

# Water Entry and Related Problems



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## Abstract

In this thesis we formalize, reconcile and generalize some existing mathematical models for deep- and shallow-water entry at small and zero deadrise angles for normal and oblique impact velocities. Our method throughout is to exploit the existence of one or more small parameters via the method of matched asymptotic expansions.

In the first chapter we describe some motivating solid-fluid impact events, summarize the literature on which this thesis builds and discuss our main modelling assumptions. In Chapter 2, the small-deadrise-angle deep-water entry model is systematically derived, a new expression for the first-order force on the body is found and the slender body limit of an exact three-dimensional solution is used to discuss the validity of strip theory. In Chapter 3, the small-deadrise-angle shallow-water entry model is systematically derived. The deep water Wagner and shallow water Korobkin theories are reconciled. New models and analytic results for the three-dimensional case are presented. In Chapter 4, models for the impact of a flat-bottomed body on deep- and shallow-water are reviewed and conjectures are made concerning their reconciliation. In Chapter 5, the theories of Chapters 2, 3 and 4 are used as building blocks to formulate and analyse some conjectured models for oblique shallow water impacts at small and zero deadrise angles. In the final chapter the main results are summarized and directions for further work indicated.

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# Chapter 1

## Introduction

### 1.1 Motivation

This thesis is concerned with the dynamics of high-velocity solid-fluid impacts, which occur in many settings throughout the physical sciences. The initial stage of impact has been the subject of much research over the past seventy years since the independent pioneering work of von Kármán [33] and Wagner [71] on the hydrodynamics of an alighting sea plane. The most well-studied scenario is the impact of the forebody of a ship on the sea surface, which can cause localized and eventually catastrophic damage to the hull [31, 49, 75]. This phenomenon is called ‘ship slamming’ in the naval architecture literature and typically takes place in deep water.

There are also many examples in which the effect of finite depth is crucial. For example, landslides into offshore lakes are catastrophic if they generate a tsunami [32, 37]. Another example of shallow water impact is the initial motion of a surf-skimmer or ‘skimboard’ [65]. This is a rigid circular disc about one metre in diameter that is able to carry its rider for quite long distances (5-10m) on very shallow water (1-2cm).

In all the above scenarios it is essential, for design or hazard management reasons, to predict the pressure distribution and the force on the impacting solid body and the induced fluid flow.

The paradigm water entry problem is the study of the normal impact of a rigid body on an idealized fluid in which surface tension, gravity, viscosity, compressibility and air cushioning effects are neglected. An excellent review is given by Korobkin & Pukhnachov [41]. Early theoreticians focused on the self-similar wedge entry problem [17, 43, 71]. In the last thirty years, numerical solutions and experiments (see Greenhow [25] for a review) have confirmed Wagner’s hypothesis [71] that in the limit of small *deadrise angle* (meaning that angle between the tangent to the profile and the initially flat free surface is everywhere small), the bulk fluid motion is approximately as if it was being loaded by an “expanding flat plate”, the free surface turning over in small high-pressure regions on the body that eject thin ‘jets’ up the sides of the impacting wedge. In the late 1980’s this key observation was formalized using the method of matched asymptotic expansions [8, 9, 31]. In the last ten years effort have been directed toward reducing the expense of numerical schemes (see, for example, [11, 60]) and recently Korobkin [40] used an inverse method to find the exact solution to the linearized outer expanding flat plate solution for several fully three-dimensional body profiles, including an elliptic paraboloid entering with constant normal velocity.

The effect of finite depth on the small deadrise angle model has received relatively little attention. The seminal theoretical paper is by Korobkin [37] who used the method of matched asymptotic expansions to find the lowest order flow for penetrations of the same order as the liquid depth. This results in a completely different asymptotic decomposition than for infinite depth Wagner theory.

The corresponding theories when the impactor has zero deadrise angle over a segment and finite deadrise angle elsewhere, for example a flat-bottomed wedge, have received a little more theoretical and some experimental interest. They are typified by the work of Yakimov [79] on the deep water case and by Korobkin [39] on the shallow water case. The latter agrees well with the experiments of Bukreev & Gusev [4].

## 1.2 Aim and structure of thesis

The aim of this thesis is to formalize, reconcile and generalize some existing mathematical models for deep and shallow water entry at small and zero deadrise angles. Our method throughout is to exploit the existence of one or more small parameters via the method of matched asymptotic expansions. The small parameters we employ are the deadrise and incidence angles and the inverse aspect ratios of the penetration and water depth to the lengthscale of the impacting body.

In the remainder of this chapter we give a short account of the main modelling assumptions and a statement of originality.

In Chapter 2 we present an account of the small-deadrise-angle deep-water entry model. We show that a small modification of previous asymptotic analyses [8, 9, 31] is required to perform the matching systematically and therefore formally derive the Wagner condition [71]. We find that we need a new intermediate region and show that this results in the first-order correction to the force on the body. Simple extensions are discussed. Finally, we find the slender body limit of the explicit solution for the entry of an elliptic paraboloid due to Korobkin [40] and discuss the implications of the new ‘leading-edge’ eigensolution for strip theory.

In Chapter 3 we analyse the small-deadrise-angle shallow-water entry model. The impact is shown to occur over four distinct temporal stages. We find the spatial decomposition of each stage and perform the temporal matching between neighbouring stages, which allows a quantitative description of the jet formation mechanism. We find that we can reconcile the infinite depth Wagner theory [71], valid at sufficiently small times when the effect of the base is negligible, with the finite depth Korobkin theory [37], valid at order unity times when the penetration depth is comparable to the layer depth. Three-dimensional extensions to the model are discussed for which the lowest order impact problem and its small time limit are derived. We will see that the latter describes the change in the geometry of the turnover curve as we pass from the very-small-time Wagner regime to the order-unity-time Korobkin regime. Finally, we present some analytic results for these two novel codimension-one free boundary problems and discuss the slender body limit.

In Chapter 4, we analyse the impact of a flat-bottomed body on water of infinite and finite depth. We use the method of matched asymptotic expansions to verify that, at small times, the impact of a flat-bottomed wedge is characterized by the similarity solutions to a local canonical problem at each corner. We show that, in contrast to the impacts considered in Chapters 2 and 3, the leading-order force on the body is non-zero initially and receives contributions from both the outer and inner regions. An argument based

entirely on the location of the contact point between the body and free surface is used to explore the existence of different solutions depending on the size of the deadrise angle. Korobkin’s theory for the order-unity-time shallow-water entry of a flat-bottomed wedge of large deadrise angle is reviewed [39]. Preliminary ideas for the unification of these small and large time theories are presented. Finally, the generalization of Korobkin’s large-deadrise-angle theory to the three-dimensional case is reviewed and simple extensions are discussed.

In Chapter 5, we use the theories of Chapters 2, 3 and 4 as building blocks to formulate some conjectured models for impacts with a tangential or forward velocity component. We begin with a brief review and discussion of how infinite depth models for oblique water entry at small deadrise angles are related to “planing” models at small angles of incidence. We investigate the effect of finite depth on these models when the forward velocity has a comparable effect to the normal body velocity in each of the four temporal stages of impact used in Chapter 3. The small time asymptotics of these models are briefly discussed. The generalization to three dimensions for both the oblique impact and planing models is discussed. The proposed three-dimensional shallow water planing model is derived and its steady states for fixed penetration depth are conjectured to be the three-dimensional generalization of Tuck’s two-dimensional surf-skimmer model [65]. We then present some new results for the two-dimensional oblique shallow water entry of a flat plate at zero deadrise angle.

Finally, in Chapter 6 we summarize the main results and suggest some directions for further work.

### 1.3 Modelling assumptions

In this thesis we neglect surface tension, gravity, viscosity, compressibility and air cushion effects. A large percentage of studies of high-velocity fluid-solid impact make the intuitive assumption that the liquid inertia dominates these forces over the typical duration of the impact [41]. In this section we briefly investigate the conditions under which this core modelling assumption is valid.

The dimensionless parameters that determine the importance of surface tension, gravity, viscosity and compressibility effects relative to the liquid inertia are the Weber, Froude, Reynolds and Mach numbers. They are given by

$$\text{We} = \frac{\rho_w L U^2}{\gamma}, \quad \text{Fr} = \frac{U}{\sqrt{gL}}, \quad \text{Re} = \frac{\rho_w R L}{\mu}, \quad \text{Ma} = \frac{U}{c_w}, \quad (1.1)$$

where  $L$  and  $U$  are the typical length and velocity scales of the induced fluid motion,  $\rho_w \approx 10^{-3} \text{kgm}^{-3}$  is the density of water,  $\mu \approx 10^{-3} \text{Kgm}^{-1}\text{s}^{-1}$  is the viscosity of water,  $g \approx 9.8 \text{ms}^{-2}$  is the acceleration due to gravity,  $\gamma \approx 7.5 \times 10^{-4} \text{Nm}^{-1}$  is the surface tension of the water-air interface and  $c_w \approx 1.4 \times 10^3 \text{ms}^{-2}$  is the local sound speed in water.

When the impact occurs on deep water, the size of  $L$  and  $U$  may be estimated from the typical size  $L_i$  and velocity  $U_i$  of the impactor. For a ship of width  $L_i \approx 10 \text{m}$  slamming onto the ocean with relative speed  $U_i \approx 30 \text{ms}^{-1}$ , we find

$$\text{We} \approx 10^{10}, \quad \text{Fr}^2 \approx 10, \quad \text{Re} \approx 10^8, \quad \text{Ma} \approx 10^{-2}. \quad (1.2)$$

When the impact is on water of characteristic depth  $H$ , we take  $L \approx H$ . Assuming the flow is incompressible, which must be verified *a posteriori*, conservation of mass implies

$U \approx HU_i/L_i$ . For a surf-skimmer of typical size  $L_i \approx 1\text{m}$ , impacting on shallow water of typical depth  $H \approx 2 \times 10^{-2}\text{m}$ , with speed  $U_i \approx 1\text{ms}^{-1}$ , we find

$$\text{We} \approx 10^8, \text{Fr}^2 \approx 10^2, \text{Re} \approx 10^6, \text{Ma} \approx 10^{-1}. \quad (1.3)$$

We see that typically  $\text{Re} \gg 1$ ,  $\text{Fr}^2 \gg 1$ ,  $\text{We} \gg 1$  and  $\text{Ma} \ll 1$ . This suggests that, although these scenarios occur on different length and time scales, they all have the common property that over the typical duration of the impact, the impact velocity is sufficiently large that it is realistic to neglect viscosity, gravity and surface tension, but sufficiently small that it is reasonable to assume the flow is incompressible. Korobkin & Pukhnachov [41] and Wilson [75] provide more convincing evidence for impacts at small deadrise angles by considering the affect on the fine flow structure predicted by Wagner theory.

Finally, we consider the affect of the air. The air pressure is  $\rho_a U_a^2$ , where  $\rho_a \approx 1\text{kgm}^{-3}$  is the density of air and  $U_a$  is the air speed. Since,  $\rho_a/\rho_w \approx 10^{-3}$  and  $U_a \approx U_i$  when the air layer is sufficiently thick, the air pressure is typically much smaller than the water pressure,  $\rho_w U^2$ ; however, just before impact the air speed becomes large because the air layer becomes thin, so air cushioning is always ultimately important in practice in any impact. To the author's knowledge there is no satisfactory model for air cushioning. Hence we give a brief discussion of the difficulties in Appendix B.

## 1.4 Statement of originality

In Chapter 2, originality is claimed for the formal derivation of the Wagner condition in section 2.2.3, the composite expansion for the pressure on the body in section 2.2.5 and the expansion of Korobkin's exact solution for an entering elliptic paraboloid in the slender body limit in section 2.4.4. In Chapter 3, the small and intermediate time asymptotic solutions in sections 3.3 and 3.4 are new work. The use of the velocity potential enables a straightforward generalization of Korobkin's shallow water impact model to three-dimensions in section 3.7, which is all new work except where stated. It appears that the formal derivation of the inner corner problem in section 4.1.1 is new, as are most of the results and conjectures in the rest of section 4.1. The end of section 4.2.1 and section 4.2.2 are new work. Except for section 5.1 and where stated, the proposed models and analysis in Chapter 5 are original.

## Chapter 2

# Infinite depth water entry at small deadrise angles

In this chapter we consider the simplest infinite depth water entry model in which the body has small deadrise angle. In section 2.1 we formulate and nondimensionalize the two-dimensional model problem. In section 2.2 we show that a slight modification of previous asymptotic analyses [8, 9, 31] is required to perform the leading-order matching and therefore formally derive the Wagner condition [71]. In section 2.3 we consider some simple extensions. In section 2.4 we consider the difficult three-dimensional generalization and find the slender body limit of Korobkin's exact solution for an entering elliptic paraboloid [40], which we then use to discuss the validity of strip theory. In section 2.5, we describe the variational formulation of the leading-order outer problem. Finally, we give a brief review of stability and water exit in section 2.6.

### 2.1 Formulation and nondimensionalization

Consider the two-dimensional normal impact of a rigid, strictly convex, smooth and symmetric body travelling at speed  $U$  through a vacuum, on an inviscid incompressible fluid of constant density  $\rho$ , that is initially at rest in the lower half plane  $z < 0$ . Suppose we are interested in penetrations of depth  $L$  and choose the moment of impact to be at time  $t = 0$  at the origin by defining the body profile to be

$$z = Lf\left(\frac{\epsilon x}{L}\right) - Ut, \quad (2.1)$$

where  $f(0) = 0$  and  $\epsilon > 0$ . The geometry before impact is shown in Figure 2.1. The *deadrise angle* is defined to be the angle  $\theta$  between the downward pointing unit normal to the body  $\mathbf{m}$  and  $-\mathbf{k}$ .

The flow is initially irrotational so by Kelvin's circulation theorem it will remain irrotational. Therefore there exists a velocity potential  $\phi(x, z, t)$  such that the fluid velocity

$$\mathbf{u} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial z}\mathbf{k}. \quad (2.2)$$

The incompressibility condition  $\nabla \cdot \mathbf{u} = 0$  implies Laplace's equation in the fluid

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \quad (2.3)$$

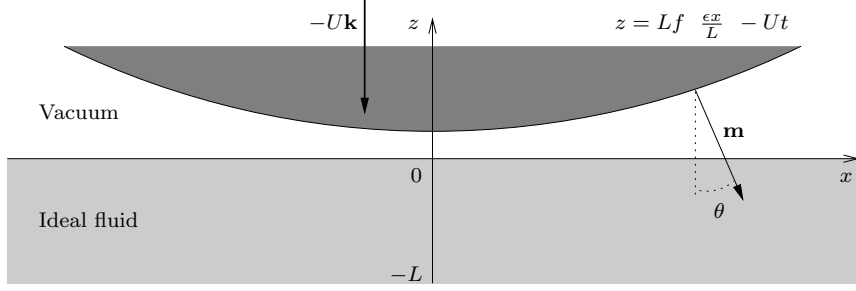


Figure 2.1: Geometry of the two-dimensional model problem just before impact.

The kinematic boundary condition on the body says there is zero normal flow through a rigid boundary and is given by

$$\epsilon f' \left( \frac{\epsilon x}{L} \right) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = U \quad \text{on} \quad z = Lf \left( \frac{\epsilon x}{L} \right) - Ut, \quad (2.4)$$

where, here and hereafter, prime denotes differentiation with respect to the argument. The kinematic boundary condition on the free surface says a fluid particle on the surface remains there and is given by

$$\frac{\partial \phi}{\partial n} = v_n \quad \text{or} \quad \frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = h(x, t), \quad (2.5)$$

where  $v_n$  is the outward pointing normal velocity of the free surface  $z = h(x, t)$ , which may be multivalued and has outward pointing unit normal  $\mathbf{n}$ .

A normal force balance on the free surface implies the pressure  $p$  there is equal to the external pressure in the vacuum, i.e.  $p = 0$  on  $z = h(x, t)$ . The pressure is given by Bernoulli's equation

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0. \quad (2.6)$$

The Bernoulli condition on the free surface is therefore

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0 \quad \text{on} \quad z = h(x, t). \quad (2.7)$$

We must also specify initial conditions at the instant of impact and the far field behaviour of the solution. There is zero flow, i.e.  $\nabla \phi = \mathbf{0}$  at  $t = 0$ , so suppose, without loss of generality,

$$\phi(x, z, 0) = 0. \quad (2.8)$$

The free surface is undisturbed at  $t = 0$  so

$$h(x, 0) = 0. \quad (2.9)$$

The fluid velocity must tend to zero at large distances from the body so

$$\phi \rightarrow 0 \quad \text{as} \quad x^2 + z^2 \rightarrow \infty, \quad (2.10)$$

to be consistent with (2.8). Finally, the free surface must be undisturbed at large distances from the body so

$$h \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad (2.11)$$

We nondimensionalize the full impact problem (2.3 - 2.11) by defining

$$\begin{aligned} x &= Lx^*, \quad z = Lz^*, \quad t = \frac{L}{U}t^*, \\ \phi &= LU\phi^*, \quad h = Lh^*, \quad p = \rho U^2 p^*, \end{aligned}$$

to obtain the dimensionless model problem (dropping stars)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid}, \quad (2.12)$$

$$\epsilon f'(\epsilon x) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = 1 \quad \text{on} \quad z = f(\epsilon x) - t, \quad (2.13)$$

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = h(x, t), \quad (2.14)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0 \quad \text{on} \quad z = h(x, t), \quad (2.15)$$

together with initial and far field conditions

$$\phi(x, z, 0) = 0, \quad h(x, 0) = 0, \quad (2.16)$$

$$\phi, \quad h \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad (2.17)$$

Bernoulli's equation (2.6) for the pressure becomes

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0. \quad (2.18)$$

The problem (2.12 - 2.17) is an unsteady, nonlinear, elliptic, free boundary problem with no known explicit solutions. It has been the subject of much research [30, 38, 41] beginning with the independent pioneering work of Von Kármán [33] and Wagner [71] on the forces on an alighting seaplane.

The existence of a free surface and a singularity in the flow variables at  $x = y = t = 0$  severely hamper a numerical analysis [26] and mean that questions of wellposedness are difficult; the only rigorous result is the existence of the similarity solution for the entry of a wedge [13]. When the body has small deadrise angle  $\theta$  (shown in Figure 2.1), the surface motion is violent and accompanied by small regions of rapid and large change. [8, 9, 31] show that these circumstances support an asymptotic analysis, although, as noted in [9], they give no formal justification of the so-called *Wagner condition* (which we describe below), because their asymptotic decomposition of the flow structure is slightly incomplete.

Provided  $f'$  is of order unity

$$\theta \sim \epsilon f'(\epsilon x) + O(\epsilon^2) \quad \text{as} \quad \epsilon \rightarrow 0. \quad (2.19)$$

Hence, our fundamental small-deadrise assumption is  $\epsilon \ll 1$  and an asymptotic solution is derived with core small parameter  $\epsilon$ . We note that for the smooth strictly convex profile considered here, the small deadrise angle assumption is equivalent to assuming the penetration depth  $L$  (i.e. the vertical length scale over which we are interested) is much smaller than the typical radius of curvature of the body near the point of impact.

## 2.2 Asymptotic solution for small deadrise angles

We begin with an overview of how the flow field in  $x > 0$  decomposes into the four regions shown in Figure 2.2 (similarly in  $x < 0$  by symmetry). We emphasise the decomposition is a very small modification of that proposed by [8, 9, 31], though it allows us to match all the regions using the classical procedure [67] and implies the first-order force calculated by [8] is incorrect by an order of magnitude.

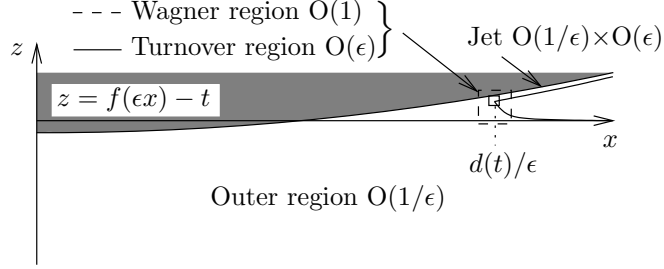


Figure 2.2: The asymptotic structure of the infinite depth case.

The free surface ‘turns over’ in the small *jet root region* of size  $O(\epsilon)$  located on the body to form a long, thin, fast moving jet running along the body in the *jet region* of extent  $O(1/\epsilon)$  and thickness  $O(\epsilon)$ . To lowest order the fluid response in the large *outer region* of size  $O(1/\epsilon)$  is as if the body were an expanding flat plate between the turnover points moving with velocity  $-\mathbf{k}$ . To match the outer and inner jet root regions of [31], it is necessary to introduce an intermediate inner region of size  $O(1)$  as shown. Matching this region to the outer solution yields the leading-order law of motion of the *turnover point* (where the free surface is vertical), which was first suggested by Wagner [71] and has therefore become known as the *Wagner condition*. We therefore refer to the intermediate inner region as the *Wagner region*. Then, matching the Wagner and the jet root regions yields the boundary data for the leading-order jet problem, which therefore decouples from the leading-order outer solution. This implies the jet exerts only a second-order influence on the motion of the turnover point.

In the rest of this section we review the leading-order solutions in the outer, jet root and jet regions and show how they match together through the new Wagner region, whose eigensolution is trivial compared to the solutions in the other regions.

### 2.2.1 The outer region

At order unity times the body intersects  $z = 0$  at  $x \sim O(1/\epsilon)$ , so we expect the turnover points to have  $x$ -coordinates  $x = \pm d(t)/\epsilon$ , where  $d(t)$  is order unity as  $\epsilon \rightarrow 0$ , and that the outer length scale is  $1/\epsilon$ . This is verified *a posteriori*. By the kinematic boundary condition on the body (2.13), the fluid velocity is of the same order as the unit impact velocity, so the scaled outer variables  $\hat{x}$ ,  $\hat{z}$  and  $\hat{\phi}$  are defined by

$$x = \frac{1}{\epsilon} \hat{x}, \quad z = \frac{1}{\epsilon} \hat{z}, \quad \phi = \frac{1}{\epsilon} \hat{\phi}. \quad (2.20)$$

The body position becomes  $\hat{z} = \epsilon(f(\hat{x}) - t)$ , so we expect the outer lower free surface (below the turnover points) to be defined by

$$\hat{z} = \epsilon \hat{h}(\hat{x}, t). \quad (2.21)$$

This assumption is again verified *a posteriori* so the body position and free surface are uniformly close to  $\hat{z} = 0$ , which therefore supports the linearization of (i) the kinematic boundary condition on the body (2.13) in  $|x| < d(t)$  and (ii) the lower free surface boundary conditions (2.14, 2.15) in  $|x| > d(t)$ . Performing this linearization, we seek a regular perturbation solution for  $\hat{\phi}$ ,  $\hat{h}$  and  $d$  in the form

$$\hat{\phi} \sim \hat{\phi}_0 + \epsilon \hat{\phi}_1 + \dots, \quad (2.22)$$

$$\hat{h} \sim \hat{h}_0 + \epsilon \hat{h}_1 + \dots, \quad (2.23)$$

$$d \sim d_0 + \epsilon d_1 + \dots, \quad (2.24)$$

though these asymptotic expansions require formal justification by a higher order analysis that we do not attempt. The leading-order outer problem is (dropping the hats)

$$\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad \text{in } z < 0, \quad (2.25)$$

$$\frac{\partial \phi_0}{\partial z} = -1 \quad \text{on } z = 0, |x| < d_0(t), \quad (2.26)$$

$$\frac{\partial \phi_0}{\partial z} = \frac{\partial h_0}{\partial t} \quad \text{on } z = 0, |x| > d_0(t), \quad (2.27)$$

$$\phi_0 = 0 \quad \text{on } z = 0, |x| > d_0(t), \quad (2.28)$$

$$\phi_0 \rightarrow 0 \quad \text{as } x^2 + z^2 \rightarrow \infty, \quad (2.29)$$

$$h_0 \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (2.30)$$

$$\phi_0, h_0, d_0 = 0 \quad \text{at } t = 0, \quad (2.31)$$

where (2.28) is obtained by integrating the leading-order Bernoulli condition on the free surface  $\partial \phi_0 / \partial t = 0$  and applying the zero initial condition (2.31).

Hence, we have a linear quasi-static mixed boundary value problem for  $\phi_0$  on the lower half-plane, in which the only geometric unknowns are the leading-order  $x$ -coordinates of the turnover or *free* points  $x = \pm d_0(t)$ . The boundary data changes from Neumann to Dirichlet at these points, so  $\phi_0$  will have a singularity there. To obtain a unique solution to (2.25 - 2.31) we must first specify this singularity, and then we must find  $d_0$ , i.e. find the leading-order law of motion of the turnover points in the  $x$ -direction. Equations (2.25 - 2.31) together with these conditions form a *codimension-two* free boundary problem [29], the free points  $x = \pm d_0(t)$  having two dimensions fewer than the space in which the motion occurs. The law of motion of the free points alone makes the codimension-two problem nonlinear.

Finally, we remark that on physical grounds we can write down two inequalities

$$h_0(x, t) \leq f(x) - t \quad \text{on } z = 0, |x| > d_0(t), \quad (2.32)$$

$$\frac{\partial \phi_0}{\partial t} < 0 \quad \text{on } z = 0, |x| < d_0(t). \quad (2.33)$$

The first says that the free surface lies below the body and the second is an integration of Bernoulli's equation that says the pressure is positive on the body. The inequalities are not required for the methodology we will adopt, but are essential to a variational formulation (see section 2.5 below) and therefore important to questions of wellposedness and the construction of efficient numerical algorithms [12, 29, 31, 35, 75].

### 2.2.1.1 Determination of the singularity at the free point

The complex velocity  $\partial w_0 / \partial \zeta$  has an inverse square root singularity at the free points [8, 9, 31], thus

$$w_0(\zeta, t) \sim S(t) \cdot R(\zeta, d_0(t)) \quad \text{as } \zeta = x + iz \rightarrow \pm d_0(t), \quad (2.34)$$

where  $w_0 = \phi_0 + i\psi_0$  is the complex potential,  $\psi_0$  is the stream function,  $S(t)$  is a complex function to be determined and  $R(\zeta, d_0(t)) := (\zeta^2 - d_0(t)^2)^{1/2}$  has a branch cut on the real axis between  $\zeta = \pm d_0(t)$  and  $R(2^{1/2}d_0(t)) = d_0(t)$ . As described in [67], there are at least four methods to convince oneself that this is the correct singularity at the free points. All require a local analysis at a free point. We could

- (1) insist, on physical grounds, that the spacially integrated kinetic energy is bounded [75];
- (2) apply Van Dyke's minimum singularity maxim [19];
- (3) assume the velocity potential  $\phi_0$  has derivatives that are square integrable (i.e.  $\phi_0 \in \mathcal{H}^1$ ); then the trace theorem for Sobolev spaces guarantees  $\phi_0$  cannot have any singularities of negative power [50];
- (4) match into an inner solution using the method of matched asymptotic expansions.

Note that the first three methods *suggest* the form of the singularity, which can *only* be derived systematically by the fourth. Fortunately, this last method is practical here through the Wagner and jet root regions as described in sections 2.2.2 and 2.2.3. In many mixed boundary value problems the local problem at the point separating different boundary conditions cannot be solved explicitly, so approaches such as (1 - 3) are necessary to make progress [29].

### 2.2.1.2 Solution methods

The mixed boundary value problem is summarized in Figure 2.3.

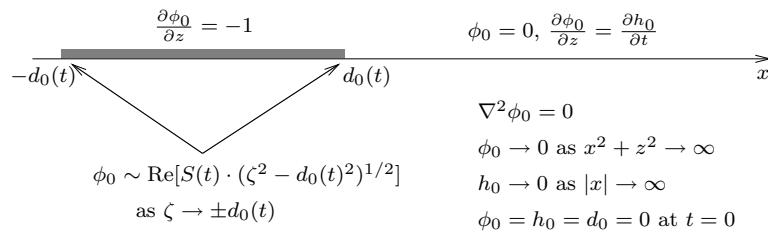


Figure 2.3: The leading-order outer problem.

There are at least four solution methods.

- (1) Formulate and solve the corresponding Riemann-Hilbert problem [19].
- (2) Use Cauchy's integral formula to formulate the potential problem as a Cauchy singular integral equation [1] for  $\partial \phi_0 / \partial z(x, 0, t)$  on  $|z| > d_0(t)$ .
- (3) Conformally map the hodograph plane  $\partial w_0 / \partial \zeta$  onto the complex fluid plane  $\zeta = x + iz$  and solve the resulting complex differential equation for  $w_0$ .

- (4) Spot that the potential problem is the same for a flat plate moving with unit velocity through a plane of incompressible, inviscid, ideal fluid at rest at infinity [48, 71]. The boundary condition  $\phi_0 = 0$  on  $|x| > d_0(t)$  corresponds to the symmetric equipotentials that extend from the corners of the plate to infinity along an axis of symmetry of the flow.

Methods (1) and (2) are equivalent, while (3) and (4) are conceptually easy. The first three methods are applicable to the small time asymptotics of the finite depth case, though the most straightforward method is (1). We therefore use method (1) here by way of introduction.

### 2.2.1.3 Solution by the Riemann-Hilbert method

The complex potential  $w_0(\zeta, t)$  is analytic on  $\Im(\zeta) < 0$  and the boundary conditions on  $\Im(\zeta) = 0-$  may be written in terms of the complex velocity

$$G_0(\zeta, t) = \frac{\partial w_0}{\partial \zeta}.$$

Therefore, by analytically continuing  $w_0$  into  $\Im(\zeta) > 0$ , using the Schwarz reflection formula

$$w_0(\zeta, t) = -\overline{w_0(\bar{\zeta}, t)} \quad \text{for } \Im(\zeta) > 0,$$

and taking care of the resulting discontinuity in  $G_0$  across the real axis, viz.

$$G_0(x + i0, t) = \begin{cases} -G_0(x - i0, t) + 2i & \text{for } |x| < d_0(t), \\ G_0(x - i0, t) & \text{for } |x| > d_0(t), \end{cases} \quad (2.35)$$

the mixed boundary value problem can be formulated as a Riemann problem for  $G_0$ , which is shown in Figure 2.4.

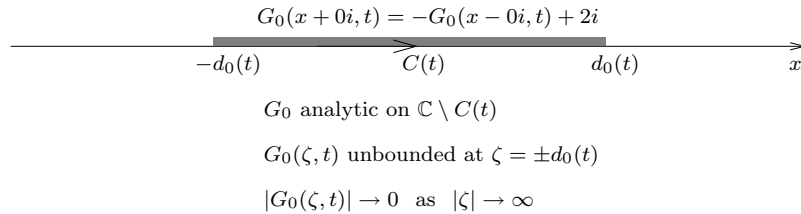


Figure 2.4: The Riemann problem for  $G_0(\zeta)$ .

The rigorous theory of the Riemann problem is described by Gakhov [16] and discussed for this particular type of problem by Gillow [19]. In our problem there is one open, simple, smooth contour  $C(t)$  across which  $G_0$  is discontinuous, so the index is one. The least singular solution that is unbounded at  $\zeta = \pm d_0(t)$  and zero at infinity is

$$G_0(\zeta, t) = \frac{iA(t) + I(\zeta, t)}{R(\zeta, t)}, \quad (2.36)$$

where  $A(t)$  is a real function to be determined, the function  $R(\zeta, t) = (\zeta^2 - d_0(t)^2)^{1/2}$  was introduced in section 2.2.1.1 (and defined so that  $R \sim \zeta$  as  $|\zeta| \rightarrow \infty$ ) and the integral  $I$ , which takes its Cauchy principal value on  $C(t)$ , is defined by

$$I(\zeta, t) = \frac{1}{2\pi i} \int_{C(t)} \frac{R(\xi - 0i, t)}{\xi - \zeta} 2i d\xi. \quad (2.37)$$

On physical grounds, we demand that the spacially integrated kinetic energy is bounded, which implies  $A(t) = 0$ . We evaluate the integral by contour integration, using the fact  $R(\xi - i0, t) = -i(d_0(t)^2 - \xi^2)^{1/2}$  for  $\xi \in (-d_0(t), d_0(t))$ , to obtain

$$G_0(\zeta, t) = i \left( 1 - \frac{\zeta}{R(\zeta, t)} \right). \quad (2.38)$$

Finally, we integrate (2.38), respecting the far field condition (2.29), to find (letting the stream function  $\psi_0 \rightarrow 0$  as  $x^2 + z^2 \rightarrow \infty$ , without loss of generality)

$$w_0(\zeta, t) = i(\zeta - R(\zeta, t)), \quad (2.39)$$

and therefore

$$\phi_0(x, z, t) = -z - \Re \left( \sqrt{d_0(t)^2 - (x + iz)^2} \right), \quad (2.40)$$

where the square-root has a branch cut on the real axis between  $x = \pm d_0(t)$  and  $(d_0(t)^2 - (x + iz)^2)^{1/2} = d_0(t)$  when  $x + iz = 0 - i0$ . Figure 2.5 shows a plot of the leading-order outer velocity field.

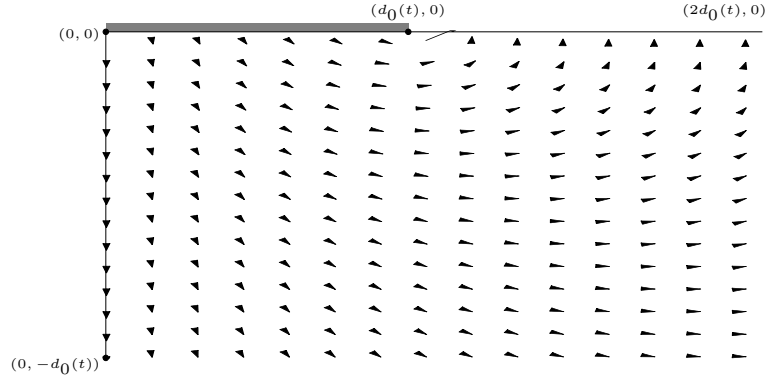


Figure 2.5: Plot of the leading-order outer velocity field.

#### 2.2.1.4 The leading-order outer free surface elevation

Having determined  $\phi_0$ , the leading-order kinematic boundary condition (2.27) yields

$$\frac{\partial h_0}{\partial t} = -1 + \frac{x}{\sqrt{x^2 - d_0(t)^2}} \quad \text{on } x > d_0(t). \quad (2.41)$$

We integrate and apply the initial condition (2.31) to obtain the leading-order outer free surface elevation

$$h_0(x, t) = -t + \int_0^t \frac{x}{(x^2 - d_0(\tau)^2)^{1/2}} d\tau \quad \text{on } x > d_0(t), \quad (2.42)$$

which is bounded at  $x = \pm d_0(t)$  where its first order partial derivatives have inverse square-root singularities. Specifically, a standard asymptotic analysis [27] of the integral in (2.42) implies

$$h_0(d_0(t) + X, t) \sim h_0(d_0(t), t) - \frac{1}{\dot{d}_0(t)} \sqrt{2d_0(t)X} + O(X) \quad \text{as } X \rightarrow 0+. \quad (2.43)$$

The leading-order outer solution is complete except for the leading-order  $x$ -coordinates of the turnover points  $x = \pm d_0(t)$ . These are determined by the Wagner condition, which we introduce shortly. First, we find the leading-order outer pressure.

### 2.2.1.5 The leading-order outer pressure

We write Bernoulli's equation (2.18) in outer variables (2.20) and drop hats to find

$$p + \frac{1}{\epsilon} \frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0. \quad (2.44)$$

Then, we expand  $p$  as an asymptotic series in the form

$$p \sim \frac{1}{\epsilon} p_0 + p_1 + \cdots, \quad (2.45)$$

substitute the asymptotic expansions (2.22, 2.45) for  $p$  and  $\phi$ , respectively, into Bernoulli's equation (2.44) and equate coefficients to find

$$p_0 = -\frac{\partial \phi_0}{\partial t}, \quad (2.46)$$

$$p_1 = -\frac{\partial \phi_1}{\partial t} - \frac{1}{2} \left[ \left( \frac{\partial \phi_0}{\partial x} \right)^2 + \left( \frac{\partial \phi_0}{\partial z} \right)^2 \right]. \quad (2.47)$$

Hence, by (2.40), we find

$$p(x, z, t) \sim \frac{1}{\epsilon} \Re \left( \frac{d_0(t) \dot{d}_0(t)}{\sqrt{d_0(t)^2 - (x + iz)^2}} \right) + O(1). \quad (2.48)$$

The outer pressure on the body in the interior region  $|x| < d_0(t)$  is therefore

$$p(x, 0, t) \sim \frac{1}{\epsilon} \frac{d_0(t) \dot{d}_0(t)}{\sqrt{d_0(t)^2 - x^2}} + O(1), \quad (2.49)$$

which we use to construct a leading-order composite expansion of the pressure on the body in section 2.2.6.

### 2.2.1.6 The Wagner condition

The matching condition to determine  $d_0(t)$  was first suggested by Wagner [71] and requires that the leading-order free surface elevation (2.42) in the outer problem as  $x \rightarrow d_0(t)+$  is equal to the body position there, viz.

$$h_0(d_0(t), t) = f(d_0(t)) - t \quad \text{for all } t \geq 0, \quad (2.50)$$

in agreement with the asymptotic structure depicted in Figure 2.2. We will shortly explain this condition in section 2.2.2.

### 2.2.1.7 The law of motion of the free point

Substituting (2.42) into (2.50) implies that

$$\int_0^t \frac{d_0(t)}{(d_0(t)^2 - d_0(\tau)^2)^{1/2}} d\tau = f(d_0(t)). \quad (2.51)$$

This singular integral equation has a unique closed form solution if  $d_0(t)$  is strictly monotonic increasing (see, for example, Sneddon [59]). In this case, writing  $t$  as a function of  $d_0$ , (2.51) becomes the Abel integral equation

$$\int_0^{d_0} \frac{t'(\sigma)}{(d_0^2 - \sigma^2)^{1/2}} d\sigma = \frac{f(d_0)}{d_0}, \quad (2.52)$$

which can be solved by writing the left hand side as a convolution integral, taking the Laplace transform, re-arranging for the transform of  $t'$  and then inverting to obtain

$$t'(\sigma) = \frac{2}{\pi} \frac{d}{d\sigma} \int_0^\sigma \frac{f(\xi)}{(\sigma^2 - \xi^2)^{1/2}} d\xi. \quad (2.53)$$

We integrate with respect to  $\sigma$  to find

$$\frac{\pi t}{2} = \int_{d_0(0)}^{d_0(t)} \frac{f(\xi)}{(d_0(t)^2 - \xi^2)^{1/2}} d\xi, \quad (2.54)$$

and then substitute the initial condition  $d_0(0) = 0$  and  $\xi = d_0(t) \sin \theta$  to obtain the leading-order law of motion of the free point

$$\frac{\pi t}{2} = \int_0^{\pi/2} f(d_0(t) \sin \theta) d\theta. \quad (2.55)$$

Differentiating we find

$$\frac{\pi}{2} = \dot{d}_0(t) \int_0^{\pi/2} f'(d_0(t) \sin \theta) \sin \theta d\theta, \quad (2.56)$$

so  $\dot{d}_0$  is non-zero and finite if  $f'(x) > 0$  on  $x > 0$ . We now match the outer and Wagner regions and therefore formally justify the Wagner condition.

### 2.2.2 The Wagner region

The leading-order elevation of the outer free surface is of order unity relative to the original dimensionless variables and the leading-order outer solution breaks down at the free points  $\hat{x} = d_0(t)$ ,  $\hat{z} = 0$  (re-introducing hats on the outer variables). We therefore define the Wagner region coordinates  $\bar{x}$  and  $\bar{z}$  by

$$x = \frac{d(t)}{\epsilon} + \bar{x}, \quad z = f(d(t)) - t + \bar{z}, \quad (2.57)$$

to retain the balance in Laplace's equation. The one-term inner expansion of the one-term outer velocity potential is found by substituting these scalings into (2.40), through the outer scalings (2.20), viz.  $\epsilon^{-1} \hat{\phi}_0(d + \epsilon \bar{x}, \epsilon[f(d) - t + \bar{z}], t)$ , and expanding to obtain

$$-\frac{1}{\epsilon^{1/2}} \Re \left( i \sqrt{2d_0(d_1 + \bar{x} + i[f(d_0) - t + \bar{z}])} \right). \quad (2.58)$$

Therefore, by Van Dyke's matching principle [67], the Wagner region velocity potential is  $O(\epsilon^{-1/2})$  so we scale

$$\phi = \frac{\bar{\phi}}{\epsilon^{1/2}}. \quad (2.59)$$

For a non-trivial balance in the kinematic boundary condition (2.14), the free surface elevation in the Wagner region is  $O(\epsilon^{1/2})$  so we scale

$$h = f(d) - t + \epsilon^{1/2}\bar{h}. \quad (2.60)$$

We substitute (2.57, 2.59, 2.60) into the model problem (2.12 - 2.15) to find in the fluid,

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\phi}}{\partial \bar{z}^2} = 0, \quad (2.61)$$

on the body  $\bar{z} = f(d + \epsilon\bar{x}) - f(d)$ ,

$$\epsilon f'(d + \epsilon\bar{x}) \frac{\partial \bar{\phi}}{\partial \bar{x}} - \frac{\partial \bar{\phi}}{\partial \bar{z}} = \epsilon^{1/2}, \quad (2.62)$$

and on the free surface  $\bar{z} = \epsilon^{1/2}\bar{h}(\bar{x}, t)$ ,

$$-d \frac{\partial \bar{h}}{\partial \bar{x}} + \epsilon^{1/2}(f'(d)d - 1) + \epsilon^{1/2} \frac{\partial \bar{\phi}}{\partial \bar{x}} \frac{\partial \bar{h}}{\partial \bar{x}} + \epsilon \frac{\partial \bar{h}}{\partial t} - \frac{\partial \bar{\phi}}{\partial \bar{z}} = 0, \quad (2.63)$$

$$-d \frac{\partial \bar{\phi}}{\partial \bar{x}} + \epsilon \frac{\partial \bar{\phi}}{\partial t} + \frac{\epsilon^{1/2}}{2} \left[ \left( \frac{\partial \bar{\phi}}{\partial \bar{x}} \right)^2 + \left( \frac{\partial \bar{\phi}}{\partial \bar{z}} \right)^2 \right] = 0. \quad (2.64)$$

Expanding  $d$  as in (2.24), the body position becomes

$$\bar{z} \sim \epsilon\bar{x}f'(d_0) + O(\epsilon^2). \quad (2.65)$$

Hence, the body and free surface are uniformly close to  $\bar{z} = 0$ , so we linearize the boundary conditions (2.62 - 2.64) onto  $\bar{z} = 0$ , expand  $\bar{\phi}$  and  $\bar{h}$  as asymptotic series in powers of the small parameter  $\epsilon^{1/2}$  and expand  $d$  as in (2.24) to obtain the leading-order Wagner region problem

$$\frac{\partial^2 \bar{\phi}_0}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\phi}_0}{\partial \bar{z}^2} = 0 \quad \text{in } \bar{z} < 0, \quad (2.66)$$

$$\frac{\partial \bar{\phi}_0}{\partial \bar{z}} = 0 \quad \text{on } \bar{z} = 0, \bar{x} < 0, \quad (2.67)$$

$$d_0 \frac{\partial \bar{h}_0}{\partial \bar{x}} + \frac{\partial \bar{\phi}_0}{\partial \bar{z}} = 0 \quad \text{on } \bar{z} = 0, \bar{x} > 0, \quad (2.68)$$

$$\frac{\partial \bar{\phi}_0}{\partial \bar{x}} = 0 \quad \text{on } \bar{z} = 0, \bar{x} > 0. \quad (2.69)$$

The matching condition for the velocity potential is given by (2.58) as

$$\bar{\phi}_0 \sim -\Re \left( i \sqrt{2d_0(d_1 + i(f(d_0) - t) + \bar{x} + i\bar{z})} \right) \quad \text{as } |\bar{x} + i\bar{z}| \rightarrow \infty. \quad (2.70)$$

To find the two-term inner expansion of the one-term outer free surface elevation, we substitute the Wagner region scalings (2.58, 2.60) into the leading-order outer free surface

elevation (2.42), through the outer scalings (2.20), viz.  $\epsilon^{-1/2}[\hat{h}_0(d + \epsilon\bar{x}, t) - f(d) + t]$ , then expand  $d$  as in (2.24) and the integral using (2.43) to obtain

$$\epsilon^{-1/2}[\hat{h}_0(d_0, t) - f(d_0) + t] - \frac{1}{\dot{d}_0} \sqrt{2d_0(d_1 + \bar{x})}.$$

Matching with the Wagner region order unity free surface elevation, we obtain the Wagner condition (2.50) from the  $O(\epsilon^{-1/2})$  term, while the  $O(1)$  term yields the second matching condition for the leading-order Wagner region problem (2.66 - 2.69), viz.

$$\bar{h}_0 \sim -\frac{1}{\dot{d}_0} \sqrt{2d_0(d_1 + \bar{x})} \quad \text{as } \bar{x} \rightarrow \infty. \quad (2.71)$$

The Wagner region problem (2.66 - 2.71) is a homogenous mixed boundary value problem. The singularity at the origin (where the boundary conditions (2.67, 2.69) change type) is determined by matching with the inner-inner jet root region. As demonstrated by [31], the jet root is driven by a square root singularity in the far field potential, so matching requires the Wagner region to have the same behaviour at  $\bar{x} = \bar{z} = 0$ . Hence, the unique eigensolution to (2.66, 2.67, 2.69, 2.70) is

$$\bar{\phi}_0 = -\Re \left( i \sqrt{2d_0(\bar{x} + i\bar{z})} \right). \quad (2.72)$$

The free surface is then given by (2.68, 2.71) as

$$\bar{h}_0 = -\frac{1}{\dot{d}_0} \sqrt{2d_0\bar{x}}. \quad (2.73)$$

This completes the leading-order Wagner region solution, which essentially translates the location of the square root singularity in the potential by an order unity distance (with respect to the original dimensionless coordinates) from  $x + iz = d_0/\epsilon$  to  $x + iz = d_0/\epsilon + d_1 + i(f(d_0) - t)$  to within  $O(\epsilon^2)$ .

### 2.2.3 The jet root region

The free surface turns over in the jet root region and the velocity of the turnover point is  $O(1/\epsilon)$ , so the local fluid velocity is  $O(1/\epsilon)$ , relative to the original dimensionless coordinates. Hence, to retain a balance in Laplace's equation and match with the Wagner region velocity potential we scale

$$\bar{x} = \epsilon X, \quad \bar{z} = \epsilon Z, \quad \bar{\phi} = \epsilon^{1/2} (\dot{d}X + \Phi), \quad \bar{h} = \epsilon^{1/2} H, \quad (2.74)$$

where the  $\dot{d}X$  term simplifies the algebra and  $H$  is multivalued. To find the leading-order jet root problem we proceed as in the outer and Wagner regions, i.e.

- (1) substitute the scalings (2.74) into the full Wagner region problem (2.61 - 2.64) to find the first order terms are  $O(\epsilon)$ ;
- (2) expand  $H \sim H_0 + \epsilon H_1 + \dots$ ,  $\Phi \sim \Phi_0 + \epsilon \Phi_1 + \dots$  and  $d$  as in (2.24);
- (3) linearize the boundary conditions onto their leading-order locations, i.e. the kinematic boundary condition on the body onto  $Z = 0$  (because the body is flat up to  $O(\epsilon)$  by (2.65, 2.74)) and the free surface boundary conditions onto  $Z = H_0$ ;

- (4) match the Wagner and jet root region velocity potential and free surface elevation to obtain the far field behaviour in the jet root region.

The resulting leading-order jet root region problem is a quasi-static Kelvin-Helmholtz travelling-wave cavity flow and is summarized in Figure 2.6. It begins as a left-flowing uniform stream of speed  $\dot{d}_0(t)$  at infinity. The top layer of unknown thickness  $H_J(t)$  (above the dividing streamline) is then diverted backward into a right-flowing stream attached to the body, also of thickness  $H_J(t)$  by conservation of mass. This right-flowing stream is the jet root and has uniform velocity  $\dot{d}_0(t)$  in the  $X$ -direction, by the Bernoulli condition on the free surface.

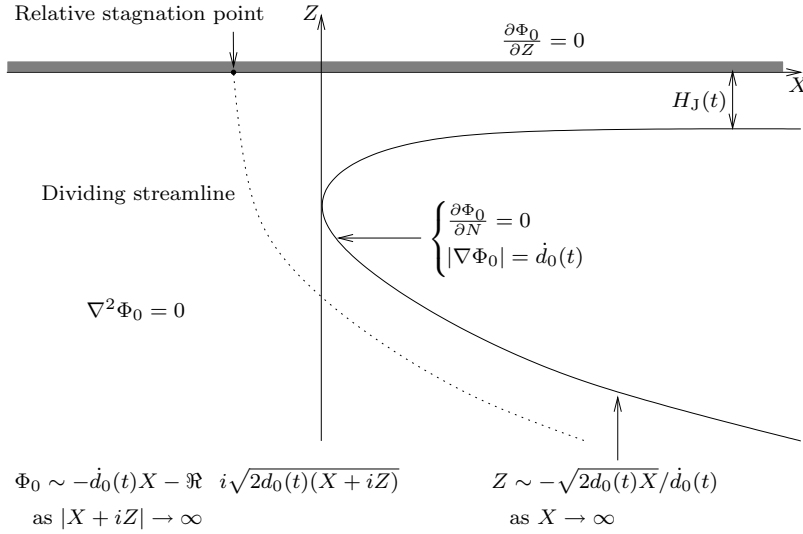


Figure 2.6: The leading-order jet root region problem.

The solution in the inner region therefore quantifies the formation and evolution of the splash jet and is found using conformal mapping and hodograph techniques [8, 9, 31, 64], which imply

$$\Phi_0 = \frac{\dot{d}_0 H_J}{\pi} \Re[\mathcal{F}(s)], \quad (2.75)$$

where  $\mathcal{F}$  is defined implicitly by the pair of equations

$$\mathcal{F} = s - \log s, \quad X + iZ = X_0 - \frac{H_J}{\pi} (s + 4\sqrt{s} + \log s), \quad (2.76)$$

and the fluid domain corresponds to the lower-half of the complex  $s$ -plane. The arbitrary real constant  $X_0$  represents the fact that the solution is unique up to an arbitrary translation in the  $X$ -direction, i.e. a higher order analysis is necessary to find the  $O(\epsilon)$  correction to the location of the relative stagnation point and therefore the order unity correction to the  $x$ -coordinate of the turnover point  $d_1$ . We emphasise that there is no solution with a different singularity at infinity. This has repercussions for linearized planing theory at small incidence angles, which we consider in Chapter 5.

In the far field (2.75, 2.76) imply

$$\Phi_0 \sim -\dot{d}_0 X - \Re \left[ 4i\dot{d}_0 \sqrt{H_J(X + iZ)/\pi} \right].$$

Comparing this with the far field matching condition in Figure 2.6, we see that the inner solution determines the asymptotic jet thickness  $H_J(t)$ , in terms of the speed of the uniform far field stream  $\mathcal{U} = \dot{d}_0(t)$  and the coefficient of the square root in the far field potential  $\mathcal{S} = \sqrt{2d_0(t)}$ , viz.

$$H_J(t) = \frac{\pi \mathcal{S}^2}{16\mathcal{U}^2} = \frac{\pi d_0(t)}{8\dot{d}_0(t)^2}. \quad (2.77)$$

### 2.2.3.1 The leading-order pressure in the inner regions

Let  $\bar{p}$  be the Wagner region pressure; then, writing Bernoulli's equation (2.18) in Wagner region variables (2.57, 2.59) we find

$$\bar{p} = \frac{\dot{d}}{\epsilon^{3/2}} \frac{\partial \bar{\phi}}{\partial \bar{x}} + \frac{1}{2\epsilon} \left[ \left( \frac{\partial \bar{\phi}}{\partial \bar{x}} \right)^2 + \left( \frac{\partial \bar{\phi}}{\partial \bar{z}} \right)^2 \right] - \frac{1}{\epsilon^{1/2}} \frac{\partial \bar{\phi}}{\partial t}$$

We expand  $\bar{p} \sim \epsilon^{-3/2} \bar{p}_0 + \epsilon^{-1} \bar{p}_1 + O(\epsilon^{-1/2})$ , substitute the asymptotic series (2.24, 2.59) for  $d$  and  $\bar{\phi}$  respectively, and equate coefficients to find

$$\bar{p}_0 = \dot{d}_0 \frac{\partial \bar{\phi}_0}{\partial \bar{x}}, \quad (2.78)$$

$$\bar{p}_1 = \dot{d}_0 \frac{\partial \bar{\phi}_1}{\partial \bar{x}} - \frac{1}{2} \left[ \left( \frac{\partial \bar{\phi}_0}{\partial \bar{x}} \right)^2 + \left( \frac{\partial \bar{\phi}_0}{\partial \bar{z}} \right)^2 \right]. \quad (2.79)$$

By (2.72),

$$\bar{p}_0 = \dot{d}_0 d_0^{1/2} \cdot \Re \left[ \frac{i}{\sqrt{2(\bar{x} + i\bar{z})}} \right], \quad (2.80)$$

so the pressure on the body in the Wagner region is

$$\bar{p}(\bar{x}, 0, t) \sim \frac{\dot{d}_0 d_0^{1/2}}{\epsilon^{3/2} \sqrt{-2\bar{x}}} + O(1/\epsilon) \quad \text{for } \bar{x} < 0. \quad (2.81)$$

Similarly, in the jet root region the pressure  $P \sim \epsilon^{-2} P_0 + O(\epsilon^{-3/2})$ , where

$$P_0 = \frac{1}{2} \left[ \dot{d}_0(t)^2 - \left( \frac{\partial \Phi_0}{\partial X} \right)^2 - \left( \frac{\partial \Phi_0}{\partial Z} \right)^2 \right]. \quad (2.82)$$

By (2.75, 2.76), the leading-order pressure on the body in the jet root region is given in terms of  $s > 0$  by

$$P(X, 0, t) \sim \frac{\dot{d}_0^2}{2\epsilon^2} \left[ 1 - \left( \frac{1 - \sqrt{s}}{1 + \sqrt{s}} \right)^2 \right] + O(1/\epsilon^{3/2}), \quad (2.83)$$

where  $X$  is defined in terms of  $s$  by (2.76). The maximum leading-order pressure occurs at the relative stagnation point ( $s = 1$ ) in the jet root region (see Figure 2.6) and has magnitude  $\dot{d}_0(t)^2/2\epsilon^2$ . This is an order of magnitude larger than the leading-order pressure in the outer and Wagner regions. In Figure 2.7 we plot  $\mathcal{P}$  where the leading-order jet root pressure on the body is  $P_0(X, 0, t) = \dot{d}_0^2 \mathcal{P}(\pi X/H_J)$ .

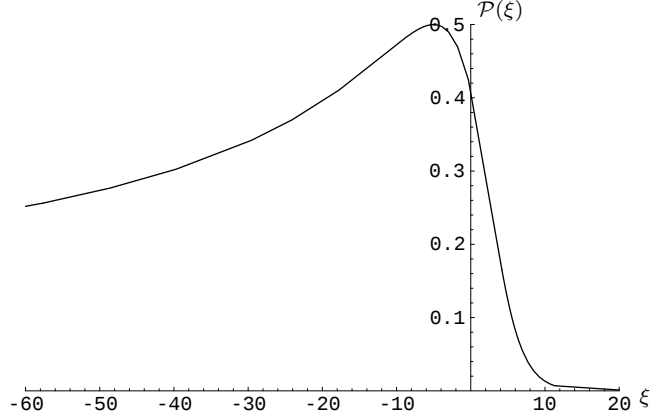


Figure 2.7: Plot of  $\mathcal{P}(\xi)$  where the leading-order jet root pressure on the body  $P_0(X, 0, t) = \dot{d}_0^2 \mathcal{P}(\xi)$  and  $\xi = \pi X / H_J$ .

We expand (2.76, 2.83) as  $s \rightarrow \infty$  to find

$$P \sim \frac{2}{\epsilon^2} \sqrt{\frac{H_J}{\pi X}} + O\left(1/\epsilon^{3/2}\right) \quad \text{as } X \rightarrow -\infty, \quad (2.84)$$

which, as required, matches with the Wagner region pressure (2.81) by (2.74, 2.77).

Below we use the leading-order Wagner and jet root pressure to construct a leading-order composite expansion of the pressure on the body. First, we briefly describe the leading-order problem in the jet region.

#### 2.2.4 The jet region

To lowest order, the slam strips off a layer of fluid from just below the free surface ahead of the impactor and ejects it along the body surface in the form of a jet. In the previous section, this process was shown to take place in a small, fast-moving region. This jet root region has high-pressure because it must reverse the direction of the incoming thin layer that lies above the dividing streamline (see Figure 2.6). Further, it was shown through a careful asymptotic analysis, that, to leading-order, the jet of thickness  $\epsilon H_J(t) = \epsilon \pi d_0(t) / 8 \dot{d}_0(t)^2$ , emanates from  $(d_0(t) + \epsilon d_1(t), \epsilon[f(d(t)) - t])$ , to within  $O(\epsilon^2)$ , with uniform cross-sectional velocity that is tangential to the body and of magnitude  $2\dot{d}_0(t)/\epsilon$ ; this is because the fluid velocity out of the jet root is  $\dot{d}_0(t)/\epsilon$  relative to the moving jet root region, which has velocity  $\dot{d}_0(t)/\epsilon$  relative to the fixed frame. Hence, the jet has length, thickness and velocity of order  $1/\epsilon$ ,  $\epsilon$  and  $1/\epsilon$ , respectively, which combined with the fact that the body has a radius of curvature at least of order  $1/\epsilon^2$ , implies the leading-order governing equations are the ‘planar’ zero-gravity shallow water equations [31, 76], viz. introducing a tilde to denote the jet variables,

$$\frac{\partial \tilde{\phi}_0}{\partial t} + \frac{1}{2} \left( \frac{\partial \tilde{\phi}_0}{\partial \tilde{x}} \right)^2 = 0, \quad (2.85)$$

$$\frac{\partial \tilde{h}_0}{\partial t} + \frac{\partial}{\partial \tilde{x}} \left( \frac{\partial \tilde{\phi}_0}{\partial \tilde{x}} \tilde{h}_0 \right) = 0, \quad (2.86)$$

where the arc length  $\tilde{x}$  along the body, the leading-order potential  $\tilde{\phi}(\tilde{x}, t)$  and the leading-order film thickness  $\tilde{h}(\tilde{x}, t)$  have been scaled with  $1/\epsilon$ ,  $1/\epsilon$  and  $\epsilon$ , respectively. Matching with the jet root region implies the boundary data is

$$\frac{\partial \tilde{\phi}_0}{\partial \tilde{x}} = 2\dot{d}_0, \quad \tilde{h}_0 = \frac{\pi d_0}{8\dot{d}_0^2} \quad \text{at} \quad \tilde{x} = d_0. \quad (2.87)$$

The first-order hyperbolic equations (2.85, 2.86) decouple and may be solved successively using the method of characteristics as described by [75]; we present an explicit example in section 2.2.6. Note that the characteristics or particle paths can never be parallel to the boundary curve  $\tilde{x} = d_0(t)$ , because the fluid particles there move with twice its speed by (2.87). The solution therefore exists and is unique sufficiently close to the turnover point. Moreover, [75] show that shocks cannot occur in the solution of (2.85) if  $\ddot{d}_0 < 0$ .

