{R} = (-2,2)$ and let $\psi : \mathcal{E} \to U$ be the smooth time-dependent flow of the restriction $X|_{J\times U}$. That is, $\mathcal{E}$ is the open subset of $J \times J \times M$ such that for all $(t_0, p) \in J \times M$, the map $\psi^{(t_0,p)}(t) := \psi(t, t_0, p)$ is the maximally extended integral curve of $X|_{J\times U}$ starting at (t_0,p). From the general theory of smooth time-dependent flows (e.g. [99, Theorem 9.48]), for all $(t_1, t_0) \in J \times J$ the set

$$U_{(t_1,t_0)} := \{p \in U : (t_1, t_0, p) \in \mathcal{E}\}$$

is open, and the map

$$\psi_{(t_1,t_0)} : U_{(t_1,t_0)} \longrightarrow U_{(t_0,t_1)}, \quad p \longmapsto \psi(t_1, t_0, p)$$

is a diffeomorphism. Moreover, since X is holomorphic, $\psi_{(t_1,t_0)}$ is a biholomorphism (this follows from the holomorphic dependence of solutions to linear systems of ODEs on the initial conditions; see, e.g., [30, Ch. 1, §8]). Since $\alpha|_Z = 0$ we have $X_{(t_0,p)} = 0$ for all $(t_0,p) \in J \times Z$, and hence $\psi(t_1, t_0, p) = p$ for all $(t_1, t_0, p) \in J \times J \times Z$.

In particular, $J \times J \times Z \subseteq \mathcal{E}$, so $U_{(1,0)}$ and $U_{(0,1)}$ contain Z. We claim that the biholomorphism $\psi_{1,0} : U_{1,0} \to U_{0,1}$ is the one we need. First, since α and ω_z are G-invariant, so is X. Hence, $U_{1,0}$ and $U_{0,1}$ are G-invariant, and $\psi_{1,0}$ is G-equivariant. Moreover, from [99, Proposition 22.15] we have for all $t_1 \in J$,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_1} \psi_{t,0}^* \omega_t &= \psi_{t_1,0}^* \left(\mathcal{L}_{X_{t_1}} \omega_{t_1} + \frac{d}{dt}\Big|_{t=t_1} \omega_t \right) = \psi_{t_1,0}^* \left(X_{t_1} \lrcorner\, d\omega_t + d(X_{t_1} \lrcorner\, \omega_{t_1}) + \eta \right) \\ &= \psi_{t_1,0}^*(d\alpha + \eta) = 0. \end{aligned}$$

Thus, $\psi_{1,0}^* \omega_1 = \psi_{0,0}^* \omega_0 = \omega_0$. □

3.3.4 Linearisation of the Holomorphic Slice Theorem

In this subsection, we explain how to put the Holomorphic Slice Theorem (Theorem 2.4.8) in a form which will be more convenient for our purpose. First, we want to linearise the slice and realise neighbourhoods of orbits in M as neighbourhoods of zero-sections of vector bundles.

Proposition 3.3.7. *Let (M, K, μ) be an integrable Hamiltonian Kähler manifold, let $p \in M^{\mu\text{-ps}}$, let $G := K_{\mathbb{C}}$, let $H := G_p$, and let $W := T_pM/T_p(G \cdot p)$. Then, there is an open ball B centred at 0 in W, a G-invariant neighbourhood U of p in M, and a G-equivariant biholomorphism $G \times_H (H \cdot B) \longrightarrow U$ mapping $[1, 0]$ to p.*

Proof. This is an intermediate step in Sjamaar's proof of the Holomorphic Slice Theorem: see the top of p. 101 in [133]. It can also be proved by linearising the action of G_p on the slice S at p [133, Theorem 1.21]. □

It will be important later to know that the open set U of the preceding proposition can be taken to be G-saturated. First, we have:

Proposition 3.3.8. *Let (M, K, μ) be an integrable Hamiltonian Kähler manifold and let $p \in M^{\mu\text{-ps}}$. Then, every G-invariant neighbourhood of p contains a neighbourhood of p which is G-saturated in $M^{\mu\text{-ss}}$.*

Proof. Our argument is similar to [73, Remark 14.24]. As observed in [72, Remark 1.1], the quotient map $\pi : M^{\mu\text{-ss}} \to M^{\mu\text{-ss}}/\!\!/G$ sends G-invariant closed subsets to closed subsets. Let U be a G-invariant neighbourhood of p in $M^{\mu\text{-ss}}$. Then, $C := M^{\mu\text{-ss}} - U$ is a G-invariant closed subset of $M^{\mu\text{-ss}}$, so $\pi(C)$ is closed in $M^{\mu\text{-ss}}/\!\!/G$. Moreover, since $G \cdot p$ is closed in $M^{\mu\text{-ss}}$, we have $\pi(p) \notin \pi(C)$. Hence, $\pi^{-1}(M^{\mu\text{-ss}}/\!\!/G - \pi(C))$ is a G-saturated neighbourhood of p contained in U. □

The set $H \cdot B$ in Proposition 3.3.7 is also H-saturated [135, Corollary 4.9] and it follows that $G \times_H (H \cdot B)$ is G-saturated in $G \times_H W$. We can then restate the Holomorphic Slice Theorem in the following form:

Theorem 3.3.9. *Let (M, K, μ) be an integrable Hamiltonian Kähler manifold. Let $p \in M^{\mu\text{-ps}}$, let $G := K_{\mathbb{C}}$, let $H := G_p$, and let $W := T_pM/T_p(G \cdot p)$. Then, there is a G-saturated neighbourhood U of p in $M^{\mu\text{-ss}}$, a G-saturated neighbourhood U' of the zero-section of the vector bundle $G \times_H W$, and a G-equivariant biholomorphism $U' \to U$ mapping $[1, 0]$ to p.*

Proof. Let $\varphi : G \times_H (H \cdot B) \to U$ be the biholomorphism of Proposition 3.3.7. By Proposition 3.3.8, there is a G-saturated neighbourhood U' of p contained in U. Let $B' \subseteq B$ be an open ball sufficiently small so that $U'' := \varphi(G \times_H (H \cdot B')) \subseteq U'$. Then, U'' is G-saturated. □

3.3.5 Proof of the Complex-Hamiltonian Local Normal Form

We now complete the proof of Theorem 3.1.3. The first step is to have an explicit expression for the complex-symplectic form η of the local model $E = G \times_H (\mathfrak{h}^\circ \times V)$ at the point $q = [1, 0, 0]$. Note that $G_q = H$, so H acts linearly on T_qE. Since the G-action is Hamiltonian, this is a complex-symplectic representation of H on T_qE. Recall that $\mathfrak{m} \subseteq \mathfrak{g}$ is the orthogonal complement to $\mathfrak{h}$.

Proposition 3.3.10. *We have $T_qE \cong \mathfrak{m} \times \mathfrak{m}^* \times V$ as complex-symplectic H-representations, where $\mathfrak{m} \times \mathfrak{m}^*$ has the canonical complex-symplectic form*

$$(\mathfrak{m} \times \mathfrak{m}^*) \times (\mathfrak{m} \times \mathfrak{m}^*) \longrightarrow \mathbb{C}, \quad ((x, \varphi), (y, \psi)) \longmapsto \psi(x) - \varphi(y).$$

Moreover, $T_q(G \cdot q) \cong \mathfrak{m} \times 0 \times 0$ under this isomorphism.

Proof. The canonical symplectic form on $T^*G = G \times \mathfrak{g}^*$ at $T_{(g,\xi)}(T^*G) = \mathfrak{g} \times \mathfrak{g}^*$ is

$$((x, \varphi), (y, \psi)) \longmapsto \psi(x) - \varphi(y) + \xi([x, y]) \tag{3.3.6}$$

(see, e.g., [1, Proposition 4.4.1]). In particular, if $\hat{q} := (1, 0, 0) \in T^*G \times V$, the symplectic form on $T^*G \times V$ at $T_{\hat{q}}(T^*G \times V) = \mathfrak{g} \times \mathfrak{g}^* \times V$ is

$$((x, \varphi, u), (y, \psi, v)) \mapsto \psi(x) - \varphi(y) + \omega_{\mathbb{C}}(u, v).$$

Now, we have $d\lambda_{\hat{q}}(x, \xi, v) = -\xi|_{\mathfrak{h}}$, so the tangent space to $\lambda^{-1}(0)$ at $\hat{q}$ is $\mathfrak{g} \times \mathfrak{h}^\circ \times V$. Moreover, $T_{\hat{q}}(H \cdot \hat{q}) = \mathfrak{h} \times 0 \times 0$, so

$$T_q(\lambda^{-1}(0)/H) = T_{\hat{q}}\lambda^{-1}(0)/T_{\hat{q}}(G \cdot \hat{q}) = \mathfrak{g}/\mathfrak{h} \times \mathfrak{h}^\circ \times V.$$

Identifying $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{m}$ and $\mathfrak{h}^\circ$ with $\mathfrak{m}^*$ gives the result. □

Now, we need to recall a result from representation theory. Recall that a subspace U in a symplectic vector space (R, ω) is called **symplectic** if $U \cap U^\omega = 0$ (or equivalently ω restricts to a symplectic form on U).

Proposition 3.3.11 (See, e.g., [91, §2])**.** *Let H be a complex reductive group. Every finite-dimensional complex-symplectic representation $H \to \mathrm{Sp}(R, \omega)$ is of the form*

$$R = U_1 \oplus \cdots \oplus U_m \oplus (V_1 \oplus W_1) \oplus \cdots \oplus (V_n \oplus W_n),$$

where:

(i) *U_i, V_i and W_i are the irreducible H-submodules of R;*

(ii) *U_i and $V_i \oplus W_i$ are symplectic subspaces;*

(iii) *every skew-symmetric H-invariant bilinear form on V_i or W_i is zero (so V_i and W_i are dual to each other);*

(iv) *the space of skew-symmetric H-invariant bilinear forms on U_i is one-dimensional.*

Moreover, if two symplectic H-representations (R_1, ω_1) and (R_2, ω_2) are isomorphic as H-modules then they are isomorphic as symplectic H-representations, i.e. there is an H-equivariant linear isomorphism $\varphi : R_1 \to R_2$ such that $\varphi^*\omega_2 = \omega_1$. □

We deduce from this proposition a first linear approximation of the hyperkähler local normal form.

Lemma 3.3.12. *Let H be a complex reductive group acting linearly on a finite-dimensional complex vector space R. Suppose that ω and η are two H-invariant complex-symplectic forms on R, and $S \subseteq R$ is an H-invariant subspace that is isotropic with respect to both ω and η. Then, there exists an H-equivariant linear isomorphism $\varphi : R \to R$ that restricts to the identity on S and such that $\varphi^*\eta = \omega$.*

Proof. Let $R = U_1 \oplus \cdots \oplus U_m \oplus (V_1 \oplus W_1) \oplus \cdots \oplus (V_n \oplus W_n)$ be the decomposition of Proposition 3.3.11 with respect to ω. The space S is H-invariant, so it is a direct sum of irreducible H-submodules. But S is isotropic, so it contains no symplectic subspace, and hence is a direct sum of V_i's and W_i's, no two of which occur in the same pair. Thus, after relabeling, we may assume that $S = V_1 \oplus \cdots \oplus V_k$ for some $k \leq n$.

Let us call the U_i-factors the **ω-symplectic-H-summands** (which are symplectic by (ii)) and the V_i- and W_i-factors the **ω-isotropic-H-summands** (which are isotropic by (iii)). We can also consider the decomposition of R in Proposition 3.3.11 with respect to η. By (iii) the η-isotropic-H-summands are the same as the ω-isotropic-H-summands. Hence, the η-symplectic-H-summands are also the same as the ω-symplectic-H-summands. Thus the decomposition of R into H-invariant symplectic subspaces with respect to η is of the form $U_1 \oplus \cdots \oplus U_m \oplus (P_1 \oplus Q_1) \oplus \cdots \oplus (P_n \oplus Q_n)$ where the P_i's and Q_i's are some reorderings of the V_i's and W_j's. Since $S = V_1 \oplus \cdots \oplus V_k$ is also isotropic with respect to η, no two V_i and V_j for $i, j \in \{1, \ldots, k\}$ occur in the same pair $P_l \oplus Q_l$. Thus, we may assume that $P_i = V_i$ for $i = 1, \ldots, k$. Let $R_1 = (V_1 \oplus W_1) \oplus \cdots \oplus (V_k \oplus W_k)$ and $R_2 = R_1^\omega$ so that $R = R_1 \oplus R_2$. Similarly, let $R_1' = (V_1 \oplus Q_1) \oplus \cdots \oplus (V_k \oplus Q_k)$ and $R_2' = (R_1')^\eta$ so that $R = R_1' \oplus R_2'$. It suffices to find an H-equivariant isomorphism $\psi : R_1 \to R_1'$ which is the identity on the V_i's and such that $\psi^*\eta = \omega$. Indeed, in that case, $R_2 \cong R/R_1 \cong R/R_1' \cong R_2'$ as H-modules so also as symplectic representations (by the last part of Proposition 3.3.11), and then we have an isomorphism $R_1 \oplus R_2 \to R_1' \oplus R_2'$ with the desired properties. To find $\psi : R_1 \to R_1'$, note that ω provides an isomorphism $W_i \to V_i^*$ and η provides an isomorphism $V_i^* \to Q_i$. Let $\gamma_i : W_i \to Q_i$ be the composition and let $\beta_i : V_i \oplus W_i \to V_i \oplus Q_i$ be the product of the identity on V_i with γ_i. Then, β_i is an H-invariant isomorphism such that $\beta_i^*\eta|_{V_i \oplus Q_i} = \omega|_{V_i \oplus W_i}$. Putting those β_i's together we get an H-equivariant isomorphism $\psi : R_1 \to R_1'$ which is the identity on the V_i's. Moreover, $\psi^*\eta = \omega$ since the factors $V_1 \oplus W_1, \ldots, V_k \oplus W_k$ are ω-orthogonal (since if $A \subseteq R_1$ is an H-invariant symplectic subspace then $R_1 = A \oplus A^\omega$ and A^ω is H-invariant so A^ω is the sum of the irreducible H-submodules of R_1 complementary to A) and similarly the factors $V_1 \oplus Q_1, \ldots, V_k \oplus Q_k$ are η-orthogonal. □

Lemma 3.3.13. *Let $H \to \mathrm{Sp}(R, \omega)$ be a complex-symplectic representation and $S \subseteq R$ an H-invariant isotropic subspace. Then, $R/S \cong S^* \times S^\omega/S$ as H-modules.*

Proof. Let $R \to S^*$ be the composition of the isomorphism $R \to R^*$ induced by ω with the restriction map $R^* \to S^*$. Let $R \to S^\omega$ be the projection along the H-invariant complement of S^ω in R (by complete reducibility). These maps give an H-equivariant surjective map $R \to S^* \times S^\omega/S$ with kernel $S^\omega \cap S$. Since S is isotropic, we have $S^\omega \cap S = S$. □

Proof of Theorem 3.1.3 (Complex-Hamiltonian Local Normal Form). Since $T_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{h}$ is isotropic in T_pM, Lemma 3.3.13 implies that $T_pM/T_p(G \cdot p) \cong \mathfrak{h}^\circ \times V$, where V is the complex-symplectic slice at p. Thus, by the Linearised Holomorphic Slice Theorem (Theorem 3.3.9), a G-saturated neighbourhood of p in $M^{\mu_\mathbb{R}\text{-ss}}$ is G-equivariantly biholomorphic to a G-saturated neighbourhood of $q = [1, 0, 0]$ in the local model $E = G \times_H (\mathfrak{h}^\circ \times V)$. Note that by Proposition 3.3.10, $T_q(G \cdot q)$ is also isotropic with respect to the canonical complex-symplectic form on E. Note also that any G-invariant neighbourhood of the zero-section $0_E = G \cdot q$ of E contains a G-saturated neighbourhood, namely $G \times_H (H \cdot B)$ for a sufficiently small open ball B. Hence, it suffices to show that, for any two G-invariant complex-symplectic forms $\omega_{\mathbb{C}}$ and η on a G-invariant neighbourhood of the zero-section $0_E = G \cdot q$ in E such that $T_q 0_E$ is isotropic with respect to both, there is a G-equivariant biholomorphism on a possibly smaller neighbourhood of 0_E which pulls back η to $\omega_{\mathbb{C}}$. By the holomorphic Darboux–Weinstein Theorem, it suffices to find such a biholomorphism that makes them match on 0_E. This is carried out by the linear algebra developed in Lemma 3.3.12, as we now explain.

By Lemma 3.3.12, there exists an H-equivariant linear isomorphism $\varphi : T_q E \to T_q E$ which restricts to the identity on $T_q 0_E$ and such that $\varphi^* \eta = \omega_{\mathbb{C}}$. We have $T_q E = \mathfrak{m} \times \mathfrak{m}^* \times V$ and $T_q 0_E = \mathfrak{m} \times 0 \times 0$, so φ is of the form

$$\varphi : \mathfrak{m} \times \mathfrak{m}^* \times V \longrightarrow \mathfrak{m} \times \mathfrak{m}^* \times V, \quad \varphi(x, \xi, v) = (x + A(\xi, v), B(\xi, v)),$$

where $A : \mathfrak{m}^* \times V \to \mathfrak{m}$ and $B : \mathfrak{m}^* \times V \to \mathfrak{m}^* \times V$ are some linear maps, with B invertible. Then,

$$\psi : E \longrightarrow E, \quad \psi([g, \xi, v]) = [g e^{A(\xi, v)}, B(\xi, v)]$$

is a G-equivariant biholomorphism with $d\psi_q = \varphi$. In particular, $\psi^* \eta|_q = \omega_{\mathbb{C}}|_q$ and, since $\omega_{\mathbb{C}}$ and η are G-invariant and ψ is G-equivariant, this implies that $\psi^* \eta|_{g \cdot q} = \omega_{\mathbb{C}}|_{g \cdot q}$ for all $g \in G$, i.e. $\psi^* \eta|_{0_E} = \omega_{\mathbb{C}}|_{0_E}$.

We can now apply the holomorphic Darboux–Weinstein Theorem. This shows the existence of a G-equivariant complex-symplectic isomorphism $f : U \to U'$ such that $f(p) = q$, where U is a G-saturated neighbourhood of p in $M^{\mu_{\mathbb{R}}\text{-ss}}$ and U' a G-saturated neighbourhood of q in E. It remains to show that $\kappa \circ f = \mu_{\mathbb{C}}$. Since $(\kappa \circ f)(p) = 0 = \mu_{\mathbb{C}}(p)$ and since moment maps are unique up to a constant (see, e.g., [27, Ch. 26]) it suffices to show that $\kappa \circ f$ is a moment map for the G-action on $M^{\mu_{\mathbb{R}}\text{-ss}}$. This follows from the fact that f is a G-equivariant complex-symplectic isomorphism. □

3.4 Local structure of singular hyperkähler quotients

Throughout this section, $((M, g, \mathsf{I}, \mathsf{J}, \mathsf{K}), K, \mu)$ is a tri-Hamiltonian hyperkähler manifold whose K-action is I-integrable. The goal of this section is to use the local normal form of §3.3 to prove Theorem 3.1.5 in the introduction which describes the local complex-symplectic structure of the singular hyperkähler quotient $M /\!\!/\!\!/_{\mu} K$. In particular, we endow $M /\!\!/\!\!/_\mu K$ with the structure of a complex-analytic space and show that the orbit-type partition is a complex-analytic Whitney stratification.

3.4.1 Complex-analytic structure

Let us first explain how the results on analytic Hilbert quotients of §2.4 help us define a complex-analytic structure on $M /\!\!/\!\!/_\mu K$. We use the notation of §3.3; in particular, $G \coloneqq K_\mathbb{C}$, $\mu_\mathbb{C} \coloneqq \mu_\mathsf{J} + i\mu_\mathsf{K}$, and $\mu_\mathbb{R} \coloneqq \mu_\mathsf{I}$. First, note that $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} \coloneqq \mu_\mathbb{C}^{-1}(0) \cap M^{\mu_\mathbb{R}\text{-ss}}$ is a G-invariant closed complex-analytic subspace of $M^{\mu_\mathbb{R}\text{-ss}}$. Hence, its image $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!\!/ G$ in $M^{\mu_\mathbb{R}\text{-ss}} /\!\!/ G$ is a closed complex-analytic subspace, and the restriction

$$\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} \longrightarrow \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!\!/ G$$

is an analytic Hilbert quotient (by Proposition 2.4.3(ii)). The space $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!\!/ G$ has a G-orbit-type partition as in §2.4.4 and $M /\!\!/\!\!/_\mu K$ has a K-orbit-type partition into hyperkähler manifolds by Theorem 3.1.1. By Heinzner–Loose's Theorem 2.4.5 and Sjamaar's Theorem 2.4.10(i), we get:

Proposition 3.4.1. *We have $\mu^{-1}(0) \subseteq \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}$, and this inclusion descends to an isomorphism*

$$M /\!\!/\!\!/_\mu K \longrightarrow \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!\!/ G$$

of partitioned spaces. □

In particular, $M /\!\!/\!\!/_\mu K$ has the structure of a complex-analytic space. We denote the structure sheaf by $\mathcal{O}_\mathsf{I}$.

3.4.2 Linear hyperkähler quotients

Let us first consider the case of a linear hyperkähler quotient; this example will be important later. Let V be a quaternionic vector space, i.e. a real vector space endowed with three endomorphisms $\mathsf{I}, \mathsf{J}, \mathsf{K}$ such that $\mathsf{I}^2 = \mathsf{J}^2 = \mathsf{K}^2 = \mathsf{IJK} = -1$. Then, $V \cong \mathbb{H}^n$ for some n, so we may endow V with a real inner-product $\langle\cdot,\cdot\rangle$ such that $\mathsf{I}, \mathsf{J}, \mathsf{K}$ are skew-symmetric. This makes V into a hyperkähler manifold, with Kähler forms $\omega_\mathsf{I}(u, v) = \langle \mathsf{I}u, v\rangle$, etc. Let L be a compact Lie group acting linearly on V by preserving $\langle\cdot,\cdot\rangle$ and $\mathsf{I}, \mathsf{J}, \mathsf{K}$. Then, there is a hyperkähler moment map, namely,

$$\phi : V \longrightarrow \mathfrak{l}^* \times \mathfrak{l}^* \times \mathfrak{l}^*, \quad \phi(v)(x_1, x_2, x_3) = \frac{1}{2}(\omega_\mathsf{I}(x_1 \cdot v, v), \omega_\mathsf{J}(x_2 \cdot v, v), \omega_\mathsf{K}(x_3 \cdot v, v)).$$

Moreover, the L-action extends to an I-complex-linear action of $H \coloneqq L_\mathbb{C}$, and the underlying complex-Hamiltonian manifold is simply $(V, \mathsf{I}, \omega_\mathbb{C}, H, \Phi_V)$ where $\omega_\mathbb{C}$ is the complex-symplectic bilinear form $\omega_\mathsf{J} + i\omega_\mathsf{K} : V \times V \to \mathbb{C}$ and Φ_V is the canonical complex moment map of Proposition 3.1.2. By the Kempf–Ness Theorem [85], every point in V is ϕ_{I}-semistable (see, e.g., [116, Proposition 3.9]), so the complex-analytic space $(V /\!/\!/_{\phi_V} L, \mathcal{O}_{\mathsf{I}})$ is simply the analytification of the affine GIT quotient $\Phi_V^{-1}(0) /\!/ H = \operatorname{Spec} \mathbb{C}[\Phi_V^{-1}(0)]^H$.

Conversely, if H is any complex reductive group and $H \to \operatorname{Sp}(V, \omega_{\mathbb{C}})$ is a complex-symplectic representation (e.g. a complex-symplectic slice) then $V \cong \mathbb{C}^{2n} \cong \mathbb{H}^n$ for some n, so we may endow V with the structure of a quaternionic vector space invariant under the action of a maximal compact subgroup L of H (by averaging). Hence, the GIT quotient $\Phi_V^{-1}(0) /\!/ H$ can always be viewed as a hyperkähler quotient.

3.4.3 Local holomorphic structure

Let $q \in M /\!/\!/_{\mu} K$ and let $p \in \mu^{-1}(0)$ be a point above q. Let $H := G_p$ and let $V := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p)$ be the complex-symplectic slice at p. Let $\Phi_V : V \to \mathfrak{h}^*$ be the canonical complex moment map of Proposition 3.1.2. The first step in proving Whitney conditions will be to use the Complex-Hamiltonian Local Normal Form (Theorem 3.1.3) to show that q has a neighbourhood U which is isomorphic as a complex-analytic and partitioned space to a neighbourhood U' of 0 in the GIT quotient $\Phi_V^{-1}(0) /\!/ H$. Since Whitney conditions are local, this will reduce the problem to the space $\Phi_V^{-1}(0) /\!/ H$ near 0. However, note here that the natural partition of $\Phi_V^{-1}(0) /\!/ H$ is by H-orbit-types rather than G-orbit-types. To show that the biholomorphism $U \to U'$ is an isomorphism of partitioned spaces, we will first need to show that once we refine the partitions into the connected components of the pieces, the G-orbit-type partition of $\Phi_V^{-1}(0) /\!/ H$ (i.e. saying that two points $p, q \in \Phi_V^{-1}(0)^{\mathrm{ps}}$ have the same orbit-type if H_p and H_q are conjugate by an element of G rather than H) is identical to the standard H-orbit-type partition. This will follow from a result of Palais [123] which says that when a compact Lie group K acts on a completely regular space X, every $x \in X$ has a neighbourhood V such that if $y \in V$ then K_y is conjugate to a subgroup of K_x:

Lemma 3.4.2. *Let (M, K, μ) be a Hamiltonian Kähler manifold and let K' be a compact Lie group containing K as a Lie subgroup. Then, the K- and K'-orbit-type partitions of $M /\!/_{\mu} K$ coincide. Moreover, if (M, K, μ) is integrable, then the $K'_{\mathbb{C}}$- and $K_{\mathbb{C}}$-orbit-type partitions of $M^{\mu\text{-ss}} /\!/ K_{\mathbb{C}}$ also coincide.*

Proof. Let $X = \mu^{-1}(0)$ and let $\pi : X \to X/K$ be the quotient map. Let $S \subseteq X/K$ be a K'-orbit-type piece, i.e. a connected component of a set of the form $X_{(H)_{K'}}/K$ for some closed subgroup $H \subseteq K$, where $(H)_{K'}$ is the conjugacy class of H in K'. We have $S = \pi(T)$ for some connected component T of $X_{(H)_{K'}}$. Fix $x \in T$. We want to show that if $y \in T$ then K_x and K_y (which are conjugate in K') are in fact conjugate in K. Let

$$A \coloneqq \{y \in T : K_y \text{ is conjugate to } K_x \text{ in } K\}.$$

It suffices to show that A is both open and closed in T. *Closed:* Let $y \in \overline{A} \cap T$ and write $y = \lim_{n\to\infty} y_n$ with $y_n \in A$. Then, there exist $k_n \in K$ such that $k_n K_x k_n^{-1} = K_{y_n}$ for all n. Since K is compact, we may assume that $\lim_{n\to\infty} k_n = k$ for some $k \in K$. Then, $kK_xk^{-1} \subseteq K_y$ by continuity of the action. Moreover, kK_xk^{-1} and K_y are isomorphic since they are conjugate in K' and since they have finitely many connected components, the inclusion $kK_xk^{-1} \subseteq K_y$ implies that $kK_xk^{-1} = K_y$. Thus, A is closed. *Open:* Let $y \in A$. By Palais [123, Corollary 2 on p. 313] there is a neighbourhood V of y in X such that if $z \in V$ then K_z is conjugate (in K) to a subgroup of K_y. Then, $V \cap T$ is a neighbourhood of y in T and $V \cap T \subseteq A$, so A is open in T.

The second statement amounts to show that if H and L are two closed subgroups of a compact Lie group R, then H and L are conjugate in R if and only if $H_{\mathbb{C}}$ and $L_{\mathbb{C}}$ are conjugate in $R_{\mathbb{C}}$. This follows from Mostow's decomposition, as explained by Sjamaar [133, Proof of Theorem 2.10, first paragraph]. □

Now, by picking a quaternionic structure on the complex-symplectic slice V as explained in §3.4.2, we can apply this result to $(V, K_p, \phi_{\mathsf{I}})$ and infer that the G- and H-orbit-type partitions of $\Phi_V^{-1}(0) /\!\!/ H$ coincide. This will be used for the last part of the following result.

Proposition 3.4.3. *Let $q \in M/\!\!/\!\!/_{\mu} K$. Take a point $p \in \mu^{-1}(0)$ above q, let $H \coloneqq G_p = (K_p)_{\mathbb{C}}$, and let $V \coloneqq (T_p(G\cdot p))^{\omega_{\mathbb{C}}}/T_p(G\cdot p)$. Then, there is a neighbourhood U of q in $M/\!\!/\!\!/_{\mu} K$, an open ball $B \subseteq V$ around 0, and a biholomorphism (with respect to $\mathcal{O}_{\mathsf{I}}$) from U to the image of $(H\cdot B) \cap \Phi_V^{-1}(0)$ in the GIT quotient $\Phi_V^{-1}(0)/\!\!/H = \operatorname{Spec} \mathbb{C}[\Phi_V^{-1}(0)]^H$ which maps q to the image of $0 \in \Phi_V^{-1}(0)$. Moreover, this biholomorphism is an isomorphism of partitioned spaces.*

Proof. Let $E = G \times_H (\mathfrak{h}^\circ \times V)$. Since H is reductive and acts freely on $G \times (\mathfrak{h}^\circ \times V)$, E is an affine variety. Moreover, the moment map $\kappa : E \to \mathfrak{g}^*$ is algebraic, so $\kappa^{-1}(0)$ is an affine variety in E and we can consider the GIT quotient $\kappa^{-1}(0) /\!\!/ G = \operatorname{Spec}(\mathbb{C}[\kappa^{-1}(0)]^G)$. We claim that $\kappa^{-1}(0)/\!\!/G \cong \Phi_V^{-1}(0)/\!\!/H$ as affine varieties. Indeed, we have $\kappa^{-1}(0) = G \times_H \Phi_V^{-1}(0)$, so the inclusion $\Phi_V^{-1}(0) \to \kappa^{-1}(0) : v \mapsto [1, v]$ descends to a morphism $\psi : \Phi_V^{-1}(0)/\!\!/H \to \kappa^{-1}(0)/\!\!/G$. Also, the projection $\kappa^{-1}(0) = G \times_H \Phi_V^{-1}(0) \to \Phi_V^{-1}(0) /\!\!/ H$ onto the second factor descends to a morphism $\kappa^{-1}(0) /\!\!/ G \to \Phi_V^{-1}(0) /\!\!/ H$ which is an inverse of ψ.

Now, for an element $[g, v] \in G \times_H \Phi_V^{-1}(0) = \kappa^{-1}(0)$ we have $G_{[g,v]} = gH_v g^{-1}$, so ψ is an isomorphism of partitioned spaces with the G-orbit-type partitions on both sides. As explained above, Lemma 3.4.2 implies that the G-orbit-type partition on $\Phi_V^{-1}(0)/\!\!/ H$ coincides with the H-orbit-type partition.

Let $U \subseteq M^{\mu_{\mathbb{R}}\text{-ss}}$, $U' \subseteq E$ and $f : U \to U'$ be as in the Complex-Hamiltonian Local Normal Form (Theorem 3.1.3), where $U' = G \times_H (H \cdot B)$ for some open ball B around zero in $\mathfrak{m}^* \times V$. Then, $W := U \cap \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ is a G-saturated open subset of $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$, and so is $W' := U' \cap \kappa^{-1}(0)$ in $\kappa^{-1}(0)$. Moreover, by Proposition 2.4.3(i), the image $W/\!\!/ G$ of W in $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!\!/ G$ is open and $W \to W/\!\!/ G$ is an analytic Hilbert quotient. Similarly, $W' \to W'/\!\!/G \subseteq \kappa^{-1}(0)/\!\!/G$ is an analytic Hilbert quotient. Since $f : U \to U'$ is a G-equivariant biholomorphism with $\kappa \circ f = \mu_{\mathbb{C}}$, it restricts to a G-equivariant biholomorphism $W \to W'$ and hence to a biholomorphism $W/\!\!/ G \to W'/\!\!/ G$ which respects the G-orbit-type partitions. Moreover, under the isomorphism $\kappa^{-1}(0)/\!\!/G \cong \Phi_V^{-1}(0)/\!\!/ H$ above we have an isomorphism $W'/\!\!/G \cong (H \cdot B \cap \Phi_V^{-1}(0))/\!\!/ H$ of complex-analytic and partitioned spaces. □

3.4.4 The orbit-type pieces are complex submanifolds

As a first application of Proposition 3.4.3, we will show that the pieces in the orbit-type partition are complex submanifolds with respect to $\mathcal{O}_{\mathsf{l}}$; this is one of the requirements in the definition of complex-analytic Whitney stratifications.

We shall achieve this by describing the orbit-type partition of $\Phi_V^{-1}(0)/\!\!/ H$, where H is a complex reductive group, $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ a complex-symplectic representation, and Φ_V the canonical complex moment map. First, note that the set V^H of fixed points of H in V is a complex-symplectic subspace. Let W be its symplectic complement, so that $V = W \oplus V^H$. Then, W is complex-symplectic and H-invariant, so it provides a complex-symplectic representation of H. The complex moment map $\Phi_W : W \to \mathfrak{h}^*$ associated with this representation is simply the restriction of Φ_V to W, so we have the decomposition

$$\Phi_V^{-1}(0)/\!\!/ H = (\Phi_W^{-1}(0)/\!\!/ H) \times V^H.$$

For each $L \subseteq H$, let $(\Phi_W^{-1}(0)/\!\!/ H)_{(L)}$ be the image of $\Phi_W^{-1}(0)^{\mathrm{ps}}_{(L)}/H$ under the bijection $\Phi_W^{-1}(0)^{\mathrm{ps}}/H \to \Phi_W^{-1}(0)/\!\!/ H$. Then, the pieces of the orbit-type partition of $\Phi_V^{-1}(0)/\!\!/ H$ are the connected components of the sets of the form $(\Phi_W^{-1}(0)/\!\!/ H)_{(L)} \times V^H$.

Lemma 3.4.4. *The orbit-type piece of $\Phi_V^{-1}(0)/\!\!/ H$ containing 0 is $\{0\} \times V^H$.*

Proof. Note that $V_{(H)} = V^H$ since if $v \in V$ and $H_v = gHg^{-1}$ for some $g \in H$, then $gHg^{-1} \subseteq H$, and since gHg^{-1} and H are isomorphic Lie groups with finitely many connected components this implies $gHg^{-1} = H$ and hence $H_v = H$. In particular, $W_{(H)} = W \cap V^H = 0$, so the piece containing 0 is $(\Phi_W^{-1}(0)/\!\!/H)_{(H)} \times V^H = \{0\} \times V^H$. □

Proposition 3.4.5. *The pieces of the orbit-type partition of $M/\!\!/\!\!/_\mu K$ are non-singular complex-analytic subspaces with respect to $\mathcal{O}_\mathsf{I}$.*

Proof. By Lemma 3.4.4 and Proposition 3.4.3, the embedding of a K-orbit-type piece in $M/\!\!/\!\!/_\mu K$ is locally biholomorphic to the embedding of $\{0\} \times V^H$ in $(\Phi_W^{-1}(0)/\!\!/ H) \times V^H$. □

3.4.5 Compatibility with the hyperkähler structures

We show that for each orbit-type stratum $S \subseteq M/\!\!/\!\!/_\mu K$, the sheaf $\mathcal{O}_\mathsf{I}$ is compatible with the complex structure I_S on S, where $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ is its hyperkähler structure as in Theorem 3.1.1.

Proposition 3.4.6. *The inclusion $S \hookrightarrow M/\!\!/\!\!/_\mu K$ is holomorphic with respect to I_S and $\mathcal{O}_\mathsf{I}$.*

Proof. We want to show that the composition $S \hookrightarrow M/\!\!/\!\!/_\mu K \to \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$ is holomorphic, where S has the complex structure I_S. Since $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$ is a closed complex-analytic subspace of $M^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$, it suffices to show that the composition $S \to \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/G \hookrightarrow M^{\mu_\mathbb{R}\text{-ss}}/\!\!/G$ is holomorphic, which is the same as the composition $S \hookrightarrow M/\!\!/\!\!/_\mu K \hookrightarrow M/\!\!/_{\mu_\mathbb{R}} K \to M^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$. Let $H \subseteq K$ be as above so that S is a connected component of $\mu^{-1}(0)_{(H)}/K$. Then, S is a subset of a connected component T of $\mu_\mathbb{R}^{-1}(0)_{(H)}/K$. Moreover, T is a stratum in the Kähler quotient $M/\!\!/_{\mu_\mathbb{R}} K$ and, from the definition of the Kähler structure on T given in §2.4 and the definition of I_S given above, the inclusion $S \hookrightarrow T$ is holomorphic. Hence, it suffices to show that the composition $T \hookrightarrow M/\!\!/_{\mu_\mathbb{R}} K \to M^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$ is holomorphic, and this follows from Theorem 2.4.10(iii). □

3.4.6 The frontier condition

In this section, we prove that the orbit-type partition of $M/\!\!/\!\!/_\mu K$ is a decomposition in the sense of Definition 2.1.3 (this is a requirement in the definition of Whitney stratified spaces). Since K is compact, $\mu^{-1}(0)/K$ satisfies the local condition, so the only thing left to show is the frontier condition. This will be achieved by the local model of Proposition 3.4.3, so we first need to discuss how the frontier condition can be inferred locally.

Given a partitioned space $(X, \mathcal{P})$ we will denote by $\mathcal{P}^\circ$ the refinement of $\mathcal{P}$ obtained by decomposing every piece of $\mathcal{P}$ into its connected components. In particular, the orbit-type partition of $M /\!\!/\!\!/_\mu K$ which we are considering is the refinement $\mathcal{P}^\circ$ of $\mathcal{P} := \{\mu^{-1}(0)_{(K_p)}/K : p \in \mu^{-1}(0)\}$. Also, we will say that a partitioned space $(X, \mathcal{P})$ is **conical** at a stratum $S \in \mathcal{P}$ if $S \subseteq \overline{T}$ for all $T \in \mathcal{P}$.

The following lemma provides a local criterion for partitioned spaces to satisfy the frontier condition.

Lemma 3.4.7. *Let $(X, \mathcal{P})$ be a partitioned space. Suppose that every point $x \in X$ has a neighbourhood U such that if $S \in \mathcal{P}$ is the stratum containing x, then $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. Then, $\mathcal{P}^\circ$ satisfies the frontier condition.*

Proof. Let $S, T \in \mathcal{P}$ and let $S = \bigsqcup_i S_i$, $T = \bigsqcup_j T_j$ be their decompositions into connected components. Suppose that $S_{i_0} \cap \overline{T_{j_0}} \neq \emptyset$ for some i_0, j_0. We want to show that $S_{i_0} \subseteq \overline{T_{j_0}}$. The set $R := S_{i_0} \cap \overline{T_{j_0}}$ is closed in S_{i_0}, so it suffices to show that R is also open in S_{i_0}. Let $x \in R$. Take a neighbourhood U of x in X such that $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. We claim that $S_{i_0} \cap U \subseteq R$, or equivalently, $S_{i_0} \cap U \subseteq \overline{T_{j_0}}$. If $T \cap U = \bigsqcup_k C_k$ is the decomposition of $T \cap U$ into connected components, then, since $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$, we have $S \cap U \subseteq \overline{C_k}$ for all k. But the set of connected components of $T \cap U$ is the union of the set of connected components of $T_j \cap U$ for all j, so there exists k_0 such that $C_{k_0} \subseteq T_{j_0} \cap U$ and hence $S_{i_0} \cap U \subseteq S \cap U \subseteq \overline{C_{k_0}} \subseteq \overline{T_{j_0}}$. □

Proposition 3.4.8. *The orbit-type partition of $M /\!\!/\!\!/_\mu K$ satisfies the frontier condition and hence is a decomposition.*

Proof. Let $q \in M /\!\!/\!\!/_\mu K$, let V, H and $B \subseteq V$ be as in Proposition 3.4.3, and let $U = (H \cdot B) \cap \Phi_V^{-1}(0) /\!\!/ H$. We denote by $[v]$ the image of a point $v \in \Phi_V^{-1}(0)$ in the GIT quotient $\Phi_V^{-1}(0) /\!\!/ H$. Then, q has a neighbourhood isomorphic to U as partitioned spaces, with an isomorphism sending q to $[0]$. Let $\mathcal{P}$ be the orbit-type partition of $\Phi_V^{-1}(0) /\!\!/ H$ and let $S \in \mathcal{P}$ be the piece containing $[0]$. By Lemma 3.4.7, it suffices to show that $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. By Lemma 3.4.4, $S = \{[0]\} \times V^H$ so $S \cap U = \{[0]\} \times (V^H \cap B)$ is connected. To show that $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$, let $T' \in (\mathcal{P}|_U)^\circ$. Then, T' is a connected component of $T \cap U$, where $T := (\Phi_W^{-1}(0) /\!\!/ H)_{(L)} \times V^H$ for some $L \subseteq H$. We need to show that $S \cap U \subseteq \overline{T'}$. Let $([0], v) \in S \cap U$, where $v \in V^H \cap B$. Take any point $([w], u)$ of T', where $w \in (\Phi_W^{-1}(0)^{\mathrm{ps}})_{(L)}$, $u \in V^H$, and $w + u \in H \cdot B$. It suffices to find a continuous path $\gamma : (0, 1] \to T \cap U$ such that $\gamma(1) = ([w], u)$ and $\lim_{t \to 0} \gamma(t) = ([0], v)$. Let $h \in H$ be such that $w + u \in h^{-1}B$. Then, $hw + u \in B$. We also have $v \in B$, so there exists $t_0 > 0$ small enough so that $t_0 hw + v \in B$ and hence $t_0 w + v \in H \cdot B$. Now, $\Phi_W(tw) = t^2 \Phi_W(w) = 0$ and hence $([tw], v) \in T \cap U$ for all $t > 0$ and $([tw], v) \to ([0], v)$ as $t \to 0$. Moreover, since B is convex, the straight line from $t_0 w + v$ to $w + u$ will stay in $(H \cdot B) \cap ((\Phi_W^{-1}(0)^{\mathrm{ps}})_{(L)} \times V^H)$ and hence $([t_0 w], v)$ and $([w], u)$ are in the same path component T' of $T \cap U$. $\square$

3.4.7 Whitney conditions

We show that the orbit-type partition of $M /\!\!/\!\!/_\mu K$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}_\mathsf{l}$ and hence a stratification in the sense of Definition 2.1.5. Our proof is very similar to that of Sjamaar–Lerman [134, §6]. Let us first recall the following result of Whitney.

Lemma 3.4.9 (Whitney [145, Lemma 19.3])**.** *Let S and T be disjoint complex submanifolds of a complex-analytic space X with $S \subseteq \overline{T}$ and* $\dim S < \dim T$. *There is a complex-analytic subspace A of S with* $\dim A < \dim S$ *such that T is regular over $S \setminus A$.* $\square$

Corollary 3.4.10. *Let X be a complex-analytic space and $T \subseteq X$ a complex submanifold with* $\dim T > 0$. *Then, T is regular over $\{x\}$ for all $x \in \overline{T} \setminus T$.*

Proof. Use Lemma 3.4.9 with $S = \{x\}$. (Remark: Whitney [145] defines a set A to have $\dim A < 0$ if and only if $A = \emptyset$, see page 500, lines 24–25 in that paper.) $\square$

Lemma 3.4.11. *Let X be a complex-analytic space and $S, T \subseteq X$ disjoint complex submanifolds such that T is regular over S. Then, for all $n \geq 0$, $T \times \mathbb{C}^n$ is regular over $S \times \mathbb{C}^n$ in $X \times \mathbb{C}^n$.*

Proof. This follows directly from the definition since $T_{(s,z)}(S \times \mathbb{C}^n) = T_s S \times \mathbb{C}^n$. $\square$

Proposition 3.4.12. *The orbit-type partition of $M /\!\!/\!\!/_\mu K$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}_\mathsf{l}$. In particular, it is a stratification in the sense of* Definition 2.1.5.

Proof. By Proposition 3.4.3, the problem reduces to checking Whitney conditions for the H-orbit-type partition of $\Phi_V^{-1}(0) /\!\!/ H$ at $[0]$. By §3.4.4, we have $\Phi_V^{-1}(0) /\!\!/ H = (\Phi_W^{-1}(0) /\!\!/ H) \times V^H$ and by Lemma 3.4.11 it suffices to check Whitney condition for $\Phi_W^{-1}(0) /\!\!/ H$ at $[0]$. But the piece containing $[0]$ is the singleton $\{[0]\}$, so this follows from Corollary 3.4.10. $\square$

3.4.8 Poisson structure

We show that there is a natural Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ making $M /\!\!/\!\!/_{\mu} K$ a stratified symplectic space as in Sjamaar–Lerman's work (§2.3) but in a complex-analytic sense:

Definition 3.4.13. A **stratified complex-symplectic space** is a complex-analytic space $(X, \mathcal{O}_X)$ with a complex-analytic Whitney stratification $\mathcal{P}$, a complex-symplectic structure on each stratum, and a sheaf of Poisson brackets on $\mathcal{O}_X$ such that the embeddings $S \hookrightarrow X$ for $S \in \mathcal{P}$ are holomorphic Poisson maps.

The definition of the Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ is as follows. Let $U \subseteq M /\!\!/\!\!/_{\mu} K$ be open, let $f, g \in \mathcal{O}_{\mathsf{I}}(U)$ and let $q \in U$. To define $\{f, g\}(q)$, let $S \subseteq M /\!\!/\!\!/_{\mu} K$ be the orbit-type stratum containing q and let $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ be its hyperkähler structure. Then, $(\omega_S)_{\mathbb{C}} \coloneqq \omega_{\mathsf{J}_S} + i\omega_{\mathsf{K}_S}$ is a complex-symplectic form on (S, I_S). By Proposition 3.4.6, the restrictions $f|_{S\cap U}$, $g|_{S\cap U}$ are I_S-holomorphic, and hence we can take their Poisson bracket $\{f|_{S\cap U}, g|_{S\cap U}\} : S \cap U \to \mathbb{C}$ with respect to $(\omega_S)_{\mathbb{C}}$ and define $\{f, g\}(q) \coloneqq \{f|_{S\cap U}, g|_{S\cap U}\}(q)$. This defines a function $\{f, g\} : U \to \mathbb{C}$ pointwise, and the goal is to show that it is holomorphic, i.e. $\{f, g\} \in \mathcal{O}_{\mathsf{I}}(U)$.

In what follows, we identify S with a G-orbit-type stratum in $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$, i.e. S is a connected component of $(\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G)_{(H)}$ for some reductive subgroup $H \subseteq G$. By the definition of the G-orbit-type partition, the map $(\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}})_{(H)} \to (\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G)_{(H)}$ is surjective (note that on the left-hand side we use polystable points), so S is the image under the quotient map $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} \to \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$ of an open subset Z of $(\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}})_{(H)}$.