To lowest order the spray jet is only affected by the body profile through  $d_0(t)$ . However, proceeding to fourth-order in the analysis [75] show that the pressure is given by  $\tilde{p} \sim \epsilon \tilde{p}_0 + O(\epsilon^2)$ , where

$$\tilde{p}_0 = -\kappa_0 \left( \frac{\partial \tilde{\phi}_0}{\partial \tilde{x}} \right)^2 (\tilde{h}_0 - \tilde{m}).$$

Here the leading-order curvature of the body is  $\kappa_0 = f''(\tilde{x})$  (scaled with  $\epsilon^2$ ) and  $\tilde{m}$  is the normal distance from the body (scaled with  $\epsilon$ ). This is the pressure required to balance the centrifugal acceleration of flow around the slightly curved body and is at least two orders of magnitude smaller than the pressure in the outer, Wagner and jet root regions. For our strictly convex body,  $\kappa_0 > 0$ , so the leading-order pressure is negative, suggesting that the sheet will separate from the body. However, [68] show that even small amounts of surface tension can have a significant effect on the existence and location of the separation point, so we cannot rely on the sign of our pressure to make predictions. Given the separation point in steady two-dimensional jet flow, [34] describe the mechanism through which the jet leaves the surface provided the jet remains uniformly slender. Once separation has occurred the particle paths are straight in the zero gravity model.

## 2.2.5 Verification

The matching presented in sections 2.2.2, 2.2.3 and 2.2.4 verifies the assumptions made at the beginning of section 2.2.1, i.e. the turnover points lie an  $O(1/\epsilon)$  distance apart, with  $O(1)$  first-order correction, and the outer free surface is an  $O(1)$  distance from the undisturbed free surface, at least for times of order unity.

The jet roots do not traverse the origin so a discontinuity in the slope of  $f$  is permissible at  $x = 0$ . This is fortunate because it is only for the particular case of the wedge  $f(x) = |x|$ , that the above asymptotic analysis has been confirmed by a more rigorous analysis: Fraenkel & McLeod [13] proved existence of the corresponding similarity solution and the limit of their rigorous asymptotics as the deadrise angle tends to zero is in agreement with the above results.

In the next section we construct a uniformly valid composite expansion for the pressure exerted on the body in the interior region and therefore find expressions for the leading- and first-order force on the body.

### 2.2.6 The pressure and force on the body

Having obtained the leading-order outer, Wagner, jet root and jet solutions, a uniformly valid composite expansion may be constructed, which reduces to the outer expansion when expanded asymptotically in outer variables as  $\epsilon \rightarrow 0$  and similarly in the other regions. There are many ways to construct the composite expansion because it is not unique [67]. The simplest is *additive composition* where the sum of the expansion in each region is corrected by subtracting their common part in all intermediate matching regions. For example, assuming that the outer expansions (2.22, 2.24) are correct, which can only be verified by a higher order analysis that we do not attempt, the composite pressure  $p_c$  on the body in  $|x| < d(t)/\epsilon$  has contributions from the outer, Wagner and jet root regions, thus

$$p_c \sim \frac{P_0 - \epsilon^2 \text{1.t.i.}(1.\text{t.o.})[\bar{p}]}{\epsilon^2} + \frac{\bar{p}_0 - \epsilon^{3/2} \text{1.t.i.}(1.\text{t.o.})[\hat{p}]}{\epsilon^{3/2}} + \frac{\hat{p}_0}{\epsilon} + \dots, \quad (2.88)$$

where the leading-order pressure term in the jet root, Wagner and outer regions are  $P_0$ ,  $\bar{p}_0$  and  $\hat{p}_0$  respectively,  $\text{1.t.i.}(1.\text{t.o.})[\bar{p}]$  is the one-term-inner jet root region expansion of the one-term-outer Wagner pressure (of order  $\epsilon^{-2}$ ) and  $\text{1.t.i.}(1.\text{t.o.})[\hat{p}]$  is the one-term-inner Wagner region expansion of the one-term-outer pressure (of order  $\epsilon^{-3/2}$ ). The leading-order composite pressure profiles at times  $t = 1, 2, 3, 4, 5$  are plotted for the parabolic body  $f(x) = x^2$  in Figure 2.8b for  $\epsilon = 0.01$ .

The leading-order pressures in the outer, Wagner, jet root and jet regions are of order  $1/\epsilon$ ,  $1/\epsilon^{3/2}$ ,  $1/\epsilon^2$  and  $\epsilon$ , respectively, while their lateral extents are of order  $1/\epsilon$ ,  $1$ ,  $\epsilon$  and  $1/\epsilon$ , respectively. Hence, the leading-order force on the body (per unit length in the perpendicular direction) is of order  $1/\epsilon^2$  due to the leading-order outer pressure. The next order term is of order  $1/\epsilon^{3/2}$  due to the Wagner region pressure, so the higher order pressure analysis of [9] for the entry of a parabolic body is incorrect because of the exclusion of this contribution. Specifically, [9] take the first-order correction to be of order  $1/\epsilon$  with contributions from (a) the first-order outer pressure of order 1 on a domain of order  $1/\epsilon$  and (b) the leading-order jet root pressure of order  $1/\epsilon^2$  on a domain of order  $\epsilon$ . To find (a) and (b), [9] made several approximations and their final analytic result rapidly overestimates the experimental impact force. The new correction term of order  $1/\epsilon^{3/2}$  might provide an explanation for their disagreement with experiment because it is negative; expanding the force on the body  $F(t)$  as

$$F \sim \frac{F_0}{\epsilon^2} + \frac{F_1}{\epsilon^{3/2}} + \dots, \quad (2.89)$$

the leading-order term is given by (2.49) as

$$F_0(t) = \int_{-d_0(t)}^{d_0(t)} \frac{d_0(t)\dot{d}_0(t)}{\sqrt{d_0(t)^2 - x^2}} dx = \pi d_0(t)\dot{d}_0(t), \quad (2.90)$$

while it is easy to show that the composite expansion (2.88) implies that the first-order term is given by (2.49, 2.81) as

$$\begin{aligned} F_1(t) &= 2 \int_{-\infty}^0 \bar{p}_0 - \epsilon^{3/2} \text{1.t.i.}(1.\text{t.o.})[\hat{p}] d\bar{x} \\ &= 2 \int_{-\infty}^0 \frac{d_0\dot{d}_0}{\sqrt{-2d_0\bar{x}}} - \Re \left[ \frac{d_0\dot{d}_0}{\sqrt{-2d_0(t)(d_1 + \bar{x} + i(f(d_0) - t))}} \right] d\bar{x} \\ &= -2d_0(t)^{1/2}\dot{d}_0(t) \cdot \Im \left[ \sqrt{d_1(t) + i(f(d_0(t)) - t)} \right], \end{aligned} \quad (2.91)$$

by contour integration. To find the first-order correction to the force it is necessary to find  $d_1$  and therefore we must solve the first-order outer problem. An integral representation of the first-order complex velocity is possible using the Riemann-Hilbert method. We have not attempted to perform the subsequent matching with the inner regions, but remark that the *ad hoc* procedure of [9], in which the outer and inner pressures are patched together to find  $d_1$ , may offer significant insight.

### 2.2.7 Example: parabolic body

For a parabolic body  $f(x) = x^2$ , we find in the outer region

$$\hat{\phi}_0 = -\hat{z} - \Re \left( \sqrt{2t - (\hat{x} + i\hat{z})^2} \right), \quad (2.92)$$

$$\hat{h}_0(\hat{x}, t) = \hat{x}^2 - |\hat{x}| \sqrt{\hat{x}^2 - 2t} - t \quad \text{for } |\hat{x}| > d_0(t), \quad (2.93)$$

$$d_0(t) = \sqrt{2t}, \quad (2.94)$$

$$\hat{p}_0(\hat{x}, 0, t) = \begin{cases} (2t - \hat{x}^2)^{-1/2} & \text{for } |\hat{x}| < d_0(t), \\ 0 & \text{for } |\hat{x}| > d_0(t). \end{cases} \quad (2.95)$$

Profiles of the leading-order outer free surface elevation (for  $\epsilon = 0.01$ ) are plotted in Figure 2.8(a).

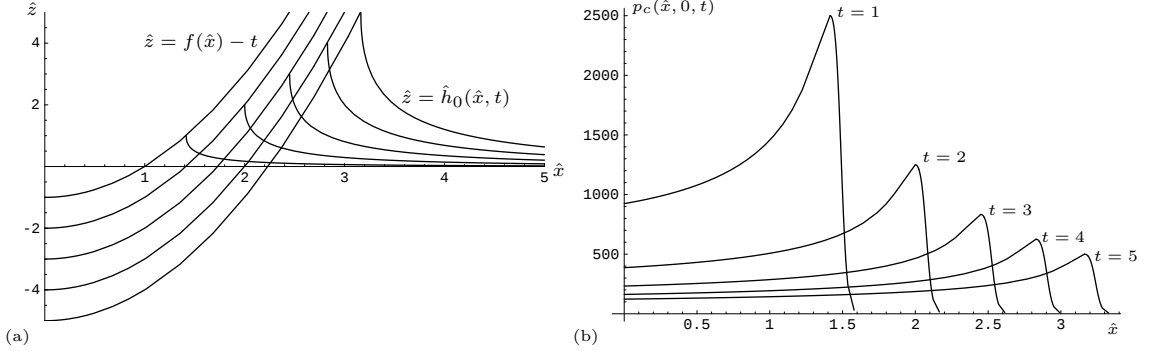


Figure 2.8: Plot of the leading-order (a) outer free surface elevation and body profile and (b) composite pressure on the body  $p_c(x, 0, t)$  (for  $\epsilon = 0.01$ ) at times  $t = 1, 2, 3, 4, 5$ .

In the Wagner region we find

$$\bar{p}_0(\bar{x}, 0, t) = \begin{cases} (-4t\bar{x}^2)^{-1/2} & \text{for } \bar{x} < 0, \\ 0 & \text{for } \bar{x} > 0, \end{cases} \quad (2.96)$$

while in the jet root region we have

$$H_0(t) = \pi \left( \frac{t}{2} \right)^{3/2}, \quad (2.97)$$

$$P_0(X_0 - (t/2)^{3/2}(s + 4\sqrt{4} + \log s), t) = \frac{1}{4t} \left[ 1 - \left( \frac{1 - \sqrt{s}}{1 + \sqrt{s}} \right) \right], \quad (2.98)$$

where  $s > 0$  and  $X_0$  is undetermined. In the right-hand jet we have

$$\tilde{\phi}_0(\tilde{x}, t) = \frac{\tilde{x}^2}{2t}, \quad \tilde{h}_0(\tilde{x}, t) = 2\pi \frac{t^4}{\tilde{x}^5}, \quad \tilde{p}_0(\tilde{x}, 0, t) = -4\pi \frac{t^2}{\tilde{x}^3} \quad \text{for } \tilde{x} \geq d_0(t); \quad (2.99)$$

we note that the leading-order force is given by

$$F_0(t) = \pi. \quad (2.100)$$

Profiles of the composite pressure on the body (for  $\epsilon = 0.01$ ) are plotted in Figure 2.8(b).

## 2.3 Extensions to the two-dimensional model

Many straightforward extensions suggest themselves. Wilson [75] considered the inclusion of gravity and surface tension, asymmetric bodies, partially submerged bodies, a non-planar initial free surface, fluid-fluid impact, variable impact speed and flat bottomed bodies, which we discuss further in Chapter 4. Morgan [50] considered  $O(1)$  and  $O(1/\epsilon)$  forward velocities, which we discuss in Chapter 5. There also exist many tough open problems, such as the fully three-dimensional case, which we consider in section 2.4.

In this section we briefly discuss a few of the more seemingly straightforward extensions, which turn out to be more difficult and interesting than they first appear.

### 2.3.1 Piecewise smooth bodies

If we allow the body to be piecewise smooth, the asymptotic structure breaks down as soon as a jet reaches a discontinuity. Since the jets do not influence the leading-order outer solution, it will remain valid until a jet root region reaches a discontinuity and the motion of the jet may be considered separately. It is not clear whether a jet root will retain its structure through a discontinuity, or separate, without performing a local analysis in space and time.

### 2.3.2 Variable impact speed and cavitation

Suppose the body is given by  $z = f(\epsilon x) - s(t)$ , where the penetration depth  $s(t)$  is differentiable, monotonic increasing and zero at  $t = 0$ . Making a change of variables from  $(x, y, t)$  to  $(x, y, s)$  with  $\phi = \dot{s} \tilde{\phi}(x, y, s)$  and  $h = \tilde{h}(x, s)$ , the leading-order outer problem (2.25 - 2.31) is the same with  $s$  replacing  $t$ . This is because time is a parameter in the outer problem except in the kinematic boundary condition on the free surface (2.27), which becomes

$$\dot{s} \frac{\partial \tilde{\phi}_0}{\partial z} = \frac{\partial}{\partial t} \tilde{h}_0(x, s(t)) = \frac{\partial \tilde{h}_0}{\partial s} \dot{s} \implies \frac{\partial \tilde{\phi}_0}{\partial z} = \frac{\partial \tilde{h}_0}{\partial s} \quad \text{on } z = 0.$$

Hence, for example, if  $f(x) = x^2$  then

$$d_0(s(t)) = \sqrt{2s(t)}, \quad (2.101)$$

by (2.94). Previous analyses with [3] and without [19, 75] this change of variables did not consider the interesting effect on the outer pressure<sup>1</sup>. By (2.40, 2.46) the leading-order outer pressure on the body in  $|x| < d_0(s)$  is

$$p_0(x, 0, t) = -\frac{\partial \phi_0}{\partial t} = -\frac{\partial}{\partial t}(\dot{s} \tilde{\phi}_0) = \frac{\ddot{s}(d_0^2 - x^2) + d_0 d_0' \dot{s}^2}{\sqrt{d_0^2 - x^2}}, \quad (2.102)$$

---

<sup>1</sup>Wilson [75] showed that the leading-order flow in the jet is unaltered, while the leading-order jet pressure on the body is modified by the addition of an inertial term proportional to  $\ddot{s}$  and the local jet thickness. This means that decelerating the body reduces the leading-order jet pressure and therefore encourages separation.

where  $d_0'(s) = \dot{d}_0(s(t))$ . The  $\ddot{s}$  term allows the pressure to become negative if the body decelerates sufficiently rapidly. Specifically, the minimum pressure  $p_0^*(t)$  occurs on the boundary of an incompressible, irrotational, inviscid flow [2] and the pressure is zero on the free surface, so (2.102) implies

$$p_0^*(t) = \min(p_0(0, 0, t), 0) = \min(d_0^2 \ddot{s} + d_0 d_0' \dot{s}^2, 0), \quad (2.103)$$

i.e. the pressure first becomes negative when  $\ddot{s}$  first becomes less than  $-d_0' \dot{s}^2 / d_0$ . For example, if  $f(x) = x^2$  the criterion for negative pressure is  $2s\ddot{s} + \dot{s}^2 < 0$  by (2.101). So, if we choose  $s(t) = 1 - e^{-t}$ , the pressure first vanishes at  $x = 0$ ,  $t = \log(3/2) \approx 0.405$  and is zero on the curve

$$x^2 = 2 - 3e^{-t} = d_0(s(t))^2 - e^{-t}. \quad (2.104)$$

Figure 2.9 shows a plot of the pressure in the  $(x, t)$ -plane, which takes its minimum value  $p_0^*(t) \approx -0.293$  at  $t = \log((7 + \sqrt{13})/4) \approx 0.975$ .

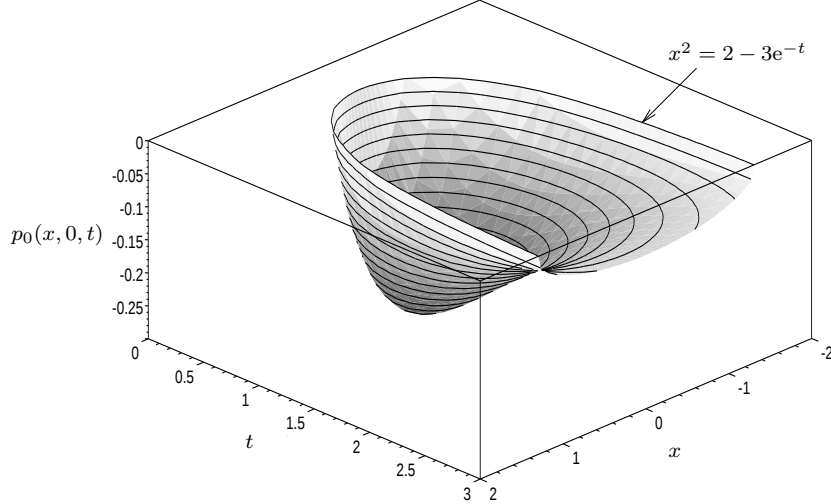


Figure 2.9: Plot of the leading-order outer pressure  $p_0(x, 0, t)$  on the parabolic body  $f(x) = x^2$  for impact speed  $\dot{s}(t) = e^{-t}$ .

If the pressure in a high Reynolds number flow drops below the vapour pressure<sup>2</sup>, microscopic pockets of vapour will suddenly expand to form small cavities of vapour. This is the phenomenon of cavitation [15] and occurs on the boundary of an inviscid flow where the lowest pressures occur [2]. The analysis above shows that the outer pressure first becomes negative on the body on the line of symmetry if the body decelerates sufficiently rapidly, so one might expect the flow could begin to cavitate there. Unfortunately, there are no experimental studies in which the body deceleration is externally controlled; most focus on a free falling body [26], which we discuss below, or the cushioning air layer trapped between the body and free surface just before impact [76], which we discuss in Appendix B. However, it is easy to think up a situation where the body is forced to decelerate faster than the fluid pressure could sustain, such as an alighting seaplane, which experiences a significant increase in lift as it alights due to the extreme “ground-effect” [63]. Cavitation can cause extensive damage to a surface because very large fluid velocities and forces are

<sup>2</sup>The vapour pressure is strictly negative because we chose the pressure on the free surface to be zero.

produced when a cavity collapses. It is therefore important to investigate the formation and evolution of cavities formed by rapid deceleration.

### 2.3.3 Free falling body

Consider a free falling body for which the leading-order law of motion is

$$-\hat{m}\ddot{s} = F_0(t) - \hat{g}, \quad (2.105)$$

provided  $\hat{m} = m/\epsilon^2 \rho L^2$  and  $\hat{g} = gL/\epsilon^2 V^2$  are order unity. Here,  $m$  is the mass of the body per unit length and the leading-order force per unit length is

$$F_0(t) = \int_{-d_0}^{d_0} p_0(x, 0, t) dx = \frac{1}{2} \pi d_0(s)^2 \ddot{s} + \pi d_0(s) d_0'(s) \dot{s}^2, \quad (2.106)$$

by (2.102). For the locally parabolic body  $f(x) = x^2$  we substitute (2.101, 2.106) into (2.105) and integrate to find

$$s(t) = \frac{\hat{m}}{\pi} \left( \sqrt{1 + \frac{2\pi \dot{s}(0)t}{\hat{m}} + \frac{\pi \hat{g} t^2}{\hat{m}^2}} - 1 \right). \quad (2.107)$$

The body has pseudo terminal velocity  $\sqrt{\hat{g}/\pi}$  (because  $d_0(s)d_0'(s)$  is constant), and substitution into (2.103) shows that  $p_0^* > 0$  for  $t \geq 0$ . Generalizing, if  $f(x) \sim O(|x|^\alpha)$  as  $|x| \rightarrow \infty$  where  $\alpha \geq 1$ , then

$$s(t) \sim \begin{cases} O(t^{2\alpha/(2+\alpha)}) & \text{for } \alpha < 2, \\ O(t^{3\alpha/(2+2\alpha)}) & \text{for } \alpha > 2, \end{cases} \quad (2.108)$$

which means the body decelerates at sufficiently large times except for  $\alpha = 2$ . The leading-order body force  $F_0$  is therefore positive which suggests  $p_0^* > 0$ , though we do not pursue the analysis further.

## 2.4 The three-dimensional model

We introduce the horizontal  $y$ -coordinate and suppose that the rigid, strictly convex and smooth body has equation  $z = f(\epsilon x, \epsilon y) - t$  where  $f(0, 0) = 0$ . The two-dimensional asymptotic structure described in section 2.2 generalizes directly to the three-dimensional case. The extra dimension implies the free surface now turns over to form a high velocity spray sheet. We denote the projection of all turnover points onto the  $(x, y)$ -plane by  $\omega(\epsilon x, \epsilon y) = t$ , having assumed this ‘turnover curve’  $\partial\Omega(t)$  passes any given point at most once. Proceeding as in the two-dimensional case, we find the structure and matching hold provided the turnover curve  $\partial\Omega(t)$  is (i) smooth and has (ii) radius of curvature much larger than the order unity cross-sectional size of the inner Wagner region, both of which can only be verified *a posteriori*. In this case, the flow in the Wagner and jet root regions is quasi-two-dimensional in all planes perpendicular to the turnover curve, with exactly the same structure, matching and solution as in the two-dimensional case; the flow is simply parametrized by the instantaneous arc length  $s$  along  $\partial\Omega(t)$ . The resulting leading-order codimension-two<sup>3</sup> free

<sup>3</sup>The turnover curve has dimension two fewer than the three-dimensional space in which the motion occurs.

boundary problem for the leading-order outer velocity potential  $\phi_0(x, y, z, t)$ , free surface elevation  $z = h_0(x, y, t)$  and turnover curve  $\omega_0(x, y) = t$  (in the  $z$ -plane) is

$$\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad \text{for } z < 0, \quad (2.109)$$

$$\frac{\partial \phi_0}{\partial z} = -1 \quad \text{for } z = 0, \omega_0(x, y) < t, \quad (2.110)$$

$$\frac{\partial \phi_0}{\partial z} = \frac{\partial h_0}{\partial t} \quad \text{for } z = 0, \omega_0(x, y) > t, \quad (2.111)$$

$$\phi_0 = 0 \quad \text{for } z = 0, \omega_0(x, y) > t, \quad (2.112)$$

$$\phi_0 \rightarrow 0 \quad \text{as } x^2 + y^2 + z^2 \rightarrow \infty, \quad (2.113)$$

$$h_0 \rightarrow 0 \quad \text{as } x^2 + y^2 \rightarrow \infty, \quad (2.114)$$

together with initial conditions

$$\phi_0(x, y, z, 0) = h_0(x, y, 0) = \omega_0(0, 0) = 0. \quad (2.115)$$

In addition, we must (i) demand that  $\phi_0$  behaves as the square root of distance  $R$  from the turnover curve  $\omega_0(x, y) = t$  as  $R \rightarrow 0$  in all planes  $\Pi(x, y, t)$  perpendicular to it, viz.

$$\begin{aligned} \phi_0(X, Y, Z, t) &\sim S(x, y, t) R^{1/2} \text{ as } R^2 = (X - x)^2 + (Y - y)^2 + Z^2 \rightarrow 0 \\ &\text{with } (X, Y, Z) \in \Pi(x, y, t) \text{ for all } (x, y) \in \partial\Omega_0(t), \end{aligned} \quad (2.116)$$

and (ii) specify the law of motion of the turnover curve through the Wagner condition

$$h_0(x, y, t) = f(x, y) - t \quad \text{on } \omega_0(x, y) = t. \quad (2.117)$$

The force on the body  $F(t) \sim F_0/\epsilon^3 + F_1/\epsilon^{5/2} + \dots$  where

$$F_0 = - \iint_{\Omega_0(t)} \frac{\partial \phi_0}{\partial t} dx dy, \quad (2.118)$$

and  $F_1(t)$  is a function of the order unity correction to the location of the turnover curve. As in the two-dimensional case, this necessitates the solution of the first-order outer problem.

The third dimension makes the hunt for analytic solutions much more difficult. In particular, the only known exact solutions are for a body with rotational symmetry [2, 56, 75] and recently for bodies that yield elliptic turnover curves [40]. For comparison with the corresponding finite depth solutions in the next chapter, we review the direct solution method for the axisymmetric case and the inverse method for the entry of an elliptic paraboloid, for which we also take the slender limit for comparison with strip theory. First though we briefly analyse the spray sheet problem.

#### 2.4.1 The spray sheet

The spray sheet is ejected from the jet root region which has size of  $O(\epsilon)$  and lies on the body an  $O(1)$  distance from the leading-order turnover curve,  $\omega_0(x, y) = t$ ,  $z = 0$ . In particular, to lowest order the spray sheet is ejected with velocity

$$\frac{2v_n}{\epsilon} \mathbf{n},$$

where  $v_n$  is the normal velocity of the leading-order turnover curve, which has outward pointing unit normal  $\mathbf{n}$  (in the  $(x, y)$ -plane) and thickness

$$\epsilon \frac{\pi S^2}{16v_n^2},$$

where  $S(x, y, t)$  is given by (2.116). The spray sheet therefore has thickness of order  $\epsilon$  and length of order  $1/\epsilon$ , which is much smaller than the impactor's radius of curvature of order  $1/\epsilon^2$  or larger. Hence, as in the two-dimensional case in section 2.4.4, the flow in the spray sheet is governed by the planar zero-gravity shallow-water equations. If  $(\hat{x}, \hat{y})$  are surface based orthogonal curvilinear coordinates (scaled with  $1/\epsilon$ ), then the leading-order potential  $\hat{\phi}_0(\hat{x}, \hat{y}, t)$  (scaled with  $1/\epsilon$ ) and the normal distance of the free surface from the body  $\hat{h}_0(\hat{x}, \hat{y}, t)$  (scaled with  $\epsilon$ ) satisfy

$$\frac{\partial \hat{\phi}_0}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \hat{\phi}_0}{\partial \hat{x}} \right)^2 + \left( \frac{\partial \hat{\phi}_0}{\partial \hat{y}} \right)^2 \right] = 0, \quad (2.119)$$

$$\frac{\partial \hat{h}_0}{\partial t} + \frac{\partial}{\partial \hat{x}} \left( \frac{\partial \hat{\phi}_0}{\partial \hat{x}} \hat{h}_0 \right) + \frac{\partial}{\partial \hat{y}} \left( \frac{\partial \hat{\phi}_0}{\partial \hat{x}} \hat{h}_0 \right) = 0. \quad (2.120)$$

Without loss of generality, we may take the coordinates  $(\hat{x}, \hat{y})$  to be cartesian with the same origin and orientation as in the outer problem. Then, matching with the jet root region implies the boundary data is (dropping hats on  $x$  and  $y$ )

$$\nabla \hat{\phi}_0 = 2v_n \mathbf{n} \quad \text{and} \quad \hat{h}_0 = \frac{\pi S^2}{16v_n^2} \quad \text{on} \quad \omega_0(x, y) = t. \quad (2.121)$$

The first order hyperbolic equations (2.119, 2.120) are similar to the eikonal and amplitude equation in the ray theory approach to geometrical optics [6]. In particular, they decouple and may be solved successively using Charpit's method [54], in which the characteristics are particle paths.

Given sufficient regularity, the boundary data (2.121) is sufficient to determine the solution sufficiently near the turnover curve, because the particle paths can never be parallel to the boundary surface  $t = \omega_0(x, y)$  in  $(x, y, t)$ -space (the particles have twice the velocity of the turnover curve) and the sheet does not exist at  $t = 0$  (so there are no characteristics emanating from  $t = 0$  in  $(x, y, t)$ -space). Specifically, if we parametrize the turnover curve by  $\mathbf{x}_0(\mu, \nu)$  as shown in Figure 2.10, in which  $\mu$  is a spacial coordinate and  $\nu$  is time, and employ Charpit's method [54], with  $\tau$  parametrizing distance along the straight characteristics, we find

$$\mathbf{x} = \mathbf{x}_0(\mu, \nu) + 2v_n(\mu, \nu)\tau\mathbf{n}(\mu, \nu), \quad (2.122)$$

$$t = \nu + \tau, \quad (2.123)$$

$$\hat{\phi}_0 = \hat{\phi}_0(\mu, \nu, 0) + 2v_n(\mu, \nu)^2\tau, \quad (2.124)$$

$$\hat{h}_0 = \frac{\pi S(\mathbf{x}_0, \nu)^2}{16v_n(\mu, \nu)^2} \exp \left( - \int_0^\tau \nabla^2 \hat{\phi}_0(\mu, \nu, \tau') d\tau' \right). \quad (2.125)$$

Here, the boundary value of the potential  $\hat{\phi}_0(\mu, \nu, 0)$  is found by integrating  $\partial \hat{\phi}_0 / \partial x$ ,  $\partial \hat{\phi}_0 / \partial y$  and  $\partial \hat{\phi}_0 / \partial t$ , which are obtained from the boundary data (2.121) and the partial differential equation (2.119).

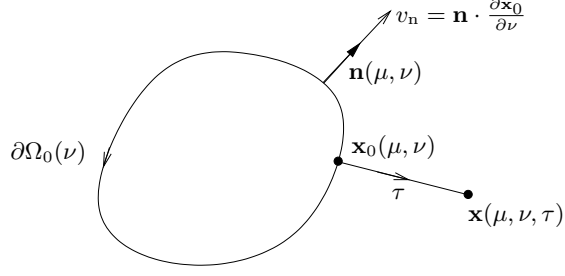


Figure 2.10: Terminology for the spray sheet boundary data.

The domain of definition of the solution is bounded by the curves on which the Jacobian,

$$J(\mu, \nu, \tau) = \frac{\partial(x, y, t)}{\partial(\mu, \nu, \tau)}, \quad (2.126)$$

vanishes. However, if the turnover curve is not convex, then the characteristics may cross within the domain of definition, as in the simple sandpile model [54]. Hence, for the remainder of this section, we confine ourselves to the case in which the turnover curve is convex and briefly investigate the solution and its domain of definition. We emphasise that this is highly speculative, because we also assume that the spray sheet does not separate.

Whenever the Jacobian vanishes the characteristics intersect and the solution becomes multivalued. We expect the sheet thickness to become unbounded as the particle paths collide by conservation of mass. This is exactly what happens to the amplitude as a caustic is approached in the ray theory of geometric optics [6]. Following [6], we differentiate the coordinates  $(x, y, t)$ , defined by (2.122, 2.123), with respect to  $x$ ,  $y$  and  $t$  to find after manipulation

$$\nabla \cdot \nabla \phi = \frac{\partial}{\partial \tau} \log J(\mu, \nu, \tau),$$

and therefore by (2.125)

$$h(\mu, \nu, \tau) = h(\mu, \nu, 0) \frac{J(\mu, \nu, 0)}{J(\mu, \nu, \tau)}, \quad (2.127)$$

which verifies our assertion.

In the two-dimensional and axisymmetric cases, (2.122 - 2.125) and the Serret-Frenet formulae imply the Jacobian

$$J = v_n - 2 \left( \kappa v_n^2 + \frac{\partial v_n}{\partial \nu} \right) \tau + 4 \kappa v_n \frac{\partial v_n}{\partial \nu} \tau^2,$$

where  $\kappa \leq 0$  is the curvature of the convex turnover curve, which has acceleration  $\partial v_n / \partial \nu$  in the normal direction. Hence, the Jacobian is always positive and no shocks form if the convex turnover curve is expanding ( $v_n \geq 0$ ) and decelerating ( $\partial v_n / \partial \nu \leq 0$ ).

To lowest order the spray sheet is only affected by the body profile through the location of the turnover curve because this implicitly determines the strength of the square root singularity  $S(x, y, t)$  in (2.116). However, the leading-order pressure is given by

$$p \sim \epsilon \left[ \kappa_1 \left( \frac{\partial \hat{\phi}_0}{\partial x} \right)^2 + \kappa_2 \left( \frac{\partial \hat{\phi}_0}{\partial y} \right)^2 \right] (\hat{h}_0 - m),$$

where  $\kappa_1, \kappa_2$  are the principal curvatures of the body surface (scaled with  $\epsilon^2$ ) in the  $x$ - and  $y$ -directions, respectively, and  $m$  is the normal distance from the body surface (scaled with  $\epsilon$ ). We conclude that the remarks made at the end of section 2.2.4 concerning the two-dimensional case also hold in the three-dimensional case.

### 2.4.2 Bodies with rotational symmetry

Taking cylindrical polar coordinates  $(r, \theta, z)$  and a body  $f = f(r)$ , Schmieden [56] solved the leading-order outer problem (2.109 - 2.117) summarized in Figure 2.11 for  $\phi_0(r, z, t)$ ,  $h_0(r, t)$  and the leading-order location of the turnover curve  $r = d_0(t)$ . In addition, at the turnover curve we require (2.116), which becomes

$$\phi_0 \sim O(R^{1/2}) \quad \text{as} \quad R^2 = (r - d_0(t))^2 + z^2 \rightarrow 0, \quad (2.128)$$

while the Wagner condition (2.117) becomes

$$h_0(d_0(t), t) = f(d_0(t)) - t \quad \text{for} \quad t \geq 0. \quad (2.129)$$

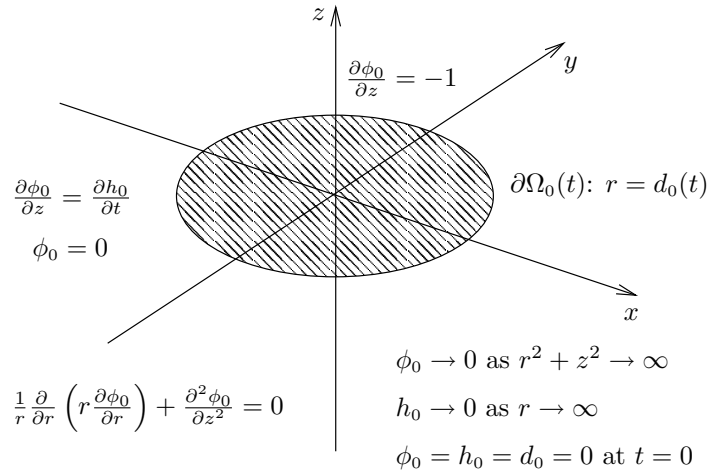


Figure 2.11: The axisymmetric leading-order outer problem.

Seeking a separable solution in  $r$  and  $z$  implies the general axisymmetric solution to Laplace's equation in  $z < 0$  is

$$\phi_0(r, z, t) = \int_0^\infty C(k, t) e^{-kz} J_0(kr) dk, \quad (2.130)$$

where  $J_\nu$  is the Bessel function of the first kind of order  $\nu$ . The function  $C$  is determined by substituting (2.130) into the (2.110, 2.111) to obtain the dual pair of integral equations, viz.

$$\int_0^\infty k C(k, t) J_0(kr) dk = -1 \quad \text{for} \quad r < d_0(t), \quad (2.131)$$

$$\int_0^\infty C(k, t) J_0(kr) dk = 0 \quad \text{for} \quad r > d_0(t), \quad (2.132)$$

Solving these dual integral equations for  $C$ , as described by [59], we obtain

$$\phi_0(r, z, t) = - \left( \frac{2d_0(t)^3}{\pi} \right)^{1/2} \int_0^\infty \frac{e^{-kz}}{k^{1/2}} J_{3/2}(kr) J_0(kr) dk, \quad (2.133)$$

and therefore for  $r > d_0(t)$ ,

$$\frac{\partial \phi_0}{\partial z}(r, 0, t) = \frac{2}{\pi} \left[ \frac{d_0(t)}{(r^2 - d_0(t)^2)^{1/2}} - \sin^{-1} \left( \frac{d_0(t)}{r} \right) \right]. \quad (2.134)$$

The free surface  $h_0$  is given by (2.111, 2.114), so the Wagner condition (2.129) yields the law of motion

$$\frac{2}{\pi} \int_0^t \left\{ \frac{d_0(\tau)}{(d_0(t)^2 - d_0(\tau)^2)^{1/2}} - \sin^{-1} \left( \frac{d_0(\tau)}{d_0(t)} \right) \right\} d\tau = f(d_0(t)) - t, \quad (2.135)$$

which may be solved [75], as in section 2.2.1.7, to give

$$t = \int_0^{\pi/2} f(d_0(t) \sin \theta) \sin \theta d\theta, \quad (2.136)$$

having applied the zero initial condition (2.115). By (2.46), the leading-order outer pressure on the body is  $p_0/\epsilon$  where

$$p_0(r, 0, t) = \frac{2}{\pi} \frac{d_0(t) \dot{d}_0(t)}{(d_0(t)^2 - r^2)^{1/2}} \quad \text{for } r < d_0(t), \quad (2.137)$$

which is  $2/\pi$  times the two-dimensional result (2.48) with  $r$  replacing  $x$ . The leading-order force on the body is  $F_0/\epsilon^3$  where

$$F_0(t) = \int_0^{2\pi} \int_0^{d_0(t)} p_0(r, 0, t) r dr d\theta = 4d_0(t)^2 \dot{d}_0(t). \quad (2.138)$$

The next order term due to the Wagner region is  $F_1/\epsilon^{5/2}$  where

$$F_1(t) = -2\pi d_0^{3/2} \dot{d}_0 \cdot \Im \left[ \sqrt{d_1 + i(f(d_0) - t)} \right], \quad (2.139)$$

assuming the first-order corrections to the outer velocity potential,  $\phi(r, z, t)$ , and location of the turnover curve,  $r = d(t)$ , are  $O(\epsilon)$ , viz.  $\phi \sim \phi_0 + \epsilon \phi_1 + \dots$  and  $d \sim d_0 + \epsilon d_1 + \dots$ .

### 2.4.3 Entry of an elliptic paraboloid

Recently Korobkin [40] used an inverse method to construct an exact similarity solution for the entry of the elliptic paraboloid

$$f(x, y) = \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2, \quad (2.140)$$

where  $a > b$  say, moving with unit velocity in the negative  $z$ -direction. The key to the inverse method is to specify the location of the turnover curve and then construct the body profile using the Wagner condition (2.117). Suppose the turnover curve may be written as  $\omega_0(x, y) = t$ , then (2.117) implies the body profile is

$$f(x, y) = \omega_0(x, y) + h_0(x, y, \omega_0(x, y)). \quad (2.141)$$

The method works *provided* the potential problem (2.109, 2.110, 2.112, 2.113, 2.116) for  $\phi_0$  can be solved on all contours of  $\omega_0(x, y)$ , so that  $h_0$  exists. Of course we must then check that the resulting body shape  $f(x, y)$  is physically acceptable; smooth and monotonic increasing with radial distance from the origin is usually sufficient.

Korobkin derived the solution to the potential problem with the general elliptic turnover curve

$$\left(\frac{x}{A(t)}\right)^2 + \left(\frac{y}{B(t)}\right)^2 = 1, \quad (2.142)$$

where  $A(t) > B(t)$  are specified, by taking the flat-disc limit of the solution for the ellipsoid,

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 + \left(\frac{z}{C}\right)^2 = 1,$$

moving with unit velocity in the negative  $z$ -direction in an unbounded ideal fluid at rest at infinity; see for example [48]. Since  $z = \pm C$  when  $x = y = 0$ , the appropriate limit is  $C \rightarrow 0$  and the result is

$$\phi_0(x, y, z, t) = \frac{AB^2z}{2\mathcal{E}(E)} \int_{\lambda(x, y, z)}^{\infty} \frac{d\xi}{\xi^{3/2} \sqrt{(A^2 + \xi)(B^2 + \xi)}}, \quad (2.143)$$

where  $E(t) = \sqrt{1 - B^2/A^2}$  is the eccentricity of the turnover curve,  $\lambda(x, y, z)$  is the non-negative root of the cubic equation

$$\frac{x^2}{A^2 + s} + \frac{y^2}{B^2 + s} + \frac{z^2}{s} = 1, \quad (2.144)$$

and

$$\mathcal{E}(E) = \int_0^{\pi/2} \sqrt{1 - E^2 \sin^2 \theta} d\theta,$$

is an elliptic integral of the first kind. The three roots  $\nu$ ,  $\mu$  and  $\lambda$  of (2.144) lie in  $[-A^2, -B^2]$ ,  $[B^2, 0]$  and  $[0, \infty)$ , respectively, and form a system of (orthogonal) confocal ellipsoidal coordinates. Successively fixing  $\nu$ ,  $\mu$  and  $\lambda$  results in a hyperboloid of two sheets, a hyperboloid of one sheet and an ellipsoid, respectively, in  $(x, y, z)$ -space. These quadratic surfaces are orthogonal and span  $(x, y, z)$ -space as the roots vary over their respective ranges.

One may check that the solution does indeed solve the potential problem (2.109, 2.110, 2.112, 2.113, 2.116). The body profile is constructed using (2.141) by calculating (i) the vertical velocity  $\partial\phi_0/\partial z$  on the free surface ( $z = 0$ ,  $x^2/A^2 + y^2/B^2 > 1$ ) from (2.143) and therefore (ii) the free surface elevation  $h_0$  through the kinematic boundary condition (2.111) and the initial condition (2.115).

Korobkin [40] considers several examples but we consider the specific case in which

$$A(t) = A_0 t^{1/2}, \quad B(t) = B_0 t^{1/2},$$

where  $A_0$ ,  $B_0$  and therefore the eccentricity  $E_0 = \sqrt{1 - B_0^2/A_0^2}$  are constant. The body shape is then the elliptic paraboloid (2.140) with major and minor semi-axes given by

$$a = A_0 [1 + (1 - E_0^2)\mathcal{D}(E_0)/\mathcal{E}(E_0)]^{-1/2}, \quad (2.145)$$

$$b = B_0 [2 - (1 - E_0^2)\mathcal{D}(E_0)/\mathcal{E}(E_0)]^{-1/2}, \quad (2.146)$$

where  $\mathcal{D}(E_0) = (\mathcal{K}(E_0) - \mathcal{E}(E_0))/E_0^2$  and

$$\mathcal{K}(E_0) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - E_0^2 \sin^2 \theta}},$$

is an elliptic integral of the first kind. The eccentricity  $e$  of a cross-section of the impacting elliptic paraboloid (2.140) is therefore given by

$$e^2 = 1 - b^2/a^2 = 1 - (1 - E_0^2) \frac{1 + (1 - E_0^2)\mathcal{D}(E_0)/\mathcal{E}(E_0)}{2 - (1 - E_0^2)\mathcal{D}(E_0)/\mathcal{E}(E_0)}. \quad (2.147)$$

We plot  $E_0$  against  $e$  in Figure 2.12(a) which shows  $e > E_0$  for  $0 < e < 1$ , so the elliptic turnover curve is less elongated than the elliptic cross-section of the body (i.e.  $b/a < B_0/A_0$ ). This is illustrated in Figure 2.12(b) where we plot the turnover curve and water line (where the body intersects  $z = 0$ ) at time  $t = 1$  for  $A_0 = 1.0$ ,  $E_0 = 0.99$ ; by (2.147) the eccentricity of the water line is  $e \approx 0.9947$ .

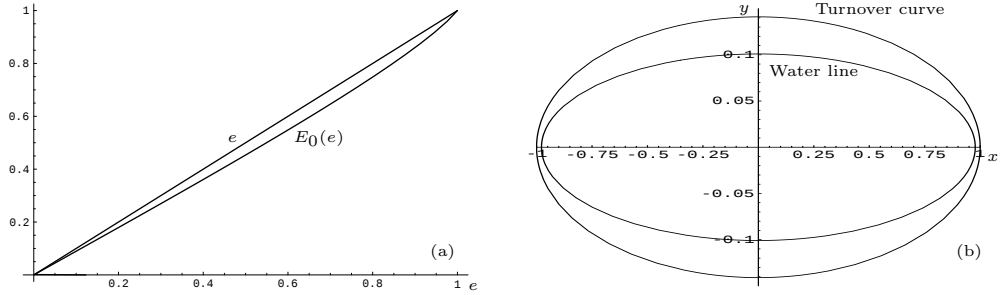


Figure 2.12: (a) Plot of the eccentricity  $E_0$  of the turnover curve as a function of the eccentricity  $e$  of a cross-section of the entering elliptic paraboloid. (b) Plot of the turnover curve and waterline ( $f(x, y) = t$ ) at time  $t = 1$  for  $A_0 = 1.0$  and  $E_0 = 0.99$ .

The solution is a similarity solution in which distance scale with  $t^{1/2}$ . The potential on the body in the interior region,  $x^2/A_0^2 + y^2/B_0^2 < t$ , is

$$\phi_0(x, y, 0, t) = -\frac{B_0}{\mathcal{E}(E_0)} \sqrt{t - \frac{x^2}{A_0^2} - \frac{y^2}{B_0^2}}, \quad (2.148)$$

so the leading-order force on the body [40] is  $F_0/\epsilon^3$  where

$$F_0(t) = - \iint_{x^2/A_0^2 + y^2/B_0^2 < t} \frac{\partial \phi_0}{\partial t}(x, y, 0, t) dx dy = \frac{\pi A_0 B_0^2 t^{1/2}}{\mathcal{E}(E_0)}. \quad (2.149)$$

The results (2.148, 2.149) are in agreement [40] with both the axisymmetric case as  $E_0 \rightarrow 0$  and the two-dimensional case as  $B_0 \rightarrow 0$ ,  $E_0 \rightarrow 1$  in cross-sections *not too close* to  $x = \pm A_0 t^{1/2}$ . We now quantify this distance by formally expanding the exact solution (2.143) as  $k \rightarrow 0$  where  $B = kA$ .

#### 2.4.4 Entry of a slender elliptic paraboloid

We will show that the outer flow structure due to (2.143) decomposes into the four regions shown in Figure 2.13 as  $k \rightarrow 0$  where  $B = kA$ . The turnover curve is

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{kA}\right)^2 = t, \quad z = 0,$$

so there is a thin *strip* region of length of  $O(A)$  and thickness of  $O(kA)$ , in which the leading-order flow is quasi-two-dimensional in all planes perpendicular to the  $x$ -axis with  $|x| < A$ . On the lengthscale  $A$  of the outer region the strip region is a cut along the  $x$ -axis between  $x = \pm A$ . To lowest order the outer flow is driven by matching with the strip region, which results in a continuous distribution of dipoles on the cut. The leading-order outer and strip solutions breakdown near to  $x = \pm A$ , where fully three-dimensional inner regions are required. The variation of the body in the inner region must therefore be of the same order in both the  $x$ - and  $y$ -directions, which implies the inner regions have size of order  $k^2A$ . We will show that the leading-order solutions in these regions match, which implies the asymptotic decomposition is complete to lowest order. In section 2.4.5, we discuss the implications for strip theory for general slender body profiles.

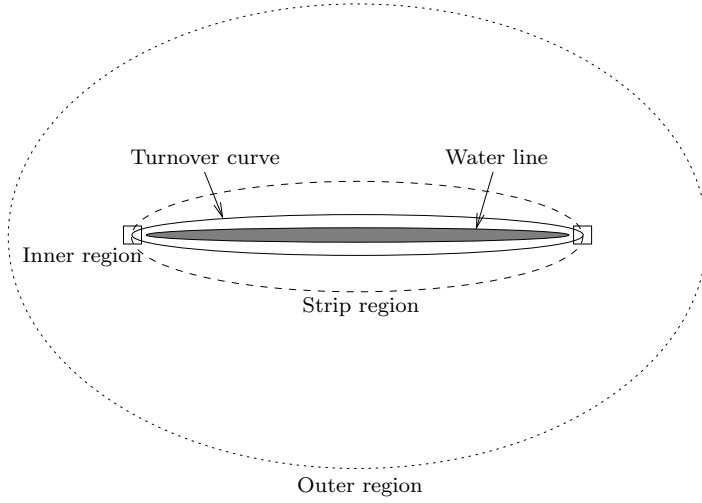


Figure 2.13: Asymptotic structure of the leading-order outer problem for the entry of a slender elliptic paraboloid.

#### 2.4.4.1 Scalings

To find the scalings for the confocal ellipsoidal coordinates  $(\lambda, \mu, \nu)$  in each region, we demand a non-trivial balance in the following geometric identities

$$x^2 = \frac{(A^2 + \lambda)(A^2 + \mu)(A^2 + \nu)}{A^2(A^2 - B^2)}, \quad (2.150)$$

$$y^2 = -\frac{(B^2 + \lambda)(B^2 + \mu)(B^2 + \nu)}{B^2(A^2 - B^2)}, \quad (2.151)$$

$$z^2 = \frac{\lambda\mu\nu}{A^2B^2}. \quad (2.152)$$

To find the resulting leading-order coordinate system in each region we use the characteristic equation (2.144), whose three roots are  $(\lambda, \mu, \nu)$  with

$$-A^2 \leq \nu \leq -B^2 \leq \mu \leq 0 \leq \lambda. \quad (2.153)$$

Since  $B = kA$  where  $k \ll 1$  we must scale  $\mu$  with  $k^2A$  in all regions.

#### 2.4.4.2 The outer region

The outer region scalings are

$$\begin{aligned} x &= Ax_1, & y &= Ay_1, & z &= Az_1, \\ \lambda &= A^2\lambda_1, & \mu &= k^2 A^2\mu_1, & \nu &= A^2\nu_1, \end{aligned} \quad (2.154)$$

where by (2.153),

$$-1 \leq \nu_1, \mu_1 \leq 0 \leq \lambda_1. \quad (2.155)$$

We substitute the scalings (2.154, 2.154) into the coordinate definitions (2.150 - 2.152) and expand all coordinates as asymptotic series in powers of  $k^2$ , viz.

$$x_1 \sim x_{1,0} + k^2 x_{1,1} + \dots, \text{ etc,}$$

to find at lowest order

$$x_1^2 = (1 + \lambda_1)(1 + \nu_1), \quad (2.156)$$

$$y_1^2 = -\lambda_1(1 + \mu_1)\nu_1, \quad (2.157)$$

$$z_1^2 = \lambda_1\mu_1\nu_1, \quad (2.158)$$

where, here and throughout the rest of section 2.4.4, we drop second subscripts. Scaling  $s = A^2 s_1$ , where  $s_1$  is order unity, the characteristic equation (2.144) becomes to lowest order

$$\frac{x_1^2}{1 + s_1} + \frac{y_1^2 + z_1^2}{s_1} = 1. \quad (2.159)$$

Therefore, successively fixing  $\lambda_1 \in [0, \infty)$  and  $\nu_1 \in [-1, 0]$  results in an ellipsoid and a hyperboloid of two sheets, respectively, in  $(x_1, y_1, z_1)$ -space. Similarly, scaling  $s = k^2 A^2 s_1$ , where  $s_1$  is order unity, we find at lowest order

$$\frac{y_1^2}{1 + s_1} + \frac{z_1^2}{s_1} = 0, \quad (2.160)$$

so fixing  $\mu_1 \in [-1, -1]$  results in two planes that intersect on the  $x_1$ -axis and make angles  $\theta$  and  $\pi - \theta$  with the positive  $y_1$ -axis in the first and second quadrant, where

$$\theta = \tan^{-1} \sqrt{\frac{-\mu_1}{1 + \mu_1}} \in [0, \pi/2]. \quad (2.161)$$

The quadratic surfaces span  $(x_1, y_1, z_1)$ -space as the roots vary over their respective ranges (2.155) and  $(\lambda, \nu, \theta)$  are orthogonal prolate spheroidal coordinates.

To find the leading-order outer velocity potential we begin by substituting the outer scalings (2.154) into (2.143) to find, upon substituting  $\xi = A^2\eta$ ,

$$\phi_0 = \frac{k^2 A z_1}{2\mathcal{E}(\sqrt{1 - k^2})} \int_{\lambda_1}^{\infty} \frac{d\eta}{\eta^{3/2} \sqrt{(1 + \eta)(k^2 + \eta)}}. \quad (2.162)$$

Hence, we scale

$$\phi_0 = k^2 A \phi_0^{(1)} \quad (2.163)$$

and then expand the integral as an asymptotic series in powers of  $k^2$  to find at lowest order (dropping second subscripts)

$$\phi_0^{(1)} = \frac{z_1}{2} \int_{\lambda_1}^{\infty} \frac{d\eta}{\eta^2 \sqrt{1+\eta}},$$

provided  $\lambda_1 \gg k^2$ . Evaluating the integral, we find the leading-order outer velocity potential

$$\phi_0^{(1)} = \frac{1}{2} \left( \sqrt{(1+1/\lambda_1)\mu_1\nu_1} - \sqrt{\lambda_1\mu_1\nu_1} \sinh^{-1}(1/\sqrt{\lambda_1}) \right). \quad (2.164)$$

It is straightforward to verify that  $\phi_0^{(1)}$  is an eigensolution of the leading-order outer potential problem, which is obtained by substituting the scalings (2.154, 2.163) and  $B = kA$  into the linearized potential problem (2.109, 2.112, 2.116) and expanding as usual; thus

$$\frac{\partial^2 \phi_0^{(1)}}{\partial x_1^2} + \frac{\partial^2 \phi_0^{(1)}}{\partial y_1^2} + \frac{\partial^2 \phi_0^{(1)}}{\partial z_1^2} = 0 \quad \text{for } z_1 < 0, \quad (2.165)$$

$$\phi_0^{(1)} = 0 \quad \text{on } z_1 = 0 \text{ away from the cut}, \quad (2.166)$$

$$\phi_0^{(1)} \rightarrow 0 \quad \text{as } x_1^2 + y_1^2 + z_1^2 \rightarrow \infty. \quad (2.167)$$

We now find the local behaviour on the  $x_1$ -axis. Expanding (2.156 - 2.158, 2.164) as  $\lambda_1 \rightarrow 0$  for fixed  $\mu_1, \nu_1 \in (-1, 0)$  and *then* expanding as  $y_1^2 + z_1^2 \rightarrow 0$  we find

$$\phi_0^{(1)} \sim \frac{(1-x_1^2)z_1}{2(y_1^2+z_1^2)} \quad \text{for } |x_1| < 1. \quad (2.168)$$

Similarly, expanding as  $\nu_1 \rightarrow 0$  for fixed  $\mu_1 \in (-1, 0)$ ,  $\lambda_1 \in (0, \infty)$  and then expanding as  $y_1^2 + z_1^2 \rightarrow 0$ , we find

$$\phi_0^{(1)} \sim \frac{z_1}{2} \left( \frac{x_1}{x_1^2-1} - \sinh^{-1} \frac{1}{\sqrt{x_1^2-1}} \right) \quad \text{for } |x_1| > 1. \quad (2.169)$$

Finally, expanding as  $\lambda_1 \sim \nu_1 \rightarrow 0$  for fixed  $\mu_1 \in (-1, 0)$ ,

$$\phi_0^{(1)} \sim \frac{\sqrt{\lambda_1\mu_1\nu_1}}{2} \left( \frac{1}{\lambda_1} + \frac{1}{2} \log \lambda_1 \right) + O\left(\sqrt{\lambda_1\mu_1\nu_1}\right), \quad (2.170)$$

(which will be required for matching with the inner region below) and then expanding as  $(x_1 \mp 1)^2 + y_1^2 + z_1^2 \rightarrow 0$ , we find

$$\phi_0^{(1)} \sim \frac{z_1}{(x_1 - 1 + \sqrt{(x_1 \mp 1)^2 + y_1^2 + z_1^2})}. \quad (2.171)$$

The local leading-order velocity potentials (2.168, 2.169, 2.171) are eigensolutions of the corresponding leading-order local problem, i.e. they are harmonic on  $z_1 < 0$  and satisfy  $\phi_0^{(1)} = 0$  on  $z_1 = 0$  away from the cut. Therefore, it is reasonable that the outer solution will be in the form of a continuous distribution of dipoles on the cut with axes in the  $z_1$ -direction. It is necessary to match with the leading-order strip region potential through (2.168) and with the leading-order inner region potential through (2.170).

#### 2.4.4.3 The strip region

The strip region scalings are

$$\begin{aligned} x &= Ax_2, & y &= kAy_2, & z &= kAz_2, \\ \lambda &= k^2 A^2 \lambda_2, & \mu &= k^2 A^2 \mu_2, & \nu &= A^2 \nu_2, \end{aligned} \quad (2.172)$$

where by (2.153),

$$-1 \leq \nu_2, \mu_2 \leq 0 \leq \lambda_2. \quad (2.173)$$

We expand as in the outer region to find at lowest order

$$x_2^2 = 1 + \nu_2, \quad (2.174)$$

$$y_2^2 = -(1 + \lambda_2)(1 + \mu_2)\nu_2, \quad (2.175)$$

$$z_2^2 = \lambda_2 \mu_2 \nu_2. \quad (2.176)$$

Scaling  $s = A^2 s_2$ , where  $s_2$  is order unity, the characteristic equation (2.144) becomes to lowest order

$$x_2^2 + \frac{y_2^2}{1 + s_2} + \frac{z_2^2}{s_2} = 1. \quad (2.177)$$

Therefore, successively fixing  $\lambda_2 \in [0, \infty)$  and  $\mu_2 \in [-1, 0]$  results in an ellipsoid and a hyperboloid of one sheet, respectively, in  $(x_2, y_2, z_2)$ -space. It is clear from (2.174) that fixing  $\nu_2 \in [-1, 0]$  results in two planes perpendicular to the  $x_2$ -axis. The quadratic surfaces and the planes span  $|x_2| < 1$  and  $(\lambda_2, \mu_2, \nu_2)$  are elliptical cylindrical coordinates.

To find the leading-order strip velocity potential we substitute the strip region scalings (2.172) into the exact velocity potential (2.143) to find, upon substituting  $\xi = k^2 A^2 \eta$ ,

$$\phi_0 = \frac{kAz_2}{2\mathcal{E}(\sqrt{1-k^2})} \int_{\lambda_2}^{\infty} \frac{d\eta}{\eta^{3/2} \sqrt{(1+k^2\eta)(1+\eta)}}. \quad (2.178)$$

Hence, we scale

$$\phi_0 = kA\phi_0^{(2)} \quad (2.179)$$

and then expand the integral using a standard asymptotic analysis [27] to find at lowest order

$$\phi_0^{(2)} = \sqrt{(1 + \lambda_2)\mu_2\nu_2} - \sqrt{\lambda_2\mu_2\nu_2}. \quad (2.180)$$

Using the identity  $\sqrt{x/2 + (x^2 + y^2)/2} = \Re(\sqrt{x + iy})$  for real  $x$  and  $y$ , we manipulate the coordinates (2.174 - 2.176) to find

$$\phi_0^{(2)} = \Re \left[ \sqrt{1 - x_2^2 - (y_2 + iz_2)^2} \right], \quad (2.181)$$

and therefore  $\phi_0^{(2)}$  is an eigensolution of the quasi-two-dimensional potential problem (2.25, 2.26, 2.28, 2.29, 2.34) with  $d_0 = 1 - x_2^2$ . Hence, as required, the strip potential has a dipole in the far field ( $\lambda_2 \rightarrow \infty$ ) with axis in the  $z_2$ -direction and strength  $\sqrt{1 - x_2^2}/2$  that enable matching with the outer potential through (2.168). The leading-order strip solution breaks down near to  $x_2 = \pm 1$  where the inner regions describe the leading-order flow.

We note that the strip region turnover curve has equation  $x_2^2 + y_2^2 = 1$  to lowest order, while the waterline has equation  $x_2^2 + 2y_2^2 = 1$  obtained by expanding (2.145, 2.146) as  $k \rightarrow 0$ , viz.

$$a \rightarrow A_0 \left( 1 - \frac{1}{2} k^2 \log k \right), \quad b \rightarrow k A_0 \left( 2 + \frac{1}{2} k^2 \log k \right) \quad \text{as } k \rightarrow 0,$$

where we recall that the turnover curve (2.142) has major and minor semi-axes  $A(t) = A_0 t^{1/2}$ ,  $B(t) = k A(t)$ , respectively. The presence of the  $k^2 \log k$  terms at first order implies the waterline does not enter the inner region of size  $k^2$  at  $x_2 = \pm 1$ , as illustrated in Figure 2.13.

#### 2.4.4.4 The inner region

The right-hand inner region scalings are

$$\begin{aligned} x &= A(1 + k^2 x_3), & y &= k^2 A y_3, & z &= k^2 A z_3, \\ \lambda &= k^2 A^2 \lambda_3, & \mu &= k^2 A^2 \mu_3, & \nu &= k^2 A^2 \nu_3, \end{aligned} \quad (2.182)$$

where by (2.153),

$$-\nu_3 \leq -1 \leq \mu_3 \leq 0 \leq \lambda_3. \quad (2.183)$$

We expand as in the outer and strip regions to find at lowest order

$$x_3 = \frac{1}{2}(1 + \lambda_3 + \mu_3 + \nu_3), \quad (2.184)$$

$$y_3^2 = -(1 + \lambda_3)(1 + \mu_3)(1 + \nu_3), \quad (2.185)$$

$$z_3^2 = \lambda_3 \mu_3 \nu_3. \quad (2.186)$$

Scaling  $s = k^2 A^2 s_3$ , where  $s_3$  is order unity, the characteristic equation (2.144) becomes, to lowest order,

$$2x_3 + \frac{y_3^2}{1 + s_3} + \frac{z_3^2}{s_3} = 1. \quad (2.187)$$

Therefore, successively fixing  $\lambda_3 \in [0, \infty)$ ,  $\mu_3 \in [-1, 0]$  and  $\nu_3 \in (-\infty, -1]$  results in an elliptic paraboloid extending in the negative  $x_3$ -direction, a hyperbolic paraboloid and an elliptic paraboloid extending in the positive  $x_3$ -direction, respectively. The quadratic surfaces span  $(x_3, y_3, z_3)$ -space and  $(\lambda_2, \mu_2, \nu_2)$  are orthogonal confocal paraboloidal coordinates.

To find the leading-order inner velocity potential we proceed as above from the exact solution (2.143); scaling  $\phi_0 = k A \phi_0^{(3)}$ , we find

$$\phi_0^{(3)} = \sqrt{(1 + \lambda_3)\mu_3\nu_3} - \sqrt{\lambda_3\mu_3\nu_3}, \quad (2.188)$$

provided  $\lambda_3 \ll 1/k^2$ . The inner region turnover curve corresponds to  $\lambda_3 = \mu_3 = 0$  and is therefore the parabola

$$2x_3 + y_3^2 = 1. \quad (2.189)$$

As shown above, the waterline does not enter the inner region to lowest order.

Again a straightforward calculation shows that  $\phi_0^{(3)}$  is an eigensolution of the local leading-order potential problem, which is obtained by substituting the scalings (2.182) and  $B = kA$  into the full potential problem (2.109, 2.110, 2.112, 2.116) and expanding  $\phi_0^{(3)}$  as usual, thus

$$\frac{\partial^2 \phi_0^{(3)}}{\partial x_3^2} + \frac{\partial^2 \phi_0^{(3)}}{\partial y_3^2} + \frac{\partial^2 \phi_0^{(3)}}{\partial z_3^2} = 0 \quad \text{for } z_3 < 0, \quad (2.190)$$

$$\frac{\partial \phi_0^{(3)}}{\partial z_3} = -1 \quad \text{for } z_3 = 0, 2x_3 + y_3^2 < 1, \quad (2.191)$$

$$\bar{\phi}_0 = 0 \quad \text{for } z_3 = 0, 2x_3 + y_3^2 > 1, \quad (2.192)$$

together with (2.116) on the turnover curve (2.189).

As  $\lambda_3 \sim \nu_3 \rightarrow \infty$  for fixed  $\mu_3 \in (-1, 0)$ ,

$$\phi_0^{(3)} \sim \sqrt{\frac{\mu_3 \nu_3}{\lambda_3}}, \quad (2.193)$$

which matches with the first term of the leading-order outer potential (2.170), as required. Similarly, the leading-order inner velocity potential (2.188) matches with the leading-order strip velocity potential (2.180), which completes the leading-order matching.

#### 2.4.5 Strip theory

Suppose now the body profile is

$$z = f(\epsilon x, \epsilon y) - t \quad \text{where } \epsilon \ll \varepsilon \ll 1, \quad (2.194)$$

and  $f$  is smooth, strictly convex and such that  $f(0, 0) = 0$ . The variation in the  $x$ -direction is much slower than in the  $y$ -direction. This is the typical situation for ship hulls with the  $x$ -axis running from stern to bow and the  $y$ -axis running from starboard to port (on the left as one faces the bow at the front). The analysis of the previous section for the entry of a slender elliptic paraboloid showed to lowest order

- (1) the three-dimensional impact reduces to a sequence of quasi-two-dimensional impacts in each vertical body cross-section;
- (2) each of the quasi-two-dimensional impacts begins when the cross-section hits the undisturbed free surface  $z = 0$ .

These are the two core assumptions of strip theory which is popular in ship hydrodynamics [47, 72, 73, 74] where they are used to analyse more realistic slender hulls. For example, if  $f(x, y) = x^\alpha + y^2$  ( $\alpha \geq 1$ ), then the cross-section  $x = x_0/\epsilon$  first hits  $z = 0$  when  $t = t_0 = x_0^\alpha$ . The quasi-two-dimensional leading-order outer problem in the  $(x_0/\epsilon, y/\epsilon, z/\epsilon)$ -cross-section is (2.25 - 2.29) with body profile  $z = f(x_0, y) - t = y^2 - (t - t_0)$  and initial conditions  $\phi_0(y, z, t_0; x_0) = h_0(y, t_0; x_0) = d_0(t_0; x_0) = 0$ . Hence, the leading-order  $y$ -coordinates of turnover points in the  $x = x_0/\epsilon$  cross-section are  $y = \pm d_0(t; x_0)/\epsilon$  where, by (2.94),

$$d_0(t; x_0) = \sqrt{2(t - t_0)}.$$

Writing  $t_0 = x_0^\alpha$  and rearranging we find the leading-order equation of the turnover curve

$$(\epsilon x)^\alpha + \frac{(\epsilon y)^2}{2} = t,$$

which therefore extends along the  $x$ -axis between  $x = \pm t^{1/\alpha}/\epsilon$ , where strip theory breaks down and inner regions are required. The leading-order strip solution also breaks down at distances of order  $1/\epsilon$  from the  $x$ -axis in each cross-section and does not exist in  $|x| > t^{1/\alpha}/\epsilon$ ; an outer region of size order  $1/\epsilon$  is also required.

The strip theory approach is popular for two reasons. First, the leading-order force on the body is due to the leading-order pressure in the strip region, which is relatively easy to analyse because it decouples from the problems in the inner and outer regions, i.e. no information flows out of the outer and inner regions into the strip region<sup>4</sup>. Second, the asymptotics' agreement with experiment is reasonable.

We have formally confirmed that the two strip theory assumptions hold for the entry of a slender elliptic paraboloid. Further, the highest pressure on the body occurs in the inner regions at the ends of the strip region, where the speed of the turnover curve takes its maximum value. Therefore, strip theory does not predict the flow structure in the neighbourhood of the pressure maxima, where damage is most likely to occur.

Our main contribution here is to have found the leading-order inner eigensolution at the ends of the strip region for the entry of a slender elliptic paraboloid, which is the leading-order inner eigensolution for *any* smooth slender body with parabolic port-starboard cross-section. We have also shown that the leading-order outer solution is due to a continuous distribution of dipoles on a cut along the  $x$ -axis corresponding to the strip region, with strength and direction determined exactly by strip theory. We expect this to be true for the general profile (2.194).

## 2.5 Variational formulation

Obtaining a general solution to the leading-order outer problem (2.109 - 2.117) is difficult except for simple body shapes. Fortunately, [31, 35] show that it is possible to construct a variational inequality for the displacement potential

$$\Phi_0(x, y, z, t) = - \int_0^t \phi_0(x, y, z, \tau) d\tau, \quad (2.195)$$

Since  $\phi_0$  satisfies Laplace's equation (2.109) then so does  $\Phi_0$ ,

$$\nabla^2 \Phi_0 = 0 \quad \text{in } z < 0. \quad (2.196)$$

The boundary conditions (2.111, 2.112) imply

$$\frac{\partial \Phi_0}{\partial z} = -h_0 \quad \text{for } z = 0, \omega_0(x, y) > t, \quad (2.197)$$

$$\Phi_0 = 0 \quad \text{for } z = 0, \omega_0(x, y) > t. \quad (2.198)$$

However, for  $t < \omega_0(x, y)$ ,

$$\begin{aligned} \frac{\partial \Phi_0}{\partial z} &= - \int_0^t \frac{\partial \phi_0}{\partial z} d\tau \\ &= - \int_0^{\omega_0} \frac{\partial h_0}{\partial t} d\tau + \int_{\omega_0}^t d\tau \\ &= -h_0(x, y, \omega_0) + (t - \omega_0) \\ &= t - f(x, y), \end{aligned} \quad (2.199)$$

---

<sup>4</sup>Exactly the same relationship exists between the outer and the jet root solution in the full three-dimensional leading-order water entry problem.

by (2.195) on the first line, (2.110, 2.111) on the second, integration on the third and the Wagner condition (2.117) on the fourth. Hence,

$$\frac{\partial \Phi_0}{\partial z} \rightarrow w_0 - f = -h_0 \quad \text{as } t \rightarrow \omega_0 \pm,$$

and therefore  $\Phi_0$  has continuous first derivatives everywhere. The free boundary problem (2.196 - 2.199), together with the three-dimensional generalizations of (2.32, 2.33), imply that  $\Phi_0$  satisfies the linear complementarity problem

$$\frac{\partial^2 \Phi_0}{\partial x^2} + \frac{\partial^2 \Phi_0}{\partial y^2} + \frac{\partial^2 \Phi_0}{\partial z^2} = 0 \quad \text{in } z < 0, \quad (2.200)$$

$$\Phi_0 \left( \frac{\partial \Phi_0}{\partial z} + f - t \right) = 0 \quad \text{on } z = 0, \quad (2.201)$$

$$\Phi_0 \geq 0 \quad \text{on } z = 0, \quad (2.202)$$

$$\frac{\partial \Phi_0}{\partial z} + f - t \geq 0 \quad \text{on } z = 0. \quad (2.203)$$

The existence of a variational inequality for the complementarity problem (2.200 - 2.203) is proved as follows [35, 75]. Let  $\mathcal{H}^1$  be the real Sobolev space of square integrable functions and let  $\mathcal{V} \subset \mathcal{H}^1$  be a convex set whose elements are non-negative on  $z < 0$ . Define the symmetric continuous bilinear form  $a : \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathbb{R}$  by

$$a(u, v) = \iiint_{z < 0} \nabla u \cdot \nabla v \, dx dy dz, \quad (2.204)$$

and the continuous linear map  $l : \mathcal{H}^1 \rightarrow \mathbb{R}$  by

$$l(u) = \iint_{z=0} (t - f(x, y)) u(x, y, 0) \, dx dy. \quad (2.205)$$

Then for any  $v \in \mathcal{V}$ ,

$$a(\Phi_0, v - \Phi_0) = \iint_{z=0} (v - \Phi_0) \frac{\partial \Phi_0}{\partial z} \, dx dy,$$

by (2.204), Green's theorem and (2.200). We deduce that  $\Phi_0$  satisfies the variational inequality

$$a(\Phi_0, v - \Phi_0) \geq l(v - \Phi_0) \quad \text{for all } v \in \mathcal{V}, \quad (2.206)$$

by (2.203, 2.205). See [12] for a proof of the converse, i.e. that the variational inequality (2.206) implies the complementarity problem (2.200 - 2.203), and also that these formulations are equivalent to the classical problem (2.109 - 2.117) if and only if  $\Phi_0$  has continuous first derivatives.

The formulation provides a framework in which to discuss wellposedness and an efficient means of obtaining a numerical solution. The Hilbert space projection theorem [12] says there exists a unique solution to (2.206), provided the bilinear form  $a(\cdot, \cdot)$  is coercive, i.e. there exists a constant  $c > 0$  such that

$$a(v, v) \geq c \iiint_{z < 0} v^2 \, dx dy dz \quad \text{for all } v \in \mathcal{V}. \quad (2.207)$$

Unfortunately,  $a(\cdot, \cdot)$  is only coercive if the fluid region is finite, which can be overcome by imposing an artificial boundary at a large distance from the body. Moreover, since the bilinear form  $a(\cdot, \cdot)$  is symmetric, it can be shown [12] that the variational inequality (2.206) is equivalent to the minimization

$$J(\Phi_0) \leq J(v) \quad \text{for all } v \in \mathcal{V},$$

where

$$J(v) = \frac{1}{2}a(v, v) - l(v) = \frac{1}{2} \iiint_{z < 0} |\nabla v|^2 \, dx dy dz + \iint_{z=0} (f - t) u \, dx dy.$$

This problem may be solved efficiently using finite elements in a fixed and bounded domain; the location of the free boundary is built into the solution procedure by using the well-known iterative method of projected successive over relaxation [10]. Such a numerical solution was found by [31, 75] and the results show good agreement with the asymptotic results.

## 2.6 Stability and water exit

A local-in-space and -time linear stability analysis of the two-dimensional leading-order outer problem (2.25 - 2.31, 2.50) to perturbations of size  $\delta \ll 1$  in the  $y$ -direction was performed by [19, 31]. The existence of the small parameter  $\delta$  and a (square root) singularity in leading-order outer mixed boundary value problem allowed an analysis using matched asymptotic expansions. Proceeding to two-terms in both the inner and outer solutions and matching resulted in an implicit dispersion relation, which showed that the leading-order outer problem is linearly stable if and only if the turnover curve is advancing, i.e. the time reversal of the entry problem is linearly unstable. This suggests that modelling the water exit of a partially submerged hull by the time reversal of a water entry problem is illposed, even if we can find the initial condition for the entry problem that results in the specified initial condition for the exit problem. In addition there is the following evidence for the validity of this conjecture [31].

- Replacing  $t$  with  $-t$  in the water entry model (2.25 - 2.31, 2.50) changes the sign of the leading-order pressure (2.46), which may cause the flow to separate or cavitate or both. Indeed, another solution to the exit problem is that the body is simply removed, leaving the fluid at rest at  $t = 0$ .
- The procedure to find the free surface elevation in the two-dimensional case described in section 2.2.1.4 is not reversible because integration of the kinematic boundary condition (2.27) requires  $d_0(t)$  to be increasing.

Very little is known about the regularization of the water exit problem through a better mathematical interpretation or the addition of other physical effects to the model [5].

## Chapter 3

# Finite depth water entry at small deadrise angles

In this chapter we consider the effect of finite depth on the small deadrise angle water entry model in the previous chapter. We begin with the two-dimensional case. In section 3.1 we describe how the impact breaks down into four distinct temporal stages. We find the leading-order flow in each of these stages and match neighbouring stages in sections 3.2 - 3.5. In section 3.6 we briefly consider some simple extensions. Finally, in section 3.7 we consider the three-dimensional generalization.

### 3.1 The four stages of impact

Our fundamental assumption is that the radius of curvature of the body near the point of impact is much larger than the layer depth  $H$ , rather than the penetration depth  $L$  as in Chapter 2. We therefore suppose that the fluid initially occupies the (dimensional) layer  $-H < z < 0$  and that the (dimensional) body profile is

$$z = Hf\left(\frac{\epsilon x}{H}\right) - Ut, \quad (3.1)$$

where  $\epsilon \ll 1$ . The dimensional model problem (2.3 - 2.11) is unchanged except now we set  $L = H$  and introduce the kinematic boundary condition on the base

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -H. \quad (3.2)$$

Korobkin [37] used the method of matched asymptotic expansions to analyse the flow when the depth of penetration  $Ut$  is comparable to the layer depth  $H$ . The flow in  $x > 0$  was shown to decompose into the four regions sketched in Figure 3.1. As in the infinite depth case the free surface turns over to form a long fast moving jet running along the body in the jet region. Here, however, the size of the jet root region, and therefore the thickness of the jet, is now comparable to the layer depth  $H$ . The jet root flow is driven by the high velocity *inviscid squeeze film* flow out of the *interior region*, which here and hereafter, we define to be the region of fluid lying between vertical lines drawn through the turnover points, whose  $x$ -coordinate we denote by  $x = Hd(Ut/H)/\epsilon$ . The flow in the *exterior region*, lying outside the interior region and below the lower free surface, was argued to be zero by introducing gravity and applying shallow water theory as follows. The speed of the jet root

region  $U\dot{d}/\epsilon$  is much larger than the critical speed  $\sqrt{gH}$  at which information can propagate into the thin layer, *so* the fluid remains at rest. Although this is a plausible reason for the exterior flow to be small, it was not confirmed by matching with the jet root region, which is a trivial task and reveals the flow is exponentially small in  $\epsilon$ , corresponding to small amplitude zero gravity water waves.

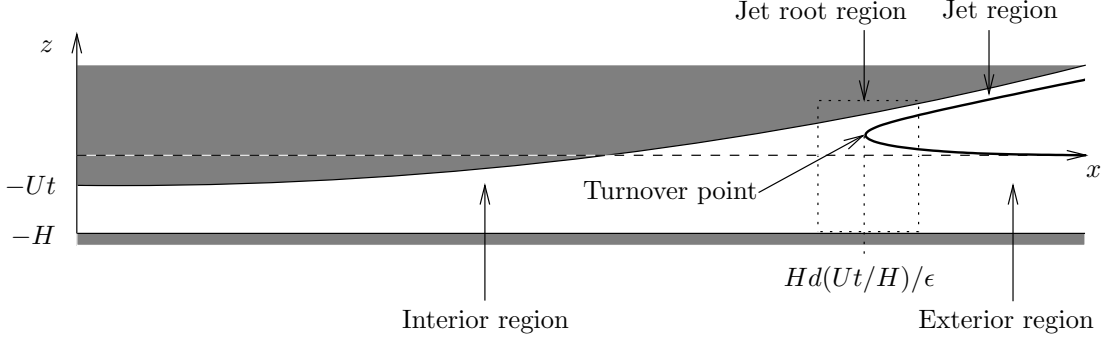


Figure 3.1: Finite depth flow structure at order unity time [37].

The law of motion of the turnover point, which now lies a distance of  $O(H)$  away from the body, is found by matching the interior, exterior and jet regions with the inner jet root region and then solving the jet root problem. Therefore, in contrast to the infinite depth case, information flows out of the jet root into the outer regions and the jet now has a leading order influence on the motion of the turnover point.

As  $t \rightarrow 0$ , the separation of the turnover points  $C(t) = 2Hd(Ut/H)/\epsilon \rightarrow 0$ , so the asymptotic structure described above breaks down and the use of zero initial conditions must be justified. Physical intuition suggests that at sufficiently small penetration depths the effect of the base is negligible, so that we arrive back at the infinite depth case. Therefore, to fully describe the evolution of the flow and justify the initial conditions, we must unify the infinite depth Wagner theory described in the previous chapter and the finite depth Korobkin theory described here. To do so we follow Korobkin [37] by comparing the size of the three length scales that define the problem, namely the penetration depth  $Ut$ , the distance between the turnover points  $C(t)$  and the fixed layer depth  $H$ . The small deadrise angle assumption implies  $Ut \ll C(t)$  for all  $t > 0$ , so, comparing these lengths with  $H$ , we expect there to be four stages of impact<sup>1</sup>, which are described below and illustrated in Figure 3.2.

- (1) At the first stage the penetration depth is sufficiently small that that the separation of the turnover points is much smaller than the layer depth; thus  $Ut \ll C(t) \ll H$ . The effect of the base is negligible and to lowest order the flow is governed by the infinite depth Wagner theory.
- (2) At the second stage the penetration depth is small compared to the layer depth, but is large enough that the separation of the turnover points is comparable to the layer depth; thus  $Ut \ll C(t) \sim H$ . The base now influences the leading-order bulk flow.
- (3) At the third stage the penetration depth is still much smaller than the layer depth, but sufficiently large that the separation of the turnover points is much larger than the layer depth; thus  $Ut \ll H \ll C(t)$ .

<sup>1</sup>Korobkin [37] characterized the impact solely by stages 1, 2 and 4.

- (4) At the fourth stage the penetration depth is comparable to the layer depth, both of which are much smaller than the distance between the turnover points; thus  $Ut \sim H \ll C(t)$ .

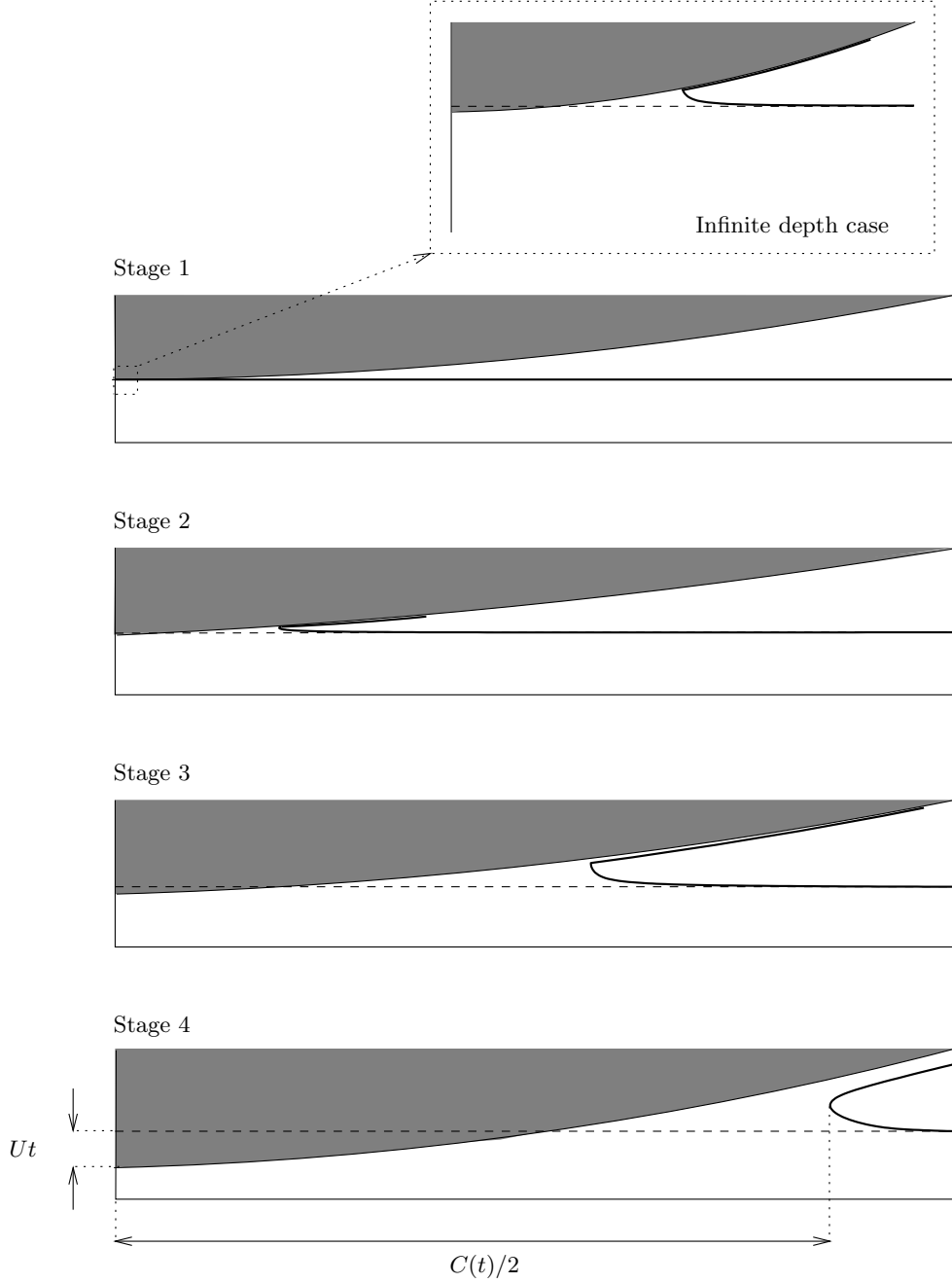


Figure 3.2: Schematic of the four stages of impact; see page 54 for the explanation.

To quantify the above scenario and find the timescale of each stage, we follow Korobkin [37] and use exactly the same geometric argument as in the infinite depth case to estimate the order of magnitude of the separation of the turnover points. First though, we nondimensionalize the model problem (2.3 - 2.11, 3.2) by scaling distances with  $H$  (rather than the penetration depth  $L$  as in the infinite depth case), time with  $H/U$ , the velocity potential

with  $HU$  and the pressure with  $\rho U^2$ . Then, taking the parabolic body profile  $f(x) = x^2$  for simplicity<sup>2</sup>, we obtain the dimensionless model problem

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid,} \quad (3.3)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -1, \quad (3.4)$$

$$2\epsilon^2 x \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = 1 \quad \text{on } z = \epsilon^2 x^2 - t, \quad (3.5)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = h(x, t), \quad (3.6)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0 \quad \text{on } z = h(x, t), \quad (3.7)$$

together with initial and far field conditions,

$$\phi(x, z, 0) = 0, \quad h(x, 0) = 0, \quad (3.8)$$

$$\phi, h \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (3.9)$$

The body  $z = \epsilon^2 x^2 - t$  intersects the undisturbed free surface when  $x = \sqrt{t}/\epsilon$ , so we expect the separation of the turnover points to be of order  $\sqrt{t}/\epsilon$  for  $t \geq 0$ . The separation is therefore comparable to the unit layer depth when  $t \sim O(\epsilon^2)$ , at which time the penetration depth  $t \ll 1$ . Hence, the (dimensionless) timescale of stage 2 is of order  $\epsilon^2$ . Similarly, the timescale of stage 4 is of order unity. Therefore, the small time limit of stage 2, at times  $t \ll \epsilon^2$ , is stage 1, while the intermediate time limit between stages 2 and 4 is stage 3, at times  $t$  such that  $\epsilon^2 \ll t \ll 1$ . We summarize these timescales and the corresponding dimensional values of the penetration depth  $Ut$ , the turnover point separation  $C(t) \sim O(H\sqrt{t}/\epsilon)$  and the layer depth  $H$  in Table 3.1.

|    |                          |        |                      |
|----|--------------------------|--------|----------------------|
| 1: | $t \ll \epsilon^2$       | $\iff$ | $Ut \ll C(t) \ll H$  |
| 2: | $t \sim \epsilon^2$      | $\iff$ | $Ut \ll C(t) \sim H$ |
| 3: | $\epsilon^2 \ll t \ll 1$ | $\iff$ | $Ut \ll H \ll C(t)$  |
| 4: | $t \sim 1$               | $\iff$ | $Ut \sim H \ll C(t)$ |

Table 3.1: Dimensionless timescales and dimensional geometric interpretation of the four stages of impact.

Hence, we expect the duration of each stage to be much smaller than that of the following stage, so that the initial  $x$ -coordinates of the turnover points are zero to lowest order in each stage, i.e. we may consider each stage separately with zero initial conditions [37]. However, we emphasise that in order to confirm these timescales and therefore the temporal decomposition of the impact into four stages, we must ensure the spatial decomposition in each stage matches into its neighbouring stages. Specifically, we must show that the small time limit of stage 2 is stage 1 and that stages 2 and 4 match through stage 3.

We make the following comments as a prelude to the asymptotic solution of the four stages of impact.

---

<sup>2</sup>The case in which  $f(x)$  is smooth and convex away from the origin where  $f(x) \sim O(|x|^\alpha)$  for  $\alpha \geq 1$  is a straightforward generalization. It is described in section 3.6, along with other simple extensions, including variable impact speed.

- Since the depth of penetration is much smaller than the layer depth in stages 1, 2 and 3, the body and the lower free surface are uniformly close to the undisturbed free surface. Therefore, in an asymptotic solution as  $\epsilon \rightarrow 0$ , the kinematic boundary condition on the body in the interior region and the lower free surface boundary conditions in the exterior region can be linearized onto the known location of the undisturbed free surface,  $z = 0$ . The only geometric unknown is therefore the leading-order location of the turnover or free points separating the interior and exterior regions. In stages 1 and 2 this leads to a codimension-two free boundary problem [29], the free points having two dimensions fewer than the space in which the motion occurs. In stage 3, the distance between the free points is much larger than the layer depth. The resulting large-aspect-ratio limit of the interior region implies the leading-order flow is uniform across the layer, so the dimensionality of the leading-order field equation is reduced by one to one. The result is a codimension-one free boundary problem, the free points having zero dimension which is one fewer than the space in which the motion occurs.
- Since the depth of penetration is comparable to the layer depth in stage 4, it is necessary to account for a leading-order deformation of the fluid domain. However, as described above, Korobkin [37] showed that a complete asymptotic solution is possible because the large aspect ratio of the interior region implies the leading-order flow is uniform across the layer. The result is therefore a codimension-one free boundary problem, similar to but necessarily more complicated than the one at stage 3. The crucial technical feature of our analysis of stage 4 is to work exclusively with the velocity potential, which enables a straightforward generalization to the three-dimensional case that is not so obvious if one follows [37] and works with the fluid velocity and pressure through Euler's equations.

### 3.2 Very small time

The timescale of stage 1 is  $t \sim \epsilon^2/\Delta^2$  where  $\Delta \gg 1$ , so the separation of the turnover points is expected to be of order  $\sqrt{t}/\epsilon \sim 1/\Delta$  and we therefore scale

$$x = \bar{x}/\Delta, \quad z = \bar{z}/\Delta, \quad t = (\epsilon/\Delta)^2 \bar{t}, \quad d = \bar{d}(\bar{t})/\Delta, \quad \phi = \bar{\phi}/\Delta, \quad h = (\epsilon/\Delta)^2 \bar{h}, \quad (3.10)$$

to retain a balance in the kinematic boundary conditions on the body and free surface (3.5, 3.6). The body profile becomes  $\bar{z} = (\bar{\epsilon}\bar{x})^2 - \bar{t}$ , where the “scaled deadrise angle” is  $\bar{\epsilon} = \epsilon\Delta^{-1/2} \ll \epsilon$ . Further, the base becomes  $\bar{z} = -\Delta \ll -1$ , which we therefore ignore to lowest order, while the full governing equations (3.3 - 3.9) are unchanged to lowest order *except*  $\epsilon$  is replaced by  $\bar{\epsilon}$ . Hence, stage 1 is exactly the infinite depth case to lowest order and the leading-order flow characteristics are therefore easily obtained from the previous chapter by replacing  $\epsilon$  with  $\bar{\epsilon}$  throughout. In particular, equations (2.92 - 2.94) imply the leading-order solution is

$$\bar{d}_0(\bar{t}) = \sqrt{2\bar{t}}, \quad (3.11)$$

$$\bar{\phi}_0 = -\bar{z} - \text{Re} \left( \sqrt{2\bar{t} - (\bar{x} + i\bar{z})^2} \right), \quad (3.12)$$

$$\bar{h}_0 = \bar{x}^2 - |\bar{x}| \sqrt{\bar{x}^2 - 2\bar{t} - \bar{t}} \quad \text{for } |\bar{x}| > \bar{d}_0(\bar{t}). \quad (3.13)$$

### 3.3 Small time

The timescale of stage 2 is  $\epsilon^2$ , so the separation of the turnover points is expected to be of order  $\sqrt{t}/\epsilon \sim O(1)$  and we therefore scale

$$x = \tilde{x}, \quad z = \tilde{z}, \quad t = \epsilon^2 \tilde{t}, \quad d = \epsilon \tilde{d}(\tilde{t}), \quad \phi = \tilde{\phi}, \quad h = \epsilon^2 \tilde{h}. \quad (3.14)$$

We write the leading-order stage 1 solution (3.11 - 3.13) in terms of the stage 2 variables (3.14), viz.

$$\bar{d}_0(\Delta^2 \tilde{t})/\Delta, \quad \bar{\phi}_0(\Delta \tilde{x}, \Delta \tilde{z}, \Delta^2 \tilde{t})/\Delta, \quad \bar{h}_0(\Delta \tilde{x}, \Delta^2 \tilde{t})/\Delta^2,$$

and expand as  $\Delta \rightarrow \infty$  to find the one-term-outer expansions of the one-term-inner variables are (3.11 - 3.13) with tilde replacing barred variables (because they are a similarity solution). Matching therefore requires

$$\tilde{d}(\tilde{t}) \sim \sqrt{2\tilde{t}}, \quad (3.15)$$

$$\tilde{\phi}(\tilde{t}^{1/2} \tilde{X}, \tilde{t}^{1/2} \tilde{Z}, \tilde{t}) \sim -\tilde{t}^{1/2} \left[ \tilde{Z} + \text{Re} \left( \sqrt{2 - (\tilde{X} + i\tilde{Z})^2} \right) \right], \quad (3.16)$$

$$\tilde{h}(\tilde{t}^{1/2} \tilde{X}, \tilde{t}) \sim \tilde{t} \left[ \tilde{X}^2 - \tilde{X} \sqrt{\tilde{X}^2 - 2} - 1 \right], \quad (3.17)$$

as  $\tilde{t} \rightarrow 0$  where  $\tilde{X}, \tilde{Z}$  are order unity. Hence, as conjectured in section 3.1, stage 2 has zero initial conditions to lowest order. Substituting the scalings (3.14) into the dimensionless model problem (3.3 - 3.9), we find (dropping tildes)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid}, \quad (3.18)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -1, \quad (3.19)$$

$$2\epsilon^2 x \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = -1 \quad \text{on } z = \epsilon^2(x^2 - t), \quad (3.20)$$

$$\frac{\partial h}{\partial t} + \epsilon^2 \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = \epsilon^2 h(x, t), \quad (3.21)$$

$$\frac{\partial \phi}{\partial t} + \frac{\epsilon^2}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0 \quad \text{on } z = \epsilon^2 h(x, t), \quad (3.22)$$

$$\phi, h, d = 0 \quad \text{at } t = 0, \quad (3.23)$$

$$\phi, h \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (3.24)$$

The body  $z = \epsilon^2(x^2 - t)$  and free surface  $z = \epsilon^2 h$  lie uniformly close to  $z = 0$ , so, linearizing the three boundary conditions on the body and lower free surface (3.5 - 3.7) and expanding  $\phi, h$  and  $d$  as asymptotic series in powers of  $\epsilon^2$ , we obtain the leading-order outer codimension-two free boundary problem shown in Figure 3.3.

Of course this is a simple modification of the infinite depth leading-order outer problem (2.25 - 2.31) to take account of the base. Hence, the leading-order outer problem has the asymptotic structure shown in Figure 3.4 and the complex potential  $w_0$  has the following behaviour at the free points

$$w_0(\zeta, t) \sim S(t) \cdot \sqrt{\zeta^2 - d_0(t)^2} \quad \text{as } \zeta = x + iz \rightarrow \pm d_0(t), \quad (3.25)$$

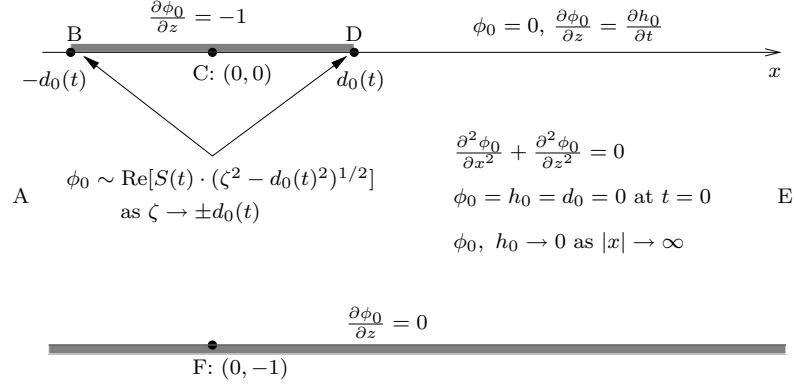


Figure 3.3: The stage 2 leading-order outer problem. The labels A, B, C, D, E, and F are used in section 3.3.1.

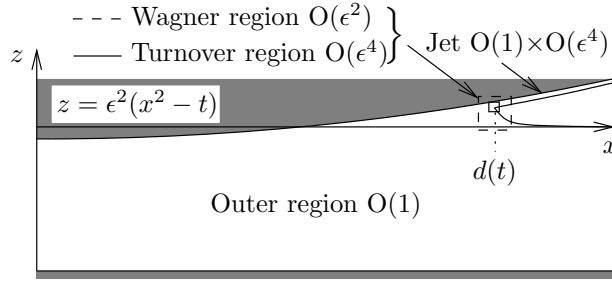


Figure 3.4: The asymptotic structure of stage 2.

where  $S(t)$  is a complex function to be determined and we choose the branch of the square root as in the infinite depth outer solution (2.40).

Matching the outer region and Wagner region of size  $O(\epsilon^2)$  implies the Wagner condition (2.50). Matching the jet root region of size  $O(\epsilon^4)$  and the Wagner region implies the asymptotic jet thickness (scaled with  $\epsilon^4$ ) is

$$H_0(t) = \frac{\pi d_0(t) S(t)^2}{8 \dot{d}_0(t)^2}, \quad (3.26)$$

by (2.77). The jet region of thickness  $O(\epsilon^4)$  and extent  $O(1)$  has exactly the structure described in section 2.2.4 and does not affect the leading-order outer solution. As in the infinite depth case, we find  $S(t)$  by solving the outer problem in Figure 3.3 for  $\phi_0$ , then integrate the leading-order kinematic boundary condition in the free surface to find  $h_0$  and finally apply the Wagner condition (2.50) to find the law of motion of the free point.

### 3.3.1 Solution of the leading-order outer problem

We map the strip  $ABCDEF A$  of the  $\zeta$ -plane (in Figure 3.3) onto the lower half of the  $\nu$ -plane (in Figure 3.5) with the conformal map

$$\nu = \xi + i\eta = \frac{e^{\pi\zeta} - a(t)}{1/a(t) - a(t)} \quad \text{where} \quad a(t) = e^{-\pi d_0(t)}. \quad (3.27)$$

The linear translation and uniform scaling map the free points B and D to the points at  $\nu = 0$  and  $\nu = 1$ , respectively, which fixes the the bounds of the final integral solution.

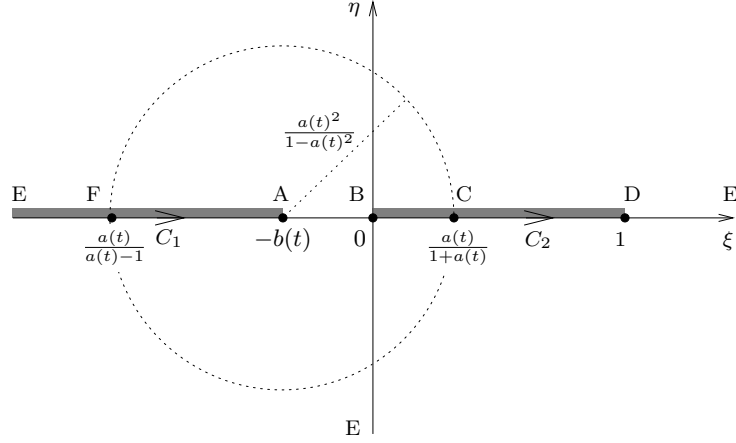


Figure 3.5: The  $\nu$ -plane.

The complex potential  $W_0(\nu, t) = \Phi_0 + i\Psi_0$  in the  $\nu$ -plane is equal to the complex potential  $w_0(\zeta, t)$  at corresponding points in the  $\zeta$ -plane, while the complex velocity is given by

$$\frac{\partial w_0}{\partial \zeta} = \pi(\nu + b(t))G_0(\nu, t) \quad \text{where} \quad b(t) = \frac{a(t)^2}{1 - a(t)^2} \quad \text{and} \quad G_0 = \frac{\partial w_0}{\partial \nu}. \quad (3.28)$$

We proceed as in section 2.2.1.3 by analytically continuing  $W_0$  into  $\Im(\nu) > 0$  using the Schwarz reflection formula (2.35) and taking care of the resulting discontinuity in  $G_0$  across the real axis. This shows that  $G_0$  is analytic on the  $\nu$ -plane except across the contours  $C_1$  and  $C_2$  (shown in Figure 3.5) where

$$G_0(\xi + 0i, t) = \begin{cases} -G_0(\xi - 0i, t) & \text{for } \xi \in (-\infty, -b(t)), \\ -G_0(\xi - 0i, t) + \frac{2i}{\pi(\xi + b(t))} & \text{for } \xi \in (0, 1). \end{cases} \quad (3.29)$$

By (3.24, 3.25), the complex velocity is zero at infinity (A, E) and unbounded at the turnover points (B, D) so

$$G_0(\nu, t) \rightarrow \infty \quad \text{as } \nu \rightarrow 0 \quad \text{and} \quad 1, \quad (3.30)$$

$$|(\nu + b(t))G_0(\nu, t)| \rightarrow 0 \quad \text{as } \nu \rightarrow -b(t) \quad \text{and} \quad |\nu| \rightarrow \infty. \quad (3.31)$$

In order to satisfy the boundary condition at  $\nu = -b$ , it is necessary to specify that  $G_0$  is unbounded there. The index is therefore two because there are two simple, smooth contours  $C_1$  and  $C_2$  across which  $G_0$  is discontinuous and  $G_0$  is unbounded at their ends in the finite  $\nu$ -plane. Hence, the least singular solution unbounded at  $\nu = -b(t)$ , 0, 1 and zero at infinity is (see, for example, [16, 19])

$$G_0(\nu, t) = \frac{A(t) + B(t)\nu + I(\nu, t)}{R(\nu, t)}.$$

Here,  $A(t)$  and  $B(t)$  are complex functions to be determined. The function  $R$  has branch cuts  $C_1$  and  $C_2$  and is defined by

$$R(\nu, t) = (\nu + b(t))^{1/2} \nu^{1/2} (\nu - 1)^{1/2}, \quad (3.32)$$

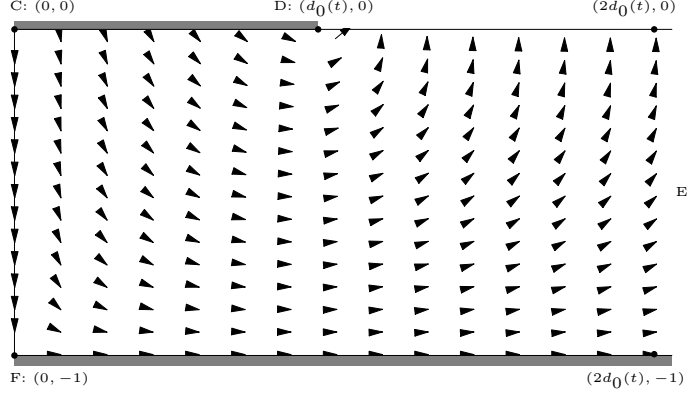


Figure 3.6: The stage 2 leading-order outer velocity field.

where the square roots are chosen appropriately. The integral  $I$ , which takes its Cauchy principal value on  $C_1 \cup C_2$ , is defined by

$$I(\nu, t) = \frac{1}{2\pi i} \int_{C_1} \frac{R(\xi - 0i, t)}{\xi - \nu} \frac{2i}{\pi(\xi + b(t))} d\xi. \quad (3.33)$$

Since  $I(\nu, t) \sim O(1/\nu)$  and  $R(\nu, t) \sim O(\nu^{3/2})$  as  $|\nu| \rightarrow \infty$ , the boundary condition at E ( $|nu| \rightarrow \infty$ ) implies that  $B(t) = 0$ . As  $\nu \rightarrow -b(t)$  at A, we find

$$(\nu + b(t))G_0(\nu, t) \sim -2^{-1/2}(I(-b(t)) + A(t))(\nu + b(t))^{1/2},$$

so the boundary condition automatically holds there. To find  $A(t)$  use the fact that C (at  $\nu = -c(t) := -a(t)/(1 - a(t))$ ) is a stagnation point and (3.28), thus

$$0 = \frac{\partial w_0}{\partial \zeta}(-i, t) = \pi(-c(t) + b(t))(I(-c(t), t) + A(t)),$$

which implies  $A(t) = -I(-c(t), t)$ .

Summarizing, the solution is defined by the pair of equations

$$\frac{\partial w_0}{\partial \zeta}(\nu, t) = \frac{1}{\pi Q(\nu, b(t))} \int_0^1 Q(\xi, b(t)) \left( \frac{1}{\xi + c(t)} - \frac{1}{\xi - \nu} \right) d\xi, \quad (3.34)$$

$$\zeta = \frac{1}{\pi} \log \frac{a(t)}{b(t)} (\nu + b(t)), \quad (3.35)$$

where

$$Q(\xi, b(t)) = \frac{\xi^{1/2}(1 - \xi)^{1/2}}{(\xi + b(t))^{1/2}}, \quad (3.36)$$

and we recall

$$a(t) = e^{-\pi d_0(t)}, \quad b(t) = (e^{2\pi d_0(t)} - 1)^{-1}, \quad c(t) = (e^{\pi d_0(t)} - 1)^{-1}. \quad (3.37)$$

Here, the square roots in (3.36) are chosen appropriately and the integral in (3.34) is taken along the real axis. In Figure 3.6 we plot the symmetric velocity field, which was obtained by writing the integral  $I$  in (3.33) as a sum of elliptic integrals of the first, second and third kind in the Legendre normal form; see, for example, [22].

### 3.3.1.1 Behaviour at the free points and infinity

By symmetry, it is sufficient to consider the behaviour at  $D$  ( $\zeta = d_0(t)$ ,  $\nu = 1$ ). We expand (3.34, 3.35) as  $\nu \rightarrow 1$  to find

$$\frac{\partial w_0}{\partial \zeta}(\nu, t) \sim \frac{(1+b(t))^{1/2}}{\pi(1-\nu)^{1/2}} \int_0^1 \frac{\xi^{1/2}(1-\xi)^{1/2}}{(\xi+b(t))^{1/2}} \left( \frac{1}{\xi+c(t)} + \frac{1}{1-\xi} \right) d\xi + O(1),$$

and

$$\zeta - d_0(t) \sim \frac{\nu - 1}{\pi(1+b(t))} + O((\nu - 1)^2),$$

which implies

$$w_0 \sim -iS(d_0(t))\sqrt{\zeta - d_0(t)} \quad \text{as } \zeta \rightarrow d_0(t), \quad (3.38)$$

where the coefficient of the square root  $S$  is solely a real function of  $d_0(t)$ , as in stage 1, and is the positive real integral

$$S(d_0(t)) = \frac{2}{\pi^{3/2}}(1+c(t)) \int_0^1 \frac{\xi^{1/2}}{(\xi+b(t))^{1/2}(1-\xi)^{1/2}(\xi+c(t))} d\xi. \quad (3.39)$$

Similarly, at  $E$  we find there is exponential decay, viz.

$$w_0 \sim \left( \frac{2i}{\pi^2} \left( \frac{a(t)}{b(t)} \right)^{1/2} \int_0^1 \frac{\xi^{1/2}(1-\xi)^{1/2}}{(\xi+b(t))^{1/2}(\xi+c(t))} d\xi \right) \exp(-\pi\zeta/2) \quad \text{as } \text{Re}(\zeta) \rightarrow \infty,$$

in contrast to the infinite depth case in which the complex potential (2.40) decays as  $O(r^{-1})$  as  $r = |x + iz| \rightarrow \infty$ .

### 3.3.1.2 Leading-order free surface elevation

On the equipotential  $DE$  the normal velocity is given by (3.34, 3.35) as

$$\frac{\partial \phi_0}{\partial z}(x, 0, t) = \frac{1}{\pi} \frac{(\nu + b(t))^{1/2}}{\nu^{1/2}(\nu - 1)^{1/2}} \int_0^1 \frac{\xi^{1/2}(1-\xi)^{1/2}}{(\xi+b(t))^{1/2}} \left( \frac{1}{\xi+c(t)} + \frac{1}{\nu-\xi} \right) d\xi, \quad (3.40)$$

where

$$\nu = \frac{e^{\pi x} - a(t)}{1/a(t) - a(t)} > 1 \quad \text{for } x > d_0(t). \quad (3.41)$$

By the leading-order kinematic boundary condition on  $DE$  the free surface elevation is

$$h_0(x, t) = \int_0^t \frac{\partial \phi_0}{\partial z}(x, 0, \tau) d\tau \quad \text{for } x > d_0(t). \quad (3.42)$$

The free surface profile  $h_0$  is bounded and strictly monotonic decreasing on  $x \geq d_0(t)$ , with an inverse square root singularity in its first partial derivatives at  $x = d_0(t)$  and exponential decay at infinity, in contrast to the algebraic decay in the infinite depth case (2.42).

### 3.3.1.3 Law of motion of the free point

Substituting the free surface elevation  $h_0$  from (3.42) into the Wagner condition (2.50) results in a singular integral equation for the plate semi-width  $d_0(t)$ , viz.

$$d_0(t)^2 - t = h_0(d_0(t), t) = \int_0^t \frac{\partial \phi_0}{\partial z}(d_0(t), 0, \tau) d\tau, \quad (3.43)$$

where the velocity is given by (3.40). To evaluate  $h_0$  at a single point it is necessary to compute the above integral, which has square root singularities on  $\tau = t$ . The numerics are therefore gruesome, so we do not pursue them further than suggesting a simple numerical scheme based on the small and large time asymptotics. Matching with stage 1 in section 3.2 revealed that

$$d_0(t) \sim \sqrt{2t} \quad \text{as } t \rightarrow 0. \quad (3.44)$$

Similarly, matching with stage 3 in section 3.4.4 below shows that

$$d_0(t) \sim \sqrt{3t} \quad \text{as } t \rightarrow \infty. \quad (3.45)$$

Hence, writing

$$d_0(t) = (\sqrt{3} + (\sqrt{2} - \sqrt{3})\lambda(t))\sqrt{t}, \quad (3.46)$$

we must find the function  $\lambda(t)$  with  $\lambda(0) = 1$  and  $\lambda(\infty) = 0$ , which characterizes the form and time scale of the transition from stage 1 to stage 3. A simple numerical scheme might be to guess a parametrized form for  $\lambda$  and then minimise some normalized measure of the residue in the law of motion (3.43).

Korobkin [35] proposed a numerical approach based on the theory of variational inequalities in his Lagrangian formulation of the infinite depth leading-order outer problem; clearly it is straightforward to modify the direct variational formulation of the two-dimensional infinite depth leading-order outer problem (see section 2.5) to account for the presence of the base. This is probably the most efficient numerical scheme, even though we have found an analytic solution! We remark that Lin & Ho [44] used a boundary element method to study the full nonlinear wedge entry problem on a layer, but did not consider the small deadrise angle limit.

### 3.3.2 Pressure and force on the body

Proceeding as in the infinite depth case, we find the outer pressure is

$$p(x, z, t) \sim -\frac{1}{\epsilon^2} \frac{\partial \phi_0}{\partial t} - \frac{1}{\epsilon} \left( \frac{\partial \phi_1}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi_0}{\partial x} \right)^2 + \left( \frac{\partial \phi_0}{\partial z} \right)^2 \right] \right) + O(1). \quad (3.47)$$

The leading-order pressure has a square root singularity at the free points and exhibits exponential decay at infinity. Table 3.2 shows the order of magnitude of the pressure, wetted length and resulting force on the body per unit length in the  $y$ -direction for each region, except for the exterior region where the pressure is exponentially small and for which there is no contact with the body.

As in the infinite depth case, the leading-order force on the body (of order  $1/\epsilon^2$  here) is due solely to the interior pressure, while the first order correction (of order  $1/\epsilon$  here) is due solely to the Wagner region. As discussed above, computations are too complicated for us to attempt a numerical solution, but, guided by hindsight, we expect the composite pressure on the body to have profiles of the form sketched in Figure 3.7.

| Region   | Pressure       | Wetted length | Force          |
|----------|----------------|---------------|----------------|
| Interior | $1/\epsilon^2$ | 1             | $1/\epsilon^2$ |
| Wagner   | $1/\epsilon^3$ | $\epsilon^2$  | $1/\epsilon$   |
| Jet root | $1/\epsilon^4$ | $\epsilon^4$  | 1              |
| Jet      | $\epsilon^2$   | 1             | $\epsilon^2$   |

Table 3.2: Order of magnitudes of the pressure, wetted length and force on the body in stage 2.

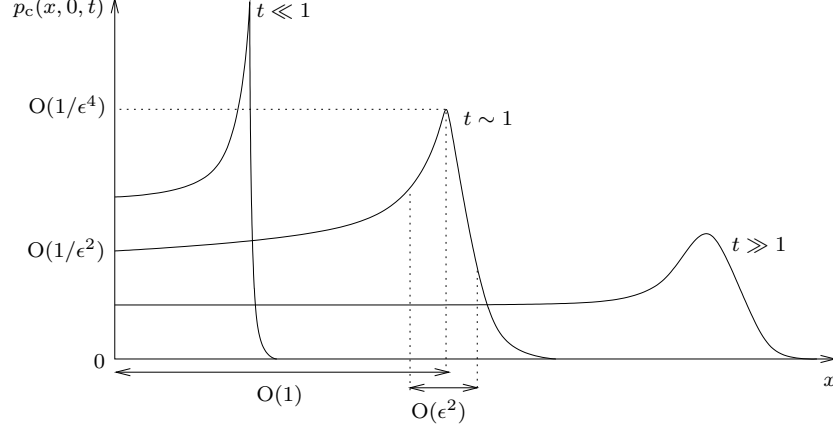


Figure 3.7: Schematic of the stage 2 composite pressure profiles on the body.

### 3.3.3 Small time limit of the small time solution

In this section we verify that the small time limit of the stage 2 complex potential, free surface elevation and  $x$ -coordinate of the turnover point are exactly (3.15 - 3.17) and find the correction due to the finite but large depth  $\Delta \gg 1$  in stage 1. By (3.10, 3.14), we scale into stage 1 from stage 2 by setting

$$\zeta = \bar{\zeta}/\Delta, \quad d_0(t) = \bar{d}_2(\bar{t})/\Delta, \quad w_0 = \bar{w}_0/\Delta, \quad h_0 = \bar{h}_0/\Delta^2. \quad (3.48)$$

To expand the complex velocity (3.34, 3.35) as  $\Delta \rightarrow \infty$  for fixed  $\bar{\zeta}$  and  $\bar{t}$ , we successively expand its components. By (3.37),

$$b(\bar{t}/\Delta^2) \sim \frac{\Delta}{2\pi\bar{d}_0(\bar{t})} + O(1), \quad c(\bar{t}/\Delta^2) \sim \frac{\Delta}{\pi\bar{d}_0(\bar{t})} + O(1), \quad (3.49)$$

so by (3.27),

$$\nu(\bar{\zeta}/\Delta, \bar{d}_0/\Delta) \sim \frac{\bar{\zeta} + \bar{d}_0}{2\bar{d}_0} + O(1/\Delta). \quad (3.50)$$

Therefore by (3.36),

$$Q(\bar{\zeta}/\Delta, b(\bar{t}/\Delta^2)) \sim \left( \frac{\pi}{2\bar{d}_0\Delta} \right)^{1/2} (\bar{\zeta}^2 - \bar{d}_0^2)^{1/2} + O(1/\Delta^{3/2}), \quad (3.51)$$

where the square root is defined as in the infinite depth complex potential (2.40). To expand the integral in (3.34), we first expand  $(\xi + b(\bar{t}/\Delta^2))^{1/2}$  using (3.49) to obtain

$$\int_0^1 \frac{Q(\xi, b(\bar{t}/\Delta^2))}{\xi - \nu} d\xi \sim \left( \frac{2\pi\bar{d}_0}{\Delta} \right)^{1/2} \int_0^1 \frac{\xi^{1/2}(1-\xi)^{1/2}}{\xi - \nu} d\xi + O(1/\Delta^{3/2}),$$

and then expand  $\nu(\bar{\zeta}/\Delta, \bar{d}_0/\Delta)$  using (3.50) to obtain, after substituting  $\xi = (\tau + \bar{d}_0)/2\bar{d}_0$ ,

$$\int_0^1 \frac{Q(\xi, b(\bar{t}/\Delta^2))}{\xi - \nu(\bar{\zeta}/\Delta, \bar{d}_0/\Delta)} d\xi \sim \left( \frac{\pi}{2\bar{d}_0\Delta} \right)^{1/2} \int_{-\bar{d}_0}^{\bar{d}_0} \frac{(\bar{d}_0^2 - \tau^2)^{1/2}}{\tau - \bar{\zeta}} d\tau + O(1/\Delta^{3/2}). \quad (3.52)$$

The integral therein is exactly the integral in (2.37), thus

$$\int_0^1 \frac{Q(\xi, b(\bar{t}/\Delta^2))}{\xi - \nu(\bar{\zeta}/\Delta, \bar{d}_0/\Delta)} d\xi \sim \left( \frac{\pi^3}{2\bar{d}_0\Delta} \right)^{1/2} [(\bar{\zeta}^2 - \bar{d}_0^2)^{1/2} - \bar{\zeta}] + O(1/\Delta^{3/2}). \quad (3.53)$$

The constant integral in the complex velocity (3.34) is therefore

$$\int_0^1 \frac{Q(\xi, b(\bar{t}/\Delta^2))}{\xi + c(\bar{t}/\Delta^2)} d\xi \sim \left( \frac{\pi^3}{2\bar{d}_0\Delta} \right)^{1/2} i[\Delta - (\Delta^2 + \bar{d}_0^2)^{1/2}] + O(1/\Delta^{3/2}), \quad (3.54)$$

which is  $O(1/\Delta^{3/2})$ . Finally, substituting (3.51, 3.53, 3.54) into (3.34) we obtain

$$\frac{\partial \bar{w}_0}{\partial \bar{\zeta}} \sim i \left( 1 - \frac{\bar{\zeta}}{(\bar{\zeta}^2 - \bar{d}_0^2)^{1/2}} \right) + O(1/\Delta), \quad (3.55)$$

which implies

$$\bar{w}_0 \sim i[\bar{\zeta} - (\bar{\zeta}^2 - \bar{d}_0^2)^{1/2}] + O(1/\Delta). \quad (3.56)$$

The kinematic boundary condition on the free surface is unchanged, viz.

$$\frac{\partial \bar{\phi}_0}{\partial \bar{z}} = \frac{\partial \bar{h}_0}{\partial \bar{t}} \quad \text{on} \quad \bar{z} = 0, |\bar{x}| > \bar{d}_0(\bar{t}),$$

and therefore yields  $\bar{h}_0$  as in section 2.2.1.4. Then, the Wagner condition  $\bar{h}_0(\bar{d}_0(\bar{t}), \bar{t}) = \bar{d}_0(\bar{t})^2 - \bar{t}$  yields the law of motion of the free point as in section 2.2.1.7. Thus, for the parabolic body  $f(x) = x^2$  we find

$$\bar{d}_0(\bar{t}) \sim (2\bar{t})^{1/2} + O(1/\Delta). \quad (3.57)$$

Hence, the leading-order  $1/\Delta^2$ -timescale limit of the stage 2 solution is exactly (3.15 - 3.17). Further, the effect of the base enters the stage 1 problem at order  $1/\Delta$ , which confirms that it is valid to ignore the base and consider the infinite depth problem in stage 1 to lowest order in  $1/\Delta$ .