Lemma 3.4.14. *The set Z is a complex submanifold of M, the map $\pi : Z \to S$ is a holomorphic submersion, and $\pi^*(\omega_S)_{\mathbb{C}} = i^*\omega_{\mathbb{C}}$ where $i : Z \hookrightarrow M$.*

Proof. By the Complex-Hamiltonian Local Normal Form (Theorem 3.1.3), the embedding of Z in M is locally biholomorphic to the embedding of $G/H \times V^H$ in $G \times_H (\mathfrak{h}^\circ \times V)$ and π is locally biholomorphic to the projection $G/H \times V^H \to V^H$. This proves the first and second assertions. For the third assertion, we first note that since the pullback of the symplectic forms $\omega_{\mathsf{I}_S}, \omega_{\mathsf{J}_S}, \omega_{\mathsf{K}_S}$ on $\mu^{-1}(0)_{(H)}$ are the restriction of the symplectic forms $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$ on M, we have $j^*(\pi^*(\omega_S)_{\mathbb{C}}) = j^*(i^*\omega_{\mathbb{C}})$ where $j : \mu^{-1}(0)_{(H)} \hookrightarrow Z$. Since j descends to a diffeomorphism $\mu^{-1}(0)_{(H)}/K \to (\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G)_{(H)}$ we get that for all $p \in \mu^{-1}(0)_{(H)}$, $T_pZ = T_p\mu^{-1}(0)_{(H)} + T_p(G \cdot p)$. Hence, the result follows by the same argument as in the proof of Proposition 3.2.2. □

Lemma 3.4.15. *Let $f : U \to \mathbb{C}$ be a holomorphic G-invariant function on an open set $U \subseteq M$, and let Ξ_f be the holomorphic vector field on U dual to df under $\omega_{\mathbb{C}}$. Then, Ξ_f is tangent to Z, i.e. $\Xi_f(p) \in T_pZ$ for all $p \in Z \cap U$.*

Proof. Let $\mathfrak{m} = \mathfrak{h}^{\perp}$ as in §3.3.2. By the local normal form we may assume that $M = G \times_H (\mathfrak{m}^* \times V)$, $p = [1, 0, 0]$ and $Z = G/H \times V^H$. By Lemma 3.3.10, $T_pM = \mathfrak{m} \times \mathfrak{m}^* \times V$, $Z = \mathfrak{m} \times 0 \times V^H$, and $T_p(G \cdot p) = \mathfrak{m} \times 0 \times 0$. Let $(x, \xi, v) := \Xi_f(p) \in \mathfrak{m} \times \mathfrak{m}^* \times V$. Then,

$$df_p(y, \eta, w) = \eta(x) - \xi(y) + \omega_{\mathbb{C}}(v, w)$$

for all $(y, \eta, w) \in \mathfrak{m} \times \mathfrak{m}^* \times V$. Since f is G-invariant, we have $df_p(\mathfrak{m} \times 0 \times 0) = 0$, so $\xi = 0$. Also, G-equivariance implies that for all $w \in V$ and $h \in H$ we have $df_p(0, 0, h \cdot w) = df_p(0, 0, w)$, so $\omega_{\mathbb{C}}(v, h \cdot w) = \omega_{\mathbb{C}}(v, w)$. Since $\omega_{\mathbb{C}}$ is H-invariant, this implies $\omega_{\mathbb{C}}(h^{-1}v - v, w) = 0$ for all $w \in V$ and $h \in H$, so $v \in V^H$. Thus, $\Xi_f(p) = (y, 0, v) \in \mathfrak{m} \times 0 \times V^H = T_pZ$. □

Lemma 3.4.16. *For all open set $U \subseteq M/\!\!/\!\!/_{\mu} K$ and $f, g \in \mathcal{O}_{\mathsf{l}}(U)$, we have $\{f, g\} \in \mathcal{O}_{\mathsf{l}}(U)$.*

Proof. We identify $M/\!\!/\!\!/_{\mu} K$ with $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!\!/G$. Let $\Pi : \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} \to \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!\!/G$ be the quotient map. Then, $\{f, g\} \in \mathcal{O}_{\mathsf{l}}(U)$ if and only if the pullback $\Pi^*\{f, g\} : \Pi^{-1}(U) \to \mathbb{C}$ is holomorphic. This is a local statement, so we may assume that $\Pi^{-1}(U) = \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} \cap U'$ for some G-invariant open set $U' \subseteq M^{\mu_{\mathbb{R}}\text{-ss}}$ such that $\Pi^* f$ and $\Pi^* g$ extend to holomorphic G-invariant functions $\hat{f}, \hat{g} : U' \to \mathbb{C}$. Then, it suffices to show that $\Pi^*\{f, g\} = \{\hat{f}, \hat{g}\}|_{\Pi^{-1}(U)}$. Since $\hat{f}$, $\hat{g}$ and $\omega_{\mathbb{C}}$ are G-invariant, so is $\{\hat{f}, \hat{g}\}$. Thus, it suffices to show that $\Pi^*\{f, g\}(p) = \{\hat{f}, \hat{g}\}(p)$ for every polystable point $p \in \Pi^{-1}(U) \cap \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}}$. We have $p \in Z$ for some Z as above. Let $S = \Pi(Z)$, $\pi = \Pi|_Z : Z \to S$ and $i : Z \to M$, as before. Then, we have $d\pi(\Xi_{\hat{f}}(p)) = \Xi_f(\pi(p))$, where Ξ_f is the Hamiltonian vector field of f on $U \cap S$, since for all $v \in T_pZ$,

$$\begin{aligned}(\omega_S)_{\mathbb{C}}(d\pi(\Xi_{\hat{f}}(p)), d\pi(v)) &= \omega_{\mathbb{C}}(\Xi_{\hat{f}}(p), v) = d\hat{f}_p(v) = df_{\pi(p)}(d\pi(v)) \\ &= (\omega_S)_{\mathbb{C}}(\Xi_f(\pi(p)), d\pi(v)).\end{aligned}$$

Thus,

$$\begin{aligned}\{f, g\}(\Pi(p)) &:= \eta(\Xi_f(\pi(p)), \Xi_f(\pi(p))) = \eta(d\pi(\Xi_{\hat{f}}(p)), d\pi(\Xi_{\hat{f}}(p))) \\ &= \omega_{\mathbb{C}}(\Xi_{\hat{f}}(p), \Xi_{\hat{g}}(p)) = \{\hat{f}, \hat{g}\}(p).\end{aligned}$$

So $\Pi^*\{f, g\} = \{\hat{f}, \hat{g}\}|_{\Pi^{-1}(U)}$ and hence $\{f, g\} \in \mathcal{O}_{\mathsf{l}}(U)$. □

It is clear from its construction that the Poisson bracket is uniquely determined by the property that the inclusions of the strata are Poisson maps. Thus, we have:

Proposition 3.4.17. *There is a unique Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ such that for every $S \subseteq M /\!/\!/_\mu K$ in the orbit-type partition, the inclusion $S \hookrightarrow M /\!/\!/_\mu K$ is a Poisson map with respect to $(\omega_S)_\mathbb{C} = \omega_{\mathsf{J}_S} + i\omega_{\mathsf{K}_S}$. Thus, $(M /\!/\!/_\mu K, \mathcal{O}_{\mathsf{I}})$ is a stratified complex-symplectic space.* □

3.4.9 Local model

Let H be a complex reductive group and $H \to \mathrm{Sp}(V, \omega_\mathbb{C})$ a complex-symplectic representation. Then, as explained in §3.4.2, we can view the affine GIT quotient $V_0 := \Phi_V^{-1}(0) /\!/ H$ as a hyperkähler quotient. Hence, if $\mathcal{O}_{V_0}$ denotes the underlying complex-analytic structure of V_0, then $(V_0, \mathcal{O}_{V_0})$ together with the H-orbit-type partition is a stratified complex-symplectic space. In particular, there is a canonical Poisson bracket on $\mathcal{O}_{V_0}$ (which does not depend on the choice of quaternionic structure, as can be seen from its construction). Moreover, $\mathcal{O}_{V_0}(V_0)$ contains $\mathbb{C}[\Phi_V^{-1}(0)]^G$ and it is easy to see from the proof of Proposition 3.4.16 that this Poisson bracket restricts to the usual one on $\mathbb{C}[\Phi_V^{-1}(0)]^G$. Recall from Proposition 3.4.3 that $\Phi_V^{-1}(0) /\!/ H$ provides a local model for the complex-analytic structure of $M /\!/\!/_\mu K$. Here we show that $\Phi_V^{-1}(0) /\!/ H$ is also a local model for the Poisson structure.

Proposition 3.4.18. *Let $q \in M /\!/\!/_\mu K$. Let $p \in \mu^{-1}(0)$ be a point above q, let $H := G_p$, let V be the complex-symplectic slice at p, and let $\Phi_V : V \to \mathfrak{h}^*$ the canonical complex-moment map. Then, H is a complex reductive group and q has a neighbourhood which is isomorphic as a stratified complex-symplectic space to a neighbourhood of $[0]$ in $\Phi_V^{-1}(0) /\!/ H$.*

Proof. By Proposition 3.4.3, all it remains to check is that the biholomorphism $U \to U'$ respects the Poisson brackets. Let η be the complex-symplectic form on the local model $E = G \times_H (\mathfrak{h}^\circ \times V)$ and $\kappa : E \to \mathfrak{g}^*$ the complex moment map. Since the hyperkähler local normal form for (M, K, μ) is an isomorphism of complex-symplectic manifolds, we only need to show that the isomorphism $\kappa^{-1}(0) /\!/ G = \Phi_V^{-1}(0) /\!/ H$ of affine varieties (see proof of Proposition 3.4.3) respects the Poisson brackets. This follows from the fact that V is a complex-symplectic submanifold of E via the embedding $\iota : V \hookrightarrow G \times_H (\mathfrak{h}^\circ \times V)$, $v \mapsto [1, 0, v]$ and that the isomorphism in question descends from this map. □

3.4.10 The abelian case

Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold, where K is a compact abelian Lie group (not necessarily connected) whose action on M is I-integrable. The goal of this section is to use the local model in §3.4.9 and the recent paper [24] to show that $(M/\!\!/\!\!/_\mu K, \mathcal{O}_{\mathsf{l}})$ is a normal complex-analytic space (Corollary 3.1.6). Let us first recall the result of [24] which is relevant to this discussion:

Proposition 3.4.19 (Bulois [24, Proposition 3.4])**.** *Let W be a finite-dimensional complex representation of a connected complex torus $T \cong (\mathbb{C}^*)^r$. Endow $W \oplus W^*$ with the canonical symplectic form and consider the moment map*

$$\Phi : W \oplus W^* \longrightarrow \mathfrak{t}^*, \quad \Phi(w,\xi)(x) = \xi(x \cdot w) \quad (w \in W, \xi \in W^*, x \in \mathfrak{t} := \mathrm{Lie}(T))$$

for the diagonal action of T on $W \oplus W^*$. Then, the affine GIT quotient

$$\Phi^{-1}(0)/\!\!/ T = \mathrm{Spec}(\mathbb{C}[\Phi^{-1}(0)]^T)$$

is normal. □

It suffices to slightly extend this result:

Corollary 3.4.20. *Let $T \to \mathrm{Sp}(V,\omega)$ be a complex-symplectic representation of a* (*not necessarily connected*) *abelian complex reductive group T. Then, $\Phi^{-1}(0)/\!\!/ T$ is normal, where Φ is the canonical moment map given by $\Phi(v)(x) = \frac{1}{2}\omega(x \cdot v, v)$.*

Proof. Since irreducible representations of an abelian complex reductive group are 1-dimensional, Proposition 3.3.11 shows that there is a representation W of T such that $V = W \oplus W^*$ with the canonical symplectic form as in Proposition 3.4.19. Now, since T is abelian, we have $T = T^\circ \times A$ for some connected torus $T^\circ \cong (\mathbb{C}^*)^r$ and some finite abelian group A. Moreover, $\mathrm{Lie}(T) = \mathrm{Lie}(T^\circ)$ and the moment map Φ for the T-action is equal to the moment map for the T°-action. Thus, $\Phi^{-1}(0)/\!\!/ T^\circ$ is normal by Bulois' result, and hence so is $\Phi^{-1}(0)/\!\!/ T = (\Phi^{-1}(0)/\!\!/ T^\circ)/\!\!/ A$. □

By Theorem 3.1.5(iv) and Corollary 3.4.20, $(M/\!\!/\!\!/_\mu K, \mathcal{O}_{\mathsf{l}})$ is normal.

Chapter 4

A Kempf–Ness type theorem

4.1 Introduction

4.1.1 Overview

Let G be a complex reductive group acting on a smooth complex-algebraic variety M and let $\mathcal{L}$ be a linearisation of this action. Suppose that M has a Kähler structure that is preserved by a maximal compact subgroup $K \subseteq G$ and that the K-action is Hamiltonian with moment map $\mu : M \to \mathfrak{k}^*$. Then, both the GIT quotient $M /\!\!/_{\mathcal{L}} G$ and the Kähler quotient $M /\!\!/_{\mu} K$ exist. Since the analytification of a GIT quotient is an analytic Hilbert quotient, if $M^{\mu\text{-ss}} = M^{\mathcal{L}\text{-ss}}$, then $M /\!\!/_{\mu} K \cong M /\!\!/_{\mathcal{L}} G$ as complex-analytic and partitioned spaces (by Theorem 2.4.5 and Theorem 2.4.10). Such identification is usually called a "Kempf–Ness type theorem" (after [85]) and various instances are known in the literature. They provide a beautiful link between symplectic and algebraic geometry and have many applications. For example, the topology of the GIT quotient $M /\!\!/_{\mathcal{L}} G$ can be better understood by passing to the symplectic quotient $M /\!\!/_{\mu} K$; see, e.g., Kirwan [88] and Schwarz [129]. In this thesis, we use Kempf–Ness type theorems in the other direction; namely, we aim to

describe some specific symplectic reductions using algebraic geometry.

Thus, it is essential to allow the symplectic structure and the moment map to be as general as possible. The goal of this chapter is to state and prove a version of the Kempf–Ness Theorem sufficiently general for our purpose (Theorem 4.1.2). We will apply it in the next chapter to describe hyperkähler quotients of certain moduli spaces of solutions to Nahm's equations. This chapter is based on the first part of the author's paper [108].

4.1.2 Statement of result

In its simplest form, the Kempf–Ness Theorem says that if $M \subseteq \mathbb{C}^n$ is a smooth complex affine variety endowed with the standard symplectic structure and K is a closed subgroup of the unitary group $\mathrm{U}(n)$ preserving M, then there is a moment map μ for the action of K on M (the restriction of (2.3.1)) such that the symplectic reduction $M /\!\!/_{\mu} K$ is homeomorphic to the affine GIT quotient $M /\!\!/ G \coloneqq \operatorname{Spec} \mathbb{C}[M]^G$ (see, e.g., [129, Corollary 4.7]).

This theorem admits many generalisations and variants; for instance, there are versions for projective manifolds [117, §2] [88, §8] [111, §8.2] [61, §4] [133, §2.2]. Another important version—which is closer to the spirit of this thesis—is when M is an affine variety as above, but we shift the moment map. More precisely, if $\chi : K \to S^1$ is a character, then $\xi \coloneqq i\, d\chi \in \mathfrak{k}^*$ is central (i.e. fixed by the coadjoint action), so we can consider the symplectic reduction $\mu^{-1}(\xi)/K$. Then, King [86, §6] (see also Hoskins [82]) showed that $\mu^{-1}(\xi)/K$ is homeomorphic to the twisted GIT quotient $M /\!\!/_{\chi} G$, i.e. the GIT quotient of M by G with respect to the trivial line bundle $M \times \mathbb{C}$ with the G-action $g \cdot (p, z) = (g \cdot p, \chi(g) z)$. In other words,

$$\mu^{-1}(\xi)/K \cong M /\!\!/_{\chi} G \coloneqq \operatorname{Proj} \bigoplus_{n=0}^{\infty} \mathbb{C}[M]^{G,\chi^n},$$

where $\mathbb{C}[M]^{G,\chi^n}$ is the set of $u \in \mathbb{C}[M]$ such that $u(g \cdot p) = \chi(g)^n u(p)$ for all $p \in M$ and $g \in G$. This is a good quotient for the action of G on the set of semistable points

$$M^{\chi\text{-ss}} \coloneqq \{p \in M : \exists n \geq 1 \text{ and } u \in \mathbb{C}[M]^{G,\chi^n} \text{ such that } u(p) \neq 0\}. \tag{4.1.1}$$

There is another useful generalisation, which is to consider symplectic reduction with respect to a symplectic form on $M \subseteq \mathbb{C}^n$ which is not necessarily the standard one. More precisely, take a Kähler potential on M, i.e. a smooth function $f : M \to \mathbb{R}$ such that the 2-form $\omega \coloneqq 2i\partial\bar{\partial} f$ is symplectic. Then, if f is K-invariant, proper, and bounded below, there is still a moment map μ (not necessarily the same as above) such that $M /\!\!/_{\mu} K \cong M /\!\!/ G$ (see, e.g., [69, Lemma 6.1] or [106, Proposition 4.2]). The standard version can be recovered by taking $f = \frac{1}{2}\| \cdot \|^2$.

It is then natural to try to combine these two versions, i.e. to shift the moment map μ associated with a Kähler potential f on M by $\xi \coloneqq i\, d\chi$, where $\chi : K \to S^1$ is a character. However, this requires more care into the relationship between f and the algebraic structure of M. In general, $\mu^{-1}(\xi)/K$ can fail to be homeomorphic to $M /\!\!/_{\chi} G$, even if f is K-invariant, proper, and bounded below (see §4.6 for a counterexample). The goal of this chapter is to give a sufficient condition for this homeomorphism to hold. Namely, we show that it is enough to require that e^f

dominates polynomials on M, which is denoted by $\mathbb{C}[M] \subseteq o(e^f)$. Informally, this means that

$$\lim_{p\to\infty} \frac{u(p)}{e^{f(p)}} = 0, \quad \text{for all } u \in \mathbb{C}[M].$$

More precisely:

Definition 4.1.1. Let M be a topological space and $u, v : M \to \mathbb{R}$ two functions. We say that v **dominates** u, denoted by $u \in o(v)$, if for all $c > 0$ there exists a compact set $C \subseteq M$ such that $|u(p)| \leq cv(p)$ for all $p \in M \setminus C$.

This definition generalises the familiar notion of "little-o" for functions $\mathbb{R} \to \mathbb{R}$. The inclusion $\mathbb{C}[M] \subseteq o(e^f)$ holds in the standard case since $f = \frac{1}{2}\|\cdot\|^2$. The main theorem of this chapter is the following.

Theorem 4.1.2. *Let M be a smooth complex affine variety, K a compact Lie group acting on M such that the action map $K \times M \to M$ is real algebraic, $f : M \to \mathbb{R}$ a K-invariant Kähler potential such that $\mathbb{C}[M] \subseteq o(e^f)$, and $\chi : K \to S^1$ a character. Then:*

(1) *The action of K on M extends to a complex-algebraic action of the complexification $G := K_{\mathbb{C}}$.*

(2) *Let $\mathfrak{k} := \mathrm{Lie}(K)$ and let I be the complex structure on M. Then, the map*

$$\mu : M \longrightarrow \mathfrak{k}^*, \quad \mu(p)(x) = df(\mathsf{I}x^\#_p), \quad (p \in M, x \in \mathfrak{k})$$

is a moment map for the action of K on M with respect to $\omega := 2i\partial\bar{\partial}f$.

(3) *Let $\xi := ai\,d\chi \in \mathfrak{k}^*$ for $a > 0$ and let $\mu_\xi := \mu - \xi$. Then, $M^{\mu_\xi\text{-ss}} = M^{\chi\text{-ss}}$. In particular, $\mu^{-1}(\xi) \subseteq M^{\chi\text{-ss}}$ and this inclusion descends to an isomorphism $M /\!\!/_{\mu_\xi} K \cong M /\!\!/_\chi G$ of complex-analytic and partitioned spaces.*

Parts (1) and (2) are well-known and do not require $\mathbb{C}[M] \subseteq o(e^f)$. The main content of this theorem is the equality $M^{\mu_\xi\text{-ss}} = M^{\chi\text{-ss}}$. The proof is largely inspired by King [86]. If $\chi = 1$, the condition $\mathbb{C}[M] \subseteq o(e^f)$ can be replaced by the weaker condition that f is proper and bounded below (see, e.g., [69, Lemma 6.1] or [106, Proposition 4.2]).

The rest of this chapter is devoted to the proof of Theorem 4.1.2. We first get a general form of the Kempf–Ness Theorem for complex-algebraic varieties with an integral Kähler form (Theorem 4.3.4). This is mainly an adaptation of the original work of Kempf–Ness [85] in the context of polarised Kähler manifolds as in Guillemin–Sternberg [61, §4] and using the Heinzner–Loose Theorem (Theorem 2.4.5) to omit the compactness assumption. We then apply it to affine varieties using ideas of King [86].

4.2 Moment maps from linearisations

As explained in the introduction, a Kempf–Ness type theorem is a statement of the form $M^{\mu\text{-ss}} = M^{\mathcal{L}\text{-ss}}$, i.e. *analytic semistability = algebraic semistability*. This brings naturally the question of how to get a moment map from a linearisation. There is a well-known process for this, which we review in this section.

Let M be a complex manifold and $\mathcal{L}$ a unitary line bundle on M, i.e. a holomorphic line bundle endowed with a hermitian fibre metric. Recall that $\mathcal{L}$ is called **positive** if its curvature $\Theta_{\mathcal{L}} \in \Omega^2(M; i\mathbb{R})$ has the property that the real $(1,1)$-form $\frac{i}{2\pi}\Theta_{\mathcal{L}}$ is Kähler. Since $\frac{i}{2\pi}\Theta_{\mathcal{L}}$ is a representative of the first Chern class of $\mathcal{L}$, a Kähler form ω on M is equal to $\frac{i}{2\pi}\Theta_{\mathcal{L}}$ for some $\mathcal{L}$ if and only if ω is **integral**, i.e. its cohomology class is in the image of the natural map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$. A **polarised Kähler manifold** is a triple $(M, \omega, \mathcal{L})$ where ω is a Kähler form and $\mathcal{L}$ is a positive line bundle such that $\omega = \frac{i}{2\pi\hbar}\Theta_{\mathcal{L}}$ for some $\hbar > 0$. The positive real number $\hbar$ will be fixed throughout this section.

Let $(M, \omega, \mathcal{L})$ be a polarised Kähler manifold and G a complex reductive group acting on $\mathcal{L}$ by bundle automorphisms (covering some action on M) in such a way that a maximal compact subgroup $K \subseteq G$ preserves the hermitian fibre metric. This implies that the Kähler form $\omega = \frac{i}{2\pi\hbar}\Theta_{\mathcal{L}}$ is K-invariant. Moreover, there is a canonical moment map for the action of K on M with respect to ω. This is well-known in the literature, but we will provide a proof for completeness and for fixing the notation; see, e.g., Guillemin–Sternberg [61, §3], Woodward [146, Proposition 3.2.9], Thomas [142, §4], Donaldson–Kronheimer [41, §6.5.1], or Sjamaar [133, §2.2].

Let $\mathcal{L}^*$ be the line bundle dual to $\mathcal{L}$. Then, $\mathcal{L}^*$ inherits a hermitian metric from $\mathcal{L}$ and also an action of G. Let $\dot{\mathcal{L}}^*$ be the complement of the zero-section in $\mathcal{L}^*$ and define $\hat{\mu} : \dot{\mathcal{L}}^* \to \mathfrak{k}^*$ by

$$\hat{\mu}(v)(x) = \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} \log \|\exp(itx) \cdot v\|^2, \quad (v \in \dot{\mathcal{L}}^*, x \in \mathfrak{k}). \tag{4.2.1}$$

Then, $\hat{\mu}$ is $\mathbb{C}^*$-invariant and hence descends to a smooth map $\mu : M \to \mathfrak{k}^*$.

Proposition 4.2.1. *The map μ is a moment map for the action of K on M with respect to ω.*

Proof. Let $\rho : \dot{\mathcal{L}}^* \to M$ be the projection and let

$$\varphi : \dot{\mathcal{L}}^* \longrightarrow \mathbb{R}, \quad \varphi(v) = \log \|v\|^2.$$

The curvature of $\mathcal{L}^*$ is $-\Theta_{\mathcal{L}}$, so (by definition of the canonical connection on a unitary line bundle) we have $\rho^*(-\Theta_{\mathcal{L}}) = \bar{\partial}\partial\varphi$ and hence

$$\bar{\partial}\partial\varphi = 2\pi\hbar i \rho^*\omega. \tag{4.2.2}$$

Let $x \in \mathfrak{k}$. We want to show that $d\mu^x = x^\# \lrcorner\, \omega$, where $\mu^x : M \to \mathbb{R} : p \mapsto \mu(p)(x)$. By definition of μ, we have $(\mu^x \circ \rho)(v) = \frac{1}{4\pi\hbar} d\varphi(\mathsf{I}^\#_v)$. Also, since φ is K-invariant, we have $d\varphi(x^\#) = 0$, so

$$\partial\varphi(x^\#) = -\frac{i}{2} d\varphi(\mathsf{I}x^\#) = -2\pi\hbar i \mu^x \circ \rho. \tag{4.2.3}$$

Now, since K acts by biholomorphisms on $\mathcal{L}^*$, the Lie derivative $\mathscr{L}_{x^\#}$ commutes with ∂ and hence $\mathscr{L}_{x^\#}\partial\varphi = \partial(\mathscr{L}_{x^\#}\varphi) = 0$. Thus, using Cartan's magic formula, (4.2.2), and (4.2.3), we get

$$\begin{aligned} 0 &= d(x^\# \lrcorner\, \partial\varphi) + x^\# \lrcorner\, d\partial\varphi \\ &= d(\partial\varphi(x^\#)) + x^\# \lrcorner\, \bar\partial\partial\varphi \\ &= d(-2\pi\hbar i\mu^x \circ \rho) + x^\# \lrcorner\, 2\pi\hbar i \rho^*\omega \\ &= 2\pi\hbar i \rho^* \left(-d\mu^x + x^\# \lrcorner\, \omega\right). \end{aligned}$$

Hence, $d\mu^x = x^\# \lrcorner\, \omega$.

To show equivariance, let $p \in M$ and take a point $\hat{p} \in \dot{\mathcal{L}}^*$ above p. We want to show that $\mu(k \cdot p)(x) = \mu(p)(\mathrm{Ad}_{k^{-1}} x)$ for all $k \in K$ and $x \in \mathfrak{k}$, or equivalently,

$$\frac{d}{dt}\Big|_{t=0} \log \| \exp(itx) \cdot k \cdot \hat{p}\|^2 = \frac{d}{dt}\Big|_{t=0} \log \| \exp(it\,\mathrm{Ad}_{k^{-1}} x) \cdot \hat{p}\|^2.$$

This is clear since $\exp(it\,\mathrm{Ad}_{k^{-1}} x) = k^{-1}\exp(itx)k$ and the fibre metric is K-invariant.

□

Therefore, (M, K, μ) is an integrable Hamiltonian Kähler manifold. Let us fix a terminology for the kind of Hamiltonian manifolds obtained in this way:

Definition 4.2.2. A **polarised Hamiltonian Kähler manifold** is a tuple $(M, \omega, K, \mathcal{L}, \mu)$ where $(M, \omega, \mathcal{L})$ is a polarised Kähler manifold, K is a compact Lie group acting on $\mathcal{L}$ by preserving the unitary structure and extending to an action of $G = K_{\mathbb{C}}$, and μ is the associated moment map as in Proposition 4.2.1.

Given a character $\chi : G \to \mathbb{C}^*$, we can **twist** the action of G on $\mathcal{L}$ by defining a new action $(g, v) \mapsto \chi(g)\, g \cdot v$ on $\mathcal{L}$. We denote by $\mathcal{L}_\chi$ the line bundle $\mathcal{L}$ with this twisted G-action. The action of K on $\mathcal{L}_\chi$ still preserves the fibre metric, so the discussion above implies that we have a new moment map μ_χ for the action of K on M. The next proposition identifies this map.

Proposition 4.2.3. *Let $(M, \omega, K, \mathcal{L}, \mu)$ be a polarised Hamiltonian Kähler manifold and $\chi : G \to \mathbb{C}^*$ a character. Then, the moment map associated with $\mathcal{L}_\chi$ is the shift $\mu_\xi := \mu - \xi$ where $\xi := \frac{i}{2\pi\hbar} d\chi \in \mathfrak{k}^*$. Thus, $(M, \omega, K, \mathcal{L}_\chi, \mu_\xi)$ is a polarised Hamiltonian Kähler manifold.*

Remark 4.2.4. Since $\chi(K)$ is a compact subgroup of $\mathbb{C}^*$ we have $\chi(K) \subseteq S^1$. Hence, $d\chi(\mathfrak{k}) \subseteq i\mathbb{R}$ and $\frac{i}{2\pi\hbar} d\chi \in \mathfrak{k}^*$. Moreover, $\frac{i}{2\pi\hbar} d\chi$ is central.

Proof. Let $\hat{\mu}_\chi : \dot{\mathcal{L}}_\chi^* \to \mathfrak{k}^*$ be the lift of the moment map associated with $\mathcal{L}_\chi$ as in (4.2.1). The new action of G on $\mathcal{L}^*$ is $(g, v) \mapsto \chi(g)^{-1} g \cdot v$, so for all $x \in \mathfrak{k}$ and $v \in \dot{\mathcal{L}}^*$, we have

$$\begin{aligned}
\hat{\mu}_\chi(v)(x) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} \log \|\chi(\exp(itx))^{-1} \exp(itx) \cdot v\|^2 \\
&= \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} \left(-\log |\chi(\exp(itx))|^2 + \log \|\exp(itx) \cdot v\|^2\right) \\
&= \hat{\mu}(v)(x) - \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} |\chi(\exp(itx))|^2 \\
&= \hat{\mu}(v)(x) - \frac{1}{4\pi\hbar}(d\chi(ix) + \overline{d\chi(ix)}) \\
&= \hat{\mu}(v)(x) - \frac{i}{2\pi\hbar} d\chi(x).
\end{aligned}$$

Hence $\mu_\chi = \mu - \xi$. □

4.3 The general Kempf–Ness Theorem

In this section, we state and prove a general form of the Kempf–Ness Theorem for polarised Hamiltonian Kähler manifolds. The following definition will be one of the assumptions of the theorem:

Definition 4.3.1. Let M be a complex-algebraic variety with an algebraic action of a complex reductive group G and let $\mathcal{L}$ be a linearisation of this action. We say that $(M, G, \mathcal{L})$ satisfies the **geometric criterion** if a point $p \in M$ is $\mathcal{L}$-semistable if and only if for any non-zero lift $\hat{p}$ of p in $\mathcal{L}^*$, the closure of the orbit $G \cdot \hat{p} \subseteq \mathcal{L}^*$ is disjoint from the zero-section.

This definition is motivated by the following two examples:

Example 4.3.2. Let $M \subseteq \mathbb{CP}^n$ be a projective variety with a G-action coming from a representation $G \to \mathrm{GL}(n+1, \mathbb{C})$ and let $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^n}(1)|_M$ with the natural G-action. It is a standard result that $(M, G, \mathcal{L})$ satisfies the geometric criterion [111, Proposition 2.2].

Example 4.3.3. Let M be a complex affine variety with an algebraic action of G and let $\mathcal{L}_\chi$ be the trivial line bundle on M with the G-action twisted by a character χ. Then, as observed by King [86, Lemma 2.2], $(M, G, \mathcal{L}_\chi)$ also satisfies the geometric criterion.

We will only use the affine case, but it is interesting to know that these two examples are special cases of a more general result for projective-over-affine varieties (see, e.g., [62, §4] or [126, §1]).

Our goal is to prove the following Kempf–Ness type theorem.

Theorem 4.3.4. *Let $(M, \omega, K, \mathcal{L}, \mu)$ be a polarised Hamiltonian Kähler manifold such that $(M, G, \mathcal{L})$ is complex-algebraic and satisfies the geometric criterion. Suppose also that the norm-squared function $\mathcal{L}^* \to \mathbb{R}$, $v \mapsto \|v\|^2$ is proper on all closed G-orbits that are disjoint from the zero-section. Then, $M^{\mathcal{L}\text{-ss}} = M^{\mu\text{-ss}}$.*

This result might be known to some experts, but we have not found a proof nor a statement at this level of generality. Special cases are found in [117, 88, 61, 133, 86, 3]; see also [41, 89, 142, 146] for good expositions. For completeness, we provide a full proof of Theorem 4.3.4. The proof resembles the original one of Kempf–Ness [85], but in a more abstract setting. We could have obtained the affine case (Theorem 4.1.2) more directly, but this more general result might be of independent interest, and the proof does not require much more effort.

The rest of this subsection is devoted to the proof of Theorem 4.3.4. For $v \in \dot{\mathcal{L}}^*$, define the *Kempf–Ness function*

$$F_v : G \longrightarrow \mathbb{R}, \quad F_v(g) = \frac{1}{4\pi\hbar} \log \|g \cdot v\|^2.$$

Let $p \in M$ and fix a non-zero lift $\hat{p}$ of p in $\mathcal{L}^*$. Our first goal is:

Proposition 4.3.5. *The following are equivalent:*

(1) *$p \in M^{\mu\text{-ps}}$;*

(2) *$F_{\hat{p}}$ has a critical point;*

(3) *$F_{\hat{p}}$ has a global minimum;*

(4) *$G \cdot \hat{p}$ is closed in $\mathcal{L}^*$.*

We will show

$$(4) \implies (1) \implies (2) \implies (3) \implies (4).$$

Lemma 4.3.6. *For all $x \in \mathfrak{k}$, we have*

$$\frac{d}{dt} F_{\hat{p}}(e^{itx}) = \mu(e^{itx} \cdot p)(x) \tag{4.3.1}$$

$$\frac{d^2}{dt^2} F_{\hat{p}}(e^{itx}) = \|x^{\#}_{e^{itx} \cdot p}\|^2, \tag{4.3.2}$$

where the latter is the norm of $x^{\#}_{e^{itx} \cdot p}$ with respect to the Kähler metric on M. In particular,

$$(dF_{\hat{p}})_1(ix) = \mu(p)(x). \tag{4.3.3}$$

Proof. This is a simple computation. The first identity follows from the definition of μ and the fact that $F_{\hat{p}}(e^{i(t_0+t)x}) = F_{e^{it_0x}\cdot\hat{p}}(e^{itx})$. To get the second identity, note that

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=t_0} F_{\hat{p}}(e^{itx}) &= \frac{d}{dt}\Big|_{t=t_0} \mu(e^{itx}\cdot p)(x) = \frac{d}{dt}\Big|_{t=0} \mu^x(e^{itx}\cdot e^{it_0x}\cdot p)\\ &= d\mu^x(\mathsf{I}x^{\#}_{e^{it_0x}\cdot p}) = \omega(x^{\#}_{e^{it_0x}\cdot p}, \mathsf{I}x^{\#}_{e^{it_0x}\cdot p})\\ &= \|x^{\#}_{e^{it_0x}\cdot p}\|^2.\end{aligned}$$

□

Lemma 4.3.7. (4) ⇒ (1), *i.e. if* $G\cdot\hat{p}$ *is closed in* $\mathcal{L}^*$ *then* $p \in M^{\mu\text{-ps}}$.

Proof. Since $G\cdot\hat{p}$ is closed, $\|\cdot\|^2$ is proper on $G\cdot\hat{p}$ by assumption. Hence, $\|G\cdot\hat{p}\|^2$ is closed in $\mathbb{R}$, so it attains a minimum $\|g\cdot\hat{p}\|^2$ for some $g \in G$. Then, g is a minimum of $F_{\hat{p}}$ so $(dF_{\hat{p}})_g = 0$. Note that $F_{g\cdot\hat{p}} = F_{\hat{p}} \circ R_g$ where $R_g : G \to G$ is right multiplication by g, so $(dF_{g\cdot\hat{p}})_1 = (dF_{\hat{p}})_g \circ (dR_g)_1 = 0$. By (4.3.3), this implies $\mu(g\cdot p) = 0$ and hence $p \in M^{\mu\text{-ps}}$ by (2.4.1). □

Lemma 4.3.8. (1) ⇒ (2), *i.e. if* $p \in M^{\mu\text{-ps}}$ *then* $F_{\hat{p}}$ *has a critical point.*

Proof. By (2.4.1), if $p \in M^{\mu\text{-ps}}$ then there exists $g \in G$ such that $\mu(g\cdot p) = 0$. Then, $(dF_{g\cdot\hat{p}})_1(i\mathfrak{k}) = 0$ by (4.3.3). Also, $(dF_{g\cdot\hat{p}})_1(\mathfrak{k}) = 0$ since $F_{g\cdot\hat{p}}$ is K-invariant, so $(dF_{g\cdot\hat{p}})_1 = 0$. Since $F_{g\cdot\hat{p}} = F_{\hat{p}} \circ R_g$, we get $(dF_{\hat{p}})_g = 0$. □

Note that the group G_p acts naturally on the fibre $\mathcal{L}^*_p$. Since $\mathcal{L}^*_p$ is one-dimensional, this action must be multiplication by a non-zero complex number, i.e. we have a character

$$\lambda_p : G_p \longrightarrow \mathbb{C}^*$$

such that $g\cdot v = \lambda_p(g)v$ for all $v \in \mathcal{L}^*_p$ and $g \in G_p$. Let $\mathfrak{k}_p := \mathrm{Lie}(K_p)$ and $\mathfrak{g}_p := \mathrm{Lie}(G_p)$. Then, $i\mathfrak{k}_p \subseteq \mathfrak{g}_p$ so from the polar decomposition we get that $(K_p)_{\mathbb{C}} \subseteq G_p$.

Lemma 4.3.9. *If* $(dF_{\hat{p}})_1 = 0$ *then* $|\lambda_p(h)| = 1$ *for all* $h \in (K_p)_{\mathbb{C}}$.

Proof. Let $h \in (K_p)_{\mathbb{C}}$. We have $\|h\cdot\hat{p}\| = \|\lambda_p(h)\hat{p}\| = |\lambda_p(h)|\|\hat{p}\|$ so it suffices to show that $\|h\cdot\hat{p}\| = \|\hat{p}\|$. Write $h = ke^{ix}$ for $k \in K_p$ and $x \in \mathfrak{k}_p$. Then, $\|h\cdot\hat{p}\| = \|e^{ix}\cdot\hat{p}\|$. Thus, by (4.3.1) and (4.3.3),

$$\frac{d}{dt}\frac{1}{4\pi\hbar}\log\|e^{itx}\cdot\hat{p}\|^2 = \mu(e^{itx}\cdot p)(x) = \mu(p)(x) = (dF_{\hat{p}})_1(ix) = 0,$$

so $\|e^{itx}\cdot\hat{p}\|^2$ is independent of t. In particular, $\|h\cdot\hat{p}\| = \|e^{ix}\cdot\hat{p}\| = \|\hat{p}\|$. □

The polar decomposition $K \times \mathfrak{k} \cong G$ has a generalisation due to Mostow [109, 110] which we will need here (see also [74, Theorem VI.1.4], [73, Corollary 9.5] or [68, Theorem 2.4.9]).

Proposition 4.3.10 (Mostow Decomposition)**.** *Let K be a compact Lie group and $H \subseteq K$ a closed subgroup. Let $\mathfrak{k} := \mathrm{Lie}(K)$, $\mathfrak{h} := \mathrm{Lie}(H)$, and let $\mathfrak{h}^\perp$ be the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{k}$ with respect to a K-invariant inner-product. Let $K \times_H \mathfrak{h}^\perp$ be the quotient of $K \times \mathfrak{h}^\perp$ by the H-action $h \cdot (k, x) = (kh^{-1}, \mathrm{Ad}_h x)$. Then, the map*

$$K \times_H \mathfrak{h}^\perp \longrightarrow K_{\mathbb{C}}/H_{\mathbb{C}}, \quad [k, x] \longmapsto ke^{ix}H_{\mathbb{C}}$$

is a diffeomorphism. □

Lemma 4.3.11. (2) $\Rightarrow$ (3), *i.e. if $F_{\hat{p}}$ has a critical point, then it has a global minimum.*

Proof. Suppose that $(dF_{\hat{p}})_g = 0$. By using that $F_{g\cdot\hat{p}} = F_{\hat{p}} \circ R_g$ and replacing $\hat{p}$ by $g \cdot \hat{p}$, we may assume without loss of generality that $(dF_{\hat{p}})_1 = 0$. For $x \in \mathfrak{k}$, we have $\|x_p^\#\|^2 = 0$ if and only if $x_p^\# = 0$ if and only if $x \in \mathfrak{k}_p =: \mathrm{Lie}(K_p)$. Thus, (4.3.2) implies that for all $x \in \mathfrak{k} \setminus \mathfrak{k}_p$ the function

$$F_{\hat{p},x} : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longrightarrow F_{\hat{p}}(e^{itx})$$

is convex on $\mathbb{R}$ and strictly convex near $t = 0$. Moreover, since $(dF_{\hat{p}})_1 = 0$, we get that $t = 0$ is a global minimum of $F_{\hat{p},x}$. Now, by Mostow decomposition (Proposition 4.3.10) any $g \in G$ can be decomposed as $g = ke^{ix}h$ for $k \in K$, $x \in \mathfrak{k}_p^\perp$ and $h \in (K_p)_{\mathbb{C}} \subseteq G_p$. By Lemma 4.3.9, we have $\|ke^{ix}h \cdot \hat{p}\| = \|e^{ix} \cdot \hat{p}\|$, so $F_{\hat{p}}(g) = F_{\hat{p}}(e^{ix}) \geq F_{\hat{p}}(1)$, and hence 1 is a global minimum of $F_{\hat{p}}$. □

Lemma 4.3.12. *Let V be a finite-dimensional real vector space and $f : V \to \mathbb{R}$ a C^2 function. Suppose that $f(0)$ is a minimum of f and that for all non-zero $v \in V$, the function $f_v : \mathbb{R} \to \mathbb{R}$, $f_v(t) := f(tv)$ satisfies $f_v''(t) \geq 0$ for all $t \in \mathbb{R}$ and $f_v''(0) > 0$. Then, for any norm $\|\cdot\|$ on V, there are constants $c_0, c_1 > 0$ such that $\|v\| \leq c_0 + c_1 f(v)$ for all $v \in V$. In particular, f is proper.*

Proof. The function $F : V \times \mathbb{R} \to \mathbb{R}$, $F(v, t) = f_v''(t)$ is continuous, $F(v, 0) > 0$ for all non-zero $v \in V$, and $S := \{v \in V : \|v\| = 1\}$ is compact, so there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$f_v''(t) \geq \varepsilon, \quad \text{for all } t \in [0, \delta] \text{ and } v \in S.$$

Since $f(0)$ is a minimum, $f_v'(0) = 0$ and hence for all $t \geq \delta$ and $v \in S$ we have

$$f_v'(t) = \int_0^t f_v''(s)ds = \int_0^\delta f_v''(s)ds + \int_\delta^t f_v''(s)ds \geq \int_0^\delta \varepsilon\, ds = \varepsilon\delta$$

so

$$f(tv) = f(\delta v) + \int_\delta^t f_v'(s)ds \geq f(\delta v) + \int_\delta^t \varepsilon\delta\, ds = f(\delta v) + (t - \delta)\varepsilon\delta.$$

Thus, for all $w \in V$ with $\|w\| \geq \delta$ we have $f(w) \geq f(\delta w/\|w\|) + (\|w\| - \delta)\varepsilon\delta$, or

$$\varepsilon\delta\|w\| \leq \varepsilon\delta^2 + f(w) - f(\delta w/\|w\|).$$

Let $m = \sup\{|f(v)| : \|v\| \leq \delta\}$. Then, $-f(\delta w/\|w\|) \leq m$, so for all $w \in V$ such that $\|w\| \geq \delta$, we have

$$\varepsilon\delta\|w\| \leq \varepsilon\delta^2 + m + f(w).$$

If $\|w\| \leq \delta$ then this is also true since $|f(w)| \leq m$ so $m + f(w) \geq 0$ and hence

$$\varepsilon\delta\|w\| \leq \varepsilon\delta^2 \leq \varepsilon\delta^2 + m + f(w).$$

Thus,

$$\|v\| \leq \delta + \frac{m}{\varepsilon\delta} + \frac{1}{\varepsilon\delta} f(v), \quad \text{for all } v \in V,$$

and hence the first part of the lemma holds with $c_0 = \delta + \frac{m}{\varepsilon\delta}$ and $c_1 = \frac{1}{\varepsilon\delta}$. To show that f is proper, let $C \subseteq \mathbb{R}$ be compact. It suffices to show that $f^{-1}(C)$ is bounded. But $C \subseteq [a, b]$ for some $a, b \in \mathbb{R}$, so if $v \in f^{-1}(C)$ then $f(v) \leq b$ and hence $\|v\| \leq c_0 + c_1 b$. □

Lemma 4.3.13. (3) $\Rightarrow$ (4), *i.e. if $F_{\hat{p}}$ has a global minimum, then $G \cdot \hat{p}$ is closed in $\mathcal{L}^*$.*

Proof. Let $\varphi : \dot{\mathcal{L}}^* \to \mathbb{R}$ be defined by $\varphi(v) = \frac{1}{4\pi\hbar} \log \|v\|^2 = F_v(1)$. It suffices to show that the restriction of φ to $G \cdot \hat{p}$ is proper (in general, if $\varphi : x \to \mathbb{R}$ is a continuous function on a metrisable topological space X which is proper on a subset $A \subseteq X$, then A is closed). Without loss of generality, $F_{\hat{p}}$ has a global minimum at 1. Define

$$f : \mathfrak{k}_p^\perp \longrightarrow \mathbb{R}, \quad f(x) = F_{\hat{p}}(e^{ix}).$$

Then, f attains a global minimum at 0. By Lemma 4.3.12 and Lemma 4.3.6, f is proper. By Mostow's decomposition and Lemma 4.3.9, the map

$$\alpha : S^1 \times K \times \mathfrak{k}_p^\perp \longrightarrow G \cdot \hat{p}, \quad (z, k, x) \longmapsto zke^{ix} \cdot \hat{p}$$

is surjective. Since S^1 and K are compact and f is proper, the function $\psi : S^1 \times K \times \mathfrak{k}_p^\perp \to \mathbb{R}$, $\psi(z, k, x) = f(x)$ is also proper. Note that $\varphi \circ \alpha = \psi$. Since ψ is proper and α is surjective, this implies that φ is proper on $G \cdot \hat{p}$. Indeed, for all $C \subseteq \mathbb{R}$, the surjectivity of α implies that $\varphi^{-1}(C) \cap G \cdot \hat{p} = \alpha(\psi^{-1}(C))$. □

This concludes the proof of Proposition 4.3.5. In particular, if $p \in \mu^{-1}(0)$ then $G \cdot \hat{p}$ is closed in $\mathcal{L}^*$ and hence, by the geometric criterion, $p \in M^{\mathcal{L}\text{-ss}}$. Thus, the following general fact implies that $M^{\mu\text{-ss}} \subseteq M^{\mathcal{L}\text{-ss}}$.

Lemma 4.3.14. *Let* (M, K, μ) *be an integrable Hamiltonian Kähler manifold. Then,* $M^{\mu\text{-ss}}$ *is the smallest* G*-invariant open set of* M *containing* $\mu^{-1}(0)$*.*

Proof. Let $U \subseteq M$ be G-invariant, open, and containing $\mu^{-1}(0)$. We want to show that $M^{\mu\text{-ss}} \subseteq U$. Let $p \in M^{\mu\text{-ss}}$. Then, by definition, there exists $q \in \overline{G \cdot p} \cap \mu^{-1}(0)$, where $q = \lim_{n\to\infty} g_n \cdot p$ for some $g_n \in G$. But $q \in \mu^{-1}(0) \subseteq U$ and U is open, so there exists $N \geq 0$ such that $g_n \cdot p \in U$ for all $n \geq N$. Then, $p \in U$ since U is G-invariant. □

We can now conclude the proof of Theorem 4.3.4, i.e. that $M^{\mathcal{L}\text{-ss}} = M^{\mu\text{-ss}}$.

Proof of Theorem 4.3.4. As we just explained, Lemma 4.3.14 and Proposition 4.3.5 imply that $M^{\mu\text{-ss}} \subseteq M^{\mathcal{L}\text{-ss}}$. Now, let $p \in M^{\mathcal{L}\text{-ss}}$. Then, $\overline{G \cdot \hat{p}} \subseteq \mathcal{L}^*$ is disjoint from the zero-section. Since the G-action on $\mathcal{L}$ is algebraic, there exists a closed orbit $G \cdot \hat{q} \subseteq \overline{G \cdot \hat{p}}$ (see, e.g., [20, Proposition 1.8]). Since $\overline{G \cdot \hat{p}}$ is disjoint from the zero-section, $\hat{q}$ is non-zero and hence the point $q \in M$ below $\hat{q}$ is in $M^{\mu\text{-ps}}$ by Proposition 4.3.5. Thus, $G \cdot q \cap \mu^{-1}(0) \neq \emptyset$. We have $q \in \overline{G \cdot p}$ so $\overline{G \cdot p} \cap \mu^{-1}(0) \neq \emptyset$, and hence $p \in M^{\mu\text{-ss}}$. □

4.4 The compact case

As an aside, we remark that we can now easily recover the original Kempf–Ness Theorem for projective manifolds [88, 117], and also its generalisation to transcendental Kähler structures due to Sjamaar [133, Theorem 2.18]:

Corollary 4.4.1. *Let* $(M, \omega, K, \mathcal{L}, \mu)$ *be a polarised Hamiltonian Kähler manifold. If* M *is compact, then* $M^{\mu\text{-ss}} = M^{\mathcal{L}\text{-ss}}$ *and hence* $M/\!/_\mu K \cong M/\!/_\mathcal{L} G$ *as complex-analytic and partitioned spaces.*

Proof. By Kodaira's embedding theorem, M is projective algebraic, say $M \subseteq \mathbb{CP}^n$, and G acts by a representation $G \to \mathrm{GL}(n+1, \mathbb{C})$. Thus, $(M, G, \mathcal{L})$ satisfies the geometric criterion and we only need to show that the norm-squared function $N : \mathcal{L}^* \to \mathbb{R}$ is proper on closed G-orbits disjoint from the zero-section. But N is even proper on $\mathcal{L}^*$ since M is compact. □

4.5 The affine case

We now come to the main goal of this chapter, which is to prove Theorem 4.1.2.

Let us first discuss part (1). Let M be a smooth complex affine variety and K a compact Lie group acting on M. It is well known that if the action map $K \times M \to M$

is real algebraic, then this action extends to a complex-algebraic action of $G := K_{\mathbb{C}}$. This follows from the fact that for every $u \in \mathbb{C}[M]$, the linear span of $K \cdot u$ is finite-dimensional and hence M can be embedded in a finite-dimensional complex representation R of K. Then, we use the universality property of complexifications, which says that the representation $K \to \mathrm{GL}(R)$ extends uniquely to a representation $K_{\mathbb{C}} \to \mathrm{GL}(R)$ (the author learned this argument in [69, p. 226]).

Now, let $\chi : G \to \mathbb{C}^*$ be a character (equivalently, $\chi : K \to S^1$), let $\mathcal{L}$ be the trivial line bundle on M with the trivial G-action $g \cdot (p, z) = (g \cdot p, z)$, and let $\mathcal{L}_\chi$ be $\mathcal{L}$ with the G-action twisted by χ, i.e. $g \cdot (p, z) = (g \cdot p, \chi(g) z)$. Then, King [86, §2] observed that the GIT quotient $M /\!\!/_{\mathcal{L}_\chi} G$ can be described as the variety $M /\!\!/_\chi G := \operatorname{Proj} \bigoplus_{n=0}^\infty \mathbb{C}[M]^{G,\chi^n}$ and that $M^{\mathcal{L}_\chi\text{-ss}} = M^{\chi\text{-ss}} := \{p \in M : \exists n \geq 1, u \in \mathbb{C}[M]^{G,\chi^n}$ such that $u(p) \neq 0\}$, where $\mathbb{C}[M]^{G,\chi^n}$ is the set of $u \in \mathbb{C}[M]$ such that $u(g \cdot p) = \chi(g)^n u(p)$ for all $g \in G$ and $p \in M$.

Let $f : M \to \mathbb{R}$ be a K-invariant Kähler potential on M and let $\omega := 2i\partial\bar{\partial} f$ be the associated Kähler form. We want to apply Proposition 4.2.1 to get a moment map, so we need a fibre metric on $\mathcal{L}$ such that $\omega = \frac{i}{2\pi\hbar}\Theta_{\mathcal{L}}$. Since $\mathcal{L}$ is trivial, we may define a metric by

$$\langle (p, u), (p, v) \rangle = u\bar{v}e^{-4\pi\hbar f(p)}, \quad (p \in M, u, v \in \mathbb{C})$$

where $\hbar > 0$ is arbitrary.

Lemma 4.5.1. $\omega = \frac{i}{2\pi\hbar}\Theta_{\mathcal{L}}$

Proof. This follows directly from the definition of the canonical connection on a unitary line bundle (e.g. Wells [144, §III.2]). Indeed, from [144, p. 78, eq. (2.1)], the connection 1-form associated with the trivial frame is $\theta = e^{4\pi\hbar f}\partial e^{-4\pi\hbar f} = -4\pi\hbar\partial f$, and hence by [144, Proposition 2.2], the curvature is $\Theta_{\mathcal{L}} = \bar{\partial}\theta = -4\pi\hbar\bar{\partial}\partial f$. Hence, $\frac{i}{2\pi\hbar}\Theta_{\mathcal{L}} = 2i\partial\bar{\partial} f = \omega$. □

Thus, $(M, \omega, \mathcal{L})$ is a polarised Kähler manifold. The fibre metric on $\mathcal{L}$ is K-invariant, so we have a moment map $\mu : M \to \mathfrak{k}^*$ by Proposition 4.2.1. Explicitly:

Lemma 4.5.2. *We have* $\mu(p)(x) = df(\mathsf{I}x_p^\#)$ *for all* $p \in M$ *and* $x \in \mathfrak{k}$.

Proof. The fibre metric on $\mathcal{L}^* = M \times \mathbb{C}$ is $\langle (p, u), (p, v) \rangle = u\bar{v}e^{4\pi\hbar f(p)}$. Every $p \in M$ has a canonical lift $\hat{p} = (p, 1)$. Then, by definition of μ, we have

$$\begin{aligned}\mu(p)(x) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} \log \|(e^{itx} \cdot p, 1)\|^2 = \frac{d}{dt}\Big|_{t=0} \frac{1}{4\pi\hbar} \log e^{4\pi\hbar f(e^{itx} \cdot p)} \\ &= \frac{d}{dt}\Big|_{t=0} f(e^{itx} \cdot p) = df(\mathsf{I}x_p^\#).\end{aligned}$$

□

In particular, we proved part (2) of Theorem 4.1.2.

Now, by Proposition 4.2.3, the moment map associated with $\mathcal{L}_\chi$ is $\mu_\xi := \mu - \xi$, where $\xi := \frac{i}{2\pi\hbar} d\chi$. Hence, $(M, \omega, K, \mathcal{L}_\chi, \mu_\xi)$ is a polarised Hamiltonian Kähler manifold. To finish the proof Theorem 4.1.2, it suffices to show that if $\mathbb{C}[M] \subseteq o(e^f)$ (Definition 4.1.1) then $M^{\chi\text{-ss}} = M^{\mu_\xi\text{-ss}}$ (using $\hbar = \frac{1}{2\pi a}$). We do this using the general Kempf–Ness Theorem (Theorem 4.3.4). As explained earlier, $(M, G, \mathcal{L})$ satisfies the geometric criterion [86, Lemma 2.2]. Hence, it suffices to show that, if $\mathbb{C}[M] \subseteq o(e^f)$, then the norm-squared function

$$N : \mathcal{L}_\chi^* = M \times \mathbb{C} \longrightarrow \mathbb{R}, \quad (p, z) \longmapsto |z|^2 e^{4\pi\hbar f(p)}$$

is proper on every closed G-orbit disjoint from the zero-section. We prove this by the following two lemmas.

Lemma 4.5.3. *If $\mathbb{C}[M] \subseteq o(e^f)$, then $\mathbb{C}[M] \subseteq o(e^{\alpha f})$ for all $\alpha > 0$.*

Proof. Let $c > 0$ and let $u \in \mathbb{C}[M]$. We want to show that there exists a compact set $C \subseteq M$ such that $|u(p)| \leq c e^{\alpha f(p)}$ for all $p \in M \setminus C$. If $\alpha \geq 1$, this follows immediately from $\mathbb{C}[M] \subseteq o(e^f)$. Hence, suppose that $0 < \alpha < 1$ and let $n \in \mathbb{Z}$ be such that $1 < 1/\alpha < n$. Since $1 \in \mathbb{C}[M] \subseteq o(e^f)$, there is a compact set $C_1 \subseteq M$ such that $1 \leq c^{1/\alpha} e^{f(p)}$ for all $p \in M \setminus C_1$. Hence, $1 \leq c e^{\alpha f(p)}$ for all $p \in M \setminus C_1$. Now, $u^n \in \mathbb{C}[M] \subseteq o(e^f)$ so there is a compact set $C_2 \subseteq M$ such that $|u(p)|^n \leq c^{1/\alpha} e^{f(p)}$ for all $p \in M \setminus C_2$. Let $C := C_1 \cup C_2$ and let $p \in M \setminus C$. Then, either $|u(p)| \leq 1$ and hence $|u(p)| \leq 1 \leq c e^{\alpha f(p)}$, or $|u(p)| > 1$ and hence $|u(p)| = (|u(p)|^{1/\alpha})^\alpha < (|u(p)|^n)^\alpha \leq (c^{1/\alpha} e^{f(p)})^\alpha = c e^{\alpha f(p)}$. □

Lemma 4.5.4. *If $S \subseteq M \times \mathbb{C}$ is Zariski-closed and disjoint from the zero-section, then $N|_S$ is proper.*

Proof. The proof is adapted from King [86, Lemma 6.3]. Since f is bounded below and the Kähler form $\omega = 2i\partial\bar{\partial} f$ is unaffected by adding to f a constant, we may assume that f maps M to $[0, \infty)$. To show that $N|_S$ is proper, it suffices to show that for all $B > 0$ there exists a compact subset $C \subseteq M$ and $r > 0$ such that $N^{-1}([0, B]) \cap S \subseteq C \times \overline{\mathbb{D}}_r$, where $\overline{\mathbb{D}}_r$ is the closed disc of radius r centred at 0 in $\mathbb{C}$. Since S is Zariski-closed and disjoint from $M \times \{0\}$ there is a polynomial $u \in \mathbb{C}[M \times \mathbb{C}]$ such that $u(M \times \{0\}) = 0$ and $u(S) = 1$. Then, u must be of the form

$$u(p, z) = z u_1(p) + \cdots + z^n u_n(p)$$

for some $u_k \in \mathbb{C}[M]$. By Lemma 4.5.3, we have $u_k \in o(e^{2\pi\hbar k f})$ for all k, so there is a compact subset $C \subseteq M$ such that

$$\frac{1}{2n} e^{2\pi\hbar k f(p)} > B^{k/2} |u_k(p)|$$

for all $p \in M \setminus C$ and $k = 1, \dots, n$. Now, let $(p, z) \in N^{-1}([0, B]) \cap S$. Then, $|z|^2 \leq |z|^2 e^{4\pi\hbar f(p)} = N(p, z) \leq B$ so $z \in \overline{\mathbb{D}}_{\sqrt{B}}$. We claim that $p \in C$. Indeed, we have $N(p, z) = |z|^2 e^{4\pi\hbar f(p)} \leq B$ so $|z|^2 \leq e^{-4\pi\hbar f(p)} B$ and hence $|z|^k = (|z|^2)^{k/2} \leq (e^{-4\pi\hbar f(p)} B)^{k/2} = e^{-2\pi\hbar k f(p)} B^{k/2}$. Thus, if $p \notin C$ we get

$$|u(p, z)| \leq \sum_{k=1}^{n} |z|^k |u_k(p)| \leq \sum_{k=1}^{n} e^{-2\pi\hbar k f(p)} B^{k/2} |u_k(p)| < \sum_{k=1}^{n} \frac{1}{2n} = \frac{1}{2},$$

contradicting that $(p, z) \in S$ since $u(S) = 1$. □

By Theorem 4.3.4, this implies that $M^{\mathcal{L}\text{-ss}} = M^{\mu_\xi\text{-ss}}$. Since GIT quotients are analytic Hilbert quotients and by the uniqueness of categorical quotients, this concludes the proof of Theorem 4.1.2.

4.6 A counterexample

We show that the assumption on the relationship between $\mathbb{C}[M]$ and f in Theorem 4.1.2 is not superfluous. Namely, we give an example of an affine variety M with an action of a complex reductive group $G = K_{\mathbb{C}}$ and a K-invariant Kähler potential f that is proper and bounded below, but there exists a character χ such that if $\xi = i\, d\chi$ then $\mu^{-1}(\xi)/K$ and $M /\!\!/_\chi G$ are not homeomorphic. More precisely, we will have $\mu^{-1}(\xi)/K = \emptyset$ and $M /\!\!/_\chi G = \{\mathrm{pt}\}$.

Consider $K = S^1$ acting on $M = \mathbb{C}^*$ by multiplication. An S^1-invariant Kähler potential on $\mathbb{C}^*$ is a function of the form

$$f : \mathbb{C}^* \longrightarrow \mathbb{R}, \quad f(z) = r(|z|^2),$$

for any smooth function $r : (0, \infty) \to \mathbb{R}$ such that $t\ddot{r}(t) + \dot{r}(t) > 0$ for all $t > 0$. Indeed, we have

$$\omega := 2i\partial\bar{\partial} f = 4(t\ddot{r}(t) + \dot{r}(t)) dx \wedge dy,$$

where $z = x + iy \in \mathbb{C}^*$ and $t = |z|^2$. Every character of $K = S^1$ is of the form

$$\chi : S^1 \longrightarrow S^1, \quad \chi(z) = z^n$$

for some $n \in \mathbb{Z}$. We consider the GIT quotient $M /\!\!/_\chi G = \mathbb{C}^* /\!\!/_{\mathcal{L}_n} \mathbb{C}^*$, where $\mathcal{L}_n$ is the trivial line bundle $\mathcal{L}_n = \mathbb{C}^* \times \mathbb{C}$ over $\mathbb{C}^*$ with the linearisation $\lambda \cdot (z, u) = (\lambda z, \lambda^n u)$. We have $(\mathbb{C}^*)^{\mathcal{L}_n\text{-ss}} = \mathbb{C}^*$, so $M /\!\!/_\chi G$ is a single point (for any χ).

The moment map associated with the potential f as in Theorem 4.1.2(2) is

$$\mu : \mathbb{C}^* \longrightarrow \mathbb{R}, \quad \mu(z) = -2|z|^2 \dot{r}(|z|^2),$$

using the isomorphism $\mathfrak{k}^* = (i\mathbb{R})^* \to \mathbb{R} : \xi \mapsto \xi(i)$. Note that under this isomorphism, the central element $\xi := i\, d\chi \in \mathfrak{k}^*$ is $-n \in \mathbb{R}$. Thus,

$$\mu^{-1}(\xi)/K = \{t \in (0, \infty) : 2t\dot{r}(t) = n\}.$$

Therefore, to get a counterexample, it suffices to find a proper function $r : (0, 1) \to \mathbb{R}$ such that $t\ddot{r}(t) + \dot{r}(t) > 0$ and $2t\dot{r}(t)$ is bounded. An example of such a function is

$$r(t) = \sqrt{1 + (\log t)^2}.$$

Indeed, we have

$$t\ddot{r}(t) + \dot{r}(t) = \frac{1}{t(1 + (\log t)^2)^{3/2}} \quad \text{and} \quad 2t\dot{r}(t) = \frac{2\log t}{\sqrt{1 + (\log t)^2}}.$$

Since $|2t\dot{r}(t)| \leq 2$, we obtain a counterexample with $n = 3$.

Note that not every polynomial is in $o(e^f)$. For example, $e^{\sqrt{1+(\log|z|^2)^2}} < |z|^3$ for large z, so the polynomial $z^3 \in \mathbb{C}[\mathbb{C}^*]$ is not in $o(e^f)$.

Chapter 5

Nahm's equations and GIT

5.1 Introduction

Nahm's equations are a system of ordinary differential equations associated with any compact Lie group K. They come naturally from gauge theory as the reduction of the self-dual Yang-Mills equations on the principal K-bundle on $\mathbb{R}^4$ to one dimension. More concretely, they are

$$\begin{aligned} \dot{A}_1 + [A_0, A_1] + [A_2, A_3] &= 0 \\ \dot{A}_2 + [A_0, A_2] + [A_3, A_1] &= 0 \\ \dot{A}_3 + [A_0, A_3] + [A_1, A_2] &= 0 \end{aligned} \tag{5.1.1}$$

where the A_i's are differentiable maps from an interval $I \subseteq \mathbb{R}$ to the Lie algebra $\mathfrak{k}$ of K and $\dot{A}_i$ is the derivative of A_i.

The group of gauge transformations acts naturally on the set of solutions so we can speak of the *moduli space* of solutions to Nahm's equations on I. By choosing different intervals I and imposing suitable boundary conditions, we can get many smooth moduli spaces associated with K. Moreover, Nahm's equations can be interpreted as the three components of a hyperkähler moment map for the action of the group of gauge transformations, so, when smooth, these moduli spaces have natural hyperkähler structures coming from an infinite-dimensional version of the hyperkähler quotient construction.

This has been a very useful tool for constructing hyperkähler structures on complex-symplectic varieties associated with a complex reductive group $G = K_{\mathbb{C}}$. For example, the cotangent bundle T^*G [95], coadjoint orbits in $\mathfrak{g}^*$ [98], and $G \times \mathbb{C}^{\operatorname{rk} G}$ [11] have been shown to be hyperkähler by diffeomorphisms with moduli spaces of solutions to Nahm's equations on the intervals $[0, 1]$, $[0, \infty)$, and $[0, 1)$, respectively. The case $I = (0, 1)$ is also interesting and will be discussed in Chapter 8.

In this chapter, we focus on the moduli space $\mathcal{M}_I$ of solutions to Nahm's equations on a compact interval $I = [a, b]$. Kronheimer [95] showed that $\mathcal{M}_I$ is a smooth hyperkähler manifold diffeomorphic to T^*G. Moreover, this diffeomorphism endows T^*G with a hyperkähler structure $(g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ whose underlying complex-symplectic structure $(\mathsf{I}, \omega_{\mathsf{J}} + i\omega_{\mathsf{K}})$ is the canonical one obtained by viewing T^*G as the cotangent bundle of a complex manifold. There is also an SO(3)-hyperkähler rotation on $\mathcal{M}_I$, so $\mathcal{M}_I$ is biholomorphic to T^*G with respect to any element of its two-sphere of complex structures. We will give a more detailed review of the construction of $\mathcal{M}_I$ and its basic properties in §5.2.

The moduli space $\mathcal{M}_I \cong T^*G$ is interesting for its large group of symmetries: the action of $K \times K$ on G by left and right multiplications lifts to an action on T^*G which preserves the hyperkähler structure. Moreover, Dancer–Swann [34] found a hyperkähler moment map for this action. Explicitly, the moment map evaluates a solution A to Nahm's equations (5.1.1) at the endpoints of the interval $I = [a, b]$ on which it is defined and identifies $\mathfrak{k} \cong \mathfrak{k}^*$ with a K-invariant inner-product:

$$\nu : \mathcal{M}_I \longrightarrow (\mathfrak{k}^* \times \mathfrak{k}^*)^3, \quad A \longmapsto \begin{pmatrix} A_1(a) & A_2(a) & A_3(a) \\ -A_1(b) & -A_2(b) & -A_3(b) \end{pmatrix}.$$

Under the diffeomorphism $\mathcal{M}_I \cong T^*G$, the complex part $\nu_{\mathbb{C}} := \nu_{\mathsf{J}} + i\nu_{\mathsf{K}}$ is the canonical moment map for the action of $G \times G$ on T^*G, i.e. (3.3.2). On the other end, the real part $\nu_{\mathbb{R}} := \nu_{\mathsf{I}}$ has no known explicit expression as a map on T^*G; the problem is that the diffeomorphism $T^*G \to \mathcal{M}_I$ relies on an existence result in analysis and hence is not explicit. The map $\nu_{\mathbb{R}} : T^*G \to \mathfrak{k}^*$ is most likely transcendental, unlike $\nu_{\mathbb{C}} : T^*G \to \mathfrak{g}^*$.

The $K \times K$-action commutes with the SO(3)-hyperkähler rotation and is the restriction of the $G \times G$-action on T^*G, so $(T^*G, K \times K, \nu)$ is a fully integrable tri-Hamiltonian hyperkähler manifold. Hence, for any closed subgroup H of $K \times K$ and $\xi \in Z_{\mathfrak{h}^*} \otimes \mathbb{R}^3$ (where $Z_{\mathfrak{h}^*} \subseteq \mathfrak{h}^*$ is the set of fixed points of the coadjoint action), the hyperkähler quotient of T^*G by H at level ξ is a stratified hyperkähler space. The goal of this chapter is to apply the Kempf–Ness type theorem proved in the previous chapter (Theorem 4.1.2) to obtain explicit expressions for these stratified hyperkähler spaces as quasi-projective algebraic varieties. We will use this in Chapter 6 to give detailed descriptions of special cases of these spaces.

The main technical result of this chapter is thus to prove the existence of a global Kähler potential satisfying the assumptions of Theorem 4.1.2:

Theorem 5.1.1. *Let K be a compact connected Lie group, let $G := K_{\mathbb{C}}$, and let $(g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ be Kronheimer's hyperkähler structure on T^*G. Then, the assumptions* *of* Theorem 4.1.2 *are satisfied for the action of* $K \times K$ *on the Kähler manifold* (T^*G, g, I)*, i.e. there is a* $K \times K$*-invariant global Kähler potential* $f : T^*G \to \mathbb{R}$ *for* (g, I) *such that* $\mathbb{C}[T^*G] \subseteq o(e^f)$*. Moreover, the moment map associated with* f *is the real part of Dancer–Swann's hyperkähler moment map, i.e.* $\nu_{\mathbb{R}}(p)(z) = df(\mathsf{I}z_p^{\#})$ *for all* $p \in T^*G$ *and* $z \in \mathfrak{k} \times \mathfrak{k}$*.*

The difficulty with this theorem is to relate the analytic structure on $\mathcal{M}_I$ with the algebraic structure on T^*G; this requires some analysis. We then easily deduce the expected identification of hyperkähler and complex-symplectic quotients:

Theorem 5.1.2. *Let* H *be a closed subgroup of* $K \times K$ *and let*

$$\mu := \nu|_{\mathfrak{h}} : T^*G \longrightarrow \mathfrak{h}^* \otimes \mathbb{R}^3$$

be the corresponding hyperkähler moment map. Let $\chi : H \to S^1$ *be a character and let* $\xi_{\mathbb{R}} := ai\, d\chi$ *for any* $a > 0$*. Then,*

$$(T^*G)^{(\mu_{\mathbb{R}} - \xi_{\mathbb{R}})\text{-ss}} = (T^*G)^{\chi\text{-ss}},$$

where the right-hand side is defined as in (4.1.1)*. Hence, for any* $\xi_{\mathbb{C}} \in Z_{\mathfrak{h}^*_{\mathbb{C}}}$*, we have* $\mu^{-1}(\xi) \subseteq \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})^{\chi\text{-ss}}$*, where* $\xi := (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in \mathfrak{h}^* \otimes \mathbb{R}^3$*, and this inclusion descends to an isomorphism*

$$T^*G /\!\!/\!\!/_{\mu_\xi} H \cong \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) /\!\!/_{\chi} H_{\mathbb{C}} = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} \mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})]^{H_{\mathbb{C}}, \chi^n} \right)$$

of complex-analytic and partitioned spaces (*where* $\mu_\xi := \mu - \xi$)*.*

Corollary 5.1.3. *In the notation of* Theorem 5.1.2*, if* $\eta := (0, \xi_{\mathbb{C}})$*, then* $T^*G /\!\!/\!\!/_{\mu_\eta} H$ *is the affine variety* $\operatorname{Spec} \mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})]^{H_{\mathbb{C}}}$ *and there is a projective morphism* $T^*G /\!\!/\!\!/_{\mu_\xi} H \to T^*G /\!\!/\!\!/_{\mu_\eta} H$*.*

This chapter has two sections. In §5.2, we review the construction of the moduli space $\mathcal{M}_I$ of solutions to Nahm's equations which endows T^*G with the structure of a tri-Hamiltonian hyperkähler manifold; this material will also be used in Chapter 7. In §5.3, we prove Theorem 5.1.1 and deduce Theorem 5.1.2. This chapter is based on the second part of the author's paper [108].

5.2 Background on Nahm's equations

Let K be a compact Lie group and let $G := K_{\mathbb{C}}$. Recall from §3.3 that the action of $G \times G$ on G by left and right multiplications lifts to a complex-Hamiltonian action on T^*G with a complex moment map $\nu_{\mathbb{C}} : T^*G \to \mathfrak{g}^* \times \mathfrak{g}^*$ (3.3.2). The goal of this section is to recall how Nahm's equations endow T^*G with the structure of a tri-Hamiltonian hyperkähler manifold whose underlying complex-Hamiltonian manifold is $(T^*G, G \times G, \nu_{\mathbb{C}})$. The original sources are Kronheimer [95] and Dancer–Swann [34]. See also Bielawski [11, 12] and Takayama [138].

5.2.1 The moduli space

Since K is compact, we may assume it is a closed subgroup of $\mathrm{U}(n)$ for some n and G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$. Fix a K-invariant inner-product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ and a compact interval $I = [a, b]$.

Consider the vector space $\mathcal{P}$ of all C^1 maps $I \to \mathfrak{k}$; the **path space** of $\mathfrak{k}$. Let $\mathcal{P}_{\mathbb{H}} := \mathcal{P} \otimes \mathbb{H}$ be the quaternionic vector space of all quadruples $A = (A_0, A_1, A_2, A_3) = A_0 + A_1 i + A_2 j + A_3 k$ with $A_i \in \mathcal{P}$. Then, $\mathcal{P}_{\mathbb{H}}$ is a Banach manifold with tangent spaces $T_A \mathcal{P}_{\mathbb{H}} \cong \mathcal{P}_{\mathbb{H}}$ for all $A \in \mathcal{P}_{\mathbb{H}}$ and we have three complex structures $\mathsf{I}, \mathsf{J}, \mathsf{K}$ on $\mathcal{P}_{\mathbb{H}}$ induced by multiplications on the left by i, j, and k. There is also a natural inner-product on $\mathcal{P}_{\mathbb{H}}$ given by

$$\langle X, Y \rangle = \int_a^b \sum_{i=0}^{3} \langle X_i(t), Y_i(t) \rangle dt, \qquad (X, Y \in \mathcal{P}_{\mathbb{H}}),$$

and this endows $\mathcal{P}_{\mathbb{H}}$ with a Riemannian metric. Moreover, it is compatible with the three complex structures of $\mathcal{P}_{\mathbb{H}}$, making it a hyperkähler Banach manifold.

Next, consider the group $\mathcal{K}$ of all C^2 maps $\gamma : I \to K$. It acts on $\mathcal{P}_{\mathbb{H}}$ by

$$\gamma \cdot (A_0, A_1, A_2, A_3) = (\gamma A_0 \gamma^{-1} - \dot{\gamma}\gamma^{-1}, \gamma A_1 \gamma^{-1}, \gamma A_2 \gamma^{-1}, \gamma A_3 \gamma^{-1}).$$

and this preserves the hyperkähler structure. By viewing A as a connection $\sum_{i=0}^3 A_i dx^i$ on the trivial principal K-bundle over $I \times \mathbb{R}^3$ and γ as a bundle automorphism, this is a gauge transformation, i.e. $\gamma \cdot A$ is the pullback of A by γ.

Let $\mathcal{K}^0$ be the subgroup of $\mathcal{K}$ consisting of all $\gamma \in \mathcal{K}$ with $\gamma(a) = \gamma(b) = 1$. Then, there is a hyperkähler moment map for the action of $\mathcal{K}^0$ on $\mathcal{P}_{\mathbb{H}}$ given by

$$\kappa : \mathcal{P}_{\mathbb{H}} \longrightarrow \mathcal{P} \otimes \mathbb{R}^3, \quad (A_0, A_1, A_2, A_3) \longmapsto \begin{pmatrix} \dot{A}_1 + [A_0, A_1] + [A_2, A_3] \\ \dot{A}_2 + [A_0, A_2] + [A_3, A_1] \\ \dot{A}_3 + [A_0, A_3] + [A_1, A_2] \end{pmatrix},$$

where $\dot{A}_i$ is the derivative of A_i with respect to $t \in I$. The three ODEs that we get by setting $\kappa(A) = 0$ are called **Nahm's equations** and

$$\mathcal{M}_I := \kappa^{-1}(0)/\mathcal{K}^0$$

is called the **moduli space of solutions to Nahm's equations on** I. Note that there is a residual action of the group $\mathcal{K}/\mathcal{K}^0 = K \times K$ on $\mathcal{M}_I$.

Theorem 5.2.1 (Kronheimer [95]). *The space $\mathcal{M}_I$ is a finite-dimensional smooth manifold, and the hyperkähler quotient construction endows it with a complete hyperkähler structure $(g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ invariant under $K \times K$. Moreover, there is a $K \times K$-equivariant isomorphism of complex-symplectic manifolds*

$$\varphi : \mathcal{M}_I \longrightarrow T^*G, \tag{5.2.1}$$

where $\mathcal{M}_I$ has the complex-symplectic structure $(\mathsf{I}, \omega_\mathsf{J} + i\omega_\mathsf{K})$ and T^*G the canonical complex-symplectic form $-d\theta$ where θ is the tautological one-form. □

The isomorphism (5.2.1) will be described in more details in §5.2.3. There is also a canonical hyperkähler moment map for the action of $K \times K$ on $\mathcal{M}_I$:

Theorem 5.2.2 (Dancer–Swann [34, §3]). *By identifying $\mathfrak{k}^*$ with $\mathfrak{k}$ by the chosen K-invariant inner-product, the map*

$$\nu : \mathcal{M}_I \longrightarrow (\mathfrak{k}^* \times \mathfrak{k}^*)^3, \quad A \longmapsto \begin{pmatrix} A_1(a) & A_2(a) & A_3(a) \\ -A_1(b) & -A_2(b) & -A_3(b) \end{pmatrix}$$

is a hyperkähler moment map for the action of $K \times K$ on $\mathcal{M}_I$. Moreover, under the isomorphism $\varphi : \mathcal{M}_I \to T^*G$, the complex part $\nu_\mathbb{C} := \nu_\mathsf{J} + i\nu_\mathsf{K}$ is the canonical moment map (3.3.2) *for the action of $G \times G$ on T^*G.* □

Now, SO(3) also acts naturally on $\mathcal{P}_\mathbb{H}$ by rotating A_1, A_2, A_3 while keeping A_0 fixed. This action preserves the metric and the Nahm equations, and hence descends to an SO(3)-hyperkähler rotation on $\mathcal{M}_I$ commuting with the $K \times K$-action (see Dancer–Swann [34, §2]). Moreover, since the action of $K \times K$ extends to an I-holomorphic action of $G \times G$, $(\mathcal{M}_I, K \times K, \nu)$ is a fully integrable tri-Hamiltonian hyperkähler manifold.

5.2.2 Global Kähler potentials

Consider the U(1) subgroup of $\mathrm{SO}(3) \subseteq \mathrm{GL}(3, \mathbb{R})$ which fixes the z-axis while rotating the xy-plane. Then, the restriction of the SO(3)-action on $\mathcal{M}_I$ to U(1) preserves the Kähler structure (g, K). Moreover, there is a moment map for this action:

Proposition 5.2.3 (Dancer–Swann [34, §3]). *The function*

$$\rho_3 : \mathcal{M}_I \longrightarrow \mathbb{R}, \quad A \longmapsto \frac{1}{2}\int_a^b \|A_1(t)\|^2 + \|A_2(t)\|^2 \, dt$$

is a moment map for the action of $\mathrm{U}(1)$ *on the Kähler manifold* $(\mathcal{M}_I, g, \mathsf{K})$. □

By a general result of Hitchin–Karlhede–Lindström–Roček [77, §3(E)], this map is also a global Kähler potential for both (g, I) and (g, J), i.e.

$$\omega_{\mathsf{I}} = 2i\partial_{\mathsf{I}}\bar{\partial}_{\mathsf{I}}\rho_3 \quad \text{and} \quad \omega_{\mathsf{J}} = 2i\partial_{\mathsf{J}}\bar{\partial}_{\mathsf{J}}\rho_3.$$

Similarly, we can consider the other two $\mathrm{U}(1)$ subgroups of $\mathrm{SO}(3)$ fixing the complex structures I and J on $\mathcal{M}_I$, respectively. They have moment maps

$$\rho_1(A) = \frac{1}{2}\int_a^b \|A_2(t)\|^2 + \|A_3(t)\|^2 \, dt \quad \text{and} \quad \rho_2(A) = \frac{1}{2}\int_a^b \|A_1(t)\|^2 + \|A_3(t)\|^2 \, dt.$$

Then, ρ_1 is a Kähler potential for (g, J) and (g, K), and ρ_2 for (g, I) and (g, K).

In particular, ρ_2 and ρ_3 are both Kähler potentials for (g, I). However, neither of them is proper, so they cannot be used in our version of the Kempf–Ness theorem (the condition $\mathbb{C}[M] \subseteq o(e^f)$ in Theorem 4.1.2 implies that f is proper). But we will show in §5.3.2 that their average $F := \frac{1}{2}(\rho_2 + \rho_3)$ has all the desired properties and hence is the Kähler potential which we will use for the main theorem of this chapter (Theorem 5.1.1).

We summarise this discussion in the following proposition:

Proposition 5.2.4. *The function*

$$F : \mathcal{M}_I \longrightarrow \mathbb{R}, \quad A \longmapsto \frac{1}{4}\int_a^b \left(2\|A_1(t)\|^2 + \|A_2(t)\|^2 + \|A_3(t)\|^2\right) dt$$

is a $K \times K$*-invariant global Kähler potential for* $(\mathcal{M}_I, g, \mathsf{I})$. □

5.2.3 The isomorphism $\varphi : \mathcal{M}_I \to T^*G$

Let us now describe in more details the isomorphism $\varphi : \mathcal{M}_I \to T^*G$ appearing in Kronheimer's Theorem 5.2.1. We follow the presentation of Dancer–Swann [34].

For the rest of this section, *holomorphic* will mean holomorphic with respect to the complex structure I. Let $\mathcal{G}$ be the group of C^2 maps $g : I \to G$, and let $\mathcal{G}^0$ be the group of $g \in \mathcal{G}$ with $g(a) = g(b) = 1$. Let $\mathcal{P}_{\mathbb{C}} = \mathcal{P} \otimes \mathbb{C}$ be the vector space of C^1 maps $I \to \mathfrak{g}$. By identifying $\mathcal{P}_{\mathbb{H}}$ with $\mathcal{P}_{\mathbb{C}} \times \mathcal{P}_{\mathbb{C}}$ via

$$(A_0, A_1, A_2, A_3) \longmapsto (\alpha, \beta) = (A_0 + iA_1, A_2 + iA_3),$$

the action of $\mathcal{K}$ on $\mathcal{P}_{\mathbb{H}}$ extends to a holomorphic action of $\mathcal{G}$, namely

$$g \cdot (\alpha, \beta) = (g\alpha g^{-1} - \dot{g}g^{-1}, g\beta g^{-1}).$$

By a result of Donaldson [40, Proposition 2.8], the infinite-dimensional hyperkähler quotient $\kappa^{-1}(0)/\mathcal{K}^0$ is biholomorphic to $\kappa_{\mathbb{C}}^{-1}(0)/\mathcal{G}^0$, where $\kappa_{\mathbb{C}} := \kappa_{\mathsf{J}} + i\kappa_{\mathsf{K}}$ (this can be viewed as an infinite-dimensional version of the Kempf–Ness Theorem). Under the identification of $\mathcal{P}_{\mathbb{H}}$ with $\mathcal{P}_{\mathbb{C}} \times \mathcal{P}_{\mathbb{C}}$, this map is simply

$$\kappa_{\mathbb{C}}(\alpha, \beta) = \dot{\beta} + [\alpha, \beta].$$

The equation $\dot{\beta} + [\alpha, \beta] = 0$ is thus called the **complex Nahm equation**, and we call $\mathcal{N}_I := \kappa_{\mathbb{C}}^{-1}(0)/\mathcal{G}^0$ the **complex Nahm moduli space on I**.

The advantage of this description is that the equation $\dot{\beta} + [\alpha, \beta] = 0$ is now easily trivialisable by the action of $\mathcal{G}$. Indeed, we can always find a transformation in $\mathcal{G}$ sending α to 0, and then β becomes constant. Simply take $g : I \to G$ to be the unique solution to the linear ODE

$$\dot{g} = g\alpha, \quad g(b) = 1.$$

Then, we have $g \cdot (\alpha, \beta) = (0, x)$, where $x = \beta(b) \in \mathfrak{g}$, and this gives a biholomorphism

$$\mathcal{N}_I \xrightarrow{\cong} G \times \mathfrak{g}, \quad (\alpha, \beta) \longmapsto (g(a)^{-1}, \beta(b)). \tag{5.2.2}$$

Now, using the invariant inner-product on $\mathfrak{k}$, we get a complex-linear G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ and hence $G \times \mathfrak{g} \cong G \times \mathfrak{g}^* \cong T^*G$, using left translations as in (3.3.1). Thus, we have a sequence of biholomorphisms

$$\varphi : \mathcal{M}_I \xrightarrow{\cong} \mathcal{N}_I \xrightarrow{\cong} G \times \mathfrak{g} \xrightarrow{\cong} G \times \mathfrak{g}^* \xrightarrow{\cong} T^*G.$$

Moreover, this is an isomorphism of complex-symplectic manifolds, i.e. $\varphi^*(-d\theta) = \omega_{\mathbb{C}}$, where θ is the tautological holomorphic 1-form on T^*G and $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$.

5.2.4 $\mathcal{M}_I$ as an open subset of $K \times \mathfrak{k}^3$

The space $\mathcal{M}_I$ has also a convenient description as an open subset of $K \times \mathfrak{k}^3$, as shown in Dancer–Swann [34, Theorem 3]. This finite-dimensional description will be useful for our analysis, so we review it here.

For each solution $A \in \mathcal{A}$ to Nahm's equations on I, there is a unique $\gamma \in \mathcal{K}$ such that $\gamma(b) = 1$ and which transforms the first coordinate to zero, i.e. $\gamma \cdot A = (0, P_1, P_2, P_3)$ for some $P_i : I \to \mathfrak{k}$. The map γ is simply the solution to the linear initial value problem $\dot{\gamma} = \gamma A_0$, $\gamma(b) = 1$. Now, the P_i's are solutions to the so-called **reduced Nahm equations**

$$\begin{aligned} \dot{P}_1 + [P_2, P_3] &= 0 \\ \dot{P}_2 + [P_3, P_1] &= 0 \\ \dot{P}_3 + [P_1, P_2] &= 0. \end{aligned} \tag{5.2.3}$$

The main advantage of this reduced form is that the number of equations is equal to the number of unknowns, so a solution $P = (P_1, P_2, P_3)$ is completely determined by its value at the endpoint $P(b) \in \mathfrak{k}^3$. Thus, for each $x \in \mathfrak{k}^3$, we let

$$P^{x,b} := \text{the unique solution to (5.2.3) with } P(b) = x, \tag{5.2.4}$$

and

$$V_I := \{x \in \mathfrak{k}^3 : P^{x,b} \text{ is defined at least on } I = [a, b]\}.$$

Then, we have a bijection

$$\mathcal{M}_I \longrightarrow K \times V_I, \quad A \longmapsto (\gamma(a)^{-1}, A_1(b), A_2(b), A_3(b)), \tag{5.2.5}$$

where γ is the unique element of $\mathcal{K}$ such that $\dot{\gamma} = \gamma A_0$ and $\gamma(b) = 1$. Its inverse is

$$K \times V_I \longrightarrow \mathcal{M}_I, \quad (k, x) \longmapsto \gamma_k \cdot (0, P_1^{x,b}, P_2^{x,b}, P_3^{x,b}),$$

where γ_k is any smooth map $I \to K$ with $\gamma_k(a) = k$ and $\gamma_k(b) = 1$.

Theorem 5.2.5 (Dancer–Swann [34, §3]). *The set V_I is a star-shaped open subset of $\mathfrak{k}^3$ and the map $\mathcal{M}_I \to K \times V_I$ given by* (5.2.5) *is a diffeomorphism. Moreover, this map is equivariant with respect to the $K \times K$-action on $\mathcal{M}_I$ and the action on $K \times V_I$ given by*

$$(k_1, k_2) \cdot (k, x) = (k_1 k k_2^{-1}, \mathrm{Ad}_{k_2} x).$$

Also, the hyperkähler moment map for the action of $K \times K$ on $\mathcal{M}_I$ is

$$K \times V_I \longrightarrow (\mathfrak{k}^* \times \mathfrak{k}^*) \otimes \mathbb{R}^3, \quad (k, x) \longmapsto (\mathrm{Ad}_k P^{x,b}(a), -x),$$

after identifying $\mathfrak{k} \cong \mathfrak{k}^*$ with the K-invariant inner-product. □

The fact that V_I is star-shaped is a consequence of the following simple observation, which will also be useful to us later:

Lemma 5.2.6. *For all $x \in \mathfrak{k}^3$ and $s \in \mathbb{R}$, we have*

$$P^{sx,b}(t) = sP^{x,b}(b + s(t - b)).$$

In particular, if $P^{x,b}$ is defined on $[a, b]$ then $P^{sx,b}$ is defined on $[b + s^{-1}(a - b), b]$. □

5.3 Proofs

The goal of this section is to prove Theorem 5.1.1 and Theorem 5.1.2 in the introduction. To ease the notation, we will work with the interval $I = [0,1]$ and drop all subscripts I (a solution A on $[a,b]$ can always be rescaled to $[0,1]$ by $t \mapsto (b-a)A((b-a)t+a)$). We split Theorem 5.1.1 in the following two slightly more precise propositions:

Proposition 5.3.1. *Let $F : \mathcal{M} \to \mathbb{R}$ be the global Kähler potential for $(\mathcal{M}, g, \mathsf{I})$ defined in* Proposition 5.2.4 *and let f be the corresponding function on T^*G using the diffeomorphism φ of §5.2.3. Then, $\mathbb{C}[T^*G] \subseteq o(e^f)$.*

Proposition 5.3.2. *Let f be as in* Proposition 5.3.1 *and let $\nu_\mathbb{R}$ be the first component of Dancer–Swann's hyperkähler moment map defined in* Theorem 5.2.2*, viewed as a map on T^*G via φ. Then,*

$$\nu_\mathbb{R}(p)(z) = df(\mathsf{I} z_p^\#), \quad \textit{for all } p \in T^*G \textit{ and } z \in \mathfrak{k} \times \mathfrak{k},$$

where I is the natural complex structure on T^*G, and $z^\#$ is the vector field on T^*G generated by z under the $K \times K$-action.

5.3.1 The diffeomorphism $K \times V \cong G \times \mathfrak{g}$

It will be more convenient to work with $K \times V$ rather than $\mathcal{M}$ using the diffeomorphism $K \times V \to \mathcal{M}$ of Theorem 5.2.5. Also, since the isomorphism $G \times \mathfrak{g} \cong T^*G$ is complex-algebraic, we can work directly with $G \times \mathfrak{g}$. Hence, our results depend strongly on the diffeomorphism

$$\psi : K \times V \longrightarrow G \times \mathfrak{g}$$

obtained by the composition $K \times V \to \mathcal{M} \to \mathcal{N} \to G \times \mathfrak{g}$. It will, therefore, be useful to have a more explicit description of ψ, and this is the goal of this section.

Since $I = [0,1]$, we use the notation $P^x := P^{x,1}$ for the unique solution to the reduced Nahm equations (5.2.3) with $P(1) = x$ as defined in (5.2.4). As before, the three components of P^x are denoted by (P_1^x, P_2^x, P_3^x).

Since K is compact, the exponential map $\mathfrak{k} \to K$ is surjective, so it suffices to describe ψ for elements of the form (e^y, x).

Lemma 5.3.3. *For all $y \in \mathfrak{k}$ and $x \in V$, we have*

$$\psi(e^y, x) = (g(0)^{-1}, x_2 + ix_3),$$

where $g : [0,1] \to G$ is the unique solution to the linear ODE

$$\dot{g}(t) = g(t)(y + i \operatorname{Ad}_{e^{(1-t)y}} P_1^x(t)), \quad g(1) = 1. \tag{5.3.1}$$

Proof. The diffeomorphism $K \times V \to \mathcal{M}$ is given by $(k, x) \mapsto \gamma_k \cdot (0, P^x)$, where γ_k is any smooth map $[0,1] \to K$ with $\gamma_k(0) = k$ and $\gamma_k(1) = 1$. In particular, when $k = e^y$, we can take $\gamma_k(t) = e^{(1-t)y}$ and hence $(e^y, x) \mapsto e^{(1-t)y} \cdot (0, P^x(t)) = (y, \mathrm{Ad}_{e^{(1-t)y}} P^x(t))$. The image of the latter in $\mathcal{N}$ is $(y + i\,\mathrm{Ad}_{e^{(1-t)y}} P_1^x(t), \mathrm{Ad}_{e^{(1-t)y}}(P_2^x(t) + iP_3^x(t)))$. The definition of the isomorphism $\mathcal{N} \to G \times \mathfrak{g}$ in (5.2.2) concludes the proof. □

It will also be useful to know the derivative of that map at the origin:

Lemma 5.3.4. *Under the identifications $T_{(1,0)}(K \times V) = \mathfrak{k}^4$ and $T_{(1,0)}(G \times \mathfrak{g}) = \mathfrak{g}^2$, we have $d\psi_{(1,0)}(x_0, x_1, x_2, x_3) = (x_0 + ix_1, x_2 + ix_3)$.*

Proof. Let $y = x_0$ and $x = (x_1, x_2, x_3)$ and denote by g_y^x the unique solution to (5.3.1) with $g_y^x(1) = 1$. Using Lemma 5.2.6, we get that $g_{sy}^{sx}(t) = g_y^x(1 + s(t-1))$ for all $s \in \mathbb{R}$. Thus,

$$\begin{aligned} d\psi_{(1,0)}(y, x) &= \frac{d}{ds}\Big|_{s=0} \psi(e^{sy}, sx) = \frac{d}{ds}\Big|_{s=0} (g_{sy}^{sx}(0)^{-1}, sx_2 + isx_3) \\ &= \frac{d}{ds}\Big|_{s=0} (g_y^x(1-s)^{-1}, sx_2 + isx_3) \\ &= (\dot{g}_y^x(1), x_2 + ix_3) \\ &= (x_0 + ix_1, x_2 + ix_3). \end{aligned}$$ □

5.3.2 Growth rate of the potential (proof of Proposition 5.3.1)

We now prove Proposition 5.3.1, which says that $\mathbb{C}[T^*G] \subseteq o(e^f)$. This is the core section of this chapter.

Although the condition $\mathbb{C}[T^*G] \subseteq o(e^f)$ implies that f is proper, the first step is to show that f is indeed proper. Equivalently, we need to show that $F : \mathcal{M} \to \mathbb{R}$ is proper, which, using the diffeomorphism $K \times V \to \mathcal{M}$, is the same as showing properness of

$$\rho : V \longrightarrow \mathbb{R}, \quad \rho(x) = \frac{1}{4} \int_0^1 \left(2\|P_1^x(t)\|^2 + \|P_2^x(t)\|^2 + \|P_3^x(t)\|^2\right) dt.$$

We will mainly work with ρ rather than F. To prove that ρ is proper, we need the following four lemmas.

Lemma 5.3.5. *Let P be a solution to the reduced Nahm equations* (5.2.3) *on an interval I. Then, $\|P_i\|^2$ is convex on I for all i.*

Proof. We have $\frac{d}{dt}\|P_1\|^2 = 2\langle P_1, \dot{P}_1\rangle = 2\langle P_1, [P_2, P_3]\rangle$ so

$$\begin{aligned} \frac{d^2}{dt^2}\|P_1\|^2 &= 2\langle \dot{P}_1, [P_2, P_3]\rangle + 2\langle P_1, [\dot{P}_2, P_3] + [P_2, \dot{P}_3]\rangle \\ &= 2\langle \dot{P}_1, [P_2, P_3]\rangle + 2\langle [P_3, P_1], \dot{P}_2\rangle + 2\langle [P_1, P_2], \dot{P}_3\rangle \\ &= 2\|[P_2, P_3]\|^2 + 2\|[P_3, P_1]\|^2 + 2\|[P_1, P_2]\|^2. \end{aligned} \tag{5.3.2}$$

Similar statements hold for $\|P_2\|^2$ and $\|P_3\|^2$. $\square$

Lemma 5.3.6. *Let P be a solution to the reduced Nahm equations* (5.2.3) *on an open interval (a,b). If $\int_a^b \|P_i\|^2 dt < \infty$ for some i, then P can be extended to a larger interval (a',b') with $a' < a$ and $b' > b$.*

Proof. Let $\Phi : \mathfrak{k}^3 \to \mathfrak{k}^3$, $\Phi(x_1,x_2,x_3) = ([x_3,x_2],[x_1,x_3],[x_2,x_1])$ so that the reduced Nahm equations become $\dot{P}(t) = \Phi(P(t))$. To show that P extends past b, we want to show that there exists $t_0 < b$ such that the unique solution S to the initial value problem $\dot{S}(t) = \Phi(S(t))$, $S(t_0) = P(t_0)$ exists on $[t_0, t_0+\varepsilon)$ for some $\varepsilon > b - t_0$. From the existence and uniqueness theorem for first-order systems of ODEs, ε can be taken to be c/M where $c > 0$ is arbitrary and

$$M \coloneqq \sup\{\|\Phi(x)\| : \|x - P(t_0)\| \le c\};$$

see, for example, [30, Ch. 1, Theorem 2.3 and 5th paragraph of page 19]. Note that Φ is a homogeneous polynomial of degree 2, so there exists $\alpha > 0$ such that $\|\Phi(x)\| \le \alpha\|x\|^2$ for all $x \in \mathfrak{k}^3$. Hence, $M \le \alpha(\|P(t_0)\| + c)^2$ and it suffices to show that there exist $t_0 < b$ and $c > 0$ such that

$$\frac{c}{\alpha(\|P(t_0)\| + c)^2} > b - t_0. \tag{5.3.3}$$

We have $\frac{d}{dt}\|P_1\|^2 = 2\langle P_1, [P_2,P_3]\rangle = 2\langle [P_1,P_2],P_3\rangle = \frac{d}{dt}\|P_3\|^2$ and similarly $\frac{d}{dt}\|P_2\|^2 = \frac{d}{dt}\|P_1\|^2$, so the maps $\|P_i\|^2$ differ by constants. Hence, since $\int_a^b \|P_i\|^2 dt < \infty$ for some i, we also have $\int_a^b \|P\|^2 dt < \infty$. Moreover, $\|P\|^2$ is convex (Lemma 5.3.5) so the fact that $\int_a^b \|P\|^2 dt < \infty$ implies that there exists $\delta > 0$ such that $\|P(t)\|^2 < \frac{1}{b-t}$ for all $t \in (b-\delta, b)$. Let $c > \alpha$. By taking δ small enough, we may assume that $\alpha(1 + c\sqrt{\delta})^2 < c$. Then, for all $t_0 \in (b-\delta, b)$ we have

$$\alpha(b-t_0)(\|P(t_0)\| + c)^2 < \alpha(b-t_0)\left(\frac{1}{\sqrt{b-t_0}} + c\right)^2 = \alpha\left(1 + c\sqrt{b-t_0}\right)^2 < c$$

and hence (5.3.3) follows. Hence, P can be extended past b, and a similar argument shows that it can be extended before a. $\square$

Lemma 5.3.7. *Let $x \in \mathfrak{k}^3$ and $s > 0$ be such that $sx \in V$. Then, P^x is defined on $[1-s,1]$ and*

$$\rho(sx) = \frac{s}{4}\int_{1-s}^{1} \left(2\|P_1^x(t)\|^2 + \|P_2^x(t)\|^2 + \|P_3^x(t)\|^2\right) dt.$$

Proof. We have $P^{sx}(t) = sP^x(1 + s(t-1))$ by Lemma 5.2.6 and the result follows by a change of variable. $\square$

Lemma 5.3.8. *There exists a compact set $C \subseteq V$ and a constant $\alpha > 0$ such that*

$$\|x\| \leq \alpha\rho(x)$$

for all $x \in V \setminus C$.