### 3.4 Intermediate time

In section 3.1 we argued that the extent of the interior region in stage 3 is large compared to the penetration depth (which only becomes comparable to the layer depth in stage 4). Therefore, in order to match into stage 3, we require the stage 2 turnover point  $d_0(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . It is not immediately clear how to expand the complex velocity (3.34, 3.35) as  $d_0(t) \rightarrow \infty$ , so we perform the large-time, large-aspect-ratio asymptotics directly on the stage 2 outer problem shown in Figure 3.3.

To investigate the large time behaviour and retain a balance in the kinematic boundary conditions on the body and free surface and in the Wagner condition (2.50) we scale

$$t = \bar{t}/\delta^2, \quad x = \bar{x}/\delta, \quad z = \bar{z}, \quad d_0(t) = \bar{d}_0(\bar{t})/\delta, \quad \phi_0 = \bar{\phi}_0, \quad h_0 = \bar{h}_0/\delta^2, \quad (3.58)$$

where  $\delta \ll 1$ . The stage 2 leading-order outer problem shown in Figure 3.3 becomes (dropping bars)

$$\delta^2 \frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad \text{for } -1 < z < 0, \quad (3.59)$$

$$\frac{\partial \phi_0}{\partial z} = 0 \quad \text{on } z = -1, \quad (3.60)$$

$$\frac{\partial \phi_0}{\partial z} = -1 \quad \text{on } z = 0, \quad |x| < d_0(t), \quad (3.61)$$

$$\frac{\partial \phi_0}{\partial z} = \frac{\partial h_0}{\partial t} \quad \text{on } z = 0, \quad |x| > d_0(t), \quad (3.62)$$

$$\phi_0 = 0 \quad \text{on } z = 0, \quad |x| > d_0(t), \quad (3.63)$$

together with the usual initial and far field conditions (3.8, 3.9), the Wagner condition (2.50) and inverse square root singularities in the first derivatives of  $\phi_0$  and  $h_0$  at the free points  $x = \pm d_0(t)$ .

We will show that the outer flow field in  $x \geq 0$  decomposes into the three regions shown in Figure 3.8, where for convenience we employ the symmetry condition

$$\frac{\partial \phi_0}{\partial x} = 0 \quad \text{on } x = 0. \quad (3.64)$$

By matching through the inner transition region of order unity in size, we will show the flow is uniform and exponentially small in the interior and exterior regions, respectively, which are of order  $1/\delta$  in length and order unity in height.

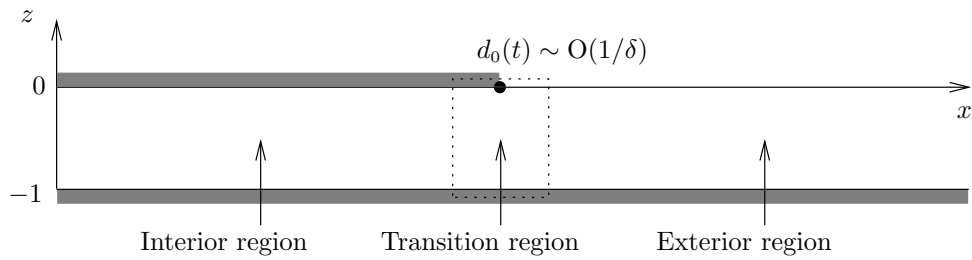


Figure 3.8: The large time asymptotic structure of the stage 2 leading-order outer problem on the timescale  $1/\delta$  where  $\delta \ll 1$ .

### 3.4.1 Interior region

The general solution of the interior region problem (3.59, 3.60, 3.61, 3.64) for  $\phi_0 = \phi_0^{(I)}$  on  $0 \leq x < d_0(t)$  is

$$\phi_0^{(I)} = \frac{1}{2} \left( \frac{x^2}{\delta^2} - (z+1)^2 \right) + \sum_{m=0}^{\infty} A_m \cosh(m\pi x/\delta) \cos(m\pi z), \quad (3.65)$$

where  $A_m$  are unknown functions of  $t$  and  $\delta$ . Matching with the exterior region solution through the transition region will confirm that the sum of eigensolutions is exponentially small as  $\delta \rightarrow 0$ . The flow is therefore uniform across the layer to lowest order.

### 3.4.2 Exterior region

The general solution of the exterior region problem (3.59, 3.60, 3.62, 3.63) for  $\phi_0 = \phi_0^{(E)}$  and  $h_0 = h_0^{(E)}$  on  $x > d_0(t)$  is

$$\phi_0^{(E)} = \sum_{m=0}^{\infty} B_m \exp(-(m+1/2)\pi x/\delta) \cos((m+1/2)\pi z), \quad (3.66)$$

$$h_0^{(E)} = \sum_{m=0}^{\infty} C_m \exp(-(m+1/2)\pi x/\delta) \cos((m+1/2)\pi z), \quad (3.67)$$

where  $B_m$  and

$$C_m = \pi(m+1/2) \int_0^t B_m(\tau, \delta) d\tau$$

are unknown functions of  $t$  and  $\delta$ . Again, matching with the interior region solution through the transition region will confirm that the sums of eigenfunctions are exponentially small as  $\delta \rightarrow 0$ .

### 3.4.3 Transition region

#### 3.4.3.1 Scalings and leading-order problem

To transform (3.59) into Laplace's equation near the free point  $x = d_0(t)$  and retain a balance in the Wagner condition (2.50) and the kinematic boundary condition (3.62) on the free surface we scale

$$\bar{x} = \bar{d}_0(\bar{t}) + \delta X, \quad \bar{z} = Z, \quad \bar{w}_0 = W_0/\delta, \quad \bar{h}_0 = H_0, \quad (3.68)$$

where the transition region complex potential  $W_0 = \Phi_0 + i\Psi_0$ , say. We substitute the scalings (3.68) into the codimension-two free boundary problem (3.59 - 3.63) to obtain the scaled problem (dropping bars)

$$\frac{\partial^2 \Phi_0}{\partial X^2} + \frac{\partial^2 \Phi_0}{\partial Z^2} = 0 \quad \text{for} \quad -1 < Z < 0, \quad (3.69)$$

$$\frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on} \quad Z = -1, \quad (3.70)$$

$$\delta + \frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on} \quad Z = 0, X < 0, \quad (3.71)$$

$$\delta \frac{\partial H_0}{\partial t} + \dot{d}_0(t) \frac{\partial H_0}{\partial X} - \frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on} \quad Z = 0, X > 0, \quad (3.72)$$

$$\Phi_0 = 0 \quad \text{on} \quad Z = 0, X > 0. \quad (3.73)$$

We expand  $\Phi_0$ ,  $H_0$  and  $d_0$  as asymptotic series in powers of  $\delta$  in the form

$$\Phi_0 \sim \Phi_{0,0} + \delta \Phi_{0,1} + \cdots, \quad (3.74)$$

$$H_0 \sim H_{0,0} + \delta H_{0,1} + \cdots, \quad (3.75)$$

$$d_0 \sim d_{0,0} + \delta d_{0,1} + \cdots, \quad (3.76)$$

to find leading-order problem is homogeneous (dropping the second subscript):

$$\frac{\partial^2 \Phi_0}{\partial X^2} + \frac{\partial^2 \Phi_0}{\partial Z^2} = 0 \quad \text{for } -1 < Z < 0, \quad (3.77)$$

$$\frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on } Z = -1, \quad (3.78)$$

$$\frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on } Z = 0, X < 0, \quad (3.79)$$

$$\frac{\partial \Phi_0}{\partial Z} + \dot{d}_0(t) \frac{\partial H_0}{\partial X} = 0 \quad \text{on } Z = 0, X > 0, \quad (3.80)$$

$$\Phi_0 = 0 \quad \text{on } Z = 0, X > 0, \quad (3.81)$$

together with the Wagner condition (2.50). This becomes

$$H_0(0, t) = d_0(t)^2 - t, \quad (3.82)$$

and there are inverse square root singularities in the first derivatives of  $\Phi_0$  and  $H_0$  at the origin and boundary conditions at infinity, which we now derive by matching with the interior and exterior regions.

### 3.4.3.2 Leading-order matching

We write the interior velocity potential (3.65) in transition region variables (3.68), viz.  $\delta\phi_0^{(I)}(d_0(t) + \delta X, Z, t)$ , and expand as  $\delta \rightarrow 0$  to obtain

$$\begin{aligned} & \frac{d_0(t)^2 + A_{0,0}(t)}{2\delta} + [d_0(t)X + A_{0,1}(t)] + \frac{\delta}{2} [(X^2 - (Z+1)^2) + A_{0,2}(t)] \\ & + \sum_{m=1}^{\infty} \delta A_m(t, \delta) \cosh(m\pi d_0(t)/\delta + m\pi X) \cos(m\pi Z), \end{aligned} \quad (3.83)$$

where we have expanded

$$A_0(t, \delta) \sim \frac{A_{0,0}(t)}{\delta^2} + \frac{A_{0,1}(t)}{\delta} + A_{0,2}(t, \delta). \quad (3.84)$$

We match (3.83) with the inner velocity potential defined by (3.68, 3.74) to find

$$A_{0,0}(t) = -d_0(t)^2, \quad (3.85)$$

$$\Phi_0 \sim d_0(t)X + A_{0,1}(t) \quad \text{as } X \rightarrow -\infty. \quad (3.86)$$

Similarly, we expand  $\delta\phi_0^{(E)}(d_0(t) + \delta X, Z, t)$  as  $\delta \rightarrow 0$  to obtain

$$\sum_{m=0}^{\infty} \delta B_m(t, \delta) \exp(-(m+1/2)\pi(d_0(t)/\delta + X)) \cos((m+1/2)\pi Z), \quad (3.87)$$

and match to find

$$\Phi_0 \sim 0 \quad \text{as } X \rightarrow \infty. \quad (3.88)$$

By exactly the same argument

$$H_0 \sim 0 \quad \text{as } X \rightarrow \infty. \quad (3.89)$$

### 3.4.3.3 Solution

The leading-order transition region problem (3.77 - 3.81) is a homogeneous, quasi-static mixed boundary value problem. As expected it is exactly the Wagner region problem with a base (see section 2.2.2). The unique eigensolution may be found by conformally mapping the strip  $-1 < Z < 0$  onto the lower half of the complex  $\zeta$ -plane through

$$\zeta = \xi + i\eta = \exp(\pi(X + iZ)), \quad (3.90)$$

to obtain the following mixed boundary value problem for the complex velocity,  $U_0 - iV_0$ , say, which is holomorphic on  $\eta < 0$ :

$$V_0 = 0 \quad \text{on} \quad \xi < 1, \quad \eta = 0, \quad (3.91)$$

$$U_0 = 0 \quad \text{on} \quad \xi > 1, \quad \eta = 0, \quad (3.92)$$

$$U_0 - iV_0 \sim \begin{cases} d_0(t) & \text{as } \zeta \rightarrow 0, \\ O((1 - \zeta)^{-1/2}) & \text{as } \zeta \rightarrow 1, \\ 0 & \text{as } |\zeta| \rightarrow \infty. \end{cases} \quad (3.93)$$

Here,  $(1 - \zeta)^{-1/2}$  is real on the lower edge of the branch cut  $\zeta = \xi < 1$ . The transformed problem has unique eigensolution

$$U_0 - iV_0 = \frac{d_0(t)}{\sqrt{1 - \zeta}}. \quad (3.94)$$

We substitute this into (3.94) and integrate, respecting (3.88), to find the velocity potential is

$$\Phi_0 = d_0(t)X - \frac{2}{\pi}d_0(t) \log \left| 1 + \sqrt{1 - \exp(\pi(X + iZ))} \right|. \quad (3.95)$$

Figure 3.9 shows a plot of the velocity field.

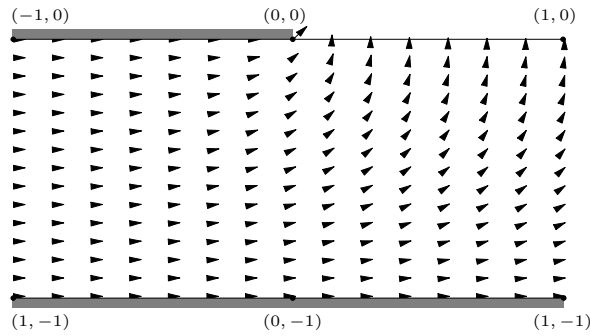


Figure 3.9: The leading-order transition region velocity field.

### 3.4.3.4 Higher-order matching

We expand (3.95) as  $X \rightarrow -\infty$  to find

$$\Phi_0 \sim d_0(t) \left( X - \frac{2}{\pi} \log 2 \right) + O(d_0(t)e^{\pi X}, d_0(t)e^{2\pi X}, \dots). \quad (3.96)$$

Hence, by (3.86),

$$A_{0,1}(t) = -\frac{2d_0(t)}{\pi} \log 2, \quad (3.97)$$

and by (3.83),

$$A_m(t, \delta) \sim O(\delta^{-1} d_0(t) \exp(-m\pi d_0(t)/\delta)) \quad \text{for } m \geq 1, \quad (3.98)$$

which verifies the assumption in section 3.4.1, that the sum of eigensolutions in the interior region velocity potential (3.65) is exponentially small.

Similarly, as  $X \rightarrow \infty$  we find

$$\Phi_0 \sim O(d_0(t)e^{-\pi X/2}, d_0(t)e^{-2\pi X/2}, \dots). \quad (3.99)$$

Hence, by (3.87),

$$B_m(t, \delta) \sim O(\delta^{-1} d_0(t) \exp((m+1/2)\pi d_0(t)/\delta)) \quad \text{for } m \geq 0, \quad (3.100)$$

which verifies the assumption in section 3.4.2, that the exterior region velocity potential (3.66) is exponentially small.

Summarizing, in the interior region where  $x \in (0, d_0(t))$  we have

$$\phi_0^{(I)} \sim \frac{1}{2\delta^2} (x^2 - d_0(t)^2) - \frac{2d_0(t)}{\pi\delta} \log 2 + O(1), \quad (3.101)$$

while in the exterior region where  $x \in (d_0(t), \infty)$  we have

$$\phi_0^{(E)} \sim O(\delta^{-1} d_0(t) \exp(\pi(d_0(t) - x)/2\delta) \cos(\pi z/2)). \quad (3.102)$$

### 3.4.3.5 Determination of the free surface elevation

By (3.80, 3.90, 3.94) we obtain

$$-\dot{d}_0(t) \frac{\partial H_0}{\partial X} = \frac{\partial \Phi_0}{\partial Z} = \frac{d_0(t)}{\sqrt{\exp(\pi X) - 1}} \quad \text{for } X > 0. \quad (3.103)$$

We integrate over  $(X, \infty)$ , with the zero matching condition (3.88) at infinity, to find the leading-order elevation

$$H_0(X, t) = \frac{d_0(t)}{\dot{d}_0(t)} \left( 1 - \frac{2}{\pi} \tan^{-1} \sqrt{\exp(\pi X) - 1} \right) \quad \text{for } X > 0, \quad (3.104)$$

which we plot in Figure 3.10.

### 3.4.4 Law of motion of the free point

Substituting the free surface elevation (3.104) at  $X = 0$  into the Wagner condition (3.82), we obtain the first order ordinary differential equation

$$\frac{dd_0}{dt} = \frac{d_0(t)}{d_0(t)^2 - t}, \quad (3.105)$$

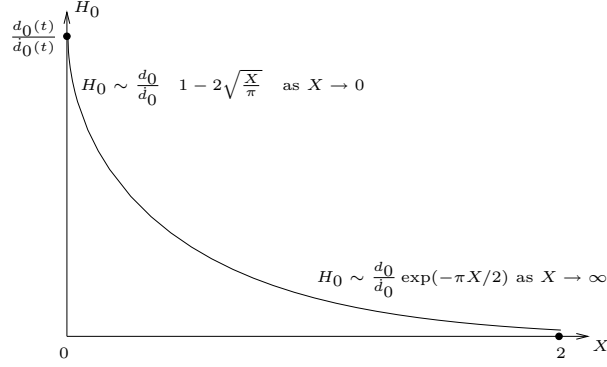


Figure 3.10: The leading-order free surface elevation in the transition region.

Fortunately this may be written as a complete differential and integrated to obtain

$$\int_{d_0(0)}^{d_0(t)} \xi^2 d\xi = \left[ t\xi \right]_{\xi=d_0(0)}^{\xi=d_0(t)}. \quad (3.106)$$

The initial  $x$ -coordinate of the turnover point  $d_0(0)$  is shown to be zero by matching with stage 2 in exactly the same way as we matched stages 1 and 2 in section 3.2. Hence,

$$d_0(t) = \sqrt{3t}. \quad (3.107)$$

### 3.4.5 Large time limit of the small time solution

With the hindsight of the above asymptotic solution it is straightforward to take the large time limit of the small time complex velocity (3.34, 3.35) in the transition region. Substituting the transition region scalings (3.68) into (3.34, 3.35) through (3.58), we find the transition region complex velocity

$$\frac{\partial \Phi}{\partial X} - i \frac{\partial \Phi}{\partial Z} = \frac{\delta \left( \nu + \frac{\varepsilon^2}{1-\varepsilon^2} \right)^{1/2}}{\pi \nu^{1/2} (1-\nu)^{1/2}} (I_1 + I_2), \quad (3.108)$$

where

$$I_1 = \int_0^1 \frac{\xi^{1/2} (1-\xi)^{1/2}}{\left( \xi + \frac{\varepsilon^2}{1-\varepsilon^2} \right)^{1/2}} \frac{d\xi}{\xi + \frac{\varepsilon}{1-\varepsilon}}, \quad (3.109)$$

$$I_2 = \int_0^1 \frac{\xi^{1/2} (1-\xi)^{1/2}}{\left( \xi + \frac{\varepsilon^2}{1-\varepsilon^2} \right)^{1/2}} \frac{d\xi}{\xi - \nu}, \quad (3.110)$$

$$\nu = \frac{\exp(\pi(X + iZ)) - \varepsilon^2}{1 - \varepsilon^2}, \quad (3.111)$$

and the small parameter (re-introducing bars)

$$\varepsilon = \exp \left( -\pi \bar{d}_0(\bar{t}) / \delta \right). \quad (3.112)$$

To evaluate the asymptotic limit of  $I_1$  as  $\varepsilon \rightarrow 0$ , we split the range of integration into  $(0, \delta_1)$ ,  $(\delta_1, \delta_2)$  and  $(\delta_2, 1)$ , where

$$\varepsilon^2 \ll \delta_1 \ll \varepsilon \ll \delta_2 \ll 1. \quad (3.113)$$

In the first interval we find, upon substituting  $\xi = \varepsilon^2 \tau$ ,

$$\int_0^{\delta_1} \frac{\xi^{1/2}(1-\xi)^{1/2}}{\left(\xi + \frac{\varepsilon^2}{1-\varepsilon^2}\right)^{1/2}} \frac{d\xi}{\xi + \frac{\varepsilon}{1-\varepsilon}} = \varepsilon \int_0^{\delta_1/\varepsilon^2} \frac{\tau^{1/2}(1-\varepsilon^2\tau)^{1/2}}{\left(\tau + \frac{1}{1-\varepsilon^2}\right)^{1/2}} \frac{d\tau}{\varepsilon\tau + 1} \sim O(\delta_1/\varepsilon).$$

Similarly, in the second interval we substitute  $\xi = \varepsilon\tau$  to find

$$\begin{aligned} \int_{\delta_1}^{\delta_2} \frac{\xi^{1/2}(1-\xi)^{1/2}}{\left(\xi + \frac{\varepsilon^2}{1-\varepsilon^2}\right)^{1/2}} \frac{d\xi}{\xi + \frac{\varepsilon}{1-\varepsilon}} &= \int_{\delta_1/\varepsilon}^{\delta_2/\varepsilon} \frac{\tau^{1/2}(1-\varepsilon\tau)^{1/2}}{\left(\tau + \frac{\varepsilon}{1-\varepsilon^2}\right)^{1/2}} \frac{d\tau}{\tau + 1} \\ &\sim \int_{\delta_1/\varepsilon}^{\delta_2/\varepsilon} \frac{d\tau}{\tau + 1} + O(\delta_2) \\ &= \log(1 + \delta_2/\varepsilon) - \log(1 + \delta_1/\varepsilon) + O(\delta_2). \end{aligned}$$

Finally, in the third interval we find

$$\begin{aligned} \int_{\delta_2}^1 \frac{\xi^{1/2}(1-\xi)^{1/2}}{\left(\xi + \frac{\varepsilon^2}{1-\varepsilon^2}\right)^{1/2}} \frac{d\xi}{\xi + \frac{\varepsilon}{1-\varepsilon}} &\sim \int_{\delta_2}^1 \frac{(1-\xi)^{1/2}}{\xi} d\xi + O(\varepsilon) \\ &\sim 2 \tanh^{-1} \sqrt{1-\delta_2} - 2\sqrt{1-\delta_2} + O(\varepsilon) \\ &\sim \log \frac{1}{\delta_2} + O(1). \end{aligned}$$

Adding these three asymptotic expansions and employing (3.113) implies

$$I_1 \sim \log\left(\frac{1}{\varepsilon} + \frac{1}{\delta_2}\right) + O(1) \sim \log \frac{1}{\varepsilon} + O(1).$$

Hence, in contrast to the small time limit found in section 3.3.3, the dominant contribution to the large time complex velocity is due to the constant integral  $I_1$ , because  $I_2$  is order unity as  $\varepsilon \rightarrow 0$  by (3.110). Expanding (3.111) we find

$$\nu \sim \exp(\pi(X + iZ)) + O(\varepsilon^2),$$

and therefore the complex velocity (3.108) becomes

$$\begin{aligned} \frac{\partial \Phi}{\partial X} - i \frac{\partial \Phi}{\partial Z} &\sim \frac{\delta}{\pi \sqrt{1 - \exp(\pi(X + iZ))}} (I_1 + I_2) \\ &\sim \frac{\bar{d}_0(\bar{t})}{\pi \sqrt{1 - \exp(\pi(X + iZ))}} + O(1), \end{aligned}$$

because  $\log(1/\varepsilon) = \pi \bar{d}_0(\bar{t})/\delta$  by (3.112). This is *exactly* the leading-order transition region complex velocity (3.94), which confirms the validity of our large time asymptotic solution of the lowest order small time mixed boundary value problem (in Figure 3.3) and therefore the law of motion of the free point (3.107).

### 3.4.6 Inner regions

For clarity we re-introduce bars on the stage 2 large-time variables defined by (3.58). Then, by (3.95),

$$W_0 \sim -i\bar{S}(\bar{d}_0(\bar{t}))\sqrt{X+iZ} \quad \text{as } X+iZ \rightarrow 0, \quad (3.114)$$

where

$$\bar{S}(\bar{d}_0(\bar{t})) = \frac{2\bar{d}_0'(\bar{t})}{\pi^{1/2}}. \quad (3.115)$$

This is in agreement with the small  $\delta$  asymptotics of (3.38) because, splitting the range of integration in (3.39) at  $\xi = c$  and Taylor expanding over the two ranges  $(0, c)$  and  $(c, 1)$  as  $b, c \rightarrow 0$ , implies the square root in (3.38) has coefficient

$$S(\bar{d}_0(\bar{t})/\delta) \sim \frac{2\bar{d}_0'(\bar{t})}{\pi^{1/2}\delta} \quad \text{as } \delta \rightarrow 0. \quad (3.116)$$

Thus

$$w_0(\bar{d}_0(\bar{t})/\delta + X + iZ, \bar{t}/\delta^2) \sim -i\frac{2\bar{d}_0'(\bar{t})}{\pi^{1/2}\delta}\sqrt{X+iZ}, \quad (3.117)$$

with  $X + iZ \rightarrow 0$  and then  $\delta \rightarrow 0$ , which is in agreement with (3.114) by the transition region scalings (3.68).

Hence, the strength of the inverse square root singularity at the free point scales with  $1/\delta$  at large times  $t = O(1/\delta^2)$ , which implies the asymptotic structure sketched in Figure 3.11.

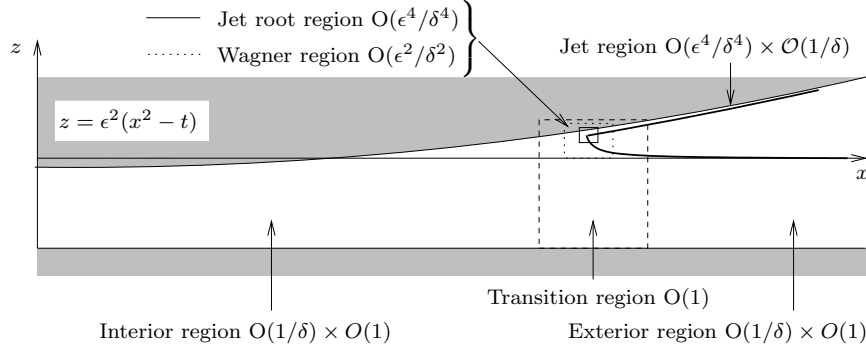


Figure 3.11: The asymptotic structure of stage 2 at times  $t = O(1/\delta^2)$  for  $\delta \ll 1$  (not to scale).

Relative to the original dimensionless variables, the Wagner and jet root regions have size  $O(\epsilon^2/\delta^2)$  and  $O(\epsilon^4/\delta^4)$ , respectively, so the jet region has thickness  $O(\epsilon^4/\delta^4)$  and length  $O(1/\delta)$ . Relative to the original dimensionless variables the asymptotic jet thickness is

$$\epsilon^4 H_0(\bar{t}/\delta^2) \sim \frac{\epsilon^4}{\delta^4} \left( \frac{\bar{d}_0'(\bar{t})}{2\bar{d}_0'(\bar{t})} \right)^2 \quad \text{as } \delta \rightarrow 0, \quad (3.118)$$

by (2.77, 3.116).

### 3.4.7 Pressure and force on the body

We proceed as in stage 1 to find the interior region pressure  $\bar{p}^{(1)} \sim \bar{p}_0^{(1)}/\epsilon^2 + O(1/\epsilon)$ , where by (3.101),

$$\bar{p}_0^{(1)} = -\frac{\partial \bar{\phi}_0^{(1)}}{\partial \bar{t}} = \bar{d}_0(\bar{t})\bar{d}'_0(\bar{t}), \quad (3.119)$$

i.e. for the parabolic body  $f(\bar{x}) = \bar{x}^2$  the leading-order interior pressure is constant, viz.  $\bar{p}_0^{(1)} = 3/2$ . The transition region pressure  $P \sim P_0/\epsilon^2 + O(1/\epsilon)$ , where by (3.95),

$$P_0 = \bar{d}_0(\bar{t})\bar{d}'_0(\bar{t})\text{Re} \left( (1 - \exp(\pi(X + iZ)))^{-1/2} \right), \quad (3.120)$$

Note that, as required, the leading-order interior pressure matches with the leading-order transition pressure, viz.

$$P_0 \sim \bar{d}_0(\bar{t})\bar{d}'_0(\bar{t}) = \bar{p}_0^{(1)}(\bar{d}_0(\bar{t}), \bar{t}) \quad \text{as } X \rightarrow -\infty. \quad (3.121)$$

Hence, the boundary condition  $\bar{\phi}_0^{(1)}(\bar{d}_0(\bar{t}), \bar{t}) = 0$ , deduced from (3.85, 3.101), is equivalent to a pressure matching condition. This may be seen directly by differentiating it to obtain

$$\frac{\partial \bar{\phi}_0^{(1)}}{\partial \bar{t}} + \bar{d}'_0 \frac{\partial \bar{\phi}_0^{(1)}}{\partial \bar{x}} = 0 \quad \text{at } \bar{x} = \bar{d}_0(\bar{t}), \quad (3.122)$$

and recalling  $\bar{p}_0^{(1)} = -\partial \bar{\phi}_0^{(1)}/\partial \bar{t}$  and  $\partial \bar{\phi}_0^{(1)}/\partial \bar{x}(\bar{d}_0(\bar{t}), \bar{t}) = \bar{d}_0(\bar{t})$ .

As  $X \rightarrow \infty$ , we enter the exterior region in which the pressure is exponentially small. Table 3.3 shows the order of magnitude of the pressure, wetted length and resulting force on the body (per unit length in the  $y$ -direction) for each region, except the exterior for which there is no contact with the body.

| Region     | Pressure              | Wetted length         | Force                 |
|------------|-----------------------|-----------------------|-----------------------|
| Interior   | $1/\epsilon^2$        | $1/\delta$            | $1/\delta\epsilon^2$  |
| Transition | $1/\epsilon^2$        | 1                     | $1/\epsilon^2$        |
| Wagner     | $\delta/\epsilon^3$   | $\epsilon^2/\delta^2$ | $1/\delta\epsilon$    |
| Jet root   | $\delta^2/\epsilon^4$ | $\epsilon^4/\delta^4$ | $1/\delta^2$          |
| Jet        | $\epsilon^2/\delta^2$ | $1/\delta$            | $\epsilon^2/\delta^3$ |

Table 3.3: Order of magnitudes of the pressure, wetted length and force on the body in stage 3 for which  $\epsilon \ll \delta \ll 1$ .

As in stages 1 and 2, the leading-order force on the body of order  $1/\delta\epsilon^2$  is due solely to the interior pressure, while the first order correction of order  $1/\epsilon^2$  is now due to the transition pressure (rather than the Wagner pressure (2.78)). Profiles of the composite pressure  $\bar{p}_c$  on the parabolic body  $f(\bar{x}) = \bar{x}^2$  are sketched in Figure 3.12.

### 3.4.8 Summary

The large time limit of stage 2 is valid for times of order  $\epsilon^2/\delta^2$ , which is exactly the stage 3 timescale provided  $\epsilon \ll \delta \ll 1$ . Thus, Figure 3.11 shows the asymptotic structure of stage

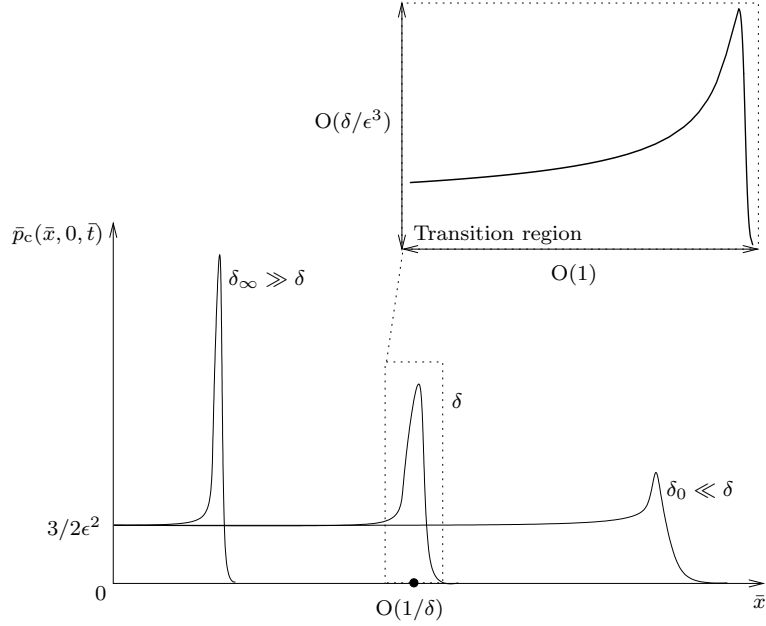


Figure 3.12: Schematic of stage 3 composite pressure profiles on the body at times  $\epsilon^2/\delta^2$  for  $\epsilon \ll \delta_0 \ll \delta \ll \delta_\infty \ll 1$ . The box contains a plot of the the composite pressure in the transition region for  $\epsilon = 0.01$  and  $\delta = 0.1$ .

3. By (3.64, 3.65, 3.85, 3.101, 3.105) the stage 3 interior codimension-one<sup>3</sup> free boundary problem<sup>4</sup> is

$$\frac{\partial^2 \bar{\phi}_0}{\partial \bar{x}^2} = 1 \quad \text{for } 0 < \bar{x} < \bar{d}_0, \quad (3.123)$$

$$\frac{\partial \bar{\phi}_0}{\partial \bar{x}} = 0 \quad \text{for } \bar{x} = 0, \quad (3.124)$$

$$\bar{\phi}_0 = 0 \quad \text{for } \bar{x} = \bar{d}_0, \quad (3.125)$$

$$\frac{\partial \bar{\phi}_0}{\partial \bar{x}} - \frac{d\bar{d}_0}{d\bar{t}} (\bar{d}_0(\bar{t})^2 - \bar{t}) = 0 \quad \text{for } \bar{x} = \bar{d}_0. \quad (3.126)$$

The evolution of the stage 2 codimension-two problem to this stage 3 codimension-one problem occurs because of the evolution of the large aspect ratio interior region and the resulting formation of the order unity size transition region. Six remarks should be made:

(1) The consistency condition for (3.123 - 3.126) is

$$\bar{d}_0 = \int_0^{\bar{d}_0} \frac{\partial^2 \bar{\phi}_0}{\partial \bar{x}^2} d\bar{x} = \left[ \frac{\partial \bar{\phi}_0}{\partial \bar{x}} \right]_{\bar{x}=0}^{\bar{x}=\bar{d}_0} = \frac{d\bar{d}_0}{d\bar{t}} (\bar{d}_0(\bar{t})^2 - \bar{t}). \quad (3.127)$$

which is simply the stage 3 law of motion of the turnover point.

(2) The consistency condition (3.127) may be deduced directly by applying a global conservation of mass argument to the transition region, taking care of the matching conditions at infinity and the inverse square root singularity at the origin.

<sup>3</sup>The free point now has only one dimension fewer than the space in which the motion occurs.

<sup>4</sup>The form of the kinematic boundary condition (3.126) is chosen because it enables a simple generalization to the three-dimensional case in section 3.7.4.

(3) The stage 3 law of motion (3.106) may be written

$$\int_0^{\bar{d}_0(\bar{t})} (f(\xi) - \bar{t}) d\xi = 0, \quad (3.128)$$

which means that all the displaced fluid remains in the interior region to lowest order. Clearly this agrees with the asymptotic structure depicted in Figure 3.11 for  $\epsilon^2 \ll \delta \ll 1$ .

(4) There is zero flow in the exterior region to lowest order so the boundary conditions (3.125, 3.126) at  $\bar{x} = d_0(\bar{t})$  are generalized “Rankine-Hugoniot conservation conditions” [54], viz.

$$\left[ \bar{\phi}_0 \right]_{\bar{d}_0-}^{\bar{d}_0+} = 0, \quad (3.129)$$

$$\left[ \frac{\partial \bar{\phi}_0}{\partial \bar{x}} \right]_{\bar{d}_0-}^{\bar{d}_0+} = -\frac{d\bar{d}_0}{d\bar{t}} (\bar{d}_0^2 - \bar{t}); \quad (3.130)$$

as shown in the previous section, (3.129) represents continuity of pressure across the ‘shock’ or transition region, while (3.130) represents conservation of mass.

(5) The large time asymptotics of stage 2 are an example of the long wavelength approximation of a codimension-two free boundary problem, as described by Howison *et. al.* [29]. There is hardly any rigorous theory *unless* the leading-order approximation can be formulated as a variational inequality [12], which we discuss further in section 3.7.4.7 for the three dimensional case.

(6) As shown in Figure 3.11, at times of order  $\epsilon^2/\delta^2$ , the sizes of the transition, Wagner and jet root regions are of order 1,  $\epsilon^2/\delta^2$  and  $\epsilon^4/\delta^4$ , respectively. Hence, as  $\delta \rightarrow \epsilon$  and we approach order unity time, we expect the inner regions to merge into a jet root region of order unity size with pressure of order  $1/\epsilon^2$  by Table 3.3. This confirms the scenario of Korobkin [37] with asymptotic structure sketched in Figure 3.1, which we now describe.

## 3.5 Time of order unity

The flow field in  $x > 0$  decomposes into the four regions shown in Figure 3.1. The (dimensionless) interior, exterior and jet regions have length  $O(1/\epsilon)$  and thickness  $O(1)$ , which is the size of the jet root region.

### 3.5.1 Interior and exterior regions

We proceed as in previous stages with the scaled outer variables

$$x = \frac{1}{\epsilon} \hat{x}, \quad z = \hat{z}, \quad t = \hat{t}, \quad \phi = \hat{\phi}, \quad h = \hat{h}, \quad (3.131)$$

by substituting into (3.3 - 3.9) and dropping hats to find

$$\epsilon^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid}, \quad (3.132)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -1, \quad (3.133)$$

$$2\epsilon^2 x \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z} = 1 \quad \text{on } z = x^2 - t, \quad (3.134)$$

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{on } x = 0, \quad (3.135)$$

$$\frac{\partial \phi}{\partial z} - \frac{\partial h}{\partial t} - \epsilon^2 \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} = 0 \quad \text{on } z = h(x, t), \quad (3.136)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \epsilon^2 \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right) = 0 \quad \text{on } z = h(x, t), \quad (3.137)$$

$$\phi = h = d = 0 \quad \text{at } t = 0, \quad (3.138)$$

$$\phi, h \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (3.139)$$

The interior region problem (3.132, 3.133, 3.134, 3.135), for  $\phi = \phi^{(I)}$  on  $0 \leq x < d_0(t)$ , is a simple generalization of the stage 3 interior region problem. The high velocities generated under the impactor, together with (3.132, 3.134), suggest that

$$\phi^{(I)} \sim \frac{1}{\epsilon^2} \phi_0^{(I)}(x, t) + \frac{1}{\epsilon} \phi_1^{(I)}(t) + \phi_2^{(I)}(x, z, t) + O(\epsilon). \quad (3.140)$$

where the solvability conditions for  $\phi_0^{(I)}$  and  $\phi_1^{(I)}$  imply that

$$\phi_0^{(I)} = \int_{x_0(t)}^x \frac{\xi}{l(\xi, t)} d\xi, \quad (3.141)$$

$$\phi_2^{(I)} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{x}{l(x, t)} \right) (z+1)^2 + \phi_1^{(I)}(t), \quad (3.142)$$

where  $x_0(t)$  and  $\phi_1^{(I)}(t)$  are order unity functions to be determined and the layer thickness is

$$l(x, t) = 1 + x^2 - t. \quad (3.143)$$

As in stage 3, the lowest order flow is uniform across the layer.

Similarly, the exterior region problem (3.132, 3.133, 3.136, 3.137, 3.139), for  $\phi = \phi^{(E)}$  on  $x > d_0(t)$ , is a simple generalization of the stage 3 exterior region problem with solution

$$\phi^{(E)} \sim \frac{1}{\epsilon} A(t) \exp((d_0(t) - x)/2), \quad (3.144)$$

where  $A(t)$  is an order unity function that we leave undetermined. As in stage 3, the solution is exponentially small corresponding to small amplitude zero gravity water waves.

### 3.5.2 Jet root region

To match with the outer regions the inner jet root region must have size of order unity, which is also in accordance with the large time behaviour of the stage 3 inner regions. The inner velocity must be comparable to the velocity of the turnover point of  $O(1/\epsilon)$ . The jet root region scalings are therefore

$$x = \frac{d(t)}{\epsilon} + X, \quad z = Z, \quad \phi = \frac{1}{\epsilon} \left( \dot{d}(t)X + \Phi(X, Z, t) \right), \quad h = H(X, t), \quad (3.145)$$

where the introduction of the  $\dot{d}(t)X/\epsilon$  term in the inner velocity potential simplifies the algebra.

We substitute the scalings (3.145) into (3.3 - 3.7) to obtain the following jet root problem. In the fluid,

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Z^2} = 0, \quad (3.146)$$

on the base  $Z = -1$ ,

$$\frac{\partial \Phi}{\partial Z} = 0, \quad (3.147)$$

on the body  $Z = (d(t) + \epsilon X)^2 - t$ ,

$$\frac{\partial \Phi}{\partial Z} = -\epsilon + \epsilon^2 2(d(t) + \epsilon X) \frac{\partial \Phi}{\partial X}, \quad (3.148)$$

and on the free surface  $Z = H(X, t)$ , which is multivalued,

$$\frac{\partial \Phi}{\partial Z} \mp \frac{\partial \Phi}{\partial X} \frac{\partial H}{\partial X} = \epsilon \frac{\partial H}{\partial t}, \quad (3.149)$$

$$\left( \frac{\partial \Phi}{\partial X} \right)^2 + \left( \frac{\partial \Phi}{\partial Z} \right)^2 - \dot{d}(t)^2 = -\frac{\epsilon}{2} \left( \ddot{d}(t)X + \frac{\partial \phi}{\partial t} \right). \quad (3.150)$$

In (3.149) we take the minus(plus) sign on the segment of the free surface  $Z = H(x, t)$  with upward(downward) and outward pointing normal.

We expand  $\Phi$ ,  $H$  and  $d$  as asymptotic series in powers of  $\epsilon$  as usual. The body is flat to lowest order because

$$Z = (d(t) + \epsilon X)^2 - t \sim (d_0(t)^2 - t) + \epsilon 2X d_0(t) + O(\epsilon^2),$$

so we linearize the kinematic boundary condition on the body (3.148) onto  $Z = d_0(t)^2 - t$  and the free surface conditions (3.149, 3.150) onto  $Z = H_0(X, t)$ . Finally, we substitute the asymptotic expansions for  $\Phi$ ,  $H$  and  $d$  into the resulting scaled problem to obtain the leading-order jet root region problem

$$\frac{\partial^2 \Phi_0}{\partial X^2} + \frac{\partial^2 \Phi_0}{\partial Z^2} = 0 \quad \text{in the fluid}, \quad (3.151)$$

$$\frac{\partial \Phi_0}{\partial Z} = 0 \quad \text{on } Z = -1 \quad \text{and } d_0(t)^2 - t, \quad (3.152)$$

$$\frac{\partial \Phi_0}{\partial N} = 0 \quad \text{on } Z = H_0(X, t) \quad (3.153)$$

$$\left( \frac{\partial \Phi_0}{\partial X} \right)^2 + \left( \frac{\partial \Phi_0}{\partial Z} \right)^2 - \dot{d}_0(t)^2 = 0 \quad \text{on } Z = H_0(X, t), \quad (3.154)$$

where  $\mathbf{N}$  is the outward pointing unit normal to the fluid domain,  $\Omega$ , say. Comparing this (see Figure 3.13) with the leading-order jet root problem in stages 1, 2 and 3 sketched in Figure 2.6, Laplace's equation, the body boundary condition and the free surface boundary conditions are exactly the same. However, in stage 4 there is a base and the flow is driven by matching with the interior and exterior regions, rather than a singularity in the outer problem, as in stages 1, 2 and 3.

### 3.5.2.1 Matching with the interior and exterior regions

We write the interior velocity potential (3.140) in jet root variables (3.145), viz.  $\epsilon(\phi^{(I)}(d(t) + \epsilon X, Z, t) - \dot{d}X)$ , and expand as  $\epsilon \rightarrow 0$  to obtain

$$\frac{1}{\epsilon} \int_{x_0(t)}^{d_0(t)} \frac{\xi}{l(\xi, t)} d\xi + \left( \frac{d_0}{l(d_0, t)} - \dot{d}_0 \right) X + O(\epsilon). \quad (3.155)$$

We substitute the outer coordinates (3.131) into the inner jet root velocity potential (3.145), expand and write in terms of the inner coordinates  $(X, Z)$  to obtain

$$\Phi_0(X, Z, t) + O(\epsilon). \quad (3.156)$$

Matching (3.155) and (3.156), we obtain without loss of generality,

$$x_0(t) = d_0(t), \quad (3.157)$$

and

$$\Phi_0 \sim \left( \frac{d_0}{l(d_0, t)} - \dot{d}_0 \right) X + \Phi_0^{(I)}(t) \quad \text{as } X \rightarrow -\infty, \quad -1 < Z < d_0(t)^2 - t, \quad (3.158)$$

where  $\Phi_0^{(I)}(t)$  is order unity. Similarly, assuming the exterior solution is exponentially small, which is verified *a posteriori* as in stage 3, we find

$$\Phi_0 \sim -\dot{d}_0 X + \Phi_0^{(E)}(t) \quad \text{as } X \rightarrow \infty, \quad -1 < Z < 0, \quad (3.159)$$

$$H_0 \sim 0 \quad \text{as } X \rightarrow \infty, \quad (3.160)$$

where  $\Phi_0^{(E)}(t) = O(1)$ . We leave  $\Phi_0^{(I)}(t)$  and  $\Phi_0^{(E)}(t)$  undetermined.

### 3.5.2.2 Flow structure and matching with the jet region

The flow structure in the jet root region is shown in Figure 3.13. The incoming stream, of speed  $\dot{d}_0$  and of height  $1 < l(d_0, t)$ , from the exterior region at  $X = \infty$  is split into two. The fluid above the dividing streamline is turned backwards into a right-flowing stream. This jet root has unknown thickness  $H_J(t)$  and speed equal to that of the incoming stream, i.e.  $\dot{d}_0$ , by Bernoulli's equation on the free surface (3.154). The matching condition with the outer jet region is therefore

$$\Phi_0 \sim \dot{d}_0 X + \Phi_0^{(J)}(t) \quad \text{as } X \rightarrow \infty, \quad d_0(t)^2 - t - H_J(t) < Z < d_0(t)^2 - t, \quad (3.161)$$

$$H_0 \sim d_0(t)^2 - t - H_J(t) \quad \text{as } X \rightarrow \infty, \quad (3.162)$$

where  $\Phi_0^{(J)}(t)$  is an order unity matching function that we leave undetermined. The fluid below the dividing streamline flows onward toward the interior region, where it fills the channel of height  $l(d_0, t)$  but moves with reduced speed  $\dot{d}_0 - d_0/l(d_0, t)$ .

### 3.5.2.3 Global arguments

Tuck & Dixon [65] found the exact solution to this problem by hodograph methods in an analysis of the flow past a two-dimensional skimming plate in shallow water, which we

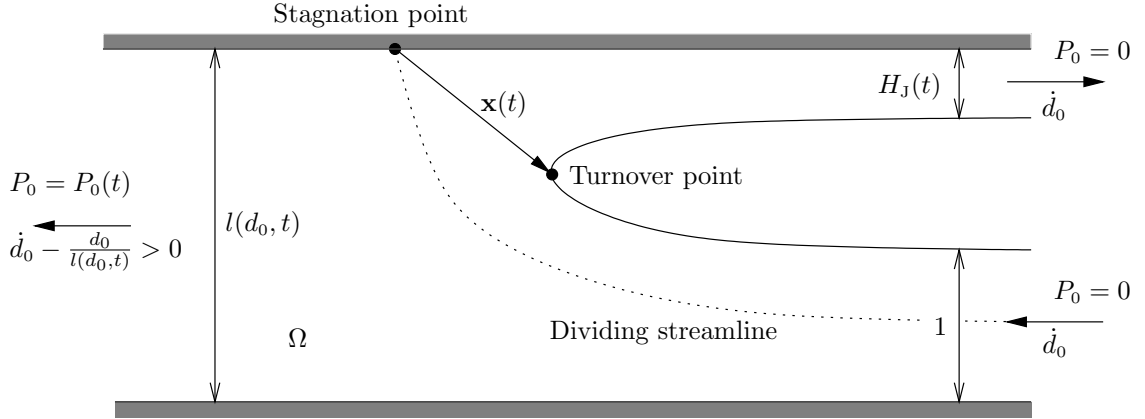


Figure 3.13: Schematic of the flow morphology in the stage 4 jet root region.

describe in section 5.2.2.2. However, we will now see that global conservation of mass and momentum together with the leading-order Bernoulli equation<sup>5</sup>

$$P_0 + \frac{1}{2}|\nabla\Phi_0|^2 = \frac{1}{2}\dot{d}_0^2, \quad (3.163)$$

are sufficient to determine the unknown asymptotic jet thickness  $J_0(t)$ , outflow speed  $\dot{d}_0 - d_0/l(d_0, t)$  and outflow pressure  $P_0(t)$  at  $X = -\infty$ .

### Conservation of mass

As in stage 3 (c.f. remark (2) on page 78), we apply Green's Theorem to  $\nabla\Phi_0$  on the fluid domain  $\Omega$  to find

$$0 = \iint_{\Omega} \nabla^2 \Phi_0 dX dZ = \int_{\partial\Omega} \frac{\partial\Phi_0}{\partial N} dS,$$

where  $\mathbf{N}$  is the outward pointing unit normal to the boundary  $\partial\Omega$  parametrized by arc length  $S$ . The normal velocity  $\partial\Phi_0/\partial N$  is zero on the base, wall and free surface by (3.152, 3.153), so, taking care of the boundary conditions at infinity (3.158, 3.161, 3.161) we deduce the global conservation of mass equation

$$\dot{d}_0 \cdot 1 = \dot{d}_0 \cdot H_J + \left( \dot{d}_0 - \frac{d_0}{l(d_0, t)} \right) \cdot l(d_0, t). \quad (3.164)$$

### Conservation of momentum

By Bernoulli's equation (3.163) and Green's theorem we find

$$0 = \int \int_{\Omega} \nabla \cdot \left( P_0 + \frac{1}{2}|\nabla\Phi_0|^2 \right) dX dZ = \int_{\partial\Omega} \left( P_0 + \frac{1}{2}|\nabla\Phi_0|^2 \right) \mathbf{N} dS.$$

We substitute the following integral identity<sup>6</sup> for harmonic  $\Phi_0$ , that relates the fluid speed to the normal velocity on the boundary

$$\int_{\partial\Omega} \frac{1}{2}|\nabla\Phi_0|^2 \mathbf{N} dS = \int_{\partial\Omega} \frac{\partial\Phi_0}{\partial N} \nabla\Phi_0 dS,$$

<sup>5</sup>Derived by writing Bernoulli's equation (2.18) in jet root region variables (3.145), substituting the usual asymptotic expansion for  $\Phi$  and expanding  $P \sim P_0/\epsilon^2 + O(1/\epsilon)$ .

<sup>6</sup>We have been unable to prove this identity in a coordinate free way.

to obtain

$$0 = \int_{\partial\Omega} P_0 \mathbf{N} + \frac{\partial\Phi_0}{\partial N} \nabla\Phi_0 dS. \quad (3.165)$$

The normal velocity  $\partial\Phi_0/\partial N$  is zero on the base, wall and free surface by (3.152, 3.153). The pressure at infinity in the incoming stream and jet is zero by Bernoulli's equation (3.163) and the matching conditions (3.159, 3.161). The pressure  $P_0(t)$  at infinity in the outgoing stream is found by applying Bernoulli's equation (3.163) on the base, together with the matching condition (3.158), viz.

$$P_0(t) = \frac{1}{2} \left( \dot{d}_0^2 - \left( \dot{d}_0 - \frac{d_0}{l(d_0, t)} \right)^2 \right). \quad (3.166)$$

Hence, taking care of the boundary conditions at infinity (3.158, 3.159, 3.161), the  $X$ -component of (3.165) implies the equation representing global conservation of horizontal momentum

$$\dot{d}_0^2 \cdot 1 = -\dot{d}_0^2 \cdot H_J + \left( P_0(t) + \left( \dot{d}_0 - \frac{d_0}{l(d_0, t)} \right)^2 \right) \cdot l(d_0, t). \quad (3.167)$$

### Solution

The three equations (3.164, 3.166, 3.167) for the jet height  $H_J(t)$ , outflow speed  $\dot{d}_0 - d_0/l(d_0, t)$  and outflow pressure  $P_0(t)$ , may be reduced to a quadratic equation for the outflow speed. The physically relevant solution is

$$H_J(t) = \left( l(d_0, t)^{1/2} - 1 \right)^2, \quad (3.168)$$

$$\dot{d}_0 - \frac{d_0}{l(d_0, t)} = \dot{d}_0 \left( \frac{2}{l(d_0, t)^{1/2}} - 1 \right), \quad (3.169)$$

$$P_0(t) = \frac{d_0 \dot{d}_0}{l(d_0, t)^{3/2}}. \quad (3.170)$$

Note that the outflow speed  $\dot{d}_0 - d_0/l(d_0, t)$  is less than the inflow speed  $\dot{d}_0$ .

The jet region has length and thickness of order  $1/\epsilon$  and unity, respectively, and therefore has exactly the same structure as in all previous stages analysed in section 2.4.1.

### 3.5.3 Law of motion of the free point

The law of motion of the free point is simply equation (3.169) which rearranges to

$$\frac{dd_0}{dt} = \frac{d_0(t)}{2l(d_0(t), t)^{1/2}(l(d_0(t), t)^{1/2} - 1)}, \quad (3.171)$$

where we recall the layer thickness is  $l(d_0(t), t) = 1 + d_0(t)^2 - t$ . Matching with the stage 3 solution implies

$$d_0(t) \sim \sqrt{3t} \quad \text{as } t \rightarrow 0, \quad (3.172)$$

which guarantees stage 4 has zero initial conditions to lowest order.

The ordinary differential equation (3.171) was solved numerically using the small time asymptotics (3.172) to start the fourth-fifth order Runge-Kutta scheme. Figure 3.14 shows plots of  $d_0(t)$  and  $(3t)^{1/2} \sim d_0(t)$  as  $t \rightarrow 0$ .

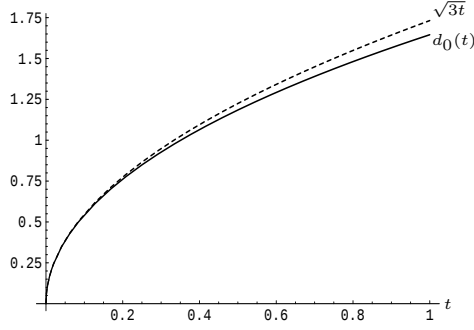


Figure 3.14: Plot of the location of the stage 4 turnover point for the parabolic body  $f(x) = x^2$ .

### 3.5.4 Pressure and force on the body

We proceed as in stage 1 by writing Bernoulli's equation (2.6) in outer variables (3.145) to find the interior region pressure  $p^{(i)} \sim p_0^{(i)}(x, t)/\epsilon^2 + O(1/\epsilon)$ , where, by (3.141),

$$p_0^{(i)} = -\frac{\partial \phi_0^{(i)}}{\partial t} - \frac{1}{2} \left( \frac{\partial \phi_0^{(i)}}{\partial x} \right)^2, \quad (3.173)$$

$$= \frac{d_0 \dot{d}_0}{l(d_0, t)} - \int_{d_0}^x \frac{\xi}{l(\xi, t)^2} d\xi - \frac{1}{2} \left( \frac{x}{l(x, t)} \right)^2. \quad (3.174)$$

As required the leading-order pressure in the interior region  $p_0^{(i)}$  as  $x \rightarrow d_0 -$  is equal to the leading-order transition region pressure  $P_0$  as  $X \rightarrow -\infty$ , viz.

$$p_0^{(i)}(d_0, t) = P_0(t), \quad (3.175)$$

by (3.166, 3.174). Further, as  $t \rightarrow 1 -$ ,

$$p_0^{(i)}(0, t) \sim -\frac{1}{2} \log(1 - t), \quad (3.176)$$

which implies compressibility effects will become important at the origin where the incompressible water entry model will break down [40]. As  $X \rightarrow \infty$ , we enter the exterior region in which the pressure is exponentially small. Table 3.4 shows the order of magnitude of the pressure, wetted length and resulting force on the body per unit length in the  $y$ -direction for each region, except the exterior for which there is no contact with the body. These scalings are retrieved from the stage 3 scalings shown in Table 3.3 as  $\delta \rightarrow \epsilon$ . Profiles of the leading-order composite pressure  $p_c$  on the parabolic body  $f(x) = x^2$  are plotted in Figure 3.15 for  $\epsilon = 0.01$ .

| Region   | Pressure       | Wetted length | Force          |
|----------|----------------|---------------|----------------|
| Interior | $1/\epsilon^2$ | $1/\epsilon$  | $1/\epsilon^3$ |
| Jet root | $1/\epsilon^2$ | 1             | $1/\epsilon^2$ |
| Jet      | 1              | $1/\epsilon$  | $1/\epsilon$   |

Table 3.4: Order of magnitudes of the pressure, wetted length and force on the body in stage 4.

As in previous stages, the leading-order force on the body is due solely to the interior pressure. Writing  $F(t) \sim F_0(t)/\epsilon^2 + F_1(t)/\epsilon + O(1)$ , (3.174) implies

$$F_0(t) = \int_{-d_0(t)}^{d_0(t)} \frac{\partial \phi_0^{(i)}}{\partial t}(\xi, t) + \frac{1}{2} \left( \frac{\partial \phi_0^{(i)}}{\partial x}(\xi, t) \right)^2 d\xi, \quad (3.177)$$

which, for the parabolic body  $f(x) = x^2$ , is equal to

$$\frac{d_0(t)^3}{(1 + d_0(t)^2 - t)^{3/2}((1 + d_0(t)^2 - t)^{1/2} - 1)} + \frac{1}{2\sqrt{1-t}} \arctan\left(\frac{d_0(t)}{\sqrt{1-t}}\right), \quad (3.178)$$

so that,

$$F_0(t) \sim \begin{cases} 3(3t)^{1/2} & \text{as } t \rightarrow 0+, \\ 1/2\sqrt{1-t} & \text{as } t \rightarrow 1-. \end{cases}$$

We plot the stage 4 leading-order force  $F_0(t)$  in Figure 3.16. The leading-order force  $F_0(t)$  is a convex function until  $t \approx 0.5$ , at which time the presence of the base becomes significant at the origin. The location of the inflection point would be a convenient figure to compare with experiment. Finally, we note that the first-order correction to the force on the body  $F_1/\epsilon$  is due to both the first-order interior and jet root pressures.