Proof. Let $\varepsilon > 0$ be such that $S_\varepsilon := \{y \in \mathfrak{k}^3 : \|y\| = \varepsilon\} \subseteq V$. Then, $\rho(S_\varepsilon)$ is compact and hence attains a minimum $M \geq 0$. We have $\rho(x) = 0$ if and only if $x = 0$, so, in fact, $M > 0$. We claim that the lemma holds with $\alpha := \frac{\varepsilon}{M}$ and $C := \{x \in \mathfrak{k}^3 : \|x\| \leq \varepsilon\}$. To show this, let $x \in V$ with $\|x\| > \varepsilon$ and write $x = sy$ where $y \in S_\varepsilon$ and $s > 1$. Then, by Lemma 5.3.7, we have

$$\begin{aligned}
\rho(x) &= \frac{s}{4}\int_{1-s}^{1} \left(2\|P_1^y(t)\|^2 + \|P_2^y(t)\|^2 + \|P_3^y(t)\|^2\right) dt \\
&\geq \frac{s}{4}\int_{0}^{1} \left(2\|P_1^y(t)\|^2 + \|P_2^y(t)\|^2 + \|P_3^y(t)\|^2\right) dt \\
&= s\rho(y) \\
&\geq sM,
\end{aligned}$$

and hence $\|x\| = \varepsilon s \leq \varepsilon\frac{\rho(x)}{M} = \alpha\rho(x)$. □

Proposition 5.3.9. *The function ρ is proper, and hence so is f.*

Proof. We need to show that $\rho^{-1}([0, c])$ is compact for all $c > 0$. By Lemma 5.3.8, $\rho^{-1}([0, c])$ is bounded, and hence it suffices to show that it is closed in $\mathfrak{k}^3$ (which does not follow from continuity since the domain V of ρ is open in $\mathfrak{k}^3$). Let $x_n \in V$ be a sequence which converges to some $x \in \mathfrak{k}^3$ and satisfies $\rho(x_n) \leq c$ for all n. Suppose, for contradiction, that $x \notin V$. Since V is star-shaped about 0, we must have $\{t > 0 : tx \in V\} = (0, s)$ for some $0 < s \leq 1$. Moreover,

$$\rho(tx) = \frac{t}{4}\int_{1-t}^{1} \left(2\|P_1^x(u)\|^2 + \|P_2^x(u)\|^2 + \|P_3^x(u)\|^2\right) du,$$

must tend to infinity as $t \to s$, as otherwise P^x is defined on $[1 - s, 1]$ by Lemma 5.3.6 and hence $P^{sx}(t)$ is defined on $[0, 1]$ by Lemma 5.2.6, which implies that $sx \in V$. Thus, $\rho(tx) \to \infty$ as $t \to s$, so, in particular, there exists $t < s$ such that $\rho(tx) > c$. But $tx \in V$, so there exists $r > 0$ such that the open ball $B_r(tx)$ is contained in V and $\rho(y) > c$ for all $y \in B_r(tx)$. Let n be such that $x_n \in B_r(x)$. Then, $|tx_n - tx| < tr < r$ so $\rho(tx_n) > c$. But since $t < 1$ we have

$$\begin{aligned}
\rho(x_n) &= \frac{1}{4}\int_{0}^{1} \left(2\|P_1^{x_n}(u)\|^2 + \|P_2^{x_n}(u)\|^2 + \|P_3^{x_n}(u)\|^2\right) du \\
&\geq \frac{t}{4}\int_{1-t}^{1} \left(2\|P_1^{x_n}(u)\|^2 + \|P_2^{x_n}(u)\|^2 + \|P_3^{x_n}(u)\|^2\right) du \\
&= \rho(tx_n) \\
&> c,
\end{aligned}$$

a contradiction. Thus, $x \in V$, and hence $\rho^{-1}([0,c])$ is closed in $\mathfrak{k}^3$. □

Now, we view T^*G as an affine variety in $\mathbb{C}^N$ for some $N > 0$ and endow $\mathbb{C}^N$ with a norm $|\cdot|$. Then, the inclusion $\mathbb{C}[T^*G] \subseteq o(e^f)$ will be a consequence of the following estimate.

Proposition 5.3.10. *There exist $b, c > 0$ and a compact set $B \subseteq T^*G$ such that*

$$|p|^2 \leq b e^{c\sqrt{f(p)}}, \quad \text{for all } p \in T^*G \setminus B.$$

Proof. We can view the diffeomorphism $\psi : K \times V \to G \times \mathfrak{g}$ as taking values in $\mathbb{C}^N$. Hence, the proposition can be reformulated as saying that there exist $b, c > 0$ and a compact set $D \subseteq V$ such that

$$|\psi(k,x)|^2 < b e^{c\sqrt{\rho(x)}} \tag{5.3.4}$$

for all $(k,x) \in K \times (V \setminus D)$.

Let us first be more explicit about $\mathbb{C}^N$ and the choice of a norm on it. Throughout this proof, we view G as a subgroup of $\mathrm{SL}(n,\mathbb{C})$ for some $n > 0$ and K as a subgroup of $\mathrm{SU}(n)$. Then, $G \times \mathfrak{g}$ can be viewed as an affine variety in $\mathbb{C}^{2n^2} = \mathfrak{gl}(n,\mathbb{C}) \times \mathfrak{gl}(n,\mathbb{C})$. Take the standard norm $|\cdot|$ on $\mathfrak{gl}(n,\mathbb{C})$, i.e. $|x|^2 = \sum_{ij} |x_{ij}|^2$, and the product norm on $\mathfrak{gl}(n,\mathbb{C}) \times \mathfrak{gl}(n,\mathbb{C})$. The latter is the norm we use to estimate ψ. First, we have the following basic fact.

Lemma 5.3.11. *$|xy| \leq n^2 |x||y|$ for all $x, y \in \mathfrak{gl}(n,\mathbb{C})$.*

Proof. Indeed,

$$|xy|^2 = \sum_{i,j}\left(\sum_k x_{ik}y_{kj}\right)^2 \leq \sum_{i,j}\left(\sum_k |x||y|\right)^2 = n^4|x|^2|y|^2. \qquad \square$$

We will also need to following standard result from the theory of ODEs.

Lemma 5.3.12 (Grönwall [59])**.** *Let $u : [0,t_0] \to \mathbb{R}$ be differentiable. If there is a continuous function $\beta : [0,t_0] \to \mathbb{R}$ such that $\dot{u}(t) \leq \beta(t)u(t)$ for all $t \in [0,t_0]$, then*

$$u(t) \leq u(0)e^{\int_0^t \beta(s)ds}$$

for all $t \in [0,t_0]$. □

Finally, we observe the following fact.

Lemma 5.3.13. *Let K be a compact connected Lie group. Then, there exists a compact set $B \subseteq \mathfrak{k}$ such that $K = \exp(B)$.*

Proof. Let $T \subseteq K$ be a maximal torus and let $\mathfrak{t} \subseteq \mathfrak{k}$ be its Lie algebra. Then, the restriction of exp to $\mathfrak{t}$ is of the form

$$\mathbb{R}^m \longrightarrow (S^1)^m, \quad (\theta_1, \ldots, \theta_m) \longmapsto (e^{i\theta_1}, \ldots, e^{i\theta_m}).$$

Thus, $B' := [0, 2\pi]^m$ is a compact subset of $\mathfrak{t}$ such that $\exp(B') = T$. Now, let $B := K \cdot B' = \{\mathrm{Ad}_k x : k \in K, x \in B'\}$. Then, B is compact since K and B' are compact. Since every element of K is conjugate to an element of T (see, e.g., [90, Theorem 4.36]), we have $\exp(B) = K$. □

With these preliminaries, we can now prove the proposition. Let B be as in Lemma 5.3.13 and let $r > 0$ be such that B is contained in the ball of radius r centred at 0 in $\mathfrak{k}$ in the norm $|\cdot|$. The restriction of $|\cdot|$ to $\mathfrak{k}$ might not be the same as the norm $\|\cdot\|$ induced by the K-invariant inner-product, but since $\mathfrak{k}$ is finite-dimensional, there exist $c_0, c_1 > 0$ such that $c_0\|x\| \leq |x| \leq c_1\|x\|$ for all $x \in \mathfrak{k}$. Let

$$b := 2ne^{2n^2 r} \quad \text{and} \quad c := 2\sqrt{2}n^2 c_1,$$

where the integer n is the same as the one used for the embedding $G \subseteq \mathrm{SL}(n, \mathbb{C})$. By Lemma 5.3.8, there exist a compact set $C \subseteq V$ and $\alpha > 0$ such that $|x|^2 \leq \alpha\rho(x)^2$ for all $x \in V \setminus C$. Note that the set

$$\{t \in [0, \infty) : \frac{b}{2} e^{c\sqrt{t}} \leq \alpha t^2\}$$

is contained in an interval $[0, \beta]$ for some $\beta > 0$, so

$$\begin{aligned} D &:= \{x \in V : \frac{b}{2} e^{c\sqrt{\rho(x)}} \leq |x|^2\} \\ &\subseteq C \cup \{x \in V : \frac{b}{2} e^{c\sqrt{\rho(x)}} \leq \alpha\rho(x)^2\} \\ &\subseteq C \cup \rho^{-1}([0, \beta]). \end{aligned}$$

Since ρ is proper (Lemma 5.3.9), $\rho^{-1}([0, \beta])$ is compact, and hence D is also compact. We claim that (5.3.4) holds with those b, c and D.

Let $(k, x) \in K \times V$ and write $k = e^y$ for some $y \in B$ (so $|y| \leq r$). By Lemma 5.3.3, we have

$$\psi(k, x) = (g(0)^{-1}, x_2 + ix_3),$$

where $g : [0, 1] \to G$ is the unique solution to the linear ODE

$$\dot{g}(t) = g(t)(y + i\,\mathrm{Ad}_{e^{(1-t)y}} P_1^x(t)), \quad g(1) = 1.$$

Since $|\psi(k,x)|^2 = |g(0)^{-1}|^2 + |x_2 + ix_3|^2$, we first need to estimate $|g(0)^{-1}|^2$. To do so, we will apply Grönwall's Lemma to the function

$$u : [0,1] \longrightarrow \mathbb{R}, \quad u(t) = |h(t)|^2,$$

where $h(t) \coloneqq g(1-t)^{-1}$. We have

$$\dot{h}(t) = g(1-t)^{-1}\dot{g}(1-t)g(1-t)^{-1} = (y + i\,\mathrm{Ad}_{e^{ty}} P_1^x(1-t))h(t).$$

Hence, by the Cauchy-Schwarz inequality and Lemma 5.3.11,

$$\begin{aligned}
\dot{u}(t) &= 2\langle h(t), \dot{h}(t)\rangle \\
&= 2\langle h(t), (y + i\,\mathrm{Ad}_{e^{ty}} P_1^x(1-t))h(t)\rangle \\
&\leq 2|h(t)||(y + i\,\mathrm{Ad}_{e^{ty}} P_1^x(1-t))h(t)| \\
&\leq 2n^2|y + i\,\mathrm{Ad}_{e^{ty}} P_1^x(1-t)||h(t)|^2 \\
&\leq 2n^2(|y| + |\,\mathrm{Ad}_{e^{ty}} P_1^x(1-t)|)u(t) \\
&\leq 2n^2(r + c_1\|P_1^x(1-t)\|)u(t).
\end{aligned}$$

By Grönwall's Lemma,

$$\begin{aligned}
|g(0)^{-1}|^2 = u(1) &\leq u(0)\exp\left(2n^2\int_0^1 (r + c_1\|P_1^x(1-s)\|)ds\right) \\
&= n\exp\left(2n^2 r + 2n^2 c_1\int_0^1 \|P_1^x(s)\|ds\right).
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
\int_0^1 \|P_1^x(s)\|ds &\leq \left(\int_0^1 \|P_1^x(s)\|^2 ds\right)^{1/2}\left(\int_0^1 ds\right)^{1/2} \\
&\leq \left(\frac{1}{2}\int_0^1 (2\|P_1^x(s)\|^2 + \|P_2^x(s)\|^2 + \|P_3^x(s)\|^2)ds\right)^{1/2} \\
&= \sqrt{2\rho(x)},
\end{aligned}$$

so

$$|g(0)^{-1}|^2 \leq n\exp\left(2n^2 r + 2\sqrt{2}n^2 c_1\sqrt{\rho(x)}\right) = \frac{b}{2}e^{c\sqrt{\rho(x)}}.$$

Therefore,

$$|\psi(k,x)|^2 = |g(0)^{-1}|^2 + |x_2 + ix_3|^2 \leq \frac{b}{2}e^{c\sqrt{\rho(x)}} + |x|^2.$$

Then, when $x \in V \setminus D$, we have

$$|\psi(k,x)|^2 \leq \frac{b}{2}e^{c\sqrt{\rho(x)}} + |x|^2 < be^{c\sqrt{\rho(x)}},$$

where the last inequality follows from the definition of D. □

Lemma 5.3.14. *The function e^f dominates $\alpha e^{\beta\sqrt{f}}$ for all $\alpha, \beta > 0$.*

Proof. This follows directly from properness of f. Indeed, for all $\gamma > 0$, the set

$$B \coloneqq \{t \in [0, \infty) : \alpha e^{\beta\sqrt{t}} \geq \gamma e^t\}$$

is compact, so $C \coloneqq f^{-1}(B)$ is also compact. If $p \notin C$ we have $f(p) \notin B$ and hence $\alpha e^{\beta\sqrt{f(p)}} < \gamma e^{f(p)}$. $\square$

Proof of Proposition 5.3.1. Let $\gamma > 0$ and $u \in \mathbb{C}[T^*G]$. We want to show that there exists a compact set $C \subseteq T^*G$ such that $|u(p)| \leq \gamma e^{f(p)}$ for all $p \in T^*G \setminus C$. We view T^*G as an affine variety in $\mathbb{C}^N$ for some $N > 0$ and write $u(p) = \sum a_{i_1 \dots i_N} p_1^{i_1} \cdots p_N^{i_N}$, where $p = (p_1, \dots, p_N) \in \mathbb{C}^N$. Then, $|u(p)| \leq \sum_{k=0}^n a_k |p|^k$ for some $a_k > 0$ and $n \geq 0$. By Proposition 5.3.10, there exists a compact set $B \subseteq T^*G$ and $b, c > 0$ such that $|p|^2 \leq b e^{c\sqrt{f(p)}}$ for all $p \in T^*G \setminus B$. Hence, for all $p \in T^*G \setminus B$ we have $|u(p)| \leq \sum_{k=0}^n a_k b^{\frac{k}{2}} e^{\frac{ck}{2}\sqrt{f(p)}}$. By Lemma 5.3.14, for all $k \in \{0, \dots, n\}$ there exists a compact set C_k such that if $p \notin C_k$ then $a_k b^{\frac{k}{2}} e^{\frac{ck}{2}\sqrt{f(p)}} \leq \frac{\gamma}{n+1} e^{f(p)}$. Let $C = B \cup C_0 \cup \dots \cup C_n$. Then, for all $p \notin C$, we have $|u(p)| \leq \sum_{k=0}^n \frac{\gamma}{n+1} e^{f(p)} = \gamma e^{f(p)}$. $\square$

5.3.3 The moment map (proof of Proposition 5.3.2)

We now prove Proposition 5.3.2, which says that the map

$$\nu_{\mathbb{R}} : \mathcal{M} \longrightarrow \mathfrak{k}^* \times \mathfrak{k}^*, \quad \nu_{\mathbb{R}}(A) = (A_1(0), -A_1(1))$$

satisfies

$$\nu_{\mathbb{R}}(A)(z) = dF(\mathsf{I} z_A^\#), \quad \text{for all } A \in \mathcal{M} \text{ and } z \in \mathfrak{k} \times \mathfrak{k},$$

where $F : \mathcal{M} \to \mathbb{R}$ is the global Kähler potential of Proposition 5.2.4.

We know by Lemma 4.5.2 that $dF(\mathsf{I} z_A^\#)$ defines a moment map. Since $\nu_{\mathbb{R}}(0) = (0, 0)$ and moment maps are unique up to an additive constant, it suffices to show that $dF(\mathsf{I} z_0^\#) = 0$ for all $z \in \mathfrak{k} \times \mathfrak{k}$. Equivalently, we show that $d\tilde{\rho}(\mathsf{I} z_0^\#) = 0$, where $\tilde{\rho} : K \times V \to \mathbb{R}$ is the pullback of F by the diffeomorphism $K \times V \to \mathcal{M}$, i.e. $\tilde{\rho}(k, x) = \rho(x)$. If $z = (z_1, z_2) \in \mathfrak{k} \times \mathfrak{k}$ then on $T_{(1,0)}T^*G = \mathfrak{g} \times \mathfrak{g}$ we have

$$\mathsf{I}(z_1, z_2)^\#_{(1,0)} = \left.\frac{d}{dt}\right|_{t=0} (\exp(itz_1)\exp(-itz_2), 0) = (i(z_1 - z_2), 0).$$

Thus, by Lemma 5.3.4, it suffices to show that $d\tilde{\rho}_{(1,0)}$ vanishes on $0 \times \mathfrak{k} \times 0 \times 0 \subseteq \mathfrak{k}^4$. Let $x \in \mathfrak{k}^3$ be arbitrary. Then, for $s > 0$ small enough we have $sx \in V$ and, by

Lemma 5.2.6, $P^{sx}(t) = sP^x(1 - s(t-1))$. Thus,

$$\begin{aligned} d\tilde{\rho}_{(1,0)}(0, x_1, x_2, x_3) &= \frac{d}{ds}\bigg|_{s=0} \tilde{\rho}(1, sx) \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{1}{4}\int_0^1 \left(2\|P_1^{sx}(t)\|^2 + \|P_2^{sx}(t)\|^2 + \|P_3^{sx}(t)\|^2\right) dt \\ &= \frac{d}{ds}\bigg|_{s=0} \frac{s^2}{4}(\text{some smooth function of } s) \\ &= 0. \end{aligned}$$

This concludes the proof of Proposition 5.3.2. □

5.3.4 Hyperkähler quotients (proof of Theorem 5.1.2)

We now use Theorem 5.1.1 to deduce Theorem 5.1.2, i.e. $T^*G /\!/\!/_{\mu_\xi} H \cong \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) /\!/_\chi H_{\mathbb{C}}$ as complex-analytic and partitioned spaces:

Proof of Theorem 5.1.2. We have $\mu_{\mathbb{R}}(p)(z) = df(\mathsf{I} z_p^{\#})$ for all $p \in T^*G$ and $z \in \mathfrak{h}$, so $(T^*G)^{(\mu_{\mathbb{R}} - \xi_{\mathbb{R}})\text{-ss}} = (T^*G)^{\chi\text{-ss}}$ by Theorem 4.1.2, and hence $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})^{(\mu_{\mathbb{R}} - \xi_{\mathbb{R}})\text{-ss}} = \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})^{\chi\text{-ss}}$. Now, $T^*G /\!/\!/_{\mu_\xi} H$ is the analytic Hilbert quotient $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})^{(\mu_{\mathbb{R}} - \xi_{\mathbb{R}})\text{-ss}} /\!/ H_{\mathbb{C}}$, and GIT quotients are analytic Hilbert quotients, so the result follows from uniqueness of categorical quotients. □

Chapter 6

Stratified hyperkähler spaces from semisimple Lie algebras

6.1 Introduction

We have seen in the previous chapters that Lie theory provides a large class of stratified hyperkähler spaces. For any compact Lie group K, closed subgroup H of $K \times K$, and $\xi \in Z_{\mathfrak{h}^*} \otimes \mathbb{R}^3$, the hyperkähler quotient $T^*G /\!/\!/_{\mu_\xi} H$ is a stratified hyperkähler space (where $G \coloneqq K_{\mathbb{C}}$ and μ_ξ is as in Theorem 5.1.2). In this chapter, we analyse in detail a special case, namely, when $\xi = 0$ and $H = T_K \times T_K$ for a maximal torus $T_K \subseteq K$. This space will be shown to depend only on the Lie algebra $\mathfrak{g}$ of $G \coloneqq K_{\mathbb{C}}$, and hence will be denoted by

$$\mathcal{D}(\mathfrak{g}) \coloneqq T^*G /\!/\!/_{\mu} (T_K \times T_K).$$

The main goal of this chapter is to describe the stratification poset of $\mathcal{D}(\mathfrak{g})$, which turns out to be very rich and admits an explicit combinatorial description: it is isomorphic to the poset of root subsystems of $\mathfrak{g}$. In particular, we can compute the stratification structure of $\mathcal{D}(\mathfrak{g})$ explicitly for any given $\mathfrak{g}$; see §6.5 for examples. This chapter is based on the author's paper [106].

Our motivation for studying these singular spaces is Dancer [32, pp. 88–89], who studied the case where $K = \mathrm{SU}(2)$, i.e. the hyperkähler quotient of $T^*\mathrm{SL}(2, \mathbb{C})$ by $\mathrm{U}(1) \times \mathrm{U}(1)$ at level zero. He showed that this space is isomorphic to the D_2-surface $x^2 - zy^2 = z$: a complex surface with two isolated singularities. A corollary of our results is the following generalisation of Dancer's example: for any complex semisimple Lie algebra $\mathfrak{g}$, $\mathcal{D}(\mathfrak{g})$ is a normal irreducible complex affine variety, its most singular stratum is a finite set in bijection with the Weyl group of $\mathfrak{g}$, and its smooth locus is a connected open dense subset of complex dimension $2(\dim \mathfrak{g} - 2\,\mathrm{rk}\,\mathfrak{g})$.

The strata can also be described individually. We will also show that the stratum associated with a root subsystem $\Psi \subseteq \Phi$ is isomorphic as a hyperkähler manifold to a disjoint union of copies of the smooth locus of $\mathcal{D}(\mathfrak{g}_\Psi)$, where $\mathfrak{g}_\Psi$ is the semisimple Lie algebra with root system Ψ. Moreover, the number of copies is known and will be described Lie theoretically. We will also discuss resolutions of singularities of $\mathcal{D}(\mathfrak{g})$ obtained by shifting the real part of the moment map.

All these results rely heavily on the Kempf–Ness type theorem for T^*G proved in the last chapter, which enables us to study $\mathcal{D}(\mathfrak{g})$ and its orbit-type partition algebraically.

The chapter is organised as follows. In §6.2, we give background material on root subsystems of semisimple Lie algebras. We then give precise statements of our results in §6.3 and prove them in §6.4. Finally, we study examples in §6.5.

6.2 Background on root subsystems

A crucial ingredient in the description of the stratification poset of $\mathcal{D}(\mathfrak{g})$ is the notion of root subsystems and regular subalgebras, which we review in this section. This material goes back to Dynkin [43] and can also be found in textbook form in Onishchick–Vinberg [122, Chapter 6].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{t}$, $\Phi \subseteq \mathfrak{t}^*$ the set of roots, and $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the Cartan decomposition. A subset Ψ of Φ is called a **root subsystem** if

(1) $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi \Longrightarrow \alpha + \beta \in \Psi$,

(2) $\alpha \in \Psi \Longleftrightarrow -\alpha \in \Psi$.

Equivalently, Ψ is a root subsystem if $(\operatorname{span}_\mathbb{Z} \Psi) \cap \Phi = \Psi$. A root subsystem is always a root system itself. We use the notation $\Psi \leq \Phi$ to say that Ψ is a root subsystem of Φ. This gives a *partial order* on the set of root subsystems.

Root subsystems are closely related to the notion of regular subalgebras. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called **regular** if there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $[\mathfrak{t}, \mathfrak{h}] \subseteq \mathfrak{h}$. We denote by $\mathcal{C}_\mathfrak{g}$ the set of conjugacy classes of regular semisimple subalgebras of $\mathfrak{g}$. Since all Cartan subalgebras are conjugate, every element of $\mathcal{C}_\mathfrak{g}$ has a representative which is regular with respect to a fixed Cartan subalgebra $\mathfrak{t}$.

Proposition 6.2.1. *The set of semisimple subalgebras of* $\mathfrak{g}$ *that are regular with respect to* $\mathfrak{t}$ *is in one-to-one correspondence with the set of root subsystems of* Φ*. The* *correspondence associates to* $\Psi \leq \Phi$ *the subalgebra*

$$\mathfrak{g}_\Psi := \mathfrak{t}_\Psi \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha, \tag{6.2.1}$$

where $\mathfrak{t}_\Psi$ *is the span of the coroots* h_α *for* $\alpha \in \Psi$. □

Recall that if k is the number of simple factors of $\mathfrak{g}$, there is an $(\mathbb{R}_{>0})^k$ family of non-degenerate symmetric invariant bilinear forms on $\mathfrak{g}$ which are positive definite on the real span of the coroots. We call those bilinear forms **admissible**. Equivalently, a bilinear form is admissible if its restriction to some (and hence all) compact real form(s) is negative definite. For example, the Killing form is admissible.

Proposition 6.2.2. *Any admissible bilinear form on* $\mathfrak{g}$ *remains admissible on* $\mathfrak{g}_\Psi$. *Also,* $\mathfrak{t}_\Psi$ *is a Cartan subalgebra of* $\mathfrak{g}_\Psi$, (6.2.1) *is the corresponding Cartan decomposition, and the map* $\mathfrak{t}^* \to \mathfrak{t}_\Psi^*$ *restricts to an isomorphism of abstract root systems from* $\operatorname{span}_{\mathbb{R}} \Psi$ *to the root system of* $\mathfrak{g}_\Psi$ *with respect to the above bilinear form.* □

Let $W_\mathfrak{g} = W_\Phi = W(\mathfrak{g}, \mathfrak{t})$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$. It acts on the set of root subsystems by $w \cdot \Psi := \{w \cdot \alpha : \alpha \in \Psi\}$.

Proposition 6.2.3. *Let* Ψ_1 *and* Ψ_2 *be two root subsystems. Then,* $\mathfrak{g}_{\Psi_1} = \mathfrak{g}_{\Psi_2}$ *if and only if there exists* $w \in W_\Phi$ *such that* $w \cdot \Psi_1 = \Psi_2$. *Thus, the map* $\Psi \mapsto \mathfrak{g}_\Psi$ *descends to a bijection* $\{$*root subsystems of* $\Phi\}/W_\Phi \to \mathcal{C}_\mathfrak{g}$. □

There is also a natural partial order on $\mathcal{C}_\mathfrak{g}$ induced by inclusion: we say that $[\mathfrak{h}_1] \leq [\mathfrak{h}_2]$ if there exists an inner automorphism φ of $\mathfrak{g}$ such that $\varphi(\mathfrak{h}_1) \subseteq \mathfrak{h}_2$.

Let $\mathfrak{h}$ be a regular semisimple subalgebra of $\mathfrak{g}$, say $\mathfrak{h} = \mathfrak{g}_\Psi$ for some $\Psi \subseteq \Phi$. Then, the Weyl group $W_\mathfrak{h}$ of $\mathfrak{h}$ can be viewed as a subgroup of $W_\mathfrak{g}$, namely the group generated by the simple reflections s_α for $\alpha \in \Psi$. In particular, the index $|W_\mathfrak{g} : W_\mathfrak{h}| := |W_\mathfrak{g}|/|W_\mathfrak{h}|$ is a well-defined positive integer. We define the **embedding number** of $\mathfrak{h}$ in $\mathfrak{g}$ to be

$$m_\mathfrak{g}(\mathfrak{h}) = |W_\mathfrak{g} : W_\mathfrak{h}||\{w \cdot \Psi : w \in W_\mathfrak{g}\}|, \tag{6.2.2}$$

where the second factor is the number of root subsystems in the $W_\mathfrak{g}$-orbit of Ψ. The embedding number is a positive integer which depends on the particular way in which $\mathfrak{h}$ embeds in $\mathfrak{g}$. It depends only on the conjugacy class of $\mathfrak{h}$, and hence $m_\mathfrak{g}$ descends to a map $m_\mathfrak{g} : \mathcal{C}_\mathfrak{g} \to \mathbb{N}$. Note that we always have $m_\mathfrak{g}(\mathfrak{g}) = 1$ and $m_\mathfrak{g}(0) = |W_\mathfrak{g}|$. These numbers will be important for our study of the stratified space $\mathcal{D}(\mathfrak{g})$, as they will count the number of connected components of the strata.

6.3 Statement of results

Let G be a connected complex semisimple Lie group, $K \subseteq G$ a maximal compact subgroup, and $T_K \subseteq K$ a maximal torus. Let $\mu : T^*G \to (\mathfrak{t}_\mathfrak{k}^* \times \mathfrak{t}_\mathfrak{k}^*) \otimes \mathbb{R}^3$ be the hyperkähler moment map for the action of $T_K \times T_K$ on T^*G obtained by restricting Dancer–Swann's hyperkähler moment map ν in Proposition 5.2.2. Define

$$\mathcal{D}(G) \coloneqq T^*G /\!\!/\!\!/_\mu (T_K \times T_K).$$

Since the action of $K \times K$ on T^*G is fully integrable and commutes with the SO(3)-hyperkähler rotation, $\mathcal{D}(G)$ is a stratified hyperkähler space all of whose complex-analytic structures are isomorphic. Without loss of generality, we focus on describing the one $\mathcal{O}_\mathsf{I}$ descending from the standard complex structure on T^*G.

Note that $\mathcal{D}(G)$ does not depend on the choice of maximal compact subgroup K and maximal torus T_K since both are unique up to conjugation. On the other hand, the hyperkähler structure depends on the choice of invariant inner-product on $\mathfrak{k} \coloneqq \mathrm{Lie}(K)$, or equivalently, on the choice of admissible bilinear form on $\mathfrak{g}$. A crucial ingredient in the description of $\mathcal{D}(G)$ is that it depends *only* on $\mathfrak{g}$ and its bilinear form:

Theorem 6.3.1. *Let $\tilde{G}$ be another complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Then, there is an isomorphism of stratified hyperkähler spaces $\mathcal{D}(\tilde{G}) \cong \mathcal{D}(G)$.*

Hence, we will use the notation $\mathcal{D}(\mathfrak{g})$ thereafter.

As a stratified space, $\mathcal{D}(\mathfrak{g})$ has an associated stratification poset $(\mathcal{P}, \leq)$ (see §2.1). Since $\mathcal{D}(\mathfrak{g})$ depends only on $\mathfrak{g}$, this poset is an interesting structure canonically associated with the semisimple Lie algebra $\mathfrak{g}$, and it is natural to ask if it can be described by Lie theory alone. This is indeed the case: it is isomorphic to the poset of root subsystems of Φ. Let us now explain how to get this isomorphism.

Let $\mathcal{O} \subseteq \mathfrak{g}$ be a regular semisimple adjoint orbit and consider the action of $T \coloneqq (T_K)_\mathbb{C}$ on the cotangent bundle $T^*\mathcal{O} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ by $t \cdot (x, \eta) = (\mathrm{Ad}_t\, x, \mathrm{Ad}_t^*\, \eta)$. Let $\mathfrak{t}^\circ$ be the annihilator of $\mathfrak{t}$ in $\mathfrak{g}^*$. Using the Kempf–Ness type theorem for T^*G (Theorem 5.1.2) and the normality of hyperkähler quotients by abelian groups (Corollary 3.1.6), we will get the following result.

Proposition 6.3.2. *The hyperkähler quotient $\mathcal{D}(\mathfrak{g})$ is isomorphic as a complex-analytic and partitioned space to the GIT quotient $(T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!\!/ T$, and the latter is a normal and irreducible complex affine variety.*

In particular, the orbit-type strata of $\mathcal{D}(\mathfrak{g})$ are of the form $(T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ))^{\mathrm{ps}}_Z/T$ for some closed subgroups Z of T (we use the subscript Z instead of (Z) since T is abelian). Let Φ be the root system of $\mathfrak{g}$ and view it as a subset of $\mathrm{Hom}(T, \mathbb{C}^*)$. For every root subsystem $\Psi \leq \Phi$, let

$$\begin{aligned} Z_\Psi &\coloneqq \{t \in T : \alpha(t) = 1 \text{ for all } \alpha \in \Psi\} \\ \mathcal{D}(\mathfrak{g})_\Psi &\coloneqq (T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ))^{\mathrm{ps}}_{Z_\Psi}/T. \end{aligned} \tag{6.3.1}$$

Theorem 6.3.3. *The map*

$$\{\text{root subsystems of } \Phi\} \longrightarrow \{\text{orbit-type strata of } \mathcal{D}(\mathfrak{g})\}, \quad \Psi \longmapsto \mathcal{D}(\mathfrak{g})_\Psi$$

is an isomorphism of posets.

In particular, there is a top stratum $\mathcal{D}(\mathfrak{g})^{\mathrm{top}} \coloneqq \mathcal{D}(\mathfrak{g})_\Phi$ corresponding to the root system Φ itself and a bottom stratum $\mathcal{D}(\mathfrak{g})^{\mathrm{bottom}} \coloneqq \mathcal{D}(\mathfrak{g})_\emptyset$ corresponding to the trivial root subsystem $\emptyset$. Recall from §6.2 that any admissible bilinear form on $\mathfrak{g}$ remains admissible on $\mathfrak{g}_\Psi$. Hence, it induces a natural hyperkähler structure on $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}}$. Let W be the Weyl group of $\mathfrak{g}$ and W_Ψ the Weyl group of $\mathfrak{g}_\Psi$.

Theorem 6.3.4.

(1) $\mathcal{D}(\mathfrak{g})^{\mathrm{top}}$ *is a connected open dense subset of real dimension* $4(\dim \mathfrak{g} - 2\,\mathrm{rk}\,\mathfrak{g})$.

(2) $\mathcal{D}(\mathfrak{g})^{\mathrm{bottom}}$ *is a finite set of* $|W|$ *elements.*

(3) *For all* $\Psi \leq \Phi$*, the stratum* $\mathcal{D}(\mathfrak{g})_\Psi$ *is isomorphic as a hyperkähler manifold to a disjoint union of* $|W : W_\Psi|$ *copies of* $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}}$.

A stratification can always be refined arbitrarily by taking submanifolds of the strata. It is thus desirable to get a stratification as coarse as possible. We describe next one way of coarsening the orbit-type stratification of $\mathcal{D}(\mathfrak{g})$. Recall from §6.2 that $\mathcal{C}_\mathfrak{g}$ is the set of conjugacy classes of regular semisimple subalgebras of $\mathfrak{g}$ and $m_\mathfrak{g} : \mathcal{C}_\mathfrak{g} \to \mathbb{N}$ is the map of embedding numbers (6.2.2). For each $[\mathfrak{h}] \in \mathcal{C}_\mathfrak{g}$, let

$$\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} \coloneqq \bigcup_{[\mathfrak{g}_\Psi] = [\mathfrak{h}]} \mathcal{D}(\mathfrak{g})_\Psi.$$

Theorem 6.3.5. *The partition* $\mathcal{P} = \{\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} : [\mathfrak{h}] \in \mathcal{C}_\mathfrak{g}\}$ *is a stratification of* $\mathcal{D}(\mathfrak{g})$ *and the map* $\mathcal{C}_\mathfrak{g} \to \mathcal{P}$, $[\mathfrak{h}] \mapsto \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ *an isomorphism of posets. Moreover,* $\mathcal{D}(\mathfrak{g})_{[\mathfrak{g}]} = \mathcal{D}(\mathfrak{g})^{\mathrm{top}}$, $\mathcal{D}(\mathfrak{g})_{[0]} = \mathcal{D}(\mathfrak{g})^{\mathrm{bottom}}$ *and* $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ *is isomorphic as a hyperkähler manifold to a disjoint union of* $m_\mathfrak{g}(\mathfrak{h})$ *copies of* $\mathcal{D}(\mathfrak{h})^{\mathrm{top}}$.

We will give examples of this stratification in §6.5.

Next, we construct resolutions of singularities by shifting the real part of the moment map. For each $\zeta \in \mathfrak{t}_\mathfrak{k}^*$, let $\mathcal{D}_\zeta(\mathfrak{g})$ be the hyperkähler quotient of T^*G by $T_K \times T_K$ with the moment map shifted by $((\zeta, 0), (0, 0), (0, 0)) \in (\mathfrak{t}_\mathfrak{k}^* \times \mathfrak{t}_\mathfrak{k}^*) \otimes \mathbb{R}^3$. Let

$$\mathfrak{t}_\circ^* := \mathfrak{t}^* - \bigcup_{\Psi < \Phi} \operatorname{span} \Psi. \tag{6.3.2}$$

Theorem 6.3.6. *We have $\mathfrak{t}_\circ^* \neq \emptyset$ if and only if $\mathfrak{t}_\circ^* \cap \mathfrak{t}_\mathfrak{k}^* \neq \emptyset$ if and only if $\mathfrak{g}$ is of type A. In that case, for all $\zeta \in \mathfrak{t}_\circ^* \cap \mathfrak{t}_\mathfrak{k}^*$, $\mathcal{D}_\zeta(\mathfrak{g})$ is a smooth hyperkähler manifold. Moreover, there is a character $\chi : T_K \to S^1$ such that the element $\zeta := i\, d\chi \in \mathfrak{t}_\mathfrak{k}^*$ lies in $\mathfrak{t}_\circ^*$ and the natural map $\pi : \mathcal{D}_\zeta(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ is a complex resolution of singularities.*

6.4 Stratification of $\mathcal{D}(\mathfrak{g})$ into hyperkähler manifolds

6.4.1 $\mathcal{D}(G)$ depends only on the Lie algebra of G

In this section, we prove Theorem 6.3.1, which says that $\mathcal{D}(G) := T^*G /\!/\!/_\mu (T_K \times T_K)$ depends only on the Lie algebra $\mathfrak{g}$ of G.

The first step is the explicit algebraic expression for $\mathcal{D}(G)$ claimed in Proposition 6.3.2. By Theorem 5.1.2 we have $\mathcal{D}(G) = \mu_\mathbb{C}^{-1}(0) /\!/ (T \times T)$, and from (3.3.2),

$$\mu_\mathbb{C}^{-1}(0) = \{(g, \xi) \in G \times \mathfrak{t}^\circ : \operatorname{Ad}_g^* \xi \in \mathfrak{t}^\circ\}.$$

Let $\mathcal{O}$ be a regular semisimple orbit in $\mathfrak{g}$, say $\mathcal{O} = G \cdot \tau$ for some $\tau \in \mathfrak{t}^{\mathrm{reg}}$. Then, the map $G \to \mathcal{O} : g \mapsto \operatorname{Ad}_g \tau$ is a principal T-bundle, and the associated vector bundle $G \times_T \mathfrak{t}^\circ$ is isomorphic to the cotangent bundle $T^*\mathcal{O}$ via $(g, \xi) \mapsto (\operatorname{Ad}_g \tau, \operatorname{Ad}_g^* \xi)$ (see, e.g., [29, Lemma 1.4.9]). Thus, we have

$$\mathcal{D}(G) = \{(g, \xi) \in G \times_T \mathfrak{t}^\circ : \operatorname{Ad}_g^* \xi \in \mathfrak{t}^\circ\} /\!/ T = (T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!/ T,$$

where T acts on $T^*\mathcal{O} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ by the adjoint and coadjoint actions. Moreover, the action of T on $G \times \mathfrak{t}^\circ$ is free, so the $(T \times T)$-orbit-type partition of $\mathcal{D}(G)$ coincides with the T-orbit-type partition of $(T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!/ T$. (Another way of viewing this is to use reduction in stages [134, §4] to get $\mathcal{D}(G) = (T^*G /\!/\!/ (1 \times T_K)) /\!/\!/ (T_K \times 1)$ and note that by Theorem 5.1.2, $T^*G /\!/\!/ (1 \times T_K) = G \times_T \mathfrak{t}^\circ = T^*\mathcal{O}$.) Since $T_K \times T_K$ is abelian, Corollary 3.1.6 implies that $\mathcal{D}(G)$ is normal as a complex-analytic space and hence also as a complex affine variety. It is also irreducible since $T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)$ is clearly connected. This concludes the proof of Proposition 6.3.2.

We already see that $\mathcal{D}(G)$ depends only on $\mathfrak{g}$ as an algebraic variety since $T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)$ depends only on $\mathfrak{g}$ and T acts by the adjoint representation. It remains to show that the hyperkähler structures on the strata are also independent of coverings:

Proof of Theorem 6.3.1. It suffices to consider the case where $\tilde{G}$ is the universal cover of G. Let $\pi : \tilde{G} \to G$ be the covering map. Let $\tilde{K}$ and K be maximal compact subgroups of $\tilde{G}$ and G respectively with the same Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Then, the map $F : \tilde{G} \times \mathfrak{g} \to G \times \mathfrak{g}$, $(g, x) \mapsto (\pi(g), x)$ is hyperkähler since it descends from the identity map on the space of solutions to Nahm's equations on $\mathfrak{k}$. Let $T_{\tilde{K}}$ and T_K be maximal tori in $\tilde{K}$ and K respectively, with the same Lie algebra $\mathfrak{t}_{\mathfrak{k}} \subseteq \mathfrak{k}$. Then, F is equivariant with respect to the covering $T_{\tilde{K}} \times T_{\tilde{K}} \to T_K \times T_K$. Moreover, if $\tilde{\mu}$ and μ are the moment maps for the actions of $T_{\tilde{K}} \times T_{\tilde{K}}$ and $T_K \times T_K$ respectively, then $\mu \circ F = \tilde{\mu}$, so F maps $\tilde{\mu}^{-1}(0)$ to $\mu^{-1}(0)$, and hence descends to a continuous map $\bar{F} : \mathcal{D}(\tilde{G}) \to \mathcal{D}(G)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(\tilde{G}) & \xrightarrow{\quad \bar{F} \quad} & \mathcal{D}(G) \\ \downarrow{\cong} & & \downarrow{\cong} \\ (T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!\!/ \tilde{T} & \xrightarrow{\cong} & (T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!\!/ T \end{array}$$

Since the bottom map and two vertical maps are isomorphisms of complex-analytic and partitioned spaces, so is $\bar{F}$. Moreover, since $F : T^*\tilde{G} \to T^*G$ is a complex-symplectomorphism, $\bar{F}$ restricts to complex-symplectomorphisms on the strata. Hence, by definition of the Poisson brackets, $\bar{F}$ is a Poisson map. By the SO(3)-hyperkähler rotation, $\bar{F}$ is an isomorphism of stratified hyperkähler spaces. □

We may now use the notation $\mathcal{D}(\mathfrak{g})$ instead of $\mathcal{D}(G)$.

6.4.2 Stratification poset

In this section, we prove Theorem 6.3.3, which identifies the stratification poset of $\mathcal{D}(\mathfrak{g})$ with the poset of root subsystems.

By Proposition 6.3.2, the $(T_K \times T_K)$-orbit-type partition of $\mathcal{D}(\mathfrak{g})$ is the same as the T-orbit-type partition of $(T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)) /\!\!/ T$. It will be convenient to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the non-degenerate invariant bilinear form, and hence $\mathfrak{t}^\circ$ with $\mathfrak{t}^\perp \subseteq \mathfrak{g}$. Then, $\mathcal{D}(\mathfrak{g}) = (T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)) /\!\!/ T$, where T acts by the adjoint action on both factors. Thus, the orbit-type partition of $\mathcal{D}(\mathfrak{g})$ is of the form $\mathcal{P} = \{\mathcal{D}(\mathfrak{g})_Z : Z \in \mathcal{I}\}$ for some collection $\mathcal{I}$ of closed subgroups of T, where $\mathcal{D}(\mathfrak{g})_Z = (T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)^{\mathrm{ps}}_Z / T$. Our first objective is to identify the collection $\mathcal{I}$ precisely. Let Φ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. We use the same notation as in (6.3.1) for Z_Ψ and $\mathcal{D}(\mathfrak{g})_\Psi$.

Lemma 6.4.1. *The stabiliser subgroup of an element $(x, y) \in \mathfrak{g} \times \mathfrak{g}$ under the adjoint T-action is $Z_{\Phi(x,y)}$, where*

$$\Phi(x, y) := \Phi \cap \operatorname{span}_{\mathbb{Z}}\{\alpha \in \Phi : (x_\alpha, y_\alpha) \neq (0, 0)\}.$$

Moreover, $\Phi(x, y)$ is a root subsystem of Φ.

Proof. Let $\Phi'(x, y) = \{\alpha \in \Phi : (x_\alpha, y_\alpha) \neq (0, 0)\}$. The weight spaces for the action of T on $\mathfrak{g} \times \mathfrak{g}$ are $\mathfrak{g}_\alpha \times \mathfrak{g}_\alpha$ for $\alpha \in \Phi \cup \{0\}$. Thus, t fixes (x, y) if and only if $t \in Z_{\Phi'(x,y)}$, so we want to show that $Z_{\Phi'(x,y)} = Z_{\Phi(x,y)}$. Since $\Phi'(x, y) \subseteq \Phi(x, y)$, we have $Z_{\Phi(x,y)} \subseteq Z_{\Phi'(x,y)}$. Conversely, let $t \in Z_{\Phi'(x,y)}$ and $\beta \in \Phi(x, y)$. Then, $\beta = \sum_{\alpha\in\Phi'(x,y)} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{Z}$, so $\beta(t) = \prod_{\alpha\in\Phi'(x,y)} \alpha(t)^{n_\alpha} = 1$.

Any set of the form $\Phi \cap \operatorname{span}_{\mathbb{Z}} \Psi$ for some subset $\Psi \subseteq \Phi$ is a root subsystem, so $\Phi(x, y)$ is a root subsystem. □

This means that there is some collection $\mathcal{J}$ of root subsystems of Φ such that the orbit-type partition is $\mathcal{P} = \{\mathcal{D}(\mathfrak{g})_\Psi : \Psi \in \mathcal{J}\}$. The next lemma shows that $\mathcal{J}$ is, in fact, the set of *all* root subsystems of Φ.

Lemma 6.4.2. *$\mathcal{D}(\mathfrak{g})_\Psi \neq \emptyset$ for any root subsystem $\Psi \leq \Phi$.*

Proof. Take $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Psi$ and let $x := \sum_{\alpha\in\Psi} x_\alpha$. Then, $(\tau, x) \in T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)$. Recall that a point is polystable if and only if 0 is in the interior of the convex hull of its set of weights [125, Proposition 6.15]. Thus, (τ, x) is polystable since its set of weights is precisely $\Psi \cup \{0\}$ and we have $\alpha \in \Psi \iff -\alpha \in \Psi$. Moreover, $\Phi(\tau, x) = \Phi \cap \operatorname{span}_{\mathbb{Z}} \Psi = \Psi$, so $(\tau, x) \in (T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp))^{\mathrm{ps}}_{Z_\Psi}/T = \mathcal{D}(\mathfrak{g})_\Psi$. □

In other words, we have shown that the map $\Psi \mapsto \mathcal{D}(\mathfrak{g})_\Psi$ in Theorem 6.3.3 is surjective. The next lemma shows that it is injective.

Lemma 6.4.3. *Let Ψ_1 and Ψ_2 be two root subsystems. Then, $\mathcal{D}(\mathfrak{g})_{\Psi_1} = \mathcal{D}(\mathfrak{g})_{\Psi_2}$ if and only if $\Psi_1 = \Psi_2$.*

Proof. The only non-trivial part is to show that if $Z_{\Psi_1} = Z_{\Psi_2}$ then $\Psi_1 = \Psi_2$. It suffices to show that if Ψ is a root subsystem and $\alpha \in \Phi$ is such that $\alpha(t) = 1$ for all $t \in Z_\Psi$, then $\alpha \in \Psi$. By viewing α as an element of $\mathfrak{t}^*$, we want to show that if $\alpha(h) \in 2\pi i\mathbb{Z}$ for all $h \in \mathfrak{t}$ such that $\{\beta(h) : \beta \in \Psi\} \subseteq 2\pi i\mathbb{Z}$, then $\alpha \in \Psi$. Let $\beta_1, \ldots, \beta_k \in \Psi$ be a set of simple roots and complete it to a basis $\{\beta_1, \ldots, \beta_n\}$ for $\mathfrak{t}^*$. Let $h_1, \ldots, h_n$ be the dual basis in $\mathfrak{t}$. Then, $\alpha = a_1\beta_1 + \cdots + a_n\beta_n$ and since $\beta_i(2\pi i h_j) \in 2\pi i\mathbb{Z}$, the assumption on α implies that $2\pi i a_j = \alpha(2\pi i h_j) \in 2\pi i\mathbb{Z}$ so $a_j \in \mathbb{Z}$. Moreover, for all $h \in \operatorname{span}\{h_{k+1}, \ldots, h_n\}$ we have $\beta(h) = 0$ for all $\beta \in \Psi$, so $\alpha(h) = (a_{k+1}\beta_{k+1} + \cdots + a_n\beta_n)(h) \in 2\pi i\mathbb{Z}$. Thus, $a_{k+1}\beta_{k+1} + \cdots + a_n\beta_n = 0$ and hence $\alpha \in (\operatorname{span}_{\mathbb{Z}} \Psi) \cap \Phi = \Psi$. □

Thus, $\Psi \mapsto \mathcal{D}(\mathfrak{g})_\Psi$ is a bijection from the set of root subsystems of Φ to the set of orbit-type strata of $\mathcal{D}(\mathfrak{g})$. It remains to show that it is an isomorphism of posets. The crucial ingredient is the following result:

Proposition 6.4.4. *For all $\Psi \leq \Phi$ we have*

$$\overline{\mathcal{D}(\mathfrak{g})_\Psi} = \bigcup_{\chi \leq \Psi} \mathcal{D}(\mathfrak{g})_\chi.$$

Moreover, $\overline{\mathcal{D}(\mathfrak{g})_\Psi}$ is also the Zariski-closure of $\mathcal{D}(\mathfrak{g})_\Psi$, and $\mathcal{D}(\mathfrak{g})_\Psi$ is Zariski-locally-closed.

Proof. Let $\mathfrak{g}'_\Psi := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ (which is slightly different than the $\mathfrak{g}_\Psi$ defined in §6.2), $M := T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)$, and $M_\Psi := M \cap (\mathfrak{g}'_\Psi \times \mathfrak{g}'_\Psi)$. We claim that $M_{Z_\Psi} \subseteq M_\Psi$. Indeed, if $(x, y) \in M_{Z_\Psi}$ then, by Lemma 6.4.1, $Z_\Psi = Z_{\Phi(x,y)}$ and, by the proof of Lemma 6.4.3, this implies that $\Psi = \Phi(x, y)$. Thus, $\{\alpha \in \Phi : (x_\alpha, y_\alpha) \neq (0, 0)\} \subseteq \Psi$, so $(x, y) \in M_\Psi$. Hence, $\mathcal{D}(\mathfrak{g})_\Psi = M^{\mathrm{ps}}_{Z_\Psi}/T = (M_\Psi)^{\mathrm{ps}}_{Z_\Psi}/T$. Moreover, $(M_\Psi)^{\mathrm{ps}}_{Z_\Psi}$ is Zariski-open in M_Ψ since $(M_\Psi)^{\mathrm{ps}}_{Z_\Psi} = (M_\Psi)^{\mathrm{s}} \cap (M_\Psi)_{Z_\Psi}$, where $(M_\Psi)^{\mathrm{s}}$ is the set of stable points (which is always Zariski-open) and Z_Ψ is the kernel of the action of T on M_Ψ so $(M_\Psi)_{Z_\Psi}$ is Zariski-open in M_Ψ (see [125, Proposition 7.2]). Hence, $\overline{\mathcal{D}(\mathfrak{g})_\Psi} = M_\Psi /\!\!/ T$ is Zariski-closed in $\mathcal{D}(\mathfrak{g})$, and so $\overline{\mathcal{D}(\mathfrak{g})_\Psi}$ is the Zariski-closure of $\mathcal{D}(\mathfrak{g})_\Psi$. Since $(M_\Psi)^{\mathrm{ps}}_{Z_\Psi}$ is Zariski-open and saturated in M_Ψ, its image $\mathcal{D}(\mathfrak{g})_\Psi$ in $M^{\mathrm{ps}} /\!\!/ T = \overline{\mathcal{D}(\mathfrak{g})_\Psi}$ is Zariski-open, and hence $\mathcal{D}(\mathfrak{g})_\Psi$ is Zariski-locally-closed. Now, by decomposing $M_\Psi /\!\!/ T$ by T-orbit-types, we get

$$\overline{\mathcal{D}(\mathfrak{g})_\Psi} = \bigcup_{\chi \leq \Psi} (M_\Psi)^{\mathrm{ps}}_{Z_\chi}/T = \bigcup_{\chi \leq \Psi} M^{\mathrm{ps}}_{Z_\chi}/T = \bigcup_{\chi \leq \Psi} \mathcal{D}(\mathfrak{g})_\chi. \qquad \square$$

In particular, this proposition shows that $\Psi_1 \leq \Psi_2$ if and only if $\mathcal{D}(\mathfrak{g})_{\Psi_1} \leq \mathcal{D}(\mathfrak{g})_{\Psi_2}$, so the map $\Psi \mapsto \mathcal{D}(\mathfrak{g})_\Psi$ is an isomorphism of posets, which concludes the proof of Theorem 6.3.3.

6.4.3 Description of the strata

Now that we know that the orbit-type partition of $\mathcal{D}(\mathfrak{g})$ is of the form $\{\mathcal{D}(\mathfrak{g})_\Psi : \Psi \leq \Phi\}$, our next goal is to describe the strata $\mathcal{D}(\mathfrak{g})_\Psi$ individually.

First, note that since Φ is the greatest element in the poset of root subsystems, $\mathcal{D}(\mathfrak{g})_\Phi$ is dense in $\mathcal{D}(\mathfrak{g})$. We denote this top stratum by $\mathcal{D}(\mathfrak{g})^{\mathrm{top}} := \mathcal{D}(\mathfrak{g})_\Phi$.

Proposition 6.4.5. *The stratum $\mathcal{D}(\mathfrak{g})^{\mathrm{top}}$ is Zariski-open, connected, and of real dimension* $4(\dim \mathfrak{g} - 2 \operatorname{rk} \mathfrak{g})$.

Proof. By Proposition 6.4.4, $\mathcal{D}(\mathfrak{g})^{\text{top}}$ is Zariski-open in $\mathcal{D}(\mathfrak{g})$ and hence is connected since $\mathcal{D}(\mathfrak{g})$ is irreducible. The set of stable points of $T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^{\perp})$ is non-empty (for example, it contains (τ, x) if $x_\alpha \neq 0$ for all $\alpha \in \Phi$), so the complex-algebraic dimension of $\mathcal{D}(\mathfrak{g})$ is $\dim_{\mathbb{C}}(T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^{\perp})) - \operatorname{rk} \mathfrak{g}$. Moreover, the smooth map $T\mathcal{O} \to \mathfrak{t}$, $(x, y) \mapsto \pi_{\mathfrak{t}}(y)$ (where $\pi_{\mathfrak{t}}$ is the orthogonal projection $\mathfrak{g} \to \mathfrak{t}$) has full rank at a generic point, so

$$\dim_{\mathbb{C}}(T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^{\perp})) = \dim_{\mathbb{C}}(T\mathcal{O}) - \operatorname{rk} \mathfrak{g} = 2 \dim \mathfrak{g} - 3 \operatorname{rk} \mathfrak{g},$$

and hence $\dim_{\mathbb{C}} \mathcal{D}(\mathfrak{g}) = 2 \dim \mathfrak{g} - 4 \operatorname{rk} \mathfrak{g}$. □

Let Ψ be a root subsystem of Φ, W_Ψ be the Weyl group of Ψ, and $|W_\Phi : W_\Psi|$ the index of W_Ψ in W_Φ. The rest of this section is devoted to the proof that $\mathcal{D}(\mathfrak{g})_\Psi$ is a disjoint union of $|W_\Phi : W_\Psi|$ copies of $\mathcal{D}(\mathfrak{g}_\Psi)^{\text{top}}$ (Theorem 6.3.4).

Lemma 6.4.6. *The connected subgroup G_Ψ of G with Lie algebra $\mathfrak{g}_\Psi$ is closed and has a compact real form $K_\Psi \subseteq K$.*

Proof. Since the Killing form of $\mathfrak{g}$ remains nondegenerate on $\mathfrak{g}_\Psi$ (Proposition 6.2.2) we have $\mathfrak{g} = \mathfrak{g}_\Psi \oplus \mathfrak{g}_\Psi^{\perp}$, and it follows that $\mathfrak{g}_\Psi = (\mathfrak{k}_\Psi)_{\mathbb{C}}$, where $\mathfrak{k}_\Psi = \mathfrak{g}_\Psi \cap \mathfrak{k}$. Moreover, semisimple subgroups of compact Lie groups are closed, so the connected subgroup K_Ψ of K with Lie algebra $\mathfrak{k}_\Psi$ is closed. Hence, $G_\Psi := (K_\Psi)_{\mathbb{C}}$ is a closed connected subgroup of G with Lie algebra $\mathfrak{g}_\Psi$. □

Recall that $\mathfrak{t}_\Psi$ is the span of the coroots of Ψ and is a Cartan subalgebra of $\mathfrak{g}_\Psi$. Let

$$\mathfrak{z}_\Psi := \operatorname{Lie}(Z_\Psi) = \{h \in \mathfrak{t} : \alpha(h) = 0, \forall \alpha \in \Psi\},$$

so that $\mathfrak{t} = \mathfrak{z}_\Psi \oplus \mathfrak{t}_\Psi$. Then, $\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ and is a reductive Lie algebra with semisimple factor $\mathfrak{g}_\Psi$.

Lemma 6.4.7. *The intersection $\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)$ has $|W_\Phi : W_\Psi|$ connected components, each of the form $\zeta + \mathcal{O}_\Psi$ for some $\zeta \in \mathfrak{z}_\Psi$, and $\mathcal{O}_\Psi$ a regular semisimple G_Ψ-orbit in $\mathfrak{g}_\Psi$.*

Proof. Since $\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)$ is G_Ψ-invariant, it is a union of G_Ψ-orbits. But G_Ψ acts trivially on $\mathfrak{z}_\Psi$, so they are of the form $\zeta + \mathcal{O}_\Psi$, where $\zeta \in \mathfrak{z}_\Psi$ and $\mathcal{O}_\Psi$ is a regular semisimple G_Ψ-orbit in $\mathfrak{g}_\Psi$. From the classification of semisimple orbits [31, §2.2], $\mathcal{O}_\Psi$ intersects $\mathfrak{t}_\Psi$ in precisely $|W_\Psi|$ elements. Hence $(\zeta + \mathcal{O}_\Psi) \cap \mathfrak{t}$ contains exactly $|W_\Psi|$ elements. But $(\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)) \cap \mathfrak{t} = \mathcal{O} \cap \mathfrak{t}$ contains exactly $|W_\Phi|$ elements, so there must be exactly $|W_\Phi|/|W_\Psi| = |W_\Phi : W_\Psi|$ of these G_Ψ-orbits. Moreover, semisimple orbits are closed, so $\zeta + \mathcal{O}_\Psi$ is closed in $\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)$. Since there are only finitely many of them, they are also open in $\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)$, and hence they are the connected components. $\square$

Proposition 6.4.8. *For all $\Psi \leq \Phi$, $\mathcal{D}(\mathfrak{g})_\Psi$ is isomorphic as a hyperkähler manifold to a disjoint union of $|W_\Phi : W_\Psi|$ copies of $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}}$.*

Proof. Let T_{K_Ψ} be the maximal torus in K_Ψ with Lie algebra $\mathfrak{t}_\Psi \cap \mathfrak{k}$, and let $T_\Psi := (T_{K_\Psi})_\mathbb{C}$ be the maximal torus in G_Ψ with Lie algebra $\mathfrak{t}_\Psi$. Recall from the proof of Proposition 6.4.4 that $\mathcal{D}(\mathfrak{g})_\Psi = (M_\Psi)^{\mathrm{ps}}_{Z_\Psi}/T$, where $M := T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)$ and $M_\Psi := M \cap ((\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi) \times (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi))$. Thus, by Lemma 6.4.7, $\mathcal{D}(\mathfrak{g})_\Psi$ is a disjoint union of sets of the form $(T(\zeta + \mathcal{O}_\Psi) \cap (\mathfrak{g} \times \mathfrak{t}^\perp))^{\mathrm{ps}}_{Z_\Psi}/T$ for $\zeta \in \mathfrak{z}_\Psi$ and $\mathcal{O}_\Psi \subseteq \mathfrak{g}_\Psi$. These sets are homeomorphic to the top stratum $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}} = (T\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp))^{\mathrm{ps}}_{Z_\Psi \cap T_\Psi}/T_\Psi$ via the map $(x, y) \mapsto (\zeta + x, y)$, and hence it suffices to show this map is hyperkähler.

Since $\mathcal{O}_\Psi$ is semisimple, we have $\mathcal{O}_\Psi = G_\Psi \cdot h$ for some $h \in \mathfrak{t}_\Psi$. Thus, $\zeta + h \in (\zeta + \mathcal{O}_\Psi) \cap \mathfrak{t} \subseteq \mathcal{O} \cap \mathfrak{t}$ and hence $\zeta + h = w \cdot \tau$ for some $w \in W$. Let $k \in N_K(T_K)$ be a representative of w. Then, the composition

$$F : G_\Psi \times \mathfrak{g}_\Psi \hookrightarrow G \times \mathfrak{g} \longrightarrow G \times \mathfrak{g}, \quad (g, x) \longmapsto (gk, \mathrm{Ad}_{k^{-1}} x)$$

is hyperkähler since this is just the action of $(1, k^{-1}) \in K \times K$. Moreover, it is equivariant with respect to the Lie group homomorphism $T_{K_\Psi} \times T_{K_\Psi} \to T_K \times T_K$, $(s, t) \mapsto (s, k^{-1}tk)$ and maps the zero level set of the moment for $T_{K_\Psi} \times T_{K_\Psi}$ to that of $T_K \times T_K$. Therefore, it descends to a continuous map $\bar{F} : \mathcal{D}(\mathfrak{g}_\Psi) \to \mathcal{D}(\mathfrak{g})$. For all $g \in G_\Psi$ we have $\zeta + \mathrm{Ad}_g h = \mathrm{Ad}_g(\zeta + h) = \mathrm{Ad}_{gk} \tau$, so the diagram

$$\begin{array}{ccc} T^*G_\Psi /\!\!/\!\!/ (T_{K_\Psi} \times T_{K_\Psi}) & \xrightarrow{\bar{F}} & T^*G /\!\!/\!\!/ (T_K \times T_K) \\ \downarrow \cong & & \downarrow \cong \\ (T\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp)) /\!\!/ T_\Psi & \longrightarrow & (T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)) /\!\!/ T \end{array}$$

commutes, where the bottom map is $(\mathrm{Ad}_g h, \mathrm{Ad}_g x) \mapsto (\zeta + \mathrm{Ad}_g h, \mathrm{Ad}_g x)$ and the vertical maps are the isomorphisms of Proposition 6.3.2. The restriction of the bottom map to $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}}$ is the map considered earlier, and hence $\mathcal{D}(\mathfrak{g}_\Psi)^{\mathrm{top}}$ is hyperkähler isomorphic to the connected components of $\mathcal{D}(\mathfrak{g})_\Psi$ corresponding to $\zeta + \mathcal{O}_\Psi$. $\square$

Corollary 6.4.9. *The stratum $\mathcal{D}(\mathfrak{g})^{\mathrm{bottom}} := \mathcal{D}(\mathfrak{g})_\emptyset$ is a finite set of $|W_\Phi|$ elements.*

Proof. By Proposition 6.4.8, $\mathcal{D}(\mathfrak{g})^{\mathrm{bottom}}$ is a disjoint union of $|W_\Phi|$ copies of $\mathcal{D}(\mathfrak{g}_\emptyset)^{\mathrm{top}}$ $= \mathcal{D}(0)^{\mathrm{top}}$, and $\mathcal{D}(0)^{\mathrm{top}}$ is just a point by Proposition 6.4.5. $\square$

Proposition 6.4.5, Proposition 6.4.8 and Corollary 6.4.9 together prove Theorem 6.3.4.

6.4.4 A coarser stratification

The goal of this section is to prove Theorem 6.3.5 about the coarser partition $\mathcal{P} = \{\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} : [\mathfrak{h}] \in \mathcal{C}_{\mathfrak{g}}\}$. Let $[\mathfrak{h}] = [\mathfrak{g}_\Psi] \in \mathcal{C}_{\mathfrak{g}}$, where $\Psi \leq \Phi$. Then, by Proposition 6.2.3, we have

$$\mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\Psi]} = \bigcup_{w \in W_\Phi} \mathcal{D}(\mathfrak{g})_{w\cdot\Psi}.$$

Let $\{w \cdot \Psi : w \in W_\Phi\} = \{w_1 \cdot \Psi, \ldots, w_n \cdot \Psi\}$, where the $w_i \cdot \Psi$'s are distinct. Then, $n|W_\Phi : W_\Psi|$ is the embedding number $m_{\mathfrak{g}}(\mathfrak{g}_\Psi)$ introduced in §6.2, and

$$\mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\Psi]} = \bigcup_{i=1}^{n} \mathcal{D}(\mathfrak{g})_{w_i\cdot\Psi}.$$

Lemma 6.4.10. *The above union is a topological disjoint union.*

Proof. It suffices to show that if $u, v \in W_\Phi$ and $\overline{\mathcal{D}(\mathfrak{g})_{u\cdot\Psi}} \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} \neq \emptyset$ then $u\cdot\Psi = v\cdot\Psi$. By Proposition 6.4.4, we have

$$\overline{\mathcal{D}(\mathfrak{g})_{u\cdot\Psi}} \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} = \bigcup_{\chi \leq u\cdot\Psi} \mathcal{D}(\mathfrak{g})_\chi \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi},$$

so there exists $\chi \leq u \cdot \Psi$ such that $\mathcal{D}(\mathfrak{g})_\chi \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} \neq \emptyset$. Hence, $v \cdot \Psi = \chi \subseteq u \cdot \Psi$ so $v \cdot \Psi = u \cdot \Psi$. □

In particular, Lemma 6.4.10 says that each piece $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ in $\mathcal{P}$ is a topological manifold and is locally closed. Moreover, by combining with Proposition 6.4.8, we get that $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ has $m_{\mathfrak{g}}(\mathfrak{h})$ connected components, each isomorphic to $\mathcal{D}(\mathfrak{h})^{\text{top}}$ as a hyperkähler manifold. Now, recall that $\mathcal{C}_{\mathfrak{g}}$ has a partial order $\leq$ induced by inclusion.

Lemma 6.4.11. *For all $[\mathfrak{h}] \in \mathcal{C}_{\mathfrak{g}}$, we have*

$$\overline{\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}} = \bigcup_{[\mathfrak{q}] \leq [\mathfrak{h}]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{q}]}.$$

Proof. Let Ψ be such that $[\mathfrak{h}] = [\mathfrak{g}_\Psi]$. If we let $[\Psi]$ be the equivalence class of Ψ in $\{\text{root subsystems of } \Phi\}/W_\Phi$, we get

$$\begin{aligned}\overline{\mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\Psi]}} &= \bigcup_{w\in W} \overline{\mathcal{D}(\mathfrak{g})_{w\cdot\Psi}} = \bigcup_{w\in W} \bigcup_{\chi \leq w\cdot\Psi} \mathcal{D}(\mathfrak{g})_\chi = \bigcup_{[\chi]\leq[\Psi]} \bigcup_{w\in W} \mathcal{D}(\mathfrak{g})_{w\cdot\chi} \\ &= \bigcup_{[\chi]\leq[\Psi]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\chi]} = \bigcup_{[\mathfrak{q}]\leq[\mathfrak{h}]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{q}]}.\end{aligned}$$ □

It is then immediate that $[\mathfrak{h}_1] \leq [\mathfrak{h}_2]$ if and only if $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}_1]} \leq \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}_2]}$, so the map $\mathcal{C}_{\mathfrak{g}} \to \mathcal{P}$, $[\mathfrak{h}] \mapsto \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ is an isomorphism of posets. This concludes the proof of Theorem 6.3.5.

6.4.5 Resolution of singularities

In this section, we prove Theorem 6.3.6, which constructs a resolution of singularities of $\mathcal{D}(\mathfrak{g})$ when $\mathfrak{g}$ is of type A.

The idea is to shift the real part of the moment map by an element ζ of the open set $\mathfrak{t}^*_\circ$ defined in (6.3.2) to get a smooth hyperkähler manifold $\mathcal{D}_\zeta(\mathfrak{g})$, i.e. $\mathcal{D}_\zeta(\mathfrak{g})$ is the hyperkähler quotient of T^*G by $T_K \times T_K$ with the moment map shifted by $((\zeta, 0), (0, 0), (0, 0)) \in (\mathfrak{t}^*_\mathfrak{k} \times \mathfrak{t}^*_\mathfrak{k}) \otimes \mathbb{R}^3$. However, this set is non-empty only when $\mathfrak{g}$ is of type A; in general, we cannot avoid at least orbifold singularities by this method.

Proposition 6.4.12. *The set $\mathfrak{t}^*_\circ$ is non-empty if and only if $\mathfrak{g}$ is of type A. In that case, $\mathfrak{t}^*_\circ \cap \mathfrak{t}^*_\mathfrak{k}$ is also non-empty.*

Proof. Call a semisimple Lie algebra **good** if its associated set $\mathfrak{t}^*_\circ$ is non-empty. Then, if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple Lie algebras such that either $\mathfrak{g}_1$ or $\mathfrak{g}_2$ is not good, then the sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is also not good. Thus, it suffices to show that simple Lie algebras of type A are good, but those of type B, C, D, E, F, G are not good. Note that a semisimple Lie algebra $\mathfrak{g}$ is not good if and only if it has a regular semisimple subalgebra $\mathfrak{h}$ such that $\mathrm{rk}(\mathfrak{h}) = \mathrm{rk}(\mathfrak{g})$. Hence, the result follows from the classification of regular semisimple subalgebras, as carried out by Dynkin [43]; see also [122, Ch. 6 and Table 5].

The last part follows from the fact that $\mathfrak{t}^*_\mathfrak{k}$ is a real form of $\mathfrak{t}^*$ and for any root $\alpha \in \Phi$, $i\alpha$ restricts to an element of $\mathfrak{t}^*_\mathfrak{k}$. □

Let $\pi_\mathfrak{t} : \mathfrak{g} \to \mathfrak{t}$ be the projection with respect to the root space decomposition. To show that $\mathcal{D}_\zeta(\mathfrak{g})$ is smooth, we need the following simple lemma.

Lemma 6.4.13. *For all $\Psi \leq \Phi$ we have $\pi_\mathfrak{t}([\mathfrak{g}, \mathfrak{g}_\Psi]) \subseteq \mathfrak{t}_\Psi$.*

Proof. Let $x = x_0 + \sum_{\alpha\in\Phi} x_\alpha \in \mathfrak{g}$ and $y = y_0 + \sum_{\beta\in\Psi} y_\beta \in \mathfrak{g}_\Psi$, where $x_0, y_0 \in \mathfrak{t}$ and $x_\alpha, y_\alpha \in \mathfrak{g}_\alpha$. Then,

$$[x, y] = \sum_{\beta\in\Psi} \beta(x_0) y_\beta - \sum_{\alpha\in\Phi} \alpha(y_0) x_\alpha + \sum_{\alpha\in\Phi,\beta\in\Psi} [x_\alpha, y_\beta].$$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ we have

$$\pi_\mathfrak{t}([x, y]) = \sum_{\alpha\in\Phi,\beta\in\Psi,\alpha+\beta=0} [x_\alpha, y_\beta] = \sum_{\beta\in\Psi} [x_{-\beta}, y_\beta].$$

Now, $[x_{-\beta}, y_\beta] \in \mathbb{C}h_\alpha$ so $\pi_\mathfrak{t}([x, y]) \subseteq \sum_{\alpha\in\Psi} \mathbb{C}h_\alpha = \mathfrak{t}_\Psi$. □

Proposition 6.4.14. *For all $\zeta \in \mathfrak{t}^*_\circ \cap \mathfrak{t}^*_\mathfrak{k}$, the hyperkähler quotient $\mathcal{D}_\zeta(\mathfrak{g})$ is smooth and connected.*

Proof. By the SO(3)-hyperkähler rotation, it suffices to show smoothness of the hyperkähler quotient of T^*G by $T_K \times T_K$ with the moment map shifted instead by $((0,0),(0,\zeta),(0,0)) \in (\mathfrak{t}_{\mathfrak{k}}^* \times \mathfrak{t}_{\mathfrak{k}}^*) \otimes \mathbb{R}^3$. We denote the latter by $\mathcal{D}'_\zeta(\mathfrak{g})$. By the Kempf–Ness type theorem for T^*G (Theorem 5.1.2), it is an affine GIT quotient

$$\mathcal{D}'_\zeta(\mathfrak{g}) = (T\mathcal{O} \cap (\mathfrak{g} \times (\zeta + \mathfrak{t}^\perp)))/\!\!/ T$$

where now ζ is viewed as an element of $\mathfrak{t}_\circ := \mathfrak{t} - \bigcup_{\Psi<\Phi} \mathfrak{t}_\Psi$.

We want to show that if $p \in T\mathcal{O} \cap (\mathfrak{g} \times (\zeta + \mathfrak{t}^\perp))$, then $T_p = Z_G$. Every $p \in T\mathcal{O}$ is of the form $p = (x, [y,x])$ for some $x \in \mathcal{O}$ and $y \in \mathfrak{g}$. Hence, it suffices to show that if $T_{(x,[y,x])} \neq Z_G$, then $\pi_{\mathfrak{t}}([y,x]) \subseteq \mathfrak{t}_\Psi$ for some $\Psi < \Phi$. But $T_{(x,[y,x])} = T_x \cap T_{[y,x]}$, so if $T_{(x,[y,x])} \neq Z_G$, then $T_x \neq Z_G$, and hence $T_x = T_\Psi$ for some $\Psi < \Phi$. In particular, $x \in \mathfrak{g}_\Psi$ so, by Lemma 6.4.13, $\pi_{\mathfrak{t}}([y,x]) \subseteq \pi_{\mathfrak{t}}([\mathfrak{g}, \mathfrak{g}_\Psi]) \subseteq \mathfrak{t}_\Psi$, and hence $[y,x] \notin \zeta + \mathfrak{t}^\perp$. Therefore, $\mathcal{D}_\zeta(\mathfrak{g})$ is smooth.

To show that $\mathcal{D}_\zeta(\mathfrak{g})$ is connected, note that $T\mathcal{O} \cap (\mathfrak{g} \times (\zeta + \mathfrak{t}^\perp))$ is a smooth affine bundle over $\mathcal{O}$ and hence is connected. □

Proposition 6.4.15. *Suppose that $G = \mathrm{SL}(n,\mathbb{C})$. There exists a character $\chi : T \to \mathbb{C}^*$ such that the element $\xi := i\, d\chi \in \mathfrak{t}_{\mathfrak{k}}^*$ lies in $\mathfrak{t}_\circ^*$.*

Proof. We may assume that $T \subseteq \mathrm{SL}(n,\mathbb{C})$ is the group of diagonal matrices of determinant 1. Then, for example, the character $\chi : \mathrm{diag}(t_1,\ldots,t_n) \mapsto t_1$ has the desired property. □

Now, let $\chi : T \to \mathbb{C}^*$ and $\zeta \in \mathfrak{t}_\circ^* \cap \mathfrak{t}_{\mathfrak{k}}^*$ be as in Proposition 6.4.15. Then, if $\mu_{\mathbb{C}} : T^*G \to \mathfrak{t}^* \times \mathfrak{t}^*$ denotes the complex moment map for the action of $T \times T$ on T^*G, we have $\mathcal{D}_\zeta(\mathfrak{g}) = \mu_{\mathbb{C}}^{-1}(0)/\!\!/_\chi (T \times T)$ and $\mathcal{D}(\mathfrak{g}) = \mu_{\mathbb{C}}^{-1}(0)/\!\!/ (T \times T)$. In particular, the inclusion $\mu_{\mathbb{C}}^{-1}(0)^{\chi\text{-ss}} \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)$ descends to a morphism

$$\pi : \mathcal{D}_\zeta(\mathfrak{g}) \longrightarrow \mathcal{D}(\mathfrak{g})$$

of complex-algebraic varieties.

Proposition 6.4.16. *The map π is a resolution of singularities, i.e.*

(i) *π is proper;*

(ii) *π restricts to an isomorphism $\pi^{-1}(\mathcal{D}(\mathfrak{g})^{\mathrm{top}}) \xrightarrow{\cong} \mathcal{D}(\mathfrak{g})^{\mathrm{top}}$ of smooth complex-algebraic varieties;*

(iii) *$\pi^{-1}(\mathcal{D}(\mathfrak{g})^{\mathrm{top}})$ is open dense in $\mathcal{D}_\zeta(\mathfrak{g})$.*

Proof. The proof is quite standard (see, e.g., Kirilov [87, §9.4]) but we recall the arguments for completeness. Let $X := \mu_{\mathbb{C}}^{-1}(0)$ and $H := T \times T$. Then, π is the projective morphism $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathbb{C}[X]^{H,\chi^n}) \to \mathrm{Spec}(\mathbb{C}[X]^H)$ and hence is proper. Now, let

$$X^{\mathrm{reg}} := \{x \in X : H \cdot x \text{ is closed and } H_x = Z_G \times Z_G\}.$$

Then, X^{reg} is a non-singular Zariski-open subset of X (see, e.g., Lemma 3.4.14 for the non-singularity). Since closed orbits are embedded, we see that $X^{\mathrm{reg}} \subseteq X^{\chi\text{-ss}}$. Moreover, since X^{reg} is H-saturated in $X^{\chi\text{-ss}}$, the image $(X/\!\!/_{\chi} H)^{\mathrm{reg}}$ of X^{reg} in $X/\!\!/_{\chi} H$ is Zariski-open and the restriction $X^{\mathrm{reg}} \to (X/\!\!/_{\chi} H)^{\mathrm{reg}}$ is a good quotient. Similarly, the image $(X/\!\!/H)^{\mathrm{reg}}$ of X^{reg} in $X/\!\!/H$ is Zariski-open and the restriction $X^{\mathrm{reg}} \to (X/\!\!/H)^{\mathrm{reg}}$ is a good quotient. Thus, π restricts to an isomorphism $(X/\!\!/_{\chi} H)^{\mathrm{reg}} \to (X/\!\!/ H)^{\mathrm{reg}}$. Since $(X/\!\!/_{\chi} H)^{\mathrm{reg}}$ is a Zariski-open subset of a non-singular connected space $\mathcal{D}_{\zeta}(\mathfrak{g})$, it is also dense in $\mathcal{D}_{\zeta}(\mathfrak{g})$. Now, it suffices to observe that $(X/\!\!/ H)^{\mathrm{reg}} = \mathcal{D}(\mathfrak{g})^{\mathrm{top}}$ and $\pi^{-1}((X/\!\!/ H)^{\mathrm{reg}}) = (X/\!\!/_{\chi} H)^{\mathrm{reg}}$. □

6.5 Examples

In this section, we look at specific examples of $\mathcal{D}(\mathfrak{g})$ and describe their stratification into hyperkähler manifolds explicitly. We will focus on the coarser stratification whose set of strata is isomorphic to the partially ordered set $\mathcal{C}_{\mathfrak{g}}$ of conjugacy classes of regular semisimple subalgebras (Theorem 6.3.5). We will draw Hasse diagrams where each node is of the form mL, where L is the Lie algebra isomorphism class of a conjugacy class $[\mathfrak{h}] \in \mathcal{C}_{\mathfrak{g}}$ and m its embedding number. In other words, a node of the form mL in the Hasse diagram represents a stratum which is isomorphic as a hyperkähler manifold to a disjoint union of m copies of $\mathcal{D}(L)^{\mathrm{top}}$. We write isomorphism classes of Lie algebras multiplicatively; for example, $L = A_1^2 B_2$ is $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(5, \mathbb{C})$.

6.5.1 $\mathfrak{sl}(2, \mathbb{C})$

The root system Φ of $\mathfrak{sl}(2, \mathbb{C})$ embeds in $\mathbb{R}^2$ as:

Thus, the root subsystems are just Φ itself and the empty set. They both determine a conjugacy class in $\mathcal{C}_{\mathfrak{g}}$, and the embedding numbers are $m(\mathfrak{sl}(2, \mathbb{C})) = 1$ and $m(0) = |W| = 2$, so the Hasse diagram is:

We can see from this diagram that $\mathcal{D}(\mathfrak{sl}(2,\mathbb{C}))$ consists of a smooth open dense subset and two isolated singularities. This agrees with Dancer's computation [32, pp. 88–99] that $\mathcal{D}(\mathfrak{sl}(2,\mathbb{C}))$ is the D_2-surface, i.e. the affine variety in $\mathbb{C}^3$ cut out by the equation $x^2 - zy^2 = z$. We can also compute an explicit algebraic expression for the resolution of singularities:

Proposition 6.5.1. *We have*

$$\mathcal{D}(\mathfrak{sl}_2) = \{(x,y,z) \in \mathbb{C}^3 : x^2 - zy^2 = z\}$$
$$\mathcal{D}_\zeta(\mathfrak{sl}_2) = \{(x,y,z,[u:v]) \in \mathbb{C}^3 \times \mathbb{CP}^1 : xu - vy^2 = v, zu = xv\}$$

and the map $\mathcal{D}_\zeta(\mathfrak{sl}_2) \to \mathcal{D}(\mathfrak{sl}_2)$ *is the projection onto the* (x,y,z)*-coordinates.*

Proof. By Proposition 6.3.2, $\mathcal{D}(\mathfrak{sl}_2)$ is isomorphic to $(T\mathcal{O} \cap (\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{t}^\perp)) /\!\!/ \mathbb{C}^*$, where $\mathcal{O} \subseteq \mathfrak{sl}(2,\mathbb{C})$ is a regular semisimple orbit, $\mathbb{C}^* = T$ is the subgroup of diagonal matrices in $\mathrm{SL}(2,\mathbb{C})$, and

$$\mathfrak{t} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in \mathbb{C} \right\}, \qquad \mathfrak{t}^\perp = \left\{ \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} : y,z \in \mathbb{C} \right\}.$$

We first need an algebraic expression for $\mathcal{O}$. Let

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, $\tau \in \mathfrak{t}$ is regular semisimple, so we can take $\mathcal{O}$ to be the $\mathrm{SL}(2,\mathbb{C})$-orbit of τ. Then, $\mathcal{O}$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ which have the same minimal polynomial as τ, and this gives us the algebraic expression

$$\mathcal{O} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x^2 + yz = 1 \right\}.$$

Thus, we get

$$T\mathcal{O} \cap (\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{t}^\perp) = \left\{ \left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} \right) : x^2 + yz = 1, yw + zv = 0 \right\}. \tag{6.5.1}$$

Let

$$A := \mathbb{C}[x,y,z,v,w]/\langle x^2 + yz - 1, yw + zv \rangle$$

so that $T\mathcal{O} \cap (\mathfrak{sl}_2 \times \mathfrak{t}^\perp) = \mathrm{Spec}(A)$. Now, $\mathbb{C}^*$ acts on $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ by

$$t \cdot \left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \right) = \left(\begin{pmatrix} x & t^2 y \\ t^{-2} z & -x \end{pmatrix}, \begin{pmatrix} u & t^2 v \\ t^{-2} w & -u \end{pmatrix} \right)$$

and hence the ring of invariant polynomials is isomorphic to

$$\mathbb{C}[a,b,p,q,r,s]/(ps - qr),$$

where $a = x$, $b = u$, $p = yz$, $q = yw$, $r = zv$ and $s = vw$. Combining this with (6.5.1), we see that $\mathcal{D}(\mathfrak{sl}_2)$ is the affine variety with coordinate ring

$$\mathbb{C}[a, p, q, r, s]/(a^2 + p - 1, q + r, ps - qr).$$

The three equations in the ideal are easily reducible to one equation

$$q^2 - sa^2 + s = 0.$$

Hence, by letting $x = q$, $y = ia$ and $z = -s$, we get

$$x^2 - zy^2 = z.$$

This shows that $\mathcal{D}(\mathfrak{sl}_2)$ is the Kleinian D_2-singularity. Now, we compute $\mathcal{D}_\zeta(\mathfrak{sl}_2)$, where $\zeta = i\,d\chi$ and $\chi : \mathbb{C}^* \to \mathbb{C}^*$ is the identity character. By Theorem 5.1.2, this is $\mathrm{Proj}(\tilde{A})$ where

$$\tilde{A} \coloneqq \bigoplus_{n=0}^{\infty} A_n, \quad A_n \coloneqq \{f \in A : f(x, \lambda^2 y, \lambda^{-2} z, \lambda^2 v, \lambda^{-2} w) = \lambda^n f(x, y, z, v, w)\}.$$

Clearly, $A_n = 0$ for all odd n. Moreover, the map

$$\begin{aligned} \mathbb{C}[a, p, q, r, s, \mu, \nu] &\longrightarrow \tilde{A} \\ a &\longmapsto x \\ p &\longmapsto yz \\ q &\longmapsto yw \\ r &\longmapsto zv \\ s &\longmapsto vw \\ \mu &\longmapsto y \\ \nu &\longmapsto v \end{aligned}$$

where a, p, q, r, s are in degree 0 and μ, ν in degree 2, is a surjective homomorphism of graded rings. The kernel is

$$\langle a^2 + p - 1, q + r, ps - qr, p\nu - r\mu, q\nu - s\mu \rangle.$$

After eliminating the variables p and r, we see that

$$\mathcal{D}_\zeta(\mathfrak{sl}_2) = \{(a, q, s, [\mu : \nu]) \in \mathbb{C}^3 \times \mathbb{CP}^1 : q\mu = (a^2 - 1)\nu, s\mu = q\nu\}.$$

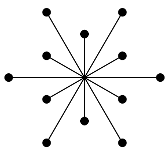
Setting $x = q$, $y = ia$, and $z = -s$, $u = \mu$ and $v = -\nu$, we get

$$\mathcal{D}_\zeta(\mathfrak{sl}_2) = \{(x, y, z, [u : v]) \in \mathbb{C}^3 \times \mathbb{CP}^1 : xu - vy^2 = v, zu = xv\}$$

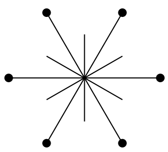
and the map $\mathcal{D}_\zeta(\mathfrak{sl}_2) \to \mathcal{D}(\mathfrak{sl}_2)$ is the projection onto the (x, y, z) coordinates. □

6.5.2 The exceptional Lie algebra $\mathfrak{g}_2$

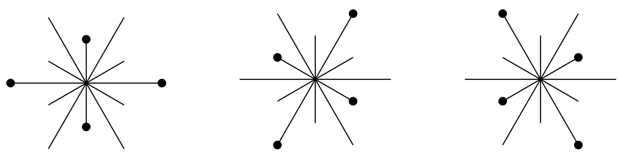
The root system of $\mathfrak{g}_2$ embeds in $\mathbb{R}^2$ as:



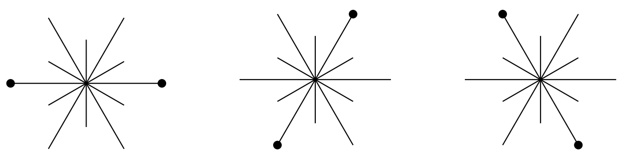
The six long roots form an A_2 subsystem:



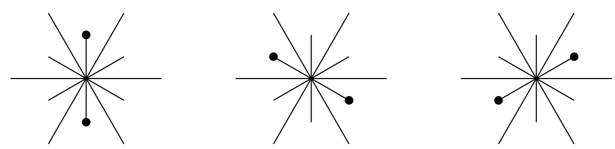
It is fixed by the Weyl group, so the embedding number is $|W_{G_2}|/|W_{A_2}| = 12/6 = 2$. There are three A_1^2 subsystems:



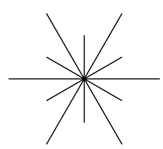
They are in the same W-orbit, so they determine a single conjugacy class in $\mathcal{C}_{\mathfrak{g}_2}$ with embedding number $3|W_{G_2}|/|W_{A_1^2}| = 3 \cdot 12/4 = 9$. There are three A_1 subsystems formed by the long roots:



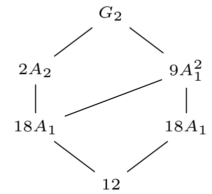
They are in the same W-orbit and the embedding number is $3 \cdot 12/2 = 18$. Similarly, the short roots form three A_1 subsystems



which are W-conjugate and have embedding number 18. Finally, the empty set



forms a root subsystem with embedding number $|W_{G_2}| = 12$. Therefore, the stratification structure of $\mathcal{D}(\mathfrak{g}_2)$ can be written symbolically as:

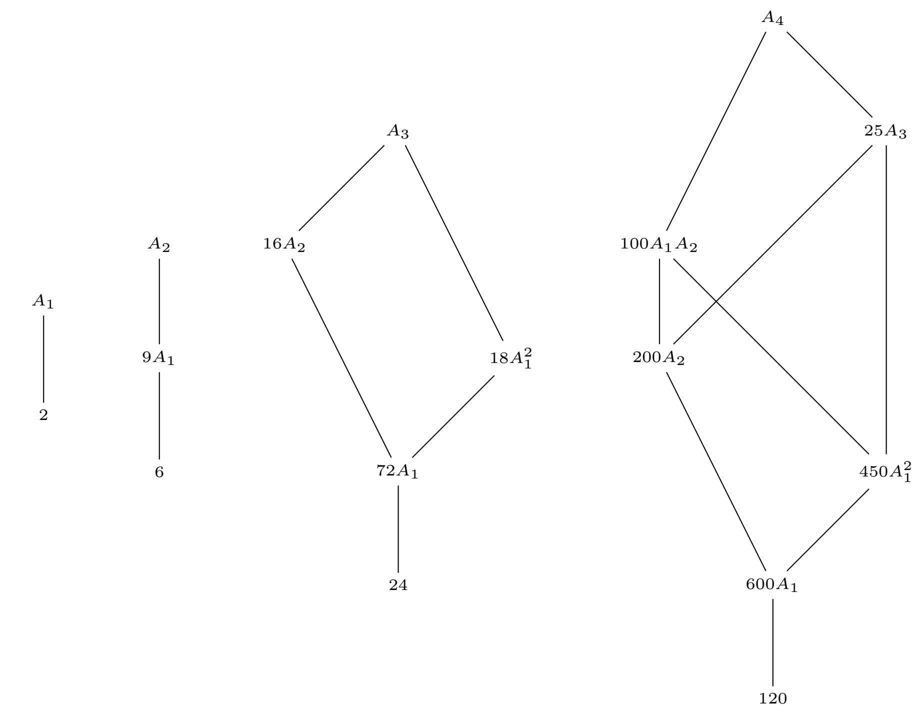


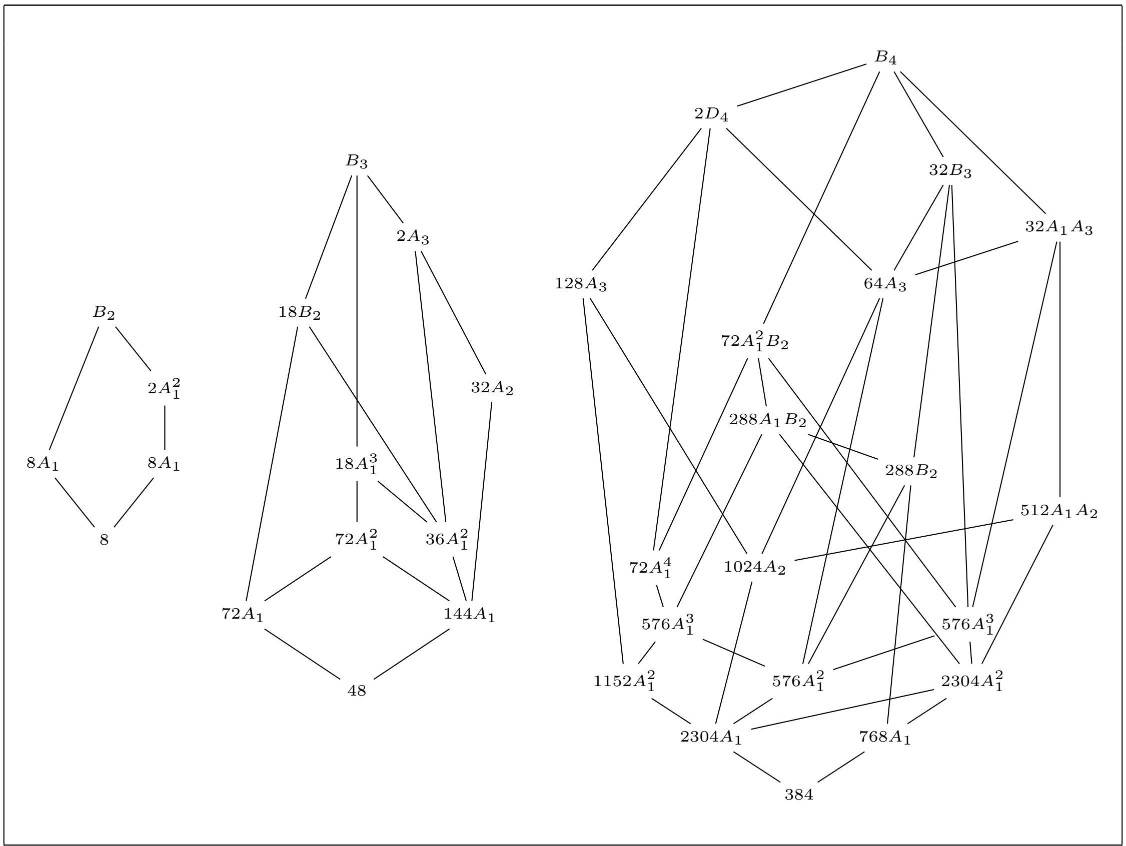
For example, we see from this diagram that on the boundary of the open dense stratum $\mathcal{D}(\mathfrak{g}_2)^{\text{top}}$ there are two copies of the space $\mathcal{D}(\mathfrak{sl}(3,\mathbb{C}))^{\text{top}}$. Also, the set of most singular points is a finite set of 12 elements.

6.5.3 All simple Lie algebras of rank ≤ 4

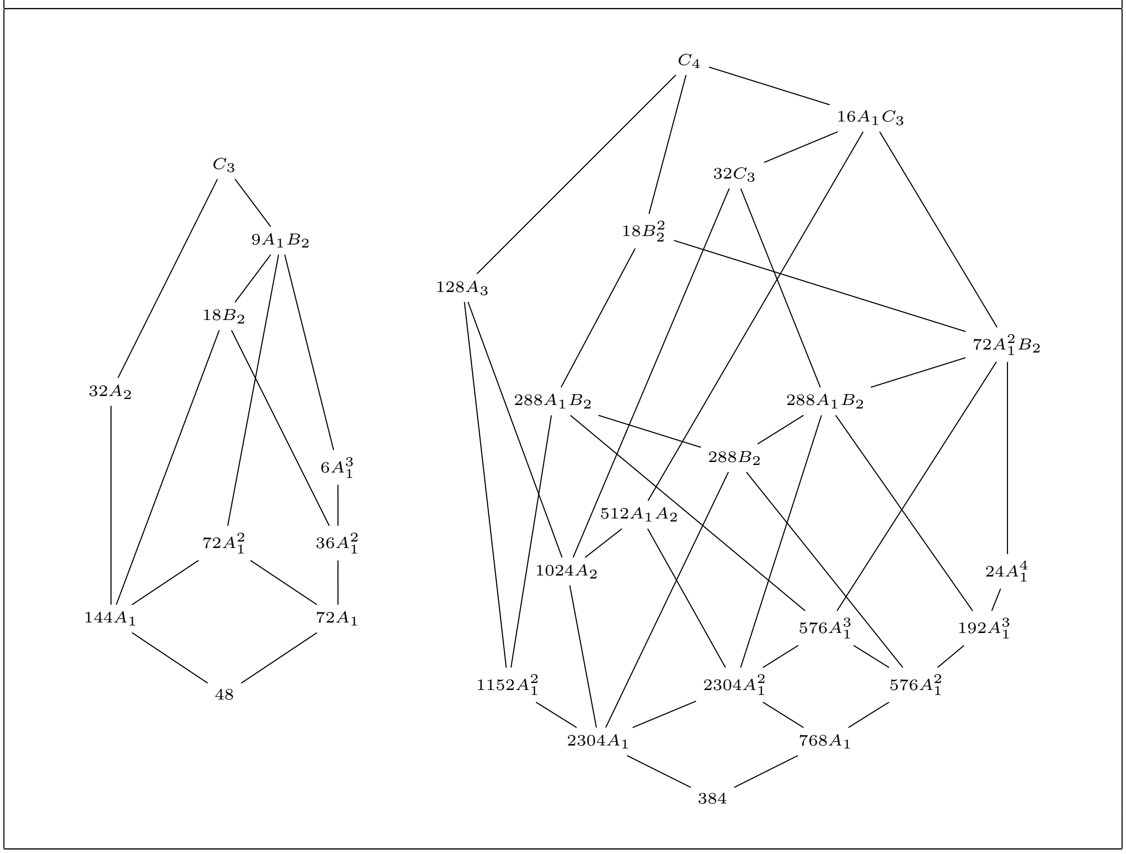
The computations outlined for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}_2$ can be implemented on a computer. We have done that for all simple Lie algebras of rank ≤ 4. The diagrams are arranged so that a node $m'L'$ is higher than a node mL if and only if $\dim \mathcal{D}(L')^{\text{top}} > \dim \mathcal{D}(L)^{\text{top}}$.