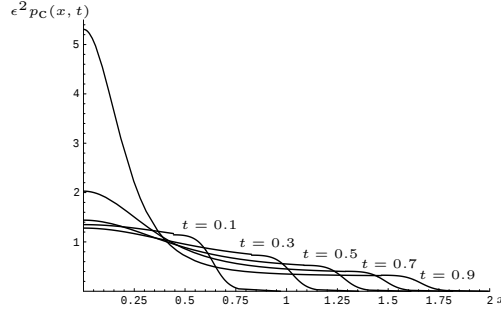


Figure 3.15: Schematic of stage 4 composite pressure profiles on the parabolic body  $f(x) = x^2$  for  $\epsilon = 0.01$ .

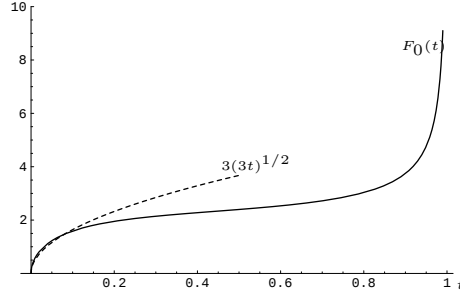


Figure 3.16: The stage 4 leading-order force  $F_0(t)$  on the parabolic body  $f(x) = x^2$ .

### 3.5.5 Small time limit of the order unity time solution

Consider the behaviour of the free surface in the jet root region (see Figure 3.13) at small times. The asymptotic jet thickness  $H_J(t) \sim t^2$  as  $t \rightarrow 0$  by (3.168, 3.172), while [65] show that the position vector of the turnover point from the relative stagnation point

$$\mathbf{x}(t) \sim \left( \frac{6}{\pi} \mathbf{i} - \frac{1}{8} \tan^{-1} \frac{1}{\sqrt{2}} \mathbf{k} \right) t^2 \quad \text{as } t \rightarrow 0$$

Further, the elevation of the body in the jet root region  $z \sim d_0(t)^2 - t = 2t$  as  $t \rightarrow 0$ . Hence, the jet root structure collapses to the stage 3 inner region structure (sketched in Figure

3.11) as  $t \rightarrow 0$ . Similarly, the stage 4 outer regions tend to the stage 3 outer regions as  $t \rightarrow 0$ . For example, the small time limit of the stage 4 interior problem (3.179 - 3.182) is exactly the stage 3 interior problem (3.123 - 3.126).

### 3.5.6 Summary

The stage 4 analysis shows that the stage 3 transition and inner regions merge to form an order unity jet root region. To lowest order the flow in the interior region is uniform across the layer and governed by the following codimension-one free boundary problem for the leading-order velocity potential  $\phi_0(x, t)$ ,

$$\frac{\partial}{\partial x} \left( l(x, t) \frac{\partial \phi_0}{\partial x} \right) = 1 \quad \text{for } 0 < x < d_0(t), \quad (3.179)$$

$$\frac{\partial \phi_0}{\partial x} = 0 \quad \text{on } x = 0, \quad (3.180)$$

$$\phi_0 = 0 \quad \text{on } x = d_0(t), \quad (3.181)$$

$$\frac{\partial \phi_0}{\partial x} - 2 \frac{dd_0}{dt} (1 - l(d_0, t)^{-1/2}) = 0 \quad \text{on } x = d_0(t). \quad (3.182)$$

As in stage 3, the exponentially small exterior velocity potential implies the boundary conditions (3.181, 3.182) are generalised Rankine-Hugoniot conditions at the shock  $x = d_0(t)$ . Equation (3.179) represents conservation of mass in the interior region, (3.181) represents continuity of pressure across the shock, viz.

$$\left[ \phi_0 \right]_{d_0-}^{d_0+} = 0, \quad (3.183)$$

and (3.182) represents conservation of mass across the shock, viz.

$$\left[ l(d_0, t) \frac{\partial \phi_0}{\partial x} \right]_{d_0-}^{d_0+} = -2\dot{d}_0 (l(d_0, t) - l(d_0, t)^{1/2}), \quad (3.184)$$

where the right hand side represents the mass flux lost to the jet. The consistency condition again represents global leading-order conservation of mass, viz.

$$d_0 = \int_0^{d_0} \frac{\partial}{\partial x} \left( l \frac{\partial \phi_0}{\partial x} \right) dx = \left[ l \frac{\partial \phi_0}{\partial x} \right]_{x=0}^{x=d_0} = -2\dot{d}_0 (l(d_0, t) - l(d_0, t)^{1/2})$$

and therefore

$$\frac{d}{dt} \int_0^{d_0} (f(x) - t) dx = \int_0^{d_0} -1 dx + (l(d_0) - 1)\dot{d}_0 = -H_0 \dot{d}_0,$$

in which the last term is exactly the mass flux lost to the jet. Here, we used (3.179, 3.180, 3.182) in the first equation and the transport theorem, the first equation and the asymptotic spray sheet thickness (3.168) in the second one.

## 3.6 Extensions to the model

One could in theory generalize to finite depth all the extensions to the infinite depth case discussed in section 2.3. Progress may be hampered or helped by the presence of the base. In this section we consider two of the more straightforward extensions to the model. In the next section we generalise to three-dimensions.

### 3.6.1 Variable impact speed

Making the change of variables defined in section 2.3.2 implies the asymptotic solution in each stage generalises directly to an arbitrary monotonic time-dependent impact velocity  $\dot{s}(t)$ . The temporal decomposition into the four stages of impact is still based on the ratio of the penetration depth  $s(t)$ , the distance between the turnover points (of  $O(\sqrt{s}/\epsilon)$  for a parabolic body) and the unit layer depth. The set of profiles through which the free surface and velocity field pass are functions of the body profile only and, in particular, independent of the impact velocity. Therefore, generalizations of the theory to account for the deceleration of the body due to the fluid pressure, as studied by Korobkin [37] for stage 4, are straightforward extensions of the model problem. As in the infinite depth case, the pressure becomes negative and cavitation occurs if the body decelerates sufficiently rapidly. We do not attempt to model cavitation and refer to the related work of [58] on cavitation in inviscid negative squeeze films. The simplest model is motivated by the experiments of [45, 46], in which a thin layer of fluid is dynamically loaded in tension by rapidly pulling apart the two plates enclosing the film. It assumes the cavity (i) nucleates at the point where the vapour pressure is first reached, (ii) remains at the vapour pressure and (iii) moves with the fluid. This model predicts that the flow will supercavitate, i.e. the pressure drops below the vapour pressure inside the fluid. A more sophisticated model is required and the use of a mushy region [55] is being investigated [58].

### 3.6.2 More general bodies

Suppose the body  $f(x)$  is symmetric, smooth and convex except at the origin where  $f(x) \sim O(|x|^\alpha)$  with  $\alpha \geq 1$ . The interior region now has length of order  $t^{1/\alpha}/\epsilon$  for  $t \geq 0$ , so the stage 2 timescale is  $\epsilon^\alpha$ . The structure is therefore exactly the same as for the parabolic body because the body is uniformly close to the undisturbed free surface throughout the interior region in stages 1, 2 and 3, viz. the body elevation at the turnover point where  $x \sim O(t^{1/\alpha}/\epsilon)$  is  $z = \epsilon^\alpha x^\alpha - t \sim O(t) \ll 1$ . The stage 1 and 3 turnover points grow like  $t^{1/\alpha}$ , the coefficients being determined by the corresponding law of motion (2.55, 3.127), while the stage 4 law of motion (3.171) is unchanged except  $l(x, t) = 1 + f(x) - t$ .

Suppose now the body is asymmetric, smooth and convex except at the origin and  $f(x) \sim a_\pm |x|^{\alpha_\pm}$  as  $x \rightarrow 0$ , where  $a_+ \neq a_-$  are order unity. If  $\alpha_- = \alpha_+$  the generalization is straightforward. If  $\alpha_- \neq \alpha_+$  then only stage 4 is straightforward because the distances of the turnover points from the origin have different orders of magnitude at small times. For example, if  $\alpha_- = 1$  and  $\alpha_+ = 2$  the distance of the left turnover point from the origin is  $O(t/\epsilon)$ , while the distance to the right turnover point is  $O(\sqrt{t}/\epsilon)$ . This means that at small times the left turnover point is effectively pinned at the origin on the lengthscale of the motion of the right turnover point, i.e. the right turnover point is in its stage 3 before the left turnover point enters its stage 2.

## 3.7 The three-dimensional model

In this section we generalize the finite depth water entry model to three dimensions. We introduce the horizontal  $y$ -coordinate and body profile

$$z = f(\epsilon x, \epsilon y) - t \tag{3.185}$$

to the dimensionless model problem (3.3 - 3.9) exactly as in the infinite depth case reviewed in section 2.4. For simplicity we consider the entry of the elliptic paraboloid (2.140) so that the turnover curve,  $\omega(\epsilon x, \epsilon y) = t$  say, lies a distance of  $O(\sqrt{t}/\epsilon)$  away from the origin in all directions. The temporal and spatial decomposition of the flow into four stages proceeds exactly as in the two-dimensional case *provided* the turnover curve  $\partial\Omega(t)$  is (i) smooth and has (ii) radius of curvature much larger than the size of the largest inner region, i.e. the Wagner region in stages 1 and 2, the transition region in stage 3 and the jet root region in stage 4. As in the infinite depth generalization to three dimensions, these can only be verified *a posteriori*. Further, the flow in the inner regions in the neighbourhood of the turnover curve are quasi-two-dimensional in all planes perpendicular to the turnover curve, with exactly the same structure, matching and solution as in the two-dimensional case; the flow is simply parametrized by the instantaneous arc length  $s$  along the turnover curve  $\partial\Omega(t)$ , which we suppose has outward pointing unit normal  $\mathbf{n}$  and normal velocity  $v_n$  as sketched in Figure 3.17. We therefore present only the leading-order outer problem in each stage.

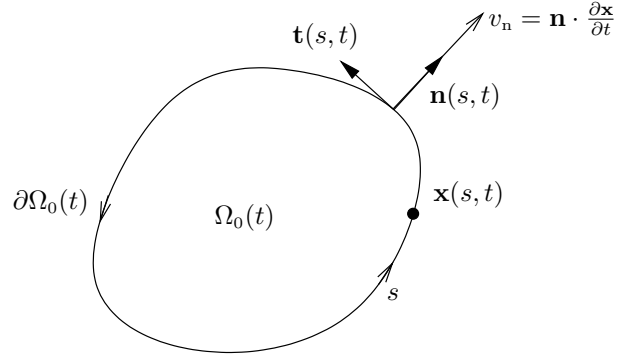


Figure 3.17: Geometry of the turnover curve.

Only the infinite depth case reviewed in section 2.4 appears in the literature. The third dimension and finite depth make the hunt for analytic and numerical solutions of the leading-order outer problems difficult. We present the analytic progress we have made.

### 3.7.1 Very small time

As in the two-dimensional case, the leading-order stage 1 outer problem is the leading-order infinite depth outer problem (2.109 - 2.117). The infinite depth deadrise angle  $\epsilon$  is simply replaced by the finite depth scaled deadrise angle  $\bar{\epsilon} = \epsilon\Delta^{-1/2}$ , as in section 3.2, where  $\epsilon^2/\Delta^2$  is the timescale of stage 1 provided  $\Delta \gg 1$ . In section 2.4.3 Korobkin's inverse method [40] was used to construct a similarity solution for the entry of the elliptic paraboloid (2.140), in which distances scale with the square root of time. This guarantees the matching with stage 2 proceeds as in section 3.2, which therefore has zero initial conditions to lowest order.

### 3.7.2 Small time

We take the leading-order outer stage 1 problem, add a base at  $z = -1$  and its associated kinematic boundary condition (3.19) to obtain the leading-order outer stage 2 problem. The leading-order asymptotic spray sheet thickness, pressure and force on the body are calculated exactly as in stage 1.

There are no known exact solutions to the three-dimensional finite depth codimension-two free boundary problem, though some progress is possible in the axisymmetric case. A numerical analysis may be based on the schemes discussed in section 3.3.1.3 for the two-dimensional case. Again, the variational formulation is probably the best bet and novel numerical schemes may be based on the small and large time asymptotics. Specifically,

$$\omega_0 \sim \left(\frac{x}{a_0}\right)^2 + \left(\frac{y}{b_0}\right)^2 \quad \text{as } x^2 + y^2 \rightarrow 0,$$

where  $a_0$  and  $b_0$  are given by (2.145, 2.146). Similarly, below we will show that matching with stage 3 implies

$$\omega_0 \sim \left(\frac{x}{A_0}\right)^2 + \left(\frac{y}{B_0}\right)^2 \quad \text{as } x^2 + y^2 \rightarrow \infty,$$

where  $A_0$  and  $B_0$  are functions of  $a$  and  $b$  that we shall determine.

### 3.7.2.1 Bodies with rotational symmetry

Suppose  $f = f(r)$  and  $\partial\Omega(t)$  is  $r = (x^2 + y^2)^{1/2} = d_0(t)$ . Vorovich & Yudovich [70] considered the stage 2 leading-order outer problem in order to solve the leading-order outer small time model for the normal impact of a circular disc of fixed radius  $d_0(t)$ . They reduced the problem to an integral equation as follows. Proceed as in the infinite depth case in section 2.4.2 by separating the variables to find the general axisymmetric solution on a layer of depth  $h$  is

$$\phi_0(r, z, t) = \int_0^\infty A(k, t) \frac{\cosh k(z + h)}{\cosh(kh)} J_0(kr) dk. \quad (3.186)$$

The boundary conditions (2.110, 2.112) on  $z = 0$  imply the dual pair of integral equations

$$\int_0^\infty k A(k, t) J_0(kr) \tanh(kh) dk = -1 \quad \text{for } r < d_0(t), \quad (3.187)$$

$$\int_0^\infty A(k, t) J_0(kr) dk = 0 \quad \text{for } r > d_0(t), \quad (3.188)$$

which may be combined [59] to form a single integral equation for  $A$ , namely

$$\frac{\pi}{2} A(k, t) = \frac{d}{dk} \left( \frac{\sin k}{k} \right) + \int_0^1 \int_0^\infty A(\kappa, t) (1 - \tanh(\kappa h)) \sin \kappa u \sin ku \, d\kappa du.$$

Vorovich & Yudovich [70] obtained the series solution for  $h/d_0 > \frac{2}{\pi} \log 2$ , which is of little use here. Chebakov [7, 42] found the asymptotic solution as  $h \rightarrow 0$  by a direct analysis on the leading-order linearized potential problem. Their analysis is similar to the analysis we performed in two-dimensions for the stage 3 outer problem in section 3.4; the disc and therefore the transition region is fixed, so the free surface elevation is found by integrating the linearized kinematic boundary condition (2.111), rather than travelling wave version (3.80) due to the rapidly expanding disc.

### 3.7.3 Intermediate time

#### 3.7.3.1 Leading-order interior region problem

To lowest order the interior region flow is uniform across the layer and the two-dimensional leading-order interior region problem (3.123, 3.125, 3.126) generalizes to

$$\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2} = 1 \quad \text{for } \omega_0(x, y) < t, \quad (3.189)$$

$$\phi_0 = 0 \quad \text{on } \omega_0(x, y) = t, \quad (3.190)$$

$$\frac{\partial \phi_0}{\partial n} - v_n(f(x, y) - t) = 0 \quad \text{on } \omega_0(x, y) = t, \quad (3.191)$$

together with initial condition

$$\omega_0(0, 0) = 0, \quad (3.192)$$

derived by matching with stage 2 as in the two-dimensional case. Poisson's equation (3.189) represents leading-order conservation of mass in the layer. The first boundary condition (3.190) represents continuity of pressure both normally and tangentially across the inner transition region in the neighbourhood of the turnover curve, while the second boundary condition (3.191) represents normal conservation of mass across this region.

The six remarks made in section 3.4.8 generalize directly. For example, the consistency condition for Poisson's equation (3.189) becomes

$$\Omega_0(t) = \iint_{\Omega_0(t)} 1 \, dx \, dy = \int_{\partial\Omega_0(t)} \frac{\partial \phi_0}{\partial n} \, ds = \int_{\partial\Omega_0(t)} v_n(f(x, y) - t) \, ds,$$

by Green's theorem and (3.191). Then, by the transport theorem

$$\begin{aligned} \frac{d}{dt} \iint_{\Omega_0(t)} (f(x, y) - t) \, dx \, dy &= \iint_{\Omega_0(t)} \frac{\partial}{\partial t} (f(x, y) - t) \, dx \, dy \\ &\quad + \int_{\partial\Omega_0(t)} v_n(f(x, y) - t) \, ds, \end{aligned}$$

which is identically zero by the consistency condition and therefore represents leading-order conservation of mass, viz.

$$\iint_{\Omega_0(t)} (f(x, y) - t) \, dx \, dy = 0, \quad (3.193)$$

by the zero initial condition (3.192). Hence, exactly as in the two-dimensional case the displaced fluid lies in the interior region to lowest order.

The asymptotic spray sheet thickness (scaled with  $\epsilon^4/\delta^4$ ) is

$$H_0(x, y, t) = \left( \frac{1}{2v_n} \frac{\partial \phi_0}{\partial n} \right)^2 = \frac{1}{4} (f(x, y) - t)^2 \quad \text{for } (x, y) \in \partial\Omega_0(t). \quad (3.194)$$

The leading-order pressures in all regions and therefore the leading and first-order force are the same as in the two-dimensional case, although now, of course, we must integrate over the extra dimension.

### 3.7.3.2 Related free boundary problems

Questions concerning the wellposedness of (3.189 - 3.192) and regularity of the free boundary  $\partial\Omega_0(t)$  are difficult. The most similar well studied problem is the Hele-Shaw squeeze film free boundary problem with zero surface tension [57], viz.

$$\frac{\partial}{\partial x} \left( l^2 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( l^2 \frac{\partial p}{\partial y} \right) = \frac{1}{l} \frac{\partial l}{\partial t} \quad \text{for } \omega(x, y) < t, \quad (3.195)$$

$$p = 0 \quad \text{on } \omega(x, y) = t, \quad (3.196)$$

$$\frac{\partial p}{\partial n} + \frac{v_n}{l^2} = 0 \quad \text{on } \omega(x, y) = t, \quad (3.197)$$

together with initial conditions for  $p$  and  $\omega$  at  $t = 0$ . Here,  $p(x, y, t)$  is the pressure,  $l(x, y, t)$  is the plate separation,  $\omega(x, y) = t$  is the free boundary (which is assumed to pass any given point at most once) with outward pointing unit normal  $\mathbf{n}$  and normal velocity  $v_n$ . The fluid velocity is given by Darcy's law  $\mathbf{u}(x, y, t) = -l^2 \nabla p$ .

If the gap width  $l$  is constant and the flow is driven by a point source or sink, (3.195 - 3.196) is the classical Hele-Shaw problem, which is also a model for groundwater flows and is a special case of the Stefan model for melting or solidification of a pure material; see [20] for a comprehensive “bibliography of free and moving boundary problems for Hele-Shaw and Stokes flow”. The classical Hele-shaw problem captures many of the essential features of these more complex situations and is therefore of enormous practical and theoretical importance. However, the non-constant gap width and the resulting presence of a continuously distributed source term on the right hand side of (3.195), both of which vary in space and time, makes analytic progress difficult; complex variable methods are no longer applicable unless  $l$  is solely a function of time. In this case, [57] establish existence, uniqueness and regularity of solutions with analytic data by studying the complex moments of the domain  $\Omega(t)$ , namely

$$\iint_{\Omega(t)} \zeta^m dx dy,$$

where  $\zeta = x + iy$  for  $m = 0, 1, 2, \dots$ . Unfortunately, the presence of the body profile in our kinematic boundary condition (3.191) prohibits such an analysis. The following result solely carries over from [57]:

$$\begin{aligned} \frac{d}{dt} \iint_{\Omega_0(t)} (f(x, y) - t) \zeta^m dx dy &= \iint_{\Omega_0(t)} -\zeta^m dx dy + \int_{\partial\Omega_0(t)} (f(x, y) - t) \zeta^m v_n ds \\ &= \iint_{\Omega_0(t)} -\zeta^m + \nabla \cdot (\zeta^m \nabla \phi_0) dx dy \\ &= \iint_{\Omega_0(t)} \nabla \cdot (\phi_0 \nabla \zeta^m) dx dy \\ &= \int_{\partial\Omega_0(t)} \phi_0 \frac{\partial \zeta^m}{\partial n} ds \\ &= 0 \end{aligned}$$

where we have used the transport theorem in the first equality, (3.191) and Green's theorem in the second one, (3.189) and the fact  $\zeta^m$  is harmonic in the third one, Green's theorem in the fourth and finally (3.190) in the fifth one. Hence, by the initial condition (3.192),

$$\iint_{\Omega_0(t)} (f(x, y) - t) \zeta^m dx dy = 0$$

which implies (i) the conservation of mass (as in (3.193)) for  $m = 0$  and (ii) the centre of mass of the interior region remains at the origin for  $m = 1$ .

We have been unable to make further direct analytic progress on the stage 3 problem (3.189 - 3.192) for general  $f(x, y)$ . Fortunately, the form of the problem supports an inverse method, which allows us to construct explicit analytic solutions, and a variational formulation. Before describing these we detail the axisymmetric case.

### 3.7.3.3 Bodies with radial symmetry

Suppose  $f = f(r)$ ,  $\phi_0 = \phi_0(r, t)$  and the turnover curve  $\partial\Omega_0(t)$  is  $r = d_0(t)$ . The solution of (3.189, 3.190) is

$$\phi_0 = \frac{1}{4}(r^2 - d_0(t)^2). \quad (3.198)$$

We substitute this into (3.191), integrate and apply the zero initial condition (3.192) to obtain the law of motion of the turnover curve

$$\int_0^{d_0(t)} (f(\xi) - t) \xi \, d\xi = 0, \quad (3.199)$$

which is of course the axisymmetric consistency condition (3.193). In particular, if  $f(r) = (r/a)^2$ , then

$$d_0(t) = a(2t)^{1/2}, \quad (3.200)$$

$$p_0(t) = \frac{1}{2}d_0(t)\dot{d}_0(t) = \frac{a^2}{2}, \quad (3.201)$$

$$F_0(t) = \frac{1}{2}\pi d_0(t)^3\dot{d}_0(t) = \pi a^3 t, \quad (3.202)$$

while for the two-dimensional parabola  $f(x) = x^2$  we recall that  $d_0(t) = (3t)^{1/2}$ ,  $p_0(t) = 3/2$  and  $F_0(t) = 3(3t)^{1/2}$ .

### 3.7.3.4 The inverse method

Here we suppose the order unity time dimensionless body profile and stage 3 scalings may be chosen to yield the stage 3 model problem (3.189 - 3.192) for general body profile  $f(x, y)$ . We can then use an inverse method to construct solutions to (3.189 - 3.192) and check *a posteriori* that (i) the stage 3 scalings exist and (ii) the body profile is physically acceptable; smooth and monotonic increasing with radial distance from the origin is usually sufficient. The key to the inverse method is to spot that

$$\phi_0 = \mathcal{F}(\omega_0(x, y)) - \mathcal{F}(t), \quad (3.203)$$

automatically satisfies the first boundary condition (3.190). Here we demand that the twice continuously differentiable function  $\mathcal{F} : [0, \infty) \rightarrow \mathfrak{R}$  is invertible so that

$$\omega_0(x, y) = \mathcal{F}^{-1}(\mathcal{G}(x, y)), \quad (3.204)$$

where  $\mathcal{G} = \mathcal{F}(\omega_0(x, y))$ , say. The velocity potential (3.203) satisfies Poisson's equation (3.189) provided

$$\frac{\partial^2 \mathcal{G}}{\partial x^2} + \frac{\partial^2 \mathcal{G}}{\partial y^2} = 1. \quad (3.205)$$

Suppose the turnover curve  $\partial\Omega_0(t)$  is traced out by  $\mathbf{x}(s, t)$ , where we recall  $s$  is arc length increasing in an anti-clockwise sense (see Figure 3.17). Then, differentiating  $t = \omega_0(\mathbf{x}(s, t))$  with respect to  $s$  and  $t$  implies that

$$1 = \frac{\partial \mathbf{x}}{\partial t} \cdot \nabla \omega_0 \quad \text{and} \quad \nabla \omega_0 = |\nabla \omega_0| \mathbf{n}, \quad (3.206)$$

and therefore

$$1 = |\nabla \omega_0| v_n, \quad (3.207)$$

where we recall  $\partial\Omega_0(t)$  has outward pointing unit normal  $\mathbf{n}$  and normal velocity  $v_n$ . Hence, the velocity potential (3.203) satisfies the second boundary condition (3.191) provided the body is given by

$$f = \omega_0 + \mathcal{F}'(\omega_0) |\nabla \omega_0|^2, \quad (3.208)$$

from which we may check assumptions (i) and (ii).

The leading-order interior region pressure is  $p_0/\epsilon^2$  where

$$p_0(x, y, t) = -\frac{\partial \phi_0}{\partial t} = \mathcal{F}'(t), \quad (3.209)$$

so the leading-order force on the body is  $F_0/\epsilon^2\delta^2$ , where  $\epsilon^2/\delta^2$  is the stage 3 timescale for  $\epsilon \ll \delta \ll 1$  and

$$F_0(t) = \text{Area}[\Omega_0(t)] \cdot \mathcal{F}'(t). \quad (3.210)$$

The first-order force is  $O(1/\epsilon^2\delta)$  due to the transition region of arc length of order  $1/\delta$ . The asymptotic spray sheet thickness (3.194) becomes

$$H_0(x, y, t) = \frac{1}{4} |\nabla \mathcal{G}|^4, \quad (3.211)$$

by (3.208). Hence, the possibility exists for the spray sheet thickness to vanish. It will only do so at critical points of  $\mathcal{G}$  where  $\nabla \mathcal{G} = \mathbf{0}$ , at which time the asymptotics breakdown locally. We discuss this interesting phenomenon below, having derived the intermediate time solution for the elliptic paraboloid (2.140), for which we know assumptions (i) and (ii) hold.

### 3.7.3.5 Entry of an elliptic paraboloid

A particular solution of (3.205) is

$$\mathcal{G} = \frac{1}{2} [kx^2 + (1-k)y^2], \quad (3.212)$$

where  $k \in [0, 1]$ . Let  $\mathcal{F}(\xi) = \xi/\alpha$  where  $\alpha > 0$ , then (3.204, 3.208) imply

$$\phi_0(x, y, t) = \frac{1}{\alpha} (\omega_0(x, y) - t), \quad (3.213)$$

$$\omega_0(x, y) = \left(\frac{x}{A_0}\right)^2 + \left(\frac{y}{B_0}\right)^2 \quad (3.214)$$

$$f(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2, \quad (3.215)$$

where

$$\frac{1}{A_0^2} = \frac{\alpha k}{2}, \quad (3.216)$$

$$\frac{1}{B_0^2} = \frac{\alpha(1-k)}{2}, \quad (3.217)$$

$$\frac{1}{a^2} = \alpha \left( \frac{k}{2} + k^2 \right), \quad (3.218)$$

$$\frac{1}{b^2} = \alpha \left( \frac{1-k}{2} + (1-k)^2 \right). \quad (3.219)$$

Hence, as in the infinite depth case an impacting elliptic paraboloid has a similarity solution in which distances scale with the square root of time and the turnover curve is an ellipse. The constant  $\alpha(a, b) > 0$  allows the impactor to be any elliptic paraboloid with  $a \geq b$  and is the positive root of the quartic in  $\alpha$  obtained by eliminating  $k$  from (3.218, 3.219). If a cross-section of the elliptic paraboloid has eccentricity  $e = \sqrt{1 - b^2/a^2}$  and the turnover curve has eccentricity  $E_0 = \sqrt{1 - B_0^2/A_0^2}$ , then eliminating  $\alpha$  and  $k$  from (3.216 - 3.219) we obtain

$$E_0 = 1 + e^2/6 - \sqrt{1 - e^2 + e^4/36}, \quad (3.220)$$

which we plot in Figure 3.18a. As in stage 1,  $e > E_0$  for  $0 < e < 1$ , so the elliptic turnover curve is less elongated than a cross-section of the impacting elliptic paraboloid, as depicted in Figure 3.18b for  $E_0 = 0.99$ ,  $e \approx 0.9983$ ,  $\alpha = 1$ .

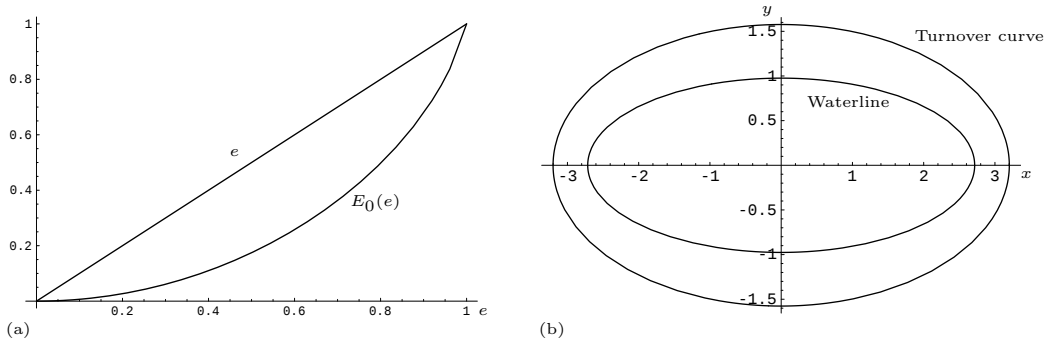


Figure 3.18: (a) Plot of the eccentricity  $E_0$  of the turnover curve as a function of the eccentricity  $e$  of the cross-section of the entering elliptic paraboloid in stage 3. (b) Plot of the turnover curve and waterline ( $f(x, y) = t$ ) at  $t = 1$  for  $E_0 = 0.99$  and  $\alpha = 1$ .

The leading-order interior pressure and force on the body are given by (3.209, 3.210) as

$$p_0(x, y, t) = 1/\alpha, \quad (3.221)$$

$$F_0(t) = \pi A_0 B_0 t / \alpha, \quad (3.222)$$

in contrast to the stage 1 leading-order force (2.149) which scales with the square root of time. The asymptotic jet thickness is given by (3.211, 3.212, 3.213, 3.216, 3.217) as

$$H_0(x, y, t) = \frac{4t^2}{\alpha^4} \text{ on } \omega_0(x, y) = t, \quad (3.223)$$

i.e. the asymptotic spray sheet thickness is uniform on the turnover curve in contrast to stage 1 [40].

Finally, we note that as in stage 1, the solution is in agreement with both the axisymmetric case as  $E_0 \rightarrow 0$  and the two-dimensional case as  $B_0 \rightarrow 0$ ,  $E_0 \rightarrow 1$  in cross-sections *not too close* to  $x = \pm A_0 t^{1/2}$ , which again allows an analysis of the accuracy of a strip theory approach for slender bodies.

### 3.7.3.6 Splitting the spray sheet

In this section we consider the splitting of the spray sheet within the framework of the inverse method and present an explicit example. We suppose  $\mathcal{G}$  is sufficiently regular that, by (3.205),

- (1)  $\nabla \mathcal{G}$  is harmonic.

Then, since  $\nabla \mathcal{G}$  is not identically zero by (3.205), the uniqueness theorem [1] implies that the asymptotic spray sheet thickness (3.211) cannot vanish at time  $t > 0$  on any subset of the leading-order turnover curve  $\partial\Omega_0(t)$  that contains a limit point.

On physical grounds we also demand that

- (2)  $\omega_0$  and its first partial derivatives are bounded;
- (3)  $\mathcal{F}'(t)$  is non-zero and bounded for  $t > 0$ .

By (3.208), condition (2) guarantees that we do not violate the small deadrise angle assumption. By (3.209), condition (3) guarantees that the leading-order pressure in the interior region is non-zero and bounded. Since  $\mathcal{G} = \mathcal{F}(\omega_0)$ ,

$$\nabla \mathcal{G} = \mathcal{F}' \nabla \omega_0, \quad (3.224)$$

so conditions (1 - 3) imply that the asymptotic spray sheet thickness (3.211) vanishes at the point  $\mathbf{x}^* \in \partial\Omega_0(\omega_0(\mathbf{x}^*))$  if and only if  $\nabla \omega_0(\mathbf{x}^*) = \mathbf{0}$ . Moreover, since  $\phi_0$  and its first partial derivatives are bounded by (3.203) and conditions (2, 3), the normal velocity of the turnover curve,  $v_n$ , must be unbounded at such points by (3.207). Hence, for solutions of the form (3.203) generated by the inverse method and satisfying conditions (1 - 3), the spray sheet splits at time  $t^* > 0$  at  $\mathbf{x}^* \in \partial\Omega_0(t^*)$  if and only if the curve  $\mathcal{G}(\mathbf{x}) = \mathcal{F}(t^*)$  is not smooth at  $\mathbf{x} = \mathbf{x}^*$ . Further, the body is horizontal, viz.  $\nabla f(\mathbf{x}^*) = \mathbf{0}$  by (3.208), and hits the undisturbed free surface, viz.  $f(\mathbf{x}^*) = t^*$  by (3.194), at such points, which is in accordance with physical intuition. The asymptotics break down near to the critical point  $\mathbf{x} = \mathbf{x}^*$  near to the critical time  $t = t^*$  when the fluid velocity becomes comparable to the sound speed.

The inverse method allows us to construct the body profile that results in a given turnover curve  $\partial\Omega_0(t^*)$  at time  $t^* > 0$ , provided we can solve (3.205) with the additional boundary condition

$$\mathcal{G}(\mathbf{x}) = \mathcal{F}(t^*) \quad \text{for } \mathbf{x} \in \partial\Omega_0(t^*), \quad (3.225)$$

in which case the spray sheet splits at non-smooth points on  $\partial\Omega_0(t^*)$ .

Suppose we specify  $\partial\Omega_0(t^*)$  to be a simple closed rectilinear curve, then (3.205, 3.225) have a polynomial solution if and only if  $\partial\Omega_0(t^*)$  is an equilateral triangle [53]. In this

case, if  $\partial\Omega_0(t^*)$  is the equilateral triangle with unit side bounded by  $y = -1/2\sqrt{3}$  and  $y \pm \sqrt{3}x = 1/\sqrt{3}$  (so there is symmetry in the  $y$ -axis and the centroid is at the origin), then

$$\mathcal{G}(x, y) = t^* - \frac{1}{2\sqrt{3}}(y + 1/2\sqrt{3})(y + \sqrt{3}x - 1/\sqrt{3})(y - \sqrt{3}x - 1/\sqrt{3}). \quad (3.226)$$

Let  $\mathcal{F}$  be the identity function, then  $\omega_0 = \mathcal{G}$  vanishes at the origin provided the critical time  $t^* = 1/36$  and by (3.208) the body is the quartic

$$f(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{2}{\sqrt{3}}y(3x^2 - y^2) + \frac{3}{2}x^2y^2 + \frac{3}{4}(x^4 + y^4). \quad (3.227)$$

In Figure 3.19, we plot the contours  $\omega_0(x, y) = t$  for six equally spaced times from  $t = 0$  to the critical time  $t = t^*$  and the asymptotic spray sheet thickness on the horizontal edge of  $\Omega_0(t^*)$ , namely

$$H_0(x, -1/2\sqrt{3}, t^*) = \frac{9}{16384}(1 - 4x^2)^4 \quad \text{for } |x| < 1/2.$$

The velocity of the fluid ejected into the spray sheet is normal to the turnover curve, so when the sheet splits at a corner of  $\partial\Omega_0(t^*)$  the edges of the split sheet will span an angle of  $2\pi/3$ , as depicted in Figure 3.19.

We plot  $z = f(x, y)$  on  $\Omega_0(t^*)$  in Figure 3.20. Although the body profile is not convex on  $\Omega_0(t^*)$ , it is monotonic increasing with radial distance from the origin and is therefore physically acceptable. Specifically, if we choose the full dimensionless order unity time profile to be

$$z = \frac{\delta\epsilon}{2}(x^2 + y^2) + \frac{2\delta^{1/2}\epsilon^{5/2}}{\sqrt{3}}y(3x^2 - y^2) + \frac{3\epsilon^4}{2}x^2y^2 + \frac{3\epsilon^4}{4}(x^4 + y^4),$$

where  $\epsilon \ll \delta \ll 1$ , then substituting the stage 3 scalings

$$x = \bar{x}/\sqrt{\epsilon\delta}, \quad y = \bar{y}/\sqrt{\epsilon\delta}, \quad z = \bar{z}, \quad t = (\epsilon/\delta)^2\bar{t},$$

we obtain the stage 3 body profile

$$\bar{z} = (\epsilon/\delta)^2(f(\bar{x}, \bar{y}) - \bar{t})$$

and therefore confirm assumptions (i) and (ii) made at the start of section 3.7.3.4.

Finally, we remark that the inverse method may be a useful design tool because, within the constraints discussed above, it allows us to specify the location of the free surface and in particular at which points the spray sheet will split. For example the 1mm scale structure of tyre tread is designed to eject water into grooves that run around the tyre, thereby increasing contact with the road and therefore the odds against hydroplaning. The above analysis suggests the non-smooth points of  $\mathcal{G}$  should be aligned with the grooves thereby forcing the impacting fluid into them.

### 3.7.3.7 Variational formulation of the outer problem

The velocity potential  $\phi_0(x, y, t)$  is defined on  $\omega_0(x, y) < t$ , so the leading-order stage 3 displacement potential is defined by

$$\Phi_0(x, y, t) = - \int_{\omega_0(x, y)}^t \phi_0(x, y, \tau) d\tau. \quad (3.228)$$

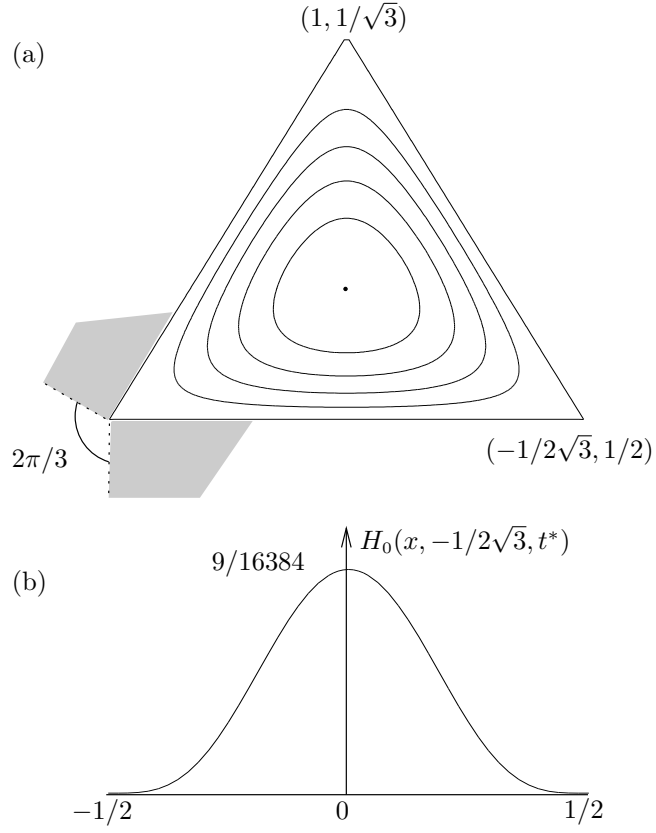


Figure 3.19: (a) Contour plot of the turnover curve  $\omega_0(x, y) = t$  for  $t = 0, t^*/5, 2t^*/5, 3t^*/5, 4t^*/5, t^*$ . (b) Plot of the asymptotic spray sheet thickness  $H_0$  on the horizontal edge of the equilateral triangle  $\Omega_0(t^*)$ .

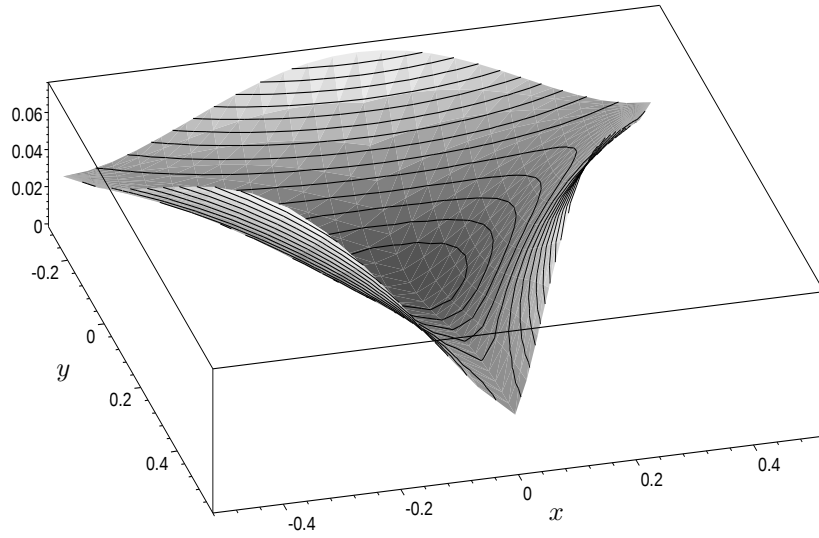


Figure 3.20: Plot of the body profile  $z = f(x, y)$  on  $\Omega_0(t^*)$ .

A simple calculation using (3.190) shows that on the interior region  $\omega_0 < t$ ,

$$\nabla^2 \Phi_0 = - \int_{\omega_0}^t \nabla^2 \phi_0(x, y, \tau) d\tau + [\nabla \omega_0 \cdot \nabla \phi_0]_{\omega_0=t}.$$

Since  $\phi_0$  satisfies Poisson's equation (3.189) and on  $\omega_0 = t$ ,

$$\nabla \omega_0 \cdot \nabla \phi_0 = |\nabla \omega_0| \mathbf{n} \cdot \nabla \phi_0 = f - \omega_0,$$

by (3.206) and (3.190, 3.207), we deduce

$$\nabla^2 \Phi_0 = \omega_0 - t + (f - \omega_0) = f - t \quad \text{for } \omega_0 < t. \quad (3.229)$$

Also, by definition,

$$\Phi_0 = 0 \quad \text{on } \omega_0 = t, \quad (3.230)$$

and (3.190) implies

$$\frac{\partial \Phi_0}{\partial n} = 0 \quad \text{on } \omega_0 = t. \quad (3.231)$$

In addition, if we demand on physical grounds<sup>7</sup>,

$$\Phi_0 \geq 0 \quad \text{for } \omega_0 \leq t, \quad (3.232)$$

$$f - t \geq 0 \quad \text{for } \omega_0 \geq t, \quad (3.233)$$

then we may derive a variational inequality directly from the free boundary problem (3.229 - 3.231) as follows. We begin by extending  $\Phi_0$  to be zero on the exterior region, viz.

$$\Phi_0 = 0 \quad \text{for } \omega_0 > t. \quad (3.234)$$

As in section 2.6, we let  $\mathcal{H}^1$  be the real Sobolev space of square integrable functions and we let  $\mathcal{V} \subset \mathcal{H}^1$  be a set whose elements are non-negative. We define the symmetric continuous bilinear form  $a : \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathbb{R}$  by

$$a(u, v) = \iint_{\mathbb{R}^2} \nabla u \cdot \nabla v \, dx dy \quad (3.235)$$

and the continuous linear map  $l : \mathcal{H}^1 \rightarrow \mathbb{R}$  by

$$l(u) = \iint_{\mathbb{R}^2} (t - f(x, y)) u(x, y) \, dx dy. \quad (3.236)$$

Then for any  $v \in \mathcal{V}$ ,

$$a(\Phi_0, v - \Phi_0) = - \iint_{\omega_0 < t} (v - \Phi_0) \nabla^2 \Phi_0 \, dx dy,$$

---

<sup>7</sup>By (3.119), condition (3.232) guarantees that the leading-order pressure is positive on the interior region. The second condition (3.233) follows from the two assumptions that the turnover curve lies at or above the undisturbed level of the free surface, which was implicitly assumed in our stage 3 asymptotic decomposition, and that the body profile is monotonic increasing with radial distance from the origin.

by (3.234), Green's theorem on the interior region and (3.231). Then, by (3.229),

$$a(\Phi_0, v - \Phi_0) = \iint_{\omega_0 < t} (v - \Phi_0)(t - f) dx dy.$$

Hence, by (3.232 - 3.234), we deduce the variational inequality

$$a(\Phi_0, v - \Phi_0) \geq l(v - \Phi_0) \quad \text{for all } v \in \mathcal{V}. \quad (3.237)$$

Since  $f - t$  is negative inside the waterline, the variational inequality (3.237) is not amenable to classical theory and we cannot apply the projection theorem to deduce existence and uniqueness [10, 12]. We do not pursue the investigation further, but remark that it would be interesting to see if the free boundary problem (3.229 - 3.231) or the variational inequality (3.237) can be derived from the large-time, large-aspect-ratio limits of the corresponding stage 2 problems (c.f. section 2.6).

### 3.7.3.8 Stability Analysis

We now perform a local-in-time linear stability analysis of the one-dimensional stage 3 model (3.123 - 3.126) to perturbations in the  $y$ -direction. We perturb the right-hand free boundary such that it now has position

$$x \sim d_0(t) + \varepsilon e^{iky + \sigma(k, t)}, \quad (3.238)$$

where  $\varepsilon$  is a prescribed small number, the wave number  $k > 0$  and we wish to find the instantaneous linear growth rate  $\dot{\sigma}$ . The symmetry condition (3.124) implies the perturbed potential is

$$\phi_0(x, t) + \varepsilon \mathcal{X}(x) e^{iky + \sigma(t)}, \quad (3.239)$$

where  $\mathcal{X}(x) = \cosh(kx)$  and  $A$  is a constant. Expanding the three-dimensional boundary conditions (3.190, 3.191) on (3.238) and retaining terms up to order  $\varepsilon$ , we find the instantaneous linear growth rate at wavenumber  $k > 0$  is

$$\dot{\sigma}(k, t) = -k \dot{d}_0 \tanh(k d_0) - \frac{d}{dt} \log |f(d_0(t)) - t|. \quad (3.240)$$

The two-dimensional solution is linearly stable if and only if  $\dot{\sigma} < 0$ . The first term of the right-hand side of (3.240) is negative if and only if  $\dot{d}_0 > 0$  because  $k > 0$ , while the second term is negative if and only if  $\dot{d}_0 f'(d_0) > 1$ . Hence, the stage 3 linear stability condition is

$$\dot{d}_0 > 1/f'(d_0), \quad (3.241)$$

which is equivalent to saying that the turnover points (which lie on the body to lowest order) must be moving up with respect to the fixed frame. If  $1/f'(d_0) > \dot{d}_0 > 0$ , the solution is linearly unstable at large wavenumbers. If  $\dot{d}_0 < 0$ , the solution is linearly unstable. The solution is stable for the body profile  $f(x) = kx^\alpha$  for all  $k > 0$  and  $\alpha \geq 1$  because

$$d_0(t) = \left( \frac{(\alpha + 1)t}{k} \right)^{1/\alpha} \implies \frac{d}{dt} (f(d_0(t)) - t) = \alpha \geq 1,$$

where we have used the stage 3 law of motion (3.128).

We remark that, as expected, information is lost in the the local-in-space and -time linear stability analyses of [5, 19, 31], which were described in section 2.6 and show that the local solution is linearly stable if and only if  $\dot{d}_0 > 0$ .

### 3.7.4 Time of order unity

#### 3.7.4.1 Leading-order interior region problem

To lowest order the interior region flow is uniform across the layer of thickness  $l(x, y, t) = 1 + f(x, y) - t$  and the two-dimensional leading-order interior region problem (3.179, 3.181, 3.182) generalizes to

$$\frac{\partial}{\partial x} \left( l \frac{\partial \phi_0}{\partial x} \right) + \frac{\partial}{\partial y} \left( l \frac{\partial \phi_0}{\partial y} \right) = 1 \quad \text{for } \omega_0(x, y) < t, \quad (3.242)$$

$$\phi_0 = 0 \quad \text{on } \omega_0(x, y) = t, \quad (3.243)$$

$$\frac{\partial \phi_0}{\partial n} - 2v_n \left( 1 - \frac{1}{\sqrt{l}} \right) = 0 \quad \text{on } \omega_0(x, y) = t, \quad (3.244)$$

together with initial condition

$$\omega_0(0, 0) = 0, \quad (3.245)$$

derived by matching with stage 3 as in the two-dimensional case. The governing equations have exactly the same physical meaning as in stage 3, although here mass is lost to the jet through (3.243). As in the two-dimensional case, the consistency condition represents global leading-order conservation of mass in the interior region, viz.

$$\frac{d}{dt} \iint_{\Omega_0(t)} (f - t) dx dy = - \int_{\partial\Omega_0(t)} H_0 v_n ds,$$

in which the right hand side is the mass flux lost to the spray sheet.

The only ‘exact’ analytic solution is for a body with axial symmetry, which we present below. There is no known theory concerning questions of existence, uniqueness or regularity of the free boundary, nor is there an obvious variational statement. A numerical approach is required which we do not attempt. However, a local-in-space and-time linear stability analysis of the two-dimensional solution for a symmetric body profile to perturbations in the perpendicular direction is straightforward, exactly as in stage 3; we briefly present the results.

#### 3.7.4.2 Bodies with rotational symmetry

Suppose  $l = 1 + f(r) - t$ ,  $\phi_0 = \phi_0(r, t)$  and the turnover curve  $\partial\Omega_0(t)$  is  $r = d_0(t)$ . The solution of (3.242, 3.243) is

$$\phi_0(r, t) = \int_0^{d_0(t)} \frac{\xi}{2l(\xi, t)} d\xi, \quad (3.246)$$

which, together with (3.244), implies the law of motion of the turnover curve

$$\frac{dd_0}{dt} = \frac{d_0(t)}{4l(d_0(t), t)^{1/2}(l(d_0(t), t)^{1/2} - 1)}. \quad (3.247)$$

The only difference with the two-dimensional case is the factor of two in the denominator of the velocity potential and the speed of the turnover point.

### 3.7.4.3 Stability analysis

We proceed as in the stage 3 stability analysis in section 3.7.3.8. Substituting the perturbations (3.238, 3.239) into the fully two-dimensional stage 4 model (3.242 - 3.244), the leading-order potential  $\phi_0(x, t)$  and right-hand turnover curve  $x = d_0(t)$  satisfy the equations governing the one-dimensional stage 4 model (3.179 - 3.182), while the first-order problem implies

$$\frac{d}{dx} \left( l \frac{d\mathcal{X}}{dx} \right) - k^2 l \mathcal{X} = 0 \quad \text{for } 0 < x < d_0, \quad (3.248)$$

$$\frac{d\mathcal{X}}{dx} = 0 \quad \text{at } x = 0, \quad (3.249)$$

$$\mathcal{X} + \frac{\partial \phi_0}{\partial x} = 0 \quad \text{at } x = d_0, \quad (3.250)$$

$$\frac{d\mathcal{X}}{dx} - 2\dot{\sigma} \left( 1 - \frac{1}{\sqrt{l}} \right) - \frac{\dot{d}_0}{l^{3/2}} \frac{\partial l}{\partial x} + \frac{\partial^2 \phi_0}{\partial x^2} = 0 \quad \text{at } x = d_0, \quad (3.251)$$

where we recall the layer thickness  $l(x, t) = 1 + f(x) - t$ . We scale

$$\mathcal{X} = -\frac{\partial \phi_0}{\partial x}(d_0, t) \tilde{\mathcal{X}}, \quad x = d_0 \tilde{x}, \quad k = d_0 \tilde{k}, \quad (3.252)$$

to find that the instantaneous linear growth rate at wavenumber  $\tilde{k} > 0$  is given by (dropping tildes on these variables)

$$\dot{\sigma}(k, t) = -\frac{\dot{d}_0}{d_0} \frac{d\mathcal{X}}{dx}(1) - \frac{\sqrt{l(d_0, t)}}{2(\sqrt{l(d_0, t)} - 1)} \left[ \frac{\dot{d}_0}{l(d_0, t)^{3/2}} \frac{\partial l}{\partial x}(d_0, t) - \frac{\partial^2 \phi_0}{\partial x^2}(d_0, t) \right], \quad (3.253)$$

where  $\mathcal{X}$  satisfies the Liouville equation

$$\frac{d}{dx} \left( \tilde{l} \frac{d\mathcal{X}}{dx} \right) - k^2 \tilde{l} \mathcal{X} = 0 \quad \text{for } 0 < x < 1, \quad (3.254)$$

with the boundary conditions

$$\frac{d\mathcal{X}}{dx}(0) = 0 \quad \text{and} \quad \mathcal{X}(1) = 1, \quad (3.255)$$

and the scaled layer depth is  $\tilde{l}(x, t) = 1 + f(d_0 x) - t$ .

Since  $\tilde{l} > 0$  for  $t \in [0, 1)$ , the Liouville equation (3.254) has exponential behaviour, i.e. integrating over  $(x_1, x_2)$  implies

$$\left[ \tilde{l} \frac{d\mathcal{X}}{dx} \right]_{x_1}^{x_2} = k^2 \int_{x_1}^{x_2} \tilde{l} \mathcal{X} dx; \quad x_2 > x_1, \quad \tilde{l} > 0,$$

so  $\mathcal{X}'' > 0$  almost everywhere for  $\mathcal{X} > 0$  and  $\mathcal{X}'' < 0$  almost everywhere for  $\mathcal{X} < 0$ ; see, for example, [51]. By (3.255), we deduce that  $d\mathcal{X}/dx(1) > 0$  for all  $k > 0$ . Hence, the first term of the right-hand side of (3.253) is negative if and only if  $\dot{d}_0 > 0$ , so the solution is linearly stable if and only if  $\dot{d}_0 > 0$  and the second term in (3.253) is negative; by (3.179, 3.182), this term is equal to

$$-\frac{1}{2\sqrt{l(d_0, t)}(\sqrt{l(d_0, t)} - 1)} \left[ \dot{d}_0 \left( 2 - \frac{1}{\sqrt{l(d_0, t)}} \right) \frac{\partial l}{\partial x}(d_0, t) - 1 \right]. \quad (3.256)$$

We conclude that, as in the stage 3 analysis in section 3.7.4.8, the two-dimensional solution is linearly stable if and only if  $\dot{d}_0$  lies above a critical speed, viz. by (3.256)

$$\dot{d}_0 > \left( \frac{\sqrt{l(d_0, t)}}{2\sqrt{l(d_0, t)} - 1} \right) \frac{1}{f'(d_0)}, \quad (3.257)$$

which is the stage 4 stability condition for a general symmetric body profile  $f(x)$ . We note that, as required, the small time limit of the stage 4 stability condition (3.257) is exactly the stage 3 stability condition (3.241).

In Figure 3.21 we plot  $d_0$ ,  $\dot{d}_0$  and the critical speed of the turnover point, i.e. the right-hand side of (3.257), for the parabola  $f(x) = x^2$ , which shows that the solution is linearly stable throughout the impact; the stability as  $t \rightarrow 0$  is guaranteed by the stage 3 analysis in section 3.7.3.8.

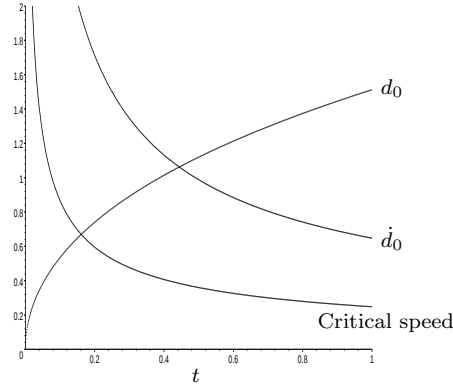


Figure 3.21: Plot of  $d_0$ ,  $\dot{d}_0$  and the critical speed for the parabola  $f(x) = x^2$ .

### 3.7.5 Extensions to more general bodies

Suppose the body profile (3.185) is such that

$$f(x, y) \sim a|x|^\alpha + b|y|^\beta \quad \text{as } x^2 + y^2 \rightarrow 0, \quad (3.258)$$

where  $a, b > 0$  are order unity and  $\alpha \geq \beta \geq 1$ , without loss of generality. If  $\alpha = \beta$ , then the temporal decomposition into four stages proceeds as before. The stage 2 timescale is  $\epsilon^\alpha$  and the leading-order outer stage 1 and 3 problems have a similarity solution in which distances scale with  $t^{1/\alpha}$ . If  $\alpha > \beta$  then the temporal decomposition is not so clear. The waterline, i.e. the intersection of the body with  $z = 0$ , has extent of order  $t^{1/\alpha}/\epsilon$  and  $t^{1/\beta}/\epsilon$  in the  $x$ - and  $y$ -directions, respectively. Since  $\alpha > \beta$ , we expect the turnover curve to extend much further in the  $x$ -direction at small times, so that the strip theory assumptions (i) and (ii) in section 2.4.5 may be employed. Since these fully determine the lowest order solution in the strips region, we may crudely decompose the flow into four stages by comparing the size  $t^{1/\beta}/\epsilon$  of the strip cross-section with the unit layer depth, exactly as in the case  $\alpha = \beta$ .

However, if we wish to find the full flow structure, i.e. recalling the strip theory terminology in Figure 2.13, the flow in

- the outer region of size  $O(t^{1/\alpha}/\epsilon)$ ,

- the inner tip regions at the end of the strip of size<sup>8</sup>  $O(t^{1/\gamma}/\epsilon)$  as  $t \rightarrow 0$ , say, where  $\gamma(\alpha, \beta) < \beta$ ,
- the intermediate regions that might be necessary to match these two regions with the strip region,

then we must also compare the size of the outer, inner and intermediate regions with the unit layer depth, resulting in a finer temporal decomposition. The analysis is beyond the scope of this thesis.

If the body profile is  $z = f(\epsilon x, \epsilon y) - t$ , where  $\epsilon \ll \varepsilon \ll 1$ , then strip theory is straightforward on the order unity time problem, except in the inner “leading-edge” regions. However, the small time temporal decomposition of this problem, which has not been studied, immediately promises some interesting behaviour. For example, if (3.258) holds at the origin with  $\alpha > \beta$ , then the strip will extend in the  $x$ -direction at sufficiently small times and then swap to the  $y$ -direction by order unity times, the interior region having comparable extent in the  $x$ - and  $y$ -directions on some intermediate timescale.

### 3.8 What happens next?

Finally, we remark that we have left open the problem of what happens after the impactor hits the base. Such a study would entail some careful cavitation modelling for the dry region that is revealed by the ejected liquid.

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<sup>8</sup>The size of the inner region is determined by demanding equal variation in the  $x$ - and  $y$ -directions and is therefore a function of the body profile parameters and is much smaller than the width of the strip region.

## Chapter 4

# Water entry of flat bottomed bodies

So far, our analysis of deep and shallow water entry problems has been for piecewise smooth bodies at small and strictly positive deadrise angles except on a set of zero measure. In this chapter we consider the impact of piecewise smooth, flat-bottomed bodies on water of infinite and finite depth.