Hasse diagrams

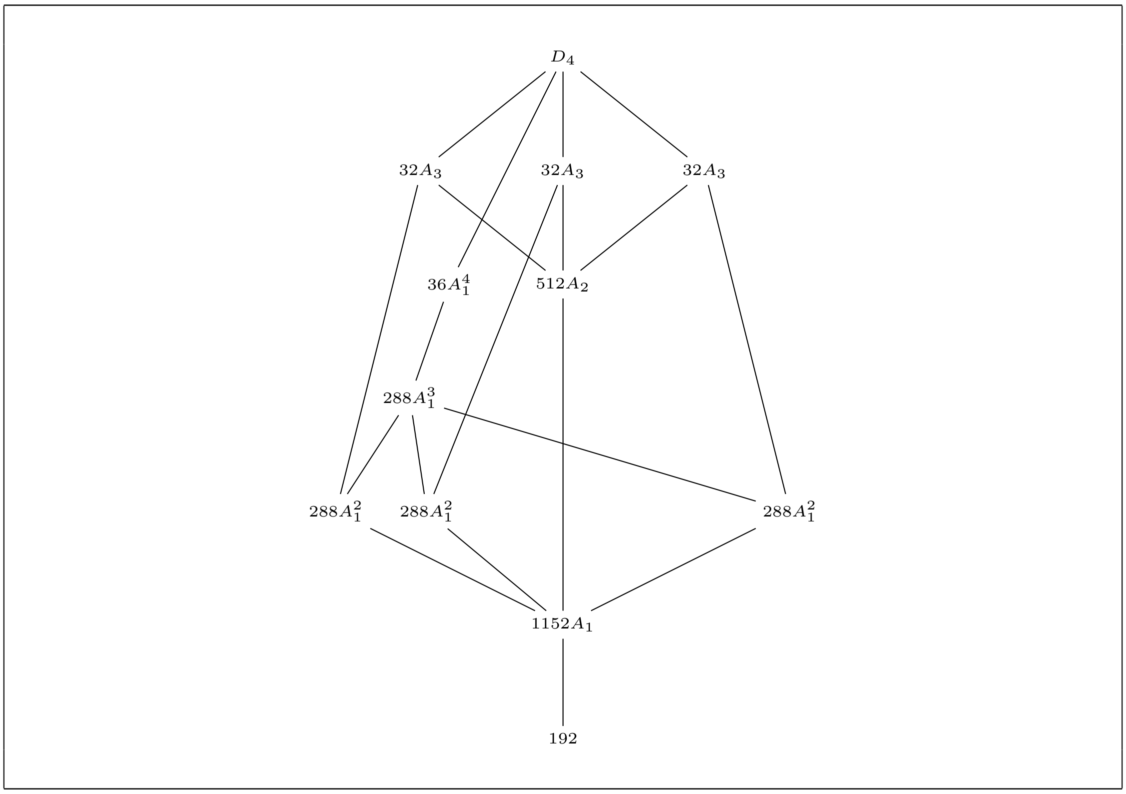


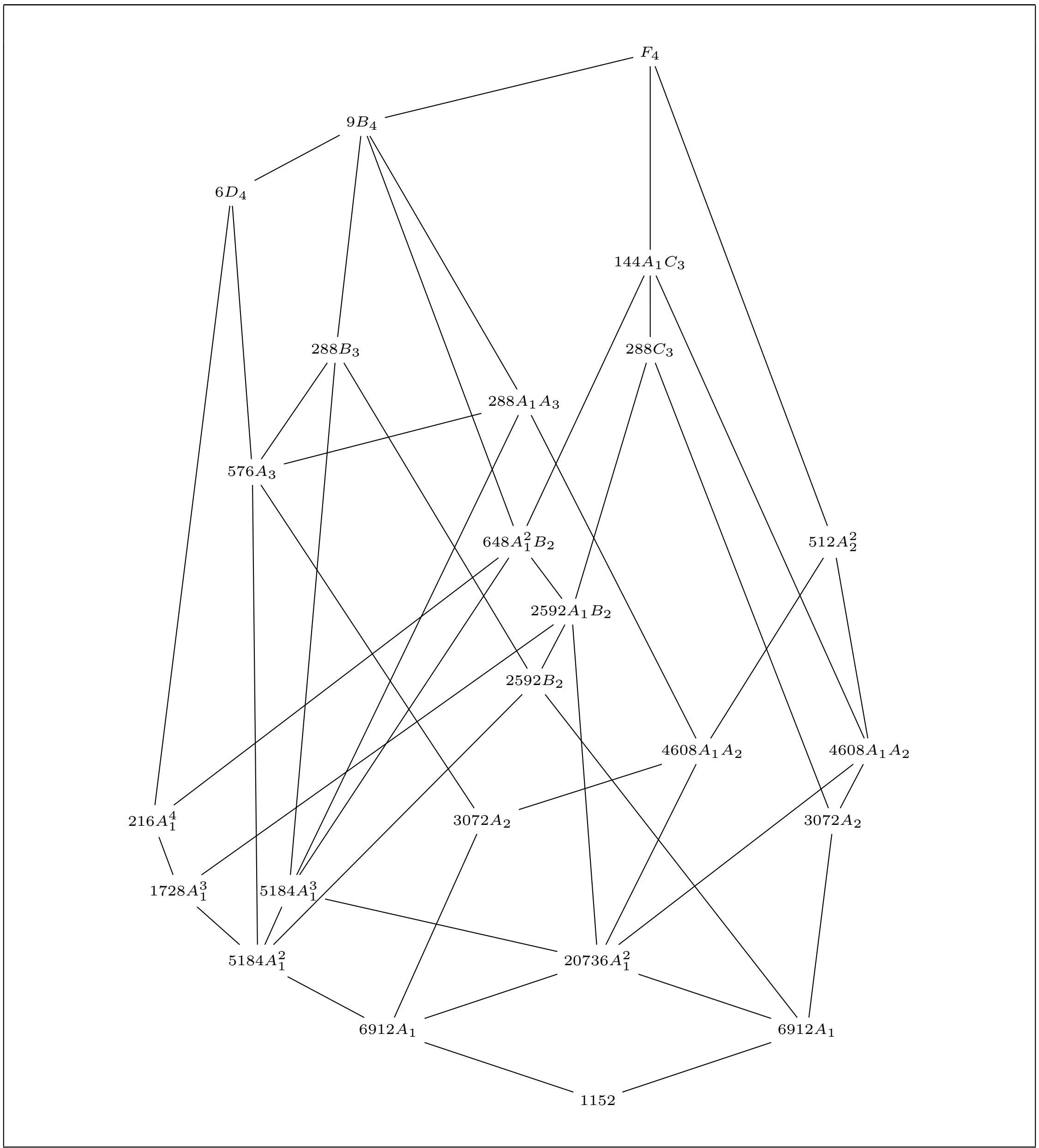


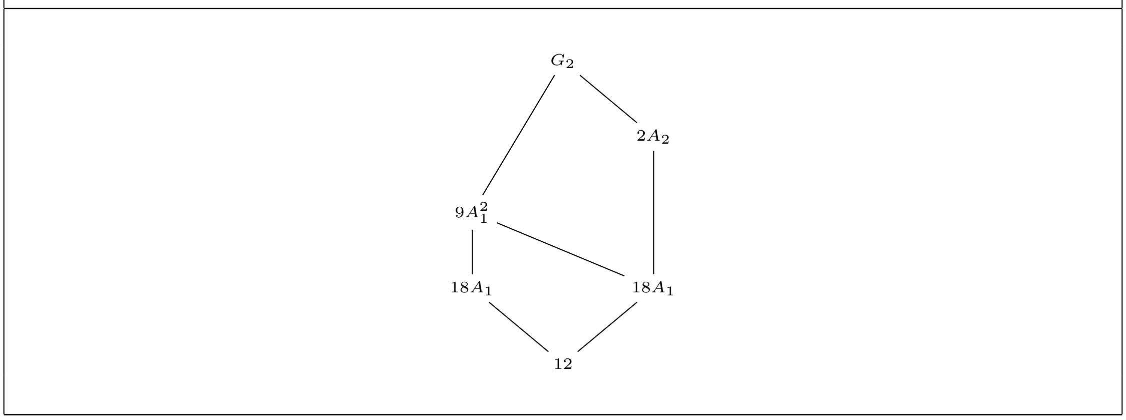
B_2
$2A_1^2$
$8A_1$
$8A_1$
8
B_3
$2A_3$
$18B_2$
$32A_2$
$18A_1^3$
$72A_1^2$
$36A_1^2$
$72A_1$
$144A_1$
48
B_4
$2D_4$
$32B_3$
$32A_1A_3$
$128A_3$
$64A_3$
$72A_1^2B_2$
$288A_1B_2$
$288B_2$
$512A_1A_2$
$72A_1^4$
$1024A_2$
$576A_1^3$
$576A_1^3$
$1152A_1^2$
$576A_1^2$
$2304A_1^2$
$2304A_1$
$768A_1$
384



C_3
$9A_1B_2$
$18B_2$
$32A_2$
$6A_1^3$
$72A_1^2$
$36A_1^2$
$144A_1$
$72A_1$
48
C_4
$16A_1C_3$
$32C_3$
$18B_2^2$
$128A_3$
$72A_1^2B_2$
$288A_1B_2$
$288A_1B_2$
$288B_2$
$512A_1A_2$
$1024A_2$
$24A_1^4$
$576A_1^3$
$192A_1^3$
$1152A_1^2$
$2304A_1^2$
$576A_1^2$
$2304A_1$
$768A_1$
384


D_4
$32A_3$
$32A_3$
$32A_3$
$36A_1^4$
$512A_2$
$288A_1^3$
$288A_1^2$
$288A_1^2$
$288A_1^2$
$1152A_1$
192


F_4
$9B_4$
$6D_4$
$144A_1C_3$
$288B_3$
$288C_3$
$288A_1A_3$
$576A_3$
$648A_1^2B_2$
$512A_2^2$
$2592A_1B_2$
$2592B_2$
$4608A_1A_2$
$4608A_1A_2$
$216A_1^4$
$3072A_2$
$3072A_2$
$1728A_1^3$
$5184A_1^3$
$5184A_1^2$
$20736A_1^2$
$6912A_1$
$6912A_1$
1152


G_2
$2A_2$
$9A_1^2$
$18A_1$
$18A_1$
12

Chapter 7

Groupoid structures from Nahm's equations

7.1 Introduction

Recall that a groupoid G is a generalisation of the notion of a group, where the product of two elements $a, b \in \mathsf{G}$ is not always defined. More specifically, there are two maps s and t—called the source and target maps—from G to a set X such that ab is defined only if $\mathsf{t}(a) = \mathsf{s}(b)$ (with some compatibility conditions; see §7.2). An important example is the fundamental groupoid $\pi_1(M)$ of a topological space M. Here, elements of $\pi_1(M)$ are fixed-point homotopy classes of paths $[0,1] \to M$ and the maps s, t are evaluations at 0 and 1 respectively: we can form the product pq of two paths p, q that satisfy $\mathsf{t}(p) = \mathsf{s}(q)$ by concatenation of paths.

In this chapter, we investigate possible groupoid structures emerging from the moduli space $\mathcal{M}_{[a,b]}$ of solutions to Nahm's equations on a compact interval $[a,b]$. The key observation is that if $a < b < c$, then elements of $\mathcal{M}_{[a,b]}$ and $\mathcal{M}_{[b,c]}$ which agree at b can be combined to form elements of $\mathcal{M}_{[a,c]}$, and this gives a well-defined smooth map

$$\mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]} \longrightarrow \mathcal{M}_{[a,c]}, \tag{7.1.1}$$

where $\mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]}$ is the set of pairs (A, B) with $A_i(b) = B_i(b)$ for $i = 1, 2, 3$. We call (7.1.1) the **concatenation map**. The idea is to use this map as an analogy for concatenation of paths in the fundamental groupoid and try to obtain non-trivial Lie groupoid structures from this. Here, a *Lie* groupoid is one whose source and target maps are smooth. We consider three different approaches:

1. We can use the concatenation map to obtain a partial product on the space $\mathcal{M} := \mathcal{M}_{[0,1]}$ itself with source and target maps s and t defined to be the evaluation of (A_1, A_2, A_3) at $t = 0$ and $t = 1$, respectively. More precisely, we have a translation map $\mathcal{M}_{[0,1]} \to \mathcal{M}_{[1,2]}$ and a scaling map $\mathcal{M}_{[0,2]} \to \mathcal{M}_{[0,1]}$, so the concatenation map $\mathcal{M}_{[0,1]} * \mathcal{M}_{[1,2]} \to \mathcal{M}_{[0,2]}$ gives a partial product

$$\mathcal{M} * \mathcal{M} \longrightarrow \mathcal{M}, \quad (A, B) \longmapsto AB$$

defined for all $A, B \in \mathcal{M}$ such that $\mathsf{t}(A) = \mathsf{s}(B)$. Unfortunately, this operation does not make $\mathcal{M}$ into a groupoid; the compatibility conditions are not satisfied. In fact, we will show that $\mathcal{M}$ does not admit any groupoid structure whose source and target maps are evaluations of (A_1, A_2, A_3) at $t = 0$ and $t = 1$, unless the compact Lie group K in the definition of $\mathcal{M}$ is abelian.

2. The second approach is similar to the first one, but exploiting the natural biholomorphism between $\mathcal{M}_I$ and the moduli space $\mathcal{N}_I$ of solutions to the *complex* Nahm equation (§5.2.1). The concatenation map (7.1.1) turns into a map

$$\mathcal{N}_{[a,b]} * \mathcal{N}_{[b,c]} \longrightarrow \mathcal{N}_{[a,c]}$$

where $\mathcal{N}_{[a,b]} * \mathcal{N}_{[b,c]}$ is the set of pairs (A, B) with $A_2(b) + iA_3(b) = B_2(b) + iB_3(b)$. Now, using a translation map and a scaling map as in the first approach, we get a product on $\mathcal{N}_{[0,1]}$ defined for pairs (A, B) such that $A_2(1) + iA_3(1) = B_2(0) + iB_3(0)$. In this case, we *do* get a Lie groupoid structure. Moreover, under the biholomorphism $\mathcal{N} \cong T^*G$, we recover the well-known canonical Lie groupoid structure on T^*G.

3. The third approach has a different flavour. Instead of trying to construct a groupoid structure on each $\mathcal{M}_I$ individually, we look at the union of all moduli spaces $\mathcal{M}_I$. More precisely, for each $r \in \mathbb{R}$, let I_r be the closed interval with endpoints 0 and r, and let $\mathcal{M}_r := \mathcal{M}_{I_r}$ (we will give a precise definition of $\mathcal{M}_{\{0\}}$). Then, the disjoint union $\mathcal{W} := \bigsqcup_{r \in \mathbb{R}} \mathcal{M}_r$ is a smooth manifold, and we will construct a Lie groupoid structure on $\mathcal{W}$ with source and target maps evaluations of (A_1, A_2, A_3) at $t = 0$ and $t = r$ respectively and product given by concatenation. We will also show that $\mathcal{W}$ can be interpreted as the deformation space of K in $\mathcal{M}$.

7.2 Background on groupoids

In this section, we recall the definition of a groupoid and mention three examples which will be important in this chapter. This material can be found, for example, in Mackenzie [102] or Cannas da Silva–Weinstein [28].

Definition 7.2.1. A **groupoid** consists of two sets G and X together with the following structure maps.

(a) A pair of surjective maps $\mathsf{s}, \mathsf{t} : \mathsf{G} \to \mathsf{X}$ called the **source** and **target** maps, respectively.

(b) A map $\mathsf{G} * \mathsf{G} \to \mathsf{G}$, $(p, q) \mapsto pq$ called the **partial product**, where

$$\mathsf{G} * \mathsf{G} = \{(p, q) \in \mathsf{G} \times \mathsf{G} : \mathsf{t}(p) = \mathsf{s}(q)\}.$$

(c) A map $1 : \mathsf{X} \to \mathsf{G}$, $x \mapsto 1_x$ called the **object inclusion map**.

These maps are required to satisfy the following five compatibility conditions.

(i) $\mathsf{s}(pq) = \mathsf{s}(p)$ and $\mathsf{t}(pq) = \mathsf{t}(q)$ for all $(p, q) \in \mathsf{G} * \mathsf{G}$.

(ii) $p(qr) = (pq)r$ for all $p, q, r \in G$ such that $\mathsf{t}(p) = \mathsf{s}(q)$ and $\mathsf{t}(q) = \mathsf{s}(r)$.

(iii) $\mathsf{s}(1_x) = \mathsf{t}(1_x) = x$ for all $x \in \mathsf{X}$.

(iv) $1_{\mathsf{s}(p)}p = p$ and $p1_{\mathsf{t}(p)} = p$ for all $p \in \mathsf{G}$.

(v) For each $p \in \mathsf{G}$ there exists $p^{-1} \in \mathsf{G}$ such that $\mathsf{t}(p^{-1}) = \mathsf{s}(p)$, $\mathsf{s}(p^{-1}) = \mathsf{t}(p)$, and $p^{-1}p = 1_{\mathsf{t}(p)}$, $pp^{-1} = 1_{\mathsf{s}(p)}$.

It is called a **Lie groupoid** if G and X are smooth manifolds, the maps in (a), (b) and (c) are smooth, and s and t are submersions.

Example 7.2.2 (Fundamental groupoid)**.** Let M be a topological space and $\pi_1(M)$ the set of continuous paths $[0, 1] \to M$ identified up to endpoint preserving homotopies. Let $\mathsf{s}, \mathsf{t} : \pi_1(M) \to M$ be evaluations at 0 and 1, respectively. Define a partial product $\pi_1(M) * \pi_1(M) \to \pi_1(M)$ by concatenation of paths and reparametrisation. This gives $\pi_1(M)$ the structure of a groupoid, called the **fundamental groupoid**.

Example 7.2.3 (Action groupoid)**.** Let G be a Lie group acting smoothly on a smooth manifold M. There is a canonical Lie groupoid structure on $G \times M$, called the **action groupoid** [102, Example 1.1.9]. The source and target maps are

$$\mathsf{s}, \mathsf{t} : G \times M \longrightarrow M, \quad \mathsf{s}(g, p) = g \cdot p, \quad \mathsf{t}(g, p) = p,$$

the partial product is

$$(g, p)(h, q) = (gh, q), \qquad \text{if } p = h \cdot q$$

the object inclusion map is $1_p = (1, p)$, and the inverse is $(g, p)^{-1} = (g^{-1}, g \cdot p)$.

Example 7.2.4 (Cotangent groupoid)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, G acts on $\mathfrak{g}^*$ via the coadjoint representation, so $G \times \mathfrak{g}^*$ is a Lie groupoid by the preceding example. Hence, by using the natural diffeomorphism $T^*G \cong G \times \mathfrak{g}^*$ obtained by left translations, we get a canonical Lie groupoid structure on T^*G, called the **cotangent groupoid** [102, Example 1.1.17 and Example 1.2.8].

7.3 The concatenation map

Let K be a compact Lie group and $\mathcal{M}_{[a,b]}$ the corresponding moduli space of solutions to Nahm's equations on $[a,b]$ as introduced in §5.2. Let $\mathcal{K}_{[a,b]}$ be the group of smooth maps $\gamma : [a,b] \to K$ and $\mathcal{K}^0_{[a,b]}$ the subgroup of $\gamma \in \mathcal{K}_{[a,b]}$ with $\gamma(a) = \gamma(b) = 1$. Recall that the action of $K \times K \cong \mathcal{K}_{[a,b]}/\mathcal{K}^0_{[a,b]}$ on $\mathcal{M}_{[a,b]}$ is tri-Hamiltonian with hyperkähler moment map

$$\mu : A \longmapsto \begin{pmatrix} A_1(a) & A_2(a) & A_3(a) \\ -A_1(b) & -A_2(b) & -A_3(b) \end{pmatrix}.$$

Let $a < b < c$ be real numbers. Then, K acts on $\mathcal{M}_{[a,b]}$ as the right factor of $K \times K$ and on $\mathcal{M}_{[b,c]}$ as the left factor, so we have an action of K on $\mathcal{M}_{[a,b]} \times \mathcal{M}_{[b,c]}$ with hyperkähler moment map

$$\Phi : \mathcal{M}_{[a,b]} \times \mathcal{M}_{[b,c]} \longrightarrow \mathfrak{k}^3, \quad \Phi(A,B) = (B_1(b) - A_1(b), B_2(b) - A_2(b), B_3(b) - A_3(b)).$$

Let

$$\mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]} := \Phi^{-1}(0),$$

and

$$\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]} := \Phi^{-1}(0)/K.$$

Since the action of K on $\mathcal{M}_{[a,b]} \times \mathcal{M}_{[b,c]}$ is free and tri-Hamiltonian, both $\mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]}$ and $\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]}$ are smooth manifolds, and there is a hyperkähler structure on the latter.

Proposition 7.3.1. *If A and B are solutions to Nahm's equations on $[a,b]$ and $[b,c]$ respectively and $A_i(b) = B_i(b)$ for $i = 1,2,3$, then there exist $f \in \mathcal{K}^0_{[a,b]}$ and $g \in \mathcal{K}^0_{[b,c]}$ such that $f \cdot A$ and $g \cdot B$ fit into a single smooth solution to Nahm's equations on $[a,c]$. Moreover, this gives a well-defined smooth map*

$$\mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]} \longrightarrow \mathcal{M}_{[a,c]} \tag{7.3.1}$$

called the **concatenation map***, which descends to a hyperkähler isomorphism*

$$\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]} \xrightarrow{\cong} \mathcal{M}_{[a,c]}.$$

Let us first recall some notation from §5.2.4. If $x \in \mathfrak{k}^3$ and $b \in \mathbb{R}$, we let $P^{x,b}$ be the maximally extended solution to the reduced Nahm equations (5.2.3) with $P(b) = x$. Also, $V_{[a,b]}$ is the set of $x \in \mathfrak{k}^3$ such that $P^{x,b}$ is defined at least on $[a, b]$. Then, $V_{[a,b]}$ is a star-shaped open subset of $\mathfrak{k}^3$ and there is a canonical diffeomorphism $\mathcal{M}_{[a,b]} \cong K \times V_{[a,b]}$.

Proof of Proposition 7.3.1. Let $f : [a, b] \to K$ and $g : [b, c] \to K$ be smooth maps such that

$$\begin{aligned} A &= f \cdot (0, P_1, P_2, P_3), \quad f(b) = 1 \\ B &= g \cdot (0, Q_1, Q_2, Q_2), \quad g(b) = 1, \end{aligned}$$

where P and Q are solutions to the reduced Nahm equations (5.2.3). Since $A_i(b) = B_i(b)$ for $i = 1, 2, 3$ and $f(b) = g(b) = 1$, we have $P(b) = Q(b)$, so P and Q fit into a single smooth solution R to the reduced Nahm equations on $[a, c]$ (smoothness follows from uniqueness of solutions to systems of ODEs with given initial conditions). Now, take a smooth map $h : [a, c] \to K$ such that $h(a) = f(a)$, $h(b) = 1$ and $h(c) = g(c)$, and let

$$U := h \cdot (0, R_1, R_2, R_3).$$

Then, U is smooth because h and R are. Moreover, we have $(h|_{[a,b]})f^{-1} \in \mathcal{K}^0_{[a,b]}$, $(h|_{[b,c]})g^{-1} \in \mathcal{K}^0_{[b,c]}$, and

$$\begin{aligned} (h|_{[a,b]})f^{-1} \cdot A &= (h|_{[a,b]}) \cdot (0, P_1, P_2, P_3) = U|_{[a,b]} \\ (h|_{[b,c]})g^{-1} \cdot B &= (h|_{[b,c]}) \cdot (0, Q_1, Q_2, Q_3) = U|_{[b,c]}. \end{aligned}$$

This proves the first part of the proposition.

As an element of $\mathcal{M}_{[a,c]}$, U depends only on the equivalence classes of A and B in $\mathcal{M}_{[a,b]}$ and $\mathcal{M}_{[b,c]}$ respectively, so this gives a well defined map

$$\varphi : \mathcal{M}_{[a,b]} * \mathcal{M}_{[b,c]} \longrightarrow \mathcal{M}_{[a,c]}.$$

To see that φ is smooth, we argue as follows. By Theorem 5.2.5, under the diffeomorphisms $\mathcal{M}_{[a,b]} \cong K \times V_{[a,b]}$ and $\mathcal{M}_{[b,c]} \cong K \times V_{[b,c]}$, the hyperkähler moment map Φ becomes

$$\Phi : (K \times V_{[a,b]}) \times (K \times V_{[b,c]}) \longrightarrow K \times V_{[a,c]}, \quad \Phi((k, x), (l, y)) = \mathrm{Ad}_l\, P^{y,c}(b) - x.$$

We claim that the concatenation map φ is

$$(K \times V_{[a,b]}) * (K \times V_{[b,c]}) \longrightarrow K \times V_{[a,c]}, \quad ((k, x), (l, y)) \longmapsto (kl, y), \qquad (7.3.2)$$

which is smooth. To prove (7.3.2), let $((k,x),(l,y)) \in (K \times V_{[a,b]}) * (K \times V_{[b,c]})$. Then, (k,x) corresponds to $\gamma_k \cdot (0, P^{x,b}) \in \mathcal{M}_{[a,b]}$ where γ_k is any smooth map $[a,b] \to K$ with $\gamma_k(a) = k$ and $\gamma_k(b) = 1$. Similarly, (l,y) corresponds to $\gamma_l \cdot (0, P^{y,c}) \in \mathcal{M}_{[b,c]}$ for any $\gamma_l : [b,c] \to K$ with $\gamma_l(b) = l$ and $\gamma_l(c) = 1$. Now, if l also denotes the constant map $[b,c] \to K : t \mapsto l$, we have $\gamma_l \cdot (0, P^{y,c}) = \gamma_l l^{-1} \cdot (0, \mathrm{Ad}_l P^{y,c})$ and $\gamma_l l^{-1}(b) = 1$. Moreover, $\mathrm{Ad}_l P^{y,c}(b) = P^{x,b}(b)$, so $\mathrm{Ad}_l P^{y,c}$ and $P^{x,b}$ fit into a solution R to the reduced Nahm equations on $[a,c]$. Then, by definition of the concatenation map, $((k,x),(l,y))$ is mapped to $h \cdot (0,R)$, where $h(a) = k$ and $h(c) = l^{-1}$. Since $h \cdot (0,R) = hl \cdot (0, \mathrm{Ad}_{l^{-1}} R)$ and $hl(c) = 1$, the corresponding element of $K \times V_{[a,c]}$ is $(h(a), \mathrm{Ad}_{l^{-1}} R(c)) = (kl, y)$. This proves (7.3.2) and hence that the concatenation map is smooth.

Now, by Theorem 5.2.5, the diagonal action of K on $(K \times V_{[a,b]}) * (K \times V_{[b,c]})$ is

$$m \cdot ((k,x),(l,y)) = ((km^{-1}, \mathrm{Ad}_m x), (ml, y)).$$

Hence, φ is K-invariant and descends to a smooth map $\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]} \to \mathcal{M}_{[a,c]}$, which is a diffeomorphism with inverse

$$K \times V_{[a,c]} \longrightarrow (K \times V_{[a,b]}) *_K (K \times V_{[b,c]}), \quad (k,y) \longmapsto ((k, P^{y,c}(b)), (1,y)).$$

It can be verified directly from the definitions of the hyperkähler structures on $\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]}$ and $\mathcal{M}_{[a,c]}$ that the map $\mathcal{M}_{[a,b]} *_K \mathcal{M}_{[b,c]} \to \mathcal{M}_{[a,c]}$ is an isomorphism of hyperkähler manifolds. But a simpler argument is to use the SO(3)-hyperkähler rotation to reduce to check that, under the isomorphisms $\mathcal{M}_I \cong T^*G$, the corresponding map $T^*G *_G T^*G \to T^*G$ is a complex-symplectomorphism. This map is given by $[(g,x),(h,y)] \mapsto (gh, y)$ and from the explicit expression (3.3.6) for the complex-symplectic form on T^*G, we easily verify that it is a complex-symplectomorphism. □

It will be convenient later to record the following corollary of the proof.

Lemma 7.3.2. *Under the diffeomorphisms* $\mathcal{M}_{[a,b]} \cong K \times V_{[a,b]}$ *and* $\mathcal{M}_{[b,c]} \cong K \times V_{[b,c]}$, *the concatenation map* (7.3.1) *is*

$$\begin{array}{ccc} (K \times V_{[a,b]}) * (K \times V_{[b,c]}) & \longrightarrow & K \times V_{[a,c]} \\ ((k,x),(l,y)) & \longmapsto & (kl, y), \end{array}$$

where the domain is the set of pairs $((k,x),(l,y))$ *such that* $x = \mathrm{Ad}_l P^{y,c}(b)$. □

7.4 A non-existence result

There are natural translation maps

$$\mathcal{M}_{[a,b]} \longrightarrow \mathcal{M}_{[a+c,b+c]}, \quad A(t) \longmapsto A(t-c)$$

and scaling maps

$$\mathcal{M}_{[a,b]} \longrightarrow \mathcal{M}_{[0,1]}, \quad A(t) \longmapsto (b-a)A(a+t(b-a)).$$

Hence, the concatenation map $\mathcal{M}_{[0,1]} * \mathcal{M}_{[1,2]} \to \mathcal{M}_{[0,2]}$ gives a partial product on $\mathcal{M} \coloneqq \mathcal{M}_{[0,1]}$, i.e. a smooth map

$$\mathcal{M} * \mathcal{M} \longrightarrow \mathcal{M}, \quad (A,B) \longmapsto AB,$$

where

$$\mathcal{M} * \mathcal{M} = \{(A,B) \in \mathcal{M} \times \mathcal{M} : \mathsf{t}(A) = \mathsf{s}(B)\},$$

and

$$\begin{aligned} \mathsf{s} : \mathcal{M} &\longrightarrow \mathfrak{k}^3, \quad \mathsf{s}(A) = (A_1(0), A_2(0), A_3(0)), \\ \mathsf{t} : \mathcal{M} &\longrightarrow \mathfrak{k}^3, \quad \mathsf{t}(A) = (A_1(1), A_2(1), A_3(1)). \end{aligned} \tag{7.4.1}$$

However, this product is not associative. The easiest way to see this is to translate it to a partial product on $K \times V$, where $V \coloneqq V_{[0,1]}$. The scaling map $\mathcal{M}_{[a,b]} \to \mathcal{M}_{[0,1]}$ is given by $K \times V_{[a,b]} \to K \times V : (k,x) \mapsto (k,(b-a)x)$, so the partial product on $K \times V$ is

$$(k,x) \cdot (l,y) = (kl, 2y).$$

In particular,

$$\Big((k,x)\cdot(l,y)\Big)\cdot(m,z) = (kl,2y)\cdot(m,z) = (klm,2z),$$

while

$$(k,x)\cdot\Big((l,y)\cdot(m,z)\Big) = (k,x)\cdot(lm,2z) = (klm,4z).$$

Therefore, this product cannot be used to give $\mathcal{M}$ the structure of a groupoid. In fact, this is not a surprise since, if $\mathfrak{k}$ is not abelian, it is not possible to have any groupoid structure on $\mathcal{M}$ whose source and target maps are evaluations of (A_1, A_2, A_3) at the endpoints:

Theorem 7.4.1. *Suppose that there is a groupoid structure on $\mathcal{M}$ whose source and target maps are* (7.4.1)*. Then, $\mathfrak{k}$ is abelian.*

The main ingredient of the proof is the following observation:

Lemma 7.4.2. *Let P be a solution to the reduced Nahm equations on a compact interval $[a,b]$. If $P(a)$ and $P(b)$ are conjugate, then P is constant.*

Proof. Without loss of generality, $[a,b]=[0,1]$, so $P(1)=\operatorname{Ad}_k P(0)$ for some $k\in K$. Define $Q:[1,2]\to\mathfrak{k}^3$ by $Q(t)=\operatorname{Ad}_k P(t-1)$. Then, Q is a solution to the reduced Nahm equations with $Q(1)=P(1)$, so, by uniqueness of solutions, P extends to $[0,2]$. Moreover, $P(2)=\operatorname{Ad}_k P(1)$ and $P(1)=\operatorname{Ad}_k P(0)$, so $\|P(0)\|^2=\|P(1)\|^2=\|P(2)\|^2$. Since $\|P\|^2$ is convex (Lemma 5.3.5), it must be constant. By (5.3.2), we get $[P_i,P_j]=0$ for all i,j and hence $\dot{P}=0$. □

Proof of Theorem 7.4.1. This follows from property (iii) in the definition of a groupoid, which says that $\mathsf{s}(1_x)=\mathsf{t}(1_x)=x$ for all x in the image of s and t. Using the diffeomorphism $K\times V\cong\mathcal{M}$ of Theorem 5.2.5, the source and target maps are $\mathsf{s}(k,x)=\operatorname{Ad}_k P^x(0)$ and $\mathsf{t}(k,x)=x$, where $P^x:=P^{x,1}$. Hence, $\operatorname{im}\mathsf{t}=V\subseteq\mathfrak{k}^3$ and the object inclusion map must be of the form

$$1:V\longrightarrow K\times V,\quad 1_x=(\gamma(x),x)$$

for some map $\gamma:V\to K$. Then, for all $x\in V$, we have $\mathsf{s}(1_x)=x$, so $\operatorname{Ad}_{\gamma(x)}P^x(0)=x=P^x(1)$, and hence P^x is constant by Lemma 7.4.2. In particular, $[x_i,x_j]=0$ for all i,j. Now, if $y,z\in\mathfrak{k}$ are arbitrary, we can find $\delta>0$ such that $x:=(0,\delta y,\delta z)\in V$, and then the above discussion shows that $[y,z]=0$. □

7.5 A Lie groupoid structure on $\mathcal{M}$

Let $\mathcal{N}_I$ be the moduli space of solutions to the complex Nahm equation $\dot{\beta}+[\alpha,\beta]=0$ on $I=[a,b]$. Recall, from §5.2.3, that we have a biholomorphism $\mathcal{M}_I\cong\mathcal{N}_I$. With this identification, we can get a complex version of the concatenation map. That is, if (α_1,β_1) and (α_2,β_2) are solutions to $\dot{\beta}+[\alpha,\beta]=0$ on $[a,b]$ and $[b,c]$ respectively and $\beta_1(b)=\beta_2(b)$, then there exist $g_1\in\mathcal{G}^0_{[a,b]}$ and $g_2\in\mathcal{G}^0_{[b,c]}$ such that $g_1\cdot(\alpha_1,\beta_1)$ and $g_2\cdot(\alpha_2,\beta_2)$ fit into a single smooth solution to $\dot{\beta}+[\alpha,\beta]=0$ on $[a,c]$. The proof is similar. Moreover, this gives a well-defined smooth map making the following diagram commute:

$$\begin{array}{ccc}\mathcal{M}_{[a,b]}*\mathcal{M}_{[b,c]} & \longrightarrow & \mathcal{M}_{[a,c]}\\ \downarrow & & \downarrow\\ \mathcal{N}_{[a,b]}*\mathcal{N}_{[b,c]} & \longrightarrow & \mathcal{N}_{[a,c]},\end{array}$$

where

$$\mathcal{N}_{[a,b]} * \mathcal{N}_{[b,c]} := \{(A,B) \in \mathcal{N}_{[a,b]} \times \mathcal{N}_{[b,c]} : A_2(b) + iA_3(b) = B_2(b) + iB_3(b)\}.$$

Now, let $\mathcal{N} := \mathcal{N}_{[0,1]}$. Using the translation map

$$\mathcal{N} \longrightarrow \mathcal{N}_{[1,2]}, \quad (\alpha(t), \beta(t)) \longmapsto (\alpha(t-1), \beta(t-1))$$

and the scaling map

$$\mathcal{N}_{[0,2]} \longrightarrow \mathcal{N}, \quad (\alpha(t), \beta(t)) \longmapsto (2\alpha(2t), \beta(2t)),$$

we get from $\mathcal{N}_{[0,1]} * \mathcal{N}_{[1,2]} \to \mathcal{N}_{[0,2]}$ a holomorphic map

$$\mathcal{N} * \mathcal{N} \longrightarrow \mathcal{N},$$

where

$$\mathcal{N} * \mathcal{N} := \{(A,B) \in \mathcal{N} \times \mathcal{N} : \mathsf{t}(A) = \mathsf{s}(B)\}$$

and

$$\mathsf{s} : \mathcal{N} \longrightarrow \mathfrak{g}, \quad A \longmapsto A_2(0) + iA_3(0)$$
$$\mathsf{t} : \mathcal{N} \longrightarrow \mathfrak{g}, \quad A \longmapsto A_2(1) + iA_3(1).$$

Under the biholomorphisms $\mathcal{N}_I \cong G \times \mathfrak{g}$ given by (5.2.2), these maps are

$$\mathsf{s}(g,x) = \mathrm{Ad}_g\, x, \quad \mathsf{t}(g,x) = x,$$

and the partial product $\mathcal{N} * \mathcal{N} \to \mathcal{N}$ is

$$(g,x)(h,y) = (gh, y), \quad \text{if } x = \mathrm{Ad}_h\, y.$$

But those are precisely the source and target maps and partial product of the action groupoid of G on $\mathfrak{g}$ (Example 7.2.3). Using the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ coming from the invariant inner-product, this then corresponds to the cotangent groupoid T^*G (Example 7.2.4). Hence, concatenation of paths and rescaling makes $\mathcal{M}$ into a Lie groupoid over $\mathfrak{g}$ isomorphic to the cotangent groupoid T^*G.

7.6 The union of all Nahm moduli spaces as a Lie groupoid

For all $r \in \mathbb{R}$, let I_r be the closed interval with endpoints 0 and r, i.e. $I_r = [0,r]$ if $r > 0$, $I_r = \{0\}$ if $r = 0$, and $I_r = [r,0]$ if $r < 0$. For $r \neq 0$, let $\mathcal{M}_r := \mathcal{M}_{I_r}$. Also, we interpret $\mathcal{M}_0 = \mathcal{M}_{\{0\}}$ to be $K \times \mathfrak{k}^3$. To see why this is a sensible definition of $\mathcal{M}_{\{0\}}$, recall that we have diffeomorphisms $\mathcal{M}_r \cong K \times V_{I_r}$, where

$$V_{I_r} \coloneqq \{x \in \mathfrak{k}^3 : P^{x,r} \text{ is defined at least on } I_r\}.$$

Clearly, $V_{\{0\}} = \mathfrak{k}^3$, since for all $x \in \mathfrak{k}^3$, $P^{x,0}$ is defined in a small neighbourhood of 0. This makes a precise reason for why $\lim_{r\to 0} \mathcal{M}_r = K \times \mathfrak{k}^3$.

Moreover, these diffeomorphisms allow us to identify the disjoint union

$$\mathcal{W} \coloneqq \bigsqcup_{r\in\mathbb{R}} \mathcal{M}_r$$

with an open subset of $K \times \mathfrak{k}^3 \times \mathbb{R}$. Indeed, the set

$$W \coloneqq \{(x,r) \in \mathfrak{k}^3 \times \mathbb{R} : P^{x,r} \text{ is defined on } I_r\}$$

is open in $\mathfrak{k}^3 \times \mathbb{R}$, and we have a bijection

$$\mathcal{W} = \bigsqcup_{r\in\mathbb{R}} \mathcal{M}_r \xrightarrow{\cong} K \times W, \quad A \in \mathcal{M}_r \longmapsto (\gamma(0)^{-1}, A_1(r), A_2(r), A_3(r), r),$$

where $\gamma : I_r \to K$ is the unique solution to $\dot{\gamma} = \gamma A_0$ with $\gamma(r) = 1$. In particular, $\mathcal{W}$ is a smooth manifold.

Let $\mathcal{W}_+$ be the open subset of $\mathcal{W}$ given by $\mathcal{W}_+ = \bigsqcup_{r>0} \mathcal{M}_r$. For all $r, s > 0$, the concatenation map (7.3.1) gives smooth maps

$$\mathcal{M}_r * \mathcal{M}_s \longrightarrow \mathcal{M}_{r+s},$$

where

$$\mathcal{M}_r * \mathcal{M}_s = \{(A,B) \in \mathcal{M}_r \times \mathcal{M}_s : \mathsf{t}(A) = \mathsf{s}(B)\}$$

and $\mathsf{s}, \mathsf{t} : \mathcal{M}_r \to \mathfrak{k}^3$ are

$$\mathsf{s}(A) = (A_1(0), A_2(0), A_3(0)), \quad \mathsf{t}(A) = (A_1(r), A_2(r), A_3(r)).$$

We can then join all these maps together to get maps

$$\mathsf{s}, \mathsf{t} : \mathcal{W}_+ \longrightarrow \mathfrak{k}^3,$$

and a partial product

$$\mathcal{W}_+ * \mathcal{W}_+ \longrightarrow \mathcal{W}_+,$$

where $\mathcal{W}_+ * \mathcal{W}_+ \coloneqq \{(A,B) \in \mathcal{W}_+ \times \mathcal{W}_+ : \mathsf{t}(A) = \mathsf{s}(B)\}$.

In fact, these operations extend naturally to $\mathcal{W}$ to endow it with the structure of a Lie groupoid. To see this, we translate to $K \times W$. By Theorem 5.2.5, the maps s and t above are the restrictions of

$$\mathsf{s} : K \times W \longrightarrow \mathfrak{k}^3, \quad \mathsf{s}(k, x, r) = \mathrm{Ad}_k P^{x,r}(0)$$
$$\mathsf{t} : K \times W \longrightarrow \mathfrak{k}^3, \quad \mathsf{t}(k, x, r) = x,$$

and, by Lemma 7.3.2, the partial product $\mathcal{W}_+ * \mathcal{W}_+ \to \mathcal{W}_+$, is the restriction of

$$(K \times W) * (K \times W) \longrightarrow K \times W, \quad (k, x, r)(l, y, s) = (kl, y, r + s).$$

It is then easy to check that these extensions satisfy the axioms of a Lie groupoid with object inclusion map

$$1 : \mathfrak{k}^3 \longrightarrow K \times W, \quad x \longmapsto (1, x, 0),$$

and inverse

$$(k, x, r)^{-1} = (k^{-1}, \mathrm{Ad}_k P^{x,r}(0), -r).$$

Thus, we get a Lie groupoid structure on the union $\mathcal{W} = \bigsqcup_{r \in \mathbb{R}} \mathcal{M}_r$ of all Nahm moduli spaces that is compatible with concatenation of paths. Recall that on each $\mathcal{M}_r$ individually, it was not possible to construct a groupoid structure with source and target evaluations of (A_1, A_2, A_3) at the endpoints precisely because the object inclusion map could not exist on $\mathcal{M}_r$ for $r \neq 0$. Here we resolved the problem by allowing the object inclusion to lie in the limit space $\lim_{r \to 0} \mathcal{M}_r$.

7.7 Deformation space interpretation

There is a natural smooth embedding of the compact Lie group K in $\mathcal{M} = \mathcal{M}_{[0,1]}$. In terms of Nahm's equations, this embedding is

$$K \hookrightarrow \mathcal{M}, \quad k \longmapsto \gamma_k \cdot (0, 0, 0, 0),$$

where γ_k is any smooth map $[0, 1] \to K$ with $\gamma_k(0) = k$ and $\gamma_k(1) = 1$. Under the diffeomorphism $\mathcal{M} \cong K \times V$, this is the inclusion $K \hookrightarrow K \times \{0\} \hookrightarrow K \times V$. In terms of T^*G, it is the inclusion $K \hookrightarrow G \hookrightarrow T^*G$, where G is contained in T^*G as the zero-section.

In this section, we observe that the Lie groupoid $\mathcal{W}$ defined in the previous section can be interpreted as the deformation space of K in $\mathcal{M}$. Let us first recall that the **deformation space** of a smooth embedded submanifold N of a smooth manifold M (as it appears in [76, §3.1]; see also [64, §3] and [15, §2.2]) is a smooth manifold $\mathscr{D}(M, N)$ which is uniquely characterised by the following properties:

(i) There is a set-theoretic identification

$$\mathscr{D}(M,N) = (\nu(M,N) \times \{0\}) \sqcup (M \times \mathbb{R}^{\times}),$$

where $\mathbb{R}^{\times} := \mathbb{R} \setminus \{0\}$ and $\nu(M,N)$ is the normal bundle of N in M, i.e. $\nu(M,N) := TM|_N/TN$.

(ii) Define

$$\pi : \mathscr{D}(M,N) \longrightarrow \mathbb{R}$$

by mapping $\nu(M,N)\times\{0\}$ to 0 and $(p,r) \in M\times\mathbb{R}^{\times}$ to r. Then, π is a smooth submersion.

(iii) Define

$$\kappa : \mathscr{D}(M,N) \longrightarrow M$$

by mapping $\nu(M,N)\times\{0\}$ to $N \subseteq M$ via the bundle map and $(p,r) \in M\times\mathbb{R}^{\times}$ to p. Then, κ is smooth.

(iv) For any smooth function $f : M \to \mathbb{R}$ which vanishes on N, define

$$\widehat{f} : \mathscr{D}(M,N) \longrightarrow \mathbb{R}$$

by mapping a vector $v \in \nu(M,N)$ to $df(v)$ (this is well-defined since $f|_N = 0$) and a point $(p,r) \in M \times \mathbb{R}^{\times}$ to $\frac{1}{r}f(p)$. Then, $\widehat{f}$ is smooth.

The deformation space $\mathscr{D}(M,N)$ always exists, and the above properties uniquely determine its topology and smooth structure. Our goal is to interpret the Lie groupoid $\mathcal{W}$ as a deformation space:

Theorem 7.7.1. *We have $\mathcal{W} = \mathscr{D}(\mathcal{M},K)$ and the smooth submersion*

$$\pi : \mathcal{W} \longrightarrow \mathbb{R}$$

maps $\mathcal{M}_r$ to r for all $r \in \mathbb{R}$.

Proof. For all $r \in \mathbb{R}^{\times}$, there is a natural diffeomorphism

$$\mathcal{M}_r \longrightarrow \mathcal{M}, \quad A(t) \longmapsto rA(rt).$$

Moreover, the normal bundle of K in $K\times V = \mathcal{M}$ is $K\times\mathfrak{k}^3 = \mathcal{M}_0$ with the projection onto K. Hence, there is an identification

$$\mathcal{W} \cong (\nu(\mathcal{M},K) \times \{0\}) \sqcup (\mathcal{M} \times \mathbb{R}^{\times}),$$

and it suffices to show that properties (ii)-(iv) are satisfied. This identification forces the map

$$\pi : \mathcal{W} \longrightarrow \mathbb{R}$$

to send $\mathcal{M}_r$ to r for all $r \in \mathbb{R}$. Hence, by identifying $\mathcal{W}$ as the open set $K \times W \subseteq K \times \mathfrak{k}^3 \times \mathbb{R}$, π is the restriction of the projection $(k, x, r) \mapsto r$, so is a smooth submersion. Thus (ii) is satisfied.

Now, by identifying $\mathcal{M}$ with $K \times V \subseteq K \times \mathfrak{k}^3$ and $\mathcal{W}$ with $K \times W \subseteq K \times \mathfrak{k}^3 \times \mathbb{R}$, the map $\kappa : \mathcal{W} \to \mathcal{M}$ is the restriction of the map

$$K \times \mathfrak{k}^3 \times \mathbb{R} \longrightarrow K \times \mathfrak{k}^3, \quad (k, x, r) \longmapsto (k, rx),$$

and hence (iii) also holds.

Finally, let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function which vanishes on K and define $\widehat{f} : \mathcal{W} \to \mathbb{R}$ as in (iv). Then, under the identifications $\mathcal{M} = K \times V \subseteq K \times \mathfrak{k}^3$ and $\mathcal{W} = K \times W \subseteq K \times \mathfrak{k}^3 \times \mathbb{R}$, $\widehat{f}$ is given by

$$\widehat{f}(k, x, r) = \begin{cases} \frac{1}{r} f(k, rx) & ; r \neq 0 \\ \frac{d}{dt}\Big|_{t=0} f(k, tx) = \lim_{t \to 0} \frac{1}{t} f(k, tx) & ; r = 0, \end{cases}$$

which is smooth since f is (this can be seen from Taylor's theorem). □

Chapter 8

Universal centralisers, Higgs bundles, and mirror symmetry

8.1 Introduction

Let G be a complex semisimple group. In this chapter, we study links between the following three subjects:

(1) The theory of G-Higgs bundles and its relationship with mirror symmetry.

(2) The universal centraliser of G: a complex-algebraic variety which plays a fundamental rôle in geometric representation theory and also appears in physics as the Coulomb branch of pure 3d $\mathcal{N} = 4$ supersymmetric gauge theory on $\mathbb{R}^3 \times S^1$.

(3) The hyperkähler quotient of T^*G by the diagonal subgroup of $K \times K$.

Recall, following Hitchin [78, 79], that a ***G*-Higgs bundle** over a Riemann surface X of genus at least 2 is a principal G-bundle P over X together with a **Higgs field**, i.e. a section Φ of $\mathrm{ad}(P) \otimes K$ where $K = T^*X$ is the canonical bundle. With a suitable notion of semistability (a generic condition), there is a quasi-projective moduli space $\mathcal{M}_G$ of semistable G-Higgs bundles over X [131, 132, 121]. Moreover, the smooth locus of $\mathcal{M}_G$ caries a hyperkähler structure obtained from a homeomorphism with a certain moduli space of solutions to the self-dual Yang–Mills equations on X which—just like Nahm's equations—can be viewed as an infinite-dimensional hyperkähler quotient.

The space $\mathcal{M}_G$ also comes with a canonical projective morphism $h : \mathcal{M}_G \to \mathcal{A}_G$, called the **Hitchin map**, where $\mathcal{A}_G$ is a finite-dimensional complex vector space [79].

Explicitly, h is defined by taking a basis $u_1, \ldots, u_r$ of generators for $\mathbb{C}[\mathfrak{g}]^G$ (which is a free polynomial algebra), let $\mathcal{A}_G \coloneqq \bigoplus_{k=0}^{r} H^0(X, K^{\deg u_i})$, and

$$h : \mathcal{M}_G \longrightarrow \mathcal{A}_G, \quad h(P, \Phi) = (u_1(\Phi), \ldots, u_r(\Phi)).$$

This morphism is a **completely integrable system**, in the sense that

(i) $\dim \mathcal{A}_G = \frac{1}{2} \dim \mathcal{M}_G$;

(ii) for any choice of basis for $\mathcal{A}_G$, the corresponding holomorphic functions $\mathcal{M}_G \to \mathbb{C}$ are Poisson commuting;

(iii) there is an open dense subset on which h is a submersion.

Furthermore, the generic fibres of h are abelian varieties. Now, this is where mirror symmetry comes in. By considering the Langlands dual group $\check{G}$, there is an isomorphism $\mathcal{A}_G \cong \mathcal{A}_{\check{G}}$, so we can see $\mathcal{M}_G$ and $\mathcal{M}_{\check{G}}$ as fibrations on a common base $\mathcal{A}$:

$$\begin{array}{ccc} \mathcal{M}_G & & \mathcal{M}_{\check{G}} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \check{h}} & \\ & \mathcal{A} & \end{array}$$

Then, for a generic $x \in \mathcal{A}$, the fibres $h^{-1}(x)$ and $\check{h}^{-1}(x)$ are *dual* abelian varieties. This was first proved by Hausel–Thaddeus [65] for $G = \mathrm{SL}(n, \mathbb{C})$ (so $\check{G} = \mathrm{PGL}(n, \mathbb{C})$) and by Donagi–Pantev [39] for the general case. This is thus an instance of SYZ mirror symmetry [137]: the set $\check{h}^{-1}(x)$ parametrises flat $\mathrm{U}(1)$-connections on $h^{-1}(x)$, and conversely.

Moreover, $\mathcal{M}_G$ and $\mathcal{M}_{\check{G}}$ also provide examples of topological mirror symmetry: the Hodge numbers behave as we would expect for mirror pairs. More precisely, by defining the **E-polynomial**

$$\mathrm{E}(\mathcal{M}_G) \coloneqq \sum_{p,q} h^{p,q}(\mathcal{M}_G) u^p v^q \in \mathbb{Z}[u, v]$$

where $h^{p,q}(\mathcal{M}_G) \in \mathbb{Z}$ are the stringy mixed Hodge numbers, we have

$$\mathrm{E}(\mathcal{M}_G) = \mathrm{E}(\mathcal{M}_{\check{G}}),$$

at least when $G = \mathrm{SL}(n, \mathbb{C})$ (the general case is still open) [65, 58]. Here, the stringy mixed Hodge numbers are a natural modification of Deligne's compactly supported mixed Hodge numbers [36, 37] which behave better for singular varieties such as $\mathcal{M}_G$. For nonsingular varieties, the two notions coincide. The equality $\mathrm{E}(\mathcal{M}_{\mathrm{SL}(n,\mathbb{C})}) = \mathrm{E}(\mathcal{M}_{\mathrm{PGL}(n,\mathbb{C})})$ is a highly non-trivial fact which was first proved by

Hausel–Thaddeus [65] for $n = 2, 3$ and more recently by Groechenig–Wyss–Ziegler [58] for all n.

Another important aspect of $\mathcal{M}_G$ and $\mathcal{M}_{\check{G}}$ is their close relationship with the geometric Langlands program, as observed by Kapustin–Witten [84]. In this work, they consider special submanifolds of $\mathcal{M}_G$ which they call **branes**. In a Kähler manifold, an **A-brane** is a Lagrangian submanifold, and a **B-brane** is a complex submanifold. Hence, in a hyperkähler manifold, such as $\mathcal{M}_G$, we can consider, for example, BAA branes, which are submanifolds that are complex with respect to I and Lagrangian with respect to ω_{J} and ω_{K}. Similarly, there are BBB branes, ABA branes, etc. It is expected that there is some equivalence of categories between branes in $\mathcal{M}_G$ and branes in $\mathcal{M}_{\check{G}}$ (or more precisely, branes endowed with some special sheaves) which exchanges BAA-branes on one side with BBB-branes on the other. The study of branes in $\mathcal{M}_G$ has thus captured much attention recently [6, 7, 81, 16, 18, 19, 21, 46, 47, 50, 75] and some also considered branes inside other hyperkähler manifolds [17, 48, 49, 83]. One way of constructing branes in $\mathcal{M}_G$, due to Baraglia–Schaposnik [7], is to consider a real form $S \subseteq G$. Such an S induces an involution $\Theta : \mathcal{M}_G \to \mathcal{M}_G$, and the set $\mathcal{M}_G^{\Theta}$ of fixed points is a BAA brane. Moreover, there is a candidate for the BBB brane in $\mathcal{M}_{\check{G}}$ corresponding to $\mathcal{M}_G^{\Theta}$ under Langlands duality: it is $\mathcal{M}_{\check{H}}$, where $\check{H} \subseteq \check{G}$ is the so-called Nadler group [113] associated with $S \subseteq G$; see [7, §7] or [21, Ch. 6].