### 4.1 Infinite depth water entry

Consider the impact of a rectangle with bottom  $z = -t$  for  $|x| < 1$  on the lower half-space  $z < 0$ . The dimensionless model problem (2.12 - 2.17) is unchanged upon setting  $f \equiv 0$ . Indeed, scaling

$$x = x', \quad z = z', \quad t = \delta t', \quad \phi = \phi', \quad h = \delta h', \quad (4.1)$$

where the timescale  $\delta \ll 1$ , dropping primes and expanding

$$\phi \sim \mathcal{H}(t)\phi_0(x, z) + \delta\phi_1(x, z, t) + \dots, \quad h \sim h_0(x, t) + \delta h_1(x, t) + \dots, \quad (4.2)$$

where  $\mathcal{H}(t)$  is the heaviside function, we find that the leading-order outer problem is exactly (2.25 - 2.31) with  $d_0 \equiv 1$ . Hence, if we suppose the leading-order outer velocity potential has the same behaviour at the two points where the boundary conditions change type, then

$$\phi_0 = \Re \left( i\zeta - i\sqrt{\zeta^2 - 1} \right), \quad (4.3)$$

where  $\zeta = x + iz$ . The free surface elevation becomes

$$h_0 = t \left( \frac{|x|}{\sqrt{x^2 - 1}} - 1 \right) \quad \text{for } |x| > 1, \quad (4.4)$$

by the kinematic boundary condition (2.27). Hence, in contrast to Wagner theory, an infinite elevation is predicted at the two fixed points  $(\pm 1, 0)$  in the leading-order outer solution. The presence of these singularities necessitates the existence of inner regions whose solution describes the local flow structure. The inner regions must be small in size compared to the order unity length of the base, so the inner problem cannot contain a lengthscale. Therefore, in the spirit of the ‘wedge entry’ problem [43], we anticipate the

existence of a similarity formulation of the leading-order inner problem. This is confirmed by Yakimov [79], though his analysis is ad hoc and requires formal justification with the method of matched asymptotic expansions. We show that this is straightforward. However, all questions concerning the existence, uniqueness and geometry of the local problem need further discussion. We subsequently discuss these questions in the context of infinite and finite depth water entry of flat-bottomed wedges of small and order unity deadrise angle.

#### 4.1.1 The inner problem

We consider the inner problem near  $\zeta = 1$ . To match with the outer solution and obtain a non-trivial balance in the kinematic and Bernoulli conditions (2.14, 2.15) we scale

$$x = 1 + \delta^{2/3}\bar{x}, \quad z = \delta^{2/3}\bar{z}, \quad \phi = \delta^{1/3}\bar{\phi}, \quad h = \delta^{1/3}\bar{h}, \quad t = \delta\bar{t}. \quad (4.5)$$

Note that we do not change to a moving frame as we did in the local analyses of the turnover regions for bodies of small but finite deadrise angle (in Chapters 2 and 3). This precludes any possibility of a local Helmholtz flow. Indeed, substituting the scalings (4.5) into the full dimensionless model problem (2.12 - 2.15) we find (dropping bars)

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0 \quad \text{in the fluid,} \\ \delta^{1/3} + \frac{\partial \phi}{\partial z} &= 0 \quad \text{on } z = -\delta^{1/3}t, \quad x < 0, \\ \frac{\partial \phi}{\partial x} &= 0 \quad \text{on } x = 0, \quad z > 0, \\ \frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial z} &= 0 \quad \text{on } z = h(x, t), \\ \frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] &= 0 \quad \text{on } z = h(x, t). \end{aligned}$$

Then, expanding  $\phi$  and  $h$  as asymptotic series in powers of  $\delta^{1/3}$  we find the leading-order problem is

$$\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad \text{in the fluid,} \quad (4.6)$$

$$\frac{\partial \phi_0}{\partial z} = 0 \quad \text{on } z = 0, \quad x < 0, \quad (4.7)$$

$$\frac{\partial \phi_0}{\partial x} = 0 \quad \text{on } x = 0, \quad z > 0, \quad (4.8)$$

$$\frac{\partial h_0}{\partial t} + \frac{\partial \phi_0}{\partial x} \frac{\partial h_0}{\partial x} - \frac{\partial \phi_0}{\partial z} = 0 \quad \text{on } z = h_0(x, t), \quad (4.9)$$

$$\frac{\partial \phi_0}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi_0}{\partial x} \right)^2 + \left( \frac{\partial \phi_0}{\partial z} \right)^2 \right] = 0 \quad \text{on } z = h_0(x, t), \quad (4.10)$$

together with the following far field conditions obtained by matching with the outer solution (4.3, 4.4),

$$\phi_0 \sim \Re(-i\sqrt{2\zeta}) \quad \text{as } |\zeta| \rightarrow \infty, \quad (4.11)$$

$$h_0 \sim t/\sqrt{2x} \quad \text{as } x \rightarrow \infty. \quad (4.12)$$

The initial conditions are therefore

$$\phi_0(x, z, 0) = \Re(-i\sqrt{2\zeta}), \quad h_0(x, 0) = 0. \quad (4.13)$$

Note that, in contrast to Wagner theory, an intermediate region is not required to match the leading-order outer and inner regions because the inner region of size  $O(\delta^{2/3})$  is much larger than the penetration depth of  $O(\delta)$ .

The local leading-order problem is invariant under the transformation (4.5) with  $x$  replacing  $x - 1$  and therefore exhibits a similarity solution if we follow [79] and write

$$x = t^{2/3}\xi, \quad z = t^{2/3}\eta, \quad \phi_0 = t^{1/3}\Phi(\xi, \eta), \quad h_0 = t^{2/3}H(\xi), \quad (4.14)$$

which yields the following paradigm zero-gravity free boundary problem:

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \quad \text{in the fluid}, \quad (4.15)$$

$$\frac{\partial \Phi}{\partial \eta} = 0 \quad \text{on } \eta = 0, \quad \xi < 0, \quad (4.16)$$

$$\frac{\partial \Phi}{\partial \xi} = 0 \quad \text{on } \xi = 0, \quad 0 < \eta < H(0), \quad (4.17)$$

where  $H(0) \geq 0$ ; the free boundary conditions on  $\eta = H(\xi)$ , which may be multivalued,

$$\frac{1}{3}\Phi + \frac{1}{2} \left( \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \left( \frac{\partial \Phi}{\partial \eta} \right)^2 \right) = \frac{2}{3} \left( \xi \frac{\partial \Phi}{\partial \xi} + \eta \frac{\partial \Phi}{\partial \eta} \right), \quad (4.18)$$

$$\frac{\partial \Phi}{\partial \eta} - \frac{2}{3}\eta = \left( \frac{\partial \Phi}{\partial \xi} - \frac{2}{3}\xi \right) \frac{\partial H}{\partial \xi}; \quad (4.19)$$

and finally the matching conditions

$$\Phi \sim \Re \left( -i\sqrt{2(\xi + i\eta)} \right) \quad \text{as } \xi^2 + \eta^2 \rightarrow \infty, \quad (4.20)$$

$$H \sim 1/\sqrt{2\xi} \quad \text{as } \xi \rightarrow \infty. \quad (4.21)$$

The free surface meets the vertical wall at  $(0, H(0))$  and it is not clear whether  $H(0)$  is zero, finite or infinite. However, it would be physically reasonable to assume that the free surface leaves from the corner and tangential to the flat bottom, i.e. the smooth separation condition

$$H(0) = \frac{dH}{d\xi}(0) = 0. \quad (4.22)$$

Then, a local analysis at the origin (up to  $O(r)$  in the governing equations) yields

$$\Phi \sim c_0 + c_1 r \cos \theta + c_2 r^{3/2} \sin(3\theta/2) + c_3 r^2 \cos 2\theta + O(r^{5/2}), \quad (4.23)$$

$$H \sim a_0 \xi^{3/2} + O(\xi^2), \quad (4.24)$$

where  $\xi + i\eta = r \exp(i\theta)$  and  $c_1 > 0$  and  $c_2$  are globally determined constants, in terms of which

$$a_0 = \frac{4c_2}{15c_1}, \quad c_0 = -\frac{3c_1^2}{2}, \quad c_3 = \frac{1}{6} - \frac{1369}{3600} \frac{c_2^2}{c_1} \quad (4.25)$$

i.e. the free surface leaves the base tangentially at  $(0, 0)$  where it has an inverse square root curvature singularity.

No other theory is available for this problem but, if it does have a solution, the free boundary must have at least one maximum because it is positive for large  $\xi$  by the far field condition (4.21). Hence, if  $H(\xi)$  has no minima the configuration is as shown in Figure 4.1a. Moreover, fluid is squeezed out from beneath the impactor, and this generates the single maximum shown in the free surface elevation. By contrast, the jets in the Wagner and Korobkin flows considered in Chapters 2 and 3 comprise fluid from outside the region beneath the impactor.

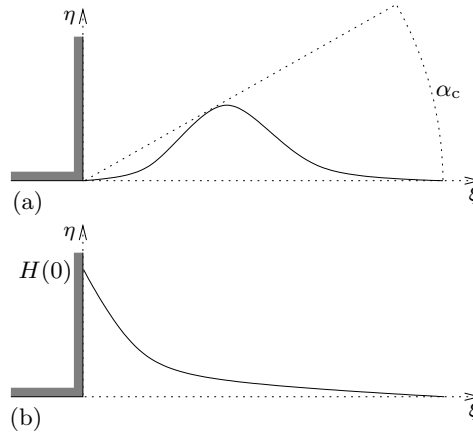


Figure 4.1: Two feasible free surface morphologies in the inner region for the entry of a rectangle.

A numerical analysis is required for which the boundary element method might be most appropriate<sup>1</sup>. The main difficulty is taking care of the singularities at the corner and in the far field. We will review related numerical evidence for the existence of this ‘humped’ similarity solution in section 4.3.1, where we consider the straightforward three-dimensional generalization.

Even if the humped similarity solution does exist, it is unlikely that it is the only mathematically possible solution near the corner of the impactor. For example, we could conjecture that surface tension could cause the fluid ejected from beneath the impactor to adhere to the vertical segment of the wall near the corner, with separation further up the side (i.e.  $H(0)$  non-zero and finite), as in the ‘teapot’ flows of [68]. The free boundary would then form a jet as depicted in Figure 4.1b, although this may seem unlikely in view of the large velocities at the corner. At any event, the photographs in [4, 79], which appear to show tangential separation and a localised vertical jet some distance from the body, suggest that this does not occur.

#### 4.1.2 Entry of a flat-bottomed wedge

The above analysis may also apply to the entry of a flat-bottomed wedge with deadrise angle  $\alpha < \pi/2$ . The only difference may occur in the inner corner region where the wall now makes an angle  $\alpha$  with the horizontal. Assuming the scenario sketched in Figure 4.1a is relevant for large enough  $\alpha$ , we now consider what might happen as  $\alpha$  decreases. There is a critical angle  $\alpha_c$  at which the impactor first touches the free surface, as in Figure 4.1a,

<sup>1</sup>Such an analysis is currently being pursued by Vanden-Broeck.

and we might conjecture that, for  $\alpha < \alpha_c$ , there exists solely the similarity solution in which the flow adheres to the wall, thus forming a jet as in Figure 4.2.

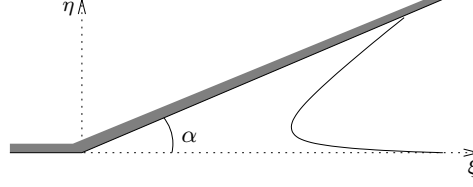


Figure 4.2: A feasible free surface morphology in the inner region for the entry of a flat-bottomed wedge of deadrise angle  $\alpha < \alpha_c$ .

Further evidence for the existence of such a similarity solution is found by considering the small deadrise angle limit in which  $\alpha \ll 1$ . The right-hand wall of the flat-bottomed wedge has equation

$$z = \alpha(x - 1) - t \quad \text{for } x > 1, \quad (4.26)$$

so assuming the Wagner theory of Chapter 2 holds at times  $t = \delta t'$ , with the jet root attached to the side wall of the body, the right-hand turnover point has leading-order  $x$ -coordinate  $\delta d_0(t')/\alpha$ , where  $d_0(t') = O(1)$  as  $\alpha \rightarrow 0$ . Thus, the bottom of the wedge and the interior region between the turnover points are comparable in length when  $\delta = \alpha$ , without loss of generality. In this case,  $d_0(t')$  is determined by the Wagner condition (2.50) with body profile  $f(x) = x - 1 - t'$  for  $x > 1$  and initial condition  $d_0(0) = 1$ , which imply [31]

$$\sqrt{d_0(t')^2 - 1} + \sin^{-1} \left( \frac{1}{d_0(t')} \right) = \frac{\pi}{2}(1 + t'),$$

and therefore

$$d_0(t') \sim 1 + \frac{1}{2} \left( \frac{3\pi t'}{2} \right)^{2/3} + O((t')^{4/3}) \quad \text{as } t' \rightarrow 0+. \quad (4.27)$$

The 2/3-power behaviour suggests that such a flow would be able to be matched with the jet similarity solution as  $t' \rightarrow 0$ .

Further, we note that

$$d_0(t') \sim \frac{\pi}{2}(1 + t') + O(1/t') \quad \text{as } t' \rightarrow \infty.$$

Hence, as expected, the effect of the flat bottom is negligible as  $t' \rightarrow \infty$  and therefore on order unity timescales, since, for a wedge of deadrise angle  $\alpha \ll 1$ , the right-hand turnover point has  $x$ -coordinate  $\pi t/2\alpha + O(1)$  at order unity times  $t$ .

Note that the ‘hump’ and jet similarity solutions are the only possible scenarios for a flat-bottomed wedge because the walls are straight and all distances scale with  $t^{2/3}$ . However, if the wall is not straight and impinges on the region  $\theta < \alpha_c$  away from the origin, then, as depicted in Figure 4.3, the free surface will hit the wall in finite time in the region  $\theta < \alpha_c$ . There is then the possibility of re-attachment and the formation of a trapped bubble.

Establishing the existence and character of the bifurcation that occurs as  $\alpha$  passes through  $\alpha_c$  is an interesting open question of practical importance.

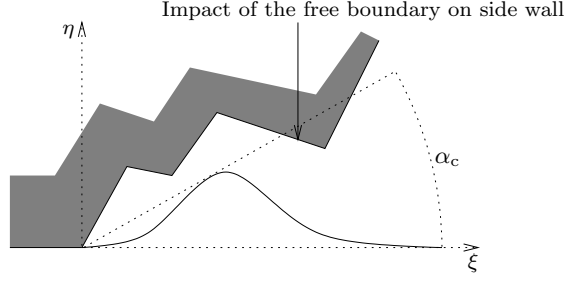


Figure 4.3: Schematic of the possible free surface morphology if the side wall of the flat-bottomed body impinged on the region  $\theta < \alpha_c$ .

### 4.1.3 The pressure and force on the body

In this section we consider the order of magnitude of the pressure acting on the body in the outer and inner regions and their contribution to the force on the body.

Substituting the outer scalings (4.1) into Bernoulli's equation (2.18), we find (dropping primes)

$$p + \frac{1}{\delta} \frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] = 0. \quad (4.28)$$

Substituting the expansion (4.2) implies

$$p \sim \frac{1}{\delta} \mathcal{H}'(t) \phi_0 - \frac{\partial \phi_1}{\partial t} - \frac{1}{2} \left[ \left( \frac{\partial \phi_0}{\partial x} \right)^2 + \left( \frac{\partial \phi_0}{\partial z} \right)^2 \right] + o(1), \quad (4.29)$$

where the time derivative of the heaviside function,  $\mathcal{H}'(t)$ , is the Dirac delta function. If the proposed asymptotic expansion for the velocity potential (4.2) is correct, then it is necessary to solve the first-order outer problem to find the leading-order outer pressure at time  $t > 0$ .

Substituting the inner scalings (4.5) into Bernoulli's equation (2.18) and expanding the inner potential as in section 4.1.2, i.e. in powers of  $\delta^{1/3}$ , we find the inner pressure

$$\bar{p} \sim -\frac{\bar{p}_0}{\delta^{2/3}} + O(1/\delta^{1/3}), \quad (4.30)$$

where we have reintroduced bars for clarity. In terms of the similarity variables (4.15), the leading-order inner pressure is

$$\begin{aligned} \bar{p}_0 &= -\frac{\partial \bar{\phi}_0}{\partial t} - \frac{1}{2} \left[ \left( \frac{\partial \bar{\phi}_0}{\partial \bar{x}} \right)^2 + \left( \frac{\partial \bar{\phi}_0}{\partial \bar{z}} \right)^2 \right] \\ &= \frac{1}{t^{2/3}} \left[ \frac{2}{3} \left( \xi \frac{\partial \Phi}{\partial \xi} + \eta \frac{\partial \Phi}{\partial \eta} \right) - \frac{1}{3} \Phi - \frac{1}{2} \left( \left( \frac{\partial \Phi}{\partial \xi} \right)^2 + \left( \frac{\partial \Phi}{\partial \eta} \right)^2 \right) \right]. \end{aligned} \quad (4.31)$$

The leading-order pressure in the outer and inner regions are of order unity and  $1/\delta^{2/3}$ , respectively, while their lateral extents are of order unity and  $\delta^{2/3}$ , respectively. Hence, the leading-order force on the body (per unit length in the perpendicular direction) is of order unity and has contributions from *both* the outer and inner regions, which is again in contrast to the Wagner and Korobkin flows considered in Chapters 2 and 3. This reveals

the principal difference between the impact of a flat-bottomed body considered here and those for which impact is made instantaneously at only one point as considered in Chapters 2 and 3.

Lacking a numerical solution in the inner region, we do not pursue the investigation further.

#### 4.1.4 The effect of finite depth

In this section we consider the effect of finite depth on the above small time analysis. We suppose the half-space is replaced by a layer of depth  $\beta$ , as depicted in Figure 4.4, the usual model problem having been nondimensionalized with respect to half the width of the base of the normally entering flat-bottomed wedge. In the outer region away from the corners, the boundary conditions may be linearized onto the  $x$ -axis provided the timescale  $\delta \ll \beta$ . However, for sufficiently small layer depths  $\beta$  and for sufficiently large timescales  $\delta$ , the basin must affect the inner corner regions and therefore invalidate the similarity formulation that we considered in section 4.1.2. In this section we will show that this occurs at times of order  $\beta^2$ , i.e. the inner corner regions are as in section 4.1.2 provided  $\delta \ll \beta^2$ . Our argument will be based entirely on the leading-order outer solution and is therefore independent of the precise form of the flow in the inner regions at the corners, which was considered in section 4.1.2 and is summarized in Figure 4.4.

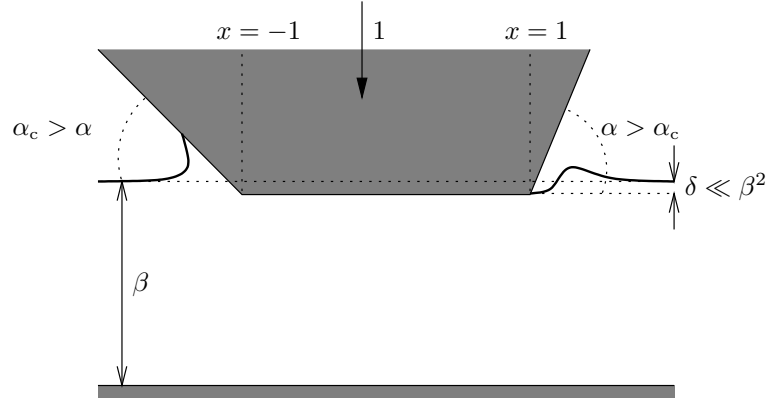


Figure 4.4: Entry of a flat-bottomed body on a layer of depth  $\beta$  on the timescale  $\delta \ll \beta^2$ .

The outer solution (4.3) is simply replaced by the stage 2 outer solution (3.34 - 3.37), in which we must scale distances with the layer depth  $\beta$ . Hence, the velocity potential,  $\phi_0$ , and free surface elevation,  $h_0 = t\partial\phi_0/\partial z(x, 0, t)$ , have square root and inverse square root singularities, respectively, at the corners, viz. by (3.38)

$$\phi_0 \sim \operatorname{Re} \left( -ik(\beta)\sqrt{2(\zeta - 1)} \right) \quad \text{as } \zeta \rightarrow 1, \quad (4.32)$$

$$h_0 \sim \frac{tk(\beta)}{\sqrt{2(x - 1)}} \quad \text{as } x \rightarrow 1, \quad (4.33)$$

where the coefficient  $k(\beta)$  is solely a function of the layer depth  $\beta$ . The analysis of section 3.3.1.1 implies

$$k(\beta) = S(1/\beta)/\sqrt{2}, \quad (4.34)$$

where  $S$  is the positive real integral (3.39). Note that the analysis of section 3.3.3 implies

$$k(\beta) \sim 1 + O(1/\beta) \quad \text{as } \beta \rightarrow \infty, \quad (4.35)$$

so, as required, the finite depth solution is in agreement with the infinite depth solution as the layer depth  $\beta$  tends to infinity. We note two points concerning the small depth limit. First, the analysis of section 3.4.5 implies

$$k(\beta) \sim \frac{2}{\sqrt{\pi\beta}} + O(1) \quad \text{as } \beta \rightarrow 0. \quad (4.36)$$

Second, the analysis of sections 3.4 implies that in the limit as  $\beta \rightarrow 0$ , the outer solution decomposes into essentially three regions, namely, the interior, exterior and transition regions, which are shown in Figure 4.5 and are described in detail in sections 3.4.1, 3.4.2 and 3.4.3, respectively.

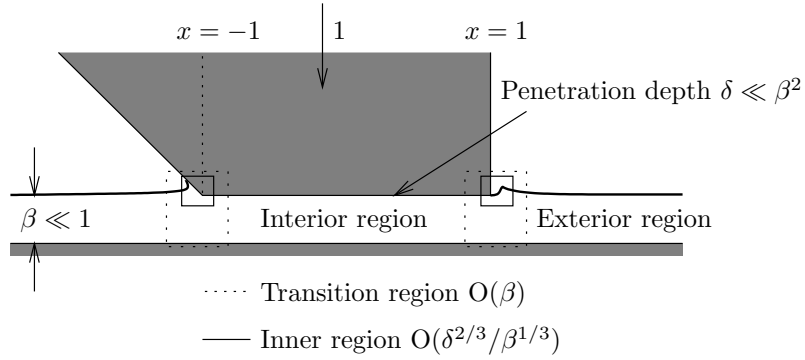


Figure 4.5: Asymptotic structure of the outer solution for the impact of a flat-bottomed body on shallow water of depth  $\beta \ll 1$  at timescales  $\delta \ll \beta^2$ .

The inner corner problem is therefore unchanged, i.e. it is exactly (4.15 - 4.21), provided we also scale distances with  $k^{2/3}$  and the potential with  $k^{4/3}$  in (4.14), which eliminates  $k$  from the far field conditions derived from (4.32, 4.33). Since  $k(\beta)$  is a strictly increasing function of  $\beta$ , the sole effect of introducing a layer of depth  $\beta$  on the inner region is to increase its size to  $(k(\beta)\delta)^{2/3}$ . Hence, as  $\beta \rightarrow 0$ , the asymptotic structure depicted in Figure 4.5 is valid so long as the inner region of size  $(k(\beta)\delta)^{2/3}$  is much smaller than the layer depth  $\beta$ , i.e. for timescales  $\delta \ll \beta^2$  by (4.36).

We emphasise that, as in the infinite depth case considered in section 4.1.3, we conjecture that the humped similarity solution (in Figure 4.1a) exists and is relevant for sufficiently large deadrise angles, while the jet similarity solution (in Figure 4.2) exists and is relevant for sufficiently small deadrise angles. In the next section we consider the fate of these proposed asymptotic solutions.

## 4.2 Shallow water entry

In this section we consider the impact of the flat-bottomed wedge considered in section 4.1.2 on shallow water of depth  $\beta \ll 1$  at times of order  $\beta$ . On this timescale the penetration depth is comparable to the layer depth. Hence, it is necessary to account for a leading-order deformation of the fluid domain as in the Korobkin flows of section 3.5. In section 4.2.1 we consider the impact of a rectangle and argue that the asymptotic solution due to

Korobkin [39] may be valid for deadrise angles  $\alpha > \pi/4$ . In section 4.2.2, we suppose that Korobkin's theory in section 3.5 is applicable to the impact of a flat-bottomed wedge of deadrise angle  $\alpha$  of the same order as the (dimensionless) layer depth  $\beta$ . In both scenarios we briefly discuss the initiation of the flow with regard to the analysis performed in the previous section, which is summarized in Figure 4.5.

#### 4.2.1 Right deadrise angle

Korobkin [39] proposed the scenario depicted in Figure 4.6 for the normal impact of a rectangle of width 2 moving with unit velocity into a shallow fluid layer of depth  $\beta \ll 1$ , at times of order  $\beta$  when the penetration depth is comparable to the layer depth. The flow is characterized by the spouts of water (VI in Figure 4.6) that are ejected from small jet root regions (IV in Figure 4.6) that travel away from the body. Hence the flow is very different from the flow induced by a body of small deadrise angle impacting on shallow water, in which the jet roots are attached to the body and eject jets that, at least initially, run along the body; see Figure 3.1 and section 3.5.

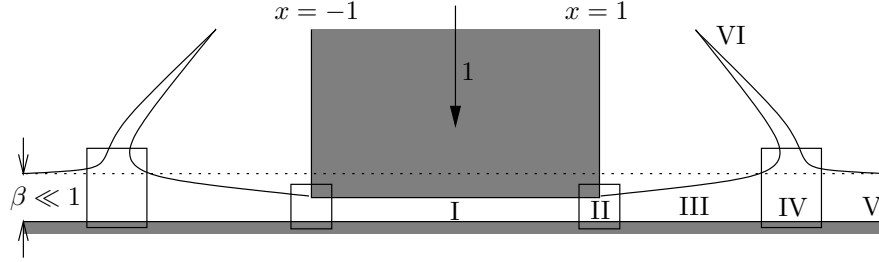


Figure 4.6: Order  $\beta$  time asymptotic structure for the impact of a rectangle on shallow water of depth  $\beta \ll 1$ .

We suppose that the shallow fluid layer initially lies in  $0 < z < \beta$  and that impact begins at  $t = 0$ . Rescaling the vertical coordinate  $z$  and time  $t$  with  $\beta$ , the bottom of the rectangle has equation

$$z = 1 - t \quad \text{for } |x| < 1.$$

By symmetry we consider the flow in  $x \geq 0$ . Beneath the impactor, in region I, the usual inviscid squeeze film solution applies, and the analogue of (3.140, 3.141) is  $\phi \sim \phi_0(x, t)/\beta + O(1)$ , where by symmetry

$$\phi_0(x, t) = \frac{x^2}{2(1-t)} + A(t), \quad (4.37)$$

and  $A(t)$  is to be determined.

Assuming that in the small inner region II the free surface leaves the corner tangentially, the only leading-order solution there that conserves mass, momentum and energy is the trivial solution with uniform flow. Hence, the boundary conditions for region I obtained by matching with region II are zero pressure at  $x = \pm 1$ , viz. by (3.173)

$$\frac{\partial \phi_0}{\partial t} + \frac{1}{2} \left( \frac{\partial \phi_0}{\partial x} \right)^2 = 0 \quad \text{at } x = \pm 1,$$

which implies  $A(t) = -1/(1-t)$  and so, summarizing, the leading-order inviscid squeeze film velocity (scaled with  $1/\beta$ ) and pressure (scaled with  $1/\beta^2$ ) are

$$\frac{\partial \phi_0}{\partial x}(x, t) = \frac{x}{1-t}, \quad p_0(x, t) = \frac{1-x^2}{(1-t)^2}, \quad (4.38)$$

respectively. Region II ejects a thin stream of fluid with high velocity (scaled with  $1/\beta$ ) and therefore the flow in region III is governed, to lowest order, by the zero-gravity shallow-water equations for the extentional fluid velocity,  $u_0(x, t)$ , and the stream height,  $h_0(x, t)$ , viz. differentiating (2.85, 2.86),

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = 0, \quad (4.39)$$

$$\frac{\partial h_0}{\partial t} + \frac{\partial(u_0 h_0)}{\partial x} = 0, \quad (4.40)$$

with boundary conditions

$$u_0(1, t) = \frac{1}{1-t}, \quad h_0(1, t) = 1-t,$$

and solution, by the method of characteristics (see Figure 4.7a),

$$u_0(x, t) = \frac{2-x}{1-t}, \quad h_0(x, t) = \frac{1-t}{(2-x)^2}, \quad (4.41)$$

assuming we are not too close to  $x = 2$ . This solution cannot, however, extend as far as  $x = 2$  because the jet described by (4.41) collides with the initially quiescent layer V, which we expect to move exponentially slowly as in the order unity time exterior region in Chapter 3.

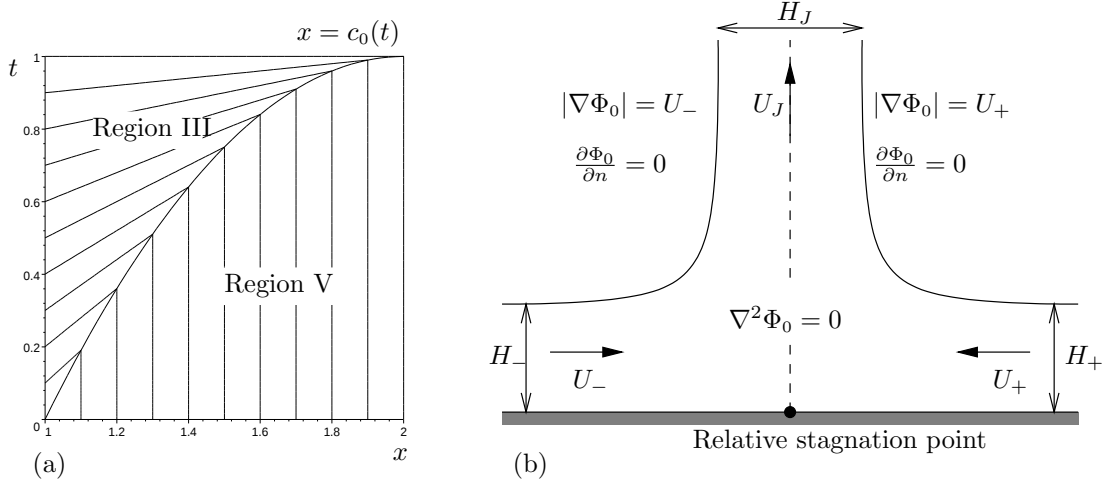


Figure 4.7: (a) Characteristics in regions III and V intersect at the shock  $x = c_0(t)$ . (b) Leading-order jet interaction problem in region IV. The scalings  $x = c_0(t) + \beta X$ ,  $y = \beta Y$  and  $\phi = \dot{c}_0(t)X + \Phi$  imply the leading-order problem shown. In addition  $U_- = u_0(c_0, t) - \dot{c}_0$ ,  $U_+ = \dot{c}_0$ ,  $H_- = h_0(c_0, t)$ ,  $H_+ = 1$  and we derive  $U_J$ ,  $H_J$  in the text.

To lowest order the interaction between III and V is a classical jet interaction problem [48], as depicted in Figure 4.7b. This takes place in a small region IV of size order  $\beta$

centred on  $x = c(t)$ , in which the two incoming jets from regions III and V collide to eject a “waterspout” in region VI. Matching guarantees the disturbance is exponentially small in region V and that the leading-order pressure is zero across the waterspout, which therefore does not support a vortex sheet; through the Bernoulli conditions on the free surfaces the second implies  $U_- = U_+$ , i.e.

$$\dot{c}_0(t) = \frac{1}{2}u_0(c_0(t), t). \quad (4.42)$$

Hence, with the initial condition  $c_0(0) = 1$ , the position of the shock is

$$c_0(t) = 2 - \sqrt{1 - t},$$

which we plot in Figure 4.7a. The leading-order jet root pressure  $P_0$  is of order  $1/\beta^2$  and given by Bernoulli’s equation  $P_0 + |\nabla\Phi|^2/2 = \dot{c}_0^2/2$ ; as noted by Korobkin [39], this is comparable to the inviscid squeeze film pressure below the impactor and takes its maximum,  $\dot{c}_0^2/2$ , at the relative stagnation point on the base, as depicted in Figure 4.7b.

Since  $H_- = h_0(c_0(t), t) = 1$ , a waterspout of initial dimensional thickness  $2\beta$  is ejected from  $x = c_0(t)$  with dimensional vertical velocity  $U_J/\beta = \dot{c}_0/\beta$ , to lowest order. At time  $t$ , the fluid ejected at time  $\tau$  reaches  $(c_0(\tau) + \dot{c}_0(\tau)(t - \tau), \dot{c}_0(\tau)(t - \tau))$  and therefore the waterspout centreline is the hyperbola

$$(2 - x)^2 - z^2 = 1 - t \quad \text{for } 1 + t/2 < x < c_0(t).$$

Hence, the waterspout approaches the body with an inclination of  $\pi/4$  as time increases, which is in accordance with the photographs in [4]. The leading-order waterspout thickness  $\beta h_0(z, t)$  is found by the usual conservation of mass argument across the thin jet [39], which yields

$$h_0(z, t) = 2A(z, t)^2 e^{1-A(z, t)} \quad \text{for } 0 < z < t/2,$$

where

$$A(z, t) = 1 + \frac{z(z + \sqrt{1 - t + z^2})}{1 - t + z(z + \sqrt{1 - t + z^2})}.$$

We plot waterspout profiles in Figure 4.8 for  $t = 0.1, 0.3, 0.5, 0.7, 0.9$ . The tip of the waterspout lies on the line  $z = x - 1$ , which suggests that, provided we assume smooth separation at the corners, the analysis also holds for flat bottomed wedges with deadrise angle greater than  $\pi/4$ .

The small time behaviour of the detached waterspout is

$$c_0(t) \sim 1 + t/2 + O(t^2) \quad \text{as } t \rightarrow 0, \quad (4.43)$$

so the asymptotic structure depicted in Figure 4.7 breaks down at times of order  $\beta^2$  (recall the rectangle hits the base at time equal to  $\beta$ ). This leaves open a major problem concerning the initiation of the flow.

In section 4.1.4, we showed that for times much smaller than  $\beta^2$ , the motion is to leading order confined to the corners of the impactor as in Figure 4.5, basal effects only being important insofar as they determine the global constants that complete the specification of the local corner problems.

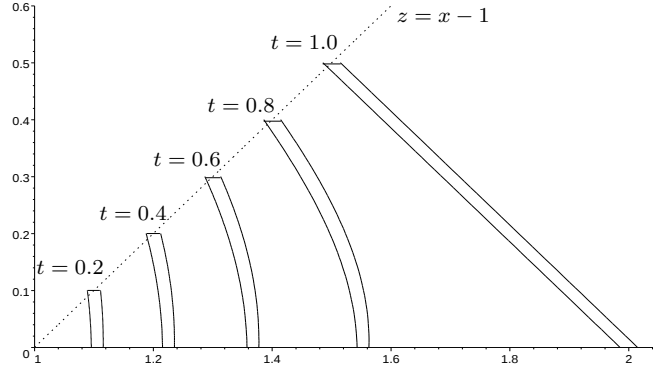


Figure 4.8: Leading-order waterspout profiles for  $t = 0.2, 0.4, 0.6, 0.8, 1.0$ .

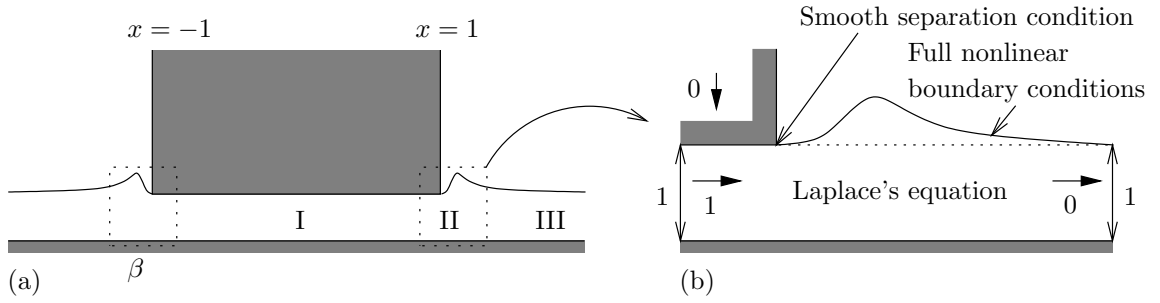


Figure 4.9: (a) Proposed asymptotic structure on the order  $\beta^2$  timescale. (b) Leading-order corner problem in region II, which is subject to zero initial conditions.

At times of order  $\beta^2$  the kinematic boundary condition on the bottom of the rectangle may still be linearized onto  $z = 0$ , while the analysis of section 4.1.4 suggests the transition region and inner corner region merge (see Figure 4.5). We therefore propose the asymptotic decomposition depicted in Figure 4.9a.

In region I beneath the impactor the small time inviscid squeeze film solution (3.65) applies with  $\delta = \beta$  and is of course just the small time limit of (4.38). To lowest order the fluid is ejected from beneath the body with speed equal to  $1/\beta$ , so the body has zero velocity to lowest order in the small corner region II where the free boundary conditions take their full nonlinear form. Since the ejected fluid particles travel at most a distance of order  $\beta$  over the timescale  $\beta^2$ , we expect the flow in region III to be exponentially small resulting in the zero matching condition as  $X \rightarrow \infty$  in Figure 4.9b.

It would be interesting to see if there was numerical evidence to show that the humped similarity solution (at times  $t \ll \beta^2$ ) moves out and grows into the waterspout depicted in Figure 4.6 (at times  $t$  such that  $\beta^2 \ll t \sim \beta$ ) through this inner region (at times  $t \sim \beta^2$ ).

#### 4.2.2 Small deadrise angle

We now consider the shallow water impact of the flat-bottomed wedge considered in section 4.1.3; see Figure 4.10. We suppose that the wedge has small deadrise angle  $\alpha$  comparable to the (dimensionless) layer depth  $\beta \ll 1$ .

For simplicity, we let  $\alpha = \beta$ . Let us assume that Korobkin's theory of section 3.5 describes the order  $\beta$  time asymptotic solution. Then the law of motion of the turnover point with leading-order  $x$ -coordinate  $d_0(t)/\alpha$  is (3.171) with  $l(x, t) = 1 + (x - 1) - t$  for

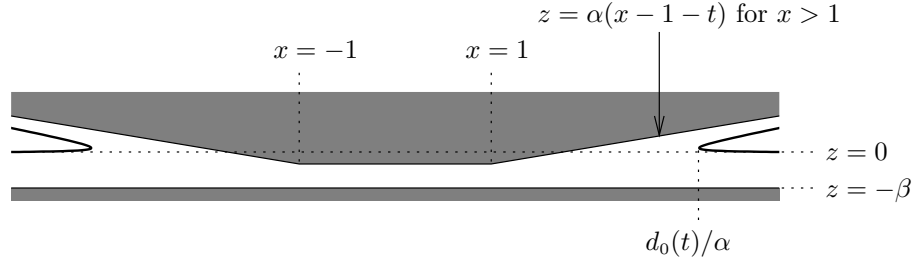


Figure 4.10: Order  $\beta$  time entry of a flat-bottomed wedge of deadrise angle  $\alpha = \beta$  on shallow water of depth  $\beta \ll 1$ .

$x \geq 1$  and initial condition  $d_0(0) = 1$ . It is then straightforward to show

$$d_0(t) \sim 1 + (3/2)^{1/2} t^{2/3} + O(t^{4/3}) \quad \text{as } t \rightarrow 0.$$

As in (4.27) in section 4.1.2, it is possible that such a flow would be able to be matched with the flow analysed in section 4.1.4, in which the similarity solution takes the form of a jet as depicted in Figure 4.2.

### 4.3 Three-dimensional models

In this section we briefly consider the generalization to the normal impact of a flat-bottomed cylinder with arbitrary cross-section  $\Omega$  on water of infinite, finite and small depths.

#### 4.3.1 Infinite and finite depth models

As in the two-dimensional case, the infinite depth leading-order small  $\delta$ -timescale problem is exactly the outer Wagner problem (2.109 - 2.114, 2.116), with the free turnover curve  $\Omega_0(t)$  replaced by the fixed cross-section  $\Omega$ . The theory reviewed in section 2.4 allows us to write down the outer solution for circular and elliptic cross-sections  $\Omega$ . The inner region now lies an order  $\delta^{2/3}$  distance from the boundary  $\partial\Omega$  of  $\Omega$  and the structure and matching proceed exactly as in the two-dimensional case, provided, as in all three-dimensional generalizations in this thesis, the boundary  $\delta\Omega$  is smooth and has radius of curvature much larger than the cross-sectional size of the inner region. Again the lowest order inner region flow is quasi-two-dimensional in all planes perpendicular to  $\partial\Omega$ . Therefore, all the implications concerning the impact of flat-bottomed bodies with walls that are not vertical (c.f. section 4.1.2) and the generalizations to the small time asymptotics of the finite depth case (c.f. section 4.1.3) are valid. The results in section 3.7 allow us to construct the corresponding outer three-dimensional solutions and strip theory may be employed, as described in sections 2.4.5 and 3.7.6, for bodies with long and thin cross-sections. As described in section 3.7.3.1, in the axisymmetric version of section 4.1.4, i.e. the small time asymptotics of a flat circular disc impacting on water of finite depth, the leading-order outer solution away from the corner regions was found by Vorovich & Yodovich [70] for large layer depths and by Chebakov [7] for small layer depths.

Finally, we note that Gaudet [18] used the boundary integral method to compute the order unity time solution to the full free boundary problem in the infinite depth, axisymmetric, small gravity case assuming a smooth separation condition on the edge of the circular disc. His solution, which shows the formation and growth of a single ‘humped’ profile,

is in excellent agreement with the outer solution (2.133) away from the corner and with the experiments of [21]. This provides evidence for the existence of the humped similarity solution in Figure 4.1a.

### 4.3.2 Shallow water model

The generalization of the order  $\beta$  time asymptotics of the normal impact of a rectangle onto shallow water of depth  $\beta \ll 1$ , which we reviewed in section 4.2.1, to the three-dimensional case was briefly investigated by Korobkin [39]. The inviscid squeeze film flow beneath the impactor, in region I in Figure 4.6, implies the leading-order velocity potential  $\phi(x, y, t)$  satisfies

$$\nabla^2 \phi = \frac{1}{1-t} \quad \text{in } \Omega, \quad (4.44)$$

and then matching with region II in a neighbourhood of the corner, through a quasi-two-dimensional region of cross-sectional extent  $\beta$  in which the lowest order flow is uniform in all planes perpendicular to  $\partial\Omega$ , yields the zero pressure matching condition

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 = 0 \quad \text{on } \partial\Omega. \quad (4.45)$$

Korobkin [39] solved (4.44, 4.45) when the cross-section is a circle and an ellipse, for which  $\phi = A(t) + B(t)x^2 + C(t)y^2$  where  $A$ ,  $B$  and  $C$  are uniquely determined.

However, the interesting three-dimensional exterior flow was not analysed. In the shallow water region III the zero-gravity shallow water equations (2.119, 2.120) hold; we drop subscripts and for clarity introduce hats on the leading-order exterior dependent variables. Matching with region II implies the leading-order potential  $\hat{\phi}(x, y, t)$  and free surface elevation  $\hat{h}(x, y, t)$  are equal to the outgoing potential on the boundary from region I and the gap thickness, respectively, viz.

$$\hat{\phi} = \phi, \quad h = 1 - t \quad \text{on } \partial\Omega. \quad (4.46)$$

The method of characteristics described in section 2.4.1 then allows us to solve for  $\hat{\phi}$  and  $\hat{h}$ , though here the fluid is not necessarily ejected with velocity normal to the boundary. The shock condition (4.42) becomes

$$v_n = \frac{1}{2} \mathbf{n} \cdot \nabla \hat{\phi}, \quad (4.47)$$

on the two-dimensional waterspout root  $\partial D(t)$ , which has normal velocity  $v_n$  and outward pointing unit normal  $\mathbf{n}$ , as depicted in Figure 4.11. The initial condition for the waterspout is  $\partial D(0) = \partial\Omega$ .

For example, if  $\Omega$  is the unit disc, the flow is axisymmetric and it is straightforward to show that the ejected fluid velocity, and therefore the normal shock speed, is exactly half what it was in the two-dimensional case, i.e. the radial distance of  $\partial D$  from the origin is  $(3 - \sqrt{1-t})/2$ . Further the free surface elevation  $h = 1$  on the shock  $\partial D$ , so the waterspout is again ejected vertically.

Given the initial velocity and thickness of the fluid sheet ejected into the waterspout, the location of the centre-surface of the sheet may be written down in parametric form, because each fluid particle follows a straight path. The thickness may be found by the usual conservation of mass argument across a thin sheet [28]. In the axisymmetric case the

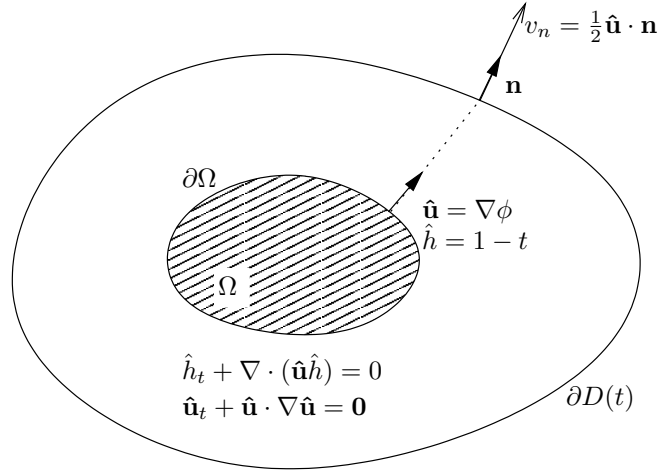


Figure 4.11: Plane view of the asymptotic regions in the shallow water entry of a flat bottomed body  $\Omega$ . The exterior region fluid velocity is  $\hat{\mathbf{u}} = \nabla \hat{\phi}$ , for which the zero-gravity shallow water equations and boundary data are shown.

locus of the centre-line in each radial plane is exactly half that of the two-dimensional centre-line, which is a convenient figure to check with experiment. More complicated cross-sections  $\Omega$  result in shallow-water and waterspout problems requiring a numerical solution.

## Chapter 5

# Oblique water entry; some conjectured models and their implications

So far, our analysis of deep and shallow water entry problems for bodies of small and zero deadrise angle has been for an impact velocity normal to the initially planar free surface. In this chapter we use the theories of Chapters 2, 3 and 4 as building blocks to formulate some conjectured models for impacts with a tangential or forward velocity component. Nearly all of the resulting models require formal justification and further analysis; we record them for future work and present our preliminary observations.

In section 5.1 we review and discuss models for infinite depth oblique water entry at small deadrise angles and for planing at small incidence angles. In section 5.2 we consider the effect of finite depth on these models. In section 5.3 we consider the oblique shallow water entry of a flat plate at zero deadrise angle. We focus on the two-dimensional case throughout except in section 5.2.3 where we consider three-dimensional shallow-water planing at small incidence angles.

### 5.1 Infinite depth models

#### 5.1.1 Oblique water entry at small deadrise angles

Suppose we superimpose a dimensionless forward velocity  $U\mathbf{i}$  onto the two-dimensional, infinite depth, small deadrise angle scenario introduced in section 2.1. The convex, piecewise smooth body profile becomes

$$z = f(\epsilon x - \epsilon U t) - t, \quad (5.1)$$

where  $f(0) = 0$  and we recall that the small deadrise angle assumption corresponds to  $\epsilon$  being small. On the wetted part of the body the kinematic boundary condition (2.13) becomes

$$\frac{\partial \phi}{\partial z} = \epsilon \frac{\partial \phi}{\partial x} f'(\epsilon x - \epsilon U t) - \epsilon U f'(\epsilon x - \epsilon U t) - 1. \quad (5.2)$$

For order unity times, the contribution to the normal impact velocity from the terms on the right-hand side of (5.2) are  $O(\epsilon)$ ,  $O(\epsilon U)$  and  $O(1)$ , respectively. Hence the forward motion

can only affect the leading-order outer problem if  $U \geq O(1/\epsilon)$ . We let  $U_0 = \epsilon U$  and consider the three cases  $U_0 \ll 1$ ,  $U_0 = O(1)$  and  $U_0 \gg 1$ .

#### 5.1.1.1 Case A: Small forward velocity $U_0 \ll 1$

If  $U_0 \ll 1$  and  $t = O(1)$ , then  $\epsilon f'(\epsilon x - \epsilon U t)$ ,  $\epsilon U f'(\epsilon x - \epsilon U t) \ll 1$  and the forward motion does not affect the kinematic boundary condition on the body (5.2) to lowest order. Hence, the Wagner theory of section 2.2 applies and the leading-order outer problem (2.25 - 2.31, 2.34, 2.50) is unchanged. As noted by [50], this explains why an impact splash is largely independent of the oblique impact speed. Korobkin [36] deduced the same result using Lagrangian variables.

If  $f'$  is bounded, then  $\epsilon f'(\epsilon x - \epsilon U t)$  and  $\epsilon U f'(\epsilon x - \epsilon U t)$  remain uniformly small and the solution is uniformly valid for all  $t > 0$ ; see (5.4) for an explicit justification of this statement for the wedge  $f(x) = |x|$ . However, if  $f'$  is unbounded, since the turnover points  $x = \pm d_0 \rightarrow \pm\infty$  as  $t \rightarrow \infty$  by (2.56), the contribution to the normal impact velocity from the forward motion will eventually become comparable to the contribution from the normal body motion, and therefore have an order unity effect.

#### 5.1.1.2 Case B: Critical forward velocity $U_0 = O(1)$

If  $U_0$  and  $t$  are order unity, then  $\epsilon f'(\epsilon x - \epsilon U t) \ll 1$  and  $\epsilon U f'(\epsilon x - \epsilon U t) = O(1)$ , so the forward motion has a leading order effect through the second term on the right-hand side of (5.2). Since the Wagner theory of section 2.2 is applicable when  $U_0 = O(\epsilon)$ , we expect it to also apply at sufficiently small forward velocities  $U_0 = O(1)$  at order unity times. We now assume that it does, describe how to find the leading-order outer solution and then discuss the conditions necessary for it to be physically acceptable.

By (5.1, 5.2), the outer scalings are exactly as in the normal impact case, i.e. (2.20), so denoting the left (−) and right (+) turnover points by  $x = d_{\pm}(t)/\epsilon$ , we proceed as in section 2.2.1 to obtain the leading-order outer oblique water entry problem shown in Figure 5.1, in which we have dropped hats and subscripts on the leading-order velocity potential,  $\phi(x, z, t)$ , and free surface elevation,  $h(x, t)$ .

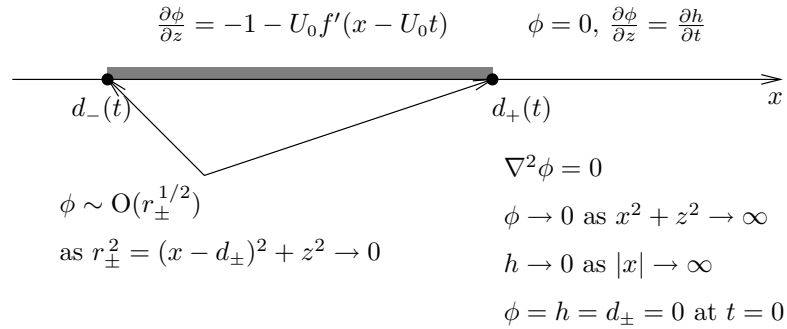


Figure 5.1: The leading-order outer oblique water entry problem.

The complex velocity for this Riemann-Hilbert problem with bounded spatially integrated kinetic energy is

$$\frac{1}{\pi \sqrt{(\zeta - d_-)(\zeta - d_+)}} \int_{d_-}^{d_+} \frac{\sqrt{(\xi - d_-)(\xi - d_+)}}{\xi - \zeta} (1 + U_0 f'(x - U_0 t)) d\xi,$$

where  $\zeta = x + iz$  and the integral is taken along the real  $\zeta$ -axis. Provided the turnover points are advancing, i.e.  $\dot{d}_- < 0 < \dot{d}_+$ , the kinematic boundary condition (2.27) may be integrated, subject to the zero initial condition (2.31), to obtain the leading-order outer free surface elevation  $h(x, t)$  and therefore the coupled leading-order laws of motion of the free points through the Wagner conditions

$$h(d_{\pm}, t) = f(d_{\pm} - U_0 t) - t \quad \text{for } t \geq 0. \quad (5.3)$$

The above procedure must in general be implemented numerically, though analytic progress may be made in the limit  $U_0 \rightarrow 0$ . For example, Morgan [50] showed that if  $f(x) = |x|$ , then as  $U_0 \rightarrow 0$ ,

$$d_{\pm} \sim \pm \left( \frac{\pi}{2} + \frac{U_0}{4\pi} (16 - 4\pi \mp \pi^2) \right) t. \quad (5.4)$$

Having computed the solution it is necessary to verify that it is physically acceptable. The review of water exit and stability in section 2.7 suggests we should demand that

- (1) the turnover points are advancing;
- (2) the leading-order outer pressure is positive on the body in the interior region.

The first condition, i.e.  $\dot{d}_- < 0 < \dot{d}_+$ , implies the solution is linearly stable. The second condition, i.e.  $\partial\phi/\partial t < 0$  for  $z = 0$  and  $d_-(t) < x < d_+(t)$ , guarantees the flow does not cavitate<sup>1</sup>. Both conditions can only be checked *a posteriori*. The analysis of section 2.3.2 concerning normal water entry with variable impact speed showed that the pressure can become negative on the body if it decelerates sufficiently rapidly, which suggests that the second condition does not necessarily hold if the first one does. It is not clear whether the second condition implies the first *a priori*.

Note that we do not demand that the body is entering<sup>2</sup> on the interior region, i.e.  $-1 - U_0 f'(x - U_0 t) < 0$  for  $d_-(t) < x < d_+(t)$ , because there is no evidence that this is unphysical if condition (2) holds, although exit may be a sign of instability. Suppose the body is entering everywhere at time  $t$ , then the analysis of section 2.3.2 suggests that condition (2) does not necessarily hold at time  $t$ . Moreover, it is not possible to deduce that condition (1) holds at time  $t$  *a priori*. To see this, we differentiate (5.3) and rearrange to find that the normal impact velocity at the trailing turnover point is

$$-1 - U_0 f'(d_- - U_0 t) = -\dot{d}_- f'(d_- - U_0 t) + dh_0(d_-, t)/dt;$$

the sign of  $\dot{d}_-$  cannot be determined without finding  $h_0$ ,  $d_-$  and  $d_+$ .

Conditions (1) and (2) hold if  $U_0 = 0$  so, by continuity, it is not unreasonable to expect them to hold for sufficiently small forward velocities  $U_0 = O(1)$  at times  $t = O(1)$ . However, by (5.2), we expect the normal impact velocity to be negative for sufficiently large  $U_0 = O(1)$  somewhere on the trailing segment of the body where  $f'(x - U_0 t) < 0$ . As discussed above, it is not clear how this will effect the two conditions, though one might expect that as the segment of the body that is exiting grows, so too does the likelihood of a growing region of negative pressure, leading to cavitation and a breakdown of the Wagner theory. It is not clear what happens when the theory breaks down or if it is bound to do so at sufficiently large times. In section 5.1.2 we review and discuss a way to avoid these problems.

<sup>1</sup>Note that if the leading-order pressure is positive on the body in the outer region, then it is also positive on the body in the intermediate and jet root regions at the turnover points.

<sup>2</sup>For example, if  $f(x) = |x|$ , then  $-1 - U_0 f'(x - U_0 t) = -1 - U_0 \operatorname{sgn}(x - U_0 t) > 0$  for  $x < U_0 t$ , if and only if  $U_0 > 1$ , i.e. the wedge is entering everywhere for all  $t$ , if and only if the forward velocity is smaller than the normal body velocity.

### 5.1.1.3 Case C: Large forward velocity $U_0 \gg 1$

If  $U_0 \gg 1$  and  $t = O(1)$ , then  $\epsilon U f'(\epsilon x - \epsilon U t) \gg 1$  and the forward motion dominates the contribution from the normal motion of the body. Since the impactor translates a horizontal distance  $U = U_0/\epsilon$  much greater than the extent of its segment in  $z < 0$ , of order  $1/\epsilon$ , the Wagner theory of section 2.2 cannot apply at order unity times and a completely new asymptotic expansion is required.

### 5.1.2 Planing at small incidence angles

To avoid the problems caused by high forward velocities that were discussed in section 5.1.1.2, one procedure is to remove the entire trailing half of the body to the left of the keel<sup>3</sup>; see, for example, [66, 69, 78]. It is then assumed that the trailing edge is sharp, so that the free surface separates smoothly from the body there, i.e. the free surface leaves the sharp trailing edge tangential to the body, and that Wagner theory describes the flow structure in the neighbourhood of the turnover point. The proposed configuration (see, for example, [66]) in the specific limit in which the body has variable normal impact speed  $\dot{s}(t)$  and constant forward velocity  $U = U_0/\epsilon$ , where  $\dot{s}$  and  $U_0$  are order unity, is shown in Figure 5.2. In such “unsteady planing” problems, the distance from the trailing edge to the turnover point, namely  $d(t)/\epsilon$ , is called the wetted length, rather than the entire contact set between the body and the fluid.

In this section we justify the use of Wagner theory near to the turnover point and analyse the effect of the smooth separation condition on the oblique water entry model in section 5.1.1.2.

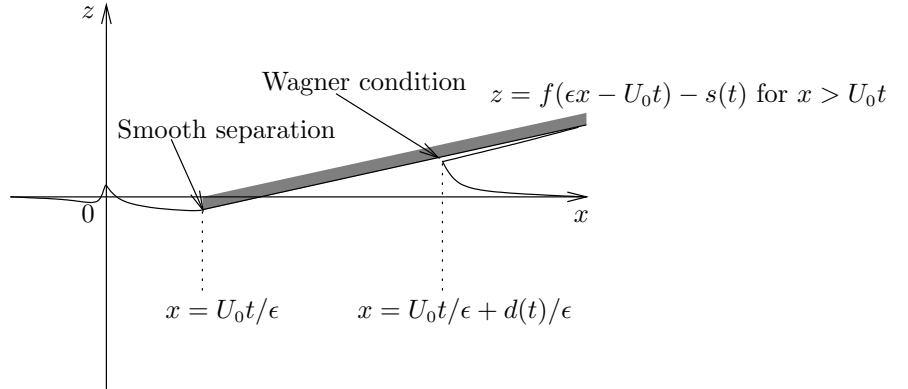


Figure 5.2: Schematic of the proposed unsteady planing model relative to the original dimensionless coordinates. The blip in the free surface near the origin is discussed in the text.

In order to be able to consider steady state solutions with  $\dot{s} = 0$ , we work in a frame of reference fixed with respect to the horizontal motion of the body by substituting

$$x = U_0 t / \epsilon + \hat{x} / \epsilon, \quad z = \hat{z} / \epsilon, \quad \phi = (U_0 \hat{x} + \hat{\phi}) / \epsilon, \quad h = \hat{h}, \quad (5.5)$$

into the dimensionless model problem (2.12, 2.14 - 2.15) with the kinematic boundary condition

$$\frac{\partial \phi}{\partial z} = \epsilon \frac{\partial \phi}{\partial x} f'(\epsilon x - \epsilon U t) - \epsilon U f'(\epsilon x - \epsilon U t) - \dot{s}(t), \quad (5.6)$$

<sup>3</sup>The practical applications are limited; the most well studied scenario is an alighting seaplane.

on the body profile

$$z = f(\epsilon x - \epsilon U t) - s(t) \quad \text{for } x > U t, \quad (5.7)$$

where  $s(0) = 0$ .

We proceed as in section 2.2.1 to obtain the leading-order outer “unsteady planing” problem (dropping hats and subscripts on  $\phi$ ,  $h$  and the wetted length  $d(t)$ )

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in } z < 0, \quad (5.8)$$

$$\frac{\partial \phi}{\partial z} + \dot{s} + U_0 f'(x) = 0 \quad \text{on } z = 0, \quad 0 < x < d, \quad (5.9)$$

$$\frac{\partial \phi}{\partial z} - \frac{\partial h}{\partial t} + U_0 \frac{\partial h}{\partial x} = 0 \quad \text{for } z = 0, \quad x < 0, \quad x > d, \quad (5.10)$$

$$\frac{\partial \phi}{\partial t} - U_0 \frac{\partial \phi}{\partial x} - U_0^2 = 0 \quad \text{for } z = 0, \quad x < 0, \quad x > d, \quad (5.11)$$

In the moving frame the fluid velocity before impact is  $-U_0 \mathbf{i}$  and the free surface is undisturbed, so the initial and far field conditions are, without loss of generality,

$$\phi + U_0 x, \quad h, \quad d = 0 \quad \text{at } t = 0, \quad (5.12)$$

$$\phi + U_0 x \rightarrow 0 \quad \text{as } x^2 + z^2 \rightarrow \infty, \quad (5.13)$$

$$h \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (5.14)$$

The leading-order kinematic boundary condition on the body does not depend on  $\partial \phi / \partial x$ , so we may subtract the uniform flow  $-U_0 \mathbf{i}$  by writing

$$\phi = -U_0 x + \bar{\phi}, \quad (5.15)$$

so that  $\bar{\phi}$  is the velocity potential for the flow induced by the plate. The resulting unsteady planing problem is shown in Figure 5.3, in which we have dropped the bar on the velocity potential.

The behaviour at the fixed point  $(x, z) = (0, 0)$ , at the free point  $(x, z) = (d(t), 0)$  and at infinity, the pressure boundary condition, and the initial and far field conditions need further discussion. We begin with the steady problem and explain why it is not possible to satisfy the zero far field conditions in Figure 5.3.

### 5.1.2.1 Steady state solutions

We now seek steady state solutions to the planing problem in Figure 5.3 at constant penetration depth,  $s = s_0 > 0$ . Setting  $\phi = \phi(x, z)$ ,  $h = h(x)$  and  $d = \text{constant}$ , the time dependent terms in Figure 5.3 are zero and therefore the pressure boundary condition becomes  $\phi(x, 0) = 0$  for  $x < 0$ ,  $x > d$  by (5.11, 5.15).

Demanding smooth separation at the trailing edge and matching with an inner region of size  $O(1)$  there, in which the lowest order flow is uniform with velocity  $-U_0 \mathbf{i}$  relative to the fixed frame, we find

$$\phi \sim O(r^{3/2}) \quad \text{as } r^2 = x^2 + z^2 \rightarrow 0, \quad (5.16)$$

$$h(0) = f(0) - s_0. \quad (5.17)$$

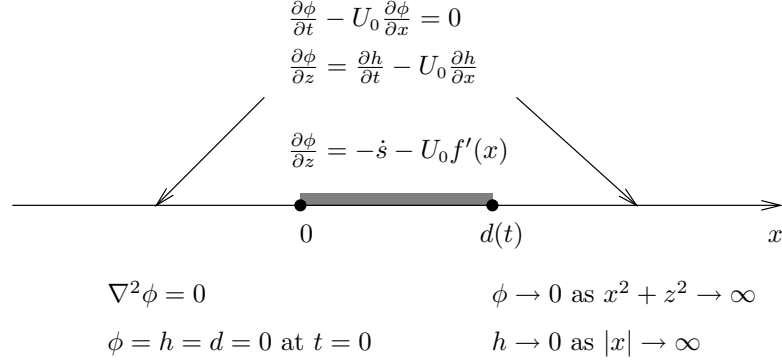


Figure 5.3: Field equation and boundary conditions for the proposed leading-order outer codimension-two free boundary value problem for two-dimensional unsteady planing with variable impact speed  $\dot{s}(t)$  and constant forward velocity  $U_0/\epsilon$ . The conditions necessary to close the problem are discussed in the text.

The outer free surface has an inverse square root curvature singularity at the trailing edge; see, for example, [61]. Note that demanding  $\phi$  to be differentiable across the trailing edge is equivalent to demanding smooth separation of the *outer* free surface there<sup>4</sup>.

At the free point,  $(x, z) = (d(t), 0)$ , the only allowable singularity in the potential is a square root, viz.

$$\phi \sim O(r_d^{1/2}) \quad \text{as } r_d^2 = (x - d)^2 + z^2 \rightarrow 0, \quad (5.18)$$

because the flow in the inner jet root region in section 2.2.3 is unique up to an arbitrary  $O(\epsilon)$  translation along the body. Thus, matching the outer solution with this small  $O(\epsilon)$  high pressure jet root region, through an intermediate region of size  $O(1)$ , we deduce that Wagner theory is also valid for steady planing with the usual Wagner condition

$$h(d) = f(d) - s_0. \quad (5.19)$$

Since the complex velocity has  $-1/2$ - and  $1/2$ -power singularities at the turnover point and trailing edge, respectively, the index of the corresponding Riemann-Hilbert problem is two [16], so the potential is unbounded at infinity, i.e. it is impossible to satisfy the condition that the potential tends to zero in the far field. Indeed, the Riemann-Hilbert method implies the solution is the classical steady airfoil flow:

$$\frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial z} = \frac{U_0}{\pi} \sqrt{\frac{\zeta}{\zeta - d}} \int_0^d \sqrt{\frac{\xi - d}{\xi}} \frac{f'(\xi)}{\xi - \zeta} d\xi, \quad (5.20)$$

where  $\zeta = x + iz$ , the square roots are chosen appropriately and the integral is taken along the real  $\zeta$ -axis. Hence, there is a line vortex at infinity, viz.

$$\phi \sim -\Re \left( \frac{i\Gamma}{2\pi} \log \zeta \right) \quad \text{as } |\zeta| \rightarrow \infty, \quad (5.21)$$

of strength

$$\Gamma = 2U_0 \int_0^d \sqrt{\frac{d - \xi}{\xi}} f'(\xi) d\xi. \quad (5.22)$$

<sup>4</sup>To see this we take the limits of the kinematic boundary conditions on the body and on the free surface at the trailing edge, viz.  $f'(0+) = \partial\phi/\partial z(0+, 0)$  and  $h'(0-) = \partial\phi/\partial z(0-, 0)$ .

We now discuss the implications of this behaviour.

The line vortex at infinity implies the so-called Green's paradox (see Appendix A), that the free surface tends to minus infinity logarithmically in the far field, viz. by the steady version of the kinematic boundary condition on the free surface and (5.21),

$$h(x) \sim -\frac{\Gamma}{2\pi U_0} \log |x| \quad \text{as } |x| \rightarrow \infty, \quad (5.23)$$

so the linearization of the boundary conditions onto the  $x$ -axis breaks down at distances of  $O(e^{1/\epsilon})$  from the origin. This means that we cannot set the penetration depth of the trailing edge below the undisturbed level of the free surface and that we cannot invoke the Wagner condition (5.19) to determine the wetted length  $d(t)$ . Hence we have found a one-parameter family of solutions and this, as noted by Fridman [14], is an intrinsic feature of steady two-dimensional infinite depth planing problems without gravity. Several authors, including [62, 77], have shown that introducing gravity in the far field removes Green's paradox and enables a unique solution to be determined by matching the above 'inner' solution with an 'outer' solution containing gravity at large distances; essentially the 'outer' solution determines  $\Gamma$  in (5.21) or (5.23) and hence  $d$  from (5.22). Green [24] showed that, if a base is introduced, then there is no Green's paradox because the incoming and outgoing free surfaces asymptote to parallel straight lines; by conservation of mass, the downstream depth is simply the upstream depth minus the thickness of the spray jet; see Figure 5.6. We will consider depth effects further in section 5.2.

In Appendix A we show that, for a flat plate  $f(x) = x$ , there is good agreement between the above solution the small incidence angle asymptotics of Green's exact solution [24] for the Kelvin-Helmholtz flow past a planing semi-infinite flat plate at an arbitrary incidence angle to the uniform far field velocity on infinite depth fluid (see Figure A.1a). This is a special case of Green's solution for a flat plate of finite length, in which the angle that the ejected jet makes to the plate is undetermined (see figure A.1a). In Appendix A, we also show that as the incidence angle  $\epsilon$  of a finite length plate to the far field velocity tends to zero, the jet root region, where the free surface turns over, is of size  $O(\epsilon^2)$  and lies at the leading edge. Thus, even if we take the above linearized solution and 'fix' the location of the turnover point at the leading edge of a body of specified length, the solution is not unique, although the outer problem is now at least closed.

### 5.1.2.2 Unsteady planing

In this section we suppose that conditions (5.17 - 5.19) also apply to the unsteady planing problem in Figure 5.3 and investigate the consequences. Our analysis uses the Riemann-Hilbert method and the theory of index and is equivalent to the analysis of Ulstein & Faltinsen [66], who employed the analogy with unsteady airfoil theory and the discrete vortex method to investigate the solution numerically.

The first point to note is that, on physical grounds, we must ensure that the zero far field conditions in Figure 5.3 are satisfied. However, the index argument in the previous section also applies here, so, for a solution to exist, there must be  $n \geq 1$  singularities on the  $x$ -axis, at  $x_j \in (-\infty, 0) \cup (d, \infty)$ , say, such that

$$\frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial z} \sim O((\zeta - x_j)^{-\gamma_j}) \quad \text{as } \zeta \rightarrow x_j,$$

where  $\zeta = x + iz$  and  $\sum_{j=1}^{j=n} \gamma_j = 1$ . We argue that there is a single line vortex at  $x = -U_0 t$ , i.e.  $n = 1$ ,  $x_1 = -U_0 t$ ,  $\gamma_1 = 1$ , as follows. Integrating the pressure boundary condition

implies

$$\phi = \mathcal{G}(x + U_0 t) \quad \text{on } z = 0, \quad x < 0, \quad x > d,$$

where  $\mathcal{G}$  is to be determined. The initial condition  $\phi(x, 0, 0) = 0$  implies

$$\phi = 0 \quad \text{on } z = 0, \quad x < -U_0 t, \quad x > d,$$

However, on the trailing segment  $-U_0 t < x < 0$  of the  $x$ -axis,  $\mathcal{G}$  is determined by the potential at the trailing edge, viz.

$$\phi = \mathcal{G}(x + U_0 t) = \phi(0, 0, (x + U_0 t)/U_0) \quad \text{on } z = 0, \quad -U_0 t < x < 0. \quad (5.24)$$

Hence, there can only be a singularity at  $X = -U_0 t$ , which implies there is a line vortex there; by (5.24),

$$\phi \sim -\Re \left( \frac{k}{2\pi} \log(\zeta + U_0 t) \right) \quad \text{as } \zeta \rightarrow -U_0 t, \quad (5.25)$$

where the strength  $k = \mathcal{G}(0+) = \phi(0, 0, 0+) \neq 0$ . We conclude that the smooth separation condition at the trailing edge generates a “starting vortex” at  $t = 0$ , as in unsteady linearized airfoil theory [52]. The local problem at  $\zeta = -U_0 t$  is open; hence the small blip at the origin in Figure 5.3.

As described in detail by [52, 66], the unsteady planing problem in Figure 5.3 is much more difficult to solve than the oblique water entry problem in Figure 5.1 because the function  $\mathcal{G}$  in (5.24) must also be determined.

Finally, we remark that the existence of a line vortex at  $\zeta = -U_0 t$  in the outer solution prompts the conjecture that, if  $s(t) \rightarrow s_0$  as  $t \rightarrow \infty$ , then the solution to the unsteady planing problem converges to the steady state solution in the previous section with a circulation  $\Gamma = k$  and

$$d = \lim_{t \rightarrow \infty} d(t). \quad (5.26)$$

## 5.2 Finite depth models

### 5.2.1 Oblique water entry at small deadrise angles

In this section we consider basal effects on the two-dimensional oblique water entry model discussed in section 5.1.1. The dimensionless model problem is (3.3, 3.4, 3.6 - 3.9) with the body profile (5.1) on which the kinematic boundary condition is (5.2). As in Chapter 3, we initially suppose the body is the parabola  $f(x) = x^2$ ; it will become apparent that the generalization to symmetric, smooth body profiles is straightforward. However, if  $f$  is symmetric and  $f(x) \sim O(|x|)$  as  $x \rightarrow 0$ , we will see that the generalization is quite different.

On the timescale  $\delta \ll 1$ , the penetration depth is  $O(\delta)$ , so we scale

$$t = \delta \bar{t}, \quad x = \delta \bar{x}, \quad z = \delta \bar{z},$$

to find the oblique impact profile (5.1) becomes

$$\bar{z} = f(\bar{\epsilon} \bar{x} - \bar{\epsilon} U \bar{t}) - \bar{t}, \quad (5.27)$$

where the scaled deadrise angle  $\bar{\epsilon} = \epsilon\delta^{1/2}$ . The base becomes  $\bar{z} = -1/\delta$  and provided we scale

$$\phi = \delta\bar{\phi}, \quad h = \delta\bar{h},$$

the field, boundary, initial and far field conditions (3.3, 3.4, 3.6 - 3.9) are unchanged, except  $\bar{\epsilon}$  replaces  $\epsilon$  in the body profile (5.27) and its kinematic boundary condition (5.2) becomes

$$\frac{\partial\bar{\phi}}{\partial\bar{z}} = \bar{\epsilon}\frac{\partial\bar{\phi}}{\partial\bar{x}}f'(\bar{\epsilon}\bar{x} - \bar{\epsilon}U\bar{t}) - \bar{\epsilon}Uf'(\bar{\epsilon}\bar{x} - \bar{\epsilon}U\bar{t}) - 1. \quad (5.28)$$

Recall that the four stages of impact discussed in Chapter 3 have timescales  $\delta$  such that  $\delta \ll 1$ ,  $\delta \sim \epsilon^2$ ,  $\epsilon \ll \delta \ll 1$  and  $\delta \sim 1$ , obtained by comparing the penetration depth of order unity, to the distance between the turnover points of order  $1/\bar{\epsilon}$ , and the layer depth of order  $1/\delta$ . Also, recalling the analysis of section 5.2.1, it is now straightforward to categorize into case A, B or C, the effect of the forward motion on the  $\delta$ -timescale leading-order outer problem depending on whether  $\bar{\epsilon}U$  is small, order unity or large<sup>5</sup>, viz.

- (1) If  $\bar{\epsilon}U \ll 1$ , then we revert to the scenario of case A on the  $\delta$ -timescale and therefore for all earlier times.
- (2) If  $\bar{\epsilon}U$  is order unity, then we revert to the scenario of case B on the  $\delta$ -timescale, while case A holds at all smaller timescales and case C holds at all larger timescales. For example, if  $\epsilon^2U$  is order unity, then we are in the scenario of case B in stage 2, while the forward motion does not affect stage 1 to lowest order, but has an extreme effect on stages 3 and 4.
- (3) If  $\bar{\epsilon}U \gg 1$ , then we are in the scenario of case C on the  $\delta$ -timescale and therefore for all later times.

Since there is no leading-order effect in case A and case C is the nontrivial large forward velocity limit of case B, we focus on the critical case B in which  $U_0 = \bar{\epsilon}U = O(1)$  in *each* stage. This is possible because remark (2) implies that the forward motion has no leading-order effect on any earlier stage, so that the analysis of Chapter 3 implies we have zero initial conditions for case B in each stage. Further, remark (2) implies that in stages 1, 2 and 3 we expect the asymptotic solution, obtained by modifying the normal impact solution to account for the comparable effect of the forward impact velocity, to break down at sufficiently large times because the effect of the forward motion is extreme (case C) in any later stage. Finally, we note that the local-in-space and -time linear stability analyses of [19, 31] (that were discussed in section 2.6) apply to stages 1, 2 and 3 because the inner regions are governed by Wagner theory.

We now briefly derive and discuss the proposed leading-order case B models in each stage. We begin with stages 1 and 2. We then consider stage 4 and use it to derive the proposed stage 3 model. We choose this ordering because, in Chapter 3, we showed that for normal impacts the leading-order stage 3 interior region problem (3.189 - 3.192) is most easily recovered by taking the small-time limit of the leading-order stage 4 interior region problem (3.242 - 3.245), rather than the large-time, large-aspect-ratio limit of the stage 2 problem in Figure 3.3. The same is true for oblique impacts in case B.

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<sup>5</sup>If  $f(x) \sim O(|x|^\alpha)$  as  $x \rightarrow 0$ , then the scaled deadrise angle  $\bar{\epsilon} = \epsilon\delta^{1/\alpha}$ . Hence, remarks (1-3) hold for all  $\alpha > 1$ . However, if  $\alpha = 1$ , then  $\bar{\epsilon} = \epsilon$  and the forward motion has the same effect on each stage, which depends solely on the size of  $\epsilon U$  as in the infinite depth case.

### 5.2.1.1 Very small and small time

The very small time ( $\delta \ll \epsilon^2$ ) deep water limit for forward velocities  $U_0 = \delta^{1/2}\epsilon U = O(1)$  was discussed in section 5.1.1.2 and it is conceptually straightforward to generalize to the small time ( $\delta \sim \epsilon^2$ ) finite depth case B for  $U_0 = \epsilon^2 U = O(1)$ , by adding a base at  $z = -1$  to the outer problem in Figure 5.1. Matching with stage 1, in which the forward motion has no leading-order effect, implies the initial conditions are  $d_{\pm}(0) = 0$  and

$$d_{\pm}(t) \sim \pm\sqrt{2t} \quad \text{as } t \rightarrow 0,$$

from the normal impact analysis in section 2.2.6. The conformal mapping and Riemann-Hilbert techniques in section 3.3.1 may be used to write down the complex velocity and therefore derive the coupled Wagner conditions as in section 5.1.1.2. A numerical analysis is required, though here we expect the solution to violate condition (1) or (2) on page 108 at sufficiently large times, no matter how small  $U_0 = O(1)$ , as we enter stage 3, case C.

### 5.2.1.2 Time of order unity

The critical forward velocity regime is  $U_0 = \epsilon U = O(1)$  because  $\bar{\epsilon} = \epsilon$  when  $\delta = 1$ . The layer thickness (3.143) becomes

$$l(x, t) = 1 + f(x - U_0 t) - t, \quad (5.29)$$

where  $f(x) = x^2$ , so the right-hand side of the conservation equation (3.179) becomes  $-\partial l / \partial t$ , which is simply minus the normal impact velocity. Denoting the turnover points by  $x = d_{\pm}(t)$  as usual, the free boundary conditions (3.181, 3.182) imply the proposed leading-order stage 4 oblique water entry model is (dropping subscripts on  $\phi$ )

$$\frac{\partial}{\partial x} \left( l(x, t) \frac{\partial \phi}{\partial x} \right) - 1 - U_0 f'(x - U_0 t) = 0 \quad \text{for } d_-(t) < x < d_+(t), \quad (5.30)$$

$$\phi = 0 \quad \text{at } x = d_{\pm}(t), \quad (5.31)$$

$$\frac{\partial \phi}{\partial x} - 2\dot{d}_{\pm}(1 - l(d_{\pm}, t)^{-1/2}) = 0 \quad \text{at } x = d_{\pm}(t). \quad (5.32)$$

By remark (2) on page 114, the forward motion has no leading order effect on the earlier stages, so the analysis of section 3.4 implies the initial conditions are  $d_{\pm}(0) = 0$  and

$$d_{\pm}(t) \sim \pm\sqrt{3t} \quad \text{as } t \rightarrow 0. \quad (5.33)$$

Integrating the field equation (5.30) subject to the boundary conditions (5.31) we find

$$\phi(x, t) = \int_{d_-}^x \frac{\xi + f(\xi - U_0 t) + A(t)}{1 + f(\xi - U_0 t) - t} d\xi, \quad (5.34)$$

where  $A(t)$  is determined by

$$0 = \int_{d_-}^{d_+} \frac{\xi + f(\xi - U_0 t) + A(t)}{1 + f(\xi - U_0 t) - t} d\xi. \quad (5.35)$$

Hence, the kinematic boundary conditions (5.32) imply

$$2\dot{d}_{\pm} \left( 1 - \frac{1}{\sqrt{1 + f(d_{\pm} - U_0 t) - t}} \right) = \frac{d_{\pm} + f(d_{\pm} - U_0 t) + A(t)}{1 + f(d_{\pm} - U_0 t)}, \quad (5.36)$$

which may be expanded subject to  $d_{\pm}(0) = 0$  to verify (5.33). The two coupled first-order ordinary differential equations for the leading-order location of the turnover points (5.36) must be solved numerically using their small time asymptotics (5.33) to begin the scheme, as in the normal impact case in section 3.5.3. Having computed the solution we must check that it is physically acceptable as discussed in section 5.1.1.2 for the infinite depth case. A linear stability analysis is required. The stability analysis of the normal impact case in section 3.7.5.3 suggests that there is a critical speed for each of the turnover points above which the solution is linearly stable. Fortunately, an explicit example of a breakdown of the solution is captured by the stage 3 model as follows.

### 5.2.1.3 Intermediate time

To find the proposed stage 3 oblique water entry model we take the small time limit of the proposed stage 4 problem (5.30 - 5.32) by scaling

$$t = \delta T, \quad x = \varepsilon X, \quad (5.37)$$

where the timescale  $\delta$  is such that  $\epsilon^2 \ll \delta \ll 1$  and the horizontal stage 3 lengthscale  $\varepsilon(\delta) \ll 1$  is determined by demanding

$$f(x - U_0 t) = f(\varepsilon X - \delta U_0 T) \sim \delta F(X - U_1 T) \quad \text{as } \delta \rightarrow 0, \quad (5.38)$$

where  $F$  is the first term in the Taylor expansion of  $f$  at the origin and the stage 3 forward velocity  $U_1 = U_0 \delta / \varepsilon$  is retained to derive the proposed stage 3, case B problem. For the parabola,  $f(x) = x^2 = F(x)$ , so there is equality in (5.38) and  $\varepsilon = \delta^{1/2}$ ,  $U_1 = U_0 \delta^{1/2} \ll 1$ .

By (5.37) we scale the  $x$ -coordinates of the turnover points as

$$d_{\pm}(t) = \varepsilon D_{\pm}(T), \quad (5.39)$$

and since

$$\frac{\partial l}{\partial t} \sim -1 - U_1 F'(X - U_1 T) \quad \text{as } \delta \rightarrow 0, \quad (5.40)$$

we scale

$$\phi = \varepsilon^2 \Phi \quad (5.41)$$

for a nontrivial balance in the governing equation (5.30). Note that (5.40) confirms that the forward motion is negligible<sup>6</sup> to lowest order in stage 3 if  $U_0 = O(1)$  because  $U_1 = U_0 \delta^{1/2}$ . Finally, since

$$1 - \frac{1}{\sqrt{l}} \sim \frac{\delta}{2} (F(X - U_1 T) - T) \quad \text{as } \delta \rightarrow 0,$$

we obtain the proposed leading-order stage 3 oblique water entry problem

$$\frac{\partial^2 \Phi}{\partial X^2} - 1 - U_1 F'(X - U_1 T) = 0 \quad \text{for } D_-(T) < T < D_+(T), \quad (5.42)$$

$$\Phi = 0 \quad \text{at } X = D_{\pm}, \quad (5.43)$$

$$\frac{\partial \Phi}{\partial X} - \dot{D}_{\pm} (F(D_{\pm} - U_1 T) - T) = 0 \quad \text{at } X = D_{\pm}, \quad (5.44)$$

$$D_{\pm} = 0 \quad \text{at } t = 0. \quad (5.45)$$

---

<sup>6</sup>By footnote 5 on page 138, this is true for all symmetric, smooth body profiles. However, if  $f$  is symmetric and  $f(x) \sim O(|x|)$  as  $x \rightarrow 0$ , then  $\varepsilon = \delta$ , so  $U_1 = U_0 = O(1)$  and therefore the forward motion has an order unity effect in stages 3 and 4.

The zero initial conditions follow from remark (2) on page 114 and the analysis in section 3.5.3 implies for the parabola  $F(X) = X^2$ ,

$$D_{\pm} \sim \pm\sqrt{3T} \quad \text{as } T \rightarrow 0. \quad (5.46)$$

As mentioned above, we expect this model to be linearly unstable if the turnover points are not advancing. However, the stability analysis of the normal impact case in section 3.7.4.8 showed that the solution becomes unstable as soon as a turnover point begins to move downward relative to the fixed frame. It is not unreasonable to expect this result to hold for oblique impacts. Hence, we conjecture that the time,  $T = T_1(U_1)$ , say, at which a turnover point first reverses direction, is an upper bound on the time at which the solution is linearly stable, although our best guess is that this occurs at the time,  $T = T_2(U_1)$ , say, at which  $-1 - U_1 F'(D_{\pm} - U_1 T)$  first vanishes.

For the parabola  $F(X) = X^2$ , the solution to (5.42 - 5.45) is

$$\Phi = \frac{U_1}{3} (X^3 - D_-^3) + \frac{1}{2} (1 - 2U_1^2 T) (X^2 - D_-^2) - A(T) (X - D_-), \quad (5.47)$$

where

$$A(T) = \frac{U_1}{3} (D_-^2 + D_- D_+ + D_+^2) + \frac{1}{2} (1 - 2U_1^2 T) (D_- + D_+), \quad (5.48)$$

and the laws of motion of the free points are

$$((D_{\pm} - U_1 T)^2 - T) \dot{D}_{\pm} = U_1 D_{\pm}^2 + (1 - 2U_1^2 T) D_{\pm} - A(T), \quad (5.49)$$

which may be expanded subject to  $D_{\pm}(0) = 0$  to verify (5.46).

In Figure 5.4a we plot the  $X$ -coordinates of the turnover points,  $D_{\pm}$ , and of the keel of the impactor,  $U_1 T$ , for  $U_1 = 1$ . The trailing turnover point,  $D_-$ , reverses direction at time  $T \approx 0.750$ , over which time the leading turnover point is advancing, so  $T_1(1) \approx 0.750$ .

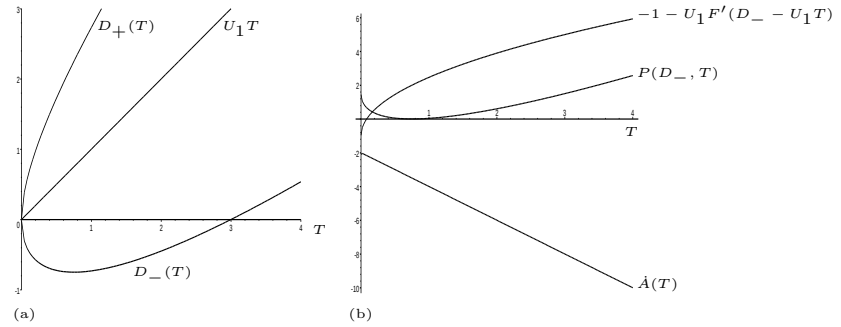


Figure 5.4: (a) Plot of  $D_{\pm}(T)$  and  $U_1 T$  for  $U_1 = 1$ . The two coupled first-order ordinary differential equations (5.49) were solved numerically using the small time asymptotics (5.46) to start the fourth-fifth order Runge-Kutta scheme. (b) Plot of  $\dot{A}(T)$ ,  $P(D_-(T), T) = C(T)$  and  $-1 - U_1 F'(D_- - U_1 T)D_-(T)$ ,  $D_+(T)$  for  $U_1 = 1$ .

The leading-order pressure in the interior region (scaled with  $1/\epsilon^2$ ) is

$$P = -\frac{\partial \Phi}{\partial T} = U_1^2 (X^2 - D_-^2) - \dot{A}(T) (X - D_-) + C(T), \quad (5.50)$$

where

$$C(T) = (U_1 D_-^2 + (1 - 2U_1^2 T) D_- - A(T)) \dot{D}_-.$$

We plot  $\dot{A}(T)$ , the leading-order outer pressure at the trailing turnover point,  $C(T)$ , and the normal impact velocity at the trailing turnover point,  $-1 - F'(D_- - U_1 T)$ , for  $U_1 = 1$  in Figure 5.4b. Since  $\dot{A} < 0$  for  $T < T_1(1)$ , the pressure is positive on the interior region and vanishes first at time  $T = T_1(1)$ ; the latter is in accordance with (5.50, 5.51), which implies  $P(D_-, T) = 0$  if  $\dot{D}_- = 0$ . The body begins to exit at, and therefore sufficiently near, the trailing turnover point at time  $T = T_2(1) \approx 0.064$ .

In Figure 5.5 we plot the time  $T_1(U_1)$  at which the trailing turnover point has zero velocity and the time  $T_2(U_1)$  at which the normal impact velocity vanishes there. We see that the above remarks hold for all  $U_1 > 0$ .

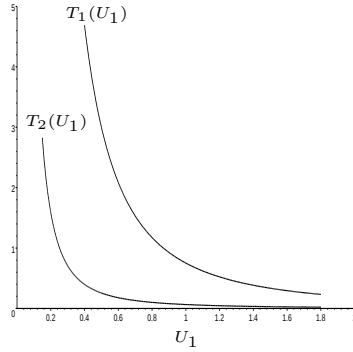


Figure 5.5: Plot of  $T_1(U_1)$  and  $T_2(U_1)$  for  $U_1 > 0$ .

### 5.2.2 Planing at small incidence angles

In this section we consider the effect of finite depth on the planing analysis of section 5.1.2. As noted in section 5.1.2.1, Green [23] used the hodograph method to find the exact solution to the Kelvin-Helmholtz flow past a semi-infinite flat plate at arbitrary incidence angle,  $\alpha$ , as depicted in Figure 5.6. The solution determines the downstream height,  $h_2$ , and jet thickness,  $h_3$ , in terms of the upstream height,  $h_1$ , the penetration depth,  $s_0$ , and the angle of incidence,  $\alpha$ . Global conservation of mass implies  $h_3 = h_1 - h_2$  and Green's paradox does not apply. This is true for all steady and unsteady planing problems on a stream of finite depth.

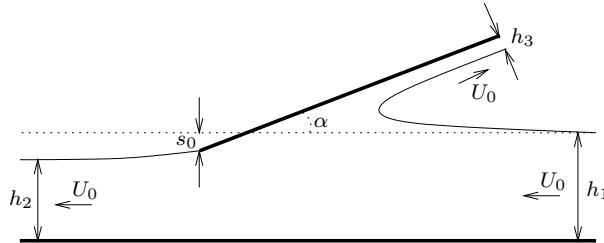


Figure 5.6: Geometry of Green's exact solution [23] for the planing of a flat finite plate on a stream of finite depth. We must also specify Laplace's equation  $\nabla^2 \phi = 0$  in the fluid, the kinematic condition  $\partial \phi / \partial n = 0$  on the plate and free boundaries and the pressure condition  $|\nabla \phi| = U_0$  on the free boundaries

Green's exact solution is cumbersome and not suitable for numerical computations. It is therefore useful to find simpler and more general models by introducing one or more small

parameters into the geometry and using the method of matched asymptotic expansions. Tuck [65] found the leading-order solution in the small-incidence-angle shallow-water limit with the penetration depth comparable to the layer depth, which we describe shortly. In this section we propose models for the unsteady version of this problem and its small penetration depth limit and present our preliminary observations.

The dimensionless model problem is (3.3, 3.4, 3.6 - 3.9) with the kinematic boundary condition (5.6) on the body profile (5.7). As in section 5.1.2, we assume smooth separation of the free surface at the sharp trailing edge  $(x, z) = (Ut, -s(t))$ .

In the small incidence angle limit  $\epsilon \ll 1$ , the temporal decomposition is exactly as in section 5.2.1 with the wetted length from the trailing edge to the turnover point replacing the distance between the two oblique water entry turnover points. Hence, the three key remarks made on page 114 apply and we therefore consider the specific limit in each stage in which the effect of the forward motion is comparable to the effect of the normal body motion. We discuss the small time asymptotics of these proposed models subsequently.

### 5.2.2.1 Very small and small penetration

The very small penetration ( $s \ll \epsilon^2$ ), deep water limit was discussed in section 5.1.2.1 and, as for the oblique water entry model in section 5.2.1.1, is conceptually straightforward to generalize to the small penetration ( $s \sim \epsilon^2$ ), finite depth case by adding a base at  $z = -1$  to the outer problem in Figure 5.3. Recall that the index of the oblique water entry model is two because there are two simple, smooth contours in the transformed plane (c.f. Figure 3.5) and the potential is unbounded at their ends in the finite plane and therefore zero at infinity; see section 3.3.1. Here, the smooth separation condition implies the potential is now bounded at the end corresponding to the trailing edge, so a line vortex must be induced as in the infinite depth case, resulting in a complicated problem for the potential as discussed in section 5.1.2.

As in section 5.2.1, we consider the proposed stage 4 model first and use it to derive the proposed stage 3 model subsequently.

### 5.2.2.2 Penetration of order unity

For penetrations of order unity we expect Korobkin's analysis reviewed in section 4.3.1 to apply at and behind the trailing edge, because the body and therefore the free surface are flat to lowest order in the "gap-sized" region at the trailing edge (region II in Figures 4.6 and 5.7). We therefore envisage the asymptotic decomposition depicted in Figure 5.7, in which Korobkin's shallow water impact theories of sections 3.5 and 4.2.1 are applied ahead and behind the inviscid squeeze film in region I, respectively. The proposed flow in each of the nine regions are summarized in Figure 5.7.

Hence, taking the dimensionless stage 4 oblique water entry model (5.30 - 5.32), we replace the left turnover point with a sharp trailing edge  $x = U_0 t$ , at which a zero pressure boundary condition is applied, and then move to a frame of reference fixed with respect to the horizontal motion of the body by setting

$$x = U_0 t + \hat{x}, \quad \phi = U_0 \hat{x} + \hat{\phi}, \quad d_+ = U_0 t + \hat{d}, \quad (5.51)$$

to find the following proposed unsteady planing model beneath the impactor (dropping

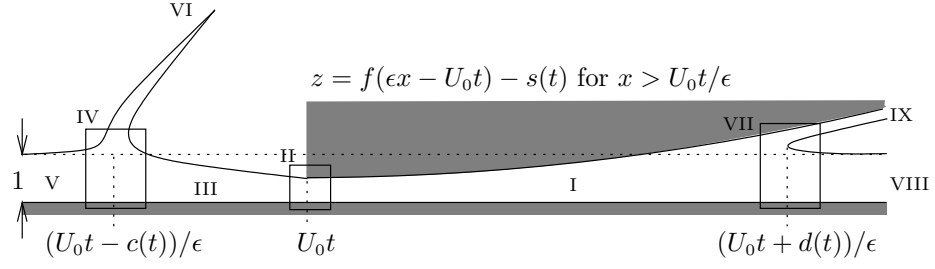


Figure 5.7: Proposed asymptotic structure of the unsteady planing problem at  $O(1)$  penetration. To lowest order the flow in each region is as follows. I inviscid squeeze film flow; II uniform flow; III zero-gravity shallow-water flow; IV Kelvin-Helmholtz flow as in Figure 4.7b; V exponentially small flow; VI thin waterspout jet governed by zero-gravity jet equations; VII Kelvin-Helmholtz flow as in Figure 3.13; VIII exponentially small flow; IX thin spray sheet governed by zero-gravity shallow-water equations. See the text for details.

hats):

$$\frac{\partial}{\partial t} \left( l \frac{\partial \phi}{\partial x} \right) - \dot{s} = 0 \quad \text{for } 0 < x < d(t), \quad (5.52)$$

$$\frac{\partial \phi}{\partial x} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{1}{2} U_0^2 = 0 \quad \text{at } x = 0, \quad (5.53)$$

$$\phi + U_0 x = 0 \quad \text{at } x = d(t), \quad (5.54)$$

$$\frac{\partial \phi}{\partial x} - U_0 \left( 1 - \frac{2}{\sqrt{l}} \right) - 2\dot{d} \left( 1 - \frac{1}{\sqrt{l}} \right) = 0 \quad \text{at } x = d(t), \quad (5.55)$$

where the layer thickness is  $l(x, t) = 1 + f(x) - s(t)$ .

In region III behind the impactor the thin ejected stream satisfies the zero-gravity shallow-water equations (2.85, 2.86), with boundary data determined by matching with region II. Exactly as in section 4.2.1, region III terminates at the waterspout root IV, where it interacts with the exponentially small flow in region V. The waterspout root, at  $x = U_0 t - c(t)$  relative to the fixed frame, is governed by equation (4.42). Hence, in the moving frame (5.51), we set  $h(x, t) = \hat{h}(\hat{x}, t)$  to find the following proposed unsteady planing model behind the impactor (retaining hats on  $\phi$  and  $h$  to denote exterior dependent variables):

$$\frac{\partial \hat{\phi}}{\partial x} + \frac{1}{2} \left( \frac{\partial \hat{\phi}}{\partial x} \right)^2 - \frac{1}{2} U_0^2 = 0 \quad \text{for } -c < x < 0, \quad (5.56)$$

$$\frac{\partial \hat{h}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \hat{\phi}}{\partial x} \hat{h} \right) = 0 \quad \text{for } -c < x < 0, \quad (5.57)$$

with the boundary data

$$\hat{\phi}(0, t) = \phi(0, t), \quad \hat{h}(0, t) = 1 - s(t), \quad (5.58)$$

and shock condition

$$\dot{c} = \frac{1}{2} U_0 + \frac{1}{2} \frac{\partial \hat{\phi}}{\partial x}(c, t). \quad (5.59)$$

The initiation of this flow is discussed in section 5.2.2.4; as in Chapters 3 and 4 we expect the initial conditions to be  $d(0) = c(0) = 0$ . We make the following preliminary observations. The models (5.52 - 5.56) and (5.57 - 5.59) decouple and may be solved in this order. The analysis proceeds exactly as in section 5.2.1.2 and 4.2.1, respectively. The term multiplying  $\dot{d}$  in the kinematic condition (5.55) is zero at  $t = 0$ , so the small time asymptotics, which are derived in the next section, are required to begin the numerical solution. The linear stability of the turnover point and the waterspout root are interesting open questions.

Finally, we note that the steady state version of (5.52 - 5.56) with  $s = s_0 = \text{constant}$  and  $d = \text{constant}$  is exactly the model that Tuck [65] used to analyse the flow past a two-dimensional skimming plate at small incidence angle on shallow water. His asymptotic decomposition is exactly as in Figure 5.7, except now there is uniform flow behind the impactor with velocity  $-U_0$  and height  $1 - s_0$ . We therefore conjecture that the unsteady planing model (5.52 - 5.59), with  $s(t) \rightarrow s_0$  as  $t \rightarrow \infty$ , tends to Tuck's travelling wave solution sufficiently near to the plate with

$$d = \lim_{t \rightarrow \infty} d(t).$$

However, we expect the flow in region III behind the impactor to tend to uniform flow with velocity  $-U_0$  (and height  $1 - s_0$ ), so that  $\dot{c} \rightarrow 0$  as  $t \rightarrow \infty$  by (5.59). This implies that the flux ejected into the waterspout VI tends to zero as  $t \rightarrow \infty$  and that the asymptotics break down locally as gravity becomes important. What happens next is an interesting open question.

### 5.2.2.3 Intermediate penetration

To find the intermediate penetration ( $\epsilon^2 \ll s \ll 1$ ) planing problem one may proceed as in section 5.2.1.3; substituting the small penetration scalings

$$s = \delta S, \quad x = \varepsilon X, \quad \phi = \varepsilon^2 \Phi, \quad (5.60)$$

where  $\varepsilon(\delta) = \delta^{1/2}$  for  $f(x) = x^2$ , the stage 4 pressure condition at the trailing edge (5.53) implies to lowest order

$$\frac{\partial \Phi}{\partial T} = 0 \quad \text{at} \quad X = U_1 T, \quad (5.61)$$

where  $U_1 = \delta^{1/2} U_0$ . As in section 5.2.1.3, we now treat  $U_1$  as order unity and note that setting  $U_1 = 0$  yields the small time asymptotics to the stage 4 planing model (5.52 - 5.56).

Moving to a frame of reference fixed with respect to the horizontal motion of the body by setting

$$X = U_1 \hat{X}, \quad \Phi = U_1 \hat{\Phi} + \hat{\Phi}, \quad D_+ = U_1 T + \hat{D}, \quad (5.62)$$

the stage 3 oblique water entry model (5.42 - 5.45) implies the proposed stage 3 planing model is (dropping hats)

$$\frac{\partial^2 \Phi}{\partial X^2} - \dot{S} - U_1 F'(X) = 0 \quad \text{for} \quad 0 < X < D(T), \quad (5.63)$$

$$\frac{\partial \Phi}{\partial T} - U_1 \frac{\partial \Phi}{\partial X} - U_1^2 = 0 \quad \text{at} \quad X = 0, \quad (5.64)$$

$$\Phi + U_1 X = 0 \quad \text{at} \quad X = D(T), \quad (5.65)$$

$$\frac{\partial \Phi}{\partial X} + U_1(1 + F(X) - S) + \dot{D}(F(X) - S) = 0 \quad \text{at} \quad X = D(T), \quad (5.66)$$

and as in the previous section we expect the initial condition to be

$$D(0) = 0, \quad (5.67)$$

though this must be justified as in Chapter 3, which is discussed further in section 5.2.2.4. The observations made at the end of the previous section concerning the analysis of the stage 4 interior region model (5.52 - 5.56) apply here.

The equations governing the trailing spray sheet are derived by substituting the small penetration scalings (5.60) and  $h = 1 + \delta H$  into the zero-gravity shallow-water equations (2.85, 2.86). In the moving frame (5.62) they become (substituting  $H = \hat{H}$  and retaining hats on the exterior dependent variables)

$$\frac{\partial \hat{\Phi}}{\partial T} - U_1 \frac{\partial \hat{\Phi}}{\partial X} = U_1^2, \quad (5.68)$$

$$\frac{\partial \hat{H}}{\partial T} - U_1 \frac{\partial \hat{H}}{\partial X} = -\frac{\partial^2 \hat{\Phi}}{\partial X^2}. \quad (5.69)$$

Matching with the interior region flow as in stage 4 implies the boundary data is

$$\hat{\Phi}(0, T) = \Phi(0, T), \quad \hat{H}(0, T) = -S(T). \quad (5.70)$$

The method of characteristics implies the solution to (5.68 - 5.70) is

$$\hat{\Phi}(X, T) = \Phi(0, T + X/U_1) - U_1 X, \quad (5.71)$$

$$\hat{H}(X, T) = -S(T + X/U_1) + \frac{X}{U_1^3} \frac{\partial^2 \Phi}{\partial T^2}(0, T + X/U_1), \quad (5.72)$$

valid for  $-U_0 T < X < 0$ . As in stages 1 and 2, the interaction between the fluid displaced by the sharp trailing edge and the fluid behind the impactor is not understood. This is in contrast to the stage 4 planing model (5.52 - 5.59), in which the interaction ejects a waterspout.

#### 5.2.2.4 Small time behaviour

We emphasize that the problems in sections 5.2.2.1 - 5.2.2.3 are *not* an evolution sequence. The small time asymptotics of each of the planing models must be considered separately in order to justify the zero initial conditions. Remark (2) on page 114 implies the forward motion is negligible at smaller times, so the sharp trailing edge enters the water normally to lowest order and therefore the analysis of section 4.1 may apply at the trailing edge at sufficiently small times, i.e. the leading-order outer potential has a square root singularity at the sharp trailing edge at the origin and, assuming smooth separation there, the inner flow may take the form depicted in Figure 4.1a. The crucial difference here is that the outer potential is time-dependent because, in the small deadrise angle limit, Wagner theory is expected to determine the leading-order location of the right turnover point,  $x = d(t)$ , say. Hence, the similarity formulation of the leading-order inner problem at the trailing edge is different because the far field matching conditions depend on  $d(t)$ . Thus, as in all of the above planing models, information flows from the leading jet root region into the “wake” behind the body. As in section 4.2.1, it would be interesting to see if the small time solution alluded to here can be reconciled with the stage 4 unsteady planing model (5.52 - 5.59).

### 5.2.3 Three dimensional models

As in sections 3.7 and 4.3 for the normal impact cases, it is straightforward to generalize the proposed oblique water entry and planing models of sections 5.2.1 and 5.2.2 to three dimensions because we have worked exclusively with the velocity potential. In this section we illustrate this point by considering the three-dimensional version of the stage 4 planing problem in section 5.2.2.2.

#### 5.2.3.1 Unsteady planing at penetrations of order unity

We first scale into the outer region and move into a frame of reference with velocity  $U_0 \mathbf{i}$ . Exactly how we introduce a sharp trailing edge to the resulting stage 4 body profile,  $z = f(x, y) - s(t)$ , say, is crucial. As in the two-dimensional unsteady planing problem considered in section 5.1.2.2, we demand that it lies in  $x \geq 0$  to guarantee that we avoid negative normal impact velocities on the body. However, we must now choose the projection of the trailing edge onto the  $(x, y)$ -plane,  $\partial\Omega_T$ , say. We may choose it to run from  $y = -\infty$  to  $y = \infty$ , for example along the keel  $x = 0$ . At the other extreme, we may suppose the body profile is bounded in the  $x$ - and  $y$ -directions. For example we may choose the body to lie within  $x^2 + y^2 < 1$ , in which case  $\partial\Omega_T = \{(x, y) : x^2 + y^2 = 1, x < 0\}$ ; here and hereafter, all relevant figures will be depicted for such a “surf-skimmer”.

Regardless of the chosen body profile and sharp trailing edge, the leading free turnover curve,  $\Gamma_L(t)$  say, must in general intersect the sharp trailing edge at two or more points, which we label P and call transition points. If the body is convex, we expect there to be two transition points as depicted in Figure 5.8. The transition points are, by definition, confined to lie on the sharp trailing edge, though their location must be determined as part of the solution. We let  $\Gamma_T(t)$  denote the ‘active’ segment of the sharp trailing edge  $\partial\Omega_T$  between the turnover points.

The three-dimensional generalization of the unsteady planing problem (5.30 - 5.32) for the leading-order inviscid squeeze film potential  $\phi = \phi(x, y, t)$  in the interior region  $\mathcal{W}(t)$ , between the active segment of the sharp trailing edge  $\Gamma_T(t)$  and the leading free turnover curve  $\Gamma_L(t)$  (see Figure 5.8), is

$$\frac{\partial}{\partial x} \left( l \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( l \frac{\partial \phi}{\partial y} \right) - \dot{s} = 0 \quad \text{for } (x, y) \in \mathcal{W}(t), \quad (5.73)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 - U_0^2 \right) = 0 \quad \text{for } (x, y) \in \Gamma_T(t), \quad (5.74)$$

$$\phi + U_0 x = 0 \quad \text{for } (x, y) \in \Gamma_L(t), \quad (5.75)$$

$$\frac{\partial \phi}{\partial n} - U_0 \cos \theta \left( 1 - \frac{2}{\sqrt{l}} \right) - 2v_n \left( 1 - \frac{1}{\sqrt{l}} \right) = 0 \quad \text{for } (x, y) \in \Gamma_L(t), \quad (5.76)$$

where the layer thickness  $l(x, y, t) = 1 + f(x, y) - s(t)$  and  $\theta$  is the angle between the  $x$ -axis and the outward pointing unit normal  $\mathbf{n}$  to  $\Gamma_L(t)$ , which has normal velocity  $v_n$  (relative to the moving frame). The comments in section 5.2.2.4 concerning the initial conditions apply here, though we expect  $\mathcal{W}(0) = (0, 0)$ , so that the small time asymptotics are necessary to begin any numerical scheme.

A thin sheet of fluid  $\mathcal{V}(t)$  is ejected from the sharp trailing edge  $\Gamma_T(t)$ ; see Figure 5.8 for a plane view of this and other relevant notation and Figure 5.10 for a possible streamline configuration and free surface geometry. As described in section 4.3, it is governed by

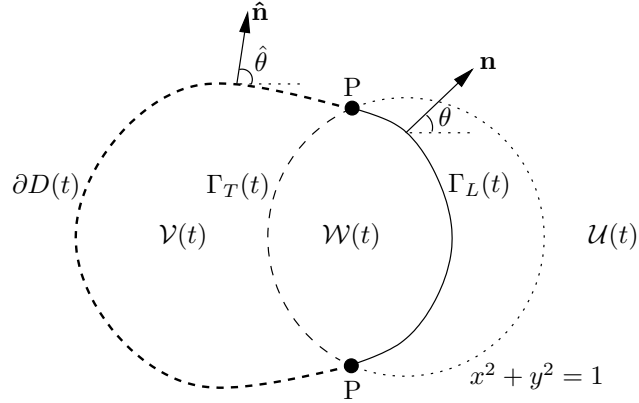


Figure 5.8: Plane view of the geometry of the three-dimensional planing problem for penetrations of order unity. See the text for details of the notation.

the zero-gravity shallow-water equations (2.119, 2.120); relative to the moving frame and introducing hats on exterior variables these become

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \left( \left( \frac{\partial \hat{\phi}}{\partial x} \right)^2 + \left( \frac{\partial \hat{\phi}}{\partial y} \right)^2 \right) - \frac{1}{2} U_0^2 = 0 \quad \text{for } (x, y) \in \mathcal{V}(t), \quad (5.77)$$

$$\frac{\partial \hat{h}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \hat{\phi}}{\partial x} \hat{h} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \hat{\phi}}{\partial y} \hat{h} \right) = 0 \quad \text{for } (x, y) \in \mathcal{V}(t). \quad (5.78)$$

As in section 4.3.1 the boundary data (c.f. (4.46)) is

$$\hat{\phi}(x, y, t) = \phi(x, y, t), \quad \hat{h}(x, y, t) = 1 + f(x, y) - s(t) \quad \text{for } (x, y) \in \Gamma_T(t). \quad (5.79)$$

Further, this ‘trailing sheet’ interacts with the uniform incoming flow  $-U_0 \mathbf{i}$  in the exterior region  $\mathcal{U}(t)$ , forming a waterspout sheet that is ejected out of the root  $\partial D(t)$ . The waterspout root  $\partial D(t)$  is governed by equation (4.47), which, under the transformation to the moving frame, viz.  $x = U_0 t + \hat{x}$ ,  $\phi = U_0 \hat{x} + \hat{\phi}$ , becomes

$$\hat{v}_{\hat{n}} + \frac{1}{2} U_1 \cos \hat{\theta} = \frac{1}{2} \frac{\partial \hat{\phi}}{\partial \hat{n}}, \quad (5.80)$$

where  $\hat{\theta}$  is the angle between the  $x$ -axis and the outward pointing unit normal  $\hat{\mathbf{n}}$  to  $\partial D(t)$ , which has normal velocity  $\hat{v}_{\hat{n}}$ . The initial condition is expected to be  $\partial D(0) = \Gamma_T(0) = (0, 0)$ .

The evolution of the fully two-dimensional waterspout root or shock is an interesting open question. In particular, if the transition points  $P$  (where the turnover curve  $\Gamma_L(t)$  intersects the active trailing edge  $\Gamma_T(t)$ ) lie at the undisturbed water level, i.e.  $l(P, t) = 0$ , then the waterspouts (and the spray sheet ejected from  $\Gamma_L(t)$ ) will have zero thickness there initially and therefore we might expect the scenario depicted in Figure 5.9a, in which the dotted lines represent the curves traced out by the ends of the waterspout. However, if the transition points lie above the undisturbed water level, i.e.  $l(P, t) > 0$ , then the waterspouts (and the spray sheet) will have finite thickness at a transition point and we might expect the scenario depicted in Figure 5.9b. For bodies not symmetric in the  $x$ -axis, there exists the possibility of a combination of these scenarios. We remark that there is also the possibility of converging trailing sheet streamlines forming another waterspout root if  $\Gamma_T$  were convex, say.

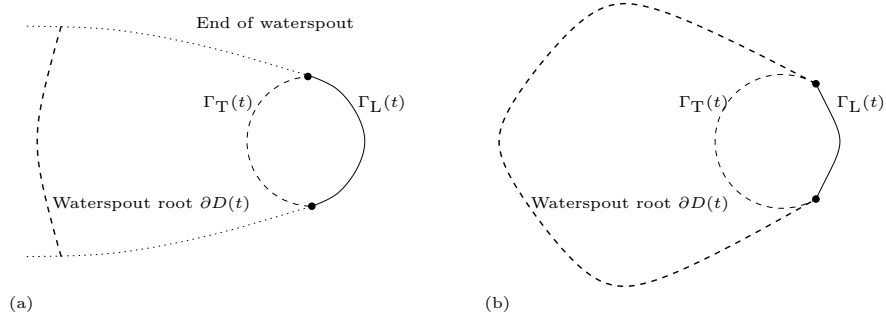


Figure 5.9: Plane view of the possible external configuration in the order unity penetration depth unsteady planing model.

### 5.2.3.2 Steady planing at penetrations of order unity

We now consider the steady states of the proposed three-dimensional planing model, which are obtained by fixing the penetration  $s(t) \equiv s_0 \in (0, 1)$  and removing the time-dependent terms from the interior region problem (5.73 - 5.76) and the trailing sheet problem (5.77 - 5.79). By doing so, our straightforward generalization of Korobkin's two-dimensional small-deadrise-angle shallow-water impact theory to the three-dimensional case in section 3.7 has allowed us to find the three-dimensional generalization of Tuck's two-dimensional shallow-water surf-skimmer model [65], which we discussed in section 5.2.2.2.

In the exterior region, in a the frame moving with the skimmer, a streamline of the incoming uniform flow  $\nabla\hat{\phi} = -U_0\mathbf{i}$ ,  $\hat{h} = 1$  ahead of the skimmer either hits the turnover curve  $\Gamma_L$ , a transition point  $P$ , the trailing sheet ejected from behind the skimmer<sup>7</sup>, which is governed by the steady versions of (5.77 - 5.79), or nothing. Hence we propose the scenario depicted in Figure 5.10.

The pressure boundary condition (5.75) implies the tangential derivative of the velocity potential is equal to  $U_0 \sin \theta$  on the turnover curve  $\Gamma_L$ , which we recall has outward pointing unit normal  $\mathbf{n}$  that makes an angle  $\theta$  with the  $x$ -axis, so

$$\begin{aligned} \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 &= U_0^2 \sin^2 \theta + \left(\frac{\partial\phi}{\partial n}\right)^2 \\ &= U_0^2 [1 + 4 \cos^2 \theta (1/\sqrt{l} - 1)/\sqrt{l}] \quad \text{for } (x, y) \in \Gamma_L, \end{aligned}$$

by the steady version of (5.76). Hence, the fluid velocity is continuous across a transition point  $P$  if  $l(P) = 1$ , i.e. the transition point  $P$  lies in the plane  $z = 0$ , or  $\theta(P) = \pm\pi/2$ , i.e. the tangent to the turnover curve is in the  $x$ -direction at  $P$ , which implies  $P$  must lie at an extremity of the body in the  $y$ -direction.

The interaction between the trailing sheet,  $\mathcal{V}$ , and the incoming sheet,  $\mathcal{U}$ , now takes place across a waterspout root  $\partial D$  that is fixed in the moving frame. For continuity of pressure across the ejected waterspout the incoming velocity components  $U_{\pm}$  in Figure 4.7b must be equal. Since  $U_{\pm} = U_0 \cos \hat{\theta}_{\pm}$ , where  $\hat{\theta}_{\pm}$  is the acute angle between the normal  $\hat{\mathbf{n}}$  to the shock and intersecting streamlines from the far field (+) and spray sheet (-), as depicted in Figure 5.11, we deduce that  $\hat{\theta}_{+} = \hat{\theta}_{-}$ , i.e. the shock bisects the angle between incoming streamlines. Also, since the sheets have speed equal to  $U_0$  everywhere, by the steady version of (5.77), the incoming tangential velocities  $U_0 \sin \hat{\theta}_{\pm}$  are equal and therefore

<sup>7</sup>In Tuck's two-dimensional case [65] this is the trivial uniform flow with a with layer depth equal to the body-to-basin depth at the sharp trailing edge; see section 5.2.2.2

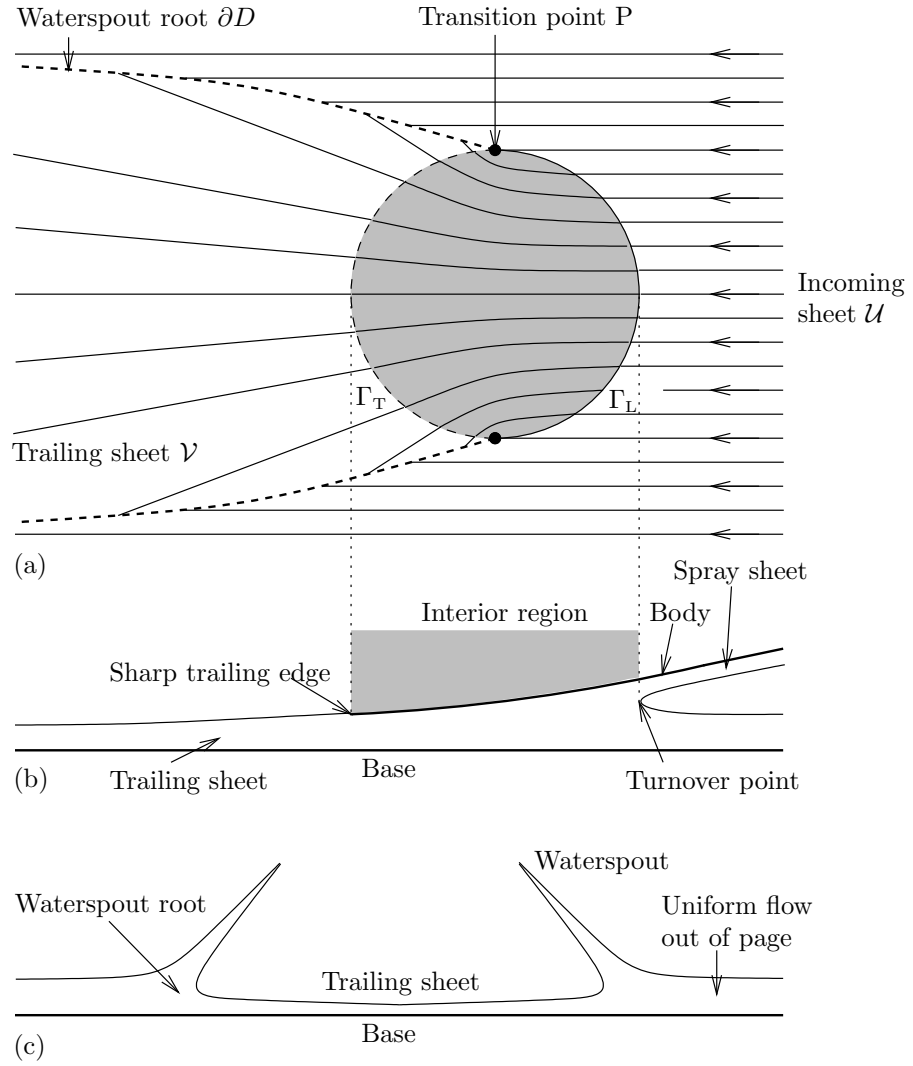


Figure 5.10: (a) Plane view of a possible streamline configuration for the order unity penetration depth skimming problem. (b) Side view of the proposed flow geometry in the plane  $y = 0$ . (c) View from behind of the proposed flow geometry in the planes  $x = \text{constant}$  behind the trailing edge. As described in the text, the flow in all vertical planes perpendicular to the waterspout root  $\partial D$  has the flow geometry sketched in Figure 4.7b.

the waterspout does not support a vortex sheet, as in the two-dimensional normal impact of a rectangle on shallow water in section 4.3.