In this chapter, we want to point out analogies between the above discussion and another interesting space associated with G: the **universal centraliser**

$$\mathfrak{Z}_G \coloneqq \{(g, x) \in G \times \mathfrak{g}^{\mathrm{reg}} : \mathrm{Ad}_g\, x = x\}/G,$$

where G acts by conjugation on both factors. This space has the structure of a smooth complex affine variety and has been studied from the point of view of geometric representation theory: it appears, in particular, in the celebrated work of Ngô [119, 120] on the fundamental lemma, and—in a different context—in Bezrukavnikov–Finkelberg–Mirković [10]. It is also an important ingredient in the work of Teleman [140, 141], where it appears as a "classifying space for topological 2-dimensional gauge theories" (and called the *BFM space*, in reference to the former paper). It has also been recently recognised by Braverman–Finkelberg–Nakajima [22] (see also [23]) as the Coulomb branch of pure 3d $\mathcal{N} = 4$ gauge theory for the Langlands dual group $\check{G}$ on $\mathbb{R}^3 \times S^1$ (it is denoted by $\mathcal{M}_C(\check{G}, 0)$ there) and has been shown to carry a non-trivial hyperkähler structure which also comes from the self-dual Yang–Mills equations, or more precisely, the Nahm equations. Moreover, there is a canonical fibration

$$\varpi : \mathfrak{Z}_G \longrightarrow \mathbb{C}^{\mathrm{rk}\,\mathfrak{g}}, \quad \varpi(g, x) = (u_1(x), \ldots, u_r(x))$$

which is known to be a completely integrable system. The map ϖ is not a projective morphism, but the generic fibres are complex-algebraic tori, i.e. products of copies of $\mathbb{C}^*$.

The resemblance with the Hitchin map is striking, but we are not aware of any work in the literature which studied this analogy. We observe that the Lie algebra element x in a pair $[g, x] \in \mathfrak{Z}_G$ can be thought of as a "Higgs field over a point" and ϖ as the corresponding Hitchin map. This idea will be put on a firm ground by showing that, in a precise sense, $\mathfrak{Z}_G$ is a *fine moduli space of regular marked G-Higgs bundles over a point.* Here, a **marking** on a G-Higgs bundle (P, Φ) over any complex manifold is an automorphism of (P, Φ). (The group element g in the pair $[g, x]$ corresponds to the marking.)

We also investigate whether $\mathfrak{Z}_G$ and $\mathfrak{Z}_{\check{G}}$ can be seen as mirror pairs in some sense. We have $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\check{\mathfrak{g}}$, so $\mathfrak{Z}_G$ and $\mathfrak{Z}_{\check{G}}$ fibre onto a common base $\mathbb{C}^r$, and there is a similar diagram:

$$\begin{array}{ccc} \mathfrak{Z}_G & & \mathfrak{Z}_{\check{G}} \\ & {\scriptstyle\varpi}\searrow \quad \swarrow{\scriptstyle\check{\varpi}} & \\ & \mathbb{C}^r & \end{array}$$

We will see that, for a generic $x \in \mathbb{C}^r$, the fibres $\varpi^{-1}(x)$ and $\check{\varpi}^{-1}(x)$ are dual complex-algebraic tori. We can then think of $\check{\varpi}^{-1}(x)$ as parametrising flat $\mathbb{C}^*$-connections (rather than flat $\mathrm{U}(1)$-connections) on $\varpi^{-1}(x)$, and conversely. This is thus, in some sense, a non-compact version of SYZ mirror symmetry.

Moreover, the theory of branes inside $\mathcal{M}_G$ has interesting analogues for $\mathfrak{Z}_G$. Given a real form $S \subseteq G$, we will construct an involution $\Theta : \mathfrak{Z}_G \to \mathfrak{Z}_G$ similar to the one considered by Baraglia–Schaposnik [7] for $\mathcal{M}_G$, and whose set of fixed points is also a BAA brane. The space $\mathfrak{Z}_G$ has an $\mathrm{SU}(2)$-hyperkähler rotation, so this also provides examples of ABA and AAB branes inside $\mathfrak{Z}_G$.

On the other hand, we will show that the E-polynomials of $\mathfrak{Z}_G$ and $\mathfrak{Z}_{\check{G}}$ are not equal in general, contrary to the case of $\mathcal{M}_G$ and $\mathcal{M}_{\check{G}}$. This discrepancy might be traced to the non-compactness of the fibres, so it is natural to ask if there is some partial compactification $\overline{\mathfrak{Z}}_G$ for which topological mirror symmetry would hold. There is already a canonical choice for $\overline{\mathfrak{Z}}_G$ which has been studied in the literature [5], but, unfortunately, we will see that, even with this, the E-polynomials do not agree.

Thus, although the Hodge numbers do not match, many non-trivial properties of $\mathcal{M}_G$ have analogues in $\mathfrak{Z}_G$, and we hope that by removing complications due to the geometry of the curve in that way—i.e. considering Higgs bundles over a point—the space $\mathfrak{Z}_G$ can still provide some toy model for mirror symmetry.

Now, leaving mirror symmetry and Higgs bundles aside, we also study the relationship between $\mathfrak{Z}_G$ and the hyperkähler quotient $T^*G /\!/\!/_\mu \Delta K$, where K is a maximal compact subgroup of G and ΔK is the diagonal subgroup of $K \times K$; the latter is a stratified hyperkähler space by the results of Chapter 3. It is known that $\mathfrak{Z}_G$ is isomorphic to the complex-symplectic reduction of $(T^*G)^{\mathrm{reg}} \cong G \times \mathfrak{g}^{\mathrm{reg}}$ by the diagonal subgroup ΔG in $G \times G$, so it is natural to ask how $\mathfrak{Z}_G$ and $T^*G /\!/\!/_\mu \Delta K$ are related. The results of Chapter 5 imply that $T^*G /\!/\!/_\mu \Delta K$ is the affine GIT quotient $\mathcal{Z} /\!/ G$, where $\mathcal{Z} := \{(g, x) \in G \times \mathfrak{g} : \mathrm{Ad}_g x = x\}$, so there is a natural morphism $\mathfrak{Z}_G \to T^*G /\!/\!/_\mu \Delta K$. However, even though $\mathfrak{Z}_G$ is smooth, we will show that it is not true that this map realises $\mathfrak{Z}_G$ as a subset of the smooth locus of $T^*G /\!/\!/_\mu \Delta K$. The map is a birational equivalence, but it is neither injective nor surjective. The spaces $\mathfrak{Z}_G$ and $T^*G /\!/\!/_\mu \Delta K$ can be thought of as genuinely different ways of completing their common subset to an affine variety.

The chapter is organised as follows. In §8.2, we review some facts about the universal centraliser $\mathfrak{Z}_G$. In §8.3, we explain how to see $\mathfrak{Z}_G$ as a moduli space of regular marked G-Higgs bundles over a point. Then, §8.4 is about the analogy between mirror symmetry for Higgs bundles and the fibrations $\mathfrak{Z}_G \to \mathbb{C}^r \leftarrow \mathfrak{Z}_{\check{G}}$. In §8.5, we discuss the relationship between $\mathfrak{Z}_G$ and $T^*G /\!/\!/_\mu \Delta K$.

8.2 Background on the universal centraliser

In this section, we review the construction of the universal centraliser of a complex semisimple group G and some of its properties. We refer to [140, §5.1] [141, §3] [22, Appendix A] [38, §11] [8, §2.1] [52, §2] [127, §3] [5, §2] for other reviews on this subject or to the original papers [10] [120, §2] [119, §3].

There are several equivalent descriptions of the universal centraliser of G, each revealing a different geometric structure. The most basic one is as the quotient space

$$\mathfrak{Z} := \{(g, x) \in G \times \mathfrak{g}^{\mathrm{reg}} : \mathrm{Ad}_g x = x\}/G,$$

where G acts on $G \times \mathfrak{g}^{\mathrm{reg}}$ by the adjoint action on both factors. Here $\mathfrak{g}^{\mathrm{reg}}$ is the set of **regular** elements of $\mathfrak{g}$, i.e. elements $x \in \mathfrak{g}$ whose centraliser $\mathfrak{g}^x := \{y \in \mathfrak{g} : [x, y] = 0\}$ has dimension $\mathrm{rk}\,\mathfrak{g}$. We will also sometimes use the notation $\mathfrak{Z}_G$ to specify the group G (which will always be complex semisimple). This describes $\mathfrak{Z}$ as a topological space, but it also has the structure of a complex-algebraic variety. Let

$$\mathcal{Z} := \{(g, x) \in G \times \mathfrak{g} : \mathrm{Ad}_g x = x\},$$

and $\mathscr{Z}^{\mathrm{reg}} := \mathscr{Z} \cap (G \times \mathfrak{g}^{\mathrm{reg}})$, so that $\mathfrak{Z} = \mathscr{Z}^{\mathrm{reg}}/G$. Then, $\mathscr{Z}^{\mathrm{reg}}$ is a Zariski-open subset of the affine variety $\mathscr{Z}$, and the categorical quotient $\mathscr{Z}^{\mathrm{reg}} /\!\!/ G$ exists as a complex-algebraic variety. But all G-orbits in $\mathscr{Z}^{\mathrm{reg}}$ have the same dimension, so they are closed in $\mathscr{Z}^{\mathrm{reg}}$, and hence this categorical quotient is a geometric quotient, namely $\mathfrak{Z}$.

In fact, the categorical quotient $\mathfrak{Z} = \mathscr{Z}^{\mathrm{reg}} /\!\!/ G$ is a smooth complex affine variety. To see this, it is useful to recall the notion of the Kostant slice [93] [29] [127]. Let (e, f, h) be a **principal $\mathfrak{sl}_2$-triple**, i.e. a triple of elements of $\mathfrak{g}$ such that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$ and $e \in \mathfrak{g}^{\mathrm{reg}}$ (and hence also $f, h \in \mathfrak{g}^{\mathrm{reg}}$). Equivalently, this is a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ such that $\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{g}^{\mathrm{reg}}$. These triples exist and are unique up to conjugation [92, §5]. The **Kostant slice** (also known as the Kostant *section*) is the affine space

$$\Sigma := f + \mathfrak{g}^e,$$

where $\mathfrak{g}^e$ is the centraliser of e in $\mathfrak{g}$. Kostant [93, §4] showed that $\Sigma \subseteq \mathfrak{g}^{\mathrm{reg}}$, that the composition $\Sigma \hookrightarrow \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}^{\mathrm{reg}}/G$ is a homeomorphism, and that the composition $\Sigma \hookrightarrow \mathfrak{g} \to \mathfrak{g} /\!\!/ G := \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$ is an isomorphism of complex affine varieties. Thus, we have a commutative diagram:

$$\begin{array}{ccccc} \Sigma & \hookrightarrow & \mathfrak{g}^{\mathrm{reg}} & \hookrightarrow & \mathfrak{g} \\ & \searrow^{\cong} & \downarrow & & \downarrow \\ & & \mathfrak{g}^{\mathrm{reg}}/G & \xrightarrow{\cong} & \mathfrak{g} /\!\!/ G \end{array} \tag{8.2.1}$$

The Kostant slice Σ can then be viewed as a global slice for the action of G on $\mathfrak{g}^{\mathrm{reg}}$.

There is a similar picture for $\mathfrak{Z}$, where the rôle of Σ is played by

$$\mathscr{Z}_\Sigma := \mathscr{Z} \cap (G \times \Sigma).$$

This set is a nonsingular affine variety in $T^*G = G \times \mathfrak{g}$, and the composition $\mathscr{Z}_\Sigma \hookrightarrow \mathscr{Z}^{\mathrm{reg}} \to \mathfrak{Z}$ is an isomorphism of complex-algebraic varieties [10, §2.2]. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{Z}_\Sigma & \hookrightarrow & \mathscr{Z}^{\mathrm{reg}} & \hookrightarrow & \mathscr{Z} \\ & \searrow^{\cong} & \downarrow & & \downarrow \\ & & \mathfrak{Z} & \longrightarrow & \mathscr{Z} /\!\!/ G \end{array} \tag{8.2.2}$$

However, unlike Kostant's diagram (8.2.1), the composition $\mathscr{Z}_\Sigma \hookrightarrow \mathscr{Z} \to \mathscr{Z} /\!\!/ G$ is *not* an isomorphism, so $\mathfrak{Z}$ cannot be identified with $\mathscr{Z} /\!\!/ G$. The map $\mathfrak{Z} = \mathscr{Z}^{\mathrm{reg}} /\!\!/ G \to \mathscr{Z} /\!\!/ G$ is a birational equivalence, but it is neither injective nor surjective; this will be shown in §8.5. A subtle point to keep in mind here is that the categorical quotient $\mathscr{Z}^{\mathrm{reg}} /\!\!/ G$ is not the same as the image of $\mathscr{Z}^{\mathrm{reg}}$ in $\mathscr{Z} /\!\!/ G$ since $\mathscr{Z}^{\mathrm{reg}}$ is not saturated in $\mathscr{Z}$.

There is also a natural complex-symplectic structure on $\mathfrak{Z}$ [10, §2.4]. Indeed, by identifying $G \times \mathfrak{g}$ with T^*G, an equivalent description of $\mathfrak{Z}$ is as the complex-symplectic reduction of the open subset $(T^*G)^{\mathrm{reg}} \subseteq T^*G$ of regular covectors by the diagonal subgroup of $G \times G$. We have that $\mathcal{Z}^{\mathrm{reg}}$ is the zero level set of the complex moment map, both $\mathcal{Z}^{\mathrm{reg}}$ and $\mathcal{Z}^{\mathrm{reg}} /\!\!/ G$ are smooth, and $\mathcal{Z}^{\mathrm{reg}} \to \mathcal{Z}^{\mathrm{reg}} /\!\!/ G$ is a surjective holomorphic submersion, so the complex-symplectic version of the Marsden–Weinstein Theorem applies. Moreover, this complex-symplectic structure is compatible with a hyperkähler structure; this will be explained in §8.2.1 below.

The universal centraliser $\mathfrak{Z}$ also comes with an interesting fibration. Let

$$\mathfrak{c} := \mathfrak{g} /\!\!/ G = \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$$

and let

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{c}$$

be the quotient map. Then, projection onto the Lie algebra factor gives a natural morphism

$$\varpi : \mathfrak{Z} \longrightarrow \mathfrak{c}, \quad [g, x] \longmapsto \pi(x).$$

The algebra $\mathbb{C}[\mathfrak{g}]^G$ is freely generated with $\operatorname{rk} \mathfrak{g}$ generators, so $\mathfrak{c} \cong \mathbb{C}^{\operatorname{rk} \mathfrak{g}}$ (but not canonically), and hence ϖ can be seen as a map

$$\mathfrak{Z} \longrightarrow \mathbb{C}^{\operatorname{rk} \mathfrak{g}}.$$

This map is a completely integrable system, in the sense that it is a surjective submersion, the $\operatorname{rk} \mathfrak{g}$ holomorphic functions $\mathfrak{Z} \to \mathbb{C}$ are Poisson commuting with respect to the natural complex-symplectic structure on $\mathfrak{Z}$, and all fibres are complex Lagrangian submanifolds [140, §5.1].

Using the isomorphisms $\mathfrak{Z} \cong \mathcal{Z}_\Sigma$ and $\mathfrak{c} \cong \Sigma$, it is easy to describe the fibres of ϖ. Indeed, ϖ is the projection $\mathcal{Z}_\Sigma \to \Sigma$ on the right factor, so, for each $x \in \Sigma$, the fibre $\varpi^{-1}(x)$ is the centraliser G^x of x in G. Moreover, every regular element in $\mathfrak{g}$ is conjugate to a unique element of Σ, so $\mathfrak{Z}$ parameterises the set of conjugacy classes of centralisers of regular elements; this justifies the name. Recall also that centralisers of regular elements are abelian (this can be inferred from Kostant [93, Proposition 14] using that G/Z_G is isomorphic to the adjoint group of $\mathfrak{g}$), so each fibre of ϖ is an abelian algebraic group of dimension $\operatorname{rk} \mathfrak{g}$. In particular, when $x \in \mathfrak{c}$ corresponds to a regular semisimple element in Σ (a generic condition), $\varpi^{-1}(x)$ is a maximal torus in G.

8.2.1 Hyperkähler structure

Let K be a maximal compact subgroup of G and $\mathfrak{k} := \mathrm{Lie}(K)$. Although $\mathfrak{Z}$ is a complex-symplectic quotient of an open subset of T^*G by a subgroup of $G \times G$, this quotient is not isomorphic to a hyperkähler quotient of T^*G by a subgroup of $K \times K$ via the Kempf–Ness Theorem. In fact, the hyperkähler quotient of T^*G by the diagonal subgroup of $K \times K$ is the space $\mathcal{Z}/\!\!/G$ above (see §8.5). The hyperkähler structure on $\mathfrak{Z}$ comes from a different moduli space of solutions to Nahm's equations, where we take solutions on the open interval $(0,1)$ rather than $[0,1]$ and impose some boundary conditions.

Recall that every homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ is the complexification of a homomorphism $\mathfrak{su}(2) \to \mathfrak{k}$. This means that if (e,f,h) is a regular $\mathfrak{sl}_2$-triple as above, then there exist $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{k}$ such that

$$e = -\sigma_2 + i\sigma_3, \quad f = \sigma_2 + i\sigma_3, \quad h = -2i\sigma_1.$$

Then, $\sigma_1, \sigma_2, \sigma_3$ are elements of $\mathfrak{k}$ such that

$$[\sigma_1, \sigma_2] = \sigma_3, \quad [\sigma_2, \sigma_3] = \sigma_1, \quad [\sigma_3, \sigma_1] = \sigma_2,$$

and $\sigma_2 + i\sigma_3 \in \mathfrak{g}^{\mathrm{reg}}$. Such a triple $(\sigma_1, \sigma_2, \sigma_3)$ is called a **principal $\mathfrak{su}(2)$-triple**. Let $\mathcal{A}_{\mathfrak{k}}$ be the set of $\mathfrak{k}$-valued solutions to Nahm's equations on $(0,1)$ that are analytic near 0 and 1 and have simple poles of residues $(0, \sigma_1, \sigma_2, \sigma_3)$ at 0 and 1. Let $\mathcal{K}_0$ be the group of smooth maps $k : [0,1] \to K$ which are analytic near 0 and 1 and satisfy $k(0) = k(1) = 1$. Then, $\mathcal{K}_0$ acts on $\mathcal{A}_{\mathfrak{k}}$ as in §5.2.1. The work of Bielawski [11] (specifically Corollary 4.1) shows that the quotient

$$\mathcal{N}_K := \mathcal{A}_{\mathfrak{k}}/\mathcal{K}_0$$

is a finite-dimensional hyperkähler manifold diffeomorphic to $\mathfrak{Z}$ (this is also explained in [22, Appendix A]). Just like in §5.2.1, the hyperkähler structure comes from viewing this as an infinite-dimensional version of the hyperkähler quotient construction, and it depends on a choice of K-invariant inner-product on $\mathfrak{k}$. To see the diffeomorphism $\mathcal{N}_K \cong \mathfrak{Z}$, we first identify $\mathcal{A}_{\mathfrak{k}}/\mathcal{K}_0$ with the complex-symplectic quotient $\mathcal{A}_{\mathfrak{g}}/\mathcal{G}_0$, where $\mathcal{A}_{\mathfrak{g}}$ is the set of $\mathfrak{g}$-valued solutions to the complex Nahm equation (i.e. $\dot{\beta} + [\alpha, \beta] = 0$) on $(0,1)$ with simple poles of residues $(i\sigma_1, \sigma_2 + i\sigma_3)$ at 0 and 1, and $\mathcal{G}_0$ is the group of smooth maps $[0,1] \to G$ with $g(0) = g(1) = 1$. Fix a smooth map

$$D : (0,1) \longrightarrow G$$

such that $D(t) = \exp(\frac{1}{2}\log(t)h)$ near $t = 0$, and $D(t) = \exp(\frac{1}{2}\log(1-t)h)$ near $t = 1$. Define

$$\mathcal{Z}_\Sigma \longrightarrow \mathcal{A}_\mathfrak{g}/\mathcal{G}_0, \quad (g, x) \longmapsto D \cdot \gamma_g \cdot (0, x), \tag{8.2.3}$$

where γ_g is any smooth map $[0, 1] \to G$ with $\gamma_g(0) = 1$ and $\gamma_g(1) = g$. Then, this map is a diffeomorphism (see the proof of Lemma 3.3 in Bielawski [11]).

Just like for the cotangent bundle T^*G, there is an isometric SU(2)-action on $\mathfrak{Z}$ which rotates the complex structures. To see this, let $\mathfrak{su}_\sigma$ be the subalgebra of $\mathfrak{k}$ spanned by $(\sigma_1, \sigma_2, \sigma_3)$ and let SU_σ be the subgroup of K with Lie algebra $\mathfrak{su}_\sigma$. By letting $\mathcal{P}$ be the space of smooth maps $(0, 1) \to \mathfrak{k}$, we may view $\mathcal{A}_\mathfrak{k}$ as a subset of $\mathcal{P} \times (\mathcal{P} \otimes \mathfrak{su}_\sigma)$. As such, there is an action of SU_σ on $\mathcal{N}_K = \mathcal{A}_\mathfrak{k}/\mathcal{K}_0$ given by

$$b \cdot (T_0, \sum T_i \otimes \sigma_i) = (\mathrm{Ad}_b\, T_0, \sum (\mathrm{Ad}_b\, T_i) \otimes (\mathrm{Ad}_b\, \sigma_i)).$$

This preserves the boundary conditions and Nahm's equations, and hence provides an isometric action of SU_σ which rotates the complex structures. The action of the U(1)-subgroup of SU_σ given by $e^{\mathbb{R}\sigma_1}$ complexifies to a $\mathbb{C}^*$-action which, on $\mathfrak{Z} = \mathcal{Z}^{\mathrm{reg}}/G$, is the natural $\mathbb{C}^*$-scaling $z \cdot [g, x] = [g, zx]$.

8.2.2 Examples

The cases $G = \mathrm{SL}(n, \mathbb{C})$ and $G = \mathrm{PGL}(n, \mathbb{C})$

The space $\mathfrak{Z}_{\mathrm{PGL}(n,\mathbb{C})}$ is biholomorphic to the moduli space of centred SU(2)-monopoles of charge n on $\mathbb{R}^3$ and $\mathfrak{Z}_{\mathrm{SL}(n,\mathbb{C})}$ to its universal (n-fold) cover. It has even been proved that the hyperkähler metric on $\mathfrak{Z}_{\mathrm{PGL}(n,\mathbb{C})}$ coming from Nahm's equations coincides with the one coming from the Bogomolny equations [114] [22, Appendix A]. Let us see these biholomorphisms more concretely in the case $n = 2$:

The case $G = \mathrm{SL}(2, \mathbb{C})$

In this case (see [10, §3.2]), the Kostant slice is

$$\Sigma = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} : \delta \in \mathbb{C} \right\},$$

and

$$\mathcal{Z}_\Sigma = \left\{ \left(\begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix}, \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} \right) : \xi, \eta, \delta \in \mathbb{C},\ \xi^2 - \delta\eta^2 = 1 \right\}.$$

Thus, $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ can be identified with the affine variety in $\mathbb{C}^3$ cut out by the equation $\xi^2 - \delta\eta^2 = 1$. This is indeed the familiar complex equation for the Atiyah–Hitchin manifold [2].

The case $G = \mathrm{PGL}(2,\mathbb{C})$

Since $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, we have $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} = \mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}/\mathbb{Z}_2$. Thus, from the computation of $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ above we see that

$$\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} = \mathrm{Spec}(\mathbb{C}[\xi,\eta,\delta]/(\xi^2 - \delta\eta^2 - 1))^{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ acts by $(\pm 1)\cdot(\xi,\delta,\eta) = (\pm\xi,\pm\eta,\delta)$. The invariant polynomials are generated by $u = \xi^2$, $v = \eta^2$, $w = \xi\eta$ and δ, and the relations by $uv = w^2$ and $u - \delta v = 1$. Thus, after eliminating u, we find that

$$\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} = \{(v,w,\delta) \in \mathbb{C}^3 : w^2 - \delta v^2 = v\}.$$

This expression can also be inferred from [22, Lemma 6.9(1)] using that $\mathfrak{Z} = \mathcal{M}_C(\check{G}, 0)$.

8.3 The moduli space of regular marked G-Higgs bundles over a point

We now formalise the analogy between the universal centraliser and Higgs bundles by showing that $\mathfrak{Z}$ is a fine moduli space of regular marked G-Higgs bundles over a point. The moduli problem will be defined analogously to Nitsure [121] in the case of semistable Higgs bundles over a curve. We refer to Newstead [118] for the basic theory of moduli spaces.

Let G be a complex semisimple group. Recall, as in the introduction, that a G-Higgs bundle over a compact Riemann surface X is a holomorphic principal G-bundle $P \to X$ together with a section Φ of $\mathrm{ad}(P) \otimes T^*X$. There are two natural ways of generalising this notion. One is to replace T^*X by a fixed line bundle L on X, so that a Higgs field now takes values in $\mathrm{ad}(P) \otimes L$. Another is to keep T^*X but let X be any complex manifold with the constraint that a Higgs field Φ must satisfy $[\Phi,\Phi] = 0$. Of course, the latter is sensible only if $\dim(X) > 0$ since otherwise, $T^*X = 0$ and hence all Higgs fields are zero. However, keeping the philosophy of the former generalisation, we may define Higgs bundles on a 0-dimensional manifold X by replacing T^*X by a (trivial) line bundle, i.e. a Higgs field on $P \to X$ is simply a section of $\mathrm{ad}(P)$. With obvious notions of morphisms of G-Higgs bundles in each of the above senses, we get categories G-$\mathbf{Higgs}_X$ for all G and X. We define a **marking** on a G-Higgs bundle (P,Φ) to be an automorphism of (P,Φ) in the corresponding category. There is a category G-$\mathbf{MHiggs}_X$ of marked G-Higgs bundles on X and a forgetful functor

$$G\text{-}\mathbf{MHiggs}_X \longrightarrow G\text{-}\mathbf{Higgs}_X.$$

In both these categories, there are natural subcategories of regular objects defined as follows. For any principal G-bundle P over a variety S, there is an open set $\mathrm{ad}(P)^{\mathrm{reg}}$ of regular elements in $\mathrm{ad}(P)$ defined by those points $[p,x] \in \mathrm{ad}(P) = P \times_G \mathfrak{g}$ such that $x \in \mathfrak{g}^{\mathrm{reg}}$; this is well-defined since $\mathfrak{g}^{\mathrm{reg}}$ is G-invariant. We say that $(P,\Phi) \in G$-$\mathbf{Higgs}_X$ is **regular** if Φ takes values in $\mathrm{ad}(P)^{\mathrm{reg}} \otimes K$ (where $K = L$ or T^*X) and let G-$\mathbf{Higgs}_X^{\mathrm{reg}}$ be the full subcategory of regular G-Higgs bundles over X. (This induces an abstract G-Higgs bundle over X in the sense of Donagi–Gaitsgory [38].) Similarly, there is a notion of the category G-$\mathbf{MHiggs}_X^{\mathrm{reg}}$ of regular marked G-Higgs bundles over X.

We now specialise to the case where X is just a point and denote by G-$\mathbf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}}$ and G-$\mathbf{MHiggs}_{\mathrm{pt}}^{\mathrm{reg}}$ the corresponding categories. Thus, an object of G-$\mathbf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}}$ is a pair (P,Φ), where P is a principal bundle over a point, and Φ is an element of $\mathrm{ad}(P) = P \times_G \mathfrak{g}^{\mathrm{reg}}$. A morphism from (P,Φ) to (Q,Ψ) is a homomorphism $f : P \to Q$ of G-bundles such that the natural vector bundle homomorphism $\mathrm{ad}(P) \to \mathrm{ad}(Q)$ pushes Φ to Ψ. A marking on a Higgs bundle (P,Φ) is an automorphism φ of (P,Φ), and a morphism from (P,Φ,φ) to (Q,Ψ,ψ) is a morphism f from (P,Φ) to (Q,Ψ) such that $f \circ \varphi = \psi \circ f$.

Although Higgs bundles over a point may appear rather trivial at first, we will see that the moduli space of regular marked Higgs bundles over a point exists and is a non-trivial hyperkähler variety: the universal centraliser $\mathfrak{Z}$.

As for any moduli problem, we need to define a notion of families of objects in G-$\mathbf{MHiggs}_{\mathrm{pt}}^{\mathrm{reg}}$ and G-$\mathbf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}}$ parametrised by a variety S. Following Nitsure [121], we say that a **family** of objects in G-$\mathbf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}}$ parametrised by S is a principal G-bundle P over S together with a section Φ of $\mathrm{ad}(P)^{\mathrm{reg}}$. For each point $x \in S$, we get an object of G-$\mathbf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}}$ by considering the fibre P_x and the element $\Phi(x) \in \mathrm{ad}(P)_x^{\mathrm{reg}} = \mathrm{ad}(P_x)^{\mathrm{reg}}$. As for Higgs bundles over a curve, we say that two families (P,Φ) and (Q,Ψ) over S are **equivalent** if $(P_x,\Phi(x)) \cong (Q_x,\Psi(x))$ in G-$\mathbf{Higgs}_{\mathrm{pt}}$ for all $x \in S$. If $f : S \to T$ is a morphism of algebraic varieties, and (Q,Ψ) is a family parametrised by T, then the pullback $(f^*Q, f^*\Psi)$ is a family parametrised by S. Moreover, by associating to each variety S the set of equivalence classes of families parametrised by S, we get a contravariant functor

$$G\text{-}\mathsf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}} : \mathbf{Var} \longrightarrow \mathbf{Set},$$

where **Var** is the category of complex-algebraic varieties and **Set** the category of sets. Recall that a **fine moduli space** is a variety that represents this functor, i.e. an object $M \in \mathbf{Var}$ and an isomorphism of functors G-$\mathsf{Higgs}_{\mathrm{pt}}^{\mathrm{reg}} \to \mathrm{Hom}(-,M)$. In the case of marked G-Higgs bundles, we say that a **family** of objects in G-$\mathbf{MHiggs}_{\mathrm{pt}}^{\mathrm{reg}}$

parametrised by S is a triple (P, Φ, φ), where P is principal G-bundle over S, Φ is a section of $\mathrm{ad}(P)^{\mathrm{reg}}$, and φ is an automorphism of (P, Φ). Again, two families are **equivalent** if they induce isomorphic marked G-Higgs bundles at each point $x \in S$. Then, we have a contravariant functor

$$G\text{-}\mathsf{MHiggs}^{\mathrm{reg}}_{\mathrm{pt}} : \mathbf{Var} \longrightarrow \mathbf{Set},$$

which we want to represent by a variety. The goal of this section is to show:

Theorem 8.3.1. *The variety $\mathfrak{c} = \mathfrak{g}/\!/G$ is a fine moduli space of regular G-Higgs bundles over a point, the variety $\mathfrak{Z}$ is a fine moduli space of regular marked G-Higgs bundles over a point, and the forgetful functor corresponds to the fibration $\varpi : \mathfrak{Z} \to \mathfrak{c}$.*

The rest of this section is devoted to the proof of Theorem 8.3.1. It is essentially an application of Kostant's results on Σ and $\mathfrak{c}$. We need to know that $\Sigma \to \mathfrak{g}^{\mathrm{reg}}/G$ is a bijection, that $\Sigma \to \mathfrak{c}$ is an isomorphism of affine varieties, and that for each $x \in \mathfrak{g}^{\mathrm{reg}}$ the centraliser G^x of x in G is abelian; the rest is unravelling the definitions.

Let us first recall the following basic facts.

Proposition 8.3.2. *Let $P \to S$ be a principal G-bundle. Then, the vector space of sections of* $\mathrm{ad}(P)$ *is isomorphic to the vector space of G-equivariant morphisms $P \to \mathfrak{g}$. Also, the group of automorphisms of P is isomorphic to the group of G-equivariant morphisms $P \to G$ (with the conjugation action on the right-hand side).* □

More specifically, a G-equivariant map $\Phi : P \to \mathfrak{g}$ induces a section of $\mathrm{ad}(P)$ by using that the map $P \to \mathrm{ad}(P) = P \times_G \mathfrak{g}$, $p \mapsto [p, \Phi(p)]$ is G-invariant and hence descends to a section $S \to \mathrm{ad}(P)$. Similarly, for each automorphism $\varphi : P \to P$, there is a unique G-equivariant morphism $\psi : P \to G$ such that $\varphi(p) = \psi(p) \cdot p$ for all $p \in P$.

Thus, a regular G-Higgs bundle over a point can be described as a pair (P, Φ), where P is a complex-algebraic variety with a free and transitive G-action and Φ is a G-equivariant map $P \to \mathfrak{g}^{\mathrm{reg}}$. Then, a marking on (P, Φ) is a G-equivariant map $\varphi : P \to G$ such that $\mathrm{Ad}_\varphi \Phi = \Phi$. Moreover, in this setting, a morphism between (P, Φ) and (Q, Ψ) is a G-equivariant morphism $f : P \to Q$ such that $\Psi \circ f = \Phi$, and a morphism between marked objects (P, Φ, φ) and (Q, Ψ, ψ) is a morphism $f : (P, \Phi) \to (Q, \Psi)$ such that $\psi \circ f = \varphi$. A family of G-Higgs bundles over a point parametrised by S is now a pair (P, Φ), where P is a principal G-bundle over S and Φ a G-equivariant map $P \to \mathfrak{g}$. A family of marked G-Higgs bundles adds the data of a G-equivariant map $\varphi : P \to G$ such that $\mathrm{Ad}_\varphi \Phi = \Phi$. We will use this new point of view for the rest of this section.

To prove the first part of the theorem, we show that there is a family $(U_{\mathfrak{c}}, \Omega_{\mathfrak{c}})$ of regular G-Higgs bundles over a point parametrised by $\mathfrak{c}$ that has the universal property, i.e. for every family (P, Φ) parametrised by a variety S, there is a unique morphism $f : S \to \mathfrak{c}$ such that (P, Φ) is equivalent to $(f^*U_{\mathfrak{c}}, f^*\Omega_{\mathfrak{c}})$. To define $(U_{\mathfrak{c}}, \Omega_{\mathfrak{c}})$, let

$$\sigma : \mathfrak{c} \longrightarrow \Sigma$$

be the inverse of Kostant's isomorphism $\Sigma \hookrightarrow \mathfrak{g} \to \mathfrak{c}$. Take the trivial G-bundle $U_{\mathfrak{c}} = \mathfrak{c} \times G$ over $\mathfrak{c}$ and the Higgs field $\Phi_{\mathfrak{c}} : U_{\mathfrak{c}} \to \mathfrak{g}^{\mathrm{reg}}$, $\Phi_{\mathfrak{c}}(x, g) = \mathrm{Ad}_g\, \sigma(x)$.

Lemma 8.3.3. *The family $(U_{\mathfrak{c}}, \Psi_{\mathfrak{c}})$ has the universal property, and hence $\mathfrak{c}$ is a fine moduli space of regular G-Higgs bundles over a point.*

Proof. Let (P, Φ) be a family of regular G-Higgs bundles over a point parametrised by S. Then, the composition of $\Phi : P \to \mathfrak{g}$ with $\pi : \mathfrak{g} \to \mathfrak{c}$ is G-invariant and hence descends to a morphism $f : S \to \mathfrak{c}$. We claim that (P, Φ) is equivalent to the pullback family $(f^*U_{\mathfrak{c}}, f^*\Omega_{\mathfrak{c}})$. This amounts to show that for all $x \in S$, the regular G-Higgs bundle (P_x, Φ_x) over $\{x\}$ is isomorphic to (G, Ψ_f), where $\Psi_f : G \to \mathfrak{g}^{\mathrm{reg}}$ is defined by $\Psi_f(g) = \mathrm{Ad}_g\, \sigma(f(x))$. Pick any $p \in P_x$. Then, $\Phi_x(p)$ and $\sigma(f(x))$ are both elements of $\mathfrak{g}^{\mathrm{reg}}$ and $\pi(\Phi_x(p)) = f(x) = \pi(\sigma(f(x)))$ so, by Kostant's isomorphism diagram (8.2.1), $\Phi_x(p)$ and $\sigma(f(x))$ are conjugate. Since $\Phi_x : P_x \to \mathfrak{g}$ is G-equivariant, this implies that there exists $p_x \in P_x$ such that $\Phi_x(p_x) = \sigma(f(x))$. Define $F : G \to P_x$ by $F(g) = g \cdot p_x$. Then, F is a G-equivariant morphism and $\Phi_x \circ F(g) = \Phi_x(g \cdot p_x) = \mathrm{Ad}_g\, \Phi_x(p_x) = \mathrm{Ad}_g\, \sigma(f(x)) = \Psi_f(g)$ so $\Phi_x \circ F = \Psi_f$. Thus, F is an isomorphism of G-Higgs bundles over a point and, since x is arbitrary, this implies that (P, Φ) is equivalent to $(f^*U_{\mathfrak{c}}, f^*\Omega_{\mathfrak{c}})$.

Now, we need to show that the morphism $f : S \to \mathfrak{c}$ is unique. Suppose that $h : S \to \mathfrak{c}$ is another morphism such that $(h^*U_{\mathfrak{c}}, h^*\Omega_{\mathfrak{c}})$ is equivalent to (P, Φ). Pick $x \in S$. Then, $(G, \Psi_h) \cong (P_x, \Phi_x)$ where as above $\Psi_h : G \to \mathfrak{g}^{\mathrm{reg}}$ is defined by $\Psi_h(g) = \mathrm{Ad}_g\, \sigma(h(x))$. Any such isomorphism must be of the form $H : G \to P_x$, $H(g) = g \cdot q_x$ for some $q_x \in P_x$. Then, $\Phi_x \circ H = \Psi_h$. Pick $g \in G$ such that $p_x = g \cdot q_x$ where p_x is as above. Then, $\mathrm{Ad}_g\, \sigma(h(x)) = \Psi_h(g) = \Phi_x(H(g)) = \Phi_x(p_x) = \sigma(f(x))$ and by applying π on both sides we get $h(x) = f(x)$. □

We now construct a universal family $(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ of regular marked G-Higgs bundles over a point parametrised by $\mathfrak{z}$. For this part, we identify $\mathfrak{z}$ with the affine variety $\mathcal{Z}_\Sigma$. Let $U_{\mathfrak{z}}$ be the trivial principal G-bundle over $\mathfrak{z}$, let $\Omega_{\mathfrak{z}} : U_{\mathfrak{z}} = \mathfrak{z} \times G \to \mathfrak{g}^{\mathrm{reg}}$ be defined by $\Omega_{\mathfrak{z}}((a, x), g) = \mathrm{Ad}_g\, x$, and let $\omega_{\mathfrak{z}} : U_{\mathfrak{z}} = \mathfrak{z} \times G \to G$ be the map $\omega_{\mathfrak{z}}((a, x), g) = gag^{-1}$.

Lemma 8.3.4. *The family $(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ has the universal property, and hence $\mathfrak{z}$ is a fine moduli space of regular marked G-Higgs bundles over a point.*

Proof. The proof is similar to the case of $\mathfrak{c}$. Let (P, Φ, φ) be a family of regular marked G-Higgs bundles over a point parametrised by S. Then, the map $\varphi \times \Phi : P \to \mathcal{Z}^{\mathrm{reg}}$ is G-equivariant and hence descends to a morphism $f : S \to \mathcal{Z}^{\mathrm{reg}}/G \to \mathcal{Z}_\Sigma = \mathfrak{z}$. We claim that $f^*(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ is equivalent to (P, Φ, φ). Write $f : S \to \mathcal{Z}_\Sigma$ as $f(x) = (\alpha(x), \beta(x))$ where $\alpha : S \to G$ and $\beta : S \to \Sigma$. Let $x \in S$. Then, we want to show that (P_x, Φ_x, φ_x) is isomorphic to (G, Ψ_f, ψ_f) where $\Psi_f : G \to \mathfrak{g}$, $\Psi_f(g) = \mathrm{Ad}_g \beta(x)$ and $\psi_f : G \to G$, $\psi_f(g) = g\alpha(x)g^{-1}$. By definition of f, there exists $p_x \in P_x$ such that $(\varphi(p_x), \Phi(p_x)) = f(x) \in \mathcal{Z}_\Sigma$. Let $F : G \to P_x$, $F(g) = g \cdot p_x$. Then, F is a G-equivariant morphism. Moreover, $\Phi_x \circ F(g) = \Phi_x(g \cdot p_x) = \mathrm{Ad}_g \Phi_x(p_x) = \mathrm{Ad}_g \beta(x) = \Psi_f(g)$ so $\Phi_x \circ F = \Psi_f$, and $\varphi_x \circ F(g) = \varphi_x(g \cdot p_x) = g\varphi_x(p_x)g^{-1} = g\alpha(x)g^{-1} = \psi_f(g)$ so $\varphi_x \circ F = \psi_f$. This means that F is an isomorphism and since x is arbitrary, the two families (P, Φ, φ) and $f^*(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ are equivalent.

We now need to show that the morphism $f : S \to \mathfrak{z}$ is unique. Suppose that there is another morphism $h : S \to \mathfrak{z}$ such that $h^*(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ is equivalent to (P, Φ, φ). Write $h(x) = (\mu(x), \nu(x))$ where $\mu : S \to G$ and $\nu : S \to \Sigma$. Take $x \in S$. Then, $(G, \Psi_h, \psi_h) \cong (P_x, \Phi_x, \varphi_x)$ where as above $\Psi_h : G \to \mathfrak{g}^{\mathrm{reg}}$, $\Psi_h(g) = \mathrm{Ad}_g \nu(x)$ and $\psi_h : G \to G$, $\psi_h(g) = g\mu(x)g^{-1}$. This isomorphism must be of the form $H : G \to P_x$ where $H(g) = g \cdot q_x$ for some $q_x \in P_x$, and $\Phi_x \circ H = \Psi_h$, $\varphi_x \circ H = \psi_h$. Take $g \in G$ such that $p_x = g \cdot q_x$, where p_x is as above. Then, $\mathrm{Ad}_g \nu(x) = \Psi_h(g) = \Phi_x(H(g)) = \Phi_x(g \cdot q_x) = \Phi_x(p_x) = \beta(x)$. Thus, $\nu(x)$ and $\beta(x)$ are conjugate elements of Σ and by Kostant's theorem this implies that $\nu(x) = \beta(x)$. Similarly, $g\mu(x)g^{-1} = \psi_h(g) = \varphi_x(H(g)) = \varphi_x(g \cdot q_x) = \varphi_x(p_x) = \alpha(x)$. But $\mathrm{Ad}_{\mu(x)} \nu(x) = \nu(x)$ and $\mathrm{Ad}_g \nu(x) = \beta(x) = \nu(x)$ and the centraliser $G^{\nu(x)}$ is abelian since $\nu(x)$ is regular, so this implies that $g\mu(x)g^{-1} = \mu(x)$ and hence $\mu(x) = \alpha(x)$. Thus, $h(x) = f(x)$. □

In particular, $\mathfrak{z}$ is in bijection with the set of isomorphism classes of regular marked G-Higgs bundle over a point, where a point $(a, x) \in \mathfrak{z}$ is mapped to the restriction of $(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})$ at (a, x). Similarly, $\mathfrak{c}$ is in bijection with the set of isomorphism classes of regular G-Higgs bundle over a point, where $x \in \mathfrak{c}$ is mapped to the restriction of $(U_{\mathfrak{c}}, \Omega_{\mathfrak{c}})$ at x. Moreover, the unmarked G-Higgs bundle underlying $(U_{\mathfrak{z}}, \Omega_{\mathfrak{z}}, \omega_{\mathfrak{z}})|_{(a,x)}$ is simply $(U_{\mathfrak{c}}, \Omega_{\mathfrak{c}})|_{\pi(x)}$, so the forgetful functor corresponds to the morphism $\varpi : \mathfrak{z} \to \mathfrak{c}$, $(a, x) \mapsto \pi(x)$. This concludes the proof of Theorem 8.3.1.

8.4 Universal centralisers and mirror symmetry

8.4.1 The dual fibration

Let G be a complex semisimple algebraic group, let $\check{G}$ be its Langlands dual group, and let $\mathfrak{Z}$ and $\check{\mathfrak{Z}}$ be the corresponding universal centralisers. Then, we have two fibrations $\varpi : \mathfrak{Z} \to \mathfrak{c}$ and $\check{\varpi} : \check{\mathfrak{Z}} \to \check{\mathfrak{c}}$, where $\mathfrak{c} = \mathfrak{g}/\!\!/G$ and $\check{\mathfrak{c}} = \check{\mathfrak{g}}/\!\!/\check{G}$. Let $r := \operatorname{rk}\mathfrak{g} = \operatorname{rk}\check{\mathfrak{g}}$. In this section, we explain how to define isomorphisms $\mathfrak{c} \cong \mathbb{C}^r \cong \check{\mathfrak{c}}$ and a Zariski-open set $\mathbb{C}^r_\circ \subseteq \mathbb{C}^r$ such that the induced fibrations

$$\begin{array}{ccc} \mathfrak{Z} & & \check{\mathfrak{Z}} \\ & {\scriptstyle\varpi}\searrow \quad \swarrow{\scriptstyle\check{\varpi}} & \\ & \mathbb{C}^r & \end{array}$$

have the property that for all $x \in \mathbb{C}^r_\circ$, the fibres $\varpi^{-1}(x)$ and $\check{\varpi}^{-1}(x)$ are dual complex-algebraic tori.

Let us first recall the construction of the Langlands dual group. Fix a maximal torus $T \subseteq G$ and let $\mathfrak{t} \subseteq \mathfrak{g}$ be the corresponding Cartan subalgebra. Then, the root system Φ of $\mathfrak{g}$ with respect to $\mathfrak{t}$ can be seen as a subset of the character lattice $X^*(T) := \operatorname{Hom}(T, \mathbb{C}^*)$ and the coroot system $\check{\Phi}$ as a subset of the cocharacter lattice $X_*(T) := \operatorname{Hom}(\mathbb{C}^*, T)$. The **root datum** of G is the quadruple $(\Phi, X^*(T), \check{\Phi}, X_*(T))$, and it completely determines G up to isomorphisms. Conversely, any abstract root datum (see, e.g., [136, §7.4]) comes from a complex semisimple group in that way. By swapping $X^*(T)$ with $X_*(T)$ and Φ with $\check{\Phi}$, we get a new root datum $(\check{\Phi}, X_*(T), \Phi, X^*(T))$, and the corresponding complex semisimple group is called the **Langlands dual group** of G and is denoted by $\check{G}$. For example, the Langlands dual group of $\mathrm{SL}(n, \mathbb{C})$ is $\mathrm{PGL}(n, \mathbb{C})$.

Let $\check{\mathfrak{g}}$ be the Lie algebra of $\check{G}$, let $\check{T} \subseteq \check{G}$ be a maximal torus, and let $\check{\mathfrak{t}} \subseteq \check{\mathfrak{g}}$ be its Lie algebra. Then, $\check{\mathfrak{t}}$ can be identified with the dual vector space $\mathfrak{t}^*$ since $X_*(\check{T}) = X^*(T)$ and there are natural identifications $\mathfrak{t}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ and $\check{\mathfrak{t}} = X_*(\check{T}) \otimes_{\mathbb{Z}} \mathbb{C}$. This duality also appears at the level of the groups T and $\check{T}$, in the following sense. Recall that a complex-algebraic torus S can be recovered from its cocharacter lattice $X_*(S)$. Indeed, $X_*(S) \otimes_{\mathbb{Z}} \mathbb{C} = \operatorname{Lie}(S)$ and there is a short exact sequence $0 \to X_*(S) \to \operatorname{Lie}(S) \overset{\exp}{\to} S \to 1$. This gives an equivalence of categories between complex-algebraic tori and lattices (i.e. finitely generated free abelian groups). Thus, every complex-algebraic torus S has a **dual torus**, which is simply the torus associated with the dual lattice $\operatorname{Hom}(X_*(S), \mathbb{Z}) \cong X^*(S)$ (the isomorphism $\operatorname{Hom}(X_*(S), \mathbb{Z}) \cong X^*(S)$ is canonical; it comes from the perfect pairing

$\mathrm{Hom}(\mathbb{C}^*, S) \times \mathrm{Hom}(S, \mathbb{C}^*) \to \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z})$. Now, we have $X_*(\check{T}) = X^*(T)$ and $X^*(\check{T}) = X_*(T)$, so T and $\check{T}$ are indeed dual complex-algebraic tori.