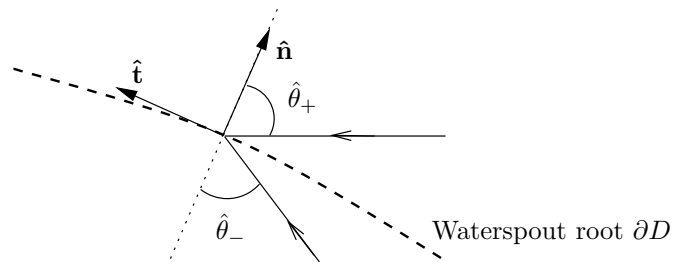


Figure 5.11: Terminology for the steady waterspout shock problem.

The shock condition may also be derived succinctly from the unsteady law (5.80) because  $\hat{v}_{\hat{n}} = 0$ , viz. by the steady version of (5.77) and the fact the fluid is moving away from the trailing edge  $\Gamma_T$ , we deduce that the trailing sheet velocity at the waterspout root  $\partial D$  is

$$\nabla \hat{\phi} = U_0(\cos \hat{\theta} \hat{\mathbf{n}} + \sin \hat{\theta} \hat{\mathbf{t}}) = U_0(\cos 2\hat{\theta} \hat{\mathbf{i}} + \sin 2\hat{\theta} \hat{\mathbf{j}}),$$

which implies  $\hat{\mathbf{n}}$  bisects  $\hat{\mathbf{i}}$  and  $\nabla \hat{\phi}$  and therefore  $\hat{\theta}_+ = \hat{\theta}_-$ .

The waterspout centre-surface is a ruled surface determined by the velocity of the fluid particles ejected from the waterspout root. The waterspout thickness is determined by the usual conservation of mass argument for a thin sheet; see, for example, [28].

### 5.3 Oblique shallow water entry of a flat plate at zero deadrise angle

In section 4.3.1 we reviewed Korobkin's analysis of the normal impact of a rectangle on shallow water of depth  $\beta \ll 1$ , at unit velocity and at times of order  $\beta$ ; see Figure 4.6. The analysis also applies to the impact of a flat plate at zero deadrise angle, provided we assume smooth separation at its ends. In this section we consider the effect of a forward velocity on the latter scenario; the theory may also hold for a flat-bottomed wedge of sufficiently large deadrise angle, with appropriate modifications in the waterspouts and their roots.

The inviscid squeeze film velocity below the normally impacting plate (region I in Figure 4.6) is of order  $1/\beta$ , so the forward velocity  $U$  has a comparable effect to the squeeze velocity if  $U = U_0/\beta$ , where  $U_0 = O(1)$ .

We will show that there is an interesting competition between the effect of the normal and forward velocities at the leading edge and in particular

- a detached waterspout is formed behind the plate for all  $U_0 > 0$ ;
- below the critical forward velocity  $U_0 = 1/2$ , a detached waterspout is also formed ahead of the plate, as depicted in Figure 5.12a;
- above the critical forward velocity  $U_0 = 1/2$ , the pressure can only be non-negative everywhere if the right-hand waterspout root is initially attached to the leading edge, as depicted on Figure 5.12b.

Under the non-negative pressure assumption, we then argue that there is a change in geometry of the “unsteady ploughing flow” in Figure 5.12b, to Korobkin's detached waterspout regime in Figure 5.12a at a critical time.

To be consistent with section 4.3.1, we suppose that the base lies at  $z = 0$  and that the shallow fluid layer initially lies in  $0 < z < \beta$ , where the layer depth  $\beta \ll 1$ . Rescaling the vertical coordinate  $z$  and time  $t$  with  $\beta$ , the plate has equation

$$z = 1 - t \quad \text{for} \quad -1 + U_0 t < x < 1 + U_0 t.$$

We denote the leading (+) and trailing (−) ends of the plate by  $x_{\pm} = \pm 1 + U_0 t$ .

Beneath the impactor, in region I, the potential  $\phi \sim \phi_0(x, t)/\beta + O(1)$  as in section 4.3.1, where  $\phi_0$  is now given by

$$\phi_0(x, t) = \frac{(x - x_0(t))^2}{2(1 - t)} + A(t); \tag{5.81}$$

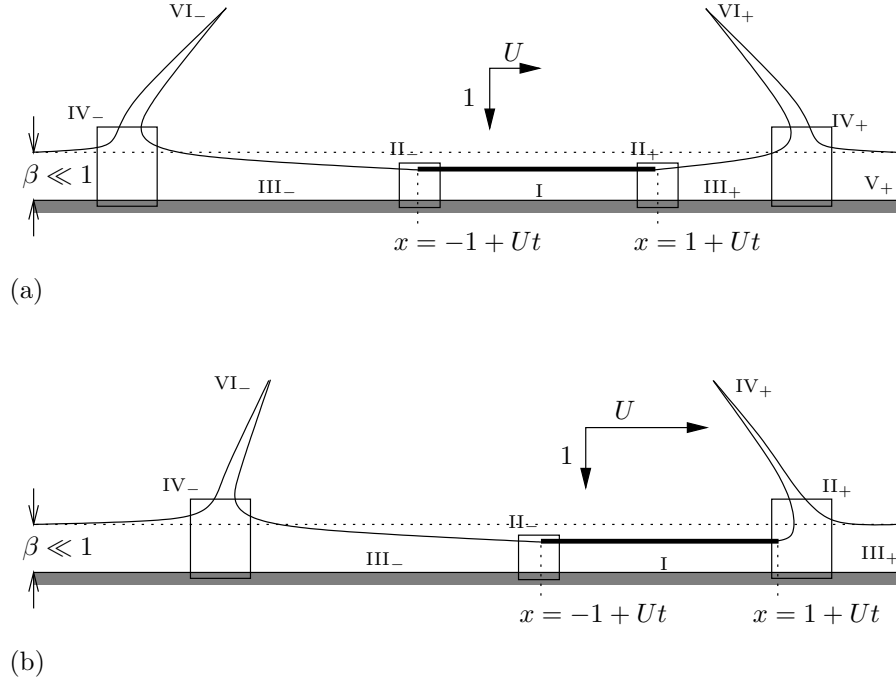


Figure 5.12: Proposed initial flow geometries with forward velocity  $U = U_0/\beta$  in the cases (a) detached leading waterspout ( $U_0 < 1/2$ ) and (b) attached leading waterspout ( $U_0 > 1/2$ ).

$A(t)$  and the location of the stagnation point  $x = x_0(t)$  (provided  $x_- \leq x_0 \leq x_+$ ) are to be determined.

There is outflow at both ends  $x = x_{\pm}$  when  $U_0 = 0$ , so by continuity we might expect there to be outflow at both ends for sufficiently small  $U_0$ , *provided* we demand smooth separation. We now investigate the consequences of this assumption for forward velocities  $U_0 > 0$ .

### 5.3.1 Asymptotic solution in Korobkin's regime

Assuming smooth separation at both ends, the inner corner regions  $\text{II}_{\pm}$  in Figure 5.12a are the same as in the normal impact case and matching therefore requires the inviscid squeeze film flow to have zero leading-order pressure at both edges, viz.

$$p_0 = -\frac{\partial \phi_0}{\partial t} - \frac{1}{2} \left( \frac{\partial \phi_0}{\partial x} \right)^2 = 0, \quad (5.82)$$

at  $x = x_{\pm}$ . Substituting (5.81) into (5.82) we obtain two coupled equations for the leading-order unknowns  $x_0(t)$  and  $A(t)$ . Eliminating  $A(t)$  we find a first-order ordinary differential equation for the location of the stagnation point  $x = x_0(t)$ , viz.

$$(1-t)\dot{x}_0 + 2x_0 = 2U_0t. \quad (5.83)$$

Subject to the initial condition  $x_0(0) = 0$ , its solution is

$$x_0(t) = U_0 t^2.$$

The leading-order pressure is given by

$$p_0 = \frac{(1-t)(x-x_0)\dot{x}_0 - (x-x_0)^2}{(1-t)^2} - \dot{A}, \quad (5.84)$$

where

$$\dot{A} = \frac{(1-t)(x_- - x_0)\dot{x}_0 - (x_- - x_0)^2}{(1-t)^2}. \quad (5.85)$$

We leave  $A(t)$  undetermined, since for our purposes it is sufficient to note that the pressure is a quadratic in  $x$ , that vanishes at  $x = x_{\pm}$  and has a positive maximum at the centre of the rectangle  $x = U_0 t$ .

The boundary data for the zero gravity shallow-water regions  $\text{III}_{\pm}$  in Figure 5.12a ahead (+) and behind (−) the plate are therefore

$$u_0(x_{\pm}(t), t) = \frac{x_{\pm}(t) - x_0(t)}{1-t}, \quad h_0(x_{\pm}(t), t) = 1-t; \quad (5.86)$$

the parametric solution of (4.39, 4.40) is

$$u_0(x, t) = u_{\pm}(\tau) = \frac{x_{\pm}(\tau) - x_0(\tau)}{1-\tau} \quad \text{on} \quad x - x_{\pm}(\tau) = u_{\pm}(\tau) \cdot (t - \tau). \quad (5.87)$$

However, the important quantities are the relative fluid velocities at the trailing (−) and leading (+) edge, which by (5.86) are

$$u_0(x_{\pm}(t), t) - U_0 = \frac{\pm 1}{1-t} - U_0(1-t).$$

We conclude the following.

- There is outflow at the trailing edge  $x = x_-$ , i.e.  $u_0(x_-, t) - U_0 < 0$ , for all  $U_0 > 0$ ,  $t \in [0, 1)$ .
- Similarly, there is outflow at the leading edge  $x = x_+$  for all  $U_0, t \in [0, 1)$ . However, if  $U_0 > 1$  there is inflow at the leading edge, which reverses direction at time  $t = 1 - U_0^{-1/2}$ , i.e. there is outflow at later times as depicted in Figure 5.13.

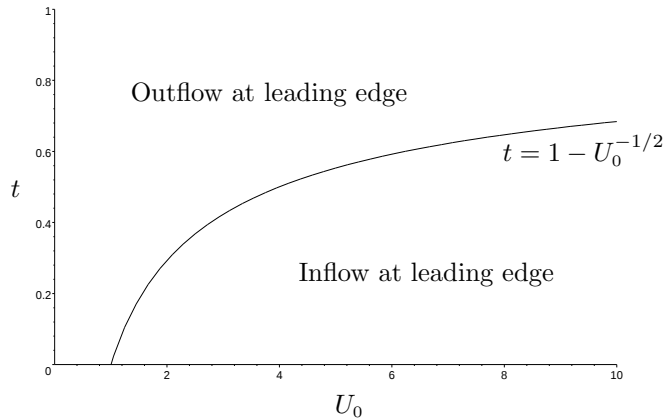


Figure 5.13: Plot of the critical time  $t = 1 - U_0^{-1/2}$  at which the flow reverses direction at the leading edge  $x = x_+$ .

### 5.3.1.1 When is Korobkin's solution valid?

Since there is initially outflow at the leading edge if and only if the forward velocity is less than the normal impact speed ( $U_0 < 1$ ), one might expect Korobkin's decomposition in Figure 5.12a to be valid in this case, with the obvious minor algebraic modifications to account for the asymmetry in the exterior flows. However, crucially the shock condition (4.42) says the velocity of a waterspout root is half the incoming fluid velocity from the neighbouring shallow-water region. The initial shallow-water velocity at the edges is  $u_0(x_{\pm}(0), 0) = u_{\pm}(0) = \pm 1$ . Hence, the waterspout root always escapes from the trailing end, but cannot escape from the leading edge unless  $U_0 < 1/2$ , in which case we expect Korobkin's detached waterspout decomposition to be valid. Note that once a waterspout root has escaped, its motion is entirely determined by the flow in the adjacent shallow-water region  $\text{III}_{\pm}$  in Figure 5.12a.

If  $U_0 > 1/2$ , the forward velocity is sufficiently large that the inviscid squeeze film cannot initially cause the formation of a detached waterspout root at the leading edge. Thus, Korobkin's decomposition is invalid and we envisage the ploughing scenario depicted in Figure 5.12b, in which the waterspout root is fixed at the leading edge in the "gap-sized" inner region  $\text{II}_{+}$ . We now investigate the consequences of assuming this scenario for  $U_0 > 0$  and the nonuniqueness that the above conjectures imply when  $0 < U_0 < 1/2$ .

### 5.3.2 Asymptotic solution in the ploughing regime

We scale into the small inner region  $\text{II}_{+}$  in Figure 5.12 (that is fixed at the leading edge) by scaling

$$x = U_0 t' + \beta X, \quad z = \beta Z, \quad t = \beta t', \quad \phi = U_0 X + \Phi, \quad h = \beta H, \quad p = P, \quad (5.88)$$

and expand  $\Phi$ ,  $H$ ,  $P$  as asymptotic series in powers of  $\beta$  to obtain the leading-order problem shown in Figure 5.14 (dropping the primes and subscripts). The Bernoulli condition (iii) in Figure 5.14 implies that the ploughing scenario cannot occur for  $U_0 = 0$ . Assuming the flow is exponentially small in the region  $\text{III}_{+}$  ahead of the plate and matching with the three neighbouring outer regions, we obtain the far field conditions shown. The ejected jet has thickness  $H_j$ , is in the direction of the unit vector  $\mathbf{v}$  and makes an angle  $\gamma$  with the plate. The dividing streamline is depicted for the inflow case, i.e.  $u_0(x_+(t), t) > U_0$ , in which the relative stagnation point lies on the base. If  $u_0(x_+, t) = U_0$ , there is no dividing streamline and the relative stagnation point lies at  $X = -\infty$ , where the flow tends to zero exponentially. In the outflow case, i.e.  $u_0(x_+(t), t) < U_0$ , the dividing streamline runs from the relative stagnation point on the plate to  $(X, Z) = (\infty, H_j)$ .

The solution was found by Green [23] using the hodograph method. However, we will now see that, as in the stage 4 jet root problem in section 3.5.2.3, global conservation of mass,  $X$ -momentum and Bernoulli's equation<sup>8</sup>

$$P + \frac{1}{2} |\nabla \Phi|^2 = \frac{1}{2} U_0^2, \quad (5.89)$$

together with the solution for the leading-order flow (5.81) beneath the impactor, subject to appropriate boundary conditions, are sufficient to determine the four unknowns  $H_j(t)$ ,  $u_0(x_+, t)$ ,  $p_0(x_+, t)$  and  $\gamma(t)$ .

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<sup>8</sup>Derived by writing Bernoulli's equation (2.18) in jet root region variables (5.88), expanding as usual and dropping primes and subscripts.

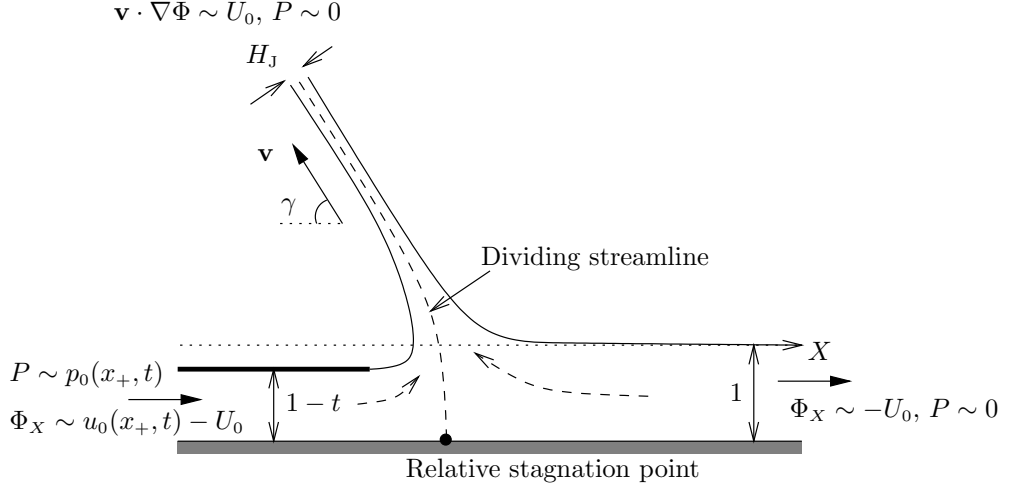


Figure 5.14: The inner region  $\Pi_+$  at the leading edge in Figure 5.12b, which also satisfies (i) Laplace's equation  $\nabla^2 \Phi = 0$  in the fluid; (ii) the zero normal flow kinematic boundary condition  $\Phi_N = 0$  on the solid walls and free surface; (iii) the Bernoulli condition  $|\nabla \Phi| = U_0$  on the free surface.

Global conservation of mass implies

$$-(1-t) \cdot (u_0(x_+, t) - U_0) + U_0 \cdot H_J = U_0 \cdot 1, \quad (5.90)$$

while the lowest order Bernoulli equation (5.89) on the base implies

$$p_0(x_+, t) + \frac{1}{2} (u_0(x_+, t) - U_0)^2 = \frac{1}{2} U_0^2. \quad (5.91)$$

In addition, global conservation of  $X$ -momentum implies

$$U_0^2 \cdot 1 = (p_0(x_+, t) + (u_0(x_+, t) - U_0)^2) \cdot (1-t) + U_0^2 \cdot H_J \cdot \cos \gamma, \quad (5.92)$$

because the jet makes an angle  $\gamma$  with the plate in the far field.

Hence, we have obtained three equations (5.90, 5.91, 5.92) for the four unknowns. As noted above, the fourth equation is obtained by solving for the leading-order flow (5.81) beneath the impactor, subject to the pressure matching conditions (5.82) on  $x = x_-$  and (5.91) on  $x = x_+$ . As in section 5.3.1.1, this yields a first-order ordinary differential equation for the leading-order location of the relative stagnation point  $x_0 = x_1 - U_0 t$ , where

$$4(1-t)\dot{x}_1 + (6 + 2U_0(1-t))x_1 + x_1^2 = 2U_0(1-t) - 1. \quad (5.93)$$

This is subject to the initial condition  $x_1(0) = 0$  and was solved using a fourth-fifth order Runge-Kutta procedure. In Figure 5.15 we use (5.86) to plot the relative fluid velocity at the leading (+) and trailing (-) edges for forward velocities  $U_0 = 0, 0.5, 1.0, 1.5, 2.0$ . Note that (5.86) also implies  $u_0(x_{\pm}, t) - U_0 \sim O(\pm 1/(1-t))$  as  $t \rightarrow 1-$ .

The solution depends continuously on the forward velocity  $U_0$ , so we conclude the following.

- As expected there is always outflow at the trailing edge for all  $U_0 > 0$ .
- There is always outflow at the leading edge for  $U_0 < 1$ . However, for  $U_0 > 1$ , there is inflow at the leading edge for  $t < t^*(U_0) < 1$  and, no matter how large the forward

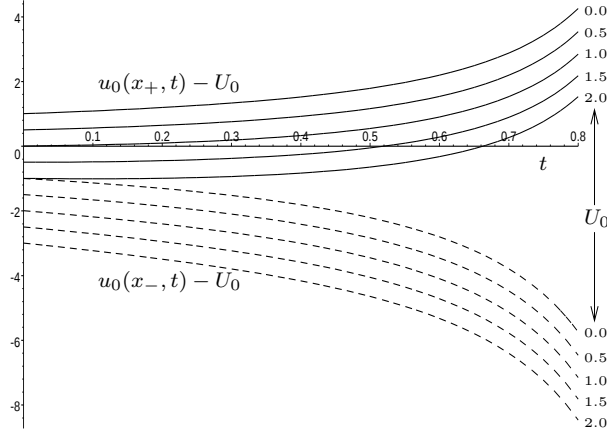


Figure 5.15: Plot of the relative fluid velocities  $u_0(x_{\pm}, t) - U_0$  for  $U_0 = 0, 0.5, 1.0, 1.5, 2.0$ .

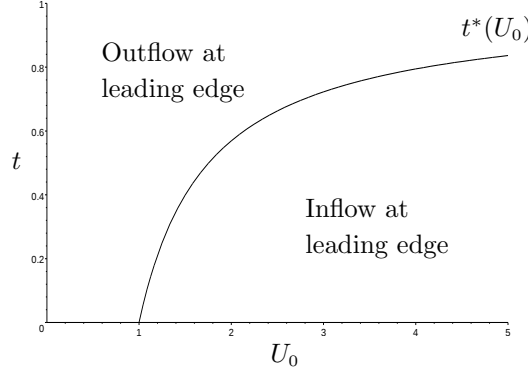


Figure 5.16: Plot of the critical time  $t^*(U_0)$  at which the flow reverses direction at the leading edge.

velocity  $U_0$ , the inviscid squeeze film induced velocity eventually forces outflow for  $t > t^*(U_0)$ . We plot the critical time  $t^*(U_0)$  at which the flow reverses direction at the leading edge in Figure 5.16.

The leading-order pressure beneath the impactor is again given by (5.84, 5.85), except now  $x_0 = x_1 - U_0 t$  is determined by (5.93).

As  $U_0 \rightarrow \infty$ , a matched asymptotic expansion analysis shows that

$$x_0(t) - U_0 t = x_1(t) \sim \begin{cases} 1 - \exp(-\tau/U_0) & \text{for } t = \tau/U_0, \tau = O(1), \\ 1 - 4/U_0(1 - t) & \text{for } O(1/U_0) \ll t \ll 1 - O(1/U_0), \\ X(\tau) & \text{for } t = 1 - \tau/U_0, \tau = O(1), \end{cases}$$

where

$$\frac{dX}{d\tau} = \frac{1}{2}(X - 1) + \frac{1}{4\tau}(1 + 6X + X^2) \quad \text{with matching condition } X(\infty) = 1.$$

This is harder to solve than the original ordinary differential equation (5.93) because

$$X(\tau) \sim \begin{cases} (2\sqrt{2} - 3) + (2 - \sqrt{2})\tau & \text{as } \tau \rightarrow 0, \\ 1 - \int_{\tau}^{\infty} e^{(\tau-s)/2} \frac{ds}{4s} & \text{as } \tau \rightarrow \infty. \end{cases}$$

However, we can now easily deduce the first order behaviour of  $t^*(U_0)$  for large  $U_0$ , because the flow reverses in the right hand boundary layer as  $U_0 \rightarrow \infty$ . Expanding  $u_0(x_+(1 - \tau/U_0), 1 - \tau/U_0) = U_0$ , we find  $t^*(U_0) \sim 1 - \tau^*/U_0$  as  $U_0 \rightarrow \infty$ , where  $\tau^*$ , which we leave undetermined, satisfies  $X(\tau^*) = 1 - \tau^*$ .

### 5.3.2.1 When is the ploughing solution valid?

The ploughing scenario yields a possible solution for all  $U_0 > 0$ , so it appears there are two feasible solutions for  $0 < U_0 < 1/2$ . However, as noted in section 4.3, there is good agreement with experiment in the normal impact case, so our immediate aim is to explain, within the framework of inviscid small aspect ratio theory, why we can rule out the ploughing scenario at zero and therefore sufficiently small forward velocities.

Returning to the ploughing solution analysed in the previous section, the simplest argument considers the leading-order inviscid squeeze film pressure at the leading edge,  $p_0(x_+(t), t)$ , which is equal to

$$\frac{1}{2} (U_0^2 - (1 - U_0)^2) \quad \text{at } t = 0, \quad (5.94)$$

by (5.91) and because  $u_0(x_+(0), 0) = 1$ . Therefore, if  $U_0 < 1/2$ , the pressure is initially negative near to the leading edge and decreases with time, tending to minus infinity like  $O(-1/(1-t)^2)$  as  $t \rightarrow 1-$ . If  $U_0 > 1/2$ , the pressure is initially positive near to the leading edge and increases to a maximum at time  $t = t^*(U_0)$  and then decreases, reaching zero at time  $t = t^{**}(U_0)$ , when  $u_0(x_+(t), t) = 2U_0$  by (5.91), and tending to minus infinity as before. In Figure 5.17 we plot the relative fluid velocity  $u_0(x_+, t) - U_0$  and pressure  $p_0(x_+, t)$  at the leading edge for  $U_0 = 1/4, 4$ , which are typical for the cases  $U_0 < 1/2$  and  $U_0 > 1/2$ , respectively. In Figure 5.18 we plot the critical time  $t^{**}(U_0)$  at which the leading edge pressure vanishes.

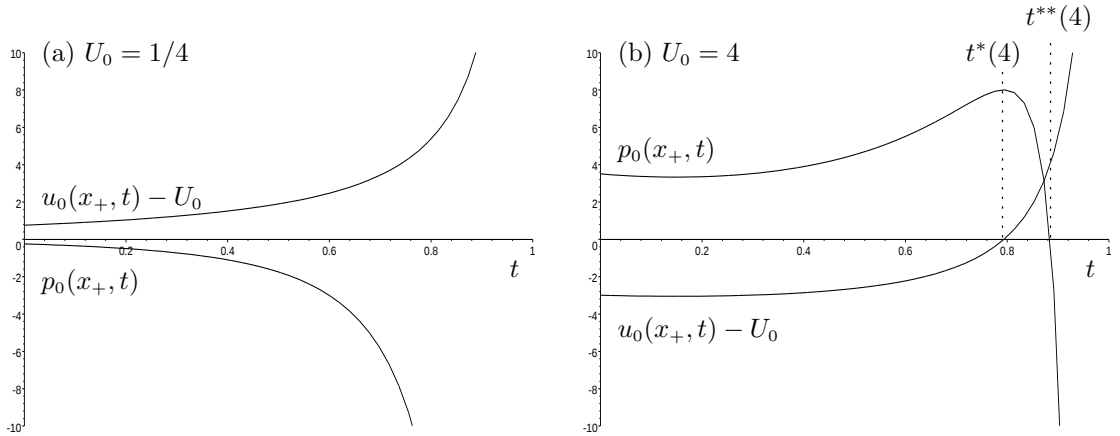


Figure 5.17: Plot of the relative fluid velocity  $u_0(x_+, t) - U_0$  and pressure  $p_0(x_+, t)$  at the leading edge for (a)  $U_0 = 1/4$  and (b)  $U_0 = 4$ .

As discussed in section 5.3.2, cavitation occurs at sufficiently strictly negative pressures. Therefore, if  $U_0 < 1/2$ , the ploughing model predicts that the flow near to the leading edge will cavitate immediately or soon after impact. If  $U_0 > 1/2$ , the flow does not cavitate immediately, but must do so near to the leading edge at, or soon after, the time  $t = t^{**}(U_0)$  at which the pressure vanishes. In both these cases, it is difficult to justify cavitation

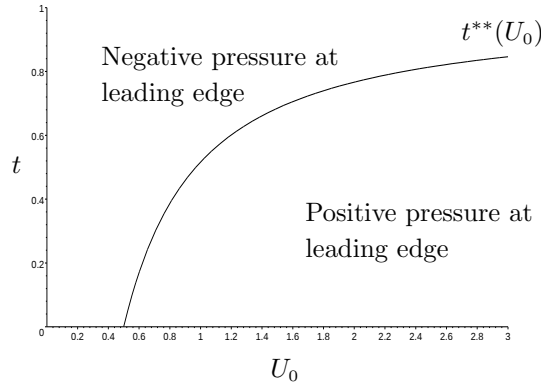


Figure 5.18: Plot of the critical time  $t^{**}(U_0)$  at which the pressure at the leading edge vanishes.

on physical grounds, so if we demand that the pressure is non-negative everywhere, we immediately rule out the ploughing solution for the case  $U_0 < 1/2$ .

The only leading-order inner flow at an end of the plate that maintains a zero pressure matching condition for the lowest order flow beneath the impactor is uniform flow (as in regions  $\text{II}_{\pm}$  in Figure 5.12a). In the case  $U_0 > 1/2$ , we therefore envisage a jump to the detached waterspout regime (in Figure 5.12a) at time  $t = t^{**}(U_0)$  when the pressure at the leading edge vanishes. As shown above, the attached waterspout  $\text{II}_{+}$  cannot escape from the leading corner unless  $u_0(x_+, t) \geq 2U_0$ , so  $t = t^{**}(U_0)$  is also the earliest time at which the transition can occur.

After the proposed change in the flow regime, the  $x$ -coordinate of the relative stagnation point  $x_0(t)$  is given by (5.83). If we demand  $x_0$  to be continuous across the jump, then so are the lowest order inviscid squeeze film velocity and pressure, and therefore so is the jet angle  $\gamma$  (determined by (5.92)), although there is a discontinuity in their time derivatives.

### 5.3.3 Large time behaviour and steady states

Suppose now the penetration depth is  $s(t)$ , rather than  $t$ , where  $s(0) = 0$ ,  $s$  is continuous and monotonic increasing and  $s(t) \rightarrow s_0 \in (0, 1)$  as  $t \rightarrow \infty$ . The above analysis proceeds with significant algebraic modifications and the critical forward velocity is now a function of  $s$ ,  $\dot{s}$  and  $\ddot{s}$ , rather than being a constant. As  $t \rightarrow \infty$ , the inviscid squeeze film velocity induced by the motion of the plate tends to zero. Hence, only the ploughing regime in Figure 5.12b is valid at sufficiently large times and if the leading waterspout escapes as in Figure 5.12a at an earlier time, it would have to reattach when the inviscid squeeze film velocity is not strong enough to support it. This argument also implies that any steady state travelling wave solutions, in which the flow is zero behind the leading edge of the plate relative to the fixed frame, must be in the ploughing regime; by (5.82 - 5.92), the jet thickness is  $H_j = s_0$  and the jet angle  $\gamma = 0$ . We conjecture that the above solution for  $U_0 > 0$  tends to this steady state as  $t \rightarrow \infty$ , except with a trailing waterspout shock far from the plate that ‘decays’ as  $t \rightarrow \infty$ , as discussed in section 5.2.2.2.

## Chapter 6

# Conclusion

In this final chapter we briefly summarise the main results that we have obtained. We then highlight some of the outstanding problems requiring further work.

### 6.1 Review of thesis

In this thesis we have formalised three existing models and derived several new models for deep and shallow water entry at small and zero deadrise angles for normal and oblique impact velocities. Our method throughout has been to exploit the existence of one or more small parameters via the method of matched asymptotic expansions. The small parameters we employed are the deadrise and incidence angles and the inverse aspect ratios of the penetration and water depth to the length scale of the impacting body.

Chapter 1 began with a description of some of the many solid-fluid impact scenarios that motivate this thesis. We then summarised relevant work which has been carried out by other authors and investigated conditions under which surface tension, gravity, viscous, compressible and air cushioning effects can be neglected.

In Chapter 2 we presented an account of the small deadrise angle water entry model. We showed that a small modification of previous asymptotic analyses [8, 9, 31] is required to perform the matching systematically and therefore formally derive the Wagner condition [71]. We found we needed a trivial eigensolution in the new intermediate region in order to match Wagner’s outer expanding plate solution [71] to the small turnover region on the body. This results in the first-order correction to the force on the body compared to the first-order analysis of [8] and might provide an explanation for their disagreement with experiment. Simple extensions were discussed including the possibility of cavitation if the body decelerates sufficiently rapidly. Finally, we found the slender body limit of the explicit solution for the entry of an elliptic paraboloid due to Korobkin [40] and discussed the implications of the new ‘leading-edge’ eigensolution for strip theory.

In Chapter 3 we analysed the small deadrise angle shallow water entry model. The impact was shown to occur over four distinct temporal stages characterised by the ratio of the three length scales inherent in the model, namely, the layer depth, the penetration depth and the distance between the points where the free surface turns over to form the jets. The key observation that enabled a complete asymptotic analysis is that the latter distance is always much larger than the penetration depth because the body is shallow. We found the spatial decomposition of each stage and performed the temporal matching between neighbouring stages, which allowed a quantitative description of the jet formation

mechanism and we were therefore able to reconcile the infinite depth Wagner theory [71], valid at sufficiently small times when the effect of the base is negligible, with the finite depth Korobkin theory [37], valid at order unity times when the penetration depth is comparable to the layer depth. Three-dimensional extensions to the model were discussed for which the lowest order impact problem and its small time limit were derived. The latter describes the change in the geometry of the turnover curve as we pass from the very-small-time Wagner regime to the order-unity-time Korobkin regime. Although no general theory is available for these two novel codimension-one free boundary problems, some analytic progress was made. In both regimes the symmetric one-dimensional solution was shown to be linearly stable to perturbations in the perpendicular direction provided the speeds of the turnover points are greater than a strictly positive critical speed that depends on their location. In the second problem an inverse method was developed to analyse the small-time impact of an elliptic paraboloid on shallow water and to find a criteria for the splitting of the ejected spray sheet.

In Chapter 4 we analysed the impact of a flat-bottomed body on water of infinite and finite depth. We used the method of matched asymptotic expansions to verify that, at small times, the impact of a flat-bottomed wedge is characterised by the similarity solutions to a local canonical problem at each corner. We showed that, in contrast to the impacts considered in Chapters 2 and 3, the leading-order force on the body is non-zero initially and receives contributions from both the outer and inner regions. Again no general theory is available for this problem, but an argument based entirely on the location of the contact point between the body and free surface was used to explore the existence of different shaped solutions depending on the size of the deadrise angle. Korobkin’s theory for the order unity time shallow water entry of a flat-bottomed wedge of large deadrise angle was reviewed. This theory proposes the formation of a detached ‘waterspout’ or spray jet ejected from a jet root region away from the body, in sharp contrast to the corresponding small deadrise angle case analysed in Chapter 3, in which the jet root is attached to the body. Preliminary ideas for the unification of these small and large time theories were presented. Finally, the generalisation of Korobkin’s large-deadrise-angle theory to the three-dimensional case was reviewed and simple extensions were briefly discussed.

In Chapter 5 we used the theories of Chapters 2, 3 and 4 as building blocks to formulate some conjectured models for impacts with a tangential or forward velocity component. We began with a brief review and discussion of the intricately related infinite depth models for oblique water entry at small deadrise angles and planing at small angles of incidence. We investigated the effect of finite depth on these models when the forward velocity has a comparable effect to the normal body velocity in each of the four temporal stages of impact used in Chapter 3. The small time asymptotics of these models were briefly discussed. The generalisation to three dimensions was shown to be conceptually straightforward for both the oblique impact and planing models, except in the latter case where we showed that there is the additional open problem of finding the location of the so-called “transition points”, where the turnover curve intersects the sharp trailing edge. To illustrate these points, the proposed three-dimensional shallow water planing model was derived. Its steady states for fixed penetration depth were conjectured to be the three-dimensional generalisation of Tuck’s two-dimensional surf-skimmer model [65]. We then considered the oblique shallow water entry of a flat two-dimensional plate at zero deadrise angle. We showed that, although it is straightforward to generalise Korobkin’s analysis of the two-dimensional shallow-water entry of a rectangle (in Chapter 4) to account for forward velocities comparable to the “inviscid squeeze film” induced flow, there is an interesting quantifiable competition between

the forward and normal velocities. We then briefly considered the large time behaviour when the penetration depth of the plate tends to a constant and travelling waves at fixed penetration depth.

## 6.2 Future work

We have shown that there are many open questions that need to be answered before the shallow water impact theories considered in this thesis can be considered complete. Whenever these open questions have arisen we have tried to suggest possible conjectures. In this section we recapitulate the prominent areas which emerged and which require further work.

- In section 2.2.5 we showed that the first-order force (2.91) on a two-dimensional impactor of small deadrise angle is a function of the first-order correction of the  $x$ -coordinates of the turnover points, namely  $x = \pm d_1$ . To put this expression to use a higher-order analysis is required to find  $d_1$ .
- In section 2.3.2 we showed that large negative pressures can be generated on a body of small deadrise angle if it decelerates sufficiently rapidly. Some careful cavitation modelling is therefore required, which for example, might offer insight into how the Basilisk lizard can run on water by rapidly withdrawing its feet before the cavity collapses [5, 21].
- As discussed in section 3.3.1.3, it would be interesting to perform the numerics on the two-dimensional stage 2 law of motion of the turnover point (3.43) to find the precise form of the transition from stage 1 to stage 3.
- It would be interesting to see if any well-posedness theory could be developed for the three-dimensional stage-3 normal impact model (3.189 - 3.192) from (i) the conservation of the modified Richardson moments described in section 3.7.4.2, (ii) the inverse method described in section 3.7.4.4 and (iii) the variational formulation described in section 3.7.4.7.
- Numerical solutions are required for the three-dimensional stage-4 normal impact model (3.242 - 3.245) for non-axisymmetric body profiles. The boundary integral method might be most appropriate using the stage 3 theory to begin the scheme.
- The paradigm zero-gravity free boundary problem (4.15 - 4.22) requires a numerical solution, which would provide evidence for the existence of a critical deadrise angle, above which a ‘humped’ similarity solution exists. As mentioned in section 4.1.1, this is currently being attempted by Vanden-Broeck using the boundary integral method.
- As discussed in section 4.2.1, it would be interesting to see if there was numerical evidence to show that this humped similarity solution evolves into Korobkin’s detached waterspout solution in Figure 4.6.
- As emphasised throughout Chapter 5, nearly all of the proposed models therein require formal justification and further analysis. For example, an analysis of the small time asymptotics of the two-dimensional planing models proposed in section 5.2.2 is required to justify the use of zero initial conditions. Such an analysis would use a combination of the small time asymptotics employed in Chapters 3 and 4.

- It would be interesting to investigate the linear stability of (i) the waterspout root in sections 4.2.1, 5.2.2.2 and 5.3.1.1, (ii) the oblique shallow water entry problems in section 5.2.1 and (iii) the shallow water planing problems in section 5.2.2
- Before attempting a numerical solution of the three-dimensional shallow water planing problems considered in section 5.2.3, the local behaviour at the transition points, where the turnover curve intersects the sharp trailing edge, must be determined.
- It would be interesting to investigate the three-dimensional generalisation of the shallow water impact of a flat plate at zero deadrise considered in section 5.3.

There are also many important extensions that we have not touched on in this thesis. For impacts at small and zero deadrise angle, one effect that is always important in practice is that of the air cushioning layer. Hence a preliminary investigation of the effect of finite depth on the simplest air cushioning model due to [31, 76] is presented in Appendix B.

## Appendix A

# Infinite depth steady planing

### A.1 Green's solution

Green's [24] solution to the Kelvin-Helmholtz cavity flow shown in Figure A.1a is

$$\frac{d\zeta}{dw} = \frac{at - 1 - \sqrt{(1-a^2)(1-t^2)}}{t-a}, \quad (\text{A.1})$$

$$\frac{dw}{dt} = \frac{\delta}{\pi(b-a)} \frac{t-a}{t-b}, \quad (\text{A.2})$$

where  $w$  is the complex potential,  $\zeta = x + iz$  and the plate BD is along the  $x$ -axis. The  $t$ -plane is shown in Figure A.1b. The constants  $a \in [-1, 1]$ ,  $b \in [1, \infty)$  are determined by the angles  $\alpha, \gamma$  that the plate (i.e. the  $x$ -axis) makes with the far-field velocity in the bulk of the fluid and in the jet, respectively, viz.

$$a = \cos \alpha, \quad (\text{A.3})$$

$$b = \frac{1 + \cos \alpha \cos \gamma}{\cos \alpha + \cos \gamma}. \quad (\text{A.4})$$

The jet has asymptotic thickness  $\delta > 0$ , which, by conservation of mass, is equal to the distance between the free surface EFG and the dividing streamline as  $x \rightarrow \infty$ .

Expanding (A.1, A.2) implies the farfield behaviour of the complex potential is

$$w \sim -e^{i\alpha}\zeta - i\Gamma \log \zeta + O(1) \quad \text{as } |\zeta| \rightarrow \infty, \quad (\text{A.5})$$

where

$$\Gamma = \frac{\delta \sin \alpha}{\pi(b-a)} > 0. \quad (\text{A.6})$$

The presence of the line vortex implies the free surfaces AB and FG tend to minus infinity logarithmically in the far field, viz.

$$y \sim -x \tan \alpha - \Gamma \log |x| + O(1) \quad \text{as } |x| \rightarrow \infty. \quad (\text{A.7})$$

This is Green's paradox and we discuss its implications concerning the uniqueness of Green's solution subsequently.

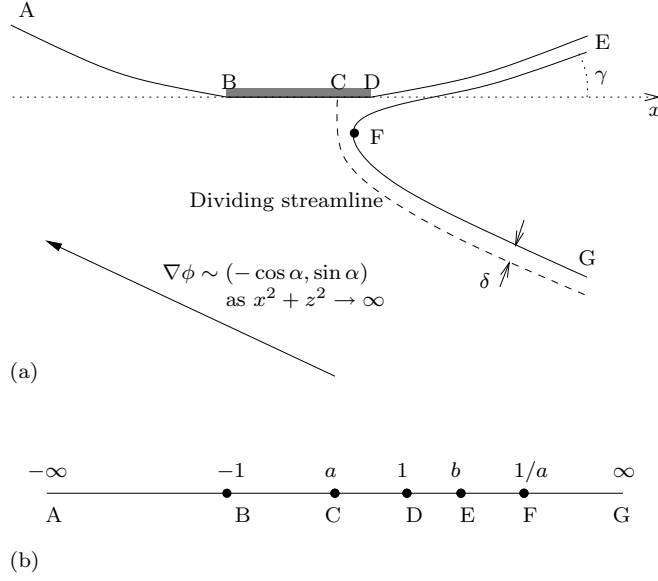


Figure A.1: (a) The  $z$ -plane. We must also specify Laplace's equation  $\nabla^2\phi = 0$  in the fluid, the kinematic condition  $\partial\phi/\partial n = 0$  on the plate and free boundaries and the pressure condition  $|\nabla\phi| = 1$  on the free boundaries. (b) The  $t$ -plane.

Combining (A.1, A.2), Green [24] obtained an expression for  $d\zeta/dt$ , which he integrated along the plate BD to find its parametric equation

$$\begin{aligned} \zeta = \text{constant} + \frac{\delta}{\pi(b-a)} & \left[ at - (1-ab)\log(b-t) \right. \\ & \left. + \sqrt{1-a^2} \left( b\sin^{-1}t - \sqrt{1-t^2} + \sqrt{b^2-1}\sin^{-1}\frac{1-bt}{b-t} \right) \right]. \end{aligned} \quad (\text{A.8})$$

Hence, the length of the plate  $|BD| = z(+1) - z(-1)$  is

$$|BD| = \frac{\delta}{\pi(b-a)} \left( \pi \sin\alpha(b - \sqrt{b^2-1}) + 2a - (1-ab)\log\frac{b-1}{b+1} \right). \quad (\text{A.9})$$

Similarly, the so-called *wetted length*  $|BC| = z(a) - z(-1)$  from the trailing edge B to the stagnation point C is given by

$$\begin{aligned} |BC| = \frac{\delta}{\pi(b-a)} & \left[ 2a^2 + a - 1 - (1-ab)\log\frac{b-a}{b+1} \right. \\ & \left. + \sqrt{1-a^2} \left( b(\pi - \alpha) + \sqrt{b^2-1} \left( \sin^{-1}\frac{1-ab}{b-a} - \frac{\pi}{2} \right) \right) \right]. \end{aligned} \quad (\text{A.10})$$

If we specify the length of the plate  $|BD|$  and its angle of incidence  $\alpha$ , then the jet angle  $\gamma$  is left undetermined, so Green has found a one-parameter family of solutions. This is essentially because Green's paradox implies it is impossible to set the penetration depth of the trailing edge above the undisturbed level of the free surface.

In the special case  $\gamma = 0$ , (A.4) implies  $b = 1$ , so (A.9) implies that the plate BD extends along the  $x$ -axis to positive infinity. It is therefore *not* possible to find a solution for a plate

of finite length with a smoothly separating jet and  $\gamma = 0$ . In this case, we instead work with the wetted length (A.10), which becomes

$$|\text{BC}| = \frac{\delta}{\pi(1-a)} \left[ 2a^2 + a - 1 - (1-a) \log \frac{1-a}{2} + \sqrt{1-a^2}(\pi - \alpha) \right]. \quad (\text{A.11})$$

## A.2 Small angles of incidence

We now investigate the limit  $\alpha \rightarrow 0$  in the cases  $\gamma > 0$  and  $\gamma = 0$ . The first point to note in both cases is that (A.6) implies

$$\Gamma \sim \frac{(1 + \cos \gamma)\delta}{\pi\alpha} + O(\delta\alpha^3) \quad \text{as } \alpha \rightarrow 0, \quad (\text{A.12})$$

so that Green's paradox persists, although, by (A.7), the free surfaces AB and FG lie within  $o(1)$  of the  $x$ -axis for distances from the trailing edge much smaller than  $O(1/\alpha)$  as  $\alpha \rightarrow 0$ . This is in contrast to the linearized solution in section 5.1.2.2, in which the linearization is valid at distances from the trailing edge much smaller than  $O(e^{1/\epsilon})$ , where  $\epsilon \ll 1$  is the small incidence angle.

We now consider the relative locations of the stagnation point C (at  $t = a$ ), the end of the plate D (at  $t = 1$ ) and the turnover point F (at  $t = 1/a$ ).

### Non-parallel jet $\gamma > 0$

If  $\gamma > 0$ , then  $b > 1$  and we obtain the following expansions in the limit  $\alpha \rightarrow 0$ :

$$b \sim 1 + \frac{1}{2} \frac{1 - \cos \gamma}{1 + \cos \gamma} \alpha^2 + O(\alpha^4), \quad (\text{A.13})$$

$$|\text{BD}| \sim \frac{2(1 + \cos \gamma)\delta}{\pi\alpha^2} + \frac{(1 + \cos \gamma)\delta}{\alpha} + O(1), \quad (\text{A.14})$$

$$|\text{CD}| \sim \frac{\delta}{2\pi} [5(1 + \cos \gamma) + 2(\log 2 - \log(1 - \cos \gamma)) \cos \gamma - 2(\pi - \gamma) \sin \gamma] + O(\alpha^2), \quad (\text{A.15})$$

by (A.4, A.9, A.10). Hence, the jet thickness  $\delta$  and the distance between the stagnation point C and the leading edge D are  $O(|\text{BD}|\alpha^2)$ . This also guarantees, by conservation of mass between the dividing streamline and free surface EFG (where the fluid speed is equal to unity), that the turnover point F is a distance  $O(|\text{BD}|\alpha^2)$  from the stagnation point C *and* the end of the plate D. Thus, the free surface EFG turns over in a small jet root region of size  $O(|\text{BD}|\alpha^2)$  fixed at the leading edge of the plate and the thin ejected jet may still asymptote to any angle  $\gamma \in (0, \pi)$ .

### Parallel jet $\gamma = 0$

If  $\gamma = 0$ , then  $b = 1$  and, as shown above, the plate extends along the positive  $x$ -axis to infinity. The wetted length (A.11) becomes

$$|\text{BC}| \sim \frac{4\delta}{\pi\alpha^2} + \frac{2\delta}{\beta} + O(\delta \log \alpha) \quad \text{as } \alpha \rightarrow 0. \quad (\text{A.16})$$

Hence, as in the previous section, the fact the jet thickness is  $O(|\text{BC}|\alpha^2)$  implies the free surface EFG turns over in a small jet root region of size  $O(|\text{BC}|\alpha^2)$  on the body.

### A.3 Comparison with linearized planing theory

A simple calculation shows that for a flat plate, the asymptotic jet thickness predicted by Wagner theory (see (2.77)) and the strength of the line vortex in the far field predicted by linearized planing theory in Section 5.1.2.2 (see (5.21, 5.22)), both agree, to lowest order, with the small- $\alpha$  asymptotics of the exact solution when  $\gamma = 0$  (see (A.5, A.12, A.16)). Moreover, when the plate has finite length,  $d$ , Fridman [14] found the solution to the inner jet root problem in Wagner theory when it lies at the leading end of the plate, from which the free surface is assumed to separate smoothly. In this case, matching with the outer solution in section 5.1.2.2 implies that the asymptotic jet thickness  $\delta \sim \pi d \alpha^2 / 2(1 + \cos \gamma)$  as  $\alpha \rightarrow 0$  for  $\gamma > 0$  (and  $\gamma = 0$ ), where the angle that the jet makes with the plate in the far field,  $\gamma$ , is left undetermined. To lowest order, this is also in agreement with the small- $\alpha$  asymptotics of the exact solution by (A.5, A.12, A.14).

## Appendix B

### An open problem: Air cushioning

The most likely explanation for the discrepancy between the pressure predicted by the infinite depth Wagner theory of Chapter 2 and experiment are conjectured to be due to air entrapment [31, 76], in which a pocket of air is trapped between the free surface and the impactor as depicted in Figure B.1.

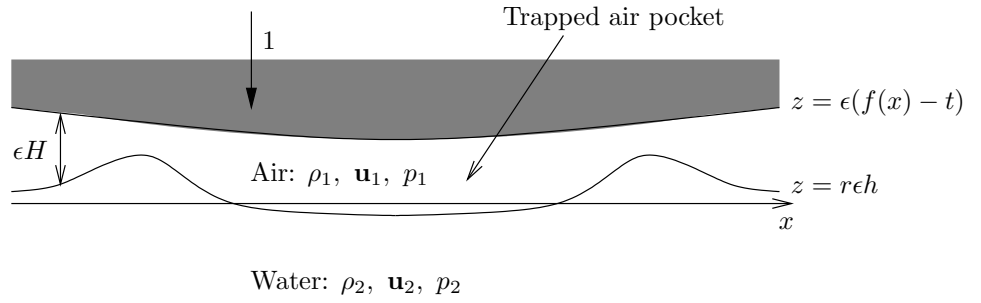


Figure B.1: Air cushion geometry. See text for the terminology.

Howison *et al.* [31] and Wilson [76] modelled the initial formation of the trapped pocket of air in the configuration shown in Figure B.1. At sufficiently small times the Reynolds and Mach numbers in both fluids are large and small, respectively, so the fluids are taken to be inviscid and incompressible. Assuming the inverse aspect ratio of the air layer  $\epsilon$  and the air-water density ratio  $r = \rho_1/\rho_2 = \theta\epsilon$  where  $\theta = O(1)$  as  $\epsilon \rightarrow 0$ , the pressure and free surface elevation must be scaled with  $\rho_1 V^2/\epsilon$  and  $r$ , respectively, to obtain the leading order water and air problems shown in Figure B.2.

The solution to the leading order water problem in Figure B.2(a) is

$$\phi(x, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{\partial h}{\partial t}(\xi, t) \log |(\xi - x)^2 + z^2| d\xi.$$

Hence, the pressure gradient acting on the air layer was found in terms of the Hilbert transform  $\mathcal{H}$  of the free surface acceleration, viz.

$$\frac{\partial p_2}{\partial x}(x, 0, t) = \frac{\partial^2 \phi}{\partial x \partial t}(x, 0, t) = -\mathcal{H} \left( \frac{\partial^2 h}{\partial t^2} \right) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 h}{\partial t^2}(\xi, t) \frac{d\xi}{\xi - x}. \quad (\text{B.1})$$

The solution to the “inviscid squeeze film” air problem in Figure B.2(b) was found by integrating the incompressibility equation,  $u_x + w_z = 0$ , across the air layer of thickness

$$\begin{array}{c}
\frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial t}, p_2 = p_1 \\
\hline
\nabla^2 \phi = 0 \\
\phi \sim 0 \text{ as } x^2 + z^2 \rightarrow \infty \\
\text{(a)} \quad h \sim 0 \text{ as } |x| \rightarrow \infty
\end{array}$$
  

$$\begin{array}{c}
w = f'(x)u - t \text{ on } z = f(x) - t \\
\hline
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial p_1}{\partial x}, \frac{\partial p_1}{\partial z} = 0 \\
\hline
\text{(b)} \quad w = \theta \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \right), p_1 = p_2 \text{ on } z = \theta h
\end{array}$$

Figure B.2: The leading order (a) water and (b) air problems.

$H(x, t) = f(x) - t - \theta h(x, t)$ , to obtain the leading order conservation of mass and  $x$ -momentum equations, viz.

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} (uH) = 0, \quad (\text{B.2})$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial p_1}{\partial x}. \quad (\text{B.3})$$

By the pressure matching condition,  $p_1(x, t) = p_2(x, 0, t)$ , the right-hand side of (B.3) is given by (B.1), viz.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \mathcal{H} \left( \frac{\partial^2 h}{\partial t^2} \right). \quad (\text{B.4})$$

Howison *et al.* [31] and Wilson [76] investigated the air cushioning model (B.2, B.4) analytically in the limits as  $t \rightarrow 0$  and  $\theta \rightarrow 0$  and numerically for various body profiles, which all predicted the formation of a trapped air pocket as depicted in Figure B.1. The model was argued to become invalid in the thin gaps at the ends of the air pocket as the air speed approaches the sound speed, and therefore it cannot be used to predict the size of the entrapped air pocket. To the author's knowledge no satisfactory model for the effect of compressibility exists. Further, no theory is available concerning the well-posedness of the intego-differential equations (B.2, B.4). The crucial question is whether the Kelvin-Helmholtz instability associated with the density difference between the two inviscid fluids is stabilized by the slamming pressure  $p_2(x, 0, t)$  over the timescale of the slam.

We now briefly discuss depth effects, which yields another worrying result concerning the wellposedness of such models.

Suppose we add a base  $z = -\beta$  to the water problem in Figure B.2, where  $\beta = O(1)$  as the inverse aspect ratio of the air layer  $\epsilon \rightarrow 0$ . The kinematic boundary condition  $\partial \phi / \partial z = 0$  on the base  $z = -\beta$  implies the solution (see, for example, [54]) is now given by

$$\frac{\partial \phi}{\partial z}(x, z, t) = \int_{-\infty}^{\infty} -\frac{\partial h}{\partial t}(\xi, t) G(\xi - x, z) d\xi,$$

where the Green's function is

$$G(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx/\beta} \frac{\sinh k(1 + z/\beta)}{\sinh k} dk.$$

Hence, the pressure gradient acting on the air layer becomes

$$\begin{aligned} \frac{\partial p_2}{\partial x}(x, 0, t) &= -\frac{\partial^2 \phi}{\partial x \partial t}(x, 0, t) \\ &= -\int_{-\infty}^{\infty} \frac{\partial^2 h}{\partial t^2}(\xi, t) \frac{1}{2\pi i \beta} \int_{-\infty}^{\infty} e^{-ikx/\beta} \coth k dk d\xi \\ &= -\int_{-\infty}^{\infty} \frac{\partial^2 h}{\partial t^2}(\xi, t) \frac{1}{2\beta} \coth \frac{\pi}{2\beta}(\xi - x) d\xi, \end{aligned} \quad (\text{B.5})$$

where we have used contour integration to evaluate the interior integral in the second line. Note that this agrees with (B.1) in the limit  $\beta \rightarrow \infty$ . It may be possible to proceed with an analytic and numerical analysis of (B.3, B.4, B.5) as in the infinite depth case. However, for our purposes we simply note that in the small depth limit  $\beta \rightarrow 0$ , an analysis along the lines of section 3.4.1 suggests that (B.5) becomes

$$\frac{\partial p_2}{\partial x}(x, 0, t) \sim \frac{1}{\beta} \int_0^x \frac{\partial^2 h}{\partial t^2}(\xi, t) d\xi.$$

Hence, we must scale  $h = \beta\eta$ , so that the air layer thickness becomes  $H = f(x) - t - \beta\theta\eta$  and the shallow-water air cushioning model is (B.2) together with

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \int_0^x \frac{\partial^2 \eta}{\partial t^2}(\xi, t) d\xi. \quad (\text{B.6})$$

Seeking a regular perturbation solution to the shallow-water air cushioning model (B.2, B.6), we find

$$u \sim \frac{x}{1-t}, \quad \eta \sim \eta(x, 0) + \left(2 + \frac{\partial \eta}{\partial t}(x, 0)\right) t + 2 \log(1-t) \quad \text{as } \beta \rightarrow 0,$$

by symmetry. The air velocity  $u$  is the inviscid squeeze film velocity (4.38) because the free surface is effectively a rigid wall on  $z = 0$  to lowest order. The presence of the  $\log(1-t)$  term implies the free surface elevation  $\eta$  tends to minus infinity as  $t \rightarrow 1-$ , which might imply the formation of a trapped air pocket. However, (B.2) implies

$$\int_0^x \frac{\partial H}{\partial t}(\xi, t) d\xi = u(0, t)H(0, t) - u(x, t)H(x, t).$$

By symmetry  $u(0, t) = 0$ , so, using subscripts to denote differentiation, (B.2, B.6) may be written

$$\begin{pmatrix} 0 & 1 \\ \beta\theta + H & u \end{pmatrix} \begin{pmatrix} u \\ H \end{pmatrix}_t + \begin{pmatrix} H & u \\ \beta\theta u & 0 \end{pmatrix} \begin{pmatrix} u \\ H \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (\text{B.7})$$

because  $H_{tt} = -\beta\theta\eta_{tt}$ . The system (B.7) has characteristics

$$\frac{dx}{dt} = \frac{1 \pm \sqrt{-H/\beta\theta}}{u}.$$

Since  $H > 0$ , the shallow-water air cushioning model (B.2, B.6) is elliptic and therefore illposed, because it is unstable to arbitrary small wavelengths. It is not clear if there are any implications concerning the well-posedness of the infinite and finite depth models considered above. We conclude that even the simplest air cushioning model presents considerable mathematical challenges.

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