Let $W = N_G(T)/T$ and $\check{W} = N_{\check{G}}(\check{T})/\check{T}$ be the Weyl groups of G and $\check{G}$ respectively. Then, the bijection $\Phi \to \check{\Phi}$, $\alpha \mapsto \check{\alpha}$ induces a bijection from the set of root reflections in W to the set of root reflections in $\check{W}$ and extends to an isomorphism $W \to \check{W}$. Moreover, the isomorphism $\kappa : \mathfrak{t} \to \check{\mathfrak{t}}$ given by the Killing form κ of $\mathfrak{g}$ is equivariant with respect to $W \to \check{W}$, so it descends to an isomorphism $\mathfrak{t}/W \to \check{\mathfrak{t}}/\check{W}$. Hence, by the Chevalley restriction theorem, we have a canonical isomorphism

$$\varphi : \mathfrak{c} \longrightarrow \check{\mathfrak{c}}.$$

Let $\mathfrak{g}^{\mathrm{rs}}$ be the set of regular semisimple elements of $\mathfrak{g}$ and let $\mathfrak{c}_\circ \subseteq \mathfrak{c}$ be the image of $\mathfrak{g}^{\mathrm{rs}}$ under the GIT quotient $\mathfrak{g} \to \mathfrak{c}$. Since $\mathfrak{g}^{\mathrm{rs}}$ is Zariski-open in $\mathfrak{g}$ (see, e.g., [29, Lemma 3.1.5]) and G-saturated (semisimple orbits are closed), $\mathfrak{c}_\circ$ is also Zariski-open in $\mathfrak{c}$. Equivalently, under the Chevalley isomorphism $\mathfrak{c} \cong \mathfrak{t}/W$, the set $\mathfrak{c}_\circ$ corresponds to $\mathfrak{t}_\circ/W$, where $\mathfrak{t}_\circ \subseteq \mathfrak{t}$ is the complement of the root hyperplanes. Let $\check{\mathfrak{c}}_\circ \subseteq \check{\mathfrak{c}}$ be defined analogously. Since the isomorphism $\kappa : \mathfrak{t} \to \check{\mathfrak{t}}$ preserves the root hyperplanes, we have $\varphi(\mathfrak{c}_\circ) = \check{\mathfrak{c}}_\circ$. Moreover, for all $x \in \mathfrak{c}_\circ$, the fibre $\varpi^{-1}(x)$ is the centraliser of a regular semisimple element, and hence is a maximal torus in G. Similarly, $\check{\varpi}^{-1}(\varphi(x))$ is a maximal torus in $\check{G}$, and since all maximal tori are conjugate, the discussion above shows that $\varpi^{-1}(x)$ and $\check{\varpi}^{-1}(\varphi(x))$ are dual complex-algebraic tori. We summarise this discussion in the following theorem.

Theorem 8.4.1. *Let G be a complex semisimple group, let $\check{G}$ be its Langlands dual, and let $\varpi : \mathfrak{Z} \to \mathfrak{c}$ and $\check{\varpi} : \check{\mathfrak{Z}} \to \check{\mathfrak{c}}$ be the corresponding universal centralisers. Then, there are Zariski-open sets $\mathfrak{c}_\circ \subseteq \mathfrak{c}$ and $\check{\mathfrak{c}}_\circ \subseteq \check{\mathfrak{c}}$ and an isomorphism $\varphi : \mathfrak{c} \to \check{\mathfrak{c}}$ such that $\varphi(\mathfrak{c}_\circ) = \check{\mathfrak{c}}_\circ$ and for all $x \in \mathfrak{c}_\circ$ the fibres $\varpi^{-1}(x)$ and $\check{\varpi}^{-1}(\varphi(x))$ are dual complex-algebraic tori.* □

8.4.2 Branes

Recall, as stated in the introduction, that an A-brane in a Kähler manifold is a Lagrangian submanifold and a B-brane is a complex submanifold. Thus, if $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ is a hyperkähler manifold, a BAA-brane is a submanifold that is complex with respect to I and Lagrangian with respect to ω_{J} and ω_{K}. Equivalently, a BAA-brane is a complex-Lagrangian submanifold with respect to $(\mathsf{I}, \omega_{\mathsf{J}} + i\omega_{\mathsf{K}})$. Similarly, a BBB-brane is a hyper-complex submanifold. An important tool for constructing such branes is by taking fixed point sets of special involutions:

Proposition 8.4.2. *Let* $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ *be a hyperkähler manifold and* $f : M \to M$ *a smooth involution* (*i.e.* $f^2 = \mathrm{Id}$) *that is isometric with respect to* g*. Let* M^f *be the set of fixed points of* f*. If* f *is holomorphic with respect to* I*, and anti-holomorphic with respect to* J *and* K*, then* M^f *is a* BAA*-brane. If* f *is holomorphic with respect to* I*,* J*, and* K*, then* M^f *is a* BBB*-brane.* □

The proof is not difficult and well-known; see, e.g., [6] [48, §2.2] [49, §2.3]. The goal of this section is to construct such involutions on the universal centraliser $\mathfrak{Z}$ of a complex semisimple group G, and hence provide interesting branes inside $\mathfrak{Z}$. Moreover, since $\mathfrak{Z}$ has an SU(2)-hyperkähler rotation, this also gives ABA-branes and AAB-branes. The process which we use to construct those branes is analogous to the one used by Baraglia–Schaposnik [7] to construct branes in the moduli space $\mathcal{M}_G$ of G-Higgs bundles over a Riemann surface using real forms.

Let $\mathrm{Out}(G)$ be the group of outer automorphisms of G, i.e.

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Int}(G),$$

where $\mathrm{Aut}(G)$ is the group of automorphisms of G (as a complex-algebraic group) and $\mathrm{Int}(G)$ is the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms of the form $a \mapsto gag^{-1}$ for some $g \in G$. Let $\mathrm{Out}_2(G) \subseteq \mathrm{Out}(G)$ be the subgroup of elements of order 2. Then, $\mathrm{Out}_2(G)$ is a subgroup of the group of involutions of the Dynkin diagram of $\mathfrak{g}$, and is equal to it when G is simply connected [90, Theorem 7.8].

The group $\mathrm{Out}_2(G)$ has a close relationship with real forms of G (i.e. anti-holomorphic involutions). Indeed, for any real form $\sigma : G \to G$, there exists a compact real form ρ (one whose set of fixed points is a maximal compact subgroup) which commutes with σ (see, e.g., [74]). Then, $\theta := \sigma\rho = \rho\sigma$ is an involution of G and hence descends to an element of $\mathrm{Out}_2(G)$. Moreover, compact real forms are unique up to conjugation, so this gives a well-defined map from the set of real forms to $\mathrm{Out}_2(G)$. This map is surjective, and two real forms σ_1 and σ_2 map to the same element if and only if they are inner-equivalent, i.e. $\sigma_1\sigma_2^{-1} \in \mathrm{Int}(G)$. Hence, $\mathrm{Out}_2(G)$ parametrises the set of inner-equivalent classes of real forms (see, e.g., [51, §2]).

Recall from §8.2.1 that $\mathfrak{Z}$ is diffeomorphic to the moduli space $\mathcal{N}_K = \mathcal{A}_{\mathfrak{k}}/\mathcal{K}_0$ of solutions to Nahm's equations on $(0, 1)$ with simple poles of residues $(0, \sigma_1, \sigma_2, \sigma_3)$ at 0 and 1, where $(\sigma_1, \sigma_2, \sigma_3)$ is a principal $\mathfrak{su}(2)$-triple such that

$$h = -2i\sigma_1, \quad e = -\sigma_2 + i\sigma_3, \quad f = \sigma_2 + i\sigma_3.$$

Moreover, there is a natural hyperkähler metric on $\mathcal{N}_K$ depending on a choice of K-invariant inner-product on $\mathfrak{k}$. Here we consider the one induced by the Killing form.

Theorem 8.4.3. *Let $\theta \in \mathrm{Out}_2(G)$ and define involutions $\Theta_\pm$ on $\mathfrak{Z} = \mathscr{Z}^{\mathrm{reg}}/G$ by*

$$\Theta_\pm : \mathfrak{Z} \longrightarrow \mathfrak{Z}, \quad [a,x] \longmapsto [\theta(a), \pm\theta(x)].$$

Then, the set of fixed points of Θ_+ and Θ_- are BBB *and* BAA *branes respectively.*

Note that $\Theta_\pm$ are well-defined, since if $g \in G$ then $[g\theta(a)g^{-1}, \pm \mathrm{Ad}_g \theta(x)] = [\theta(a), \pm\theta(x)]$ for all $(a,x) \in \mathscr{Z}^{\mathrm{reg}}$.

The rest of this section is devoted to the proof of Theorem 8.4.3. We show that there is a corresponding involution on $\mathcal{N}_K$ which satisfies the conditions of Proposition 8.4.2. To define this involution, we first need to find special representatives of the element $\theta \in \mathrm{Out}_2(G)$.

Proposition 8.4.4. *Any element of $\mathrm{Out}_2(G)$ has two representatives $\tau_\pm \in \mathrm{Aut}(G)$ such that $\tau_\pm^2 = 1$, $\tau_\pm(K) = K$, and $\tau_\pm(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \pm\sigma_2, \pm\sigma_3)$.*

We begin with two lemmas.

Lemma 8.4.5. *If $(\sigma_1, \sigma_2, \sigma_3)$ and $(\sigma_1', \sigma_2', \sigma_3')$ are two principal $\mathfrak{su}(2)$-triples in $\mathfrak{k}$, then there exists $k \in K$ such that $\mathrm{Ad}_k(\sigma_1', \sigma_2', \sigma_3') = (\sigma_1, \sigma_2, \sigma_3)$.*

Proof. Recall that the map

$$\mathrm{Hom}(\mathfrak{su}(2), \mathfrak{k})/K \longrightarrow \mathrm{Hom}(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{g})/G$$

is bijective (see, e.g., [97, Appendix]). Then, the result follows from the fact that all principal $\mathfrak{sl}(2,\mathbb{C})$-triples in $\mathfrak{g}$ are conjugate (Kostant [92, Corollary 3.7 and Corollary 5.5]). □

Lemma 8.4.6. *Let (e,f,h) be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ and let $g \in G$ be such that $(\mathrm{Ad}_g e, \mathrm{Ad}_g f, \mathrm{Ad}_g h) = (e,f,h)$. Then, g is in the centre Z_G of G.*

Proof. By Kostant [92, Theorem 5.3] and the fact that all principal $\mathfrak{sl}_2$-triples are conjugate, there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $h \in \mathfrak{t}$ and $e = \sum_{\alpha \in \Pi} e_\alpha$ for some system of simple roots Π and non-zero root vectors $e_\alpha \in \mathfrak{g}_\alpha$ with respect to $\mathfrak{t}$. Now, h is regular semisimple, so its centraliser G^h is the maximal torus T with Lie algebra $\mathfrak{t}$. Since $g \in G^h = T$, we have

$$\mathrm{Ad}_g e = \sum_{\alpha \in \Pi} \alpha(g) e_\alpha,$$

where $\alpha(g) \in \mathbb{C}^*$ is the evaluation of the root α viewed as a character $T \to \mathbb{C}^*$. But $\mathrm{Ad}_g e = e$, so $\alpha(g) = 1$ for all $\alpha \in \Pi$, and hence also $\alpha(g) = 1$ for all roots α. Hence, Ad_g acts trivially on $\mathfrak{g}$, so $g \in \ker(\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})) = Z_G$. □

Proof of Proposition 8.4.4. A representative of an element of $\mathrm{Out}_2(G)$ is an element $\theta \in \mathrm{Aut}(G)$ such that $\theta^2 \in \mathrm{Int}(G)$. We want to show that there exists $h \in G$ such that $\theta_h := h\theta h^{-1}$ preserves K, $\theta_h^2 = 1$ and $\theta_h(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \pm\sigma_2, \pm\sigma_3)$. First, note that $\theta(K)$ is a maximal compact subgroup of G. Since all maximal compact subgroups are conjugate, there exists g such that $\theta_g(K) = K$. Now, $(\theta_g(\sigma_1), \pm\theta_g(\sigma_2), \pm\theta_g(\sigma_3))$ are two principal $\mathfrak{su}(2)$-triples in $\mathfrak{k}$, so by Lemma 8.4.5, there exist $k_\pm \in K$ such that $\theta_{k_\pm \cdot g}(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \pm\sigma_2, \pm\sigma_3)$. Note that $\tau := \theta_{k_\pm \cdot g}$ still preserves K and $\tau^2 = \mathrm{Ad}_h$ for some $z \in G$. We claim that $z \in Z_G$ and hence $\tau^2 = 1$. Indeed, $\tau^2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \sigma_2, \sigma_3)$ so $\tau^2(e, f, h) = (e, f, h)$ and by Lemma 8.4.6 this implies $z \in Z_G$. □

Let $\tau_\pm \in \mathrm{Aut}(G)$ be involutions of G as in Proposition 8.4.4. Then, $\tau_\pm$ restrict to involutions of K, and hence we can define involutions $\mathcal{T}_\pm$ on $\mathcal{A}_\mathfrak{k}$ (the space of solutions to Nahm's equations on $(0, 1)$ with simple poles of residues $(0, \sigma_1, \sigma_2, \sigma_3)$ at 0 and 1) via

$$\mathcal{T}_\pm(T_0, T_1, T_2, T_3) = (\tau_\pm(T_0), \tau_\pm(T_1), \pm\tau_\pm(T_2), \pm\tau_\pm(T_3)).$$

The pole condition is preserved since $\tau_\pm(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \pm\sigma_2, \pm\sigma_3)$. Moreover, if $k \in \mathcal{K}_0$, then

$$\mathcal{T}_\pm(k \cdot (T_0, T_1, T_2, T_3)) = \tau_\pm(k) \cdot \mathcal{T}_\pm(T_0, T_1, T_2, T_3),$$

so $\mathcal{T}_\pm$ descend to involutions

$$\mathcal{T}_\pm : \mathcal{N}_K \longrightarrow \mathcal{N}_K.$$

Lemma 8.4.7. *The sets of fixed points of $\mathcal{T}_\pm$ are, respectively,* BBB *and* BAA *branes.*

Proof. Since every automorphism of $\mathfrak{k}$ preserves the Killing form, $\mathcal{T}_\pm$ are Riemannian isometries. The action of the complex structure I on a tangent vector $X \in T_T\mathcal{A}_\mathfrak{k}$ is

$$\mathsf{I} \cdot (X_0, X_1, X_2, X_3) = (-X_1, X_0, -X_3, X_2),$$

so $\mathcal{T}_\pm$ are I-holomorphic. Now, the action of J on X is

$$\mathsf{J} \cdot (X_0, X_1, X_2, X_3) = (-X_2, X_3, X_0, -X_1),$$

so $\mathcal{T}_+$ is J-holomorphic and $\mathcal{T}_-$ is J-anti-holomorphic. Similarly, $\mathcal{T}_-$ is K-holomorphic and $\mathcal{T}_+$ is K-anti-holomorphic. By Proposition 8.4.2, this proves the lemma. □

It remains to show:

Lemma 8.4.8. *Under the diffeomorphism $\mathcal{N}_K \to \mathfrak{Z}$, the involutions $\mathcal{T}_\pm$ correspond to the involutions $\Theta_\pm$.*

Proof. Let $\mathcal{T}'_\pm$ be the involutions on $\mathcal{A}_\mathfrak{g}/\mathcal{G}_0$ (see §8.2.1) defined by $\mathcal{T}'_\pm(\alpha,\beta) = (\tau_\pm(\alpha), \pm\tau_\pm(\beta))$ (this is well-defined since $\tau_\pm(e,f,h) = (\pm e, \pm f, h)$ and hence $\mathcal{T}'_\pm$ preserve the pole condition). They correspond to the involutions $\mathcal{T}_\pm$ under the diffeomorphism $\mathcal{N}_K = \mathcal{A}_k/\mathcal{K}_0 \to \mathcal{A}_\mathfrak{g}/\mathcal{G}_0$, so it suffices to show that $\Theta_\pm$ correspond to $\mathcal{T}'_\pm$ under the diffeomorphism $\mathcal{Z}_\Sigma \to \mathcal{A}_\mathfrak{g}/\mathcal{G}_0$ given by (8.2.3). First, since $\tau_\pm(e) = \pm e$ and $\tau_\pm(f) = \pm f$, we get that both $\pm\tau$ preserve the Kostant slice $\Sigma = f + \mathfrak{g}^e$. Hence, the involutions $\Theta_\pm$ as maps on $\mathcal{Z}_\Sigma \cong \mathfrak{Z}$ are given by

$$\Theta_\pm : \mathcal{Z}_\Sigma \longrightarrow \mathcal{Z}_\Sigma, \quad \Theta(g,x) = (\tau_\pm(g), \pm\tau_\pm(x)).$$

Now, let $(g,x) \in \mathcal{Z}_\Sigma$. Then, (g,x) is mapped to $D \cdot \gamma_g \cdot (0,x) \in \mathcal{A}_\mathfrak{g}/\mathcal{G}_0$ and $(\tau_\pm(g), \pm\tau_\pm(x))$ is mapped to $D\cdot\tau_\pm(\gamma_g)\cdot(0,\pm\tau_\pm(x))$. Since $\tau_\pm(h) = h$ we have $\tau_\pm(D) = D$ and hence $D\cdot\tau_\pm(\gamma_g)\cdot(0,\pm\tau_\pm(x)) = \tau_\pm(D\cdot\gamma_g)\cdot(0,\pm\tau_\pm(x)) = \mathcal{T}'_\pm(D\cdot\gamma_g\cdot(0,x))$. □

This concludes the proof of Theorem 8.4.3. To produce ABA-branes and AAB-branes, we can use hyperkähler rotation. Equivalently, we can find an automorphism $\tau : \mathfrak{k} \to \mathfrak{k}$ such that $\tau(\sigma_1,\sigma_2,\sigma_3) = (-\sigma_1,\sigma_2,-\sigma_3)$, and then the set of fixed points of the involution

$$\mathcal{N}_K \longrightarrow \mathcal{N}_K, \quad (T_0,T_1,T_2,T_3) \longmapsto (\tau(T_0), -\tau(T_1), \tau(T_2), -\tau(T_3)).$$

is an ABA-brane. We can also construct AAB-branes by the same method.

8.4.3 E-polynomials and partial compactification

Unlike the other sections, this one illustrates an unpleasant *difference* between universal centralisers and moduli spaces of Higgs bundles over a Riemann surface. We show that the E-polynomials of $\mathfrak{Z}$ and $\check{\mathfrak{Z}}$ do not behave as we would expect for mirror pairs: they are not equal. A proof that they are not equal *for all G* seems difficult, so we shall content ourself with the more modest result that *there exists G* such that they differ:

Proposition 8.4.9. *The E-polynomial of $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ is $u^2v^2 + uv$ while that of $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}$ is u^2v^2.*

Since $\mathfrak{Z}$ is a smooth complex-algebraic variety, the stringy mixed Hodge numbers considered by Hausel–Thaddeus [65] and are the same as Deligne's compactly supported mixed Hodge numbers [36, 37], so we will only discuss the latter. In fact, for this section, the only thing we need to know about the E-polynomials is the following result (see [100, Proposition 2.3 and Proposition 2.4] or [112, Theorem 2.2 and Lemma 2.3]).

Lemma 8.4.10.

(1) $\mathrm{E}(\mathbb{C}^n) = u^n v^n$ *for all* $n \geq 0$.

(2) *If* $Z = \bigsqcup_{i=1}^n Z_i$ *is a decomposition of a complex-algebraic variety* Z *into locally closed subvarieties* Z_i, *then* $\mathrm{E}(Z) = \sum_{i=1}^n \mathrm{E}(Z_i)$.

(3) *If* X *and* Y *are two complex-algebraic varieties, then* $\mathrm{E}(X \times Y) = \mathrm{E}(X)\mathrm{E}(Y)$.

□

This is a very useful tool for computing E-polynomials; for example, from the decomposition $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$, we get that $\mathrm{E}(\mathbb{CP}^n) = 1 + uv + \cdots + u^n v^n$.

Proof of Proposition 8.4.9. Recall from §8.2.2 that

$$\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})} = \{(\xi, \eta, \delta) \in \mathbb{C}^3 : \xi^2 - \delta\eta^2 = 1\}.$$

Let Z_1 be the subset of $(\xi, \eta, \delta) \in \mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ such that $\eta = 0$ and Z_2 its complement. Clearly, $Z_1 \cong \mathbb{C} \times \mathbb{Z}_2$, so

$$\mathrm{E}(Z_1) = \mathrm{E}(\mathbb{C})\mathrm{E}(\mathbb{Z}_2) = 2uv.$$

Also, if $\eta \neq 0$ then $\delta = (\xi^2 - 1)/\eta^2$, so $Z_2 \cong \mathbb{C} \times \mathbb{C}^*$. Note that $uv = \mathrm{E}(\mathbb{C}) = \mathrm{E}(\mathbb{C}^* \sqcup \{0\}) = \mathrm{E}(\mathbb{C}^*) + 1$, so $\mathrm{E}(\mathbb{C}^*) = uv - 1$ and hence

$$\mathrm{E}(Z_2) = \mathrm{E}(\mathbb{C})\mathrm{E}(\mathbb{C}^*) = uv(uv - 1).$$

Putting these together,

$$\mathrm{E}(\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}) = \mathrm{E}(Z_1) + \mathrm{E}(Z_2) = 2uv + uv(uv - 1) = u^2v^2 + uv.$$

Now, for

$$\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} = \{(v, w, \delta) \in \mathbb{C}^3 : w^2 - \delta v^2 = v\},$$

we have

$$W_1 \coloneqq \{(v, w, \delta) \in \mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} : v = 0\} \cong \mathbb{C}$$

and

$$W_2 \coloneqq \{(v, w, \delta) \in \mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} : v \neq 0\} \cong \mathbb{C} \times \mathbb{C}^*,$$

so

$$\mathrm{E}(\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}) = \mathrm{E}(W_1) + \mathrm{E}(W_2) = uv + uv(uv - 1) = u^2v^2.$$

□

The failure of the E-polynomials to be equal might be traced to the fact that, contrary to the Hitchin system, the fibration $\mathfrak{Z} \to \mathfrak{c}$ is not a projective morphism: the fibres are not compact. So it is natural to ask if we can compactify the fibres in such a way as to recover topological mirror symmetry. There is a natural choice of partial compactification which has recently been studied by Balibanu [5]. The idea is to use the wonderful compactification $\overline{G}$ of G, which is a natural projective variety with a $G \times G$-action and containing G as an open dense orbit. Indeed, $\mathfrak{Z} = \mathcal{Z}_\Sigma$ sits naturally as a subvariety of $\overline{G} \times \Sigma$, so we can define $\overline{\mathfrak{Z}}$ to be the closure of $\mathcal{Z}_\Sigma$ is $\overline{G} \times \Sigma$. Then, the fibration $\mathfrak{Z} \to \mathfrak{c}$ extends naturally to a morphism $\overline{\mathfrak{Z}} \to \mathfrak{c}$ whose fibres are closed subvarieties of $\overline{G}$, hence compact. Then Balibanu [4, 5] showed (among other things) that when G is of adjoint type, $\overline{\mathfrak{Z}}$ is smooth [5] and that the fibre of $\overline{\mathfrak{Z}} \to \mathfrak{c}$ at zero (the analogue of the "global nilpotent cone" in Hitchin systems) is isomorphic to the Peterson variety [4]. However, as we now show, even with this partial compactification the E-polynomials do not agree:

Proposition 8.4.11. *The E-polynomial of $\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ is $u^2v^2 + 2uv$ while that of $\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}$ is $u^2v^2 + uv$.*

Here $\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ and $\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}$ are both smooth (see below), so the E-polynomials can still be interpreted in the sense of Deligne. Let us first recall the wonderful compactifications of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PGL}(2,\mathbb{C})$ (see, e.g., [124, p. 50]). For $\mathrm{SL}(2,\mathbb{C})$, it is

$$\overline{\mathrm{SL}(2,\mathbb{C})} = \left\{[a:b:c:d:t] \in \mathbb{CP}^4 : ad - bc = t^2\right\}$$

with embedding

$$\mathrm{SL}(2,\mathbb{C}) \longhookrightarrow \overline{\mathrm{SL}(2,\mathbb{C})}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [a:b:c:d:1].$$

For $\mathrm{PGL}(2,\mathbb{C})$, it is simply $\mathbb{CP}^3$ with embedding

$$\mathrm{PGL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 \longhookrightarrow \mathbb{CP}^3, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [a:b:c:d].$$

Lemma 8.4.12. *The partial compactification of $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ is*

$$\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})} = \{([\xi:\eta:t],\delta) \in \mathbb{CP}^2 \times \mathbb{C} : \xi^2 - \delta\eta^2 = t^2\}$$

and $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ embeds as the subset of $([\xi:\eta:t],\delta) \in \overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ such that $t \neq 0$.

Proof. We have that

$$\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})} = \{(\xi,\eta,\delta) \in \mathbb{C}^3 : \xi^2 - \delta\eta^2 = 1\}$$

embeds in $\overline{\mathrm{SL}(2,\mathbb{C})} \times \Sigma \subseteq \mathbb{CP}^4 \times \mathbb{C}$ via

$$\iota : (\xi, \eta, \delta) \longmapsto ([\xi : \delta\eta : \eta : \xi : 1], \delta).$$

The image of ι is equal to

$$\{([a : b : c : d : t], \delta) \in \mathbb{CP}^4 \times \mathbb{C} : ad - bc = t^2, a = d, b = \delta c, t \neq 0\}$$

and its closure is

$$\begin{aligned}&\{([a : b : c : d : t], \delta) \in \mathbb{CP}^4 \times \mathbb{C} : ad - bc = t^2, a = d, b = \delta c\}\\ &= \{([a : c : t], \delta) \in \mathbb{CP}^2 \times \mathbb{C} : a^2 - \delta c^2 = t^2\}\end{aligned}$$

□

Corollary 8.4.13. *The E-polynomial of $\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ is $u^2v^2 + 2uv$.*

Proof. The complement of $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}$ in $\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ is $\{([\xi : \eta], \delta) \in \mathbb{CP}^1 \times \mathbb{C} : \xi^2 - \delta\eta^2 = 0\}$. Note that η cannot vanish, as otherwise ξ would vanish as well. Thus,

$$\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})} - \mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})} = \{(\xi, \delta) \in \mathbb{C} \times \mathbb{C} : \xi^2 = \delta\} = \mathbb{C}.$$

From Proposition 8.4.9, we get that

$$\mathrm{E}(\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}) = \mathrm{E}(\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}) + \mathrm{E}(\mathbb{C}) = (u^2v^2 + uv) + uv = u^2v^2 + 2uv.$$

□

Now, for $\mathrm{PGL}(2, \mathbb{C})$, we have:

Lemma 8.4.14. *The partial compactification of $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}$ is $\mathbb{CP}^1 \times \mathbb{C}$ and $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}$ embeds as the subset of $([\xi : \eta], \delta) \in \mathbb{CP}^1 \times \mathbb{C}$ such that $\xi^2 - \delta\eta^2 \neq 0$.*

Proof. We have that $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})} = \mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}/\mathbb{Z}_2$ embeds in $\overline{\mathrm{PGL}(2,\mathbb{C})} \times \Sigma = \mathbb{CP}^3 \times \mathbb{C}$ via

$$\iota : (\xi, \eta, \delta) \longmapsto ([\xi : \delta\eta : \eta : \xi], \delta).$$

The image of ι is

$$\{([a : b : c : d], \delta) \in \mathbb{CP}^3 \times \mathbb{C} : a = d, \delta c = b, ad - bc \neq 0\}$$

and hence its closure is

$$\{([a : b : c : d], \delta) \in \mathbb{CP}^3 \times \mathbb{C} : a = d, \delta c = b\} = \mathbb{CP}^1 \times \mathbb{C}.$$

The embedding of $\mathfrak{Z}_{\mathrm{SL}(2,\mathbb{C})}/\mathbb{Z}_2$ in $\mathbb{CP}^1 \times \mathbb{C}$ is thus given by

$$(\xi, \eta, \delta) \longmapsto ([\xi, \eta], \delta)$$

and the image is easily seen to be the set of $([\xi, \eta], \delta)$ such that $\xi^2 - \delta\eta^2 \neq 0$. □

Corollary 8.4.15. *The E-polynomial of* $\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}$ *is* $u^2v^2 + uv$.

Proof. The complement of $\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}$ in $\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}$ is

$$\{([\xi : \eta], \delta) \in \mathbb{CP}^1 \times \mathbb{C} : \xi^2 - \delta\eta^2 = 0\}$$

which is isomorphic to $\mathbb{C}$ as in the proof of Corollary 8.4.13. Thus,

$$\mathrm{E}(\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}) = \mathrm{E}(\mathfrak{Z}_{\mathrm{PGL}(2,\mathbb{C})}) + \mathrm{E}(\mathbb{C}) = u^2v^2 + uv. \qquad \square$$

Corollary 8.4.13 and Corollary 8.4.15 together prove Proposition 8.4.11, and hence the E-polynomials of $\overline{\mathfrak{Z}}_{\mathrm{SL}(2,\mathbb{C})}$ and $\overline{\mathfrak{Z}}_{\mathrm{PGL}(2,\mathbb{C})}$ do not match.

8.5 The relationship between $(T^*G)^{\mathrm{reg}} /\!\!/_{\mu_{\mathbb{C}}} \Delta G$ and $T^*G /\!\!/\!\!/_{\mu} \Delta K$

Let ΔK be the diagonal subgroup of $K \times K$. Then, ΔK acts non-freely on T^*G, and hence the hyperkähler quotient $T^*G /\!\!/\!\!/_{\mu} \Delta K$ is a stratified hyperkähler space and a complex affine variety by Theorem 5.1.2. This is closely related to the description of $\mathfrak{Z}$ as the complex-symplectic reduction of $(T^*G)^{\mathrm{reg}}$ by ΔG, and it is natural to ask if $\mathfrak{Z}$ is the smooth locus of $T^*G /\!\!/\!\!/_{\mu} \Delta K$ or is at least contained in it. More precisely, by Theorem 5.1.2, we have $T^*G /\!\!/\!\!/_{\mu} \Delta K = \mathcal{Z} /\!\!/ G$ (see Proposition 8.5.1 below) so there is a natural map $\mathfrak{Z} \to T^*G /\!\!/\!\!/_{\mu} \Delta K$ (which is the bottom map in the commutative diagram (8.2.2)), and we would like to know if it realises $\mathfrak{Z}$ as a subset of the smooth locus of $T^*G /\!\!/\!\!/_{\mu} \Delta K$. We show that this is not the case. The map $\mathfrak{Z} \to T^*G /\!\!/\!\!/_{\mu} \Delta K$ is neither injective nor surjective, and its image is neither contained in the smooth locus nor contains it. On the other hand, it is a birational equivalence; it restricts to an isomorphism from the open dense subset $\mathfrak{Z}^{\mathrm{rs}} := \{[a, x] \in \mathfrak{Z} : x \in \mathfrak{g}^{\mathrm{rs}}\}$ to a proper subset of the smooth locus of $T^*G /\!\!/\!\!/_{\mu} \Delta K$. In particular, $\mathfrak{Z}^{\mathrm{rs}}$ has two *a priori* different hyperkähler structures.

The situation is best illustrated with an example: take $G = \mathrm{SL}(2, \mathbb{C})$. Then, $\mathfrak{Z}$ and $T^*G /\!\!/\!\!/_{\mu} \Delta K$ are the hypersurfaces in $\mathbb{C}^3$ given respectively by

$$\xi^2 - \delta\eta^2 = 1 \quad \text{and} \quad \xi^2 - \delta\eta^2 = \delta.$$

Note that they are part of the dihedral series of varieties $\xi^2 - \delta\eta^2 = \delta^{n-1}$ with $n = 1$ and $n = 2$. We then see that $\mathfrak{Z}$ is smooth (as expected) while $T^*G /\!\!/\!\!/_{\mu} \Delta K$ has two isolated singularities at $(\xi, \eta, \delta) = (0, \pm i, 0)$. The canonical morphism $\mathfrak{Z} \to T^*G /\!\!/\!\!/_{\mu} \Delta K$ is given by

$$(\xi, \eta, \delta) \longmapsto (-i\delta\eta, i\xi, \delta).$$

(Interestingly, this transformation of $\mathbb{C}^3$ maps the D_n-variety to the D_{n+1}-variety for all $n \geq 1$.) One can see directly from this example that $\mathfrak{Z} \to T^*G /\!/\!/_\mu K$ is not an embedding nor is surjective, but restricts to an isomorphism on the subset of points with $\delta \neq 0$. The fibres at the singular points $(0, \pm i, 0)$ are copies of $\mathbb{C}$. Also, if $\eta \in \mathbb{C} - \{\pm i\}$ then $(0, \eta, 0) \in T^*G /\!/\!/_\mu \Delta K$ but is not in the image $\mathfrak{Z}$.

We now tackle the general case. Let us first explain how our Kempf–Ness type theorem for T^*G in Chapter 5 identifies $T^*G /\!/\!/_\mu \Delta K$ with $\mathcal{Z} /\!/ G$:

Proposition 8.5.1. *Let $\mu : T^*G \to \mathfrak{k}^* \otimes \mathbb{R}^3$ be the hyperkähler moment map for the action of ΔK on T^*G. Then, $\mu_{\mathbb{C}}^{-1}(0) = \mathcal{Z}$ and the inclusion $\mu^{-1}(0) \subseteq \mathcal{Z}$ descends to an isomorphism $T^*G /\!/\!/_\mu \Delta K \to \mathcal{Z} /\!/ G$ of complex-analytic and partitioned spaces.*

Proof. By identifying T^*G with $G \times \mathfrak{g}$, the complex moment map is

$$\mu_{\mathbb{C}} : G \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \mu_{\mathbb{C}}(g, x) = \mathrm{Ad}_g\, x - x$$

(see (3.3.2)). Thus, $\mu_{\mathbb{C}}^{-1}(0) = \mathcal{Z}$ and using Theorem 5.1.2 with $H = \Delta K$ and $\xi = 0$ finishes the proof. □

Therefore, $T^*G /\!/\!/_\mu \Delta K$ is a complex affine variety, and there is a natural morphism

$$\psi : \mathfrak{Z} \longrightarrow T^*G /\!/\!/_\mu \Delta K,$$

namely, the one descending from the composition $\mathcal{Z}^{\mathrm{reg}} \hookrightarrow \mathcal{Z} \to \mathcal{Z} /\!/ G$ (the bottom map of (8.2.2)). Moreover, $T^*G /\!/\!/_\mu \Delta K = \mathcal{Z}^{\mathrm{ps}}/G$ and $\mathfrak{Z} = \mathcal{Z}^{\mathrm{reg}}/G$. Another closely related space is the open dense subset $\mathcal{Z}^{\mathrm{rs}}/G \subseteq \mathfrak{Z}$, where

$$\mathcal{Z}^{\mathrm{rs}} = \mathcal{Z} \cap (G \times \mathfrak{g}^{\mathrm{rs}}).$$

The central result which we need to clarify the relationship between $\mathcal{Z}^{\mathrm{ps}}/G$ and $\mathcal{Z}^{\mathrm{reg}}/G$ is the following.

Proposition 8.5.2. *$\mathcal{Z}^{\mathrm{reg}} \cap \mathcal{Z}^{\mathrm{ps}} = \mathcal{Z}^{\mathrm{rs}}$.*

We prove this by the following three lemmas.

Lemma 8.5.3. *We have $\mathcal{Z}^{\mathrm{rs}} \subseteq \mathcal{Z}^{\mathrm{ps}}$.*

Proof. Let $(g, x) \in \mathcal{Z}$ be such that $x \in \mathfrak{g}^{\mathrm{rs}}$. To show that $G \cdot (g, x)$ is closed in $\mathcal{Z}$, we argue as in Kostant [93, Lemma 5 (2)]. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing x and let T be the maximal torus of G with Lie algebra $\mathfrak{t}$. By the Iwasawa decomposition (see, e.g., [25, Theorem 26.3]), we have $G = KAN$, where K is a maximal compact subgroup of G, A is a subgroup of T, and N is a unipotent complex affine group. We claim that A fixes (g,x). Let $a \in A$. Then, $a \in T$ and $x \in \mathfrak{t}$, so $\operatorname{Ad}_a x = x$, and hence $a \in G^x$. Since $g \in G^x$ and G^x is abelian (because $x \in \mathfrak{g}^{\mathrm{rs}}$), we also have $aga^{-1} = g$, and hence $g \cdot (a,x) = (a,x)$. Moreover, A normalises N, so we get that $G \cdot (g,x) = KN \cdot (g,x)$. Now, orbits of unipotent groups on quasi-affine varieties are closed (see, e.g., [20, Proposition 4.10]) and K is compact, so $G \cdot (g,x) = K \cdot (N \cdot (g,x))$ is closed. □

Lemma 8.5.4. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and x a regular nilpotent element of $\mathfrak{g}$. Then, there exists $h \in \mathfrak{g}$ such that $[h,x] = 2x$ and*

$$\lim_{t\to\infty} e^{-t\operatorname{ad}_h} y = 0, \quad \textit{for all } y \in \mathfrak{g}^x. \tag{8.5.1}$$

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

be the corresponding Cartan decomposition (where $\Phi \subseteq \mathfrak{h}^*$). Let Φ^+ be a system of positive roots and $\Pi \subseteq \Phi^+$ the corresponding set of simple roots. For each $\alpha \in \Pi$, choose an $\mathfrak{sl}(2,\mathbb{C})$-triple $(e_\alpha, f_\alpha, h_\alpha)$ with $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$. Choose $h \in \mathfrak{h}$ such that $\alpha(h) = 2$ for all $\alpha \in \Pi$. Since $\{h_\alpha : \alpha \in \Pi\}$ spans $\mathfrak{h}$, there exists $a_\alpha \in \mathbb{C}$ such that $h = \sum_{\alpha\in\Pi} a_\alpha h_\alpha$. Let

$$e := \sum_{\alpha\in\Pi} e_\alpha \quad \text{and} \quad f := \sum_{\alpha\in\Pi} a_\alpha f_\alpha.$$

Then, (e,f,h) is a principal $\mathfrak{sl}_2$-triple and (see Kostant [92, §5] or Collingwood–McGovern [31, proof of Theorem 4.1.6] or Chriss–Ginzburg [29, Proof of Theorem 3.7.13]). Since all regular nilpotent elements are conjugate, we may assume that $x = e$. Hence, $[h,x] = 2x$ and we claim that h satisfies (8.5.1). Let $y \in \mathfrak{g}^x$. We have $\mathfrak{g}^e \subseteq \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ by [92, Theorem 6.7] (it is also easy to compute directly), so $y = \sum_{\alpha\in\Phi^+} y_\alpha$, for some $y_\alpha \in \mathfrak{g}_\alpha$. Hence,

$$e^{-t\operatorname{ad}_h} y = \sum_{\alpha\in\Phi^+} e^{-t\alpha(h)} y_\alpha.$$

Since $\alpha(h) = 2$ for all $\alpha \in \Pi$, we have $\alpha(h) > 0$ for all $\alpha \in \Phi^+$, so the right-hand side goes to 0 as $t \to \infty$. □

Lemma 8.5.5. *Let $(g,x) \in \mathcal{Z}^{\mathrm{reg}}$ and let $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ be the Jordan decomposition of x. Then, there exists $a \in G$ such that $G \cdot (a, x_{\mathrm{s}})$ is the unique closed orbit in $\overline{G \cdot (g,x)}$. More precisely, we can write $g = ze^{y'+y''}$, where $z \in Z_G$, $y' \in Z_{\mathfrak{g}^{x_{\mathrm{s}}}}$ and $y'' \in [\mathfrak{g}^{x_{\mathrm{s}}}, \mathfrak{g}^{x_{\mathrm{s}}}] \cap \mathfrak{g}^{x_{\mathrm{n}}}$, and then $a = ze^{y'}$.*

Proof. Recall that $\mathfrak{g}^{x_s}$ is a reductive Lie algebra. Thus, it decomposes as $\mathfrak{g}^{x_s} = Z_{\mathfrak{g}^{x_s}} \oplus [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]$, where $Z_{\mathfrak{g}^{x_s}}$ is the centre of $\mathfrak{g}^{x_s}$ and $[\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]$ is semisimple. Let $\tilde{\mathfrak{g}} := [\mathfrak{g}^{x_s}, \mathfrak{g}^{x_s}]$. Since x is regular, x_n is a regular nilpotent element of $\tilde{\mathfrak{g}}$ (see Kostant [93, Proposition 13 and Remark 10]). By Lemma 8.5.4, there exists $h \in \tilde{\mathfrak{g}}$ such that $[h, x_n] = 2x_n$, and

$$\lim_{t\to\infty} \mathrm{Ad}_{e^{-th}}\, y = 0, \quad \text{for all } y \in \tilde{\mathfrak{g}}^{x_n}.$$

Since G^x/Z_G is connected [93, Proposition 14], we may write $g = zg_0$ for $z \in Z_G$ and g_0 in the identity component of G^x. Since G^x is abelian and $\mathrm{Lie}(G^x) = \mathfrak{g}^x$, there exists $y \in \mathfrak{g}^x$ such that $g_0 = e^y$. We have $\mathfrak{g}^x = \mathfrak{g}^{x_s} \cap \mathfrak{g}^{x_n}$ [93, 3.4.9], so $\mathfrak{g}^x = Z_{\mathfrak{g}^{x_s}} \oplus \tilde{\mathfrak{g}}^{x_n}$. Thus, $y = y' + y''$ for some $y' \in Z_{\mathfrak{g}^{x_s}}$ and $y'' \in \tilde{\mathfrak{g}}^{x_n}$. Now, since $h \in \mathfrak{g}^{x_s}$ we have $[h, y'] = 0$, so $\mathrm{Ad}_{e^{-ih}}\, y' = y'$ and hence

$$e^{-th} g e^{th} = z e^{-th} g_0 e^{th} = z e^{\mathrm{Ad}_{e^{-th}}\, y} \to z e^{y'}$$

as $t \to \infty$. Also, $[h, x_s] = 0$ and $[h, x_n] = 2x_n$ so

$$\mathrm{Ad}_{e^{-th}}\, x = x_s + e^{-2t} x_n \to x_s$$

as $t \to \infty$. Therefore, if $a := ze^{y'}$, then $(a, x_s) \in \overline{G \cdot (g, x)}$. Moreover, $(a, x_s) \in \mathcal{Z}^{\mathrm{ps}}$ by Lemma 8.5.3. □

Proof of Proposition 8.5.2. It only remains to show that $\mathcal{Z}^{\mathrm{reg}} \cap \mathcal{Z}^{\mathrm{ps}} \subseteq \mathcal{Z}^{\mathrm{rs}}$. Let $(g, x) \in \mathcal{Z}^{\mathrm{reg}} \cap \mathcal{Z}^{\mathrm{ps}}$ and let $a \in G$ be as in Lemma 8.5.5. Since $G \cdot (g, x)$ is closed in $\mathcal{Z}$, we have $G \cdot (g, x) = G \cdot (a, x_s)$. In particular, x is conjugate to x_s and hence is semisimple. □

Before explaining how the identity $\mathcal{Z}^{\mathrm{reg}} \cap \mathcal{Z}^{\mathrm{ps}} = \mathcal{Z}^{\mathrm{rs}}$ clarifies the relationship between $\mathfrak{Z}$ and $T^*G /\!\!/\!\!/_{\mu}\, \Delta K$, we first observe the following interesting consequence.

Corollary 8.5.6. *We have $(T^*G)^{\mathrm{reg}} /\!\!/\!\!/_{\mu} \Delta K = (T^*G)^{\mathrm{rs}} /\!\!/\!\!/_{\mu} \Delta K$ as subsets of $T^*G /\!\!/\!\!/_{\mu} \Delta K$, and they are strictly contained in the smooth locus.*

Proof. By Proposition 8.5.1, we have $\mu^{-1}(0) \subseteq \mathcal{Z}^{\mathrm{ps}}$, so $\mu^{-1}(0) \cap (T^*G)^{\mathrm{reg}} \subseteq \mathcal{Z}^{\mathrm{ps}} \cap \mathcal{Z}^{\mathrm{reg}} = \mathcal{Z}^{\mathrm{rs}}$ and hence $\mu^{-1}(0) \cap (T^*G)^{\mathrm{reg}} = \mu^{-1}(0) \cap (T^*G)^{\mathrm{rs}}$. This proves the first statement. For the second statement, use that the inclusion $\mu^{-1}(0) \subseteq \mathcal{Z}^{\mathrm{ps}}$ descends to an isomorphism $T^*G /\!\!/\!\!/_{\mu}\, \Delta K \to \mathcal{Z}^{\mathrm{ps}}/G$ of partitioned spaces. Since $\mathcal{Z}^{\mathrm{rs}} \subseteq \mathcal{Z}^{\mathrm{ps}}$, the embedding of $(T^*G)^{\mathrm{rs}} /\!\!/\!\!/_{\mu}\, \Delta K$ in $T^*G /\!\!/\!\!/_{\mu}\, \Delta K$ corresponds to the embedding of $\mathcal{Z}^{\mathrm{rs}}/G$ in $\mathcal{Z}^{\mathrm{ps}}/G$. Now, for all $(g, x) \in \mathcal{Z}^{\mathrm{rs}}$, its stabiliser subgroup is G^x which is a maximal torus, so they are all conjugate, and hence $\mathcal{Z}^{\mathrm{rs}}/G$ is in the smooth locus. On the other hand, not every point of the smooth locus is in $\mathcal{Z}^{\mathrm{rs}}/G$; for example, if $t \in G$ is regular semisimple, then $(t, 0) \in \mathcal{Z}^{\mathrm{ps}} - \mathcal{Z}^{\mathrm{rs}}$ also has stabiliser conjugate to a maximal torus. □

Lemma 8.5.5 has another consequence:

Proposition 8.5.7. *The fibre of* $\psi : \mathfrak{Z} \to T^*G /\!\!/\!\!/_\mu \Delta K$ *at* 0 *is isomorphic to the connected component of* G^f *where* $f \in \mathfrak{g}$ *is any regular nilpotent element.*

Proof. The map ψ is the composition of $\mathcal{Z}_\Sigma \hookrightarrow \mathcal{Z}$ with the quotient map $\pi : \mathcal{Z} \to \mathcal{Z} /\!\!/ G$. Thus, $\psi^{-1}(0)$ is the set $\pi^{-1}(\pi(1,0)) \cap \mathcal{Z}_\Sigma$. The identity component G_0^f of G^f embeds in $\mathcal{Z}_\Sigma$ via $g \mapsto (g, f)$, and we want to identify its image with $\pi^{-1}(\pi(1,0)) \cap \mathcal{Z}_\Sigma$. Let $g \in G_0^f$. Since G^f is abelian, we may write $g = e^y$ for some $y \in \mathfrak{g}^f$. By Lemma 8.5.5, the unique closed orbit in $\overline{G \cdot (g, f)}$ is $G \cdot (1,0) = \{(1,0)\}$ so $\pi(g,f) = \pi(1,0)$ and hence $G_0^f \subseteq \pi^{-1}(\pi(1,0)) \cap \mathcal{Z}_\Sigma$. Conversely, let $(g,x) \in \mathcal{Z}_\Sigma$ be such that $\pi(g,x) = \pi(1,0)$. Since $x \in \Sigma$, it is regular, so we may apply Lemma 8.5.5 again, which says that $x_{\mathrm{s}} = 0$ so x is nilpotent. But the only nilpotent element of Σ is f, so $x = f$ and hence $g \in G^f$. We must have $g \in G_0^f$ as otherwise $\pi(g,x) = \pi(z,0) \neq \pi(1,0)$ for some $z \in Z_G$. □

The main conclusion is now:

Proposition 8.5.8. *The natural morphism* $\psi : \mathfrak{Z} \to T^*G /\!\!/\!\!/_\mu \Delta K$ *is a birational equivalence; it restricts to an isomorphism from* $\mathfrak{Z}^{\mathrm{rs}}$ *to* $(T^*G)^{\mathrm{rs}} /\!\!/\!\!/_\mu \Delta K$*. However, it is neither injective nor surjective, and the image of* $\mathfrak{Z}$ *in* $T^*G /\!\!/\!\!/_\mu \Delta K$ *is neither contained in the smooth locus nor contains it.*

Proof. Under the isomorphism $T^*G /\!\!/\!\!/_\mu \Delta K \to \mathcal{Z} /\!\!/ G$, the set $(T^*G)^{\mathrm{rs}} /\!\!/\!\!/_\mu \Delta K$ corresponds to the image $\mathcal{Z}^{\mathrm{rs}} /\!\!/ G$ of $\mathcal{Z}^{\mathrm{rs}}$ in $\mathcal{Z} /\!\!/ G$. Since $\mathfrak{Z}^{\mathrm{rs}}$ and $\mathcal{Z}^{\mathrm{rs}} /\!\!/ G$ are smooth, it suffices to show that the restriction $\mathfrak{Z}^{\mathrm{rs}} = \mathcal{Z}^{\mathrm{rs}}/G \to \mathcal{Z}^{\mathrm{rs}} /\!\!/ G$ is bijective. This follows from the inclusion $\mathcal{Z}^{\mathrm{rs}} \subseteq \mathcal{Z}^{\mathrm{ps}}$, since then $(\mathcal{Z}^{\mathrm{rs}})^{\mathrm{ps}} = \mathcal{Z}^{\mathrm{rs}}$ and hence the map $\mathcal{Z}^{\mathrm{rs}}/G \to \mathcal{Z}^{\mathrm{rs}} /\!\!/ G$ is bijective by the general theory. By Corollary 8.5.7, ψ is not injective and $\psi(\mathfrak{Z})$ is not contained in the smooth locus. To conclude the proof, it suffices to find a point in the smooth locus of $\mathcal{Z} /\!\!/ G$ which is not in the image of ψ. Take a regular semisimple element $t \in G$ and consider $(t, 0) \in \mathcal{Z}^{\mathrm{ps}}$. Its stabiliser is a maximal torus, so $(t, 0)$ maps to the smooth locus of $\mathcal{Z} /\!\!/ G$. But if $\pi(t, 0) \in \psi(\mathfrak{Z})$, then $t \in Z_G$ by Lemma 8.5.5, which contradicts that t is regular. □

To summarise more informally, we have $T^*G /\!\!/\!\!/_\mu \Delta K = \mathcal{Z}^{\mathrm{ps}}/G$ and $\mathfrak{Z} = \mathcal{Z}^{\mathrm{reg}}/G$, and both contain $\mathcal{Z}^{\mathrm{rs}}/G$ as an open dense subset, but they are different ways of completing $\mathcal{Z}^{\mathrm{rs}}/G$ to an affine variety.

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